Chapter 4 - Atomic Structure

4-1: The fact that most particles pass through undeflected means that there is not much to deflect these particles; most of the volume of an atom is empty space, and gases and metals are overall electrically neutral.

4-3: For a "closest approach", the incident proton must be directed "head-on" to the nucleus, with no angular momentum with respect to the nucleus (an "impact parameter" of zero; see the Appendix to Chapter 4). In this case, at the point of closest approach the proton will have no kinetic energy, and so the potential energy at closest approach will be the initial kinetic energy, taking the potential energy to be zero in the limit of very large separation. Equating these energies,

$$\mathrm{KE}_{\,\mathrm{initial}} = \frac{Z\,e^2}{4\pi\,\epsilon_0\,r_{\mathrm{min}}}, \qquad \text{or}$$

$$\begin{aligned} r_{\mathrm{min}} = \left(\frac{1}{4\pi\,\epsilon_0}\right)\frac{Z\,e^2}{\mathrm{KE}_{\,\mathrm{initial}}} &= \left(8.988\times10^{9}\ \mathrm{N\cdot m^2/C^2}\right)\frac{(79)\left(1.602\times10^{-19}\ \mathrm{C}\right)^2}{(1.602\times10^{-13}\ \mathrm{J})} \\ &= 1.14\times10^{-13}\ \mathrm{m}. \end{aligned}$$

4-5: The wavelengths in the Brackett series are given in Equation (4.9); the shortest wavelength (highest energy) corresponds to the largest value of n. For $n \to \infty$,

$$\lambda \to \frac{16}{R} = \frac{16}{1.097\times10^{7}\ \mathrm{m}^{-1}} = 1.46\times10^{-6}\ \mathrm{m} = 1.46\ \mu\mathrm{m}.$$

4-7: While the kinetic energy of any particle is positive, the potential energy of any pair of particles that are mutually attracted is negative. For the system to be bound, the total energy, the sum of the positive kinetic energy and the total negative potential energy, must be negative. For a classical particle subject to an inverse-square attractive force (such as two oppositely charged particles or two uniform spheres subject to gravitational attraction) in a circular orbit, the potential energy is twice the negative of the kinetic energy.

4-9: (a) The velocity v_1 is given by Equation (4.4), with $r = r_1 = a_0$. Combining to find v_1^2,

$$v_1^2 = \frac{e^2}{4\pi\,\epsilon_0\,m\,a_0} = \frac{e^2}{4\pi\,\epsilon_0\,m\left(\dfrac{h^2\,\epsilon_0}{\pi\,m\,e^2}\right)} = \frac{e^4}{4\,\epsilon_0^2\,h^2}, \qquad \text{so}$$

$$\frac{v_1}{c} = \frac{e^2}{2\,\epsilon_0\,h}\,\frac{1}{c} = \alpha.$$

(b) From the above,

$$\alpha = \frac{\left(1.602\times10^{-19}\ \text{C}\right)^2}{2\ (8.854\times10^{-12}\ \text{C}^2/\,(\text{N}\cdot\text{m}^2))\,(6.626\times10^{-34}\ \text{J}\cdot\text{s})\,(2.998\times10^{8}\ \text{m/s})}$$
$$= 7.296\times10^{-3},$$

so that $\dfrac{1}{\alpha} = 137.1$ to four significant figures.

A close check of the units is worthwhile; treating the units as algebraic quantities, the units as given in the above calculation are

$$\frac{\left[\text{C}^2\right]}{\dfrac{\left[\text{C}^2\right]}{[\text{N}]\ [\text{m}^2]}\ [\text{J}]\ [\text{s}]\ \dfrac{[\text{m}]}{[\text{s}]}} = \frac{[\text{N}\cdot\text{m}]}{[\text{J}]} = 1.$$

Thus, α is a dimensionless quantity, and will have the same numerical value in any system of units.

The most accurate (November, 2001) value of $1/\alpha$ is

$$\frac{1}{\alpha} = 137.03599976,$$

accurate to better than 4 parts per *billion*. For the most accurately known values of this or other physical constants, see, for instance, the **Particle Data Group** tables of **Constants, Units, Atomic and Nuclear Properties**, available at

http://pdg.lbl.gov/2001/contents_sports.html

(c) Using the above expression for α and Equation (4.13) with $n = 1$ for a_0,

$$\alpha\,a_0 = \frac{e^2}{2\,\epsilon_0\,h\,c}\,\frac{h^2\,\epsilon_0}{\pi\,m\,e^2} = \frac{1}{2\pi}\,\frac{h}{m\,c} = \frac{\lambda_C}{2\pi},$$

where the Compton wavelength λ_C is given by Equation (2.22).

