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An Infinite-Dimensional Generalization of the Jung theorem

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Abstract—A complete characterization of the extremal subsets of Hilbert spaces, which is an infinite-dimensional generalization of the classical Jung theorem, is given. The behavior of the set of points near the Chebyshev sphere of such a subset with respect to the Kuratowski and Hausdorff measures of noncompactness is investigated.

Key words: *Jung theorem, Jung constant, extremal subset of a Hilbert space, Chebyshev sphere, Kuratowski and Hausdorff noncompactness measures.*

1. INTRODUCTION

Let $(X, \|\cdot\|)$ be a Banach space. For a nonempty bounded subset A in X and a nonempty subset B in X, we use the following notation: $d(A) := \sup\{\|x-y\| \colon x, y \in A\}$ for the diameter of A;

$$r_B(A) := \inf_{y \in B} \sup_{x \in A} \|x-y\|$$

is the relative Chebyshev radius of A in B; $r(A) := r_{\overline{\mathrm{co}}A}(A)$, where $\overline{\mathrm{co}}A$ denotes the closed convex hull of A; and

$$C_B(A) := \left\{ y \in B \colon \sup_{x \in A} \|x-y\| = r_B(A) \right\}$$

is the set of Chebyshev centers of A in B.

The *Jung constant* of the space X is defined as

$$J(X) := \sup\{r_X(A) \colon A \subset X,\ d(A) = 1\}.$$

In studying uniformly normal structures, another important geometric constant, the *relative Jung constant* of X, is also considered (see [1]); it is defined as

$$J_s(X) := \sup\{r(A) \colon A \subset X,\ d(A) = 1\}.$$

It is known that if X is a Hilbert space, then $C_X(A)$ consists of only one point, which belongs to the closed convex hull $\overline{\mathrm{co}}A$ of A (see [2]). Therefore, in this case, we have $J(X) = J_s(X)$. The classical Jung theorem asserts that

$$J(E^n) = J_s(E^n) = \sqrt{\frac{n}{2(n+1)}}$$

for Euclidean n-space $X = E^n$ [3], [4]. The Jung constant $J(H)$ of a Hilbert space (in the infinite-dimensional case) was calculated in [5] (cf. [6], [1], [7]):

$$J(H) = J_s(H) = \frac{1}{\sqrt{2}}.$$

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Definition. We say that a bounded subset A of X consisting of more than one point is *extremal* (*relatively extremal*) if $r_X(A) = J(X)d(A)$ (respectively, $r(A) = J_s(X)d(A)$).

Note that if X is a Hilbert space, then these two notions coincide; thus, in this case, we consider only extremal subsets. It follows from the second part of the Jung theorem cited above that a bounded subset A in E^n is extremal if and only if A contains a regular n-simplex with edges of length $d(A)$. In the infinite-dimensional case ($X = H$), Gulevich [8] obtained the following partial result: If A is a relatively compact subset in a Hilbert space and $d(A) > 0$, then

$$r(A) < \frac{1}{\sqrt{2}} d(A).$$

In other words, an extremal subset of a Hilbert space cannot be relatively compact.

The objective of this paper is to completely characterize the extremal subsets of Hilbert spaces, that is, to obtain an infinite-dimensional generalization of the second part of the classical Jung theorem.

Main theorem. *If A is an extremal subset in a Hilbert space H and $d(A) = \sqrt{2}$, then $\chi(A) = 1$. Moreover, for each $\varepsilon \in (0, \sqrt{2})$ and any positive integer p, there exists a p-simplex $\Delta(\varepsilon, p)$ with vertices in A and edges of length at least $\sqrt{2} - \varepsilon$.*

Conversely, if $d(A) = \sqrt{2}$ and, for each $\varepsilon \in (0, \sqrt{2})$ and any positive integer p, there exists a p-simplex $\Delta(\varepsilon, p)$ with vertices in A and edges of length at least $\sqrt{2} - \varepsilon$, then A is an extremal subset.

In the statement of the main theorem, $\chi(A)$ denotes the Hausdorff noncompactness measure of A, i.e., the greatest lower bound of positive numbers r for which A can be covered by finitely many balls of radius r centered on H. Using an observation from [7] (the mushroom lemma), we also obtain a result concerning the set of points near the Chebyshev sphere of an extremal subset with respect to the noncompactness measure. This result shows that the main contribution to the noncompactness measure is made by these points of the extremal subset.

