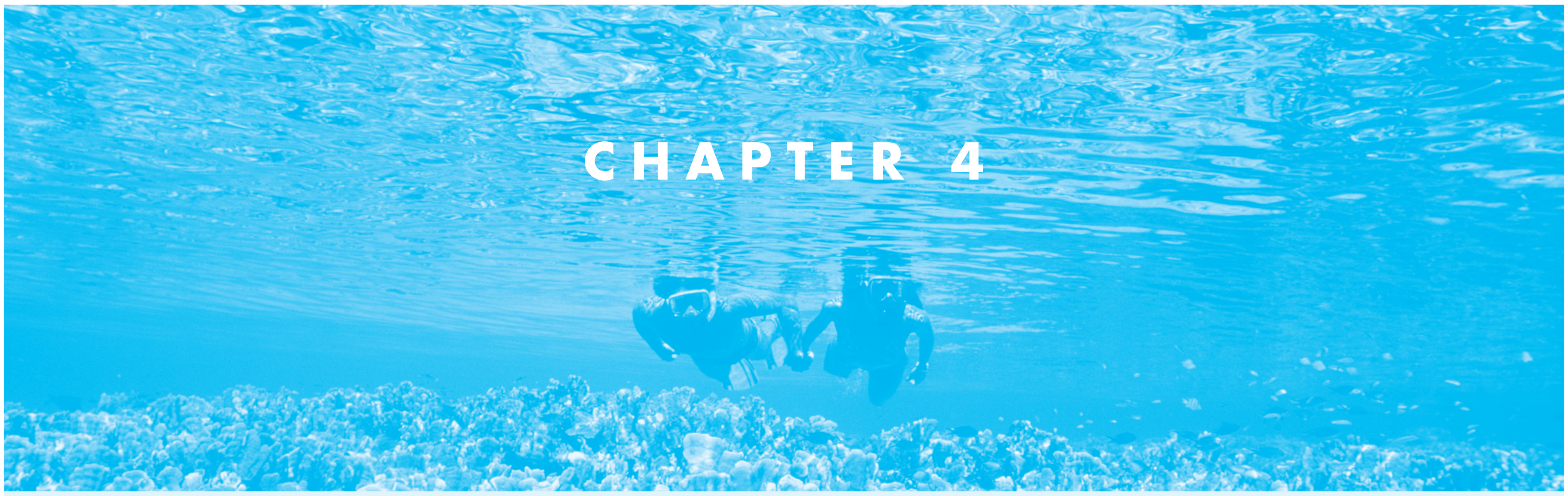


CHAPTER 4

THE LAWS OF EXPONENTS AND LOGARITHMS: MEASURING THE UNIVERSE

OVERVIEW

Most of the examples we've studied so far have come from the social sciences. In order to delve into the physical and life sciences, we need to compactly describe and compare the extremes in deep time and deep space. In this chapter, we introduce the tools that scientists use to represent very large and very small quantities.

After reading this chapter, you should be able to

- write expressions in scientific notation
- convert between English and metric units
- simplify expressions using the rules of exponents
- compare numbers of widely differing sizes
- calculate logarithms base 10 and plot numbers on a logarithmic scale

4.1 *The Numbers of Science: Measuring Time and Space*

On a daily basis we encounter quantities measured in tenths, tens, hundreds, or perhaps thousands. Finance or politics may bring us news of "1.3 billion people living in China" or "a federal debt of over $7 trillion." In the physical sciences the range of numbers encountered is much larger. *Scientific notation* was developed to provide a way to write numbers compactly and to compare the sizes found in our universe, from the largest object we know—the observable universe—to the tiniest—the minuscule quarks oscillating inside the nucleus of an atom. We use examples from deep space and deep time to demonstrate powers of 10 and the use of scientific notation.

Powers of 10 and the Metric System

The international scientific community and most of the rest of the world use the *metric system,* a system of measurements based on the meter (which is about 39.37 inches, a little over 3 feet). In daily life Americans have resisted converting to the metric system and still use the *English system* of inches, feet, and yards. Table 4.1 shows the conversions for three standard metric units of length: the meter, the kilometer, and the centimeter. For a more complete conversion table, see the inside back cover.

Conversions from Metric to English for Some Standard Units

Metric Unit	Abbreviation	In Meters	Equivalent in English Units	Informal Conversion
meter	m	1 m	3.28 ft	The width of a twin bed, a little more than a yard
kilometer	km	1000 m	0.62 mile	A casual 12-minute walk, a little over half a mile
centimeter	cm	0.01 m	0.39 in	The length of a black ant, a little under half an inch

Table 4.1

Deep space

The Observable Universe. Current measurements with the most advanced scientific instruments generate a best guess for the radius of the observable universe at about 100,000,000,000,000,000,000,000,000 meters, or "one hundred trillion trillion meters." Obviously, we need a more convenient way to read, write, and express this number. To avoid writing a large number of zeros, exponents can be used as a shorthand:

For an appreciation of the size of things in the universe, we highly recommend the video by Charles and Ray Eames and related book by Philip and Phylis Morrison titled *Powers of Ten: About the Relative Size of Things.*

10^{26} can be written as a 1 with twenty-six zeros after it.

10^{26} means: $10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10$, the product of twenty-six 10s.

10^{26} is read as "10 to the twenty-sixth" or "10 to the twenty-sixth power."

So the estimated size of the radius of the observable universe is 10^{26} meters. The sizes of other relatively large objects are listed in Table 4.2.[1]

The Relative Sizes of Large Objects in the Universe

Object	Radius (in meters)
Milky Way	$1{,}000{,}000{,}000{,}000{,}000{,}000{,}000 = 10^{21}$
Our solar system	$1{,}000{,}000{,}000{,}000 = 10^{12}$
Our sun	$1{,}000{,}000{,}000 = 10^{9}$
Earth	$10{,}000{,}000 = 10^{7}$

Table 4.2

[1]The rough estimates for the sizes of objects in the universe in this section are taken from Timothy Ferris, *Coming of Age in the Milky Way* (New York: Doubleday, 1988).

Us. Human beings are roughly in the middle of the scale of measurable objects in the universe. Human heights, including children's, vary from about one-third of a meter to 2 meters. In the wide scale of objects in the universe, a rough estimate for human height is 1 meter.

To continue the system of writing all sizes using powers of 10, we need a way to express 1 as a power of 10. Since $10^3 = 1000$, $10^2 = 100$, and $10^1 = 10$, a logical way to continue would be to say that $10^0 = 1$. Since reducing a power of 10 by 1 is equivalent to dividing by 10, the following calculations give justification for defining 10^0 as equal to 1.

$$10^2 = \frac{10^3}{10} = \frac{(10)(10)(\cancel{10})}{\cancel{10}} = 100$$

$$10^1 = \frac{10^2}{10} = \frac{(10)(\cancel{10})}{\cancel{10}} = 10$$

$$10^0 = \frac{10^1}{10} = \frac{\cancel{10}}{\cancel{10}} = 1$$

By using negative exponents, we can continue to use powers of 10 to represent numbers less than 1. For consistency, reducing the power by 1 should remain equivalent to dividing by 10. So, continuing the pattern established above, we define $10^{-1} = 1/10$, $10^{-2} = 1/10^2$, and so on. For any positive integer, n, we define

$$10^{-n} = \frac{1}{10^n}$$

DNA Molecules. A DNA strand provides genetic information for a human being. It is made up of a chain of building blocks called nucleotides. The chain is tightly coiled into a double helix, but stretched out it would measure about 0.01 meter in length. How does this DNA length translate to a power of 10? The number 0.01, or one-hundredth, equals $1/10^2$. We can write $1/10^2$ as 10^{-2}. So a DNA strand, uncoiled and measured lengthwise, is approximately 10^{-2} meters, or one centimeter.

Table 4.3 shows the sizes of some objects relative to the size of human beings.

The Relative Sizes of Small Objects in the Universe

Object	Radius (in meters)	
Human beings	$1 = \frac{10}{10} =$	10^0
DNA molecules	$0.01 = \frac{1}{100} = \frac{1}{10^2} =$	10^{-2}
Living cells	$0.000\ 01 = \frac{1}{100{,}000} = \frac{1}{10^5} =$	10^{-5}
Atoms	$0.000\ 000\ 000\ 1 = \frac{1}{10{,}000{,}000{,}000} = \frac{1}{10^{10}} =$	10^{-10}

Table 4.3

The following box gives the definition for various powers of 10.

