9

Geometric Models

9.1 Introduction to Coordinate Systems

What is a coordinate system? In simple terms, a coordinate system provides a way to *locate* and *identify* points in the plane. In the usual *rectangular* or *Cartesian coordinate system,* every point can be pictured in either of two ways. First, a point P with coordinates (x_0, y_0) can be thought of as lying at the corner of a unique rectangle whose opposite corner is at the origin and two of whose sides lie along the two coordinate axes, as illustrated in Figure 9.1(a). The base of this rectangle is x_0, and its height is y_0. Second, the point P can be thought of as the intersection of two perpendicular lines, one parallel to the y-axis at a distance of x_0 from it and the other parallel to the x-axis at a height of y_0 from it, as illustrated in Figure 9.1(b).

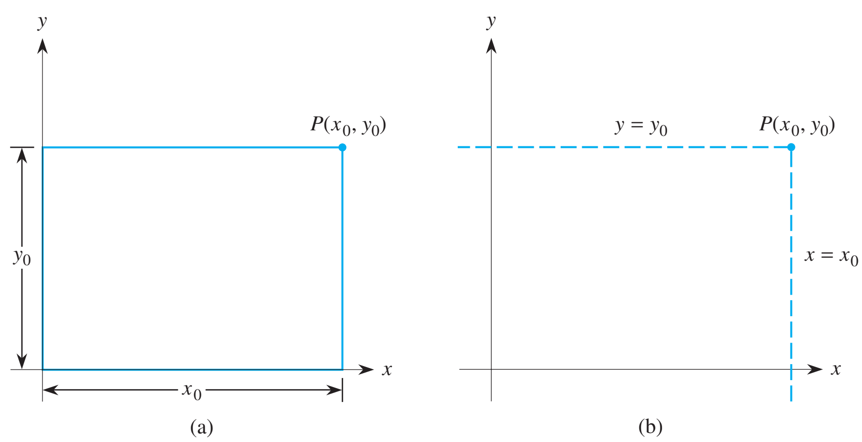


FIGURE 9.1

Mathematicians have found that, in many situations, rectangular coordinates are not the most natural or the most effective way to locate points and have developed alternative coordinate systems. One such approach involves the use of two axes that are not perpendicular, but rather meet at the origin at some angle other than a right angle. Points in such a slanted coordinate system can be located at the opposing vertex of a parallelogram, as illustrated in Figure 9.2.

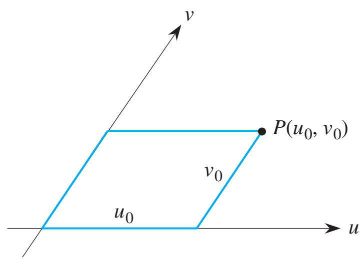


FIGURE 9.2

Another approach is to locate a point by using a circle centered at the origin O instead of a rectangle. To do so requires specifying both the radius of the circle and an angle θ to indicate where on the circle the point is located. This approach leads to the **polar coordinate system,** which is illustrated in Figure 9.3. We investigate this coordinate system in Sections 9.6 and 9.7.

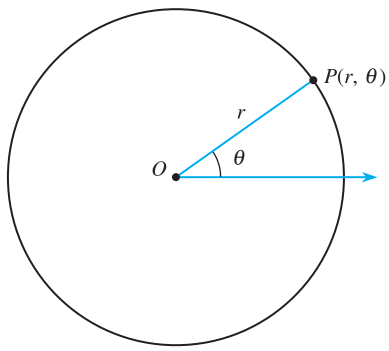


FIGURE 9.3

Other approaches are used for particular applications that involve locating points lying on some ellipse centered at the origin (an elliptic coordinate system), on some parabola (a parabolic coordinate system), or on a hyperbola (a hyperbolic coordinate system), as illustrated in Figures 9.4(a–c), respectively. In fact, the long range navigation (LORAN) system used by navigators in ships and planes to locate their positions is based on the fact that every point in a plane can be interpreted as lying at the intersection of two hyperbolas in a hyperbolic coordinate system.

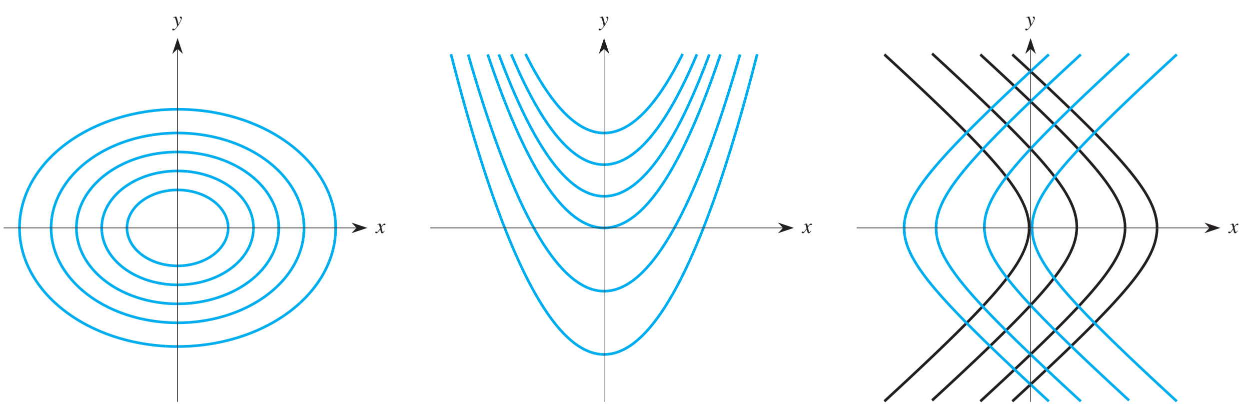


FIGURE 9.4

In this chapter, we first develop several special characteristics of the rectangular coordinate system and then show how to represent some extremely important curves. Later we explore other ways to represent functions.

9.2 Analytic Geometry

One of the most useful and far-reaching developments in mathematics is Rene Descartes's idea of representing algebraic concepts geometrically. This approach, known as **analytic geometry,** lets you visualize the mathematics graphically to complement the algebraic approach that is based on symbols. Everything we have done involving graphs of functions is an outgrowth of Descartes's ideas. In this section we examine some additional ideas involving points, lines, and circles in the plane.

We begin by considering the two points A at (x_0, y_0) and B at (x_1, y_1) in the plane. You already know how to find an equation of the line through them by using either the point–slope form

$$y - y_0 = m(x - x_0)$$

or the slope–intercept form

$$y = mx + b,$$

where the slope of the line is

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

Alternatively, we have the implicit form for the equation of a line,

$$ax + by = c,$$

where c/a and c/b represent the x- and the y-intercepts of the line, respectively.

Distance Between Points

We now ask: What is the distance between the points A at (x_0, y_0) and B at (x_1, y_1)? We write this distance as $|AB|$. Figure 9.5 shows that the points A and B determine a right triangle ABC; the coordinates of point C are (x_1, y_0) because C is at the same horizontal distance as B (measured from the y-axis) and at the same vertical height as A (measured from the x-axis). Moreover, the horizontal distance from A to C is $x_1 - x_0$; it is the change, or difference, in the x-coordinates. Similarly, the vertical distance from C to B is $y_1 - y_0$; it is the change in the y-coordinates. Consequently, the distance from A to B is the length of the hypotenuse of this right triangle. The Pythagorean theorem therefore gives us the **distance formula.**

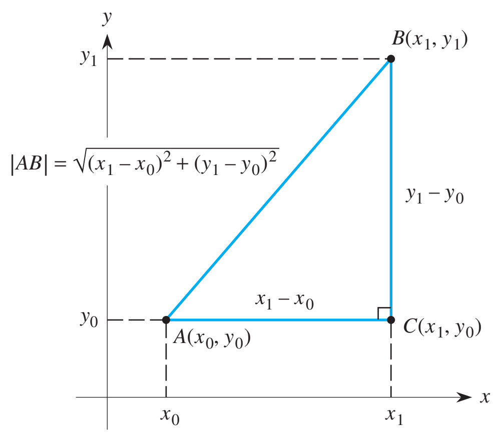


FIGURE 9.5

Distance Formula

The distance from point A at (x_0, y_0) to point B at (x_1, y_1) is

$$|AB| = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}.$$

EXAMPLE 1

Find the distance from the point A at $(2, 5)$ to the point B at $(6, 8)$.

Solution Applying the distance formula gives

$$|AB| = \sqrt{(6-2)^2 + (8-5)^2}$$
$$= \sqrt{16 + 19} = \sqrt{25} = 5 \text{ units}.$$

Consider again the two points A at (x_0, y_0) and B at (x_1, y_1) in the plane. Suppose that we want to determine the *midpoint* M of the line segment connecting A to B. Figure 9.6 shows that the points A and B determine a right triangle ABC and that the points A and M determine a smaller right triangle AMD. These two right triangles

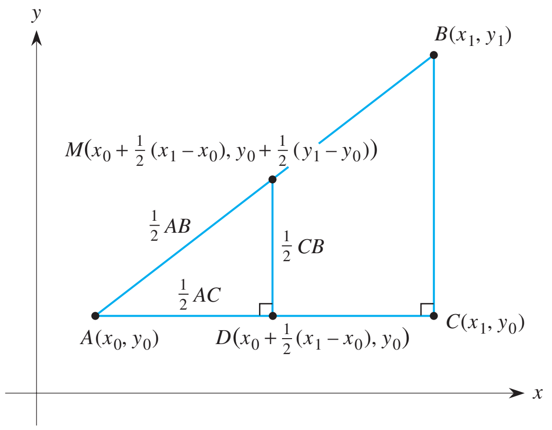


FIGURE 9.6

are similar, and hence their corresponding sides are proportional (see Appendix A4). Because M is halfway from A to B, we see that D is halfway from A to C, and the height DM is half the height CB. Thus the x-coordinate at D (and hence also at M) is

$$x = x_0 + \frac{1}{2}(x_1 - x_0).$$

Similarly, because the height DM is half the height CB, the y-coordinate at M is

$$y = y_0 + \frac{1}{2}(y_1 - y_0).$$

We can rewrite these expressions as

$$x_0 + \frac{1}{2}(x_1 - x_0) = x_0 + \frac{1}{2}x_1 - \frac{1}{2}x_0 = \frac{1}{2}(x_1 + x_0)$$

and

$$y_0 + \frac{1}{2}(y_1 - y_0) = y_0 + \frac{1}{2}y_1 - \frac{1}{2}y_0 = \frac{1}{2}(y_1 + y_0).$$

Thus the coordinates of the midpoint M of a line segment are simply the averages of the x-coordinates and the y-coordinates of the endpoints, respectively.

Midpoint Formula

The midpoint M of the line segment from A at (x_0, y_0) to B at (x_1, y_1) is at

$$x = x_0 + \frac{1}{2}(x_1 - x_0), \qquad y = y_0 + \frac{1}{2}(y_1 - y_0)$$

or

$$x = \frac{x_1 + x_0}{2}, \quad y = \frac{y_1 + y_0}{2}.$$

EXAMPLE 2

Find the midpoint of the line segment joining A at $(1, 11)$ and B at $(3, 7)$.

Solution The coordinates of the midpoint are

$$\begin{aligned} x &= x_0 + \frac{1}{2}(x_1 - x_0) \\ &= 1 + \frac{1}{2}(3 - 1) = 1 + 1 = 2 \end{aligned}$$

and

$$\begin{aligned} y &= y_0 + \frac{1}{2}(y_1 - y_0) \\ &= 11 + \frac{1}{2}(7 - 11) = 11 + \frac{1}{2}(-4) = 9. \end{aligned}$$

Alternatively,

$$x = \frac{x_1 + x_2}{2} = \frac{3 + 1}{2} = 2 \quad \text{and} \quad y = \frac{y_1 + y_2}{2} = \frac{7 + 11}{2} = 9.$$

Figure 9.7 shows the solution.

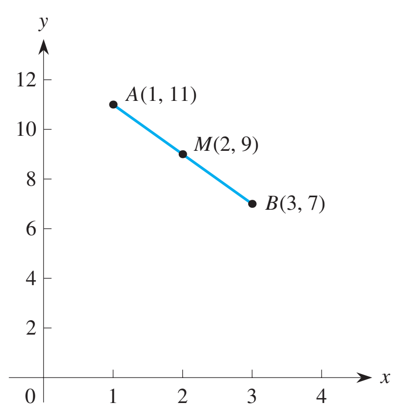


FIGURE 9.7

We might also want to determine a point at some other fraction of the distance from A to B. To do so, we simply extend the preceding argument to determine a point P at any distance from A to B. Suppose that we want the point one-quarter of the way from A to B. We then have

$$x = x_0 + \frac{1}{4}(x_1 - x_0) \quad \text{and} \quad y = y_0 + \frac{1}{4}(y_1 - y_0).$$

Think About This

Verify that this quarter-distance formula is correct by using an argument comparable to the one used for the midpoint formula.

In general, if we want the point P at a fraction t of the distance from A to B, it will be located at

$$x = x_0 + t\cdot(x_1 - x_0)$$
$$y = y_0 + t\cdot(y_1 - y_0),$$

as shown in Figure 9.8. Incidentally, if $t > 1$, we get a point on the line *beyond* B, and if $t < 0$, we get a point on the line *before* A.

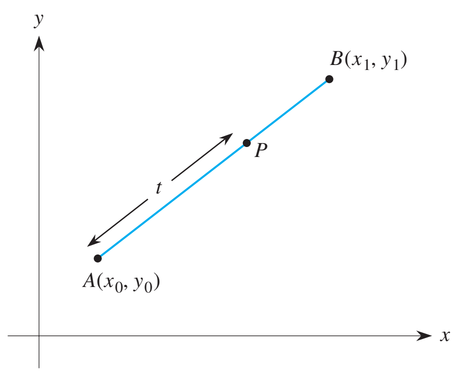


FIGURE 9.8

EXAMPLE 3

a. Find the point P located three-fifths of the way from A at $(-1, 3)$ to B at $(4, 13)$.

b. Find the point Q located seven-fifths of the way from A to B.

Solution

a. For P with $t = 3/5$,

$$x = x_0 + t\cdot(x_1 - x_0) = -1 + \frac{3}{5}[4 - (-1)] = 2 \quad \text{and}$$

$$y = y_0 + t\cdot(y_1 - y_0) = 3 + \frac{3}{5}(13 - 3) = 9.$$

Verify by plotting the points $(-1, 3)$, $(4, 13)$, and $(2, 9)$ that P is located three-fifths the way from A to B.

b. Similarly, for Q with $t = 7/5$,

$$x = x_0 + t\cdot(x_1 - x_0) = -1 + \frac{7}{5}[4 - (-1)] = 6 \quad \text{and}$$

$$y = y_0 + t\cdot(y_1 - y_0) = 3 + \frac{7}{5}(13 - 3) = 17.$$

Plot Q at $(6, 17)$ and note that this point lies on the line through A and B and is beyond B.

We could continue this example with different values of t to find other points on the line. In fact, every value of t determines a unique point on the line joining A at (x_0, y_0) and B at (x_1, y_1). Therefore the *two* equations for x and y give us a different way of representing the line. They are known as a *parametric representation* or *parametric equations* of the line, and the quantity t is called a *parameter.*

Parametric equations of the line through (x_0, y_0) and (x_1, y_1) are

$$x = x_0 + (x_1 - x_0)t \quad \text{and}$$
$$y = y_0 + (y_1 - y_0)t.$$

Note that this parametric form involves two interrelated equations for the line, not a single equation as in the point–slope form. It is possible to eliminate the parameter t to produce a single equation for the line. We ask you to do so in the Problems at the end of this section. However, the parametric form can provide valuable information.

The Equation of a Circle

We apply the concept of distance between two points in the plane to define a circle. A **circle** is the set of all points in the plane at a fixed distance from a fixed point. The fixed distance is called the **radius,** and the fixed point is called the **center.**

This definition allows us to find a general equation for any circle. Let r be the radius and let point C with coordinates (x_0, y_0) be the center of a circle. A point P with coordinates (x, y) lies on this circle provided that its distance from the center C is r, as shown in Figure 9.9. The distance formula gives the equivalent expression

$$|CP| = \sqrt{(x - x_0)^2 + (y - y_0)^2} = r.$$

We can eliminate the square root in this equation by squaring both sides and thus obtain the *standard form* for the equation of a circle.

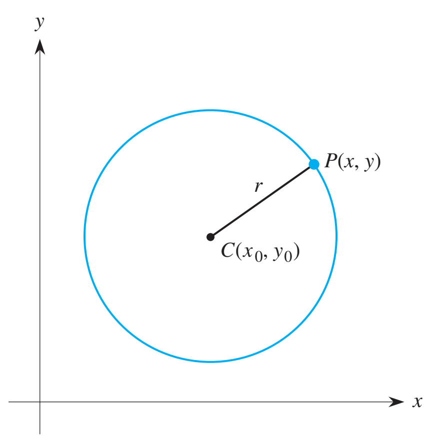


FIGURE 9.9

The equation of the circle with radius r centered at (x_0, y_0) is

$$(x - x_0)^2 + (y - y_0)^2 = r^2.$$

For instance, the circle of radius 8 centered at $(5, 2)$ has the equation

$$(x - 5)^2 + (y - 2)^2 = 8^2 = 64,$$

whereas the equation of the circle of radius 3 centered at $(-5, 0)$ is

$$(x + 5)^2 + y^2 = 9.$$

As a special case, the equation of a circle of radius r centered at the origin is

$$x^2 + y^2 = r^2.$$

Note that the equation of a circle does not represent a function. Picture any vertical line that passes through the circle but is not tangent to the circle—it intersects the circle twice and so the circle fails the vertical line test. That is, each such value of x has two corresponding values of y, which violates the definition of a function, as shown in Figure 9.10.

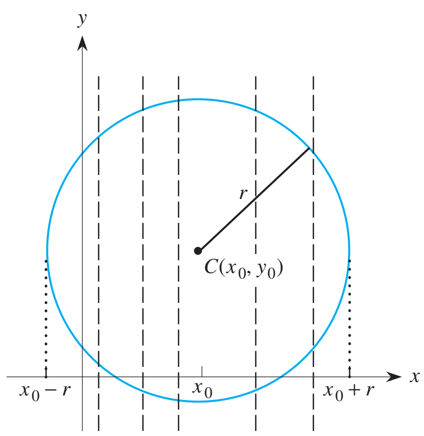


FIGURE 9.10

We get a similar result algebraically. For example, the circle of radius 10 centered at the origin has the equation

$$x^2 + y^2 = 100.$$

If we select $x = 6$, say, then

$$36 + y^2 = 100 \quad \text{so that} \quad y^2 = 64,$$

which has the solutions

$$y = 8 \quad \text{or} \quad y = -8.$$

Again, two values of y correspond to one value of x. Even though a circle does not represent a function, it is nonetheless a very important curve. We discuss several other curves that do not represent functions in Sections 9.3 and 9.4.

Let's start with the equation of the circle of radius 8 centered at $(5, 2)$:

$$(x - 5)^2 + (y - 2)^2 = 8^2 = 64.$$

We expand the left-hand side and combine like terms to get

$$x^2 - 10x + 25 + y^2 - 4y + 4 = 64$$

or

$$x^2 + y^2 - 10x - 4y - 35 = 0,$$

which is an equivalent, although different, representation for the same circle. Clearly, we could do the same with the equation of any circle,

$$(x - x_0)^2 + (y - y_0)^2 = r^2,$$

which is centered at (x_0, y_0) with radius r. Expanding the left-hand side, we get

$$x^2 - 2x_0x + x_0^2 + y^2 - 2y_0y + y_0^2 = r^2$$

or, equivalently,

$$x^2 + y^2 - 2x_0x - 2y_0y + x_0^2 + y_0^2 - r^2 = 0.$$

Because x_0, y_0, and r are constants, we can write this equation in the alternative form

$$x^2 + y^2 + Cx + Dy + E = 0,$$

where we have introduced the new constants

$$C = -2x_0, \qquad D = -2y_0, \quad \text{and} \quad E = x_0^2 + y_0^2 - r^2.$$

Such an equation, known as the **general equation of the circle,** represents a circle for any choice of constants C, D, and E such that the radius of the circle is positive.

Suppose that we start with an equation such as

$$x^2 + y^2 - 10x - 4y - 35 = 0,$$

which we know from the preceding derivation is the equation of a circle. Examples 4 and 5 demonstrate how to work backward from this equation to determine the center and radius of the circle.

EXAMPLE 4

Show that

$$x^2 + y^2 - 10x - 4y - 35 = 0$$

is the equation of a circle by finding its center and radius.

Solution To solve this problem, we use the technique of *completing the square* (see Appendix A8) in both the x- and y-terms on the left-hand side and obtain

$$\begin{aligned} x^2 + y^2 - 10x - 4y - 35 &= (x^2 - 10x) + (y^2 - 4y) - 35 \\ &= [(x^2 - 10x + 25) - 25] + [(y^2 - 4y + 4) - 4] - 35 \\ &= [(x - 5)^2 - 25] + [(y - 2)^2 - 4] - 35 \\ &= (x - 5)^2 + (y - 2)^2 - 64 = 0. \end{aligned}$$

Adding 64 to both sides, we get

$$(x - 5)^2 + (y - 2)^2 = 64 = 8^2.$$

This is the equation of the circle with radius 8 and center at (5, 2).

