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Representation of results of particle size analysis —

Part 2: Calculation of average particle sizes/diameters and moments from particle size distributions

Représentation de données obtenues par analyse granulométrique —

Partie 2: Calcul des tailles/diamètres moyens des particules et des moments à partir de distributions granulométriques

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Foreword

ISO (the International Organization for Standardization) is a worldwide federation of national standards bodies (ISO member bodies). The work of preparing International Standards is normally carried out through ISO technical committees. Each member body interested in a subject for which a technical committee has been established has the right to be represented on that committee. International organizations, governmental and non-governmental, in liaison with ISO, also take part in the work. ISO collaborates closely with the International Electrotechnical Commission (IEC) on all matters of electrotechnical standardization.

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Draft International Standards adopted by the technical committees are circulated to the member bodies for voting. Publication as an International Standard requires approval by at least 75 % of the member bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this part of ISO 9276 may be the subject of patent rights. ISO shall not be held responsible for identifying any or all such patent rights.

International Standard ISO 9276-2 was prepared by Technical Committee ISO/TC 24, *Sieves, sieving and other sizing methods*, Subcommittee SC 4, *Sizing by methods other than sieving*.

ISO 9276 consists of the following parts, under the general title *Representation of results of particle size analysis*:

— *Part 1: Graphical representation*

— *Part 2: Calculation of average particle sizes/diameters and moments from particle size distributions*

— *Part 3: Calculation of means and moments of particle size distributions*

— *Part 4: Characterization of a classification process.*

— *Part 5: Validation of calculations relating to particle size analyses using the logarithmic normal probability distribution*

Introduction

In particle size analysis, particulate matter is often characterized based on representative samples of the population with the final aim of linking the size information with some other important physical property such as strength, flowability, solubility, etc. In general, a correlation between the physical property and the size of the particles, the so-called property function, can be obtained if an average particle size has been derived or calculated from the measured distribution of sizes.

A unique definition of the average size, $\bar{x}_{k,r}$, is given in this part of ISO 9276, using the so-called moments, $M_{k,r}$, of a size distribution. Apart from average sizes, moments are also used in the calculation of volume related surface area, the spread and other statistical measures of a particle size distribution.

Representation of results of particle size analysis —

Part 2:
Calculation of average particle sizes/diameters and moments from particle size distributions

1 Scope

The object of this part of ISO 9276 is to provide the relevant equations for the calculation of average particle sizes or average particle diameters and moments from a given particle size distribution. It is assumed that the size distribution is available as a histogram. It is nevertheless also possible to apply the same mathematical treatment if the particle size distribution is represented by an analytical function.

It is furthermore assumed in this part of ISO 9276 that the particle size x of a particle of any other shape may also be represented by the diameter of an equivalent sphere, e.g. a sphere having the same volume as the particle concerned.

2 Normative references

The following normative documents contain provisions which, through reference in this text, constitute provisions of this part of ISO 9276. For dated references, subsequent amendments to, or revisions of, any of these publications do not apply. However, parties to agreements based on this part of ISO 9276 are encouraged to investigate the possibility of applying the most recent editions of the normative documents indicated below. For undated references, the latest edition of the normative document referred to applies. Members of ISO and IEC maintain registers of currently valid International Standards.

ISO 565:1990, *Test sieves — Metal wire cloth, perforated metal plate and electroformed sheet — Nominal sizes of openings*.

ISO 9276-1, *Representation of results of particle size analysis — Part 1: Graphical representation*.