Powers of 10

When n is a positive integer:

$10^n = \underbrace{10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10}_{n \text{ factors}}$ which can be written as 1 followed by n zeros.

$10^0 = 1$

$10^{-n} = \frac{1}{10^n}$ which can be written as a decimal point followed by $n-1$ zeros and a 1.

Multiplying by 10^n is equivalent to moving the decimal point to the right n places.

Multiplying by 10^{-n} is equivalent to dividing by 10^n, or moving the decimal point to the left n places.

The metric language

By international agreement, standard prefixes specify the power of 10 that is attached to a specific unit of measure. They indicate the number of times the basic unit has been multiplied or divided by 10. Usually these prefixes are attached to metric units of measure, but they are occasionally used with the English system. Table 4.4 gives prefixes and their abbreviations for certain powers of 10. A more complete table is on the inside back cover.

Prefixes for Powers of 10

pico-	p	10^{-12}	(unit)		10^0
nano-	n	10^{-9}	kilo-	k	10^3
micro-	μ	10^{-6}	mega-	M	10^6
milli-	m	10^{-3}	giga-	G	10^9
centi-	c	10^{-2}	tera-	T	10^{12}

Table 4.4

EXAMPLE 1 Indicate the number of meters in each unit of measure: cm, mm, Gm.

SOLUTION

$$1 \text{ cm} = 1\ \textit{centi}\text{meter} = 10^{-2} \text{ m} = \frac{1}{10^2} \text{ m} = \frac{1}{100} \text{ m} = 0.01 \text{ meter}$$

$$1 \text{ mm} = 1\ \textit{milli}\text{meter} = 10^{-3} \text{ m} = \frac{1}{10^3} \text{ m} = \frac{1}{1000} \text{ m} = 0.001 \text{ meter}$$

$$1 \text{ Gm} = 1\ \textit{giga}\text{meter} = 10^9 \text{ m} = 1{,}000{,}000{,}000 \text{ meters}$$

EXAMPLE 2 Translate the following underlined expressions.

A standard CD holds about 700 *mega*bytes of information.

Translation: $700 \cdot 10^6$ bytes or 700,000,000 bytes

A calculator takes about one *milli*second to add or multiply two 10-digit numbers.

Translation: $1 \cdot 10^{-3}$ second or 0.001 second

In Tokyo on January 11, 1999, the NEC company announced that it had developed a *pico*second pulse emission, optical communications laser.

Translation: $1 \cdot 10^{-12}$ second or 0.000 000 000 001 second

It takes a New York City cab driver one *nano*second to beep his horn when the light changes from red to green.

Translation: $1 \cdot 10^{-9}$ second or 0.000 000 001 second

Scientific Notation

In the previous examples we estimated the sizes of objects to the nearest power of 10 without worrying about more precise measurements. For example, we used a gross estimate of 10^7 meters for the measure of the radius of Earth. A more accurate measure is 6,368,000 meters. This number can be written more compactly using *scientific notation* as

$$6.368 \cdot 10^6 \text{ meters}$$

The number 6.368 is called the *coefficient*. The absolute value of the coefficient must always lie between 1 and 10. The power of 10 tells us how many places to shift the decimal point of the coefficient in order to get back to standard decimal form. Here, we would multiply 6.368 times 10^6, which means we would move the decimal place six places to the right, to get 6,368,000 meters.

Any nonzero number, positive or negative, can be written in scientific notation, that is, written as the product of a coefficient N multiplied by 10 to some power, where $1 \leq |N| < 10$. Thus 2 million, 2,000,000, and $2 \cdot 10^6$ are all equivalent representations of the same number. The one you choose depends on the context.

In the following examples, you'll learn how to write numbers in scientific notation. Later we'll use scientific notation to simplify operations with very large and very small numbers.

Scientific Notation

A number is in *scientific notation* if it is in the form

$N \cdot 10^n$ where

N is called the *coefficient* and $1 \leq |N| < 10$

n is an integer

EXAMPLE 3 The distance to Andromeda, our nearest neighboring galaxy, is 15,000,000,000,000,000,000,000 meters. Express this number in scientific notation.

SOLUTION The coefficient needs to be a number between 1 and 10. We start by identifying the first nonzero digit and then placing a decimal point right after it to create the coefficient of 1.5. The original number written in scientific notation will be of the form

$$1.5 \cdot 10^?$$

What power of 10 will convert this expression back to the original number? The original number is larger than 1.5, so the exponent will be positive. If we move the decimal place 22 places to the right, we will get back 1.5,000,000,000,000,000,000,000.

This is equivalent to multiplying 1.5 by 10^{22}. So, in scientific notation, 15,000,000,000,000,000,000,000 is written as

$$1.5 \cdot 10^{22}$$

EXAMPLE 4 The radius of a hydrogen atom is 0.000 000 000 052 9 meter across. Express this number in scientific notation.

SOLUTION The coefficient is 5.29. The original number written in scientific notation will be of the form

$$5.29 \cdot 10^?$$

What power of 10 will convert this expression back to the original number? The original number is smaller than 5.29, so the exponent will be negative. If we move the decimal place 11 places to the left, we will get back 0.000 000 000 05.2 9. This is equivalent to dividing 5.29 by 10^{11} or multiplying it by 10^{-11}:

$$\begin{aligned} 0.000\,000\,000\,052\,9 &= \frac{5.29}{10^{11}} \\ &= 5.29\left(\frac{1}{10^{11}}\right) \\ &= 5.29 \cdot 10^{-11} \end{aligned}$$

This number is now in scientific notation.[2]

EXAMPLE 5 Express $-0.000\,000\,000\,052\,9$ in scientific notation.

SOLUTION In this case the coefficient, -5.29, is negative. Notice that the absolute value of the coefficient, $|-5.29|$, is equal to 5.29, which is between 1 and 10. In scientific notation, $-0.000\,000\,000\,052\,9$ is written as

$$-5.29 \cdot 10^{-11}$$

Converting from Standard Decimal Form to Scientific Notation

Place a decimal point to the right of the first nonzero digit, creating the coefficient N, where

$$1 \le |N| < 10.$$

Determine n, the power of 10 needed to convert the coefficient back to the original number.

Write in the form $N \cdot 10^n$, where the exponent n is an integer.

Examples: $346{,}800{,}000 = 3.468 \cdot 10^8$ $\quad 0.000\,008\,4 = 8.4 \cdot 10^{-6}$

The poem "Imagine" offers a creative look at the Big Bang.

Deep time

The Big Bang. In 1929 the American astronomer Edwin Hubble published an astounding paper claiming that the universe is expanding. Most astronomers and cosmologists now agree with his once-controversial theory and believe that approximately 13.7 billion years ago the universe began an explosive expansion from an infinitesimally small point. This event is referred to as the "Big Bang," and the universe has been expanding ever since it occurred.[3] Scientific notation can be used to record the progress of the universe since the Big Bang Theory, as shown in Table 4.5.

The Tale of the Universe in Scientific Notation

Object	Age (in years)
Universe	13.7 billion = 13,700,000,000 = $1.37 \cdot 10^{10}$
Earth	4.6 billion = 4,600,000,000 = $4.6 \cdot 10^9$
Human life	100 thousand = 100,000 = $1.0 \cdot 10^5$

Table 4.5

[2]Most calculators and computers automatically translate a number into scientific notation when it is too large or small to fit into the display. The notation is often slightly modified by using the letter E (short for "exponent") to replace the expression "times 10 to some power." So $3.0 \cdot 10^{26}$ may appear as 3.0 E+26. The number after the E tells how many places, and the sign (+ or −) indicates in which direction to move the decimal point of the coefficient.

[3]Depending on its total mass and energy, the universe will either expand forever or collapse back upon itself. However, cosmologists are unable to estimate the total mass or total energy of the universe, since they are in the embarrassing position of not being able to find about 90% of either. Scientists call this missing mass *dark matter* and missing energy *dark energy,* which describes not only their invisibility but also the scientists' own mystification.

Carl Sagan's video *Cosmos* and book *Dragons of Eden* condense the life of the universe into one calendar year.

