ESSAY 7

Pragmaticism as an Anti-Foundationalist Philosophy of Mathematics

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According to Charles Peirce's pragmaticism, mathematical reasoning concerns the creations of the mind. It emphasises experimentation and observation on diagrammatic and iconic representations of these creations, and does not presuppose mathematical foundations. This paper contrasts Peirce's pragmaticism with a number of recent philosophies of mathematics, including intuitionism, structuralism, fictionalism, platonism, and quasi-empiricism. I argue that pragmaticism, as an anti-foundationalist philosophy of mathematics, is a positive thesis

about such diagrammatic and iconic representations drawing from actual mathematical practice.

Peirce's remarks on the philosophy of mathematics are not well known, and appear among his actual mathematical and logical works. Many of them have not been made public to date. Some of these remarks Peirce never explicated with any particular regard for the philosophical aspects of mathematics. Rather, they relate to what he sees essential in actual mathematical practices, and as such are not separate from his extensive research on logic, semeiotic, phenomenology, and the methodology of special sciences. After all, the entire industry christened as the 'philosophy of mathematics' has been post-Peircean.

This ought not to block our philosophical road to Peirce's inquisitive thoughts in the least, however. In this paper I wish to establish that what Peirce has to say about the philosophy of mathematics is very significant. My task is to study what he had in the offing in comparison with what we nowadays regard as pivotal research questions falling within the philosophy of mathematics. Such a study enables us to put his views into a sharper focus than what might be possible by merely trying to understand his views in their own right, or as if his handiwork were only of some historical or exegetic interest. As a by-product I hope to communicate how the comparison enables us to appreciate when something goes wrong in the received theories concerning the philosophy of mathematics and mathematical practice.

1 Pragmaticism as an Anti-Foundationalist Philosophy of Mathematics

I have argued in Pietarinen [forthcoming] that Peirce's pragmaticism articulates a self-standing philosophy of mathematics which differs markedly from the foundationalist philosophies such as logicism, intuitionism, and platonism. It differs from Hilbert's axiomatic programme and quasi-empiricism in notable respects, too. A term of art that might be used to characterise Peirce's position is *anti-foundationalism*. As such it does not explain much, however. In which sense is pragmaticism not a foundationalist philosophy of mathematics? What is Peirce's philosophy of mathematics? What is its positive contribution? These questions define my agenda in the present treatise.

A couple of remarks that Peirce makes on the nature of mathematics are particularly relevant to our concerns. In 1896 he noted the following:

> It is an error to make mathematics consist exclusively in the tracing out of necessary consequences. For the framing of the hypothesis of the two-way spread of imaginary quantity, and the hypothesis of Riemann surfaces, were certainly mathematical achievements. Mathematics is, therefore, the study of the substance of hypotheses, or mental creations, with a view to the drawing of necessary conclusions.
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> (NEM 4:268, 1896, *On Quantity, with Special Reference to Collectional and Mathematical Infinity*)

The substance of mathematics refers to mental creations. Despite this allusion, which might, sight unseen, suggest that Peirce's thought is allied with intuitionistic philosophy, I shall argue that there are compelling reasons to take Peirce's views to be quite remote from the concerns of intuitionism. After all, intuitionism is a foundationalist philosophy which takes mathematical objects to be created or constructed by our mental and cognitive processes. It is the aspects of mathematical reasoning, Peirce explains a couple of years later, that are hypothetical and whose constructions refer to the "creations of the mind":

> Mathematical reasoning holds. Why should it not? It relates only to the creations of the mind, concerning which there is no obstacle to our learning whatever is true of them . . . It is fallible, as everything human is fallible. Twice two may perhaps not be four. But there is no more satisfactory way of assuring ourselves of anything than the mathematical way of assuring ourselves of mathematical theorems. No aid from the science of logic is called for in that field.
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> (CP 2.192, c.1902, *General and Historical Survey of Logic: Why Study Logic?*)

Mathematics and logic are strictly separate sciences, as Peirce always insists, having learned that notion from his father, Benjamin Peirce. Mathematicians *reason*, while logicians are engaged in the *study* of the processes and theories of reasoning.

That proper mathematical reasoning "relates only to the creations of the mind" does not mean, unlike in intuitionism, that it is mathematical objects that are created in the minds of those who practice mathematics. It means that in studying the outcomes of the mental processes correlated with mathematical reasoning we are at once engaged in studying those aspects of the mathematical structures in which real mathematical objects are seen to figure. Mathematical signs have objects and interpretants just as any other kinds of signs do. Unapproachable by the intuitionist philosophy of mathematics, Peirce succeeds in avoiding unwarranted appeals to psychologism and mentalism.

Another observation cropping up in the previous quotation is the fallible character of mathematics. I will comment on fallibilism in mathematics in a moment; what is clear by now is that Peirce's disdain for foundationalist approaches to mathematics does not imply rejecting or denying anything of the relevance and reality of mathematical *facts* just as it does not mean rejecting or denying the reality or objectivity of mathematical *objects*. The denial of anything of this kind would mean subscribing to a substantial foundational position, such as nominalism, in order to be able to explain at all of what such a negation is in fact taken to consist. Accordingly, one conclusion I argue for in this paper is that pragmatistic stance disavows those off-shoots of structuralist philosophies of mathematics that see mathematical objects are objects of fiction (Field [1980]).

Pragmatistic philosophy recognises that (i) laws of logic, just as laws of nature, are subject to change, and that (ii) no single science is capable of founding all others. In mathematics, logic serves no foundational purpose.[1] In fact, in Peirce's classification of the sciences, mathematics and phenomenology are sciences that precede normative logic (Pietarinen [2006*a*]), and mathematics is not grounded on anything else. In a very concrete sense, thus, it is mathematics that provides the *a priori* for metaphysical investigations.

