SPARKCHARTS™

CALCULUS I

BACKGROUND AND FUNCTIONS

WHAT IS CALCULUS?

Calculus is the study of "nice"—smoothly changing—functions.

- **Differential calculus** studies how quickly functions are changing at particular points.
- **Integral calculus** studies areas enclosed by curves.
- The **Fundamental Theorem of Calculus** connects the two.

FUNCTIONS

WHAT IS A FUNCTION?

A **function** is a rule for churning out values: for every value you plug in, there's a unique value that comes out.

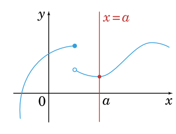
- The set of all the values that can be plugged in is the **domain.**
- The set of all the values that can be output is the **range.**

WRITING FUNCTIONS DOWN

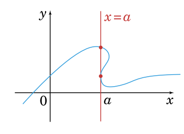
A **table** is a list that keeps track of input values (such as ages of a cactus) and corresponding output values (such as the number of needles on the cactus) of a function. There may not be a universal equation that describes such a function.

An **equation** such as $f(x) = x^2 + 1$ describes how to numerically manipulate the **independent variable** (often x) to get the output value $f(x)$. If $y = x^2 + 1$, then y is a "function of x," and it is the **dependent variable.**

A **graph** represents a function visually. If $y = f(x)$, then plotting many points $(x, f(x))$ on the plane will give a picture of the function. Usually, the independent variable is represented horizontally, and the dependent variable vertically. A graph represents a function as long as it passes the **vertical line test:** for every x-value, there is at most one y-value.



Function



Not a function

TYPES OF FUNCTIONS

For more on functions, see the SparkChart on Pre-Calculus.

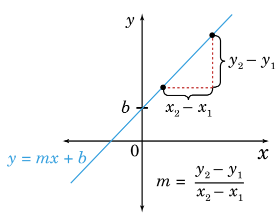
CONSTANT FUNCTIONS—Horizontal lines

A **constant function** $y = c$ has only one output value. Its graph is a horizontal line at height c.

LINEAR FUNCTIONS—Straight lines

A **linear function** can be expressed in the easy-to-graph slope-intercept form $y = mx + b$, where b is the **y-intercept** (the value of $f(0)$) and m is the slope. The slope of a straight line measures how steep it is; if (x_1, y_1) and (x_2, y_2) are two points on the line, then the slope is

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}.$$

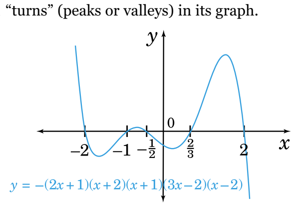


POLYNOMIAL FUNCTIONS

A general **polynomial** function can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

The polynomial has **degree** n if $a_n \neq 0$. All polynomials are defined for all real numbers. If n is odd, then the range is all real numbers, too. If n is even, then the polynomial reaches some maximum if $a_n < 0$ or some minimum if $a_n > 0$. The y-intercept is the constant term a_0. A polynomial of degree n has at most n **roots** or **zeroes**—values of x where the graph crosses the x-axis—and at most $n - 1$ "turns" (peaks or valleys) in its graph.



Polynomial of degree 5
Roots: $-2, -1, -\frac{1}{2}, \frac{2}{3}, 2$
4 "turns"

SPECIAL CASES OF POLYNOMIALS—Lines and parabolas

A polynomial of degree 0 is the constant function $f(x) = a$ for $a \neq 0$.

A polynomial of degree 1 is a linear function.

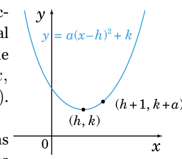
A polynomial of degree 2 is a **quadratic**; it can be written in the form $f(x) = ax^2 + bx + c$. It has 0, 1, or 2 roots, which are found using the **Quadratic Formula:**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- If $\sqrt{b^2 - 4ac} < 0$, then the quadratic has no real roots.
- If $\sqrt{b^2 - 4ac} = 0$, then the quadratic has 1 real root.
- If $\sqrt{b^2 - 4ac} > 0$, then the quadratic has 2 real roots.

The graph of a quadratic function is a **parabola.** The general quadratic can be expressed in the form $f(x) = a(x - h)^2 + k$, where $h = -\frac{b}{2a}$ and $k = f(h)$.

- The **vertex** is at (h, k).
- If $a > 0$, the parabola opens up. If $a < 0$, the parabola opens down.
- If $|a|$ is large, the parabola is narrow. If $|a|$ is small, the parabola is wide.



Parabola. Here $a > 0$.

RATIONAL FUNCTIONS

A **rational function** is a quotient of two polynomials: $f(x) = \frac{p(x)}{q(x)}$. It is defined everywhere except at the roots of $q(x)$. The zeroes of $f(x)$ are those roots of $p(x)$ that are not also roots of $q(x)$.