Table 4.5 tells us that humans, *Homo sapiens sapiens,* first walked on Earth about 100,000 or $1.0 \cdot 10^5$ years ago. In the life of the universe, this is almost nothing. If all of time, from the Big Bang to today, were scaled down into a single year, with the Big Bang on January 1, our early human ancestors would not appear until less than 4 minutes before midnight on December 31, New Year's Eve.

Algebra Aerobics 4.1

1. Express as a power of 10:
- **a.** 10,000,000,000
- **b.** 0.000 000 000 000 01
- **c.** 100,000
- **d.** 0.000 01

2. Express in standard notation (without exponents):
- **a.** 10^{-8} **c.** 10^{-4}
- **b.** 10^{13} **d.** 10^{7}

3. Express as a power of 10 and then in standard notation:
- **a.** A nanosecond in terms of seconds
- **b.** A kilometer in terms of meters
- **c.** A gigabyte in terms of bytes

4. Rewrite each measurement in meters, first using a power of 10 and then using standard notation:
- **a.** 7 cm **b.** 9 mm **c.** 5 km

5. Avogadro's number is $6.02 \cdot 10^{23}$. A mole of any substance is defined to be Avogadro's number of particles of that substance. Express this number in standard notation.

6. The distance between Earth and its moon is 384,000,000 meters. Express this in scientific notation.

7. An angstrom (denoted by Å), a unit commonly used to measure the size of atoms, is 0.000 000 01 cm. Express its size using scientific notation.

8. The width of a DNA double helix is approximately 2 nanometers, or $2 \cdot 10^{-9}$ meter. Express the width in standard notation.

9. Express in standard notation:
- **a.** $-7.05 \cdot 10^{8}$ **c.** $5.32 \cdot 10^{6}$
- **b.** $-4.03 \cdot 10^{-5}$ **d.** $1.021 \cdot 10^{-7}$

10. Express in scientific notation:
- **a.** −43,000,000 **c.** 5,830
- **b.** −0.000 008 3 **d.** 0.000 000 024 1

11. Express as a power of 10:
- **a.** $\frac{1}{100,000}$
- **b.** $\frac{1}{1,000,000,000}$

Exercises for Section 4.1

1. Write each expression as a power of 10.
- **a.** $10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$
- **b.** $\frac{1}{10 \cdot 10 \cdot 10 \cdot 10 \cdot 10}$
- **c.** one billion
- **d.** one-thousandth
- **e.** 10,000,000,000,000
- **f.** 0.000 000 01

2. Express in standard decimal notation (without exponents):
- **a.** 10^{-7} **c.** -10^{8} **e.** 10^{-3}
- **b.** 10^{7} **d.** -10^{-5} **f.** 10^{5}

3. Express each in meters, using powers of 10. (See inside back cover.)
- **a.** 10 cm **c.** 3 terameters
- **b.** 4 km **d.** 6 nanometers

4. Express each unit using a metric prefix. (See inside back cover.)
- **a.** 10^{-3} seconds
- **b.** 10^{3} grams
- **c.** 10^{2} meters

5. Computer storage is often measured in gigabytes and terabytes. Write these units as powers of 10.

6. Express each of the following using powers of 10.
- **a.** 10,000,000,000,000 **d.** $\frac{1}{10 \cdot 10 \cdot 10 \cdot 10}$
- **b.** 0.000 000 000 001 **e.** one million
- **c.** $10 \cdot 10 \cdot 10 \cdot 10$ **f.** one-millionth

7. Write each of the following in scientific notation:
- **a.** 0.000 29 **d.** 0.000 000 000 01 **g.** −0.0049
- **b.** 654.456 **e.** 0.000 002 45
- **c.** 720,000 **f.** −1,980,000

8. Why are the following expressions *not* in scientific notation? Rewrite each in scientific notation.
- **a.** $25 \cdot 10^{4}$ **c.** $0.012 \cdot 10^{-2}$
- **b.** $0.56 \cdot 10^{-3}$ **d.** $-425.03 \cdot 10^{2}$

9. Write each of the following in standard decimal form:
- **a.** $7.23 \cdot 10^{5}$ **d.** $1.5 \cdot 10^{6}$
- **b.** $5.26 \cdot 10^{-4}$ **e.** $1.88 \cdot 10^{-4}$
- **c.** $1.0 \cdot 10^{-3}$ **f.** $6.78 \cdot 10^{7}$

10. Express each in scientific notation. (Refer to the chart in Exploration 4.1.)

a. The age of the observable universe

b. The size of the first living organism on Earth

c. The size of Earth

d. The age of Pangaea

e. The size of the first cells with a nucleus

11. Determine if the expressions are true or false. If false, change the right-hand side to make the expression true.

a. $0.00\,756 = 7.56 \cdot 10^{-2}$

b. $3.432 \cdot 10^5 = 343{,}200$

c. 49 megawatts $= 4.9 \cdot 10^6$ watts

d. $1{,}596{,}000{,}000 = 1.5 \cdot 10^9$

e. 5 megapixels $= 5.0 \cdot 10^6$ pixels

f. 6 picoseconds $= 6.0 \cdot 10^{12}$ seconds

12. Express each quantity in scientific notation.

a. The mass of an electron is about 0.000 000 000 000 000 000 000 000 001 67 gram.

b. One cubic inch is approximately 0.000 016 cubic meter.

c. The radius of a virus is 0.000 000 05 meter.

13. Evaluate:

a. $|9|$ **b.** $|-9|$ **c.** $|-1000|$ **d.** $-|-1000|$

14. Determine the value of each expression.

a. $|-5 - 3|$ **c.** $|2 - 6|$

b. $|6 - 2|$ **d.** $-2|-1+3| + |-5|$

15. Determine which statements are true.

a. $|a - b| = |b - a|$

b. $|-7a| = 7a$

c. $2|-1+4| = 2|-1|+2|4|$

d. $|-2p| = |-2| \cdot |p|$

16. What values for x would make the following true?

a. $|x| = 7$ **c.** $|x - 2| = 7$ **e.** $|2 - x| = 7$

b. $|x - 1| = 5$ **d.** $|2x| < 0$ **f.** $|2x| = 8$

17. Substitute the value $x = 5$ into the statement. Then replace the ? with the sign ($>$, $<$, or $=$) that would make the statement true. Then repeat for $x = -5$.

a. $|x - 1|$ **?** 5 **c.** $|x - 1|$ **?** 0 **e.** $|2x - 1|$ **?** 11

b. $2|3-x|$ **?** 10 **d.** $|-x|$ **?** 4 **f.** $|-x|$ **?** 6

18. a. Generate a small table of values and plot the function $y = |x|$ for $-5 \le x \le 5$.

b. On the same graph, plot the function $y = |x - 2|$.

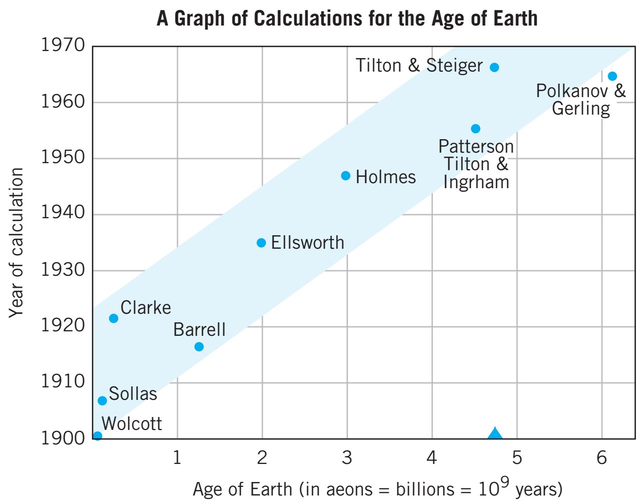
19. The accompanying amusing graph shows a roughly linear relationship between the "scientifically" calculated age of Earth and the year the calculation was published. For instance, in about 1935 Ellsworth calculated that Earth was about 2 billion years old. The age is plotted on the horizontal axis and the year the calculation was published on the vertical axis. The triangle on the horizontal coordinate represents the presently accepted age of Earth.

a. Who calculated that Earth was less than 1 billion years old? Give the coordinates of the points that give this information.

b. In about what year did scientists start putting the age of Earth at over a billion years? Give the coordinates that represent this point.

c. On your graph sketch an approximation of a best-fit line for these points. Use two points on the line to calculate the slope of the line.

d. Interpret the slope of that line in terms of the year of calculation and the estimated age of Earth.



