SPARKCHARTS™

ALGEBRA I

SPARK CHARTS

NUMBER SYSTEMS

The **natural numbers** are the numbers we count with: $1, 2, 3, 4, 5, 6, \ldots, 27, 28, \ldots$

The **whole numbers** are the numbers we count with and zero: $0, 1, 2, 3, 4, 5, 6, \ldots$

The **integers** are the numbers we count with, their **negatives,** and zero: $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

—The **positive integers** are the natural numbers.

—The **negative integers** are the "minus" natural numbers: $-1, -2, -3, -4, \ldots$

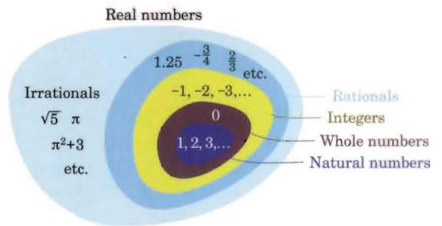
The **rational numbers** are all numbers expressible as $\frac{\text{integer}}{\text{integer}}$ fractions. The fractions may be **proper** (less than one; **Ex:** $\frac{1}{3}$) or **improper** (more than one; **Ex:** $\frac{21}{17}$). Rational numbers can be positive (**Ex:** $5.125 = \frac{41}{8}$) or negative (**Ex:** $-\frac{3}{4}$). All integers are rational: **Ex:** $4 = \frac{4}{1}$.

The **real numbers** can be represented as points on the number line. All rational numbers are real, but the real number line has many points that are "between" rational numbers and are called **irrational. Ex:** $\sqrt[3]{2}$, π, $\sqrt{3} - 9$, $0.12112111211112\ldots$.

The **imaginary** numbers are square roots of negative numbers. They don't appear on the real number line and are written in terms of $i = \sqrt{-1}$. **Ex:** $\sqrt{-49}$ is imaginary and equal to $i\sqrt{49}$ or $7i$.

The **complex numbers** are all possible sums of real and imaginary numbers; they are written as $a + bi$, where a and b are real and $i = \sqrt{-1}$ is imaginary. All reals are complex (with $b = 0$) and all imaginary numbers are complex (with $a = 0$).

The Fundamental Theorem of Algebra says that every polynomial of degree n has exactly n complex roots (counting multiple roots).



Venn Diagram of Number Systems

SETS

A **set** is any collection—finite or infinite—of things called **members** or **elements.** To denote a set, we enclose the elements in braces. **Ex:** $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the (infinite) set of natural numbers. The notation $a \in \mathbb{N}$ means that a is in $\mathbb{N}$, or a "is an element of" $\mathbb{N}$.

DEFINITIONS

—**Empty set** or **null set:** $\emptyset$ or $\{\}$: The set without any elements. Beware: the set $\{0\}$ is a set with one element, 0. It is not the same as the empty set.

—**Union** of two sets: $A \cup B$ is the set of all elements that are in either set (or in both). **Ex:** If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cup B = \{1, 2, 3, 4, 6\}$.

—**Intersection** of two sets: $A \cap B$ is the set of all the elements that are both in A and in B. **Ex:** If $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$, then $A \cap B = \{2\}$. Two sets with no elements in common are **disjoint**; their intersection is the empty set.

—**Complement** of a set: $\bar{A}$ is the set of all elements that are not in A. **Ex:** If we're talking about the set $\{1, 2, 3, 4, 5, 6\}$, and $A = \{1, 2, 3\}$, then $\bar{A} = \{4, 5, 6\}$. It is always true that $A \cap \bar{A} = \emptyset$ and $A \cup \bar{A}$ is everything.

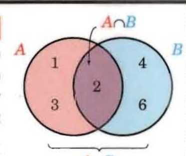
—**Subset:** $A \subset C$: A is a subset of C if all the elements of A are also elements of C.

Ex: If $A = \{1, 2, 3\}$ and $C = \{-2, 0, 1, 2, 3, 4, 5, 8\}$, then $A \subset C$.

VENN DIAGRAMS

A **Venn Diagram** is a visual way to represent the relationship between two or more sets. Each set is represented by a circle-like shape; elements of the set are pictured inside it. Elements in an overlapping section of two sets belong to both sets (and are in the intersection).

Counting elements: (size of $A \cup B$) = (size of A) + (size of B) − (size of $A \cap B$).



Venn Diagram:
$A = \{1, 2, 3\}$,
$B = \{2, 4, 6\}$,
$A \cup B = \{1, 2, 3, 4, 6\}$,
$A \cap B = \{2\}$.

PROPERTIES OF ARITHMETIC OPERATIONS

PROPERTIES OF REAL NUMBERS UNDER ADDITION AND MULTIPLICATION

Real numbers satisfy 11 properties: 5 for addition, 5 matching ones for multiplication, and 1 that connects addition and multiplication. Suppose a, b, and c are real numbers.

Property	Addition (+)	Multiplication (× or ·)
Commutative	$a + b = b + a$	$a \cdot b = b \cdot a$
Associative	$(a + b) + c = a + (b + c)$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
Identities exist	0 is a real number. $a + 0 = 0 + a = a$. 0 is the **additive identity.**	1 is a real number. $a \cdot 1 = 1 \cdot a = a$. 1 is the **multiplicative identity.**
Inverses exist	$-a$ is a real number. $a + (-a) = (-a) + a = 0$. Also, $-(-a) = a$.	If $a \neq 0$, $\frac{1}{a}$ is a real number. $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$. Also, $\frac{1}{\frac{1}{a}} = a$.
Closure	$a + b$ is a real number.	$a \cdot b$ is a real number.

Distributive property (of additon over multiplication) $a \cdot (b + c) = a \cdot b + a \cdot c$; $(b + c) \cdot a = b \cdot a + c \cdot a$

There are also two (derivative) properties having to do with zero.

Multiplication by zero: $a \cdot 0 = 0 \cdot a = 0$.

