Harry moved to Los Gatos, California, in the early 1970's after the untimely death of his wife, Marian, whose devotion to mathematics rivaled that of Harry. He chose Los Gatos since that was the home of his daughter Margery Dodge. Tragedy marred his life again when she died in 1977. The affection which Horace, Margery's husband, and their children always had for Harry grew only stronger after her passing. Never in my experience have I seen such devotion and admiration of a father-in-law by his son-in-law.

Harry continued his active interest in and thoughtful advice to the MAA almost to the day of his death. On January 8, 1981, Horace Dodge drove Harry to the national meeting of the MAA being held at the time in San Francisco. The Board of Governors was in session when he arrived in San Francisco. It had acted on a long list of items in the morning. The meeting was considered one of the most successful ever held: the discussions were full and informative, and they led to unanimous decisions on important matters of business. By the middle of the afternoon the Board had developed a deep sense of satisfaction. Suddenly, the door opened, and Harry entered. The time and the occasion were right; President Dorothy Bernstein suspended business and introduced him as Mr. MAA amid applause from members of the Board.

Later, that evening, Harry had dinner with G. Baley Price, David P. Roselle, and Alfred B. Willcox in the Captain's Table dining room of the San Francisco Hilton. For the following account of this dinner, I am indebted to Baley Price: "The dining room was delightful; it was spacious, quiet and peaceful, and lighted and decorated in superb good taste. The meal was delicious, the conversation was pleasant; and the waiter, sensing that the occasion was something special, employed all his skills and graces to make the dinner a real party. He provided special treats from the chef, he displayed tempting desserts, and he responded with obvious pleasure to the group's every wish. After dessert, he brought, with a great flourish, a large silver bowl streaming white clouds of vapor from dry ice; the silver bowl contained four chocolate mints! The four guests enjoyed the dinner immensely; it was the end of a perfect day. Harry seemed in excellent health and spirits; when he bade the three good-night, little did they realize that they would not see him again."

HENRY L. ALDER

MATHEMATICAL MODELS: A SKETCH FOR THE PHILOSOPHY OF MATHEMATICS

SAUNDERS MAC LANE
Department of Mathematics, The University of Chicago, Chicago, IL 60637

1. Introduction. The aim of this note is to encourage a renewed study of the philosophy of mathematics, a subject dormant since about 1931. This date marks the end of a period of activity centered around what seemed to be a "crisis" in the foundations. That crisis, initiated by paradoxes such as the Russell paradox of the "set of all sets not members of themselves," led to the development of three competing schools in the philosophy of mathematics: Logicism, Formalism, and Intuitionism. In addition there were long-standing, more general philosophical doctrines: Platonism and Empiricism. Empiricism holds that mathematics is simply another branch of science, and so concludes that mathematics deals directly with the real world. However, none of these schools or doctrines in the philosophy of mathematics provides a satisfactory analysis of the nature of mathematics. Their deficiencies are eloquently analyzed in a recent article

The author received the Association's Award for Distinguished Service in 1975. See this MONTHLY, vol. 82, pp. 107–108.—*Editors* by N. D. Goodman [4]. Other current publications in the philosophy of mathematics, from Putnam [6] to Quine [7] to Wang [10], show little new insight and no new input from mathematics.

To develop a fresh view of the philosophy of mathematics, we begin by looking at the actual state of mathematics. We do this on the grounds that a sound philosophy of mathematics ought to start with a description of what is really there. As N. D. Goodman said [4]: "Mathematics can only flourish if there is a common conception of what we are about (and only if there is an agreement that the different structures that we study are aspects of one reality)."

The various earlier philosophies of mathematics listed above each arose out of the dominant aspects of mathematics as then understood. For example, Platonism arose in Greece and applied to mathematics there because it fitted Greek geometry; it has been popular among mathematicians recently because it fitted well with the view that mathematics derives from axioms for sets. Logicism arose together with the discovery and formalization of mathematical logic. Intuitionism was the child of emphasis on numbers as the starting point of mathematics and on intuition as a basis of topology. Formalism arose with the development of axiomatic methods and the discovery that proof theory might give consistency proofs for abstract mathematics. Empiricism sprang from the 19th-century view of mathematics as almost coterminal with theoretical physics; it was much influenced by Kant's dichotomy between analytic and synthetic.

Now we search for a philosophy of mathematics better attuned to the present state of the subject.

