# Nonsinusoidal Circuits

# 25

## Objectives

- *Become familiar with the components of the Fourier series expansion for any sinusoidal or nonsinusoidal function.*
- *Understand how the appearance and time axis plot of a waveform can identify which terms of a Fourier series will be present.*
- *Be able to determine the response of a network to any input defined by a Fourier series expansion.*
- *Learn how to add two or more waveforms defined by Fourier series expansions.*

## 25.1 INTRODUCTION

Any waveform that differs from the basic description of the sinusoidal waveform is referred to as **nonsinusoidal.** The most obvious and familiar are the dc, square-wave, triangular, sawtooth, and rectified waveforms in Fig. 25.1.

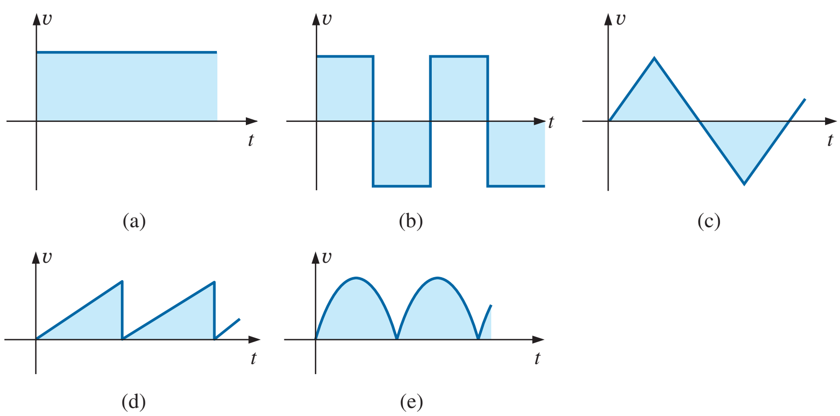


**FIG. 25.1**

*Common nonsinusoidal waveforms: (a) dc; (b) square-wave; (c) triangular; (d) sawtooth; (e) rectified.*

The output of many electrical and electronic devices are nonsinusoidal, even though the applied signal may be purely sinusoidal. For example, the network in Fig. 25.2 uses a diode to clip off the negative portion of the applied signal in a process called *half-wave rectification,* which is used in the development of dc levels from a sinusoidal input. You will find in your electronics courses that the diode is similar to a mechanical switch, but it is different because it can conduct current in only one direction. The output waveform is definitely nonsinusoidal, but note that it has the same period as the applied signal and matches the input for half the period.

This chapter demonstrates how a nonsinusoidal waveform like the output in Fig. 25.2 can be represented by a series of terms. It also explains how to determine the response of a network to such an input.

NON

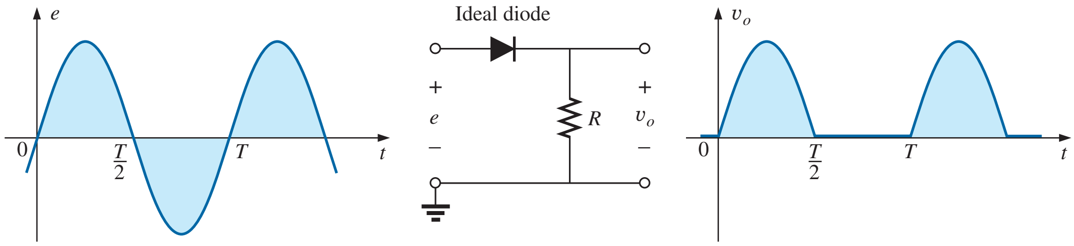


**FIG. 25.2**
*Half-wave rectifier producing a nonsinusoidal waveform.*

## 25.2 FOURIER SERIES

**Fourier series** refers to a series of terms, developed in 1822 by Baron Jean Fourier (Fig. 25.3), that can be used to represent a nonsinusoidal periodic waveform. In the analysis of these waveforms, we solve for each term in the Fourier series:

$$f(t) = \underbrace{A_0}_{\substack{\text{dc or}\\ \text{average value}}} + \underbrace{A_1 \sin \omega t + A_2 \sin 2\omega t + A_3 \sin 3\omega t + \cdots + A_n \sin n\omega t}_{\text{sine terms}} + \underbrace{B_1 \cos \omega t + B_2 \cos 2\omega t + B_3 \cos 3\omega t + \cdots + B_n \cos n\omega t}_{\text{cosine terms}} \tag{25.1}$$

Depending on the waveform, a large number of these terms may be required to approximate the waveform closely for the purpose of circuit analysis.

As shown in Eq. (25.1), the Fourier series has three basic parts. The first is the dc term $A_0$, which is the average value of the waveform over one full cycle. The second is a series of sine terms. There are no restrictions on the values or relative values of the amplitudes of these sine terms, but each will have a frequency that is an integer multiple of the frequency of the first sine term of the series. The third part is a series of cosine terms. There are again *no* restrictions on the values or relative values of the amplitudes of these cosine terms, but each will have a frequency that is an integer multiple of the frequency of the first cosine term of the series. For a particular waveform, it is quite possible that all of the sine *or* cosine terms are zero. Characteristics of this type can be determined by simply examining the nonsinusoidal waveform and its position on the horizontal axis.

The first term of the sine and cosine series is called the **fundamental component.** It represents the minimum frequency term required to represent a particular waveform, and it also has the same frequency as the waveform being represented. A fundamental term, therefore, must be present in any Fourier series representation. The other terms with higher-order frequencies (integer multiples of the fundamental) are called the **harmonic terms.** A term that has a frequency equal to twice the fundamental is the second harmonic; three times, the third harmonic; and so on.

### Average Value: $A_0$

The dc term of the Fourier series is the average value of the waveform over one full cycle. If the net area above the horizontal axis equals that

**FIG. 25.3**
*Baron Jean Fourier.*
Courtesy of the Smithsonian Institution
Photo No. 56,822

**French** (Auxerre, Grenoble, Paris)
**(1768–1830)**
**Mathematician, Egyptologist,** and **Administrator**
**Professor of Mathematics,** École Polytechnique

Best known for an infinite mathematical series of sine and cosine terms called the *Fourier series* which he used to show how the conduction of heat in solids can be analyzed and defined. Although he was primarily a mathematician, a great deal of Fourier's work revolved around real-world physical occurrences such as heat transfer, sunspots, and the weather. He joined the École Polytechnique in Paris as a faculty member when the institute first opened. Napoleon requested his aid in the research of Egyptian antiquities, resulting in a three-year stay in Egypt as Secretary of the Institut d'Égypte. Napoleon made him a baron in 1809, and he was elected to the Académie des Sciences in 1817.

below in one full period, $A_0 = 0$, and the dc term does not appear in the expansion. If the area above the axis is greater than that below over one full cycle, $A_0$ is positive and will appear in the Fourier series representation. If the area below the axis is greater, $A_0$ is negative and will appear with the negative sign in the expansion.

## Odd Function (Point Symmetry)

***If a waveform is such that its value for $+t$ is the negative of that for $-t$, it is called an odd function or is said to have point symmetry.***

Fig. 25.4(a) is an example of a waveform with point symmetry. Note that the waveform has a peak value at $t_1$ that matches the magnitude (with the opposite sign) of the peak value at $-t_1$. For waveforms of this type, all the parameters $B_{1\to\infty}$ of Eq. (25.1) will be zero. In fact,

***waveforms with point symmetry can be fully described by just the dc and sine terms of the Fourier series.***

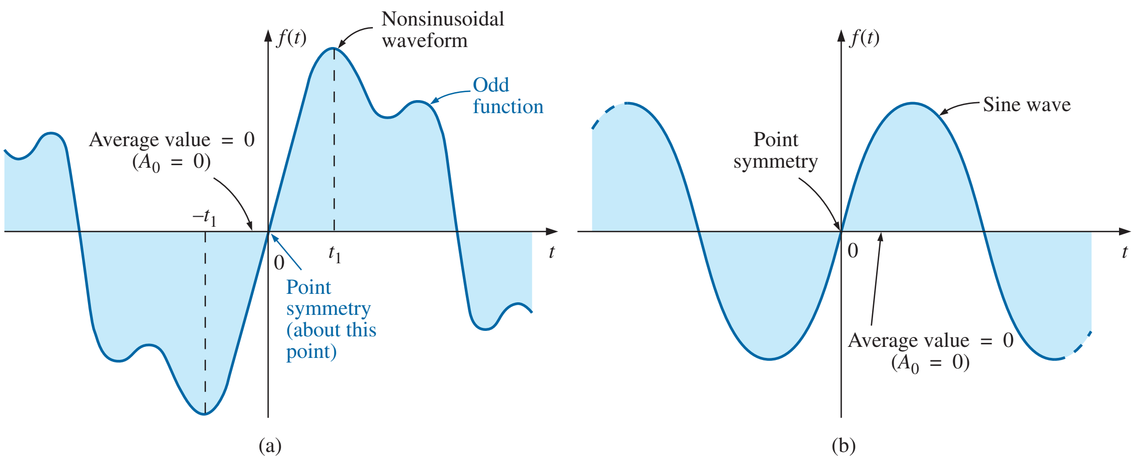


**FIG. 25.4**
*Point symmetry.*

Note in Fig. 25.4(b) that a sine wave is an odd function with point symmetry.

For both waveforms in Fig. 25.4, the following mathematical relationship is true:

$$f(t) = -f(-t) \quad \text{(odd function)} \tag{25.2}$$

In words, it states that the magnitude of the function at $+t$ is equal to the negative of the magnitude at $-t$ [$t_1$ in Fig. 25.4(a)].