Zero product property: If $ab = 0$ then $a = 0$ or $b = 0$ (or both).

INEQUALITY SYMBOLS

Sign	Meaning	Example
$<$	less than	$1 < 2$ and $4 < 56$
$>$	greater than	$1 > 0$ and $56 > 4$
$\neq$	not equal to	$0 \neq 3$ and $-1 \neq 1$
$\leq$	less than or equal to	$1 \leq 1$ and $1 \leq 2$
$\geq$	greater than or equal to	$1 \geq 1$ and $3 \geq -29$

The sharp end always points toward the smaller number; the open end toward the larger.

PROPERTIES OF EQUALITY AND INEQUALITY

Trichotomy: For any two real numbers a and b, exactly one of the following is true: $a < b$, $a = b$, or $a > b$.

Other properties: Suppose a, b, and c are real numbers.

Property	Equality (=)	Inequality (< and >)
Reflexive	$a = a$	
Symmetric	If $a = b$, then $b = a$.	
Transitive	If $a = b$ and $b = c$, then $a = c$.	If $a < b$ and $b < c$, then $a < c$.
Addition and subtraction	If $a = b$, then $a + c = b + c$ and $a - c = b - c$.	If $a < b$, then $a + c < b + c$ and $a - c < b - c$.
Multiplication and division	If $a = b$, then $ac = bc$ and $\frac{a}{c} = \frac{b}{c}$ (if $c \neq 0$).	If $a < b$ and $c > 0$, then $ac < bc$ and $\frac{a}{c} < \frac{b}{c}$. If $a < b$ and $c < 0$, then switch the inequality: $ac > bc$ and $\frac{a}{c} > \frac{b}{c}$.

LINEAR EQUATIONS IN ONE VARIABLE

A **linear equation** in one variable is an equation that, after simplifying and collecting like terms on each side, will look like $ax + b = c$ or like $ax + b = cx + d$. Each side can involve xs added to real numbers and multiplied by real numbers but not multiplied by other xs.

Ex: $\frac{3}{4}\left(-\frac{x}{2} - 3\right) + x = 9 - \left(x - \frac{5}{8}\right)$ is a linear equation. But $x^2 + 9 = 3$ and $x(x + 4) = 2$ and $\sqrt{x} = 5$ are not linear.

Linear equations in one variable will always have (a) exactly one real number solution, (b) no solutions, or (c) all real numbers as solutions.

FINDING A UNIQUE SOLUTION

Ex: $\frac{3}{4}\left(-\frac{x}{2} - 3\right) + x = 9 - \left(x - \frac{5}{8}\right)$.

1. Get rid of fractions outside parentheses.

Multiply through by the LCM of the denominators.

Ex: Multiply by 4 to get $3\left(-\frac{x}{2} - 3\right) + 4x = 36 - 4\left(x - \frac{5}{8}\right)$.

2. Simplify using order of operations (PEMDAS).

Use the distributive property and combine like terms on each side. Remember to distribute minus signs.

Ex: Distribute the left-side parentheses: $-\frac{3}{2}x - 9 + 4x = 36 - 4\left(x - \frac{5}{8}\right)$.

Combine like terms on the left side: $\frac{5}{2}x - 9 = 36 - 4\left(x - \frac{5}{8}\right)$.

Distribute the right-side parentheses: $\frac{5}{2}x - 9 = 36 - 4x + \frac{5}{2}$.

Combine like terms on the right side: $\frac{5}{2}x - 9 = \frac{77}{2} - 4x$.

3. Repeat as necessary to get the form $ax + b = cx + d$.

Multiply by 2 to get rid of fractions: $5x - 18 = 77 - 8x$.

4. Move variable terms and constant terms to different sides.

Usually, move variables to the side that had the larger variable coefficient to begin with. Equation should look like $ax = b$.

Add $8x$ to both sides to get $5x + 8x - 18 = 77$ or $13x - 18 = 77$.

Add 18 to both sides to get $13x = 77 + 18$ or $13x = 95$.

5. Divide both sides by the variable's coefficient. *Stop if* $a = 0$.

Divide by 13 to get $x = \frac{95}{13}$.

6. Check the solution by plugging into the original equation.

Does $\frac{3}{4}\left(\frac{-95}{2 \cdot 13} - 3\right) + \frac{95}{13} = 9 - \left(\frac{95}{13} - \frac{5}{8}\right)$? Yes! Hooray.

DETERMINING IF A UNIQUE SOLUTION EXISTS

—The original equation has **no solution** if, after legal transformations, the new equation is false.

Ex: $2 = 3$ or $3x - 7 = 2 + 3x$.

—**All real numbers** are solutions to the original equation if, after legal transformations, the new equation is an identity.

Ex: $2x = 3x - x$ or $1 = 1$.

Any linear equation can be simplified into the form $ax = b$ for some a and b. If $a \neq 0$, then $x = \frac{b}{a}$ (exactly one solution). If $a = 0$ but $b \neq 0$, then there is no solution. If $a = b = 0$, then all real numbers are solutions.

INEQUALITIES IN ONE VARIABLE

Use the same procedure as for equalities, except *flip the inequality when multiplying or dividing by a negative number.* **Ex:** $-x > 5$ is equivalent to $x < -5$.

—The inequality may have **no solution** if it reduces to an impossible statement. **Ex:** $x + 1 > x + 9$ reduces to $1 > 9$.

—The inequality may have **all real numbers** as solutions if it reduces to a statement that is always true. **Ex:** $5 - x \geq 3 - x$ reduces to $5 \geq 3$ and has infintely many solutions.

