8

More About the Trigonometric Functions

8.1 Relationships Among Trigonometric Functions

In many applications of trigonometry, particularly in calculus, it often is necessary to transform one trigonometric function into another using an appropriate *trigonometric identity.* Recall that an identity is a relationship that is true for *all* values of the variable. For instance, the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1 \qquad (1)$$

that we discussed in Section 6.4 holds for every value of x.

However, suppose that we ask whether $\sin x + \cos x$ equals 1. Figure 8.1 shows a portion (one complete cycle) of the graph of $y = \sin x + \cos x$. Note that the function is not identically equal to 1 because its graph is not a horizontal line of height 1. Although there are several specific values of x for which $\sin x + \cos x$ equals 1 (such as $x = 0$, $x = \pi/2$ and $x = 2\pi$), the relationship does not hold for *every* value of x. So $\sin x + \cos x = 1$ is not an identity, but simply an equation that holds for some specific values of the variable.

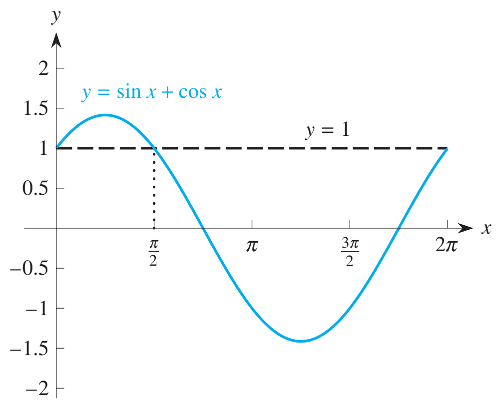


FIGURE 8.1

Identities Involving the Sine and Cosine

Consider again the Pythagorean identity (1). We can use it to transform sines to cosines with

$$\cos^2 x = 1 - \sin^2 x$$

so that

$$\cos x = \pm\sqrt{1 - \sin^2 x}.$$

Similarly, we can transform cosines to sines by using

$$\sin^2 x = 1 - \cos^2 x$$

so that

$$\sin x = \pm\sqrt{1 - \cos^2 x}.$$

Each of these equations holds for all values of the variable x, so each is an identity.

The Reflection Identities

We explore several other useful relationships among the trigonometric functions here. Two properties of the sine and cosine functions are

$$\sin(-x) = -\sin x \qquad \textbf{(2)}$$

$$\cos(-x) = \cos x, \qquad \textbf{(3)}$$

for any x. These two relationships, known as the **reflection identities,** are easy to see graphically. The graph of the cosine function is symmetric about the vertical y-axis, as illustrated in Figure 8.2. That is, for any positive value of x, the height of the cosine function is the same to the left of the y-axis (at $-x$) as it is at the same distance to the right of the y-axis (at x). Thus

$$\cos(-x) = \cos x$$

for any value of x. We discussed this same type of behavior in Section 2.7 for power functions with even powers such as $f(x) = x^2$ and $g(x) = x^4$. For this reason, the cosine function is called an *even function.*

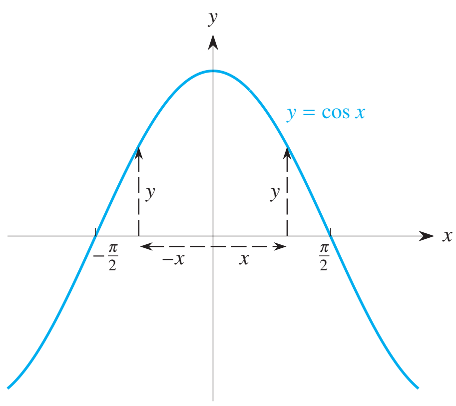


FIGURE 8.2

However, the sine curve is not symmetric about the y-axis. Rather, if you move a distance of x to the left of the y-axis and consider the height to the sine curve, it is equivalent, but opposite in sign, to the height you get if you move the same distance x to the right of the y-axis, as shown in Figure 8.3. Thus

$$\sin(-x) = -\sin x,$$

for any value of x. We encountered this type of behavior with power functions such as $g(x) = x^3$ when the power is odd. As a result, the sine function is called an *odd function.*

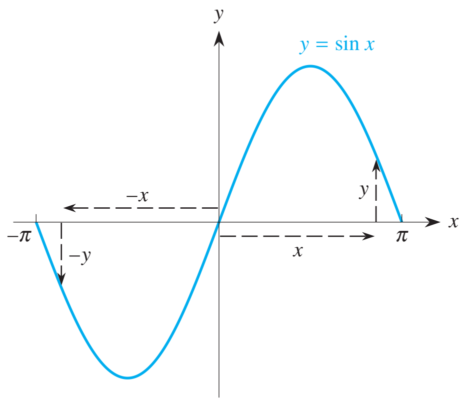


FIGURE 8.3

We discuss even and odd functions again in Section 8.2 when we describe connections between polynomial functions and trigonometric functions.

Think About This Write a reflection identity for the tangent function.

The Double-Angle Identities

We next consider some additional relationships involving the sine and cosine. The Pythagorean identity says that $\sin^2 x + \cos^2 x = 1$, which is equivalent to $\cos^2 x + \sin^2 x = 1$. What happens if we take the difference instead of the sum? Figure 8.4 shows the graph of $y = \cos^2 x - \sin^2 x$, for x between 0 and 2π. It is a sinusoidal curve that oscillates between -1 and 1 and completes two full cycles between 0 and 2π, so it has a period of π and a frequency of 2. But these features exactly describe the function $y = \cos 2x$. Therefore it seems that

$$\cos^2 x - \sin^2 x = \cos 2x,$$

or, equivalently,

$$\cos 2x = \cos^2 x - \sin^2 x. \qquad \textbf{(4)}$$

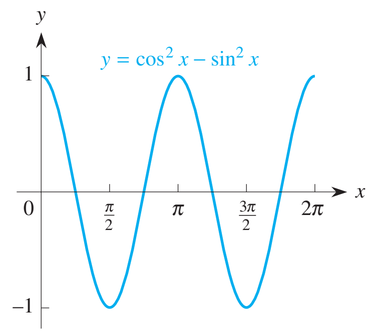


FIGURE 8.4

You can verify this relationship numerically by substituting any value of x into Equation (4). Alternatively, you can verify this relationship graphically by examining the graphs of the functions $y = \cos 2x$ and $y = \cos^2 x - \sin^2 x$.

We can rewrite Equation (4) in several alternative, but equivalent, forms by making use of the Pythagorean identity (1). Thus

$$\cos 2x = \cos^2 x - \sin^2 x = (1 - \sin^2 x) - \sin^2 x$$

so that

$$\cos 2x = 1 - 2\sin^2 x. \tag{4a}$$

Similarly, we can rewrite Equation (4) as

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x - (1 - \cos^2 x)$$

so that

$$\cos 2x = 2\cos^2 x - 1. \tag{4b}$$

Verify the identities in Equations (4a) and (4b) visually by using your function grapher and numerically by substituting several different values for x into the equations.

We next consider $\sin 2x$. Suppose that we want to express $\sin 2x$ in an equivalent form that does not show the frequency 2 explicitly. Is it possible that $\sin 2x$ and $2 \sin x$ are equivalent? Graph the two functions and you'll see that they cannot be the same. The first, $y = \sin 2x$, is a sinusoidal curve with an amplitude of 1 and a frequency of 2, so its values oscillate between -1 and 1 and it completes two full cycles between $x = 0$ and $x = 2\pi$. The second function, $y = 2 \sin x$, is a sinusoidal curve with an amplitude of 2 and a frequency of 1, so its values oscillate between -2 and 2 and it completes one full cycle between 0 and 2π.

The actual relationship for $\sin 2x$ is

$$\sin 2x = 2 \sin x \cos x. \tag{5}$$

You can verify Equation (5) graphically on your function grapher. When you graph the two functions $y = \sin 2x$ and $y = 2 \sin x \cos x$ simultaneously, you will see only one graph—the second traces precisely over the first. You can also verify this result numerically: Pick any value for x and evaluate $\sin 2x$ and $2 \sin x \cos x$. The results will be identical for every value of x, thus supporting the fact that Equation (5) is an identity.

The identities in Equations (4), (4a), (4b), and (5) are known as the **double-angle identities** for the sine and cosine.

The Sum and Difference Identities

The double-angle identities in Equations (4) and (5) actually are special cases of more general identities known as the sum and difference identities for sine and cosine that are formally derived in any trigonometry text. The **sum identities** are

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \tag{6}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y. \tag{7}$$

To show how the double-angle identities are derived from these formulas, we set $y = x$ in Equations (6) and (7). For instance, in Equation (6),

$$\sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x,$$

giving

$$\sin(2x) = 2 \sin x \cos x.$$

The same process in Equation (7) produces the double-angle formula for the cosine.

Similarly, we can replace y with $-y$ in the two sum identities, Equations (6) and (7), and then use the reflection identities to derive the **difference identities** for the sine and cosine:

$$\begin{aligned}\sin(x - y) &= \sin x \cos(-y) + \cos x \sin(-y) \\ &= \sin x \cos y - \cos x \sin y \qquad \textbf{(8)}\end{aligned}$$

$$\begin{aligned}\cos(x - y) &= \cos x \cos(-y) - \sin x \sin(-y) \\ &= \cos x \cos y + \sin x \sin y. \qquad \textbf{(9)}\end{aligned}$$

EXAMPLE 1

Show that $\cos(x - \pi/2) = \sin x$ for all x by using the difference identity for the cosine.

Solution Using the difference identity in Equation (9), we have

$$\begin{aligned}\cos\left(x - \frac{\pi}{2}\right) &= \cos x \cos \frac{\pi}{2} + \sin x \sin \frac{\pi}{2} \\ &= \cos x \cdot (0) + \sin x \cdot (1) = \sin x.\end{aligned}$$

EXAMPLE 2

Reduce $\sin 3x$ to an equivalent expression involving only sines and not including any multiple angles.

Solution We write

$$\begin{aligned}\sin 3x = \sin(2x + x) &= \sin 2x \cos x + \cos 2x \sin x && \text{Sum identity} \\ &= (2 \sin x \cos x)\cos x + (1 - 2 \sin^2 x)\sin x && \text{Double angle identity} \\ &= 2 \sin x \cos^2 x + \sin x - 2 \sin^3 x && \\ &= 2 \sin x \cdot (1 - \sin^2 x) + \sin x - 2 \sin^3 x && \text{Pythagorean identity} \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x && \\ &= 3 \sin x - 4 \sin^3 x. &&\end{aligned}$$

EXAMPLE 3

Reduce $\sin 4x$ to an equivalent expression involving sines and cosines that has no multiple angles.

Solution Following the approach in Example 2, we write

$$\begin{aligned}\sin 4x &= \sin(3x + x) \\ &= \sin 3x \cos x + \cos 3x \sin x. && \text{Sum identity}\end{aligned}$$

But, this expression involves expanding cos $3x$, and we haven't worked that out yet. (You are asked to do so in a problem at the end of this section.) Alternatively, we could start with

$$\begin{aligned}
\sin 4x &= \sin(2x + 2x) = \sin[2(2x)] \\
&= 2 \sin 2x \cos 2x && \text{Double angle identity} \\
&= 2(2 \sin x \cos x)(\cos^2 x - \sin^2 x) && \text{Double angle identities} \\
&= 4 \sin x \cos^3 x - 4 \sin^3 x \cos x.
\end{aligned}$$

The Half-Angle Identities

Occasionally we face the reverse problem of starting with powers of the sine or cosine—say, $\sin^3 x$ or $\cos^4 x$—and having to eliminate all powers by rewriting the expression in terms of sines and cosines with multiple angles. To eliminate the powers, we make use of two additional identities. Starting with the double-angle identity in Equation (4a),

$$\cos 2x = 1 - 2 \sin^2 x,$$

we have

$$2 \sin^2 x = 1 - \cos 2x$$

so that

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x). \tag{10}$$

Similarly, if we start with the double-angle identity in Equation (4b),

$$\cos 2x = 2 \cos^2 x - 1,$$

we get

$$2 \cos^2 x = 1 + \cos 2x$$

or

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x). \tag{11}$$

The identities in Equations (10) and (11) are the **half-angle identities.** Verify them graphically on your function grapher. We illustrate their use in Example 4.

EXAMPLE 4

Rewrite $\cos^4 x$ in terms of cosines of multiple angles by eliminating all exponents.

Solution Using Equation (11), we have

$$\cos^4 x = (\cos^2 x)^2 = \left[\frac{1}{2}(1 + \cos 2x)\right]^2$$

$$= \frac{1}{4}[1 + 2\cos 2x + \cos^2(2x)].$$

This expression involves $\cos^2(2x)$, so we apply Equation (11) again to get

$$\cos^4 x = \frac{1}{4}\left\{1 + 2\cos 2x + \frac{1}{2}[1 + \cos 2(2x)]\right\}$$
$$= \frac{1}{4}\left(1 + 2\cos 2x + \frac{1}{2} + \frac{1}{2}\cos 4x\right)$$
$$= \frac{1}{4}\left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)$$
$$= \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

For easy reference, we list all the fundamental trigonometric identities involving the sine and cosine functions. These identities reappear both in this course and in later mathematics and associated courses.

Trigonometric Identities

Pythagorean identity:	$\sin^2 x + \cos^2 x = 1$	(1)
Reflection identities:	$\sin(-x) = -\sin x$	(2)
	$\cos(-x) = \cos x$	(3)
Double-angle identities:	$\cos 2x = \cos^2 x - \sin^2 x$	(4)
	$= 1 - 2\sin^2 x = 2\cos^2 x - 1$	(4a,b)
	$\sin 2x = 2\sin x\cos x$	(5)
Sum identities:	$\sin(x + y) = \sin x\cos y + \cos x\sin y$	(6)
	$\cos(x + y) = \cos x\cos y - \sin x\sin y$	(7)
Difference identities:	$\sin(x - y) = \sin x\cos y - \cos x\sin y$	(8)
	$\cos(x - y) = \cos x\cos y + \sin x\sin y$	(9)
Half-angle identities:	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$	(10)
	$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$	(11)

Using the Trigonometric Identities

Suppose that a projectile, such as a cannonball or a high-pressure stream of water, is shot off with an initial velocity v_0 at an angle θ with the horizontal, as shown in Figure 8.5. The distance R that the cannonball or the water travels—its *range*—depends on the angle θ. For very small angles, the range is minimal because gravity pulls the object to the ground quickly. For very large angles (close to 90°), the object is shot almost vertically upward and comes back to the ground fairly near the point at which it was released. For moderately sized angles, the range is considerably larger.

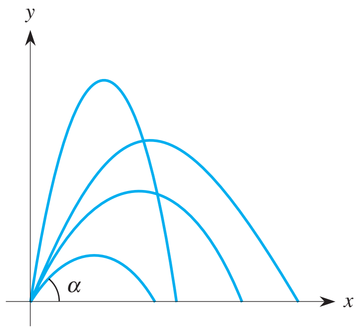


FIGURE 8.5

It is shown in physics that the range R is given by

$$R = \frac{2v_0^2 \sin\theta\cos\theta}{g},$$

where g is the acceleration due to gravity.

