Chapter 1 - Relativity

Problems that involve relativistic effects at speeds much smaller than the speed of light, or the equivalence of special relativity and Newtonian mechanics at low speeds, often require finding differences such as

$$1-\sqrt{1-\frac{v^2}{c^2}} \qquad \text{or} \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}-1$$

when $v \ll c$.

These are both differences between quantities that are equal to 1 in the limit as $v \to 0$, but as the quantities are not the same for $v \neq 0$, we are interested in how the differences depend on v (more specifically, the ratio v/c) in the limit $v \ll c$.

There are many ways to find the functional form of these differences; four familiar methods are explained here.

I - Binomial Theorem for Non-integral Exponents

This is the method used in Section 1.8.

A familiar form of the binomial theorem is

$$(1+x)^\alpha = 1+\alpha\, x+\frac{\alpha(\alpha-1)}{2}\,x^2+\frac{\alpha(\alpha-1)(\alpha-2)}{2\cdot 3}\,x^3+\cdots.$$

If α is a nonnegative integer, the coefficients of the powers of x are the usual binomial coefficients, and the series truncates. However, if $|x|<1$, the series will converge for other values of α, particularly negative integers or fractions. Specifially, if $\alpha=-\frac{1}{2}$,

$$\begin{aligned}(1+x)^{-1/2} &= 1+\left(-\frac{1}{2}\right)x+\frac{(-1/2)(-3/2)}{2}\,x^2+\frac{(-1/2)(-3/2)(-5/2)}{6}\,x^3+\cdots\\ &= 1-\frac{1}{2}\,x+\frac{3}{8}\,x^2-\frac{5}{16}\,x^3+\cdots.\end{aligned}$$

When $x=-\left(\frac{v^2}{c^2}\right)$, this becomes

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1+\frac{1}{2}\,\frac{v^2}{c^2}+\frac{3}{8}\,\frac{v^4}{c^4}+\frac{5}{16}\,\frac{v^6}{c^6}+\cdots.$$

Similarly, when $\alpha = 1/2$,

$$(1+x)^{1/2} = 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^2 + \frac{1}{16}\,x^3 + \cdots \qquad \text{and}$$

$$\sqrt{1-\frac{v^2}{c^2}} = 1 - \frac{1}{2}\frac{v^2}{c^2} - \frac{1}{8}\frac{v^4}{c^4} - \frac{1}{16}\frac{v^6}{c^6} + \cdots.$$

In the limit $v \ll c$, then,

$$1-\sqrt{1-\frac{v^2}{c^2}} \approx \frac{1}{2}\frac{v^2}{c^2} + \frac{1}{8}\frac{v^4}{c^4}, \qquad \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \approx \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}.$$

Note that the v^6/c^6 and higher-order terms have been neglected; in practice, the v^4/c^4 terms are seldom used.

II - Algebraic

Consider the difference

$$\begin{aligned} 1-\sqrt{1-\frac{v^2}{c^2}} &= \left(1-\sqrt{1-\frac{v^2}{c^2}}\right) \times \frac{1+\sqrt{1-\frac{v^2}{c^2}}}{1+\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1-\left(\sqrt{1-\frac{v^2}{c^2}}\right)^2}{1+\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{\frac{v^2}{c^2}}{1+\sqrt{1-\frac{v^2}{c^2}}}. \end{aligned}$$

The denominator is seen to approach 2 as $v \ll c$, and so

$$1-\sqrt{1-\frac{v^2}{c^2}} \approx \frac{1}{2}\frac{v^2}{c^2}$$

for low speed. Similarly,

$$\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 = \frac{1-\sqrt{1-\frac{v^2}{c^2}}}{\sqrt{1-\frac{v^2}{c^2}}} \approx \frac{1}{2}\frac{v^2}{c^2}$$

for low speed, where the previous result has been used.

Finding higher-order corrections by this method is possible, only slightly tedious, but fairly unenlightening. For example, for the next order, consider

$$\left(1 - \frac{1}{2}\,\frac{v^2}{c^2} - \sqrt{1 - \frac{v^2}{c^2}}\right) \times \frac{1 - \frac{1}{2}\,\frac{v^2}{c^2} + \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{1}{2}\,\frac{v^2}{c^2} + \sqrt{1 - \frac{v^2}{c^2}}}.$$

This algebraic method is equivalent to that used to find a derivative of a square root by taking a limit.