3 Symbols and abbreviated terms

For the purposes of this part of ISO 9276, the following symbols and abbreviated terms apply.

i — number of the size class with upper particle size, x_i

k — power of x

n — total number of size classes

r — type of quantity of a distribution (general description)

- $r = 0$ — type of quantity, number
- $r = 1$ — type of quantity, length

	$r = 2$	type of quantity, surface or projected area
	$r = 3$	type of quantity, volume or mass

$M_{k,r}$ complete k-th moment of a $q_r(x)$-distribution

$m_{k,r}$ complete k-th central moment of a $q_r(x)$-distribution

$q_r(x)$ density distribution

$\bar{q}_{r,i}$ average height of a density distribution in the i-th particle size interval, Δx_i

$\bar{q}_{r,i}\,(x_{i-1}, x_i)$ histogram

$Q_r(x)$ cumulative distribution

$\Delta Q_{r,i}$ difference of two values of the cumulative distribution, i.e. relative amount in the i-th particle size interval, Δx_i

s_r standard deviation of a $Q_r(x)$ distribution

s_g geometric standard deviation of a normal distribution

S surface area

S_V volume specific surface area

V particle volume

$\bar{V}$ average particle volume

x particle diameter, diameter of a sphere

x_i upper particle size of the i-th particle size interval

x_{i-1} lower particle size of the i-th particle size interval

x_{min} particle size below which there are no particles in a given size distribution

x_{max} particle size above which there are no particles in a given size distribution

$\bar{x}_{k,r}$ average particle diameter (general description)

$\bar{x}_{k,0}$ arithmetic average particle diameter (general description)

$\bar{x}_{1,0}$ arithmetic average length diameter

$\bar{x}_{2,0}$ arithmetic average surface diameter

$\bar{x}_{3,0}$ arithmetic average volume diameter

$\bar{x}_{1,r}$ weighted average particle diameter (general description)

$\bar{x}_{1,1}$ weighted average length diameter

$\bar{x}_{1,2}$ weighted average surface diameter, Sauter-Diameter

$\bar{x}_{1,3}$ weighted average volume diameter

$\bar{x}_{\text{geo},r}$ geometric mean diameter (informative only)

$\bar{x}_{\text{har},r}$ harmonic mean diameter (informative only)

$x_{50,3}$ median particle size of a cumulative volume distribution

$\Delta x_i = x_i - x_{i-1}$ width of the i-th particle size interval

z dimensionless variable of a logarithmic normal probability distribution

4 Basic definition of a moment

The complete k-th moment of a $q_r(x)$ density distribution (see [1] in the bibliography) is represented by integrals as defined in equation (1):

$$M_{k,r} = \int_{x_{\text{min}}}^{x_{\text{max}}} x^k q_r(x)\,\mathrm{d}x \tag{1}$$

where

M stands for moment;

k indicates the power of x;

r describes the type of quantity of the density distribution.

If $r = 0$, $q_0(x)$ represents a number density distribution, if $r = 3$, $q_3(x)$ represents a volume or mass density distribution.

Equation (1) describes a "complete moment" if the integral boundaries are represented by the minimum particle size x_{min} and the maximum particle size x_{max}.

A special complete moment is represented by $M_{0,r}$:

$$M_{0,r} = \int_{x_{\text{min}}}^{x_{\text{max}}} x^0 q_r(x)\,\mathrm{d}x = \int_{x_{\text{min}}}^{x_{\text{max}}} q_r(x)\,\mathrm{d}x = Q_r(x_{\text{max}}) - Q_r(x_{\text{min}}) = 1 \tag{2}$$

A moment is incomplete, if the integration is performed between two arbitrary particle diameters x_{i-1} and x_i within the given size range of a distribution $x_{\text{min}} < x_{i-1} < x < x_i < x_{\text{max}}$.

$$M_{k,r}(x_{i-1}, x_i) = \int_{x_{i-1}}^{x_i} x^k q_r(x)\,\mathrm{d}x \tag{3}$$

Apart from the moments related to the origin of the particle size axis and shown in equations (1) and (3), the so-called k-th central moment of a $q_r(x)$, density distribution, $m_{k,r}$, can be derived from a given density distribution. It is related to the weighted average particle diameter $\bar{x}_{1,r}$ [see equation (11)].

