SPARKCHARTS™

CALCULUS REFERENCE

SPARK CHARTS™

THEORY

DERIVATIVES AND DIFFERENTIATION

Definition: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

DERIVATIVE RULES

1. **Sum and Difference:** $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$
2. **Scalar Multiple:** $\frac{d}{dx}\big(cf(x)\big) = cf'(x)$
3. **Product:** $\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$

   Mnemonic: If f is "hi" and g is "ho," then the product rule is "ho d hi plus hi d ho."
4. **Quotient:** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

   Mnemonic: "Ho d hi minus hi d ho over ho ho."
5. **The Chain Rule**
   - First formulation: $(f \circ g)'(x) = f'(g(x))\,g'(x)$
   - Second formulation: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
6. **Implicit differentiation:** Used for curves when it is difficult to express y as a function of x. Differentiate both sides of the equation with respect to x. Use the chain rule carefully whenever y appears. Then, rewrite $\frac{dy}{dx} = y'$ and solve for y'.

   Ex: $x \cos y - y^2 = 3x$. Differentiate to first obtain $\frac{dx}{dx}\cos y + x\frac{d(\cos y)}{dx} - 2y\frac{dy}{dx} = 3\frac{dx}{dx}$, and then $\cos y - x(\sin y)y' - 2yy' = 3$. Finally, solve for $y' = \frac{\cos y - 3}{x \sin y + 2y}$.

COMMON DERIVATIVES

1. **Constants:** $\frac{d}{dx}(c) = 0$
2. **Linear:** $\frac{d}{dx}(mx + b) = m$
3. **Powers:** $\frac{d}{dx}(x^n) = nx^{n-1}$ (true for all real $n \neq 0$)
4. **Polynomials:** $\frac{d}{dx}(a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0) = a_n n x^{n-1} + \cdots + 2a_2 x + a_1$
5. **Exponential**
   - **Base e:** $\frac{d}{dx}(e^x) = e^x$
   - **Arbitrary base:** $\frac{d}{dx}(a^x) = a^x \ln a$
6. **Logarithmic**
   - **Base e:** $\frac{d}{dx}(\ln x) = \frac{1}{x}$
   - **Arbitrary base:** $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$
7. **Trigonometric**
   - **Sine:** $\frac{d}{dx}(\sin x) = \cos x$
   - **Cosine:** $\frac{d}{dx}(\cos x) = -\sin x$
   - **Tangent:** $\frac{d}{dx}(\tan x) = \sec^2 x$
   - **Cotangent:** $\frac{d}{dx}(\cot x) = -\csc^2 x$
   - **Secant:** $\frac{d}{dx}(\sec x) = \sec x \tan x$
   - **Cosecant:** $\frac{d}{dx}(\csc x) = -\csc x \cot x$
8. **Inverse Trigonometric**
   - **Arcsine:** $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
   - **Arccosine:** $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$
   - **Arctangent:** $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
   - **Arccotangent:** $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$
   - **Arcsecant:** $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$
   - **Arccosecant:** $\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$

INTEGRALS AND INTEGRATION

DEFINITE INTEGRAL

The **definite integral** $\int_a^b f(x)\,dx$ is the **signed area** between the function $y = f(x)$ and the x-axis from $x = a$ to $x = b$.

- **Formal definition:** Let n be an integer and $\Delta x = \frac{b-a}{n}$. For each $k = 0, 2, \ldots, n-1$, pick point x_k^* in the interval $[a + k\Delta x, a + (k+1)\Delta x]$. The expression $\Delta x \sum_{k=0}^{n-1} f(x_k^*)$ is a **Riemann sum.** The definite integral $\int_a^b f(x)\,dx$ is defined as $\lim_{n\to\infty} \Delta x \sum_{k=0}^{n-1} f(x_k^*)$.

INDEFINITE INTEGRAL

- **Antiderivative:** The function $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$.
- **Indefinite integral:** The indefinite integral $\int f(x)\,dx$ represents a **family of antiderivatives:** $\int f(x)\,dx = F(x) + C$ if $F'(x) = f(x)$.

FUNDAMENTAL THEOREM OF CALCULUS

Part 1: If $f(x)$ is continuous on the interval $[a, b]$, then the area function $F(x) = \int_a^x f(t)\,dt$ is continuous and differentiable on the interval and $F'(x) = f(x)$.