Source: American Scientist, Research Triangle Park, NC. Copyright © 1980.

4.2 *Positive Integer Exponents*

> **SOMETHING TO THINK ABOUT**
> ? *What can we say about the value of $(-1)^n$ when n is an even integer? When n is an odd integer?*

No matter what the base, whether it is 10 or any other number, repeated multiplication leads to *exponentiation*. For example,

$$3 \cdot 3 \cdot 3 \cdot 3 = 3^4$$

Here 4 is the *exponent* of 3, and 3 is called the *base*. In general, if a is a real number and n is a positive integer, then we define a^n as the product of n factors of a.

Definition of a^n

In the expression a^n, the number a is called the *base* and n is called the *exponent* or *power*.

If n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} \qquad \text{(the product of } n \text{ factors of } a)$$

Exponent Rules

In this section we'll see how the rules for manipulating expressions with exponents make sense if we remember what the exponent tells us to do to the base. First we focus on cases where the exponents are positive integers. Later, we extend these rules to cases where the exponents can be any rational numbers, such as negative integers or fractions. In later courses you will extend the rules to all real numbers.

Rules for Exponents

1. $a^n \cdot a^m = a^{(n+m)}$
2. $\dfrac{a^n}{a^m} = a^{(n-m)} \quad a \neq 0$
3. $(a^m)^n = a^{(m \cdot n)}$
4. $(ab)^n = a^n b^n$
5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n} \quad b \neq 0$

We show below how Rules 1, 3, and 5 make sense and leave Rules 2 and 4 for you to justify in the exercises.

Rule 1. To justify this rule, think about the total number of times a is a factor when a^n is multiplied by a^m:

$$a^n \cdot a^m = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} \cdot \underbrace{a \cdot a \cdots a}_{m \text{ factors}} = \underbrace{a \cdot a \cdot a \cdots a}_{n+m \text{ factors}} = a^{(n+m)}$$

Rule 3. First think about how many times a^m is a factor when we raise it to the nth power:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdots a^m}_{n \text{ factors of } a^m}$$

Use Rule 1: $$= a^{\overbrace{(m+m+\cdots+m)}^{n \text{ terms}}}$$

Represent adding m n times as $m \cdot n$ $$= a^{(m \cdot n)}$$

Rule 5. Remember that the exponent n in the expression $(a/b)^n$ applies to the whole expression within the parentheses:

$$\left(\frac{a}{b}\right)^n = \underbrace{\left(\frac{a}{b}\right) \cdot \left(\frac{a}{b}\right) \cdots \left(\frac{a}{b}\right)}_{n \text{ factors of } a/b}$$

$$= \frac{\overbrace{a \cdot a \cdots a}^{n \text{ factors of } a}}{\underbrace{b \cdot b \cdots b}_{n \text{ factors of } b}}$$

$$= \frac{a^n}{b^n}$$

EXAMPLE 1 Simplify and write as an expression with exponents:

$$7^3 \cdot 7^2 = 7^{3+2} = 7^5 \qquad (x^5)^3 = x^{5\cdot3} = x^{15}$$

$$w^3 \cdot w^5 = w^{3+5} = w^8 \qquad (11^2)^4 = 11^{2\cdot4} = 11^8$$

$$\frac{10^8}{10^3} = 10^{8-3} = 10^5 \qquad \frac{z^8}{z^3} = z^{8-3} = z^5$$

EXAMPLE 2 Simplify:

$$(3a)^4 = 3^4a^4 = 81a^4$$

$$(-5x)^3 = (-5)^3x^3 = -125x^3$$

$$\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3} = \frac{8}{27}$$

EXAMPLE 3 Simplify:

$$\left(\frac{-2a}{3b}\right)^3 = \frac{(-2a)^3}{(3b)^3} = \frac{(-2)^3a^3}{3^3b^3} = \frac{-8a^3}{27b^3}$$

$$\frac{-5(x^3)^2}{(2y^2)^3} = \frac{-5x^6}{8y^6}$$

EXAMPLE 4 **Using scientific notation to simplify calculations**

Deneb is 1600 light years from Earth. How far is Earth from Deneb when measured in miles?

SOLUTION The distance that light travels in 1 year, called a *light year,* is approximately 5.88 trillion miles.

Since $\quad 1 \text{ light year} = 5{,}880{,}000{,}000{,}000 \text{ miles}$

then the distance from Earth to Deneb is

$$\begin{aligned} 1600 \text{ light years} &= (1600) \cdot (5{,}880{,}000{,}000{,}000 \text{ miles}) \\ &= (1.6 \cdot 10^3) \cdot (5.88 \cdot 10^{12} \text{ miles}) \\ &= (1.6 \cdot 5.88) \cdot (10^3 \cdot 10^{12}) \text{ miles} \\ &\approx 9.4 \cdot 10^{3+12} \text{ miles} \\ &\approx 9.4 \cdot 10^{15} \text{ miles} \end{aligned}$$

Using ratios to compare sizes of objects

In comparing two objects of about the same size, it is common to subtract one size from the other and say, for instance, that one person is 6 inches taller than another. This method of comparison is not effective for objects of vastly different sizes. To say that the difference between the estimated radius of our solar system (1 terameter, or 1,000,000,000,000 meters) and the average size of a human (about 10^0 or 1 meter) is $1{,}000{,}000{,}000{,}000 - 1 = 999{,}999{,}999{,}999$ meters is not particularly useful. In fact, since our measurement of the solar system certainly isn't accurate to within 1 meter, this difference is meaningless. As shown in the following example, a more useful method for comparing objects of wildly different sizes is to calculate the ratio of the two sizes.

EXAMPLE 5 **The ratio of two quantities**

In April 2007, the U.S. federal government reported that the estimated gross federal debt was $8.87 trillion and the estimated U.S. population was 301 million. What was the approximate federal debt *per person*?

SOLUTION

$$\frac{\text{federal debt}}{\text{U.S. population}} = \frac{8.87 \cdot 10^{12}\text{ dollars}}{3.01 \cdot 10^{8}\text{ people}}$$

$$= \left(\frac{8.87}{3.01}\right) \cdot \left(\frac{10^{12}}{10^{8}}\right) \frac{\text{dollars}}{\text{people}} \approx 2.95 \cdot 10^{4} \frac{\text{dollars}}{\text{people}}$$

So the federal debt amounted to about $2.95 $\cdot$ 10^4 or $29,500 per person.

EXAMPLE 6 How many times larger is the sun than Earth?

SOLUTION 1 The radius of the sun is approximately 10^9 meters and the radius of Earth is about 10^7 meters. One way to answer the question "How many *times* larger is the sun than Earth?" is to form the ratio of the two radii:

$$\frac{\text{radius of the sun}}{\text{radius of Earth}} = \frac{10^9\text{ m}}{10^7\text{ m}}$$

$$= \frac{10^9\ \cancel{\text{m}}}{10^7\ \cancel{\text{m}}} = 10^{9-7} = 10^2$$

The units cancel, so 10^2 is unitless. The radius of the sun is approximately 10^2, or 100, times larger than the radius of Earth.

SOLUTION 2 Another way to answer the question is to compare the volumes of the two objects. The sun and Earth are both roughly spherical. The formula for the volume V of a sphere with radius r is $V = \frac{4}{3}\pi r^3$.

The radius of the sun is approximately 10^9 meters and the radius of Earth is about 10^7 meters. The ratio of the two volumes is

$$\frac{\text{volume of the sun}}{\text{volume of Earth}} = \frac{(4/3)\pi(10^9)^3\text{ m}^3}{(4/3)\pi(10^7)^3\text{ m}^3}$$

$$= \frac{(10^9)^3}{(10^7)^3} \quad (\textit{Note}: \tfrac{4}{3}\pi \text{ and m}^3 \text{ cancel.})$$

$$= \frac{10^{27}}{10^{21}}$$

$$= 10^6$$

So while the radius of the sun is 100 times larger than the radius of Earth, the *volume* of the sun is approximately 10^6 = 1,000,000, or 1 million, times larger than the volume of Earth!