The complete k-th central moment is defined as:

$$m_{k,r} = \int_{x_{\min}}^{x_{\max}} \left(x - \bar{x}_{1,r}\right)^k q_r(x)\,\mathrm{d}x \tag{4}$$

The incomplete k-th central moment is represented by:

$$m_{k,r}\left(x_{i-1}, x_i\right) = \int_{x_{i-1}}^{x_i} \left(x - \bar{x}_{1,r}\right)^k q_r(x)\,\mathrm{d}x \tag{5}$$

5 Average particle diameters

All average particle diameters are defined by equation (6):

$$\bar{x}_{k,r} = \sqrt[k]{M_{k,r}} \tag{6}$$

Depending on the numbers chosen for the subscripts, k and r, different average particle diameters may be defined. Since the average particle diameters calculated from equation (6) may differ considerably, the subscripts k and r should always be quoted.

The two groups of average particle diameters given in 5.1 and 5.2, should preferably be used.

5.1 Arithmetic average particle diameters

Arithmetic mean particle diameters, calculated from a number density distribution, $q_0(x)$, are:

$$\bar{x}_{k,0} = \sqrt[k]{M_{k,0}} \tag{7}$$

Counting single particles in a microscope image is a typical example of obtaining number ($r = 0$) percentages as a basis for averaging.

The recommended average particle diameters (see [2] in the bibliography) are:

arithmetic average length diameter*:*

$$\bar{x}_{1,0} = M_{1,0} \tag{8}$$

arithmetic average surface diameter*:*

$$\bar{x}_{2,0} = \sqrt[2]{M_{2,0}} \tag{9}$$

arithmetic average volume diameter:

$$\bar{x}_{3,0} = \sqrt[3]{M_{3,0}} \tag{10}$$

5.2 Weighted average particle diameters

Weighted average particle diameters are defined by:

$$\bar{x}_{1,r} = M_{1,r} \tag{11}$$

Weighing sieves before and after sieving is a typical example of obtaining mass (r = 3) percentages as a basis for averaging.

Weighted average particle diameters represent the abscissa of the centre of gravity of a $q_r(x)$ distribution. The recommended weighted average particle diameters are represented by equations (12) to (15).

The weighted average particle diameter of a number density distribution, $q_0(x)$, is equivalent to the arithmetic average length diameter [see equation (8)]. It is represented by the arithmetic average length diameter:

$$\bar{x}_{1,0} = M_{1,0} \tag{12}$$

The weighted average particle diameter of a length density distribution, $q_1(x)$, is given by the weighted average length diameter:

$$\bar{x}_{1,1} = M_{1,1} \tag{13}$$

The weighted average particle diameter of a surface density distribution, $q_2(x)$, is represented by the weighted average surface diameter:

$$\bar{x}_{1,2} = M_{1,2} \tag{14}$$

The weighted average particle diameter of a volume density distribution, $q_3(x)$, is given by the weighted average volume diameter:

$$\bar{x}_{1,3} = M_{1,3} \tag{15}$$

5.3 The calculation of $M_{k,r}$ and average particle diameters from a number or a volume density distribution, $q_0(x)$ or $q_3(x)$

In many cases of practical application, the measured data are either represented by a number density distribution, $q_0(x)$, or a volume density distribution, $q_3(x)$. The calculation of the average particle diameters described above can then be performed according to equation 16 (see [1] in the bibliography):

$$\bar{x}_{k,r} = \sqrt[k]{M_{k,r}} = \sqrt[k]{\frac{M_{k+r,0}}{M_{r,0}}} = \sqrt[k]{\frac{M_{k+r-3,3}}{M_{r-3,3}}} \tag{16}$$