Solutions given the reduced inequality and the condition:

	$a > 0$	$a < 0$	$a = 0$, $b > 0$	$a = 0$, $b < 0$	$a = 0$, $b = 0$
$ax > b$	$x > \frac{b}{a}$	$x < \frac{b}{a}$	none	all	none
$ax \geq b$	$x \geq \frac{b}{a}$	$x \leq \frac{b}{a}$	none	all	all
$ax < b$	$x < \frac{b}{a}$	$x > \frac{b}{a}$	all	none	none
$ax \leq b$	$x \leq \frac{b}{a}$	$x \geq \frac{b}{a}$	all	none	all

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"I WOULD ADVISE YOU, SIR, TO STUDY ALGEBRA... YOUR HEAD WOULD BE LESS MUDDY."

SAMUEL JOHNSON

ABSOLUTE VALUE

The **absolute value** of a number n, denoted $|n|$, is its distance from 0. It is always nonnegative. Thus $|3| = 3$ and $|-5| = 5$. Also, $|0| = 0$.

Formally,

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

- The **distance** between a and b is the positive value $|a - b| = |b - a|$. **Ex:** $|5 - 8| = 3$.
- Absolute value bars act like parentheses when determining the order of operations.

PROPERTIES OF ABSOLUTE VALUE

$|a| = |b|$ means $a = b$ or $a = -b$.

$|a| = 0$ means $a = 0$.

If $b \geq 0$, **then**

$|a| = b$ means $a = b$ or $a = -b$.

$|a| < b$ means $-b < a < b$.

$|a| > b$ means $a < -b$ or $a > b$.

If $b < 0$, **then**

$|a| \leq b$ is impossible.

$|a| > b$ means a could be anything.

SOLVING EQUATIONS WITH ABSOLUTE VALUE

- Change the equation until the absolute value expression is alone on one side.
- Feel free to factor out positive constants. **Ex:** $|2x - 4| = 6$ is equivalent to $|x - 2| = 3$. To factor out negative constants, use $|a| = |-a|$. Thus $|-x - 1| = 4$ is equivalent to $|x + 1| = +4$. (The solutions are $\{3, -5\}$; the equation $|x + 1| = -4$ has no solutions.)
- Use the Properities of Absolute Value to unravel the absolute value expression. There will be two equalities unless using the property that $|a| = 0$ implies $a = 0$.
- Solve each one separately. There may be no solutions, 1 or 2 solutions, or all real numbers may be solutions.
- Check specific solutions by plugging them in. If there are infinitely many solutions or no solutions, check two numbers of large magnitude, positive and negative.
- Be especially careful if the equation contains variables both inside and outside the absolute value bars. Keep track of which equalities and inequalities hold true in which case.
- **Ex. 1:** $|3x - 5| = 2x$. If $2x \geq 0$, then $3x - 5 = 2x$ or $3x - 5 = -2x$. The first gives $x = 5$; the second gives $x = 1$. Both work.
- **Ex. 2:** $|3x + 5| = 3x$. If $3x \geq 0$, then we can rewrite this as $3x + 5 = 3x$ or $3x + 5 = -3x$. The first, $3x + 5 = 3x$ gives no solutions. The second seems to give the solution $x = -\frac{6}{5}$. But wait! The equation $3x + 5 = -3x$ only holds if $3x \geq 0$, or $x \geq 0$. So $-\frac{6}{5}$ does not work. No solution.

SOLVING INEQUALITIES WITH ABSOLUTE VALUE

Unraveling tricks:

$|a| < b$ is true when $b \geq 0$ and $-b < a < b$.

$|a| > b$ is true when $b < 0$ OR $\{b \geq 0$ and $a > b\}$ OR $\{b \geq 0$ and $a < -b\}$.

Ex: $|10x + 1| < 7x + 3$ is equivalent to $7x + 3 \geq 0$ and $-7x - 3 < 10x + 1 < 7x + 3$. Thus the equations $7x + 3 \geq 0$, $-7x - 3 < 10x + 1$, and $10x + 1 < 7x + 3$ must all hold. Solving the equations, we see that $x \geq -\frac{3}{7}$, $x > -\frac{17}{4}$, and $x < \frac{2}{3}$. The first condition forces the second, and the solutions are all x with $-\frac{3}{7} \leq x < \frac{2}{3}$.

LEAST-THINKING METHOD

Trial and error: Unravel every absolute value by replacing every $|expression|$ with $\pm(expression)$. Find all solutions to the associated equalities. Also, find all solutions that the equation obtains by replacing the absolute value with 0. All of these are potential boundary points. Determine the solution intervals by testing a point in every interval and every boundary point. The point $x = 0$ is often good to test.

It's simplest to keep track of your information by graphing everything on the real number line.

Ex: $|10x + 1| < 7x + 3$.
Solve the three equations $10x + 1 = 7x + 3$, $-10x - 1 = 7x + 3$, and $7x + 3 = 0$ to find potential boundary points. The three points are, not surprisingly, $\frac{2}{3}$, $-\frac{17}{4}$, and $-\frac{3}{7}$. Testing the three boundary points and a point from each of the four intervals gives the solution $-\frac{3}{7} \leq x < \frac{2}{3}$.

GRAPHING ON THE REAL NUMBER LINE

The **real number line** is a pictorial representation of the real numbers; every number corresponds to a point. Solutions to one-variable equations and (especially) inequalities may be graphed on the real number line. The idea is to shade in those parts of the line that represent solutions.

Origin: A special point representing 0. By convention, points to the left of the origin represent negative numbers, and points to the right of the origin represent positive numbers.

Ray: A half-line; everything to the left or the right of a given point. The endpoint may or may not be included.

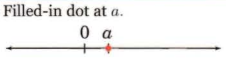
Interval: A piece of the line; everything between two endpoints, which may or may not be included.

Open (ray or interval): Endpoints not included.

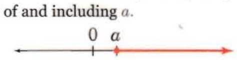
Closed (ray or interval): Endpoints included.

GRAPHING SIMPLE STATEMENTS

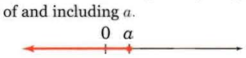
$x = a$: Filled-in dot at a.



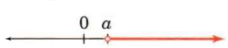
$x \geq a$: Shaded closed ray: everything to the right of and including a.