Pragmatistic philosophy of mathematics accentuates the importance

[1]Blais ([1989]) has characterised a kind of anti-foundationalist pragmatic philosophy of mathematics that might have satisfied William James but probably not Peirce, since it overlooks the finesse of Peirce's position. Patin [1957] is one of the earliest studies on Peirce's philosophy of mathematics. Its representation of the main tenets of pragmatism, intuitionism, and formalism is entirely obsolete, however.

of the actual practice of mathematics. At the same time, it displays an astute recognition of the existing aperture between the actual practices of mathematicians on the one hand and mathematical systems (axiomatic, formal systems) on the other. Since Charles, just as Benjamin Peirce, was an accomplished mathematician, he held a privileged vantage point from which to observe the existence of such a gap and the reasons for its existence. One of his observations concerns the interpretation of Aristotelian axiomatic method: that such a method is bound to be quite foreign to mathematical practices if it is taken to constitute the bulk of what mathematicians do, namely to deduce (by syllogistic reasoning or otherwise) theorems from a given collection of axioms.

Being a working mathematician, it might also appear puzzling why Peirce was not a platonist regarding the nature of mathematical knowledge and mathematical objects. Briefly, the reason is methodological: Mathematics suggests deep metaphysical questions that need serious philosophical analysis, while the quest for answers must be guided by an insight into the nature of mathematics as well as a broad application of concrete mathematical practices. I will come back to platonism in a separate section below.

Nor can Peirce's approach to the metaphysics of mathematics reasonably be claimed to be naturalistic, either. The nature of mathematical entities is wholly different from the nature of the entities, including laws or principles governing mathematical discoveries, that are established by the methods of natural sciences.[2]

What Peirce would not say, it is appropriate to emphasise, is that these mathematical practices are necessarily social. Popular treatises tend to emphasise that in order for any true scientific inquiry to make progress, it needs to go through the common and shared public investigative efforts of the entire scientific community. But mathematical

[2]Kitcher's ([1984]) naturalistic and quasi-empiricistic approach to mathematical objects is a case in point of unwarranted naturalisation of mathematics from the Peircean practice-based point of view. Hoffman ([2004]) suggests that Kitcher's approach be 'fictionalised' by turning the notions Kitcher uses as examples of idealisation in mathematics and science into characters of fiction. This proposal contains a fallacy of moving from the non-existence of idealised objects to fictional objects, however. Why it is a fallacy is shown, among others, by its unviable consequence: It would establish the fictitious character of the whole of the objects of science in the same go. For a critique of fictionalism, see a separate subsection below.

facts cannot be results of inherently social practices. What matters is that mathematicians act in accordance with the habits of reasoning in a certain way in response to certain kinds of mathematical problems. The cultivation and modification of these habits does not rest on social factors but on self-controlled action guided by the ideals that mathematicians contemplate in their minds in dealing with mathematical problems.[3] Peirce calls such an instinctive and stable faculty of reasoning the inquirer's *logica utens*. It is with such a *utens* that mathematicians are able to prove theorems, not by the *logica docens*, which is the schooled and developing faculty of theories of reasoning (Pietarinen [2005]). Peirce explains this matter in the passage immediately preceding the quotation above concerning the nature of mathematical reasoning:

> You think that your *logica utens* is more or less unsatisfactory. But you do not doubt that there is *some* truth in it. Nor do I; nor does any man. Why cannot men see that what we do not doubt, we do not doubt; so that it is false pretence to pretend to call it in question? There are certain parts of your *logica utens* which nobody really doubts . . . The truth of it is too evident. Mathematical reasoning holds. Why should it not? It relates only to the creations of the mind. (CP 2.192)

A characteristic feature of pragmaticism is that there is no and need not be any ultimate basis for knowledge. It is the basic property of Peirce's notion of *continuity*, which characterises the "true continuum," that all cognition rests on former cognitions but that there is not and need not be any first cognition or first transcendental object. Yet his intuitive concept of such a continuum has eluded mathematical definitions. We cannot even begin to survey the attempts that have been made to that effect here,[4] but let us note that Peirce thought true continuum to be created by a peculiar phenomenological insight rather than by any

[3] See here Peirce's unpublished MS 280, 1905, *The Basis of Pragmaticism*. Transcription available at http://www.helsinki.fi/~pietarin/.

[4] See e.g. Hudry [2004], Myrwold [1995], Ehrlich [forthcoming], and Stjernfelt [2007] for recent attempts and expositions.

well-definable mathematical structure. This does not mean that an approximate mathematical model could not be found for it. Such a model may well be within the reach of contemporary metamathematics, in so far as it is taken into account that the investigation must take place in the borderlands of mathematics (the 'first' science of discovery for Peirce according to this classification of the sciences) and phenomenology (the 'first' philosophy and the 'second' science of discovery).

This understanding of continuum objectifies *fallibilism*, the view that our current theories of science, including our mathematical theories concerning mathematical facts, may turn out to be false. Reasoning and observation in science is performed by us, human beings. Products of science follow from the methods employed in reasoning and observation. Science, including exact sciences, are anthropomorphic in the sense that we never acquire absolute certainty concerning the truth of our best scientific theories. Since Peirce's continuum adds to actuality all that is possible—including all possible mathematical entities, all possible objects, relations, propositions, and facts—and since possibility greatly outweighs actuality, there will be an inevitable uncertainty and vagueness in reality that cannot be disposed of even by the best theories and the best methods of exact sciences currently at hand. The point is recorded here:

> The principle of continuity is the idea of fallibilism objectified. For fallibilism is the doctrine that our knowledge is never absolute but always swims, as it were, in a continuum of uncertainty and of indeterminacy. Now the doctrine of continuity is that *all things* so swim in continua.
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> (CP 1.171, c.1897, *Notes on Scientific Philosophy*)

It is from this *synechistic* and modal nature of the true continuum that fallibilism of our best theories ensues, not vice versa. However, since the model of synechistic mathematics is something that is not and need not be 'well founded', the fallibilistic epistemology concerning mathematical truths receives its explanation without any need of seeking being founded on anything else.[5]

[5]One might contemplate models of non-well-founded set theory as the suitable candidates for Peircean continuum. Even if there be some analogy, as such non-well-

2 Which Philosophy of Mathematics Is Pragmaticism Not?

Given these introductory remarks on Peirce on the nature of mathematics, I will next contrast his views with some of the subsequent work on the nature and practice of mathematics.