- If the degree of $p(x)$ is greater than the degree of $q(x)$ $(\deg p(x) \geq \deg q(x))$, then at points where $|x|$ is very large, $f(x)$ will behave like a polynomial of degree $\deg p(x) - \deg q(x)$.
- If $\deg p(x) < \deg q(x)$, then when $|x|$ is very large, $f(x)$ will approach 0. See *Limits and Continuity.*

EXPONENTIAL AND LOGARITHMIC FUNCTIONS—Very fast or very slow growth

Simple **exponential functions** can be written in the form $y = a^x$, where the **base** a is positive (and $a \neq 1$). The function is always increasing if $a > 1$ and always decreasing if $a < 1$. The domain is all the reals; the range is the positive reals. Exponential functions grow extremely fast—faster than any polynomial. The basic shape of the graph is always the same, no matter the value of a.

Logarithmic functions have the form $y = \log_a x$. The number $\log_a b$ is "the power to which you raise a to get b":

$$\log_a x = y \text{ if and only if } a^y = x.$$

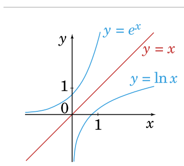
REMEMBER: Logarithms are exponents.

Logarithmic functions are defined for the positive reals only; they have the same basic shape as exponential functions but a different orientation.

Changing the base of a logarithm is the same thing as mulplying the logarithm by a constant:

$$\log_a x = (\log_a b)(\log_b x).$$

Also, $\log_a b = \frac{1}{\log_b a}$.



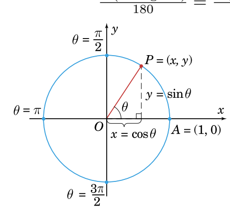
The number e is a special real number (approximately 2.71828), often used as a base for exponential functions. The logarithm base e is called the **natural logarithm** and is written $\log_e x = \ln x$. The natural log follows all logarithm rules. Any logarithmic expression can be written in terms of natural logarithms using the change of base formula:

$$\log_a x = \frac{\ln x}{\ln a}.$$

RULES FOR EXPONENTS:	RULES FOR LOGARITHMS:
$a^{\log_a b} = b$	$\log_a a^n = n$
$a^{m+n} = a^m a^n$	$\log_a(bc) = \log_a b + \log_a c$
$a^{m-n} = \frac{a^m}{a^n}$	$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$
$a^{-n} = \frac{1}{a^n}$	$\log_a \frac{1}{b} = -\log_a b$
$a^{mn} = (a^m)^n$	$\log_a b^n = n \log_a b$
$a^{\frac{1}{n}} = \sqrt[n]{a}$	$\log_a \sqrt[n]{b} = \frac{1}{n}\log_a b$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$	$\log_a \sqrt[n]{b^m} = \frac{m}{n}\log_a b$
$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$	
$(ab)^n = a^n b^n$	
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	

TRIGONOMETRIC FUNCTIONS—Sinusoidal waves

- **Angle measure:** Angles can be measured in **degrees** or in **radians:** $\frac{\theta \text{ (in degrees)}}{180} = \frac{\theta \text{ (in radians)}}{\pi}$.



- The **unit circle** is the circle of radius 1 centered at the origin $O = (0, 0)$.
- If $A = (1, 0)$, then any point $P = (x, y)$ on the circle defines the angle $\angle AOP$ of measure θ (counterclockwise). Since θ and $\theta + 2\pi$ define the same angle, all trig functions satisfy $f(x) = f(x + 2\pi)$; they are **periodic** with period 2π (or π).

Sine: $\sin\theta = y$, the y-coordinate of $P = (x, y)$. For all θ, $-1 \leq \sin\theta \leq 1$. Sine is an odd function.

Cosine: $\cos\theta = x$, the x-coordinate of $P = (x, y)$. For all θ, $-1 \leq \cos\theta \leq 1$. Cosine is an even function.

Tangent: $\tan\theta = \frac{y}{x}$, the slope of the line $\overline{OP}$.

Secant: $\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x}$

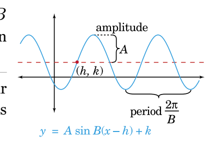
Cosecant: $\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y}$

Cotangent: $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta} = \frac{x}{y}$

Sinusoidal functions can be written in the form

$$y = A \sin B(x - h) + k.$$

- $|A|$ is the **amplitude.**
- k is the **average value:** halfway between the maximum and the minimum value of the function.
- $\frac{2\pi}{B}$ is the **period.** A larger B means more cycles in a given interval.
- h is **phase shift**, or how far the beginning of the cycle is from the y-axis.



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“BUT HOW IS ONE TO MAKE A SCIENTIST UNDERSTAND THAT THERE IS SOMETHING UNALTERABLY DERANGED ABOUT DIFFERENTIAL CALCULUS....”