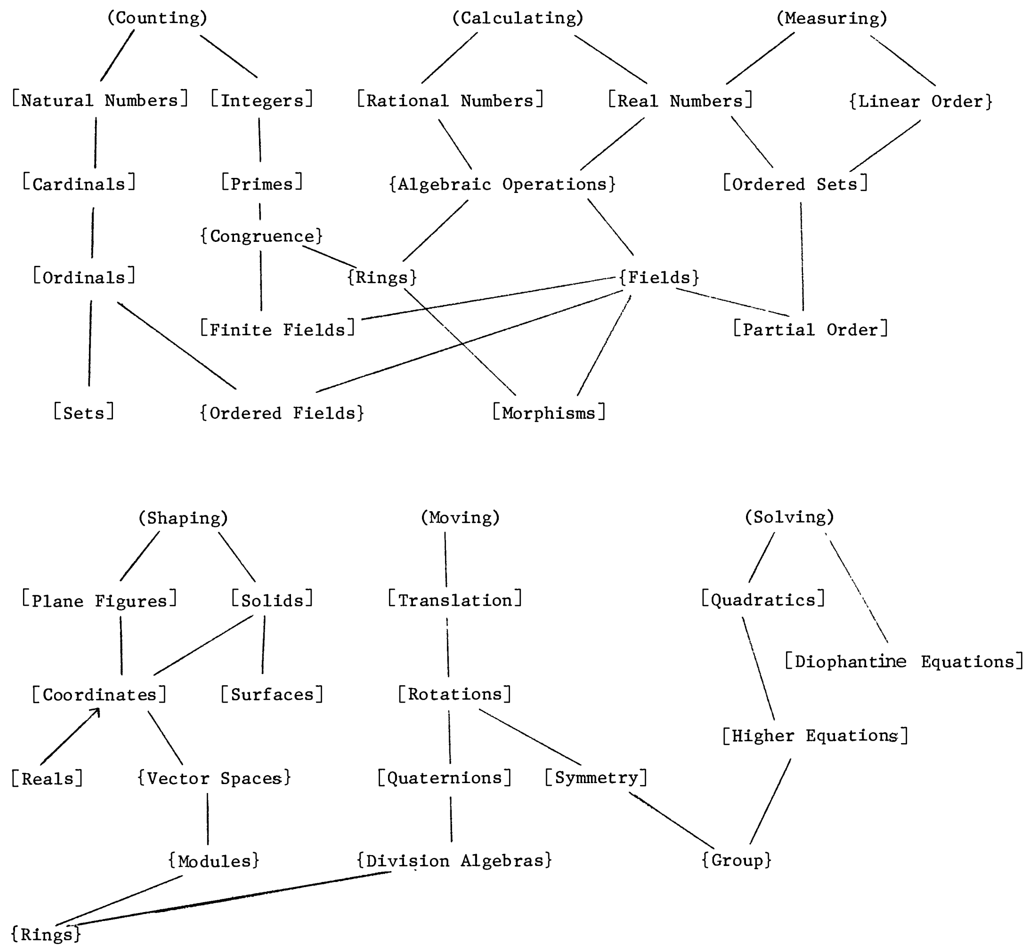
2. The Origins of Mathematics. Mathematics begins with puzzles and problems dealing with combinatoric and symbolic aspects of the general human experience. Some of these aspects turn out to be systematic and intrinsic, rather than arbitrary and tied to one context. They become the stuff of elementary mathematics. From this starting point, the subject has developed to be a deductive analysis of a large number of very different but interlocking formal structures. These structures have been derived from experience in many successive stages; by abstractions from various observations of the world, its problems, and the interconnections of these problems. These observations can be described as starting with a variety of human activities, each one of which leads more or less directly to a corresponding portion of mathematics:

Counting	:	to arithmetic and number theory;
Measuring	:	to real numbers, calculus, analysis;
Shaping	:	to geometry, topology;
Forming (as in architecture)	:	to symmetry, group theory;
Estimating	:	to probability, measure theory, statistics;
Moving	:	to mechanics, calculus, dynamics;
Calculating	:	to algebra, numerical analysis;
Proving	:	to logic;
Puzzling	:	to combinatorics, number theory;
Grouping	:	to set theory, combinatorics.

These various human activities are by no means completely separate; indeed, they interact with each other in complex ways. Table 1 gives only a partial view of this complexity, to indicate how various human activities lead to the concepts now present in algebra. The two parts of this table should (and do) fit together by many crosslinks.

On the basis of many more elaborate tables such as this, giving the origin and interconnection of mathematical ideas, we conclude that mathematics started from various human activities which suggest objects and operations (addition, multiplication, comparison of size) and thus lead to concepts (prime number, transformation) which are then embedded in formal axiomatic systems (Peano arithmetic, Euclidean geometry, the real number system, field theory, etc.). These systems turn out to codify deeper and nonobvious properties of the various originating human activities.