EXAMPLE 5

Use a trigonometric identity to simplify the formula for the range of a projectile and use the result to determine the angle that leads to the maximum range for any initial velocity.

Solution The range is

$$R = \frac{2v_0^2 \sin\theta\cos\theta}{g}.$$

Because $\sin 2\theta = 2\sin\theta\cos\theta$, this expression for the range reduces to

$$R = \frac{v_0^2 \sin 2\theta}{g}.$$

Because g and v_0 are fixed, the range is maximal when $\sin 2\theta$ is maximal and the largest value of the sine function is 1, which occurs when $2\theta = 90°$ or $\theta = 45°$. Therefore a projectile subject only to the force of gravity has a maximum range when the initial angle $\theta = 45°$.

In most derivations in physics and engineering involving wave phenomena such as electromagnetic waves (e.g., radio signals or electric currents in a circuit), sound waves, or water waves, the height y of the wave as a function of time t is usually given in the form

$$y = A\sin kt + B\cos kt,$$

where A and B are constants and k is the frequency. In Example 6, we show how this type of expression can be simplified by using a trigonometric identity to give far more insight into the behavior of the wave than this fairly complicated expression provides.

EXAMPLE 6

The equation of a wave is $y = 4 \sin t - 3 \cos t$. Use a trigonometric identity to explain the behavior of this wave.

Solution The graph of this function between $t = 0$ and $t = 2\pi$ is shown in Figure 8.6. It looks like a sine wave shifted horizontally to the right by about $\pi/6$, or 30°. Also, the amplitude of this wave seems to be about 5, compared to the amplitudes of 4 and 3 in the two terms of the function. Finally, the period of this wave seems to be about 2π. As a result, the equation of the wave $y = 4 \sin t - 3 \cos t$ appears to be equivalent to $y = C \sin(t - D)$, where the amplitude $C \approx 5$ and the phase shift D is about $\pi/6$. Let's see why.

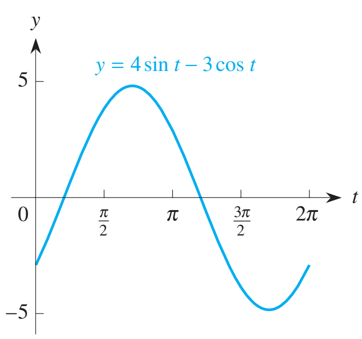


FIGURE 8.6

The seemingly equivalent form for the wave $y = C \sin(t - D)$ suggests using the difference formula for the sine, or

$$C \sin(t - D) = C \cdot (\sin t \cos D - \cos t \sin D). \qquad \textbf{(12)}$$

Also, the fact that the individual amplitudes in the original formula are 4 and 3 and the apparent amplitude we observe for the wave is about $C = 5$ suggests the Pythagorean theorem

$$\sqrt{4^2 + 3^2} = 5.$$

Factoring 5 out of the original formula for the wave yields

$$y = 4 \sin t - 3 \cos t = 5\left(\frac{4}{5} \sin t - \frac{3}{5} \cos t\right).$$

Comparing this expression to Equation (12) suggests that we make the association

$$C = 5, \qquad \cos D = \frac{4}{5}, \quad \text{and} \quad \sin D = \frac{3}{5}.$$

If $\cos D = 4/5$,

$$D = \arccos \frac{4}{5} \approx 36.87°,$$

which is close to what we predicted for the phase shift, based on the graph shown in Figure 8.6. To be sure that this result is consistent with the third condition, we see that if $\sin D = 3/5$,

$$D = \arcsin \frac{3}{5} \approx 36.87°.$$

Consequently, the wave formula

$$\begin{aligned} y &= 4\sin t - 3\cos t = 5(\cos D \sin t - \sin D \cos t) \\ &= 5\sin(t - 36.87°) \end{aligned}$$

and the original wave is the same as a pure sinusoidal function centered about $y = 0$ with amplitude 5, period $2\pi = 360°$, and a phase shift of about 36.87°, or 0.6435 radian.

Identities Involving the Tangent

Just as there are trigonometric identities relating the sine and cosine, there are identities involving the tangent function. We encountered two of them in Section 6.4. The first is the key identity relating the tangent function to the sine and the cosine:

$$\tan x = \frac{\sin x}{\cos x}.$$

We also derived an analog to the Pythagorean identity in Section 6.4:

$$\tan^2 x + 1 = \frac{1}{\cos^2 x},$$

provided that $\cos x \neq 0$.

Likewise, there are double-angle, sum, and difference formulas, and so on, for the tangent function. We investigate a double-angle identity and a sum identity in the Problems. If you're interested, you can find more details about them in any trigonometry textbook.

Problems

1. Using ideas on amplitude and frequency, explain why $\cos 3x$ cannot be identically equal to $3\cos x$.
2. Using ideas on amplitude, explain why $\cos 2x = 2\cos^2 x - 1$ is reasonable. (Recognize that such an argument is not a proof.)

Examine each equation in Problems 3–14 graphically to see if the relationship may be an identity. If it is not an identity, attempt to locate graphically or numerically at least one point that lies on both curves. If it seems to be an identity, prove it algebraically.

3. $\sin^3 x + \cos^3 x = 1$
4. $\cos 3x = \cos^3 x - \sin^3 x$
5. $\dfrac{\sin 2x}{\sin x} = 2\cos x$
6. $(1 - \cos\theta)(1 + \cos\theta) = \sin^2\theta$
7. $\sin 3x = 3\sin x$
8. $\dfrac{\cos^2\theta}{1 + \sin\theta} = 1 - \sin\theta$
9. $\dfrac{1 - \cos\alpha}{\sin\alpha} = \dfrac{\sin\alpha}{1 + \cos\alpha}$
10. $\cos 3\beta = 3\cos^3\beta - 1$
11. $\sin^2 3x + \cos 6x = \cos^2 3x$
12. $\cos^2 2x = 3(1 - \sin 2x)$
13. $\sin(\cos x) = \cos(\sin x)$
14. $\sin(\cos x) = \sin x \cos x$
15. Express $\cos 3x$ in terms of powers of $\sin x$ and $\cos x$, but with no multiple angles.
16. Express $\cos 4x$ in terms of powers of $\sin x$ and $\cos x$, but with no multiple angles.
17. Express $\cos 5x$ in terms of powers of $\sin x$ and $\cos x$, but with no multiple angles.
18. Examine the results of Problems 15–17 and the formula for $\cos 2x$. Are there any patterns in the terms? If so, what are they?
19. By setting $y = x$ in the sum identity in Equation (7), show that you get the double-angle identity in Equation (4).
20. Rewrite $\sin^4 x$ in terms of multiple angles by eliminating all exponents.

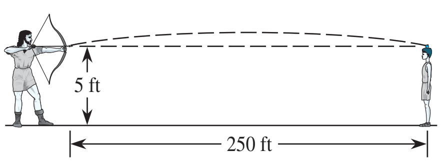
21. Rewrite $\sin^2 x \cos^2 x$ in terms of multiple angles by eliminating all exponents.

22. **a.** Sketch the graph of $\sin(x + \pi/2)$. What familiar function do you get from this phase shift?
b. Use the sum identity for the sine function to show that $\sin(x + \pi/2)$ actually equals that function.

23. **a.** Repeat Problem 22 for $\sin(x + \pi)$.
b. Repeat Problem 22 for $\cos(x + \pi/2)$.

24. In the half-angle identities in Equations (10) and (11), let $y = 2x$, so that $x = \frac{1}{2}y$. Rewrite each identity in terms of y to see why they are called half-angle identities.

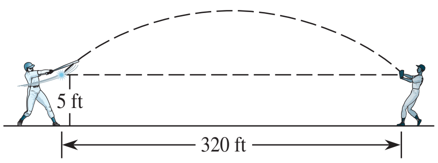
25. William Tell is about to shoot the most important arrow of his life. His son is standing 250 feet away. Tell releases the arrow at a height of 5 feet above the ground with an initial speed of 180 feet per second. The height of the center of the apple on his son's head is also 5 feet above the ground. Find algebraically two different angles α at which Tell should release the arrow in order to have it pass through the apple without hitting the boy.



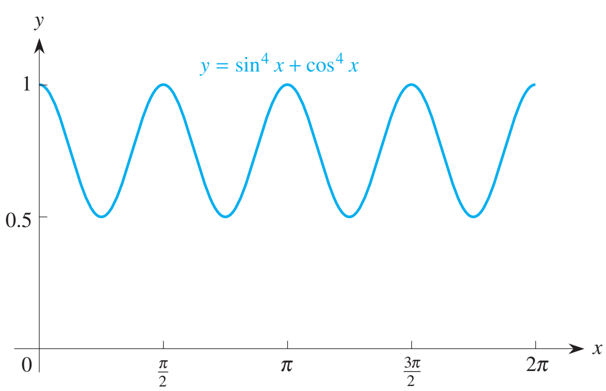
26. Suppose that William Tell's son is actually a foot shorter than in Problem 26 so that the center of the apple is now 4 feet above the ground and that the arrow comes off the bow string at a height of 5 feet. Estimate, graphically, two different angles α at which Tell should release the arrow in order for it to pass through the apple without hitting the boy.

27. In Example 6, we converted the wave $y = 4 \sin t - 3 \cos t$ to the equivalent pure sinusoidal expression $y = 5 \sin(t - 36.87°)$.
a. Convert this formula to a pure cosine curve by an appropriate horizontal shift.
b. Repeat the derivation in Example 6 by using the sum or difference identity for cosines to derive the equivalent formula as a cosine wave.
c. How does the result in part (b) compare to the result in part (a)?

28. A baseball player hits a ball with an initial velocity of 120 feet per second at a height of 5 feet above the ground. The ball is caught 320 feet from home plate by an outfielder whose glove is also 5 feet above the ground. Use the formula for the range of a projectile to determine the angle of inclination of the ball as it comes off the bat.



29. The accompanying figure shows the graph of the function $y = \sin^4 x + \cos^4 x$ from $x = 0$ to 2π. The curve suggests that the function is equivalent to some sinusoidal function.



a. By examining the graph carefully on your function grapher, estimate values for each parameter to find a sinusoidal function that seems to have the matching behavior pattern. (*Hint*: The parameters should be simple fractions or whole numbers.)
b. Superimpose the graph of your function over the graph of $y = \sin^4 x + \cos^4 x$ to verify that they do appear to be the same.
c. Use the half-angle identities for sine and cosine repeatedly to prove that $y = \sin^4 x + \cos^4 x$ does reduce to the expression you conjectured.

30. Repeat Problem 29 with the function $y = \sin^6 x + \cos^6 x$.

31. Refer to the functions shown in Problem 1 of Section 7.2 and decide which are odd, even, or neither.

32. **a.** Use some ideas from Section 5.5 on the sum of the terms in an exponential sequence to explain why you can calculate the value of

$$1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots$$

as $\dfrac{1}{1 - \sin x}$.

Are there any values of x for which this approach does not work?

b. What formula would you get for the sum of the terms

$$1 + \sin x + \sin^2 x + \cdots + \sin^n x$$

for any given positive integer n?

33. Use the result of Problem 32(b) with different values of n to calculate the value of

$$1 + \sin x + \sin^2 x + \sin^3 x + \sin^4 x + \cdots,$$

for $x = \pi/6$ correct to three decimal places. Now suppose that you want to do so for $x = \pi/3$ instead. Will you need approximately the same number of terms, more terms, or fewer terms to get the same three decimal place accuracy? Explain.

34. a. Verify graphically that

$$\tan\theta + \frac{1}{\tan\theta} = \frac{1}{\sin\theta\cos\theta},$$

for all θ for which the denominators are nonzero.

b. Show algebraically that the expression in part (a) is an identity. (*Hint*: Transform $\tan\theta$ to equivalent expressions in $\sin\theta$ and $\cos\theta$.)

35. Use appropriate trigonometric identities to show that

$$\frac{1}{\tan x} - \tan x = \frac{2}{\tan 2x}.$$

36. Use the identity in Problem 35 to derive a double-angle formula for the tangent function.

37. a. Derive the double-angle identity

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

by using the double-angle identities for sine and cosine. (*Hint:* Divide both the numerator and the denominator by $\cos^2 x$.)

b. Derive the addition identity for the tangent,

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

by using the addition formulas for sine and cosine. (*Hint:* Divide both the numerator and the denominator by $\cos x \cos y$.)

Examine each equation in Problems 38–46 graphically to see whether the relationship may be an identity. If it is not an identity, attempt to locate graphically or numerically at least one point that lies on both curves. If it seems to be an identity, prove it algebraically.

38. $1 + \dfrac{1}{\tan^2\theta} = \dfrac{1}{\sin^2\theta}$

39. $\tan 2\theta = 2\tan\theta$

40. $\tan^2 x - \sin^2 x = (\tan x \sin x)^2$

41. $1 - \tan^2 x = \dfrac{1}{\cos^2 x}$

42. $\tan\dfrac{\alpha}{2} = \dfrac{1 - \cos\alpha}{\sin\alpha}$ (*Hint*: Let $\dfrac{\alpha}{2} = \theta$.)

43. $\tan^2 x = 1 + 2\tan x$

44. $1 - \cos 2x = \tan x \sin 2x$

45. $\tan(\sin x) = \tan x \sin x$

46. $\cos(\tan x) = \tan(\cos x)$

47. What is wrong with the following "proof"?

$$\cos(\tan x) = \cos\left(\frac{\sin x}{\cos x}\right) = \sin x.$$

8.2 Approximating Sine and Cosine with Polynomials

Have you ever wondered what happens when you press either the `SIN` or `COS` key on your calculator and the value for the function appears? How does the calculator actually find the values of these functions?

Approximating the Sine Function

In this section, we consider one approach that has been used to compute function values. We begin by examining the graph of the sine function, with x measured in radians, as shown in Figure 8.7(a). We zoom in on the portion of the curve close to the origin, as marked by the box; the corresponding curve is shown in Figure 8.7(b). If we zoom in still further about the origin, as marked by the box in Figure 8.7(b), we get the portion of the sine curve shown in Figure 8.7(c). This final graph looks like a straight line rather than a portion of a curve. (In fact, if you zoom in sufficiently on any smooth curve, it will eventually look like a straight line.)

