4

Extended Families of Functions

4.1 Introduction to Polynomial Functions

Samantha has been keeping track of the price of the stock of HyperTech Corporation since her grandmother gave her several shares as a gift. She has plotted the stock values, as shown in Figure 4.1, and wants to construct a mathematical model that represents the price of the stock. Clearly, a linear function, an exponential function, a power function, or a logarithmic function is not a reasonable candidate because none have this kind of behavior pattern. To better capture the trend in the stock prices, Samantha needs a function that changes both its direction and its concavity, as illustrated in Figure 4.2.

Note that the graph increases, then decreases, and finally increases again. Thus, the graph has two turning points, one at the local maximum point and the other at the local minimum point. Also, the curve initially is concave down and then is concave up, so the graph has one point of inflection, where the concavity changes.

In this section, we introduce a new family of functions, the *polynomial functions,* that possess this type of more complicated behavior. A **polynomial function,** or **polynomial,** is any finite sum of power functions with nonnegative integer powers. For instance,

$$y = 3x - 5; \quad \textbf{(1)}$$

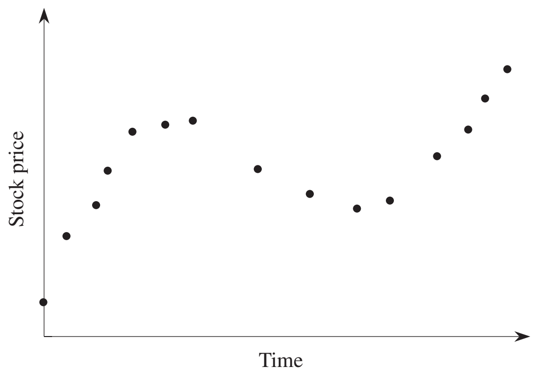


FIGURE 4.1

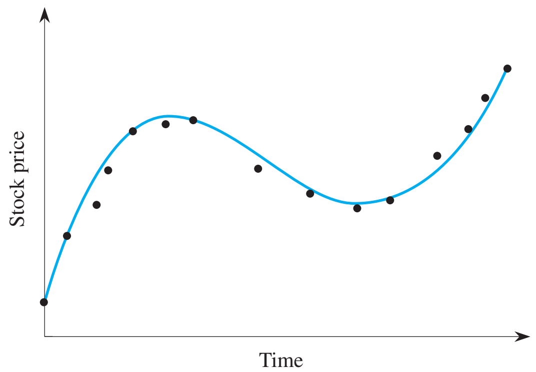


FIGURE 4.2

$$y = 6x^2 + x - 7; \quad \textbf{(2)}$$
$$y = 6 + 8x - 3x^2; \quad \textbf{(3)}$$
$$y = 4x^3 + 5x^2 - 7x + 12; \quad \textbf{(4)}$$
$$y = 10x^8 - 7x^5 + 3 \quad \textbf{(5)}$$

are all polynomials.

The **degree of a polynomial** is the highest power of the variable present. Hence, the degree of Polynomial (1) is 1, the degree of Polynomials (2) and (3) is 2, the degree of Polynomial (4) is 3, and the degree of Polynomial (5) is 8.

The constant multiples in each term in any polynomial are called its **coefficients.** In particular, the coefficient of the highest power term is the **leading coefficient.** Thus, in Polynomial (1), the coefficients are 3 and -5 and the leading coefficient is 3; in Polynomial (2), the coefficients are 6, 1, and -7 and the leading coefficient is 6. Note that, in Polynomial (3), the leading coefficient is -3 (it is *not* necessarily the first coefficient). As we show in Section 4.2, the sign of the leading coefficient determines the overall behavior of the polynomial.

Another way to describe a polynomial is to say that it is a *linear combination* of power functions because, as we noted, it is made up of a sum of power functions. In this sense, power functions are the basic building blocks we use to construct any polynomial.

If a polynomial has degree 1, it is a linear function of the form, $y = ax + b$, where a and b are constants and $a \neq 0$. Its graph is a line with slope $m = a$ and vertical intercept b.

If a polynomial has degree 2, it is called a **quadratic function** and it has the form

$$y = ax^2 + bx + c,$$

where a, b, and c are constants and $a \neq 0$. With three coefficients in the equation, the set of all quadratic functions is a three parameter family of functions. The graph of any quadratic function is a curve called a **parabola.** Such curves abound in the real world—in the path of a fly ball in baseball, in the shape of the main support cable in a suspension bridge such as the Golden Gate Bridge or the George Washington Bridge, or in the cross sections of a TV satellite dish, as depicted in Figure 4.3.

If a polynomial has degree 3, it is called a **cubic function** and its graph is called a **cubic.** In general, a cubic function has the form

$$y = ax^3 + bx^2 + cx + d,$$

FIGURE 4.3

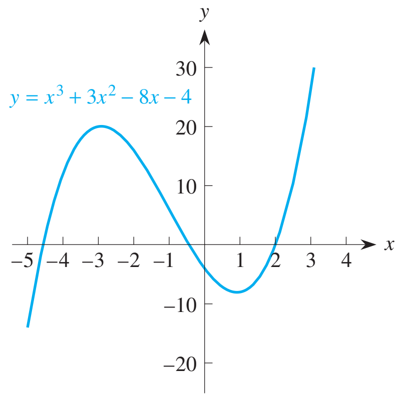


FIGURE 4.4

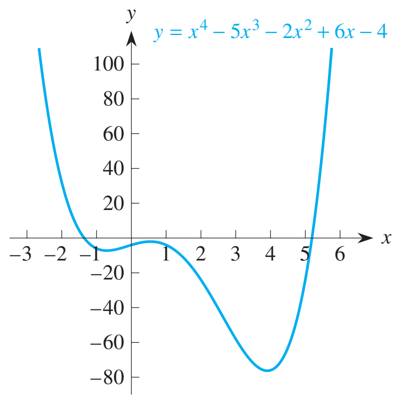


FIGURE 4.5

where a, b, c, and d are constants and $a \neq 0$. For example, the graph of the cubic $y = x^3 + 3x^2 - 8x - 4$ is shown in Figure 4.4. This graph is typical of a cubic function, having two turning points and one inflection point.

If a polynomial has degree 4, it is called a **quartic function** and its graph is called a **quartic.** In general, a quartic polynomial has the form

$$y = ax^4 + bx^3 + cx^2 + dx + e,$$

where a, b, c, d, and e are constants and $a \neq 0$. The graph of the quartic $y = x^4 - 5x^3 - 2x^2 + 6x - 4$ is shown in Figure 4.5. This graph is typical of a quartic polynomial. Notice that it has three turning points and two inflection points.

Think About This

How many parameters are there in the family of cubic polynomials? In the family of quartic polynomials? In the family of polynomials of degree n, for any n? ❒

The Zeros of a Polynomial

A key piece of information about any polynomial function is the value or values of the variable x that make the function zero. These values are known as the **zeros** of the polynomial. For instance, the zeros of the quadratic polynomial $P(x) = x^2 - 6x + 8$ are $x = 2$ and $x = 4$ because

$$P(2) = 2^2 - 6(2) + 8 = 0 \quad \text{and} \quad P(4) = 4^2 - 6(4) + 8 = 0.$$

From a different point of view, if we set the expression for the polynomial function equal to zero, we have an equation and the solutions to this equation are called the **roots.** So, corresponding to the quadratic *polynomial* $P(x) = x^2 - 6x + 8$, we have the quadratic *equation*

$$x^2 - 6x + 8 = 0.$$

Factoring this expression gives

$$(x - 2)(x - 4) = 0.$$

The two solutions of this equation, $x = 2$ and $x = 4$, are the roots of the quadratic function.

Note that a *function has zeros,* that an *equation has roots,* and that there is a direct correspondence between them. The zeros of a function f occur at precisely the same points as the roots of the equation $f(x) = 0$.

EXAMPLE 1

Find the zeros of the quadratic function $P(x) = x^2 - 5x + 6$ and the roots of the corresponding quadratic equation $x^2 - 5x + 6 = 0$, both graphically and algebraically.

Solution The graph of the quadratic function $y = x^2 - 5x + 6$ is shown in Figure 4.6. Note that the graph crosses the x-axis twice: once when $x = 2$ and again when $x = 3$. So, graphically, we conclude that these are the zeros of the function. If we consider the associated quadratic equation

$$x^2 - 5x + 6 = 0,$$

its roots are $x = 2$ and $x = 3$.

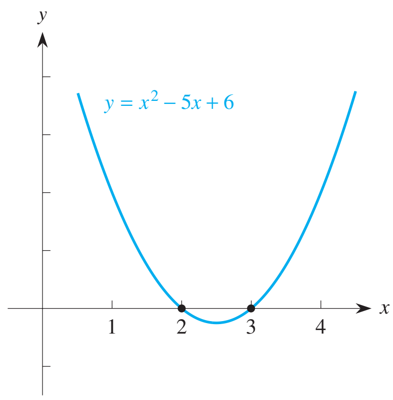


FIGURE 4.6

Alternatively, we can solve this equation algebraically. We start with the associated quadratic equation

$$x^2 - 5x + 6 = 0.$$

We can factor the quadratic expression on the left as

$$(x - 2)(x - 3) = 0.$$

Recall that, when the product of two factors is zero, one or the other or both must be zero, so we have either $x - 2 = 0$ or $x - 3 = 0$, leading to the roots $x = 2$ and $x = 3$. Because they are the roots of the quadratic equation, they are also the zeros of the quadratic polynomial $P(x) = x^2 - 5x + 6$.

If the coefficients in a quadratic are appropriately chosen, we *may* be able to find the roots of a quadratic equation by algebraic factoring, as we did in Example 1. If the coefficients are not just right—say, $4.35709x^2 + 15.46031x - 11.02013 = 0$, or even $5x^2 + 3x - 17 = 0$—the factoring approach won't work. The same principle applies to polynomials of higher degree, but the algebra typically becomes much more complicated as the degree of the polynomial increases. Consequently, factoring is far less likely to work when the degree of a polynomial is 3 or higher.

The two roots for any quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

always can be found from the *quadratic formula.*

The Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is derived in any algebra textbook.

EXAMPLE 2

Find the zeros of the quadratic polynomial $P(x) = x^2 - 3x - 5$.

Solution With $a = 1$, $b = -3$, and $c = -5$, the quadratic formula gives the roots of the associated quadratic equation $x^2 - 3x - 5 = 0$ as

$$\begin{aligned} x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)} \\ &= \frac{3 \pm \sqrt{9 - (-20)}}{2} \\ &= \frac{3 \pm \sqrt{29}}{2}. \end{aligned}$$

The result is a pair of irrational numbers. Thus, the roots are

$$x = \frac{3 + \sqrt{29}}{2} \approx 4.19258 \quad \text{and} \quad x = \frac{3 - \sqrt{29}}{2} \approx -1.19258.$$

The quadratic formula was essentially known to the ancient Babylonians, some 4000 years ago. However, not until about 1540 did Italian mathematicians Tartaglia and Cardano discover a comparable, although considerably more complicated, formula for the three roots of any cubic equation. Not long after that, another Italian mathematician, Ferrari, discovered an even more complicated formula that gives the four roots of any quartic equation. (These formulas are programmed into some calculators and software packages.) Finally, in 1824, Danish mathematician Abel proved that no general formula could exist that would give the roots of any polynomial equation of fifth or higher degree. When we encounter polynomials of higher degree, we usually have to resort to numerical methods to find the roots. We illustrate this approach in Example 3 for a polynomial of degree 3.

EXAMPLE 3

Find, correct to four decimal places, all the zeros of the cubic polynomial $y = Q(x) = x^3 + 3x^2 - 8x - 4$.

Solution The graph of this polynomial is shown in Figure 4.7. Note that it crosses the x-axis three times and that each of these points is a zero of the polynomial. By zooming in on each point in turn, using a function grapher, we estimate that the zeros are located at approximately $x = -4.56155$, $x = -0.43845$, and $x = 2.00000$. This last value for x suggests that the third zero may be $x = 2$ precisely. To determine whether that is indeed the case, we substitute $x = 2$ into the formula for the cubic and find that

$$Q(2) = (2)^3 + 3(2)^2 - 8(2) - 4 = 8 + 12 - 16 - 4 = 0,$$

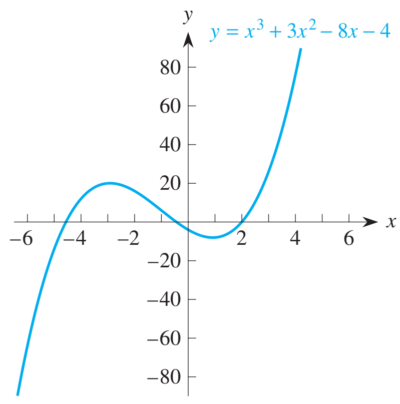


FIGURE 4.7

so $x = 2$ is precisely the zero. Because we were asked to give the three zeros correct to four decimal places, we conclude that $x \approx -4.5616$, $x \approx -0.4385$, and $x = 2$.

We found three zeros for the cubic polynomial in Example 3 based on its graph in Figure 4.7. But, how do we really know that there are no additional zeros? We could expand the viewing window and examine the graph from $x = -100$ to $x = 100$, say, or perhaps from $x = -1000$ to $x = 1000$, and maybe other zeros will come into view. Unfortunately, this kind of exploratory approach never completely closes the door on the possibility that other zeros might exist if only we look further. Instead, we need to know something about the behavior of polynomials in general, which will give us information on how many zeros a particular polynomial has and some indication of where to look for them. We discuss this in the next section.

Problems

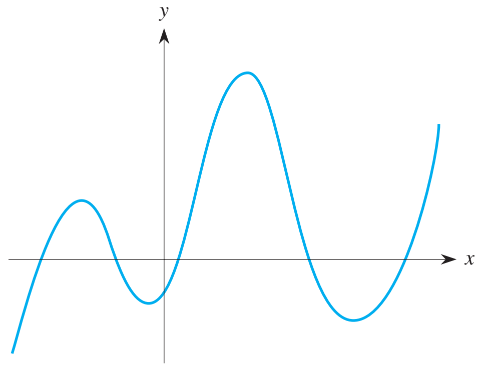
1. What is the degree of each polynomial?
   a. $P(x) = 6x^3 - 5x^2 + 8$
   b. $P(x) = 5x^4 + 6x^3 + 7x - 11$
   c. $P(x) = 6x - 5x^2$
   d. $P(x) = x^5 - x^8$
   e. $P(x) = -4x^3 - 9x^2 + 12$
   f. $P(x) = 10 - 4x + 5x^3 + 3x^6$
2. What is the leading coefficient of each polynomial in Problem 1?
3. Which values of x are zeros of the polynomial $P(x) = x^3 + 2x^2 - 3x$ and which are not?
   a. $x = 3$ **b.** $x = 2$ **c.** $x = 1$
   d. $x = 0$ **e.** $x = -1$ **f.** $x = -2$
   g. $x = -3$
4. Consider these polynomials.
   a. $P(x) = 6x^3 - 5x^2 + 8$
   b. $P(x) = -6 - 5x^2 + 8x^3$
   c. $P(x) = 8 - 5x^2 - 6x^3$
   d. $P(x) = -8x^3 - 5x^2 + 6$
   e. $P(x) = 6x^3 - 5x^2 + 8$
   f. $P(x) = 3x^2 - 5x + 4$
   g. $P(x) = -4x^2 - 5x - 8$
   h. $P(x) = 4 - 3x^2$
   i. $P(x) = 8 - 5x + 4x^2$
   j. $P(x) = 6x^4 - 5x^3 + 8x - 3$
   k. $P(x) = 3x^4 - 5x^3 + 8x - 6$
   l. $P(x) = 3 - 6x^4$

   For each polynomial in (a)–(l), indicate whether it is a
   i. quadratic polynomial,
   ii. cubic polynomial,
   iii. quartic polynomial,
   iv. quadratic polynomial whose leading coefficient is 4,
   v. cubic polynomial whose leading coefficient is -6, or
   vi. quartic polynomial whose leading coefficient is -6.

5. The table gives some values for a polynomial P. Identify possible roots of the corresponding polynomial equation.

x	-5	-4	-3	-2	-1	
$P(x)$	227	21	0	-8	0	
x	0	1	2	3	4	5
$P(x)$	16	23	0	-32	0	166

6. The figure to the right shows the graph of a polynomial. How many zeros does it have?
7. Estimate, correct to three decimal places, all the zeros of the polynomial $P(x) = 2x^3 - 6x^2 + 5x - 3$.
8. Estimate, correct to three decimal places, all the zeros of the polynomial $P(x) = x^4 - 4x^3 + 5x - 1$.
9. For the cubic polynomial in Problem 7, how many turning points are there? how many inflection points? Estimate, correct to two decimal places, all the turning points.
10. For the quartic polynomial in Problem 8, how many turning points are there? how many inflection points? Estimate, correct to two decimal places, all the turning points.



Exercising Your Algebra Skills

Add or subtract each pair of polynomials by combining like terms.

1. $(6x^3 - 5x^2 + 8) + (6x - 5x^2)$
2. $(6x^3 - 5x^2 + 8) - (6x - 5x^2)$
3. $(5x^4 + 6x^3 + 7x - 11) + (-4x^3 - 9x^2 + 12)$
4. $(5x^4 + 6x^3 + 7x - 11) - (-4x^3 - 9x^2 + 12)$
5. $(10 - 4x + 5x^3 + 3x^4) + (5x^4 + 6x^3 + 7x - 11)$
6. $(10 - 4x + 5x^3 + 3x^4) - (5x^4 + 6x^3 + 7x - 11)$

Multiply each pair of polynomials.

7. $x(3x - 5)$
8. $x(4x + 2)$
9. $x(7 + 3x)$
10. $x(6 - 5x)$
11. $(x - 1)(x - 3)$
12. $(x - 2)(x - 5)$
13. $(x - 2)(x + 3)$
14. $(x + 4)(x + 3)$
15. $(x + 5)(x - 5)$
16. $(x - 3)(x + 3)$
17. $(x + 2)(x - 2)$
18. $(x - 21)(x + 21)$

Raise each polynomial to the indicated power.

19. $(x - 1)^2$
20. $(x - 3)^2$
21. $(x + 2)^2$
22. $(2x + 5)^2$
23. $(2x - 6)^2$
24. $(x + 10)^2$

4.2 The Behavior of Polynomial Functions

The behavior of a polynomial depends on the ideas we introduced in Section 4.1: the degree, the zeros, and the sign of the leading coefficient of the polynomial. Let's see how.

Quadratic Polynomials

We begin by analyzing the behavior of quadratic functions. The graph of any quadratic function $y = ax^2 + bx + c$ is a parabola that opens either upward or downward. The sign of the leading coefficient a in

$$y = ax^2 + bx + c$$

determines whether the parabola opens upward or downward and so determines the overall behavior of the parabola. When the leading coefficient is positive, the parabola opens upward and is concave up. When the leading coefficient is negative, the parabola opens downward and is concave down. To understand why, think about what happens when x gets very large—say, $x = 100$ or $x = 1000$. Then x^2 is much larger, on the order of 10,000 or 1,000,000. Therefore the term ax^2 eventually overwhelms any contribution from the linear term bx or the constant term c. Thus, when a is positive, the quadratic term is extremely positive and the parabola opens upward. Similarly, when a is negative, the quadratic term is extremely negative and the parabola opens downward.

For instance, the parabola $y = 5x^2 - 20x - 300$ has a positive leading coefficient and so opens upward—when x becomes large, either positively or negatively, the overall effect is positive. In contrast the parabola $y = 20 - 4x^2$ has a negative leading coefficient and so opens downward—when x becomes large, either positively or negatively, the overall effect is negative. Check the graphs of both functions on your function grapher to convince yourself of the behavior in each case. Moreover, whichever way the parabola opens, as x increases indefinitely in either direction, the parabola either increases toward infinity or decreases toward negative infinity, as illustrated in Figure 4.8.

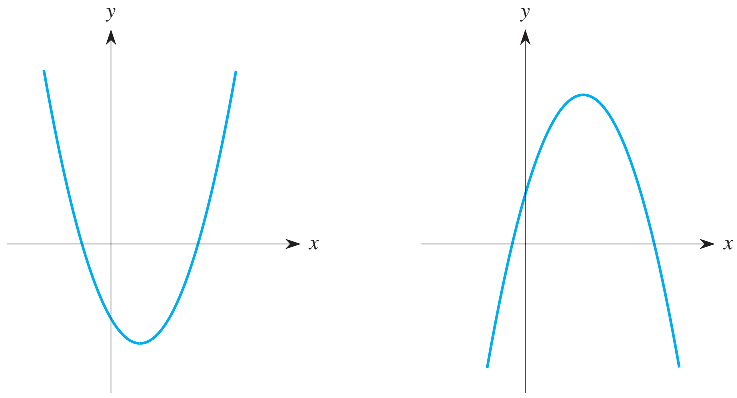


FIGURE 4.8 Leading coefficient > 0 Leading coefficient < 0

Every parabola has one turning point—also called its **vertex.** For instance, the parabola $y = x^2$ has its vertex at the origin, because that is the location of the turning point. If a parabola opens upward, the turning point corresponds to the minimum value of the function. If a parabola opens downward, the turning point corresponds to the maximum value of the function. In addition, the parabola is always symmetric about the vertical line through its turning point, so the left and right halves of the parabola are mirror images of one another. (See Appendix D for a discussion of symmetry.)

Next, let's examine the effects of the other two terms (the linear term and the constant term) in the formula for a quadratic function. Figure 4.9 shows the graphs associated with the quadratic functions $y = x^2$, $y = x^2 + 6$, $y = x^2 - 5x + 6$, and $y = x^2 + 5x + 6$. The leading term determines the basic behavior of the quadratic function, so all four open upward. However, the other terms affect the location of the graph. The constant term 6 in $y = x^2 + 6$ raises the parabola $y = x^2$ by 6 units (if the constant term is negative, the parabola would be lowered instead). Use your function grapher to experiment with this effect on the graph of the parabola by changing the constant term. For instance, how do the graphs of $y = x^2 + 5x + 7$ and $y = x^2 + 5x - 2$ compare to the graph of $y = x^2 + 5x + 6$? Be sure to look at

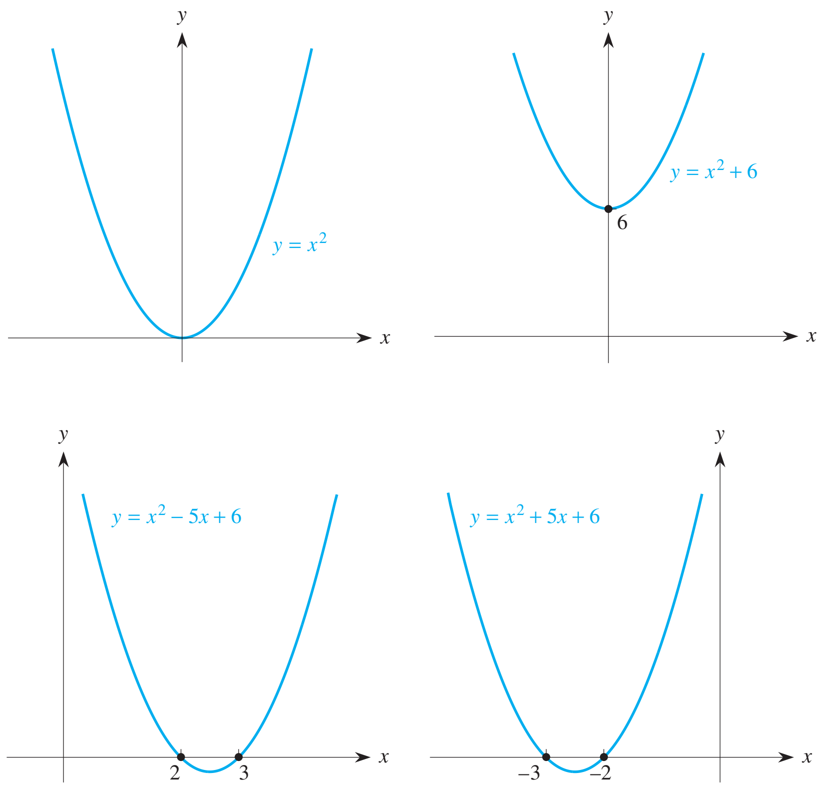


FIGURE 4.9

enough graphs to convince yourself of the effect of the constant term. We investigate this effect in detail in Section 4.7.

The effect of the linear term is more complicated because it involves both vertical and horizontal shifting of the parabola. We don't go into that here but do so in Section 4.7.

Furthermore, as we showed in Section 4.1, the two roots of any quadratic equation

$$ax^2 + bx + c = 0$$

can always be found from the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These two roots could be real numbers (as in Examples 1 and 2 in Section 4.1) or the roots could be a pair of complex numbers of the form $x = \alpha + \beta i$ and $x = \alpha - \beta i$ where $i = \sqrt{-1}$ (α and β are the Greek letters alpha and beta, respectively). A pair of complex numbers such as these is called a pair of *complex conjugates.* Complex numbers are discussed in Appendix E.

EXAMPLE 1

Find the roots of the quadratic equation $x^2 - 2x + 2 = 0$.

Solution Using $a = 1$, $b = -2$, and $c = 2$ in the quadratic formula, we get

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4 - 8}}{2}$$

$$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2}.$$

We now divide through by 2 and find that the two complex roots of the quadratic are $x = 1 + i$ and $x = 1 - i$.

Because every quadratic equation has two roots, every quadratic function has exactly two zeros. Let's see what this means in terms of the graph of the quadratic function. Consider again the quadratic polynomial $P(x) = x^2 - 5x + 6$, whose graph is shown in Figure 4.10. This polynomial has zeros at $x = 2$ and $x = 3$ because we can factor the quadratic as

$$x^2 - 5x + 6 = (x - 2)(x - 3).$$

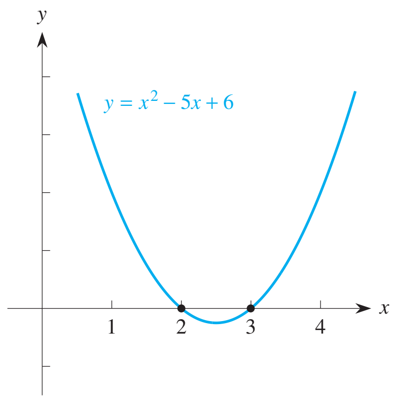


FIGURE 4.10

But the graph shows that the parabola crosses the x-axis at the points $x = 2$ and $x = 3$. Thus, just as the point at which a line crosses the x-axis gives the root of a linear equation, the points at which a parabola crosses the x-axis give the *real roots* of a quadratic equation, as illustrated in Figure 4.11 (a) and (b), respectively.

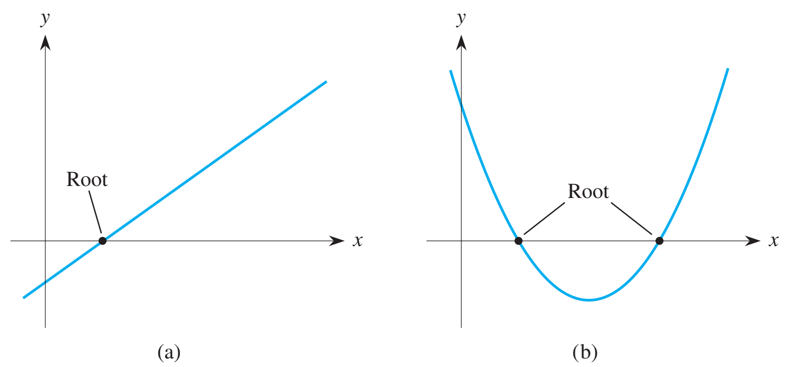


FIGURE 4.11

If we know that a parabola crosses the x-axis at a point $x = r$, then $x = r$ is a zero of the associated quadratic function and $x - r$ is a factor of the quadratic expression. You can locate the real roots of any quadratic to any desired level of accuracy with your graphing calculator or with the quadratic formula, so you can always find the linear factors.

In general, for any quadratic function $f(x) = ax^2 + bx + c$,

- the real roots of the quadratic equation $ax^2 + bx + c = 0$ correspond graphically to the points where the associated parabola crosses the x-axis, and
- the real roots of the quadratic equation $ax^2 + bx + c = 0$ correspond algebraically to the linear factors of the quadratic polynomial.

Depending on the orientation of the parabola (opening up or down) and the position of the turning point, a parabola may not touch the x-axis at all. This is the case with the graph of $y = x^2 + 6$, as shown in Figure 4.12. For such a parabola, the corresponding quadratic equation still has two roots, but they are complex roots. If a quadratic equation has complex roots, they must occur in conjugate pairs of the form $\alpha \pm \beta i$. This property follows directly from the quadratic formula for the case where the term inside the radical, $b^2 - 4ac$, is negative. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic. When the discriminant is positive, the two roots are real numbers. When the discriminant is negative, the two roots are complex numbers. Finally, when the discriminant is zero, there is a *double real root.*

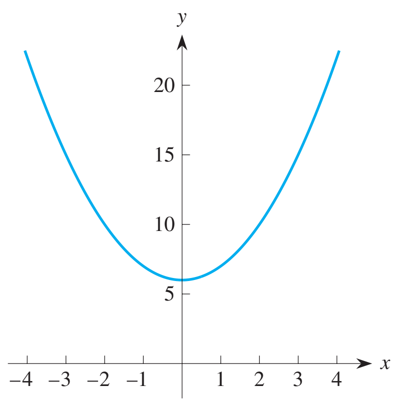


FIGURE 4.12

For instance, the discriminant for $y = x^2 + 6$ is $0^2 - 4(1)(6) = -24 < 0$, so the two roots are complex and they occur in pairs. From the quadratic formula, the roots are

$$x = \frac{-0 \pm \sqrt{0^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{\pm\sqrt{-24}}{2}$$

$$= \pm\frac{2\sqrt{-6}}{2} = \pm\sqrt{6}i.$$

We have already demonstrated that a parabola can cross the x-axis at two points (corresponding to two real roots) or that it may not ever cross the x-axis (corresponding to a pair of complex conjugate roots). A third possibility is that the parabola could be tangent to the x-axis; that is, it can touch the axis and bounce back without ever crossing the axis. For instance, consider the quadratic

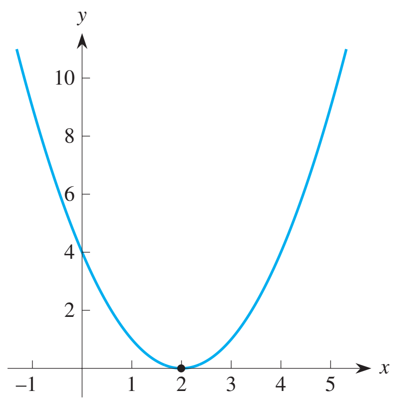


FIGURE 4.13

function $y = x^2 - 4x + 4$. If you apply the quadratic formula, you will find that the discriminant $b^2 - 4ac = (-4)^2 - 4(1)(4) = 0$ and the two roots are $x = 2$ and $x = 2$, as shown in Figure 4.13. Use your function grapher to zoom in on this point and note how the parabola just touches the x-axis at $x = 2$. Thus, the quadratic function has two roots, but because they are equal, $x = 2$ is a double root.

Think About This

Try changing the value of the constant term in $y = x^2 - 4x + 4$ slightly from 4—say, to 4.01 and then to 3.99. What happens to the graph in each case? What is the value of the discriminant in each case?

We summarize these ideas about quadratic polynomials as follows.

Characteristics of Quadratic Polynomials $y = ax^2 + bx + c$

- A quadratic polynomial has degree 2 and has precisely 2 zeros.
- A parabola has precisely one turning point, its vertex.
- A parabola opens upward if the leading coefficient a is positive; it opens downward if the leading coefficient a is negative, as shown in Figures 4.14 (a)–(f).
- The corresponding quadratic equation of degree 2 has precisely 2 roots. They may be real or complex.
- The complex roots occur in pairs of complex conjugates, $x = \alpha \pm \beta i$, where $i = \sqrt{-1}$, as illustrated in Figures 4.14 (a) and (d).
- The real roots correspond to the points where the parabola crosses the x-axis, as illustrated in Figures 4.14 (b) and (e), or where the parabola touches the x-axis, as illustrated in Figures 4.14 (c) and (f).
- You can always find the real roots graphically by using your function grapher to zoom in on the points where the parabola crosses or touches the x-axis.
- The real roots correspond to the linear factors of the quadratic expression.
- You can always find the roots, real or complex, by using the quadratic formula.

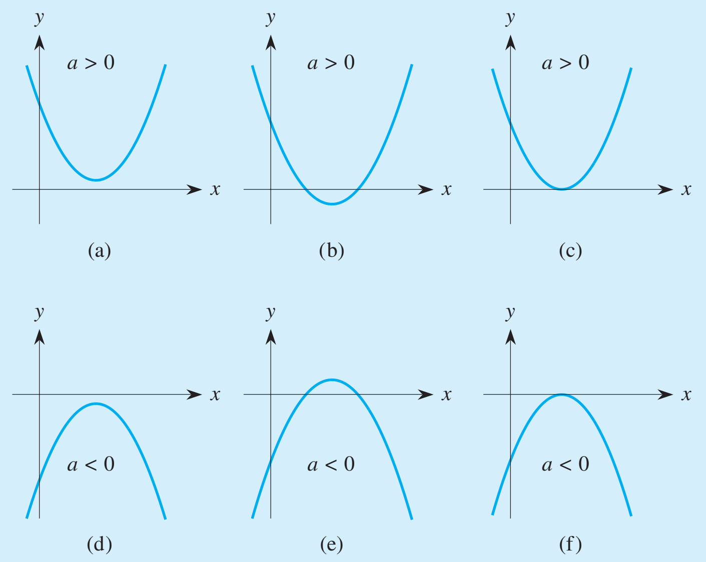


FIGURE 4.14

Cubic Polynomials

Next, we consider the characteristics of cubic polynomials having the form

$$y = ax^3 + bx^2 + cx + d,$$

where a, b, c, and d are constants and $a \neq 0$. The graph of the cubic polynomial $y = x^3 + 3x^2 - 8x - 4$ is shown in Figure 4.15, which is typical of a cubic function. The cubic rises toward positive infinity in one direction and drops toward negative infinity in the other. Also, this particular cubic has two turning points and crosses the x-axis at three points, so it has three real zeros. Moreover, the curve has one point of inflection, is concave down on one side of the point of inflection, and is concave up on the other.

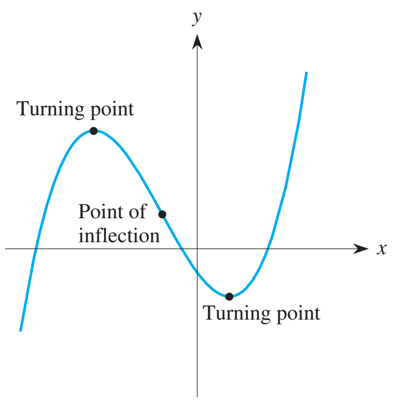


FIGURE 4.15

As with a quadratic function, the sign of the leading coefficient in a cubic always determines the overall behavior pattern of the function. If the leading coefficient is positive, the cubic increases as x increases (except possibly for a relatively small dip between the two turning points), as shown on the left in Figure 4.16. If the leading coefficient is negative, the cubic decreases as x increases (except for a possible rise between the two turning points), as shown on the right in Figure 4.16.

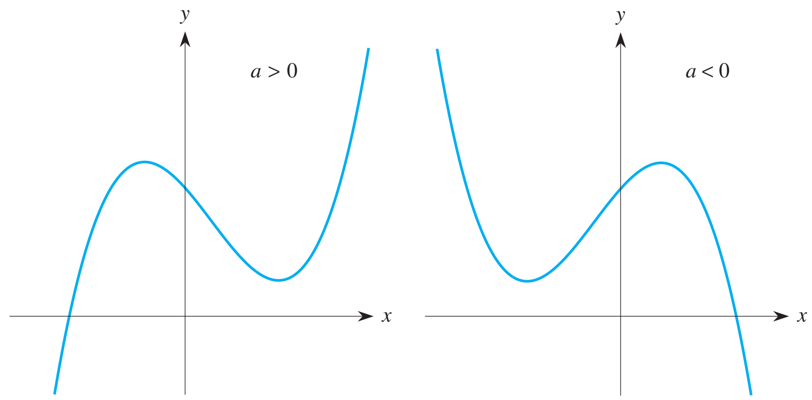


FIGURE 4.16

In general, a cubic function $y = ax^3 + bx^2 + cx + d$ has three zeros, and the corresponding cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

has three roots. The roots can all be real numbers or can consist of a single real number and a pair of complex conjugate numbers. Each of the real roots corresponds to a linear factor of the corresponding cubic expression. Any complex conjugate roots must occur in pairs and correspond to a quadratic factor of the cubic polynomial.

> The real roots of a cubic equation correspond graphically to the points at which the associated cubic curve crosses the x-axis.
>
> The real roots of a cubic equation correspond algebraically to the linear factors of the cubic polynomial.

If a cubic has three real roots, its curve crosses the x-axis at the corresponding three points. If it has only one real root, the curve crosses the x-axis only once, as shown in Figure 4.17.