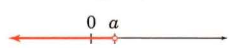
$x \leq a$: Shaded closed ray: everything to the left of and including a.



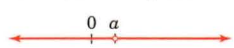
$x > a$: Shaded open ray: everything to the right of (and not including) a. An open circle around the point a represents the not-included endpoint.



$x < a$: Shaded open ray: everything to the left of (and not including) a. Open circle around a.



$x \neq a$: Everything is shaded except for a, around which there is an open circle.

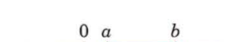


$a < x < b$; $a \leq x < b$; $a < x \leq b$; $a \leq x \leq b$: A whole range of values can be solutions. This is represented by shading in a portion of the number line:

Shaded interval between a and b.

Only works if $a < b$.

Filled-in circle if the endpoint is included, open circle if the endpoint is not included.

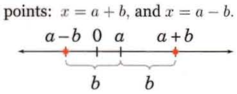


$a < x \leq b$

GRAPHING ABSOLUTE VALUE STATEMENTS

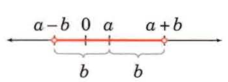
Absolute value is distance. $|x - a| = b$ means that the distance between a and x is b. Throughout, b must be non-negative.

$|x - a| = b$: The distance from a to x is b. Plot two points: $x = a + b$, and $x = a - b$.



$|x - a| < b$; $|x - a| \leq b$:

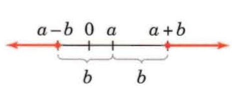
The distance from a to x is less than (no more than) b; or x is closer than b to a. Plot the interval (open or closed) $a - b < x < a + b$ (or $a - b \leq x \leq a + b$.)



$|x - a| < b$ (open interval)

$|x - a| > b$; $|x - a| \geq b$:

The distance from a to x is more than (no less than) b; or x is further than b away from a. Plot the double rays (open or closed) $x < a - b$ and $x > a + b$ (or $x \leq a - b$ and $x \geq a + b$).



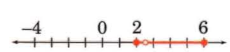
$|x - a| \geq b$

OTHER COMPOUND INEQUALITIES

Intersection: Inequalities joined by **AND.** Both (or all) of the inequalities must true.
Ex: $|x - 1| < 4$ is really $x > -3$ AND $x < 5$. Equivalently, it is $\{x : x > -3\} \cap \{x : x < 5\}$. The graph is the intersection of the graphs of both inequalities. Shade the portions that would be shaded by both if graphed independently.

Union: Inequalities joined by **OR.** At least one of the inequalities must be true.
Ex: $|x - 1| > 4$ is really $x < -3$ OR $x > 5$. Equivalently, it is $\{x : x < -3\} \cup \{x : x > 5\}$. The graph is the union of the graphs of the individual inequalities. Shade the portions that would be shaded by either one (or both) if graphed independently.

- Endpoints may disappear. **Ex:** $x > 5$ OR $x \geq 6$ just means that $x > 5$. The point 6 is no longer an endpoint.



$|x - 4| \leq 2$ AND $x \neq 2.5$

IF YOU CAN DETERMINE ALL POTENTIAL ENDPOINTS...

Plot the points and test all the intervals one by one by plugging a sample point into the equation. Also test all endpoints to determine if they're included. See example in Solving Inequalities with Absolute Value, above.

THE CARTESIAN PLANE

The **Cartesian** (or **coordinate**) **plane** is a method for giving a name to each point in the plane on the basis of how far it is from two special perpendicular lines, called **axes.**

TERMINOLOGY OF THE CARTESIAN PLANE

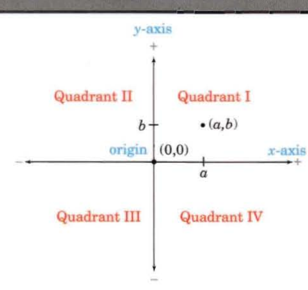
***x*-axis:** Usually, the **horizontal** axis of the coordinate plane. Positive distances are measured to the right; negative, to the left.

***y*-axis:** Usually, the **vertical** axis of the coordinate plane. Positive distances are measured up; negative, down.

Origin: $(0, 0)$, the point of intersection of the x-axis and the y-axis.

Quadrants: The four regions of the plane cut by the two axes. By convention, they are numbered counterclockwise starting with the upper right (see the diagram at right).

Point: A location on a plane identified by an **ordered pair of coordinates** enclosed in parentheses. The first coordinate is measured along the x-axis; the second, along the y-axis. **Ex:** The point $(1, 2)$ is 1 unit to the right and 2 units up from the origin. Occasionally (rarely), the first coordinate is called the **abscissa;** the second, the **ordinate.**



Cartesian plane with Quadrants I, II, III, IV; point (a, b).

Sign ($\pm$) of the x- and y-coordinates in the four quadrants:

	I	II	III	IV
x	+	−	−	+
y	+	+	−	−

LINES IN THE CARTESIAN PLANE

A straight line is uniquely identified by any two points, or by any one point and the incline, or slope, of the line.

- **Slope of a line:** The slope of a line in the Cartesian plane measures how steep it is—

CONTINUED ON OTHER SIDE

SPARKCHARTS™

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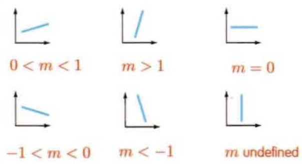
a measure of how fast the line moves "up" for every bit that it moves "over" (left or right). If (a,b) and (c,d) are two points on the line, then the slope is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{d-b}{c-a}.$$

- Any pair of points on a straight line will give the same slope value.
- **Horizontal lines** have slope 0.
- The slope of a **vertical line** is **undefined;** it is "infinitely large."
- Lines that go "up right" and "down left" (ending in I and III) have **positive slope.**
- Lines that go "up left" and "down right" (ending in II and IV) have **negative slope.**
- **Parallel lines** have the same slope.
- The slopes of **perpendicular lines** are negative reciprocals of each other: if two lines of slope m_1 and m_2 are perpendicular, then $m_1 m_2 = -1$ and $m_2 = -\frac{1}{m_1}$.