4-11: With the mass, orbital speed and orbital radius of the earth known, the earth's orbital angular momentum is known, and the quantum number that would characterize the earth's orbit about the sun would be this angular momentum divided by $\hbar$;

$$n = \frac{L}{\hbar} = \frac{m\,v\,R}{\hbar} = \frac{\left(6.0 \times 10^{24}\ \mathrm{kg}\right)\left(3.0 \times 10^{4}\ \mathrm{m/s}\right)\left(1.5 \times 10^{11}\ \mathrm{m}\right)}{(1.055 \times 10^{-34}\ \mathrm{J{\cdot}s})} = 2.6 \times 10^{74}.$$

(The number of significant figures not of concern.)

4-13: The uncertainty in position of an electron confined to such a region is, from Equation (3.22), $\Delta p \geq \frac{\hbar}{2\,a_0}$, while the magnitude of the linear momentum of an electron in the first Bohr orbit is

$$p = \frac{h}{\lambda} = \frac{h}{2\pi\,a_0} = \frac{\hbar}{a_0};$$

the value of Δp found from Equation (3.13) is half of this momentum.

4-15: The Doppler effect shifts the frequencies of the emitted light to both higher and lower frequencies to produce wider lines than atoms at rest would give rise to.

4-17: It must assumed that the initial electrostatic potential energy is negligible, so that the final energy of the hydrogen atom is $E_1 = -13.6$ eV. The energy of the photon emitted is then $-E_1$, and the wavelength is

$$\lambda = \frac{hc}{(-E_1)} = \frac{1.240 \times 10^{-6}\ \mathrm{eV{\cdot}m}}{13.6\ \mathrm{eV}} = 9.12 \times 10^{-8}\ \mathrm{m} = 91.2\ \mathrm{nm},$$

in the ultraviolet part of the spectrum (see, for instance, the back endpapers of the text).

4-19: From either Equation (4.7) with $n = 10$ or Equation (4.18) with $n_f = 1$ and $n_i = 10$,

$$\lambda = \frac{100}{99}\,\frac{1}{R} = \frac{100}{99}\,\frac{1}{1.097 \times 10^{7}\ \mathrm{m}^{-1}} = 9.21 \times 10^{-8}\ \mathrm{m} = 92.1\ \mathrm{nm},$$

which is in the ultraviolet part of the spectrum (see, for instance, the back endpapers of the text).

4-21: The electrons' energy must be at least the difference between the $n = 1$ and $n = 3$ levels,

$$\Delta E = E_3 - E_1 = -E_1 \left(1 - \frac{1}{9}\right) = (13.6 \text{ eV}) \frac{8}{9} = 12.1 \text{ eV}$$

(this assumes that few or none of the hydrogen atoms had electrons in the $n = 2$ level). A potential difference of 12.1 eV is necessary to accelerate the electrons to this energy.

4-23: The energy needed to ionize hydrogen will be the energy needed to raise the energy from the ground state to the first excited state plus the energy needed to ionize an atom in the second excited state; these are the energies that correspond to the longest wavelength (least energetic photon) in the Lyman series and the shortest wavelength (most energetic photon) in the Balmer series. The energies are proportional to the reciprocals of the wavelengths, and so the wavelength of the photon needed to ionize hydrogen is

$$\lambda = \left(\frac{1}{\lambda_{2 \to 1}} + \frac{1}{\lambda_{\infty \to 2}}\right)^{-1} = \left(\frac{1}{121.5 \text{ nm}} + \frac{1}{364.6 \text{ nm}}\right)^{-1} = 91.13 \text{ nm}.$$

As a check, note that this wavelength is R^{-1}.

4-25: (a) From Equation (4.7) with $n = n_i$,

$$\frac{1}{\lambda} = R \left(1 - \frac{1}{n_i^2}\right),$$

which is solved for

$$n_i = \left(1 - \frac{1}{\lambda R}\right)^{-1/2} = \sqrt{\frac{\lambda R}{\lambda R - 1}}.$$

(b) Either of the above forms gives n very close (four places) to 3; specifically, with the product $\lambda R = \left(102.55 \times 10^{-9} \text{ m}\right) \left(1.097 \times 10^{7} \text{ m}^{-1}\right) = 1.125$ rounded to four places as $\frac{9}{8}$, $n = 3$ exactly.