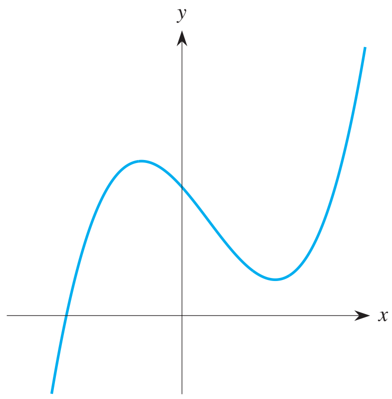


FIGURE 4.17

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EXAMPLE 2

Analyze the behavior of the cubic function

$$f(x) = (x - 1)(x + 2)(x + 5) = x^3 + 6x^2 + 3x - 10.$$

Solution The cubic has the three linear factors—$(x - 1)$, $(x + 2)$, and $(x + 5)$—so it has three real zeros: at $x = 1$, $x = -2$, and $x = -5$, corresponding to each of the three factors. Consequently, its graph crosses the x-axis at $x = 1, -2$, and -5, as shown in Figure 4.18. Further, the leading term $x^3 = 1 \cdot x^3$, being the highest power present, eventually dominates the other terms as x increases. Because the leading coefficient 1 is positive, the cubic must increase toward $+\infty$ as $x \to \infty$ and decrease toward $-\infty$ as $x \to -\infty$. Verify this result graphically by using your function grapher and numerically by substituting some large positive and negative values for x.

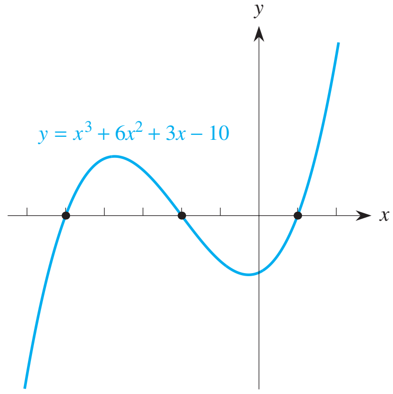


FIGURE 4.18

Although there is a formula for calculating the roots of a cubic equation, it is considerably more complicated than the quadratic formula and is seldom used. If the cubic polynomial happens to factor simply, you can find the zeros directly because each factor corresponds to a zero. However, that is not likely to happen. Usually, the simplest way to find the real roots of a cubic equation is to approximate them by using your function grapher—just keep zooming in on the points where the curve crosses the x-axis until you find the roots to whatever degree of accuracy you desire.

We summarize these ideas about cubic polynomials as follows.

Characteristics of Cubic Polynomials $y = ax^3 + bx^2 + cx + d$

- A cubic polynomial of degree 3 has precisely 3 zeros.
- A cubic has at most two turning points.
- A cubic typically has one inflection point.
- A cubic increases (rises upward) to the right as x increases if the leading coefficient a is positive; it decreases (falls downward) to the right as x increases if the leading coefficient a is negative.
- The corresponding cubic equation of degree 3 has precisely 3 roots. The roots may all be real or one real and two complex.

- The complex roots occur in pairs of complex conjugates, $\alpha \pm \beta i$, where $i = \sqrt{-1}$.
- The real roots correspond to the points where the cubic crosses the x-axis.
- You can always find the real roots graphically by using your function grapher to zoom in on the points where the cubic crosses the x-axis.
- The real roots correspond to linear factors of the cubic expression.

Figure 4.19 illustrates most of the possible cases for a cubic polynomial. In Figures 4.19 (a) and (b) there are three distinct real roots when the leading coefficient a is either positive or negative. In Figures 4.19 (c) and (d) there are three real roots, but one of them is repeated, so the x-axis is tangent to the cubic at the corresponding point. These two graphs correspond to when the leading coefficient $a > 0$; similar graphs can be drawn when $a < 0$. Figure 4.19 (e) shows a cubic with a triple real root and $a > 0$; note how the curve flattens as it crosses the x-axis. Think about the graph of $y = x^3$ as it passes through the origin. Finally, in Figures 4.19 (f)–(h) there is one real root and a pair of complex roots, again when $a > 0$. You can draw similar graphs when $a < 0$.

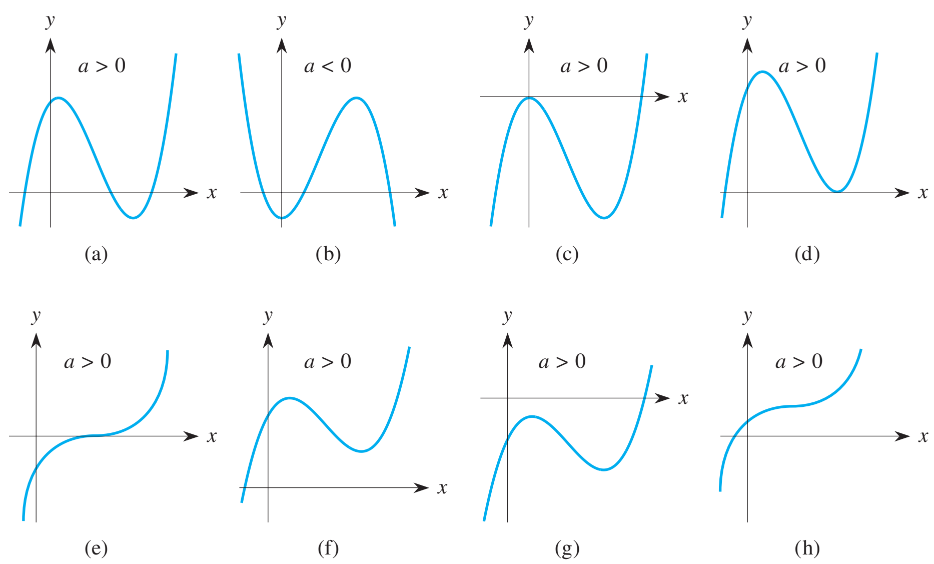


FIGURE 4.19

Moreover, it turns out that every cubic is symmetric about its inflection point. Imagine a cubic with a hinge at its inflection point—if you take either half of the curve and rotate it about that hinge, it will eventually be a perfect fit to the other half of the curve.

Think About This Prove that any cubic polynomial of the form $f(x) = ax^3$ is symmetric about its inflection point at the origin by showing that for any value of x—say, $x = h > 0$—then $f(-h) = -f(h)$.

Polynomials of Degree *n*

The ideas discussed for polynomials of degree 2 (quadratics) and degree 3 (cubics) can be extended to polynomials of any degree n, where n is a positive integer. In particular, they have the following characteristics.

Characteristics of Polynomials of Degree *n*

- A polynomial of degree n has precisely n zeros.
- A polynomial of degree n has at most $n - 1$ turning points.
- A polynomial of degree n has at most $n - 2$ points of inflection.
- If the leading coefficient is positive, the polynomial rises toward $+\infty$ to the right as $x \to \infty$, and if the leading coefficient is negative, the polynomial falls toward $-\infty$ to the right as $x \to \infty$.
- The corresponding polynomial equation of degree n has precisely n roots, which may be real or complex.
- The complex roots occur in pairs of complex conjugates, $\alpha \pm \beta i$, where $i = \sqrt{-1}$.
- The real roots correspond to the points where the curve crosses or touches the x-axis.
- You can always find the real roots graphically to any desired level of accuracy by using your function grapher to zoom in on the points where the curve crosses the x-axis.
- The real roots correspond to linear factors of the polynomial expression.

Think About This

Sketch the graph of a fifth degree polynomial with five real roots and a positive leading coefficient. Sketch the graph of a fifth degree polynomial with three real roots and a negative leading coefficient.

EXAMPLE 3

Suppose that a polynomial has roots at $x = -4, -1, 1, 3$, and 6. Find a possible formula for it and describe its behavior.

Solution The polynomial has five real roots, so its degree must be at least 5; it might be higher if there are complex roots or repeated roots. The five corresponding linear factors are $(x - (-4)) = (x + 4)$, $(x + 1)$, $(x - 1)$, $(x - 3)$, and $(x - 6)$. If these are the only roots, one possible formula for this polynomial is

$$P(x) = (x + 4)(x + 1)(x - 1)(x - 3)(x - 6),$$

although any constant multiple A of this expression would be an alternative formula.

We can determine the value of the multiple A if we know the vertical intercept of the polynomial—or any other point on the curve. If the multiple A is positive, the graph of the polynomial has the behavior shown in Figure 4.20. Note that $P(x)$ rises toward ∞

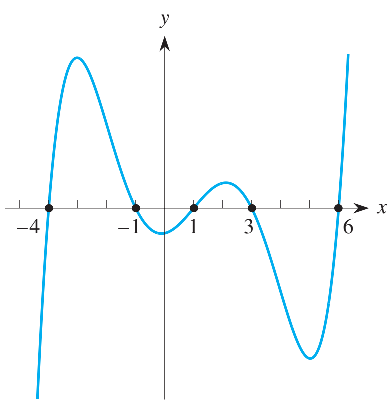


FIGURE 4.20

as $x \to \infty$ and that $P(x)$ falls toward $-\infty$ as $x \to -\infty$. Alternatively, if the constant multiple A is negative, this behavior is reversed; the graph drops toward $-\infty$ as $x \to \infty$ and rises toward $+\infty$ as $x \to -\infty$. Can you explain why this is the case?

What if a polynomial has a double or repeated factor? For instance,

$$P(x) = (x + 1)(x-2)(x-4)^2 = 0$$

has roots at $x = -1$, 2, and 4, but $x = 4$ is a double root because $(x - 4)^2 = (x - 4)(x - 4)$ is a repeated factor. Note that its graph, as shown in Figure 4.21, falls to touch the x-axis at $x = 4$ where it flattens and then rises again. Zooming in on the curve about this point reveals that the x-axis is tangent to the graph at $x = 4$, just as the x-axis is tangent to the parabola $y = x^2$ at the origin.

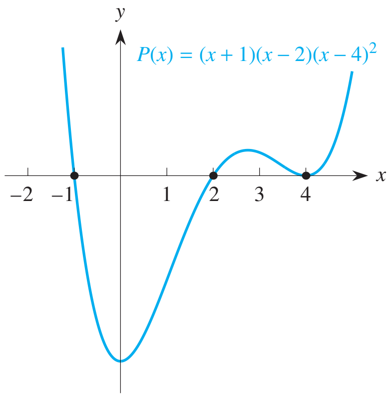


FIGURE 4.21

Think About This The polynomial $y = (x + 1)(x - 2)^3$ has a triple factor. Examine its graph to see what happens near that triple root. First, try to predict what will happen, based on your knowledge of the behavior of $y = x^3$ near the origin.

EXAMPLE 4

Use the fifth degree polynomial from Example 3 to demonstrate why it must have four turning points and three inflection points.

Solution Let's trace the polynomial's curve in Figure 4.20 slowly from left to right. The function starts rising as we move to the right and crosses the x-axis at the first root at $x = -4$. It must cross the x-axis again at $x = -1$, so there must be a turning point between these two roots. Similarly, there must be a turning point between the roots at $x = -1$ and $x = 1$, and in fact, there is a turning point between each successive pair of roots. Because there are five real roots, there must be four turning points.

Now let's consider inflection points. We begin with the first two turning points, one near $x = -3$ where the curve is concave down and the next near $x = -0.2$ where the curve is concave up. The change in concavity means that there must be an inflection point between the successive turning points. In fact, between each successive pair of turning points, there is a point of inflection. Because there are four turning points, there must be three inflection points.

Things may not be quite so simple if there are complex roots or multiple roots.

The End Behavior of a Polynomial

The end behavior of any polynomial depends on the sign of the leading term because, as x approaches $+\infty$ or $-\infty$, the leading term eventually dominates all other terms. For instance, consider the polynomial $P(x) = 2x^4 - 6x^3 + 7x - 10$. When x is very large, the term $2x^4$ will overwhelm all other terms. The graphs of the functions $P(x) = 2x^4 - 6x^3 + 7x - 10$ and $Q(x) = 2x^4$ for x between -3 and 4 and y between -25 and 100 are shown in Figure 4.22; the two curves look quite different. Figure 4.23 presents a slightly larger view, where x is between -6 and 6 and y is between -50 and 950. Here the two curves look more similar than in the preceding view. In the much larger view presented in Figure 4.24, where x is between -25 and 25 and y is between 0 and 500,000, there no longer is much difference between the two curves. The term $2x^4$ dominates the behavior of the polynomial, and the effect of the lower power terms is negligible.

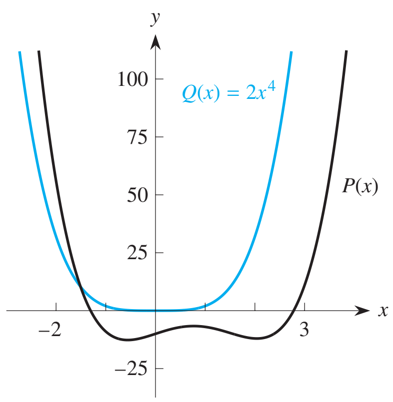


FIGURE 4.22

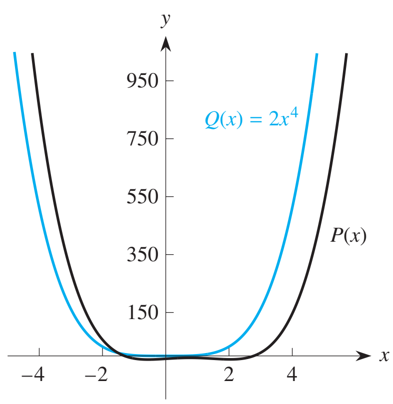


FIGURE 4.23

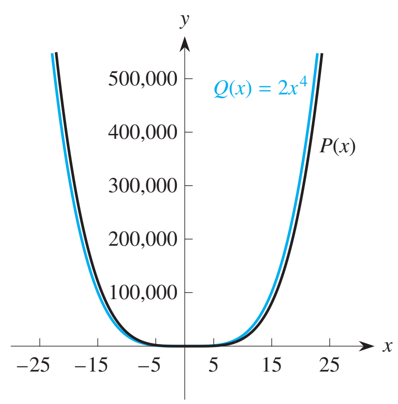


FIGURE 4.24

In general, for any polynomial, when x is large enough, the curve is indistinguishable from the curve corresponding to just the leading term. In other words, in the large, the behavior of any polynomial is virtually identical to that of the power function consisting of the leading term. You can see the *end behavior* easily on your function grapher if you use a reasonably large viewing window.

On the one hand, if the viewing window is too large, the location of the turning points and the zeros of a polynomial is a *local* aspect of the graph and can be easily missed. On the other hand, if the viewing window is too small, the overall growth pattern of the polynomial is lost. For instance, by focusing too closely on one particular turning point or root, you may lose sight of all the others. Rarely does a single view suffice to show all the important details of a function. Therefore, as a matter of routine, you should use the information given in several different views on your calculator or computer to sketch a rough hand-drawn graph of the function, called the *complete graph,* which highlights the key information, even if you intentionally do *not* draw it to scale.

We expect you to use your function grapher to produce the graph of a polynomial, but you should interpret with care what the calculator or computer shows. Usually, the important characteristics of any function—and a polynomial in particular—are

- the end behavior (is it increasing or decreasing as $x \to \infty$ or $x \to -\infty$?),
- the intervals over which the function is increasing or decreasing,
- the locations of the turning points,
- the intervals over which the function is concave up or concave down,
- the locations of the points of inflection, and
- the locations of the real zeros.

EXAMPLE 5

For the polynomial shown in Figure 4.25, answer the following questions.

a. How many real roots are there?
b. How many turning points are there?
c. How many inflection points are there?
d. What is the minimum degree of the polynomial?
e. How many complex roots does it have?
f. What is the sign of the leading coefficient?

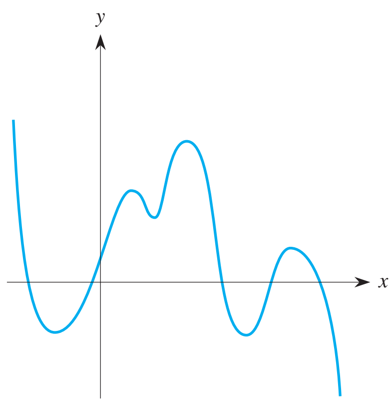


FIGURE 4.25

Solution

a.–c. The graph shown in Figure 4.25 reveals five real roots that correspond to the five points where the curve crosses the x-axis. It also shows six turning points and five inflection points.

d. Because the number of turning points is typically 1 less than the degree of the polynomial and the number of inflection points is 2 less than the degree, we conclude that the polynomial shown is at least a seventh degree polynomial.

e. Because there are five real roots and the degree of the polynomial is at least seven, there must be at least two complex roots.

f. The graph eventually falls toward $-\infty$ as $x \to \infty$, so we conclude that the leading coefficient must be negative.

EXAMPLE 6

Factor the polynomial $P(x) = x^4 - 5x^3 - 7x^2 + 8x + 3$.

Solution This polynomial is a quartic, so it has precisely four roots. We know that the linear factors of the polynomial correspond to its real roots, and the graph shown in Figure 4.26 reveals that there are four real roots. As a result, there cannot be any complex roots. We can locate each of these real roots to any desired level of accuracy, using either numerical or graphical methods. Correct to four decimal places, the roots are $x \approx -1.6272$, $x \approx -0.3105$, $x \approx 1.0000$, and $x \approx 5.9377$. The third of these results, $x \approx 1.0000$, suggests that the root might be $x = 1$ precisely. To verify that this is true, we substitute into the formula for the polynomial and find that

$$P(1) = (1)^4 - 5(1)^3 - 7(1)^2 + 8(1) + 3 = 1 - 5 - 7 + 8 + 3 = 0.$$

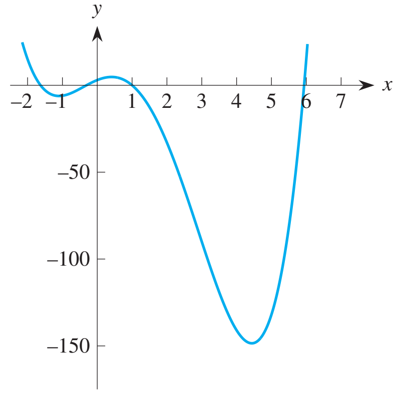


FIGURE 4.26

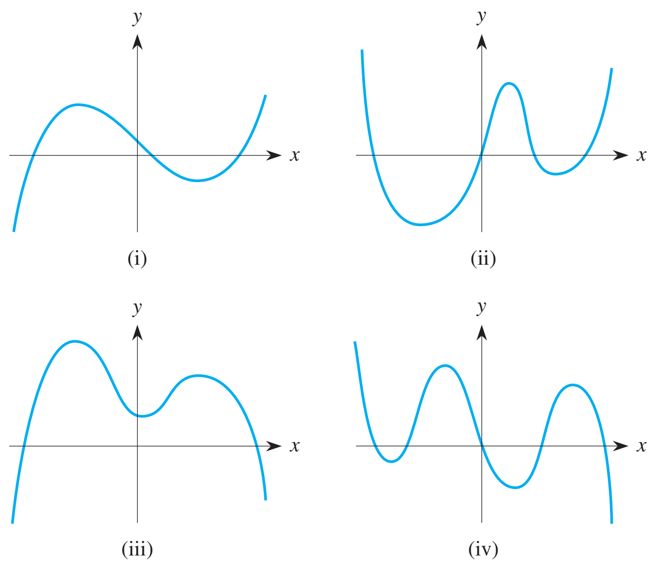
(If you do the same with the other three values, which are just approximations to the roots, the value of the polynomial will only be close to, but not quite equal to, zero.)

The corresponding linear factors are therefore roughly $(x + 1.6272)$, $(x + 0.3105)$, $(x - 1)$, and $(x - 5.9377)$, so the polynomial can be factored, approximately, as

$$P(x) = x^4 - 5x^3 - 7x^2 + 8x + 3 \approx (x + 1.6272)(x + 0.3105)(x - 1)(x - 5.9377).$$

Problems

1. The overall trend in the growth of the gross domestic product (GDP) has been upward except for a small dip. Sketch a graph representing the value of the GDP as a function of time. What type of function might model it? What can you conclude about any of the coefficients?
2. The overall pattern in the growth of the Dow-Jones Industrial Average over the past 10 years has been one of increase except for three sharp, but relatively short-term, drops. Sketch a graph representing the value of the Dow as a function of time. What type of function might model it? What can you conclude about any of the coefficients?
3. Each graph represents a polynomial. For each one:
   a. What is the minimum possible degree of the polynomial? Why?
   b. Is the leading coefficient of the polynomial positive or negative? Why?



4. Each table gives some values for a polynomial. What is the minimum degree of each polynomial? Based

on the values given, what can you conclude about the sign of the leading coefficient in each case?

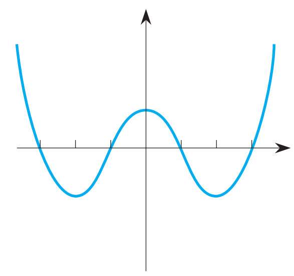
a.

x	−3	−2	−1	0	1	2	3	4	5
y	145	16	−5	−9	12	−3	21	2	−48

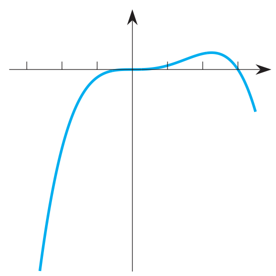
b.

x	−3	−2	−1	0	1	2	3	4	5
y	145	16	27	−11	24	16	41	−9	78

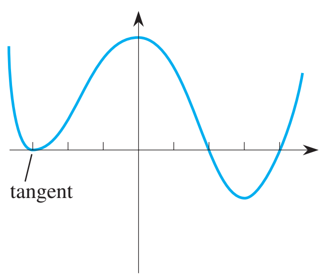
5. Match each polynomial expression a–f with its graph (i)–(vi). Use your knowledge about roots; do not use your function grapher.



(i)



(ii)



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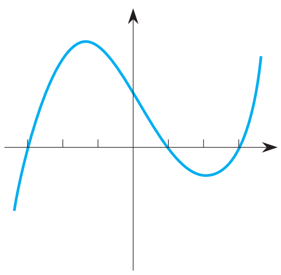
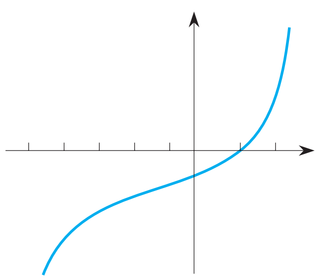
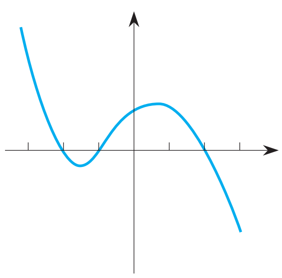


(iv)



(v)



(vi)

a. $f(x) = (x - 1)(x - 3)(x + 3)$
b. $f(x) = (x + 1)(x + 2)(2 - x)$
c. $f(x) = (x - 1)(x^2 + 4)$
d. $f(x) = (x - 1)(x + 1)(x - 3)(x + 3)$
e. $f(x) = 3x^3 - x^4$
f. $f(x) = (x - 2)(x - 4)(x + 3)^2$

Based on your knowledge about roots and factors, sketch the graph of each polynomial function in Problems 6–9. Do not use your function grapher.

6. $f(x) = (x + 2)(x - 1)(x - 3)$

7. $f(x) = 5(x^2 - 4)(x^2 - 25)$

8. $f(x) = -5(x^2 - 4)(x^2 - 25)$

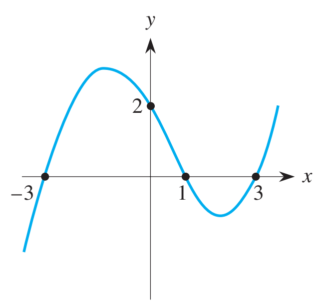
9. $f(x) = 5(x - 4)^2(x^2 - 25)$

10. The polynomial $P(x) = 2x^6 + 5x^5 - 8x^4 - 21x^3 - 12x^2 + 22x + 12$ can be factored as $P(x) = (x - 2)(x - 1)(2x + 1)(x + 3)(x^2 + 2x + 2)$.

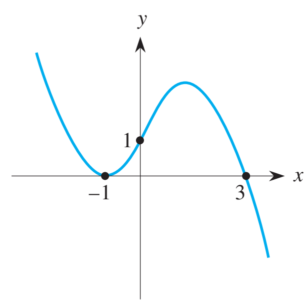
a. What is the degree of the polynomial?
b. What are the real roots? The complex roots?
c. What happens as $x \to \infty$? As $x \to -\infty$?
d. What is the maximum number of turning points you expect? Explain.
e. What is the maximum number of points of inflection? Explain.

11. Determine cubic polynomials that represent the accompanying graphs.

a.

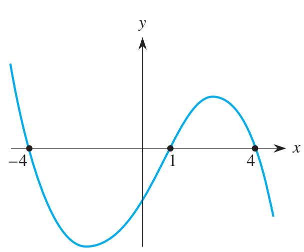


b.

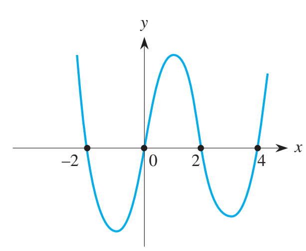


12. Each graph represents a function. For each one, **(i)** read off approximate intervals over which the function is increasing and over which it is decreasing; **(ii)** estimate intervals over which the function is concave up and concave down; and **(iii)** find a possible formula for the function.

a.



b.



13. For each polynomial, (**a**) determine the number of real roots and the number of complex roots; and (**b**) find all real roots correct to three decimal places.

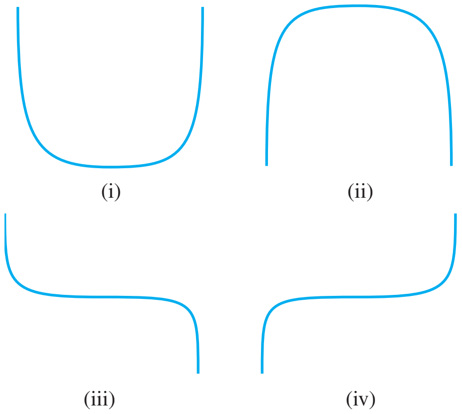
i. $P(x) = x^4 - 8x^2 - 9$

ii. $P(x) = x^5 - 4x^4 - 6x^3 + 6x^2 - 27x + 27$

iii. $P(x) = x^6 - 4x^5 + 6x^4 - 16x^3 + 11x^2 - 12x + 6$

iv. $P(x) = x^6 - 9x^5 + 26x^4 - 41x^3 + 71x^2 - 42x + 6$

14. Determine which of the graphs suggest the end behavior for each polynomial.



a. $y = 5x^5 - 8x^4 + 2x^2 + 3x - 4$

b. $y = -4x^6 + 3x^4 + 7x^3 - 8x^2 - 4x$

c. $y = 3x^8 + 4x^5 + 6x^3 - 5x^2 + 6$

d. $y = -x^7 - 4x^6 + 3x^4 - 6x^3 + 7x - 9$

e. $y = -4x^9 + 6x^6 - 5x^3 + 35$

f. $y = 100 - x^4$

g. $y = (9 - 6x^2)^2$

h. $y = (9 - 6x^3)^2$

i. $y = (9 - 6x^2)^3$

j. $y = (9 - 6x^3)^3$

15. **a.** The graph of the polynomial $P(x) = 2x^4 - 6x^3 + 7x - 10$ in Figure 4.22 suggests that there are three turning points. Use your function grapher to locate them to 3 decimal place accuracy by zooming in on the graph.

b. Estimate all intervals over which $P(x)$ is increasing or decreasing.

c. Estimate the locations of all points of inflection.

d. Estimate all intervals over which $P(x)$ is concave up or concave down.

e. Estimate all real roots.

16. Describe the end behavior of each function. Specifically for the graph of each function f, (**i**) as $x \to \infty$, does $f(x) \to \infty$ or $-\infty$? why? and (**ii**) as $x \to -\infty$, does $f(x) \to \infty$ or $-\infty$? why?

a. $f(x) = -3x^3 + 70x^2 - 20.$

b. $f(x) = 20x^4 + 3x^3 + x^2 + 1000.$

c. $f(x) = -3x^4 + 20x^3 - 5x^2 + x - 20.$

d. $f(x) = x^4 + x^5.$

e. $f(x) = 4x^4 + 5x^5 - 6x^6.$

17. Find the equation of a quadratic polynomial that has a real root at $x = 2$ and a turning point at $(1, 5)$.

18. A cubic polynomial P has turning points at $(1, 4)$ and $(5, 12)$.

a. What is the behavior of $P(x)$ as $x \to \infty$?

b. Where is the point of inflection? (*Hint*: Recall that a cubic is symmetric about its point of inflection.)

19. Suppose that a quadratic polynomial has roots at $x = 6$ and $x = -2$.

a. Write a possible formula for the quadratic function.

b. Use the fact that a quadratic is symmetric about the vertical line through its turning point to determine the x-coordinate of the turning point of this quadratic function.

c. Suppose that the quadratic has a maximum value of 20. What must be its equation?

d. Suppose that the quadratic has a minimum value of -20 instead. What must be its equation?

20. An apple is tossed from ground level straight up at time $t = 0$ with velocity 64 ft/sec. Its height at time t is $f(t) = -16t^2 + 64t$. Find the time when it hits the ground and the instant when it reaches its highest point. What is the maximum height?

21. The height s (in cm) of an object above the ground at time t (in seconds) is given by

$$s = v_0 t - \frac{1}{2}gt^2,$$

where v_0 represents the initial velocity and g is a constant, the acceleration due to gravity.

a. At what height does the object start?

b. How long is the object in the air before it hits the ground?

c. When will the object reach its maximum height?

d. What is that maximum height?

22. **a.** Sketch a smooth graph of today's air temperature from midnight to midnight.

b. When is it a minimum? A maximum?

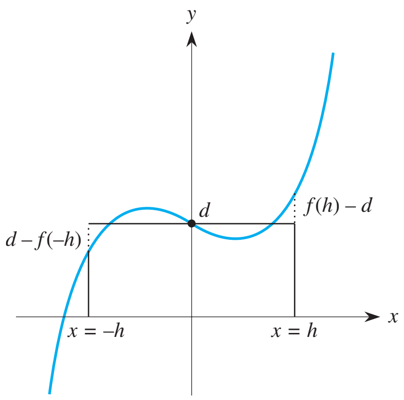
c. When does it have a point of inflection?

d. What type of polynomial might be a good match to the curve you drew?

e. What function would be a better choice if you expand the domain to include the temperatures for yesterday and tomorrow?

23. Factor the polynomial $P(x) = x^3 - 5x^2 + 3x + 7$, using zeros that are correct to two decimal places.

24. a. Prove that any cubic polynomial of the form $f(x) = ax^3 + cx$ is symmetric about its inflection point at the origin by showing algebraically that, if you take any value of x—say, $x = h > 0$—then $f(-h) = -f(h)$.



b. Prove that any cubic polynomial of the form $f(x) = ax^3 + cx + d$ is symmetric about its inflection point at $(0, d)$ by showing algebraically that, if you take any value of x—say, $x = h > 0$—then $f(h) - d = d - f(-h)$, as illustrated in the accompanying figure. (*Note:* The same ideas apply to an arbitrary cubic polynomial when the bx^2 term is present, but the proof is considerably more complicated.)

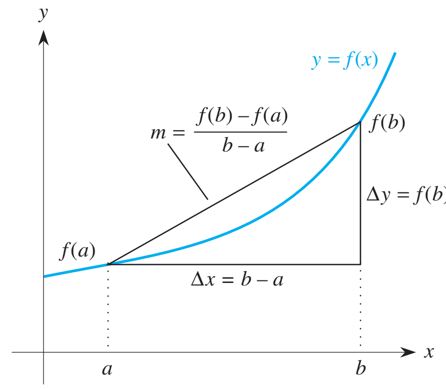
25. Recall that the *average rate of change* of a function f over an interval $x = a$ to $x = b$ (see Section 2.8) is defined as the slope of the line segment connecting the endpoints of the curve on that interval, or

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

as illustrated in the accompanying figure. The table gives some values for the function $f(x) = x^2 - 4x$.

a. Find the average rate of change of f from $x = -2$ to $x = 3$.

b. Calculate the average rate of change of f between each successive pair of points in the table; that is,



x	-2	-1	0	1	2	3
$y = f(x) = x^3 - 4x$	0	3	0	-3	0	15

between $x = -2$ and $x = -1$, between $x = -1$ and $x = 0$, and so on. What is the average value of all these slopes?

c. Extend the table to include the point $x = 4$ and repeat parts (a) and (b). Does the same result hold?

d. Extend the table farther to include $x = -3$. Show that the same result holds.

e. Does the same result hold for any function and any set of points? State this result as a potential theorem.

26. Prove the result you conjectured in part (e) of Problem 25. Let f be defined on an interval from a to b. The average rate of change of f is

$$\frac{f(b) - f(a)}{b - a}.$$

Let

$$x_0 = a, x_1, x_2, \ldots, x_n = b$$

be any set of uniformly spaced points so that

$$\Delta x = \frac{b - a}{n}.$$

27. Find all polynomials p of degree ≤ 2 that satisfy each set of conditions.

a. $p(0) = p(1) = p(2) = 1$
b. $p(0) = p(1) = 1$ and $p(2) = 2$
c. $p(0) = p(1) = 1$
d. $p(0) = p(1)$

(*Hint*: Think about the graphs.)

Exercising Your Algebra Skills

Factor each of the following polynomial expressions as completely as possible. (Note that not all are factorable.)

1. $x^2 + 7x + 12$
2. $x^2 - 4x - 5$
3. $x^2 - 7x + 12$
4. $x^2 - x - 12$
5. $x^2 + x - 12$
6. $x^2 - 4x + 4$
7. $x^2 - 6x + 9$
8. $x^2 - 25$
9. $x^2 - 100$
10. $x^2 + 36$
11. $x^3 + x^2 - 20x$
12. $x^3 - 4x^2 + 3x$
13. $x^3 + 10x^2 + 25x$
14. $x^3 - 36x$

4.3 Modeling with Polynomial Functions

As we mentioned in Example 4 of Section 3.2, one of the most famous moments in the history of science was Galileo's reported experiment of dropping various objects from the top of the 180-foot high Leaning Tower of Pisa and discovering that they fell at the same rate, regardless of their weight. Instead of looking at the speed of a falling object, we now look at the height H, in feet, of an object falling from the top of the tower at various times t, as given in the table.

Time	0	0.5	1.0	1.5	2.0	2.5	3.0
Height	180	176	164	144	116	80	36

Note how the object starts falling slowly and then accelerates. (Incidentally, these values are considerably more accurate than anything Galileo could have measured at the end of the fourteenth century.)

The ideas we introduced in Chapter 3 on fitting linear, exponential, power, and logarithmic functions to a set of data can be extended to fitting polynomial functions to data. All graphing calculators have the capability to fit quadratic, cubic, and quartic polynomials to any set of data; spreadsheets such as Excel™ can fit polynomials up to degree 6, and specialized software packages allow polynomials of any finite degree. However, the approach used to determine a best-fit polynomial is different from the types of transformations we used in Sections 3.4 and 3.5. In fact, it is based on the idea of fitting a linear function of several variables to a set of data, as we discussed in Section 3.7. As we also discussed there, the correlation coefficient does not apply directly. Instead, statisticians have developed a comparable measure of the goodness of fit, known as the *coefficient of determination,* which is denoted by R^2. Its value is provided by most calculators and software. It always lies between 0 and 1, and the closer R^2 is to 1, the better is the fit; a value of 1 indicates a perfect fit.

EXAMPLE 1

(**a**) Find an equation for the height of an object falling from the top of the 180-foot high Leaning Tower of Pisa as a function of time. (**b**) Then use the formula to calculate how long it takes for the object to hit the ground.

Solution

a. We show the scatterplot of the data for height H as a function of time t in Figure 4.27 and observe that the pattern in the data resembles the right half of a parabola with

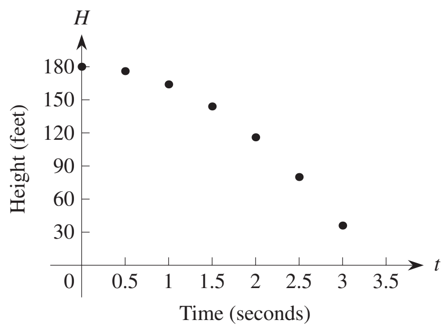


FIGURE 4.27

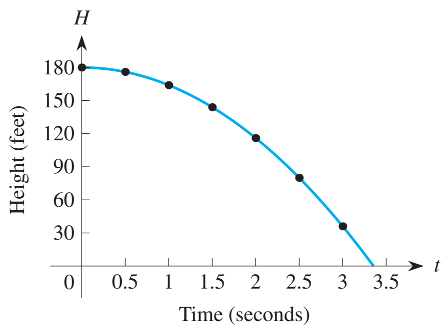


FIGURE 4.28

negative leading coefficient. Using the quadratic function regression routine on a calculator, we obtain the equation

$$H = -16t^2 + 180.$$

The corresponding value for the coefficient of determination is $R^2 = 1.00$, suggesting that the parabola apparently is a perfect fit to the data, as shown in Figure 4.28.

b. The object hits the ground when $H = 0$. To find how long it takes, we must find the value of t for which

$$H = -16t^2 + 180 = 0.$$

We can solve this quadratic equation graphically, with the quadratic formula, or by direct algebraic means. Algebraically, we add $16t^2$ to both sides of this equation to obtain

$$16t^2 = 180$$

so that

$$t^2 = \frac{180}{16} = 11.25.$$

When we take the square root of both sides, we get $t \approx \pm 3.35$. Because $t = -3.35$ seconds makes no real-world sense, we conclude that it takes about 3.35 seconds for the object to hit the ground.