Rough direction of lines with slope m:



x-Intercept: The x-coordinate of the point where a line crosses the x-axis. The x-intercept of a line that crosses the x-axis at $(a,0)$ is a. Horizontal lines have no x-intercept.

y-Intercept: The y-coordinate of the point where a line crosses the y-axis. The y-intercept of a line that crosses the y-axis at $(0,b)$ is b. Vertical lines have no y-intercept.

FINDING THE EQUATION OF A LINE

Any line in the Cartesian plane represents some linear relationship between x and y values. The relationship always can be expressed as $Ax + By = C$ for some real numbers A, B, C. The coordinates of every point on the line will satisfy the equation.

A **horizontal line** at height b has equation $y = b$.

A **vertical line** with x-intercept a has equation $x = a$.

Given slope m and y-intercept b:

Equation: $y = mx + b$.

Standard form: $mx - y = -b$.

Given slope m and any point (x_0, y_0):

Equation: $y - y_0 = m(x - x_0)$.

Standard form: $mx - y = mx_0 - y_0$.

Alternatively, write down $y_0 = mx_0 + b$ and solve for $b = y_0 - mx_0$ to get the slope-intercept form.

Given two points (x_1, y_1) and (x_2, y_2):

Find the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

Equation:

$y - y_1 = m(x - x_1) = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$, or

$y - y_2 = m(x - x_2)$.

Given slope m and x-intercept a:

Equation: $x = \frac{y}{m} + a$.

Given x-intercept a and y-intercept b:

Equation: $\frac{x}{a} + \frac{y}{b} = 1$.

Given a point on the line and the equation of a parallel line:

Find the slope of the parallel line (see Graphing Linear Equations). The slope of the original line is the same. Use point-slope form.

Given a point on the line and the equation of a perpendicular line:

Find the slope m_0 of the perpendicular line. The slope of the original line is $-\frac{1}{m_0}$. Use point-slope form.

GRAPHING LINEAR EQUATIONS

A **linear equation in two variables** (say x and y) can be manipulated—after all the x-terms and y-terms and constant terms are have been grouped together—into the form $Ax + By = C$. The graph of the equation is a straight line (hence the name).

- **Using the slope to graph:** Plot one point of the line. If the slope is expressed as a ratio of small whole numbers $\pm\frac{r}{s}$, keep plotting points r up and $\pm s$ over from the previous point until you have enough to draw the line.
- **Finding intercepts:** To find the y-intercept, set $x = 0$ and solve for y. To find the x-intercept, set $y = 0$ and solve for x.

SLOPE-INTERCEPT FORM: y = mx + b

One of the easiest-to-graph forms of a linear equation.

m is the slope.

b is the y-intercept. $(0,b)$ is a point on the line.

POINT-SLOPE FORM: y − k = m(x − h)

m is the slope.

(h,k) is a point on the line.

STANDARD FORM: Ax + Bx = C

Less thinking: Solve for y and put the equation into slope-intercept form.

Less work: Find the x- and the y-intercepts. Plot them and connect the line.

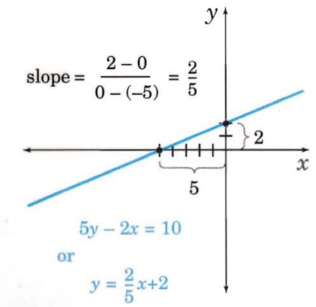
Slope: $-\frac{A}{B}$.

y-intercept: $\frac{C}{B}$.

x-intercept: $\frac{C}{A}$.

ANY OTHER FORM

If you are sure that an equation is linear, but it isn't in a nice form, find a couple of solutions. Plot those points. Connect them with a straight line. Done.



SIMULTANEOUS LINEAR EQUATIONS

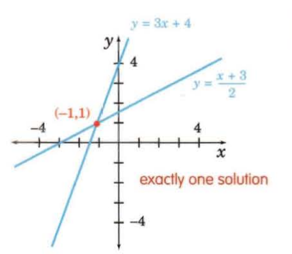
A linear equation in two variables (say $ax + by = c$, with a and c not both zero) has infinitely many ordered pair (x,y) **solutions**—real values of x and y that make the equation true. Two simultaneous linear equations in two variables will have:

- Exactly **one solution** if their graphs intersect—the most common scenario.
- **No solutions** if the graphs of the two equations are parallel.
- **Infinitely many solutions** if their graphs coincide.

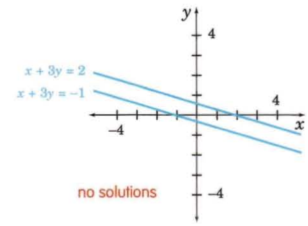
SOLVING BY GRAPHING: TWO VARIABLES

Graph both equations on the same Cartesian plane. The intersection of the graph gives the simultaneous solutions. (Since points on each graph correspond to solutions to the appropriate equation, points on *both graphs* are solutions to both equations.)

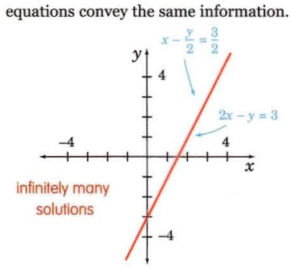
- Sometimes, the exact solution can be determined from the graph; other times the graph gives an estimate only. Plug in and check.
- If the lines intersect in exactly one point (most cases), the intersection is the unique solution to the system.



- If the lines are parallel, they do not intersect; the system has no solutions. *Parallel lines have the same slope;* if the slope is not the same, the lines will intersect.



- If the lines coincide, there are infinitely many solutions. Effectively, the two equations convey the same information.



