Mereology in Philosophy of Mathematics

Geoffrey Hellman

There is a surprising variety of programs in the philosophy and foundations of mathematics that have found mereology a useful and, in some cases, an indispensable tool. After emphasizing a number of key relevant features of mereology, we will briefly examine four such programs, including (1) Goodman and Quine's efforts to recover syntax of mathematical language as part of a finitist, formalist philosophy of mathematics; (2) Field's and Burgess' "synthetic mechanics" as an effort to recover nominalistically certain applications of mathematics in physics inspired by synthetic geometry; (3) Lewis' attempt to ground set theory on a combination of mereology and plural logic, which he called "megethology" (theory of size); (4) Hellman's modal-structuralism employing the same machinery as (3) together with modal logic, but to provide an eliminative structuralist alternative to platonist, face-value readings of abstract mathemtical theories.

It must be noted at the outset that, all by itself, mereology is too weak to provide a framework for even very elementary mathematics. It is, after all, just a set of "axioms" in the sense of defining conditions for a complete Boolean algebra, or an atomic version thereof, with minor adjustments to avoid commitment to a null entity. There is little more to this than commitment to closure of given things under the operation of "nominalistic summation", i.e. passing to the whole, or "fusion" (nominalistic "sum") constituted of given things as "parts". ("x is part of y"—written $x < y$—is the basic partial ordering relation, and may be taken as the sole extralogical primitive.[1] It is presented entirely schematically in the manner of abstract algebra.) Four elements of weakness of this machinery stand out in relation to set theory:

[1]Alernatively, it can be defined in terms of "overlaps", x **o** y, via

$$x < y \leftrightarrow \forall z(z \ \mathbf{o} \ x \rightarrow z \ \mathbf{o} \ y).$$

1. There are no existence axioms (the existence of wholes or fusions being conditional on some given entities as "parts"), in contrast with set theory's axiom of the existence of an infinite set, as well as a null set, etc.

2. Unlike sets *vis-à-vis* their members, wholes of given parts do not generally retain information recovering those parts in the sense that multiple decompositions into parts are equally correct. An important exception to this is uniqueness of decomposition into *atoms* (individuals lacking proper parts), when the wholes in question are exhausted by atoms.

3. Whereas a set of a single object, its singleton, is distinct from that object, the whole of a single object is just that object, i.e. nothing new is recognized. The operation "set of" is "creative" and iterative, leading to ever larger sets: given a set x, one passes to the set of all subsets of x (the "power-set"of x), always a set of higher cardinality by Cantor's theorem, and this goes on indefinitely; whereas given some individuals, say atoms, "taking sums" leads just to all the wholes corresponding to the non-empty sets of atoms, and then no further wholes are generated. (In effect, in mereology one applies the power-set operation just once.) So, for example, with set theory, given a single individual, one immediately has the means of introducing a structure for the natural numbers, e.g. à la Zermelo, by iterating the singleton operation, which then behaves as a "successor" operation on numbers. Mereology by itself, in comparison, gets exactly nowhere.

4. As a key example of this weakness of mereology in comparison with set theory, whereas in set theory a general theory of relations is readily developed, via any of infinitely many ways of identifying *ordered pairs* of objects as sets (so that an n-ary relation can be taken as a set of ordered n-tuples of related objects), mereology by itself has no general means of coding ordered pairs of individuals, and lacks resources for a theory of relations.

Thus, it is only *in combination with other primitives* that mereology functions in accounts of mathematics, to which we now turn.

In their "Steps Toward a Constructive Nominalism", Goodman and Quine used mereology along with a short list of syntactic primitive predicates of concrete marks or inscriptions intended to reconstruct enough formal syntax of mathematical language to serve as the basis of a formalist, nominalistic account of mathematics as a symbolic, rule-governed activity (which of course would include all theorem-proving as well as problem-posing, conjecturing, applying, etc). As they noted, standard syntax is thoroughly platonistic in that it refers to abstract types, including characters, sequences thereof, formulas, proofs, etc. Formation rules, as standardly presented, provide closure conditions, ways of forming complex formulas out of atomic ones, and the class of all (well-formed) formulas is specified as the minimal closure of the atomic ones under these conditions, i.e. the intersection of all classes containing the atomic formulas and closed under the operations provided by the rules. Similarly, the class of proofs is defined as the minimal closure of the axioms of the system under applications of the formal rules of inference. To nominalists such as Goodman and Quine, this is objectionable on two main counts: first, the objects are all abstract, and second, (minimal) closures under the typical operations imply an infinitude of objects. They took as their task a thorough-going reconstruction of syntax of a suitably powerful, regimented mathematical language and theory (a version of first-order set theory) that would avoid any commitment to abstract entities or to infinitely many things of any sort.