## Even Function (Axis Symmetry)

***If a waveform is symmetric about the vertical axis, it is called an even function or is said to have axis symmetry.***

Fig. 25.5(a) is an example of such a waveform. Note that the value of the function at $t_1$ is equal to the value at $-t_1$. For waveforms of this type, all the parameters $A_{1\to\infty}$ will be zero. In fact,

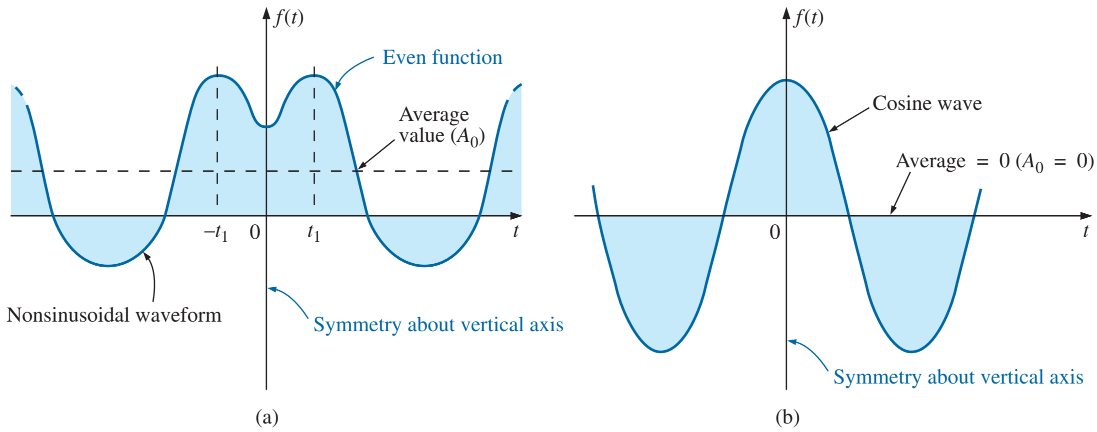


**FIG. 25.5**
*Axis symmetry.*

***waveforms with axis symmetry can be fully described by just the dc and cosine terms of the Fourier series.***

Note in Fig. 25.5(b) that a cosine wave is an even function with axis symmetry.

For both waveforms in Fig. 25.5, the following mathematical relationship is true:

$$f(t) = f(-t) \quad \text{(even function)} \tag{25.3}$$

In words, it states that the magnitude of the function is the same at $+t_1$ as at $-t$ [$t_1$ in Fig. 25.5(a)].

## Mirror or Half-Wave Symmetry

***If a waveform has half-wave or mirror symmetry as demonstrated by the waveform of Fig. 25.6, the even harmonics of the series of sine and cosine terms will be zero.***

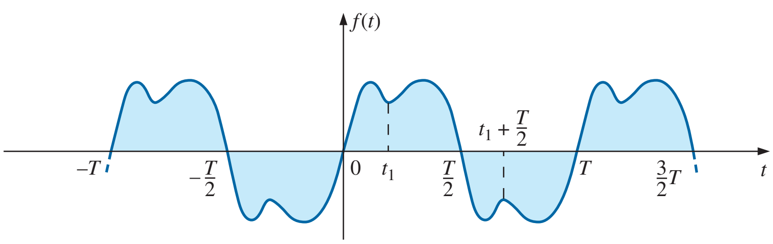


**FIG. 25.6**
*Mirror symmetry.*

In functional form, the waveform must satisfy the following relationship:

$$f(t) = -f\left(t + \frac{T}{2}\right) \tag{25.4}$$

Eq. (25.4) states that the waveform encompassed in one time interval $T/2$ will repeat itself in the next $T/2$ time interval, but in the negative sense ($t_1$ in Fig. 25.6). For example, the waveform in Fig. 25.6 from zero to $T/2$ will repeat itself in the time interval $T/2$ to $T$, but below the horizontal axis.

## Repetitive on the Half-Cycle

The repetitive nature of a waveform can determine whether specific harmonics will be present in the Fourier series expansion. In particular,

***if a waveform is repetitive on the half-cycle as demonstrated by the waveform in Fig. 25.7, the odd harmonics of the series of sine and cosine terms are zero.***

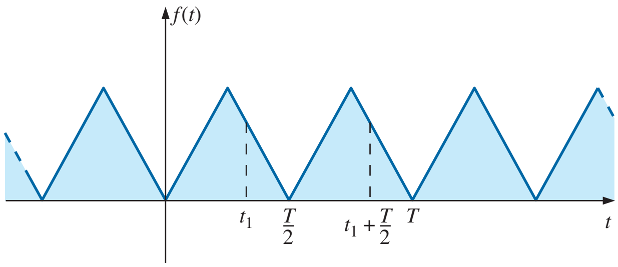


**FIG. 25.7**
*A waveform repetitive on the half-cycle.*

In functional form, the waveform must satisfy the following relationship:

$$f(t) = f\left(t + \frac{T}{2}\right) \qquad \textbf{(25.5)}$$

Eq. (25.5) states that the function repeats itself after each $T/2$ time interval ($t_1$ in Fig. 25.7). The waveform, however, will also repeat itself after each period $T$. In general, therefore, for a function of this type, if the period $T$ of the waveform is chosen to be twice that of the minimum period ($T/2$), the odd harmonics will all be zero.

## Mathematical Approach

The constants $A_0$, $A_{1\rightarrow n}$, and $B_{1\rightarrow n}$ can be determined by using the following integral formulas:

$$A_0 = \frac{1}{T}\int_0^T f(t)\,dt \qquad \textbf{(25.6)}$$

$$A_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt \qquad \textbf{(25.7)}$$

$$B_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt \qquad \textbf{(25.8)}$$

These equations have been presented for recognition purposes only; they are not used in the following analysis.

## Instrumentation

Three types of instrumentation are available that reveal the dc, fundamental, and harmonic content of a waveform: *the spectrum analyzer, wave analyzer,* and *Fourier analyzer.* The purpose of such instrumentation is not solely to determine the composition of a particular waveform but also to reveal the level of distortion that may have been introduced by a system. For instance, an amplifier may be increasing the applied signal by a factor of 50, but in the process it may have distorted the waveform in a way that is quite unnoticeable from the oscilloscope display. The amount of distortion appears in the form of harmonics at frequencies that are multiples of the applied frequency. Each of the above instruments reveal which frequencies are having the most impact on the distortion, permitting their removal with properly designed filters.

The spectrum analyzer is shown in Fig. 25.8. It has the appearance of an oscilloscope but rather than display a waveform that is voltage (vertical axis) versus time (horizontal axis), it generates a display scaled off in dB (vertical axis) versus frequency (horizontal axis). Such a display is said to be in the *frequency domain* versus the *time domain* of the standard oscilloscope. The height of the vertical line in the display of Fig. 25.8 reveals the impact of that frequency on the shape of the waveform. Spectrum analyzers are unable to provide the phase angle associated with each component.

**FIG. 25.8**
*Spectrum analyzer.*
(Courtesy of Hewlett Packard)

**EXAMPLE 25.1** Determine which components of the Fourier series are present in the waveforms in Fig. 25.9.

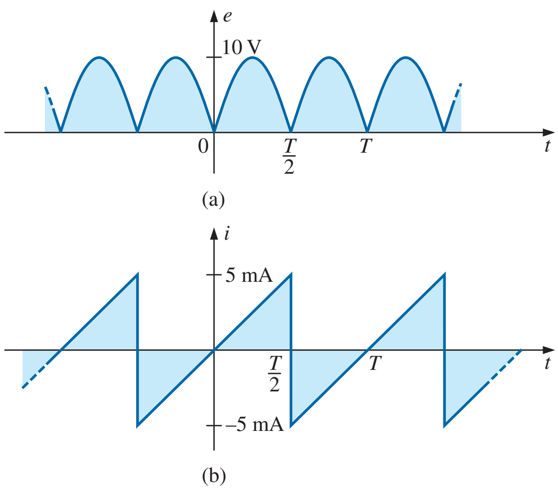


**FIG. 25.9**
*Example 25.1.*

### *Solutions:*

a. The waveform has a net area above the horizontal axis and therefore will have **a positive dc term $A_0$.**

The waveform has axis symmetry, resulting in **only cosine terms** in the expansion.

The waveform has half-cycle symmetry, resulting in **only even terms** in the cosine series.

b. The waveform has the same area above and below the horizontal axis within each period, resulting in $\boldsymbol{A_0 = 0}$.

The waveform has point symmetry, resulting in **only sine terms** in the expansion.

**EXAMPLE 25.2** Write the Fourier series expansion for the waveforms in Fig. 25.10.

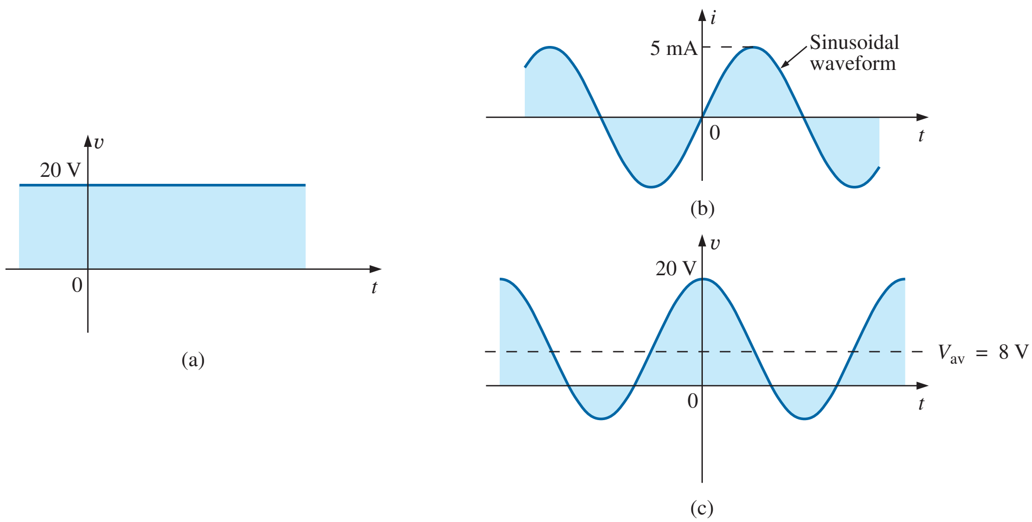


**FIG. 25.10**
*Example 25.2.*

***Solutions:***

a. $A_0 = 20 \qquad A_{1\to n} = 0 \qquad B_{1\to n} = 0$
$\boldsymbol{v = 20}$

b. $A_0 = 0 \qquad A_1 = 5 \times 10^{-3} \qquad A_{2\to n} = 0 \qquad B_{1\to n} = 0$
$\boldsymbol{i = 5 \times 10^{-3} \sin \omega t}$

c. $A_0 = 8 \qquad A_{1\to n} = 0 \qquad B_1 = 12 \qquad B_{2\to n} = 0$
$\boldsymbol{v = 8 + 12 \cos \omega t}$