SOLVING BY SUBSTITUTION: TWO VARIABLES

- Use one equation to solve for one variable (say, y) in terms of the other (x): isolate y on one side of the equation.
- Plug the expression for y into the other equation.
- Solve the resulting one-variable linear equation for x.
    - If there is no solution to this new equation, there are no solutions to the system.
    - If all real numbers are solutions to the new equation, there are infinitely many solutions; the two equations are **dependent.**
- Solve for y by plugging the x-value into the expression for y in terms of x.
- Check that the solution works by plugging it into the original equations.

Ex: $\begin{cases} x - 4y = 1 \\ 2x - 11 = 2y \end{cases}$

Using the first equation to solve for y in terms of x gives $y = \frac{1}{4}(x-1)$. Plugging in to the second equation gives $2x - 11 = 2\left(\frac{1}{4}(x-1)\right)$. Solving for x gives $x = 7$. Plugging in for y gives $y = \frac{1}{4}(7-1) = \frac{3}{2}$. Check that $(7, \frac{3}{2})$ works.

SOLVING BY ADDING OR SUBTRACTING EQUATIONS: TWO VARIABLES

Express both equations in the same form. $ax + by = c$ works well.

Look for ways to add or subtract the equations to eliminate one of the variables.

- If the coefficients on a variable in the two equations are the same, subtract the equations.
- If the coefficients on a variable in the two equations differ by a sign, add the equations.
- If one of the coefficients on one of the variables (say, x) in one of the equations is 1, multiply that whole equation by the x-coefficient in the other equation; subtract the two equations.
- If no simple combination is obvious, simply pick a variable (say, x). Multiply the first equation by the x-coefficient of the second equation, multiply the second equation by the x-coefficient of the first equation, and subtract the equations.

If all went well, the sum or difference equation is in one variable (and easy to solve if the original equations had been in $ax + by = c$ form). Solve it.

- If by eliminating one variable, the other is eliminated too, then there is no unique solution to the system. If there are no solutions to the sum (or difference) equation, there is no solution to the system. If all real numbers are solutions to the sum (or difference) equation, then the two original equations are dependent and express the same relationship between the variables; there are infinitely many solutions to the system.
- Plug the solved-for variable into one of the original equations to solve for the other variable.

Ex: $\begin{cases} x - 4y = 1 \\ 2x - 11 = 2y \end{cases}$

Rewrite to get $\begin{cases} x - 4y = 1 \\ 2x - 2y = 11 \end{cases}$.

The x-coefficient in the first equation is 1, so we multiply the first equation by 2 to get $2x - 8y = 2$, and subtract this equation from the original second equation to get:

$(2-2)x + (-2-(-8))y = 11 - 2$ or $6y = 9$, which gives $y = \frac{3}{2}$, as before.

CRAMER'S RULE

The solution to the simultaneous equations

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ is given by $x = \frac{de - bf}{ad - bc}$, $y = \frac{af - ce}{ad - bc}$

if $ad - bc \neq 0$.

MORE THAN TWO VARIABLES

There is a decent chance that a system of linear equations has a unique solution only if there are as many equations as variables.

- If there are too many equations, then the conditions are likely to be too restrictive, resulting in no solutions. (This is only actually true if the equations are "independent"—each new equation provides new information about the relationship of the variables.)
- If there are too few equations, then there will be too few restrictions; if the equations are not contradictory, there will be infinitely many solutions.
- All of the above methods can, in theory, be used to solve systems of more than two linear equations. In practice, graphing only works in two dimensions. It's too hard to visualize planes in space.
- Substitution works fine for three variables; it becomes cumbersome with more variables.
- Adding or subtracting equations (or rather, arrays of coefficients called **matrices**) is the method that is used for large systems.

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EXPONENTS AND POWERS

Exponential notation is shorthand for repeated multiplication: $3 \cdot 3 = 3^2$ and $(-2y) \cdot (-2y) \cdot (-2y) = (-2y)^3$.

In the notation a^n, a is the base, and n is the exponent. The whole expression is "a to the nth power," or the "nth power of a, or, simply, "a to the n."

a^2 is "a squared;" a^3 is "a cubed."

$(-a)^n$ is not necessarily the same as $-(a^n)$.

Ex: $(-4)^2 = 16$, whereas $-(4^2) = -16$. Following the order of operation rules, $-a^n = -(a^n)$.

RULES OF EXPONENTS

Product of powers: $a^m a^n = a^{m+n}$

If the bases are the same, then to multiply, simply add their exponents. **Ex:** $2^3 \cdot 2^8 = 2^{11}$.

Quotient of powers: $\frac{a^m}{a^n} = a^{m-n}$

If the bases of two powers are the same, then to divide, subtract their exponents.

Exponentiation powers: $(a^m)^n = a^{mn}$

To raise a power to a power, multiply exponents.

Power of a product: $(ab)^n = a^n b^n$

Quotient of a product: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Exponentiation distributes over multiplication and division, but not over addition or subtraction. **Ex:** $(2xy)^2 = 4x^2y^2$, but $(2 + x + y)^2 \neq 4 + x^2 + y^2$.

Zeroth power: $a^0 = 1$

To be consistent with all the other exponent rules, we set $a^0 = 1$ unless $a = 0$. The expression 0^0 is undefined.

Negative powers: $a^{-n} = \frac{1}{a^n}$

We define negative powers as reciprocals of positive powers. This works well with all other rules. **Ex:** $2^3 \cdot 2^{-3} = \frac{2^3}{2^3} = 1$. Also, $2^3 \cdot 2^{-3} = 2^{3+(-3)} = 2^0 = 1$.

Fractional powers: $a^{\frac{1}{n}} = \sqrt[n]{a}$

This definition, too, works well with all other rules.

ROOTS AND RADICALS

Taking roots undoes raising to powers: $\sqrt[3]{8} = 2$ because $2^3 = 8$.

The expression $\sqrt[n]{a}$ reads "the nth root of a."