Part 2: If $f(x)$ is continuous on the interval $[a, b]$ and $F(x)$ is any antiderivative of $f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

APPROXIMATING DEFINITE INTEGRALS

1. **Left-hand rectangle approximation:** $L_n = \Delta x \sum_{k=0}^{n-1} f(x_k)$
2. **Right-hand rectangle approximation:** $R_n = \Delta x \sum_{k=1}^{n} f(x_k)$
3. **Midpoint Rule:** $M_n = \Delta x \sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right)$
4. **Trapezoidal Rule:** $T_n = \frac{\Delta x}{2}\big(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\big)$
5. **Simpson's Rule:** $S_n = \frac{\Delta x}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\big)$

TECHNIQUES OF INTEGRATION

1. **Properties of Integrals**
   - **Sums and differences:** $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
   - **Constant multiples:** $\int cf(x)\,dx = c\int f(x)\,dx$
   - **Definite integrals: reversing the limits:** $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
   - **Definite integrals: concatenation:** $\int_a^p f(x)\,dx + \int_p^b f(x)\,dx = \int_a^b f(x)\,dx$
   - **Definite integrals: comparison:**
     If $f(x) \le g(x)$ on the interval $[a, b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.
2. **Substitution Rule**—a.k.a. ***u*-substitutions:** $\int f(g(x))g'(x)\,dx = \int f(u)\,du$
   - $\int f\big(g(x)\big)g'(x)\,dx = F(g(x)) + C$ if $\int f(x)\,dx = F(x) + C$.
3. **Integration by Parts**

   Best used to integrate a product when one factor ($u = f(x)$) has a simple derivative and the other factor ($dv = g'(x)\,dx$) is easy to integrate.
   - **Indefinite Integrals:**
     $\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$ or $\int u\,dv = uv - \int v\,du$
   - **Definite Integrals:** $\int_a^b f(x)g'(x)\,dx = f(x)g(x)\big]_a^b - \int_a^b f'(x)g(x)\,dx$
4. **Trigonometric Substitutions:** Used to integrate expressions of the form $\sqrt{\pm a^2 \pm x^2}$.

Expression	Trig substitution	Expression becomes	Range of θ	Pythagorean identity used
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ $dx = a\cos\theta\,d\theta$	$a\cos\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ $(-a \le x \le a)$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$ $dx = a\sec\theta\tan\theta\,d\theta$	$a\tan\theta$	$0 \le \theta < \frac{\pi}{2}$ $\pi \le \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$ $dx = a\sec^2\theta\,d\theta$	$a\sec\theta$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$

APPLICATIONS

GEOMETRY

Area: $\int_a^b \big(f(x) - g(x)\big)\,dx$ is the area bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ if $f(x) \ge g(x)$ on $[a, b]$.

Volume of revolved solid (disk method): $\pi\int_a^b \big(f(x)\big)^2\,dx$ is the volume of the solid swept out by the curve $y = f(x)$ as it revolves around the x-axis on the interval $[a, b]$.

Volume of revolved solid (washer method): $\pi\int_a^b \big(f(x)\big)^2 - \big(g(x)\big)^2\,dx$ is the volume of the solid swept out between $y = f(x)$ and $y = g(x)$ as they revolve around the x-axis on the interval $[a, b]$ if $f(x) \ge g(x)$.

Volume of revolved solid (shell method): $\int_a^b 2\pi x f(x)\,dx$ is the volume of the solid obtained by revolving the region under the curve $y = f(x)$ between $x = a$ and $x = b$ around the y-axis.

Arc length: $\int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,dx$ is the length of the curve $y = f(x)$ from $x = a$ to $x = b$.

Surface area: $\int_a^b 2\pi f(x)\sqrt{1 + \big(f'(x)\big)^2}\,dx$ is the area of the surface swept out by revolving the function $y = f(x)$ about the x-axis between $x = a$ and $x = b$.

CONTINUED ON OTHER SIDE

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MOTION

1. Position $s(t)$ *vs.* **time** t **graph:**

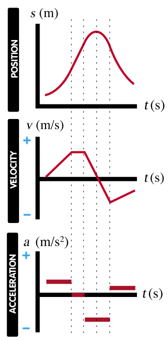
- The slope of the graph is the velocity: $s'(t) = v(t)$.
- The concavity of the graph is the acceleration: $s''(t) = a(t)$.

2. Velocity $v(t)$ *vs.* **time** t **graph:**

- The slope of the graph is the accleration: $v'(t) = a(t)$.
- The (signed) area under the graph gives the displacement (change in position):
  $$s(t) - s(0) = \int_0^t v(\tau)\,d\tau$$

3. Acceleration $a(t)$ *vs.* **time** t **graph:**

- The (signed) area under the graph gives the change in velocity: $$v(t) - v(0) = \int_0^t a(\tau)\,d\tau$$



PROBABILITY AND STATISTICS

- **Average value** of $f(x)$ between a and b is $\bar{f} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