This leads to:

$$\bar{x}_{2,0} = \sqrt{M_{2,0}} = \sqrt{\frac{M_{-1,3}}{M_{-3,3}}} \tag{17}$$

$$\bar{x}_{3,0} = \sqrt[3]{M_{3,0}} = \sqrt[3]{\frac{1}{M_{-3,3}}} \tag{18}$$

$$\bar{x}_{1,1} = M_{1,1} = \frac{M_{2,0}}{M_{1,0}} = \frac{M_{-1,3}}{M_{-2,3}} \tag{19}$$

$$\bar{x}_{1,2} = M_{1,2} = \frac{M_{3,0}}{M_{2,0}} = \frac{1}{M_{-1,3}} \tag{20}$$

$$\bar{x}_{1,3} = M_{1,3} = \frac{M_{4,0}}{M_{3,0}} \tag{21}$$

It can be seen from equations (16) to (21), that the following moments are needed if the average particle diameters defined above are to be calculated:

— from a given volume density distribution, $q_3(x)$: $M_{1,3}$; $M_{-1,3}$; $M_{-2,3}$; $M_{-3,3}$

— from a given number density distribution, $q_0(x)$: $M_{1,0}$; $M_{2,0}$; $M_{3,0}$; $M_{4,0}$

5.4 Calculation of $M_{k,r}$ from a number or a volume density distribution, $q_0(x)$ or $q_3(x)$, given as a histogram

If a density distribution is given as a histogram, $q_r(x_{i-1}, x_i)$ is constant in the particle size interval $\Delta x_i = x_i - x_{i-1}$.

Equation (1) may therefore be rewritten as follows:

$$M_{k,r} = \int_{x_{\min}}^{x_{\max}} x^k q_r(x)\,\mathrm{d}x = \sum_{i=1}^{n} \bar{q}_{r,i} \int_{x_{i-1}}^{x_i} x^k \mathrm{d}x \tag{22}$$

and, calculating the appropriate integral average of x^k with $k \neq -1$:

$$M_{k,r} = \frac{1}{k+1}\sum_{i=1}^{n} \bar{q}_{r,i}\left(x_i^{k+1} - x_{i-1}^{k+1}\right) = \frac{1}{k+1}\sum_{i=1}^{n} \Delta Q_{r,i}\left(\frac{x_i^{k+1} - x_{i-1}^{k+1}}{x_i - x_{i-1}}\right) \tag{23}$$

or with $k = -1$:

$$M_{-1,r} = \sum_{i=1}^{n} \bar{q}_{r,i} \ln\frac{x_i}{x_{i-1}} = \sum_{i=1}^{n} \Delta Q_{r,i} \frac{\ln\dfrac{x_i}{x_{i-1}}}{x_i - x_{i-1}} \tag{24}$$

with

$$\bar{q}_{r,i} = \frac{\Delta Q_{r,i}}{x_i - x_{i-1}} \tag{25}$$

The moments $M_{1,0}$, $M_{2,0}$, $M_{3,0}$, $M_{4,0}$, $M_{1,3}$, $M_{-1,3}$, $M_{-2,3}$ and $M_{-3,3}$ can therefore be calculated from equations (26) to (33):

$$M_{1,0} = \frac{1}{2}\sum_{i=1}^{n} \bar{q}_{0,i}\left(x_i^2 - x_{i-1}^2\right) = \frac{1}{2}\sum_{i=1}^{n} \Delta Q_{0,i}\left(x_i + x_{i-1}\right) \tag{26}$$

$$M_{2,0} = \frac{1}{3}\sum_{i=1}^{n} \bar{q}_{0,i}\left(x_i^3 - x_{i-1}^3\right) = \frac{1}{3}\sum_{i=1}^{n} \Delta Q_{0,i}\left(\frac{x_i^3 - x_{i-1}^3}{x_i - x_{i-1}}\right) \tag{27}$$

$$M_{3,0} = \frac{1}{4}\sum_{i=1}^{n}\bar{q}_{0,i}\left(x_i^4 - x_{i-1}^4\right) = \frac{1}{4}\sum_{i=1}^{n}\Delta Q_{0,i}\left(\frac{x_i^4 - x_{i-1}^4}{x_i - x_{i-1}}\right) \quad (28)$$