2.1 INTUITIONISM

A couple of years after Peirce, Hermann Weyl ([1987]: 119), and L.E.J. Brouwer ([1975]) arrived at a position that seems similar to the Peircean continuum: The continuum is "intuitive" and to be conceived "as-a-whole." Brouwer's view differs markedly from Peirce's in crucial respects, however. According to Brouwer, "Mathematics is a free creation, independent of experience; it develops from one single a priori Primordial Intuition" (*On the Foundations of Mathematics* 1907: 179, in Brouwer [1975]). This contrasts sharply with Peirce in three respects. First, mathematics is not independent of experience since experience and theory are not separable in pragmaticism.[6] Second, any appeal to Cartesian intuition or a Kantian transcendental object as the final resort by which mathematical knowledge is fashioned is rejected by a pragmaticist.[7] Third, the mind is not for Peirce an individual or single creating or constructing solipsist "Subject" capable of intuition, but a collective and general "creatory" (MS 318: 18) of all kinds of signs, including signs standing for mathematical ideas. Consequently, mathematics is not absolutely "free creation"; it proceeds according to the habits of thinking

foundedness it is not enough. It suffers from the dubious notion of a set which Peirce wanted to replace with the notion of a collection, which is a "derivative individual" and does not have any "characters" attached to it (*Peirce to Cantor*: 2, 21 December 1900; Pietarinen [forthcoming]). Moreover, non-well-founded sets give rise to point-like structures just like ordinary axiomatic set theories with the axiom of foundation do and thus cannot agree with the true continuum.

[6]Experience of course does not mean only sense experience.

[7]The sole idea of Descartes that both Peirce and Brouwer might have agreed with is that by logic we do not create new mathematical truths, since according to both, logic is the science which studies the processes of drawing necessary conclusions, whereas mathematics differs from logic in being the science which draws necessary conclusions.

and reasoning, which are real generals linking contexts of mathematical discovery with action.

Yet it is true that Brouwer's concept of "the Intuitive Continuum" has some resonant similarities with Peirce's. For instance, in extending the continuum with the possible alongside the actual Brouwer broadened his philosophical horizon and did not remain an actualist about mathematical objects. But for Brouwer, the possible stood for "identifiable points," whereas Peirce rejected the notion that the "*would-be*s"—those preliminary conceptions of what might or could happen when we enquire about the world, or the inner thought, or the parts of the worlds or thoughts connected with our mutually agreed universes of discourse—can possess any point-like identities.

Consequently, Murphey ([1961]) and Engel-Tiercelin ([1993]) are mistaken in holding Peirce's views to stand in strict opposition to intuitionism because of intuitionism's commitment to actuality only. In Brouwer's formulation, possibilities are admissible constituents of the continuum. However, their metaphysical status and nature was quite different from Peirce's realistic conception influenced by his reading of the scholastic philosophers. According to Peirce's scholastic realism, what is possible is as real as what is actual. Unlike Brouwer, Peirce did not wish to extend the continuum with the possibilia merely to resolve set-theoretic paradoxes.[8]

Nevertheless, some similarities do take place between intuitionism and pragmaticism (Pietarinen [2006*b*] and [forthcoming]). Both take mathematical reasoning as related to the creations of the mind. The law of excluded middle is rejected as an a priori logical principle.[9] Existence is subordinate to inferential and cognitive processes and to the activities of seeking and finding. Unlike intuitionism, however, Peirce did not consider the failure of the law of excluded middle to be a failure of

[8]Peirce's remarks on set-theoretic paradoxes come from an incomplete note "Mr. Bertrand Russell's Paradox" (MS 818, 1911, 5 pages). In that note, he takes the sentences giving rise to an alleged paradox "to be easily interpreted so as to remove their contradictory sound" (ibid.: 3). Whether he had a predicative re-interpretation in mind of what the sentences define is impossible to tell, since the note ends abruptly before we get a comprehensive exposition of the suggested solution, and the remaining pages have been lost.

[9]Peirce studies indeterminate properties in terms of the limit interpretation.

proving or being able to provide a *construction* for one of the disjuncts, but that there are situations (models) in which none of the propositions can be *asserted*.

2.2 STRUCTURALISM

Peirce's position on continuity makes his philosophy of mathematics incompatible also with mathematical structuralism. According to (*ante rem*) structuralism (Shapiro [1997]), objects of mathematics are identified by their positions or roles that they have in various kinds of mathematical structures. Hence the subject of study in the philosophy of mathematics is the general account of all kinds of structures, such as the study of classes of models or theories of manifolds.[10]

Structuralism displays some likeness to pragmaticism. What Peirce takes to be real in mathematical constructs, in the sense of real being that which is independent of what anybody thinks of it, are the relations in hypothetical creations of the mind. Likewise, structuralism, especially category-theoretic structuralism of Hellman ([2004]), takes mappings or morphisms to be the first-class entities of mathematics, and mathematical objects to emerge only derivatively. Peirce took forms of relations to be all-important in constituting the subject matter for the logical investigation of the category of secondness, namely that of what exists and what is actual. This derivative nature of objects with respect to mappings means that their identities follow from the locations or places those objects have in the "mental chart"[11] created by mathematical imagination. In this sense, therefore, to take the identities of objects to draw from the locations or places those objects have on a map, chart, or structure is to subscribe to a version of structuralism concerning the primacy of relations over objects in mathematical ontology.