TABLE 1
Origins of Concepts of Abstract Algebra



For example, the notion of a group, though axiomatically very simple, reveals common properties of motion (rotation and translation groups), of symmetry (crystal groups), and of algebraic manipulations (Galois groups, Lie groups for differential equations). Many other mathematical concepts (function, partial order) are similarly both simple in structure and pervasive in application. The simplicity and the applicability are made effective by the formal treatment of the notions involved.

In this view, mathematics is formal, but not simply "formalistic"—since the forms studied in mathematics are derived from human activities and used to understand those activities.

The actual structure of mathematical ideas is an incredibly elaborate development of this simple description. Consider just the case of algebra. Algebra first involved manipulation to solve equations. Then geometry was reduced to coordinates, and thus geometrical problems to algebraic ones. Next, simple Euclidean spaces are described by vectors in two, three, and then higher dimensions. The resulting notion of a vector space, often one equipped with a (quadratic) inner product, worked even in infinite dimensions and then served to codify some of the methods of solving functional equations. The linear transformations acting on these vector spaces could be represented by matrices, which also cropped up in group theory, in numerical analysis, and in number theory. Presently vector spaces over a field were subsumed under modules over a ring.

Such spaces and modules were needed to measure connectivity phenomena in topology. These topological concepts were then borrowed again by algebraists to become homological algebra and to settle questions in ring theory and in number theory—or at least in the higher reaches of class field theory. Groups represented by matrices had extensive applications in physics, while finite groups had an elaborate structure all their own, reflecting both geometry and number theory. Additional number systems like the complex numbers or the quaternions were impossible, in view of topological arguments—but the p-adic numbers arose from a marriage of algebraic functions and algebraic numbers. This is but a small sample of the extraordinary way in which the various ideas of mathematics interlock.

Because of this elaborate interlocking pattern of ideas, each mathematical notion is tied to its empirical origins in multiple ways. As a result, no simplistic description of mathematics is adequate.

At each stage of development in mathematics, the structure at issue can be recorded as a formal deductive system. Such a system starts with axioms on a suitable list of undefined terms; in principle, it uses explicit rules of a specified logical system in order to deduce theorems and other conclusions from these axioms. Such an emphasis on the axiomatic method was not always present in mathematics. Moreover, it may seem more natural for some parts of mathematics than for others. Nevertheless, it is now always available for any part of mathematics. Our description of such systems is intended to cover use of intuitionistic or finitistic logic as alternatives to the more classical propositional and predicate calculus.

At this point, we can make a first summary of our position. Mathematics starts from a variety of human activities, disentangles from them a number of notions which are generic and not arbitrary, then formalizes these notions and their manifold interrelations. Thus, in the narrow sense, mathematics studies formal structures by deductive methods which, because of the formal character, require a standard of precision and rigor.

3. Absolute Rigor. This use of deductive and axiomatic methods focuses attention on an extraordinary accomplishment of fundamental interest: the formulation of an exact notion of *absolute rigor*. Such a notion rests on an explicit formulation of the rules of logic and their consequential and meticulous use in deriving from the axioms at issue all subsequent properties, as strictly formulated in theorems. Moreover, each derivation can be tested and understood in its own terms, independent of any reference to examples of the activity or the reality for which the axioms were designed (even though in fact that reality is usually present and often vital in suggesting how the deduction might be made). This formal character of mathematics may serve to distinguish it from all other types of science. Once the axioms and the rules are fully formulated, everything else is built up from them, without recourse to the outside world, or to intuition, or to experiment. Examination of texts of theoretical physics, biology, or other sciences clearly indicates a real difference in this regard. Such texts do not hesitate to appeal at any time to experience or intuition, while a mathematical proof stands or falls on its own, without outside reference.

An absolutely rigorous proof is rarely given explicitly. Most verbal or written mathematical proofs are simply sketches which give enough detail to indicate how a full rigorous proof might be constructed. Such sketches thus serve to convey conviction—either the conviction that the result is correct or the conviction that a rigorous proof could be constructed. Because of the conviction that comes from sketchy proofs, many mathematicians think that mathematics does not need the notion of absolute rigor and that real understanding is not achieved by rigor.

Nevertheless, I claim that the notion of absolute rigor is present. Approximation to rigorous proofs occur in many cases, in particular in the traditional proofs of Euclidean geometry. There each statement in the proof is supported by a corresponding reason or by a reference to a previous theorem. These traditional proofs failed to reach the ideal of rigor, notably at those places where a full proof would have used the axiom of Pasch to show that a desired point of intersection is really there. Nevertheless, these proofs in geometry did provide a clear model of rigor. This model was subsequently refined in Hilbert's *Foundations of Geometry*. Both Frege, in his *Grundgesetze der Arithmetik* [3], and Whitehead and Russell, in *Principia Mathematica* [**11**], give long and careful exhibits of essentially complete and rigorous proofs. In *Principia* the idea of a rule of inference is not clearly distinguished from that of a formal axiom, but this distinction can be readily adjoined. These explicit examples were cumbersome and tedious; they clearly showed that absolute rigor was so detailed that it was a distraction and not a help to mathematical research. Nevertheless, they made the notion of absolute rigor tangible. It is the notion clearly employed in proof theory—for example, in a wide variety of completeness and incompleteness results.