EXAMPLE 5

Find the radius and the center of the circle whose equation is

$$x^2 + y^2 + 8x - 10y - 8 = 0.$$

Solution To solve this problem, we again complete the square in both the x- and y-terms on the left-hand side and obtain

$$\begin{aligned} x^2 + y^2 + 8x - 10y - 8 &= (x^2 + 8x) + (y^2 - 10y) - 8 \\ &= [(x^2 + 8x + 16) - 16] + [(y^2 - 10y + 25) - 25] - 8 \\ &= [(x + 4)^2 - 16] + [(y - 5)^2 - 25] - 8 \\ &= (x + 4)^2 + (y - 5)^2 - 49. \end{aligned}$$

Therefore the original equation becomes

$$(x + 4)^2 + (y - 5)^2 - 49 = 0.$$

Adding 49 to both sides, we get

$$(x + 4)^2 + (y - 5)^2 = 49 = 7^2,$$

which is the equation of a circle with radius 7 and center at $(-4, 5)$.

Note that not every equation of the form $x^2 + y^2 + Cx + Dy + E = 0$ represents the equation of a circle. In Examples 4 and 5, the constant term we ended up with on the right, either 64 or 49, was a positive number, so we could take the square root to get a radius. However, the constant on the right can be a negative number, in which case the equation does not represent a circle. In fact, even if the constant on the right is 0, we would not have a circle because the radius would be 0. We ask you to investigate such cases in the Problems at the end of this section.

EXAMPLE 6

In Example 4 of Section 4.6, we found two points on the line through the Earth and the moon at which the gravitational forces of the Earth and the moon on a spacecraft are exactly equal in size numerically. One point is 216 thousand miles from the Earth toward the moon and the other is 270 thousand miles from the Earth, which is 30 thousand miles beyond the moon. Find all such points in the plane containing the Earth, the moon, and the sun.

Solution We set up a coordinate system with the Earth at the origin and the moon on the horizontal axis, 240 thousand miles to the right. Suppose that the two gravitational forces are numerically equal at some other point P in the plane with coordinates (x, y), as shown in Figure 9.11. (Technically, we should consider not only the size of the two forces but also the directions in which they are exerted; that requires the notion of a vector quantity, which we discuss in Chapter 10.)

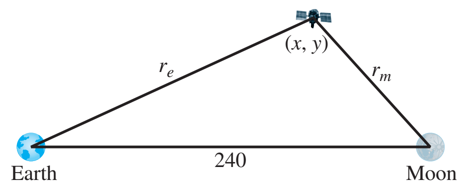


FIGURE 9.11

The distance formula gives the distance r_e from the Earth to the point P as

$$r_e = \sqrt{x^2 + y^2} \quad \text{so that} \quad r_e^{\,2} = x^2 + y^2.$$

Similarly, the distance r_m from the moon to the point P is

$$r_m = \sqrt{(x - 240)^2 + y^2} \quad \text{so that} \quad r_m^{\,2} = (x - 240)^2 + y^2.$$

Let m_0 be the mass of the spacecraft, m_1 be the mass of the Earth, and m_2 be the mass of the moon. Because the Earth is 81 times as massive as the moon, $m_1 = 81m_2$. Based on the universal law of gravitation, the size of the gravitational force F_e that the Earth exerts on the spacecraft is

$$F_e = \frac{Gm_0m_1}{r_e^2} = \frac{Gm_0 \cdot (81m_2)}{r_e^2} = \frac{81Gm_0m_2}{r_e^2}$$

and the size of the gravitational force F_m that the moon exerts on the spacecraft is

$$F_m = \frac{Gm_0m_2}{r_m^2}.$$

Equating these two expressions yields

$$\frac{81Gm_0m_2}{r_e^2} = \frac{Gm_0m_2}{r_m^2}.$$

Dividing both sides by the constant Gm_0m_2 gives

$$\frac{81}{r_e^2} = \frac{1}{r_m^2},$$

or cross-multiplying,

$$81r_m^2 = r_e^2.$$

We substitute the expressions for the two distances to get

$$81[(x - 240)^2 + y^2] = x^2 + y^2,$$

or

$$81[x^2 - 480x + (240)^2 + y^2] = x^2 + y^2$$
$$81x^2 - 81(480)x + 81(240)^2 + 81y^2 = x^2 + y^2.$$

Collecting like terms gives

$$80x^2 - 38{,}880x + 81(240)^2 + 80y^2 = 0.$$

Dividing through by the common factor 80 yields

$$x^2 - 486x + 58{,}320 + y^2 = 0,$$

which suggests the equation of a circle. We complete the square on the x terms on the left-hand side and get

$$x^2 - 486x + 58{,}320 + y^2 = x^2 - 486x + \left(\frac{486}{2}\right)^2 - \left(\frac{486}{2}\right)^2 + 58{,}320 + y^2 = 0.$$

Therefore

$$(x - 243)^2 - (243)^2 + 58{,}320 + y^2 = 0,$$

or

$$(x - 243)^2 + y^2 = (243)^2 - 58{,}320 = 729.$$

That is, in the plane formed by the Earth, the moon, and the sun, the size of the gravitational forces from the Earth and the moon are numerically equal at every point on a circle of radius $\sqrt{729} = 27$ thousand miles centered at a distance of 243 thousand miles from the Earth on a line through the moon. In fact, the center of this circle is just beyond the moon.

Incidentally, even though a circle does not fulfill the requirements of a function, if we restrict our attention to either its upper half or its lower half, the resulting semicircle does represent a function. For the circle $x^2 + y^2 = r^2$, we obtain the equations for the semicircles by solving for y:

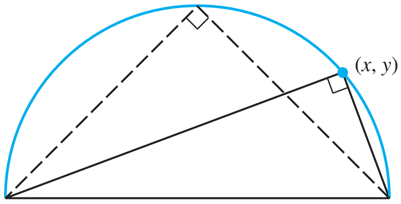
$$y^2 = r^2 - x^2 \quad \text{so that} \quad y = \pm\sqrt{r^2 - x^2}.$$

Thus the upper semicircle is the graph of the function $y = f(x) = \sqrt{r^2 - x^2}$, and the lower semicircle is the graph of the function $y = g(x) = -\sqrt{r^2 - x^2}$.

Problems

In Problems 1–6, find the distance between each pair of points.

1. $(2, 4)$ and $(5, 8)$
2. $(2, 4)$ and $(7, 16)$
3. $(4, -1)$ and $(0,4)$
4. $(2, -5)$ and $(0, 4)$
5. $(-1, 5)$ and $(3, 7)$
6. $(3, 1)$ and $(-5, -4)$
7. Find the midpoint of the line segment joining the points in Problem 1.
8. Find the midpoint of the line segment joining the points in Problem 2.
9. Find the point one-third the way from the first point to the second point in Problem 3.
10. Find the point three-fourths the way from the first point to the second point in Problem 4.
11. Find the equation of the circle that has center $(5, 2)$ and passes through the point P at $(8, -2)$.
12. Find the equation of the circle that has center $(-3, 7)$ and passes through the point P at $(2, -5)$.
13. Find the equation of the circle that has $(2, 4)$ and $(10, 4)$ as the endpoints of a diameter.
14. Find the equation of the circle that has $(-2, 3)$ and $(4, 11)$ as the endpoints of a diameter.
15. Repeat Problems 13 and 14, using the facts that any angle inscribed in a semicircle is a right angle and that perpendicular lines have slopes that are negative reciprocals.



In Problems 16–21, complete the square in both the x- and y-terms for each equation to obtain the standard form for the equation of a circle. Use it to determine the radius and center of the circle. Then draw the graph of the circle.

16. $x^2 + y^2 + 4x + 6y = 3$
17. $x^2 + y^2 + 4x + 6y = 12$
18. $x^2 + y^2 + 10x - 4y = 7$
19. $x^2 + y^2 + 10x - 4y = 71$
20. $x^2 + y^2 - 2x + 6y = -9$
21. $x^2 + y^2 - 2x + 6y + 6 = 0$
22. Determine which of the following equations represent a circle and which do not. Explain.
    a. $x^2 + y^2 - 4x - 6y + 15 = 0$
    b. $x^2 + y^2 - 4x - 6y + 13 = 0$
    c. $x^2 + y^2 - 4x - 6y + 12 = 0$
23. The equations $x = 3 + 2t$, $y = 4 - 5t$ form a parametric representation of a line.
    a. Construct a table of values for x and y corresponding to $t = -2, -1, 0, 1, \ldots, 5$.
    b. Plot these points and verify that they do seem to lie on a line.
    c. What is the slope of this line?
    d. What is a point–slope form for the equation of this line using $t = 1$?
    e. Use the midpoint formula to find the midpoint of each of the consecutive line segments determined by the entries in your table from part (a). Then use the parametric representation of the line with $t = -1.5, -0.5, 0.5, 1.5, 2.5, 3.5$, and 4.5. How do the results compare? Explain.
24. Consider again the parametric representation of the line $x = 3 + 2t$, $y = 4 - 5t$ in Problem 23. Eliminate the parameter t from the two equations by first solving the first equation for t in terms of x and then substituting the result into the second equation. How does this result compare to the result obtained in part (d) of Problem 23?
25. Start with the general parametric equations of a line

$$x = x_0 + (x_1 - x_0)t, \quad y = y_0 + (y_1 - y_0)t$$

and algebraically eliminate the parameter t. Identify the equation you produce.

26. **a.** Find the slope of the line having the parametric representation

$$x = 1 + 2t, \quad y = 2 - 3t.$$

b. Sketch the graph of this line.

27. Find the points at which the line $x = 1 + 2t$, $y = 2 - t$ intersects the circle $x^2 + y^2 = 25$. (*Hint*: First find values of the parameter t that satisfy the equation of the circle.)

28. The three points P at $(0, 2)$, Q at $(2, 4)$, and R at $(4, 0)$ are noncollinear and as such determine a circle. Find an equation of this circle. (*Hint*: Substitute the coordinates of each point into the general equation of the circle, $x^2 + y^2 + Cx + Dy + E = 0$, and then solve the resulting system of three equations in three unknowns.)

Exercising Your Algebra Skills

In Problems 1–8, complete the square for each expression.

1. $x^2 + 8x + 25$
2. $x^2 - 8x + 25$
3. $x^2 - 6x + 5$
4. $x^2 + 6x + 5$
5. $y^2 + 10y + 26$
6. $y^2 - 10y + 26$
7. $y^2 + 4y - 12$
8. $y^2 - 4y - 12$

9.3 Conic Sections: The Ellipse

When we introduced functions and their graphs in Chapter 1, we said that not every graph represents a function. In Section 9.2 we pointed out that a circle is not a function. Several other important curves that have useful and interesting properties similarly are not functions. We investigate some of these curves in this section and Section 9.4.

Consider any equation of the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A, B, C, D, and E are constants, provided that at least one of A and B is not 0. In particular, if $A = B$, we can divide through by this constant, so that the result is the equation of a circle if the coefficients lead to a positive radius. Let's see what happens when $A \neq B$.

The graph of any equation of the form $Ax^2 + By^2 + Cx + Dy + E = 0$ is known as a **conic section.** To see why, consider a slice through the double right circular cone shown in Figure 9.12. If the slicing plane is horizontal, each slice is a *circle.* However, if the slicing plane is inclined slightly from the horizontal, the curve produced is oval in shape, rather than circular, and is an *ellipse.* (Imagine a diagonal slice through a round salami.) In fact, the sharper the angle of the slice, the more elongated the ellipse will be, as shown in Figure 9.13. If the angle of slicing is increased further so that it is parallel to the "edge" of the cone, the resulting curve is a *parabola,* as shown in Figure 9.14. If the angle of the slice is increased still further, the slicing plane intersects both the upper and lower parts of the cone and produces a pair of separated curves, known as a *hyperbola,* as shown in Figure 9.15.

In summary, there are three types of conic sections: (1) the ellipse, (2) the parabola, and (3) the hyperbola. The circle is a special case of the ellipse.

The conic sections occur often in various applications of mathematics and science. For instance, the orbits of the planets about the sun are ellipses. The paths of many comets and meteoroids are hyperbolic. A cross section of the metallic reflector inside a flashlight or an automobile headlight is a parabola. The path of a

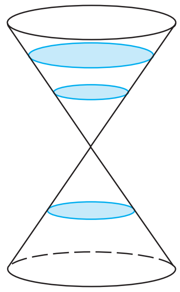


FIGURE 9.12
Slicing planes are horizontal

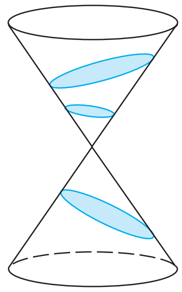


FIGURE 9.13
Slicing planes are at slight angles

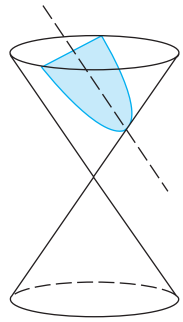


FIGURE 9.14
Slicing planes parallel to the side of the cone

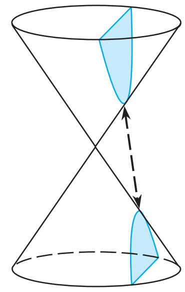


FIGURE 9.15
Slicing planes at steeper angles

thrown object, such as a perfect "spiral" pass in football or a "line drive" in baseball, is also a parabola.

Although we typically use formulas when working with the conic sections, we define them formally from a purely geometric perspective. This approach is analogous to the way we defined a circle in Section 9.2 as the set of all points at a fixed distance from a single fixed point, its center. In this section, we study the ellipse and consider the hyperbola and the parabola in Section 9.4.

The Ellipse

An **ellipse** is defined as the set of all points in the plane for which the sum of the distances to two fixed points is a constant. The two fixed points are called the **foci** (the plural of *focus*) of the ellipse. The midpoint of the line segment joining the foci is the **center** of the ellipse.

When the two foci are far apart, the resulting ellipse is very elongated. When the two foci are close together, the ellipse is close to circular and, in fact, when the two foci merge into a single point, the ellipse is a circle.

For convenience, we assume that the center of an ellipse is at the origin and that the two foci lie on the x-axis. Suppose that the foci are at F_1 with coordinates $(c, 0)$ and at F_2 with coordinates $(-c, 0)$, as illustrated in Figure 9.16. A point P

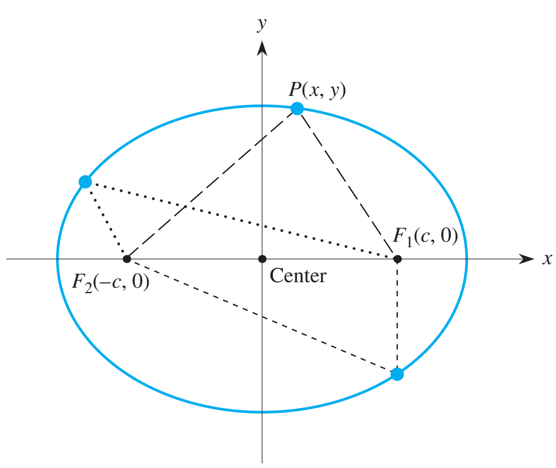


FIGURE 9.16

with coordinates (x, y) lies on the ellipse if the sum of the two distances $|F_1P|$ and $|F_2P|$ is some constant k. To make things easier, we write $k = 2a$. That is,

$$|F_1P| + |F_2P| = 2a,$$

or, equivalently,

$$\sqrt{(x - c)^2 + (y - 0)^2} + \sqrt{(x + c)^2 + (y - 0)^2} = 2a$$

or

$$\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a.$$

When we simplify this equation by eliminating both square roots (we leave the actual simplification for you to do as a problem at the end of this section), we eventually obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as the equations of the ellipse, where $b^2 = a^2 - c^2$ is a new constant. The three constants a, b, and c are related by the equation

$$a^2 = b^2 + c^2, \qquad \text{where } a > b.$$

We investigate the meaning of the constants a and b below.

To determine where the ellipse intersects the x-axis, we set $y = 0$. The equation of the ellipse then reduces to

$$\frac{x^2}{a^2} = 1 \quad \text{so that} \quad x^2 = a^2$$

from which we find that either

$$x = a \quad \text{or} \quad x = -a.$$

This result indicates that a represents the distance from the center of the ellipse to the two points where the ellipse crosses the x-axis, as illustrated in Figure 9.17. Similarly, if $x = 0$, the equation of the ellipse yields $y^2 = b^2$, from which either

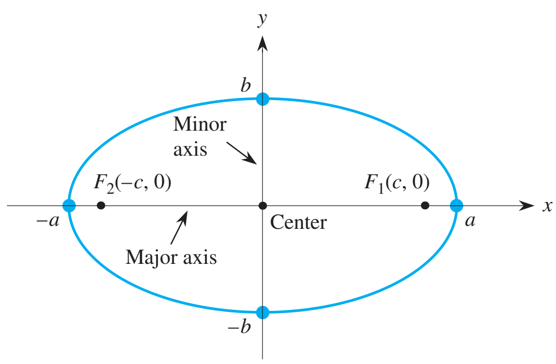


FIGURE 9.17

$$y = b \quad \text{or} \quad y = -b.$$

Thus b represents the distance from the center of the ellipse to the two points where the ellipse crosses the y-axis. The four points, $(a, 0)$, $(-a, 0)$, $(0, b)$, and $(0, -b)$, are the **vertices** of the ellipse; any one of them is a *vertex*. The lines connecting opposite vertices are called the **axes** of the ellipse. The longer axis, whether horizontal or vertical, is called the *major axis* and always contains the two foci; the shorter axis is called the *minor axis.*

In summary, a represents the distance from the center to either of the two more distant vertices along the major axis of the ellipse; b represents the distance from the center to either of the two closer vertices along the minor axis; and c represents the distance from the center to either focus of the ellipse. Because a is half the length of the major axis, we sometimes call a the length of a *semi-major axis*. Similarly, b is sometimes called the length of a *semi-minor axis*.

EXAMPLE 1

Describe and sketch the graph of the ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

Solution This ellipse is centered at the origin. Its vertices occur when $y = 0$, so $x = \pm 4$, or when $x = 0$, so $y = \pm 3$. Therefore $a = 4$ and $b = 3$, so that the major axis extends horizontally from $x = -4$ to $x = 4$, and the minor axis extends vertically from $y = -3$ to $y = 3$. See Figure 9.18. Because a, b, and c are related by $a^2 = b^2 + c^2$ we have

$$c^2 = a^2 - b^2 = 16 - 9 = 7,$$

so that $c = \pm\sqrt{7}$. Therefore the foci are located at $(\sqrt{7}, 0)$ and $(-\sqrt{7}, 0)$, giving the graph shown in Figure 9.18.

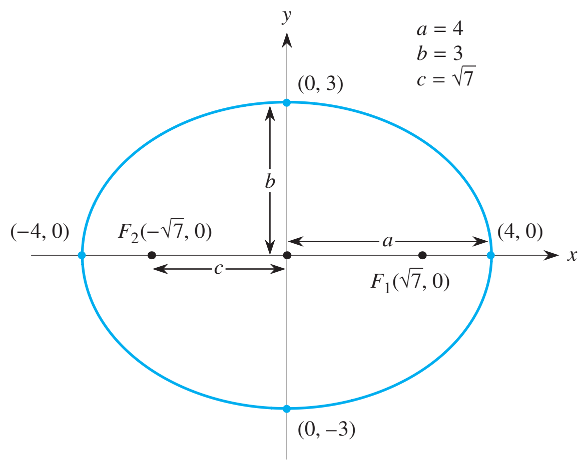


FIGURE 9.18

So far, we have considered an ellipse centered at the origin with foci along the x-axis. If we consider the analogous situation where the foci lie along the y-axis, the resulting equation for such an ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \qquad a > b.$$

Note that the constant a is still measured along the major axis and b is measured along the minor axis of the ellipse, so $a > b$. From the equation of an ellipse, we can identify its major axis immediately by observing which of the two denominators is larger.

We have only considered ellipses that are centered at the origin. In fact, an ellipse can be centered at any point (x_0, y_0). In such a case, we get the following **standard forms for the equation of an ellipse.**

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The equation of an ellipse centered at (x_0, y_0) with its major axis parallel to the x-axis is

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1.$$

The equation of an ellipse centered at (x_0, y_0) with its major axis parallel to the y-axis is

$$\frac{(x - x_0)^2}{b^2} + \frac{(y - y_0)^2}{a^2} = 1.$$

In each case,

$$a^2 = b^2 + c^2,$$

where c is the distance from the center to either focus.

EXAMPLE 2

Describe and sketch the ellipse whose equation is

$$\frac{(x - 2)^2}{4} + \frac{(y - 7)^2}{25} = 1.$$

Solution The center of this ellipse is at the point $(2, 7)$. Because $25 > 4$, the major axis is parallel to the y-axis. In particular, the major axis is on the vertical line $x = 2$, and the foci also lie on this line. The minor axis is on the horizontal line $y = 7$. Also, because $a^2 = 25$ and $b^2 = 4$, $a = 5$ and $b = 2$. Thus the length of the major axis is $2a = 10$, and the length of the minor axis is $2b = 4$. Consequently, the maximum horizontal distance from the center is 2 on either side of $x = 2$, and the maximum vertical distance from the center is 5 above and below $y = 7$. The ellipse therefore extends horizontally from $x = 0$ to $x = 4$ and extends vertically from $y = 2$ to $y = 12$. Figure 9.19 shows the graph of the ellipse.