III - Taylor Series

Letting $f(x) = (1+x)^{-1/2}$, $f(0) = 1$ and direct calculations give $f'(0) = -1/2$ and $f''(0) = 3/4$ (a generalization is not hard to do explicitly). Thus,

$$f(x) = \frac{1}{\sqrt{1+x}} \approx 1 + \left(-\frac{1}{2}\right) x + \frac{1}{2}\left(\frac{3}{4}\right) x^2 = 1 - \frac{1}{2}\,x + \frac{3}{8}\,x^2.$$

This is seen to be identical (when higher-order terms are computed) to that found by the binomial theorem, and letting $x = -v^2/c^2$ reproduces the previous result. Similary,

$$\sqrt{1+x} = (1+x)^{1/2} \approx 1 + \frac{1}{2}\,x - \frac{1}{8}\,x^4,$$

as before.

IV - Use the Machine

The mechanics of finding Taylor Series might often be left to mechanical devices. The following Maple commands reproduce the above results easily and almost immediately.

```
>g:=sqrt(1-(v/c)^2);
>series(g,v=0,8);
>series(1/g,v=0,8);
```

In the "`series`" commands above, the last argument is the order to which the series are calculated, and may be changed as desired (default is 6). Since the functions considered are even in v/c, the order is not the same as the number of terms.

1-1: All else being the same, including the rates of the chemical reactions that govern our brains and bodies, relativisitic phenomena would be more conspicuous if the speed of light were smaller. If we could attain the absolute speeds obtainable to us in the universe as it is, but with the speed of light being smaller, we would be able to move at speeds that would correspond to larger fractions of the speed of light, and in such instances relativistic effects would be more conspicuous.

1-3: Even if the judges would allow it, the observers in the moving spaceship would measure a longer time, since they would see the runners being timed by clocks that appear to run slowly compared to the ship's clocks. Actually, when the effects of length contraction are included (discussed in Section 1.4 and Appendix I), the runner's *speed* may be greater than, less than, or the same as that measured by an observer on the ground.

1-5: Note that the nonrelativistic approximation is not valid, as $v/c = 2/3$.

(a) See Example 1.1. In Equation (1.3), with t representing both the time measured by A *and* the time as measured in A's frame for the clock in B's frame to advance by t_0, we need

$$t - t_0 = t\left(1 - \sqrt{1 - \frac{v^2}{c^2}}\right) = t\left(1 - \sqrt{1 - \left(\frac{2}{3}\right)^2}\right) = t\,(0.255) = 1.00\ \text{s},$$

from which $t = 3.93$ s.

(b) A moving clock always seems to run slower. In this problem, the time t is the time that observer A measures as the time that B's clock takes to record a time change of t_0.

1-7: From Equation (1.3), for the time t on the earth to correspond to twice the time t_0 elapsed on the ship's clock,

$$\sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{2}, \qquad \text{so}$$

$$v = \frac{\sqrt{3}}{2}\,c = \frac{\sqrt{3}}{2}\,(2.998 \times 10^8\ \text{m/s}) = 2.60 \times 10^8\ \text{m/s},$$

retaining three significant figures.

1-9: The lifetime of the particle is t_0, and the distance the particle will travel is, from Equation (1.3),

$$v\,t = \frac{v\,t_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{(0.99)(2.998 \times 10^8\ \text{m/s})(1.00 \times 10^{-7}\ \text{s})}{\sqrt{1 - (0.99)^2}} = 210\ \text{m}$$

to two significant figures.

1-11: See Example 1.3; for the intermediate calculations, note that

$$\lambda = \frac{c}{\nu} = \frac{c}{\nu_0}\,\frac{\nu_0}{\nu} = \lambda_0\sqrt{\frac{1 - v/c}{1 + v/c}},$$

where the sign convention for v is that of Equation (1.8), with v positve for an approaching source and v negative for a receding source.