**EXAMPLE 25.3** Sketch the following Fourier series expansion:

$$v = 2 + 1 \cos \alpha + 2 \sin \alpha$$

***Solution:*** Note Fig. 25.11.

The solution could be obtained graphically by first plotting all of the functions and then considering a sufficient number of points on the horizontal axis; or phasor algebra could be used as follows:

$$\begin{aligned} 1 \cos \alpha + 2 \sin \alpha &= 1\text{ V} \angle 90^\circ + 2\text{ V} \angle 0^\circ = j\,1\text{ V} + 2\text{ V} \\ &= 2\text{ V} + j\,1\text{ V} = 2.236\text{ V} \angle 26.57^\circ \\ &= 2.236 \sin(\alpha + 26.57^\circ) \end{aligned}$$

and $$\boldsymbol{v = 2 + 2.236 \sin (\alpha + 26.57^\circ)}$$

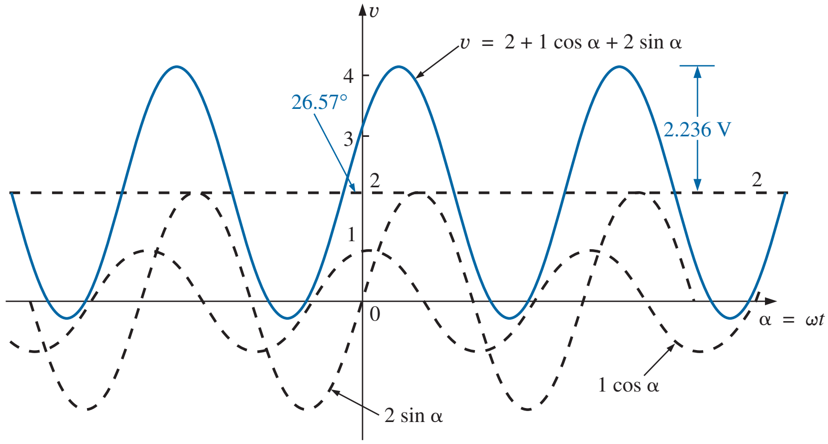


**FIG. 25.11**
*Example 25.3.*

which is simply the sine wave portion riding on a dc level of 2 V. That is, its positive maximum is 2 V + 2.236 V = 4.236 V, and its minimum is 2 V − 2.236 V = −0.236 V.

**EXAMPLE 25.4** Sketch the following Fourier series expansion:

$$i = 1 \sin \omega t + 1 \sin 2\omega t$$

***Solution:*** See Fig. 25.12. Note that in this case the sum of the two sinusoidal waveforms of different frequencies is *not* a sine wave. Recall that complex algebra can be applied only to waveforms having the *same* frequency. In this case, the solution is obtained graphically point by point, as shown for $t = t_1$.

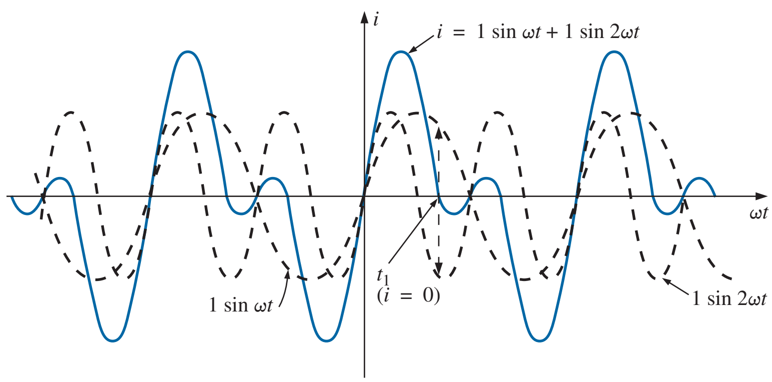


**FIG. 25.12**
*Example 25.4.*

As an additional example in the use of the Fourier series approach, consider the square wave shown in Fig. 25.13. The average value is zero, so $A_0 = 0$. It is an odd function, so all the constants $B_{1\to n}$ equal zero; only sine terms are present in the series expansion. Since the waveform satisfies the criteria for $f(t) = -f(t + T/2)$, the even harmonics are also zero.

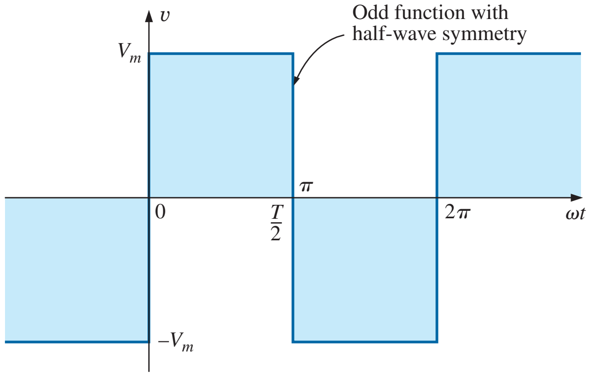


**FIG. 25.13**
*Square wave.*

The expression obtained after evaluating the various coefficients using Eq. (25.8) is

$$v = \frac{4}{\pi}V_m\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \cdots + \frac{1}{n}\sin n\omega t\right) \qquad \textbf{(25.9)}$$

Note that the fundamental does indeed have the same frequency as that of the square wave. If we add the fundamental and third harmonics, we obtain the results shown in Fig. 25.14.

Even with only the first two terms, a few characteristics of the square wave are beginning to appear. If we add the next two terms (Fig. 25.15), the width of the pulse increases, and the number of peaks increases.

As we continue to add terms, the series better approximate the square wave. Note, however, that the amplitude of each succeeding term diminishes to the point at which it is negligible compared with those of the first few terms. A good approximation is to assume that the waveform is composed of the harmonics up to and including the ninth. Any higher

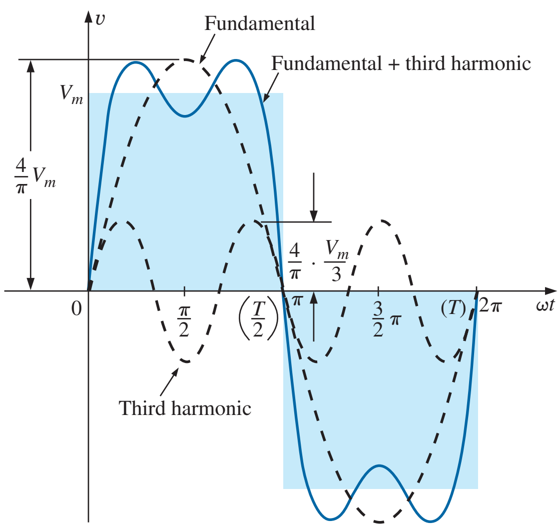


**FIG. 25.14**
*Fundamental plus third harmonic.*

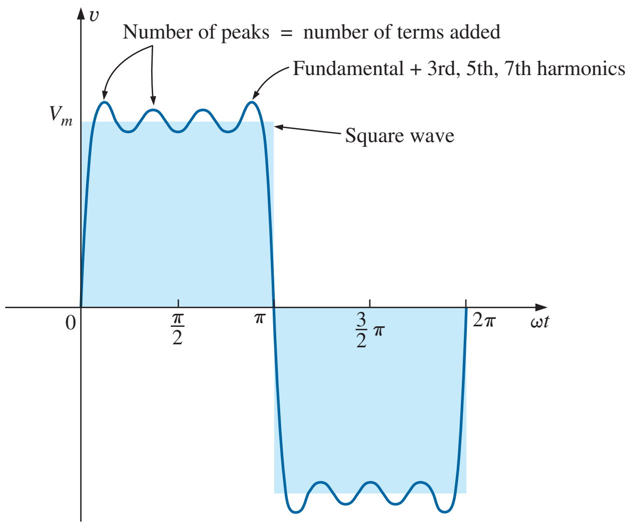


**FIG. 25.15**

*Fundamental plus third, fifth, and seventh harmonics.*

harmonics would be less than one-tenth the fundamental. If the waveform just described were shifted above or below the horizontal axis, the Fourier series would be altered only by a change in the dc term. Fig. 25.16(c), for example, is the sum of Fig. 25.16(a) and (b). The Fourier series for the complete waveform is, therefore,

$$v = v_1 + v_2 = V_m + \text{Eq. (25.9)}$$
$$= V_m + \frac{4}{\pi} V_m\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \cdots\right)$$

and $$v = V_m\left[1 + \frac{4}{\pi}\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t + \cdots\right)\right]$$

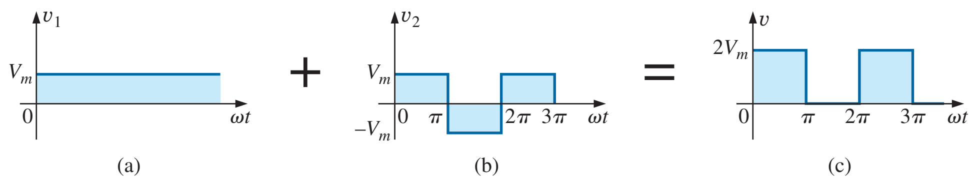


**FIG. 25.16**

*Shifting a waveform vertically with the addition of a dc term.*

The equation for the half-wave rectified pulsating waveform in Fig. 25.17(b) is

$$v_2 = 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha - \cdots \quad \textbf{(25.10)}$$

The waveform in Fig. 25.17(c) is the sum of the two in Fig. 25.17(a) and (b). The Fourier series for the waveform in Fig. 25.17(c) is, therefore,

$$v_T = v_1 + v_2 = -\frac{V_m}{2} + \text{Eq. (25.10)}$$
$$= -0.500V_m + 0.318V_m + 0.500V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$$

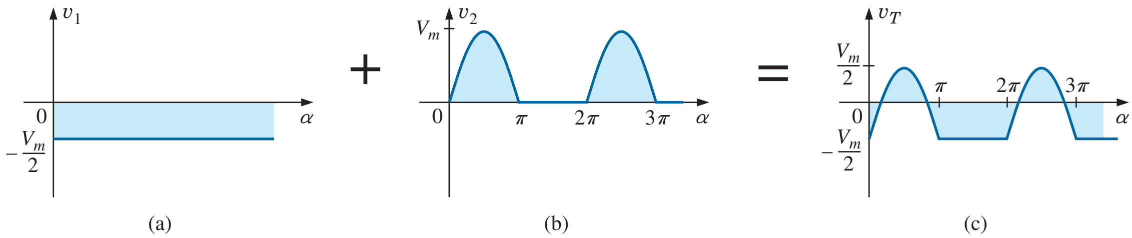


**FIG. 25.17**
*Lowering a waveform with the addition of a negative dc component.*

and $\quad v_T = -0.182V_m + 0.5V_m \sin \alpha - 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$

If either waveform were shifted to the right or left, the phase shift would be subtracted from or added to, respectively, the sine and cosine terms. The dc term would not change with a shift to the right or left.