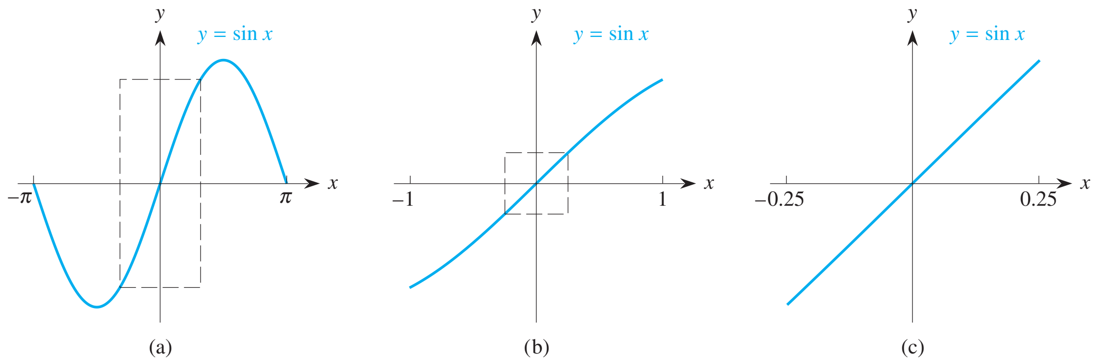


FIGURE 8.7

Linear Approximation to the Sine When x is very close to the origin, the sine curve looks like a line. Let's find the equation of this "line." Because it passes through the origin, the vertical intercept must be 0. To find the slope, we need a second point. If we trace along the sine curve very close to the origin, we find that $x = 0.001$ and $y = \sin 0.001 = 0.0009999998$ is a point on the sine curve. The slope of the line through this point and the origin is

$$m = \frac{0.0009999998 - 0}{0.001 - 0} = 0.9999998 \approx 1.$$

Therefore the equation of a line that very closely hugs the sine curve near the origin is $y = x$. We show the graph of this line, along with the sine curve, in Figure 8.8.

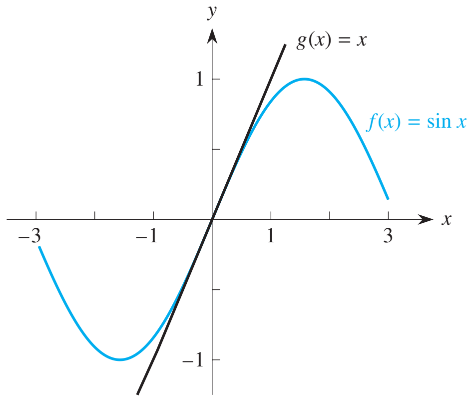


FIGURE 8.8

Observe that, when x is very close to 0, the graphs of $f(x) = \sin x$ and $g(x) = x$ are very close to one another. In fact, when x is very close to 0, the two graphs are virtually indistinguishable. That is,

$\sin x \approx x$, if x is very close to 0

Of course, as the value of x gets farther from 0, the sine curve eventually bends away from the line $y = x$.

x	$\sin x$	x	$\sin x$
0	0		
0.1	0.100	−0.1	−0.100
0.2	0.199	−0.2	−0.199
0.3	0.296	−0.3	−0.296
0.4	0.389	−0.4	−0.389
0.5	0.479	−0.5	−0.479
0.6	0.565	−0.6	−0.565
0.7	0.644	−0.7	−0.644

To show this result numerically, we look at some values of x to see how close the values along the line match the values of the sine function.

From the table at the left, we see that, when x is extremely close to 0, the value of $\sin x$ is almost identical to x itself, but the farther that x is from the origin, the less accurate the approximation. Thus, whenever x is very close to 0, we can replace $\sin x$ with x for the purposes of approximating the value of $\sin x$. For instance, to approximate $\sin(0.00243)$, we could say that

$$\sin(0.00243) \approx 0.00243.$$

Using a calculator gives $\sin(0.00243) = 0.0024299976$, so the approximation is accurate to five decimal places.

We can approximate the sine function with a linear function in a different way by using methods of linear regression. We use a set of points that lie on the sine curve $y = \sin x$ very close to the origin. They are shown rounded to 6 decimal places in Table 8.1. The line that best fits these "data" is $y = 0.9999258x$ with a correlation coefficient $r = 1.000000000$, which tells us that a line with slope of about 1 is virtually perfect. Thus we again see that when x is very close to 0, $\sin x \approx x$. However, if we move too far away from $x = 0$, the accuracy of the approximation breaks down. For instance, we would not want to approximate $\sin(0.75)$ with the value $x = 0.75$ because $\sin(0.75) = 0.6816$; the value $x = 0.75$ is too far from $x = 0$ for the approximation to be good.

TABLE 8.1

x	−0.025	−0.02	−0.015	−0.01	−0.005	0	0.005	0.01	0.015	0.02	0.025
$y = \sin x$	−0.024997	−0.019999	−0.014999	−0.01	−0.005	0	0.005	0.01	0.014999	0.019999	0.024997

This idea of approximating a function such as $y = \sin x$ with a simpler function (often a linear function) is an essential principle in mathematics. We use this principle to approximate the values of trigonometric functions because it is *impossible* to calculate them directly with algebraic methods.

Improving on the Linear Approximation to the Sine

Unfortunately, as we have noted, the linear approximation to the sine curve is only accurate if x is very close to the origin. As we take values of x farther and farther from the origin, the sine curve bends ever more sharply and eventually bends away from the line. Let's see how we can improve on the linear approximation $\sin x \approx x$ when x is somewhat farther from 0. To do so, we need a simple curve (at least one that is simpler to work with than the sine function) that bends in a similar manner. For computational purposes, the simplest curves are usually polynomials.

In Figure 8.7, we zoomed in on the sine curve very close to the origin so that the curve looked like a line. Now we zoom out a bit to see what happens for values of x from −3 to 3, as previously shown in Figure 8.8. Although the line is indistinguishable from the sine curve near the origin (roughly from $x = -0.6$ to $x = 0.6$), the sine curve bends away from the line as the first pair of turning points in the sine curve come into view. In fact, the overall shape of this portion of the sine curve is quite suggestive of a cubic polynomial with a negative leading coefficient. (Recognize that, if you zoom out a bit farther, more turning points appear and the cubic-like appearance disappears.) This result suggests that we try to approximate this portion of the sine curve with a cubic curve. We use the data in Table 8.1 that we used for the linear fit, but now fit a cubic polynomial instead. We then get the cubic

$$y = -0.1666601x^3 + 0x^2 + 0.999999999x + 0.$$

Note that (1) the constant term is 0, which assures us that the cubic passes through the origin; (2) the coefficient of the linear term is essentially 1; (3) the coefficient of the quadratic term is 0; (4) the leading coefficient is negative, which is what we expected; and (5) the value of the leading coefficient, -0.1666601, is quite close to $-1/6 = -0.16666667$. Thus a cubic polynomial that approximates the sine function is

$$T_3(x) = -\frac{x^3}{6} + x,$$

or, equivalently,

$$\sin x \approx x - \frac{x^3}{6}$$

when x is fairly close to 0.

Figure 8.9 shows both the sine curve and the cubic polynomial for x from -3.5 to 3.5. The two curves are indistinguishable from about $x = -1.2$ to $x = 1.2$, which extends over a considerably larger interval than the linear approximation, which is accurate only from about $x = -0.6$ to $x = 0.6$.

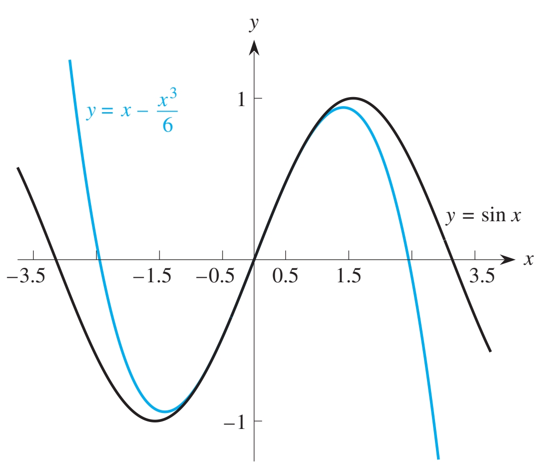


FIGURE 8.9

To illustrate the accuracy of the approximation for sin x using the cubic polynomial for values of x near 0, we try $x = 0.125$, say, and find that

$$\sin(0.125) \approx (0.125) - \frac{(0.125)^3}{6} \approx 0.1246744,$$

which agrees with the true value of $\sin(0.125) = 0.1246747$ to six decimal places. If we move farther from 0 and try $x = 0.7$, we find that

$$\sin(0.7) \approx (0.7) - \frac{(0.7)^3}{6} \approx 0.643,$$

compared to the actual value of $\sin(0.7) = 0.644$, which is correct to the nearest hundredth, so the approximation is still fairly accurate. However, the graphs in Figure 8.9 show that the two curves eventually diverge. Thus if we take x too far from 0, the accuracy of the approximation diminishes. Moreover, the farther from $x = 0$ we go, the worse the approximation is. For instance, if $x = 1$ radian $\approx 57°$,

$$\sin 1 \approx 1 - \frac{1^3}{6} \approx 0.83333,$$

compared to the correct value of $\sin 1 = 0.84147$. If $x = 1.5$ radians,

$$\sin(1.5) \approx (1.5) - \frac{(1.5)^3}{6} \approx 0.93750,$$

compared to the correct value of $\sin(1.5) = 0.99749$. If $x = 2$ radians,

$$\sin 2 \approx 2 - \frac{2^3}{6} \approx 0.66667,$$

compared to the correct value of $\sin 2 = 0.90930$. If $x = \pi$ radians,

$$\sin \pi \approx \pi - \frac{\pi^3}{6} \approx -2.02612,$$

compared to the correct value $\sin \pi = 0$. In fact, this last approximation is so bad that it gives us a value, -2.02612, outside the range of the sine function.

What if we wanted to improve on the approximation still further so that we could use it to estimate values for $\sin x$ when x is still farther from the origin? Consider the graph of the sine curve from $x = -6$ to $x = 6$ shown in Figure 8.10. It has four turning points and three inflection points, which suggests that the sine curve looks like a polynomial of degree 5. Although graphing calculators don't fit a fifth degree polynomial to a set of data, that task can be accomplished by many software packages. Using a spreadsheet, we find that the fifth degree polynomial that fits the data in Table 8.1 is

$$T_5(x) = 0.0083x^5 + 0x^4 - 0.1667x^3 + 0x^2 + 0.999999999x + 0,$$

or essentially

$$T_5(x) = 0.0083x^5 - \frac{x^3}{6} + x.$$

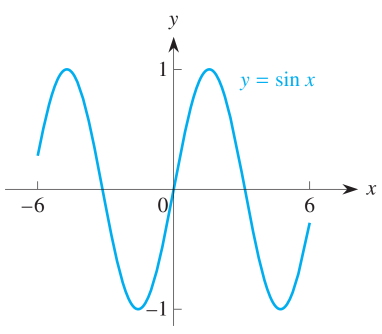


FIGURE 8.10

Note that this polynomial has a positive leading coefficient, so it increases toward the right, as we want. It also has a 0 constant coefficient, so it passes through the origin. Note also that the only change from the cubic polynomial to this fifth degree polynomial is the fifth degree term—all other terms remained the same.

Because $0.0083 \approx 1/120$, we can write this polynomial as

$$T_5(x) = \frac{x^5}{120} - \frac{x^3}{6} + x \quad \text{or} \quad T_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}.$$

A simple and interesting pattern is developing here with the coefficients: $6 = 3 \times 2 \times 1 = 3!$ and $120 = 5 \times 4 \times 3 \times 2 \times 1 = 5!$. (See Appendix A2 for a discussion of factorial notation.) So we can rewrite the approximation formula for the sine function as

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

Figure 8.11 shows the graphs of the sine function and the fifth degree polynomial for x between -4 and 4. The two curves are indistinguishable for x between roughly -2 and 2, so we have achieved a considerable improvement over the cubic approximation, which was a good match for x between roughly -1.2 and 1.2.

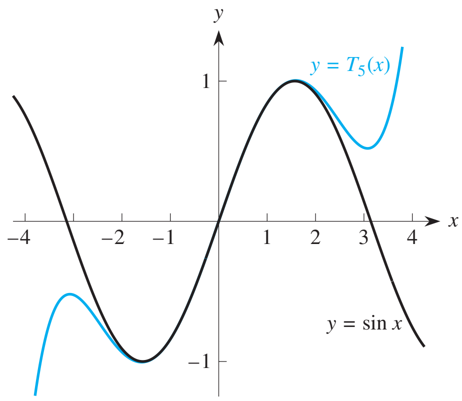


FIGURE 8.11

To verify the accuracy of this approximation, let's see how much improvement we get compared to the previous values. The results are shown in the following table.

x	$\sin x$	$T_3(x)$	$T_5(x)$
0.7	0.64422	0.643	0.64423
1	0.84147	0.83333	0.84167
1.5	0.99749	0.93750	1.00078
2	0.90930	0.66667	0.93333
π	0	-2.02612	0.52404

The fifth degree approximation is better still because we get more accurate estimates for the values of $\sin x$ over larger intervals of x-values centered at 0.

We can continue this process, using higher degree polynomials, and get even better approximations. However, before doing so, let's examine the sequence of polynomial approximations we have so far. They are

$$\sin x \approx x,$$

$$\sin x \approx x - \frac{x^3}{3!},$$

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}.$$

First, each successive polynomial involves just one additional term, compared to the preceding polynomial. Second, each polynomial involves only *odd* powers—and we know that the sine function is an *odd* function. This means that both the sine function and the approximating polynomials are symmetric about the origin. Third, the signs of successive coefficients alternate. Fourth, there is a definite pattern involving factorials in the coefficients. These polynomials are known as **Taylor polynomial approximations** after English mathematician Brook Taylor, who investigated them in the early 1700s.

Think About This Predict the next higher degree polynomial approximation to $\sin x$. How accurate is this approximation for the values $x = 0.7$, 1, 1.5, 2, and π?

Improving the Approximation Using the Behavior of sin x We could continue this process and construct Taylor polynomial approximations of higher and higher degree. However, that isn't necessary if we cleverly use some of the basic behavioral properties of the sine function. First, recall the reflection identity

$$\sin(-x) = -\sin x.$$

It allows us to approximate $\sin x$ when x is negative simply by using the corresponding positive value for x and reversing the sign of the estimate.

Second, we know that the sine function is periodic with period 2π. Therefore, if x is any number greater than $2\pi \approx 6.28$, the value of $\sin x$ is the same as the value of $\sin x_0$, where x_0 is the corresponding number between 0 and 2π radians. Consequently, we need only obtain an approximation that is accurate as far out as 2π. We can handle anything beyond that by reducing the value of x to an appropriate value x_0 between 0 and 2π by "removing" all multiples of 2π, as illustrated in Figure 8.12.

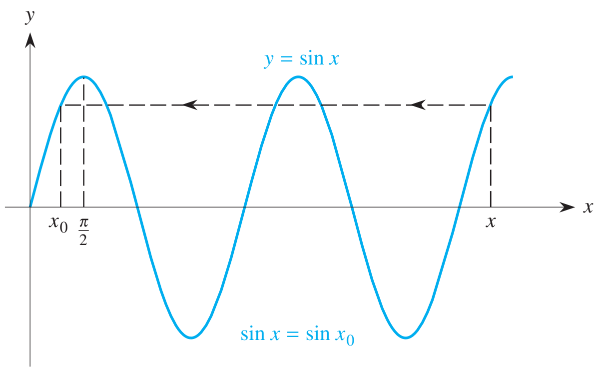


FIGURE 8.12

Now visualize the portion of the sine curve between $x = \pi$ and $x = 2\pi$, as shown in Figure 8.13. It has the same shape as the portion from $x = 0$ to $x = \pi$, only "flipped" across the x-axis. Thus, if we have a value of x between π and 2π (where $\sin x$ is negative), there is a point between 0 and π, namely at $x - \pi$, where the sine function has the same value, but with a positive sign. That is,

$$\sin x = -\sin(x - \pi).$$

So, all we need is an approximation that is sufficiently accurate for x between 0 and π.