For this problem,

$$\frac{v}{c} = -\frac{(1.50 \times 10^4 \text{ km/s})(10^3 \text{ m/km})}{(2.998 \times 10^8 \text{ m/s})} = -0.0500,$$

so that

$$\lambda = \lambda_0\sqrt{\frac{1 - v/c}{1 + v/c}} = (550 \text{ nm})\sqrt{\frac{1 + 0.0500}{1 - 0.0500}} = 578 \text{ nm}.$$

1-13: This problem may be done in several ways, all of which need to use the fact that when the frequencies due to the classical and relativistic effects are found, those frequencies, while differing by 1 Hz, will both be sufficiently close to $\nu_0 = 10^9$ Hz so that ν_0 could be used for an approximation to either.

In Equation (1.4), we have $v = 0$ and $V = -u$, where u is the speed of the spacecraft, moving away from the earth ($V < 0$). In Equation (1.6), we have $v = u$ (or $v = -u$ in Equation (1.8)). The classical and relativistic frequencies, ν_c and ν_r respectively, are

$$\nu_\mathrm{c} = \frac{\nu_0}{1 + (u/c)}, \qquad \nu_\mathrm{r} = \nu_0\sqrt{\frac{1 - (u/c)}{1 + (u/c)}} = \nu_0\,\frac{\sqrt{1 - (u^2/c^2)}}{1 + (u/c)}.$$

The last expression for ν_r is motivated by the derivation of Equation (1.6), which essentially incorporates the classical result (counting the number of ticks), and allows expression of the ratio

$$\frac{\nu_\mathrm{c}}{\nu_\mathrm{r}} = \frac{1}{\sqrt{1 - (u^2/c^2)}}.$$

Use of the above forms for the frequencies allows the calculation of the ratio

$$\frac{\Delta\nu}{\nu_0} = \frac{\nu_\mathrm{c} - \nu_\mathrm{r}}{\nu_0} = \frac{1 - \sqrt{1 - (u^2/c^2)}}{1 + (u/c)} = \frac{1 \text{ Hz}}{10^9 \text{ Hz}} = 10^{-9}.$$

Attempts to solve this equation exactly are not likely to be met with success, and even numerical solutions would require a higher precision than is commonly available. However, recognizing that the numerator $1 - \sqrt{1-(u^2/c^2)}$ is of the form that can be approximated using the methods outlined at the beginning of this chapter, we can use $1 - \sqrt{1-(u^2/c^2)} \approx (1/2)(u^2/c^2)$. The denominator will be indistinguishable from 1 at low speed, with the result

$$\frac{1}{2}\frac{u^2}{c^2} = 10^{-9},$$

which is solved for

$$u = \sqrt{2 \times 10^{-9}}\, c = 1.34 \times 10^4 \text{ m/s} = 13.4 \text{ km/s}.$$

Similar to what was done at the beginning of this chapter, the Taylor series for the desired function of u can be found by a computer. The Maple commands would be

```
>f:=(1-sqrt(1-u^2))/(1+u);
>series(f,u=0);
```

(note that for these commands, "`u`" represents the ratio of the recessional speed to the speed of light).

Mention had been made above of the limited possibility of a numerical solution. Depending on which release of Maple is used, a numerical solution is indeed possible. Maple 7 will solve the given equation with the command

```
>solve(f=1E-9);
```

with the results .00004472235955, −.00004472035955 for u/c (Maple will give both positive and negative roots, and we need to recognize which we want, as well as the limitation on precision).

1-15: The transverse Doppler effect corresponds to a direction of motion of the light source that is perpendicular to the direction from it to the observer; the angle $\theta = \pm\frac{\pi}{2}$ (or $\pm 90°$), so $\cos\theta = 0$, and $\nu = \nu_0\sqrt{1 - v^2/c^2}$, which is Equation (1.5).

For a receding source, $\theta = \pi$ (or $180°$), and $\cos\theta = -1$. The given expression becomes

$$\nu = \nu_0 \frac{\sqrt{1 - \dfrac{v^2}{c^2}}}{1 + \dfrac{v}{c}} = \nu_0 \frac{\sqrt{1 - \dfrac{v}{c}}}{\sqrt{1 + \dfrac{v}{c}}} \frac{\sqrt{1 + \dfrac{v}{c}}}{\sqrt{1 + \dfrac{v}{c}}} = \nu_0 \sqrt{\frac{1 - v/c}{1 + v/c}},$$

which is Equation (1.6).