Common Errors

The first question to ask in evaluating expressions with exponents is: To what does the exponent apply? Consider the following expressions:

1. $-a^n = -(a^n)$ ***but*** $-a^n \neq (-a)^n$ ***(unless n is odd)***

For example, in the expression -2^4, the exponent 4 applies only to 2, not to -2. The order of operations says to compute the power first, before applying the negation sign. So $-2^4 = -(2^4) = -16$. If we want to raise -2 to the fourth power, we write $(-2)^4 = (-2)(-2)(-2)(-2) = 16$.

In the expression $(-3b)^2$, everything inside the parentheses is squared. So $(-3b)^2 = (-3b)(-3b) = 9b^2$. But in the expression $-3b^2$, the exponent 2 applies only to the base b.

In the case where n is an *odd integer,* then $(-a)^n$ will equal $-(a)^n$. For example, $(-2)^3 = (-2)(-2)(-2) = -8 = -2^3$.

2. $ab^n = a(b^n)$ **and** $-ab^n = -a(b^n)$ **but** $ab^n \neq (ab)^n$

Remember, the exponent applies only to the variable to which it is attached. In the expressions ab^n and $-ab^n$, only b is raised to the nth power.

For example,

$$2 \cdot 5^3 = 2 \cdot 125 = 250 \quad \text{but} \quad (2 \cdot 5)^3 = (10)^3 = 1000$$
$$-2 \cdot 5^3 = -2 \cdot 125 = -250 \quad \text{but} \quad (-2 \cdot 5)^3 = (-10)^3 = -1000$$

You can use parentheses () to indicate when more than one variable is raised to the nth power.

3. $(ab)^n = a^n b^n$ **but** $(a + b)^n \neq a^n + b^n$ **(if** $n \neq 1$**)**

For example,

$$(2 \cdot 5)^3 = 2^3 \cdot 5^3 \quad \text{but} \quad (2 + 5)^3 \neq 2^3 + 5^3$$
$$(10)^3 = 8 \cdot 125 \qquad (7)^3 \neq 8 + 125$$
$$1000 = 1000 \qquad 343 \neq 133$$

4. $a^n \cdot a^m = a^{n+m}$ **but** $a^n + a^m \neq a^{n+m}$

For example,

$$10^2 \cdot 10^3 = 10^5 \quad \text{but} \quad 10^2 + 10^3 \neq 10^5$$
$$100 \cdot 1000 = 100{,}000 \qquad 100 + 1000 \neq 100{,}000$$

SOMETHING TO THINK ABOUT

What are some other exceptions to the generalizations made about common errors?

Common Errors Involving Exponents

In general,

$$-a^n \neq (-a)^n \qquad a^n + a^m \neq a^{n+m}$$
$$ab^n \neq (ab)^n \qquad (a + b)^n \neq a^n + b^n$$

Algebra Aerobics 4.2a

1. Simplify where possible, leaving the answer in a form with exponents:
   a. $10^5 \cdot 10^7$ **d.** $5^5 \cdot 6^7$ **g.** $3^4 + 7 \cdot 3^4$
   b. $8^6 \cdot 8^{14}$ **e.** $7^3 + 7^3$ **h.** $2^3 + 2^4$
   c. $z^5 \cdot z^4$ **f.** $5 \cdot 5^6$ **i.** $2^5 + 5^2$
2. Simplify (if possible), leaving the answer in exponent form:
   a. $\frac{10^{15}}{10^7}$ **c.** $\frac{3^5}{3^4}$ **e.** $\frac{5}{5^6}$ **g.** $\frac{2^3 \cdot 3^4}{2 \cdot 3^2}$
   b. $\frac{8^6}{8^4}$ **d.** $\frac{5}{6^7}$ **f.** $\frac{3^4}{3}$ **h.** $\frac{6}{2^4}$
3. Write each number as a power of 10, then perform the indicated operation. Write your final answer as a power of 10.
   a. $100{,}000 \cdot 1{,}000{,}000$
   b. $1{,}000 \cdot 0.000\,001$
   c. $0.000\,000\,000\,01 \cdot 0.000\,01$
   d. $\frac{1{,}000{,}000{,}000}{10{,}000}$
   e. $\frac{1{,}000{,}000}{0.001}$
   f. $\frac{0.000\,01}{0.0001}$
   g. $\frac{0.000\,001}{10{,}000}$
4. Simplify:
   a. $(10^4)^5$ **d.** $(2x)^4$ **g.** $(-3x^2)^3$
   b. $(7^2)^3$ **e.** $(2a^4)^3$ **h.** $((x^3)^2)^4$
   c. $(x^4)^5$ **f.** $(-2a)^3$ **i.** $(-5y^2)^3$
5. Simplify:
   a. $\left(\frac{-2x}{4y}\right)^3$ **c.** -5^2 **e.** $(-3yz^2)^4$
   b. $(-5)^2$ **d.** $-3(yz^2)^4$ **f.** $(-3yz^2)^3$
6. A compact disk or CD has a storage capacity of about 737 megabytes ($7.37 \cdot 10^8$ bytes). If a hard drive has a capacity of 40 gigabytes ($4.0 \cdot 10^{10}$ bytes), how many CD's would it take to equal the storage capacity of the hard drive?
7. Write as a single number with no exponents:
   a. $(3 + 5)^3$ **c.** $3 \cdot 5^2$
   b. $3^3 + 5^3$ **d.** $-3 \cdot 5^2$

Estimating Answers

By rounding off numbers and using scientific notation and the rules for exponents, we can often make quick estimates of answers to complicated calculations. In this age of calculators and computers we need to be able to roughly estimate the size of an answer, to make sure our calculations with technology make sense.

EXAMPLE 7 Estimate the value of

$$\frac{(382{,}152) \cdot (490{,}572{,}261)}{(32{,}091) \cdot (1942)}$$

Express your answer in both scientific and standard notation.

SOLUTION Round each number:

$$\frac{(382{,}152) \cdot (490{,}572{,}261)}{(32{,}091) \cdot (1942)} \approx \frac{(400{,}000) \cdot (500{,}000{,}000)}{(30{,}000) \cdot (2000)}$$

rewrite in scientific notation $\approx \dfrac{(4 \cdot 10^5) \cdot (5 \cdot 10^8)}{(3 \cdot 10^4) \cdot (2 \cdot 10^3)}$

group the coefficients and the powers of 10 $\approx \left(\dfrac{4 \cdot 5}{3 \cdot 2}\right) \cdot \left(\dfrac{10^5 \cdot 10^8}{10^4 \cdot 10^3}\right)$

simplify each expression $\approx \dfrac{20}{6} \cdot \dfrac{10^{13}}{10^7}$

we get in scientific notation $\approx 3.33 \cdot 10^6$

or in standard notation $\approx 3{,}330{,}000$

Using a calculator on the original problem, we get a more precise answer of 3,008,200.

EXAMPLE 8 As of 2007 the world population was approximately 6.605 billion people. There are roughly 57.9 million square miles of land on Earth, of which about 22% are favorable for agriculture. Estimate how many people per square mile of farmable land there are as of 2007.

SOLUTION

$$\frac{\text{size of world population}}{\text{amount of farmable land}} = \frac{6.605 \text{ billion people}}{22\% \text{ of } 57.9 \text{ million square miles}}$$

rewrite as powers of 10 $= \dfrac{6.605 \cdot 10^9 \text{ people}}{(0.22) \cdot (57.9) \cdot 10^6 \text{ mile}^2}$

round each number $\approx \dfrac{6.6 \cdot 10^9 \text{ people}}{(0.2) \cdot 60 \cdot 10^6 \text{ mile}^2}$

simplify $\approx \dfrac{6.6 \cdot 10^9 \text{ people}}{12 \cdot 10^6 \text{ mile}^2}$

we get in scientific notation $\approx 0.55 \cdot 10^3 \text{ people/mile}^2$

or in standard notation $\approx 550 \text{ people/mile}^2$

So there are roughly 550 people/mile^2 of farmable land in the world. Using a calculator and the original numbers, we get a more accurate answer of 519 people/mile^2 of farmable land, which is close to our estimate.