In the end, however, pragmaticism turns out to be incompatible with structuralism. The latter carries with it a nominalistic assumption of

[10]Since time is a continuum in which one can count even though there is no natural notion of a successor in time, mathematical structuralism should not be confounded with Kantian structuralism, a nominalistic philosophy of mathematics that Benacerraf ([1973]) has defended.

[11]MS 280, 1905, *The Basis of Pragmaticism.*

atomicity of mathematical objects. Given Peirce's characterisation of the true continuum, however, individual identities dissolve in it because it "is something whose possibilities of determination no multitude of individuals can exhaust" (CP 6.170, 1901, *Synechism*). There are neither individual points nor elements constitutive of continuity, and hence positions for those points or elements cannot be identified in such continuum. And so from the synechistic (and inter alia pragmatistic) point of view, the basic doctrine of structuralism is not satisfied. Therefore, from the overall pragmatistic point of view structuralism is bound to fail as a foundational enterprise for mathematics.

There are other versions of structuralism, in particular the modal structuralism of Hellman ([1989]), that come closer to pragmatistic concerns. Modal structuralism strives to dispense with those nominalistic assumptions of structuralism that appeal to atomistic postulates. However, it is still a version of category-theoretic structuralism, and it is likely that, because of its reliance on collections of mappings and structures as point-like constructions, Peirce would have regarded it as a form of nominalism. The argument certainly deserves longer exposition, as the conclusion hinges on the notion of morphism as an explication of general properties of mathematical domains as well as the concept of continuity involved in category theory. Nonetheless, in an undisputable sense category theory is just like set theory as the distinction is routinely drawn between small and large structures.

Hellman ([2004]) suggests overcoming these limitations and set-theoretic commitments by reformulating mathematics topos-theoretically, providing its own universe of discourse, which arguably is no longer a set-theoretic one. Without discussing here the point as to whether this logical approach to category theory can really do without set theory, we note that, interestingly, topos theory appears to provide a mathematical theory of *iconicity* in terms of homomorphism between domains and co-domains analogous to, for instance, continuous maps between topological spaces. Notable here is that the line of research that Hellmann advocates have been pursuing is a foundationalist approach to mathematics and not so much an alternative philosophy to it.

Modal structuralism, which attempts to dispense with the remaining nominalistic undercurrents of ordinary category-theoretic structuralism,[12] nevertheless coheres with Peirce's pragmaticism in certain respects. Hellman seeks to avoid the set-theoretic commitment to the totality of the universe of mathematical objects. Mathematical domains are indefinitely extendible and relative to possible worlds in the sense that mathematical constructions talk about hypothetical constructions. Mathematics, according to Hellman ([2004]: 146), concerns "what would necessarily be the case were the relevant structural conditions fulfilled." Peirce took mathematical reasoning to be of this subjunctive form while concerning necessary reasoning under such hypothetical constructions. Hellman (ibid.) takes mathematics to make "no actual commitment to objects at all, only to (propositionally) what might be the case." Like Peirce, he takes mathematics to concern possible objects. He then suggests an axiomatisation of modal existence by axioms of second-order modal logic. Like Peirce, Hellman takes actual mathematical practice to guide the axiomatisation of theories. Here it means looking at the mathematical practice that could determine what the axioms for the S5 system of second-order modal logic might look like. A question not to be forgotten is that, if those axioms and the language of S5 are couched in set-theoretic terms, does it imply that their origin is in the axiomatic set theory? Hellmann says little about the semantics for S5 for apparent reasons.

From having such a very strong logic at his disposal, Hellman encounters similar issues as Peirce did in attempting to make sense of the identities between what is actual and what is possible.[13] Hellman thinks that we cannot in fact have quantification over relations or similar higher-order entities since that would commit us to identifying actual relations with possible relations. Hence, he does not accept cross-identification with respect to higher-order notions and takes each possible world to constitute its own mathematics.

Hellman's theory opens up significant perspectives on the indispensability of higher-order notions such as relations, functions, and map-

[12]Hellman's proposal continues what Bell ([1988]) suggested in replacing the absolute universe of set theory with the plurality of the universes of topoi, each providing a possible world in which mathematics is made.

[13]See Peirce's 1906 *Prolegomena to an Apology for Pragmaticism*; CP 4.530–72; and Pietarinen [2008].

pings as the building blocks of fundamental mathematical ontology.[14] However, we might ask whether his proposal runs into serious problems when the *epistemology* of mathematics entities is at issue. Without cross-identification, we are denying that these relations, functions, or mappings are genuine many-world entities. From the metalogical perspective, we would be prevented from knowing what or which relations or functions they in fact are. Such a denial means depriving mathematical knowledge of some of its key subject matters. And there are credible grounds to take the indispensability of such higher-order notions to also mean our mathematical knowledge of them. Mathematical knowledge requires their identification. Hence, what lurks around the corner after all in Hellman's account is nominalism with respect to relations and functions.

Hellman's proposal differs from Peirce in that Hellman does not take possible objects to be real possibilities in the sense of being the real constituents of the objective world that includes mathematical entities and facts. This point follows from his rejection of the possibility of cross-identification of higher-order notions. Hellman indeed admits that his talk of possible worlds is "heuristic only" and that there are "literally no such things" such as those that merely might have existed (Hellman [2004]: 147). His second-order comprehension schema applies only "within a world" (ibid.). In contrast, Peirce had no such scruples and took real possibilities to be true constituents of the world. His scholastic realism, which lies at the core of pragmaticism, applies to mathematics just as to metaphysics. Pragmaticism, Peirce states, "is most concerned to insist" upon "the reality of some possibilities." (CP 5.453, 1905, *Issues of Pragmaticism*), which is precisely what Hellman denies.

Consequently, pragmaticism cannot be seen compatible with modal structuralism, either. However, developing a realistic alternative to Hellman's modal-structuralist mathematics, with the notion of an identification of second-order entities across possible worlds guided by mathematical practice, might turn out to be one of the closest contemporary counterparts to the pragmatistic restatement of the philosophy and practice of mathematics.