4-27: (a) A relativistic calculation would necessarily involve the change in mass of the atom due to the change in energy of the system. The fact that this mass change is too small to measure (that is, the change is measured indirectly by measuring the energies of the emitted photons) means that a nonrelativistic calculation should suffice. In this situation, the kinetic energy of the recoiling atom is

$$\mathrm{KE} = \frac{p^2}{2\,M} = \frac{(h\,\nu/c)^2}{2\,M},$$

where ν is the frequency of the emitted photon and $p = h/\lambda = h\,\nu/c$ is the magnitude of the momentum of both the photon and the recoiling atom. Equation (4.16) is then

$$E_i - E_f = h\,\nu + \mathrm{KE} = h\,\nu + \frac{(h\,\nu)^2}{2\,Mc^2} = h\,\nu\left(1 + \frac{h\,\nu}{2\,Mc^2}\right).$$

This result is equivalent to that of Problem 2-53, where $h\,\nu = E_\infty$ and the term $p^2/(2\,M)$ corresponds to $E_\infty - E$ in that problem. As in Problem 2-53, a relativistic calculation is managable; the result would be

$$E_f - E_i = h\,\nu\left(1 + \frac{1}{2}\left(1 + \frac{Mc^2}{h\nu}\right)^{-1}\right),$$

a form not often useful; see part (b).

(b) As indicated above and in the problem statement, a nonrelativistic calculation is sufficient. As in part (a),

$$\mathrm{KE} = \frac{p^2}{2\,M} = \frac{(\Delta E/c)^2}{2\,M}, \qquad \text{and}$$

$$\frac{\mathrm{KE}}{\Delta E} = \frac{\Delta E}{2\,Mc^2} = \frac{1.9\ \mathrm{eV}}{2\ (939 \times 10^6\ \mathrm{eV})} = 1.01 \times 10^{-9},$$

or 1.0×10^{-9} to two significant figures. In the above, the rest energy of the hydrogen atom is from the front endpapers.

4-29: There are many equivalent algebraic methods that may be used to derive Equation (4.19), and that result will be cited here;

$$f_n = -\frac{2\,E_1}{h}\,\frac{1}{n^3}.$$

The frequency ν of the photon emitted in going from the level $n+1$ to the level n is obtained from Equation (4.17) with $n_i = n+1$ and $n_f = n$;

$$\nu = \frac{\Delta E}{h}\left[\frac{1}{(n+1)^2} - \frac{1}{n^2}\right] = -\frac{2\,E_1}{h}\left[\frac{n + \frac{1}{2}}{n^2\,(n+1)^2}\right].$$

This can be seen to be equivalent to the expression for ν in terms of n and p that was found in the derivation of Equation (4.20), but with n replaced by $n+1$ and $p = 1$. Note that in this form, ν is positive because E_1 is negative.

From this expression,

$$\nu = -\frac{2\,E_1}{h\,n^3}\left[\frac{n^2+\frac{1}{2}\,n}{n^2+2\,n+1}\right] = f_n\left[\frac{n^2+\frac{1}{2}\,n}{n^2+2\,n+1}\right] < f_n,$$

as the term in brackets is less than 1. Similarly,

$$\nu = -\frac{2\,E_1}{h\,(n+1)^3}\left[\frac{\left(n+\frac{1}{2}\right)\,(n+1)}{n^2}\right] = f_{n+1}\left[\frac{\left(n+\frac{1}{2}\right)\,(n+1)}{n^2}\right] > f_{n+1},$$

as the term in brackets is greater than 1.

4-31: For a muonic atom, the Rydberg constant is multiplied by the ratio of the reduced masses of the muoninc atom and the hydrogen atom, $R' = R\,(m'/m_e) = 186R$, as in Example 4.7; from Equation (4.7),

$$\lambda = \frac{4/3}{R'} = \frac{4/3}{186\,(1.097\times 10^7\ \mathrm{m}^{-1})} = 6.53\times 10^{-10}\ \mathrm{m} = 0.653\ \mathrm{nm},$$

in the x-ray range.