The understanding of this notion of absolute rigor has in considerable measure led to the philosophical standpoints of logicism and formalism. In this way these two standpoints represent an important aspect of mathematical reality. However, these standpoints are one-sided. We emphasize that the choice of axioms, and the determination of directions in which they are to be developed, is in no wise determined by the formal structure, but rather by aspects of the world under study or by portions of the mathematician's insight or fancy.

This raises an important metaphysical issue: How does it happen that some important facets of the real world can in fact be accurately analyzed by austere deductions from axioms? In other words, how does it happen that logic fits the world; how can one account for the extraordinary and unexpected effectiveness of formal mathematics?

This issue can also be stated for particular cases:

- How is it that the formal calculation by Newtonian mechanics of the motions of bodies turns out to fit their actual motions?
- Why is it that the formal deduction of the possible groups of symmetry is matched by those groups as they occur in the world?
- Why is it that the theoretical properties of boundary value problems for differential equations describe so well so many aspects of electricity, optics, mechanics, hydrodynamics, and electrodynamics?
- How is it that the differential calculus seems to work both for physics and for the economists' problems of local maxima?

Such questions of the relations of formal logical deductions to actual events raise metaphysical problems to which I have no adequate answer.

In the practice of mathematics this notion of absolute rigor is certainly present; but a mathematician, in addition to being guided by his concepts of precision, is guided also by his understanding of the breadth and depth of his subject. By "breadth" I refer to the other objects within or without mathematics to which this subject applies, while the issue of "depth" involves judgment as to the choice of abstractions which will lead to the heart of the problems at issue. We can today clearly understand notions of rigor and formulate them in metamathematical terms, but there is no comparable analysis of breadth or depth of mathematical research. For example, why are the simple axioms for group theory so powerful?

One aspect of such an analysis is the choice of the direction for mathematical research: What topic should be studied next? On this there can be sharp opinions, for example, with Bourbaki. In the hands of Dieudonné this doctrine of chosen directions has become firm, not to say frozen. It reads: "To see whether you are doing good mathematics, look to see what Bourbaki notices; if your subject has not come to their favorable notice, it is not worth doing," Such a dogma can be stifling.

4. Multiple Models of Reality. Our thesis as to the nature of mathematics might be formulated thus: Mathematics deals with the construction of a variety of formal models of aspects of the world and of human experience. On the one hand, this means that mathematics is not a direct theory of some underlying platonic reality, but rather an indirect theory of formal aspects of the world (or of reality, if there is such). On the other hand, our thesis emphasizes that mathematics involves a considerable variety of models. The same experience can be modeled mathematically in more than one way.

Such variant models are well known for some basic constructions. The ordinal numbers can be constructed as equivalence classes of well-ordered sets or, following von Neumann, as certain explicit sets, with 0 taken to be the empty set and each positive ordinal the set of all smaller ordinals. Because of this alternative, there is no *unique* set-theoretic description of the ordinals. However, with either description we get ordinals with the *same* behavior.

There is a similar variety in the construction of real numbers from rationals. A real number, for Dedekind, is a cut in the rationals. For Cantor or Meray, it is an equivalence class of sets of rational numbers. For Weierstrass, it is an equivalence class of sets of rationals with a bounded sum. With any of these three constructions one obtains a complete ordered field of real numbers, different constructions yielding isomorphic fields. There is no *unique* set-theoretic model for the reals.

In these cases mathematical models are determined "up to a canonical isomorphism." Indeed, that is all that matters. More generally, many mathematical constructions can be analyzed as the construction of an object which is "universal" relative to some property (i.e., which is the value of the left adjoint to a suitable functor). By its very definition, a universal object is determined *only* up to canonical isomorphism. Thus, for example, the tensor product $V \otimes W$ of two vector spaces V and W can be exhibited by an explicit set-theoretic construction—but the construction does not matter and can be immediately forgotten once the result is proved universal. All that matters is the universal property: that any bilinear map on V and W can be written in a unique way as a linear map on the tensor product $V \otimes W$.

So far, these are cases of models which are determined "up to isomorphism" or often "up to canonical isomorphism." For many mathematical purposes though, mathematicians use axiomatic systems which have many nonisomorphic models. Thus, for group theory, the depth and power of the group axioms lie in part in the fact that these axioms have many nonisomorphic models.