ANTONIN ARTAUD

TYPES OF FUNCTIONS (CONTINUED)

TRIGONOMETRIC IDENTITIES

Sum and difference formulas

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Double-angle formulas

$\sin(2A) = 2 \sin A \cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$

$\quad = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Half-angle formulas

$$\sin \frac{A}{2} = \pm\sqrt{\frac{1-\cos A}{2}} \qquad \cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\tan \frac{A}{2} = \frac{1-\cos A}{\sin A}$$

Squared function formulas

$$\sin^2 A = \frac{1-\cos 2A}{2} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$

Pythagorean Identities

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A$

$1 + \cot^2 A = \csc^2 A$

Special trigonometric values

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	$\frac{\sqrt{0}}{2} = 0$	1	0
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = 1$	0	undefined

COMBINING FUNCTIONS

- Two functions can be combined arithmetically—**added**, **subtracted**, **multiplied**, or **divided**. The domain of the new function includes only points that are in the domain of both original functions. (A quotient function will always be undefined at the zeroes of the denominator.)
- Two functions can be **composed**: $f(g(x))$ is a function if the domain of $f(x)$ contains the range of $g(x)$. This is sometimes denoted by $(f \circ g)(x)$.

NOTE: In $(f \circ g)$, g is applied first and written second.

Ex: If $f(x) = x + 2$ and $g(x) = 4x$, then $(f \circ g)(x) = 4x + 2$.

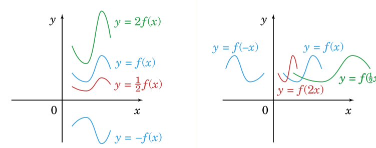
CHANGING A SINGLE FUNCTION

1. Vertical translation: Adding a constant c will translate the function vertically c units (up if c is positive, down if c is negative). The new function $y = f(x) + c$ has the same shape and the same domain as the original function.

2. Horizontal translation: The function $y = f(x - c)$ is a shift of the original function c units horizontally (to the right if c is positive, left if c is negative). The new function has the same shape and the same range as the original function.

3. Vertical stretching and **compressing:** If $c > 1$, then the function $y = cf(x)$ is a vertical stretch of the original function by a factor of c. If $c < 1$, then $y = cf(x)$is a compression of the original function by a factor of c. The horizontal distances remain unchanged.

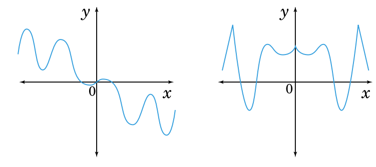
4. Horizontal stretching and **compressing:** The function $y = f\left(\frac{x}{c}\right)$ is a horizontal stretch of the original function if $c > 1$ (a compression if $c < 1$) by a factor of c. Vertical distances remain the same.



Vertical Stretch   Horizontal Stretch

5. Reflection over the axes: The function $y = -f(x)$ is a reflection of the original function over the y-axis. The function $y = f(-x)$ is a reflection of the original function over the x-axis.

- If $f(x) = f(-x)$, then $f(x)$ is called **even;** it remains unchanged when reflected over the x-axis.
Ex: $\cos x$ is an even function.
- If $f(x) = -f(-x)$, then $f(x)$ is called **odd.** A reflection over the x-axis is the same as a reflection over the y-axis. Equivalently, rotating $f(x)$ $180°$ around the origin leaves $f(x)$ unchanged.
Ex: $\sin x$ is an odd function.



Odd Function   Even Function

6. The **Inverse function**, or **reflection over the diagonal** $y = x$: Suppose the function $f(x)$ passes the "horizontal line test" in its domain: $f(x)$ never takes the same value twice. Such a function has a unique inverse $f^{-1}(x)$ whose domain is the range of $f(x)$, and vice versa.

- The inverse function has the properties that $f^{-1}(f(x)) = x$ for all x in the domain of $f(x)$ and $f\left(f^{-1}(x)\right) = x$ for all x in the domain of $f^{-1}(x)$.
- The inverse of the inverse function is the original function: $\left(f^{-1}\right)^{-1}(x) = f(x)$.
- Graphically, $y = f^{-1}(x)$ has the same shape as the orginal function, but is reflected over the slanted line $y = x$. **Ex:** $y = e^x$ and $y = \ln x$ are inverse functions. See graph in *Exponential and Logarithmic Functions.*
- If $f(x)$ takes the same value more than once, we restrict the domain before taking the inverse.
Ex: $y = x^2$ on the whole real line has no inverse, but the function $y = x^2$ on the positive reals only has the inverse $y = \sqrt{x}$.

Inverse trigonometric functions: $\sin x$, $\cos x$, and $\tan x$ are periodic and take on the same value many times. To construct inverse functions, we restrict the domain of the trigonometric functions to a single cycle.

Arcsine, the principal inverse of sine, is often denoted $\sin^{-1} x$ (not to be confused with $\frac{1}{\sin x} = \csc x$). It is defined on the interval $[-1, 1]$ and takes values in the range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Arccosine, or $\cos^{-1} x$, is defined on $[-1, 1]$ and takes values in the range $[0, \pi]$.