Let's look at the equation $H = -16t^2 + 180$ for the height at any time when the object is falling from the top of the 180-foot high tower. Note that the constant term 180 equals the height of the tower. We rewrite the function as

$$H(t) = 180 - 16t^2,$$

which indicates that the height starts at 180 feet, when $t = 0$, and decreases thereafter. In general, if an object is dropped from any initial height H_0 and is affected only by the force of gravity, its height at any time t is given by

$$H(t) = H_0 - 16t^2.$$

Now suppose that an object is not simply dropped but instead is tossed upward with some initial velocity—say, 40 ft/sec. What do we expect? Obviously, the object starts off rising until it reaches a maximum height and then falls back until it hits the ground. The larger the initial velocity, the higher the object goes. In Example 2, we construct a function to model such a situation.

EXAMPLE 2

When an object is thrown vertically upward with an initial velocity of 40 ft/sec from the top of the 180-foot high Tower of Pisa, the following set of measurements of its height as a function of time are obtained.

t	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
H	180	196	204	204	196	180	156	124	84	36

a. Find an equation of a function that can be used to model the height of the object as a function of time.

b. Estimate how long it takes for the object to reach its maximum height and what that maximum height is.

c. How long does it take for the object to fall back to the ground?

Solution

a. The scatterplot of the data shown in Figure 4.29 indicates that the pattern for the height H as a function of time t looks like a portion of a parabola with a negative leading coefficient. Using a calculator to fit a quadratic function, we find that the quadratic function that best fits the data is

$$H(t) = 180 + 40t - 16t^2.$$

Note that the coefficients of the constant and linear terms are essentially the same as the initial height 180 feet and the initial velocity 40 feet per second, respectively. Moreover, the coefficient of the quadratic term is the same, -16, as in Example 1. This function superimposed over the scatterplot shown in Figure 4.30 reveals that it is an excellent fit to the data. The associated coefficient of determination is $R^2 = 1.0$, providing additional evidence that the fit is virtually perfect.

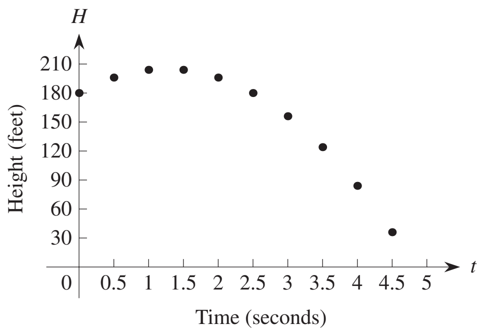


FIGURE 4.29

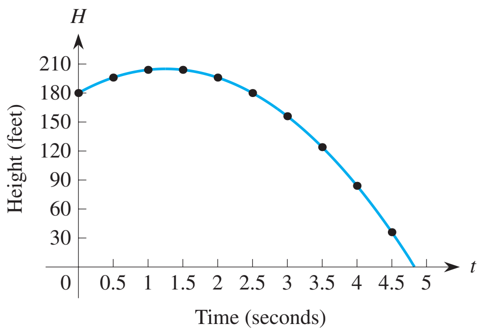


FIGURE 4.30

b. To estimate the time it takes for the object to reach its maximum height and the value for that maximum height, we need merely trace along the curve to find the coordinates of the turning point; or we can use the routine for locating the maximum for a function that is on many calculators. Either way, the coordinates are $t \approx 1.25$ seconds and $H \approx 205$ feet.

c. To find the time it takes for the object to return to the ground, we solve the equation

$$H(t) = 180 + 40t - 16t^2 = 0.$$

We can do this either graphically or by the quadratic formula. Using the quadratic formula with $a = -16$, $b = 40$, and $c = 180$ gives

$$t = \frac{-40 \pm \sqrt{40^2 - 4(-16)(180)}}{2(-16)}$$
$$= \frac{-40 \pm \sqrt{1600 + 11520}}{-32}$$
$$= \frac{-40 \pm \sqrt{13120}}{-32}.$$

Consequently, we get two possible values for t: $t \approx 4.83$ seconds and $t \approx -2.33$ seconds. The second value makes no sense physically, so the realistic solution is $t \approx 4.83$ seconds.

In general, we can say the following.

> The height of an object thrown vertically upward with initial velocity v_0 from an initial height H_0 at any time t is
>
> $$H(t) = -16t^2 + v_0 t + H_0.$$

If there is no initial velocity, so that $v_0 = 0$, this formula reduces to the expression we had previously for the height of any object falling under the influence of gravity.

The questions that we would want to answer about any object thrown upward into the air are:

1. How high does it go?
2. How long does it take to reach its maximum height?
3. How long does it take to return to the ground?

The Path of a Projectile

Picture the path of a long home run in baseball or the path of a perfect pass in football or the arch of the high-pressure stream of water from a supershooter water gun. In each case, the path looks something like the curve shown in Figure 4.31, whose shape suggests a parabola or possibly some higher degree polynomial curve with a negative leading coefficient. (If a strong wind is blowing, the path may not be quite so symmetric and the analysis of the shape of the path is considerably more complicated than that described here.)

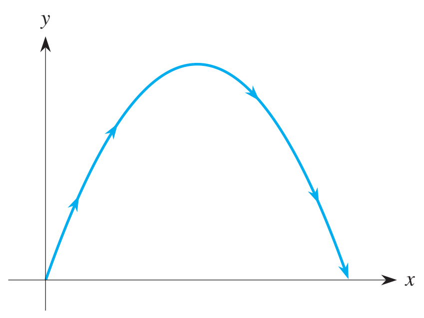


FIGURE 4.31

Using various kinds of technology, such as time-lapse photography or a video camera, we can capture a set of data on the path of such a projectile. For instance, the following set of data consists of measurements for the path of a long fly ball in baseball, where the height y of the ball depends on the distance x from home plate. Both sets of measurements are in feet.

x	0	30	60	90	120	150	180	210	240	270	300	330	360	390
y	4	37	65	88	105	117	123	124	119	109	94	73	47	16

The ball rises to a maximum height of about 124 feet. More important, the ball travels a horizontal distance of about 400 feet until it comes back down into the outfielder's glove, hits the ground or fence, or lands in the stands. To determine what happens, we need an equation for the path of the ball, which we find in Example 3.

EXAMPLE 3

(**a**) Determine the equation of a function that models the path of the baseball based on the preceding data. (**b**) If the fence 400 feet from home plate is 8 feet high, will the ball clear the fence to be a home run?

Solution

a. Because the shape of the data, as shown in the scatterplot in Figure 4.32, suggests a parabola, we begin by fitting a quadratic function to the data. The result is the quadratic function

$$y = -0.003x^2 + 1.202x + 3.936,$$

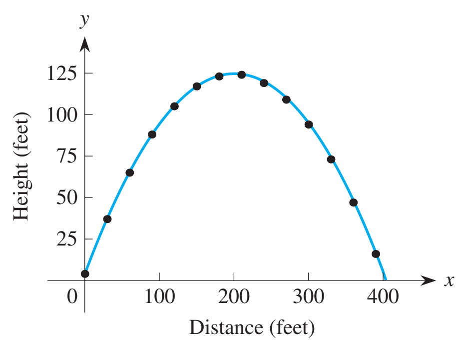


FIGURE 4.32

which is shown superimposed over the scatterplot and is an outstanding fit to the data. As expected, the leading coefficient is negative. Moreover, the corresponding value for the coefficient of determination is $R^2 = 0.9999$, which provides additional evidence that the quadratic function is an excellent model to use.

b. In order for the ball to be a home run, it must clear the 8-foot high fence when it is 400 feet from home plate. Therefore we substitute $x = 400$ into the equation of the parabola and find that

$$y = -0.003(400)^2 + 1.202(400) + 3.936 = 4.736.$$

That is, when it reaches the fence, the ball's height is somewhat less than 5 feet, so it wouldn't be a home run, as shown in the smaller view in Figure 4.33.

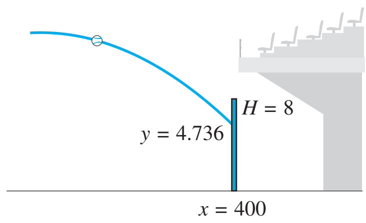


FIGURE 4.33

Fitting Polynomials to Data

The concept of fitting a polynomial function to data is one that applies in all walks of life, not just in the physical situations we encountered in Examples 1–3. We illustrate two other cases in Examples 4 and 5.

EXAMPLE 4

The table shows the accumulated total number of reported cases of AIDS in the United States since 1983.

Year	1983	1984	1985	1986	1987	1988	1989	1990
Number of AIDS Cases	4589	10,750	22,399	41,256	69,592	104,644	146,574	193,878
Year	1991	1992	1993	1994	1995	1996	1997	1998
Number of AIDS Cases	251,638	326,648	399,613	457,280	528,144	594,641	653,084	701,353

Source: U.S. Centers for Disease Control and Prevention.

Determine a function that fits the data well and interpret the behavior of the function.

Solution In Example 4 of Section 3.3, we explored the possibility that the growth in the total number of reported cases of AIDS in the United States follows an exponential pattern. The resulting best-fit exponential function, found with a calculator, was

$$A = 5413.5(1.3626)^t,$$

where t is measured in years since 1980. The corresponding correlation coefficient $r = 0.9483$ is quite close to 1, suggesting that this function is a very good fit. But, when we superimpose this exponential function over the data points, as shown in Figure 4.34, the curve doesn't fit the data well.

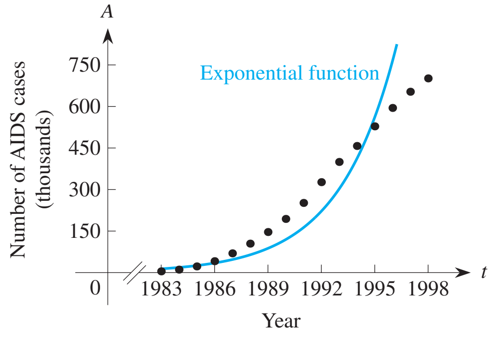


FIGURE 4.34

Alternatively, suppose that we use the capability of the calculator to fit a polynomial to this data. Most calculators allow us to fit polynomials of degree 2, 3, or 4 to a set of data, and we can easily experiment with different degrees. When we do so, we find that a cubic polynomial is an excellent fit to this set of data. The calculator gives the best cubic function, rounded to one decimal place, as

$$A = -221.9t^3 + 9261.8t^2 - 62275.9t + 122988.9,$$

where t is again the number of years since 1980. When we superimpose this polynomial over the AIDS data points shown in Figure 4.35, we get an exceptionally good fit, which certainly is a far better fit than the exponential function shown in Figure 4.34.

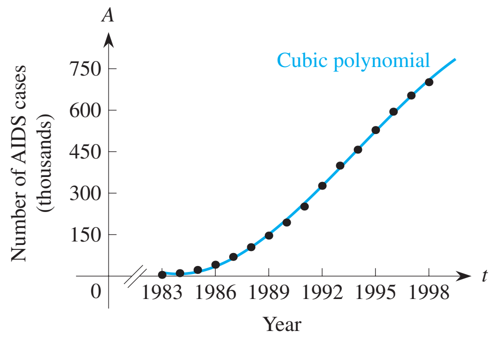


FIGURE 4.35

This graph strongly suggests that the number of cases in the spread of AIDS follows a cubic pattern. (When scientists discovered this several years ago, they were excited because polynomial growth is much slower than exponential growth, which is the trend that they too had expected.) The corresponding coefficient of determination, $R^2 = 0.99996$, provides further evidence of how well the cubic function fits the data.

We know from the formula for the cubic that the leading coefficient is negative, so the cubic will eventually approach $-\infty$. The larger view in Figure 4.36 suggests that the cubic passed its inflection point in about 1995 or 1996 and that the growth in AIDS has begun to slow somewhat since then. The graph also shows that the function will reach a turning point in about 2003.

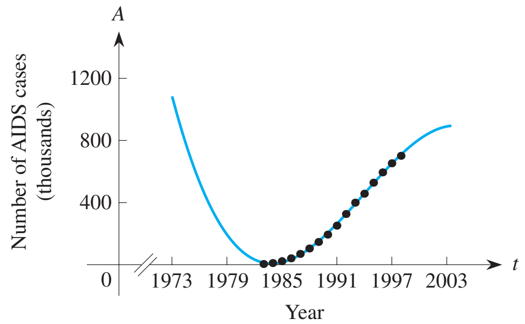


FIGURE 4.36

However, recall that the data represent the total number of AIDS cases reported in the United States, so the cubic can't actually turn and begin to decline; it can only slow and, at best, eventually level off. Thus we demonstrate again how dangerous extrapolation with a mathematical model can be. The model only describes the situation based on the data points; it is not a guarantee of the actual process, especially for extrapolating into the future or the past.

Let's look at another example of fitting polynomials to data. Figure 4.37 shows a picture of the famous Gateway Arch in St. Louis. Its shape suggests a portion of a downward opening parabola. Let's see if we can determine a specific function that best models the arch.

FIGURE 4.37

EXAMPLE 5

Determine a polynomial function that fits the Gateway Arch well.

Solution To find an appropriate function, we need some measurements for the arch. Overall, the arch stands 630 feet tall, and the distance between its two legs also is 630 feet. We superimpose a grid on the arch, as shown in Figure 4.38, and choose the coordinate system so that the vertical axis passes through the center of the arch. We then construct the following table of estimates of the height H corresponding to various horizontal distances x. We make our estimates from the middle of the arch; slightly different results might occur if we use values from the inner edge or the outer edge. We ask you to investigate these possibilities in the problems at the end of this section.

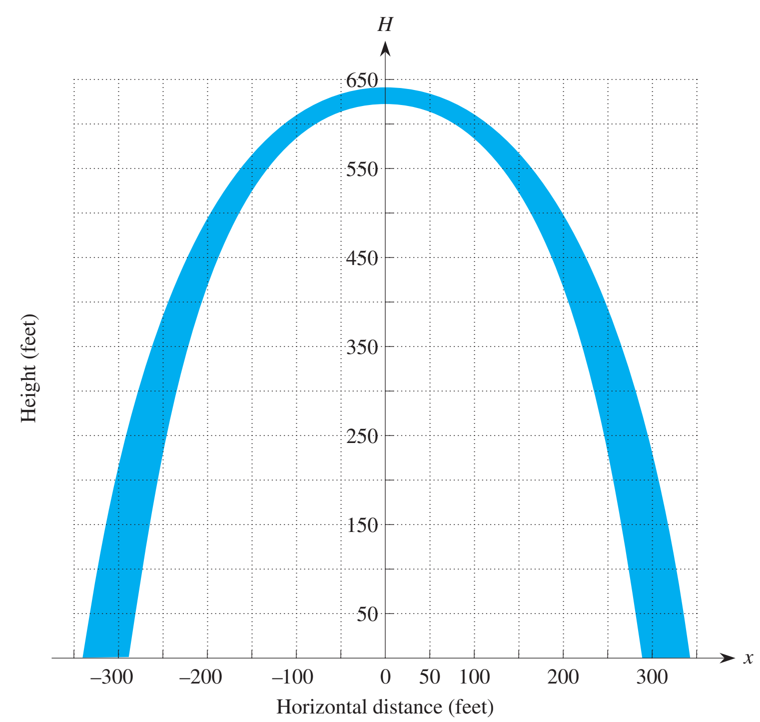


FIGURE 4.38

x	−325	−300	−250	−200	−150	−100	0	100	150	200	250	300	325
H	0	100	330	500	570	610	630	610	570	500	330	100	0

The first thing we notice from both the figure and the table is that the measurements are symmetric about the vertical axis $x = 0$. As a result, we would expect that the best-fit parabola has no x term. When we enter the data into the quadratic regression routine of a calculator, we find the quadratic function that best fits the data is

$$H = -0.0064x^2 + 0x + 699.01.$$

We plot this function over the data points, as illustrated in Figure 4.39, and conclude that it is a reasonably good fit, though certainly not a great one. Among other things, the curve rises much too high above the central data point and the pattern of data points flattens out far more than the parabola does near the center.

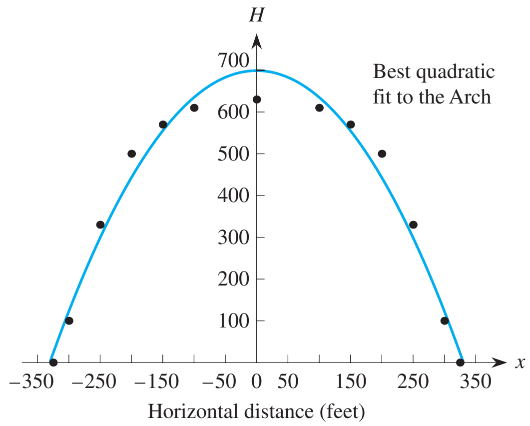


FIGURE 4.39

In our discussion of power functions with integer powers in Section 2.7, we pointed out that the higher the power, the flatter the curve as it passes through the origin. This result suggests that we should use a higher degree polynomial than a quadratic. From the basic shape of the arch, we know that a cubic would not be appropriate—it doesn't have the correct behavior. How about a quartic polynomial? When we try it, the calculator responds with the equation

$$H = (-3.27 \times 10^{-8})x^4 + 0x^3 - 0.00282x^2 + 0x + 644.25.$$

When we superimpose this function over the data points, as shown in Figure 4.40, it appears visually to be an exceptionally good fit to the shape of the arch. The coefficient of determination for this fit is $R^2 = 0.9953$, which also indicates that it is a very good fit. (Actually, the true shape of the arch is a curve known as a hyperbolic cosine, which you may encounter in calculus.)

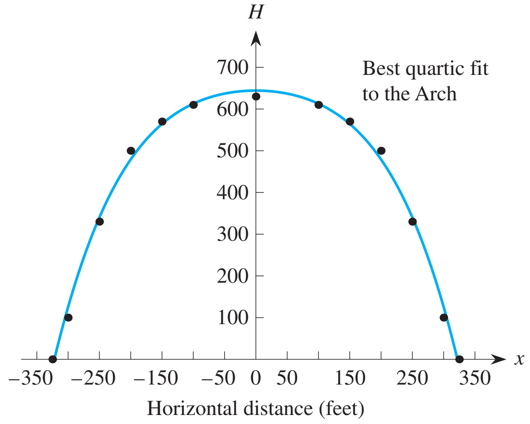


FIGURE 4.40

Problems

1. We showed in the text that the cubic function

$$y = -221.9t^3 + 9261.8t^2 - 62275.9t + 122988.9$$

is an excellent fit to the total number of reported cases of AIDS in the United States from 1983 to 1998, where t is the number of years since 1980.

   a. Based on this model, what is the prediction for the total number of cases through 2000?
   b. Check a recent copy of the *Statistical Abstract of the United States* or an almanac to see how accurate the prediction in part (a) is.
   c. If this cubic pattern continues, how many total cases would you expect by 2004?
   d. When would you expect a total of 850,000 cases of AIDS, based on this model?

2. Find the equations of the best quadratic and quartic functions to fit measurements taken at the outer edge of the Gateway Arch instead of at the middle.

3. Repeat Problem 2 with measurements taken at the inner edge of the arch instead of at the middle.

4. The table shows the percentage of the U.S. population that is foreign born in various years.

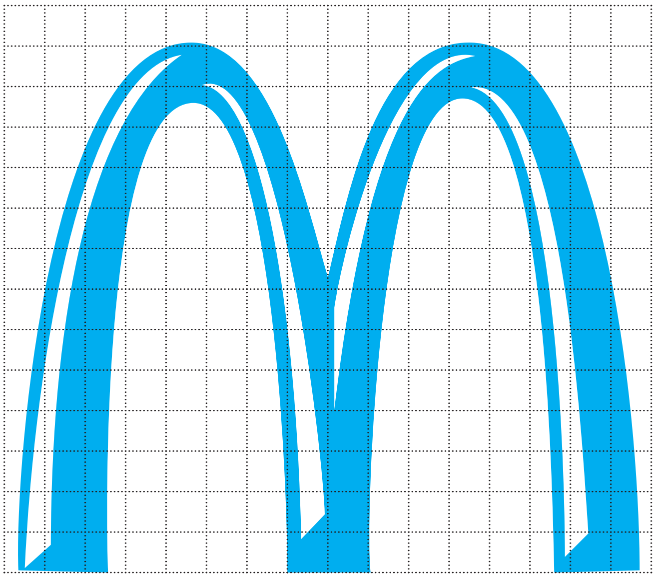
   a. What is the minimum degree polynomial that you would use to model this data?
   b. Find that polynomial and use it to estimate the time when the percentage of foreign-born people in the United States was a minimum. What was that minimum percentage?

Year	1950	1960	1970	1980	1990	2000
Percentage	6.9	5.4	4.8	6.2	7.9	10.4

Source: *2000 Statistical Abstract of the United States.*

5. The accompanying figure shows a grid superimposed on the image of the McDonald's arches.

   a. Decide on a scale that you can use to estimate measurements on the arches. (*Hint*: Think about where you want to set up your coordinate axes.)
   b. Use your estimated measurements to determine the equation of a polynomial that best fits one of the arches. (*Hint*: Think again about where you want to set up your coordinate axes.)
   c. Can you use the formula you obtained for one of the arches to construct a formula for the other arch? Explain.



6. The table gives the horsepower generated on a Chevy 383 car engine at different rpm.

Horsepower	138	172	203	216	
Revolutions per Minute	2000	2500	3000	3500	
Horsepower	209	182	144	98	42
Revolutions per Minute	4000	4500	5000	5500	6000

Source: Student project.

   a. Which variable is the independent variable and which is the dependent variable?
   b. What is the equation of the quadratic function that relates these two quantities?
   c. What does your model predict for the horsepower generated by this engine at 4800 rpm?
   d. If the engine puts out 165 horsepower, what is the possible value for the rpm according to this model?

7. Car enthusiasts know that it's not horsepower that is significant, but rather the amount of torque that an engine puts out that really matters in how quickly a car moves forward. The table gives the torque, in foot-pounds, generated at different rpm values for a Chevy 383 engine. From among the usual families of functions (linear, exponential, power,

quadratic, and cubic), find the one that seems to be the best fit to these data.

Torque	363	361	355	324	
Revolutions per Minute	2000	2500	3000	3500	
Torque	275	213	151	93	36
Revolutions per Minute	4000	4500	5000	5500	6000

Source: Student project.

8. **a.** Create a single table based on the information given in Problems 6 and 7 relating the amount of torque generated to the horsepower for the Chevy 383 engine.

b. From among the usual families of functions (linear, exponential, power, quadratic, and cubic), find the one that seems to be the best fit to this data.

9. The table shows the number of 18- to 24-year-olds in the United States in recent years. Find the quadratic function that best fits this data set and use it to predict the number of people in this age range in (**a**) 2000 and (**b**) 2005. Which prediction would you have more confidence in?

Year	1970	1975	1980
Population (millions)	24.71	28.76	30.35
Year	1985	1990	1995
Population (millions)	29.48	26.14	24.85

Source: *2000 Statistical Abstract of the United States.*

10. According to the theory of relativity, the mass M of an object increases as its velocity v increases so that $M = f(v)$. Suppose that the mass of an object is 1 unit when it is at rest ($v = 0$). The table gives the mass of the object at different speeds that are expressed as fractions of c, the speed of light (about 186,280 miles per second). Find the best quadratic fit to this set of data.

Velocity (fraction of *c*)	0	0.1	0.2
Mass	1	1.0050	1.0206
Velocity (fraction of *c*)	0.3	0.4	0.5
Mass	1.0483	1.0911	1.1547

11. While approaching the Verrazano Bridge in New York City, Ken noticed that the main cable looks like a parabola, as illustrated in the accompanying figure. As his car crawled across the bridge in heavy traffic, he estimated the following heights, in feet, of the cable above the road and the distance, in feet, starting from one of the vertical support columns.

Distance from Support Column	0	1000	2150
Estimated Height	500	150	20
Distance from Support Column	3000	4000	4300
Estimated Height	100	400	500

Find an equation of the parabola that best fits Ken's estimates. (Think how to set up the coordinate axes.)

12. The table shows the price of a barrel of oil, in dollars, in different years.

Year	1960	1970	1975	1980
Price	11	9	37	64
Year	1985	1990	1995	2000
Price	40	28	20	32

Source: Lester R. Brown et al., *Vital Signs 2000: The Environmental Trends That Are Shaping Our Future.*

a. What type of function is reasonable to use as a model for the price of oil as a function of time?

b. Find the equation of the polynomial function of appropriate degree to fit the data.

c. What does your model predict for the price of a barrel of oil in 2005?

d. Use the graph of your function to estimate the location of the turning points for the function. According to this model, what was the maximum price of a barrel of oil between 1960 and 2000 and when did it occur? What was the minimum price and when did it occur?

13. The table shows the trend in worldwide grain production (wheat, rice, and corn, primarily), in kilograms per person. The pattern in the data suggests that a quadratic function is an appropriate model for grain production per person as a function of the year.

Year	1965	1970	1975	1980
Amount per Person	270	291	303	321
Year	1985	1990	1995	1999
Amount per Person	339	335	301	309

Source: Lester R. Brown et al., *Vital Signs 2000: The Environmental Trends That Are Shaping Our Future.*

a. Find the equation of the quadratic that best fits these data.
b. Based on the model, what was the maximum level of grain production per person worldwide?
c. What does the model predict for the amount of grain produced per person in 2010?
d. Write a paragraph describing the possible reasons for this trend and the implications if the trend continues.

14. The table gives the total number, in thousands, of high school graduates in the indicated years since 1900. In Problem 12 of Section 3.3, we asked you to find the best linear, exponential, and power functions to fit these data. If you examine the data carefully, you should expect that a polynomial function would be a better fit.

Year	1900	1910	1920	
High School Grads	95	156	311	
Year	1930	1940	1950	1960
High School Grads	667	1221	1200	1858
Year	1970	1980	1990	2000
High School Grads	2889	3043	2586	2839

Source: *Digest of Education Statistics 2000,* U.S. Department of Education.

a. What degree polynomial function is a good candidate to fit these values? Explain.
b. Let t be the number of years since 1890. Determine the best polynomial function of the degree that you decided was appropriate in part (a) to model the number of high school graduates as a function of time t.
c. Use this function to predict the number of high school graduates in 2010.
d. Use this function to predict the year in which there will be 5 million high school graduates.

15. The table, collected from a chemistry lab experiment, gives the density D of water, in grams per milliliter, at various temperatures T, in °C.

Temperature, *T*	0°	4°	10°
Density, *D*	0.99987	1.00000	0.99973
Temperature, *T*	20°	30°	40°
Density, *D*	0.99823	0.99567	0.99224
Temperature, *T*	60°	80°	100°
Density, *D*	0.98324	0.97183	0.95838

Source: John R. Holum, *Elements of General and Biological Chemistry,* 8th ed. New York: John Wiley & Sons, 1991.

a. Find a quadratic function that fits these data.
b. Use your function from part (a) to find the density of water at 70°C.
c. Find the temperature at which the density of water is 0.99100 grams per milliliter.

16. The height of an object falling from an initial height of y_0 is given by the formula

$$y = y_0 - 16t^2,$$

with units of feet and seconds. What is the equivalent formula based on the metric system of units with meters and seconds? (*Hint*: 1 foot = 0.3048 meters.)

17. Galileo conducted his famous experiment in which he dropped objects from the top of the 180-foot high Leaning Tower of Pisa in about 1590. His goal was to obtain experimental data to show that all bodies fall with equal velocities. How long did it take for the objects that he dropped from the tower to hit the ground?

18. The Eiffel Tower is 300 meters tall. How long would it take an object dropped from its top to hit the ground?

4.4 The Roots of Polynomial Equations: Real or Complex?

The Roots of Quadratics

In Section 4.1, we stated that, for any quadratic equation,

$$ax^2 + bx + c = 0, \qquad a \neq 0,$$

we can always find its roots by using the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Further, the roots may be two distinct real numbers, a repeated real root, or a pair of complex conjugate numbers of the form $\alpha \pm \beta i$, where $i = \sqrt{-1}$, as illustrated in Figure 4.41.

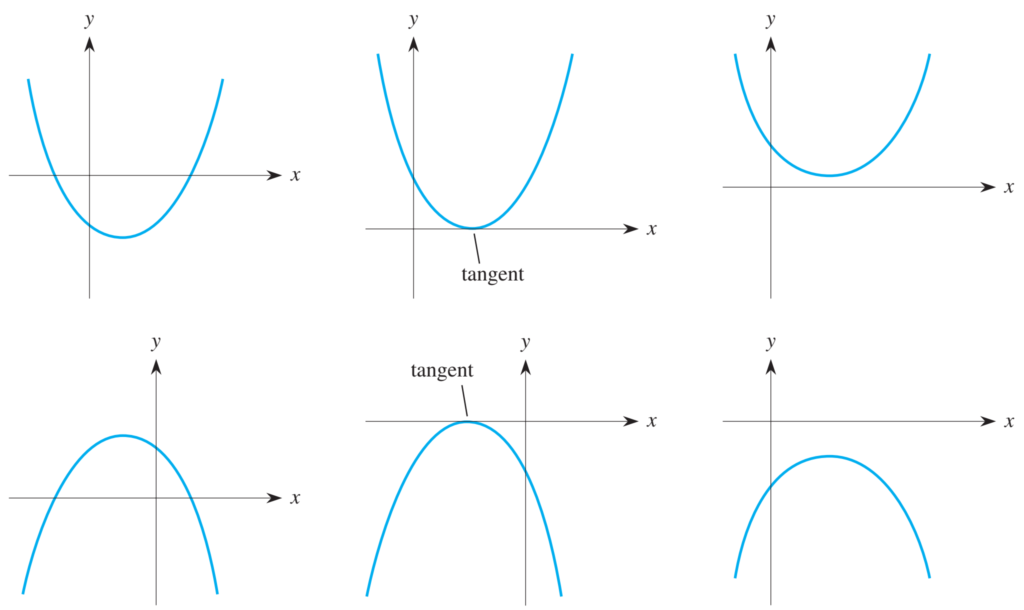


FIGURE 4.41

Most students think that complex roots occur very rarely. In this section we investigate how frequently they do arise. To do so, we consider many different quadratic equations and find the percentage of them that do have complex roots. A quadratic equation $ax^2 + bx + c = 0$ has complex roots when its discriminant, $b^2 - 4ac$, is negative. The quadratic formula then requires taking the square root of that negative discriminant to produce two complex numbers. For instance, for the quadratic equation $x^2 - 2x + 2 = 0$, the discriminant is $(-2)^2 - 4(1)(2) = -4$, so the roots will be complex. The quadratic formula gives the roots as

$$x = \frac{-(-2) \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i,$$

or $x = 1 + i$ and $x = 1 - i$. Thus we can use the sign of the discriminant as the criterion to decide whether any particular quadratic has complex roots.

To come to any meaningful conclusions about the percentage of quadratics that have complex roots, we must examine a very large number of quadratics. Doing so requires using a computer or calculator program rather than hand computation.

Even the simplest case—when the quadratic has integer coefficients—has infinitely many possible quadratics, so the best we can do is examine a finite selection of them. Let's examine all possible quadratics $y = ax^2 + bx + c$ where the coefficients a, b, and c are integers from 0 to 5, say, but $a \neq 0$. We write this in *interval notation* as $[0, 5]$. We then use a computer program that considers all possible integer values for a, b, and c in this interval and keeps track of how many of the quadratics have complex roots, using the discriminant criterion. Similarly, we can investigate all possible integer coefficients in various other intervals, the results of which are shown in Table 4.1.

TABLE 4.1

Interval for a, b, and c, $a \neq 0$	Percentage with Complex Roots
All in $[0, 5]$	70
All in $[0, 10]$	73
All in $[0, 20]$	74
All in $[0, 50]$	74.5
All in $[-3, 3]$	37.4
All in $[-5, 5]$	37.5
All in $[-10, 10]$	37.8
All in $[-20, 20]$	37.7
All in $[-50, 50]$	37.5
$[0, 5]$, $[0, 5]$, $[-5, 0]$	0
$[0, 5]$, $[-5, 0]$, $[0, 5]$	70

Therefore, rather than being a rarity, complex roots actually occur with surprising frequency. In fact, almost three-fourths of quadratics whose coefficients are all nonnegative integers have complex roots. Even allowing for negative values almost 40% have complex roots.

Think About This

There is one exception in Table 4.1. If the constant coefficient c is negative while a and b are both positive, the quadratic apparently always has two real roots. Can you explain why? Can you give another example where the quadratic always has two real roots? Look at the discriminant. (Note that we have checked only specific integer values for a and b between 0 and 5 and c between -5 and 0, so we can't generalize to what may happen over all similar intervals of values.)

We suggest that you conduct your own investigations of these ideas if an appropriate program is available or if you want to write a fairly short program for your calculator. Think about the following questions.

- With integer coefficients, what happens as the size of the interval increases? Does the frequency of complex roots stay roughly the same or does it increase or decrease significantly?
- What happens if you use different ranges of values for each coefficient?

Don't be too generous in your choices when you begin; such systematic processes tend to take a long time. For example, if you want to check all quadratics where a, b, and c are integers between 0 and 10, say, you are actually having the computer or calculator investigate 1210 different equations. (There are 10 possible values for a since the equation would not be quadratic if a were zero. There are 11 possible values for b and 11 for c, which leads to $10 \times 11 \times 11 = 1210$ different cases.) If you ask for all integers from 0 to 100 on each of the coefficients, the computer or calculator will investigate 100×101^2 different quadratics. It may take all night to complete this study of more than one million cases.

We should also find out what happens when the quadratic has noninteger coefficients, either rational numbers or irrational numbers. In such cases, we can't simply check all possible equations because there are infinitely many possibilities, even for any finite interval. Instead, we use a random selection process to generate large numbers of quadratics with randomly selected (noninteger) coefficients in desired intervals, test each for the nature of its roots, and keep track of how many of the roots are complex. (We perform just such an analysis in Supplementary Section 11.3 as part of our study of probability.)

The Roots of a Cubic Function

We next consider an arbitrary cubic equation

$$ax^3 + bx^2 + cx + d = 0,$$

where a, b, c, and d are any four real numbers and $a \neq 0$. Recall that, just as a quadratic equation has two roots, a cubic equation has three. They can be either real or complex roots. Recall also that any complex roots must occur as a pair of complex conjugates, $\alpha + \beta i$ and $\alpha - \beta i$. Thus, for any cubic equation, the three roots may be either three real numbers, or a single real number and a pair of complex conjugate numbers.

Moreover, we know that the real roots correspond geometrically to points where the cubic crosses the x-axis. If there are three distinct real roots, the cubic crosses the x-axis in three places, as illustrated in Figures 4.42(a) and 4.42(b). If there is a double real root and a separate real root, the x-axis is tangent to the cubic at the point corresponding to the double root and the curve crosses the x-axis at

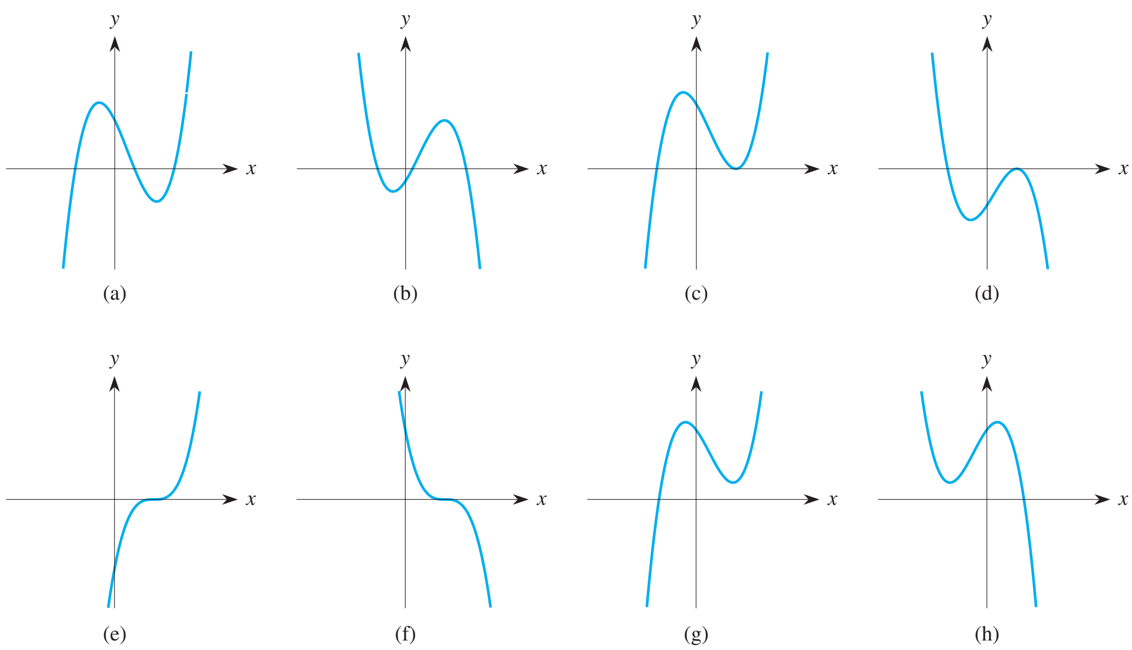


FIGURE 4.42

the point corresponding to the other real root, as depicted in Figures 4.42(c) and 4.42(d). If there is a triple real root (as with $y = x^3$), the cubic flattens as it crosses the x-axis at the single point, as shown in Figures 4.42(e) and 4.42(f). Finally, if there is a single real root and a pair of complex conjugate roots, the cubic crosses the x-axis once, as illustrated in Figures 4.42(g) and 4.42(h). Thus a cubic can have either three real roots or one real root.