[14] Colyvan ([2001]) discusses the indispensability arguments in mathematics and science.

2.3 FICTIONALISM

Burgess ([2004]: 19) claims that modal accounts of mathematics pertain to another off-shoot of structuralism known as fictionalism. From the pragmatistic perspective this is an oversimplifying assimilation. Peirce's scholastic realism takes possibilities to be just as real as actualities and having nothing to do with fiction. What is real, including a real possibility, is in fact a contrary to what is fiction: According to Peirce, "[I]f it be not real it can only be fiction: a Proposition is either True or False" (CP 4.547 and *Prolegomena*).

Yet in the literature a mini-industry has emerged holding that on the ruins of structuralism another nominalistic philosophy can be erected (Field [1980] and [1989]). The bottom line is that mathematical statements are not about just any mathematical thing at all. Mathematical statements build up narratives, and just as narrative stories of fiction, they give rise to fictional entities. Since there are no objects, mathematical statements are strictly speaking all false.

From the pragmatistic vantage point, such an extreme nominalism means blocking the road to mathematical inquiry. How can fictionalism account for progress in mathematics?[15] What is the ontology of fictional entities?[16] Peirce would have not accepted a fictionalist way of looking at mathematics. His remarks on fiction and the use and meaning of fictional names are plentiful, but they never bear on mathematics: "The fictive is," he tells, "that whose characters depend upon what characters somebody attributes to it; and the story is, of course, the mere creation of the poet's thought" (CP 5.153, 1903, *Three Types of Reasoning*). But mathematics is not a story created by somebody, namely a mathematician, just as the identity of mathematical objects cannot depend solely on what characters this somebody happens to attribute

[15] Recall here the 'no miracle' arguments in the philosophy of science. Consistency is not a sufficient requirement, since there are uncountably many 'stories' consistent with any previous installment of the 'story.' On the other hand, inconsistency is not in such bad books as philosophers of mathematics might have thought. Often, discovery and progress in mathematics consists precisely in the capability of hitting on right kinds of inconsistencies.

[16] Ontological maxims to save fictionalism, such as requirements of economy or parsimony of fictional entities do not help, because mathematics does not ascribe to such maxims, mathematics simply is not simple.

to them. The identities of mathematical objects must be free from such singular attributions.

Moreover, if the ontology of mathematics corresponds to fiction as novels or narratives correspond to fictional objects, as it according to fictionalism does, mathematics would fall short of serving as the bedrock science of discovery from which other fields of sciences of discovery, namely philosophy and the special sciences, draw their inquisitive inspiration (I am referring here to Peirce's perennial classification of sciences, see Pietarinen [2006*a*]). Given the order of dependence as the one Peirce sketched in his classification of the sciences, fictionalism forces the ontological status of the objects of all sciences to be fictional. However, according to fallibilism that characterises pragmaticism, not all scientific statements can be false in one go, although any one of them may turn out to be subject to revision or rejection in the long run.[17]

There are other arguments against fictionalism that can be marshalled. One begins with the Brouwerian premise that mathematics is not a language at all, that mathematics has nothing to do with language. Since fictionalism presupposes the possibility of narratives, and since narratives are necessarily linguistic, from the point of view of Brouwer's intuitionistic, languageless mathematics, fictionalism is unattainable.

In conclusion, fictionalism is not a plausible alternative to structuralism. It mistakes the view that mathematical reasoning concerns imaginary or phenomenal objects created by the processes of the mind for the view that takes these imaginary or phenomenal objects to be on a par with fiction.[18]

2.4 PLATONISM

Some take fictionalism, and incorrectly to my judgment, to be a contemporary recasting of platonism. My question is phrased as one concerning the relationship between platonism and pragmaticism. Since Peirce was a working mathematician and a scholastic realist about mathematical

[17]Taking refuge in conservatism about mathematics does not help here, either, since pragmaticism does not accept sciences of discovery to rest on a nominalistic basis.

[18]Thomas ([2007]) presents some more cautions reservations concerning the coherence of fictionalism in mathematics.

entities, it might appear that he would have subscribed to some version of platonism. This was not the case, however. According to that heterogeneous idea differently characterised over decades of dispute among philosophers of mathematics, what mathematics deals with are abstract objects that exist independently of our ways of conceiving them. Existence takes place in the totality of the universe within which our actual world obtains an accidental location. Abstract objects and mathematical forms linger in that external world and are up for grabs to be picked out or defined by the hardworking mathematician who is on the lookout for them. Yet they are something neither material nor mental. And if so, how can we know of such things?

According to pragmaticism, mathematical objects are, indeed, very concrete and real, tangible objects, but at the same time the manifold ways of conceiving them are amenable to constant experimentation and observation. The objects do not exist as platonic forms are taken to exist. Actuality and existence are processes undergoing continuous maintenance. They are "occurrences" in the universes of discourse.[19] Existence comes in degrees, and there are countless kinds of existences. For one, existence requires identification. Therefore Peirce, despite his deep respect for many aspects of Plato's philosophy, could not have taken seriously mathematicians' routinely expressed belief—almost a blind faith—in a pre-existing realm of mathematical abstractions that is somehow ready-made and waiting to be discovered.

That a mathematician's attitude is typically platonistic was aptly recognised by Peirce early on:

> If you enjoy the good fortune of talking with a number of mathematicians of a high order, you will find that the typical pure mathematician is a sort of Platonist. Only, he is [a] Platonist who corrects the Heraclitan error that the eternal is not continuous. The eternal is for him a world, a cosmos, in which the universe of actual existence is nothing but an arbitrary locus. The end that pure mathematics is pursuing is to discover that real potential world.
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> (CP 1.646, 1898, *Vitally Important Topics*)

[19] CP 1.214, c.1902, *A Detailed Classification of the Sciences*; CP 1.358, c.1890, *A Guess at the Riddle.*

The "real potential world" does not refer to real possibilities and real generals in the sense of Peirce's scholastic realism but to the platonistic, sense-transgressing region of the world of existence. To believe in an eternal but a "real potential world" of mathematical objects waiting to be discovered would be incompatible with the tenets of scholastic realism. According to one of them, general laws are real and operative ingredients of nature. This is not to deny that platonism could not accept change (it does, since forms may undergo change according to Plato's metaphysics), only that: (i) Non-actual but extant forms are determinate in the manner real possibilities are not; (ii) principles accounting for determinate mathematical forms are different from principles operative in Peirce's evolutionary metaphysics; and (iii) in platonism, those principles are not applicable to mathematical truths to the extent they are applicable in pragmaticism.