2. NONCOMPACTNESS MEASURES OF EXTREMAL SUBSETS

Theorem 1. *If A in an extremal subset in a Hilbert space H and $r(A) = 1$, then $\alpha(A) = \sqrt{2}$.*

Here $\alpha(A)$ denotes the *Kuratowski measure* of *noncompactness*, which is defined as the greatest lower bound of the positive numbers d for which A can be covered by finitely many subsets of diameter d.

First proof of Theorem 1. The condition $r(A) = 1$ implies

$$\bigcap_{x \in A} B\left(x, 1 - \frac{1}{n}\right) = \varnothing$$

for any integer $n \geq 2$; here $B(x, r)$ denotes the closed ball of radius r centered at x, which is weakly compact, because H is reflexive. Therefore, there exist points

$$x_{p_{n-1}+1}, \quad x_{p_{n-1}+2}, \quad \dots, \quad x_{p_n}$$

in A for which

$$\bigcap_{i=p_{n-1}+1}^{p_n} B\left(x_i, 1 - \frac{1}{n}\right) = \varnothing$$

(we assume that $p_1 = 0$).

We set

$$A_n := \{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\},$$

take a Chebyshev center c_n of each A_n in H, and let $r_n := r(A_n)$; then $r_n > 1 - 1/n$.

Let $S(c,r)$ denote the sphere of radius r centered at c. We know from the proof of the classical Jung theorem that $A \cap S(c_n, r_n) \neq \varnothing$ and $c_n \in \operatorname{co}(A_n \cap S(c_n, r_n))$. Thus, there exist $y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}$ in $A_n \cap S(c_n, r_n)$ (where $q_1 = 0$) and positive numbers

$$t_{q_{n-1}+1}, t_{q_{n-1}+2}, \dots, t_{q_n}$$

such that

$$c_n = \sum_{q_{n-1}<i\le q_n} t_i y_i, \qquad \sum_{q_{n-1}<i\le q_n} t_i = 1.$$

We claim that

$$\alpha\big(\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty}\big) = \sqrt{2}.$$

Suppose that, on the contrary,

$$\alpha\big(\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty}\big) < \sqrt{2}.$$

Then we can choose an $\varepsilon_0 \in (0, \sqrt{2}\,)$ so that

$$\alpha\big(\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty}\big) \le \sqrt{2} - \varepsilon_0;$$

hence there exist sets $D_1, D_2, \dots, D_m$ in H for which

$$d(D_i) \le \sqrt{2} - \varepsilon_0, \qquad i = 1, 2, \dots, m,$$

and

$$\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty} \subset \bigcup_{i=1}^{m} D_i.$$

For at least one set among $D_1, D_2, \dots, D_m$ (say, D_1), there exist infinitely many n such that

$$\sum_{i\in J_n} t_i \ge \frac{1}{m}, \qquad \text{where} \quad J_n := \{i \in [q_{n-1}+1, q_n] \colon y_i \in D_1\}. \tag{1}$$

For every n satisfying (1) and any $j \in J_n$, we have

$$\begin{aligned}
\sum_{q_{n-1}<i\le q_n} t_i \|y_i - y_j\|^2 &= \sum_{q_{n-1}<i\le q_n} t_i \|y_i - c_n + c_n - y_j\|^2 \\
&= \sum_{q_{n-1}<i\le q_n} t_i \big(\|y_i - c_n\|^2 + \|y_j - c_n\|^2 - 2(y_i - c_n, y_j - c_n)\big) \\
&= 2r_n^2 - 2\bigg(\sum_{q_{n-1}<i\le q_n} t_i y_i - c_n, y_j - c_n\bigg) \\
&= 2r_n^2 > 2\bigg(1 - \frac{1}{n}\bigg)^2 > 2 - \frac{4}{n},
\end{aligned}$$

where $(\,\cdot\,,\,\cdot\,)$ denotes the inner product in H.