We have demonstrated that quadratic equations are likely to have complex roots. How likely is it for a cubic equation to have complex roots? To answer this question, we again use a computer program to investigate many different cubics. First, though, we must devise a test comparable to using the sign of the discriminant in the quadratic formula to decide whether a particular cubic has complex roots.

Suppose that a cubic has three real roots. In that case, the curve crosses the x-axis at three points if the three roots are distinct, it crosses the axis at two points if there is a double real root, and it crosses the axis at one point if there is a triple real root. The cubics shown in Figure 4.43 all have the same shape; the only difference is the height of the turning points. The cubic on the left has its first turning point above the x-axis and its second below; therefore it has three real roots. The second cubic has both turning points above the x-axis and so must have one real root and a pair of complex roots. The third cubic has both turning points below the x-axis, so it also must have one real root and a pair of complex roots.

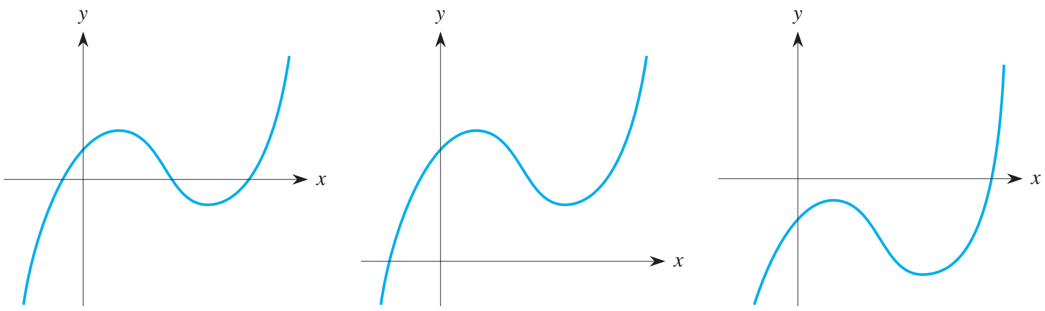


FIGURE 4.43

A further case occurs when the x-axis is tangent to the curve at one of the turning points; such a cubic has a double real root, so it cannot have a pair of complex roots and its third root must be real. The final case is when the two turning points coincide along the x-axis; this case corresponds to a triple real root. Therefore, in order to have two complex roots, a cubic must have both turning points above the x-axis or both below it.

When you study calculus, you will be able to determine that the two turning points of the cubic $y = ax^3 + bx^2 + cx + d = 0$ are located at

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a},$$

provided that $b^2 - 3ac \geq 0$. (This formula clearly resembles the quadratic formula.)

Think About This

Verify graphically that this formula gives the approximate location of the turning points of the cubic $y = x^3 - 4x^2 + 4x + 5$.

Call these two x-values x_1 and x_2. Because we know the equation of the cubic curve,

$$y = f(x) = ax^3 + bx^2 + cx + d,$$

we can determine the heights of the two turning points:

$$y_1 = f(x_1) \quad \text{and} \quad y_2 = f(x_2).$$

Once we have calculated these values, we need only check whether both are positive or both are negative to conclude that the cubic has complex roots, as illustrated in Figure 4.44. If the two y-values have opposite signs or if either is zero, the cubic has three real roots. We use this criterion in our investigation.

We apply this criterion to cubics with integer coefficients a, b, c, and d within various intervals of values. In Supplementary Section 11.3 we investigate cases with randomly generated noninteger values for a, b, c, and d within any desired intervals of values, provided that $a \neq 0$.

In Table 4.2 we list the results of performing this investigation with all possible *integer* coefficients in the indicated intervals of values. This table indicates that a cubic with integer coefficients seems even more likely to have complex roots than a quadratic does.

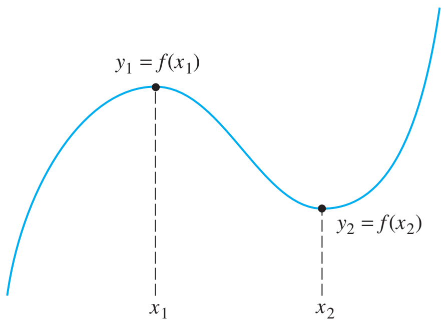


FIGURE 4.44

TABLE 4.2

Intervals for a, b, c, and d, $a \neq 0$	Percentage of Complex Roots
All in [0, 5]	94.54
All in [−3, 3]	78.43
All in [−4, 4]	78.74
All in [−5, 5]	78.93
[0, 4], [0, 4], [0, 4], [−4, 0]	88.4
[0, 4], [0, 4], [−4, 0], [0, 4]	74.8
[0, 4], [−4, 0], [0, 4], [0, 4]	88.4
[0, 4], [0, 4], [−4, 0], [−4, 0],	44
[0, 4], [−4, 0], [−4, 0], [0, 4]	44
[0, 4], [−4, 0], [−4, 0], [−4, 0]	74.8

Think About This

In intervals of the form $[-k, k]$ for all four coefficients, the proportion of complex roots seems to be essentially the same regardless of the value of k. Does that make sense? Imagine what would happen if you have a particular cubic and multiply each coefficient by 10, say. Wouldn't you expect the same type of roots? In fact, wouldn't you expect the identical roots?

Think About This

When we studied the nature of the roots of quadratics, we saw that the two roots are always real whenever $c < 0$ and $a > 0$. Are there any simple combinations of values for the coefficients a, b, c, and d in a cubic that likewise guarantee real roots? (What about $d = 0$, $c < 0$, and $a > 0$?)

It turns out that for polynomials of higher degree, the likelihood of complex roots is even greater than for quadratics or cubics, but we won't investigate these cases.

Using Information on the Nature of the Roots

We next turn to an application for which knowing the nature of the roots of a polynomial is crucial. Home thermostats and automobile cruise controls are examples of *control systems* that engineers use to control a process. In such devices, when the system deviates slightly from the specified level, it should return to that level automatically—the temperature shuts off or the car stops accelerating. Such a system is called *stable.* Often, control systems are described mathematically by a polynomial. A control system is stable if

1. all the real roots are negative, and
2. all the complex roots have negative real parts.

A control system having any positive real roots or having complex roots whose real parts are positive is *unstable.* That is, the system does not return to the specified level when small changes are introduced.

EXAMPLE

A control system is described by the cubic polynomial $P(s) = s^3 + 3s^2 + 4s + 2$. Determine whether the system is stable or unstable.

Solution The graph of this cubic polynomial is shown in Figure 4.45. Its associated cubic equation $s^3 + 3s^2 + 4s + 2 = 0$ has only one real root, so it must therefore have a pair of complex conjugate roots. Moreover, it is evident that the real root is negative. If we zoom in on the point where the curve crosses the s-axis, we find that the root appears to be located near $s = -1$. We can determine whether the root is $s = -1$ exactly by evaluating

$$\begin{aligned} P(-1) &= (-1)^3 + 3(-1)^2 + 4(-1) + 2 \\ &= -1 + 3 - 4 + 2 = 0, \end{aligned}$$

which shows that the root is precisely $s = -1$.

The problem we now face is to determine the complex roots. We know the real root $s = -1$, so the corresponding linear factor is $(s + 1)$. We can therefore factor the polynomial by dividing it by $(s + 1)$, using the technique of long division for polynomials from algebra:

$$\begin{array}{r} s^2 + 2s + 2 \\ (s+1)\overline{\big)\, s^3 + 3s^2 + 4s + 2} \\ \underline{s^3 + s^2} \qquad\quad \\ 2s^2 + 4s \qquad \\ \underline{2s^2 + 2s} \qquad \\ 2s + 2 \\ \underline{2s + 2} \\ 0 \end{array}$$

Thus, $(s^2 + 2s + 2)$ is the quadratic factor, so that the original cubic polynomial is

$$P(s) = s^3 + 3s^2 + 4s + 2 = (s + 1)(s^2 + 2s + 2).$$

We now apply the quadratic formula to find the complex roots of the quadratic factor:

$$s = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$$

$$= \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2}.$$

The two complex roots are therefore $s = -1 + i$ and $s = -1 - i$. Because the real parts of both complex roots are negative, and the real root is negative also, the control system is stable.

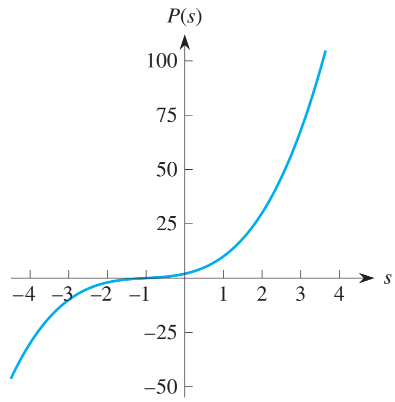


FIGURE 4.45

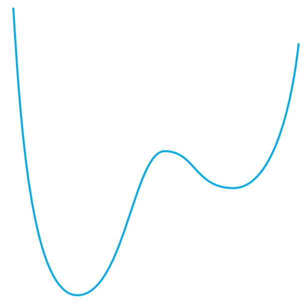
Problems

1. Consider all quadratics with $a = 1$ and the coefficient of the linear term $b = 0$ so that they take the form $y = x^2 + c$. What percentage of these quadratics should have two real roots?
2. Consider all quadratics of the form $y = ax^2 + bx$ with $c = 0$. What percentage of them should have real roots?
3. Show that, if each coefficient in the quadratic $y = ax^2 + bx + c$ is multiplied by 10, the resulting discriminant is multiplied by 100. What would you expect to happen to the discriminant if each coefficient were multiplied by the same number k? How do the roots of the two quadratics compare?
4. Consider all fourth degree polynomials of the form

$$y = ax^4 + bx^3 + cx^2 + dx + e,$$

where, for simplicity, you may consider $a > 0$.

   a. Based on the general graph shown without axes, how likely do you think it is (roughly 10%, 25%, 50%, 75%, or 90%) for such a polynomial to have four real roots? four complex roots? Explain your answers.



   b. How would your answers change if $e = 0$?
5. For each of the following cubic equations, use your function grapher to produce the graph and zoom in to estimate where the two turning points are located. Then apply the formula

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

from the text (based on $y = ax^3 + bx^2 + cx + d$) to verify that the values given by the formula match the points you found graphically.

   a. $y = x^3 + 4x^2 - 8x + 3$
   b. $y = x^3 - 7x^2 - 2x + 6$
   c. $y = 5x^3 - 3x^2 - 6x + 8$
   d. $y = -4x^3 + 3x^2 + 5x - 4$
   e. $y = -4x^3 + 3x^2 - 5x - 4$
6. **a.** Determine the location of the turning points for the cubic $y = x^3 - 3x^2 + 2x + 10$. What are the maximum and minimum values for this function?

   b. Use the fact that a cubic is symmetric about its point of inflection to determine the location of the point of inflection of the cubic in part (a).
7. If a different control system is described mathematically by each polynomial, determine whether it is stable or unstable.

a. $P(s) = s^2 + 6s + 8$
b. $P(s) = s^2 + 5s - 12$
c. $P(s) = s^2 + 5s + 3$
d. $P(s) = s^3 - 4s^2 - 12s$
e. $P(s) = s^3 + 3s^2 + 7s + 5$

4.5 Finding Polynomial Patterns

In Section 2.2, we developed a criterion for determining whether a set of m points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ follows a linear pattern when the x-values are uniformly spaced.

> A set of points lies on a line if the differences between successive y-values are all equal when the x-values are uniformly spaced. The slope of that line
>
> $$m = \frac{\Delta y}{\Delta x}$$
>
> is the constant difference between successive y-values divided by the uniform spacing between successive x-values.

x	y	Δy
0	1	
		1
1	2	
		3
2	5	
		5
3	10	
		7
4	17	
		9
5	26	

We now consider the related problem of determining whether a set of points follows a quadratic, a cubic, or a higher degree polynomial pattern. Suppose that we have the points (0, 1), (1, 2), (2, 5), (3, 10), (4, 17), and (5, 26), which actually lie on the parabola $y = x^2 + 1$. We construct the table at the left of differences of the y-values. Obviously, the Δy values are not constant. In fact, they clearly follow a linear pattern because the differences between successive Δy values (the differences of the differences) are all constant. The differences of the differences, $\Delta(\Delta y)$, are called the *second differences* and are written $\Delta^2 y$. If we extend the previous table to include the second differences of the y-values, as shown in the table below, we get a constant value for all the second differences.

In general, we have the following criterion based on uniformly spaced x-values.

> A set of points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ lies on a quadratic $y = ax^2 + bx + c$ if the second differences of the y-values are all constant when the x-values are uniformly spaced.

x	y	Δy	$\Delta^2 y$
0	1		
		1	
1	2		$2 = 3 - 1$
		3	
2	5		2
		5	
3	10		2
		7	
4	17		2
		9	
5	26		

In the problems at the end of this section we ask you to explore the significance of this constant second difference.

EXAMPLE 1

Show that the points (0, 2), (1, 0), (2, 4), (3, 14), (4, 30), and (5, 52) lie on a parabola. Then find the equation of the parabola by using regressions methods.

Solution We construct a table of second differences.

x	y	Δy	$\Delta^2 y$
0	2		
		$-2 = 0 - 2$	
1	0		$6 = 4 - (-2)$
		4	
2	4		6
		10	
3	14		6
		16	
4	30		6
		22	
5	52		

Because the differences of the differences are constant, the points follow a quadratic pattern of the form

$$y = ax^2 + bx + c,$$

where the coefficients a, b, and c must be determined.

Thinking of the points as data values and using the curve fitting routines of a calculator, we find that the quadratic function that best fits the data is

$$y = 3x^2 - 5x + 2.$$

The corresponding coefficient of determination is $R^2 = 1$, which suggests a perfect fit to the data. Figure 4.46 shows that the graph of this parabola apparently passes through all six points. Test this result by substituting each value in the formula that we created.

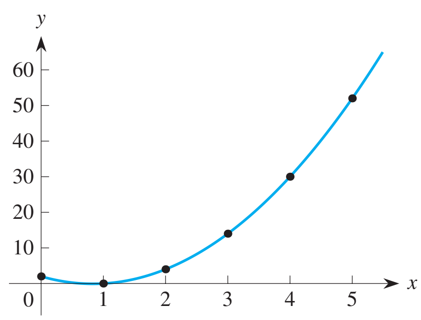


FIGURE 4.46

Alternatively, we could find the equation of the quadratic function that fits these points by using algebraic methods, as we demonstrate in Example 2.

EXAMPLE 2

Find the equation of the parabola that passes through the points (0, 2), (1, 0), (2, 4), (3, 14), (4, 30), and (5, 52), using algebraic methods.

Solution As in Example 1, we have to find the three coefficients a, b, and c in the equation of the quadratic function $y = ax^2 + bx + c$. Substituting the coordinates from the first point $x = 0$ and $y = 2$ gives

$$2 = a \cdot (0) + b \cdot (0) + c,$$

so $c = 2$ and therefore the equation of the parabola becomes $y = ax^2 + bx + 2$. Using the second point $(1, 0)$, we get

$$0 = a \cdot (1^2) + b \cdot (1) + 2 = a + b + 2,$$

and so

$$a + b = -2 \qquad \textbf{(1)}$$

Using the third point $(2, 4)$, we get

$$\begin{aligned} 4 &= a \cdot (2^2) + b \cdot (2) + 2 \\ &= 4a + 2b + 2, \end{aligned}$$

or

$$4a + 2b = 2.$$

Dividing both sides of this equation by 2 yields

$$2a + b = 1. \qquad \textbf{(2)}$$

Equations (1) and (2) are a system of two linear equations in two unknowns.

We can solve for a and b by using the usual algebraic methods. We subtract Equation (1) from Equation (2) to get

$$a = 3.$$

Substituting this value into Equation (1) gives

$$3 + b = -2 \quad \text{or} \quad b = -5.$$

So, as before, the desired quadratic is

$$y = 3x^2 - 5x + 2.$$

You can easily verify that the last three points satisfy this function.

Alternatively, we can solve this system of two equations in two unknowns by using the matrix methods described briefly in Appendix C and also find that $a = 3$ and $b = -5$. Thus the equation of the parabola again is $y = 3x^2 - 5x + 2$.

We can extend these ideas to develop similar criteria for deciding when a set of m points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ follow a polynomial pattern of degree n for any n. For instance, we have the following criterion for $n = 3$.

> A set of m points $(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)$ lies on a cubic $y = ax^3 + bx^2 + cx + d$ if the third differences (differences of the differences of the differences) of the y-values are all constant when the x-values are uniformly spaced.

Think About This

Show that the points $(-3, -17), (-2, 0), (-1, 5), (0, 4), (1, 3), (2, 8)$, and $(3, 25)$ lie on a cubic polynomial by creating a difference table that extends to the third differences.

Sums of Integers

We use the preceding ideas on differences and polynomial patterns to develop a number of formulas involving sums of numbers that arise frequently in mathematics. Among them are the sum of the first n integers

$$1 + 2 + 3 + \cdots + n$$

and the sum of the squares of the first n integers

$$1^2 + 2^2 + 3^2 + \cdots + n^2.$$

Let's begin with the expression for the sum of the integers. We let S_n denote the sum of the first n integers:

$$S_n = 1 + 2 + 3 + \cdots + n.$$

For instance, $S_4 = 1 + 2 + 3 + 4 = 10$. We want a formula for S_n for any value of n. We derive it in two ways.

The first is a particularly simple way that involves a nice trick. If

$$S_n = 1 + 2 + 3 + \cdots + (n - 2) + (n - 1) + n,$$

we can also write this sum in the reverse order as

$$S_n = n + (n - 1) + (n - 2) + \cdots + 3 + 2 + 1.$$

We now add these two equations together term by term in the following way:

$$\begin{aligned} S_n + S_n &= [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] + \cdots + [(n - 1) + 2] + [n + 1] \\ &= \underbrace{(n + 1) + (n + 1) \quad + (n + 1) + \cdots \quad + (n + 1)}_{n \text{ times}} + (n + 1). \end{aligned}$$

Because there are n of these terms on the right side, we have

$$2S_n = n(n + 1).$$

Dividing both sides by 2, we obtain

$$S_n = \frac{n(n + 1)}{2},$$

which gives the following general result.

> The sum of the first n integers is
>
> $$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}. \qquad \textbf{(3)}$$

EXAMPLE 3

Find the sum of the first 100 integers: $1 + 2 + 3 + \cdots + 100$.

Solution Using Formula (3) with $n = 100$, we get

$$1 + 2 + 3 + \cdots + 100 = \frac{100(101)}{2} = 5050.$$

We can also write Formula (3) in *summation notation* (see Appendix A3):

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Alternatively, we can derive this result by using either the ideas on fitting functions to data or algebraic methods, as shown in Example 4. The advantage of deriving this formula in other ways is that it demonstrates techniques that can be applied to more complicated cases for which the trick we used previously doesn't work.

EXAMPLE 4

Derive the formula for the sum of the first n integers by using **(a)** curve fitting methods and **(b)** algebraic methods.

Solution We again write the sum of the first n integers as $S_n = 1 + 2 + 3 + \cdots + n$. Thus the sum of the first integer is $S_1 = 1$; the sum of the first two integers is $S_2 = 1 + 2 = 3$; the sum of the first three integers is $S_3 = 1 + 2 + 3 = 6$, and when we continue $S_4 = 10$, $S_5 = 15$, $S_6 = 21$, and so on. If we form a table of second differences with these entries, we get the following.

n	S_n	ΔS_n	$\Delta^2 S_n$
1	1		
		2	
2	3		$1 = 3 - 2$
		3	
3	6		1
		4	
4	10		1
		5	
5	15		1
		6	
6	21		

The second differences $\Delta^2 S_n$ are all constant, so the desired pattern is a quadratic function of n. Thus $S_n = an^2 + bn + c$, where a, b, and c are constants that we must now determine.

a. Using the regression features of a calculator, we find that the quadratic function that best fits these points is

$$\begin{aligned} S_n &= 0.5n^2 + 0.5n + 0 \\ &= \frac{1}{2}n^2 + \frac{1}{2}n \\ &= \frac{1}{2}n(n+1), \end{aligned}$$

as before. All the points lie on this curve, as we can verify by substituting the coordinates of the points into the equation.

b. Using the point $n = 1$, $S_1 = 1$ in the quadratic function $S_n = an^2 + bn + c$, we get

$$S_1 = 1 = a \cdot (1^2) + b \cdot (1) + c,$$

and so

$$a + b + c = 1.$$

When $n = 2$, we have $S_2 = 3$ so that

$$S_2 = 3 = a \cdot (2^2) + b \cdot (2) + c,$$

and hence

$$4a + 2b + c = 3.$$

Similarly, when $n = 3$ and $S_3 = 6$, we have

$$S_3 = 6 = a \cdot (3^2) + b \cdot (3) + c,$$

so that

$$9a + 3b + c = 6.$$

We therefore have a system of three linear equations in three unknowns:

$$a + b + c = 1;$$
$$4a + 2b + c = 3;$$
$$9a + 3b + c = 6.$$

Using the matrix techniques from Appendix C, we find that

$$a = \frac{1}{2}, \quad b = \frac{1}{2}, \quad \text{and} \quad c = 0,$$

which is the same set of coefficients we found in part (a).

There is one difficulty with both derivations in Example 3. We used both methods to derive a formula for the sum of the first n integers, which is supposed to be true for *any* n. But, in fact, we based both derivations on just the first six values that we calculated for $S_1, S_2, \ldots, S_6$, which we showed followed a quadratic pattern by looking at a table of second differences. The catch is that we can't know for sure, just by looking at examples, that *all* subsequent values for S_n continue to follow a quadratic pattern. So the "proof" really isn't legitimate unless we can demonstrate that it applies to every value of n, not just the first six. We do so in Example 5.

EXAMPLE 5

Show that all the values for $S_n = 1 + 2 + 3 + \cdots + n$, for all values of n, fall in a quadratic pattern.

Solution To show that all values of S_n fall in a quadratic pattern, we must demonstrate that the second differences are always constant for any value of n. Let's consider any value of n, so that the sum of the first n integers is

$$S_n = 1 + 2 + \cdots + n.$$

If we take the next integer, $n + 1$, and form the sum of the first $n + 1$ integers, we get

$$S_{n+1} = (1 + 2 + \cdots + n) + (n + 1).$$

The difference between S_n and S_{n+1} is

$$\Delta S_n = S_{n+1} - S_n = n + 1,$$

because all other terms cancel.

Similarly, the sum of the first $n + 2$ integers is

$$S_{n+2} = (1 + 2 + \cdots + n) + (n + 1) + (n + 2).$$

The difference between this total and S_{n+1} is

$$S_{n+2} - S_{n+1} = \Delta S_{n+1} = n + 2,$$

because, again, all other terms cancel. As a result, the second difference, or difference of the differences, is just

$$\Delta S_{n+1} - \Delta S_n = (n + 2) - (n + 1) = 1,$$

for any value of n. Therefore *all* values of S_n have a constant second difference and consequently, no matter what value of n we select, the sum of all the differences must follow a quadratic pattern.

Sums of Squares of Integers

We now find a formula for the sum of the first n squares,

$$S_n = 1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2.$$

We have

$$S_1 = 1, \quad S_2 = 1^2 + 2^2 = 5, \quad S_3 = 1^2 + 2^2 + 3^2 = 14,$$
$$S_4 = 30, \quad S_5 = 55, \quad \text{and} \quad S_6 = 91,$$

and so on. For simplicity, we also use the sum of the squares of the first zero terms, $S_0 = 0^2 = 0$. Arranging these values in a table, we obtain the following.

n	S_n	ΔS_n	$\Delta^2 S_n$	$\Delta^3 S_n$
0	0	$1 = 1 - 0$	$3 = 4 - 1$	$2 = 5 - 3$
1	1	4	5	2
2	5	9	7	2
3	14	16	9	2
4	30	25	11	
5	55	36		
6	91			

The third differences $\Delta^3 S_n$ are all constant, so these data values follow a cubic pattern; that is, the formula for the sum of the squares of the first n integers is a cubic function

$$S_n = an^3 + bn^2 + cn + d.$$

Using polynomial regression, we find the cubic polynomial that fits the points $(0, 0)$, $(1, 1)$, $(2, 5)$, $(3, 14)$, $(4, 30)$, $(5, 55)$, and $(6, 91)$ has coefficients $a = 0.33333333$ (or $\frac{1}{3}$), $b = 0.5 = \frac{1}{2}$, $c = 0.16666667$ (or $\frac{1}{6}$) and $d = -3.5E(-12) = -3.5 \times 10^{-12} = -0.0000000000035$, which essentially is 0. Therefore the cubic function that fits the data is

$$S_n = \left(\frac{1}{3}\right)n^3 + \left(\frac{1}{2}\right)n^2 + \left(\frac{1}{6}\right)n + 0.$$

We factor out the common factors n and $\frac{1}{6}$ and then factor the resulting quadratic to get

$$\begin{aligned} S_n &= n\left[\left(\frac{1}{3}\right)n^2 + \left(\frac{1}{2}\right)n + \left(\frac{1}{6}\right)\right] \\ &= \left(\frac{1}{6}\right)n[2n^2 + 3n + 1] \\ &= \left(\frac{1}{6}\right)n(2n + 1)(n + 1), \end{aligned}$$

which is more commonly written as

$$S_n = \frac{n(n + 1)(2n + 1)}{6}.$$

Alternatively, we could solve for the coefficients of the cubic polynomial $S_n = an^3 + bn^2 + cn + d$ algebraically.

Using $n = 0$ and $S_0 = 0$, we get $0 = d$.

Therefore we have

$$S_n = an^3 + bn^2 + cn.$$

Further

when $n = 1$ and $S_1 = 1$:	$a + b + c = 1$
when $n = 2$ and $S_2 = 5$:	$8a + 4b + 2c = 5$
when $n = 3$ and $S_3 = 14$:	$27a + 9b + 3c = 14$

These results give a system of three equations in the three unknowns a, b, and c; we have already determined that $d = 0$. Using matrix methods to solve this system of equations, we again get $a = \frac{1}{3}$, $b = \frac{1}{2}$, and $c = \frac{1}{6}$.

In general, we have the following formula.

> The sum of the squares of the first n integers is
>
> $$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}. \qquad \textbf{(4)}$$

EXAMPLE 6

Find the sum of the squares of the first 100 integers: $1^2 + 2^2 + \cdots + 100^2$.

Solution Using Formula (4) with $n = 100$, we get

$$1^2 + 2^2 + \cdots + 100^2 = \frac{100(101)(201)}{6} = 338{,}350.$$

Note that, although Formula (4) is true for *all* values of n, we have only established it for $n = 0, 1, \ldots, 6$ by using both of these approaches. As with the sum of the first n integers, we must prove that the sum of the squares of the first n integers follows a cubic pattern for every possible value of n. We do so in Example 7.

EXAMPLE 7

Prove that the sum of the squares of the first n integers, for any n, follows a cubic pattern.

Solution To do so, we have to show that the third differences of S_n are all constant, for any value of n. We write

$$S_n = 1^2 + 2^2 + \cdots + n^2$$

so that

$$\begin{aligned} S_{n+1} &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2; \\ S_{n+2} &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 + (n+2)^2; \\ S_{n+3} &= 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 + (n+2)^2 + (n+3)^2. \end{aligned}$$

We begin by forming the first differences of each successive pair. In each case, all terms but one cancel, leaving us with

$$\begin{aligned} S_{n+1} - S_n &= \Delta S_n = (n+1)^2 = n^2 + 2n + 1; \\ S_{n+2} - S_{n+1} &= \Delta S_{n+1} = (n+2)^2 = n^2 + 4n + 4; \\ S_{n+3} - S_{n+2} &= \Delta S_{n+2} = (n+3)^2 = n^2 + 6n + 9. \end{aligned}$$

Each of these first differences is a quadratic function of n. We now form the second differences by taking the difference of each successive pair of first differences:

$$\begin{aligned} \Delta S_{n+1} - \Delta S_n &= \Delta^2 S_n = (n^2 + 4n + 4) - (n^2 + 2n + 1) = 2n + 3; \\ \Delta S_{n+2} - \Delta S_{n+1} &= \Delta^2 S_{n+1} = (n^2 + 6n + 9) - (n^2 + 4n + 4) = 2n + 5. \end{aligned}$$

Finally, we find the third differences by forming the difference between these last two expressions and get

$$\Delta^2 S_{n+1} - \Delta^2 S_n = \Delta^3 S_n = (2n + 5) - (2n + 3) = 2,$$

which is constant for *all* values of n. That is, the sum of the squares of the first n integers follows a cubic pattern for every value of n.

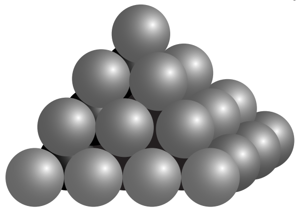
EXAMPLE 8

When cannonballs are stacked in a pyramidal pile, as shown in the accompanying figure, they are organized from the top layer down as follows: A single ball is at the top of the pile; four balls are in the second layer, arranged in a square to support the single ball on top; nine balls are in the third layer, arranged in a square of size 3 by 3 that supports the second layer; and so on. How many cannonballs are in a pile that is 10 layers high?

Solution The number of cannonballs is

$$1^2 + 2^2 + 3^2 + \cdots + 10^2.$$

We can evaluate this total using Formula (4) for the sum of the squares of the first n integers with $n = 10$. Thus



$$\begin{aligned} 1^2 + 2^2 + 3^2 + \cdots + 10^2 &= \frac{10(10+1)[2(10)+1]}{6} \\ &= \frac{10(11)(21)}{6} \\ &= 385 \text{ cannonballs}. \end{aligned}$$

Example 9 illustrates some additional applications of these ideas to find the total for a quantity when the individual amounts are known. In it we use the following basic properties of sums of numbers:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \qquad \textbf{(5)}$$

and

$$\sum_{k=1}^{n} (m \cdot a_k) = m \cdot \sum_{k=1}^{n} a_k, \qquad \text{for any constant } m. \qquad \textbf{(6)}$$

We ask you to prove these two results in the problems at the end of this section.

EXAMPLE 9

A study of the financial records of a company finds that its monthly revenues, in thousands of dollars, are modeled by the function $R(x) = 0.001x^2 + 0.02x + 32$, where x is the number of months since the start of the study and $x \geq 1$. Find the total revenue for this company over its first 10 years of operation.

Solution The 10-year period is equivalent to 120 months. We need to add the revenues $R(1)$ in month 1, $R(2)$ in month 2, $R(3)$ in month 3, . . . , $R(120)$ in month 120. Doing so, we get

$$R = R(1) + R(2) + \cdots + R(120) = \sum_{k=1}^{120} (0.001k^2 + 0.02k + 32),$$

where the variable k takes on all values between 1 and 120. Using Property (5) of sums, we simplify the preceding equation and get

$$R = \sum_{k=1}^{120} 0.001k^2 + \sum_{k=1}^{120} 0.02k + \sum_{k=1}^{120} 32,$$

Using Property (6), we get

$$R = 0.001 \sum_{k=1}^{120} k^2 + 0.02 \sum_{k=1}^{120} k + 32 \sum_{k=1}^{120} 1.$$

The first term involves the sum of the squares of the first 120 integers, so

$$\sum_{k=1}^{120} k^2 = \frac{(120)(120 + 1)(2 \cdot 120 + 1)}{6} = 583{,}220.$$

The second term involves the sum of the first 120 integers, so

$$\sum_{k=1}^{120} k = \frac{(120)(121)}{2} = 7260.$$

The third term involves the sum of 120 ones, so

$$\sum_{k=1}^{120} 1 = 120 \cdot (1) = 120.$$

Therefore the total revenue for this company over the 10-year period is

$$R = 0.001(583{,}220) + 0.02(7260) + 32(120) = 4568.42$$

thousand dollars, or about \$4.568 million.

Problems

1. In Examples 1 and 2, we found the parabola that passes through the points (0, 2), (1, 0), (2, 4), (3, 14), (4, 30), and (5, 52). Suppose now that the points are (0, 2), (1, 0), (2, 4), (3, 15), (4, 30), and (5, 52) instead.
   a. Show that these points do not lie on a parabola.
   b. Attempt to repeat the procedure used in Example 2 to see what goes wrong.
2. Determine which sets of values come from a quadratic function and which come from a cubic function. For those that come from a quadratic function, determine the equation of the quadratic.

x	$f(x)$	$g(x)$	$h(x)$	$k(x)$
0	0	1	1	3
1	−2	6	0	1
2	2	13	5	3
3	12	22	22	9
4	28	33	57	19
5	50	46	116	33

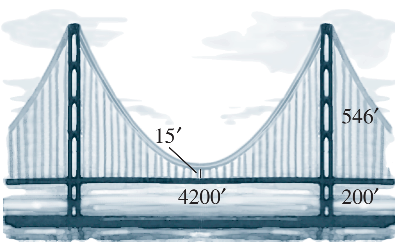
3. The following measurements were taken on a quantity that follows a cubic pattern. However, one of the values was recorded in error. Find the incorrect entry and correct it. (*Hint*: It isn't necessary to actually determine the formula for the cubic.)

x	0	1	2	3	4	5	6	7
y	40	34	24	22	40	90	184	344

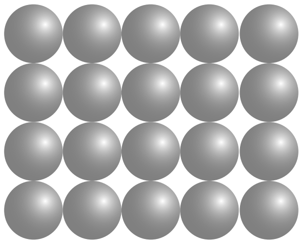
4. Consider the array of numbers known as Pascal's triangle in which each row begins and ends with 1 and each intermediate entry is simply the sum of the two numbers diagonally above it in the previous row.

1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

The rows are numbered $n = 1, 2, \ldots$ The second diagonal consists of the entries 1, 2, 3, 4, 5, and 6, . . .
   a. Find a formula for the terms in the third diagonal: 1, 3, 6, 10, 15, . . . , in terms of the row number n.
   b. Find a formula for the terms in the fourth diagonal 1, 4, 10, 20, . . . , in terms of the row number n.
5. Construct the quadratic polynomial that passes through the points (0, 1), (1, 4), and (2, 9). Use it to estimate the value of the underlying function when $x = 0.5$ and when $x = 3$.
6. The main support cable of a suspension bridge is a parabola. For the Golden Gate bridge, suppose that the cable's lowest point is 15 feet above the roadway. Use the dimensions shown in the accompanying figure to find an equation of the cable for the Golden Gate bridge.



7. Find **(a)** the sum of the first 25 integers, **(b)** the sum of the first 100 integers, and **(c)** the sum of the first 1000 integers.
8. Find **(a)** the sum of the squares of the first 25 integers and **(b)** the sum of the squares of the first 50 integers.
9. Suppose that the produce manager in a supermarket receives a delivery of 1000 large grapefruit, which he wants to display in a pyramid with a square base. How many layers are needed?
10. **a.** Find the sum of the integers from 83 through 225, inclusive.
    b. Find the sum of the squares of these integers.
11. The annual rainfall R, in inches, in a particular region in year t since the start of the last century can be modeled by the formula $R(t) = -0.02t^2 + 1.8t + 42$. Find the total rainfall from 1900 (when $t = 0$) through 2000 in that region.
12. Cannonballs are sometimes stacked in rectangular piles. The accompanying figure shows the fourth layer of a stack of n rectangular layers.
    a. Suppose that such a stack ends with a single row of two balls at the top. Devise a formula in sum-



Fourth layer

mation notation for the number of balls in a stack n layers high.

b. Use the properties of summations to expand the formula you found in part (a).

c. Suppose that a stack of cannonballs ends with a single row of three balls as the top layer. Devise a formula for the number of balls in a stack n layers high.

d. Use the summation formulas from parts (a) and (c) to predict the result if the top layer consists of a single row of four balls.

13. **a.** Consider the function $y = ax^2$. Construct a table of values for the function if $x = -2, -1, 0, 1, 2, 3$ and extend it to a table of differences until you can construct a formula for $\Delta^2 y$ for this function.

b. Repeat part (a) for the function $y = ax^3$ to devise a formula for $\Delta^3 y$.

c. Repeat part (a) for the function $y = ax^4$ to devise a formula for $\Delta^4 y$.

d. Based on your results in parts (a)–(c), predict a formula for $\Delta^5 y$ when $y = ax^5$.