Second, the platonistic belief in the pre-existing realm of mathematical abstractions is in stark violation of what Peirce dubbed *critical common sensism*, according to which indubitable beliefs, which everyone must possess, are inherently vague and which through criticism such as logical analysis and rational deliberation are subject to revision, change and possible rejection.

3 Observation and Diagrammatism

3.1 ICONS AND INDICES IN MATHEMATICS

In not accepting the platonistic conception of the cosmos, Peirce in fact came much closer to Aristotle's philosophy. Concerning experience and observation it is worth noting how he mimicked Aristotle's comment in stating that "nothing emerges in meaningful conception that first does not emerge in perceptual judgment" (EP 2:226, 1903, *Pragmatism as the Logic of Abduction*). He means that all sciences of discovery rest on observation. Observation is either *general* as in philosophy or *singular* as in mathematics and the special sciences, *degenerate* as in mathematics, or *non-degenerate* as in logic, philosophy, and the special sciences. Mathematics in particular "is observational," Peirce explains, "in so far as it makes constructions in the imagination according to abstract precepts, and then observes these imaginary objects, finding in them relations of parts not specified in the precept of construction. This is truly observation, yet certainly in a very peculiar sense; and no other kind of observation would at all answer the purpose of mathematics" (CP 1.240, 1902, *A Detailed Classification of the Sciences*).

On the face of it, this statement does remind us of intuitionism. In it as well as in pragmaticism, mathematical ontology is a hypothetical system of constructs. But pragmatistic constructs are based on very specific kinds of entities: They are *diagrams*. Building upon abstract precepts, mathematicians develop hypotheses based on observing, experiencing, and operating on the relationships exhibited in the diagram and in the domain, including empirical domains. Mathematical objects and structures are thus strictly speaking not abstract even though they are imaginary. What is abstract are the precepts that we employ to oversee the construction of mathematical forms. The diagram is a general but finite schema whose relationships can be predicated of an infinite collection of empirical objects of mathematics. The domains represented by diagrams can be infinitely extendible. This makes Peirce's philosophy quite unlike what Brouwer would have us believe.

The key feature of diagrams is that they are *iconic*: "A great distinguishing property of the Icon is that by the direct observation of it other truths concerning its objects can be discovered than those which suffice to determine its construction" (CP 2.279, 1901, *The Icon, Index and Symbol*). Peirce maintains that iconicity, which often does not involve similarity or likeness—that is, it need not exhibit any "sensual resemblance" (ibid.) between representations and things represented—lies at the bottom of all truly fertile mathematical practices.[20] Icons exhibit abstract and structural relationships and processes that preserve structural properties of the domains of investigation. Peirce provides a number of examples from algebra and geometry to support his notion of iconicity.

[20] Peirce states this in the context of his diagrammatic logical theory of existential graphs and sees it indispensable in scientific inquiry: "Diagrammatic reasoning is the only really fertile reasoning. If logicians would only embrace this method, we should no longer see attempts to base their science on the fragile foundations of metaphysics or a psychology not based on logical theory; and there would soon be such an advance in logic that every science would feel the benefit of it" (CP 4.571, 1906, *Prolegomena to an Apology for Pragmaticism*).

In addition to the pivotal role of icons in mathematical reasoning, mathematics needs indexical signs. Since indices are signs that must have objects, mathematical constructs cannot be mere fictions that are not matching up with any mathematical entities. But the observation that concerns the relationships taking place in the creations of our minds is according to Peirce a degenerate form of observation. It means that the objects of such creations need not be parts of the objective reality of the world but may also refer to images created by the previously encountered mathematical signs and their interpretations. But it would be a fallacy to hold degenerate observation to mean imagining objects of fiction consistent with antecedent narratives by mathematicians, since in the diagrammatic structures created by the mind the real relationships are being observed.

Degenerate observation is effectuated by the application of indexical signs. Indices are labels or names attached to diagrams, geometric constructions and different stages of proofs.

> But the imaginary constructions of the mathematician, and even dreams, so far approximate to reality as to have a certain degree of fixity, in consequence of which they can be recognized and identified as individuals. In short, there is a degenerate form of observation which is directed to the creations of our own minds—using the word observation in its full sense as implying some degree of fixity and quasi-reality in the object to which it endeavours to conform. Accordingly, we find that indices are absolutely indispensable in mathematics.
>
> (CP 2.305, 1901, *The Icon, Index and Symbol*)

Aside from the importance of indices in mathematics, the remark concerning the iconic forms of representation in reasoning suggests an explanation for the peculiarity of multiple conclusions in inferential reasoning. It was a problem for the Aristotelian model of scientific reasoning that syllogistics did not seem to leave room for multiple conclusions, though in actual science, such phenomena are commonplace. Those truths that "suffice to determine [the] construction" of the icon are in Peirce's terminology results of *corollarial* deductions, while "other truths" concerning the objects of the icon and the icon's "capacity of revealing unexpected truths" (CP 2.279) refer to *theorematic* deductions.[21] We often fail to see how to derive the latter kinds of conclusions if we try inferring without icons. Experimenting with icons suggests that something needs to be added to the construction of the proof that is not instantiated in the premises. And it is theorematic reasoning that Peirce takes to be the proper kind of reasoning in mathematics; it is "reasoning with specially constructed schemata" (CP 4.233, c.1903, *The Simplest Mathematics*).