For an approaching source, $\theta = 0$, $\cos\theta = 1$, and the given expression becomes

$$\nu = \nu_0 \frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{c}} = \nu_0 \frac{\sqrt{1 + \frac{v}{c}}}{\sqrt{1 - \frac{v}{c}}} \frac{\sqrt{1 - \frac{v}{c}}}{\sqrt{1 - \frac{v}{c}}} = \nu_0 \sqrt{\frac{1 + v/c}{1 - v/c}},$$

which is Equation (1.7).

1-17: The astronaut's proper length (height) is 6 ft, and this is what any observer in the spacecraft will measure. From Equation (1.9), an observer on the earth would measure

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = (6\ \text{ft}) \sqrt{1 - (0.90)^2} = 2.6\ \text{ft}.$$

1-19: The time will be the length as measured by the observer divided by the speed, or

$$t = \frac{L}{v} = \frac{L_0 \sqrt{1 - \frac{v^2}{c^2}}}{v} = \frac{(1.00\ \text{m}) \sqrt{1 - (0.100)^2}}{(0.100)(2.998 \times 10^8\ \text{m/s})} = 3.32 \times 10^{-8}\ \text{s}.$$

1-21: If the antenna has a length L' as measured by an observer on the spacecraft (L' is *not* either L or L_0 in Equation (1.9)), the projection of the antenna onto the spacecraft will have a length $L' \cos(10^\circ)$, and the projection onto an axis perpendicular to the spacecraft's axis will have a length $L' \sin(10^\circ)$. To an observer on the earth, the length in the direction of the spacecraft's axis will be contracted as described by Equation (1.9), while the length perpendicular to the spacecraft's motion will appear unchanged. The angle as seen from the earth will then be

$$\arctan\left[\frac{L' \sin(10^\circ)}{L' \cos(10^\circ) \sqrt{1 - \frac{v^2}{c^2}}}\right] = \arctan\left[\frac{\tan(10^\circ)}{\sqrt{1 - (0.70)^2}}\right] = 14^\circ.$$

The generalization of the above is that if the angle is θ_0 as measured by an observer on the spacecraft, an observer on the earth would measure an angle θ given by

$$\tan\theta = \frac{\tan\theta_0}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

1-23: The age difference will be the difference in the times that each measures the round trip to take, or

$$\Delta t = 2\,\frac{L_0}{v}\left(1-\sqrt{1-\frac{v^2}{c^2}}\right) = 2\,\frac{4\text{ yr}}{0.9}\left(1-\sqrt{1-(0.9)^2}\right) = 5\text{ yr}.$$

1-25: It is convenient to maintain the relationship from Newtonian mechanics, in that a force on an object changes the object's momentum; symbolically, $\mathbf{F} = \frac{d\mathbf{p}}{dt}$ should still be valid. In the absence of forces, momentum should be conserved in any inertial frame, and the conserved quantity is $\mathbf{p} = \gamma m\mathbf{v}$, not $m\mathbf{v}$.

1-27: For a given mass M, the ratio of the mass liberated to the mass energy is

$$\frac{M\times(5.4\times10^6\text{ J/kg})}{M\times(2.998\times10^8\text{ m/s})^2} = 6.0\times10^{-11}.$$

1-29: If the kinetic energy KE $= E_0 = mc^2$, then $E = 2\,mc^2$ and Equation (23) reduces to

$$\frac{1}{\sqrt{1-\dfrac{v^2}{c^2}}} = 2$$

($\gamma = 2$ in the notation of Section 1.7). Solving for v,

$$v = \frac{\sqrt{3}}{2}\,c = 2.60\times10^8\text{ m/s}.$$

1-31: Classically,

$$v = \sqrt{\frac{2\,\text{KE}}{m_e}} = \sqrt{\frac{2(0.100\text{ MeV})(1.602\times10^{-19}\text{ J/eV})}{(9.1095\times10^{-31}\text{ kg})}} = 1.88\times10^8\text{ m/s}.$$