14. For the sequence of numbers $\{y_0, y_1, y_2, \ldots, y_n, y_{n+1}, y_{n+2}, \ldots\}$, show that

a. $\Delta^2 y_0 = y_2 - 2y_1 + y_0$.

b. $\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n$, for any n.

15. Suppose that a set of data values (x_0, y_0), (x_1, y_1), $(x_2, y_2), \ldots$ has uniformly spaced x-values (Δx) and constant second differences $\Delta^2 y = k$ so that the points follow a quadratic pattern $y = ax^2 + bx + c$. Use the result of Problem 14 to show that the leading coefficient is

$$a = \frac{1}{2}\frac{\Delta^2 y}{(\Delta x)^2}.$$

(*Hint*: Write $x_1 = x_0 + \Delta x$ and $x_2 = x_0 + 2\Delta x$ and use the first three points to construct a system of linear equations in a, b, and c.)

16. Because the sum of the first n integers follows a quadratic pattern and the sum of the squares of the first n integers follows a cubic pattern, you might conjecture that the sum of the cubes of the first n integers

$$S_n = \sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

follows a quartic polynomial pattern.

a. Calculate values for S_n for $n = 0, 1, 2, \ldots, 7$.

b. Use a table of differences to show that these values follow a quartic pattern.

c. Find a formula for the sum of the cubes of the first n integers.

17. Find the sum of the cubes of the first 25 integers.

18. By writing out

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

$$\sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n, \quad \text{and} \quad \sum_{k=1}^{n} (m \cdot a_k),$$

show that

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

and

$$\sum_{k=1}^{n} (m \cdot a_k) = m \sum_{k=1}^{n} a_k, \qquad \text{for any constant } m.$$

19. A Pythagorean triple is a set of three integers a, b, and c that satisfy the Pythagorean theorem $a^2 + b^2 = c^2$ and hence represent the sides of a right triangle. The following is a list of the first five Pythagorean triples (a_n, b_n, c_n).

n	a_n	b_n	c_n
1	3	4	5
2	5	12	13
3	7	24	25
4	9	40	41
5	11	60	61

(There are infinitely many Pythagorean triples.) Notice that, for any n, $a_n = 2n + 1$ and $c_n = b_n + 1$.

a. Construct a table of differences to determine the pattern in the b_n terms.

b. Find a formula for b_n for each value of n, based on the pattern from part (a).

c. Show that the resulting triple (a_n, b_n, c_n), forms a Pythagorean triple for any integer n.

d. What is the next Pythagorean triple following the ones shown in the table?

4.6 Building New Functions from Old: Operations on Functions

The functions that we've considered so far, such as $y = \sqrt{x}$, $y = x^5$, $y = 10^x$, and $y = \log x$, can be thought of as building blocks from which we can construct other, more complicated functions. In a simple case, we can take the power functions $y = x$ and $y = x^2$ and the constants 3, 4, and -5 to create the quadratic function $f(x) = 3x^2 + 4x - 5$ as a linear combination of power functions. In fact, we can think of any polynomial as a linear combination of power functions. In this section, we investigate how to generate larger classes of functions by applying simple operations (e.g., addition, subtraction, multiplication, and division) to the basic families of functions that we already have discussed.

Sums and Differences

Let's begin with the sum of two functions. The function

$$f(x) = x^2 + 2^{-x}$$

is the sum of the two functions $y = x^2$ and $y = 2^{-x} = 1/2^x = \left(\frac{1}{2}\right)^x$. Their individual graphs are shown in Figure 4.47. If we "pile" one set of y-values on top of the other and add, we get the graph of the sum as shown in Figure 4.48. You can verify that this result is indeed the case by plotting the sum function on your function grapher.

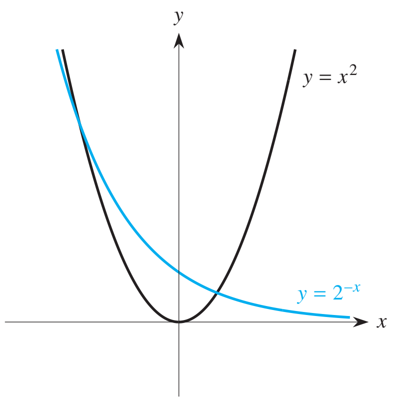


FIGURE 4.47

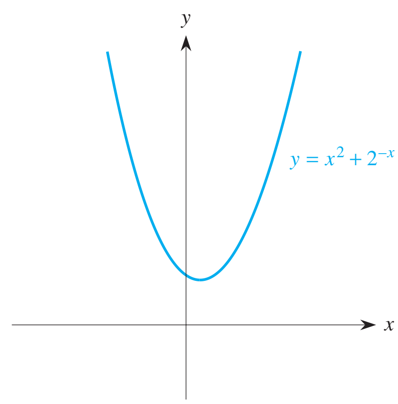


FIGURE 4.48

Note the behavior of the sum $y = x^2 + 2^{-x}$. Because $y = 2^{-x} = \left(\frac{1}{2}\right)^x$ decays rapidly as x increases, its contribution becomes less and less significant, and the quadratic term eventually dominates in the sum when x is large. As a result, toward the right, the graph quickly becomes indistinguishable from a parabola. For negative values of x, both functions become large, but 2^{-x} grows much faster than x^2 does and so the exponential term dominates on the left.

In general, we write

$$S(x) = f(x) + g(x)$$

for the sum of two functions $f(x)$ and $g(x)$. That is, for each value of x, we add the values of $f(x)$ and $g(x)$ to produce the value of $S(x)$. For instance, if $f(3) = 15$ and $g(3) = 4$, then $S(3) = f(3) + g(3) = 15 + 4 = 19$.

Similarly, we construct the difference of two functions by taking the difference between their values for each possible value of x. In general, we write

$$D(x) = f(x) - g(x)$$

for the difference of two functions. Thus, if $f(3) = 15$ and $g(3) = 4$, then $D(3) = f(3) - g(3) = 15 - 4 = 11$. Graphically, if we subtract $g(x)$ from $f(x)$ and $f(x) > g(x)$, the difference $D(x)$ is just the difference in height between the two curves for each value of x.

Products of Functions

For the product of the two functions $f(x) = x^2$ and $g(x) = 2^{-x}$, we use the same interpretation as with sums and differences of functions. Thus the product of the two functions

$$P(x) = f(x) \cdot g(x)$$

means that, for each permissible value of x, we multiply the corresponding function values. So, if $f(3) = 15$ and $g(3) = 4$, then $P(3) = f(3) \cdot g(3) = 15 \cdot 4 = 60$.

What does the graph of the product function look like? Unlike the sum and difference of two functions, there is rarely a direct graphical interpretation of the product of two functions. However, you can produce the graph of the product of two functions on your function grapher and then analyze the behavior of that graph. For instance, consider

$$P(x) = f(x) \cdot g(x) = x^2 \cdot 2^{-x}.$$

We know that, for large positive x, $y = x^2$ grows ever larger and $y = 2^{-x}$ approaches zero. We also know that an exponential function with a positive exponent grows much faster than a power function does. Similarly, an exponential function with a negative exponent decays much faster than a power function with a negative power. Together, these facts indicate that, in the product $x^2 \cdot 2^{-x}$, the exponential term drives the product toward zero as x increases. For values of $x < 0$, both functions grow without bound, so their product becomes infinitely large. By using your function grapher, you can obtain the result shown in Figure 4.49.

Let's look at a real-life example of a product of two functions. Lyme disease is caused by a bacterial infection transmitted by blood-sucking ticks. When a person is infected, the body produces antibodies to fight the bacteria. Figure 4.50 shows the level of concentration of the antibody in the bloodstream as a function of the number of weeks since the first infection. Note that the pattern is remarkably similar to

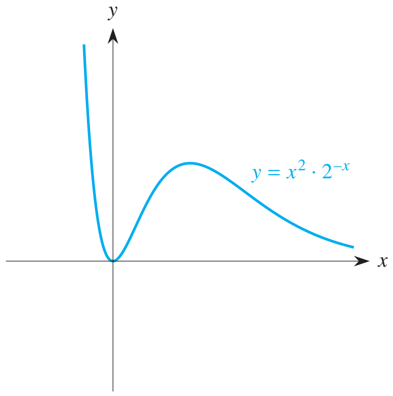


FIGURE 4.49

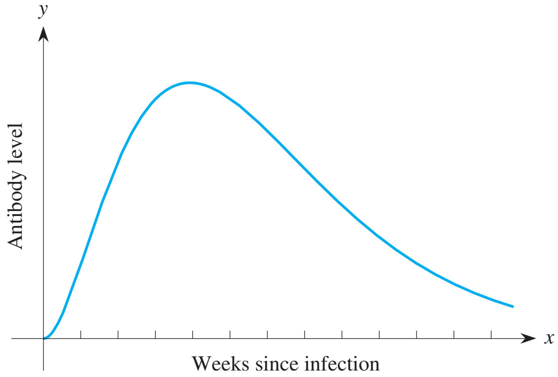


FIGURE 4.50

the behavior of the function $f(x) = x^2 \cdot 2^{-x}$ for $x > 0$. This pattern suggests that such functions, known as *surge functions,* might be appropriate as mathematical models to describe the antibody level, both for Lyme disease and possibly other infections. We explore some applications of surge functions in Section 4.9.

Quotient of Functions

When we consider the quotient of two functions

$$Q(x) = \frac{f(x)}{g(x)},$$

there is a complication that we must take into account. The quotient is undefined at any point where the denominator $g(x)$ is zero, and typically a vertical asymptote occurs there. We illustrate this behavior in Example 1.

EXAMPLE 1

Sketch the graph of the function

$$Q(x) = \frac{x^2 + 1}{x^2 - 1}.$$

Solution We begin analyzing the behavior of this function by looking at what happens when the denominator $x^2 - 1 = (x - 1)(x + 1)$ is zero. That occurs when $x = 1$ and $x = -1$, so the quotient $Q(x)$ is not defined there. When you take values of x very close to either of these two points, the corresponding values for the quotient $Q(x)$ become extremely large, positively or negatively. To see this result, first consider points near $x = 1$. Suppose that x is slightly larger than 1. So

if $x = 1.001$, then $y = Q(1.001) \approx 1000.5$;
if $x = 1.0001$, then $y = Q(1.0001) \approx 10000.5$;
if $x = 1.000001$, then $y = Q(1.000001) \approx 1{,}000{,}000.5$.

Hence, as x approaches 1 from the right (or from above) through values of x that are slightly larger than 1, y becomes ever larger and approaches ∞.

Now suppose that x is slightly smaller than 1. So

if $x = 0.999$, then $y = Q(0.999) \approx -999.5$;
if $x = 0.9999$, then $y = Q(0.9999) \approx -9999.5$;
if $x = 0.999999$, then $y = Q(0.999999) \approx -999{,}999.5$.

Hence, as x approaches 1 from the left (or from below) through values of x that are slightly smaller than 1, y approaches $-\infty$.

By a similar analysis around the point $x = -1$, you can verify that, as x approaches -1 from the right, the function approaches $-\infty$, whereas, if x approaches -1 from the left, the function approaches $+\infty$. Therefore it is not surprising that this quotient function has vertical asymptotes at $x = 1$ and $x = -1$.

We next analyze the *end behavior*—what happens to this function as x becomes large, both positively and negatively. Suppose, for instance, that $x = 1000$. The value of the function then is

$$Q(1000) = \frac{1{,}000{,}001}{999{,}999} \approx 1.000002,$$

which is extremely close to 1. Actually, adding 1 to x^2 in the numerator and subtracting 1 from x^2 in the denominator really has little effect on the value of the function when x is 1000. If x were even larger—say, 1,000,000—adding or subtracting 1 from x^2 would have a negligible effect. Thus for large values of x, the numerator is dominated by the x^2 term and the denominator is dominated by the x^2 term, so the quotient behaves like

$$Q(x) \approx \frac{x^2}{x^2} = 1$$

when x is large. As a result, this quotient function gets closer and closer to a height of 1, so that it has a horizontal asymptote of $y = 1$ as x approaches ∞.

What happens as x approaches $-\infty$? Again, for large negative values of x, the 1 in the numerator and the -1 in the denominator are negligible and x^2 again dominates both the numerator and the denominator. Thus there is also a horizontal asymptote of $y = 1$ as x approaches $-\infty$.

We next look for the points where the curve crosses the two axes. It crosses the y-axis when $x = 0$, so $Q(0) = 1/(-1) = -1$. Where does the curve cross the x-axis? For that to happen, y must equal 0, so the numerator has to be 0. Because the numerator for $Q(x)$, $x^2 + 1$, is never 0 for real values of x, the quotient cannot be 0 anywhere. Hence the curve never crosses the x-axis. The complete graph of this function is shown in Figure 4.51.

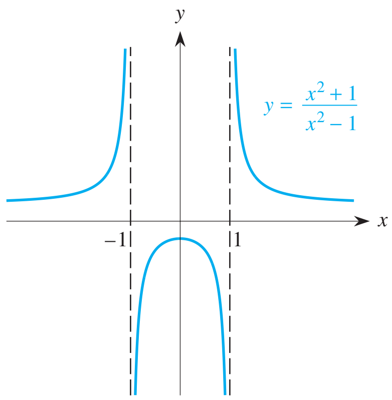


FIGURE 4.51

Rational Functions

Example 1 illustrates most of the ideas involving the behavior of quotients of functions in general and quotients of polynomials in particular. The quotient (or ratio) of two polynomials is called a **rational function.** We assume that any common factors in the numerator and the denominator have been canceled and therefore that the rational function is expressed in simplest form.

The following are some of the important facts about rational functions.

Behavior of Rational Functions $R(x) = P(x)/Q(x)$

- The zeros of the numerator $P(x)$correspond to zeros of the rational function $R(x)$; its graph crosses the x-axis at these points.

- The zeros of the denominator $Q(x)$correspond to the points where the rational function $R(x)$ is not defined; its graph usually has a vertical asymptote at these points.
- The highest power term in the numerator $P(x)$dominates the numerator for *large values of* x, either positive or negative.
- The highest power term in the denominator $Q(x)$ dominates the denominator for *large values of* x, either positive or negative.
- For *large values of* x, either positive or negative, the rational function $R(x)$ behaves like the highest power term of the numerator divided by the highest power term of the denominator. The result may be a horizontal asymptote or the values may approach ∞ or $-\infty$ as x increases either positively or negatively.

We illustrate these ideas in Examples 2 and 3.

EXAMPLE 2

Analyze the behavior of the rational function

$$R(x) = \frac{x^2 - 1}{x - 2}.$$

Solution Here, $R(x)$ has zeros when its numerator $x^2 - 1 = 0$, so that $x = \pm 1$, and the graph crosses the x-axis at these two points. Also, the denominator is zero when $x = 2$, which creates a vertical asymptote there. Suppose that x approaches 2 from the right (with values slightly larger than 2); for instance,

if $x = 2.01$, then $y = R(2.01) = 304.01$;
if $x = 2.001$, then $y = R(2.001) = 3004.001$;
if $x = 2.00001$, then $y = R(2.00001) = 300004.00001$.

Thus, when x approaches 2 from the right, $R(x)$ approaches $+\infty$. Similarly, when x approaches 2 from the left, $R(x)$ approaches $-\infty$ (try some values of x slightly less than 2—say, $x = 1.99$ or $x = 1.9999$).

You can locate the vertical asymptotes of a rational function by finding the roots of the denominator, but you must check what happens on either side (in this case at $x = 2.001$ and $x = 1.999$, for example) to determine the sign of the function on each side of the vertical asymptote. Doing so lets you decide whether the curve rises toward $+\infty$ or drops toward $-\infty$ on each side of the vertical asymptote.

Next, consider the end behavior of $R(x)$. For large values of x, the numerator is dominated by the leading x^2 term and the denominator is dominated by the leading x term. As a result, for large values of x, the quotient behaves like $y = R(x) \approx x^2/x = x$. For instance,

if $x = 10$, then $R(10) = 12.375$;
if $x = 100$, then $R(100) = 102.0306$;
if $x = 1000$, then $R(1000) = 1002.003006$.

The larger x is, the closer $R(x)$ is to x and, for large positive values of x, the graph increases toward $+\infty$.

Similarly, for large negative values, the quotient $R(x)$ behaves like $y = x^2/x = x$ and the graph tends toward $-\infty$.

Figure 4.52 displays all this behavior in the complete graph of R.

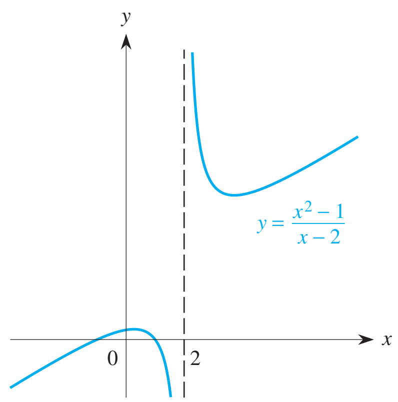


FIGURE 4.52

Think About This Graph both the original quotient function $R(x) = (x^2 - 1)/(x - 2)$ and the *limiting function* $y = x$ in the same fairly large viewing window—say, from -1000 to 1000 for both x and y—on your function grapher. What do you observe?

Getting all the important details on the behavior of a rational function from a single view in your function grapher is often almost impossible. Try it for the function $R(x)$ in Example 2 and see what types of information may be lost because of the scale you use for the domain and range.

EXAMPLE 3

Analyze the behavior of the rational function

$$S(x) = \frac{x - 2}{x^2 - 1}.$$

Solution Here, $S(x)$ was formed by interchanging the numerator and denominator of the rational function $R(x)$ in Example 2, but the behavior of the two functions is quite different.

Note that $S(x)$ has only one zero at $x = 2$ when the numerator is zero. It has two vertical asymptotes, one at $x = 1$ and the other at $x = -1$ when the denominator is zero. Let's see what happens on either side of the asymptotes. When $x = 1.001$, say, we have $S(1.001) = -499.25$, so we conclude that the curve drops toward $-\infty$ as x approaches 1 from the right. Similarly, when $x = 0.999$, we have $S(0.999) = 500.75$ and the curve rises toward $+\infty$ as x approaches 1 from the left. Similarly, when $x = -1.001$, we have $S(-1.001) = -1499.75$ and the curve drops toward $-\infty$ as x approaches -1 from the left. Also, when $x = -0.999$, $S(-0.999) = 1500.25$, and the curve rises toward $+\infty$ as x approaches -1 from the right. Use your calculator to check these conclusions numerically with other values of x on either side of $x = 1$ and on either side of $x = -1$.

Further, the numerator is dominated by x and the denominator is dominated by x^2, so for large values of x, the rational function behaves like $y = x/x^2 = 1/x$. Therefore, for large positive values of x, the function is positive and decays toward the x-axis as a horizontal asymptote. Similarly, for large negative values of x, the function is negative and rises toward the x-axis as a horizontal asymptote.

The complete graph of $S(x)$ is displayed in Figure 4.53.

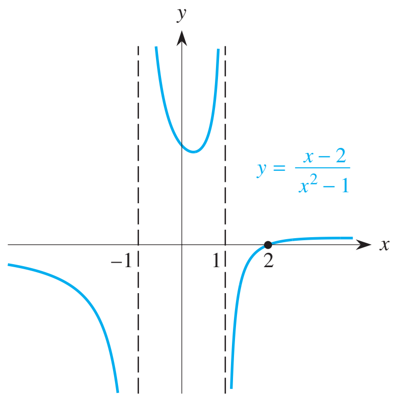


FIGURE 4.53

As before, though, we urge you to examine the behavior carefully with your function grapher to see how viewing the overall characteristics depends on the window you use.

Think About This Examine the graphs of the quotient function $S(x)$ and the limiting function $y = 1/x$ in the same large viewing window. What do you observe?

We next consider a real-world application that involves rational functions.

EXAMPLE 4

According to the law of universal gravitation, the gravitational force between any two objects of mass m_1 and m_2 is

$$F = \frac{Gm_1m_2}{r^2},$$

where r is the distance between the objects and G is the gravitational constant. Envision a spacecraft traveling from the Earth to the moon, a distance of about 240,000 miles. Because the mass of the Earth is roughly 81 times that of the moon, the Earth's gravitational effect on the spacecraft will be greater than that of the moon's until the spacecraft is quite close to the moon, when it's gravity becomes dominant. Determine the distance from the Earth when the two gravitational forces exactly balance each other.

Solution We begin with a sketch of the situation, as shown in Figure 4.54, where r represents the distance, in thousands of miles, from the Earth to the spacecraft. Hence

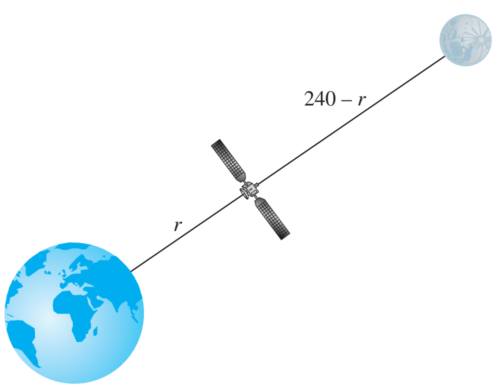


FIGURE 4.54

$240 - r$ is the distance from the moon to the spacecraft. Let m_0 be the mass of the spacecraft, m_1 be the mass of the Earth, and m_2 be the mass of the moon. The Earth's gravitational force on the spacecraft F_e is

$$F_e = \frac{Gm_0m_1}{r^2},$$

and the moon's gravitational force on the spacecraft F_m is

$$F_m = \frac{Gm_0m_2}{(240 - r)^2}.$$

Both F_e and F_m are rational functions of r. Because the Earth is 81 times as massive as the moon, $m_1 = 81m_2$. We rewrite F_e as

$$F_e = \frac{Gm_0(81m_2)}{r^2}.$$

The two gravitational forces are equal when

$$\frac{81Gm_0m_2}{r^2} = \frac{Gm_0m_2}{(240 - r)^2}.$$

Dividing both sides of this equation by Gm_0m_2 (because none of these quantities are zero) gives

$$\frac{81}{r^2} = \frac{1}{(240 - r)^2}.$$

Cross-multiplying yields

$$r^2 = 81(240 - r)^2.$$

We expand the expression on the right by squaring the binomial term and obtain

$$r^2 = 81(240^2 - 480r + r^2) = 81(240)^2 - 81(480)r + 81r^2.$$

Collecting like terms and simplifying, we have the quadratic equation

$$80r^2 - 38{,}880r + 4{,}665{,}600 = 0.$$

Dividing through by the common factor 80, we get

$$r^2 - 486r + 58{,}320 = 0.$$

Using the quadratic formula, we find that the roots of this quadratic equation are $r = 216$ and $r = 270$. These answers are distances in thousands of miles from the Earth. Because the moon is about 240 thousand miles from the Earth, the only reasonable answer is the first. Therefore the two forces balance at a point about 216 thousand miles from the Earth and about 24,000 miles this side of the moon. The second solution, 270,000 miles from the Earth, corresponds to a point beyond the moon where the effects of the moon's gravity and the Earth's gravity are numerically the same, though both forces are in the same direction.

A Function of a Function

There is yet another way in which we can construct new functions from simpler functions. In Example 5 of Section 2.2, we showed that the rate R at which a snow tree cricket chirps is a function of the temperature T, and we found a mathematical model for this relationship as the function $R = f(T) = 4T - 160$. However, the air temperature doesn't remain constant, but actually varies with the time of the day, so the temperature T is really a function of time t: $T = g(t)$. As a result, the chirp rate, though a function of the temperature T, is actually a function of time t. That is, we have two functions:

$$R = f(T) = 4T - 160 \quad \text{and} \quad T = g(t).$$

If we substitute $T = g(t)$ into the expression $R = f(T)$, we get

$$R = f(T) = f(g(t)).$$

We call this type of situation a **function of a function** or a **composite function.**

Let's look at this notion from a different perspective. Consider the function $f(x) = \sqrt{x^3 + 1}$. To see what it means, suppose that $x = 1$. Then

$$f(1) = \sqrt{1^3 + 1} = \sqrt{2}.$$

For $x = 2$,

$$f(2) = \sqrt{2^3 + 1} = \sqrt{9} = 3.$$

To evaluate this function in each case, we actually performed two *successive* steps: (1) for each value of x, we evaluated the expression $x^3 + 1$; and (2), we took the square root of the result. The reason is that we are really working with two functions successively: first the "inner" function $x^3 + 1$ and then the "outer" function $\sqrt{u}$, where $u = x^3 + 1$. The final function f is therefore a function of a function.

Let's set up the mathematical framework for this concept. Suppose that we let $y = F(u)$, where $u = G(x)$. Here, $y = F(u) = \sqrt{u}$, where in turn $u = G(x) = x^3 + 1$. Consequently,

$$y = F(u) = F(G(x)) = F(x^3 + 1) = \sqrt{x^3 + 1}.$$

Our original function f is the result of applying the functions G and F successively. This composite function $y = F(G(x))$ is sometimes written as $F \circ G$ and read "F of G".

In general, for two functions F and G, the composite function $F(G(x))$ is the result of evaluating the two functions successively, as depicted in Figure 4.55. We start with a value of x, which is carried into a value u by the first, or inner, function G, which in turn is carried to a value y by the second, or outer, function F. For this method to make sense mathematically, the domain of the outer function F must include the range of the inner function G.

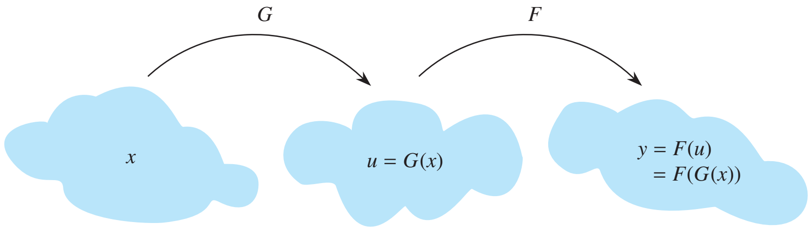


FIGURE 4.55

Using composite functions, we can construct many other types of functions by using the basic functions as building blocks.

EXAMPLE 5

Find two functions f and g so that $y = f(g(x)) = 10^{3x}$.

Solution Think about how you would evaluate this function for any value of x—first, triple the x value because of the $3x$ term and then take that power of 10. That is, the linear function $3x$ is used as the exponent for the exponential function with base 10. So the first, or inner, function is $g(x) = 3x$ followed by the second, or outer, function $y = f(x) = 10^x$. The result gives $f(g(x)) = f(3x) = 10^{3x}$, as required.

EXAMPLE 6

Find two functions f and g so that $y = f(g(x)) = \log(x^2 - 5x + 2)$.

Solution Here, the quadratic function $x^2 - 5x + 2$ is used as the argument of the log function. So the first, or inner, function is the quadratic $g(x) = x^2 - 5x + 2$ and the second, or outer, function is the log function $y = f(x) = \log x$. Using the same approach as in Example 5, we get $f(g(x)) = f(x^2 - 5x + 2) = \log(x^2 - 5x + 2)$, as required.

Are F ∘ G and G ∘ F the same?

Is the order important in forming the composition of two functions? That is, is $F \circ G$ the same as $G \circ F$? Again consider

$$f(x) = \sqrt{x^3 + 1} = F(G(x)) = F \circ G(x),$$

where

$$u = G(x) = x^3 + 1 \quad \text{and} \quad y = F(u) = \sqrt{u}.$$

If we interchange the order to form $G(F(x))$, we get

$$G(F(x)) = G(\sqrt{x}) = (\sqrt{x})^3 + 1 = x^{3/2} + 1,$$

which clearly is not the same as $F(G(x)) = \sqrt{x^3 + 1}$. By substituting a couple of values for x—say, $x = 1$ or $x = 2$, you can see that the results are numerically different. In general, except in rare cases,

$$G(F(x)) \neq F(G(x)).$$

However, if F and G are inverse functions, the equality does hold.

EXAMPLE 7

In Example 6 we chose $f(x) = \log x$ and $g(x) = x^2 - 5x + 2$. Find $f(g(x))$ and $g(f(x))$.

Solution We have

$$f(g(x)) = f(x^2 - 5x + 2) = \log(x^2 - 5x + 2),$$

whereas

$$g(f(x)) = g(\log x) = (\log x)^2 - 5\log x + 2.$$

Clearly, they are very different functions.

Applications of Composite Functions

We next consider a real-world application of composite functions in Example 8.

EXAMPLE 8

When a kicker punts a football, it's path can be modeled by the quadratic function $y = f(x) = -x^2/27 + 1.92x + 1$, where the height y and the horizontal distance downfield x from the point where the ball is kicked are measured in yards. Furthermore, the horizontal distance x from the kicker is given by $x = g(t) = 12t$, where t is measured in seconds.

a. Find an equation giving the height of the football as a composite function of time t.

b. Determine the hang-time for the football—how long it remains in the air after being punted.

Solution

a. The path of the ball is the parabola shown in Figure 4.56, where $y = f(x) = -x^2/27 + 1.92x + 1$. The graph shows that the ball carries somewhat more than 50 yards from the point where it is kicked, which is usually about 10 yards behind the line of scrimmage. Using the formula $x = g(t) = 12t$ for x as a function of t, we can form the composite function giving the height y as a function of t:

$$\begin{aligned} y = f(x) = f(g(t)) &= -\frac{(12t)^2}{27} + 1.92(12t) + 1 \\ &\approx \frac{-144t^2}{27} + 23t + 1 \\ &\approx -5.33t^2 + 23t + 1. \end{aligned}$$

Note that this is also a quadratic function of t with a negative leading coefficient.

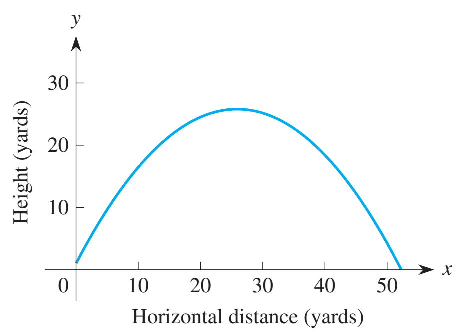


FIGURE 4.56

b. The hang-time for the football is the value of t when the ball comes back to the ground. It is the zero of the composite function, so we must solve the quadratic equation

$$-5.33t^2 + 23t + 1 = 0.$$

Equivalently, if we multiply both sides by -1, we get

$$5.33t^2 - 23t - 1 = 0.$$

Using either graphical methods or the quadratic formula, we find that $t \approx 4.36$ seconds.(A second solution to the quadratic equation gives a negative value for t, which makes no sense in this context.)

EXAMPLE 9

Two functions f and g are defined in the following table. Use the values given in the table to complete it. (If any operations are not defined, write "UNDEF.")

x	$f(x)$	$g(x)$	$f(x) - g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$	$f(g(x))$	$g(f(x))$
0	1	3					
1	0	1					
2	3	0					
3	2	2					

Solution The values of the functions for four specific values of x—namely, $x = 0, 1, 2$, and 3—are defined in the table. The first open column asks for the difference between the two functions for each value of x. For instance, when $x = 0$, the first entry for this column is $f(0) - g(0) = 1 - 3 = -2$, and so on down that column. The second open column asks for the product of the two functions for each value of x. When $x = 0$, we get $f(0) \cdot g(0) = 1 \cdot 3 = 3$, and so on down the column.

The third open column asks for the quotient of the two functions. When $x = 0$, we have $f(0)/g(0) = 1/3$, and so on. However, because $g(2) = 0$, the quotient is not defined when $x = 2$, so we enter UNDEF in the corresponding position in the table.

The fourth and fifth open columns ask for values for the composite functions $f(g(x))$ and $g(f(x))$. In the fourth column, the function g is applied first and then the function f is applied. When $x = 0$, we need to form $f(g(0))$. To do so we evaluate $g(0) = 3$ first and then take $f(g(0)) = f(3) = 2$, so the first entry in the fourth column is 2. For the next entry, we start with $x = 1$ and form $f(g(1))$. Because $g(1) = 1$, we get $f(g(1)) = f(1) = 0$. Similarly, we get the remaining two entries in this column.

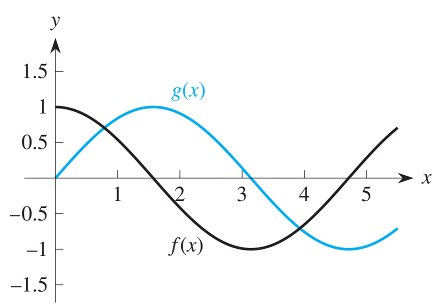
To fill in the entries in the last column, we reverse the order of operations of the two functions and apply first f, followed by g. Starting with $x = 0$, we now need $g(f(0))$. Because $f(0) = 1$, we have $g(f(0)) = g(1) = 1$. Similarly, when $x = 1$, we need $g(f(1))$. Because $f(1) = 0$, we have $g(f(1)) = g(0) = 3$. Incidentally, for each of the four values of x, $f(g(x)) \neq (g(f(x))$.

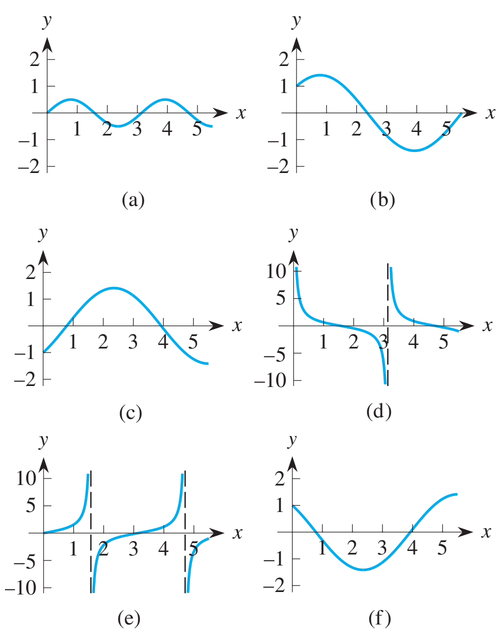
We now have the completed table.

x	$f(x)$	$g(x)$	$f(x) - g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$	$f(g(x))$	$g(f(x))$
0	1	3	-2	3	$\frac{1}{3}$	2	1
1	0	1	-1	0	0	0	3
2	3	0	3	0	UNDEF	1	2
3	2	2	0	4	1	3	0

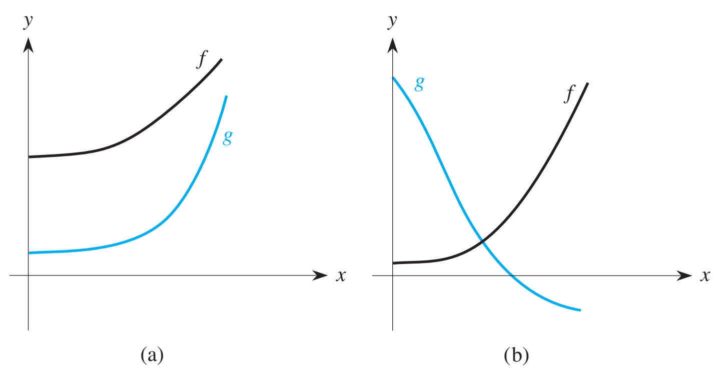
Problems

1. For $f(x) = 3x - 4$ and $g(x) = \frac{1}{x}$, find
   - **a.** $f(5) + g(5)$.
   - **b.** $f(5) - g(5)$.
   - **c.** $f(5) \cdot g(5)$.
   - **d.** $\dfrac{f(5)}{g(5)}$.
   - **e.** $f(g(5))$.
   - **f.** $g(f(5))$.
   - **g.** $f(f(5))$.
   - **h.** $g(g(5))$.
   - **i.** $f(x) + g(x)$.
   - **j.** $f(x) - g(x)$.
   - **k.** $f(x) \cdot g(x)$.
   - **l.** $\dfrac{f(x)}{g(x)}$.
   - **m.** $f(g(x))$.
   - **n.** $g(f(x))$.
   - **o.** $f(f(x))$.
   - **p.** $g(g(x))$.
2. Repeat Problem 1 for $f(x) = x^2 + 4$ and $g(x) = \sqrt{x}$.
3. Repeat Problem 1 for $f(x) = 10^x$ and $g(x) = \log x$.
4. Two functions f and g are defined in the table at the bottom. Use the values given to complete the table. If any of the entries are not defined, write "UNDEF."
5. The functions f and g have the values $f(2) = 10$, $f(4) = 20$, $f(6) = 35$, $g(2) = 8$, $g(4) = 4$, and $g(6) = 2$. Which expressions, (a)–(g), are correct, which are incorrect, and which, if any, are not defined?
   - **a.** $f(6) - f(4) = 2$
   - **b.** $f(g(6)) = 35$
   - **c.** $g(g(6)) = 8$
   - **d.** $f(2) - g(6) = 8$
   - **e.** $f(4) - g(4) = 0$
   - **f.** $f(4) \cdot g(4) = 16$
   - **g.** $f(4)/g(4) = 5$
6. Two functions f and g are given in the accompanying figure. The six graphs (a)–(f) represent $f + g$, $f - g$, $g - f$, $f \cdot g$, f/g, and g/f. Decide which is which and give reasons for your answers.