Some important links connect these views to areas of modern mathematics. For example, category-theoretic diagrams, such as topoi, are examples of iconic diagrams by which mathematical discovery can be facilitated. If the outcome of the experimentation upon the diagram is that it commutes, we are attributing a novel relational feature to it. This suggests two important issues: (i) that the commutativity of categories is an instance of theorematic reasoning and (ii) that commutativity exemplifies experimental processes by which reasoning proceeds in category-theoretic domains.

Contrary to some suggestions in the recent literature, iconic diagrams are not mere visual aids or heuristics guiding the mathematician into valid conclusions; conclusions that might be reached without such aids as well, though perhaps with more cognitive energy expended. Observations, including direct observations, concern structural similarities between the different domains that diagrams represent. Experimentation enables to expose to view and observe some hitherto unobserved relations. But diagrams need not be visual diagrams. In many instances they are icons that cannot really be visualised. This fits in well in with Peirce's broad conception of a diagram involving also other modes of representation than visual ones and appealing to other types of perception than visual perception. This happens in category theory in which

[21]The nature of theorematic reasoning has been extensively discussed in the literature (Hintikka [1980] and Webb [2006]). According to Peirce, "[A] Corollarial Deduction is one which represents the conditions of the conclusion in a diagram and finds from the observation of this diagram, as it is, the truth of the conclusion. A Theorematic Deduction is one which, having represented the conditions of the conclusion in a diagram, performs an ingenious experiment upon the diagram, and by the observation of the diagram, so modified, ascertains the truth of the conclusion" (CP 2.267, 1903, *Division of Signs*).

the iconic relationships have to do with recognitions of abstract and structural likenesses rather than similarities in appearances.

Mathematical objects are not strictly speaking abstract even though they are conceived through imaginary means, and so diagrammatic schemas cannot be held to be abstract objects, either. They are instantiated in concrete systems of relationships that are just as real, although they do not exist in the same way, as physical systems of relationships. Mathematical objects are objects of the signs to which mathematical reasoning as "creations of the mind" refer to, but they fall from the relational structures produced by cognitive processes going on in the mind. Since the mind is not an individual, mathematical objects are not merely our own ideas, although they appear to us as consequences of our cognitive activities.

Diagrammatic approach to mathematics based on iconicity of representations involves experimental reasoning and observation by human mathematicians. It is fallible just as the rest of the sciences are fallible. Although its reasonings are necessary,[22] a closer scrutiny of the methods of establishing reasoning reveals, among other things, that we make experiments upon and observations concerning *continuous* representations. And these continuous representations are iconic diagrams. Since Peirce's continuum contains real uncertainty and vagueness, those methods cannot be relinquished from producing fallible results. The degree of certainty as compared with natural or human sciences may well be higher in mathematics, and its self-correcting processes put in place faster than in other sciences, but it is fallible all the same.

A further case in point concerning the fallible character of mathematics is its hypothetical nature appealing to mental creations organised in diagrammatic images of hypothetical conditionals. Critical common sensism of course still dominates here: We must not begin to doubt anything that we have no reason to doubt.[23]

[22]Though as noted, mathematics by no means consists of performing the necessary reasoning.

[23]Cooke ([2003]: 158) argues that the context dependence of science and mathematics in the pragmatic model of inquiry supports the infallibility of mathematics, since inquirers "can never get outside their context of inquiry" to make claims about the certainty of their domains of research. But this view misses the fact that mathematicians and logicians are in fact capable of formulating metamathematical and

3.2 QUASI-EMPIRICISM

A philosophy of mathematics that has gained popularity during the last couple of decades is *quasi-empiricism.*[24] It would be now in order to contrast pragmaticism with quasi-empiricism, too. Hilary Putnam in his commentary to Peirce's *Cambridge Conference Lectures* of 1898 in fact uses the label of quasi-empiricism to tag Peirce's philosophy of mathematics (Peirce [1992]: 74).[25] True, quasi-empiricism shares some of the criticism that Peirce might have been keen to level against the up-and-coming foundational philosophies of mathematics, one of them being the sorting of mathematical truths into factual and conceptual.[26] Quasi-empiricism advances the fallibilistic view that in mathematics, axioms of an axiomatic system may be corrected according to their erroneous consequences and that the laws of logic are subject to change in the light of compelling evidence. It suggests that in mathematics, other methods of reasoning besides deduction are in operation, which of course Peirce recognised long before the coinage of quasi-empiricism.[27]

Given Peirce's attitude that all inquiry is conducted by analogous methods as laboratory activity is conducted, we might, sight unseen, in-

metalogical results concerning their domains of research, by approaching those domains piecemeal and by developing new methods to achieve the goals. To "never get outside their context of inquiry" is in fact an anti-pragmatistic claim about the universality of the notion of context (Pietarinen [2007]). After all, a reliance on the context dependence of inquiry is liable to make all science infallible.

[24]As discussed, among others, by Kitcher ([1984]), Lakatos ([1976]), Maddy ([1997]), Putnam ([1975]), and Quine ([1970]).

[25]The introduction chapter to *Reasoning and the Logic of Things* by Kenneth Laine Ketner and Hilary Putnam also attempts to characterise Peirce's position as an infinitistic version of intuitionism. My paper attempts to show that this cannot be a correct description of Peirce's position, either, since the differences are much more variegated than that of (strict) finitism vs. infinitism.

[26]Better known as the analytic/synthetic distinction. See Levy [1997] and Otte [2006] for related studies. The way in which this dichotomy fails or has been taken to fail is an interesting story of its own.