To locate the foci, we use

$$c^2 = a^2 - b^2 = 25 - 4 = 21,$$

which gives $c = \sqrt{21}$. Because the foci are on the major axis of the ellipse, they are on the vertical line $x = 2$. Thus the foci are at the points $(2, 7 + \sqrt{21})$ and $(2, 7 - \sqrt{21})$.

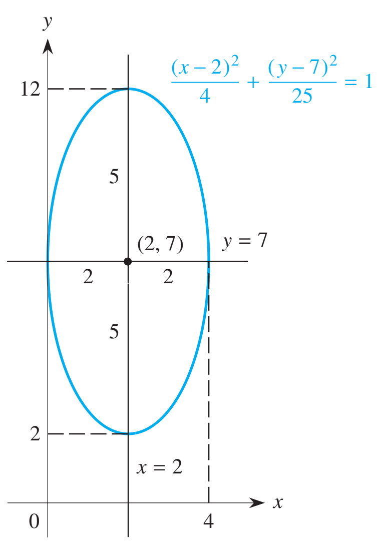


FIGURE 9.19

Suppose that we again consider the ellipse in Example 2,

$$\frac{(x-2)^2}{4} + \frac{(y-7)^2}{25} = 1.$$

We now multiply both sides of the equation by 100 to eliminate the fractions, so that

$$25(x-2)^2 + 4(y-7)^2 = 100.$$

Expanding the terms, we get

$$\begin{aligned} 25(x^2 - 4x + 4) + 4(y^2 - 14y + 49) &= 100, \\ 25x^2 - 100x + 100 + 4y^2 - 56y + 196 &= 100, \\ 25x^2 + 4y^2 - 100x - 56y + 196 &= 0. \end{aligned}$$

This last equation is an equivalent equation for the same ellipse. Often we start with such an equation and have to rewrite it algebraically to uncover the key information about the ellipse. We illustrate how to do so in Example 3.

EXAMPLE 3

Verify that the equation

$$25x^2 + 9y^2 - 50x - 36y - 164 = 0$$

represents an ellipse, and find its center, vertices, and foci. Use this information to sketch the ellipse.

Solution We first collect the terms in x and y separately and then factor out the coefficients of x^2 and y^2:

$$\begin{aligned} 25x^2 + 9y^2 - 50x - 36y - 164 &= 25x^2 - 50x + 9y^2 - 36y - 164 \\ &= [25(x^2 - 2x)] + [9(y^2 - 4y)] - 164. \end{aligned}$$

Finally, we complete the squares on both x and y to obtain

$$\begin{aligned} &25[(x^2 - 2x + 1) - 1] + 9[(y^2 - 4y + 4) - 4] - 164 \\ &\quad = 25[(x-1)^2 - 1] + 9[(y-2)^2 - 4] - 164 \\ &\quad = 25(x-1)^2 - 25 + 9(y-2)^2 - 36 - 164 \\ &\quad = 25(x-1)^2 + 9(y-2)^2 - 225. \end{aligned}$$

Therefore the original equation is equivalent to

$$25(x-1)^2 + 9(y-2)^2 - 225 = 0 \quad \text{or} \quad 25(x-1)^2 + 9(y-2)^2 = 225.$$

Dividing both sides by 225 yields

$$\frac{(x-1)^2}{9} + \frac{(y-2)^2}{25} = 1,$$

which is the standard form for the equation of an ellipse. The center is at $(1, 2)$. The major axis is vertical because $25 > 9$. Moreover, because $a = 5$, the major axis extends from $y = 2 - 5 = -3$ to $y = 2 + 5 = 7$. The minor axis is horizontal with $b = 3$, so it extends from $x = 1 - 3 = -2$ to $x = 1 + 3 = 4$. To find the foci, we solve

$$c^2 = a^2 - b^2 = 25 - 9 = 16,$$

which gives $c = 4$. Therefore the foci are 4 units above and below the center $(1, 2)$, so they are at $(1, -2)$ and $(1, 6)$. The graph of this ellipse is shown in Figure 9.20.

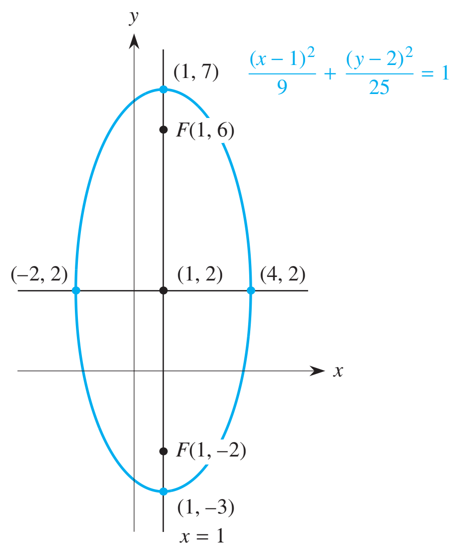


FIGURE 9.20

At the beginning of the section, we pointed out that the orbits of the planets about the sun are ellipses. More specifically, these elliptical orbits all have the sun as one of their two foci. A natural question to ask is: What is the equation of the ellipse for the orbit of the Earth? To answer it, we need two pieces of data used by astronomers to describe the orbits of the planets. The *perihelion* is the smallest distance from a planet to the sun, and the *aphelion* is the greatest distance, as depicted in Figure 9.21. For the Earth, the perihelion is approximately 147.1 million kilometers, or 91.38 million miles, and the aphelion is approximately 152.1 million kilometers, or 94.54 million miles. These two distances help identify the location of the sun on the major axis of Earth's elliptical orbit.

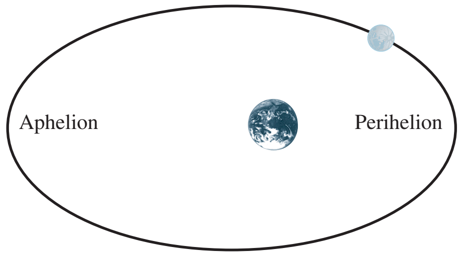


FIGURE 9.21

EXAMPLE 4

Find an equation of the Earth's orbit about the sun.

Solution We first set up a coordinate system with the sun, the other (phantom) focus, and the major axis on the x-axis, as shown in Figure 9.22. Because the perihelion and aphelion distances are almost the same, the two foci are quite close together and the orbit of the Earth is nearly circular. From Figure 9.22, we see that the distance from one vertex to the other, or $2a$, is $91.38 + 94.54 = 185.92$ million miles, so

$$a = 185.92/2 = 92.96 \text{ million miles.}$$

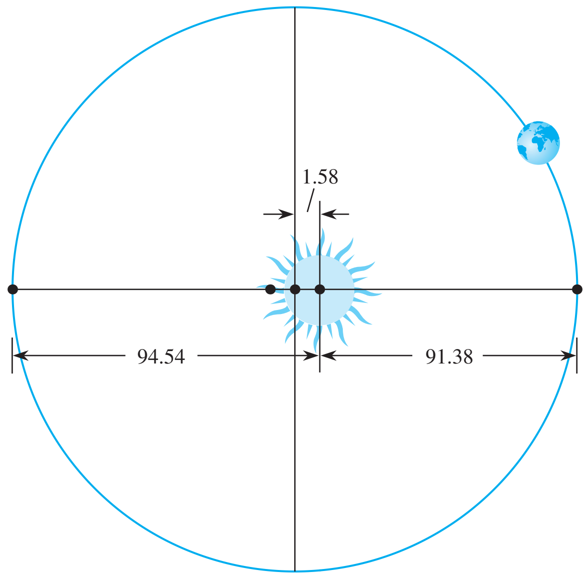


FIGURE 9.22

To find b, we first have to determine c, using the following reasoning. At perihelion, the Earth is 91.38 million miles from the sun, so the distance from the center of the ellipse to the sun (a focus) must be

$$c = 92.96 - 91.38 = 1.58 \text{ million miles.}$$

From $a^2 = b^2 + c^2$, we have

$$\begin{aligned} b^2 &= a^2 - c^2 \\ &= (92.96)^2 - (1.58)^2 = 8639.07, \end{aligned}$$

so that

$$b = \sqrt{8639.07} = 92.95 \text{ million miles.}$$

Consequently, the equation of the Earth's orbit about the sun is

$$\frac{x^2}{(92.96)^2} + \frac{y^2}{(92.95)^2} = 1.$$

As we observed previously, the Earth's orbit is very nearly circular.

The table of planetary data on the following page lists the perihelion and aphelion distances, in millions of miles, for the planets in the solar system. You can use it to compare the Earth's orbit to that of the other planets. You will use some of these entries for the Problems at the end of the section.

Reflection Property of the Ellipse

One of the most fascinating properties of an ellipse is known as the **reflection property.** Consider any line segment emanating from one of the two foci—say, F_1—as shown in Figure 9.23. It eventually intersects the ellipse and then reflects.

Planet	Perihelion	Aphelion
Mercury	28.56	43.38
Venus	66.74	67.68
Earth	91.38	94.54
Mars	128.49	154.83
Jupiter	460.43	506.87
Saturn	837.05	936.37
Uranus	1700.07	1867.76
Neptune	2771.72	2816.42
Pluto	2749.57	4582.61

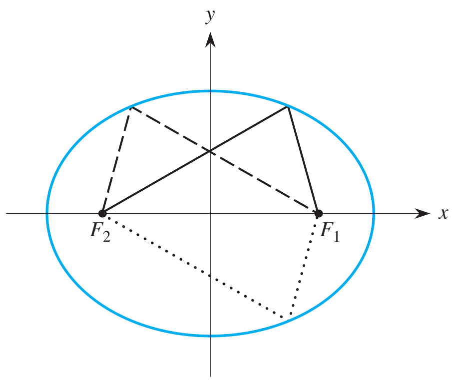


FIGURE 9.23

(According to physical principles, the angle of incidence with a tangent line equals the angle of reflection.) *Any* such reflected line segment will pass through the second focus F_2.

This property is significant because many physical phenomena, such as light and sound, travel in straight lines and reflect off solid surfaces. Thus, if a lightbulb is placed at one focus of a three-dimensional shell whose cross sections containing the major axis are all ellipses, all its light rays will bounce off the inside surface of the shell and reflect back through the other focus. The effect is similar with sound waves. Probably the best known example of this is the whispering gallery effect in the U.S. Capitol in Washington, D.C. The dome of the Capitol has the approximate shape of a three-dimensional ellipse, and there are two foci near floor level. If you stand at one of the foci and whisper, your voice is carried to the second focus across the hall and can be heard clearly by anyone standing there.

EXAMPLE 5

The distance between the foci in the "whispering gallery" of the Capitol is 38.5 feet, and the maximum height of the ceiling above ear level is 37 feet. Find the equation of an elliptical cross section of the gallery under the Capitol dome.

Solution We set up axes as shown in Figure 9.24. Because the distance between foci is 38.5 feet, $c = \frac{1}{2}(38.5) = 19.25$ feet. Also, from the maximum height of the dome, $b = 37$ feet. For an ellipse, we know that

$$a^2 = b^2 + c^2 = (37)^2 + (19.25)^2 = 1739.5625,$$

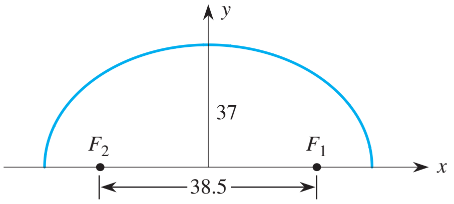


FIGURE 9.24

so $a \approx 41.7$ feet. Therefore the equation of an elliptical cross section of the Capitol whispering gallery is

$$\frac{x^2}{(41.7)^2} + \frac{y^2}{(37)^2} = 1.$$

The Average Distance from the Sun

We have stated that the orbit of each planet is an ellipse with one focus at the sun. A natural question to ask is: What is the *average* distance of a planet from the sun during a full orbit? The answer is particularly simple, yet surprising, as we demonstrate in Example 6.

EXAMPLE 6

Show that the average distance from all points on any ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is precisely equal to a, the length of the semi-major axis.

Solution We begin with the ellipse shown in Figure 9.25, with foci at F_1 and F_2. Let P_1 be any point on the right half of the ellipse. From the geometric definition of the ellipse, we know that the sum of the two distances $|F_1P_1|$ and $|F_2P_1|$ must equal the constant $2a$, or

$$|F_1P_1| + |F_2P_1| = 2a.$$

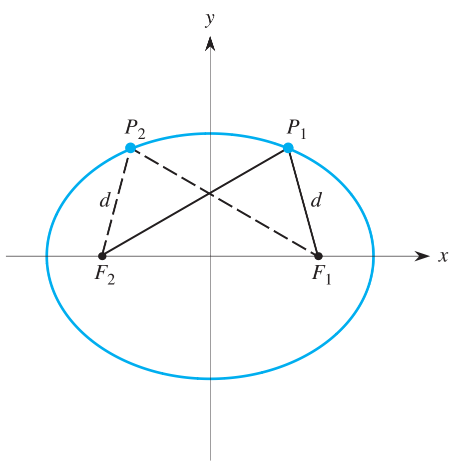


FIGURE 9.25

Using the symmetry of the ellipse, there is a comparable point P_2 on the left half of the ellipse so that

$$|F_2P_2| = |F_1P_1| \quad \text{and} \quad |F_2P_1| = |F_1P_2|.$$

Therefore

$$|F_1P_1| + |F_1P_2| = |F_1P_1| + |F_2P_1| = 2a.$$

Hence the average of these two distances from F_1 to P_1 and from F_1 to P_2 is simply a. This argument can be applied to every possible pair of matching points on the ellipse, so the average distance from F_1 to *all* points on the ellipse must be a.

Now let's apply this result to the orbits of the planets. For instance, the aphelion distance for the Earth is 94.54 million miles and the perihelion distance is 91.38 million miles. These distances can be expressed as

$$\text{Perihelion} = a - c \quad \text{and} \quad \text{Aphelion} = a + c.$$

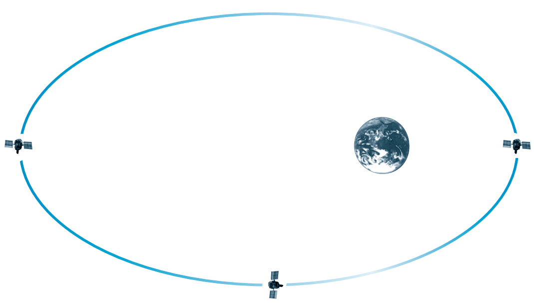
Their arithmetic average is

$$\frac{1}{2}(\text{perihelion} + \text{aphelion}) = \frac{1}{2}(a - c + a + c) = a;$$

that is, the average distance of the Earth (or any other planet) from the sun is just the average of its perihelion and aphelion distances.

Problems

1. For the satellite whose elliptic orbit about the Earth is shown in the accompanying diagram, indicate the location of the following points and give reasons for your answers.



   a. The point at which the gravitational force F exerted by the Earth on the satellite is greatest; where it is least. (Recall Newton's law of universal gravitation: $F = f(r) = GmM/r^2$, where r is the distance between the two objects.)
   b. The point at which the speed of the satellite is greatest. (*Hint*: Think of the satellite as always "falling" toward the Earth.)
   c. The point at which the speed of the satellite is least.

2. Suppose that the satellite in Problem 1 fires its retro-rockets to slow down somewhat at the point in its orbit where it is closest to the Earth. Compare the graph of the new orbit to the graph of the old one in a sketch.

3. Suppose that the satellite in Problem 1 is in a relatively low orbit about the Earth so that it encounters the upper fringe of the Earth's atmosphere. What will be the atmosphere's effect on the satellite's path? Sketch the graph of the resulting trajectory. What will happen eventually?

4. Suppose that the satellite in Problem 1 fires its booster rocket to speed up at the point in its orbit where it is closest to the Earth. Compare the graph of the new orbit to the graph of the old one in a sketch. What happens to the orbit if the booster rockets are extremely strong or continue firing for a long time?

5. Which of the nine planets in the solar system has the most circular orbit? the least circular orbit? Explain.

6. During the next few years, Pluto's orbit takes it inside the orbit of Neptune. Use the values in the table of planetary data in the text to explain why this situation can occur.

7. Use the fact that the perihelion and aphelion distances for Mercury are 46.0 and 69.8 million kilometers respectively to find the equation of the orbit of Mercury about the sun.

8. A salami is 4 inches in diameter. When the deli clerk slices it, however, the slices are at an angle of 65° to the main axis of the salami. Consequently, each slice will be in the shape of an ellipse with a minor axis of length 4 inches, as shown on the following page. Find the length of the major axis of each slice.

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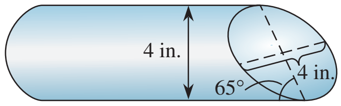
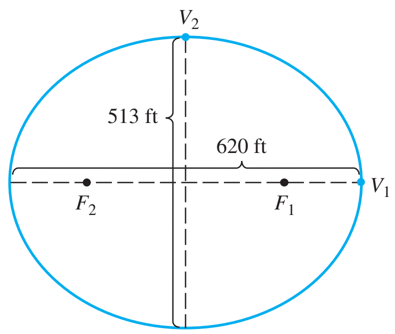


9. The Roman Coliseum is in the shape of an ellipse whose major axis measures 620 feet and whose minor axis measures 513 feet.



a. What is the equation of this ellipse?
b. How far apart are the foci?
c. How far apart are two adjacent vertices?

10. Write formulas expressing the perihelion and aphelion of an elliptic orbit in terms of the *semimajor axis* length a and the focal distance c in an ellipse.

In Problems 11 and 12, each equation represents an ellipse. In each case, identify the center, the vertices, and the foci and use the pertinent information to draw its graph.

11. $4x^2 + 9y^2 = 1$
12. $25x^2 + 4y^2 = 100$

In Problems 13–18, each equation represents an ellipse. Complete the square for x and y in each case to obtain the standard form for an ellipse. Then identify the center, the vertices, and the foci of the ellipse and use the pertinent information to draw its graph.

13. $x^2 + 4y^2 + 2x + 8y = -1$
14. $x^2 + 4y^2 + 2x + 8y = 11$
15. $x^2 + 4y^2 + 20x - 40y + 100 = 0$
16. $4x^2 + y^2 + 24x - 2y + 4 = 0$
17. $9x^2 + y^2 - 54x + 4y = -76$
18. $x^2 + 9y^2 - 6x + 36y = -36$
19. Complete the derivation of the equation of the ellipse by simplifying the equation in the text by eliminating the two square roots. (*Hint*: First isolate one of the radicals, then square both sides, and finally eliminate the remaining radical.)

9.4 Conic Sections: The Hyperbola and the Parabola

We now turn our attention to the two remaining conic sections, the hyperbola and the parabola. We begin this section by investigating the properties and some applications of the hyperbola.

The Hyperbola

We defined an ellipse geometrically as the set of all points for which the *sum* of the distances to two fixed foci is constant. In an analogous way, we define a hyperbola in terms of the *difference* of the distances to two fixed points being constant. A **hyperbola** is the set of all points for which the *difference* between the distances to two fixed points is a constant. The two points are the **foci** of the hyperbola. The midpoint of the line segment joining the foci is the **center.**

EXAMPLE 1

During a severe thunderstorm, two lightning bolts appear to strike simultaneously. You hear the thunderclap from one lightning bolt exactly 1 second after the lightning strikes at point P, and you hear the thunderclap from the second lightning bolt

2 seconds after it hits at point Q, as depicted in Figure 9.26. Sound travels at a speed of about 1100 feet per second.

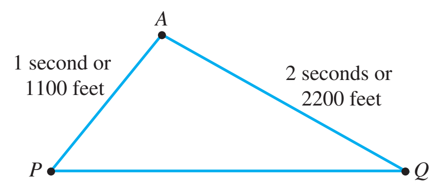


FIGURE 9.26

a. Based on this information what can you conclude about the point A, where you are?

b. Two friends of yours also see the same two lightning strikes. From Becky's location at point B, the thunder from the lightning bolt at point P takes 2 seconds to reach her and the thunder from the lightning bolt at point Q takes 3 seconds. From Carl's location at point C, the times are 4 and 5 seconds, respectively. What can you conclude about the three points A, B, and C?

Solution

a. The lightning bolts hit at points P and Q, and you're located at point A. Because it takes 1 second for the sound of the lightning bolt at P to reach A and sound travels at 1100 feet per second, the distance from P to A must be 1100 feet. Similarly, it takes 2 seconds for the sound of the strike at Q to reach A so that distance must be 2 seconds $\times$ 1100 feet/second $=$ 2200 feet.

b. Figure 9.27 shows the points A, B, and C. Reasoning as in part (a), you can conclude that Becky is 2 seconds $\times$ 1100 feet/second $=$ 2200 feet from P and 3 seconds $\times$ 1100 feet/second $=$ 3300 feet from Q. Similarly, Carl is 4400 feet from P and 5500 feet from Q.

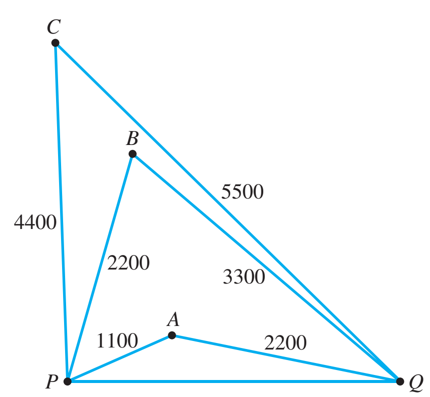


FIGURE 9.27

However, you can deduce one more piece of information: For all three, the *difference in time* between the two thunderclaps is 1 second. That is, the *difference in distance* from each of the three points A, B, and C to the points P and Q is a constant equal to 1100 feet. But if the differences in the distances from these three points to the fixed points where the lightning bolts hit are all equal, the three points A, B, and C must lie on a hyperbola whose foci are at P and Q.