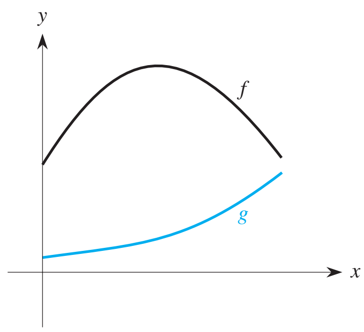


7. For the pairs of functions f and g shown, sketch the graph of the function $y = f(x) + g(x)$.

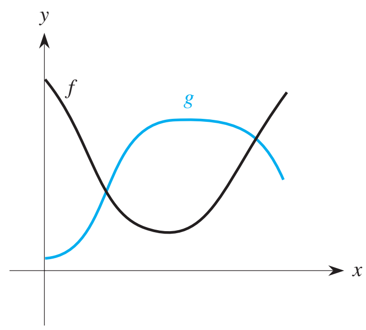


8. For the pairs of functions f and g shown, sketch the graph of the function $y = f(x) - g(x)$.

x	$f(x)$	$g(x)$	$f(x) - g(x)$	$f(x)/g(x)$	$g(x)/f(x)$	$f(x) \cdot g(x)$	$f(g(x))$	$g(f(x))$	$f^{-1}(x)$
0	1	0							
1	2	3							
2	3	1							
3	0	2							

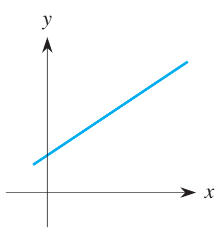


(a)

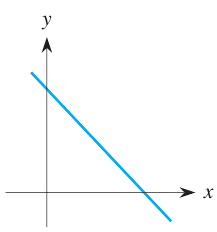


(b)

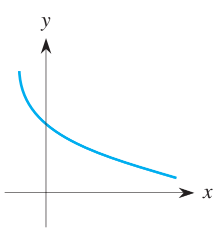
9. The graphs of three functions, **(a)**–**(c)**, are shown in the accompanying figure.



(a)



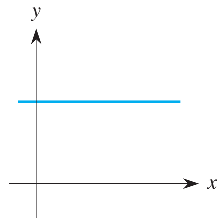
(b)



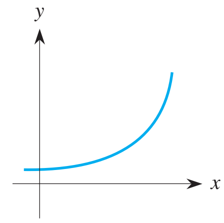
(c)

Sketch a rough graph of **(i)** 2^f, **(ii)** $\log f$, and **(iii)** f^2. If any portion of a graph is not defined, mark it on the x-axis.

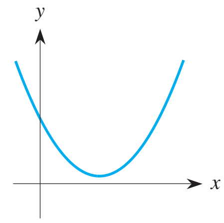
10. For each function **(a)**–**(d)**, sketch the graph of $\log f$. If any portion of a graph is not defined, mark it along the x-axis.



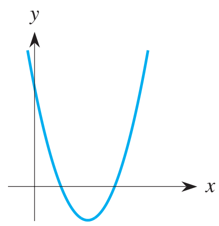
(a)



(b)

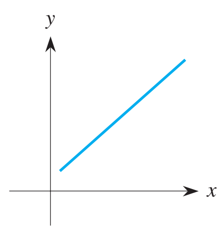


(c)

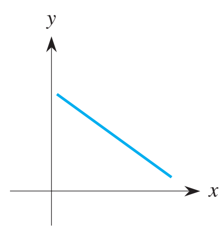


(d)

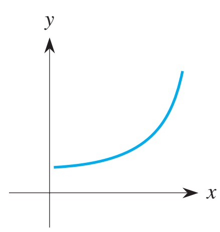
11. Often in technical books and articles, graphs are shown for $\log y$ as a function of x, as in the following graphs. In each case, given the graph of $\log y$ as a function of x, sketch the graph of y as a function of x.



(a)



(b)



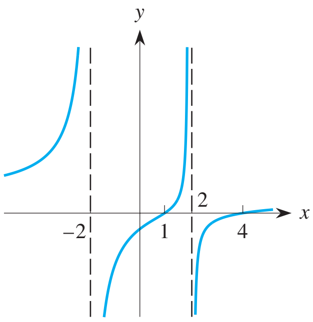
(c)

12. Match each function with its graph.

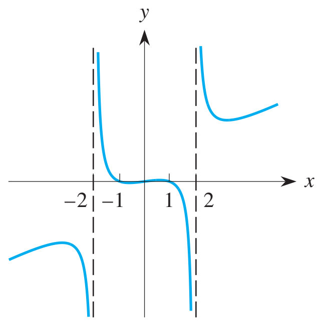
a. $y = \dfrac{x^2 - 1}{x^2 - x - 6}$ **b.** $y = \dfrac{x^2 + 1}{x^2 - x - 6}$

c. $y = \dfrac{9 - x^2}{x^2 - 4}$ **d.** $y = \dfrac{x^2 - x - 6}{x^2 - 1}$

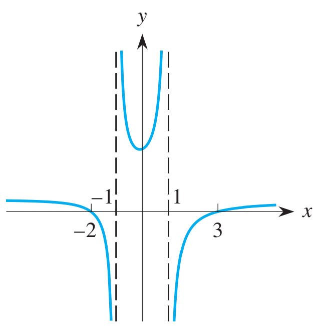
e. $y = \dfrac{x^3 - x}{x^2 - 4}$ **f.** $y = \dfrac{(x - 1)(x - 4)}{x^2 - 4}$



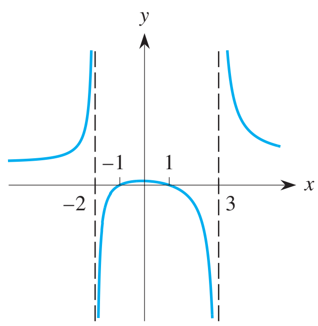
(i)



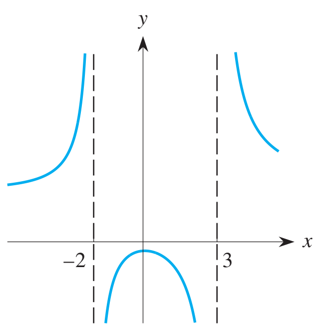
(ii)



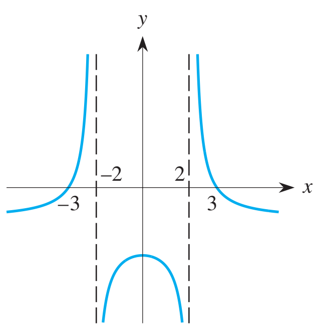
(iii)



(iv)

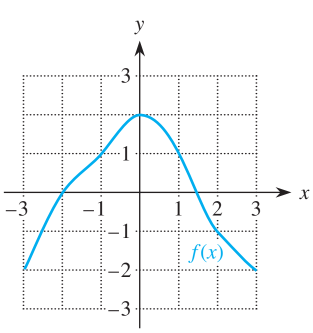


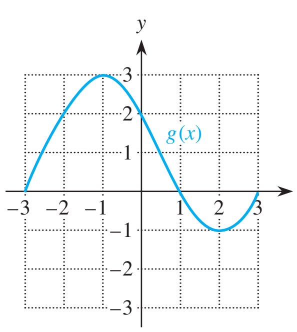
(v)



(vi)

13. For the two functions f and g that are defined by the graphs shown, find **(a)** $f(g(1))$; **(b)** $g(f(1))$; **(c)** $f(g(-1))$; and **(d)** $g(f(-1))$.





14. For the functions f and g that are defined by the graphs in Problem 13, sketch the graph of **(a)** $f(g(x))$ and **(b)** $g(f(x))$.

For Problems 15–18, determine functions F and G such that $h(x) = F(G(x))$. There are different correct answers to this question; however, do not use $F(x) = x$ or $G(x) = x$.

15. $h(x) = x^4 + 5$

16. $h(x) = (x + 5)^4$

17. $h(x) = \log(x + 3)$

18. $h(x) = 3 + \log x$

19. The time t that a traffic light should remain yellow depends on the speed limit s on the road. The function $t = 1 + s/20 + 70/s$, where t is measured in seconds and s is the speed in feet per second, is used to determine the length of the yellow cycle. Note that 30 mph = 44 ft/sec.

a. How long is the yellow cycle if the posted speed limit is 30 mph?

b. How long is the yellow cycle if the posted speed limit is 50 mph?

c. What are reasonable values for the domain and range of this function?

d. Suppose that the traffic department using this formula wants to increase the length of the yellow cycle somewhat. Should it increase or decrease the values of each of the two parameters 20 and 70 to do so?

e. Rewrite the formula for t as a rational function by combining all the terms over a common denominator.

20. According to Einstein's theory of relativity, the mass M of an object increases as its speed increases according to the formula

$$M = f(v) = \frac{M_0}{\sqrt{1 - (v^2/c^2)}} = M_0\left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$

where M_0 is the mass of the object when it is at rest ($v = 0$) and c is the speed of light (about 186,282 miles per second). Suppose that an object has a rest mass of $M_0 = 1$ unit.

a. Construct a table of values for the mass of the object for each of the following speeds expressed as a fraction of the speed of light: $v = 0$, $0.5c$, $0.9c$, $0.95c$, $0.99c$, and $0.999c$.

b. Sketch a graph showing the behavior of the mass of an object as its speed approaches the speed of light.

c. What is the mathematical significance of the speed of light? What is the physical significance of the speed of light in the context of the speeds of moving objects?

21. Some physicists hypothesize the existence of particles called *tachyons* that exist only at speeds greater than that of light. The slower that a tachyon moves, the greater is its mass; the speed of light is a lower limit on the possible speed of a tachyon. Sketch a graph of the mass as a function of speed for all possible values of $v \geq 0$. Indicate which region corresponds to normal particles and which to tachyons.

22. According to Newton's laws of motion, the speed of an object can be changed only by applying a force. Also, the greater the mass of an object, the more force is needed to accelerate it to a given velocity in a fixed amount of time. Suppose that an object is to be accelerated from speed 0 to almost the speed of light.
   a. Sketch the graph of the force needed to accelerate it as a function of the velocity v. Pay careful attention to concavity.
   b. Sketch the graph of the velocity as a function of the force needed, paying careful attention to concavity.

23. **a.** Graph the two functions $y = \sqrt{x^2 + 25}$ and $x + 5$. Are they the same?
   b. Repeat part (a) with $y = \sqrt{x^2 + 4}$ and $y = x + 2$. Are they the same?
   c. Can you find any value for a for which
$$\sqrt{x^2 + a^2} = x + a?$$

24. **a.** Graph the two functions $y = \dfrac{1}{x + 4}$ and $y = \dfrac{1}{x} + \dfrac{1}{4}$. Are they the same?
   b. Repeat part (a) with $y = \dfrac{1}{x - 5}$ and $y = \dfrac{1}{x} - \dfrac{1}{5}$. Are they the same?
   c. Can you find any value for a so that
$$\frac{1}{x + a} = \frac{1}{x} + \frac{1}{a}?$$

25. For the function $f(x) = \frac{x + 1}{x}$, **(a)** what is $f(1)$? **(b)** $f(f(1))$? **(c)** $f(f(f(1)))$? **(d)** Continue to apply the function f repeatedly to the previous result, expressing all your answers as fractions. Do you observe any pattern in the values for the numerators and denominators of the fractions that you're generating? **(e)** Now look at the decimal representations of the fractions that you generated in parts (a)–(d). Do they appear to be approaching a fixed value?

26. For any two linear functions $f(x) = ax + b$ and $g(x) = cx + d$, is $f \circ g$ the same as $g \circ f$?

27. The volume of a sphere is given by $V = \frac{4}{3}\pi r^3$ and its surface area is given by $S = 4\pi r^2$.
   a. Find a formula for the volume as a function of the surface area. Interpret the result in terms of a composite function.
   b. Find a formula for the inverse function of the function you found in part (a). What does it tell you?

28. The degree of polynomial P is m and the degree of polynomial Q is n, where $m < n$.
   a. What is the degree of $P + Q$?
   b. What is the degree of $P - Q$?
   c. What is the degree of $P \cdot Q$?
   d. What is the end behavior of P/Q?
   e. What is the end behavior of Q/P?

29. In Problem 20 of Section 1.3, we introduced a function f that represents a simple replacement code in which each letter of the alphabet is replaced by a different letter according to $f(A) = M$, $f(B) = D$, $f(C) = K$, $f(D) = V$, $f(E) = X$, $f(F) = B$, $f(G) = P$, $f(H) = T$, $f(I) = J$, $f(J) = S$, $f(K) = Z$, $f(L) = Q$, $f(M) = H$, $f(N) = O$, $f(O) = A$, $f(P) = L$, $f(Q) = W$, $f(R) = C$, $f(S) = F$, $f(T) = Y$, $f(U) = R$, $f(V) = G$, $f(W) = I$, $f(X) = U$, $f(Y) = N$, and $f(Z) = E$.
   Suppose that we now have a second such code defined by the function g:
   $g(A) = P$, $g(B) = K$, $g(C) = T$, $g(D) = E$, $g(E) = L$, $g(F) = U$, $g(G) = H$, $g(H) = N$, $g(I) = Y$, $g(J) = C$, $g(K) = R$, $g(L) = W$, $g(M) = G$, $g(N) = Z$, $g(O) = B$, $g(P) = J$, $g(Q) = A$, $g(R) = X$, $g(S) = Q$, $g(T) = D$, $g(U) = S$, $g(V) = M$, $g(W) = V$, $g(X) = I$, $g(Y) = O$, and $g(Z) = F$.
   a. Find $g(f(A))$.
   b. Find $f(g(A))$.
   c. Find $f(f(P))$.
   d. Find $g(g(K))$.
   e. Find $f^{-1}(g^{-1}(A))$.

30. The algebraic method of elimination for solving a system of linear equations involves adding a multiple of one equation to another equation to eliminate one of the variables. Consider the system of two equations in two unknowns:
$$y = 4x - 3 \qquad \textbf{(1)}$$
$$y = 7 - x. \qquad \textbf{(2)}$$
   a. Plot the two lines carefully on a sheet of graph paper and determine the point of intersection.
   b. Solve the two equations algebraically.

c. Add two times Equation (2) to Equation (1) to get a new linear function. Plot its graph on the same graph you created in part (a). What do you observe about the three lines?

d. Add three times Equation (2) to Equation (1) and plot that function on the same graph. What do you observe about the four lines?

e. Add four times Equation (2) to Equation (1) and plot that function on the same graph. What can you conclude from this result?

f. Find an appropriate multiple of Equation (2) that, when added to Equation (1), will eliminate the x term. What will the graph of the resulting line look like when x is eliminated?

4.7 Building New Functions from Old: Shifting, Stretching, and Shrinking

In Section 4.6, we created new functions from known functions by extending the standard arithmetic operations of addition, subtraction, multiplication, and division to functions. We also created new functions by using composition of functions. In this section we introduce several other ways in which we can build new functions from a single function. Suppose that we have the function $y = f(x)$. We can form a related function by changing either the independent variable x or the dependent variable y by multiplying it by a constant or by adding or subtracting a constant from it.

Shifting Functions

We can shift functions up and down or left and right. The former involves transforming the y-variable, and the latter involves transforming the x-variable.

Shifting Up and Down We first investigate the effect on any function $y = f(x)$ of adding a constant to y or subtracting a constant from y.

EXAMPLE 1

Consider $y = f(x) = x^2$ and the related functions $y = x^2 + 1$, $y = x^2 + 3$, $y = x^2 - 2$, and $y = x^2 - 5$. What is the effect of the constant in each case?

Solution All these functions are shown in Figure 4.57. Clearly, each constant shifts the basic parabola $y = x^2$ up or down by the corresponding amount that is added or subtracted. For instance, the curve $y = x^2 + 1$ lies 1 unit above $y = x^2$ for each value of x, whereas $y = x^2 - 2$ lies 2 units below it.

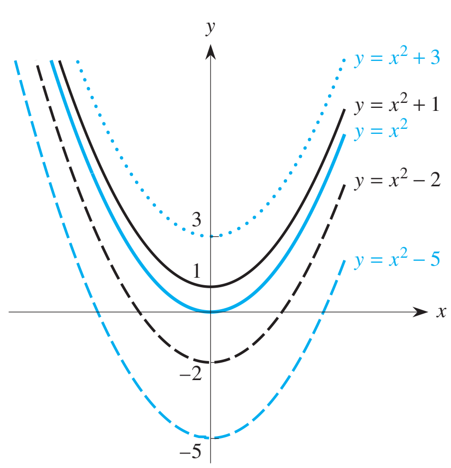


FIGURE 4.57

In general, the following principle holds for any function of x, assuming $b > 0$.

Vertical Shift

Replacing $f(x)$ with $f(x) + b$ shifts the graph of $f(x)$ *up* by the amount b.

Replacing $f(x)$ with $f(x) - b$ shifts the graph of $f(x)$ *down* by the amount b.

We can get a different feel for what is happening if we rewrite each of these expressions by moving the constant term to the left side. For instance, $y = x^2 + 1$ is equivalent to $y - 1 = x^2$, which emphasizes the fact that it is the variable y, or the height, which is being affected by the constant.

We can therefore rephrase the vertical shift principle for any function of x, assuming $b > 0$, as follows.

Vertical Shift

Replacing y with $y - b$ shifts the graph of $f(x)$ *up* by the amount b.

Replacing y with $y + b$ shifts the graph of $f(x)$ *down* by the amount b.

Shifting Left and Right Next we investigate the effect on $y = f(x)$ of adding a constant to x or subtracting a constant from x.

EXAMPLE 2

Consider $y = f(x) = x^2$ and the related functions $y = (x - 1)^2$, $y = (x - 3)^2$, and $y = (x + 2)^2$, where we replace x by $(x - 1)$, $(x - 3)$, or $(x + 2)$, respectively. What is the effect of the constant in each case?

Solution The resulting graphs are shown in Figure 4.58. Each of these changes causes a horizontal shift. For instance, $y = (x - 1)^2$ has a double zero at $x = 1$, so the graph of $y = x^2$ is shifted to the right by 1 unit. Similarly, $y = (x + 2)^2$ has a double zero at $x = -2$, so the graph of $y = x^2$ is shifted to the left by 2 units.

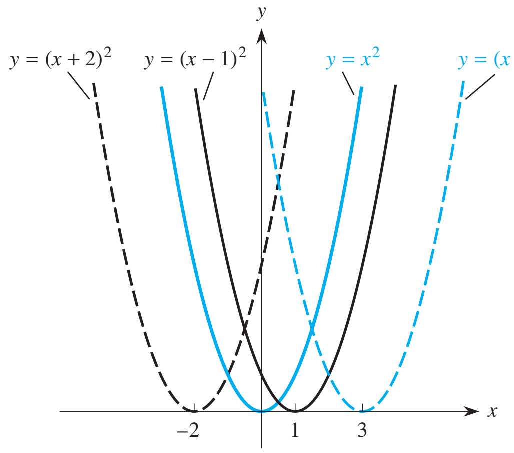


FIGURE 4.58

In general, the following principle holds for any function of x.

Horizontal Shift

Replacing x with $x - a$ shifts the graph of $f(x)$ to the *right* by the amount $a > 0$.
Replacing x with $x + a$ shifts the graph of $f(x)$ to the *left* by the amount $a > 0$.

Thus, for instance, the graph of $y = 10^{x-2}$ has the identical shape as the graph of $y = 10^x$, but is shifted to the right by 2 units. Similarly, the graph of $y = \sqrt{x+3}$ has the same shape as the graph of $y = \sqrt{x}$, but is shifted to the *left* by 3 units. Check these and other graphs on your function grapher.

In summary, when we replace x by $x - a$ or $x + a$, we are changing x and so produce a horizontal effect. When we replace y with $y - b$ or $y + b$, we produce a vertical effect.

When we combine a horizontal shift (replace x by $x - a$) and a vertical shift (replace y by $y - b$), we effectively have a diagonal shift. For example, consider the graph of $y = (x-4)^2 + 7$, or equivalently $y - 7 = (x-4)^2$. It involves a change in x (x is replaced by $x - 4$) and a change in y (y is replaced by $y - 7$). So, $y = (x-4)^2 + 7$ corresponds to shifting the parabola $y = x^2$ four units to the right and seven units up. This produces a parabola whose vertex is at $(4, 7)$, as shown in Figure 4.59.

Similarly,

$$x^2 + y^2 = r^2$$

is the equation of a circle with radius r centered at the origin (see Appendix A6). We should then expect that

$$(x-5)^2 + (y-3)^2 = r^2$$

is the graph of a circle with radius r that has been shifted 5 units to the right and 3 units up. It is therefore the equation of a circle with radius r centered at the point $(5, 3)$. The new circle is produced from the original circle by a combination of a horizontal shift (5 units to the right) and a vertical shift (3 units up), as shown in Figure 4.60.

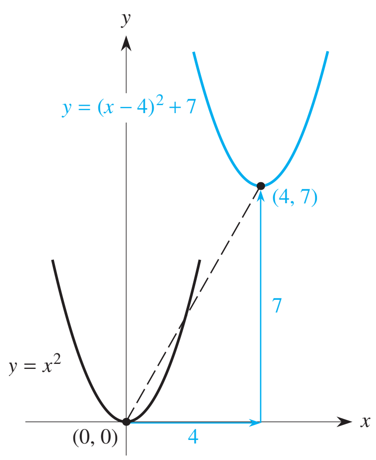


FIGURE 4.59

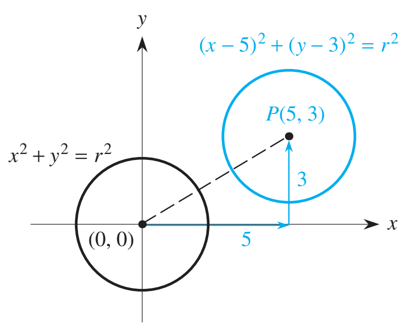


FIGURE 4.60

Stretching and Shrinking Functions

We can stretch and shrink functions vertically and horizontally. Vertical stretching and shrinking involves multiplying a function by a constant, whereas horizontal stretching and shrinking involves multiplying the independent variable by a constant.

A Constant Multiple of a Function We can also create a new function from a given function by multiplying the function, or equivalently the y-value, by a constant. For example, consider the two functions $y = 2^{-x}$ and $y = 5 \cdot 2^{-x}$. Both are exponential decay functions, as shown in Figure 4.61. The function $y = 2^{-x}$ passes through the point $(0, 1)$, whereas the transformed function $y = 5 \cdot 2^{-x}$ passes through the point $(0, 5)$, so you might be tempted to think of the second function as resulting from a vertical shift of the first. However, think about what each looks like for large values of x; both curves have the x-axis as a horizontal asymptote. Therefore the relationship between them cannot be a vertical shift. In particular, the height for every point on the curve $y = 5 \cdot 2^{-x}$ is five times the height of the corresponding point (with the same value for x) on the curve $y = 2^{-x}$. The effect of the constant multiple 5 is to increase the height all along the curve by a factor of 5. If we multiply the original function by 20, the curve will be *stretched* to a new curve that is everywhere 20 times as tall.

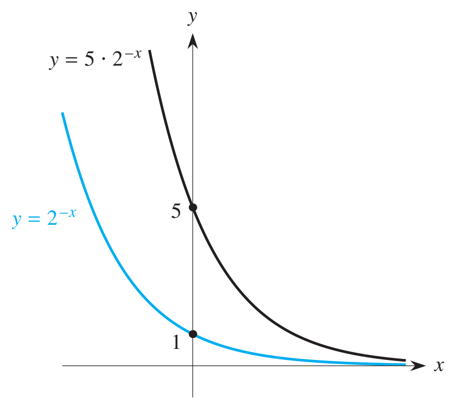


FIGURE 4.61

If instead we multiply the original function by $\frac{1}{4}$, the curve will *shrink* to a new curve that is one fourth the original height for each value of x. Finally, if we multiply the function by a negative constant, such as -3, the curve is stretched by a factor of 3, but it is also flipped upside down across the horizontal axis. Figure 4.62 shows the graphs of $y = \sqrt{x}$ and $y = -3\sqrt{x}$. Not only is the graph of the second function flipped upside down across the x-axis, but it also moves downward much faster (three times as fast) than the first function rises. Verify this result on your function grapher with some other functions.

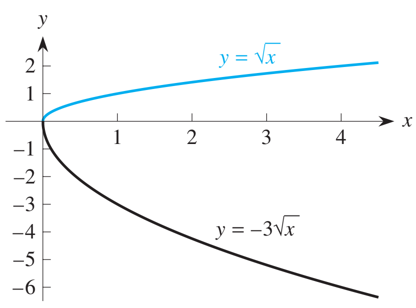


FIGURE 4.62

In general, we have the following principle.

Vertical Stretching and Shrinking

Multiplying a function by a constant changes the height of its graph by that multiple, but it does not change the general shape.

If the multiple is greater than 1, the height is increased.

If the multiple is a number between 0 and 1, the height is decreased.

If the multiple is negative, the curve is flipped over across the horizontal axis.

We illustrate an application of some of these ideas in Example 3.

EXAMPLE 3

Suppose that a chicken is taken from the freezer at 0°F and put directly into an oven kept at a constant temperature of 350°F. After 30 minutes, the temperature of the chicken is 110°F. Construct a function to model the temperature of the chicken as it cooks in the oven.

Solution The temperature of the chicken rises rapidly at first and then increases ever more slowly the closer the chicken's temperature comes to the oven temperature of 350°. Eventually, the temperature of the chicken levels off at the temperature setting for the oven. The temperature T, in °F, plotted against time t, in hours, looks like the graph shown in Figure 4.63. (This description is actually an oversimplification because the temperature rise will temporarily stop at the freezing point of 32° while the ice melts. Also, the chicken should be removed from the oven when its temperature reaches about 180°, or it will begin to burn.)

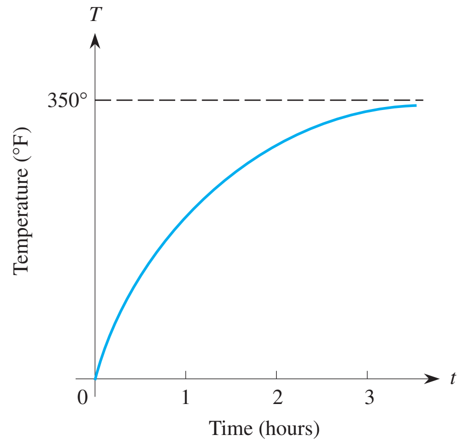


FIGURE 4.63

The horizontal line representing the temperature of 350° is a horizontal asymptote because the curve gets ever closer to this line as time goes by, but never quite reaches it. The rate at which the temperature of the chicken increases slows as it approaches 350° (if we left the chicken in the oven that long), so the curve is concave down.

We want to model this process by creating a formula giving the temperature T as a function of the time t. Simplistically we will find a mathematical model by inspecting the graph of the process and deciding which type of function has the right shape. In Section 5.4 we demonstrate how to construct such a function directly.

The graph in Figure 4.63 appears to be an exponential decay function turned upside down so that it rises toward the oven temperature 350° instead of dropping asymptotically toward the horizontal axis. We can form such a function from a pure exponential function $y = Ac^x$ by using a negative coefficient (to turn the curve upside down) and a vertical shift so that the curve approaches 350 instead of 0. Thus a formula for T might look like

$$T = 350 - Ac^t,$$

where t is in hours and $0 < c < 1$. As t increases, the term c^t approaches 0, and the entire expression $350 - Ac^t$ approaches 350.

What might be possible values for A and c? We know that at time $t = 0$, the chicken's temperature is $T = 0$ when it comes out of the freezer, so

$$T(0) = 350 - Ac^0 = 350 - A = 0.$$

Thus $A = 350$ and the formula becomes

$$T = 350 - 350(c^t).$$

Furthermore, the temperature of the chicken after half an hour is $T(\frac{1}{2}) = 110°$. This value yields

$$T(\tfrac{1}{2}) = 350 - 350(c^{1/2}) = 110.$$

So we have

$$350(c^{1/2}) = 350 - 110 = 240;$$
$$c^{1/2} = 240/350 = 0.686.$$

Squaring both sides of this equation gives

$$c = 0.47.$$

Consequently, our formula for the temperature becomes

$$T = 350 - 350(0.47)^t = 350[1 - (0.47)^t],$$

where t is measured in hours.

This function is an upside down exponential: As t increases, $(0.47)^t$ gets ever smaller, so $1 - (0.47)^t$ increases and gets ever closer to 1. That is, $1 - (0.47)^t \to 1$ as $t \to \infty$. Consequently,

$$T = 350[1 - (0.47)^t] \to 350, \qquad \text{as } t \to \infty,$$

confirming that the graph has a horizontal asymptote at $T = 350$.

Think About This

Verify the behavior of the preceding function on your function grapher. Look at the overall shape and then zoom in to verify the height of the asymptote. Estimate by eye from the graph when T reaches 180°, when it reaches 250°, and when it reaches 300°, 340°, and 349°.

In general, consider the function $y = f(t) = L + Ac^t$, where $0 < c < 1$. We know that as t increases, c^t decays toward zero so that the function approaches a limiting value of L. The question is: How does it approach L—from above or from below? First, whenever $A < 0$, the values of the function are less than L. As the term Ac^t decreases, the amount subtracted from L decreases, and the values of the function increase toward L in a concave down manner, as illustrated by the upper curve in Figure 4.64. This was the case with the temperature of the chicken in the oven. Second, whenever $A > 0$, the values of the function are greater than L and so decrease toward it in a concave up manner, as illustrated by the lower curve in Figure 4.64.

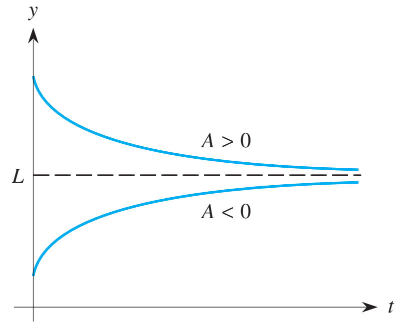


FIGURE 4.64

This type of function is used as the mathematical model for many different real-world processes.

A Constant Multiple of the Independent Variable Finally, we investigtate the effects of multiplying the independent variable x by a constant.

EXAMPLE 4

Consider the cubic function $y = f(x) = x^3 - 12x$ and the related functions $y = f(2x)$, $y = f(4x)$, and $y = f(\frac{1}{2}x)$. What is the effect of the constant multiple in each case?

Solution Figure 4.65 shows the graphs of $y = f(x) = x^3 - 12x$ and $y = f(2x) = (2x)^3 - 12(2x)$. The cubic $y = f(x) = x^3 - 12x$ passes through the origin and has two turning points. If you trace along the curve, you will find that one turning point is at $x = 2$ and the other at $x = -2$. (We could also locate the turning points by using the formula presented in Section 4.4.) The corresponding local maximum (at $x = -2$) is at a height of $y = 16$ and the local minimum (at $x = 2$) is at a height of $y = -16$.

The cubic $y = f(2x) = (2x)^3 - 12(2x) = 8x^3 - 24x$ also passes through the origin and has two turning points, one at $x = 1$ and the other at $x = -1$. The corresponding local maximum is at $y = 16$, and the local minimum is at $y = -16$. Hence the heights are the same; they just occur sooner. In fact, the curve for $y = f(2x)$ traces out the identical vertical values as $f(x)$, but does so twice as fast.

Figure 4.66 shows the graphs of $y = f(x) = x^3 - 12x$ and $y = f(4x) = (4x)^3 - 12(4x) = 64x^3 - 48x$. The local maximum for $y = f(4x)$ now occurs at $x = -\frac{1}{2}$ and the local minimum occurs at $x = \frac{1}{2}$. Again, the same heights are achieved, but the curve $y = f(4x)$ is traced out four times as fast as $y = f(x)$.

Figure 4.67, shows the graphs of $y = f(x) = x^3 - 12x$ and $y = f(\frac{1}{2}x) = (\frac{1}{2}x)^3 - 12(\frac{1}{2}x) = \frac{1}{8}x^3 - 6x$, but we had to extend the window to show the details. The function $y = f(\frac{1}{2}x)$ achieves its local maximum at $x = -4$ and its local minimum at $x = 4$. The curve $y = f(\frac{1}{2}x)$ traces out the identical heights, but does so half as fast as $y = f(x)$.

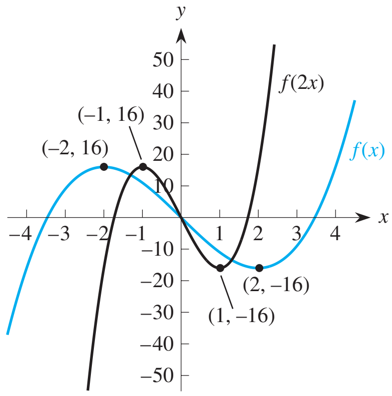


FIGURE 4.65

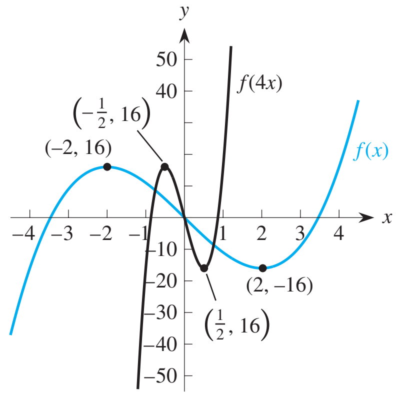


FIGURE 4.66

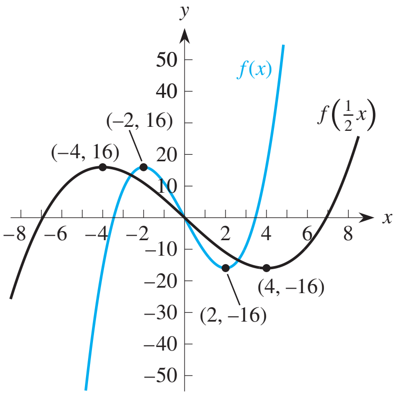


FIGURE 4.67

We summarize the ideas on stretching and shifting functions horizontally as follows.

Horizontal Stretching

Multiplying the independent variable x by a constant k changes the speed at which the graph is traced out, but it does not change the general shape.

If the multiple k is greater than 1, the graph of $y = f(kx)$ is traced out k times faster than $y = f(x)$.

If the multiple k is between 0 and 1, the graph of $y = f(kx)$ is traced out more slowly than $y = f(x)$.

If the multiple k is negative, then the curve $y = f(kx)$ is reflected across the y-axis.

EXAMPLE 5

For the function $f(x) = x^3 - 12x$, draw the graph of $f(-3x)$ and locate its turning points.

Solution Figure 4.68 shows the graphs of the two functions. The graph of $y = f(-3x)$ has the same basic shape as the graph of $y = f(x)$, but is flipped upside down across the

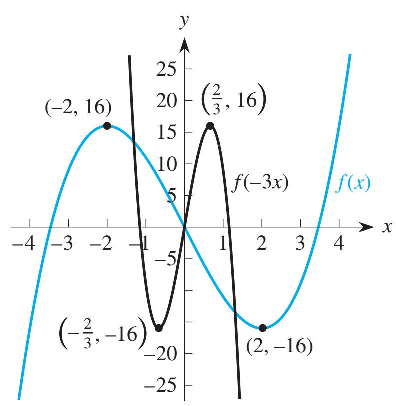


FIGURE 4.68

x-axis and is traced out 3 times as fast. As before, the turning points for $y = f(x)$ are at $(-2, 16)$ and $(2, -16)$. The turning points for $y = f(-3x)$ are at $(-\frac{2}{3}, -16)$, a local minimum, and at $(\frac{2}{3}, 16)$, a local maximum.

EXAMPLE 6

A function f is defined in the following table. Use the values given in the table to complete it. If any entries are not defined, mark them "UNDEF."

x	$f(x)$	$f(x) - 1$	$f(x - 1)$	$f(2x)$	$3f(x)$
0	1				
1	0				
2	3				
3	2				

Solution The values of the function for $x = 0, 1, 2,$ and 3 are defined in the table. The first open column asks for a vertical shift when the function's values are reduced by 1 for each value of x. For instance, when $x = 0$, the first entry is asking for $f(0) - 1 = 1 - 1 = 0$, and so on down that column.

The second open column asks for a horizontal shift of 1 unit to the right, because x is replaced by $x - 1$. Thus, when $x = 0$, we want $f(0 - 1) = f(-1)$, but there is no way to determine this value from the information given in the table; that is, the function is not defined for $x = -1$, so we record it in the table as "UNDEF." However, when $x = 1$, we want $f(1 - 1) = f(0) = 1$, and so on down the column.

The third open column asks for values when the independent variable is doubled. So, when $x = 0$, we need $f(2 \cdot 0) = f(0) = 1$; similarly, when $x = 1$, we need $f(2 \cdot 1) = f(2) = 3$. However, when $x = 2$, $f(2 \cdot 2) = f(4)$, which is not defined. When $x = 3$, $f(2 \cdot 3) = f(6)$ is also not defined.