If the half-wave rectified signal is shifted 90° to the left, as in Fig. 25.18, the Fourier series becomes

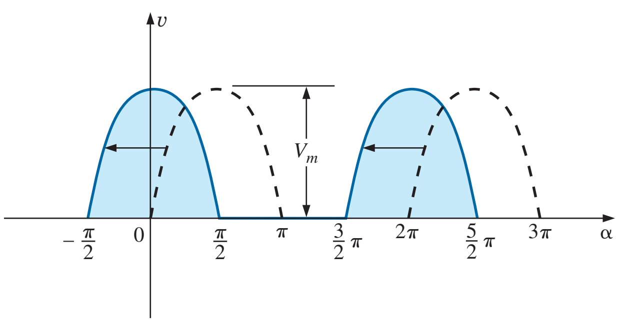


**FIG. 25.18**
*Changing the phase angle of a waveform.*

$$v = 0.318V_m + 0.500V_m \underbrace{\sin(\alpha + 90°)}_{\cos \alpha} - 0.212V_m \cos 2(\alpha + 90°) - 0.0424V_m \cos 4(\alpha + 90°) + \cdots$$

$$= 0.318V_m + 0.500V_m \cos \alpha - 0.212V_m \cos(2\alpha + 180°) - 0.0424V_m \cos(4\alpha + 360°) + \cdots$$

and $\quad v = 0.318V_m + 0.500V_m \cos \alpha + 0.212V_m \cos 2\alpha - 0.0424V_m \cos 4\alpha + \cdots$

## 25.3 CIRCUIT RESPONSE TO A NONSINUSOIDAL INPUT

The Fourier series representation of a nonsinusoidal input can be applied to a linear network using the principle of superposition. Recall that this theorem allowed us to consider the effects of each source of a circuit independently. If we replace the nonsinusoidal input with the terms of the Fourier series deemed necessary for practical considerations, we can use superposition to find the response of the network to each term (Fig. 25.19).

The total response of the system is then the algebraic sum of the values obtained for each term. The major change between using this theorem for nonsinusoidal circuits and using it for the circuits previously described is that the frequency will be different for each term in the nonsinusoidal application. Therefore, the reactances

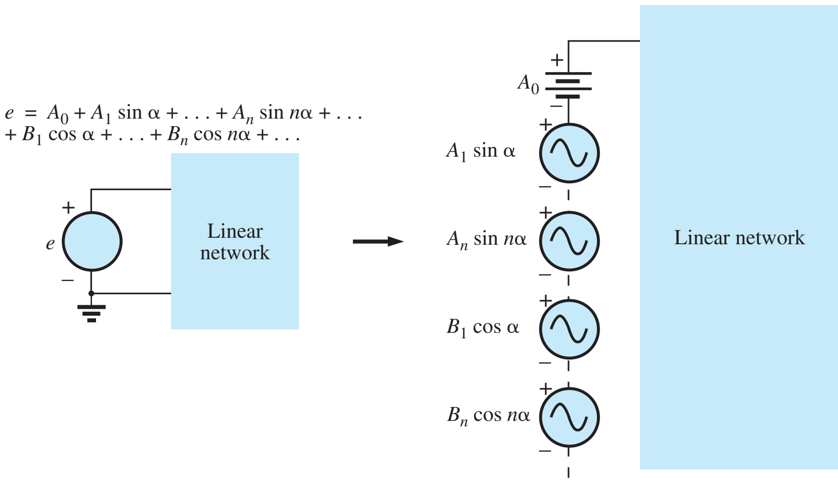


**FIG. 25.19**
*Setting up the application of a Fourier series of terms to a linear network.*

$$X_L = 2\pi f L \quad \text{and} \quad X_C = \frac{1}{2\pi f C}$$

will change for each term of the input voltage or current.

In Chapter 13, we found that the rms value of any waveform was given by

$$\sqrt{\frac{1}{T}\int_0^T f^2(t)\,dt}$$

If we apply this equation to the following Fourier series:

$$v(\alpha) = V_0 + V_{m_1}\sin\alpha + \cdots + V_{m_n}\sin n\alpha + V'_{m_1}\cos\alpha + \cdots + V'_{m_n}\cos n\alpha$$

then

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + \cdots + V_{m_n}^2 + V'^2_{m_1} + \cdots + V'^2_{m_n}}{2}} \qquad \textbf{(25.11)}$$

However, since

$$\frac{V_{m_1}^2}{2} = \left(\frac{V_{m_1}}{\sqrt{2}}\right)\left(\frac{V_{m_1}}{\sqrt{2}}\right) = (V_{1_{\text{rms}}})(V_{1_{\text{rms}}}) = V_{1_{\text{rms}}}^2$$

then

$$V_{\text{rms}} = \sqrt{V_0^2 + V_{1_{\text{rms}}}^2 + \cdots + V_{n_{\text{rms}}}^2 + V'^2_{1_{\text{rms}}} + \cdots + V'^2_{n_{\text{rms}}}} \qquad \textbf{(25.12)}$$

Similarly, for

$$i(\alpha) = I_0 + I_{m_1}\sin\alpha + \cdots + I_{m_n}\sin n\alpha + I'_{m_1}\cos\alpha + \cdots + I'_{m_n}\cos n\alpha$$

we have

$$I_{\text{rms}} = \sqrt{I_0^2 + \frac{I_{m_1}^2 + \cdots + I_{m_n}^2 + I'^2_{m_1} + \cdots + I'^2_{m_n}}{2}} \qquad \textbf{(25.13)}$$

and

$$I_{rms} = \sqrt{I_0^2 + I_{1_{rms}}^2 + \cdots + I_{n_{rms}}^2 + I'^2_{1_{rms}} + \cdots + I'^2_{n_{rms}}} \quad \textbf{(25.14)}$$

The total power delivered is the sum of that delivered by the corresponding terms of the voltage and current. In the following equations, all voltages and currents are rms values:

$$P_T = V_0 I_0 + V_1 I_1 \cos\theta_1 + \cdots + V_n I_n \cos\theta_n + \cdots \quad \textbf{(25.15)}$$

$$P_T = I_0^2 R + I_1^2 R + \cdots + I_n^2 R + \cdots \quad \textbf{(25.16)}$$

or

$$P_T = I_{rms}^2 R \quad \textbf{(25.17)}$$

with $I_{rms}$ as defined by Eq. (25.13), and, similarly,

$$P_T = \frac{V_{rms}^2}{R} \quad \textbf{(25.18)}$$

with $V_{rms}$ as defined by Eq. (25.11).

### EXAMPLE 25.5

a. Sketch the input resulting from the combination of sources in Fig. 25.20.

b. Determine the rms value of the input in Fig. 25.20.

***Solutions:***

a. Note Fig. 25.21.
b. Eq. (25.12):

$$V_{rms} = \sqrt{V_0^2 + \frac{V_m^2}{2}}$$

$$= \sqrt{(4\text{ V})^2 + \frac{(6\text{ V})^2}{2}} = \sqrt{16 + \frac{36}{2}}\text{ V} = \sqrt{34}\text{ V}$$

$$= \mathbf{5.831\ V}$$

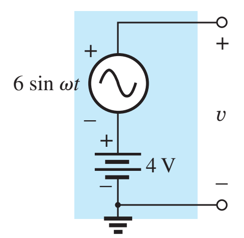


**FIG. 25.20**
*Example 25.5.*

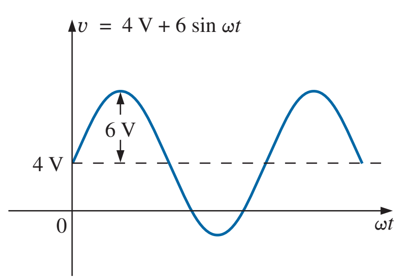


**FIG. 25.21**
*Wave pattern generated by the source in Fig. 25.20.*

It is particularly interesting to note from Example 25.5 that the rms value of a waveform having both dc and ac components is not simply the sum of the effective values of each. In other words, there is a temptation in the absence of Eq. (25.12) to state that $V_{rms} = 4\text{ V} + 0.707\,(6\text{ V}) = 8.242\text{ V}$, which is incorrect and, in fact, exceeds the correct level by some 41%.

## Instrumentation

It is important to realize that not every DMM will read the rms value of nonsinusoidal waveforms such as the one appearing in Fig. 25.21. Many are designed to read the rms value of sinusoidal waveforms only. It is important to read the manual provided with the meter to see if it is a *true rms* meter that can read the rms value of any waveform.

We learned in Chapter 13 that the rms value of a square wave is the peak value of the waveform. Let us test this result using the Fourier expansion and Eq. (25.11).

**EXAMPLE 25.6** Determine the rms value of the square wave of Fig. 25.13 with $V_m = 20$ V using the first six terms of the Fourier expansion, and compare the result to the actual rms value of 20 V.