- The **radical** is the root sign $\sqrt{\ }$.
- The expression under the radical sign is called the **radicand**. Sometimes, it is also referred to as the radical.
- The number n is the **index**. It is usually dropped for square roots: $\sqrt{a} = \sqrt[2]{a}$.
- In radical notation, n is always an integer.
- When n is even and a is positive, we have two choices for the nth root. In such cases, we agree that the expression $\sqrt[n]{a}$ always refers to the positive, or **principal**, root. **Ex:** Although $3^2 = (-3)^2 = 9$, we know that $\sqrt{9} = 3$.

SIMPLIFYING SQUARE ROOTS

A square root expression is considered **simplified** if the radical has no repeated factors. Use the rule $\sqrt{a^2b} = a\sqrt{b}$.

- Factor the radicand and move any factor that appears twice outside of the square root sign.
  Ex: $\sqrt{60} = \sqrt{2 \cdot 2 \cdot 3 \cdot 5} = \sqrt{2 \cdot 2}\sqrt{3 \cdot 5} = 2\sqrt{15}$.
- If the radicand contains a variable expression, don't lose heart: do the exact same thing. Use $\sqrt{x^{2n}} = x^n$ and $\sqrt{x^{2n+1}} = x^n\sqrt{x}$.
  Ex: $\sqrt{32x^7} = \sqrt{2^5x^7} = \sqrt{(4x^3)^2\, 2x} = 4x^3\sqrt{2x}$.

ROOT RULES

We can view roots as powers with fractional exponents; thus roots satisfy the same properties that powers do.

Radical Rule Summary

Rule	Radicals	Exponents
Root of a product	$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$(ab)^n = a^n b^n$
Root of a quotient	$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
Root of a root	$\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a}$	$(a^m)^n = a^{mn}$
Product of roots	$\sqrt[n]{a}\sqrt[m]{b} = \sqrt[mn]{a^m b^n}$	
Root of a power	$\sqrt[mn]{a^n} = \sqrt[m]{a}$	
Converting between notation	$\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$	

SIMPLIFYING HIGHER-POWERED RADICALS

An nth root expression is simplified if:

- The radicand is not divisible by any nth power. Factor the radicand and use $\sqrt[n]{a^n b} = a\sqrt[n]{b}$ as for square roots.
- The radicand is not a perfect mth power for any m that is a factor of n. Use $\sqrt[mn]{a^n} = \sqrt[m]{a}$. **Ex:** $\sqrt[6]{16} = \sqrt[6]{4^2} = \sqrt[3]{4}$.
- Two unlike roots may be joined together. Use $\sqrt[n]{a}\sqrt[m]{b} = \sqrt[mn]{a^m b^n}$. Then reduce: the final index should be the LCM of the original indices.

Ex: $\sqrt[4]{x^3}\sqrt[6]{x^5} = \sqrt[12]{x^9 x^{10}} = x\sqrt[12]{x^7}$

- When in doubt, use $\sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m = a^{\frac{m}{n}}$ to convert to fractional exponents and work with them.

Ex: $\sqrt[4]{x^3}\sqrt[6]{x^5} = \sqrt[12]{(x^3)^3}\sqrt[12]{(x^5)^2} = x^{\frac{3}{4}}x^{\frac{5}{6}} = x^{\frac{3}{4}+\frac{5}{6}} = x^{\frac{19}{12}} = x\sqrt[12]{x^7}$.

- You may lose $\pm$ sign information. **Ex:** $\sqrt[6]{(-2)^2} = \sqrt[6]{4}$ is positive, but $\sqrt[3]{-2}$ is negative.

RATIONALIZING THE DENOMINATOR

A fractional expression is considered simplified only if there are no radical signs in the denominator. Use the the rule $\frac{1}{\sqrt{d}} = \frac{\sqrt{d}}{d}$.

1. If there are radicals in the denominator, combine them into one radical expression $\sqrt{d}$.
2. Multiply the fraction by "a clever form of 1:" $\frac{\sqrt{d}}{\sqrt{d}}$. This will leave a factor of d in the denominator and, effectively, pull the radical up into the numerator.
3. Simplify the radical in the numerator and reduce the fraction if necessary.
   Ex: $\frac{5\sqrt{6}}{\sqrt{10}} = \frac{5\sqrt{6}}{\sqrt{10}} \cdot \frac{\sqrt{10}}{\sqrt{10}} = \frac{5\sqrt{60}}{10} = \frac{5 \cdot 2\sqrt{15}}{10} = \sqrt{15}$.

- If the simplified radical in the denominator is an nth root $\sqrt[n]{d}$, use the identity $\frac{1}{\sqrt[n]{d}} = \frac{\sqrt[n]{d^{n-1}}}{d}$. In other words, the clever form of 1 this time is $\frac{\sqrt[n]{d^{n-1}}}{\sqrt[n]{d^{n-1}}}$.

Ex: $\frac{\sqrt{6}}{\sqrt[3]{4}} = \frac{\sqrt{6}\sqrt[3]{16}}{\sqrt[3]{4 \cdot 16}} = \frac{\sqrt{6}\sqrt[3]{2^3 \cdot 2}}{4} = \frac{2\sqrt[6]{2^2 \cdot 6^3}}{4} = \frac{\sqrt[6]{864}}{2}$.

"Simplified" form is not necessarily simpler.

POLYNOMIALS

Polynomials are expressions obtained by adding, subtracting, and multiplying real numbers and one or several variables. Usually the variables are arranged alphabetically.

- Expressions connected by $+$ or $-$ signs are called terms. **Ex:** The polynomial $2x^3y - 7x$ has two **terms**.
- The **coefficient** of a term is the real number (non-variable) part.
- Two terms are sometimes called **like terms** if the power of each variable in the terms is the same. **Ex:** $7y^6x$ and yxy^5 are like terms. $2x^8$ and $16xy^7$ are not. Like terms can be added or subtracted into a single term.
- The **degree** of a term is the sum of the powers of each variable in the term. **Ex:** $2x^8$ and $16xy^6z$ both have degree eight.
- The degree of a polynomial is the highest degree of any of its terms.
- In a polynomial in one variable, the term with the highest degree is called the **leading term**, and its coefficient is the **leading coefficient**.