In addition to primitives for the basic symbols, e.g. "Vee x" for "x is a v-inscription", "Ac x" for "x is an accent-inscription" (where formal variables will be a v concatenated with a string of accents), "Str x" for "x is a stroke-inscription" (where the stroke will be the sole truth functional connective of alternative denial), "Ep x" for "x is an epsilon-inscription" (for set-membership), etc., they take as primitive "Cxyz" meaning "x and y and z are composed of whole characters of the language...in normal orientation and x consists of y followed by z." Informally, this makes use of mereological composition; formally that enters at a later stage when they define auxiliary notions such as that two inscriptions are "equally long" (containing the same number of characters), which is used in defining "x is the alternative denial of y and z", etc. In order to rule out infinitely long inscriptions, "composed of whole characters" should mean "of *finitely many* whole characters", i.e. the notion of "*finite*" would also be serving as a background primitive (although Goodman and Quine are silent on this important matter). Since they contrive clever alternatives to the method of minimal closure in order to define "formula" and "proof" (e.g. D20, "Quasiformula", and D29, "Proof"), they do not have that standard method of insuring that only finite strings get admitted. (Even a "v" followed by an infinite string of accents isn't ruled out as a "variable inscription" by their definitions.)[2] With these assumptions, however,

[2]Instead of explicitly taking 'finite' as primitive, Goodman and Quine could adopt some further axioms and stipulations, e.g. that every string have a terminating basic symbol, that every basic symbol in a string except the initial one have an immediate predecessor, and that a

Goodman and Quine do succeed in setting out enough formal syntax to meet their goals. Their definitions of "formula" and "proof" allow there to be just as many of either as ever get inscribed without commitment to there being infinitely many, and they must indeed have the intuitively correct shape in order to qualify.[3]

Turning to our second example, Field's *Science without Numbers* developed methods analogous to those of synthetic geometry in order to provide nominalistically acceptable, synthetic formulations of key physical theories (especially classical field theories, such as Newtonian gravitation theory), over which standard analytic applied mathematics could be demonstrated to be logically conservative (deriving only those nominalistically formulable consequences which already are derivable in the synthetic formulation). Standard reasoning in physics concerning real numbers and functions of reals representing physical magnitudes, e.g. spatial or temporal distance, the value of a potential, etc., would thereby be justified as a useful, but ultimately dispensable tool, supporting an instrumentalist view of the analytical mathematics, in a nutshell: "useful but not true". Taking inspiration from Hilbert's synthetic geometry in which primitive relations on the geometric space, such as *betweenness* and *congruence*, suffice to capture metrical relations standardly expressed in terms of real numbers and functions, Field introduced suitable synthetic relations on space-time allowing recovery, in a certain sense, of reasoning about the values of scalar and vector fields in physical theories; and he sought to argue both that this synthetic treatment of space-time is nominalistically acceptable and that the "recovery" of analytical applied mathematics established the desired logical conservativeness just mentioned as well as yielding genuine insight into the way mathematics actually applies to the physical world. The "recovery" for Field turned on proving a *representation theorem* (inspired by modern theory of measurement) roughly as follows. Let a standard model

form of induction hold, e.g. that *every* whole with the terminating symbol of a string as part which is closed under "immediate predecessor" have the initial symbol as a part. Here appeal is made to "the complete logic of Goodmanian sums", roughly equivalent to monadic second-order logic, and not recursively axiomatizable. Non-standard models are ruled out if quantification over wholes really encompasses *all* of them.