5. Foundations. In our view, the philosophy of mathematics is directed more at the understanding of the nature of mathematics than at a "foundation" of mathematics. Nevertheless, our emphasis on the fact that finished mathematics is formal is close to questions about foundations. The clear understanding of formalism in mathematics has led to a rather fixed dogmatic position which reads: Mathematics is what can be done within axiomatic set theory using classical predicate logic. I call this doctrine the *Grand Set Theoretic Foundation*.

In a preliminary version, this arose in the 19th century with Weierstrass, Dedekind, and Frege: Start with finite cardinal numbers, perhaps defined by set theory, and build up from them the natural numbers, then the integers, then the rational numbers as pairs of integers, and then the real numbers, say as Dedekind cuts in the rationals. This careful construction of the real numbers was long accepted as standard in graduate education in mathematics, even though many mathematicians did not much believe in it. They also were not always aware that this construction did not get the real numbers from natural numbers alone but had to use set theory extensively on the way. While paying lip-service to this real-number foundation, it was felt that a real number is really a point in the preexisting geometric continuum and not just a formal Dedekind cut in the rationals. This viewpoint can be expressed intuitively (as a geometric insight) or formally: Do not construct the reals, but describe them axiomatically as an ordered field, complete in the sense that every bounded set has a least upper bound.

The next step in the grand set-theoretic foundation included sets (or classes) with logic and was initiated in the work of Frege. This direction reached a crisis with the 1900 discovery of the Russell paradox of the set of all those sets not members of themselves. Actually, this paradox itself was settled very soon, in 1908, independently and in two different ways, by Russell's publication of his *Theory of types* [8] and by Zermelo's publication, also in 1908, of his *Axioms of set theory*.

It took a considerable period before this solution and system was shaken down and well formulated through the work of Skolem, Fraenkel, Paul Bernays, and others. Even in the 1940's, with the growth of abstract algebra, axiomatic set theory was not regarded as a central doctrine. It was not until about 1950 that the *Grand Set Theoretic Foundation* was finally complete and officially accepted under the slogan which might have read: "Mathematics is exactly that subject which can be developed by logical rules of proof from the Zermelo-Fraenkel axioms for set theory." This foundation scheme had its popular version in the "new math" for schools. It also had its philosophical doctrine, a version of Platonism, that the world of sets is that constructed in the standard cumulative hierarchy of all ranked sets. Here one begins with an understanding of the empty set and the ordinals and uses the power set (the set Px of all subsets of x) to construct the iterated power sets of the empty set $\varnothing$:

$$R_0 = \varnothing, R_1 = P(\varnothing), R_2 = P(P\varnothing), \ldots, R_n = P^n\varnothing, \ldots, R_\omega = \bigcup_{n \in \omega} R_n,$$

and so on through all the ordinal numbers $0, 1, 2, \ldots, 10, \ldots,$ using, at each limit ordinal, the union of all preceding sets. Thus for each ordinal α one has the collection R_α of sets of rank α. In this way, the hierarchy of sets is presented as a cumulative type theory, with R_α as all sets of type α or *less*. The Zermelo-Fraenkel axioms are then (a selection of) the facts true for all sets in this hierarchy. This is sometimes claimed to describe the ultimate Platonic reality which underlies all mathematics: Perhaps the Zermelo-Fraenkel axioms do not describe everything, but with a little more insight we will understand all the axioms necessary and then at least in principle all mathematical problems can be settled from the axioms.

It is my contention that this *Grand Set Theoretic Foundation* is a mistakenly one-sided view of mathematics and also that its precursor doctrine (Dedekind cuts) was also one-sided. This grand formulation does succeed in recording a view of mathematical rigor, but by emphasizing this it misses other important points about the nature of mathematics.

We may list various difficulties with the grand foundation as follows:

First, it does not adequately describe which are the relevant mathematical structures to be built up from the starting point of set theory. *A priori* from set theory there could be very many such structures, but in fact there are a few which are dominant, one list being provided by Bourbaki's "mother structures." Some mathematical structures (natural numbers, rational numbers, real numbers, Euclidean geometry) are intended to be unique but other structures are built to have many different models: group, ring, order and partial order, linear space and module, topological space, measure space. The "Grand Foundation" does not provide any way in which to explain the choice of these concepts (such a choice depends on the "breadth" parameters and the relation to the outside world). The grand foundation also does not recognize the common notions which appear in different types of structures, as for example the sense in which "universal" constructions (that is, adjoint functors) appear in many different places: in the construction of a vector space on a given set as basis, of a free group on a given set as generators, of the Stone-Čech compactification of a given space and of the tensor product of two vector spaces. Further development of these ideas in a positive direction seems to require the provision of a big table of mathematical structures. This table would give not only the interrelations and commonalities between the structures, but also their origins in activities arising in the world. This seems more hopeful than trying to find a formal description which will designate the mathematical structures to be studied.