On the other hand, we have

$$\begin{aligned}
\sum_{q_{n-1}<i\le q_n} t_i \|y_i - y_j\|^2 &= \sum_{i\in J_n} t_i \|y_i - y_j\|^2 + \sum_{q_{n-1}<i\le q_n, i\notin J_n} t_i \|y_i - y_j\|^2 \\
&\le (\sqrt{2} - \varepsilon_0)^2 \sum_{i\in J_n} t_i + 2\bigg(1 - \sum_{i\in J_n} t_i\bigg) \\
&= 2 - [2 - (2 - \varepsilon_0)^2]\bigg(\sum_{i\in J_n} t_i\bigg) \le 2 - [2 - (2 - \varepsilon_0)^2]\frac{1}{m}.
\end{aligned}$$

Hence

$$2-[2-(2-\varepsilon_0)^2]\frac{1}{m}>2-\frac{4}{n}$$

for the given numbers ε_0 and m and for all n satisfying (1). This contradiction shows that $\alpha(\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty}) = \sqrt{2}$. Since $d(A)=\sqrt{2}$, it follows that $\alpha(A)=\sqrt{2}$.

The second proof uses the following lemma, which is a modification of Lemma 4 (the mushroom lemma) from [7].

Lemma 2. *Let A be a nonempty bounded subset of the Hilbert space H, and let r and c denote the Chebyshev radius of A with respect to H and a Chebyshev center of A in H, respectively. Then $c \in \overline{\mathrm{co}}\, A_\varepsilon$ and $r = r(A_\varepsilon)$ for any $\varepsilon \in (0,r)$, where $A_\varepsilon := A \setminus B(c, r-\varepsilon)$.*

Proof of Lemma 2. Suppose that, on the contrary, c is not a Chebyshev center of A_ε in H; then $r_1 := r(A_\varepsilon) < r$. Choose a Chebyshev center c_1 of A_ε in H and let $c' = \alpha c_1 + (1-\alpha)c$, where $\alpha \in (0,1)$ and $0 < \|c-c'\| < \varepsilon$.

Take a point $x \in A$. If $x \in A_\varepsilon$, then

$$\|x-c'\| \le \alpha\|x-c_1\| + (1-\alpha)\|x-c\| \le \alpha r_1 + (1-\alpha)r < r.$$

If $x \in A \setminus A_\varepsilon$, then

$$\|x-c'\| \le \|x-c\| + \|c-c'\| < r-\varepsilon+\|c-c'\| < r.$$

In any case, we have $A \subset B(c',r')$ for $r' < r$. This contradiction completes the proof of the lemma.

Second proof of Theorem 1. Using Lemma 2 with $\varepsilon = 1/n$ for each integer $n \ge 2$, we obtain $c \in \overline{\mathrm{co}}(A \setminus B(c, 1-1/n))$. Hence there exist points $x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}$ in $A \setminus B(c, 1-1/n)$ and positive numbers

$$t_{p_{n-1}+1}, \quad t_{p_{n-1}+2}, \quad \dots, \quad t_{p_n}$$

(where $p_1 = 0$) for which

$$\sum_{p_{n-1}<i\le p_n} t_i = 1, \qquad \Bigl\|\sum_{p_{n-1}<i\le p_n} t_i x_i - c\Bigr\| < \frac{1}{n}.$$

Let us show that

$$\alpha\bigl(\{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\}_{n=2}^{\infty}\bigr) = \sqrt{2}.$$

Again we argue by contradiction. Suppose that

$$\alpha\bigl(\{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\}_{n=2}^{\infty}\bigr) < \sqrt{2}.$$

Then there exists an $\varepsilon_0 \in (0,\sqrt{2})$ for which

$$\alpha\bigl(\{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\}_{n=2}^{\infty}\bigr) \le \sqrt{2}-\varepsilon_0;$$

hence we can find subsets $D_1, D_2, \dots, D_m$ in H such that $d(D_i) \le \sqrt{2}-\varepsilon_0$ for $i = 1,2,\dots,m$ and

$$\{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\}_{n=2}^{\infty} \subset \bigcup_{i=1}^{m} D_i.$$

As in the first proof, there is a set (say, D_1) among $D_1, D_2, \dots, D_m$ such that

$$\sum_{i\in I_n} t_i \ge \frac{1}{m} \tag{2}$$

for infinitely many n; here

$$I_n := \{i \in [p_{n-1}+1, p_n] : x_i \in D_1\}.$$

Similarly, for each n satisfying (2) and any $j \in I_n$, we have

$$\begin{aligned}
\sum_{p_{n-1}<i\le p_n} t_i\|x_i - x_j\|^2 &= \sum_{p_{n-1}<i\le p_n} t_i\|x_i - c + c - x_j\|^2 \\
&= \sum_{p_{n-1}<i\le p_n} t_i\big(\|x_i - c\|^2 + \|x_j - c\|^2 - 2(x_i - c, x_j - c)\big) \\
&> 2\Big(1 - \frac{1}{n}\Big)^2 - 2\Big(\sum_{p_{n-1}<i\le p_n} t_i x_i - c, x_j - c\Big) \\
&\ge \Big(1 - \frac{1}{n}\Big)^2 - 2\frac{1}{n} > 2 - \frac{6}{n};
\end{aligned}$$