***Solution:***

$$\upsilon = \frac{4}{\pi}(20\text{ V})\sin\omega t + \frac{4}{\pi}\left(\frac{1}{3}\right)(20\text{ V})\sin 3\omega t + \frac{4}{\pi}\left(\frac{1}{5}\right)(20\text{ V})\sin 5\omega t + \frac{4}{\pi}\left(\frac{1}{7}\right)(20\text{ V})\sin 7\omega t + \frac{4}{\pi}\left(\frac{1}{9}\right)(20\text{ V})\sin 9\omega t + \frac{4}{\pi}\left(\frac{1}{11}\right)(20\text{ V})\sin 11\omega t$$

$$\upsilon = 25.465\sin\omega t + 8.488\sin 3\omega t + 5.093\sin 5\omega t + 3.638\sin 7\omega t + 2.829\sin 9\omega t + 2.315\sin 11\omega t$$

Eq. (25.11):

$$V_{\text{rms}} = \sqrt{V_0^2 + \frac{V_{m_1}^2 + V_{m_2}^2 + V_{m_3}^2 + V_{m_4}^2 + V_{m_5}^2 + V_{m_6}^2}{2}}$$

$$= \sqrt{(0\text{ V})^2 + \frac{(25.465\text{ V})^2 + (8.488\text{ V})^2 + (5.093\text{ V})^2 + (3.638\text{ V})^2 + (2.829\text{ V})^2 + (2.315\text{ V})^2}{2}}$$

$$= \mathbf{19.66\text{ V}}$$

The solution differs less than 0.4 V from the correct answer of 20 V. However, each additional term in the Fourier series brings the result closer to the 20 V level. An infinite number results in an exact solution of 20 V.

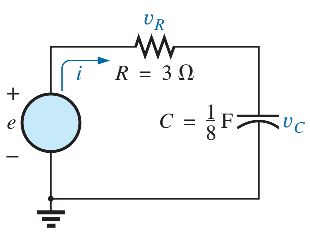


**FIG. 25.22**
*Example 25.7.*

**EXAMPLE 25.7** The input to the circuit in Fig. 25.22 is the following:

$$e = 12 + 10\sin 2t$$

a. Find the current $i$ and the voltages $\upsilon_R$ and $\upsilon_C$.
b. Find the rms values of $i$, $\upsilon_R$, and $\upsilon_C$.
c. Find the power delivered to the circuit.

***Solutions:***

a. Redraw the original circuit as shown in Fig. 25.23. Then apply superposition:

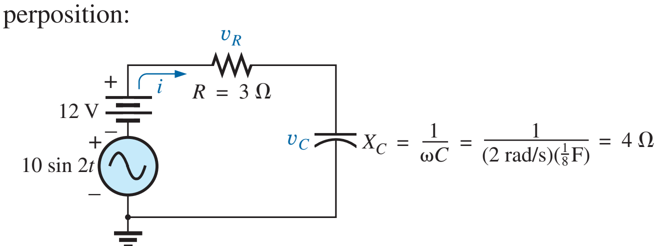


**FIG. 25.23**
*Circuit in Fig. 25.22 with the components of the Fourier series input.*

1. *For the 12 V dc supply portion of the input, $I = 0$* since the capacitor is an open circuit to dc when $\upsilon_C$ has reached its final (steady-state) value. Therefore,

$$V_R = IR = 0\text{ V} \quad \text{and} \quad V_C = 12\text{ V}$$

2. *For the ac supply,*

$$\mathbf{Z} = 3\ \Omega - j\,4\ \Omega = 5\ \Omega\ \angle -53.13°$$

and $$\mathbf{I} = \frac{\mathbf{E}}{\mathbf{Z}} = \frac{\frac{10}{\sqrt{2}}\ \text{V}\ \angle 0°}{5\ \Omega\ \angle -53.13°} = \frac{2}{\sqrt{2}}\ \text{A}\ \angle +53.13°$$

$$\mathbf{V}_R = (I\ \angle\theta)(R\ \angle 0°) = \left(\frac{2}{\sqrt{2}}\ \text{A}\ \angle +53.13°\right)(3\ \Omega\ \angle 0°)$$
$$= \frac{6}{\sqrt{2}}\ \text{V}\ \angle +53.13°$$

and

$$\mathbf{V}_C = (I\ \angle\theta)(X_C\ \angle -90°) = \left(\frac{2}{\sqrt{2}}\ \text{A}\ \angle +53.13°\right)(4\ \Omega\ \angle -90°)$$
$$= \frac{8}{\sqrt{2}}\ \text{V}\ \angle -36.87°$$

In the time domain,

$$i = \mathbf{0 + 2\sin(2}t\mathbf{ + 53.13°)}$$

Note that even though the dc term was present in the expression for the input voltage, the dc term for the current in this circuit is zero:

$$v_R = \mathbf{0 + 6\sin(2}t\mathbf{ + 53.13°)}$$

and $$v_C = \mathbf{12 + 8\sin(2}t\mathbf{ - 36.87°)}$$

b. Eq. (25.14): $I_{\text{rms}} = \sqrt{(0)^2 + \frac{(2\ \text{A})^2}{2}} = \sqrt{2}\ \text{A} = \mathbf{1.414\ A}$

Eq. (25.12): $V_{R_{\text{rms}}} = \sqrt{(0)^2 + \frac{(6\ \text{V})^2}{2}} = \sqrt{18}\ \text{V} = \mathbf{4.243\ V}$

Eq. (25.12): $V_{C_{\text{rms}}} = \sqrt{(12\ \text{V})^2 + \frac{(8\ \text{V})^2}{2}} = \sqrt{176}\ \text{V} = \mathbf{13.267\ V}$

c. $P = I_{\text{rms}}^2 R = \left(\frac{2}{\sqrt{2}}\ \text{A}\right)^2 (3\ \Omega) = \mathbf{6\ W}$

**EXAMPLE 25.8** Find the response of the circuit in Fig. 25.24 to the input shown.

$$e = 0.318E_m + 0.500E_m \sin \omega t - 0.212E_m \cos 2\omega t - 0.0424E_m \cos 4\omega t + \ldots$$

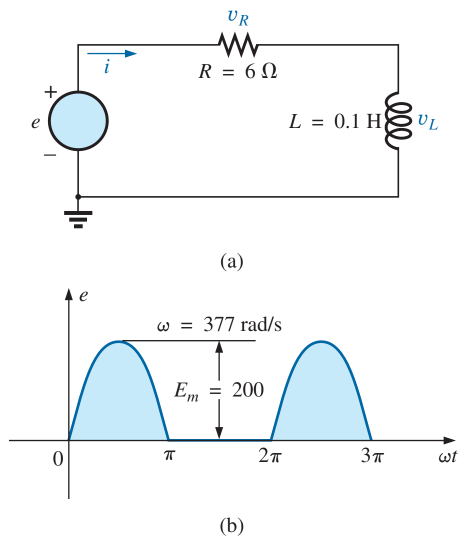


**FIG. 25.24**
*Example 25.8.*

***Solution:*** For discussion purposes, only the first three terms are used to represent $e$. Converting the cosine terms to sine terms and substituting for $E_m$ gives us

$$e = 63.60 + 100.0 \sin \omega t - 42.40 \sin(2\omega t + 90°)$$

Using phasor notation, the original circuit becomes like the one shown in Fig. 25.25.

**Applying Superposition** *For the dc term* ($E_0 = 63.6$ V):

$$X_L = 0 \qquad \text{(short for dc)}$$
$$\mathbf{Z}_T = R\ \angle 0° = 6\ \Omega\ \angle 0°$$

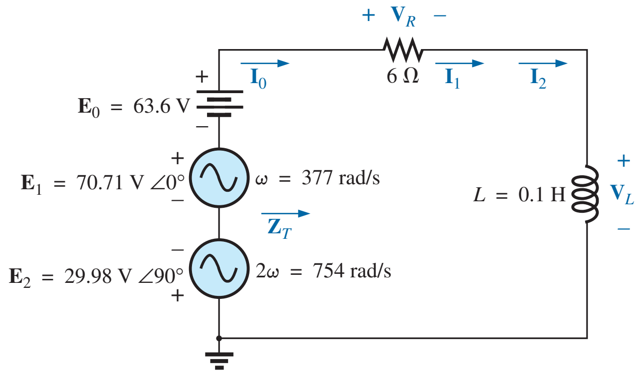


**FIG. 25.25**

*Circuit in Fig. 25.24 with the components of the Fourier series input.*

$$I_0 = \frac{E_0}{R} = \frac{63.6\text{ V}}{6\ \Omega} = 10.60\text{ A}$$
$$V_{R_0} = I_0R = E_0 = 63.60\text{ V}$$
$$V_{L_0} = 0$$

The average power is

$$P_0 = I_0^2R = (10.60\text{ A})^2(6\ \Omega) = 674.2\text{ W}$$

*For the fundamental term* ($\mathbf{E}_1 = 70.71\text{ V}\ \angle 0°$, $\omega = 377$):

$$X_{L_1} = \omega L = (377\text{ rad/s})(0.1\text{ H}) = 37.7\ \Omega$$
$$\mathbf{Z}_{T_1} = 6\ \Omega + j\,37.7\ \Omega = 38.17\ \Omega\ \angle 80.96°$$
$$\mathbf{I}_1 = \frac{\mathbf{E}_1}{\mathbf{Z}_{T_1}} = \frac{70.71\text{ V}\ \angle 0°}{38.17\ \Omega\ \angle 80.96°} = 1.85\text{ A}\ \angle -80.96°$$
$$\mathbf{V}_{R_1} = (I_1\ \angle\theta)(R\ \angle 0°) = (1.85\text{ A}\ \angle -80.96°)(6\ \Omega\ \angle 0°)$$
$$= 11.10\text{ V}\ \angle -80.96°$$
$$\mathbf{V}_{L_1} = (I_1\ \angle\theta)(X_{L_1}\ \angle 90°) = (1.85\text{ A}\ \angle -80.96°)(37.7\ \Omega\ \angle 90°)$$
$$= 69.75\text{ V}\ \angle 9.04°$$