Relativistically, solving Equation (1.23) for v as a function of KE,

$$v = c\sqrt{1-\left(\frac{m_e\,c^2}{E}\right)^2} = c\sqrt{1-\left(\frac{m_e\,c^2}{m_e\,c^2+\text{KE}}\right)^2}$$
$$= c\sqrt{1-\left(\frac{1}{1+(\text{KE}/(m_e\,c^2))}\right)^2}.$$

With $\text{KE}/(m_e\,c^2) = (0.100\text{ MeV})/(0.511\text{ MeV}) = 0.100/0.511$,

$$v = (2.998\times10^8\text{ m/s})\sqrt{1-\left(\frac{1}{1+(0.100/0.511)}\right)^2} = 1.64\times10^8\text{ m/s}.$$

The two speeds are comparable, but not the same; for larger values of the ratio of the kinetic and rest energies, larger discrepancies would be found.

1-33: Using Equation (1.22) in Equation (1.23) and solving for $\frac{v}{c}$,

$$\frac{v}{c} = \sqrt{1 - \left(\frac{E_0}{E}\right)^2} = \left[1 - \left(\frac{E_0}{E}\right)^2\right]^{1/2}.$$

With $E = 21\,E_0$, that is, $E = E_0 + (20)\,E_0$,

$$v = c\sqrt{1 - \left(\frac{1}{21}\right)^2} = 0.9989\,c.$$

(This is consistent with the expression derived in Problem 1-32.)

1-35: The difference in energies will be, from Equation (1.23),

$$m_e\,c^2\left[\frac{1}{\sqrt{1-(v_2/c)^2}} - \frac{1}{\sqrt{1-(v_1/c)^2}}\right]$$
$$= (0.511\text{ MeV})\left[\frac{1}{\sqrt{1-(2.4/3.0)^2}} - \frac{1}{\sqrt{1-(1.2/3.0)^2}}\right] = 0.294\text{ MeV},$$

to three significant figures.

1-37: Using the expression in Equation (1).20 for the kinetic energy, the ratio of the two quantities is

$$\frac{\frac{1}{2}\gamma\,m\,v^2}{\text{KE}} = \frac{1}{2}\frac{v^2}{c^2}\left(\frac{\gamma}{\gamma-1}\right) = \frac{1}{2}\frac{v^2}{c^2}\left[\frac{1}{1-\sqrt{1-\frac{v^2}{c^2}}}\right].$$

Algebraically, this quantity is not equal to 1 except at $v = 0$. For low speeds, $v \ll c$, the quantity in square brackets is approximately $\frac{1}{2}\frac{v^2}{c^2}$ (see the text at the end of Section 1.8 or the beginning of this chapter), reflecting the fact that the classical and relativistic kinetic energies have the same form in the nonrelativistic limit. However, as $v \to c$ (or $\gamma \to \infty$), the expressions are not the same, even though both $\frac{1}{2}\gamma\,m\,v^2$ and $\text{KE} = (\gamma-1)\,m\,c^2$ become infinitely large. To see this explicitly, note that the ratio $\left(\frac{\gamma}{\gamma-1}\right) \to 1$ as $\gamma \to \infty$, so that the expression approaches $\frac{1}{2}$ as $v \to c$. This is consistent with setting $v = c$ in the last expression on the right above.

1-39: Measured from the original center of the box, so that the original position of the center of mass is 0, the final position of the center of mass is

$$\left(\frac{M}{2} - m\right)\left(\frac{L}{2} + S\right) - \left(\frac{M}{2} + m\right)\left(\frac{L}{2} - S\right) = 0.$$

Expanding the products and cancelling similar terms ($\frac{M}{2}\frac{L}{2}$, mS), the result $MS = mL$ is obtained. The distance S is the product $v\,t$, where, as shown in the problem statement, $v \approx E/Mc$ (approximate in the nonrelativistic limit $M \gg E/c^2$) and $t \approx L/c$. Then,

$$m = \frac{MS}{L} = \frac{M}{L}\frac{E}{Mc}\frac{L}{c} = \frac{E}{c^2}.$$

1-41: To cross the galaxy in a matter of minutes, the proton must be highly relativistic, with $v \approx c$ (but $v < c$, of course). The energy of the proton will be $E = E_0\,\gamma$, where E_0 is the proton's rest energy and $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$. However, γ, from Equation (1.9), is the same as the ratio L_0/L, where L is the diameter of the galaxy in the proton's frame of reference, and for the highly-relativistic proton $L \approx c\,t$, where t is the time in the proton's frame that it takes to cross the galaxy. Combining,