Think About This Explain why you cannot determine the distance from P to Q in Example 1 based on the triangle APQ by using trigonometry. What additional information would you need to be able to find that distance?

The Equation of a Hyperbola We now determine the equation of a hyperbola from the geometric definition. For convenience, we place the center of a hyperbola at the origin and the foci on the x-axis at the point F_1 with coordinates $(c, 0)$ and F_2 with coordinates $(-c, 0)$, as shown in Figure 9.28. A point P with coordinates (x, y) lies on the hyperbola if, for some constant k,

$$|F_2P| - |F_1P| = k.$$

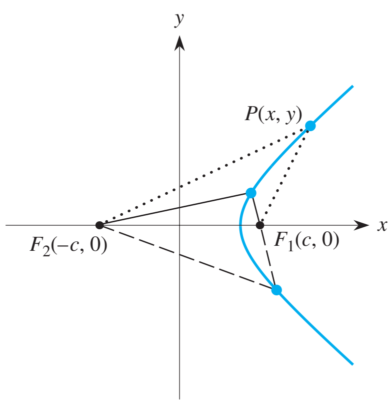


FIGURE 9.28

As with the equation of an ellipse, we let the constant $k = 2a$ for convenience. Thus

$$|F_2P| - |F_1P| = 2a,$$

so that

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a.$$

We simplify this equation by eliminating both square roots, as was done for the equation of an ellipse (see Problem 19, Section 9.3) and eventually obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$c^2 = a^2 + b^2.$$

The graph of this hyperbola is shown in Figure 9.29. Note that the hyperbola has two distinct *branches,* which is what we should expect from the discussion in Section 9.3 of slicing through a double right circular cone. The two points where this hyperbola crosses the x-axis are called its **vertices;** they correspond to the points where $y = 0$ and thus represent the points where the two branches are closest.

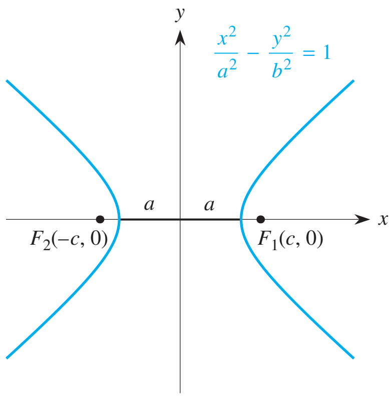


FIGURE 9.29

At the vertices, we have $y = 0$, so that

$$\frac{x^2}{a^2} = 1,$$

and therefore $x = \pm a$. Thus a represents the distance from the center to a vertex, whereas c represents the distance from the center to a focus. The line containing the foci is the **axis** of the hyperbola.

Alternatively, we can place the foci for a hyperbola on the vertical axis, as shown in Figure 9.30. The equation of such a hyperbola is

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

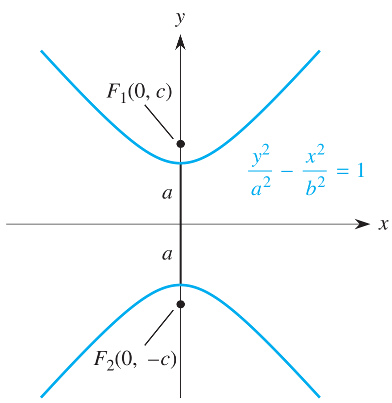


FIGURE 9.30

EXAMPLE 3

Describe the hyperbola whose equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = 1.$$

Solution The form of the equation indicates that the hyperbola is centered at the origin and that its axis is horizontal. Because $a^2 = 16$ and $b^2 = 9$, we have $a = 4$ and $b = 3$, so that

$$c^2 = a^2 + b^2 = 16 + 9 = 25,$$

and so $c = 5$. Thus the vertices are at $x = -4$ and $x = 4$, or the points $(-4, 0)$ and $(4, 0)$. The foci are the points $(-5, 0)$ and $(5, 0)$.

Think About This

Use your function grapher to see what the graph of the hyperbola in Example 3 looks like. To do so, you have to rewrite the equation by solving for y as a function of x. In particular,

$$\frac{y^2}{9} = \frac{x^2}{16} - 1 \quad \text{so that} \quad y^2 = 9\left(\frac{x^2}{16} - 1\right).$$

The upper and lower halves of the hyperbola are therefore given separately by the two functions

$$y = 3\sqrt{\frac{x^2}{16} - 1} \quad \text{and} \quad y = -3\sqrt{\frac{x^2}{16} - 1}.$$

What are the domains of the two functions?

More generally, we can consider a hyperbola as being shifted horizontally and/or vertically so that its center is at the point P with coordinates (x_0, y_0) rather than at the origin. We then have the following **standard forms for the equation of a hyperbola.**

> The equation of a hyperbola centered at (x_0, y_0) with its axis parallel to the x-axis is
>
> $$\frac{(x - x_0)^2}{a^2} - \frac{(y - y_0)^2}{b^2} = 1.$$
>
> The equation of a hyperbola centered at (x_0, y_0) with its axis parallel to the y-axis is
>
> $$\frac{(y - y_0)^2}{a^2} - \frac{(x - x_0)^2}{b^2} = 1.$$
>
> In each case,
>
> $$c^2 = a^2 + b^2.$$

Note that, in the equation of a hyperbola in standard form, the term with the positive coefficient determines the orientation. If the x^2-term is positive, the two branches open about the x-axis; if the y^2-term is positive, the two branches open about the y-axis. Also, be sure to distinguish between the equation $c^2 = a^2 + b^2$ relating the constants for a hyperbola and the equation $a^2 = b^2 + c^2$ relating the constants for an ellipse.

EXAMPLE 4

Verify that

$$x^2 - y^2 + 8x - 6y = 2$$

is an equation of a hyperbola. Find the center, vertices, and foci of the hyperbola and sketch its graph.

Solution We complete the square on both x and y so that the left-hand side becomes

$$\begin{aligned} x^2 - y^2 + 8x - 6y &= [(x^2 + 8x + 16) - 16] - [(y^2 + 6y + 9) - 9] \\ &= (x + 4)^2 - 16 - (y + 3)^2 + 9 \\ &= (x + 4)^2 - (y + 3)^2 - 7. \end{aligned}$$

The original equation therefore becomes

$$(x + 4)^2 - (y + 3)^2 - 7 = 2 \quad \text{or} \quad (x + 4)^2 - (y + 3)^2 = 9.$$

Dividing by 9, we obtain

$$\frac{(x + 4)^2}{9} - \frac{(y + 3)^2}{9} = 1.$$

Consequently, the center of the hyperbola is $(-4, -3)$ and $a = b = 3$. Furthermore, since the x^2-term is the positive one, the axis of the hyperbola is parallel to the x-axis. That is, the vertices and the foci lie on the horizontal line $y = -3$ through the center, and the hyperbola opens to the left and the right. Because $a = 3$, the vertices are 3 units left and right of the center, or at $(-7, -3)$ and at $(-1, -3)$. Also,

$$c^2 = a^2 + b^2 = 9 + 9 = 18,$$

so $c = \sqrt{18} = 3\sqrt{2}$. Thus the foci are located at F_1 at $(-4 - 3\sqrt{2}, -3)$ and F_2 at $(-4 + 3\sqrt{2}, -3)$, as shown in Figure 9.31.

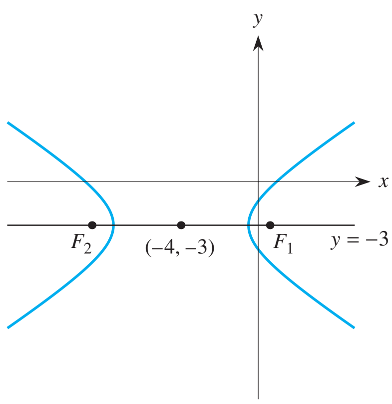


FIGURE 9.31

Based on their graphs and the vertical line test, hyperbolas (like ellipses) cannot be functions because any x-value can lead to two distinct y-values. Of course, if an x-value is beyond the limits of the ellipse, there is no corresponding y-value. Similarly, if an x-value is between the two branches of a hyperbola whose axis is horizontal, there is no corresponding y-value.

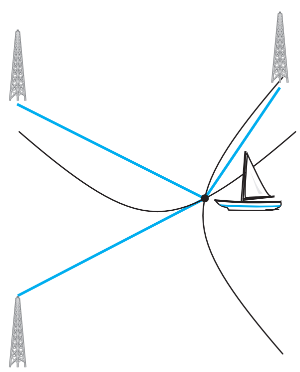


FIGURE 9.32

Applications of the Hyperbola Probably the most significant application of the hyperbola has been the long range navigation (LORAN) system used around the world by sailors to locate their positions before the advent of the global positioning system (GPS) that makes use of orbiting satellites. With LORAN, radio transmitters along coastal waters emit simultaneous radio signals that are picked up by electronic equipment on ships. As we demonstrated in Example 1 with sound waves, there will usually be a difference in the times at which a ship receives radio signals from different stations, and this difference is used to "place" the vessel on a specific hyperbola. When the same procedure is used with other radio transmitters in the LORAN network, the ship is simultaneously "placed" on a second hyperbola. Finding a point of intersection of the two hyperbolas and locating the position of the vessel is then a relatively simple matter, as illustrated in Figure 9.32.

In Example 5 we demonstrate the actual use of these ideas. To do so, we use the fact that any radio wave travels at the speed of light, or about 186,300 miles per second, or 300,000 kilometers per second.

EXAMPLE 5

A sailboat is out on Long Island Sound when a heavy fog moves in. To the south of the sailboat, on Long Island's shore, are two LORAN radio transmitters 60 km apart at points P and Q. A third transmitter is to the north on Connecticut's shore at point R, which is 40 km directly north of P. Figure 9.33 depicts this situation.

a. The receiver on the boat receives signals from P and Q that arrive 0.00016 second apart. Find an equation of the hyperbola having foci at P and Q on which the boat is located.

b. The receiver on the boat receives signals from P and R that arrive 0.0001067 second apart. Find an equation of the hyperbola having foci at P and R on which the boat is located, based on the same coordinate system used in part (a).

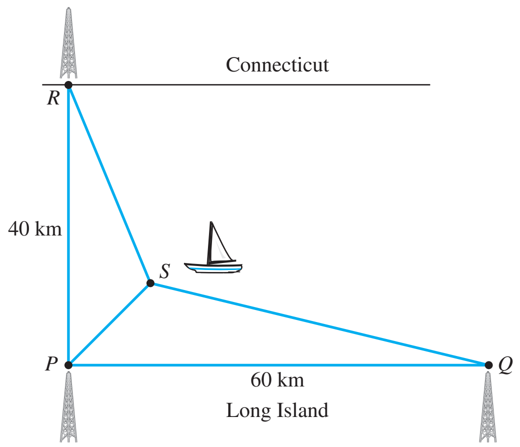


FIGURE 9.33

c. Estimate the location of the boat based on the results of parts (a) and (b).

Solution

a. Suppose that we set up a coordinate system with a horizontal axis through P and Q and the origin midway between them, as shown in Figure 9.34. In this system, the coordinates of P are $(-30, 0)$ and the coordinates of Q are $(30, 0)$. One focus of the hyperbola is at Q, so we have $c = 30$. Suppose that the sailboat is at S with coordinates (x, y) on this hyperbola. The difference in times between receipt of the two signals is 0.00016 second; using the speed of light as 300,000 km/second, this difference in times is equivalent to a difference in distance of 300,000 km/second $\times$ 0.00016 seconds $=$ 48 km. That is, $2a = 48$, so $a = 24$.

For a hyperbola $c^2 = a^2 + b^2$, so that

$$b^2 = c^2 - a^2 = (30)^2 - (24)^2 = 324 \quad \text{or} \quad b = 18.$$

Consequently, the equation of this hyperbola, which opens to the left and the right, is

$$\frac{x^2}{(24)^2} - \frac{y^2}{(18)^2} = 1.$$

b. We now consider the hyperbola with foci at P and R and center at the point $(-30, 20)$, as shown in Figure 9.35. We use a_1, b_1, and c_1 to represent the parameters for this hyperbola. Because the hyperbola opens upward and downward, its equation has the form

$$\frac{(y-20)^2}{a_1^{\,2}} - \frac{(x+30)^2}{b_1^{\,2}} = 1.$$

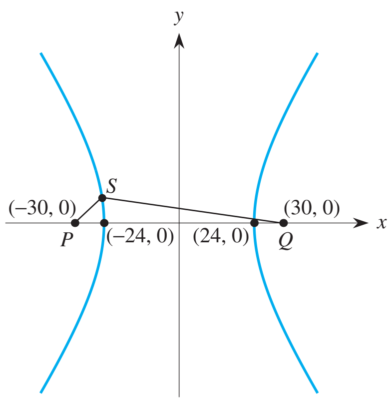


FIGURE 9.34

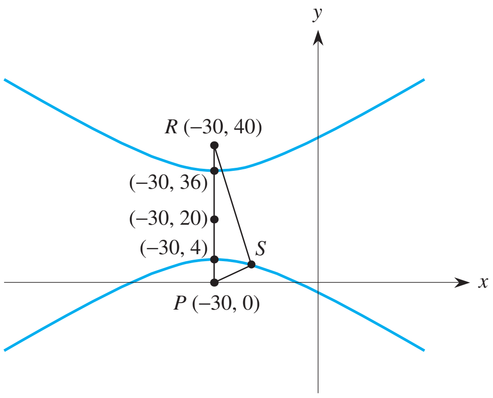


FIGURE 9.35

The distance between the foci is 40 km, so we have $c_1 = 20$.

The difference in time for receipt of the two signals is 0.0001067 second, which is equivalent to a difference in distance of $300{,}000 \times 0.0001067 = 32$ km, so $2a_1 = 32$ and $a_1 = 16$. Therefore

$$b_1^{\,2} = c_1^{\,2} - a_1^{\,2} = (20)^2 - (16)^2 = 144 \quad \text{or} \quad b_1 = 12,$$

and the equation of this hyperbola is

$$\frac{(y-20)^2}{(16)^2} - \frac{(x+30)^2}{(12)^2} = 1.$$

c. To locate the position of the sailboat, we have to find the point of intersection of the two hyperbolas. (Note that there can be as many as four points of intersection, but in practice we would know which branch of each hyperbola the boat is on, based on the strength of the signal, so the problem reduces to finding a single point of intersection.) Although we can find the point of intersection algebraically, to do so is rather complicated, so instead we estimate the point graphically. Using the equation of the first hyperbola from part (a), we have

$$\frac{y^2}{(18)^2} = \frac{x^2}{(24)^2} - 1 \quad \text{or} \quad \frac{y^2}{324} = \frac{x^2}{576} - 1$$

so that

$$y^2 = 324\left(\frac{x^2}{576} - 1\right).$$

Figure 9.36 shows that the sailboat is above the line on which the center and the foci lie, so y must be positive (otherwise the boat would be on land). We therefore take the positive square root and get

$$y = 18\sqrt{\frac{x^2}{576} - 1}.$$

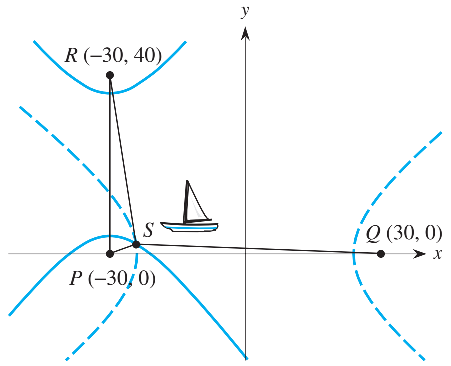


FIGURE 9.36

Similarly, starting with the equation of the second hyperbola, we have

$$\frac{(y-20)^2}{(16)^2} = \frac{(x+30)^2}{(12)^2} + 1 \quad \text{or} \quad \frac{(y-20)^2}{256} = \frac{(x+30)^2}{144} + 1$$

so that

$$(y-20)^2 = 256\left[\frac{(x+30)^2}{144} + 1\right].$$

Because the sailboat, as shown in Figure 9.35, is closer to the transmitter at P than the one at R, we need the lower branch of the hyperbola. Taking the negative square root gives

$$y - 20 = -16\sqrt{\frac{(x+30)^2}{144} + 1} \quad \text{or} \quad y = 20 - 16\sqrt{\frac{(x+30)^2}{144} + 1}.$$

To locate the sailboat, we need to find the point where the curves corresponding to these two equations intersect. From Figure 9.36 we estimate graphically that the point of intersection occurs at about $x \approx -24.18$ and $y \approx 2.21$. That is, the sailboat is located about 2.21 km off the north coast of Long Island at a position about $30 - 24.18 = 5.82$ km east of the transmitter at point P.

The Parabola

We have shown that the graph of any quadratic function $y = ax^2 + bx + c$ is a parabola opening upward or downward. However, the same parabolic shape can open to the left or the right, as shown in Figure 9.37, although neither of these graphs represents a function. To unify these ideas about parabolas, we consider the parabola from a somewhat different perspective in terms of its geometric definition as a conic section. We define a **parabola** as the set of all points in the plane for which the distance to a single fixed point is equal to the distance to a fixed line, as depicted in Figure 9.38. The fixed point is called the **focus** of the parabola. The fixed line is the **directrix** of the parabola.

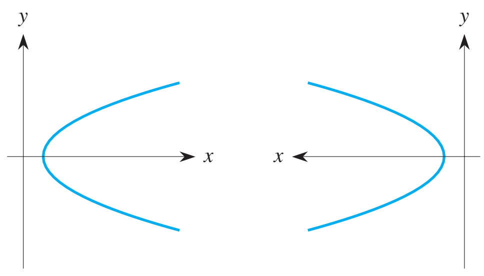


FIGURE 9.37

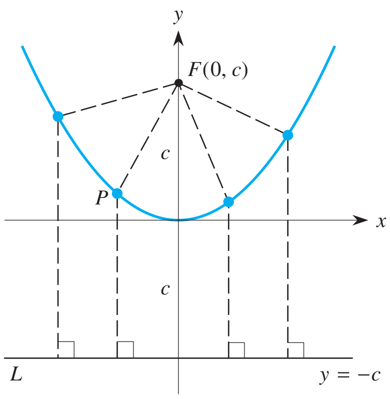


FIGURE 9.38

The Equation of a Parabola For convenience, we place the focus of the parabola at the point F on the y-axis with coordinates $(0, c)$ and let the directrix be the horizontal line $y = -c$, as shown in Figure 9.38. The graph shown corresponds to the case where $c > 0$. The parabola consists of all points P having the property that the distance from P to the focus F is equal to the vertical distance from P to the directrix line L. Thus a point P with coordinates (x, y) lies on the parabola if the distance from P to F, $\sqrt{x^2 + (y - c)^2}$, equals the distance from P to the line L, which is $y + c$; that is,

$$\sqrt{x^2 + (y - c)^2} = y + c.$$

We square both sides of this equation and get

$$x^2 + (y - c)^2 = (y + c)^2,$$

or, equivalently, when we expand the equation, we have

$$x^2 + y^2 - 2cy + c^2 = y^2 + 2cy + c^2.$$

We subtract y^2 and c^2 from both sides of this equation and obtain

$$x^2 - 2cy = 2cy.$$

Finally, we add $2cy$ to both sides and solve for y;

$$y = \frac{x^2}{4c}.$$

This is the equation of a parabola with vertex (or turning point) at the origin. If $c > 0$, the parabola opens upward. If $c < 0$, the parabola opens downward. The vertical line through the vertex is called the **axis of symmetry** of the parabola.

Alternatively, had we positioned the focus on the x-axis at F with coordinates $(c, 0)$ with a vertical directrix at $x = -c$, then we would have obtained

$$x = \frac{y^2}{4c}$$

as the equation for the parabola. This parabola likewise has its vertex at the origin; it opens to the right if $c > 0$ and opens to the left if $c < 0$. Finally, its axis of symmetry is now the horizontal line through the vertex, as depicted in Figure 9.39.

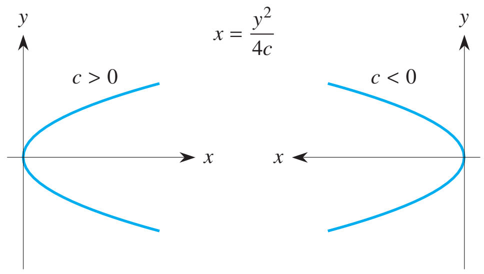


FIGURE 9.39

More generally, we can describe a parabola whose vertex is at (x_0, y_0) with the following **standard forms of the equation of a parabola.**

The equation of a parabola with its vertex at (x_0, y_0) and opening vertically is

$$y - y_0 = \frac{(x - x_0)^2}{4c}.$$

The equation of a parabola with its vertex at (x_0, y_0) and opening horizontally is

$$x - x_0 = \frac{(y - y_0)^2}{4c}.$$

Reflection Property of the Parabola Just as the ellipse has a remarkable—and useful—reflection property, the parabola has one that is even more commonly encountered. It can be shown that any ray coming into a parabola along a line parallel to the axis of symmetry of the parabola will "reflect" off the curve and pass through the focus, as shown in Figure 9.40. Alternatively, any ray emanating from the focus will reflect off the parabola and continue on a path parallel to the axis of symmetry. This reflection property is used, for example, in flashlights and in the headlights of an automobile, where the light source is located at the focus and the beams of light bounce off a parabolic reflector to concentrate more light in a particular direction. The reflection property is also used by satellite TV dishes, which are constructed in such a way that every cross section containing the axis of symmetry of the dish is a parabola, as illustrated in Figure 9.41. The TV signals coming from a satellite relay in orbit arrive at the dish along rays parallel to the axis of the dish and its parabolic cross sections. They reflect off the dish and pass through a receptor unit positioned at the focus. There the signal is collected and then transmitted to the television set.