The average power is

$$P_1 = I_1^2R = (1.85\text{ A})^2(6\ \Omega) = 20.54\text{ W}$$

*For the second harmonic* ($\mathbf{E}_2 = 29.98\text{ V}\ \angle -90°$, $\omega = 754$): The phase angle of $\mathbf{E}_2$ was changed to $-90°$ to give it the same polarity as the input voltages $\mathbf{E}_0$ and $\mathbf{E}_1$.

$$X_{L_2} = \omega L = (754\text{ rad/s})(0.1\text{ H}) = 75.4\ \Omega$$
$$\mathbf{Z}_{T_2} = 6\ \Omega + j\,75.4\ \Omega = 75.64\ \Omega\ \angle 85.45°$$
$$\mathbf{I}_2 = \frac{\mathbf{E}_2}{\mathbf{Z}_{T_2}} = \frac{29.98\text{ V}\ \angle -90°}{75.64\ \Omega\ \angle 85.45°} = 0.396\text{ A}\ \angle -174.45°$$
$$\mathbf{V}_{R_2} = (I_2\ \angle\theta)(R\ \angle 0°) = (0.396\text{ A}\ \angle -174.45°)(6\ \Omega\ \angle 0°)$$
$$= 2.38\text{ V}\ \angle -174.45°$$
$$\mathbf{V}_{L_2} = (I_2\ \angle\theta)(X_{L_2}\ \angle 90°) = (0.396\text{ A}\ \angle -174.45°)(75.4\ \Omega\ \angle 90°)$$
$$= 29.9\text{ V}\ \angle -84.45°$$

The average power is

$$P_2 = I_2^2R = (0.396\text{ A})^2(6\ \Omega) = 0.941\text{ W}$$

The Fourier series expansion for $i$ is

$$i = \mathbf{10.6 + \sqrt{2}(1.85)\sin(377\mathit{t} - 80.96°) + \sqrt{2}(0.396)\sin(754\mathit{t} - 174.45°)}$$

and

$$I_{\text{rms}} = \sqrt{(10.6\text{ A})^2 + (1.85\text{ A})^2 + (0.396\text{ A})^2} = \mathbf{10.77\ A}$$

The Fourier series expansion for $v_R$ is

$$v_R = \mathbf{63.6 + \sqrt{2}(11.10)\sin(377\mathit{t} - 80.96°) + \sqrt{2}(2.38)\sin(754\mathit{t} - 174.45°)}$$

and

$$V_{R_{\text{rms}}} = \sqrt{(63.6\text{ V})^2 + (11.10\text{ V})^2 + (2.38\text{ V})^2} = \mathbf{64.61\ V}$$

The Fourier series expansion for $v_L$ is

$$v_L = \mathbf{\sqrt{2}(69.75)\sin(377\mathit{t} + 9.04°) + \sqrt{2}(29.93)\sin(754\mathit{t} - 84.45°)}$$

and $$V_{L_{\text{rms}}} = \sqrt{(69.75\text{ V})^2 + (29.93\text{ V})^2} = \mathbf{75.90\ V}$$

The total average power is

$$P_T = I_{\text{rms}}^2 R = (10.77\text{ A})^2(6\ \Omega) = \mathbf{695.96\ W} = P_0 + P_1 + P_2$$

## 25.4 ADDITION AND SUBTRACTION OF NONSINUSOIDAL WAVEFORMS

The Fourier series expression for the waveform resulting from the addition or subtraction of two nonsinusoidal waveforms can be found using phasor algebra if the terms having the same frequency are considered separately.

For example, the sum of the following two nonsinusoidal waveforms is found using this method:

$$v_1 = 30 + 20\sin 20t + \cdots + 5\sin(60t + 30°)$$
$$v_2 = 60 + 30\sin 20t + 20\sin 40t + 10\cos 60t$$

1. dc terms:

$$V_{T_0} = 30\text{ V} + 60\text{ V} = 90\text{ V}$$

2. $\omega = 20$:

$$V_{T_{1(\max)}} = 30\text{ V} + 20\text{ V} = 50\text{ V}$$

and $$v_{T_1} = 50\sin 20t$$

3. $\omega = 40$:

$$v_{T_2} = 20\sin 40t$$

4. $\omega = 60$:

$$5\sin(60t + 30°) = (0.707)(5)\text{ V}\angle 30° = 3.54\text{ V}\angle 30°$$
$$10\cos 60t = 10\sin(60t + 90°) \Rightarrow (0.707)(10)\text{ V}\angle 90° = 7.07\text{ V}\angle 90°$$

$$\mathbf{V}_{T_3} = 3.54\text{ V}\angle 30° + 7.07\text{ V}\angle 90° = 3.07\text{ V} + j\,1.77\text{ V} + j\,7.07\text{ V} = 3.07\text{ V} + j\,8.84\text{ V}$$
$$\mathbf{V}_{T_3} = 9.36\text{ V}\angle 70.85°$$

and $$v_{T_3} = 13.24\sin(60t + 70.85°)$$

with

$$v_T = v_1 + v_2 = \mathbf{90 + 50 \sin 20}\boldsymbol{t} \mathbf{+ 20 \sin 40}\boldsymbol{t} \mathbf{+ 13.24 \sin(60}\boldsymbol{t} \mathbf{+ 70.85°)}$$

## 25.5 COMPUTER ANALYSIS

### PSpice

**Fourier Series** The computer analysis begins with a verification of the waveform in Fig. 25.15, demonstrating that only four terms of a Fourier series can generate a waveform that has a number of characteristics of a square wave. The square wave has a peak value of 10 V at a frequency of 1 kHz, resulting in the following Fourier series using Eq. (25.9) (and recognizing that $\omega = 2\pi f = 6283.19$ rad/s):

$$\begin{aligned} v &= \frac{4}{\pi}(10\text{ V})\left(\sin \omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{5}\sin 5\omega t + \frac{1}{7}\sin 7\omega t\right) \\ &= 12.732 \sin \omega t + 4.244 \sin 3\omega t + 2.546 \sin 5\omega t + 1.819 \sin 7\omega t \end{aligned}$$

Each term of the Fourier series is treated as an independent ac source as shown in Fig. 25.26 with its peak value and applicable frequency. The sum of the source voltages appears across the resistor $R$ and generates the waveform in Fig. 25.27.

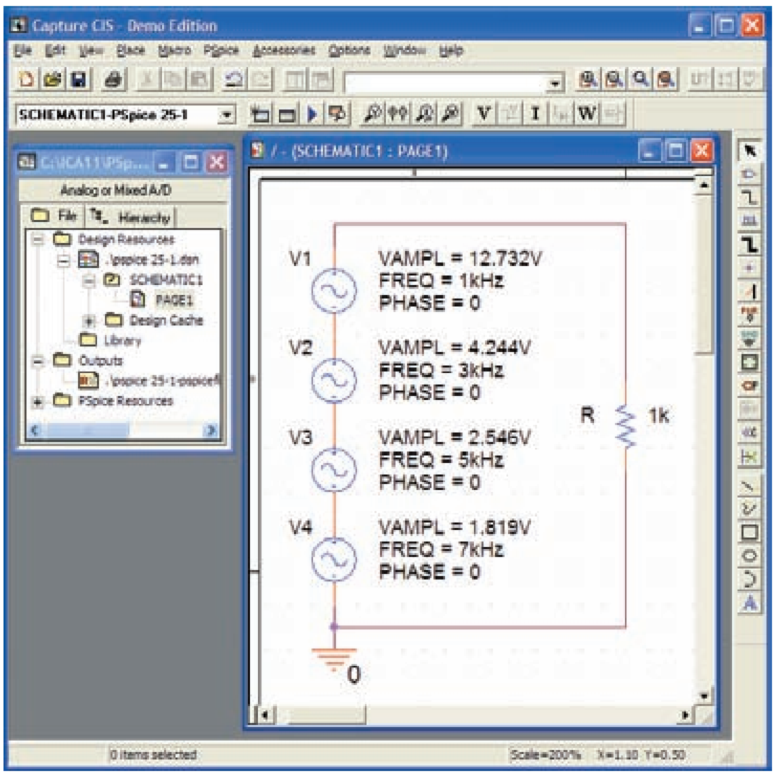


**FIG. 25.26**

*Using PSpice to apply four terms of the Fourier expansion of a 10 V square wave to a load resistor of 1 kΩ.*

Each source used **VSIN,** and since we want to display the result against time, choose **Time Domain(Transient)** in the **Simulation Settings.** For each source, select the **Property Editor** dialog box. Set **AC, FREQ, PHASE, VAMPL,** and **VOFF** (at 0 V). (Due to limited space, only **VAMPL, FREQ,** and **PHASE** are displayed in Fig. 25.26.) Under **Display,** set all of the remaining quantities on **Do Not Display.**

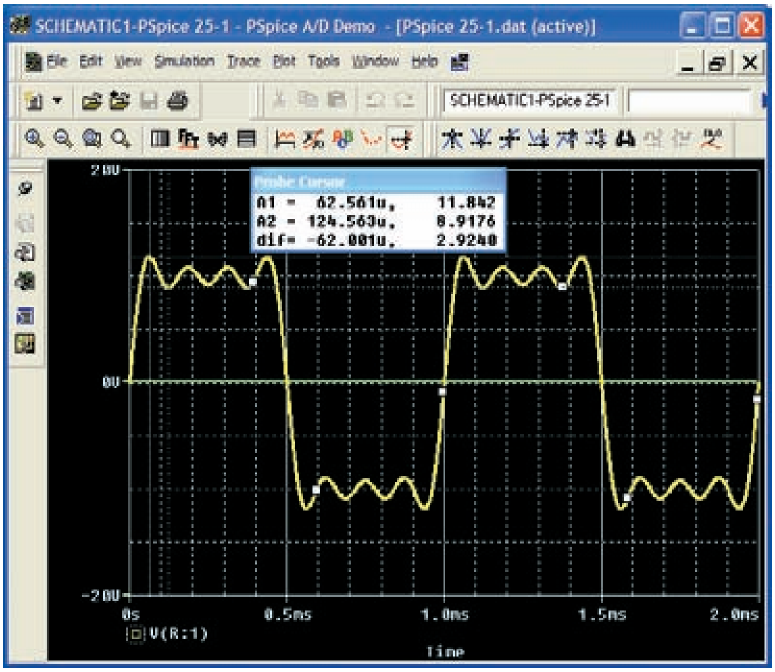


**FIG. 25.27**
*The resulting waveform of the voltage across the resistor R in Fig. 25.26.*

Set the **Run to time** at 2 ms so that two cycles of the fundamental frequency of 1 kHz appear. The **Start saving data after** remains at the default value of 0 s, and the **Maximum step size** at 1 $\mu$s, even though 2 ms/1000 = 2 $\mu$s, because we want to have additional plot points for the complex waveform. Once the **SCHEMATIC1** window appears, **Trace-Add Trace-V(R:1)-OK** results in the waveform in Fig. 25.27. To make the horizontal line at 0 V heavier, right-click on the line, select **Properties,** and then choose the green color and wider line. Click **OK,** and the wider line in Fig. 25.27 results, making it a great deal clearer where the 0 V line is located. Through the same process, make the curve yellow and wider as shown in the same figure. Using the cursors, you find that the first peak reaches 11.84 V and then drops to 8.920 V. The average value of the waveform is clearly +10 V in the positive region as shown by the dashed line entered using **Plot-Label-Line.** In every respect, the waveform is beginning to have the characteristics of a periodic square wave with a peak value of 10 V and a frequency of 1 kHz.