[27]See e.g. the entry "Logic" in Baldwin's 1901–02 *Dictionary of Philosophy and Psychology* (CP 2.216, 1901, *Why Study Logic?*), in which Peirce laments how the appeal of non-deductive methods of reasoning to mathematics has still not been properly recognised: "The generally received opinion among professors of logic is that all the above methods [of reasoning, namely abduction, deduction, and induction] may properly be used on occasion, the appeal to mathematics, however, being less generally recognized."

terpret Peirce's remarks as embracing the quasi-empiricist vision; that mathematical subjects are indeterminate, that mathematical knowledge is fallible, and that theorems are observation sentences or experimental outcomes of the actual practices and activities of our fellow mathematicians. A quasi-empiricist concludes from these that mathematical and scientific activities are remarkably closely intertwined. The assumption of pragmaticism as a quasi-empiricistic method of investigation in mathematics might also be fuelled by some of Peirce's own remarks, such as one in which he states that "[Mathematics] is supposed to have no empirical element. But this I am quite sure is a serious error" (W4: 556–7, 1884, [*On the Teaching of Mathematics*]; MS 504).

Quasi-empiricism, however, does in one respect not go far enough for Peirce, whose perspective was that all sciences are not only observational but also experimental.[28] The methods of mathematics and the methods of special sciences are continuous in the sense that every experiment is an operation of thought. Branches of sciences are of course observational and experimental in different ways, but in making new discoveries, none of them can dispense with carrying out experiments and then making observations concerning the outcomes of such experiments. The same procedure is, Peirce maintains, in operation in mathematics.

Another realm in which we can discern quasi-empiricist content of mathematical theorising concerns proofs and provability. What counts as a proof and how proofs are carried out depend on the precise nature of experiments performed upon diagrammatic constructions, which are iconic representations of inferential relations holding between premises and conclusions. Here degenerate observation concerns the similarities between the relationships that obtain in diagrams and in what diagrams are representations of. Like many other philosophers of mathematics who followed, Peirce emphasised that the importance of a proof in mathematics lies not in what the proof really is or what the systems

[28] Peirce emphasises this at several junctures. Curiously, he also held a separate lecture on the particular issue of observation in mathematics, entitled "The observational element in mathematics." This lecture was presented in a pedagogical series given in 1883–84 at the Johns Hopkins University where Peirce was an instructor in logic at that time (*Johns Hopkins University Circulars*, 3 January 1884: 32). The lecture itself has not been preserved, but MS 748c is a two-page draft of it; cf. W4: 555–8.

or constraints are within which the various proofs can be carried out, but in (i) what it is that the proofs in fact accomplish and (ii) what their general conceivable consequences will be as soon as they do accomplish their purpose.[29]

Another major moot point concerning quasi-empiricism from the vantage point of pragmaticism is its epistemological argument, originally targeted against platonism (Benacerraf [1973] and Benacerraf and Putnam [1983]). According to that argument, we can have mathematical knowledge only of objects we are causally interacting with. I have already suggested how this line of attack remains ineffectual in pragmaticism. It should now be added that, in making Benacerraf-type arguments, it is presupposed that we gain that knowledge through sense perception. That is an assumption pragmaticism does not make. Observation is not mere sense perception. It involves ratiocination.[30] There is no plain and content-free uninterpreted perception given through our senses only. Our senses work much more like "reasoning machines," as Peirce argues in another famous papers of his.[31] And mathematics and logic are both activities of making such observations. If we are willing to grant mathematics the status of one of the higher intellectual activities of humanity, then we should call to mind Peirce's line that, "[T]o say that all our knowledge relates merely to sense perception is to say that we can know nothing—not even mistakenly—about higher matters" (CP 6.492, 1908, *A Neglected Argument for the Reality of God*).[32] No escape from Benacerraf's problem to nominalistic structuralism is needed.

[29] On the other hand, quasi-empiricism tends to move too far and into the directions remote from Peirce's concerns, as for instance is the case in the naturalism of Kitcher ([1984]).

[30] According to Peirce, "The investigation of truth consists, according to the conception of logic, of two parts, observation and reasoning. This distinction is not in truth an absolute one. Modern psychology shows us that there is no such thing as pure observation free from reasoning nor as pure reasoning without any observational element" (W4: 400–1, 1883, [*Beginnings of a Logic Book*]).

[31] NEM 3:1115, 1900, *Our Senses as Reasoning Machines.*

[32] Moreover, a sore point in the epistemological argument is its appeal to the notion of causation which by no means is an essential part of pragmatistic methodology.

4 Conclusion

Peirce's pragmaticism is a noteworthy philosophy of mathematics differing markedly from the foundationalist philosophies proposed for mathematics over the course of the century that followed his investigations. But to move away from foundationalist concerns is not to move away from the indubitable yet fallible character of mathematical facts and mathematical knowledge. To do so would contravene the scholastic realist and critical common-sensist views that characterises pragmatistic thought. Pragmaticism is a method of doing philosophy of mathematics by way of drawing from actual mathematical practices and taking what mathematician do seriously.[33]

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[33]One might suspect that pragmaticism resounds with the kind of natural or naïve attitude towards mathematics taken by Wittgenstein: Mathematics is simply what mathematicians do. But neither Peirce nor Wittgenstein were that casual about the nature of mathematics. According to Peirce, studying the processes of mathematical reasoning that fall from the *logica utens* pertains to the study of normative logic and not to mathematics. Wittgenstein, contra Peirce, took mathematics, too, to be a normative field of investigation. Mathematics is not a science of discovery of new facts, but proving mathematical results means changing the rules of mathematical language games. Overall, however, Wittgenstein held an exceedingly narrow view of what mathematics can accomplish. It is impossible to reconcile Peirce's pragmatistic approach closely tied with actual mathematical investigations with Wittgenstein's finitistic and normative constructivism. Though differing with respect to the philosophy of mathematics, I have investigated some commonalities between Peirce's and Wittgenstein's philosophies of language and logic in Pietarinen [2003] and [2006*a*].

Abbreviations

CP	Peirce [1958]	EP	Peirce [1998]	MS	Peirce [1967]
NEM	Peirce [1976]	W	Peirce [1980]		

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