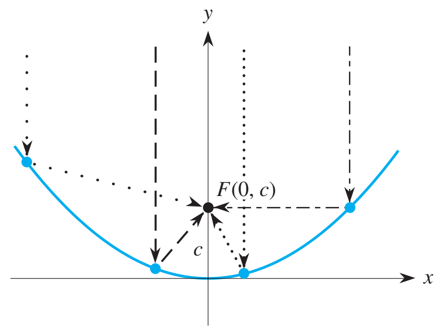


FIGURE 9.40

FIGURE 9.41

Conic Sections in General

The general equation of a conic section has the form

$$Ax^2 + By^2 + Cx + Dy + E = 0,$$

where A, B, C, D, and E are constants and at least one of A and B is nonzero. When $B = A \neq 0$, we divide both sides by A to get

$$x^2 + y^2 + \frac{C}{A}x + \frac{D}{A}y + \frac{E}{A} = 0,$$

which is the equation of a circle, provided that certain conditions are satisfied that lead to a positive radius. When $B = 0$ and $A \neq 0$, the resulting equation is quadratic in x but only linear in y and so gives the equation of a parabola opening either upward or downward. Similarly, when $A = 0$ and $B \neq 0$, we get a parabola opening either left or right. If A and B have the same sign—say, both are positive—the resulting curve is an ellipse, provided that certain conditions are satisfied. If A and B have opposite signs, the curve is a hyperbola. Thus, for instance,

$$4x^2 + 9y^2 + 8x - 36y - 5 = 0$$

represents an ellipse, whereas

$$4x^2 - 9y^2 + 8x - 36y - 5 = 0 \quad \text{and} \quad 25y^2 - 16x^2 + 10y + 8x + 3 = 0$$

both represent hyperbolas (one of which opens left and right and the other opens up and down). You have to be able to identify the type of curve from the given equation.

Finally, our discussions of conic sections have been restricted to their being in *standard position*—that is, their axes are parallel to the x- and y-axes. However, the same shapes can be rotated through some angle θ about the x-axis. When that occurs, the equation for the conic section—whether an ellipse, hyperbola, or parabola—includes a term of the form xy. For the most part, we aren't concerned with such situations here except for the case

$$xy = k,$$

where k is any constant. If we solve for y, this equation is equivalent to the power function

$$y = \frac{k}{x} = kx^{-1}.$$

This function is not defined at $x = 0$ and has two branches, the more familiar one in the first quadrant when $x > 0$ and a symmetric one in the third quadrant when $x < 0$. Together, they form the graph of a hyperbola that has been rotated from standard position through an angle of 45° or $\pi/4$ (when $k > 0$), as shown in Figure 9.42.

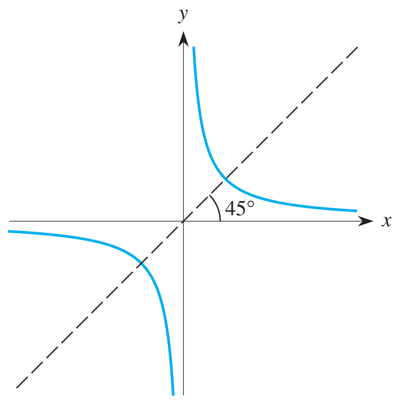
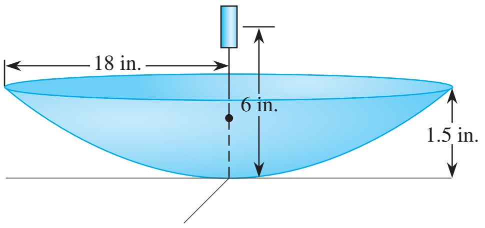


FIGURE 9.42

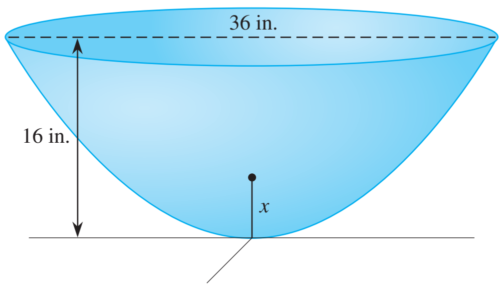
Problems

1. The small satellite TV dishes now on the market for home use have parabolic cross sections containing the axis of the dish. The focus is located at a point about 6 inches from the vertex of the parabola.



   a. Find an equation of a parabolic cross section. Assume that the dish is aimed directly upward—which makes sense only at the equator because the communications satellites are in orbit over the equator.
   b. The rim of the dish is a circle with a diameter of 18 inches at a height of about 1.5 inches above the vertex. If the dish were extended, the rim would enlarge. Find the diameter of the rim if the rim reached the height, 6 inches, of the focus.

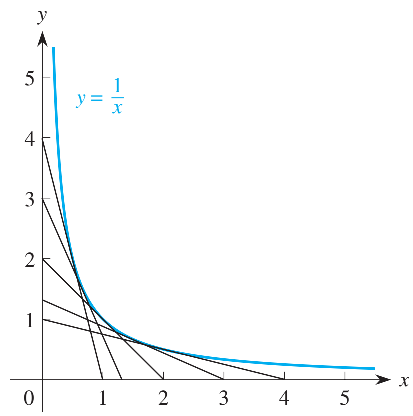
2. Suppose that a satellite receiver is 36 inches across and 16 inches deep (vertex to plane of the rim). How far from the vertex must the receptor unit be located to ensure that it is at the focus of the parabolic cross sections?



3. In this problem, we ask you to investigate the significance of the quantities a and b in the equation of a hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$
Suppose that you zoom out far enough on the graph of the hyperbola so that what you see appears to be a pair of lines that intersect at the origin.
   a. Explain why—when x and y both are very large, either positive or negative—you can ignore the number 1 in the equation.
   b. Ignore the number 1 and solve for y in terms of x to find the equations of the two lines described.
   c. What are the slopes of the two lines that the branches of the hyperbola approach?

In Problems 4–13 complete the square for x and y in each equation to obtain the standard form for a conic section. In each case, identify the conic section and use the pertinent information to draw its graph.

4. $x^2 + 4y^2 + 2x + 8y = -1$
5. $x^2 + 4y^2 + 2x + 8y = 11$
6. $x^2 - 4y^2 + 2x - 8y = 7$
7. $x^2 - 4y^2 + 2x - 8y = 19$
8. $4x^2 + y^2 + 24x - 2y + 4 = 0$
9. $4x^2 - 9y^2 - 16x - 18y = 31$
10. $9x^2 - 16y^2 - 90x + 64y = -17$
11. $4x^2 + 4y^2 - 24x + 16y + 43 = 0$
12. $9x^2 - 4y^2 + 18x - 16y = 8$
13. $9x^2 - 4y^2 + 18x - 16y = 6$
14. Explain why $x^2 + y^2 - 2x - 2y = -4$ is not the equation of a conic section.
15. In this problem, we ask you to look at the mathematics of string and wire art designs. Start with the hyperbola $xy = 1$ or $y = 1/x$ and construct a series of lines tangent to the curve, as shown in the accompanying figure. For instance, the line tangent to the curve when $x = 1$ crosses the x-axis at $x = 2$ and crosses the y-axis at $y = 2$. The line tangent to the hyperbola when $x = 2$ has an x-intercept of 4 and a y-intercept of 1. If you were to erase the curve, you would still see its outline from the tangent lines. String and wire art designers use this idea to suggest a variety of curves by using line segments made of the string or wire. The points on the axes are selected so as to follow the outline of a desired curve, such as the hyperbola.



a. Find the slope m of each of the five tangent lines shown in the figure.
b. Find a formula for the slope m as a function of the point of tangency x.

9.5 Parametric Curves

Throughout our discussion of functions, we have almost always considered expressions for which the dependent variable is given in terms of the independent variable. In some cases, however, introducing an additional variable, a *parameter,* can provide more insight into what is happening.

Parametric Representations of a Line

There are many ways to write an equation of a line, including the point–slope form, the slope–intercept form, and the normal form. However, certain questions about a line can't be answered with any of these forms. For instance, as we demonstrated in Section 9.2, if we want to locate the point that is a certain fraction of the way from the point P at (x_0, y_0) to the point Q at (x_1, y_1), it is essential to use the parametric form

$$x = x_0 + (x_1 - x_0)t$$
$$y = y_0 + (y_1 - y_0)t$$

for the line, where the parameter t takes on any value. With this form, each possible value of t gives a corresponding point on the line through P and Q.

EXAMPLE 1

Consider the parametric equations $x = 2 + 4t$ and $y = 5 - 3t$.

a. Construct a table of values for x and y corresponding to $t = -1, 0, 1, \ldots, 4$.

b. Use the table to explain why the two equations give a linear function.

c. Plot the points and draw the line that passes through them.

Solution

a. When $t = -1$, the parametric equations yield $x = 2 + 4(-1) = -2$ and $y = 5 - 3(-1) = 8$. Similarly, when $t = 0$, the equations give $x = 2 + 4(0) = 2$ and $y = 5 - 3(0) = 5$. We show the results for the given values of t in the following table:

t	-1	0	1	2	3	4
x	-2	2	6	10	14	18
y	8	5	2	-1	-4	-7

b. Note that each value of the parameter t gives rise to a pair of values x and y, which in turn produces a point (x, y). To show that these points lie on a line, we consider just the x and y values in the table. Each successive x value increases by 4 units and, simultaneously, each corresponding y value decreases by 3. Because there is a constant change in y when the x's are uniformly spaced, we conclude that these points lie on a line and so the parametric equations represent a line. In particular, the slope of this line is $\Delta y/\Delta x = -\frac{3}{4}$.

c. Figure 9.43 shows the plot of these points and the line through them.

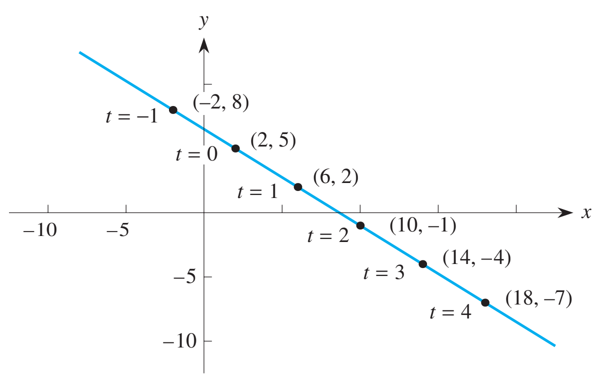


FIGURE 9.43

The pair of equations

$$x = 2 + 4t \quad \text{and} \quad y = 5 - 3t$$

giving x and y in terms of the parameter t is called a *parametric representation* or the *parametric equations* of the line. More generally, if we have a curve instead of a line, the pair of equations for x and y in terms of a parameter t is a parametric representation of the curve.

EXAMPLE 2

Eliminate the parameter t from the pair of parametric equations $x = 2 + 4t$ and $y = 5 - 3t$ and so find the slope–intercept form for the equation of this line.

Solution We start with the parametric equation $x = 2 + 4t$ and solve for t:

$$4t = x - 2 \quad \text{so that} \quad t = \frac{1}{4}(x - 2).$$

When we substitute this expression into the parametric equation for y, we get

$$y = 5 - 3t = 5 - 3\left(\frac{1}{4}\right)(x - 2) = 5 - \frac{3}{4}(x - 2),$$

or

$$y - 5 = -\frac{3}{4}(x - 2),$$

which is the point–slope form for the equation of a line with slope $-\frac{3}{4}$ that passes through the point (2, 5). This value for the slope is the same value we found in Example 1. Figure 9.43 shows that the line clearly passes through the point (2, 5), which is also a point in the table we created in Example 1.

The Path of a Projectile

Another case of a parametric representation of a function is the path of a thrown object, such as a football. The path, or trajectory, is a parabola of the form

$$y = ax^2 + bx + c.$$

However, for most real-world applications, the equation of the parabola by itself is of little value. Far more important is knowing *when* the ball, or other object, will reach a particular point. Therefore introducing time as a variable is necessary, and we do so by writing both x and y, the coordinates of each point along the parabola, in terms of a parameter t that represents time. In particular, if the object is released at time $t = 0$ from an initial height y_0 with an initial velocity v_0 at an initial angle α, as shown in Figure 9.44, then at any time t thereafter, it turns out that

$$x = (v_0 \cos \alpha)t \quad \text{and} \quad y = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + y_0.$$

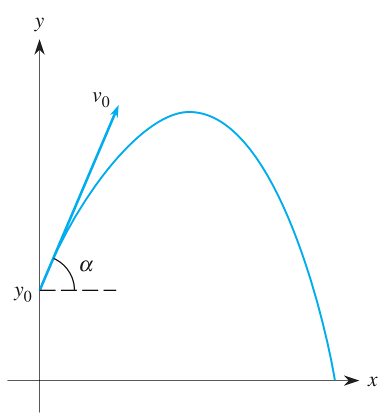


FIGURE 9.44

Each of these expressions can be thought of as a function of the parameter t, so we rewrite them as

$$x(t) = (v_0 \cos \alpha)t \quad \text{and} \quad y(t) = -\frac{1}{2}gt^2 + (v_0 \sin \alpha)t + y_0.$$

Each value of t determines a corresponding value for x (the horizontal distance) and a value for y (the vertical height), which produces a point (x, y) on the parabola. Again, the pair of equations for x and y as a function of the parameter t is a parametric representation of the curve, and the two equations are the parametric equations of the curve.

Parametric Representation of a Circle

The fact that the parametric representations of a line and of a parabola are so valuable suggests that parametric representations of other curves might also be useful. There are two key steps: (1) to decide on an appropriate parameter t, and (2) to find a way to express the usual variables x and y in terms of t.

We begin with a circle of radius r centered at the origin:

$$x^2 + y^2 = r^2.$$

Recall that we can express both x and y in terms of an angle θ drawn from the center of the circle, as shown in Figure 9.45. We write

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

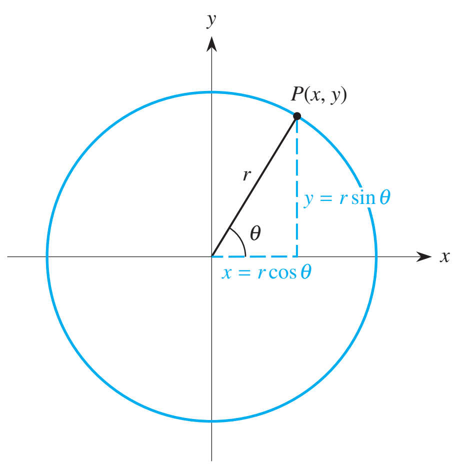


FIGURE 9.45

In retrospect, this equation is a parametric representation of the circle with the angle θ as the parameter. For each value of θ, we can calculate x and y and so get the point (x, y) on the circle. In fact, we can use any other letter, such as t, and get

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

as a parametric representation of the circle.

If we start with a parametric representation of a curve, we can sometimes *eliminate the parameter* to construct a single equation of the curve, as we did for the line in Example 2. For the circle $x = r\cos t$ and $y = r\sin t$, we eliminate the parameter t as follows:

$$x^2 + y^2 = (r\cos t)^2 + (r\sin t)^2 = r^2(\cos^2 t + \sin^2 t) = r^2.$$

Thus we are left with $x^2 + y^2 = r^2$, the usual equation for the circle.

Parametric Representation of an Ellipse

Now let's consider the ellipse centered at the origin with its major axis along the x-axis:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Its graph is shown in Figure 9.46. The question is: What might be an appropriate parameter to introduce to help describe this ellipse?

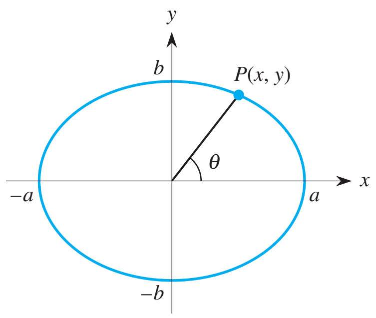


FIGURE 9.46

EXAMPLE 3

Find a parametric representation of the ellipse.

Solution Visualize a point moving around the ellipse shown in Figure 9.46. Although we can locate each point P in terms of its x- and y-coordinates, we may also be able to locate it by using the angle θ determined by P and the positive x-axis. How do we express x and y as functions of θ?

At first thought, you might be tempted to create a right triangle by dropping a perpendicular from P to the x-axis, as we did for the circle. The problem with this approach is that the length of the hypotenuse would change along with θ as the point P moves around the ellipse, unlike a circle in which the lengths of the line segments from O to P remain constant. Thus the angle θ is not a good choice for the parameter.

Nevertheless, our experience with the circle can provide some guidance. The parametric representation of a circle of radius r is

$$x = r\cos t \quad \text{and} \quad y = r\sin t.$$

If we think of the ellipse as having a "radius" of a associated with x and a "radius" of b associated with y, we might write

$$x = a\cos t \quad \text{and} \quad y = b\sin t.$$

Let's see if doing so makes sense. Suppose that (x, y) is any point on the ellipse so that x and y must satisfy

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

If we substitute our conjectured expressions for x and y into this equation, we find that

$$\frac{(a\cos t)^2}{a^2} + \frac{(b\sin t)^2}{b^2} = \frac{a^2\cos^2 t}{a^2} + \frac{b^2\sin^2 t}{b^2}$$

$$= \cos^2 t + \sin^2 t = 1.$$

Because these expressions for x and y as functions of t satisfy the equation of the ellipse, we conclude that $x = a \cos t$ and $y = b \sin t$ form a parametric representation of the ellipse with parameter t.

Using a Calculator

One of the options available on all graphing calculators is a `Parametric` mode. To use it, you need to supply an expression for x in terms of the parameter t and an expression for y in terms of t.[1] You then have to define a window not only in terms of x and y, but also in terms of an interval of values for the parameter t. Enter the parametric representation

$$x = 5 \cos t$$
$$y = 3 \sin t$$

for the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

with a range of values for the parameter from 0 to 2π in radians. Verify that the graph is indeed that of an ellipse.

To use this parametric representation, suppose that we want to know the point on the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

corresponding to a value of the parameter—say, $t = \pi/6$. We find that

$$x = 5 \cos\left(\frac{\pi}{6}\right) = 4.330$$
$$y = 3 \sin\left(\frac{\pi}{6}\right) = 1.5.$$

Alternatively, suppose you are told that the point $(4, 9/5)$ lies on the ellipse. (Verify that it does.) To find the value of the parameter t for this point, we consider

$$x = 5 \cos t = 4$$
$$y = 3 \sin t = \frac{9}{5}.$$

The first of these equations gives

$$\cos t = \frac{4}{5},$$

from which

$$t = \arccos\left(\frac{4}{5}\right) = 0.6435 \text{ radian.}$$

Verify that this value of t also satisfies the second equation $y = 3 \sin t = 9/5$.

[1]Note that in `Parametric` mode, different calculators use `t` or `T` as the "generic" variable just as different models use `x` or `X` as the "generic" variable in the usual `Function` mode.

Parametric Representations of a Hyperbola

We now consider how to write a parametric representation of the hyperbola.

EXAMPLE 4

Show that the equations

$$x = b \tan t \quad \text{and} \quad y = \frac{a}{\cos t}$$

are a parametric representation of the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

Solution If we substitute the expressions for x and y into the equation of the hyperbola, we get

$$\frac{(a/\cos t)^2}{a^2} - \frac{(b \tan t)^2}{b^2} = \frac{a^2}{a^2 \cos^2 t} - \frac{b^2 \tan^2 t}{b^2}$$

$$= \frac{1}{\cos^2 t} - \tan^2 t.$$

But, recalling the trigonometric identity

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta} \quad \text{or} \quad \frac{1}{\cos^2\theta} - \tan^2\theta = 1,$$

we see that the previous expression equals 1. Thus the two equations for x and y satisfy the equation of the hyperbola and therefore represent a pair of parametric equations of the hyperbola.

An alternative way to develop a parametric representation of the hyperbola requires introducing two new functions known as the *hyperbolic sine* and the *hyperbolic cosine.* We discuss them briefly in the Problems at the end of this section.

Parametric Representations of a Parabola

At the beginning of this section, we described how to find a parametric representation of the parabolic path of a projectile. We now consider the same situation geometrically. Actually, we can introduce a parameter in an extremely simple way. If the equation of the parabola is

$$y = ax^2 + bx + c,$$

we can let $x = t$, so that

$$y = at^2 + bt + c.$$

This approach may strike you as somewhat unfair (too easy!), but it is effective. In fact, it can be used with any function $y = f(x)$. Let's look at one of the advantages of doing so.