**Fourier Components** A frequency spectrum plot revealing the magnitude and frequency of each component of a Fourier series can be obtained by returning to **Plot** and selecting **Axis Settings** followed by **X Axis** and then **Fourier** under **Processing Options.** Click **OK,** and a number of spikes appear on the far left of the screen, with a frequency spectrum that extends from 0 Hz to 600 kHz. Select **Plot-Axis Settings** again, go to **Data Range,** and select **User Defined** to change the range to 0 Hz to 10 kHz since this is the range of interest for this waveform. Click **OK,** and the graph in Fig. 25.28 results, giving the magnitude and frequency of the components of the waveform. Using the left cursor, you find that the highest peak is 12.738 V at 1 kHz, comparing very well with the source **VI** having a peak value of 12.732 V at 1 kHz. Using the right-click cursor, you can move over to 3 kHz and find a magnitude of 4.246 V, again comparing very well with source **V2** with a peak value of 4.244 V.

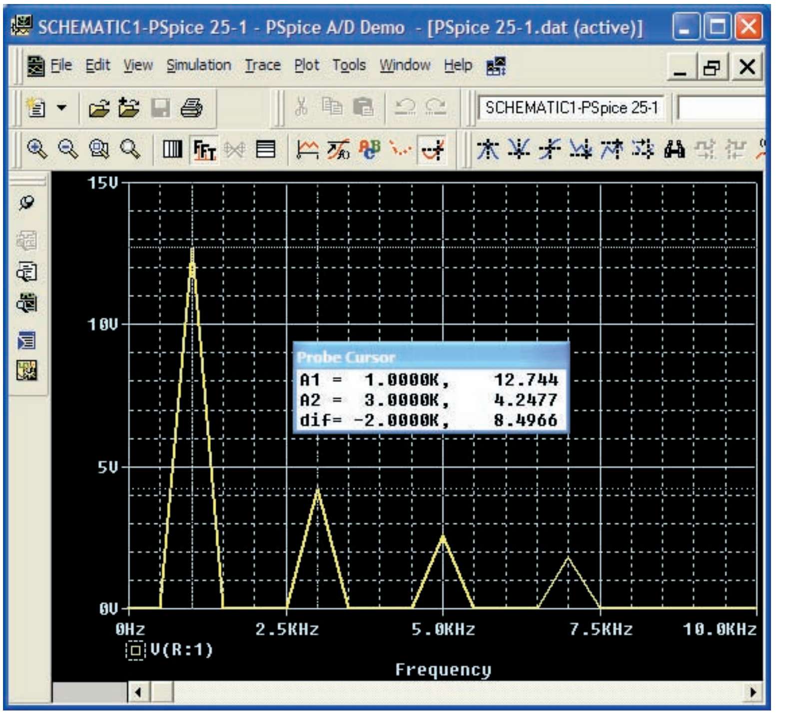


**FIG. 25.28**
*The Fourier components of the waveform in Fig. 25.27.*

# PROBLEMS

## SECTION 25.2 Fourier Series

1. For the waveforms in Fig. 25.29, determine whether the following will be present in the Fourier series representation:
   - **a.** dc term
   - **b.** cosine terms
   - **c.** sine terms
   - **d.** even-ordered harmonics
   - **e.** odd-ordered harmonics

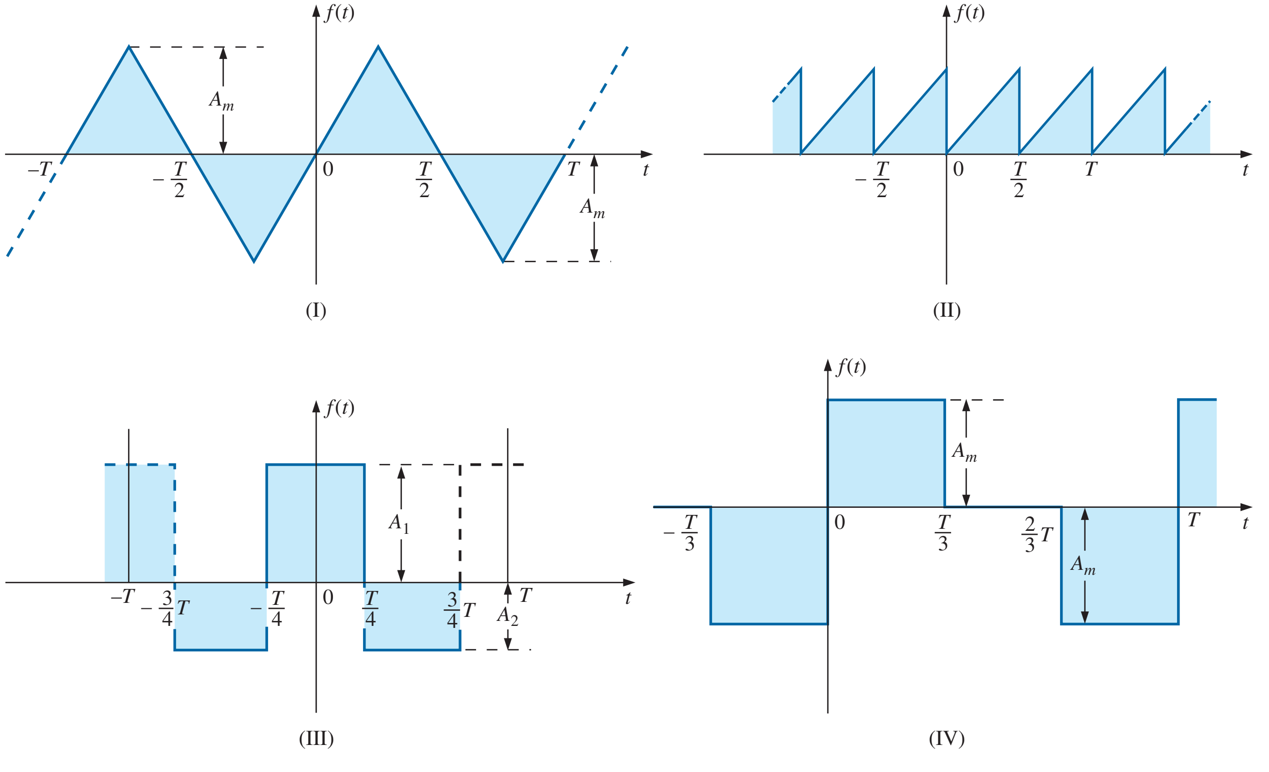

**FIG. 25.29**
*Problem 1.*

2. If the Fourier series for the waveform in Fig. 25.30(a) is

$$i = \frac{2I_m}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{35}\cos 6\omega t + \cdots\right.$$

find the Fourier series representation for waveforms (b) through (d).

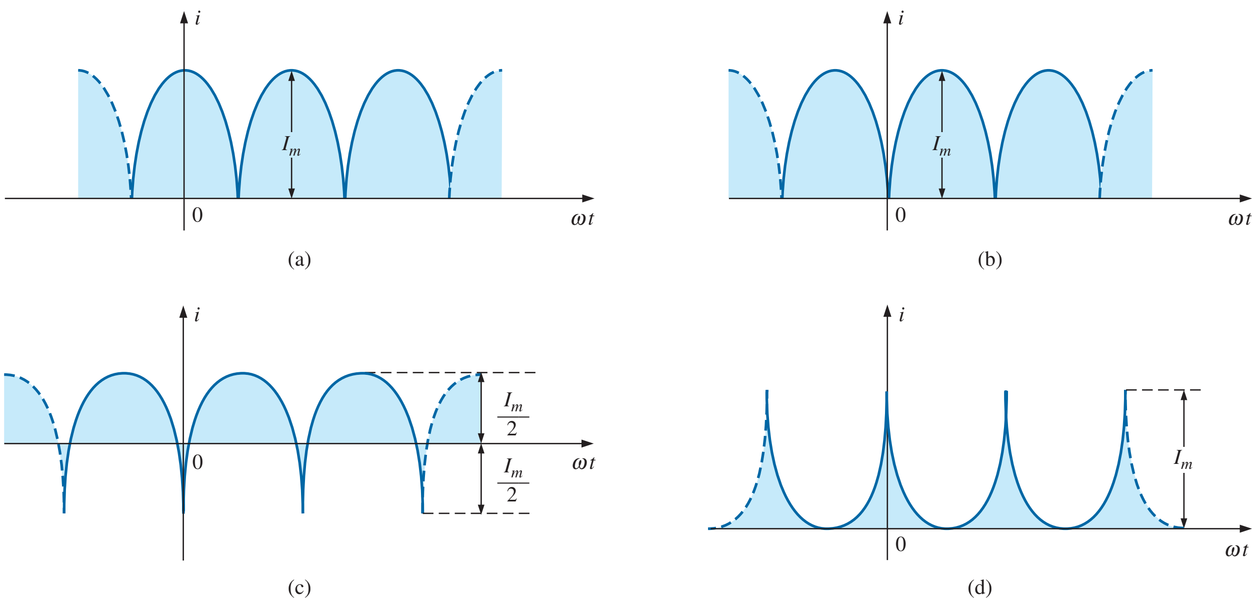


**FIG. 25.30**
*Problem 2.*

3. Sketch the following nonsinusoidal waveforms with α = $\omega t$ as the abscissa:
   **a.** $v = -4 + 2 \sin \alpha$
   **b.** $v = (\sin \alpha)^2$
   **c.** $i = 2 - 2 \cos \alpha$
4. Sketch the following nonsinusoidal waveforms with α as the abscissa:
   **a.** $i = 3 \sin \alpha - 6 \sin 2\alpha$
   **b.** $v = 2 \cos 2\alpha + \sin \alpha$
5. Sketch the following nonsinusoidal waveforms with $\omega t$ as the abscissa:
   **a.** $i = 50 \sin \omega t + 25 \sin 3\omega t$
   **b.** $i = 50 \sin \alpha - 25 \sin 3\alpha$
   **c.** $i = 4 + 3 \sin \omega t + 2 \sin 2\omega t - 1 \sin 3\omega t$