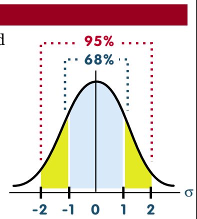
CONTINUOUS DISTRIBUTION FORMULAS

X and Y are **random variables.**

- **Probability density function** $f(x)$ of the random variable X satisfies:
  1. $f(x) \geq 0$ for all x;
  2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
- **Probability that** X **is between** a **and** b: $P(a \leq X \leq b) = \int_a^b f(x)\,dx$
- **Expected value** (a.k.a. **expectation** or **mean**) **of** X: $E(X) = \mu_X = \int_{-\infty}^{\infty} x f(x)\,dx$
- **Variance:** $\mathrm{Var}(X) = \sigma_X^2 = \int_{-\infty}^{\infty} \left(x - E(x)\right)^2 f(x)\,dx = E(X^2) - \left(E(X)\right)^2$
- **Standard deviation:** $\sqrt{\mathrm{Var}(X)} = \sigma_X$
- **Median** m satisfies $\int_{-\infty}^{m} f(x)\,dx = \int_m^{\infty} f(x)\,dx = \frac{1}{2}$.
- **Cumulative density function** ($F(x)$ is the probability that X is at most x):
  $$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$
- **Joint probability density function** $g(x, y)$ chronicles distribution of X and Y. Then
  $$f(x) = \int_{-\infty}^{\infty} g(x, y)\,dy.$$
- **Covariance:** $\mathrm{Cov}(X, Y) = \sigma_{XY} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(x - E(X)\right)\left(y - E(Y)\right) f(x, y)\,dx\,dy$
- **Correlation:** $\rho(X, Y) = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y} = \dfrac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}}$

COMMON DISTRIBUTIONS

1. **Normal distribution** (or **Bell curve**) with mean μ and variance σ: $$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
   - $P(\mu - \sigma \leq X \leq \mu - \sigma) = 68.3\%$
   - $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95.5\%$



2. χ**-square distribution:** with mean ν and variance 2ν: $$f(x) = \frac{1}{2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)} x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}$$

- **Gamma function:** $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\,dt$

MICROECONOMICS

COST

- **Cost function** $C(x)$: cost of producing x units.
- **Marginal cost:** $C'(x)$
- **Average cost function** $\overline{C}(x) = \frac{C(x)}{x}$: cost per unit when x units produced.
- **Marginal average cost:** $\overline{C}'(x)$
  If the average cost is minimized, then average cost = marginal cost.
- If $C''(x) > 0$, then to find the number of units (x) that minimizes average cost, solve for x in $\frac{C(x)}{x} = C'(x)$.

REVENUE, PROFIT

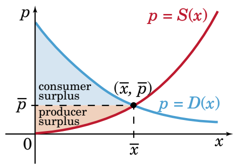
- **Demand** (or **price**) **function** $p(x)$: price charged per unit if x units sold.
- **Revenue** (or **sales**) **function:** $R(x) = xp(x)$
- **Marginal revenue:** $R'(x)$
- **Profit function:** $P(x) = R(x) - C(x)$
- **Marginal profit function:** $P'(x)$
  If profit is maximal, then marginal revenue = marginal cost.
- The number of units x maximizes profit if $R'(x) = C'(x)$ and $R''(x) < C''(x)$.

PRICE ELASTICITY OF DEMAND

- **Demand curve:** $x = x(p)$ is the number of units demanded at price p.
- **Price elasticity of demand:** $E(p) = -\frac{p\,x'(p)}{x(p)}$
  - Demand is **elastic** if $E(p) > 1$. Percentage change in p leads to larger percentage change in $x(p)$. Increasing p leads to decrease in revenue.
  - Demand is **unitary** if $E(p) = 1$. Percentage change in p leads to similar percentage change in $x(p)$. Small change in p will not change revenue.
  - Demand is **inelastic** if $E(p) < 1$. Percentage change in p leads to smaller percentage change in $x(p)$. Increasing p leads to increase in revenue.
- Formula relating elasticity and revenue: $R'(p) = x(p)\left(1 - E(p)\right)$

CONSUMER AND PRODUCER SURPLUS

- **Demand function:** $p = D(x)$ gives price per unit (p) when x units demanded.
- **Supply function:** $p = S(x)$ gives price per unit (p) when x units available.
- **Market equilibrium** is $\bar{x}$ units at price $\bar{p}$. (So $\bar{p} = D(\bar{x}) = S(\bar{x})$.)
- **Consumer surplus:**
  $CS = \int_0^{\bar{x}} D(x)\,dx - \bar{p}\bar{x} = \int_0^{\bar{x}} (D(x) - \bar{p})\,dx$
- **Producer surplus:**
  $PS = \bar{p}\bar{x} - \int_0^{\bar{x}} S(x)\,dx = \int_0^{\bar{x}} (\bar{p} - S(x))\,dx$

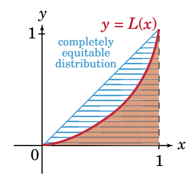


LORENTZ CURVE

The **Lorentz Curve** $L(x)$ is the fraction of income received by the poorest x fraction of the population.