Arctangent, or $\tan^{-1} x$, is defined on the whole real line and takes values in the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

LIMITS AND CONTINUITY

LIMIT OF A FUNCTION

If function $f(x)$ comes infinitely close to some value L as x gets close to a, we say that "L is the **limit** of $f(x)$ as x approaches a" and write $\lim_{x\to a} f(x) = L$. The existence or the value of $\lim_{x\to a} f(x)$ by itself says nothing at all about the existence or the value of $f(a)$. Rather, comparing the $\lim_{x\to a} f(x)$ and $f(a)$ tells about the continuity or the type of discontinuity of $f(x)$ at $x = a$.

- **One-sided limits:** We can look at the limit of $f(x)$ as x approaches a from one side only. The **left-hand limit,** $\lim_{x\to a^-} f(x)$ exists if the $f(x)$ is close to some value when x is close to and smaller than a. The **right-hand limit** $\lim_{x\to a^+} f(x)$ depends on values of $f(x)$ when x is close to and larger than a.
- If the limit of $f(x)$ as $x \to a$ exists, then so do both one-sided limits, and the three limits have the same value. Contrariwise, if both the right-hand and the left-hand limits of $f(x)$ as $x \to a^{\pm}$ exist and are equal, then the $\lim_{x\to a} f(x)$ exists and is equal to the value of the one-sided limits.

A NOTE ON INFINITY

1. The limit $\lim_{x\to a} f(x)$ can be infinite.

Ex: $\lim_{x\to 0} \frac{1}{x^2} = +\infty$. Here, the line $x = 0$ is a **vertical asymptote:** the function tends towards the line but never quite reaches it.

2. We can also look at the limit of $f(x)$ at $+\infty$ or $-\infty$. A limit at infinity, if it exists, is what $f(x)$ tends toward as $|x|$ gets very large, positively or negatively. Limits at infinity are one-sided. If $\lim_{x\to\pm\infty} f(x) = L$ exists and is finite, the line $y = L$ is a **horizontal asymptote** to the graph of $f(x)$.

Ex: $\lim_{x\to-\infty} e^x = 0$. The line $y = 0$ is a horizontal asymptote to the function $y = e^x$.

NOTE: The statement "$f(a)$ exists" implies that $f(a)$ is a finite real value. However, the statement "$\lim_{x\to a} f(x)$ exists" could mean that this limit is infinite. We'll say "$\lim_{x\to a} f(x)$ exists and is finite" to mean that the limit exists and is not infinity.

CONTINUITY AND DISCONTINUITY

- If $f(a)$ exists and is equal to $\lim_{x\to a} f(x)$, we say that $f(x)$ is **continuous** at $x = a$. If $f(x)$ is continuous at every $a \le x \le b$, we say that $f(x)$ is **continuous on the interval** $[a, b]$. If $f(x)$ is continuous at every real x, we say that $f(x)$ is continuous on the whole real line or simply continuous.
- If $\lim_{x\to a} f(x)$ exists and is finite but is not equal to $f(a)$ (which may or may not exist), we say that $f(x)$ has a **removable discontinuity** at $x = a$. **Ex:** the function $y = \frac{x^2-4}{x-2}$ is indistinguishable from $y = x + 2$ everywhere except when $x = 2$, where it is undefined. But the discontinuity at $x = 2$ could easily be "removed" by inserting a point to make the graph continuous.

Vertical asymptote: If either of the one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ exists and is infinite, then $f(x)$ has a (possibly one-sided) **vertical asymptote** at $x = a$. The two-sided $\lim_{x\to a} f(x)$ may or may not exist. Often, functions teachers use will have vertical asymptotes where the two one-sided limits tend to infinities of opposite sign.

Ex: The function $f(x) = \frac{1}{x}$ has $\lim_{x\to 0^+} f(x) = +\infty$, $\lim_{x\to 0^-} f(x) = -\infty$, and a vertical asymptote at $x = 0$.

CONTINUITY OF BASIC FUNCTIONS

- All linear, polynomial, exponential, logarithmic, and sinusoidal functions are continuous.
- Rational functions $\frac{p(x)}{q(x)}$ are continuous everywhere except at the roots of $q(x)$. At those roots of $q(x)$ which are also roots of $p(x)$, they have removable discontinuities if the multiplicity of the root in $p(x)$ is at least as great as the multiplicity of the root in $q(x)$. All other roots of $q(x)$ give vertical asymptotes. See graph on page 4 in *Sketching Graphs: Summary.*
- The trigonometric functions $\tan x = \frac{\sin x}{\cos x}$ and $\sec x = \frac{1}{\cos x}$ are continuous everywhere except at the zeroes of $\cos x$, which are odd-integer multiples of $\frac{\pi}{2}$. The trigonometric functions $\cot x = \frac{\cos x}{\sin x}$ and $\csc x = \frac{1}{\sin x}$ are continuous everwhere except at the zeroes of $\sin x$—integer multiples of π. All discontinuities for these four functions are vertical asymptotes.

TIP: In practice, functions encountered in the classroom are discontinuous only at isolated points.