Second, set theory is largely irrelevant to the practice of most mathematics. Most professional mathematicians never have occasion to use the Zermelo-Fraenkel axioms, while others do not even know them. If they did know the axioms, they would rapidly observe that most of the mathematics they do could be satisfactorily based on a much weaker set of axioms—say the

Zermelo system in which the replacement axiom of Fraenkel is dropped in favor of the weaker comprehension axiom. The comprehension axiom does allow the frequently used formation of the set of all x with a specified property $P(x)$, where with Fraenkel and Skolem a property is anything expressed by an ϵ-formula of the first-order predicate calculus. However, for mathematical purposes it suffices in most cases to use only these formulas where the quantifiers are bounded (i.e., where $\forall x$ or $\exists x$ is applied only for x in a given set).

Thus, technically, there is not one preferred system of axioms for the set theory used by mathematicians. This, however, is not the real point—which is that in practice set theory is not the grounds of all mathematics, but of just one highly specialized branch of mathematics.

The *Grand Set Theoretic Foundation* of mathematics has other, more technical disadvantages. It does not answer the difficulties presented by the Gödel incompleteness theorem. It is not strong enough to take into account some of the large constructions on the fringe of mathematics. For example, one would like to form the category of all sets (essentially the set of all sets and of all functions between sets). This can be done by speaking of the "class" of all sets. This device, however, will not yield bigger constructions, such as the category of all categories. That can be managed by a different device: Assume that there is a (Grothendieck) universe containing all (ordinary) sets, build bigger sets out of this universe, and then form the category of all categories contained in this universe. These devices to make set theory include the fringes seem artificial.

The set-theoretic approach is by no means the only possible foundation for mathematics. Another approach is to formulate axioms not on set membership, but on the composition of functions. This results in an axiomatization of the category of all sets. The resulting axioms (those for an "elementary topos") describe cartesian products, power sets, and the like, by certain "universal" properties. For this reason they probably give better insight into the conceptual form of mathematics than does set theory. There may well be other possible systematic foundations different from set-theoretic or categorical ones.

The final difficulty with the Grand Foundation is that it does not account for what E. P. Wigner has termed the unreasonable success of mathematics in its applications.

6. Cantorian Set Theory. Many students of set theory do not follow what I have called the "Grand Set Theoretic Foundation" but instead follow Cantor to emphasize the intuitive notion of a set as a collection which is a real object in its own right. For them set theory is not subsumed by the Zermelo-Fraenkel axiom system or by any other first-order formal system. It may be studied formally by other means; using infinitary languages or second-order logic. Such Cantorian sets are just as real as numbers. Indeed, one might say that number theory is formalized only in part by Peano's arithmetic in just the way set theory is formalized, but only in part, by Zermelo-Fraenkel. (There can be true properties of whole numbers not demonstrated in Peano arithmetic.)

From this point of view, set theory is just another branch of mathematics. If in this view set theory is not taken to be *the* foundation of mathematics, it can be assimilated with our proposal that mathematics consists of formal disciplines derived from a variety of human activities. Here the relevant activity is that of "collecting" things into "totalities."

However, this Cantorian point of view is often taken to concern a Platonic world of sets. This does not fit our proposal.

7. Multiple Models for Set Theory. By now there are substantially different models of set theory, satisfying one or another special axiom— the axiom of constructibility, Martin's axiom, or the axiom of determinateness. The striking result of these technical developments is that different models of set theory give different answers to specific mathematical problems. The continuum hypothesis is true on the Gödel axiom of constructibility, but false in certain Cohen models of set theory. Whitehead's problem provides another striking example. He considered a homomorphism $f: A \to G$ of one abelian group A onto another such group G, in the case when the kernel is just the (additive) group of integers. In case G is a free abelian group, the epimorphism splits (that is, there is a homomorphism $h:G \to A$ with $fh = 1$). Whitehead asked: Conversely, if such an f always splits, is G free? It now turns out that the answer may be yes or no, depending on the model of set theory. This is one of many striking cases where explicit mathematical problems have different answers, depending on the model used for set theory. (See Eklof [**2**].)

Mathematics, we hold, deals with multiple models of the world. It is not subsumed in any one big model or by any one grand system of axioms.

The idea that set theory is relative is not new; it was clearly stated for axiomatic set theory by Skolem in 1922 [**9**]. We intend simply to draw some of the philosophic consequences of that relativity. For the Platonist, there is a real world of sets, existing forever, described only approximately by the Zermelo-Fraenkel axioms or by their modifications. It may be that some final insight will give a definite axiom system, but the sets themselves are the underlying mathematical reality.