moreover,

$$\sum_{p_{n-1}<i\le p_n} t_i\|x_i - x_j\|^2 \le 2 - [2 - (\sqrt{2} - \varepsilon_0)^2]\frac{1}{m}.$$

Therefore,

$$2 - [2 - (2 - \varepsilon_0)^2]\frac{1}{m} > 2 - \frac{6}{n}$$

for all n satisfying (2). This contradiction shows that

$$\alpha\big(\{x_{p_{n-1}+1}, x_{p_{n-1}+2}, \dots, x_{p_n}\}_{n=2}^{\infty}\big) = \sqrt{2},$$

which implies $\alpha(A) = \sqrt{2}$.

An immediate corollary of Theorem 1 is Gulevich's result mentioned in the Introduction.

Corollary [8]**.** *If A is a relatively compact subset of a Hilbert space H and $d(A) > 0$, then the inequality $r(A) < (1/\sqrt{2})d(A)$ holds.*

Remark 1. In [7], another proof of the equality $J(H) = J_s(H) = 1/\sqrt{2}$ was suggested, which is essentially due to Steinlein. It is somewhat deeper; namely, it is based on the interesting inequality $\varkappa_0(X) \le (J_s(X))^{-1}$, where $\varkappa_0(X)$ is the Lifshits characteristic and $J_s(X)$ is the relative Jung constant of the Banach space X (cf. [9]). We shall return to this inequality elsewhere; in this paper, we only mention that it can be generalized to metric spaces with convexity structures.

Remark 2. It follows from Lemma 2 that A_ε is an extremal subset as well, and $\alpha(A_\varepsilon) = \sqrt{2}$ for $\varepsilon \in (0, 1)$.

Remark 3. Although $\alpha(A_\varepsilon) = \sqrt{2}$ for $\varepsilon \in (0, 1)$, it may happen that $\overline{\mathrm{co}}A \cap S(c, 1) = \varnothing$ (cf. [7]). The following question arises: What can be said about $\alpha(A \cap S(c, 1))$ provided that $A \cap S(c, 1) \ne \varnothing$? The answer is: $\alpha(A \cap S(c, 1))$ can take any values in $[0, \sqrt{2}\,]$. Below, we give some examples.

Example 1. Let $\{e_n\}_{n=1}^{\infty}$ be an infinite sequence of orthonormal vectors in a Hilbert space H. We set

$$A_1 := \bigg\{\Big(1 - \frac{1}{n}\Big)e_n\bigg\}_{n=1}^{\infty}$$

and $A_2 := \{x_1, x_2, \dots, x_n, \dots\}$, where

$$x_1 := \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2, \qquad x_2 := \frac{1}{\sqrt{2}}e_1 + \frac{1}{2}e_2 + \frac{1}{2}e_3,$$
$$x_n := \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2^2}}e_2 + \dots + \frac{1}{\sqrt{2^n}}e_n + \frac{1}{\sqrt{2^n}}e_{n+1}, \quad \dots .$$

It is easy to see that $r(A_1) = 1$, $d(A_1) = \sqrt{2}$, and 0 is a Chebyshev center of the subset A_1 in H. Moreover, $\|x_n\| = 1$ for each n,

$$\left\| x_m - \left(1 - \frac{1}{n}\right) e_n \right\| \le \sqrt{2} \qquad \text{for any } m \text{ and } n,$$

and

$$\|x_{n+p} - x_n\|^2 = \frac{1}{2^n} \to 0 \qquad \text{as} \quad n \to \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence, and $\alpha(A_2) = 0$.