[3]One should compare all this with Quine's own development of "Protosyntax" in his *Mathematical Logic* secs. 55, ff. which also uses mereology in an essential way to avoid reliance on classes and the infinite in the development of syntax for mathematics. In a footnote (14), Goodman and Quine refer back to Quine's method there of "framed ingredients" (in which special markers are used to identify the relevant parts of concatenated strings, cf. weakness 2 of mereology listed above), as a nominalistic substitute for the Fregean-Dedekindian method of minimal closure, but they criticize that method as overly contingent on "what inscriptions happen to exist in the world".

of T_{syn}, the synthetic space-time version of the original applied mathematical physical theory T, say of a single scalar field ψ, take the form (X, Seg-Cong$_X$, Scale-Betw$_X$,...), where X is the domain of space-time points, Seg-Cong$_X$ is a (4-place) relation of congruence of two line segments in X, Scale-Betw$_X$ is a relation that holds of points p, q, r when the value of ψ at q is inclusively between those at p and r. Let a model of the original, analytically formulated T be of the form $((M, d), \psi)$, where M is a manifold, d a distance function or metric on M, and ψ the scalar physical field described (typically with differential equations) in T The representation theorem for these theories then states that each standard model of T_{syn} is homomorphic to some model of T, i.e. a 1-1 mapping φ from X to M satisfies the conditions that

Seg-Cong$_X(p, q, r, s)$ if and only if $d[\varphi(p), \varphi(q)] = d[\varphi(r), \varphi(s)]$ and

Scale-Betw$_X$ (p, q, r) if and only if $\psi(\varphi(p)) \ \leq \psi(\varphi(q)) \leq \psi(\varphi(r))$.

It is especially in order to guarantee that such homomorphisms exist that Field made use of mereology. In the setting of space-time, this allows quantification over arbitrary *regions* of space-time, as well as the points, and this is then expressively equivalent to applying second-order monadic logic. This enables, for example, the expression of genuine *continuity* of space-time and of functions defined theoreon, e.g. that every "bounded, collinear (i.e. Between-related) whole of points has a least bound (from either end, that is)" —all of which can be stated geometrically. Such conditions are required for the desired representation theorem, and mereology serves as an adequate, nominalistically acceptable substitute for set theory or second-order logic. As already noted and as Field recognizes, this "complete logic of Goodmanian sums" is not recursively axiomatizable. Still, the machinery is intelligible and available to the nominalist.[4]

Now, ironically, just because of this "second-order" strength provided by mereology, enabling proof of existence of a representing homomorphism between space-time and $\mathbb{R}^4$, it turns out that analytical applied mathematics (e.g. set theory with urelements, on which Field concentrated) is *not* deductively conservative over the synthetic (geometrized) physical theory, T_{syn}. (See Shapiro (1983).) Within the latter, for example, a standard model of arithmetic can be described. Using Gödelian techniques relative to this model, the consistency of T_{syn} can then be formulated (nominalistically)—call this "Con(T_{syn})"—, but is not provable in T_{syn} (if that theory is in fact consistent), as a consequence of Gödel's (second) incompleteness theorem. However, the applied set theory *can* prove "Con(T_{syn})" because it

[4] The full strength of this second-order mereology is not required, i.e. one need not quantify over *arbitrary* regions. But enough strength is needed to transcend first-order axiomatization.

can prove the usual statement, "Con(PA)", of consistency of the Peano-Dedekind axioms and, in virtue of the representing homomorphism between space-time and $\mathbb{R}^4$, it proves the equivalence between "Con(PA)" and "Con (T_{syn})". (It is true that applied set theory is *semantically* conservative over T_{syn}, i.e. no sentence in the language of T_{syn} is *implied* by the former but not the latter; but the dispensability of abstract mathematics for *proving* physically significant results remains to be established.[5])

Interestingly, rigorous conservativeness results of the sort sought by Field have been obtained (by Burgess (1984)), but as expected, this pertains to first-order versions of T_{syn} which lack the resources for characterizing standard models of arithmetic and analysis and do not permit proof of representing homomorphisms between space-time and $\mathbb{R}^4$. Conservativeness is proved directly from properties of first-order theories and their extensions by definitions and abstractions, without recourse to Field-style representation theorems. In particular, the synthetic, geometrized physics gets by entirely with geometric-style primitives and makes no essential use of mereology, certainly not the "complete logic of Goodmanian sums". We conclude, then, that the real value of mereology in connection with Field's program is to enable one to characterize up to isomorphism key structures of interest in mathematics and physics without any reliance on the concepts of *set* and *function.* Furthermore, quite apart from the goal of logical conservativeness, the representation theorems are of value in providing insight into the applicability of mathematics to the physical world.