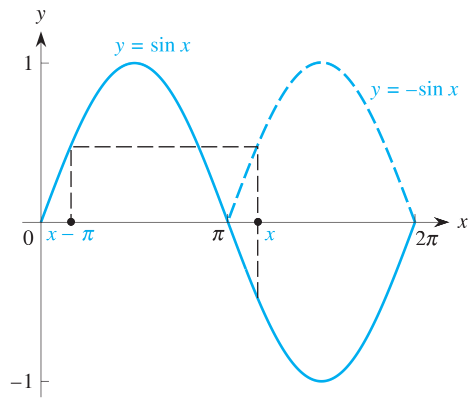


FIGURE 8.13

Think About This

Use an appropriate trigonometric identity to show that $\sin x = -\sin(x - \pi)$. ❐

Now visualize the sine curve from $x = 0$ to $x = \pi$. The two halves are symmetric, as shown in Figure 8.14. Therefore, for any point x between $\pi/2$ and π, the value of $\sin x$ is the same as that at a corresponding point between 0 and $\pi/2$. So, all we need is an approximation to $\sin x$ that is sufficiently accurate for x between 0 and $\pi/2$. The previous fifth degree polynomial $T_5(x)$ gives two-decimal accuracy for any value of x in this interval. If we want more than two-decimal accuracy, we have to use a higher degree polynomial—say, the seventh degree Taylor polynomial that we asked you to produce in a previous *Think About This* exercise.

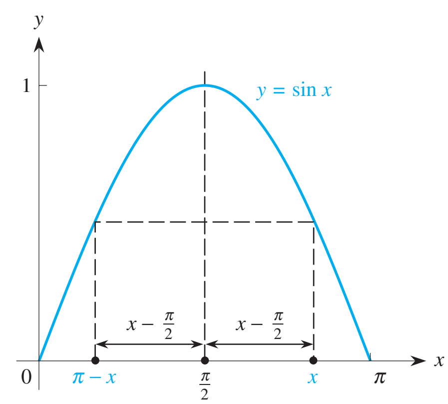


FIGURE 8.14

Think About This Try the seventh degree polynomial

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

for various values of x in the interval from 0 to $\pi/2$. Does it provide four-decimal accuracy? Is that adequate? If not, what would you do?

Approximating the Cosine Function

We now consider the comparable problem of approximating the cosine function by using a polynomial. If you zoom in on the cosine curve very close to $x = 0$, it appears indistinguishable from a horizontal line. In fact, because $\cos 0 = 1$, that line must be $y = 1$. So, for x very close to 0,

$$\cos x \approx 1.$$

However, once you move away from $x = 0$, the cosine curve bends away from the line $y = 1$.

Let's now look at the cosine curve in a somewhat wider interval about $x = 0$—say, from $x = -2$ to $x = 2$, as shown in Figure 8.15. Its overall shape suggests a parabola opening downward. Therefore we try to approximate the cosine function with a quadratic function so long as x remains fairly close to 0.

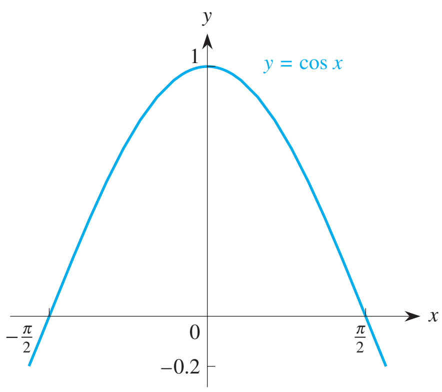


FIGURE 8.15

Approximating y = cos x Using Data Analysis There are several ways to find an equation for such a quadratic. One way is to fit a quadratic function to some set of values for $\cos x$ when x is relatively close to 0. Consider the values in Table 8.2.

TABLE 8.2

x	−0.08	−0.06	−0.04	−0.02	0	0.02	0.04	0.06	0.08
$y = \cos x$	0.9968	0.9982	0.9992	0.9998	1	0.9998	0.9992	0.9982	0.9968

Using a calculator, we find that the quadratic function that fits these data is

$$y = -0.49973x^2 + 0x + 0.99999.$$

The constant term is essentially 1 and the leading coefficient is approximately −0.5. Hence we have the following approximation to the cosine function near $x = 0$.

$$\cos x \approx 1 - \frac{x^2}{2}$$

Figure 8.16 shows the graphs of the cosine function and this quadratic Taylor approximating polynomial for x between -2 and 2. The two are virtually indistinguishable for x between about -0.8 and 0.8. For instance, $\cos(0.5) = 0.87758$ compared to the value of the approximating quadratic,

$$\cos(0.5) \approx 1 - \frac{(0.5)^2}{2} = 0.875,$$

so we have two decimal place accuracy.

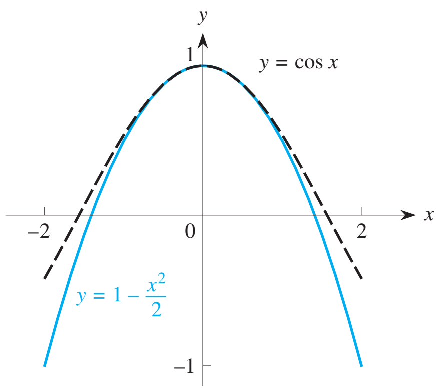


FIGURE 8.16

If we zoom out somewhat farther on the graph of the cosine curve—say, for x between -5 and 5, as shown in Figure 8.17—the cosine function no longer suggests a quadratic function. This portion of the cosine curve has four real roots, three turning points, and two inflection points, which suggest a polynomial of degree 4. Using a calculator, we find a quartic function that fits the data values in Table 8.2 is

$$y = 0.041653x^4 + 0x^3 - 0.499999x^2 + 0x + 0.999999.$$

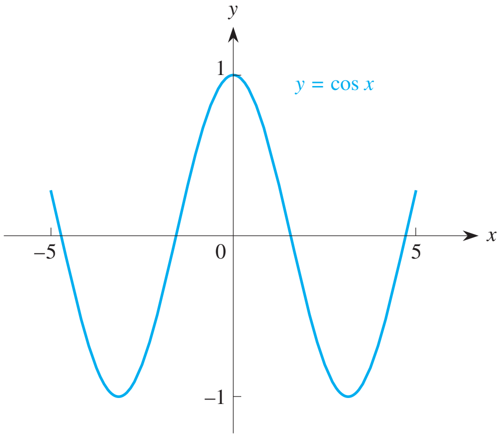


FIGURE 8.17

Note that the coefficients of both odd-powered terms are 0, the constant coefficient is essentially 1, and the quadratic coefficient is essentially $-\frac{1}{2}$. As for the leading coefficient, 0.041653,

$$\frac{1}{0.041653} = 24.00787,$$

so the leading coefficient is essentially $1/24 = 1/4!$. Therefore we have the fourth degree Taylor polynomial approximation

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!}.$$

Figure 8.18 shows the graph of this polynomial $T_4(x) = 1 - x^2/2 + x^4/4!$ and the cosine function for x between -5 and 5. Observe that $T_4(x)$ tries hard to capture the pattern in the cosine curve. In fact, the polynomial is an excellent match to the cosine for x between roughly -1.5 and 1.5. For instance, if $x = 0.5$, then $T_4(0.5) = 0.87760$, compared to $\cos(0.5) = 0.87758$; similarly, $T_4(1) = 0.541666$, compared to $\cos(1) = 0.54030$. If we choose a value of x too far from 0, the approximation breaks down. Thus $T_4(1.5) = 0.08594$, compared to $\cos(1.5) = 0.07074$.

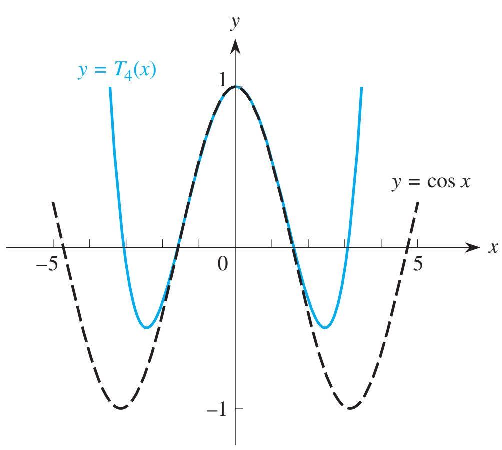


FIGURE 8.18

If we zoom out still farther on the cosine curve—say, from $x = -8$ to $x = 8$—two more turning points come into view, which suggests that we could get a better approximation to the cosine with a sixth degree polynomial. Using the values in Table 8.2 and a spreadsheet, we find, after rounding the coefficients, that

$$\cos x \approx T_6(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

Summarizing these results, the successive Taylor polynomial approximations to the cosine function are:

$$\cos x \approx 1 - \frac{x^2}{2},$$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!}$$

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!}.$$

As with the sine approximations, (1) each successive polynomial involves just one additional term; (2) each polynomial involves only *even* powers, and we know that the cosine function is an *even* function; (3) the signs of successive coefficients alternate; and (4) there is a clear pattern in the coefficients involving factorials.

The graphs of these polynomials, as well as that of the cosine curve, are shown in Figure 8.19. Note that each successive polynomial fits the cosine curve more accurately over a larger and larger interval centered at $x = 0$. You should examine these successive approximations using your function grapher.

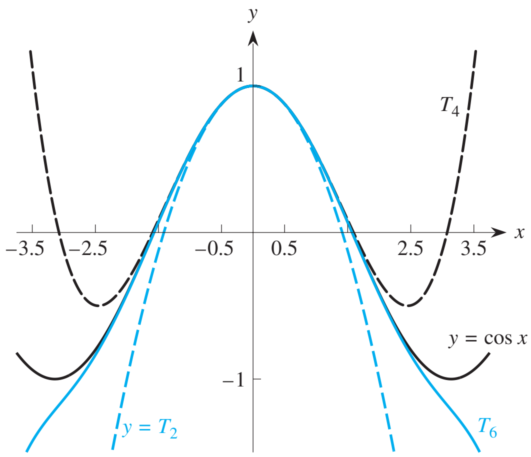


FIGURE 8.19

Think About This

How could you improve on the sixth degree polynomial approximation to the cosine? By eye, over what interval does it appear to be a good fit to the cosine curve?

Think About This

Devise a scheme to reduce any value of x to an equivalent value that allows you to use the smallest possible interval of x-values. How accurate is the fourth degree Taylor polynomial on this interval (i.e., what is the largest error between the cosine and the polynomial)? How accurate is the sixth degree polynomial?

Approximating sin *x* and cos *x* Using Trigonometric Identities

We now approach the problem of approximating the sine and cosine from a different viewpoint using several trigonometric identities. Recall the double-angle identity

$$\cos 2\theta = 1 - 2\sin^2\theta$$

from Equation (4a) of Section 8.1. If we let $x = 2\theta$ so that $x/2 = \theta$, the expression for $\cos 2\theta = \cos x$ becomes

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) = 1 - 2\left[\sin\left(\frac{x}{2}\right)\right]^2.$$

When θ is close to 0, we have $\sin\theta \approx \theta$, so that

$$\sin\left(\frac{x}{2}\right) \approx \frac{x}{2}.$$

Consequently we can approximate $\cos x$ by

$$\cos x \approx 1 - 2\left(\frac{x}{2}\right)^2 = 1 - \frac{x^2}{2}$$

when x is close to 0. Note that this approximation is identical to the quadratic Taylor polynomial that we obtained previously by using data analysis techniques.

We now use the double-angle formula for the sine,

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

with $x = 2\theta$ so that $x/2 = \theta$, which gives

$$\sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right). \tag{13}$$

We use the linear approximation for $\sin x \approx x$ and the quadratic approximation $\cos x \approx 1 - x^2/2$ to get

$$\sin\left(\frac{x}{2}\right) \approx \frac{x}{2} \quad \text{and} \quad \cos\left(\frac{x}{2}\right) \approx 1 - \frac{(x/2)^2}{2} = 1 - \frac{x^2}{8}.$$

Substituting these expressions into Equation (13), we obtain an approximation for $\sin x$ that improves on $\sin x \approx x$:

$$\begin{aligned} \sin x &\approx 2\left(\frac{x}{2}\right)\left(1 - \frac{x^2}{8}\right) = x\left(1 - \frac{x^2}{8}\right) \\ &= x - \frac{x^3}{8}. \end{aligned}$$

This is a cubic function that approximates the sine function near $x = 0$, but it is slightly different from the third degree Taylor approximation, $\sin x \approx x - x^3/6$. We compare these two cubic approximations in the Problems at the end of this section.

Figure 8.20 shows the graph of this cubic along with the sine function on the interval from -3 to 3. The two curves seem almost identical for x between -1 and 1; they are reasonably close between -2.5 and -1 and again between 1 and 2.5; but they apparently begin to diverge farther from 0. Note how much better this cubic function seems to approximate $\sin x$ than our linear approximation $\sin x \approx x$.

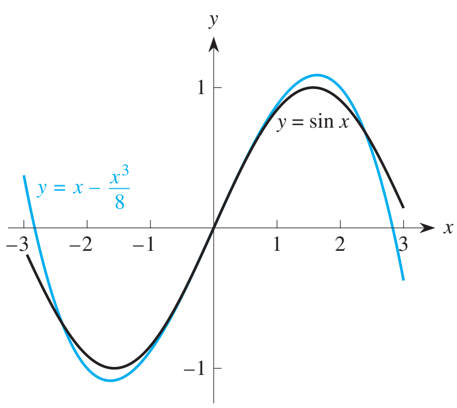


FIGURE 8.20

Think About This

Check numerically on your calculator how close the cubic is to the sine function at $x = 0.5$, at $x = 1$, and at $x = 1.5$.

We can continue this process to produce still better approximations to both the cosine and the sine functions by using the same trigonometric identities.

For instance, using the double-angle formula $\cos 2\theta = 1 - 2\sin^2\theta$ and our new approximation

$$\sin x \approx x - \frac{x^3}{8},$$

we get

$$\cos x = 1 - 2\sin^2\left(\frac{x}{2}\right) = 1 - 2\left[\sin\left(\frac{x}{2}\right)\right]^2$$
$$\approx 1 - 2\left[(x/2) - \frac{(x/2)^3}{8}\right]^2.$$

After some algebraic simplification, we eventually get

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{32} - \frac{x^6}{2048}.$$

The two graphs in Figure 8.21 illustrate that this sixth degree polynomial $P_6(x)$ is an almost perfect match to the cosine function from about $x = -1.5$ to about $x = 1.5$. It is quite accurate from about $x = -2$ to about $x = -1.5$ and from about $x = 1.5$ to about $x = 2$; thereafter its accuracy diminishes. For comparison, Figure 8.22 shows three graphs: the basic cosine curve, the initial quadratic approximation $P_2(x) = 1 - x^2/2$, and this sixth degree polynomial approximation $P_6(x)$. The higher degree polynomial is clearly a much better fit. It follows the bends of the cosine curve and stays close to it over a wider interval of x-values.