CONTINUED ON OTHER SIDE

LIMITS AND CONTINUITY (CONTINUED)

THE EPSILON–DELTA ($\varepsilon - \delta$) DEFINITIONS OF LIMITS AND CONTINUITY

Limits: $\lim_{x\to a} f(x) = L$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever x is within δ of a, $f(x)$ is within ε of L (that is, $|x-a| < \delta$ implies that $|f(x) - L| < \varepsilon$).

Continuity: Function $f(x)$ is said to be continuous at $x = a$ if and only if for every $\varepsilon > 0$, there exists some $\delta > 0$ such that whenever x_0 is within δ of a, $f(x_0)$ is within ε of $f(a)$ (that is, $|x_0 - a| < \delta$ implies that $|f(x_0) - f(a)| < \varepsilon$).

Equivalently, $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x)$ exists and is equal to $f(a)$.

LIMIT LAWS

Suppose $f(x)$ and $g(x)$ are two functions, a is a point (possibly $\pm\infty$) near which both $f(x)$ and $g(x)$ are defined. (These are only true if both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist and at least one of them is finite!)

Sum: $\lim\limits_{x\to a} \big(f(x) \pm g(x)\big) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x)$

Scalar multiple: $\lim\limits_{x\to a} \big(cf(x)\big) = c \lim\limits_{x\to a} f(x)$ Here, c is any real.

Product: $\lim\limits_{x\to a} f(x)g(x) = \left(\lim\limits_{x\to a} f(x)\right)\left(\lim\limits_{x\to a} g(x)\right)$

Quotient: If $\lim\limits_{x\to a} g(x) \neq 0$, then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$.

The Squeeze Theorem:
If $f(x) \le g(x) \le h(x)$ near $x = a$, and $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} h(x) = L$, then $\lim\limits_{x\to a} g(x)$ exists and is equal to L.

The classic application of this theorem establishes that $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Frequently-encountered limits:

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$

$\lim\limits_{x\to 0} \frac{\cos x - 1}{x} = 0$

$\lim\limits_{x\to \infty} x^n e^{-x} = 0$ for all n

TAKING DERIVATIVES

INTUITION

If a and b are two points in the domain of $f(x)$ then the **average rate of change** of $f(x)$ on the interval $[a,b]$ is $\frac{f(b)-f(a)}{b-a}$, a measure of how fast $f(x)$ has increased or decreased over the interval. This is also the slope of the line through the points $\big(a, f(a)\big)$ and $\big(b, f(b)\big)$ on the graph of $f(x)$.

The derivative of $f(x)$ at a point $x = a$ is the **instantaneous rate of change**, a measure of how fast $f(x)$ is increasing or decreasing at a. Equivalently, the derivative is the **slope of the tangent line** to the graph of $f(x)$ at the point $x = a$—the unique line through the point $\big(a, f(a)\big)$ that touches the graph at only that point near $x = a$.

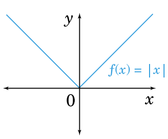
We compute the derivative $f'(a)$ by looking at the average rate of change of $f(x)$ on the interval $[a, a+h]$ and taking the limit as h goes to 0. Equivalently, $f'(a)$ is the limit as $h \to 0$ of the slope of the line through $\big(a, f(a)\big)$ and $\big(a+h, f(a+h)\big)$.

DEFINITION

If the limit $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, we say that $f(x)$ is **differentiable at** $x = a$ and the limit is the **derivative of** $f(x)$ **at** $x = a$, denoted by $f'(a)$.

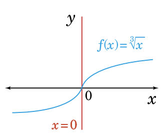
The function $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$ is the derivative function of $f(x)$. If it is defined whenever $f(x)$ is defined, then $f(x)$ is called **differentiable.**

If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at a. The converse is not true: a function can be continuous but not differentiable. There are two cases where this occurs:

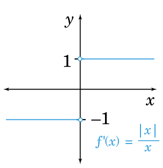


No tangent at $x = 0$

1. No tangent: Ex: $f(x) = |x|$. The function is continuous at $x = 0$ since $\lim_{x\to 0^+} |x| = \lim_{x\to 0^-} |x| = 0$, but the derivative $f'(0)$ is undefined since the left-hand slope limit, $\lim_{h\to 0^-} \frac{|h|}{h} = -1$, does not equal the right-hand slope limit, $\lim_{h\to 0^+} \frac{|h|}{h} = 1$.

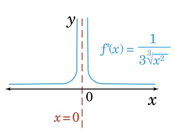


Vertical tangent at $x = 0$



$f'(0)$ undefined

2. Vertical tangent: The slope of a vertical line is "undefined." If $f(x)$ has a vertical tangent at $x = a$, then the derivative $f'(a)$ is undefined and the graph of $f'(x)$ will have a vertical asymptote at $x = a$. **Ex:** $f(x) = \sqrt[3]{x}$ has a vertical tangent at the point $(0,0)$. The derivative function, $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$, goes to infinity at 0.