If we restrict our attention to the right-hand side of the parabola, we know that the curve is strictly increasing (if $a > 0$) and so has an inverse f^{-1}. We know that the graph of the inverse function is the mirror image of the graph of f about the diagonal line $y = x$. However, in all but the simplest cases, finding an explicit, or closed form, expression for the inverse f^{-1} is not easy, or even possible. Without such a formula for f^{-1}, constructing the graph of the inverse function would normally be almost impossible.

With parametric functions, however, this becomes a simple chore. Recall the definition of a function and its inverse function. If $y = f(x)$, for each value of x, the function determines a single corresponding value for y. The inverse function undoes this process in the sense that, for each value of y, f^{-1} returns the value of x that led to y. We can draw the graph of f in the parametric form

$$x = t \quad \text{and} \quad y = f(t)$$

by using the `Parametric` mode of the graphing calculator. To produce the graph of the inverse function, all we need do is reverse the roles of x and y. That is, if we set

$$x = f(t) \quad \text{and} \quad y = t,$$

so that

$$y = t = f^{-1}(x),$$

the calculator will draw the graph of the inverse function! Try it with, say, the right side of a parabola or with an exponential function, where you know what the function and its inverse should look like.

Other Parametric Curves

Many curves that cannot be represented simply, if at all, with y as a function of x can be represented fairly readily with parametric equations. Suppose that your friend has a reflector attached to the rim of her bicycle tire. As she rides past you at a constant speed, you observe that the path of the reflector is a curve such as the one shown in Figure 9.47. This curve, showing y as a function of x, is called a **cycloid.** If the radius of the tire is a, the parametric representation of the cycloid is

$$x = at - a \sin t \quad \text{and} \quad y = a - a \cos t,$$

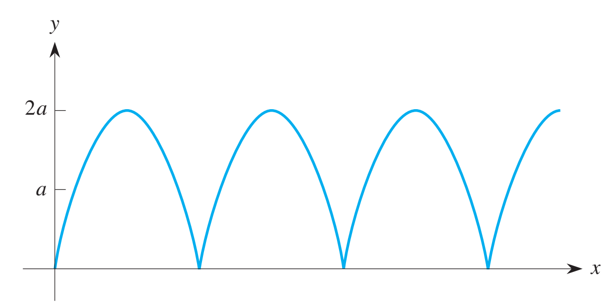


FIGURE 9.47

where the parameter t represents time. We simply cite these equations, which are typically derived in calculus, and only discuss their reasonableness here. Since the variable is time, be sure that you graph all such curves in radian mode.

Let's begin with the expression for the height $y = a - a \cos t$ of the reflector as a function of time t. The constant term a is the vertical shift, so y oscillates above and below it as the midline. The amplitude also equals a, so the height y actually oscillates between 0 and $2a$, which makes sense in terms of the physical phenomenon.

What about the expression $x = at - a\sin t$ for the horizontal distance? Note that this expression involves a sine term, which oscillates between $-a$ and a. This term is subtracted from at, which grows linearly, which again should make sense. The bicycle wheel is rolling along, so the horizontal distance traveled by the center of the wheel is simply at. Because the reflector is rotating about the rim of the tire, there must be an oscillatory adjustment to the linear distance covered.

In Problem 26 of Section 2.5, we raised the question about the shape of a water slide along which a person would slide most rapidly from one point to another; it is called the *brachistochrone* problem. From physical principles, the curve should be decreasing and concave up, so that the person gains the greatest speed at the beginning of the slide. It turns out that the specific curve along which the time needed is a minimum is an upside-down cycloid.

Let's consider another application involving a parametric representation of a curve. You have likely seen a spirograph, a toy with which you can draw intricate shapes by tracing curves as one plastic wheel rotates about another plastic wheel. Suppose that you have a large wheel of radius b and a smaller wheel of radius a that is rolling on the outside of the larger wheel, as shown in Figure 9.48. A fixed point on the outer (rolling) circle describes a curve that is known as an **epicycloid.** A parametric representation of the epicycloid is

$$x = (a+b)\cos\left(\frac{at}{b}\right) - a\cos\left[\left(\frac{a+b}{b}\right)t\right]$$

$$y = (a+b)\sin\left(\frac{at}{b}\right) - a\sin\left[\left(\frac{a+b}{b}\right)t\right].$$

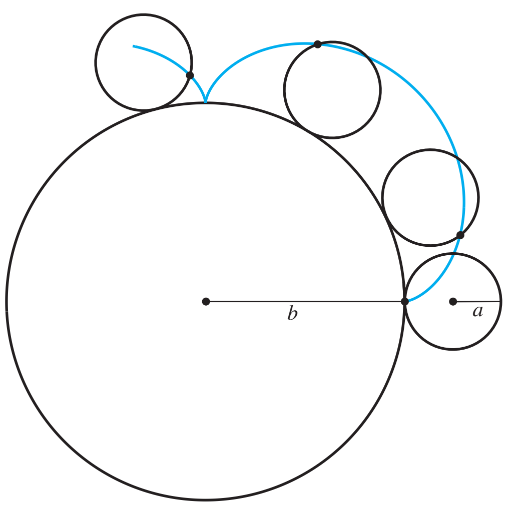


FIGURE 9.48

Let's see what the path of the fixed point on the rolling circle looks like. As the outer circle rolls on the fixed inner circle, the point on it moves back and forth, getting closer to and farther from the origin. It is closest to the origin—at a distance b—at the points where the two circles touch. It is farthest from the origin when the point is at the farthest possible position on the rolling circle, a distance of $b + 2a$. At any other time, the point is at an intermediate distance between b and $b + 2a$.

The actual shapes of epicycloids are often visually surprising and striking, as shown in Figure 9.49 for $a = 11$, $b = 28$, and t between 0 and 421π. A much simpler case is when the fixed inner circle has a radius of $b = 4$ and the rolling circle has a radius of $a = 1$, which gives the epicycloid shown in Figure 9.50 for t between 0 and $8\pi \approx 25.13$; the same curve thereafter repeats with period 8π because the identical points are repeatedly traced out. We also superimpose the inner fixed circle to indicate how the epicycloid is traced out by the fixed point as the outer (unseen) circle rolls around on the inner circle.

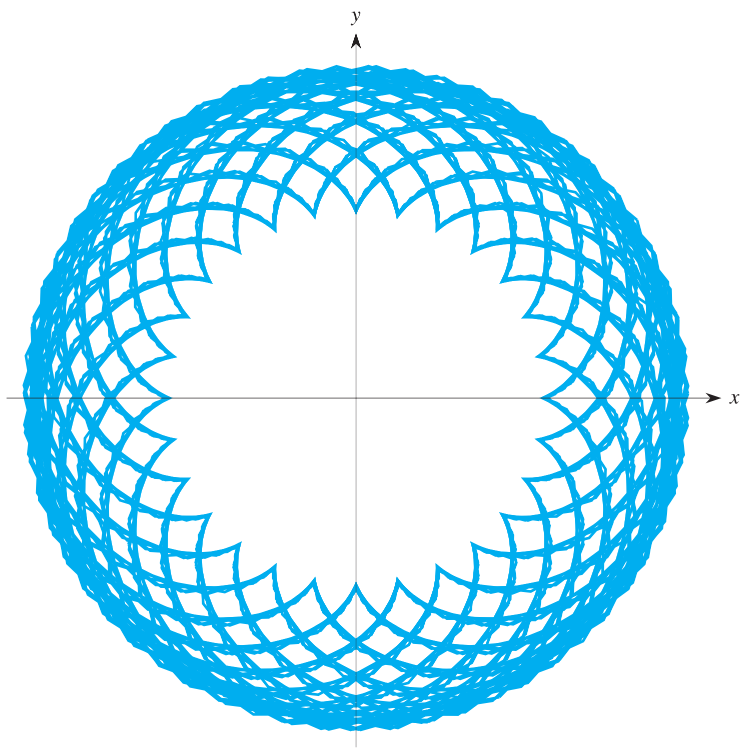


FIGURE 9.49

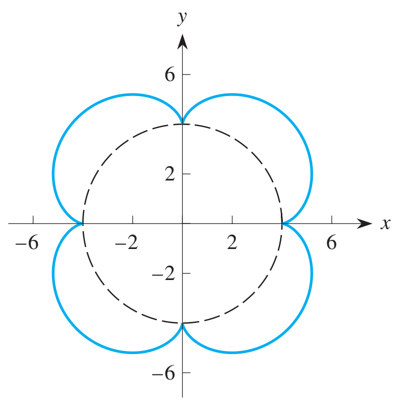


FIGURE 9.50

In the Problems for this section we ask you to experiment with the epicycloid and other curves by using parametric equations. You will see some surprising shapes if you simply try interesting combinations of functions. A favorite parametric curve that you can try is the "snowman" function whose parametric representation is

$$x = t - \frac{1}{2}\sin 10t \quad \text{and} \quad y = 5\sin t - \frac{1}{2}\cos 10t.$$

Problems

1. Consider the parametric representation of the line $x = 4 - 3t$, $y = 2 - 5t$.
   - **a.** Construct a table of points that lie on this line and find the slope of the line from the table.
   - **b.** Use the slope and a point on the line to write an equation of the line with y as a function of x.
   - **c.** How does the slope of the line relate to the coefficients in the parametric representation?
   - **d.** Eliminate the parameter t algebraically by solving for t from the first equation and substituting the result into the second.

e. Eliminate the parameter t algebraically by solving for t from the second equation and substituting the result into the first.

2. Consider the parametric representation of the line: $x = 7 - 3t$, $y = 4 + 5t$.

a. Based on the results of part (c) of Problem 1, what do you expect the slope of this line to be?
b. Create a table of values for this line and use the entries to sketch the graph of the line.
c. Find a point–slope form of the equation of the line.
d. Find an equation of the line by eliminating the parameter t algebraically.

3. Consider the curve given in parametric form $x = t^3 + 1$, $y = t^2 - 2$.

a. Create a table of values for this function by using $t = -2, -1.5, -1, \ldots, 2$ and plot the points to construct a rough sketch of the graph.
b. Draw the curve using the `Parametric` mode on your function grapher. How does the result compare to that in part (a)?
c. Eliminate the parameter t algebraically by first solving for t in terms of x.
d. Graph the function you obtained in part (c) by using the `Function` mode on your function grapher. How does it compare to your graph in part (b)?

4. Consider the curve with the parametric representation $x = t^2 + 1$, $y = t^2 - 2$.

a. Create a table of values for this function, using $t = -1, 0, 1, \ldots,$ and plot the points to construct a rough sketch of the graph. What surprising result do you get?
b. Use your function grapher in `Parametric` mode to verify that the result you obtained in part (a) is correct.
c. In terms of x and y, what are the domain and range for the curve you found in part (a)?
d. Eliminate the parameter t algebraically and explain why you got the shape you did.

5. Use the `Parametric` mode on your function grapher to draw the graph of the epicycloid with $a = 1$ and $b = 3$. Repeat with $b = 4$, $b = 5$, and $b = 6$ while keeping $a = 1$. Do you see any pattern in the periods of these curves? Do you observe any pattern in the number of loops that you get? Explain these patterns.

6. Repeat Problem 5 with $a = 1$ and $b = 6, 8, 10,$ and 12. How do the curves you get compare to the ones you obtained before?

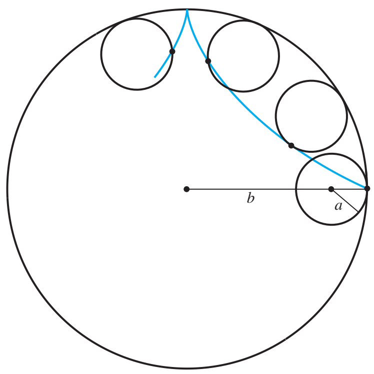
7. Repeat Problem 5 with $a = 2$ and $b = 3, 5, 7,$ and 9.

8. Figure 9.47 shows the graph of a cycloid, which is the path of a reflector mounted on the rim of a tire of radius a. Determine the coordinates of the points where the curve touches the horizontal axis.

9. A **hypocycloid** is the curve generated by a fixed point on a circle of radius a that rolls around the inside of a larger circle of radius b. The parametric equations for a hypocycloid are

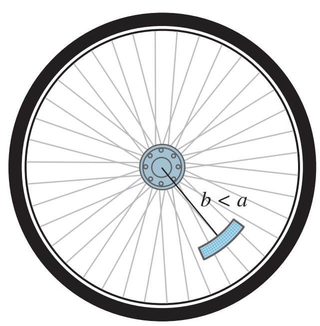
$$x = (b - a)\cos t + a\cos\left[\left(\frac{b-a}{a}\right)t\right]$$

$$y = (b - a)\sin t - a\sin\left[\left(\frac{b-a}{a}\right)t\right].$$



Use your function grapher to see the shapes that result when **(a)** $a = 1, b = 2$; **(b)** $a = 1, b = 3$; **(c)** $a = 1, b = 4$; and **(d)** $a = 2, b = 3$.

10. Suppose that a bicycle reflector is mounted partway along one of the spokes in a wheel at a distance $b < a$ from the center of the wheel. How should the parametric equations for the cycloid be modified to reflect this new position?



11. In the equations for the cycloid, $x = at - a\sin t$ and $y = a - a\cos t$, use the second equation to solve for t and then substitute the result into the first equation to eliminate the parameter and obtain x as a function of y. What does the resulting equation tell you about the path of the reflector?

9.6 The Polar Coordinate System

As we discussed in Section 9.1, the *polar coordinate system* is based on the idea that every point in the plane must lie on some circle centered at the origin. In this coordinate system, the origin is known as the **pole.** To locate a point P in such a system, we must indicate the radius r of the particular circle on which P lies, as shown in Figure 9.51. That is, does P lie on a circle of radius $r = 3$ or a circle of radius $r = 4$ or a circle of radius $r = 4.2689$? Knowing that P lies on a specific circle still does not locate the point exactly. We must also specify where the point lies on that circle. Is it at an angle of 30° or an angle of 45° or an angle of 263° measured counterclockwise from the horizontal?

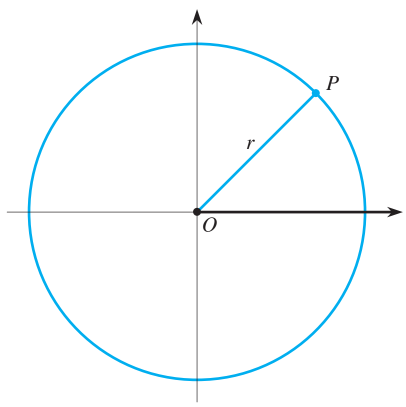


FIGURE 9.51

To formulate these ideas more precisely, let's develop some appropriate terminology. We first introduce a horizontal axis starting at the pole and pointing to the right. It is called the **polar axis** and serves as a reference. The distance from the pole to the point P, which is equivalent to the radius of a circle centered at the pole, is denoted by the coordinate r. To locate a specific point P on this circle of radius r, we must indicate how far around the circle P lies, starting from the polar axis. We measure this distance around the circle in terms of an angle coordinate θ drawn counterclockwise, or in a positive direction, from the polar axis, as shown in Figure 9.52. Thus we can locate any point in the plane if we know its distance r from the pole (to determine a circle) and the angle θ around this circle. The **polar coordinates** of the point P consist of r and θ, so we write the point as (r, θ).

For example, a point that lies 5 units from the pole at an angle of 60°, or $\pi/3$ radians, with the polar axis has polar coordinates $(5, 60°)$ or $(5, \pi/3)$, as shown in Figure 9.53. Similarly, the point Q that lies 3 units from the pole at an angle of $2\pi/3$, or 120°, has coordinates $(3, 2\pi/3)$ or $(3, 120°)$.

We can visualize the polar coordinates of a point P in the following alternative way. Any point is located on a line passing through the pole, as measured by the angle of inclination θ, which is known as the **polar angle.** The particular location of the point P along that line is determined by its distance r from the pole. Thus P can be visualized as lying at the intersection of a line through the pole and a circle centered at the pole.

This approach has an added geometric advantage. Recall from geometry that the radius drawn to any point on a circle is perpendicular to the tangent line at that point. Therefore the polar coordinates of a point are determined by the intersection

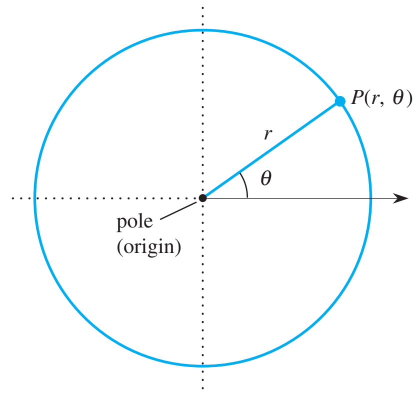


FIGURE 9.52

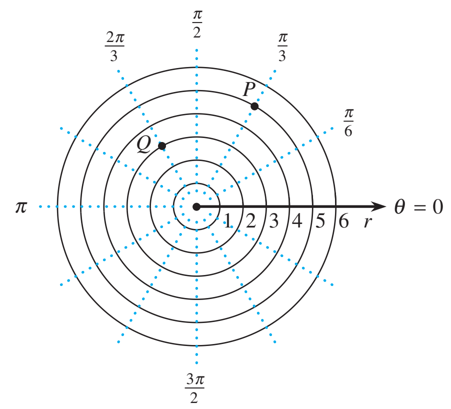


FIGURE 9.53

of two curves that we can think of as being "perpendicular" at the point. This property is analogous to what we do in rectangular coordinates where the vertical and horizontal lines that determine a point are perpendicular to each other.

Although there are many advantages to working with a polar coordinate system, it does have one disadvantage. In rectangular coordinates, every point has a unique pair of coordinates. However, every point in polar coordinates has more than one address. Consider the point 1 unit to the right of the pole on the polar axis. According to our discussion so far, you might conclude that its polar coordinates are $r = 1$ and $\theta = 0$. However, with a little thought, it should be evident that the address for this point could also be $r = 1$ and $\theta = 2\pi$, or $r = 1$ and $\theta = 4\pi$, and so on. Thus there are infinitely many polar coordinate representations of the same point.

In fact, there are still other ways to give the polar address of this point. In general, any angle θ measured counterclockwise from the polar axis is considered positive; any angle θ measured clockwise from the polar axis is considered negative, as illustrated in Figure 9.54. Thus our point on the polar axis could also be written as $(1, -2\pi)$, for instance.

Furthermore, we encounter some situations in Section 9.7 in which an angle θ gives rise to a negative value for r. Let's see what this means because a negative value of r cannot represent the radius of a circle. If $\theta = \pi/4$ and $r = 3$, we simply measure a distance of 3 units along the terminal side of the angle $\theta = \pi/4$. However, if $\theta = \pi/4$ and $r = -3$, we can locate the corresponding point by extending the terminal side of the angle backward through the pole and measuring 3 units along this extension, as illustrated in Figure 9.55.

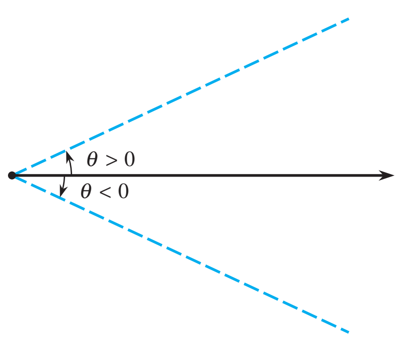


FIGURE 9.54

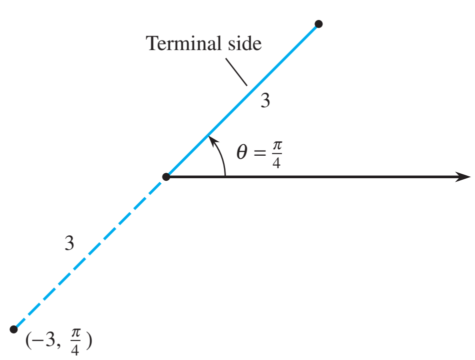


FIGURE 9.55

Let's look at this issue a bit more formally. When we draw any angle θ, it determines a terminal side OP from the pole through some point P, as illustrated in Figure 9.56. The obvious polar coordinate representation for this point is (r, θ), where r is positive because the distance is measured along the terminal side. However, we can also represent that point by considering the angle $\theta + \pi$ (corresponding to an additional rotation of π radians or 180°) and measuring a distance r from the pole in the opposite direction. In such a case, we think of r as negative and the polar coordinates of the point as $(-r, \theta + \pi)$. Thus, if a point P is located at $\theta = \pi/4$ and $r = 3$, we can consider the associated angle $\pi/4 + \pi = 5\pi/4$ and assign the coordinates $(-3, 5\pi/4)$ to the point P as well.

With these ideas in mind, we can find even more ways to write our earlier point with coordinates $r = 1$ and $\theta = 0$. For instance, we can obtain this point when $r = -1$ and $\theta = \pi$ or when $r = 1$ and $\theta = -2\pi$ or when $r = -1$ and $\theta = -\pi$, as illustrated in Figure 9.57.