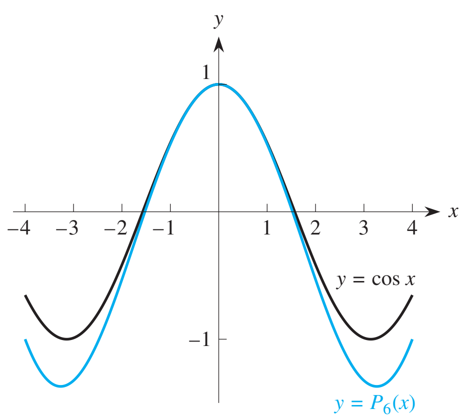


FIGURE 8.21

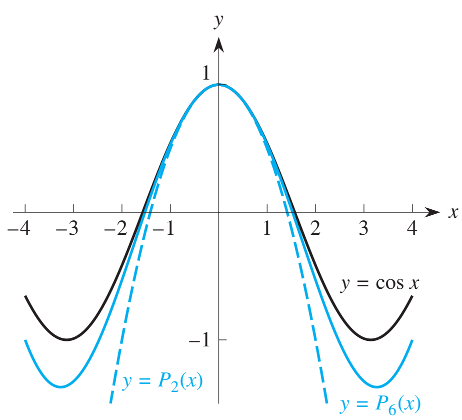


FIGURE 8.22

Approximating sin *x* and cos *x* Using Taylor Polynomials

We could continue this process to get better polynomial approximations to both sin *x* and cos *x* by using the trigonometric identities to construct polynomials of still higher degree. Unfortunately, each successive improvement is based on a series of approximations—we used $\sin x \approx x$ to generate the approximation

$$\cos x \approx 1 - \frac{x^2}{2},$$

and so on. The approximation errors in this process mount up and give less than the best possible approximation at each successive stage. For instance, we first found

$$\sin x \approx x = T_1(x)$$

and then used it to find

$$\sin x \approx x - \frac{x^3}{8} = Q_3(x).$$

Actually, as you will learn in calculus, the *best* possible cubic curve to approximate the sine curve near $x = 0$ is the Taylor polynomial of degree 3.

$$\sin x \approx x - \frac{x^3}{6} = T_3(x).$$

It is identical to the cubic polynomial we obtained earlier based on fitting a cubic function to a set of values of the sine function near 0. Figure 8.23 shows the graph of this cubic and the underlying sine curve. Figure 8.24 shows the sine curve and the two different cubic approximations: the Taylor approximation of degree three, $T_3(x) = x - x^3/6$, and the polynomial of degree three based on the trig identities, $Q_3(x) = x - x^3/8$. Note that $T_3(x)$ remains closer to the sine curve over a wider interval than $Q_3(x)$ does. Note also that $T_3(x)$ bends in such a way that it remains very close to the sine curve over a relatively large portion of its first arch.

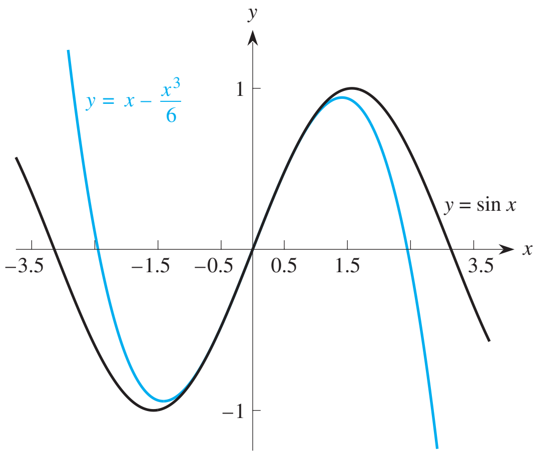


FIGURE 8.23

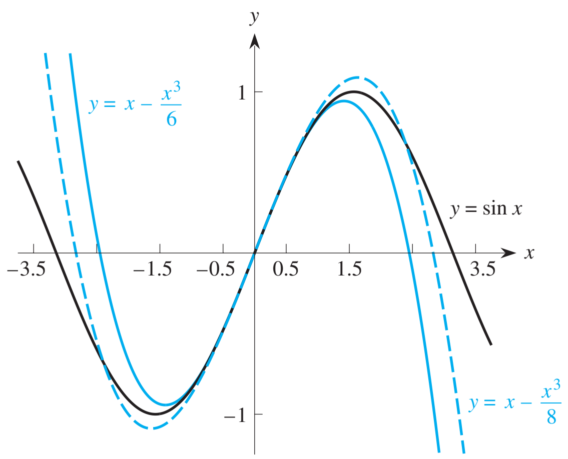


FIGURE 8.24

Use your function grapher to view what happens in a more dynamic way. In particular, examine the three curves near $x = 0$ to see that T_3 actually is closer to the sine curve than Q_3 is.

As a final note, let's look at the ideas we have developed in this section from a somewhat different perspective. Until now, we have interpreted Taylor polynomials as a means of *approximating* one function by a polynomial. An alternative interpretation is that we have been *constructing* a function (or a portion of a function) from simpler functions. That is, we have been constructing the trigonometric functions by using polynomials as the fundamental building blocks. More specifically, we have used *linear combinations* of power terms (i.e., sums of constant multiples of power terms) as these fundamental building blocks. This idea of using linear combinations of basic mathematical elements to construct more complicated mathematical structures is a continuing theme throughout mathematics.

Problems

1. Use the Taylor polynomial approximation to $f(x) = \cos x$ of degree 2 to estimate the value of the cosine function for $x = 0, 0.1, 0.2, 0.3, 0.4, 0.5$, and 0.6. Compare each estimate to the correct value.

2. Use the values you calculated in Problem 1 to construct a table that has a column containing the error in the approximation (the difference between the estimate and the correct value). Analyze the column of errors. Do they appear to grow approximately linearly? exponentially? quadratically? cubically?

3. Use the Taylor polynomial approximation to $f(x) = \sin x$ of degree 3 to construct a table of estimates for the values of the sine function when $x = 0$, 0.1, 0.2, 0.3, 0.4, 0.5, and 0.6. Calculate the errors and analyze them the same way you did in Problem 2.

4. **a.** Construct a table containing Taylor polynomial approximations of degrees $n = 1$ and $n = 3$ to $f(x) = \sin x$ for $x = -\pi/3, -\pi/4, -\pi/6, 0, \pi/6, \pi/4, \pi/3$.
   b. Add 2 columns to the table, one for $\sin x - x$ and another for $\sin x - (x - x^3/6)$, to compare the linear and cubic approximations to the correct value for each x.
   c. Use your function grapher to graph $y = \sin x - x$. What type of function does it appear to be?
   d. Use polynomial regression to find an appropriate polynomial to fit the data values of $\sin x - x$ versus x.
   e. Repeat part (c) with $y = \sin x - (x - x^3/6)$.

5. Construct a table of values of $\sin x$ for $x = -4\pi/25, -3\pi/25, -2\pi/25, \ldots, 4\pi/25$. Use your calculator to find the cubic polynomial that fits this set of sine values. How close does it come to the cubic Taylor polynomial approximation $\sin x \approx x - x^3/6$?

6. Use the Taylor polynomial approximation of degree $n = 5$ to $f(x) = \sin x$ to find a polynomial approximation of degree $n = 5$ to $g(x) = \sin(-x)$. Is the result surprising? Explain.

7. Use the Taylor polynomial approximation of degree $n = 5$ to $f(x) = \sin x$ to find a polynomial approximation of degree $n = 10$ to $g(x) = \sin(x^2)$. Graph both $g(x)$ and your approximation to it on the interval from $-\pi$ to π. Based on this graph, over what interval does your polynomial seem to be a good approximation to $g(x)$?

8. **a.** Use the Taylor polynomial approximation of degree $n = 5$ to $f(x) = \sin x$ to find a polynomial approximation of degree $n = 5$ to $h(x) = \sin 2x$.
   b. What do you get if you multiply the polynomial approximation of degree $n = 3$ to $f(x) = \sin x$ by the polynomial approximation of degree $n = 4$ to $g(x) = \cos x$?
   c. Graph $h(x)$ and twice the product of the two approximations found in part (b). What do you observe? Explain why.

9. Write the Taylor polynomial approximation of degree 3 to $f(x) = \sin x$ and the approximation of degree 4 to $g(x) = \cos x$. Square each expression and add them. What do you get? What do you think will happen if you use higher degree approximations? Explain.

10. In the text, we used the double-angle identity $\cos 2\theta = 1 - 2\sin^2\theta$ with $\theta = x/2$ to construct the approximation

$$\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{32} - \frac{x^6}{2048}.$$

Instead, use the alternative form of the identity $\cos 2\theta = \cos^2\theta - \sin^2\theta$ and any lower degree polynomials to find a different approximation to $\cos x$.

11. Repeat Problem 10 with the third form of the double-angle identity $\cos 2\theta = 2\cos^2\theta - 1$ to construct still another approximation formula for $\cos x$.

12. The function $f(x) = (\sin x)/x$ is not defined at $x = 0$.
   a. Use values of $x = 0.1, 0.01, 0.001, 0.0001, 0.00001, \ldots$ to investigate the behavior of this function close to $x = 0$. What limiting value does this function appear to approach?
   b. Use the linear Taylor polynomial approximation to $\sin x$ to explain why the limiting value you found in part (a) appears to make sense.

13. In calculus, you will have to determine the value of

$$\frac{\sin(x + \Delta x) - \sin x}{\Delta x},$$

where Δx is a very small quantity.
   a. Estimate the value of this quotient by using linear approximations to both sine expressions.
   b. Estimate the value of this quotient by using a cubic approximation to both $\sin x$ and $\sin(x + \Delta x)$.
   c. With the cubic approximation, suppose that Δx is actually 0. What does the resulting expression suggest?

14. The exponential function $f(x) = e^x$ with base $e = 2.71828\ldots$ is used extensively in mathematics and the sciences. As with the trig functions, its values are calculated using Taylor polynomial approximations:

$$e^x \approx 1 + x,$$
$$e^x \approx 1 + x + \frac{x^2}{2!},$$
$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!},$$

and so on. Use these and any further approximations that you need to approximate the values of
   a. $e^{0.1}$
   b. $e^{-0.1}$
   c. Use the given polynomials and any additional approximations to e^x that you need to estimate the value of e reasonably accurately. What degree polynomial will produce two-decimal accuracy? three-decimal accuracy? four-decimal accuracy?

8.3 Properties of Complex Numbers

One of the most amazing developments in the history of mathematics was the introduction of complex numbers to solve quadratic equations. For example, if $x^2 + 4 = 0$, then $x^2 = -4$, so that $x = \pm\sqrt{-4} = \pm 2i$, and the two roots are $x = 2i$ and $x = -2i$, where $i = \sqrt{-1}$. Similarly, from the quadratic formula, the roots of $x^2 - 2x + 10 = 0$ are

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(10)}}{2} = \frac{2 \pm \sqrt{-36}}{2} = \frac{2 \pm (6i)}{2} = 1 \pm 3i,$$

or $x = 1 - 3i$ and $x = 1 + 3i$.

In our exploration of the nature of the roots of polynomials in Section 4.4, we demonstrated that quadratic, cubic, and higher degree polynomials have a surprisingly high proportion of complex zeros. We now develop a way to visualize complex numbers that gives a deeper understanding of the processes that lead to such polynomial equations.

Any complex number $z = a + bi$ is composed of two parts, a *real part*, a, and an *imaginary part*, b. For instance, in $z = 4 + 7i$, the real part is 4 and the imaginary part is 7. We occasionally write $a = \text{Re}(z)$ and $b = \text{Im}(z)$, respectively. Note that a and b are both real numbers; it is the combination $a + bi$ that is a complex number. In the special case when $b = 0$, the complex number $z = a + bi$ reduces to a real number. In another special case where $a = 0$, the complex number z reduces to a pure imaginary number, bi.

The arithmetic of complex numbers, for the most part, is quite straightforward, and we review it briefly in Appendix E. Because $i = \sqrt{-1}$, it follows that

$$\begin{aligned} i^2 &= (\sqrt{-1})^2 = -1 \\ i^3 &= (i^2)(i) = (-1)(i) = -i \\ i^4 &= (i^2)(i^2) = (-1)(-1) = 1 \\ i^5 &= (i^4)(i) = 1(i) = i. \end{aligned}$$

In fact, all higher powers of i simply cycle through the four "values" $i, -1, -i$, and 1. That is, $i^6 = i^2 = -1$, $i^7 = i^3 = -i$, $i^8 = i^4 = 1$, $i^9 = i^5 = i$, and so on.

Visualizing complex numbers geometrically is extremely helpful. We do so by using the **complex plane,** which is a two-dimensional coordinate system designed to display a complex number $z = a + bi$. We measure the real part a horizontally and the imaginary part b vertically. In Figure 8.25 we plot the complex number $z = 2 + 5i$. Note that it lies 2 units to the right and 5 units up from the origin. Similarly, the complex numbers $1 - 3i$, and $-2 + i$ are also plotted in Figure 8.25. Any purely real number, such as 4 (which is $4 + 0i$) or -6 (which is $-6 + 0i$), lies on the horizontal axis. Any purely imaginary number, such as $4i$ (which is $0 + 4i$) or $-3i$ (which is $0 - 3i$) lies on the vertical axis.

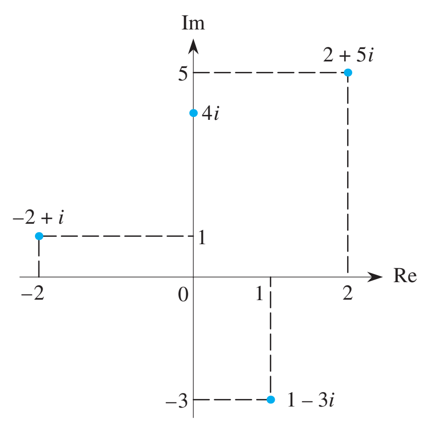


FIGURE 8.25

Suppose that $z = a + bi$ is an arbitrary complex number that we plot as a point in the complex plane. We connect the point to the origin with a line segment, which is the hypotenuse of a right triangle, as shown in Figure 8.26. The base of the triangle is a, the real part of z, and the height of the triangle is b, the size of the imaginary part of z. The Pythagorean theorem gives the length of the hypotenuse as $\sqrt{a^2 + b^2}$, which we interpret as the size of the complex number $z = a + bi$. We call it the **modulus** of the complex number and write it as

$$\|z\| = \sqrt{a^2 + b^2}.$$

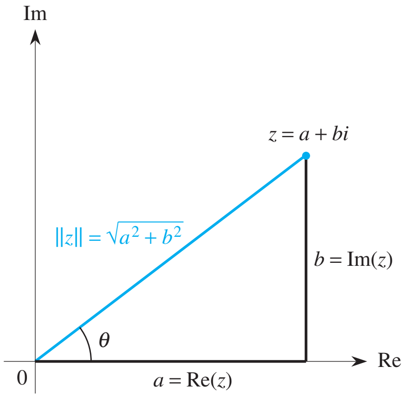


FIGURE 8.26

For instance, if $z = 4 + 3i$, then its modulus is

$$\|z\| = \sqrt{4^2 + 3^2} = \sqrt{25} = 5.$$

The complex numbers $4 - 3i$, $-4 + 3i$, and $-4 - 3i$ all have the same modulus of 5. Sketch them to verify that this is indeed the case.