In contrast to these first two programs, the programs of Lewis and Hellman both seek to respect "mathematical truth" but in contrasting ways. Lewis seeks a reconstruction of Zermelo-Fraenkel set theory which, curiously, is formulated with "nominalistically acceptable" primitives, namely a combination of mereology and plural quantification. Hellman employs this same machinery but combines it with modal logic in order to "de-ontologize" mathematics completely, while in a sense respecting its objective truth. Let us now describe the common core, the combination of mereology and plural quantification.

Plural quantifiers are common in English and are well-illustrated by exam-

[5] The Fieldian nominalist can argue here that the conservative, analytical mathematics serves as a useful instrument in helping us learn about more semantical consequences of nominalistic theories than we otherwise could know, but that this instrument still need not be regarded as asserting (platonistic) truths. But, in appealing to Field's methods of establishing the semantical conservativeness, the nominalist must still grasp the representation theorems, which are couched in platonist mathematical language (referring e.g. to homomorphisms between models of space-time and $\mathbb{R}^4$, etc.). The methods described in the next paragraph avoid such vicious circularity.

ples such as, "Some kids in the neighborhood congregate only with one another", "Any people whatever, if too closely confined under conditions of scarcity, will quarrel among one another". Such sentences cannot be accurately symbolized in the notation of first-order logic using just the predicates occurring in the English formulations. They can, however, be symbolized by adding quantification over classes, so that, e.g. the first sentence would be read as "There is a class of kids in the neighborhood such that if any member congregates with anyone, the latter is also a member". It seems that the ordinary English sentence involves us in monadic second-order logic or some set theory. The key idea behind plural quantification is to turn this on its head: "Some kids" as it functions in the first example is a *plural existential quantifier* speaking directly of many kids (without definiteness as to how many), without any singular reference to an abstract class of kids, and we understand such constructions perfectly well. Indeed, plural quantifiers can be invoked to *reduce* or *replace* class quantifiers: monadic second-order logic is ontologically innocuous, committed to no more than what the rest of the sentence already involves. Thus, given some items serving as natural numbers, the second-order principle of mathematical induction, used in characterizing the structure up to isomorphism, can be stated: "Any items among which are the zero and the successor of any among them have all the numbers among them", without any reference to sets or classes. And, in set theory itself, one may speak of "all the ordinals", or "all the cardinals", or "all the sets", without introducing an exceptional layer of "proper classes", collections which are somehow "too big" to be members of anything. And, without fear of paradox, one may speak of "all the non-self-membered collections", and so forth.[6]

This gets us the strength of monadic second-order logic, but how can we obtain polyadic or a theory of many-place relations, of the essence in formulating mathematics? Here, as Burgess, Hazen, and Lewis (1991) realized, one may combine plural quantifiers with mereology (in several different ways) to get the effect of ordered pairing of individuals (generalizing to ordered-triples, quadruples, etc.). The methods require an infinitude of sufficiently distinct things, in the simplest case, mereological atoms. This can be postulated directly using plural quantification and mereology, e.g. by

"Some X are such that an atom a is one of the X and for each of the X, x, there is a unique atom b not part of x such that the fusion $x + b$ is also one of the

[6] Boolos (1985) originally introduced plural quantifiers explicitly to avoid proper classes while justifying the use of second-order logic in set theory.

X."