$$M_{4,0} = \frac{1}{5}\sum_{i=1}^{n}\bar{q}_{0,i}\left(x_i^5 - x_{i-1}^5\right) = \frac{1}{5}\sum_{i=1}^{n}\Delta Q_{0,i}\left(\frac{x_i^5 - x_{i-1}^5}{x_i - x_{i-1}}\right) \quad (29)$$

$$M_{1,3} = \frac{1}{2}\sum_{i=1}^{n}\bar{q}_{3,i}\left(x_i^2 - x_{i-1}^2\right) = \frac{1}{2}\sum_{i=1}^{n}\Delta Q_{3,i}\left(x_i + x_{i-1}\right) \quad (30)$$

$$M_{-1,3} = \sum_{i=1}^{n}\bar{q}_{3,i}\ln\frac{x_i}{x_{i-1}} = \sum_{i=1}^{n}\Delta Q_{3,i}\frac{\ln\frac{x_i}{x_{i-1}}}{x_i - x_{i-1}} \quad (31)$$

$$M_{-2,3} = \sum_{i=1}^{n}\bar{q}_{3,i}\left(\frac{1}{x_{i-1}} - \frac{1}{x_i}\right) = \sum_{i=1}^{n}\Delta Q_{3,i}\frac{1}{x_i x_{i-1}} \quad (32)$$

$$M_{-3,3} = \frac{1}{2}\sum_{i=1}^{n}\bar{q}_{3,i}\frac{x_i^2 - x_{i-1}^2}{x_i^2 x_{i-1}^2} = \frac{1}{2}\sum_{i=1}^{n}\Delta Q_{3,i}\frac{x_i + x_{i-1}}{x_i^2 x_{i-1}^2}\,. \quad (33)$$

5.5 Calculation of volume specific surface area

Distributions of any type of quantity moments can be used in the calculation of the volume specific surface area, S_V, since S_V is inversely proportional to the weighted average surface diameter, the Sauter-Diameter, $\bar{x}_{1,2}$ [equation (14)]. It is given by:

$$S_V = \frac{6}{\bar{x}_{1,2}} \quad (34)$$

Taking into account equation (20), one arrives from surface, number or volume distributions at:

$$S_V = \frac{6}{M_{1,2}} = 6\frac{M_{2,0}}{M_{3,0}} = 6 \times M_{-1,3} \quad (35)$$

For particles other than spheres, a shape factor shall be introduced.

5.6 The variance of a particle size distribution

The spread of a size distribution may be represented by its variance, which represents the square of the standard deviation, s_r. The variance, s_r^2, of a $q_r(x)$-distribution is defined as:

$$s_r^2 = \int_{x_{min}}^{x_{max}}\left(x - \bar{x}_{1,r}\right)^2 q_r(x)\,\mathrm{d}x \quad (36)$$

Introducing complete moments, the variance can be calculated (see [2] in the bibliography) from:

$$s_r^2 = m_{2,r} = M_{2,r} - \left(M_{1,r}\right)^2 \quad (37)$$

or based on data from a histogram:

$$s_r^2 = \frac{1}{3}\left[\sum_{i=1}^{n} \bar{q}_{r,i}\left(x_i^3 - x_{i-1}^3\right)\right] - \frac{1}{4}\left[\sum_{i=1}^{n} \bar{q}_{r,i}\left(x_i^2 - x_{i-1}^2\right)\right]^2 = \frac{1}{3}\left[\sum_{i=1}^{n} \Delta Q_{r,i}\left(\frac{x_i^3 - x_{i-1}^3}{x_i - x_{i-1}}\right)\right] - \frac{1}{4}\left[\sum_{i=1}^{n} \Delta Q_{r,i}\left(x_i + x_{i-1}\right)\right]^2 \quad (38)$$