Think About This

Are there any other points in the complex plane that also have a modulus of 5? What can you say about all such complex numbers?

We again consider the complex number $z = a + bi$ and the associated right triangle in the complex plane. We now focus on the angle θ shown in Figure 8.26. By convention, θ is measured counterclockwise from the horizontal, or real, axis. Thus

$$\tan\theta = \frac{b}{a}.$$

We also have the two further relations

$$\cos\theta = \frac{a}{\|z\|} \quad \text{and} \quad \sin\theta = \frac{b}{\|z\|},$$

which lead to

$$a = \|z\|\cos\theta \quad \text{and} \quad b = \|z\|\sin\theta.$$

Consequently, we can write the original complex number z in the equivalent *trigonometric form*

$$\begin{aligned} z = a + bi &= \|z\|\cos\theta + i\|z\|\sin\theta \\ &= \|z\|(\cos\theta + i\sin\theta). \end{aligned}$$

The *trigonometric form* for the complex number $z = a + bi$ is

$$z = \|z\|(\cos\theta + i\sin\theta),$$

where

$$\|z\| = \sqrt{a^2 + b^2} \quad \text{and} \quad \tan\theta = \frac{b}{a}, a \neq 0.$$

EXAMPLE 1

Find the trigonometric form for the complex number $z = 4 + 3i$.

Solution For $z = 4 + 3i$, we have $\|z\| = 5$, so that

$$z = 4 + 3i = 5(\cos\theta + i\sin\theta),$$

where $\tan\theta = 3/4$, so that $\theta = \arctan 3/4 = 0.6435$ radian or $36.87°$.

Think About This

Use the value for the angle θ in Example 1 to show that the trigonometric form for the complex number z is identical to the original expression $4 + 3i$.

Powers of Complex Numbers

The trigonometric form for a complex number $z = a + bi$ allows us to interpret z as being located at a certain distance, the modulus, from the origin and rotated through an angle θ from the horizontal. This model gives us a way to gain some special insights into powers of complex numbers.

EXAMPLE 2

For $z = 4 + 3i$, (**a**) find z^2 algebraically and (**b**) interpret z^2 geometrically in the complex plane.

Solution

a. If $z = 4 + 3i$,

$$\begin{aligned} z^2 = (4 + 3i)^2 &= 4^2 + 2(4)(3i) + (3i)^2 && (u + v)^2 = u^2 + 2uv + v^2 \\ &= 16 + 24i + 9(i^2) \\ &= 16 + 24i - 9 && i^2 = -1 \\ &= 7 + 24i. \end{aligned}$$

This algebraic result provides no special insight into how z^2 is related to z.

b. We look at the trigonometric form for $z = 4 + 3i$. The modulus is $\|z\| = 5$, and the associated angle is $\theta = \arctan(3/4) = 0.6435$ radians, or $36.87°$, as in Example 1. Now consider the trigonometric form for z^2. Its modulus is

$$\|z^2\| = \sqrt{7^2 + 24^2} = \sqrt{49 + 576} = \sqrt{625} = 25,$$

which is the square of the modulus of the original complex number z. Next, the angle ϕ associated with z^2 is defined by

$$\tan\phi = \frac{24}{7} \quad \text{so that} \quad \phi = \arctan\frac{24}{7} \approx 1.2870 \text{ radians or } 73.74°,$$

which is exactly twice the angle θ ($=0.6435$ radians or $36.87°$) associated with z, as illustrated in Figure 8.27.

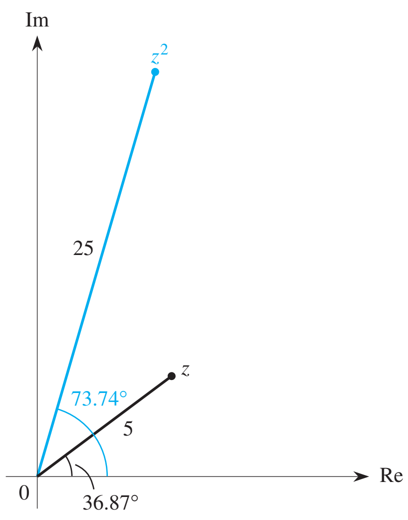


FIGURE 8.27

For this particular complex number $z = 4 + 3i$, z^2 is related to z by the process of squaring the modulus and doubling the angle. Does this rule hold in general? Let's look at two other simple cases.

EXAMPLE 3

Find the modulus and angle associated with $z^2 = (2i)^2$ and relate them to the modulus and angle associated with $z = 2i$.

Solution We know that $z = 2i = 0 + 2i$ is located at a distance of $\|z\| = 2$ from the origin with an associated angle of $\theta = \pi/2$ measured in the usual positive direction from the horizontal axis. We now consider $z^2 = (2i)^2$, which is

$$z^2 = 4i^2 = -4 = -4 + 0i.$$

This complex number has modulus 4 and associated angle π because it is on the negative real axis. That is, the modulus of $z^2 = (2i)^2$ is the square of the modulus of $z = 2i$, and the associated angle π is twice the angle $\pi/2$ associated with $z = 2i$.

EXAMPLE 4

Find the modulus and angle associated with $z^2 = (1 + i)^2$, where $z = 1 + i$, and relate them to the corresponding modulus and angle for z.

Solution For $z = 1 + i$, the modulus is

$$\|z\| = \sqrt{1^2 + 1^2} = \sqrt{2},$$

and the associated angle is $\theta = \pi/4$. Further,

$$z^2 = (1 + i)^2 = 1 + 2i + i^2 = 1 + 2i - 1 = 2i,$$

so $\|z^2\| = \sqrt{4} = 2$, the square of the modulus of z. The associated angle is $\pi/2$, or double the angle associated with z. So, again, when we square a complex number, the modulus is squared and the angle is doubled.

Let's now consider any complex number $z = a + bi$ in the equivalent trigonometric form

$$z = \|z\|(\cos\theta + i\sin\theta).$$

Squaring z gives

$$\begin{aligned} z^2 &= \|z\|^2(\cos\theta + i\sin\theta)^2 \\ &= \|z\|^2(\cos^2\theta + 2i\cos\theta\sin\theta + i^2\sin^2\theta). \end{aligned}$$

Using $i^2 = -1$ and collecting the real and imaginary terms yields

$$z^2 = \|z\|^2[(\cos^2\theta - \sin^2\theta) + 2i\cos\theta\sin\theta].$$

Now recall the double-angle identities:

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta. \end{aligned}$$

Examining the real part of the previous expression for z^2, we see that it equals $\cos 2\theta$, whereas the imaginary part equals $\sin 2\theta$. Thus we have

$$z^2 = \|z\|^2(\cos 2\theta + i\sin 2\theta).$$

Geometrically, squaring any complex number always produces a new complex number whose modulus is the square of the original modulus and whose angle is double the original angle. If the modulus of the original number is greater than 1, z^2 is a "larger" complex number, as shown in Figure 8.28(a). If $\|z\|$ is smaller than 1, z^2 is a "smaller" complex number, as shown in Figure 8.28(b).

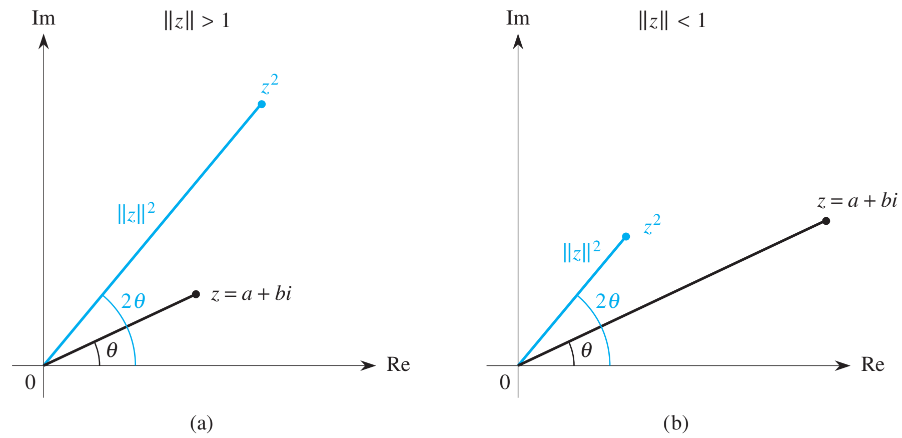


FIGURE 8.28

What about other powers of $z = a + bi$? Is there any pattern for z^n when $n > 2$?

EXAMPLE 5

Find the modulus and angle associated with z^3 when $z = 2i$.

Solution The complex number $z = 2i$ is located at a distance of 2 from the origin and at an angle of $\pi/2$. Now consider

$$z^3 = (2i)^3 = 8i^3 = -8i = 0 - 8i.$$

It is located at a distance of 8 from the origin and is rotated through an angle of $3\pi/2$, which is triple $\pi/2$. Thus the modulus of $(2i)^3$ is the cube of the modulus of $2i$, and the associated angle $3\pi/2$ is three times the angle $\pi/2$ associated with $2i$.

Let's find out whether the same pattern holds when we cube any complex number $z = a + bi$. We do so by using the trigonometric form. Because

$$z^2 = \|z\|^2(\cos 2\theta + i \sin 2\theta),$$

we have

$$\begin{aligned} z^3 &= z^2(z) \\ &= [\|z\|^2(\cos 2\theta + i \sin 2\theta)][\|z\|(\cos\theta + i\sin\theta)] \\ &= \|z\|^3[\cos 2\theta \cos\theta + i\cos 2\theta \sin\theta + i \sin 2\theta \cos\theta + i^2 \sin 2\theta \sin\theta]. \end{aligned}$$

Using $i^2 = -1$ and collecting the real and imaginary terms, we get

$$z^3 = \|z\|^3[(\cos 2\theta \cos\theta - \sin 2\theta \sin\theta) + i(\cos 2\theta \sin\theta + \sin 2\theta \cos\theta)].$$

We now use the sum identities

$$\begin{aligned} \cos(x + y) &= \cos x \cos y - \sin x \sin y \\ \sin(x + y) &= \sin x \cos y + \cos x \sin y \end{aligned}$$

with $x = 2\theta$ and $y = \theta$ to get

$$\begin{aligned} z^3 &= \|z\|^3[\cos(2\theta + \theta) + i \sin(2\theta + \theta)] \\ &= \|z\|^3(\cos 3\theta + i \sin 3\theta). \end{aligned}$$

Thus cubing any complex number always results in cubing the modulus and tripling the rotation of the original complex number. It either "lengthens" the complex number if the original modulus is greater than 1, as illustrated in Figure 8.29(a), or "contracts" it if the modulus is less than 1, as illustrated in Figure 8.29(b). If the modulus equals 1 and $\theta \neq 0$, all that happens is a rotation.

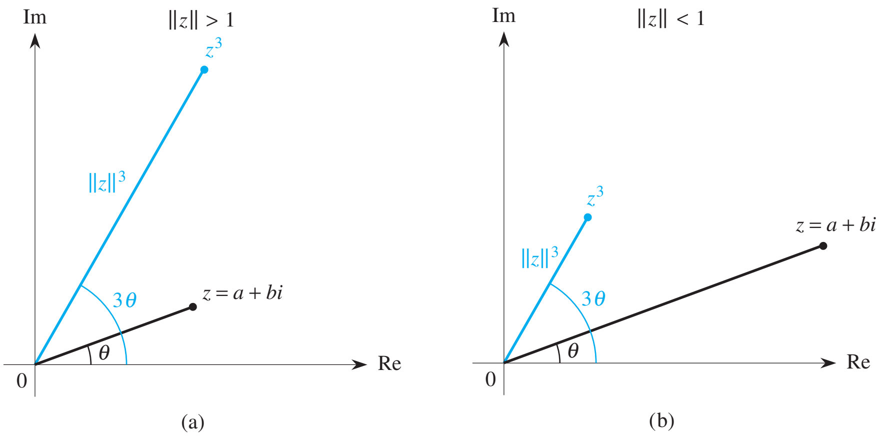


FIGURE 8.29A FIGURE 8.29B

In the Problems at the end of this section, we ask you to show that

$$z^4 = z^3(z) = \|z\|^4[\cos 4\theta + i \sin 4\theta]$$

and that, in general for any positive integer power n,

$$\begin{aligned} z^n &= z^{n-1}(z) \\ &= \|z\|^n(\cos\theta + i\sin\theta)^n \\ &= \|z\|^n(\cos n\theta + i \sin n\theta). \end{aligned}$$

This important and extremely useful result is known as **DeMoivre's theorem** after French mathematician Abraham DeMoivre who first discovered it.

DeMoivre's Theorem

If

$$z = a + bi = \|z\|(\cos\theta + i\sin\theta)$$

then

$$z^n = \|z\|^n(\cos n\theta + i\sin n\theta)$$

for any positive integer n.

Complex Conjugates

We know that complex numbers occur in complex conjugate pairs, such as $z = 3 + 5i$ and $z = 3 - 5i$ when we use the quadratic formula. If $z = a + bi$ is any complex number, we write its conjugate as $\bar{z} = a - bi$, which is shown geometrically in Figure 8.30. Clearly, z and $\bar{z}$ have the same modulus, $\sqrt{a^2 + b^2}$, so $\|\bar{z}\| = \|z\|$. Also, if the angle associated with z is θ, the angle associated with $\bar{z}$ is $-\theta$.

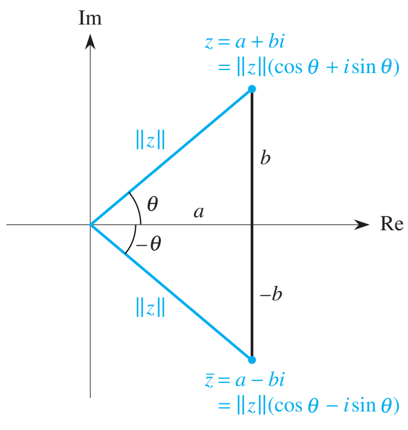


FIGURE 8.30

Using the reflection identities

$$\cos(-\theta) = \cos\theta \quad \text{and} \quad \sin(-\theta) = -\sin\theta,$$

we find that

$$\bar{z} = a - bi = \|z\|[\cos(-\theta) + i\sin(-\theta)] = \|z\|(\cos\theta - i\sin\theta).$$

Applying DeMoivre's theorem to $\bar{z}$ gives

$$\begin{aligned}(\bar{z})^n &= \|z\|^n(\cos\theta - i\sin\theta)^n \\ &= \|z\|^n(\cos n\theta - i\sin n\theta).\end{aligned}$$

A simple extension of these ideas provides a way of visualizing both the product and the quotient of any two complex numbers. We explore this approach in the Problems at the end of this section.