Now, we will explain Burgess' method (which is most easily vizualized). Much as the positive integers all can be mapped in a 1-1 fashion in two (familiar) distinct ways into some but not all of them, e.g. to all the odd numbers and to all the even numbers (in their natural order, say), there being no overlap between the odd images and the even images, we may claim that any infinitude of atoms can be mapped 1-1 to two discrete (i.e. non-overlapping) "microcosms", labelling the first-mentioned as "first", the other "second". We could then code an ordered pair $< a, b >$ of atoms as the fusion of (say) the first-image a_1 of a and the second-image b_2 of b. Quantifying plurally over such codes gets the effect of quantifying over binary relations of atoms. Moreover, any whole w of atoms—corresponding to a non-empty set of them—has as a "first image", w_1, the fusion of the first-images of its atoms and, similarly a "second image", w_2. The ordered pair $< w, v >$ of two such wholes, w and v, can then be taken as the fusion of w_1 and v_2. Plural quantification over such fusions then gets the effect of quantification over relations of arbitrary fusions of atoms. (If our atoms were labelled as natural numbers, for example, this in effect would provide relations among arbitrary real numbers!) All this depends on making sense of the initial distinct mappings of all our atoms to just some of them. But that can be understood in terms of plural quantification over "diatoms", (unordered) two-atom fusions, as spelled out in a postulate that the (postulated infinitude of) atoms have a "trisection", i.e.that they are the fusion of pairwise non-overlapping wholes, x, y, and z, and that some diatoms X map atoms of $y + z$ 1-1 into x, and similarly, *mutatis mutandis* for some diatoms Y and some diatoms Z. Let Y_x be those diatoms of Y that have an atom of x as a part, and let y_x be the fusion of those atoms of y that are part of diatoms of Y_x.Then diatoms Y_x map x 1-1 to y_x.Y_z and y_z are defined similarly. Then diatoms X along with those of Y_x map all the atoms 1-1 into $x + y_x$, and diatoms of Z and those of Y_z map all the atoms 1-1 into $z + y_z$. The ranges of these two 1-1 correspondences are discrete. Since this is all relative to given x, y, z, X, Y, Z, quantified in order, we may speak of the $x + y_x$ atoms as "first-images", and the $z + y_z$ atoms as "second-images" (deterined by the given order of plural quantifiers). Then ordered pairs of atoms and fusions of atoms are defined as indicated above.

It should be clear that mereology plays a crucial role in all this: its notions of "atom" and "diatom" impose sufficient distinctness so that relevant wholes remain distinguishable and so that many diatoms encode 1-1 mappings. Neither mereology nor plural quantifiers separately suffice for a theory of relations, but in combination they do, bringing all of mathematics within reach.

Lewis (1991, 1993) uses the language of mereology and plural quantification to recover set theory as describing part of an absolute, very large "Reality". Rather than withholding mereology from abstract classes or renouncing them entirely, as in the nominalist tradition, he accepts classes and argues that they indeed have parts, namely their subclasses, and that these are their only parts. Any singleton then counts as a mereological atom, and a class is the fusion of the singletons of its members. Indeed, one need not take "membership" as primitive but can recover set theory from structural conditions defining "s is a singleton function", viz. s is a 1-1 function whose range consists of atoms, defined on "small" fusions of singletons along with individuals (non-classes, lacking any singletons as parts), and satisfying an extremal clause: "any things that include all the individuals and are closed under application of s and under fusions of any of them includes all things." Here a fusion is "small" just in case its atoms are not in 1-1 correspondence with all the atoms. Then certain conditions are laid down (as postulates of megethology) that guarantee that the axioms of Zermelo-Fraenkel set theory are satisfied, indeed a second-order version in which proper classes are outside the domain of the singleton function.[7] For example, the Axiom of Infinity (of set theory) results from the postulate that *something small is infinite.* Analogous to "small thing" is the plural notion of "few things", corresponding 1-1 with some but not all of the atoms. The the Axiom of Power Sets is guaranteed by the postulate of megethology that *the parts of a small thing are few.* And the Axiom of Replacement is guaranteed by the postulate that *the fusion of few small things is small*; and so forth. It follows that there must be strongly inaccessibly many things (cf. Zermelo (1930)). For Lewis, this recovery is both faithful to mathematics (as he conceives it) and clarifying: classes and membership are profoundly mysterious, and a structuralist description of "singleton function" is, he thinks, an improvement. In any case, the whole reconstruction is a *tour de force* in deployment of the "nominalist" logical tools of mereology and plural quantification, about as far removed from Goodman and Quine's formalism as it could possibly be.