For the set $A := A_1 \cup A_2$, we have $r(A) = 1$ and $d(A) = \sqrt{2}$, and 0 is a Chebyshev center of A in H. Obviously,

$$A \cap S(0,1) = \overline{\mathrm{co}}\, A \cap S(0,1) = A_2.$$

Example 2. Suppose that $\{e_n\}_{n=1}^{\infty}$ and A_1 are the same as in Example 1. For each $\gamma \in (0, \sqrt{2}]$ such that $\beta := \gamma/\sqrt{2} \in (0,1]$, we choose $\lambda \in [0,1)$ so that $\lambda^2 + \beta^2 = 1$. Let $A_2 := \{y_1, y_2, \dots, y_n, \dots\}$, where

$$y_1 := \lambda e_1 + \beta e_2, \qquad y_2 := \lambda e_1 + \beta e_3, \quad \dots, \quad y_n := \lambda e_1 + \beta e_{n+1}, \quad \dots .$$

Obviously, $\|y_n\| = 1$ for each n and $\|y_n - y_m\| = \sqrt{2}\beta = \gamma$ for all $m \ne n$. For $A := A_1 \cap A_2$, we have $r(A) = 1$ and $d(A) = \sqrt{2}$; moreover, 0 is a Chebyshev center of A in H, and we have

$$A \cap S(0,1) = \overline{\mathrm{co}} A \cap S(0,1) = A_2,$$

as well as $\alpha(A_2) = \gamma$.

3. PROOF OF THE MAIN THEOREM

In the first proof of Theorem 1, we have defined a sequence $y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}$ (where $q_1 = 0$) in $A_n \cap S(c_n, r_n)$ and positive numbers $t_{q_{n-1}+1}, t_{q_{n-1}+2}, \dots, t_{q_n}$ for all $n \ge 2$ satisfying the conditions

$$c_n = \sum_{q_{n-1} < i \le q_n} t_i y_i, \qquad \sum_{q_{n-1} < i \le q_n} t_i = 1.$$

We claim that

$$\chi\big(\{y_{q_{n-1}+1}, y_{q_{n-1}+2}, \dots, y_{q_n}\}_{n=2}^{\infty}\big) = 1.$$

To prove this, suppose that A can be covered by finitely many balls $B_1, B_2, \dots, B_m$ of radius r. Then there is a ball (say, B_1) among $B_1, B_2, \dots, B_m$ such that, for infinitely many n,

$$\sum_{i \in J_n} t_i \ge \frac{1}{m}, \tag{3}$$

where

$$J_n := \{i \in [q_{n-1}+1, q_n] \colon y_i \in B_1\}.$$

As in the proof of Theorem 1, we have

$$\sum_{q_{n-1} < i \le q_n} t_i \|y_i - y_j\|^2 = 2r_n^2 > 2 - \frac{4}{n} \tag{4}$$

for each $j \in [q_{n-1}+1, q_n]$.

It follows from (4) that

$$\sum_{i \in I_{nj}} t_i < \frac{1}{\sqrt{n}}, \tag{5}$$

where

$$I_{nj} := \left\{ i \in [q_{n-1}+1, q_n] \colon \|y_i - y_j\|^2 < 2 - \frac{4}{\sqrt{n}} \right\}, \qquad j \in [q_{n-1}+1, q_n],$$

$$2(1-t_j) \ge \sum_{q_{n-1}<i\le q_n} t_i \|y_i - y_j\|^2 = 2r_n^2 > 2 - \frac{4}{n}.$$

This implies $t_j < 2/n$ for each $j \in [q_{n-1}+1, q_n]$. Therefore, if n satisfies (3), then

$$|J_n|(2/n) > \frac{1}{m},$$

or, equivalently,

$$|J_n| > \frac{n}{2m}$$

(here $|J_n|$ denotes the cardinality of J_n).

For each n satisfying (3) and any $j \in J_n$, we set

$$J_n(y_j) := \left\{ i \in J_n \colon \|y_i - y_j\|^2 \ge 2 - \frac{4}{\sqrt{n}} \right\},$$

$$\hat{J}_n(y_j) := \{ y_i \colon i \in J_n(y_j) \}.$$

Obviously, (5) implies

$$\sum_{i \in J_n \setminus J_n(y_j)} t_i < \frac{1}{\sqrt{n}}, \tag{6}$$

$$\sum_{i \in J_n(y_j)} t_i > \frac{1}{m} - \frac{1}{\sqrt{n}}. \tag{7}$$

For every positive integer p, we choose n satisfying (3) and so large that $(p+1)/\sqrt{n} \le 1/\sqrt{m}$. We claim that

$$\bigcap_{k=1}^{p} J_n(y_{i_k}) \ne \varnothing \tag{8}$$

for any $i_1, i_2, \dots, i_p \in J_n$. Indeed, otherwise, the relation $\bigcap_{k=1}^{p} J_n(y_{i_k}) = \varnothing$ would imply

$$J_n(y_{i_1}) \subset J_n \setminus \left(\bigcap_{k=2}^{p} J_n(y_{i_k}) \right) = \bigcup_{k=2}^{p} (J_n \setminus J_n(y_{i_k})).$$