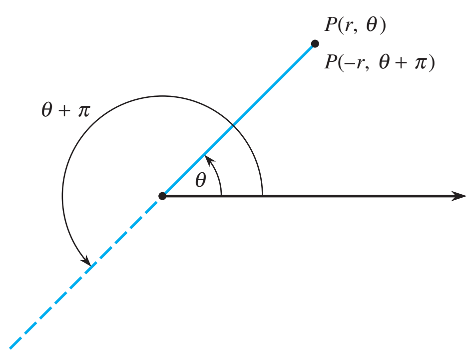


FIGURE 9.56

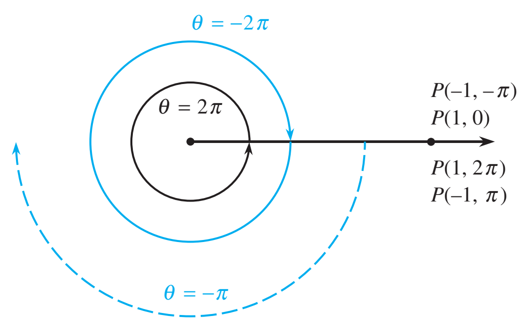


FIGURE 9.57

Think About This Can you think of any other coordinates for this point when r is negative? when θ is negative?

Hence any point in the polar coordinate system has infinitely many pairs of coordinates. Even the pole, where $r = 0$, has infinitely many representations because it can be thought of as corresponding to *any* possible angle θ.

Transforming Between Polar and Rectangular Coordinates

Often, it is useful to think of the two coordinate systems, polar and rectangular, as being superimposed. In such a case, the pole and the origin are the same point; the polar axis and the positive x-axis coincide. The question then is: How do the coordinates of a point P in one system relate to the coordinates of the same point in the other system? That is, how do we transform the rectangular coordinates (x, y) of a point into the equivalent polar coordinates (r, θ) and vice versa?

Suppose that we start with a point P having polar coordinates (r, θ) and we want to determine the corresponding rectangular coordinates x and y. From the right triangle shown in Figure 9.58, it is clear that

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$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

EXAMPLE 1

The point P has polar coordinates $r = 5$ and $\theta = \pi/3$. Find the corresponding rectangular coordinates x and y.

Solution Using the preceding two equations, we find that the rectangular coordinates are

$$x = r\cos\theta = 5\cos\frac{\pi}{3} = 5\left(\frac{1}{2}\right) = 2.5$$

and

$$y = r\sin\theta = 5\sin\frac{\pi}{3} = 5\left(\frac{\sqrt{3}}{2}\right) = 4.33.$$

For the reverse problem, suppose that we start with a point P whose rectangular coordinates are (x, y), as shown in Figure 9.59. We now want to find the corresponding polar coordinates r and θ. First, we observe that r is the distance from the pole (origin) to P. The Pythagorean theorem gives

$$r^2 = x^2 + y^2 \quad \text{so that} \quad r = \pm\sqrt{x^2 + y^2}.$$

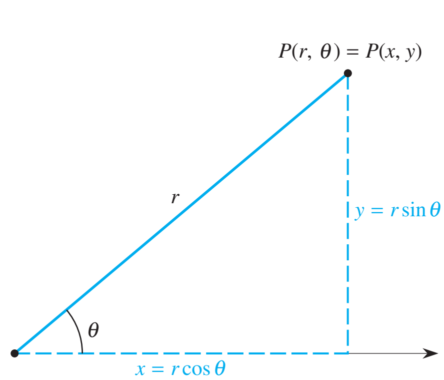


FIGURE 9.58

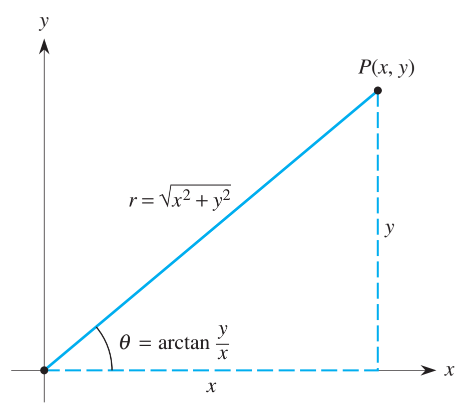


FIGURE 9.59

Next, observe that

$$\tan\theta = \frac{y}{x} \quad \text{so that} \quad \theta = \arctan\frac{y}{x}.$$

Thus, given the rectangular coordinates (x, y) of a point, we can find the polar coordinates (r, θ) by using

$$r = \pm\sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan\frac{y}{x}.$$

However, these formulas have to be used with great care, as demonstrated in Example 2.

EXAMPLE 2

If the rectangular coordinates of a point are $x = 3$ and $y = 4$, find one set of polar coordinates for that point.

Solution We find one set of polar coordinates using

$$r = \pm\sqrt{x^2 + y^2} = \pm\sqrt{3^2 + 4^2} = \pm\sqrt{25} = \pm 5$$

and

$$\theta = \arctan\frac{4}{3} = 0.927 \text{ radian,}$$

or about 53.13°. These values give rise to infinitely many possible pairs of coordinates, but not all of them are appropriate for the point P. The point P lies in the first quadrant, as shown in Figure 9.60, so one possible polar representation is $(r, \theta) = (5, 0.927)$. But the coordinates $(-5, 0.927)$ are not correct because they lie in the third quadrant. However, $(-5, \pi + 0.927)$, which is $(-5, 4.069)$ in radians or $(-5, 233.13°)$, is another possible pair of coordinates for P. But the polar coordinates $(5, \pi + 0.927)$ in radians or $(5, 233.13°)$ is not correct because it also is a point in the third quadrant. Be sure to plot the point in order to decide which value of r to match with which value of θ.

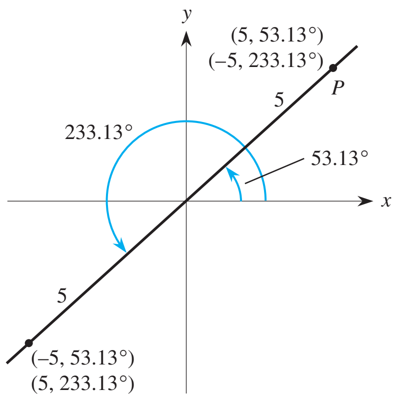


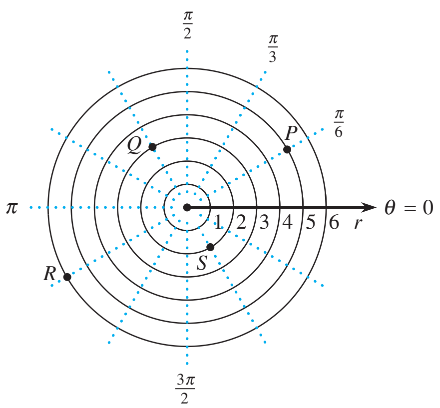
FIGURE 9.60

Think About This Explain why only even multiples of π are used for the θ-coordinate of the different representations for the point P in Example 2.

Polar coordinates are particularly useful in representing situations in which there is a single special point and all other ideas of interest are centered at that point. For instance, in physics, the total mass of a body is often assumed to be at a single point corresponding to the pole. Thus satellites can often be thought of as moving in circular orbits about a planet located at the pole of a polar coordinate system. Similarly, the magnetic field associated with a magnet can be thought of as being centered at the pole of a polar coordinate system, and all related phenomena are often best expressed in terms of polar coordinates.

Problems

1. For each of the points P, Q, R, and S shown, do the following.



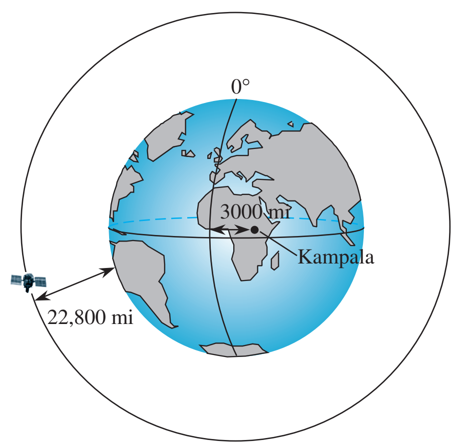
a. Write a polar coordinate representation with r and θ both positive.
b. Write a polar coordinate representation with r positive and θ negative.
c. Write a polar coordinate representation with r negative and θ positive.
d. Write a polar coordinate representation with r and θ both negative.

2. A merry-go-round at an amusement park has an inner radius of 9 feet and an outer radius of 26 feet. On the merry-go-round are five concentric circles of horses, 3 feet apart, starting with the innermost circle 10 feet from the center. Let the polar axis extend from the center of the merry-go-round to the entrance gate of the ride.
a. What are the polar coordinates of the horse in the outer, or fifth, circle that is one-third of the way around the merry-go-round to the right from the gate?
b. What are the polar coordinates of the horse in the second circle that is one-fifth of the way around to the left from the gate?

3. Transform the rectangular coordinates (x, y) in (a)–(h) to the equivalent polar coordinates. Sketch the location of each point in the polar coordinate plane.
a. $(4, 4)$
b. $(-4, 4)$
c. $(-4, -4)$
d. $(4, -4)$
e. $(3, -4)$
f. $(-3, 4)$
g. $(8, 3)$
h. $(3, 8)$

4. Transform the polar coordinates (r, θ) in (a)–(i) to equivalent rectangular coordinates. Indicate the location of each point graphically in the polar plane.
a. $(5, 0)$
b. $(5, \pi/2)$
c. $(-5, \pi/2)$
d. $(-5, 0)$
e. $(3, \pi/3)$
f. $(-3, -\pi/3)$
g. $(2, 3\pi/2)$
h. $(2, 5\pi/4)$
i. $(2, -5\pi/3)$

5. A satellite is in a circular orbit about the equator at a height of 22,800 miles above the surface of the Earth. The radius of the Earth is about 4000 miles. The longitude line running north–south from the north pole to the south pole through Greenwich, England, serves as the 0° reference. Because the circumference of the Earth is about 24,000 miles, each 15° of longitude corresponds to about 1000 miles along the equator.



a. What are the polar coordinates of the Ugandan capital Kampala, which is 3000 miles east (positive direction) of the Greenwich baseline?
b. What are the polar coordinates of the capital of Borneo, which is about 7500 miles east of the Greenwich baseline?
c. What are the polar coordinates of the satellite when it passes over Quito, Ecuador, which is 5200 miles west of the Greenwich baseline?

9.7 Families of Curves in Polar Coordinates

In Section 9.6, we introduced the notion of polar coordinates and considered coordinates (r, θ) for individual points in such a system. A far more interesting and useful question is: How do we represent curves and families of curves in polar coordinates?

Recall that, in rectangular coordinates, the curve associated with an equation $y = f(x)$ consists of all points (x, y) whose coordinates satisfy the equation. We use the comparable notion when working with polar coordinates but with the understanding that it is only necessary that one representation of a point in polar coordinates satisfies the equation.

To begin, it is usually much simpler to think of the angle θ as the independent variable and the distance r from the pole as the dependent variable. Thus for most functions in polar coordinates, we write $r = f(\theta)$ for some set of values of the angle θ. Then for each allowable value of θ, the function determines a corresponding value for r and the pair (r, θ) represents the polar coordinates of a point P in the plane. The totality of all such points determined by the equation constitutes the graph of the function. Note that writing the polar coordinates of a point as (r, θ) reverses our usual notation of writing the independent variable first and the dependent variable second, as is done in rectangular coordinates with (x, y).

Let's begin with some particularly simple cases. First, consider the equation

$$r = f(\theta) = c,$$

where c is a constant. Thus, no matter what the angle θ is, the distance r from the pole is the constant c. The set of all points that satisfy this condition forms the circle of radius c centered at the pole, as shown in Figure 9.61.

Next, consider the equation $\theta = \alpha$, where α is a constant; note that the distance r is not explicitly mentioned. No matter what distance we use, the corresponding point is always located at the angle α as measured from the polar axis. The set of all such points forms a line, inclined at the angle α which passes through the pole, as depicted in Figure 9.62. For instance, $\theta = \pi/4$ represents the line through the pole inclined at a 45° angle.

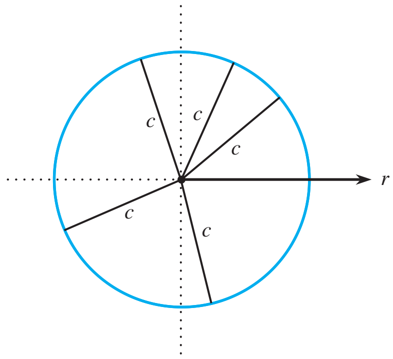


FIGURE 9.61

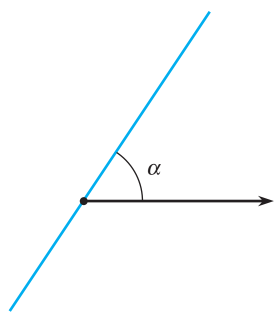


FIGURE 9.62

Shapes that are far more interesting and intricate than a circle and a line arise from relatively simple polar equations. We investigate some types of shapes and their underlying patterns for various families of polar coordinate curves. In the rest of this section, you should use your graphing calculator set in `Polar` mode or a polar graphing program for a computer.

Working with polar coordinates often has a special advantage over working with rectangular coordinates. Consider the simple curve shown in Figure 9.63, which is known as an Archimedean spiral. Its equation in polar coordinates is $r = \theta$. When $\theta = 0$, we have $r = 0$, so the curve starts at the pole. As θ increases, the distance r from the pole likewise increases, and as θ loops repeatedly around the pole, so does the curve to form the spiral shape shown.

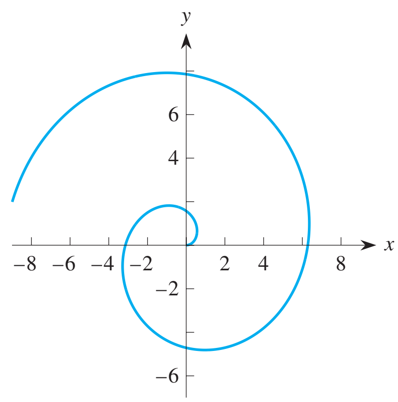


FIGURE 9.63

Now, let's find an equivalent equation in rectangular coordinates, using the transforming equations we derived in Section 9.6:

$$r^2 = x^2 + y^2 \quad \text{and} \quad \theta = \arctan \frac{y}{x},$$

and

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

Substituting the first pair of these expressions into the equation $r = \theta$, we get the rectangular equation

$$\sqrt{x^2 + y^2} = \arctan \frac{y}{x},$$

which is not particularly attractive. We can simplify this expression slightly by taking the tangent of both sides to eliminate the arctangent function:

$$\tan\left(\sqrt{x^2 + y^2}\right) = \frac{y}{x}.$$

Or, if we multiply through by x,

$$x\tan\left(\sqrt{x^2 + y^2}\right) = y.$$

Neither of these expressions is any more attractive. Moreover, we can't simplify any of these expressions to write y as a function of x or to write x as a function of y. (In fact, recall that such a curve does not represent a function.) Furthermore, none of these rectangular expressions gives any insight into the behavior of the curve, whereas the polar representation $r = \theta$ was very helpful in understanding the spinal curve shown in Figure 9.63.

Think About This What happens to the spiral if $\theta < 0$?

EXAMPLE 1

Consider the polar function $r = f(\theta) = \cos\theta$, whose graph is shown in Figure 9.64. By eye, the curve appears to be circular, appears to pass through the pole, and appears to be symmetrical about the polar axis. Show that this curve is a circle.

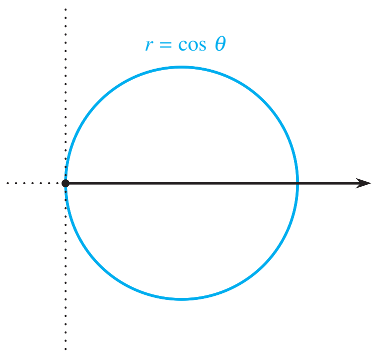


FIGURE 9.64

Solution To prove that the curve is a circle, we can try to express it in rectangular coordinates where the equation of a circle would be recognizable. Although we could attempt to substitute the transforming expressions for r and θ into the equation $r = \cos\theta$, using a little trick is much easier. We multiply both sides of the given equation $r = \cos\theta$ by r to get

$$r^2 = r\cos\theta,$$

which is equivalent to the rectangular equation

$$x^2 + y^2 = x \quad \text{or} \quad x^2 - x + y^2 = 0.$$

To determine whether this is the equation of a circle, we complete the square in the x-terms:

$$\begin{aligned}(x^2 - x) + y^2 &= \left[x^2 - x + \left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2\right] + y^2\\ &= \left(x^2 - x + \frac{1}{4}\right) - \frac{1}{4} + y^2\\ &= \left(x - \frac{1}{2}\right)^2 + y^2 - \frac{1}{4} = 0,\end{aligned}$$

so that

$$\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4}.$$

This is the equation of a circle with radius $\frac{1}{2}$ centered at the (rectangular) point $(\frac{1}{2}, 0)$. This circle is indeed symmetrical about the horizontal axis and does pass through the pole.

Think About This

1. Describe the graph of $r = 5\cos\theta$.
2. Describe the graph of $r = a\cos\theta$, for any multiple $a > 0$.
3. Describe the graph of $r = a\sin\theta$, for any multiple $a > 0$.

The Family of Rose Curves

Let's consider some related curves in polar coordinates. When we graph the equation $r = \cos 2\theta$, we obtain the result shown in Figure 9.65. This graph corresponds to angles θ ranging from 0 to 2π. If we extend the values beyond this interval, in either direction, the same points repeat, so the result is a periodic function with period 2π. If you experiment with your polar function grapher, you will notice that the graph shown is traced repeatedly when you take a large range of values for the angle θ.

Don't just look at the completed shape, but rather consider this curve and other polar coordinate curves we discuss in a dynamic manner. How are the curves produced or traced? Think of the cursor on the calculator or the computer screen as a moving point that traces the curve and observe carefully how the curve is generated.

Note that the graph shown in Figure 9.65 for $r = \cos 2\theta$ consists of four loops of equal size. (Actually, depending on the calculator or computer graphics package you use, there may be some distortion and the loops may not appear to be precisely the same size even though they are.) To get a better feel for how the particular shape evolves, watch carefully as the curve is traced in Figure 9.66. Note that it starts at the far right (corresponding to $\theta = 0$ where $r = 1$) and then loops around (portion ①) until it passes through the pole (corresponding to $\theta = \pi/4$, where $r = \cos 2(\pi/4) = \cos(\pi/2) = 0$). It then starts to form a second loop (portion ②) as r takes on negative values. Eventually, it completes the loop (portion ③) before it again passes through the pole, this time at an angle of $\theta = 3\pi/4$ so that again $r = 0$. It then begins to form the third loop (portion ④) and completes that loop (portion ⑤) when it passes through the pole, where $\theta = 5\pi/4$. It then forms a fourth loop (portions ⑥ and ⑦), for θ between $5\pi/4$ and $7\pi/4$. It finally completes the original loop (portion ⑧) as θ progresses to 2π. This curve is known as a *four leaf rose.*

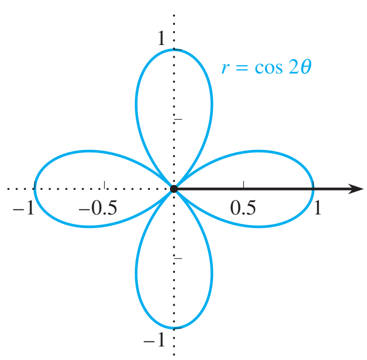


FIGURE 9.65

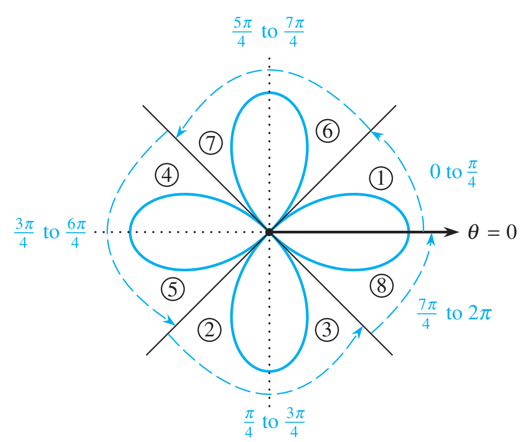


FIGURE 9.66

Think About This

1. What shape is produced if you graph $r = a \cos 2\theta$ for any multiple a?
2. Describe the graph corresponding to $r = a \sin 2\theta$ for any multiple a. How does it compare to the graph of the cosine function in part (1)?

Let's now make a relatively simple change and consider $r = \cos 3\theta$ instead of the four-leaf rose $r = \cos 2\theta$.

EXAMPLE 2

Describe the graph of $r = \cos 3\theta$.

Solution The resulting graph is shown in Figure 9.67, but we need to observe carefully how the curve is traced. First, we observe that the curve now consists of only three loops, and they are traced for values of θ between 0 and π. For any angles outside the interval $[0, \pi]$, the same points are produced, so the polar curve is periodic with period π, even though the function $f(\theta) = \cos 3\theta$ is periodic with period $2\pi/3$. Next, we observe that the curve starts when $\theta = 0$ and $r = 1$ to produce the point at the far right. It then forms a half loop and passes through the pole when $\theta = \pi/6$. The lower left full loop is traced for values of θ between $\pi/6$ and $\pi/2$. The upper left full loop is traced as θ ranges from $\pi/2$ to $5\pi/6$. The bottom half of the right-hand loop is completed as θ ranges from $5\pi/6$ to π. This curve is known as a *three-leaf rose.*

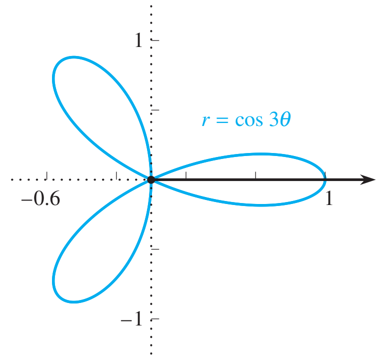


FIGURE 9.67

In general, the family of curves given by $r = \cos n\theta$ or $r = \sin n\theta$ for any positive integer n are called **rose curves.** Figures 9.68(a) and (b) show the graphs of $r = \cos 4\theta$ and $r = \cos 5\theta$; note that they contain eight and five loops, or *petals,* respectively.