In our view, such a Platonic world is speculative. It cannot be clearly explained as a matter of fact (ontologically) or as an object of human knowledge (epistemologically). Moreover, such ideal worlds rapidly become too elaborate; they must display not only the sets but all the other separate structures which mathematicians have described or will discover. The real nature of these structures does not lie in their often artificial construction from set theory, but in their relation to simple mathematical ideas or to basic human activities.

Hence, we hold that mathematics is not the study of intangible Platonic worlds, but of tangible formal systems which have arisen from real human activities.

8. Models of Geometry. Space provides a striking example of the multiple variety of mathematical models. The original human experience of space is vague and varied: Space is extensive and hollow, both fixed and the locus of movement. With Euclidean geometry it is analyzed axiomatically as a receptacle: Space is described in terms of the things (lines, triangles, circles) which can be pushed around within it. With non-Euclidean geometry came the possibility of a different deductive model of the "same" space. In a different direction, the description of the plane and the three-space by means of Cartesian coordinates led to an analysis of much more general figures within space: those given by general algebraic equations or by other functions, including in particular poorly behaved functions (curves without tangents). This analytic approach also presently indicated that those original geometric intuitions of space also applied to space of more than three dimensions—and even to infinite dimensional spaces. Here, too, space is apprehended partially in geometric terms, and partially—by vector analysis—in algebraic fashion. Thus there are many mathematical models of space (Mac Lane [**5**]).

The case of topological spaces and manifolds is especially striking. First came the general notion of a metric space, motivated, it seems, by the use of a metric for function spaces for the calculus of variations and for integral equations. Then came a striking discovery: The continuity of a real valued function f on a metric space M can be defined wholly in terms of the open subsets of M. It was this discovery, combined with the study of Riemann surfaces, which led to the definition of topological spaces. This definition represented considerable extension of the notion of geometry.

However, for other parts of geometry one needed algebraic functions or differentiable functions on the space—and these classes of functions (apparently) cannot be described just in terms of their action on subsets of that space. One must instead specify for each open subset of the space all the good functions (differentiable, analytic, or algebraic, as the case may be) on that subset. These specifications amount to defining a sheaf on the space. In this way, a differentiable manifold can be described as pieces of Euclidean space, pasted together so that the appropriate sheaves match. Similarly for algebraic geometry a "scheme" is described by pasting together suitable affine pieces so that the sheaves match. Thus the intuitive idea of a "space" for differential or algebraic geometry can be adequately formalized only by sophisticated and deep notions, such as those of sheaf theory.

9. Breadth, Clarity, and Depth. Let me return to the philosophical issues. We hold that logicism, formalism, and Platonism have been too much dominated by the notions of set theory and deductive rigor. A balanced philosophy of mathematics should complement these ideas with others. The others we tended to list as three: breadth, clarity, and depth.

All three become important because of the extent of abstraction in mathematics. Abstraction consists in formulating essential aspects of some subject matter in terms of suitable axioms. Such abstraction can take place in several successive stages, interlocking different branches of mathematics. However, to be well directed or relevant that abstraction needs these three qualities.

The breadth of a mathematical notion refers to the variety of the situations in which it is to apply and to the pertinence and relevance of the abstraction made. It carries also the caution that deductions of theorems are guided not just by rigor but by the intent of the applications or by the origin of the abstraction.

Second, abstraction has increased the need for clarity in presentation; if the object of study is abstract, it must be understood not by its intuitive content but by its precise and abstract description.

Clarity goes beyond the precision of rigor to a clear ordering of ideas. The development of abstract mathematics, especially after 1920, is in this view a reflection of the necessity for such clarity. When geometry was the geometry of a three-dimensional real world, there could be continual appeals to the real world. Now, done in greater generality, it must be done rigorously and exactly; this means also that it must be clear and perspicuous.

The depth of a mathematical notion refers to the way in which that notion gets at the nonobvious, more fundamental structures and concepts underlying the problems at issue—as group theory underlies symmetry or as uniform continuity is subtly involved in many questions of real analysis. The study of manifolds in differential geometry and in algebraic geometry offers another example of the discovery of deeper notions. Initially one thinks of a manifold as a suitable smooth set of points spread out in some given ambient space. Later on one forgets the ambient space and considers the manifold in terms of the well-behaved functions which can be defined on it, as well as the germs of such functions at each point. This study in turn leads to the sheaf of germs of well-behaved functions on the manifold, and so to the deeper ideas of sheaf theory.

The depth of a mathematical notion may well change with time. For example, in the late nineteenth century the notion of uniform continuity seemed hard. It now seems easier, and is often dismissed as being simply a change in the position of an existential quantifier.