$f'(0)$ undefined

NOTATION

Different notations for the derivative function are useful in different contexts. The most common notations in calculus are $f'(x)$, y', $\frac{d}{dx} f(x)$, and $\frac{dy}{dx}$. The last two are in **Leibniz notation;** $\frac{dy}{dx}$ evolved from $\frac{\triangle y}{\triangle x} = \frac{\text{change in } y}{\text{change in } x}$, or slope. The expressions dy and dx represent infinitesimal changes in y and x.

- Higher-order derivatives can be written in "prime" notation: $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$, or in Leibniz notation: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$.
- The derivative at a particular point a is most often expressed as $f'(a)$ or $\left.\frac{dy}{dx}\right|_{x=a}$.

METHODS AND TRICKS

Assume that $f(x)$ and $g(x)$ are two differentiable functions.

Sum and Difference: $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$

Scalar Multiple: $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$

Product: $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$

MNEMONIC: If f is "hi" and g is "ho," then the product rule is "ho d hi plus hi d ho."

Quotient: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

MNEMONIC: "Ho d hi minus hi d ho over ho ho."

The **Chain Rule** takes the derivative of composite functions. Here are two ways of writing it:

1. $(f \circ g)'(x) = f'\big(g(x)\big)\, g'(x)$.
2. If $u = g(x)$ and $y = f(u) = f\big(g(x)\big)$, then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

IMPLICIT DIFFERENTIATION

Implicit differentiation uses the product and chain rules to find slopes of curves when it is difficult or impossible to express y as a function of x. Leibniz notation may be easiest when differentiating implicitly. Take the derivative of each term in the equation with respect to x. Then rewrite $\frac{dy}{dx} = y'$ and $\frac{dx}{dx} = 1$ and solve for y'.

Ex 1: $x^2 + y^2 = 1$
Implicitly differentiating with respect to x gives the expression $2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$, which simplifies to $2x + 2yy' = 0$ or $y' = -\frac{x}{y}$. The derivative can now be found for any point on the curve, even though it is not actually a function. You will get the same result if you first solve for $y = \pm\sqrt{1-x^2}$ and keep track of the $\pm$ signs in different quadrants.

Ex 2: $x \cos y - y^2 = 3x$
Differentiate to obtain first $\frac{dx}{dx}\cos y + x\frac{d(\cos y)}{dx} - 2y\frac{dy}{dx} = 3\frac{dx}{dx}$, and then $\cos y - x \sin y y' - 2yy' = 3$. Finally, solve for $y' = \frac{\cos y - 3}{x \sin y + 2y}$.

DERIVATIVES OF BASIC FUNCTIONS

Constants:	$\frac{d(c)}{dx} = 0$	A constant function is always flat.
Linear:	$\frac{d(mx+b)}{dx} = m$	The line $y = mx + b$ has slope m.
Powers:	$\frac{d(x^n)}{dx} = nx^{n-1}$	True for all real $n \neq 0$.
Polynomial:	$\frac{d(a_n x^n + \cdots + a_2x^2 + a_1x + a_0)}{dx} = na_nx^{n-1} + \cdots + 2a_2x + a_1$	
Exponential:	$\frac{d(e^x)}{dx} = e^x$	This is why e is called the "natural" logarithm base: Ae^x are the only functions that are their own derivatives.
	$\frac{d(a^x)}{dx} = a^x \ln a$	When in doubt, convert a^x to $e^{x \ln a}$.
Logarithmic:	$\frac{d(\ln x)}{dx} = \frac{1}{x}$	Found using implicit differentiation.
	$\frac{d(\log_a x)}{dx} = \frac{1}{x \ln a}$	When in doubt, convert $\log_a x$ to $\frac{\ln x}{\ln a}$.

TAKING DERIVATIVES (CONTINUED)

Trigonometric: Found using the definition of derivative and the Squeeze Theorem. If you know the derivatives of $\sin x$ and $\cos x$, you can find all the rest using the definitions of the trigonometric functions and the quotient rule.

$$\frac{d(\sin x)}{dx} = \cos x \qquad \frac{d(\cos x)}{dx} = -\sin x$$

$$\frac{d(\tan x)}{dx} = \sec^2 x \qquad \frac{d(\cot x)}{dx} = -\csc^2 x$$

$$\frac{d(\sec x)}{dx} = \sec x \tan x \qquad \frac{d(\csc x)}{dx} = -\csc x \cot x$$

Inverse Trigonometric: A pain. Found by implicit differentiation.

$$\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d(\cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2} \qquad \frac{d(\cot^{-1} x)}{dx} = -\frac{1}{1+x^2}$$

$$\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d(\csc^{-1} x)}{dx} = -\frac{1}{x\sqrt{x^2-1}}$$

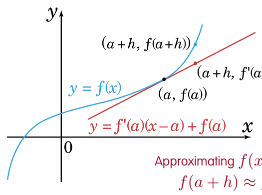
USING DERIVATIVES

THE TANGENT LINE: APPROXIMATING $f(x)$ NEAR $x = a$

The **tangent line** to a curve $y = f(x)$ at the point $(a, f(a))$ is given by the equation

$$y = f(a) + f'(a)(x-a).$$

The tangent line gives a (very) crude approximation to $f(x)$: If h is small, then $f(a+h) \approx f'(a)h + f(a)$. Useful when $f'(a)$ is known and $f(x)$ is hard to compute.