4-33: The H_α lines, corresponding to $n = 3$ in Equation (4.6), have wavelengths of $\lambda = (36/5)\,(1/R)$. For a tritium atom, the wavelength would be $\lambda_T = (36/5)\,(1/R_T)$, where R_T is the Rydberg constant evaluated with the reduced mass of the tritium atom replacing the reduced mass of the hydrogen atom. The difference between the wavelengths would then be

$$\Delta\lambda = \lambda - \lambda_T = \lambda\left[1-\frac{\lambda_T}{\lambda}\right] = \lambda\left[1-\frac{R}{R_T}\right].$$

The values of R and R_T are proportional to the respective reduced masses, and their ratio is

$$\frac{R}{R_T} = \frac{m_e\,m_H/\,(m_e+m_H)}{m_e\,m_T/\,(m_e+m_T)} = \frac{m_H\,(m_e+m_T)}{m_T\,(m_e+m_H)}.$$

Using this in the above expression for $\Delta\lambda$,

$$\Delta\lambda = \lambda\left[\frac{m_e\,(m_T-m_H)}{m_e\,(m_e+m_H)}\right] \approx \lambda\,\frac{2\,m_e}{3\,m_H},$$

where the approximations $m_e + m_H \approx m_H$ and $m_T \approx 3\,m_H$ have been used. Inserting numerical values,

$$\Delta\lambda = \frac{(36/5)}{(1.097\times 10^7\ \mathrm{m}^{-1})}\,\frac{2\,\left(9.1095\times 10^{-31}\ \mathrm{kg}\right)}{3\,\left(1.6736\times 10^{-27}\ \mathrm{kg}\right)} = 2.38\times 10^{-10}\ \mathrm{m} = 0.238\ \mathrm{nm}.$$

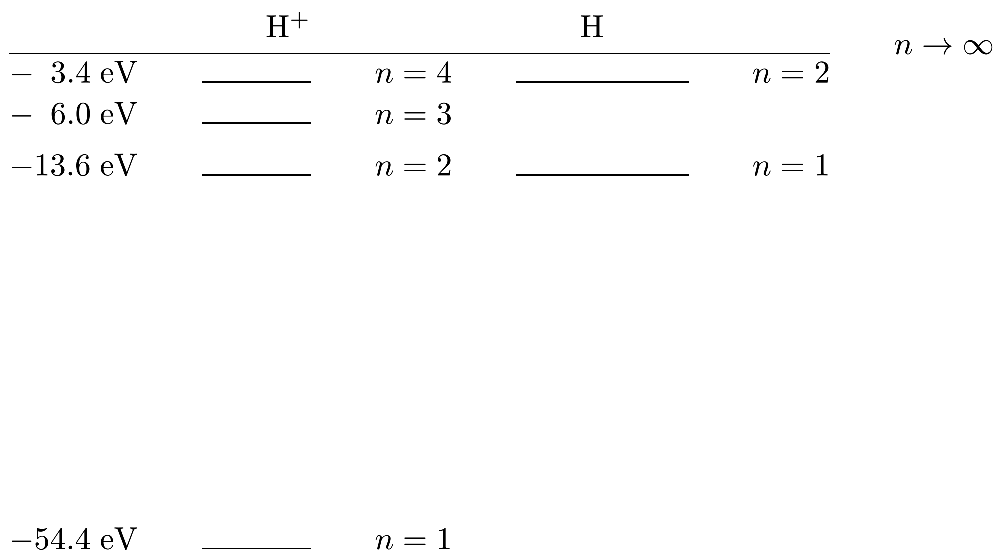
4-35: (a) The steps leading to Equation (4.15) are repeated, with Ze^2 instead of e^2 and $Z^2 e^4$ instead of e^4, giving

$$E_n = -\frac{m' \, Z^2 e^4}{8\pi \, \epsilon_0^2 \, h^2} \frac{1}{n^2},$$

where the reduced mass m' will depend on the mass of the nucleus.

(b) A plot of the energy levels is given below. The scale is close, but not exact, and of course there are many more levels corresponding to higher n. In the approximation that the reduced masses are the same, for He^+, with $Z = 2$, the $n = 2$ level is the same as the $n = 1$ level for Hydrogen, and the $n = 4$ level is the same as the $n = 2$ level for hydrogen.