$$E = E_0\,\gamma = E_0\,\frac{L_0}{L} \approx E_0\,\frac{L_0}{c\,t} \approx (10^9\ \text{eV})\frac{(10^5\ \text{ly})}{c\,(300\ \text{s})} \times (3 \times 10^7\ \text{s/yr}) = 10^{19}\ \text{eV}.$$

1-43: Taking magnitudes in Equation (1.16),

$$p = \frac{m_e\,v}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{(0.511\ \text{MeV}/c^2)(0.600\,c)}{\sqrt{1 - (0.600)^2}} = 0.383\ \text{MeV}/c.$$

1-45: When the kinetic energy of an electron is equal to its rest energy, the total energy is twice the rest energy, and Equation (1.24) becomes

$$4\,m_e^2 c^4 = m_e^2 c^4 + p^2 c^2, \quad \text{or} \quad p = \sqrt{3}\left(m_e\,c^2\right)/c = \sqrt{3}\,(511\ \text{keV}/c) = 885\ \text{keV}/c.$$

The result of Problem 1-29 could be used directly; $\gamma = 2$, $v = (\sqrt{3}/2)c$, and Equation (1.17) gives $p = \sqrt{3}\,m_e\,c$, as above.

1-47: Solving Equation (1.23) for the speed v in terms of the rest energy E_0 and the total energy E,

$$v = c\sqrt{1 - \left(\frac{E_0}{E}\right)} = c\sqrt{1 - \left(\frac{0.938}{3.500}\right)^2} = 0.963\,c,$$

numerically 2.888×10^8 m/s. (The result of Problem 1-32 does *not* give an answer accurate to three significant figures.) The value of the speed may be substituted into Equation (1.16) (or the result of Problem 1-46), or Equation (1.24) may be solved for the magnitude of the momentum,

$$p = \sqrt{(E/c)^2 - (E_0/c)^2} = \sqrt{(3.500\ \text{GeV}/c)^2 - (0.93828\ \text{GeV}/c)^2} = 3.372\ \text{GeV/c}.$$

(Although the final result is not affected, a more precise value for the proton rest mass, taken from the front endpapers, was used in the last calculation.)

1-49: From $E = m\,c^2 + \text{KE}$ and Equation (1.24),

$$\left(m\,c^2 + \text{KE}\right)^2 = m^2\,c^4 + p^2\,c^2.$$

Expanding the binomial, cancelling the $m^2\,c^4$ term, and solving for m,

$$m = \frac{(p\,c)^2 - \text{KE}^2}{2\,c^2\,\text{KE}} = \frac{(335\ \text{MeV})^2 - (62\ \text{MeV})^2}{2\,c^2\,(62\ \text{MeV})} = 874\ \text{MeV}/c^2.$$

The particle's speed may be found any number of ways; a very convenient result is that of Problem 1-46, giving

$$v = c^2\,\frac{p}{E} = c\,\frac{p\,c}{m\,c^2 + \text{KE}} = c\,\frac{335\ \text{MeV}}{874\ \text{MeV} + 62\ \text{MeV}} = 0.36\,c.$$

There's a neat algebraic "trick" that may be used in this and many similar problems. (In what follows, factors of c will not be included.) Essentially, the problem reduces mathematically to solving the two equations

$$E = m + \text{KE}, \qquad E^2 = m^2 + p^2$$

for E and m, given known values for p and KE. Rewrite the two equations as

$$E - m = \text{KE}, \qquad E^2 - m^2 = (E - m)(E + m) = p^2$$

and substitute the first into the second to obtain $E + m = \dfrac{p^2}{\text{KE}}$ (the KE $= 0$ case is trivial). Adding this to $E - m = \text{KE}$, and then subtracting the same relation gives

$$E = \frac{p^2 + \text{KE}^2}{2\,\text{KE}}, \qquad m = \frac{p^2 - \text{KE}^2}{2\,\text{KE}},$$

as obtained above.