By virtue of (6) and (7), we would have

$$\frac{1}{m} - \frac{1}{\sqrt{n}} < \sum_{\alpha \in J_n(y_{i_1})} t_\alpha \le \sum_{k=2}^{p} \sum_{\alpha \in J_n \setminus J_n(y_{i_k})} t_\alpha < (p-1)\frac{1}{\sqrt{n}}$$

and $1/m < p/\sqrt{n}$, which would contradict the choice of n and p.

It follows from (8) that if $1 \le k \le p$ and $i_1, i_2, \dots, i_k \in J_n$, then

$$\bigcap_{\alpha=1}^{k} \hat{J}_n(y_{i_\alpha}) \ne \varnothing.$$

Take $j \in J_n$ for the same n and p as above. Setting $z_1 := y_j$, we successively define

$$z_2 \in \hat{J}_n(z_1), \qquad z_3 \in \hat{J}_n(z_1) \cap \hat{J}_n(z_2), \quad \dots, \quad z_{p+1} \in \bigcap_{i=1}^{p} \hat{J}_n(z_i).$$

Obviously,

$$\|z_i - z_j\|^2 \ge 2 - \frac{4}{\sqrt{n}}$$

for all $i \ne j$ from $\{1, 2, \dots, p+1\}$. Now, given $\varepsilon \in (0, \sqrt{2})$, we choose n as above so large that

$$2 - \frac{4}{\sqrt{n}} \ge (\sqrt{2} - \varepsilon)^2.$$

We see that the points $z_1, z_2, \dots, z_{p+1}$ form a p-simplex $\Delta(\varepsilon, p)$ with edges of length at least $\sqrt{2}-\varepsilon$.

Now, let us show that $r \ge 1$ (r is the radius of the balls $B_1, B_2, \dots, B_m$). Let c' and r' denote a Chebyshev center of the simplex $\Delta(\varepsilon, p)$ in H and the Chebyshev radius of this simplex with respect to H, respectively. The proof of the classical Jung theorem implies the existence of nonnegative numbers $\alpha_1, \alpha_2, \dots, \alpha_{p+1}$ for which

$$\sum_{i=1}^{p+1} \alpha_i = 1 \qquad \text{and} \qquad c' = \sum_{i=1}^{p+1} \alpha_i z_i.$$

For every $j \in \{1, 2, \dots, p+1\}$, we have

$$\begin{aligned}\Big(2 - \frac{4}{\sqrt{n}}\Big)(1-\alpha_j) &\le \sum_{i=1}^{p+1} \alpha_i \|z_i - z_j\|^2 = \sum_{i=1}^{p+1} \alpha_i \|z_i - c' + c' - z_j\|^2 \\ &= \sum_{i=1}^{p+1} \alpha_i \big(\|z_i - c'\|^2 + \|z_j - c'\|^2\big) - \Big(\sum_{i=1}^{p+1} \alpha_i (z_i - c'), z_j - c'\Big) \le 2(p')^2.\end{aligned}$$

Thus,

$$\Big(2 - \frac{4}{\sqrt{n}}\Big) \sum_{j=1}^{p+1} (1-\alpha_j) \le 2(p+1)(r')^2,$$

or, equivalently,

$$r' \ge \sqrt{\frac{(2 - 4/\sqrt{n})p}{2(p+1)}}. \tag{9}$$

The right-hand side of (9) tends to 1 as $p \to \infty$. Obviously, $r \ge r'$, because $\Delta(\varepsilon, p) \subset B_1$. This implies $r \ge 1$, as required. Therefore, $\chi(A) = 1$.

Conversely, if $d(A) = \sqrt{2}$ and A contains a p-simplex $\Delta(\varepsilon, p)$ with edge lengths at least $\ge \sqrt{2}-\varepsilon$ for each $\varepsilon \in (0, \sqrt{2})$ and any positive integer p, then A is an extremal subset by definition.

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