Finally, the last open column asks for 3 times the value of the function. When $x = 0$, we need $3 \cdot f(0) = 3 \cdot 1 = 3$, and so on down the column. The completed table follows.

x	$f(x)$	$f(x) - 1$	$f(x - 1)$	$f(2x)$	$3f(x)$
0	1	0	UNDEF	1	3
1	0	−1	1	3	0
2	3	2	0	UNDEF	9
3	2	1	3	UNDEF	6

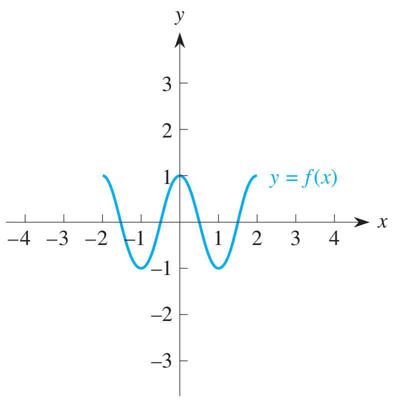
Problems

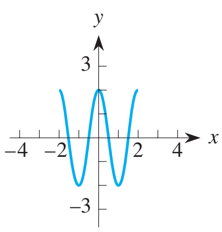
x	$f(x)$	$5f(x)$	$f(x)+3$	$f(x-1)$	$[f(x)]^2$
3	5				
4	2				
5	−1				
6	3				
7	8				

1. A function f is defined in the table above. Use the values given to complete the table. If any of the entries are not defined, write "UNDEF."

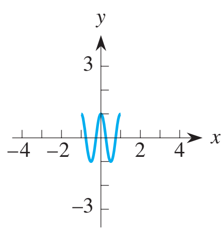
2. A function $y = f(x)$ is defined by the accompanying graph. Match each transformation of f with one of the graphs (i)–(vi).

a. $y = f(2x)$ **b.** $y = 2f(x)$
c. $y = f(x) + 2$ **d.** $y = f(x + 2)$
e. $y = f(x) - 2$ **f.** $y = f(x - 2)$

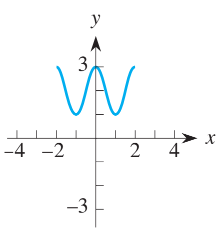




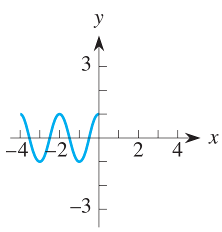
(i)



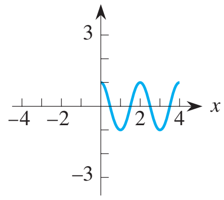
(ii)



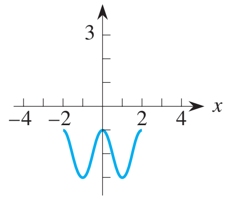
(iii)



(iv)



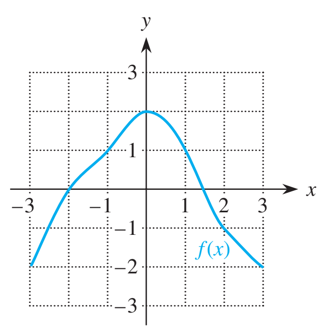
(v)

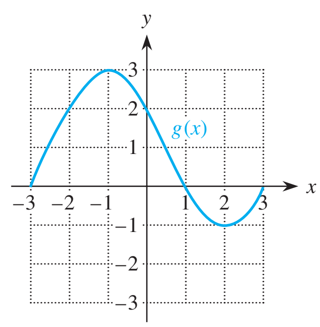


(vi)

3. For the functions f and g that are defined by the graphs shown, sketch the graph of

a. $2g(x)$. **b.** $g(2x)$.
c. $f(x + 1)$. **d.** $f(x - 1)$.
e. $f(x)+1$. **f.** $g\left(\frac{1}{2}x\right)$.





4. Consider the function $y = f(x) = x^2$.

a. Write an equation for the function that you get when you stretch the graph of f by a factor of 2 and then shift it up 3 units. Call this new function F and sketch its graph.

b. What is the equation you get if you reverse the order of the two operations in part (a)? Call this new function G and sketch it.

c. What is $F - G$?

5. a. Translate the line $y = mx$ to a line with slope m that passes through the point $(5, 12)$.

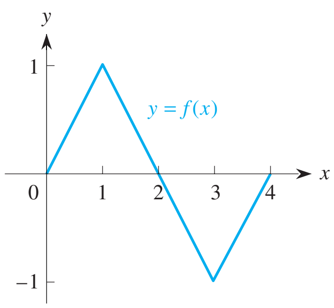
b. Repeat part (a) if the new line passes through the point (x_0, y_0). What do you call this new equation?

6. a. Translate the parabola $y = x^2$ to a parabola with vertex at $(5, 12)$.

b. Repeat part (a) if the new parabola has its vertex at the point (x_0, y_0).

7. For the function f shown, sketch the graph of

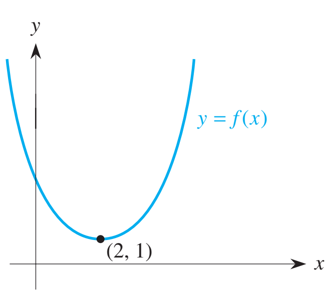
a. $y = -f(x)$ **b.** $y = 2f(x)$
c. $y = f(x) - 1$ **d.** $y = f(x - 1)$
e. $y = f(x + 1)$ **f.** $y = f(x) + 1$



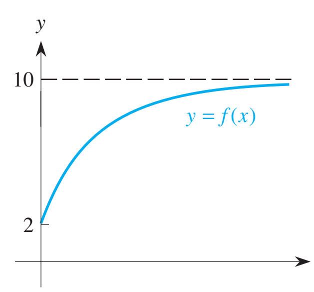
8. The graphs of three functions **(a)–(c)** are shown in the accompanying figures. Sketch the graph of

(i) $y = -f(x)$ **(ii)** $y = 2f(x)$
(iii) $y = -2f(x)$ **(iv)** $y = f(x + 2)$
(v) $y = f(x) + 2$ **(vi)** $y = f(x) - 2$
(vii) $y = f(x - 2)$

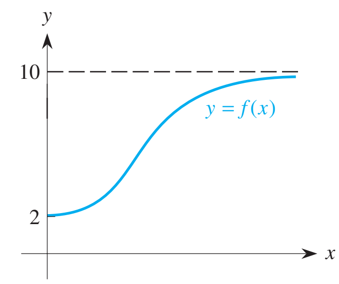
a.



b.



c.



9. Consider the function f in the table. If this function is shifted 4 units to the right and 7 units upward, construct the corresponding table for the transformed function.

x	-1	0	1	2	3
y	24	16	11	8	9
x	4	5	6	7	
y	15	27	39	35	

10. a. Use the graphs of $f(x) = x$ and $g(x) = \log x$ to sketch a rough graph of the product $P(x) = x \log x$.

b. Estimate the values of x for which $\log x < x$ and the values for which $\log x > x$.

c. Because $\log x$ grows exceedingly slowly, the product $x \log x$ grows only slightly faster than x does. Use your function grapher to decide whether $x \log x$ ever grows faster than $x^{1.5}$, than $x^{1.1}$, than $x^{1.05}$. What does this investigation suggest to you about the rate of growth of $x \log x$ compared to power functions x^p?

11. If $f(x) = x^2 - 3x + 4$ and h is a constant, find

a. $f(x) + h$ **b.** $f(x + h)$

c. $f(x + h) - f(x)$ **d.** $\dfrac{f(x + h) - f(x)}{h}$

e. What is the value of the expression in part (d) if $x = 5$ and if $h = 0.1$? if $h = 0.01$? if $h = 0.0001$?

12. a. An unbaked apple pie is taken from the counter in a kitchen where the temperature is 70°F and placed in an oven. Suppose that, after 60 minutes, the temperature of the pie is 180°F. Sketch a graph of the temperature of the pie as a function of time.

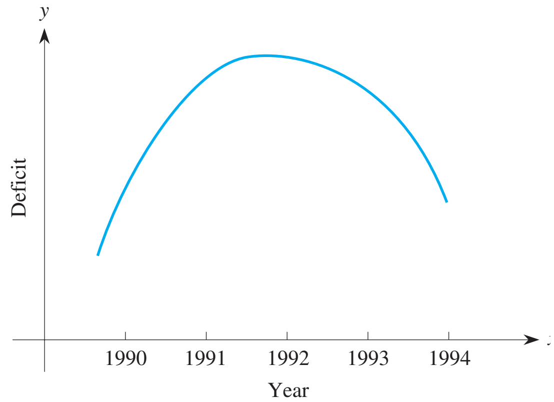
b. The pie is removed from the oven and placed back on the counter. Suppose that it takes another 60 minutes for its temperature to come back down to 70°F. Sketch a graph of the temperature of the pie as a function of time.

c. When the first pie is removed from the oven, a second, unbaked pie is put in the oven to bake. Sketch a graph of the sum of the temperatures of the two pies as a function of time over the 60-minute period.

d. Find a formula that models the temperature of the pie, while it cools, as a function of time.

13. A Thanksgiving turkey is taken from the refrigerator at a temperature of 40°F and placed in a hot oven at 350°F to cook. After 1 hour, the internal temperature of the bird is 124°F. Write a possible formula for the temperature of the turkey as a function of time, in minutes.

14. In an attempt to claim responsibility for winning the war against the growing national balance of trade deficit, the president presented a graph similar to the one shown to illustrate the trend in the *annual* deficit.



a. Based on this graph, sketch the graph of the *total* national debt as a function of time.

b. Does your graph have any points of inflection? If so, what do they represent?

c. Do you agree or disagree with the president's assertion that the war has been won? Explain.

15. Use your function grapher to graph the functions $f(x) = x^n(0.5)^x$, for $n = 1, 2, 3, 4, 5$, and estimate the location of the turning point for each curve for $x > 0$. Then perform a linear regression analysis on the x-values of these turning points, as functions of n. Is the linear fit appropriate? What does it predict for $n = 1.5$? Is it accurate compared to the actual graph?

16. Use your function grapher to graph the functions $f(x) = x^2a^x$, for $a = 0.3, 0.4, 0.5, 0.6$, and 0.7. Estimate the location of the turning point for each curve by zooming in on it. Then determine the function from among the usual families of functions—linear, exponential, and power—that best fits these data as a function of the base a.

17. Describe how you might use the results of Problems 15 and 16 to find a function of the form $f(x) = x^p a^x$ that matches the function for the level of Lyme disease antibody in the bloodstream discussed in Section 4.6 (see Figure 4.50).

18. Find conditions on the coefficients a, b, and c in $P(x) = ax^2 + bx + c$ if P is to satisfy each equation for all values of x.

a. $P(x) = P(-x)$ **b.** $P(x) = -P(x)$
c. $P(2x) = 2P(x)$

Exercising Your Algebra Skills

For the function $f(x) = x^2 - 5x + 3$, find a simplified expression for

1. $f(2x)$. **2.** $f(3x)$. **3.** $f(4x)$. **4.** $f\left(\frac{1}{2}x\right)$.
5. $f(x + 1)$. **6.** $f(x - 2)$.
7. $f(2x - 1)$. **8.** $f(x^2)$.

4.8 Using Shifting and Stretching to Analyze Data

The ideas on shifting and stretching functions in Section 4.7 can be applied to create functions that fit sets of data that do not quite fall into the standard behavior patterns, such as exponential growth or decay, that we have discussed.

Analyzing a Cooling Experiment

Suppose that an experiment is conducted to study the rate at which temperature changes. A temperature probe (a thermometer connected to a calculator) is first heated in a cup of hot water and then removed and placed in a cup of cold water, as illustrated in Figure 4.69. The temperature of the probe, in °C, is measured every second for 36 seconds and recorded in Table 4.3; the data are also displayed in the scatterplot in Figure 4.70.

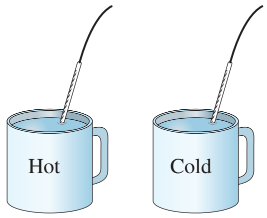


FIGURE 4.69

TABLE 4.3 *Experimental Data: Temperature (°C) versus Time*

Time	Temperature	Time	Temperature	Time	Temperature
1	42.3	13	12.51	25	9.29
2	36.03	14	11.91	26	9.16
3	30.85	15	11.54	27	9.16
4	26.77	16	11.17	28	9.04
5	23.58	17	10.67	29	8.91
6	20.93	18	10.42	30	8.83
7	18.79	19	10.17	31	8.78
8	17.08	20	9.92	32	8.78
9	15.82	21	9.8	33	8.78
10	14.77	22	9.67	34	8.78
11	13.82	23	9.54	35	8.66
12	13.11	24	9.42	36	8.66

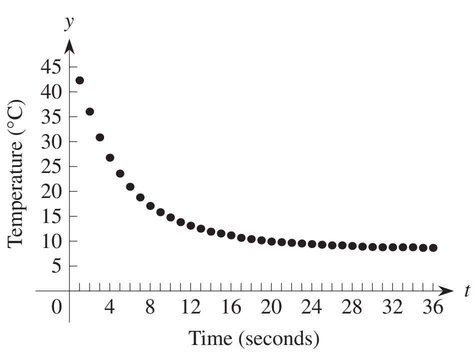


FIGURE 4.70

EXAMPLE 1

Find a function that fits the data from the temperature cooling experiment.

Solution The pattern depicted in Figure 4.70 is that of a decreasing, concave up function, so we might consider either a decaying exponential function or a power function with $p < 0$. However, a power function is not a good model for the process because it has a vertical asymptote at time $t = 0$, whereas the function we want must have a finite value when $t = 0$. So the more appropriate model would be an exponential decay function.

But there is a catch. Any exponential decay function decreases to zero, but the temperature readings decay to the temperature of the cold water (which cannot be 0°C, for then the water would be frozen). From the experimental data, the temperature of the cold water is about 8.6°C. How do we construct a function that decays to about 8.6 rather than to 0? Probably the most reasonable approach is to subtract 8.6 from each of the temperature readings to obtain a new set of data that decays to zero. This approach is equivalent to performing a vertical shift downward of 8.6 (i.e., replacing the temperature T with $T - 8.6$) to produce the transformed data shown in Table 4.4.

TABLE 4.4 *Transformed Data: $(T - 8.6)$ versus Time*

Time	$T - 8.6$	Time	$T - 8.6$	Time	$T - 8.6$
1	33.7	13	3.91	25	0.69
2	27.43	14	3.31	26	0.56
3	22.25	15	2.94	27	0.56
4	18.17	16	2.57	28	0.44
5	14.98	17	2.07	29	0.31
6	12.33	18	1.82	30	0.23
7	10.19	19	1.57	31	0.18
8	8.48	20	1.32	32	0.18
9	7.22	21	1.20	33	0.18
10	6.17	22	1.07	34	0.18
11	5.22	23	0.94	35	0.06
12	4.51	24	0.82	36	0.06

The scatterplot of the transformed data, shown in Figure 4.71, looks like an exponential decay pattern that tends toward 0. Using a calculator, we find that the exponential function that best fits the transformed data is

$$y = T - 8.6 = 35.4394(0.848)^t.$$

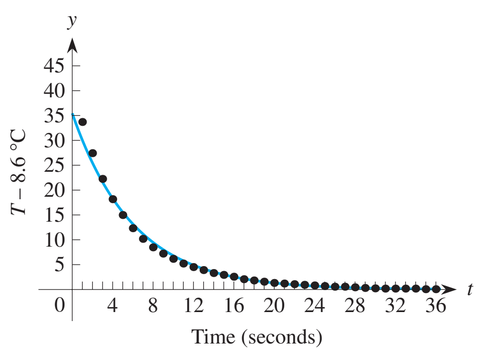


FIGURE 4.71

The graph of this function is shown superimposed over the transformed data in Figure 4.71, and there appears to be extremely close agreement. The corresponding correlation coefficient is $r = -0.9948$, which is very close to -1.

Having found the exponential function that best fits the transformed data, we now have to undo the transformation. We simply add the same amount, 8.6, to the function $y = T - 8.6$ to create the final expression

$$T(t) = 8.6 + 35.4394(0.848)^t.$$

This function is shown superimposed over the original temperature data in Figure 4.72, and it is an exceptionally good fit to the temperature readings. In particular, note how this function approaches the limiting value of about 8.6 for the temperature readings as t increases.

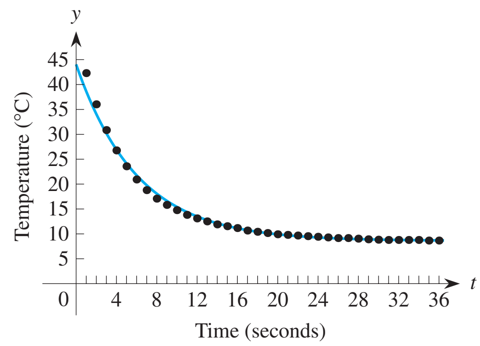


FIGURE 4.72

Analyzing the Challenger Data

At the beginning of Chapter 3, we considered data relating the number of incidents involving O-ring problems on space shuttle launches to the air temperature at launch. These data eventually were used to identify the O-rings as the likely cause of the *Challenger* disaster. We now use this set of data as a case study to illustrate the process of data analysis when it is necessary to shift the data values.

Recall that the dependent variable was the number N of O-ring problems or "incidents" as a function of launch temperature T. The data are shown in the following table.

T	53	57	58	63	66	67	67	67	68	69	70	70
N	3	1	1	1	0	0	0	0	0	0	1	1
T	70	70	72	73	75	75	76	76	78	79	80	81
N	0	0	0	0	2	0	0	0	0	0	0	0

Figure 4.73 shows the scatterplot for this data along with a curve superimposed over the data points to indicate the nature of the relationship, which appears to be a decaying exponential. However, this curve is only an artist's rendering of the apparent relationship. We want to obtain a formula for such a function.

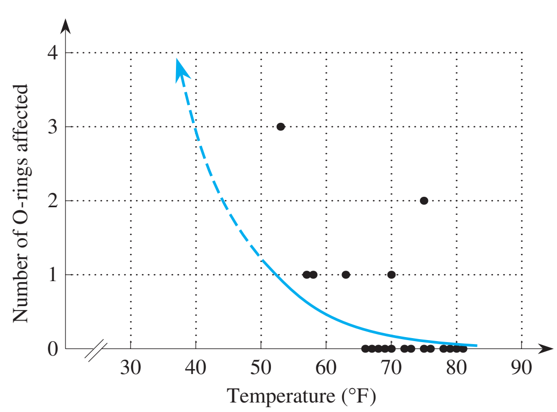


FIGURE 4.73

EXAMPLE 2

Find a function that can model the data on the number of O-ring incidents as a function of the air temperature.

Solution The decreasing, concave up pattern in the scatterplot in Figure 4.73 suggests either a decaying exponential function or a power function with $p < 0$. The power function does not make sense, however, because there is no vertical asymptote at $T = 0$. Fitting an exponential function to the set of data by using the transformation approach used by calculators and spreadsheets involves plotting the logarithm of the number of incidents $\log N$ versus the temperature T. But because the values for N include $N = 0$, we cannot take the logarithm of 0—it is not defined!

One way to circumvent this problem is to shift the data values up to avoid the zeros. The simplest approach is to increase each value of N by 1, replacing N by $N + 1$ and then comparing $N + 1$ to T. We first construct the exponential function that best fits the resulting set of data to obtain the exponential regression equation relating $N + 1$ to T. We then shift back down to obtain an expression for N in terms of T. The data values that we work with are given in the following table, and the associated scatterplot of $N + 1$ versus T is shown in Figure 4.74.

T	N	$N + 1$	T	N	$N + 1$
53	3	4	70	0	1
57	1	2	70	0	1
58	1	2	72	0	1
63	1	2	73	0	1
66	0	1	75	2	3
67	0	1	75	0	1
67	0	1	76	0	1
67	0	1	76	0	1
68	0	1	78	0	1
69	0	1	79	0	1
70	1	2	80	0	1
70	1	2	81	0	1

The resulting exponential regression equation giving $N + 1$ as a function of T is

$$N + 1 = 13.41(0.967)^T,$$

which is shown superimposed on the scatterplot, in Figure 4.74. Finally, we solve for N by subtracting 1 from both sides to get the exponential function that can be used to model N as a function of T:

$$N = 13.41(0.967)^T - 1,$$

which is shown superimposed over the original scatterplot in Figure 4.75. It appears to capture the trend in the data reasonably well.

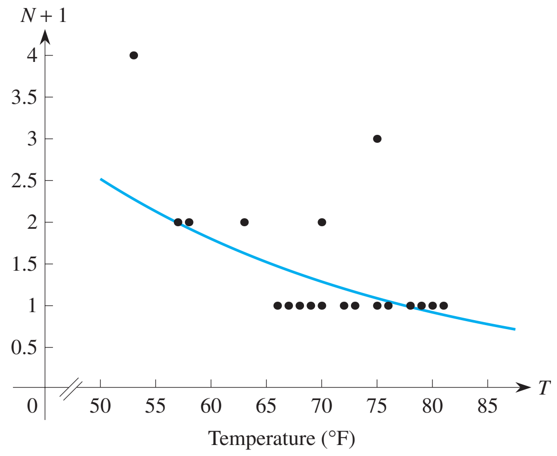


FIGURE 4.74

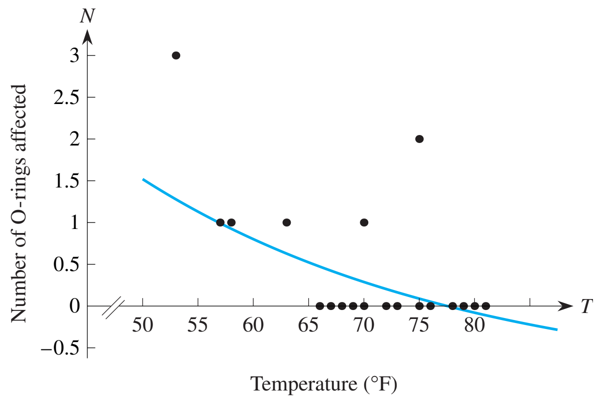


FIGURE 4.75

The graph certainly suggests that the likelihood of trouble with the O-rings will increase dramatically with falling temperature. However, we know that there is a danger in extrapolating far beyond the range of data values. But the overall trend is so dramatic and the potential loss in terms of both human life and hardware is so extreme that there shouldn't have been a launch if the data had been analyzed in this way.

Terminal Velocity in Skydiving

Matthew is a skydiving enthusiast. He knows, from his reading and from first-hand experience, that the faster he is falling, especially in a spread-eagled position, the greater the air resistance, so that eventually his speed reaches a maximum, known as the *terminal velocity.* He also found the following set of data on the downward velocity v, in feet per second, of a skydiver at different times t, in seconds, after jumping out of a plane.

t	0	1	2	3	4	5	6	7	8	9	10	11	12
v	0	16	46	76	104	124	138	148	156	163	167	171	174

Source: Student project.

EXAMPLE 3

(**a**) Find a function that fits these data on velocity as a function of time from among the usual candidates. (**b**) Improve on the fit by using an appropriate shift.

Solution

a. The data falls in an increasing, concave down pattern, as shown in Figure 4.76. The potential candidates for a function having such a pattern are either a power function with $0 < p < 1$ or a logarithmic function. However, a log function is not defined at $t = 0$. Also, both functions grow indefinitely, while the values for the skydiver's velocity approach terminal velocity, which is a horizontal asymptote. What's worse, we don't know what this limiting value for the terminal velocity is. Thus neither function can be a good fit.

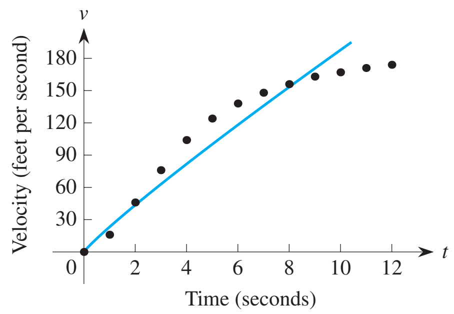


FIGURE 4.76

Moreover, we cannot use a calculator or spreadsheet program to fit a power function to the data because of the first point (0, 0)—their regression routines all use the transformation approach, which involves having to take the log of 0. However, if we delete the point (0, 0), we can fit a power function to the remaining data. Figure 4.76 shows the graph of the best-fit power function $v = 23.2t^{0.908}$ (obtained using a calculator) superimposed over the scatterplot of the data. The corresponding correlation coefficient is $r = 0.962$, which is fairly close to 1. The power function is a reasonable fit, but it clearly becomes less good when extended to the right.

b. The pattern in the data suggests an upside down exponential decay function that rises toward a horizontal asymptote. Suppose that we conjecture a value for the terminal velocity by mentally extending the preceding table. We might extrapolate that the limiting value is about 180 ft/sec. We then subtract each velocity value from this supposed limit (replacing v with $180 - v$) to obtain the transformed data shown in the following table. Effectively, this transformation is equivalent to a vertical shift with a flip across the horizontal axis due to the multiple of -1.

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$180 - v$	180	164	134	104	76	56	42	32	24	17	13	9	6

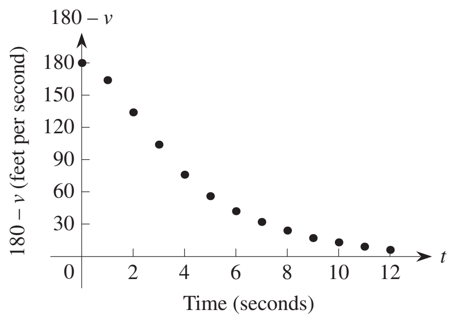


FIGURE 4.77

The scatterplot of these transformed data is shown in Figure 4.77. The decreasing, concave up pattern in this transformed data suggests either a decaying exponential function or a power function with $p < 0$; however, the latter has a vertical asymptote at zero, so it is not an appropriate candidate. A calculator gives the exponential function that best fits this transformed data as

$$y = 180 - v = 226.25(0.7492)^t$$

with a correlation coefficient of $r = -0.9963$. Figure 4.78 shows this function superimposed over the transformed data, and it is a very good fit.

We undo the transformation algebraically by solving for the velocity v to get

$$v = 180 - 226.25(0.7492)^t.$$

The graph of this function is shown superimposed over the original data in Figure 4.79, demonstrating a much better fit than the power function in Figure 4.76. This conclusion is further borne out by the correlation coefficient $r = -0.9963$, which is considerably closer to -1 than the correlation coefficient $r = 0.962$ for the power function was to 1.

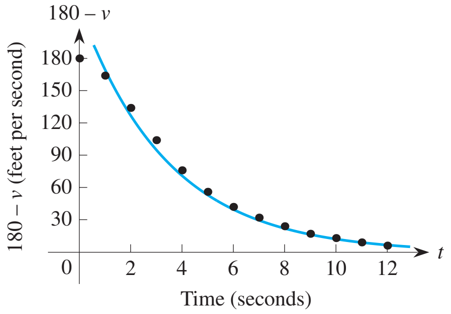


FIGURE 4.78

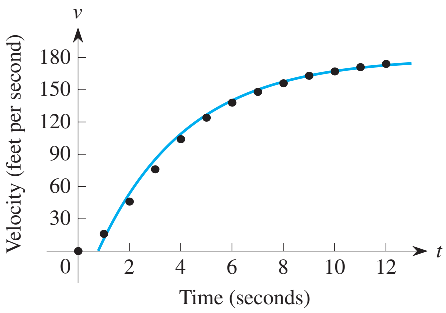


FIGURE 4.79

The value of 180 ft/sec that we chose for the terminal velocity was reasonable, but it was just an intelligent guess. Had we chosen a somewhat different value, we would have obtained a somewhat different function. With a little experimentation, you should be able to get a still better fit.

In Example 3, an exponential function was a very good fit to the transformed data, although the values for $180 - v$ did not fall precisely in an exponential decay pattern. Sometimes, a set of values fall precisely in an exponential pattern as they grow or decay toward a horizontal asymptote. The problem we face in such cases is not knowing exactly what that horizontal asymptote is, as was the case in Example 3. If the transformed data do fall in an exponential pattern, we can determine the limiting value precisely.

Suppose that a set of values $x_0, x_1, x_2, x_3, x_4, \ldots$ is such that the values either fall toward an unknown limiting value L or rise toward L in a purely exponential manner, as shown in Figure 4.80. In particular, suppose that each of the values is below the unknown horizontal asymptote L, so that the set of transformed values

$$L - x_0, \qquad L - x_1, \qquad L - x_2, \qquad L - x_3, \qquad L - x_4, \ldots$$

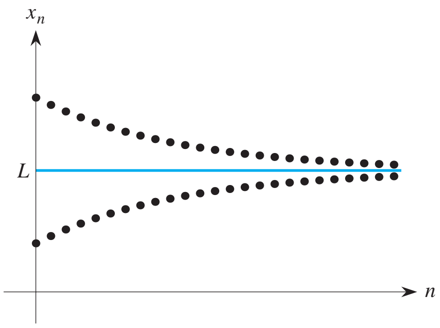


FIGURE 4.80

decays toward zero in an exponential decay pattern. As a result, we know that the successive ratios should be a constant, say k. That is,

$$\frac{L - x_1}{L - x_0} = \frac{L - x_2}{L - x_1} = \frac{L - x_3}{L - x_2} = \cdots = k,$$

where k is the constant, although unknown, ratio. Consider the first of these equalities:

$$\frac{L - x_1}{L - x_0} = \frac{L - x_2}{L - x_1}.$$

We can solve this equation algebraically for the unknown limiting value L by first cross-multiplying to get

$$(L - x_1)^2 = (L - x_0)(L - x_2).$$

Expanding these terms gives

$$L^2 - 2x_1L + x_1^2 = L^2 - x_0L - x_2L + x_0\,x_2.$$

Subtracting L^2 from both sides of this equation and then collecting like terms yields

$$(x_0 - 2x_1 + x_2)L = x_0x_2 - x_1{}^2,$$

so that

$$L = \frac{x_0x_2 - x_1^2}{x_0 - 2x_1 + x_2}, \qquad \textbf{(1)}$$

provided that the denominator $x_0 - 2x_1 + x_2 \neq 0$. In fact, if the numbers x_0, x_1, x_2, x_3, x_4, . . . approach L in an exponentially decaying manner precisely, the comparable expression—using any three successive values of the x's, not just the first three—gives the same value for L. If the values are not exact, however—even if the discrepancies are due to rounding—quite different values could arise with every group of three successive values for the x's.

EXAMPLE 4

Prozac is prescribed for individuals suffering from depression. Typically, a patient takes a dose of Prozac once a day and, for extreme depression, the dosage is 80 mg. The levels of Prozac in the blood on successive days following the start of treatment are given in the following table. (Note that the last two values are rounded to four decimal places.) It turns out (we investigate this result in detail in Section 5.1) that the level of Prozac P rises toward a horizontal asymptote in a precisely upside down exponential decay manner as a function of the number of days n. Find the value of this horizontal asymptote, assuming that the course of treatment continues.

n	0	1	2	3	4	5	6	. . .
P	80	140	185	218.75	244.0625	263.0469	277.2852	. . .

Solution We start with a scatterplot of the data, as shown in Figure 4.81, where the points appear to be approaching a horizontal asymptote at a level somewhat above 300 mg. We call this level L.

These values fall in an upside down decaying exponential pattern as they rise toward the horizontal asymptote, so we can use Equation (1) with the first three values $x_0 = 80$, $x_1 = 140$, and $x_2 = 185$ to find that

$$L = \frac{x_0x_2 - x_1^2}{x_0 - 2x_1 + x_2} = \frac{80(185) - 140^2}{80 - 2(140) + 185} = 320.$$

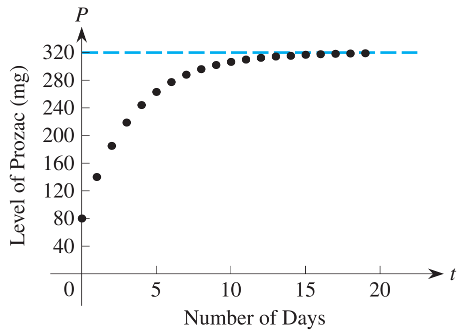


FIGURE 4.81

If instead we use the second, third, and fourth values, so that $x_0 = 140$, $x_1 = 185$, and $x_2 = 218.75$, we obtain

$$L = \frac{x_0x_2 - x_1^2}{x_0 - 2x_1 + x_2} = \frac{140(218.75) - 185^2}{140 - 2(185) + 218.75} = 320.$$

If we use the last three values shown, so that $x_0 = 244.0625$, $x_1 = 263.0469$, and $x_2 = 277.2852$, we obtain, in the same way, $L = 320.0001$. As you will see in Section 5.1 when we develop a complete mathematical model for the level of Prozac in the blood, the limiting value is 320 mg.

Horizontal Shifts

We next consider some applications involving horizontal shifts. As Example 5 demonstrates, that's just what we've been doing when we changed the scale in the independent variable.

EXAMPLE 5

The following data fall in a linear pattern. Determine the line that passes through the points **(a)** when t represents the number of years since 1980; **(b)** when t represents the number of years since 1900; **(c)** when t represents the number of years since year 0. **(d)** Explain how the three expressions compare by using ideas on shifting functions.

t	1980	1985	1990	1995	2000
y	30	40	50	60	70

Solution

a. We rescale the values of the independent variable so that t represents the number of years since 1980.

t	0	5	10	15	20
y	30	40	50	60	70

Note that each 5 years, the value of y increases by 10, so we have a line with slope

$$m = \frac{\Delta y}{\Delta t} = \frac{10}{5} = 2.$$

The equation of the line then is

$$y - 30 = 2(t - 0), \qquad t = \text{number of years since 1980.}$$

b. We now rescale the values in the table so that t represents the number of years since 1900.

t	80	85	90	95	100
y	30	40	50	60	70

These data values also lie on a line whose slope is $m = 2$, so the equation of the line is

$$y - 30 = 2(t - 80), \qquad t = \text{number of years since 1900.}$$

c. Finally, we use the original values given in the table where t represents the number of years since the year 0. The slope is still $m = 2$, so the corresponding equation of the line is

$$y - 30 = 2(t - 1980), \qquad t = \text{number of years since year 0.}$$

d. We now compare the three equations. In each case, the slope is $m = 2$ because all three lines increase at the same rate. If we expand all the equations to put them in slope–intercept form, we get

$$y = 2t + 30, \qquad y = 2t - 130, \quad \text{and} \quad y = 2t - 3930,$$

respectively. Note the great differences in the vertical intercepts for the three lines.

Let's focus on the equation in part (a), $y - 30 = 2t$, as a baseline, where t represents the number of years since 1980. We first compare it to the equation in part (b), $y - 30 = 2(t - 80)$. The second equation is the result of a horizontal shift to the right of 80 years—moving from a "starting point" of $t = 0$ in 1980 to a "starting point" of $t = 0$ in 1900. Similarly, compare the first equation to the third equation in part (c), $y - 30 = 2(t - 1980)$, which involves a horizontal shift of 1980 years to the right. So scaling the values of the independent variable is equivalent to a horizontal shift by the amount of the scaling.

Let's look at a more realistic example to see how these ideas on horizontal shifts apply when we fit an exponential function to data.

EXAMPLE 6

The following table shows the growth, in millions, of cellular phone users since 1985. Find the exponential function that best fits these values **(a)** when t represents the number of years since 1985; **(b)** when t represents the number of years since 1900; **(c)** when t represents the number of years since year 0. **(d)** Explain how the three expressions compare, using ideas on shifting functions.

t	1985	1988	1990	1991	1992	1993	1994	1995	1996	1997	1998
C	1	4	11	16	23	34	55	91	142	215	319

Source: Lester R. Brown et al., *Vital Signs 2000: The Environmental Trends That Are Shaping Our Future.*

Solution

a. We first scale the years so that t represents the number of years since 1985, giving the transformed set of data.

t	0	3	5	6	7	8	9	10	11	12	13
C	1	4	11	16	23	34	55	91	142	215	319

A calculator gives the exponential function that best fits the data as

$$C = 1.063(1.555218)^t, \qquad t = \text{number of years since 1985.}$$

The corresponding correlation coefficient is $r = 0.99925$.

b. We next scale the years in the original data so that t represents the number of years since 1900.

t	85	88	90	91	92	93	94	95	96	97	98
C	1	4	11	16	23	34	55	91	142	215	319

Again, a calculator gives the exponential function that best fits the modified data as

$$C = 5.2999 \times 10^{-17}(1.555218)^t, \qquad t = \text{number of years since 1900.}$$

The corresponding correlation coefficient again is $r = 0.99925$.

c. Finally, the exponential function that best fits the original data where t represents the number of years since the year 0 is

$$C = 2.092 \times 10^{-381}(1.555218)^t, \qquad t = \text{number of years since year 0.}$$

The corresponding correlation coefficient once more is $r = 0.99925$.

d. The growth factor, 1.555218, is the same in all three expressions. It indicates that the use of cellular phones is growing, on average, by 55.5% per year, whichever model we construct. The correlation coefficient $r = 0.99925$ is also the same in all three models. It indicates that the fit in all three cases is equally excellent. Only the constant coefficient changes from one expression to the next, and it reflects the vertical intercept of each curve.

We now look at the equation for the exponential function $C = 1.063(1.555218)^t$ in part (a), where t represents the number of years since 1985. If we perform a horizontal shift of 85 years to the right so that t represents the number of years since 1900, the formula for the function becomes

$$\begin{aligned} C = 1.063(1.555218)^{t-85} &= 1.063(1.555218)^t \cdot (1.555218)^{-85} \qquad b^{u+v} = b^u b^v \\ &= [1.063 \times (1.555218)^{-85}](1.555218)^t \\ &= 5.299988 \times 10^{-17}(1.555218)^t, \end{aligned}$$

which is virtually identical to the expression in part (b).