Annex A
(informative)

Calculation of different average particle diameters from the histogram of a given volume density distribution, numerical example

It is assumed in the following numerical example that the cumulative volume distribution follows a logarithmic normal distribution (see [5] in the bibliography):

$$q_3^*(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{z^2}{2}\right) \quad \text{(A.1)}$$

The asterisk refers to the fact that equation A.1 is formed with the dimensionless variable z having a value of:

$$z = \frac{\ln\left(x/x_{50,3}\right)}{s} = \frac{\ln\left(x/x_{50,3}\right)}{\ln s_g} \quad \text{(A.2)}$$

where:

$$s_g = e^s \quad \text{(A.3)}$$

Table A.1 has been calculated under the assumption that the distribution has a median of $x_{50,3}$ = 5 µm and a standard deviation of s = 0,50, corresponding to a geometric standard deviation s_g = 1,649. It has further been assumed that successive particle diameters follow an R10-series, that is:

$$\frac{x_i}{x_{i-1}} = \sqrt[10]{10} \quad \text{(A.4)}$$

The numbers given in Table A.1 for x_i, z_i, $Q_{3,i}$, Δx_i, $\Delta Q_{3,i}$ and $\bar{q}_{3,i}$ have been used to calculate the moments represented by equations (30) to (33). The $Q_{3,i}$-values used in the calculations have been taken from normal-distribution tables as given for example in [3] in the bibliography. The analytical values of the moments have been calculated by introducing the logarithmic normal distribution into equation (1) and integrating between x_{min} = 0 and x_{max} = ∞ . The values obtained are given in Table A.2.

Column 1 represents the values of the four moments as calculated from the analytical function. The second and the third columns represent the figures obtained in the numerical calculation using either the R10-series or the R5-series. The calculated data differ slightly from the ones given in column 1, but the agreement remains within a few percent, as shown by the deviations given in columns 4 and 5.

Table A.3 shows the average particle sizes as calculated from the moments of Table A.2, taking into account equations (17) to (21). The differences between the analytical results and the ones obtained using the R10-series and the R5-series are small. In principle, the R10-series is to be preferred over the R5-series as it causes smaller deviations.

Table A.1 — Basic data of the assumed logarithmic normal distribution for the calculation of the moments

x_i µm		z_i		$Q_{3,i}$		Δx_i µm		$\Delta Q_{3,i}$		$\bar{q}_{3,i}$ µm^{-1}	
R10	R5	R10	R5	R10	R5	R10	R5	R10	R5	R10	R5
25,00	25,00	3,22	3,22	0,999 3	0,999 3						
19,86		2,76		0,997 1		5,14		0,002 2		0,000 4	
15,78	15,78	2,30	2,30	0,989 2	0,989 2	4,08	9,22	0,007 9	0,010 1	0,001 9	0,001 1
12,53		1,84		0,967 1		3,25		0,022 1		0,006 8	
9,96	9,96	1,38	1,38	0,916 2	0,916 2	2,57	5,82	0,050 9	0,073 0	0,019 8	0,012 5
7,91		0,92		0,821 2		2,05		0,095 0		0,046 3	
6,28	6,28	0,46	0,46	0,677 2	0,677 2	1,63	3,68	0,144 0	0,239 0	0,088 3	0,064 9
4,99		0,00		0,500 0		1,29		0,177 2		0,137 4	
3,96	3,96	–0,46	–0,46	0,322 8	0,322 8	1,03	2,32	0,177 2	0,354 4	0,172 0	0,152 8
3,15		–0,92		0,178 8		0,81		0,144 0		0,177 8	
2,50	2,50	–1,38	–1,38	0,083 8	0,083 8	0,65	1,46	0,095 0	0,239 0	0,146 2	0,163 7
1,99		–1,84		0,032 9		0,51		0,050 9		0,099 8	
1,58	1,58	–2,30	–2,30	0,010 8	0,010 8	0,41	0,92	0,022 1	0,073 0	0,053 9	0,079 3
1,26		–2,76		0,002 9		0,32		0,007 9		0,024 7	
1,00	1,00	–3,22	–3,22	0,000 7	0,000 7	0,26	0,58	0,002 2	0,010 1	0,008 5	0,017 4