Algebra Aerobics 4.2b

1. Estimate the value of:
   a. $(0.000\,297\,6) \cdot (43{,}990{,}000)$
   b. $\dfrac{453{,}897 \cdot 2{,}390{,}702}{0.004\,38}$
   c. $\dfrac{0.000\,000\,319}{162{,}000}$
   d. $28{,}000{,}000 \cdot 7{,}629$
   e. $0.000\,021 \cdot 391{,}000{,}000$
2. Evaluate the following without the aid of a calculator:
   a. $(3.0 \cdot 10^3)\,(4.0 \cdot 10^2)$
   b. $\dfrac{(5.0 \cdot 10^2)^2}{2.5 \cdot 10^3}$
   c. $\dfrac{2.0 \cdot 10^5}{5.0 \cdot 10^3}$
   d. $(4.0 \cdot 10^2)^3 \cdot (2.0 \cdot 10^3)^2$
3. The radius of Jupiter, the largest of the planets in our solar system, is approximately $7.14 \cdot 10^4$ km. (If r is the radius of a sphere, the sphere's surface area equals $4\pi r^2$ and its volume equals $\frac{4}{3}\pi r^3$.) Assuming Jupiter is roughly spherical,
   a. Estimate the surface area of Jupiter.
   b. Estimate the volume of Jupiter.
   (Express your answers in scientific notation.)
4. Only about three-sevenths of the land favorable for agriculture is actually being farmed. Using the facts in Example 8, estimate the number of people per square mile of farmable land that is being used. Should your estimate be larger or smaller than the ratio of people to farmable land? Explain. (Round your answer to the nearest integer.)

Exercises for Section 4.2

1. Simplify, when possible, writing your answer as an expression with exponents:
   a. $10^4 \cdot 10^3$
   b. $10^4 + 10^3$
   c. $10^3 + 10^3$
   d. $x^5 \cdot x^{10}$
   e. $(x^5)^{10}$
   f. $4^7 + 5^2$
   g. $\dfrac{z^7}{z^2}$
   h. 256^0
   i. $\dfrac{3^5 \cdot 3^2}{3^8}$
   j. $4^5 \cdot (4^2)^3$
2. Simplify:
   a. $(-1)^4$
   b. $-(1)^4$
   c. $(a^4)^3$
   d. $-(2a^2)^3$
   e. $(2a^4)^3$
   f. $(-2a^4)^3$
   g. $(10a^2b^3)^3$
   h. $(2ab)^2 - 3a^2b^2$
3. Simplify:
   a. $(-2a)^4$
   b. $-2(a)^4$
   c. $(-x^5)^3$
   d. $(-2ab^2)^3$
   e. $(2x^4)^5$
   f. $(-4x^3)^2 + x^3(2x^3)$
   g. $(50a^{10})^2$
   h. $(3ab)^3 + ab$
4. Simplify and write each variable as an expression with positive exponents:
   a. $-\left(\dfrac{3}{5}\right)^2$
   b. $\left(\dfrac{-5a^3}{a^2}\right)^4$
   c. $\left(\dfrac{10a^3}{5b}\right)^2$
   d. $\left(\dfrac{-2x^3}{3y^2}\right)^3$
5. Simplify and write each variable as an expression with positive exponents:
   a. $-\left(\dfrac{5}{8}\right)^2$
   b. $\left(\dfrac{3x^3}{5y^2}\right)^3$
   c. $\left(\dfrac{-10x^5}{2b^2}\right)^4$
   d. $\left(\dfrac{-x^5}{x^2}\right)^3$
6. Evaluate and express your answer in standard decimal form:
   a. $-2^4 + 2^2$
   b. $-2^3 + (-4)^2$
   c. $2 \cdot 3^2 - 3(-2)^2$
   d. $-10^4 + 10^5$
   e. $10^3 + 2^3$
   f. $2 \cdot 10^3 + 10^3 + 10^2$
   g. $2 \cdot 10^3 + (-10)^3$
   h. $(1000)^0$
7. Convert each number into scientific notation then perform the indicated operation. Leave your answer in scientific notation.
   a. $2{,}000{,}000 \cdot 4000$
   b. 1.4 million ÷ 7000
   c. 50 billion · 60 trillion
   d. 2500 billion ÷ 500 thousand
8. Convert each number into scientific notation and then perform the operation without a calculator.
   a. $60{,}000{,}000{,}000 + 40{,}000{,}000{,}000$
   b. $\dfrac{(20{,}000)^6}{(400)^3}$
   c. $(2{,}000{,}000) \cdot (40{,}000)$
9. Simplify each expression using the properties of exponents.
   a. $(x^5y)(x^6)(x^2y^3)$
   b. $\dfrac{5x^6y^3}{x^2y^2}$
   c. $\left(\dfrac{-2x^5y^5}{x^2y^2}\right)^3$
   d. $(x^2)^5 \cdot (2y^2)^4$
   e. $(3x^2y^5)^4$
   f. $\left(\dfrac{3x^3y}{5xy}\right)^2$
10. Each of the following simplifications contains an error made by students on a test. Find the error and correct the simplification so that the expression becomes true.
   a. $[(x^2)^3]^5 = [x^5]^5 = x^{25}$
   b. $\dfrac{7x^2y^6}{(xy)^2} = \dfrac{7x^2y^6}{x^2y^2} = 7x^4y^8$
   c. $\left(\dfrac{4x^3y^5}{6xy^4}\right)^3 = \left(\dfrac{2x^2y}{3}\right)^3 = \frac{2}{3}x^6y^3$
   d. $(1.1 \cdot 10^6) \cdot (1.1 \cdot 10^4) = 1.1 \cdot 10^6$
   e. $\dfrac{4 \cdot 10^6}{8 \cdot 10^3} = 0.5 \cdot 10^3 = 5.0 \cdot 10^4$
   f. $6 \cdot 10^3 + 7 \cdot 10^5 = 13 \cdot 10^8$

11. Express your answer as a power of 10 and in standard decimal form. (Refer to table on inside back cover.)

a. How many times larger is a gigabyte of memory than a megabyte?

b. How many times farther is a kilometer than a dekameter?

c. How many times heavier is a kilogram than a milligram?

d. How many times longer is a microsecond than a nanosecond?

12. In 2006 the People's Republic of China was estimated to have about 1,314,000,000 people, and Monaco about 33,000. Monaco has an area of 0.75 miles^2, and China has an area of 3,705,000 miles^2.

a. Express the populations and geographic areas in scientific notation.

b. By calculating a ratio, determine how much larger China's population is than Monaco's.

c. What is the population density (the number of people per square mile) for each country?

d. Write a paragraph comparing and contrasting the population size and density for these two nations.

13. a. In 2006 Japan had a population of approximately 127.5 million people and a total land area of about 152.5 thousand square miles. What was the population density (the number of people per square mile)?

b. In 2006 the United States had a population of approximately 300 million people and a total land area of about 3620 thousand square miles. What was the population density of the United States?

c. Compare the population densities of Japan and the United States.

14. The distance that light travels in 1 year (a light year) is $5.88 \cdot 10^{12}$ miles. If a star is $2.4 \cdot 10^{8}$ light years from Earth, what is this distance in miles?

15. An average of $1.5 \cdot 10^4$ Coca-Cola beverages were consumed every second worldwide in 2005. There are $8.64 \cdot 10^4$ seconds in a day. What was the daily consumption of Coca-Cola in 2005? (*Source:* World of Coca-Cola® Atlanta)

16. Change each number into scientific notation, then perform the indicated calculation without a calculator.

a. A $600,000 lottery jackpot is divided among 300 people. What are the winnings per person?

b. A total of 2500 megawatts are used over 500 hours. What is the rate in watts per hour?

c. If there were 6 million births in 30 years, what is the birth rate per year?

17. a. For any nonzero real number a, what can we say about the sign of the expression $(-a)^n$ when n is an even integer? What can we say about the sign of $(-a)^n$ when n is an odd integer?

b. What is the sign of the resulting number if a is a positive number? If a is a negative number? Explain your answer.

18. Round off the numbers and then estimate the value of each of the following expressions without using a calculator. Show your work, writing your answers in scientific notation. If available, use a calculator to verify your answers.

a. $(2{,}968{,}001{,}000) \cdot (189{,}000)$

b. $(0.000\,079) \cdot (31{,}140{,}284{,}788)$

c. $\dfrac{4{,}083{,}693 \cdot 49{,}312}{213 \cdot 1945}$

19. Simplify each expression using two different methods, and then compare your answers.

Method I: Simplify inside the parentheses first, and then apply the exponent rule outside the parentheses.