The energy levels for H and He^+:



(c) When the electron joins the Helium nucleus, the electron-nucleus system loses energy; the emitted photon will have lost energy $\Delta E = 4\,(-13.6 \text{ eV}) = -54.4$ eV, where the result of part (a) has been used. The emitted photon's wavelength is

$$\lambda = \frac{hc}{-\Delta E} = \frac{1.240 \times 10^{-6} \text{ eV}\cdot\text{m}}{54.4 \text{ eV}} = 2.28 \times 10^{-8} \text{ m} = 22.8 \text{ nm}.$$

4-37: The minimum number of Cr^{3+} ions will be the minimum number of photons, which is the total energy of the pulse divided by the energy of each photon,

$$\frac{E}{hc/\lambda} = \frac{E\,\lambda}{hc} = \frac{(1.00 \text{ J})\,\left(694 \times 10^{-9} \text{ m}\right)}{(6.626 \times 10^{-34} \text{ J}\cdot\text{s})\,(2.998 \times 10^{8} \text{ m/s})} = 3.49 \times 10^{18} \text{ ions}.$$

4-39: Small angles correspond to particles that are not scattered much at all, and the structure of the atom does not affect these particles. To these nonpenetrating particles, the nucleus is either partially or completely screened by the atom's electron cloud, and the scattering analysis, based on a pointlike positively charged nucleus, is not applicable.

4-41: From Equation (4.29), using the value for $\dfrac{1}{4\pi\,\epsilon_0}$ given in the front endpapers,

$$\cot\frac{\theta}{2} = \frac{(5.0\ \text{eV})\left(1.602\times10^{-13}\ \text{J/MeV}\right)}{(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2)\ (79)\ (1.602\times10^{-19}\ \text{C})^2}\left(2.6\times10^{-13}\ \text{m}\right) = 11.43,$$

keeping extra significant figures. The scattering angle is then

$$\theta = 2\ \text{arccot}(11.43) = 2\arctan\left(\frac{1}{11.43}\right) = 10^\circ.$$

4-43: The fraction scattered by less than 1° is $1 - f$, with f given in Equation (4.31);

$$f = \pi\, n\, t\left(\frac{Ze^2}{4\pi\,\epsilon_0\,\text{KE}}\right)^2\cot^2\frac{\theta}{2} = \pi\, n\, t\left(\frac{1}{4\pi\,\epsilon_0}\right)^2\left(\frac{Ze^2}{\text{KE}}\right)^2\cot^2\frac{\theta}{2}$$

$$= \pi\ \left(5.90\times10^{28}\ \text{m}^{-3}\right)\left(3.0\times10^{-7}\ \text{m}\right)\left(8.988\times10^{9}\ \text{N}\cdot\text{m}^2/\text{C}^2\right)^2$$
$$\times\left(\frac{(79)\ \left(1.602\times10^{-19}\ \text{C}\right)^2}{(7.7\ \text{MeV})\ (1.602\times10^{-13}\ \text{J/MeV})}\right)^2\cot^2(0.5^\circ) = 0.16,$$

where n, the number of gold atoms per unit volume, is from Example 4.8. The fraction scattered by less than 1° is $1 - f = 0.84$.

4-45: Regarding f as a function of θ in Equation (4.31), the number of particles scattered between 60° and 90° is $f\,(60^\circ) - f\,(90^\circ)$, and the number scattered through angles greater than 90° is just $f\,(90^\circ)$, and

$$\frac{f\,(60^\circ) - f\,(90^\circ)}{f\,(90^\circ)} = \frac{\cot^2(30^\circ) - \cot^2(45^\circ)}{\cot^2(45^\circ)} = \frac{3-1}{1} = 2,$$

so twice as many particles are scattered between 60° and 90° than are scattered through angles greater than 90°.

4-47: If gravity acted on photons as if they were massive objects with mass $m = E_\nu/c^2$, the magnitude of the force F in Equation (4.28) would be

$$F = \frac{G\,M_{\text{sun}} m}{r^2};$$

the factors of r^2 would cancel, as they do for the Coulomb force, and the result is

$$2\,mc^2\,b\,\sin\frac{\theta}{2} = 2\,G\,M_{\text{sun}} m\,\cos\frac{\theta}{2} \qquad \text{and} \qquad \cot\frac{\theta}{2} = \frac{c^2\,b}{G\,M_{\text{sun}}},$$

a result that is independent of the photon's energy. Using $b = R_{\text{sun}}$,

$$\begin{aligned}\theta = 2\arctan\left(\frac{G\,M_{\text{sun}}}{c^2\,R_{\text{sun}}}\right) &= 2\arctan\left(\frac{\left(6.67\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2\right)\,\left(2.0\times10^{30}\ \text{kg}\right)}{\left(2.998\times10^{8}\ \text{m/s}\right)^2\,\left(7.0\times10^{8}\ \text{m}\right)}\right)\\ &= 2.43\times10^{-4\,\circ} = 0.87'',\end{aligned}$$

where $1^\circ = 60' = 60$ minutes $= 3600'' = 3600$ seconds.