## SECTION 25.3 Circuit Response to a Nonsinusoidal Input

6. Find the average and effective values of the following nonsinusoidal waves:
   **a.** $v = 100 + 50 \sin \omega t + 25 \sin 2\omega t$
   **b.** $i = 3 + 2 \sin(\omega t - 53°) + 0.8 \sin(2\omega t - 70°)$
7. Find the rms value of the following nonsinusoidal waves:
   **a.** $v = 20 \sin \omega t + 15 \sin 2\omega t - 10 \sin 3\omega t$
   **b.** $i = 6 \sin(\omega t + 20°) + 2 \sin(2\omega t + 30°)$
   $-1 \sin(3\omega t + 60°)$
8. Find the total average power to a circuit whose voltage and current are as indicated in Problem 6.
9. Find the total average power to a circuit whose voltage and current are as indicated in Problem 7.
10. The Fourier series representation for the input voltage to the circuit in Fig. 25.31 is

$$e = 18 + 30 \sin 400t$$

   **a.** Find the nonsinusoidal expression for the current *i*.
   **b.** Calculate the rms value of the current.
   **c.** Find the expression for the voltage across the resistor.
   **d.** Calculate the rms value of the voltage across the resistor.
   **e.** Find the expression for the voltage across the reactive element.
   **f.** Calculate the rms value of the voltage across the reactive element.
   **g.** Find the average power delivered to the resistor.

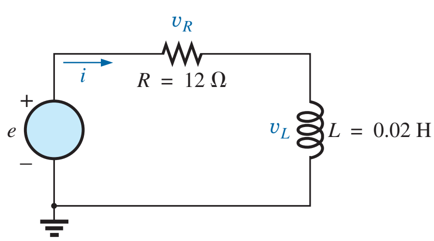


**FIG. 25.31**
*Problems 10, 11, and 12.*

**11.** Repeat Problem 10 for

$$e = 24 + 30 \sin 400t + 10 \sin 800t$$

**12.** Repeat Problem 10 for the following input voltage:

$$e = -60 + 20 \sin 300t - 10 \sin 600t$$

**13.** Repeat Problem 10 for the circuit in Fig. 25.32.

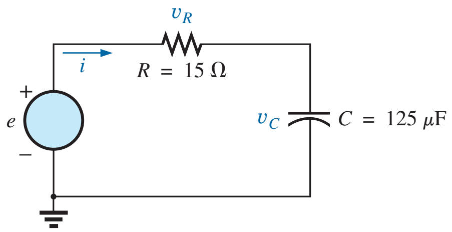


**FIG. 25.32**
*Problem 13.*

***14.** The input voltage in Fig. 25.33(a) to the circuit in Fig. 25.33(b) is a full-wave rectified signal having the following Fourier series expansion:

$$e = \frac{(2)(100\text{ V})}{\pi}\left(1 + \frac{2}{3}\cos 2\omega t - \frac{2}{15}\cos 4\omega t + \frac{2}{53}\cos 6\omega t + \cdots\right)$$

where $\omega = 377$.

**a.** Find the Fourier series expression for the voltage $v_o$ using only the first three terms of the expression.
**b.** Find the rms value of $v_o$.
**c.** Find the average power delivered to the 1 kΩ resistor.

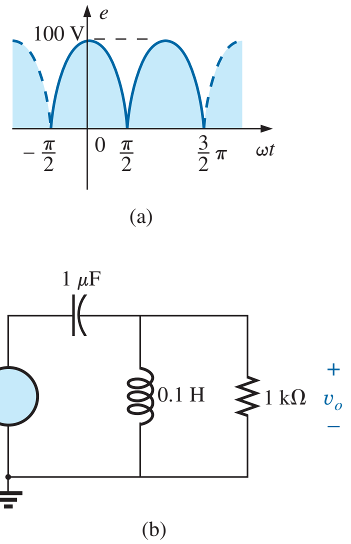


**FIG. 25.33**
*Problem 14.*

***15.** Find the Fourier series expression for the voltage $v_o$ in Fig. 25.34.

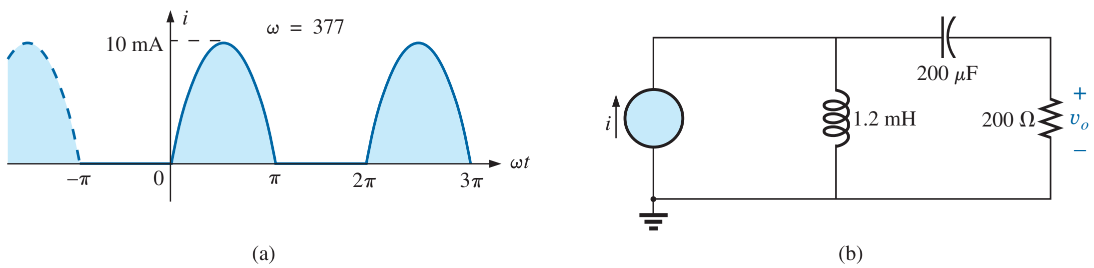


**FIG. 25.34**
*Problem 15.*

## SECTION 25.4 Addition and Subtraction of Nonsinusoidal Waveforms

**16.** Perform the indicated operations on the following nonsinusoidal waveforms:

**a.** $[60 + 70 \sin \omega t + 20 \sin(2\omega t + 90°) + 10 \sin(3\omega t + 60°)] + [20 + 30 \sin \omega t - 20 \cos 2\omega t + 5 \cos 3\omega t]$
**b.** $[20 + 60 \sin \alpha + 10 \sin(2\alpha - 180°) + 5 \cos(3\alpha + 90°)] - [5 - 10 \sin \alpha + 4 \sin(3\alpha - 30°)]$

**17.** Find the nonsinusoidal expression for the current $i_s$ of the diagram in Fig. 25.35.

$i_2 = 10 + 30 \sin 20t - 0.5 \sin(40t + 90°)$
$i_1 = 20 + 4 \sin(20t + 90°) + 0.5 \sin(40t + 30°)$

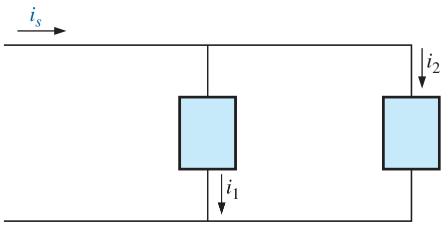


**FIG. 25.35**
*Problem 17.*

**18.** Find the nonsinusoidal expression for the voltage $e$ of the diagram in Fig. 25.36.

$v_1 = 20 - 200 \sin 600t + 100 \cos 1200t + 75 \sin 1800t$
$v_2 = -10 + 150 \sin(600t + 30°) + 50 \sin(1800t + 60°)$

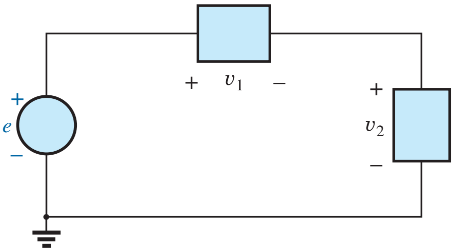


**FIG. 25.36**
*Problem 18.*

## SECTION 25.5 Computer Analysis

### PSpice

**19.** Plot the waveform in Fig. 25.11 for two or three cycles. Then obtain the Fourier components, and compare them to the applied signal.

**20.** Plot a half-rectified waveform with a peak value of 20 V using Eq. (25.10). Use the dc term, the fundamental term, and four harmonics. Compare the resulting waveform to the ideal half-rectified waveform.

**21.** Demonstrate the effect of adding two more terms to the waveform in Fig. 25.27, and generate the Fourier spectrum.

## GLOSSARY

**Axis symmetry** A sinusoidal or nonsinusoidal function that has symmetry about the vertical axis.

**Even harmonics** The terms of the Fourier series expansion that have frequencies that are even multiples of the fundamental component.

**Fourier series** A series of terms, developed in 1826 by Baron Jean Fourier, that can be used to represent a nonsinusoidal function.

**Fundamental component** The minimum frequency term required to represent a particular waveform in the Fourier series expansion.

**Half-wave (mirror) symmetry** A sinusoidal or nonsinusoidal function that satisfies the relationship

$$f(t) = -f\left(t + \frac{T}{2}\right)$$

**Harmonic terms** The terms of the Fourier series expansion that have frequencies that are integer multiples of the fundamental component.

**Nonsinusoidal waveform** Any waveform that differs from the fundamental sinusoidal function.

**Odd harmonics** The terms of the Fourier series expansion that have frequencies that are odd multiples of the fundamental component.

**Point symmetry** A sinusoidal or nonsinusoidal function that satisfies the relationship $f(\alpha) = -f(-\alpha)$.