Problems

In Problems 1–9, find the modulus and the associated angle for each complex number.

1. $z = 4 - 3i$
2. $z = 5 + 12i$
3. $z = 12 - 5i$
4. $z = -15 + 20i$
5. $z = 64 - 36i$
6. $z = 8 - 3i$
7. $z = -5 + 7i$
8. $z = 3 + \sqrt{8}i$
9. $z = -8 - \sqrt{3}i$

10–18. Find the trigonometric form for each complex number in Problems 1–9.

19–22. For each complex number in Problems 1–4, find z^2 algebraically.

23–31 For each complex number in Problems 1–9, find z^2 by using DeMoivre's theorem.

32–35. For each complex number in Problems 1–4, find z^3 algebraically.

36–44. For each complex number in Problems 1–9, find z^3 by using DeMoivre's theorem.

45. For $z = 1 + 2i$, calculate and plot z^n, for $n = 0, 1, 2, 3$, and 4.

46. Repeat Problem 45 for $z = 0.6 + 0.8i$. What difference do you observe about the behavior of the two sets of points?

47. Show that $z^4 = z^3 \cdot z = \|z\|^4(\cos 4\theta + i \sin 4\theta)$ for any complex number z.

48. Prove DeMoivre's theorem for any integer power n:

$$z^n = \|z\|^n(\cos n\theta + i \sin n\theta).$$

Hint: Write $z^n = z^{n-1} \cdot z$ and assume that

$$z^{n-1} = \|z\|^{n-1}[\cos((n-1)\theta) + i \sin((n-1)\theta)].$$

49. Suppose that you have two complex numbers $z = a + bi$ and $w = c + di$.

a. What is the product of z and w algebraically?

b. What is the product of z and w using the trigonometric forms of z and w?

c. Hypothesize and prove an extension of DeMoivre's theorem that will allow you to multiply any two complex numbers in trigonometric form. (*Hint*: Your extension should reduce to DeMoivre's theorem for z^2 when $w = z$.)

d. Apply the rule that you discovered in part (b) to find the product of

i. $z = 1 + 2i$ and $w = 1 - 2i$

ii. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w = \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

50. **a.** Hypothesize an extension of DeMoivre's theorem that will allow you to divide one complex number by another in trigonometric form.

b. Apply the rule that you proposed in part (a) to find the quotient of

i. $z = 1 + 2i$ and $w = 1 - 2i$

ii. $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $w = \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

51. **a.** Hypothesize an extension of DeMoivre's theorem that will allow you to determine the square root of a complex number z.

b. Apply the rule that you proposed in part (a) to find the square root of

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

c. Algebraically square the complex number that you obtained in part (b) to verify that it actually is the square root of the original number in part (a).

d. Can you hypothesize a further extension of DeMoivre's theorem to extract any desired root of a complex number? any desired rational power of a complex number? Explain.

52. A negative real number can be thought of as being produced by rotating the corresponding positive real number (which is located on the horizontal axis) through an angle π in the complex plane. Use this interpretation to explain why the product of two negative numbers is positive.

53. Show that, for any pair of complex conjugates $z = a + bi$ and $\bar{z} = a - bi$, $z \cdot \bar{z} = \|z\|^2$.

8.4 The Road to Chaos

In this section we investigate some fascinating results that arise from iteration processes applied to complex numbers. Let's begin with any complex number in trigonometric form—say, $z_0 = \|z_0\|(\cos \theta + i \sin \theta)$—and square it to produce $z_1 = z_0^2$. Using gord.3896.08.pgs 4/24/03 10:12 AM Page 575

DeMoivre's theorem, we know that the geometric result is a complex number whose associated angle is 2θ and whose modulus is $\|z_0\|^2$. Recall that, if $\|z_0\| > 1$, we get a rotation and an expansion to a "larger" complex number; if $\|z_0\| < 1$, we get a rotation and a contraction to a "smaller" complex number; if $\|z_0\| = 1$, we get only a rotation.

Suppose that we next square z_1 to produce $z_2 = z_1^2 = z_0^4$. If $\|z_0\| > 1$, we get a further rotation (to the angle $2 \times 2\theta = 4\theta$) and a further expansion. If $\|z_0\| < 1$, we get the same further rotation (to 4θ) and a further contraction. If $\|z_0\| = 1$, we get only the rotation (to 4θ).

What happens if we continue this process indefinitely to produce a sequence of complex numbers z_0, $z_1 = z_0^2$, $z_2 = z_1^2$, $z_3 = z_2^2, \ldots$? The geometric behavior of the terms of this sequence can be predicted easily by extending the reasoning we just used. If the modulus of the initial value z_0 is greater than 1, each successive iterate is farther from the origin in the complex plane, at a larger angle, and the sequence clearly diverges in a counterclockwise spiral pattern for $\theta > 0$, as shown in Figure 8.31(a). If $\|z_0\| < 1$, each successive term is closer to the origin; the successive iterates converge to 0 in a counterclockwise spiral pattern as each one is a further rotation of the original angle $\theta > 0$, as shown in Figure 8.31(b). Finally, if $\|z_0\| = 1$, all successive iterates fall on the boundary of the unit circle centered at the origin in the complex plane.

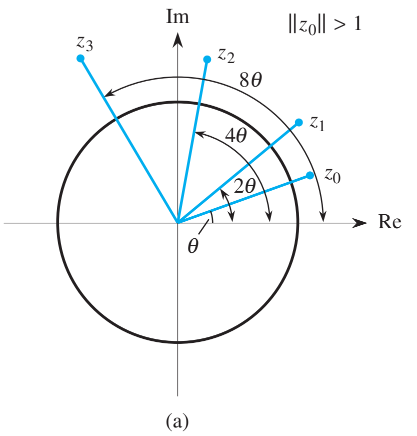


FIGURE 8.31A

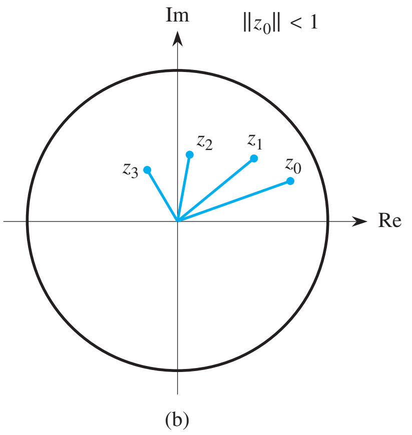


FIGURE 8.31B

The Julia Set

Let's focus on the possible initial values for z_0. Any initial point inside the unit circle starts a sequence that spirals in to the origin; any initial point on the circle itself starts a sequence that remains on the unit circle; and any initial point outside the unit circle starts a sequence that spirals away toward infinity.

We can display this graphically in the following way. Visualize the unit circle centered at the origin in the complex plane, as shown in Figure 8.32. The circle is drawn in heavy black, the interior is shaded, and the region outside the circle is unshaded. Think of the unshaded region as indicating any point that begins a sequence that diverges, the shaded region as indicating those initial points for which the sequence converges to 0, and the black as indicating those initial points for which the sequence remains *on* the circle forever. The set of initial points for which the resulting sequences do not diverge to infinity is known as the **Julia set** associated with the function $f(z) = z^2$. (It is named after French mathematician Gaston

Julia, who discovered the properties of these sets in the 1920s.) The Julia set associated with $f(z) = z^2$ consists of the unit circle and all points inside it.

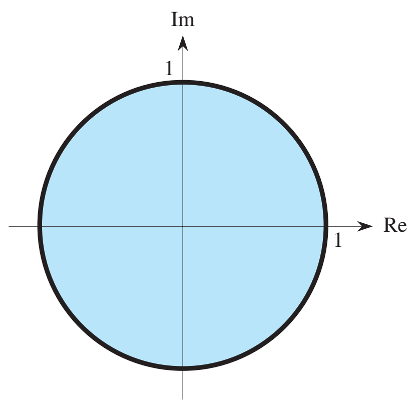


FIGURE 8.32

A relatively small change in what we have just done puts us on the road to chaos. Instead of using $f(z) = z^2$, let's see what happens if we use $f(z) = z^2 + C$, where C is any constant, either real or complex. (You may want to think of this as a family of functions for different values of C.) We take $z_1 = f(z_0) = z_0^{\,2} + C$, so that $z_2 = f(z_1) = z_1^{\,2} + C$, $z_3 = f(z_2) = z_2^{\,2} + C, \ldots$ We now consider a variety of cases with different values for C and with different starting values z_0.

Let's begin with $C = 2$. If the initial value is $z_0 = 3$,

$$z_1 = z_0^{\,2} + C = 9 + 2 = 11,$$
$$z_2 = z_1^{\,2} + C = 121 + 2 = 123,$$
$$z_3 = z_2^{\,2} + C = 123^2 + 2 = 15{,}131, \ldots,$$

and the sequence clearly diverges. Using the same $C = 2$ with some other starting values, we get

if $z_0 = 0.2$,	then	$z_1 = 2.04$,	$z_2 = 6.1616$,	$z_3 = 39.9653, \ldots;$
if $z_0 = 1 + i$,	then	$z_1 = 2 + 2i$,	$z_2 = 2 + 8i$,	$z_3 = -58 + 32i, \ldots;$
if $z_0 = 0.5 + 0.2i$,	then	$z_1 = 2.21 + 0.2i$,	$z_2 = 6.844 + 0.884i$,	$z_3 = 48.059 + 12.10i \ldots$

All three sequences seem to diverge to infinity. Of course, we can't reach such a conclusion based on just a few examples; they can, at best, suggest what may happen.

Let's use DeMoivre's theorem to analyze the behavior of the successive iterates. Suppose that z_0 is any initial value inside the unit circle, so its modulus is less than 1. When we square it, the modulus for $z_0^{\,2}$ is smaller still. However, when we add 2 to it, the point is shifted 2 units to the right, so that z_1 must be outside and to the right of the unit circle.

Now suppose that z_0 (or some subsequent iterate) is outside the unit circle. Its modulus is greater than 1, so the modulus for $z_0^{\,2}$ is larger still. When we add 2 to it, the point is again shifted 2 units to the right. For almost all possible values of z_0, the resulting z_1 will be outside the unit circle.

There are some exceptions—say, $z_0 = 1.1i$ so that

$$z_1 = (1.1i)^2 + 2 = -1.21 + 2 = 0.79.$$

However, it can be shown that, eventually all subsequent iterates will land outside the circle and ultimately diverge to infinity. (Because each iteration involves a rotation, at some stage one of the successive iterates will eventually land near the horizontal axis to the right and the following iterate will be outside and to the right of the unit circle.) Thus it turns out that, with $f(z) = z^2 + 2$, for every initial point in the complex plane, the resulting sequence diverges. The Julia set associated with the function $f(z) = z^2 + C$, when $C = 2$, will be completely empty because all initial points give rise to sequences that eventually diverge. Our diagram of this Julia set will be entirely unshaded because there are no initial points that start convergent sequences.

Similarly, if $C = 2i$, all sequences will diverge regardless of the initial value for z_0. The additive constant $2i$ results in a shift upward of 2 units in the imaginary direction. Pick several initial values for z_0 (real, imaginary, or complex) and see what happens when you calculate the successive iterates.

However, if $C = 0.2i$ and we start with $z_0 = 0.5 + 0.2i$, we obtain

$$z_1 = (0.5 + 0.2i)^2 + 0.2i = 0.21 + 0.4i;$$

$$\begin{aligned} z_2 &= -0.1159 + 0.368i; & z_6 &= -0.0394 + 0.1881i;\\ z_3 &= -0.1220 + 0.1147i; & z_7 &= -0.0338 + 0.1852i;\\ z_4 &= 0.0017 + 0.1720i; & z_8 &= -0.0332 + 0.1875i.\\ z_5 &= -0.0296 + 0.2006i; & z_9 &= -0.0341 + 0.1876i. \end{aligned}$$

The sequence apparently converges to some point in the complex plane.

Unfortunately, repeating this process for every possible starting value z_0 is not practical. Instead, we use a computer to perform such calculations for a large number of points in a grid to give a representative picture of what happens. As with the previous cases, we leave any initial point that starts a sequence that diverges to infinity unmarked to become part of the unshaded region. We put a small dot at any initial point that starts a sequence that converges to some point in the complex plane so that it will be part of the shaded Julia set.

The resulting picture of the Julia set for the function $f(z) = z^2 + 0.2i$ is shown in Figure 8.33.

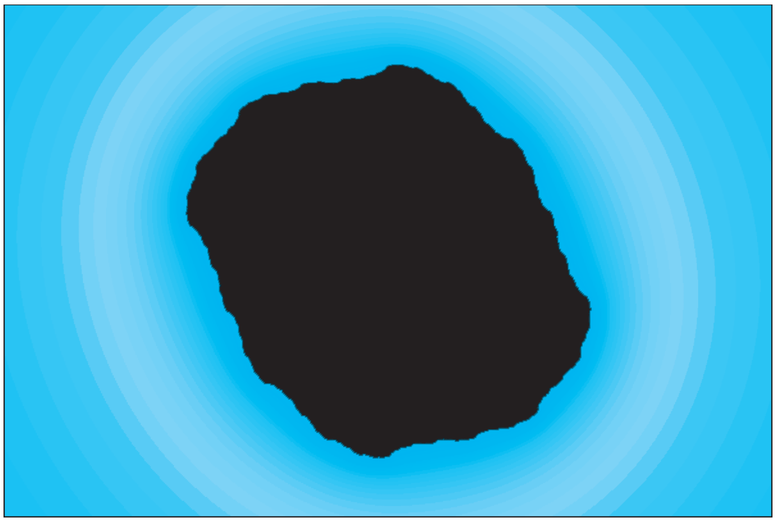


FIGURE 8.33

This picture does not indicate the limits of any of the sequences; it shows only those points that start sequences that have limits. Typically, if you take points in the interior of the shaded region, it turns out that nearby starting points tend to converge to limits that are relatively close to one another. However, if you take points near the boundary, very different results can occur. Initial points that are extremely close together can produce sequences that converge to radically different limits. The result is an instance of mathematical chaos because the behavior has no predictable patterns. Points that are very close together, provided that they are both near the boundary of the Julia set, may well lead to sequences that behave very differently. If you were to zoom in on the portion of the Julia set near these boundaries, you would see an ever more intricate design illustrating how nearby points can start sequences that either diverge or converge. They occur in a totally chaotic and unpredictable manner.