Approximating $f(x)$ near a: For small h, $f(a+h) \approx f'(a)h + f(a)$

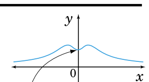
DISPLACEMENT, VELOCITY, ACCELERATION: MOTION IN ONE DIMENSION

Suppose a particle's position on a line in meters at time t seconds is determined by the function $s(t)$.

- The first derivative $s'(t)$ of the position function gives the instantaneous rate of change of motion; in other words, the particle's instantaneous **velocity** $v(t) = s'(t)$. The units for the first derivative are "meters per second," $\mathrm{m/s}$.
- The first derivative $v'(t)$ of the velocity function will give the instantaneous rate of change of velocity, or **acceleration:** $a(t) = v'(t) = s''(t)$. The units for acceleration are "meters per second per second," $\mathrm{m/s^2}$.

MAXIMA AND MINIMA

A **local minimum** (or **maximum**) is a point $(c, f(c))$ such that $f(c)$ is the least (or greatest) value of the function in some interval around c. A local minimum or maximum is also called a **relative** minimum or maximum.



local minimum

No global minumum

The **global minimum** (or **maximum**) is the point where $f(x)$ assumes its least (or greatest) value in the domain considered. The global minimum or maximum is also called the **absolute** minimum or maximum.

If the domain is a closed interval (an interval that includes its endpoints), then a continuous function will always have global minimum and maximum points, possibly at one of the endpoints. (This is the Extreme Value Theorem.)

The word **extremum** can be used to mean either minimum or maximum. The plural of extremum is **extrema**.

Critical points are points where $f'(x)$ is zero or undefined. All extrema—that is, all minima and maxima—happen either at endpoints or at critical points.

How to find extremum points:

1. Check critical points where $f(x)$ exists but $f'(x)$ is not defined. Such a point may be a local extremum, as in $f(x) = |x|$ at $x = 0$. It may be a point of discontinuity. Or it may be neither.

2. Check critical points where $f'(x) = 0$. If $f'(a) = 0$, then often, but not always, the function will have a local extremum at $x = a$.

- If the sign of $f'(x)$ switches from $+$ to $-$ at $x = a$, then $f(a)$ is a local maximum.
- If the sign of $f'(x)$ switches from $-$ to $+$ at $x = a$, then $f(a)$ is a local minimum.
- If the sign of $f'(x)$ does not switch around $x = a$, then $f(a)$ is neither a maximum nor a minimum.

Alternatively, you can use the **second derivative test**:

- If $f''(a) < 0$, then $f(a)$ is a local maximum.
- If $f''(a) > 0$, then $f(a)$ is a local minimum.
- If $f''(a) = 0$, then you must check whether $f'(x)$ switches sign around $x = a$.

TIP: Often, but not always, $f''(a) = 0$ means that $f(a)$ is neither a minimum nor a maximum. *Counterexample:* $f(x) = x^4$ has $f'(0) = f''(0) = 0$. The second derivative test tells you nothing, but the changing sign of the first derivative indicates that $f(0) = 0$ is a local minimum.

3. Check **endpoints:** If the domain is a closed interval $[a, b]$, always check $f(a)$ and $f(b)$ when looking for extrema. Also check any boundary points that are included in the domain.

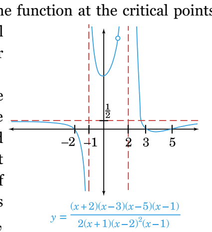
TIP: Non-included boundary points (including points where $f(x)$ is not defined, especially at vertical asymptotes) and behavior at $\pm\infty$ (including horizontal asymptotes) may affect existence of global extrema. **Ex:** $f(x) = (x^2+1)e^{-x^2}$, graphed under "Maxima and Minima", has a local minimum at $x = 0$ but no global minimum, though $f(x) > 0$ for all real x.

SKETCHING GRAPHS: SUMMARY

1. Endpoints: If the domain is an interval, evaluate the function at the endpoints. If the domain is the whole real line, establish what happens at $\pm\infty$. **Horizontal asymptotes** will appear if $\lim_{x\to\pm\infty} f(x)$ is finite. Evaluate $f(0)$ to find the y-intercept.

2. Gaps: Find all isolated points $x = a$ where $f(a)$ is not defined. For each point a, look at $\lim_{x\to a^\pm} f(x)$. A vertical asymptote will appear if $\lim_{x\to a^\pm} f(x) = \pm\infty$. A removable discontinuity (hole in the graph) will appear if $\lim_{x\to a} f(x)$ exists and is finite.

3. x-intercepts: If it is easy to determine when $f(x) = 0$, do so. If not, evaluating the function at the critical points and the endpoints will indicate where to look for zeroes.