Method II: Apply the exponent rule outside the parentheses, and then simplify.

a. $\left(\dfrac{m^2n^3}{mn}\right)^2$ **b.** $\left(\dfrac{2a^2b^3}{ab^2}\right)^4$

20. Verify that $(a^2)^3 = (a^3)^2$ using the rules of exponents.

21. Verify that $\left(\dfrac{2a^3}{5b^2}\right)^4 = \dfrac{16a^{12}}{625b^8}$ using the rules of exponents.

22. An article in the journal *Nature* (October 2000) analyzed samples of the ballast water from ships arriving in the Chesapeake Bay from foreign ports. It reported that ballast was an important factor in the global distribution of microorganisms. One gallon of ballast water contained on average 3 billion bacteria, including some that cause cholera. The scientists estimated that about 2.5 billion gallons of ballast water are discharged into the Chesapeake Bay each year. Estimate the number of bacteria per year discharged in ballast water into the Chesapeake Bay. Write your answer in scientific notation.

23. Justify the following rule for exponents. If a and b are any nonzero real numbers and n is an integer ≥ 0, then

$$(ab)^n = a^n b^n$$

24. Justify the following rule for exponents. Consider the case of $n \geq m$ and assume m and n are integers > 0. If a is any nonzero real number, then

$$\frac{a^n}{a^m} = a^{(n-m)}$$

25. In 2006 the United Kingdom generated approximately 81 terawatt-hours of nuclear energy for a population of about 60.6 million on 94,525 miles^2. In the same year the United States generated approximately 780 terawatt-hours of nuclear energy for a population of about 300 million on 3,675,031 miles^2. A terawatt is 10^{12} watts.

a. How many terawatt-hours is the United Kingdom generating per person? How many terawatt-hours is it generating per square mile? Express each in scientific notation.

b. How many terawatt-hours is the United States generating per person? How many terawatt-hours are we generating per square mile? Express each in scientific notation.

c. How much nuclear energy is being generated in the United Kingdom per square mile relative to the United States?

d. Write a brief statement comparing the relative magnitude of generation of nuclear power per person in the United Kingdom and the United States.

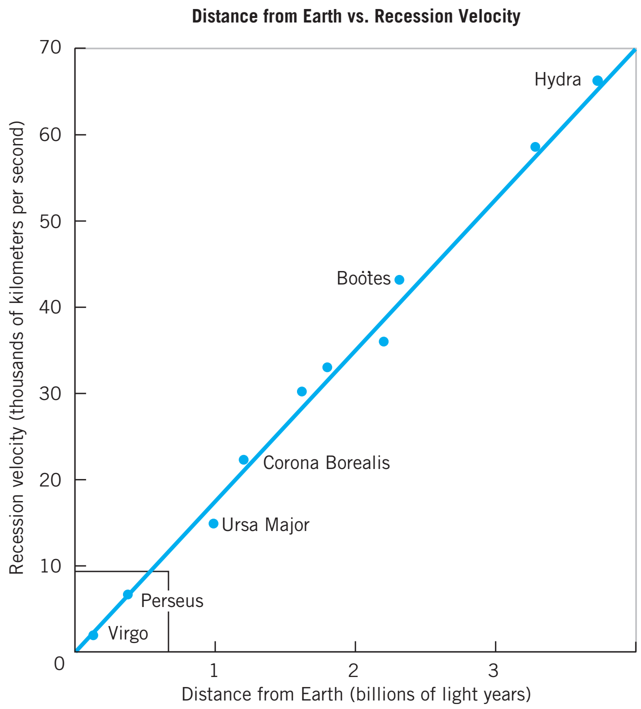
26. Hubble's Law states that galaxies are receding from one another at velocities directly proportional to the distances separating them. The accompanying graph illustrates that Hubble's Law holds true across the known universe. The plot includes ten major clusters of galaxies. The boxed area at the lower left represents the galaxies observed by Hubble when he discovered the law. The easiest way to understand this graph is to think of Earth as being at the center of the universe (at 0 distance) and not moving (at 0 velocity). In other words, imagine Earth at the origin of the graph (a favorite fantasy of humans). Think of the horizontal axis as measuring the distance of the galaxy from Earth, and the vertical as measuring the velocity at which a galaxy cluster is moving away from Earth (the recession velocity). Then answer the following questions.

a. Identify the coordinates of two data points that lie on the regression line drawn on the graph.

b. Use the coordinates of the points in part (a) to calculate the slope of the line. That slope is called the *Hubble constant*.

c. What does the slope mean in terms of distance from Earth and recession velocity?

d. Construct an equation for our line in the form $y = mx + b$. Show your work.



Source: T. Ferris, *Coming of Age in the Milky Way* (New York: William Morrow, 1988). Copyright © by Timothy Ferris. By permission of William Morrow & Company, Inc.

4.3 *Negative Integer Exponents*

The definitions for raising any base to the zero power or to a negative power follow a logic that is similar to the one used to define $10^0 = 1$ and $10^{-n} = \frac{1}{10^n}$.

> **Zero and Negative Exponents**
>
> If a is nonzero and n is a positive integer, then
>
> $$a^0 = 1$$
> $$a^{-n} = \frac{1}{a^n}$$

It is important to note that $a^1 = a$, so $a^{-1} = \frac{1}{a^1} = \frac{1}{a}$.

In the following examples, we show how to apply the five rules for exponents when the exponents are negative integers or zero.

EXAMPLE 1 Simplify $x^2 \cdot x^{-5}$.

SOLUTION Using Rule 1 for exponents,

$$x^2 \cdot x^{-5} = x^{2+(-5)} = x^{-3}$$

or we can simplify by first writing x^{-5} as $\frac{1}{x^5}$ and then use Rule 2 for exponents:

$$x^2 \cdot x^{-5} = x^2 \cdot \frac{1}{x^5} = \frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

EXAMPLE 2 Simplify. Express your answer with positive exponents.

a. $\dfrac{10^2}{10^6}$ **c.** $\dfrac{(-5)^2}{(-5)^6}$

b. $\dfrac{6^2}{6^{-7}}$ **d.** $\dfrac{x^{-2}}{x^4}$

SOLUTION Using Rule 2 for exponents,

a. $\dfrac{10^2}{10^6} = 10^{2-6} = 10^{-4} = \dfrac{1}{10^4}$

b. $\dfrac{6^2}{6^{-7}} = 6^{2-(-7)} = 6^9$

c. $\dfrac{(-5)^2}{(-5)^6} = (-5)^{2-6} = (-5)^{-4} = \dfrac{1}{(-5)^4} = \dfrac{1}{(-1)^4\,(5)^4} = \dfrac{1}{5^4}$

d. $\dfrac{x^{-2}}{x^4} = x^{-2-4} = x^{-6} = \dfrac{1}{x^6}$

EXAMPLE 3 Simplify:

a. $(13^{-8})^3$ **b.** $(w^2)^{-7}$

SOLUTION Using Rule 3 for exponents,

a. $(13^{-8})^3 = 13^{(-8)3} = 13^{-24}$ **b.** $(w^2)^{-7} = w^{2(-7)} = w^{-14}$

EXAMPLE 4 Simplify:

$$\frac{v^{-2}(w^5)^2}{(v^{-1})^4 w^{-3}}$$

SOLUTION Apply Rule 3 twice: $\dfrac{v^{-2}(w^5)^2}{(v^{-1})^4 w^{-3}} = \dfrac{v^{-2}\,w^{10}}{v^{-4}\,w^{-3}}$

Apply Rule 2 twice: $= v^{-2-(-4)}w^{10-(-3)}$

$= v^2 w^{13}$

Evaluating $\left(\frac{a}{b}\right)^{-n}$

The rule for applying negative powers is the same whether a is an integer or a fraction:

$$a^{-n} = 1/a^n \qquad \text{where } a \neq 0$$

For example,

$$\left(\frac{1}{2}\right)^{-1} = \frac{1}{(1/2)^1} = 1 \div \left(\frac{1}{2}\right) = 1 \cdot \left(\frac{2}{1}\right) = 2$$

In general, if a and b are nonzero, then

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{(a/b)^n} = 1 \div \left(\frac{a}{b}\right)^n = 1 \cdot \left(\frac{b}{a}\right)^n = \left(\frac{b}{a}\right)^n = \frac{b^n}{a^n}$$