1. **Domain** and **range** of $L(x)$ is the interval $[0, 1]$.
2. **Endpoints:** $L(0) = 0$ and $L(1) = 1$
3. Curve is nondecreasing: $L'(x) \geq 0$ for all x
4. $L(x) \leq x$ for all x

- **Coefficient of Inequality** (a.k.a. **Gini Index**):
  $$L = 2\int_0^1 \left(x - f(x)\right)dx.$$
  The quantity L is between 0 and 1. The closer L is to 1, the more equitable the income distribution.



SUBSTITUTE AND COMPLEMENTATRY COMMODITIES

X and Y are two commodities with unit price p and q, respectively.

- The amount of X demanded is given by $f(p, q)$.
- The amount of Y demanded is given by $g(p, q)$.

1. X and Y are **substitute** commodities (**Ex:** pet mice and pet rats) if $\frac{\partial f}{\partial q} > 0$ and $\frac{\partial g}{\partial p} > 0$.
2. X and Y are **complementary** commodities (**Ex:** pet mice and mouse feed) if $\frac{\partial f}{\partial q} < 0$ and $\frac{\partial g}{\partial p} < 0$.

FINANCE

- $P(t)$: the amount after t years.
- $P_0 = P(0)$: the original amount invested (the **principal**).
- r: the yearly **interest rate** (the yearly percentage is $100r\%$).

INTEREST

- **Simple interest:** $P(t) = P_0\,(1 + r)^t$
- **Compound interest**
  - Interest compounded m times a year: $P(t) = P_0\left(1 + \frac{r}{m}\right)^{mt}$
  - Interest **compounded continuously:** $P(t) = P_0 e^{rt}$

EFFECTIVE INTEREST RATES

The **effective** (or **true**) **interest rate,** r_{eff}, is a rate which, if applied simply (without compounding) to a principal, will yield the same end amount after the same amount of time.

- Interest compounded m times a year: $r_{\text{eff}} = \left(1 + \frac{r}{m}\right)^m - 1$
- Interest compounded continuously: $r_{\text{eff}} = e^r - 1$

PRESENT VALUE OF FUTURE AMOUNT

The **present value** (PV) of an amount (A) t years in the future is the amount of principal that, if invested at r yearly interest, will yield A after t years.

- Interest compounded m times a year: $PV = A\left(1 + \frac{r}{m}\right)^{-mt}$
- Interest compounded continuously: $PV = Ae^{-rt}$

PRESENT VALUE OF ANNUITIES AND PERPETUITIES

Present value of amount P paid yearly (starting next year) for t years or in perpetuity:

1. Interest compounded yearly
   - Annuity paid for t years: $PV = \frac{P}{r}\left(1 - \frac{1}{(1+r)^t}\right)$
   - Perpetuity: $PV = \frac{P}{r}$
2. Interest compounded continuously
   - Annuity paid for t years: $PV = \frac{P}{r_{\text{eff}}}\left(1 - e^{-rt}\right) = \frac{P}{e^r - 1}\left(1 - e^{-rt}\right)$
   - Perpetuity: $PV = \frac{P}{r_{\text{eff}}} = \frac{P}{e^r - 1}$

BIOLOGY

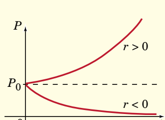
In all the following models

- $P(t)$: size of the population at time t;
- $P_0 = P(0)$, the size of the population at time $t = 0$;
- r: coefficient of rate of growth.

EXPONENTIAL (MALTHUSIAN) GROWTH / EXPONENTIAL DECAY MODEL

$\frac{dP}{dt} = rP$

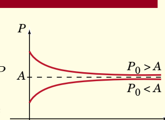
- Solution: $P(t) = P_0 e^{rt}$
- If $r > 0$, this is **exponential growth**; if $r < 0$, **exponential decay.**



RESTRICTED GROWTH (A.K.A. LEARNING CURVE) MODEL

$\frac{dP}{dt} = r\,(A - P)$

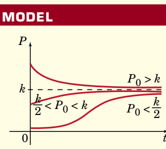
- A: long-term asymptotic value of P
- Solution: $P(t) = A + (P_0 - A)e^{-rt}$



LOGISTIC GROWTH MODEL

$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right)$

- K: the **carrying capacity**
- Solution: $$P(t) = \frac{K}{1 + \left(\frac{K - P_0}{P_0}\right)e^{-rt}}$$



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