1-51: The given observation that the two explosions occur at the same place to the second observer means that $x' = 0$ in Equation (1.41), and so the second observer is moving at a speed

$$v = \frac{x}{t} = \frac{1.00 \times 10^5 \text{ m}}{2.00 \times 10^{-3} \text{ s}} = 5.00 \times 10^7 \text{ m/s}$$

with respect to the first observer. Inserting this into Equation (1.44),

$$\begin{aligned} t' &= \frac{t - \dfrac{x^2}{t\,c^2}}{\sqrt{1 - (x/c\,t)^2}} = t\,\frac{1 - \dfrac{x^2}{c^2\,t^2}}{\sqrt{1 - \dfrac{x^2}{c^2\,t^2}}} = t\sqrt{1 - \frac{(x/t)^2}{c^2}} \\ &= (2.00 \text{ ms})\sqrt{1 - \frac{(5.00 \times 10^7 \text{ m/s})^2}{(2.998 \times 10^8 \text{ m/s})^2}} = 1.97 \text{ ms}. \end{aligned}$$

(For this calculation, the approximation $\sqrt{1 - (x/c\,t)^2} \approx 1 - (x^2/2\,c^2\,t^2)$ is valid to three significant figures.)

An equally valid method, and a good check, is to note that when the relative speed of the observers (5.00×10^7 m/s) has been determined, the time interval that the second observer measures should be that given by Equation (1.3) (but be careful of which time it t, which is t_0). Algebraically and numerically, the different methods give the same result.

1-53: (a) A convenient choice for the origins of both the unprimed and primed coordinate systems is the point, in both space and time, where the ship receives the signal. Then, in the unprimed frame (given here as the frame of the fixed stars, one of which may be the source), the signal was sent at a time $t = -r/c$, where r is the distance from the source to the place where the ship receives the signal, and the minus sign merely indicates that the signal was sent before it was received.

Take the direction of the ship's motion (assumed parallel to its axis) to be the positive x-direction, so that in the frame of the fixed stars (the unprimed frame), the signal arrives at an angle θ with respect to the positive x-direction. In the unprimed frame, $x = r\cos\theta$ and $y = r\sin\theta$. From Equation (1.41),

$$x' = \frac{x - v\,t}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{r\cos\theta - (-r/c)}{\sqrt{1 - \dfrac{v^2}{c^2}}} = r\,\frac{\cos\theta + (v/c)}{\sqrt{1 - \dfrac{v^2}{c^2}}},$$

and $y' = y = r\sin\theta$. Then,

$$\tan\theta' = \frac{y'}{x'} = \frac{\sin\theta}{(\cos\theta + (v/c))\bigg/\sqrt{1-\dfrac{v^2}{c^2}}}, \qquad \text{and}$$

$$\theta' = \arctan\left[\frac{\sin\theta\sqrt{1-\dfrac{v^2}{c^2}}}{\cos\theta + (v/c)}\right].$$

(b) From the form of the result of part (a), it can be seen that the numerator of the term in square brackets is less than $\sin\theta$, and the denominator is greater than $\cos\theta$, and so $\tan\theta' < \tan\theta$ and $\theta' < \theta$ when $v \neq 0$. Looking out of a porthole, the sources, including the stars, will appear to be in directions closer to the direction of the ship's motion than they would for a ship with $v = 0$. As $v \to c$, $\theta' \to 0$, and all stars appear to be almost on the ship's axis (farther forward in the field of view).

1-55: (a) If the man on the moon sees A approaching with speed $v = 0.800\,c$, then the observer on A will see the man in the moon approaching with speed $v = 0.800\,c$. The relative velocities will have opposite directions, but the relative speeds will be the same. The speed with which B is seen to approach A, to an observer in A, is then

$$\frac{0.800 + 0.900}{1 + (0.800)(0.900)}\,c = 0.988\,c.$$

(b) Similarly, the observer on B will see the man on the moon approaching with speed $0.900\,c$, and the apparent speed of A, to an observer on B, will be

$$\frac{0.900 + 0.800}{1 + (0.900)(0.800)}\,c = 0.988\,c.$$

(Note that Equation (1.49) is unchanged if V'_x and v are interchanged.)