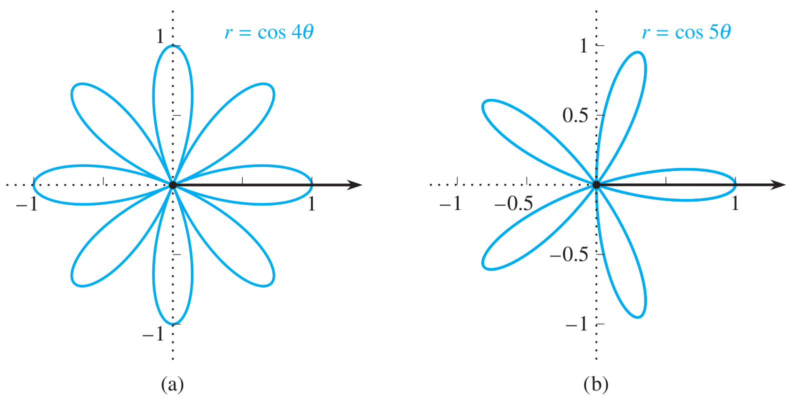


FIGURE 9.68

Think About This

1. Investigate some other cases using your polar function grapher until you can devise a rule to predict the number of petals in the rose curve $r = \cos n\theta$ for any positive integer n. Are there any numbers of petals that cannot occur in this family of rose curves? If so, what are they?
2. What can you conclude about the number of petals in the related family of rose curves given by $r = \sin n\theta$?

The Family of Cardioids

Let's consider another family of polar coordinate curves, those given in the form $r = a(1 \pm \cos\theta)$. Figure 9.69 shows the graph of $r = 1 + \cos\theta$, with $a = 1$. The heart-shaped appearance of this curve suggested its name, a *cardioid*.

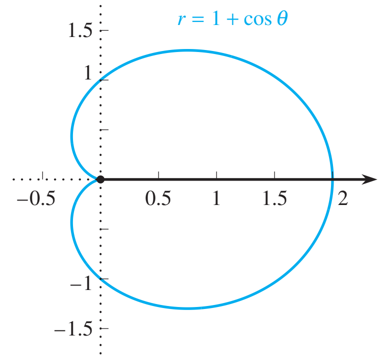


FIGURE 9.69

EXAMPLE 3

Describe how the graph of the cardioid $r = 1 + \cos\theta$ is traced.

Solution We start with $\theta = 0$ so that $r = 2$. The curve begins at the point at the far right. As θ increases to $\pi/2$, the curve arches upward. For θ between $\pi/2$ and π, the curve bends downward and eventually inward to the pole; the resulting point at $\theta = \pi$ is called a *cusp*. As the angle θ increases from π to 2π, the curve traces the mirror image of the upper half of the cardioid; this cardioid is symmetric about the polar axis with period 2π.

Those of you who have read Section 8.4 on chaos have seen that the primary central portion of the Mandelbrot set is a cardioid.

Think About This

a. What is the effect of a multiple a on the shape of the curve

$$r = a(1 + \cos\theta)?$$

b. Describe the graph of the related equation $r = 1 - \cos\theta$. How does it compare with the cardioid $r = 1 + \cos\theta$?

Think About This

Sketch some graphs of the related equations $r = 1 \pm \sin\theta$. Identify an axis of symmetry for them.

Think About This

Suppose that you combine the ideas on the rose curves and the cardioids to consider the class of polar equations of the form $r = 1 \pm \cos n\theta$ for different positive integers n. Determine a pattern regarding their shapes.

The Family of Limaçons

An extension of the cardioid known as the *limaçon* is defined by the equations $r = a \pm b \cos\theta$ and $r = a \pm b \sin\theta$. In particular, we consider two cases: $a > b$ and $a < b$. When $a = b$, both equations reduce to that of a cardioid. Figure 9.70 shows the graph of $r = 3 + 4\cos\theta$.

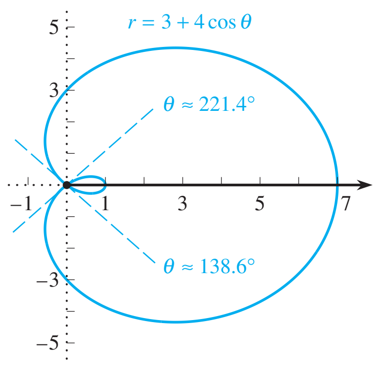


FIGURE 9.70

The curve starts at the far right, where $\theta = 0$ and $r = 7$. The curve then traces around the upper arch and eventually bends inward to pass through the pole. After passing through the pole, the curve traces the small inner loop and then passes through the pole again. It then traces the large outer loop below the polar axis, which is a mirror image of the large loop above the polar axis. The resulting curve is called a *limaçon with a loop*. (It comes from the Greek word *limax*, for snail, because the first half of the curve traced from $\theta = 0$ to $\theta = \pi$ resembles a snail-like shape.) For $\theta > 2\pi$, the curve precisely repeats this behavior.

EXAMPLE 4

At what angles does the limaçon curve $r = 3 + 4\cos\theta$ pass through the pole?

Solution The graph of the limaçon in Figure 9.70 shows two such angles—one in the "second quadrant" and the other in the "third quadrant". To find these angles, we use the fact that the pole corresponds to $r = 0$. Therefore, if we set $r = 0$, we get the equation

$$3 + 4\cos\theta = 0 \quad \text{so that} \quad \cos\theta = -\frac{3}{4}.$$

Thus one angle at which the limaçon passes through the pole must satisfy is

$$\theta = \arccos\left(-\frac{3}{4}\right) = 2.419 \text{ radians}, \quad \text{or} \quad 138.59°.$$

We use the symmetry of the cosine function to find the second solution at $\theta = 3.864$ radians, or 221.41°. These values agree with the visual estimates that can be made by looking at Figure 9.70.

Figure 9.71 shows the graph of the polar curve $r = 5 + 4\cos\theta$. This curve is known as a *limaçon without a loop*, or a *dimpled limaçon*. It is similar in appearance to a cardioid, but it does not reach the pole.

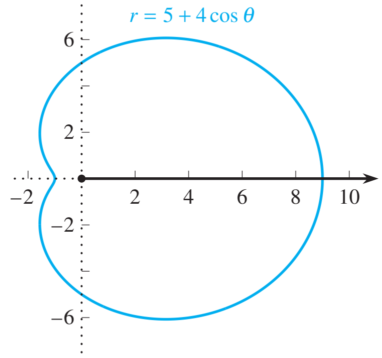


FIGURE 9.71

Think About This Can you account for the fact that the curve $r = 5 + 4\cos\theta$ never passes through the pole (where $r = 0$) and so never produces a loop?

Think About This Devise criteria based on the values of a and b in $r = a + b\cos\theta$ so that you can determine whether there is a loop. Be sure that you graph a variety of limaçons using your polar function grapher to collect enough information to know that you are correct.

Think About This Describe the shape of limaçons given by $r = a - b\cos\theta$.

Think About This What happens in the related family of limaçons given by $r = a \pm b\sin\theta$?

We urge you to experiment with the curves generated by polar coordinate equations. You can get some incredibly striking effects just by creating strange combinations of different functions. For instance, the graph of the polar function

$$r = \sin^5\theta + 8\sin\theta\cos^3\theta$$

is the butterfly shape shown in Figure 9.72.

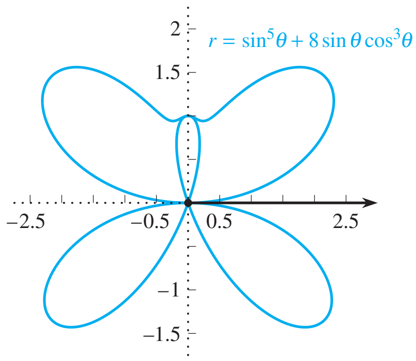


FIGURE 9.72

Think About This Explore systematically some family of polar functions—say, $r = \sin^n\theta$ for various integers n. You may well discover some fascinating new patterns and add some new items to the literature of mathematics.

Problems

In Problems 1–11, graph each polar curve using your polar function grapher. For each, use a variety of intervals for the angle θ until you obtain a "good" picture of the graph.

1. $r = \dfrac{4\sin^2\theta}{\cos\theta}$ (*Cissoid of Diocles*)
2. $r = \dfrac{1}{\sin\theta} - 2$ (*Conchoid of Nicomedes*)
3. $r = 4\sin\theta\cos^2\theta$ (*Bifolium*)
4. $r = 5\left(4\cos\theta - \dfrac{1}{\cos\theta}\right)$ (*Trisectrix*)
5. $r = \dfrac{3\sin\theta}{\theta}$ (*Cochleoid*)
6. $r = \dfrac{4}{\sqrt{\theta}}$ (*Lituus*)
7. $r = \dfrac{8}{\sin 2\theta}$ (*Cruciform*)
8. $r = \dfrac{10}{3 + 2\cos\theta}$ (*Ellipse*)
9. $r^2 = 4\cos 2\theta$ (*Lemniscate of Bernoulli*)

(*Caution*: Be sure to restrict your attention to values of θ that cause the right-hand side to be positive.)

10. $r^3 = 4\cos 3\theta$ (*Generalized Lemniscate*)

(*Caution*: Some programs and calculators are not able to evaluate the cube root of a negative number.)

11. $r = \dfrac{4}{\sin\theta}$

12–22. Repeat Problems 1–11 by changing some of the terms. What happens to the shape you produced if you use different values for the coefficients? What happens if you interchange sines and cosines? What happens if you change the multiples of θ? Keep a record of what you do and of your findings.

23. Consider the family of "hybrid rose curves"[1] given by $r = \cos(\frac{a}{b}\theta)$ for any rational number a/b.

a. By experimenting with different combinations of a and b, can you determine any rules for predicting the number of (overlapping) loops that will result? If so, state them.

b. Can you determine any rules for predicting the interval of angles θ needed to trace one complete petal of this curve? If so, state them.

c. Can you determine any rules for predicting the interval of angles θ needed to trace the entire curve? If so, state them. (*Hint*: Consider different cases, depending on whether a and b are odd or even.)

24. Consider the family of polar curves given by $r = \sin^n\theta$.

a. After graphing the curves corresponding to $n = 1$ and $n = 2$, what shape do you expect for $n = 3$? for $n = 4$?

b. Account for the fact that the shapes are not what you expected.

c. Determine a pattern for the number of loops that will correspond to any value of n.

d. What interval of angles corresponds to a complete curve? Do the same conclusions apply to $r = \cos^n\theta$?

25. Consider the family of generalized lemniscates given by $r^2 = \cos n\theta$. Can you find any pattern for the number and location of the loops that will result for any n? (*Caution*: When you try to graph these curves, you must take into account intervals of angles for which the function is well defined.)

26. Consider the family of generalized lemniscates given by $r^n = \cos n\theta$. Can you determine a pattern for the number and location of the loops that will result for any n? If so, what is it?

27. Consider the family of generalized limaçons given by $r = a + b\cos n\theta$. Can you find a pattern for the number and location of the loops that will result for any n? If so, what is it?

Chapter Summary

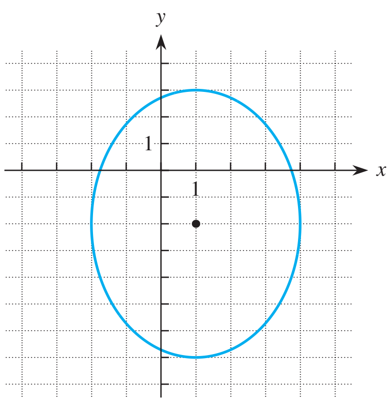
In this chapter we introduced and discussed a variety of topics related to coordinate systems in general and several specific coordinate systems in particular. This includes:

[1]These curves were studied in detail by a student, Kenneth Gordon, in the article, Investigating the petals of hybrid roses, *Mathematics and Computer Education,* 1992, 26, 66–73.

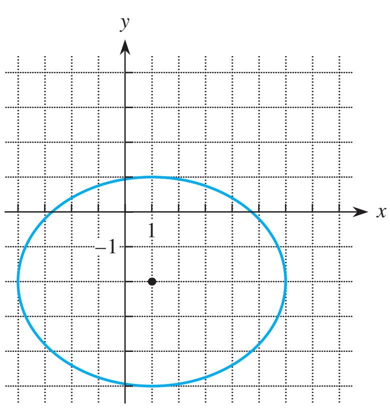
- What a coordinate system is.
- How to find the distance between points in the plane.
- How to find the midpoint of a line segment.
- How to find a point at any given distance along the line through two points.
- The parametric equations of a line.
- The equation of a circle.
- The equation of an ellipse, including finding its center, vertices, and foci.
- The reflection property of an ellipse and its applications.
- The equation of a hyperbola, including finding its center, vertices, and foci.
- Applications of the hyperbola.
- The equation of a parabola, including finding its vertex, focus, and directrix.
- The reflection property of a parabola and its applications.
- The parametric representation of curves in the plane.
- What the polar coordinate system is and how to transform between polar and rectangular coordinates.
- The behavior of families of curves in polar coordinates.

Review Problems

1. Find an equation of each ellipse shown.

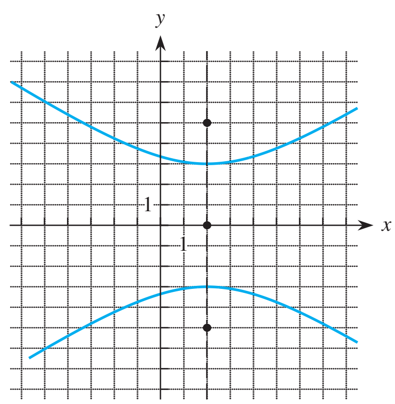


a.

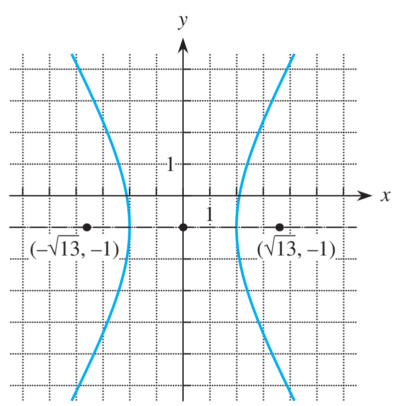


b.

2. Find an equation of each hyperbola shown.



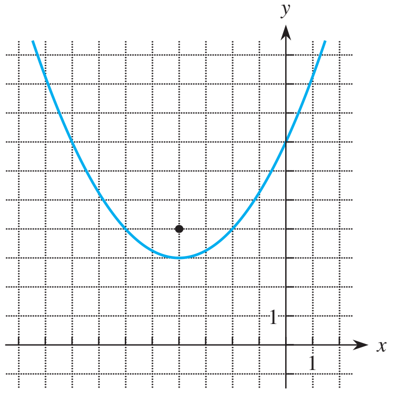
a.



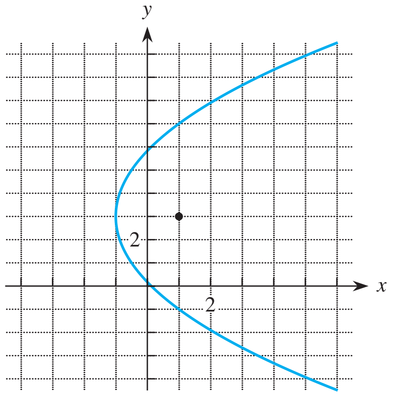
b.

3. Find an equation of each parabola shown.

a.



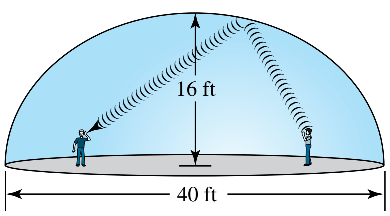
b.



4. Identify the conic whose equation is $xy = 5$ and sketch the curve.

In Problems 5–9, determine the equation for the standard form of the conic section. Identify the conic and sketch the curve. Wherever applicable, give the focus (foci), vertex (vertices), and center for each conic.

5. $x^2 - 6x + y - 34 = 0$
6. $x^2 + y^2 - 8x + 6y + 9 = 0$
7. $2x^2 + 3y^2 + 20x - 12y + 28 = 0$
8. $3x^2 - 4y^2 - 6x - 24y - 45 = 0$
9. $3y^2 - 2x^2 - 12y + 12x - 24 = 0$
10. Find the equation of the ellipse with foci $(-8, 0)$ and $(8, 0)$ and with vertices $(-10, 0)$ and $(10, 0)$.
11. Find the equation of the ellipse centered at $(-6, 3)$ with one focus at $(0, 3)$ and the minor axis 4 units long.
12. Find the equation of the hyperbola centered at $(2, 3)$ with one focus at $(2, 7)$ and the corresponding vertex at $(2, 6)$.
13. Find the equation of the hyperbola that has vertices $(0, \pm 4)$ and passes through the point $(6, \sqrt{80})$.
14. Find the equation of the set of points P with coordinates (x, y) in the plane such that the sum of the distance from $(12, 0)$ to P and the distance from $(-12, 0)$ to P is 30.
15. An ellipse passes through the point P at $(\sqrt{3}/2, 2)$ and has foci at $(-3, 0)$ and $(3, 0)$. Use the geometric definition of an ellipse to find the equation of this ellipse.
16. The ceiling of a whispering gallery is built so that the highest point of the structure is 16 feet above the floor. The floor has vertices 40 feet apart. Where along the axes should each person stand to be able to get the "whispering effect"? Ignore the height of the two people.



17. A lithotripter is a medical device used by doctors to break up kidney stones by bombarding them with intense bursts of sound waves, using the reflection property of an ellipse. The device is situated so that the sound waves emanate from one focus, reflect off an elliptic-shaped bowl, and come together to strike the kidney stone at the other focus. The distance between the two foci is 23 cm, and the distance from the source focus to the vertex on the elliptic reflector bowl is 3 cm. Find the equation of an elliptic cross section of the lithotripter bowl.
18. Let $x = 4 - t$ and $y = 2 + 3t$. Graph the points (x, y) for $t = -2, -1, 0, 1$, and 2. Find the function determined by the parametric equations.
19. Sketch the parametric curve given by

$$x = 3t \quad \text{and} \quad y = t^2 + 1, \qquad \text{for } -2 \le t \le 2.$$

20. Let $x = t^2 + 3$ and $y = t^3 - 1$.
   a. Graph the curve for $-4 \le t \le 4$.
   b. Eliminate t and write an expression for the curve in x and y.
   c. At what value of x is $y = 0$?

21. Sketch the curve

$$x = 1 - \log t \quad \text{and} \quad y = \log t, \qquad \text{for } 1 \leq t \leq 10.$$

a. Eliminate the parameter to find the expression for y as a function of x.
b. What is the largest possible domain for this function?

22. Graph the equations

$$x = \sin 2t \quad \text{and} \quad y = \cos \frac{1}{2}t,$$

for $-2\pi \leq t \leq 2\pi$.

23. Compare the graph in Problem 22 to the graphs of

a. $x = \sin 4t$, $y = \cos 2t$;
b. $x = \sin 6t$, $y = \cos 2t$;
c. $x = \sin 6t$, $y = \cos t$.
d. Determine the period of each graph in parts (a)–(c).

24. Use appropriate trigonometric identities to eliminate t and write the following expressions in terms of x and y:

$$x = \cos 2t \quad \text{and} \quad y = \sin t.$$

25. Transform each point from rectangular coordinates (x, y) to an equivalent point in polar coordinates.

a. $(3, 3)$
b. $(-1, 3)$
c. $(4, -1)$
d. $(0, 6)$.

26. Transform each point from polar coordinates (r, θ) to rectangular coordinates.

a. $(3, \pi/3)$
b. $(3, \pi/4)$
c. $(4, 3\pi/2)$
d. $(4, 5\pi/4)$
e. $(5, 5\pi/6)$
f. $(5, 2)$

27. Using polar coordinates, sketch the curve

$$r = \frac{1}{1 + \cos\theta}.$$

Convert the polar expression to rectangular coordinates and find the equation of the conic.

28. The polar equation of a well-known family of curves is

$$r = \frac{1}{\sqrt{\dfrac{\cos^2\theta}{a^2} + \dfrac{\sin^2\theta}{b^2}}}.$$

What are these curves?

In Problems 29–32, compare the graphs of each set of equations.

29. $r = \cos\theta$, $r = \cos 2\theta$, and $r = \cos 4\theta$.

30. $r = \cos\theta$, $r = \cos 3\theta$, and $r = \cos 5\theta$.

31. $r = \sin\theta$, $r = \sin 2\theta$, and $r = \sin 4\theta$.

32. $r = \sin\theta$, $r = \sin 3\theta$, and $r = \sin 5\theta$.