A striking illustration of this outcome is shown in Figure 8.34, which is the Julia set corresponding to $C = -0.2 + 0.7i$. Note how intricate the boundary appears. Figure 8.35 shows the result of zooming in on the upper left corner of the Julia set shown in Figure 8.34. Observe how roughly similar patterns repeat; this kind of repetition is typical of what happens when you zoom in repeatedly on Julia sets associated with most values for C. Also, note how much more jagged the boundary looks as more details appear in the magnified image in Figure 8.35, which also is typical.

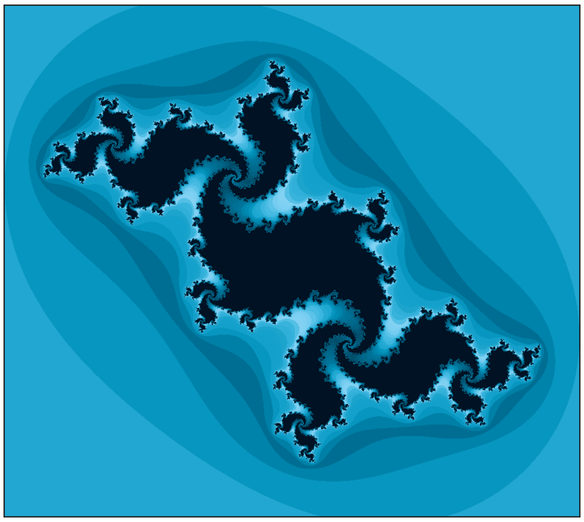


FIGURE 8.34

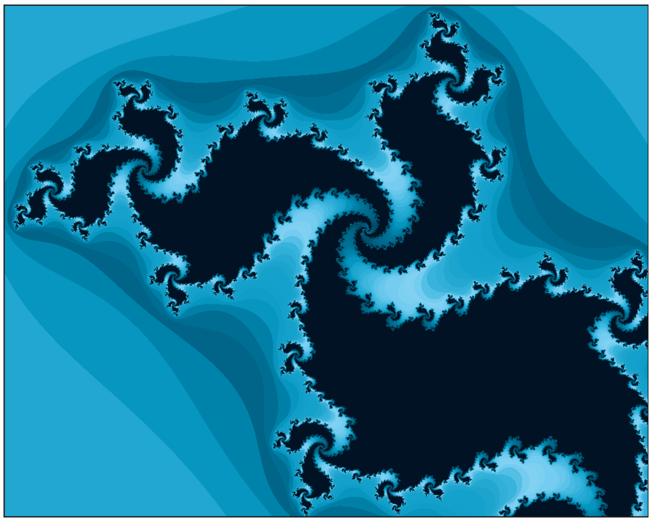


FIGURE 8.35

The Julia set associated with a complex constant C can be far more intricate than we have shown so far; it can, for instance, consist of a large variety of disconnected pieces. It may even consist of nothing but a collection of isolated points like a set of dust particles. You may want to experiment with some of these ideas yourself, using any of the many computer programs available for displaying Julia sets for iterated functions.

The Mandelbrot Set

There is a completely different way of looking at these ideas. In the discussion of Julia sets, we considered the function $f(z) = z^2 + C$, selected a particular value for C, and then examined points in the complex plane as starting points z_0 for iterated sequences. Now let's reverse this.

Suppose instead that we select a particular starting point z_0 and examine the effects of using different values for the complex constant C in $f(z) = z^2 + C$. Thus our view of the complex plane has shifted—it now represents all different constants rather than all different starting points. In particular, suppose that we select $z_0 = 0$ as the starting point for all sequences. Then, for any constant C, $z_1 = 0^2 + C = C$, $z_2 = z_1{}^2 + C = C^2 + C$, and so on. Clearly, if C is large (far from the origin in the complex plane), all successive iterates will be larger still and the successive points of the sequence will diverge. However, if C is fairly small, the successive iterates may remain close to the origin and the sequence may converge to some finite complex value.

The Julia set associated with the function $f(z) = z^2 + C$ consists of all initial points z_0 for which the sequences converge for a given constant C. Similarly, the **Mandelbrot set** associated with the function $f(z) = z^2 + C$ (named after French mathematician Benoit Mandelbrot) consists of all constants C for which the sequences starting from $z_0 = 0$ fail to diverge. For this initial point $z_0 = 0$, the Mandelbrot set illustrated in Figure 8.36 shows those constants C for which the corresponding sequences remain close to the origin. As with a Julia set, the boundary of the Mandelbrot set is an incredibly intricate structure. If you zoom in on it, as shown in Figure 8.37, you will see remarkable shapes with no predictable patterns; however, the original overall shape shown in Figure 8.36 appears to repeat at all levels of magnification. The main heart-shaped portion of the Mandelbrot set is called a *cardioid,* which we discuss in Chapter 9; the portion to the left of the cardioid is actually a circle.

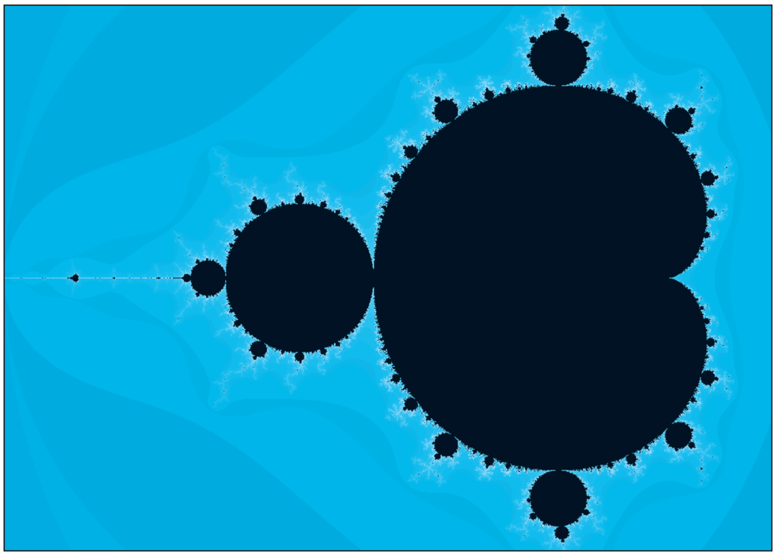


FIGURE 8.36

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FIGURE 8.37A

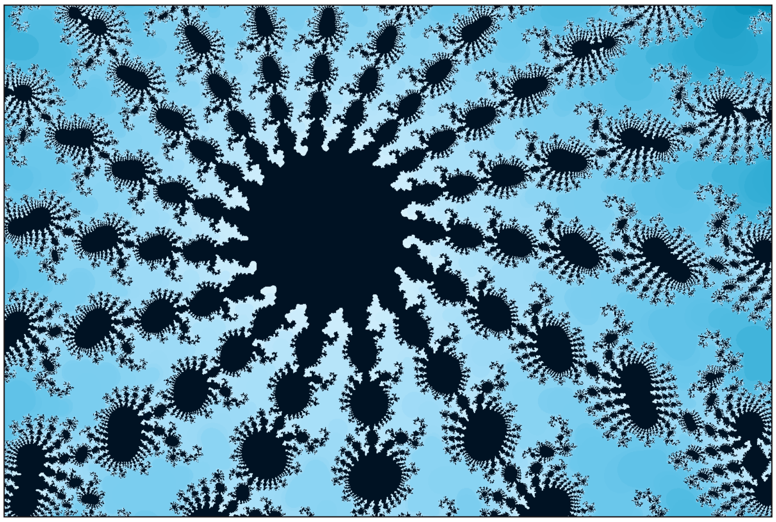

FIGURE 8.37B

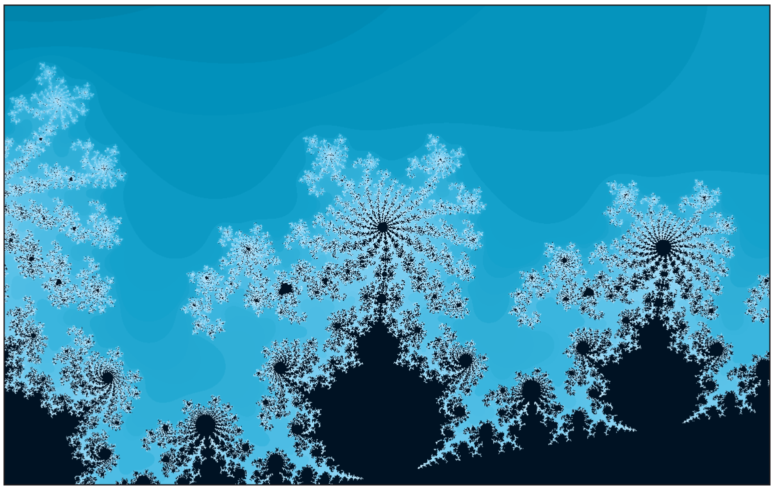

FIGURE 8.37C

These displays show the Mandelbrot set with different shadings to indicate how quickly different sequences diverge from the starting value $z_0 = 0$. When different colors are used, the results are even more dramatic. You may want to examine the Mandelbrot set, using one of the programs available for displaying it. All such programs allow you to see the details at different levels of magnification as you zoom in on the boundary. In theory, there is no limit to the degree of complexity of the boundary. Such a shape is known as a *fractal.*

Many shareware programs are available (one of the most popular is called FracInt) that will let you investigate both Julia and Mandelbrot sets. This subject is one of the most exciting areas of current mathematical research, and many new and important theorems have been proven in the last few years. These ideas have also formed the basis for many of the computer graphics images that you have undoubtedly seen in today's movies.

Problems

1. **a.** Use the quadratic formula to find a condition on those values of C for which the sequence of iterates $x = f(x) = x^2 + C$ has a real limiting value.
   b. Verify your condition in part (a) by using $C = 0.1$, starting with $x_0 = 0.5$ and performing enough iterations to see the eventual behavior.
   c. Repeat part (b), using $C = 0.4$.
2. **a.** What is the limiting value you expect if $C = 1/4$ for the sequence of function iterations based on $x = f(x) = x^2 + C$?
   b. Start the iteration process at $x_0 = 0.5 + 0.5i$ and perform enough iterations to verify that the process seems to be converging to your answer for part (a).
   c. Start the iteration process at $x_0 = 1 + i$ and perform enough iterations to determine the eventual behavior of the sequence of iterates. How could you have anticipated the result without performing the actual calculations?
3. You can think of the iteration scheme for $x = f(x) = x^2 + C$ as the difference equation $x_{n+1} = f(x_n) = x_n^2 + C$. What are the equilibrium levels for the solutions to the difference equation? Under what conditions on C will the equilibrium values be real?
4. Explain graphically the significance of C in determining whether the iteration process based on $x = f(x) = x^2 + C$ has a real limiting value by looking at the graphs of $y = x^2 + C$ and $y = x$.
5. Explain graphically why the iteration process based on the function $x = f(x) = x^3 + C$ must have at least one real limiting value.
6. Consider iterations $x = f(x)$ based on the function $f(x) = x + \sin x$.
   a. Begin the iteration process at $x_0 = 2$ and perform enough iterations to allow you to recognize the limit of the resulting sequence.
   b. Repeat part (a), starting with $x_0 = 5$.
   c. Repeat part (a), starting with $x_0 = 8$. How does the limiting value compare to π?
   d. Repeat part (c), starting with $x_0 = 15$.
   e. Based on the function f, explain why all limits will be some multiple of π.
7. Consider iterations $x = f(x)$ based on the function $f(x) = x + \cos x$. Predict the possible values that can arise for the limits based on the function f. Verify whether your predictions are correct if you start with initial values $x_0 = 1, 3, 7$, and -12.

Chapter Summary

In this chapter, we continued our discussion of trigonometric functions. In particular, we discussed the following:

- The fundamental identities that relate the sine and cosine functions.
- Some identities involving the tangent function.
- How to approximate the sine and cosine functions with polynomial functions.

- How the accuracy of a polynomial approximation depends on the degree of the polynomial.
- How to convert a complex number to its equivalent trigonometric form.
- How to construct powers of complex numbers with DeMoivre's theorem.
- The Julia set that is associated with a function $f(z)$ and the idea of chaos.
- The Mandelbrot set that is associated with a function $f(z)$.

Review Problems

Determine graphically which of the relationships in Problems 1–9 might be identities and which clearly are not identities. For those that appear to be identities, prove them algebraically.

1. $\sin x \cos^2 x + \sin^3 x = \cos x$
2. $\sin x \cos^2 x + \sin^3 x = \sin x$
3. $\sin x \cos^2 x + \sin^3 x = \cos^2 x$
4. $\sin x \cos^2 x + \sin^3 x = \sin^2 x$
5. $(\sin x + \cos x)^2 = 1 + \sin 2x$
6. $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta}$
7. $\dfrac{1}{1+\cos t} + \dfrac{1}{1-\cos t} = \dfrac{2}{\sin^2 t}$
8. $\cos^4\theta - \sin^4\theta = \cos 2\theta$
9. $\cos^6\theta - \sin^6\theta = \cos 2\theta$
10. Prove each identity.
    a. $\cos(x+y) + \cos(x-y) = 2\cos x \cos y$
    b. $\sin(x+y) + \sin(x-y) = 2\sin x \cos y$
11. Use the Taylor polynomial approximation of degree $n = 3$ for the sine function to estimate the value of the function $f(x) = \sin 3x$ at $x = 0.2$. Sketch the graph of the function and the approximating polynomial on the same set of axes.
12. Repeat Problem 11 with degree $n = 5$. Discuss any differences you observe.
13. You know that $\sin x \approx T_3(x) = x - x^3/6$ and $\cos x \approx T_2(x) = 1 - x^2/2$ when x is reasonably close to 0, so $[T_3(x)]^2 + [T_2(x)]^2$ should be fairly close to 1. Using your function grapher, estimate how far the expression $[T_3(x)]^2 + [T_2(x)]^2$ is from 1 for any value of x between -1 and 1 radian.
14. **a.** Convert the complex numbers $z = -6 + 8i$ and $w = 5 - 2i$ to trigonometric form.
    b. Use the results from part (a) to find $z \cdot w$, z/w, and w/z.
15. A complex number z has modulus 3 and an associated angle of 52°.
    a. Write the complex number in trigonometric form.
    b. Write the complex number in the usual form $z = a + bi$.
    c. Find the fifth power of this complex number z.
    d. Find the square root of this complex number z.

Use your function grapher to estimate the period for each function. Express your answers as multiples of π.

16. $f(x) = \sin 3x + \cos 2x$
17. $f(x) = \sin 3x + \cos 4x$
18. $f(x) = \sin 4x + \cos 2x$
19. $f(x) = \sin 2x + \cos 4x$
20. $f(x) = \sin\left(\dfrac{1}{2}x\right) + \cos 2x$
21. $f(x) = \sin 3x + \cos\left(\dfrac{1}{2}x\right)$
22. Based on your answers to Problems 16–21, conjecture a general rule for the period of the function $f(x) = \sin mx + \cos nx$, for any m and n.