In contrast, Hellman (1989, 1996, 2005) combined the above machinery of mereology and plural quantification with quantified S-5 modal logic to give modal-structural interpretations of mathematical theories. Infinitely many atoms are postulated only as a logico-mathematical possibility, not as actual. Then number theory, analysis, geometry, algebra and much else can be understood as proving

[7] Proper classes are admitted via the closure of "reality" under fusions of *any*—i.e. arbitrarily many—things. Consider the irony of this in light of n. 5 above.

what would be the case in any structure of the appropriate type(s) there might be (the "hypothetical component") as well as asserting that such structures are possible (the "categorical component"). Even this modalized reference to structures can be understood in terms of "suitably many things, suitably interrelated", as spelled out using mereology and plural quantification, i.e. functions and relations are "explained away" and need not be recognized as objects. At the same time, the nature of the hypothetical objects considered is entirely immaterial. Most mathematics as practiced is thus understood on its own terms rather than being embedded in set theory. Set theories can also be understood, however, in this eliminative structuralist manner, the larger cardinalities of items postulated merely as possibilities, with no single system of set theory taken as "the one true theory", i.e. mathematical pluralism is naturally accommodated. Moreover, an extendability principle (inspired by Zermelo (1930) and Putnam (1967)) is promoted: any possible structure for set theory could be properly extended. An absolutely maximal one is not recognized as a coherent possibiloity. In contrast to Lewis' absolutism, proper classes are recognized only relative to a hypothetical structure; as Zermelo argued, they become elements of still further collections in higher domains for set theory. The approach is also friendly to category and topos theory as an alternative "universal" framework for mathematics. Indeed, a relativized version of Lewis' megethology can be deployed to describe hypothetical "large domains" in which both set theory and topos theory can be developed side by side. Neither is "the true foundation" for mathematics; rather both are great frameworks unifying mathematics in their own distinctive ways.

Thus, mereology has proved an extremely useful instrument in developing a surprisingly broad range of programs in the philosophy and foundations of mathematics, some of which are flexible and powerful enough to accommodate all known mathematics.

References

[1] Boolos, G. (1985) "Nominalist Platonism" *Philosophical Review* **94**: 327-344.

[2] Burgess, J. (1984) "Synthetic Mechanics" *Journal of Philosophical Logic* **13**: 379-395.

[3] Burgess, J., Hazen, A. and Lewis. D (1991) Appendix on pairing" in Lewis (1991).

[4] Field, H. (1980) *Science without Numbers* (Princeton: Princeton University Press).

[5] Goodman, N. and Quine, W.V. (1947) "Steps toward a Constructive Nominalism" *Journal of Symbolic Logic* **12**: 105-122.

[6] Hellman, G. (1989) *Mathematics without Numbers: Towards a Modal-Structural Interpretation* (Oxford: Oxford University Press).

[7] Hellman G. (1996) "Structuralism without Structures" *Philosophia Mathematica* **4**: 100-123.

[8] Hellman, G. (2005) "Structuralism" Ch. 17 in S. Shapiro, ed. *The Oxford Handbook of Philosophy of Mathematics and Logic* (Oxford: Oxford University Press), pp. 536-562.

[9] Lewis, D. (1991) *Parts of Classes* (Oxford: Blackwell).

[10] Lewis, D. (1993) "Mathematics Is Megethology" *Philosophia Mathematica* **1**: 3-23.

[11] Putnam, H. "Mathematics without Foundations" in P. Benacerraf and H. Putnam, eds. *Philosophy of Mathematics* 2d Ed. (Cambridge: Cambridge University Press), pp. 295-311.

[12] Quine, W.V. (1951) *Mathematical Logic* (Cambridge, MA: Harvard University Press).

[13] Shapiro, S. (1983) "Conservativeness and Incompleteness" *Journal of Philosophy* **80**: 521-531.

[14] Zermelo, E. (1930) Über Grenzzahlen und Mengenbereiche: neue Untersuchungen über die Grundlagen der Mengenlehre" *Fundamenta Mathematicae* **16**: 29-47.