CLASSIFICATION OF POLYNOMIALS

By degree: Ex: $2x^5y - 4x^3y^3 + 5$ and $4y^5 - 16y^6$ are both sixth-degree polynomials.

Special names for polynomials in one variable:

By degree:	By number of terms:
degree 1: **linear**	1 term: **monomial**
degree 2: **quadratic**	2 terms: **binomial**
degree 3: **cubic**	3 terms: **trinomial**
degree 4: **quartic**	
degree 5: **quintic**	

ADDING AND SUBTRACTING POLYNOMIALS

- Only like terms can be added or subtracted together into one term:
  Ex: $3x^3y - 5xyx^2 = -2x^3y$.
- When subtracting a polynomial, it may be easiest to flip all the $\pm$ signs and add it instead.

MULTIPLYING POLYNOMIALS

The key is to multiply every term by every term, term by term. The number of terms in the (unsimplified) product is the product of the numbers of terms in the two polynomials.

- Multiplying a monomial by any other polynomial: Distribute and multiply each term of the polynomial by the monomial.
- Multiplying two binomials: Multiply each term of the first by each term of the second: $(a + b)(c + d) = ac + bc + ad + bd$.
- MNEMONIC: **FOIL:** Multiply the two **F**irst terms, the two **O**utside terms, the two **I**nside terms, and the two **L**ast terms.
- Common products:
  $(a + b)^2 = a^2 + 2ab + b^2$
  $(a - b)^2 = a^2 - 2ab + b^2$
  $(a + b)(a - b) = a^2 - b^2$
- After multiplying, simplify by combining like terms.

WORD PROBLEMS

For more on measurements, units, and percent, see the SparkChart on Math Basics.

The systematic way to solve word problems is to convert them to equations.

1. **Choose variables.** Choose wisely. Whatever you are asked to find usually merits a variable.
2. **Rewrite the statements** given in the problem as equations using your variables. Use common sense: more, fewer, sum, total, difference mean what you want them to mean. Common trigger words include:
   Of: Frequently means multiplication. **Ex:** "Half of the flowers are blue" means that if there are c flowers, then there are $\frac{1}{2}c$ blue flowers.
   Percent (%): Divide by 100. **Ex:** "12% of the flowers had withered" means that $\frac{12}{100}c$ flowers were withered.
3. **Solve the equation(s)** to find the desired quantity.
4. **Check that the answer make sense.** If the answer is $3\frac{1}{4}$ girls in the park or -3 shoes in a closet, either you made a computational mistake or the problem has no solution.

RATE PROBLEMS

Rate problems often involve speed, distance, and time. These are often good variables candidates.

Equations to use:

$(\text{distance}) = (\text{speed}) \times (\text{time})$

$(\text{speed}) = \frac{\text{distance}}{\text{time}}$

$\text{time} = \frac{\text{distance}}{\text{speed}}$

Check that the units on distance and time correspond to the units on speed. Convert if necessary:

Time:
$1\text{ min} = 60\text{ s}$; $1\text{ h} = 60\text{ min} = 3{,}600\text{ s}$

Distance: $1\text{ ft} = 12\text{ in}$; $1\text{ yd} = 3\text{ ft} = 36\text{ in}$
$1\text{ mi} = 1{,}760\text{ yd} = 5{,}280\text{ ft}$

Metric distance:
$1\text{ m} = 100\text{ cm}$; $1\text{ km} = 1000\text{ m}$
$1\text{ in} \approx 2.54\text{ cm}$; $1\text{ m} \approx 3.28\text{ ft}$; $1\text{ mi} \approx 1.61\text{ km}$

$\text{Average speed} = \frac{\text{total distance}}{\text{total time}}$

Average speed is not the average of speeds used over equal distances, it's the average of speeds used over equal time intervals.

Ex: Supercar travels at 60 mi/h for 30 min and at 90 mi/h for the rest of its 45-mile trip. How long does the trip take?

- The first part of the journey takes $30\text{ min} \times \frac{1\text{ h}}{60\text{ min}} = \frac{1}{2}\text{ h}$. During this time, Supercar travels $60\text{ mi/h} \times \frac{1}{2}\text{ h} = 30\text{ mi}$.
- The second part of the trip is $45\text{ mi} - 30\text{ mi} = 15\text{ mi}$ long. Supercar zips through this part in $\frac{15\text{ mi}}{90\text{ mi/h}} = \frac{1}{6}\text{ h}$.
- The total trip takes $\frac{1}{2}\text{ h} + \frac{1}{6}\text{ h} = \frac{2}{3}\text{ h}$, or 40 min.

What is Supercar's average speed for the trip?

$\frac{\text{Total distance}}{\text{Total time}} = \frac{45\text{ mi}}{\frac{2}{3}\text{ h}} = 67.5\text{ mi/h}$.

This may seem low, but it's right: Supercar had traveled at 60 mi/h and at 90 mi/h, but only one-fourth of the total journey time was at the faster speed.

TASK PROBLEMS

Ex: Sarah can paint a house in four days, while Justin can do it in five. How long will it take them working together?

These problems are disguised rate problems. If Sarah paints a house in 4 days, she works at a rate of $\frac{1}{4}$ house per day. Justin works at a rate of $\frac{1}{5}$ house per day. Working for x days, they have to complete one house:

$$\frac{x}{5} + \frac{x}{4} = 1.$$

Simplifying, we get $\frac{9x}{20} = 1$, or $x = \frac{20}{9} \approx 2.2$ days. This makes sense: two Sarahs can do the house in 2 days; two Justins can do it in 2.5 days; a Sarah and a Justin need some length of time in between.

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