Similarly, if we perform a horizontal shift of 1985 to the right in the equation in part (a), so that t represents the number of years since the year 0, the formula for the exponential function becomes

$$\begin{aligned} C = 1.063(1.555218)^{t-1985} &= 1.063(1.555218)^t \cdot (1.555218)^{-1985} \qquad b^{u+v} = b^u b^v \\ &= [1.063 \times (1.555218)^{-1985}](1.555218)^t \\ &= 2.0934 \times 10^{-381}(1.555218)^t, \end{aligned}$$

which again is virtually identical to the expression $C = 2.092 \times 10^{-381}(1.555218)^t$ in part (c). (The differences are due to rounding.)

This principle—scaling the values of the independent variable is equivalent to a horizontal shift in the function—applies to the linear, exponential, and polynomial families of functions, but it does *not* apply to power functions. Let's see why.

Recall that an increasing power function always passes through the origin, and that a decreasing power function always has a vertical asymptote at 0. When we fit a power function to a set of increasing data, the origin is automatically added as an extra point. Suppose that we now scale the values of the independent variable x to form a new independent variable X. When we then attempt to fit a power function to the transformed values, a different "origin" is added automatically. This new "origin" for X is much closer to the shifted data set than the origin for the original data was, as illustrated in Figure 4.82.

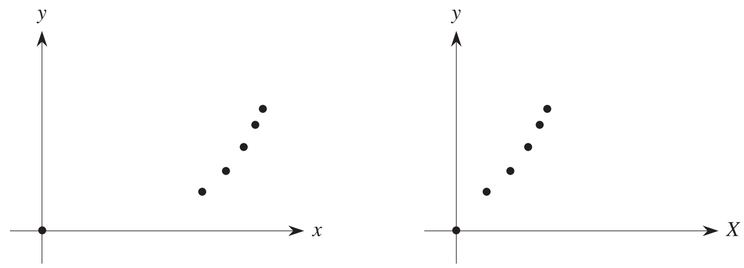


FIGURE 4.82 (a) Original x's (b) Shifted x's

Consider, for instance, the following data.

x	1	2	3	4
y	1	4	9	16

Clearly, these are points on the curve $y = x^2$ and, if we applied a power function regression routine, that is precisely the equation we would get. This curve certainly passes through the origin $(0, 0)$ for the original variable x. It also passes through each of the data points, as shown in Figure 4.83.

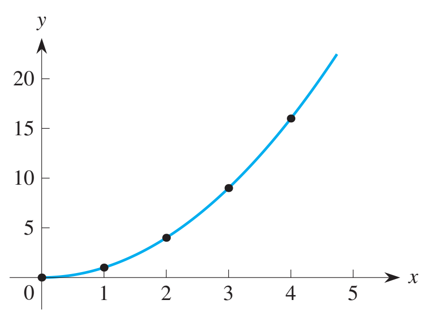


FIGURE 4.83

Let's now shift the data horizontally to the right by 10 units to get the corresponding table of values for the new variable $X = x + 10$.

X	11	12	13	14
y	1	4	9	16

If we apply a power function regression routine, we get the function $y = 1.496 \times 10^{-12}X^{11.43}$. This function passes through the new origin for X, as shown in Figure 4.84. But it misses all the data points. The resulting curve has been flattened enormously to force the new origin to become a point on the graph. As a result, the power for this transformed function is 11.43 rather than 2 for the original function $y = x^2$. Recall that, for power functions $y = x^p$, the higher the power p, the flatter the curve as it passes through the origin. Thus the resulting new power function is distorted compared to the original power function, reflecting the different origin.

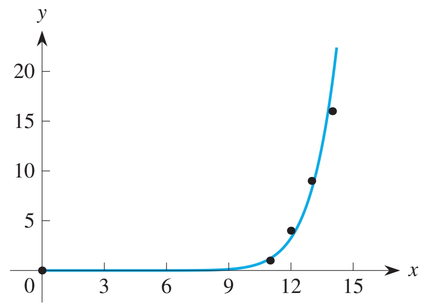


FIGURE 4.84

As we mentioned in Chapter 3, if we use different scales for the independent variable with a power function, not only does the appearance of the function change, but also, and far more important, the predictions based on using the different forms differ. Let's look at what happens when we use the two preceding functions to predict the next value in each table. In the first case, $y = x^2$. When $x = 5$, we get $y = 25$. In the second case, $y = 1.496 \times 10^{-12}X^{11.43}$. The corresponding value of X is $X = x + 10 = 15$, and we get $y = 41.462$, a dramatically different prediction. If we shifted x by other amounts, we would get still different predictions each time. The farther we shift from the original data, the worse this difference gets.

Thus, although power functions are useful, you must use them with extreme caution and careful thought.

Using Stretches

The ideas on stretching functions from Section 4.7 also have direct application when we fit functions to data. In Example 1 in Section 3.3, we created the function

$$P(t) = 3.069(1.321)^t$$

to model the growth of the U.S. population from 1780 to 1900, where P is measured in millions and t is measured in decades since 1780. Actually, we measured P in millions for convenience. If we count the number of people, the corresponding function would then be

$$P(t) = 3{,}069{,}000(1.321)^t.$$

Clearly, these two expressions differ by a factor of 1,000,000, and one function is therefore stretched into the other by this constant multiple of the function.

Moreover, when we created the function for the U.S. population, it was convenient to use t to represent the number of decades since 1780. However, it might be more meaningful to have a function in which the independent variable represents the number of years since 1780 instead. If we use the data values for P from Section 3.3 but count the years $t = 0, 10, 20, \ldots$ rather than decades, we get the function

$$P_1(t) = 3.069(1.02823)^t, \qquad t = \text{number of years since 1780},$$

compared to

$$P_2(t) = 3.069(1.321)^t, \qquad t = \text{number of decades since 1780}.$$

How do these two expressions compare? We know that each decade consists of 10 years, which suggests a constant multiple of 10 for the number of years. So, if we start with the first expression $P_1(t)$ for the population where t is measured in years and multiply the independent variable t by 10, we get

$$\begin{aligned} P_1(10t) &= 3.069(1.02823)^{10t} \\ &= 3.069[(1.02823)^{10}]^t \qquad a^{pu} = (a^p)^u \\ &= 3.069(1.321)^t, \end{aligned}$$

which is the same expression as $P_2(t) = 3.069(1.321)^t$. Whenever we convert the units for the independent variable, from years to centuries, from hours to days, from inches to centimeters, and so on, we actually are stretching or shrinking the function horizontally.

Problems

1. In the analysis of the data on the cooling experiment, we assumed that the water temperature was 8.6°C and so subtracted 8.6 from each of the data values. Assume that the water temperature is 8.65°C instead. Find the corresponding function to fit the original data. Does it appear to be a better or worse fit to the data?
2. Instead of adding 1 to each value of N, as we did with the *Challenger* data in this section, suppose that you add some other quantity (say, 2) to each value. How do the results compare to those obtained earlier?
3. A cup of hot coffee at 200°F is left on the table in a 70°F room to cool. The temperature readings on the coffee at different times as it cools to 70°F are as follows.

Time, t	0	5	10	15	20
Temperature, T	200	163	139	118	108

Find the exponential function that best fits the data.

4. While watching his VCR, Ken noticed that the counter seems to move much faster near the beginning of the tape than toward the end of the tape, so he knows that the readings are not linear. To find the actual pattern, he records the counter reading every 15 minutes and obtains the following set of data relating the counter reading to the elapsed time, in hours.

Time	0	0.25	0.50	0.75	1.0	
Reading	0	445	817	1162	1448	
Time	1.25	1.5	1.75	2.0	2.25	2.5
Reading	1732	2005	2260	2503	2721	2942

a. From among exponential, power, and logarithmic functions, find the function that best fits the data giving the VCR counter reading in terms of the elapsed time.

b. Using the function from part (a), what would you predict the reading to be after 3 hours?

c. Suppose that the label on a VCR tape indicates that a certain program Ken recorded runs from 1600 through 3400 on the counter. How long will that program run?

d. Suppose that the VCR tape is a 6-hour tape. Programs already recorded end at a counter reading of 4200. How much time is left on the tape for the next recording?

5. In Problem 23 of Section 3.3 we looked at how the boiling point (the temperature T) of water in a confined space (say, in a pressure cooker) depends on the pressure of the vapor water. The table gives the boiling point of water, in degrees Celsius, at various vapor pressures, in kilo-pascals.

a. From a scatterplot of the data of T versus P, it appears that the boiling point of water approaches a horizontal asymptote as the pressure P increases. This behavior might suggest an upside down exponential function of the form $y = A + Bc^t$, with $c < 1$. Assume that the horizontal asymptote is at $T = 110°$. Use this value to transform the data and find the corresponding exponential function.

b. Use your function from part (a) to find the boiling point of water when the vapor pressure is 6.2 kilo-pascals.

c. What vapor pressure is needed if the boiling point of the water is 105°C?

Pressure, P	0.61	1.22	2.34	4.25	7.38	12.34	19.93	31.18	47.37	70.12	101.32
Temperature, T	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°	100°

Source: *CRC Handbook of Chemistry and Physics,* 1996.

4.9 The Logistic and Surge Functions

In this section, we consider two other families of functions—the logistic and the surge functions—that frequently arise as mathematical models in a wide variety of applications.

The Logistic Function

A great many processes start out growing exponentially, but eventually other factors come into play to slow the rate of growth, causing a leveling off, as shown in Figure 4.85. Most populations grow in this manner, and many diseases spread in a comparable pattern. The use of new technological innovations, be they new electronic devices such as microwave ovens, cellular phones, or DVD players and new medical products, also spread this way. Such a pattern is called a **logistic curve,** and the associated function is called a **logistic function.**

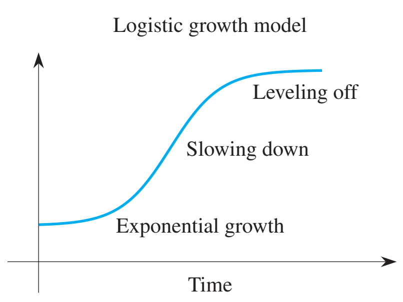


FIGURE 4.85

Logistic processes can be modeled mathematically in several different ways, and we look at one in considerable detail in Section 5.3. For now, we consider the family of functions of the form

$$f(t) = \frac{C}{1 + Ae^{-Bt}},$$

where A, B, and C are positive constants and $e = 2.71828\ldots$ is the base of the natural logarithm system that we introduced in Section 2.6. In most practical situations, the constant A typically is very large, the constant C is fairly large, and the constant B is usually between 0 and 1. Let's first analyze the behavior of this family of functions.

This function actually is the quotient of two functions, so we have to reason in the same way that we analyzed the behavior of rational functions in Section 4.6. In particular, because the numerator is a positive constant, the function has no real roots and thus never crosses the t-axis. Also, in the denominator, both the constant A and the exponential decay function e^{-Bt} are positive, so the denominator is never zero and the function has no vertical asymptotes. Furthermore, when t is negative or when t is positive and relatively small, the term Ae^{-Bt} is extremely large compared to 1. Thus the denominator behaves like $1 + Ae^{-Bt} \approx Ae^{-Bt}$, and therefore the function $f(t)$ behaves like

$$f(t) = \frac{C}{1 + Ae^{-Bt}} \approx \frac{C}{Ae^{-Bt}} = \frac{Ce^{Bt}}{A}.$$

At first (when t is small) this function grows like an exponential function: To the left, it approaches 0 as $t \to -\infty$, and to the right, as t increases, it is increasing and concave up. As t gets larger, however, the term e^{-Bt} decays toward 0, so that the function behaves as if

$$f(t) = \frac{C}{1 + Ae^{-Bt}} \approx \frac{C}{1 + A \cdot 0} = C,$$

which is a constant. That is, the function eventually (when t is larger) grows more slowly, so there is an inflection point. Beyond that point, the curve levels off and approaches a limiting value at the height of C. Thus this type of function has the shape shown in Figure 4.85 and so is called a *logistic function.* In Figure 4.86 we show the graph of the function

$$f(t) = \frac{500}{1 + 200e^{-0.5t}}.$$

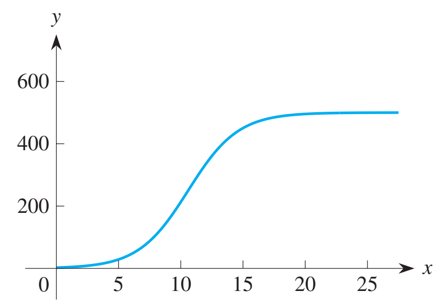


FIGURE 4.86

It has the shape of a logistic curve, eventually leveling off at a height of about 500.

In Example 1 of Section 3.3, we found that the growth in the U.S. population from 1780 to 1900 closely followed an exponential growth pattern with a growth rate of 32.1% per decade. Corresponding to the best fit exponential curve, we had a correlation coefficient of $r = 0.998$. However, we pointed out that this exponential pattern doesn't apply during the twentieth century because the growth rate has slowed dramatically for various reasons. This behavior suggests that a logistic function may be a better choice than an exponential function for modeling the U.S. population over the entire time period since 1780.

EXAMPLE 1

(**a**) Find a logistic function to fit the following data on the growth of the U.S. population, in millions, since 1780. Let t represent the number of decades since 1780. (**b**) What does the function predict about the eventual maximum population of the United States? (**c**) Use the function to predict the U.S. population in 2020.

Year	Population	Year	Population
1780	2.8	1900	76.0
1790	3.9	1910	92.0
1800	5.3	1920	105.7
1810	7.2	1930	122.8
1820	9.6	1940	131.7
1830	12.9	1950	150.7
1840	17.1	1960	179.3
1850	23.2	1970	203.3
1860	31.4	1980	226.5
1870	39.8	1990	248.7
1880	50.2	2000	281.4
1890	62.9		

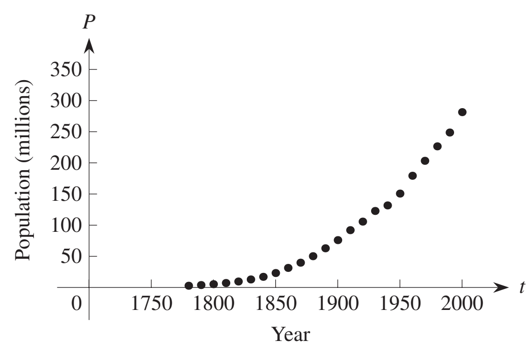


FIGURE 4.87

Solution

a. We begin with the scatterplot of the data shown in Figure 4.87, which indicates that population growth appeared to slow during the latter part of the twentieth century. The successive ratios of the population values also indicate that the rate of population growth slowed from over 20% per decade at the beginning of the twentieth century to about 10% per decade at the end.

We now want to fit a logistic curve to these data. Some calculators have the capability of fitting the best logistic function of the form discussed here to a set of data in the least squares sense. When we use this routine on the U.S. population values, we get the function

$$y = P(t) = \frac{659.45}{1 + 92.05e^{-0.198t}}.$$

This function, superimposed over the population data in Figure 4.88, appears to be an excellent fit to the data.

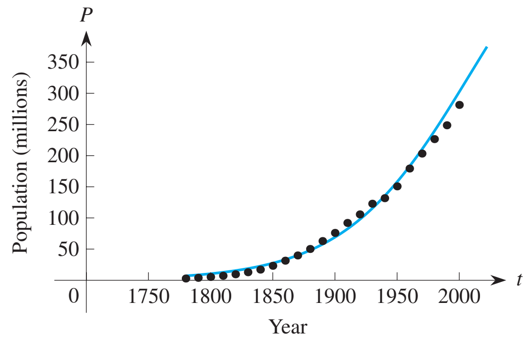


FIGURE 4.88

b. To find the limiting population predicted by this logistic function, we have to determine what happens as $t \to \infty$. As t increases, the term $e^{-0.198t}$ approaches 0, so that the quotient approaches 659.45 million people.

c. Based on this model, the population in 2020, when $t = 24$ decades, will be

$$P(24) = \frac{659.45}{1 + 92.05e^{-0.198(24)}} \approx 367.42 \text{ million people.}$$

The Surge Function

Picture what happens when a medication is first administered to a patient. The effective level of the drug in the bloodstream initially is zero. The drug level then rises rapidly toward a maximum as it is absorbed into the blood. Finally, the drug level decays slowly as it is washed out of the body by the kidneys that filter impurities from the blood. The overall pattern has the shape shown in Figure 4.89. Similarly, a new advertising campaign produces an immediate jump in sales, but

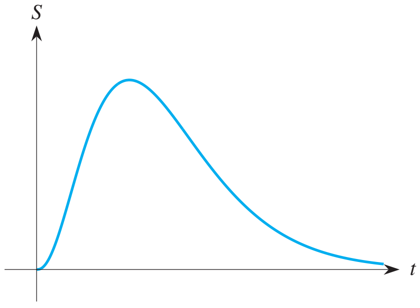


FIGURE 4.89

then the effects of the ad campaign tend to die out slowly over time. The resulting pattern can also be represented by a curve like that shown in Figure 4.89.

Both of these processes are examples of a **surge function,** which can be written as

$$S(t) = At^p b^t,$$

where A, p, and $b < 1$ are three parameters. For realistic situations, we consider only $t \geq 0$. This formula for a surge function actually is the product of a power function t^p and an exponential decay function b^t because $b < 1$. For instance, Figure 4.90 shows the graph of the surge function $S(t) = 100t^{2.5}(0.75)^t$.

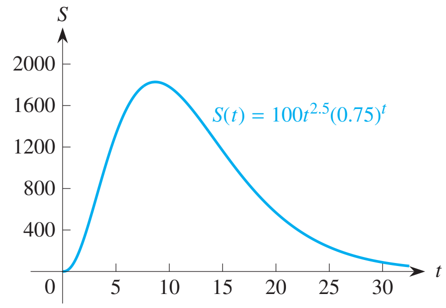


FIGURE 4.90

The coefficient A determines the maximum height of the curve. For the surge function shown in Figure 4.90, this maximum is slightly more than 1800. The power function term t^p reflects the initial impetus and, in fact, the power p determines the location of the maximum value of the function. For this surge function, the maximum occurs at about $t = 8.5$. The decaying exponential term b^t is responsible for the eventual slow decay. Also, remember that an exponential function dominates any power function for large t so that, in the product of the two functions, the exponential decay eventually overwhelms the growth in the power function term.

EXAMPLE 2

The drug L-dopa is administered to people suffering from Parkinson's disease to relieve symptoms such as extreme tremors and rigidity. To be effective, fairly high doses are required because only a small portion of a dose actually lasts in the body long enough to be effective. The side effects of the large doses can be reduced by administering another drug in conjunction with L-dopa. The following table shows the level of L-dopa L in the blood, in nanograms per milliliter, as a function of time t, in minutes.

t	0	20	40	60	80	100	120	140	160	180	200	220	240	300	360
L	0	300	2700	2950	2600	1550	1100	900	725	600	510	440	300	250	225

A plot of these points is shown in Figure 4.91, which suggests the pattern for a surge function. Find the equation of a surge function that models the data.

Solution The plot of the data indicates that the surge function reaches its maximum at about $t = 60$, where the maximum value is approximately 3000. The function also has two points of inflection, one on either side of the peak. From the table of data, the greatest increase in L occurs between $t = 20$ and $t = 40$, so we estimate that one inflection

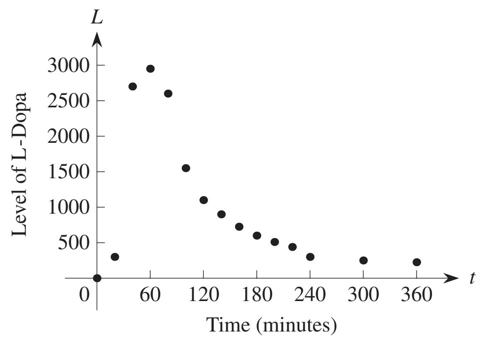


FIGURE 4.91

point occurs at $t = 30$, say. The greatest decrease in L occurs between $t = 80$ and $t = 100$, so we estimate that the other inflection point occurs at about $t = 90$.

We write $L(t) = At^p b^t$ as the general equation of a surge function, where A, p, and b are the three parameters whose values we have to determine. Unfortunately, routines to find these parameters are not built into any calculators or directly into any software packages, so we have to find an indirect way of estimating their values. To do so we apply a transformation approach similar to that used in Sections 3.4 and 3.5. Thus, if $L(t) = At^p b^t$, when we take logs of both sides, we get

$$\begin{aligned} \log L &= \log(At^p b^t) = \log A + \log t^p + \log b^t & \log(u \cdot v \cdot w) &= \log u + \log v + \log w \\ &= \log A + p \log t + t \log b. & \log(u^p) &= p \log u \end{aligned}$$

Therefore, if L is a surge function of t, $\log L$ is a linear function of t and of $\log t$. Thus, we extend the preceding table to include values for $\log t$ and $\log L$.

t	0	20	40	60	80	100	120	140	160	180	200	220	240	300	360
L	0	300	2700	2950	2600	1550	1100	900	725	600	510	440	300	250	225
log *t*	UNDEF	1.3	1.60	1.78	1.90	2.00	2.08	2.1	2.2	2.2	2.3	2.3	2.3	2.4	2.56
log *L*	UNDEF	2.4	3.43	3.47	3.41	3.19	3.04	2.9	2.8	2.7	2.7	2.6	2.4	2.4	2.35

Note that the first entries for $\log t$ and $\log L$ are marked UNDEF because the logarithmic function is not defined.

Because $\log L$ is a linear function of both t and $\log t$, we can use the values from this table in a program that performs multivariate linear regression, as discussed in Section 3.6. The linear function that best fits these data is

$$\mathtt{Y = 1.5004 - 0.00667X_1 + 1.1591X_2},$$

where $\mathtt{X_1} = t$ and $\mathtt{X_2} = \log t$. The regression equation is equivalent to

$$\log L = 1.5004 - 0.00667t + 1.1591 \log t.$$

We undo the transformation, as we did in Sections 3.4 and 3.5, by taking powers of 10 on both sides of this equation:

$$\begin{aligned} 10^{\log L} = L &= 10^{1.5004 - 0.00667t + 1.1591 \log t} & 10^{\log u} &= u \\ &= (10^{1.5004})(10^{-0.00667t})(10^{1.1591 \log t}) & 10^{u+v} &= 10^u \cdot 10^v \\ &= (31.65)(10^{-0.00667})^t \cdot (10^{\log t^{1.1591}}) & \log u^p &= p \log u \\ &= 31.65(0.9848)^t t^{1.1591}. & 10^{\log u} &= u \end{aligned}$$

Therefore our model for the surge function is

$$L(t) = 31.65t^{1.1591}(0.9848)^t.$$

The graph of this function superimposed over the original data set for the level of L-dopa in the blood is shown in Figure 4.92. It is not a particularly good fit to the data, especially for t between 0 and about 100 minutes. It reflects the overall pattern in the data but not very accurately. Moreover, the corresponding coefficient of multiple determination is $R^2 = 0.6876$, so $R = 0.8292$, which is reasonably close to 1. It is close enough for there to be a significant level of correlation.

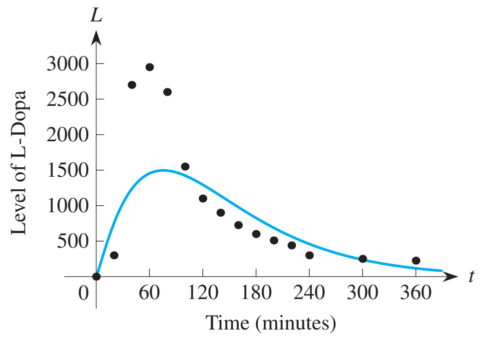


FIGURE 4.92

Problems

1. The growth pattern in human height or weight development from birth through age 18, say, usually follows a logistic growth pattern. The table gives the typical height, in centimeters, of a male and a female in the 50th percentile for height at different ages, in years.
   a. From the table, estimate the typical height of full grown males and females in the 50th percentile (assuming full growth occurs by age 18).
   b. If you have access to a calculator or software package that fits a logistic function to a set of data, find a pair of logistic functions that can be used to model the heights of both males and females as a function of age t for those in this 50th percentile group.
   c. What do the formulas from part (b) predict about the typical heights of full grown males and females in the 50th percentile?

Age	Males	Females
0	50.5	49.9
1	76.1	74.3
2	87.6	86.5
3	96.5	95.6
4	102.9	101.6
5	109.9	108.4
6	116.1	114.6
7	125.0	120.6
8	127.0	126.4
9	132.2	132.2
10	137.5	138.3
11	143.3	144.8
12	149.7	151.5
13	156.5	157.1
14	163.1	160.4
15	169.0	161.8
16	173.5	162.4
17	176.2	163.1
18	176.8	163.7

Source: U.S. Department of Health, Education, and Welfare, *NCHS Growth Curves for Children, Vital and Health Statistics, National Health Survey.* Washington, D.C.: U.S. Government Printing Office.

2. Sweden has one of the longest collections of census records of any country. The table to the right shows the Swedish population, in millions, starting in 1750 when $t = 0$ through 1920 when $t = 170$.

   a. From the table, estimate the population of Sweden at the point of inflection when $t = 110$.

   b. If you have access to a calculator or software package that fits a logistic function to a set of data, find a logistic function that can be used to model the population of Sweden as a function of time t.

   c. What does the formula from part (b) predict about the maximum possible population of Sweden?

   d. Consult the population table in Appendix G to see how well the logistic function you found in part (b) predicts the actual population in 2002.

t	Population
0	1.763
10	1.893
20	2.030
30	2.118
40	2.158
50	2.347
60	2.378
70	2.585
80	2.888
90	3.139
100	3.483
110	3.800
120	4.168
130	4.566
140	4.785
150	5.136
160	5.522
170	5.9004

Source: Raymond Pearl, *The Biology of Population Growth.* New York: Knopf, 1925.

3. Consider the surge function $S(t) = 100t^{2.5}(0.75)^t$ (see Figure 4.90). Without using your function grapher, predict how the graph of each surge function (a)–(d) compares to this function in terms of the location of the turning point and the rate at which the function decays to 0.

   a. $f(t) = 100t^3(0.75)^t$

   b. $f(t) = 100t^2(0.75)^t$

   c. $f(t) = 100t^{2.5}(0.70)^t$

   d. $f(t) = 100t^{2.5}(0.80)^t$

Chapter Summary

In this chapter, we introduced several additional families of functions and ways to build new functions out of old functions. More specifically, we described:

- How quadratic, cubic, quartic, and higher degree polynomials behave.
- How the real roots of a polynomial equation relate to the linear factors.
- How the real roots of a polynomial equation relate to the graph.
- How the number of turning points and the number of inflection points relate to the degree of a polynomial.
- How the end behavior of a polynomial depends on the sign of the leading coefficient.
- How to find the real roots of a polynomial graphically, numerically, and—in the case of quadratic functions—algebraically.

- How polynomial functions arise as models in the real world.
- How to fit polynomial functions to sets of data.
- The relative frequency with which complex roots occur.
- How to interpret the higher order differences of a set of numbers to determine polynomial patterns in sets of data.
- How to find the sum of the first n integers and the sum of the squares of the first n integers.
- What it means to add, subtract, or multiply functions to form new functions.
- The behavior of rational functions.
- What it means to have a function of a function.
- The effects of shifting, stretching, and shrinking on a function.
- How to interpret shifting and stretching of functions in terms of fitting functions to data.
- How logistic and surge functions behave.
- How to use logistic and surge functions as models.

Review Problems

Sketch the graph of each function without using your function grapher.

1. $f(x) = (x + 3)(x - 2)(x - 4)$

2. $g(x) = (2 - x)(x + 3)(x + 1)$

3. $F(x) = (x + 2)(x - 3)(x - 4)(x - 1)$

4. $G(x) = (x + 3)(x - 2)(x - 4)^2$

Factor each polynomial to determine its roots algebraically.

5. $P(x) = x^2 + x - 6$ **6.** $Q(x) = 2x^2 + 9x - 5$

7. $R(x) = x^3 - 3x^2 + 2x$

8. Use the quadratic formula to verify your answers to Problems 5–7.

9. A quadratic function f has its vertex at the point $(5, 19)$, and $f(8) = 4$. What is $f(2)$?

10. A cubic function f has its inflection point at $(6, -4)$, and $f(2) = 12$. What is $f(10)$?

11. Estimate the location of the turning points of the graph of the function $y = x^3 + 4x^2 - 5$.

12. Determine the graphs of each pair of functions f and g and use them to draw the graph of $f + g$.

a. $f(x) = x^2 - 5$, $g(x) = 3x + 2$
b. $f(x) = 2x^3 + 4$, $g(x) = x^2$

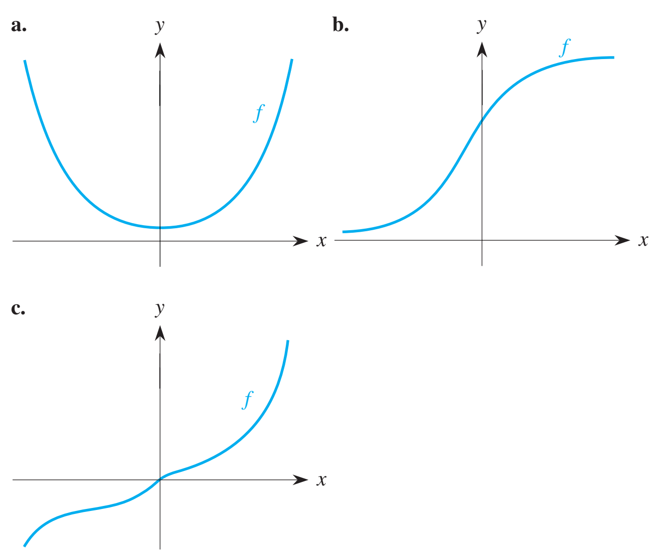
13. Analyze the behavior of each rational function including identifying all zeros, vertical asymptotes, and end behavior as x approaches ∞ and $-\infty$. Estimate all turning points graphically.

a. $R(x) = \dfrac{x^2 - 4}{x^2 + 9}$ **b.** $Q(x) = \dfrac{x^2 - 4}{x^2 - 9}$

c. $S(x) = \dfrac{x^2 + 4}{x^2 + 9}$ **d.** $T(x) = \dfrac{x^2 + 4}{x^2 - 9}$

14. For each function shown, sketch the graph of

i. $-f(x)$ **ii.** $3f(x)$
iii. $f(x) - 4$ **iv.** $f(x - 3)$
v. $f(x + 3)$ **vi.** $-f(x - 4)$



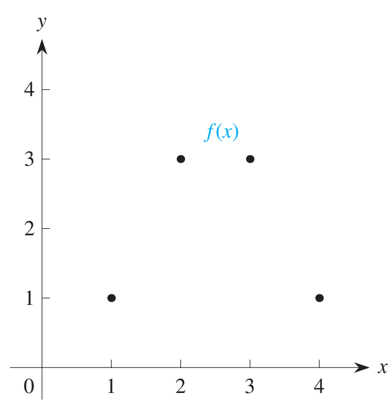
15. Suppose that $f(x) = 2x^2 + 1$ and $g(x) = (x - 1)/(x + 2)$. Find the following.

a. $f(3) + g(3)$ **b.** $f(f(3))$
c. $g(f(3))$ **d.** $g(g(3))$
e. $g(3)f(3)$ **f.** $\dfrac{f(3)}{g(3)}$
g. $f(g(x))$ **h.** $f(f(x))$
i. $g(f(x))$ **j.** $g(g(x))$
k. $g(x)f(x)$ **l.** $\dfrac{f(x)}{g(x)}$

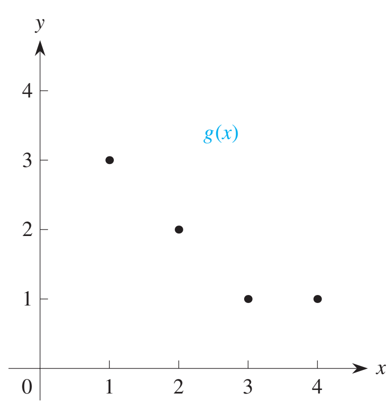
16. Suppose that $f(0) = 2$, $f(1) = 2$, $f(2) = 3$, $f(3) = 0$ and that $g(0) = 1$, $g(1) = 0$, $g(2) = 2$, $g(3) = 3$. Find the following quantities for $x = 0$, 1, 2, and 3.

a. $f(g(x))$ **b.** $g(f(x))$
c. $f(x) + g(x)$ **d.** $\dfrac{f(x)}{g(x)}$

17. Repeat Problem 16(a)–(d) for the functions f and g shown in the graphs below for $x = 1, 2, 3$, and 4.



(i)



(ii)

18. The return in dollars on an investment seems to be well approximated by the function $F(t) = 2t^2 + t + 4.2$, whereas the return on another investment is modeled by $G(t) = 7.8t + 3.5$. Determine for which values of $t > 0$ the second investment is better than the first.

19. Evaluate the sum

$$3 + 6 + 9 + 12 + 15 + \cdots + 300.$$

20. A polynomial has four turning points.

a. How many inflection points must it have? Explain.

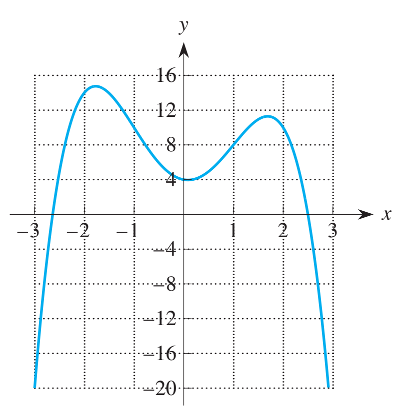
b. What is the minimum degree of the polynomial?

c. What is the minimum number of real roots that the polynomial can have? Explain your answer with a sketch of a polynomial to illustrate what can happen.

d. What is the maximum number of real roots that the polynomial can have? Explain your answer with a sketch of a polynomial to illustrate what can happen.

e. Are there any other values for the number of real roots between the minimum number in part (c) and the maximum number in part (d) that the polynomial can have? Explain your answer with a sketch of a polynomial to illustrate what can happen.

21. The accompanying figure shows the graph of a fourth degree polynomial. Use regression methods to find a possible formula for this polynomial.



22. The table that follows gives some values, rounded to the nearest integer, for a rational function.

a. Sketch a possible graph of this rational function $R(x)$.

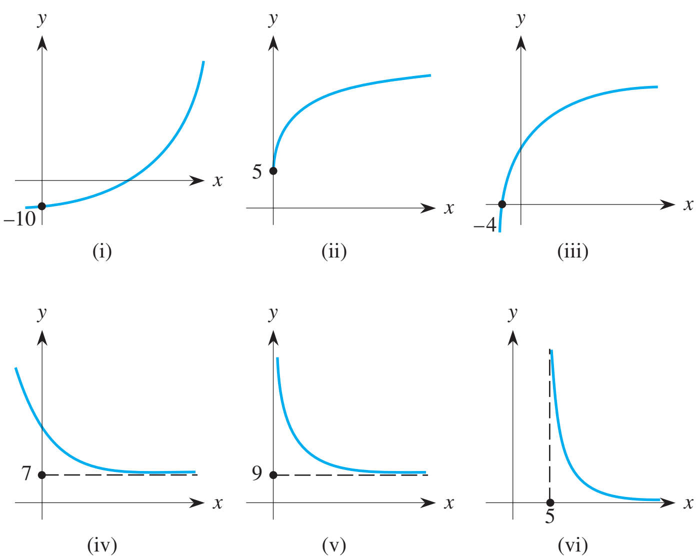
b. Find a possible formula for this rational function.

x	−4	−3	−2	−1	0	1
***R*(*x*)**	10	0	UNDEF	0	−3	0
x	2	3	4	5	6	7
***R*(*x*)**	UNDEF	0	18	UNDEF	0	21

23. Each function shown in the accompanying figure can be interpreted as a shift applied to an exponential, a power, or a logarithmic function.

  a. Identify which is which.

  b. Write a possible formula for each function.



24. The table gives the total number of cell phone subscribers, in millions, in the United States since 1990 and the average local monthly bill, in dollars, for cell phone service.

  a. Find the exponential growth function that best fits the data on the number of subscribers as a function of time since 1990.

  b. Find the exponential decay function that best fits the data on the average monthly bill as a function of time since 1990.

  c. The total industry revenue each year is the product of the number of subscribers and the average monthly bill for service. Use the results of parts (a) and (b) to write a function that models the total cell phone revenue as a function of time since 1990. What is the growth or decay factor for this revenue function?

  d. Extend the table to include a row that gives the total annual revenue in the cell phone industry. Then find the exponential function that best fits the data on the annual revenue as a function of the number of years since 1990. How does this result compare to the one you found in part (c)?

Year	1990	1991	1992	1993	1994	1995	1996	1997	1998	1999
Subscribers	4.37	6.38	8.89	13.07	19.28	28.15	38.20	48.71	60.83	76.28
Average bill	83.94	74.56	68.51	67.31	58.65	52.45	48.84	43.86	39.88	40.24

Source: *2000 Statistical Abstract of the United States.*