Table A.2 — Comparison of the analytical and the numerical calculation of the moments

	Analytical result	R10-series	R5-series	Deviation R10 %	Deviation R5 %
$M_{1,3}$, µm	5,666	5,685	5,835	0,335	2,983
$M_{-1,3}$, µm^{-1}	0,227	0,226	0,226	−0,441	−0,441
$M_{-2,3}$, µm^{-2}	0,066	0,066	0,068	0	3,030
$M_{-3,3}$, µm^{-3}	0,025	0,024	0,026	−4	4

Table A.3 — Comparison of the averaged particle size diameter as obtained from the analytical and numerical calculations

	Analytical result	R10-series	R5-series	Deviation R10 %	Deviation R5 %
$\bar{x}_{1,0}$, µm	2,640	2,717	2,604	2,917	−1,364
$\bar{x}_{2,0}$, µm	3,015	3,053	2,944	1,260	−2,355
$\bar{x}_{3,0}$, µm	3,420	3,453	3,371	0,965	−1,433
$\bar{x}_{1,1}$, µm	3,444	3,429	3,329	−0,436	−3,339
$\bar{x}_{1,2}$, µm	4,399	4,419	4,418	0,455	0,432
$\bar{x}_{1,3}$, µm	5,666	5,685	5,835	0,335	2,983

Annex B
(informative)

Further average particle diameters

Further average particle diameters cannot replace the physical meaning of arithmetic or weighed average particle diameters or the calculation of volume specific surface area.

Since the average particle diameters may differ considerably (depending on the spread of a distribution), their prerequisites and definitions should always be quoted.

B.1 Geometric mean diameter

The emergence of the geometric mean is due to the empirical finding that a particle size distribution can conform satisfactorily to the logarithmic normal probability function (see [5] in the bibliography), in which case the geometric average particle diameter represents the expectation value with the greatest probability of the logarithm of x. For a logarithmic normal probability function it is the same as the median.

Instead of the arithmetic mean, calculated from the sum of n values, divided by their number n, the geometric mean is the n-th root of the product of n values. In terms of logarithms, the logarithm of the geometric mean is calculated from the sum of the logarithms of n values, divided by their number n. The arithmetic mean is greater than the geometric mean and this difference increases as the dispersion between the two values increases.

Mathematical limit analysis of equation (6) with k approaching zero (see [4] in the bibliography), leads to the following geometric mean diameter:

$$\bar{x}_{0,r} = e^{\int_{x_{\max}}^{x_{\min}} \ln x \, q_r(x) \, \mathrm{d}x} = \bar{x}_{geo,r} \tag{B.1}$$

or, in terms of logarithms:

$$\ln \bar{x}_{geo,r} = \int_{x_{\max}}^{x_{\min}} \ln x \, q_r(x) \, \mathrm{d}x = \int_{x_{\max}}^{x_{\min}} \ln x \, q_r(\ln x) \, \mathrm{d}(\ln x) \tag{B.2}$$

B.2 Harmonic mean

The harmonic mean of a series of values is the reciprocal of the arithmetic mean of their reciprocals (see [2] in the bibliography). The harmonic mean is smaller than the geometric mean and this difference increases as the dispersion between the values becomes greater. Therefore the harmonic mean diameter can be calculated from:

$$\bar{x}_{har,r} = \frac{1}{\int_{x_{\max}}^{x_{\min}} \frac{1}{x} q_r(x) \, \mathrm{d}x} = \frac{1}{M_{-1,r}} \tag{B.3}$$

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