On the basis of this observation, we attempt a definition of mathematics about as follows:

Mathematics consists in the discovery of successive stages of the formal structures underlying the world and human activities in that world, with emphasis on those structures of broad applicability and those reflecting deeper aspects of the world.

In detail, mathematical development uses experience and intuitive insights to discover appropriate formal structures, to make deductive analyses of these structures, and to establish formal interconnections between them. In other words, mathematics studies interlocking structures. Because of the depth and of the distance from immediate concerns, mathematical treatments need be not only rigorous but also endowed with conceptual clarity.

References

1. F. Browder and S. Mac Lane, The relevance of mathematics, in Mathematics Today, ed. by Lynn Steen, Springer-Verlag, 1978, pp. 323–349.

2. Paul C. Eklof, Whitehead's problem is undecidable, this MONTHLY, 83 (1976) 775–787.

3. G. Frege, Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet, vol. 1, Jena, 1893; vol. 2, Jena, 1903.

4. N. D. Goodman, Mathematics as an objective science, this MONTHLY, 86 (1979) 540–555.

5. S. Mac Lane, Mathematical models of space, American Scientist, 68 (1980) 184–191.

6. Hilary Putnam, Philosophy of mathematics, in Current Research in the Philosophy of Science, ed. by Peter Asquith and Henry E. Kyburg, Jr., Philosophy of Science Association, East Lansing, Michigan, 1979, pp. 386–398.

7. W. V. Quine, Ontological Relativity and Other Essays, Columbia University Press, New York, 1969.

8. Bertrand Russell, Mathematical logic as based on the theory of types, Amer. J. Math., 30 (1908) 222–262.
9. Th. Skolem, Einige Bemerkungen zur axiomatische Begründung der Mengenlehre. Fifth Congress of Scandinavian Mathematicians, 1922, Helsingfors, 1923, pp. 217–232.
10. Hao Wang, From Mathematics to Philosophy, London, Routledge and Kegan Paul, 1971.
11. A. N. Whitehead and Bertrand Russell, Principia Mathematica vol. 1, Cambridge University Press, Cambridge, England, 1910; 2nd edition, 1925.
12. E. Zermelo, Untersuchungen über die Grundlagen der Mengenlehre I, Math Ann., 65 (1908) 261–281.

COMPUTER SCIENCE, MATHEMATICS, AND THE UNDERGRADUATE CURRICULA IN BOTH

ANTHONY RALSTON
Department of Computer Science, SUNY at Buffalo, Amherst, NY 14226

> There is a simple and basic fact about a computer which will, in the decades and centuries to come, affect not so much what is known in mathematics as what is thought important in it. This is its finiteness.
>
> Wallace Givens [**7** (1966)]

> We have thought of the calculus as the beginning, the gateway. But should it be?
>
> H. O. Pollak [**21** (1978)]

If new disciplines may be described as emerging from old ones, then computer science may be said to have sprung mainly from mathematics, although, of course, the influence of electrical engineering was also considerable. In recent years, however, the ties between mathematics and computer science have been steadily weakening. This has been coupled with a declining belief on the part of some (most?) computer scientists in the importance of mathematical training and mathematical tools in the education of computer scientists and in research in computer science. Some of the evidence for this "declining belief" will be considered later in this paper. Here I note only my belief that, contrariwise, the importance of mathematics in computer science is and should be growing rapidly. It is this belief which motivates much of this paper.

While thus far in the three decades of computing and in the two of computer science there has been considerable discussion of the influence, or lack of it, of mathematics on computer science, there has been little discussion of what should—or should not—be the influence of the development of computer science on mathematics research and education. The other major motivation of this paper stems from my belief that the growth of computer science should be having, but has not had, a profound effect on undergraduate mathematics education.

1. The Role of Mathematics in Computer Science Education. In the 1960's, in addition to the emerging departments of computer science themselves, there were numerous examples of departments of statistics and computer science, or applied mathematics and computer science, or computer programs and options within departments of mathematics. And this was in contrast to a

---

Anthony Ralston received his Ph.D. in Mathematics under Eric Reissner at MIT. He has held positions at Bell Laboratories and Stevens Institute of Technology and is now Professor of Computer Science at SUNY at Buffalo. He has been a visiting faculty member at the University of Leeds, the Institute of Computer Science of the University of London, and University College, London. He is a former president of the Association for Computing Machinery and of the American Federation of Information Processing Societies. Most of Professor Ralston's early research papers were in numerical analysis. In recent years, after publishing mainly on strictly computer science topics, he has "returned" to mathematics with current interests in discrete mathematics and mathematics education. He is the author or editor of eight books, including *A First Course in Numerical Analysis* (second edition, with P. Rabinowitz, McGraw-Hill, 1978) and the *Encyclopedia of Computer Science* (Van Nostrand Reinhold, 1976).