Horizontal asymptote: $y = \frac{1}{2}$

Vertical asymptote: $x = -1$ and $x = 2$

Removable discontinuity at $x = 1$

4. Rise and fall: Determine the intervals where the function is increasing and decreasing by looking at the sign of $f'(x)$. If $f'(x) > 0$, then $f(x)$ is increasing. If $f'(x) < 0$, then $f(x)$ is decreasing.

5. Local extrema: Find all local extrema by looking at the critical points where $f'(x) = 0$ or where $f'(x)$ is not defined.

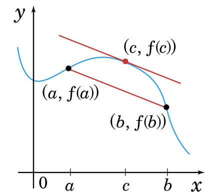
6. Concavity: Determine when the function cups up or down by looking at the sign of $f''(x)$. If $f''(x) > 0$, the function is concave up; if $f''(x) < 0$, then $f(x)$ is concave down. If $f''(a) = 0$, then the function is temporarily not curving at $x = a$; if $f''(x)$ is changing sign near $x = a$, then this is a **point of inflection** (change in concavity).

THEOREM HIGHLIGHTS

Intermediate Value Theorem: If $f(x)$ is continuous on an interval $[a, b]$, then somewhere on the interval it will achieve every value between $f(a)$ and $f(b)$: if $f(a) \le M \le f(b)$, then there exists some c in the interval $[a, b]$ (notation: $c \in [a, b]$) such that $f(c) = M$. This is a completely intuitive statement!

Rolle's Theorem: If $f(x)$ is continuous on the closed interval $[a, b]$, differentiable on the open interval (a, b), and satisfies $f(a) = f(b)$, then for some c in the interval (a, b), we have $f'(c) = 0$.

Mean Value Theorem (MVT): A generalization of Rolle's Theorem. If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that the slope of the tangent to $f(x)$ at $x = c$ is the same as the slope of the secant line through the two points $(a, f(a))$ and $(b, f(b))$: that is, $f'(c) = \frac{f(b)-f(a)}{b-a}$.



MVT: $f'(c) = \frac{f(b)-f(a)}{b-a}$

Two functions having the same derivative differ by a constant: If $f'(x) = g'(x)$, then $f(x) = g(x) + C$, for some real C. Equivalently, **a function has only one family of antiderivatives.** This theorem follows from the MVT. See the *Calculus II* SparkChart for more on antiderivatives.

Extreme Value Theorem: A function $f(x)$ continuous on the closed inteval $[a, b]$ will assume a global maximum and a global minimum somewhere on $[a, b]$.

L'HÔPITAL'S RULE

Used to evaluate indeterminate form limits: $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$. Suppose both $f(x)$ and $g(x)$ are differentiable around a and $g'(x) \neq 0$ on an interval near a (except perhaps at a).

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

OR

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$.

- L'Hôpital's Rule can also be applied if the limit is one-sided ($x \to a^\pm$).
- L'Hôpital's Rule can also be applied if the limit is taken as x approaches infinity ($x \to \pm\infty$).
- If $f'(x)$ and $g'(x)$ also satisfy the conditions for L'Hôpital's Rule, higher derivatives can be taken until the limit is well-defined.
- L'Hôpital's Rule cannot be applied to a fraction if the top limit is infinite and the bottom limit is zero, or vice versa.

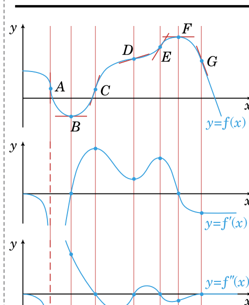
Ex: $\lim_{x\to 1} \frac{\ln x}{x-1}$. Since $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} x - 1 = 0$, use L'Hôpital's Rule:

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{d(\ln x)}{dx}}{\frac{d(x-1)}{dx}} = \lim_{x\to 1} \frac{1/x}{1} = 1.$$

- L'Hôpital's Rule can be used to evaluate other indeterminate forms, such as $\pm\infty \cdot 0$. The key is to convert the expression to $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$.

Ex: $\lim_{x\to-\infty} xe^x$. Convert to the expression $\lim_{x\to-\infty} \frac{x}{e^{-x}}$, which is an indeterminate form $\frac{-\infty}{\infty}$. Applying L'Hôpital's Rule, convert to $\lim_{x\to-\infty} \frac{1}{-e^{-x}} = 0$.

SKETCH OF A FUNCTION AND TWO OF ITS DERIVATIVES



A: Vertical tangent
B: Local minimum
C: Point of Inflection
D: Point of Inflection
E: Point of Inflection
F: Local maximum

- The function is decreasing from the y-axis to *B*, increasing from *B* to *F*, and then decreasing from *F* on.
- The function is concave down from the y-axis to *A*, concave up from *A* to *C*, concave down from *C* to *D*, concave up from *D* to *E*, concave down from *E* to *G*, and not curving from *G* on.

SPARKCHARTS™

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