EXAMPLE 5 Simplify:

a. $\left(\frac{1}{2}\right)^{-11} \cdot \left(\frac{1}{2}\right)^{-2}$ b. $\left(\frac{a}{b}\right)^{3} \cdot \left(\frac{a}{b}\right)^{-5}$

SOLUTION a. Using Rule 1 for exponents and the definition of a^{-n},

$$\left(\frac{1}{2}\right)^{-11} \cdot \left(\frac{1}{2}\right)^{-2} = \left(\frac{1}{2}\right)^{-11+(-2)}$$

$$= \left(\frac{1}{2}\right)^{-13} = \left(\frac{2}{1}\right)^{13} = 2^{13} = 8192$$

b. Using Rules 1 and 5 for exponents and the definition of a^{-n},

$$\left(\frac{a}{b}\right)^{3} \cdot \left(\frac{a}{b}\right)^{-5} = \left(\frac{a}{b}\right)^{3+(-5)}$$

$$= \left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^{2} = \frac{b^2}{a^2}$$

Algebra Aerobics 4.3

1. Simplify (if possible). Express with a single positive exponent, if possible.
   a. $10^5 \cdot 10^{-7}$
   b. $\frac{11^6}{11^{-4}}$
   c. $\frac{3^{-5}}{3^{-4}}$
   d. $\frac{5^5}{6^7}$
   e. $\frac{7^3}{7^3}$
   f. $a^{-2} \cdot a^{-3}$
   g. $3^4 \cdot 3^3$
   h. $(2^2 \cdot 3)(2^6)(2^4 \cdot 3)$
2. A typical TV signal, traveling at the speed of light, takes $3.3 \cdot 10^{-6}$ seconds to travel 1 kilometer. Estimate how long it would take the signal to travel across the United States (a distance of approximately 4300 kilometers).
3. Distribute and simplify:
   a. $x^{-2}(x^5 + x^{-6})$
   b. $-a^2(b^2 - 3ab + 5a^2)$
4. Simplify:
   a. $(10^4)^{-5}$
   b. $(7^{-2})^{-3}$
   c. $(2a^3)^{-2}$
   d. $\left(\frac{8}{x}\right)^{-2}$
   e. $(2x^{-2})^{-1}$
   f. $2(x^{-2})^{-1}$
   g. $\left(\frac{3}{2y^2}\right)^{-4}$
   h. $\frac{3}{(2y^2)^{-4}}$
5. Simplify:
   a. $\frac{t^{-3}t^0}{(t^{-4})^3}$
   b. $\frac{v^{-3}w^7}{(v^{-2})^3 w^{-10}}$
   c. $\frac{7^{-8}x^{-1}y^2}{7^{-5}xy^3}$
   d. $\frac{a(5b^{-1}c^3)^2}{5ab^2c^{-6}}$

Exercises for Section 4.3

1. Simplify and express your answer using positive exponents. Check your answers by applying the rules for exponents and doing the calculations.
   a. $10^3 \cdot 10^{-2}$
   b. $\frac{10^{-2}}{10^3}$
   c. $(10^{-3})^2$
   d. $\frac{10^3}{10^{-2}}$
2. Simplify and express your answer with positive exponents:
   a. $(x^{-3}) \cdot (x^4)$
   b. $(x^{-3}) \cdot (x^{-2})$
   c. $(x^2)^{-3}$
   d. $(n^{-2})^{-3}$
   e. $(2n^{-2})^{-3}$
   f. $n^{-4}(n^5 - n^2) + n^{-3}(n - n^4)$
3. Simplify where possible. Express your answer with positive exponents.
   a. $\frac{2^3x^4}{2^5x^8}$
   b. $\frac{x^4y^7}{x^3y^{-5}}$
   c. $\frac{x^{-2}y}{xy^3}$
   d. $\frac{(x+y)^4}{(x+y)^{-7}}$
   e. $\frac{a^{-2}bc^{-5}}{(ab^2)^{-3}c}$
4. Simplify where possible. Express your answer with positive exponents.
   a. $(3 \cdot 3^8)^{-2}$
   b. $x^3 \cdot x^{-4} \cdot x^{12}$
   c. $2^6 + 2^6 + 2^7 + 2^{-4}$
   d. $2x^{-3} + 3x(x^{-4})$
   e. $10^{-5} + 5^{-2} + 10^{10}$

5. Evaluate and write the result using scientific notation:

a. $(2.3 \cdot 10^4)(2.0 \cdot 10^6)$ **d.** $\dfrac{3.25 \cdot 10^8}{6.29 \cdot 10^{15}}$

b. $(3.7 \cdot 10^{-5})(1.1 \cdot 10^8)$ **e.** $(6.2 \cdot 10^{52})^3$

c. $\dfrac{8.19 \cdot 10^{23}}{5.37 \cdot 10^{12}}$ **f.** $(5.1 \cdot 10^{-11})^2$

6. Write each of the following in scientific notation:

a. $725 \cdot 10^{23}$ **c.** $\dfrac{1}{725 \cdot 10^{23}}$ **e.** $-725 \cdot 10^{-23}$

b. $725 \cdot 10^{-23}$ **d.** $-725 \cdot 10^{23}$

7. Change each number to scientific notation, then simplify using rules of exponents. Show your work, recording your final answer in scientific notation.

a. 10% of 0.000 01 **d.** $\dfrac{8000}{0.000\,8}$

b. $\dfrac{0.000\,05}{50{,}000}$ **e.** $\dfrac{0.006\,4}{8000}$

c. $\dfrac{3}{0.006}$ **f.** $5{,}000{,}000 \cdot 40{,}000$

8. Use scientific notation and the rules of exponents to perform the indicated operation without a calculator. Show your work, recording your answer in decimal form.

a. $\dfrac{20}{200{,}000}$ **d.** $\dfrac{10{,}000{,}000}{25{,}000}$

b. $\dfrac{0.006}{60{,}000}$ **e.** $0.06 \cdot 600$

c. $200 \cdot 0.000\,007\,5$ **f.** 10% of 0.000 05

9. For each equation determine the value of x that makes it true.

a. $10^x = 0.000\,001$ **c.** $\dfrac{1}{10^x} = 0.000\,1$

b. $10^x = \dfrac{1}{1{,}000{,}000}$ **d.** $10^{-x} = 100{,}000$

10. For each equation determine the value of x that makes it true.

a. $6.3 \cdot 10^x = 0.000\,63$ **d.** $x^3 = \dfrac{1}{1000}$

b. $10^{-3} = x$ **e.** $4^{-3} \cdot 2^{-5} = 2^x$

c. $5^x = \dfrac{1}{125}$ **f.** $9^{-1} \cdot 27^{-2} = 3^x$

11. Simplify the following expressions by using properties of exponents. Write your final answers with only positive exponents.

a. $\dfrac{(-2x^3y^{-1})^{-3}}{(x^2y^{-2})^0}$ **c.** $\left(\dfrac{3x^2y^{-5}}{5x^3y^4}\right)^{-1}$

b. $\dfrac{(-2x^3y^{-1})^{-2}}{(x^2y^{-2})^{-1}}$ **d.** $\left[\left(3x^{-1}z^4\right)^{-2}\right]^{-3}$

12. Each of the following simplifications is false. In each case identify the error and correct it.

a. $x^{-2}x^{-5} = x^{10}$

b. $\dfrac{2^{-1}x^2y^{-2}}{x^{-1}y^5} = \dfrac{x^2x^{-1}}{2y^{-2}y^5} = \dfrac{x}{2y^3}$

c. $(3x^{-1})^2 = \left(\dfrac{1}{3x}\right)^2 = \dfrac{1}{9x^2}$

d. $(x + y)^{-1} = \dfrac{1}{x} + \dfrac{1}{y}$

13. A TV signal traveling at the speed of light takes about $8 \cdot 10^{-5}$ second to travel 15 miles. How long would it take the signal to travel a distance of 3000 miles?

14. Round off the numbers and then estimate the values of the following expressions without a calculator. Show your work, writing your answers in scientific notation. If available, use a calculator to verify your answers.

a. $(0.000\,359) \cdot (0.000\,002\,47)$

b. $\dfrac{0.000\,007\,31 \cdot 82{,}560}{1{,}891{,}000}$

15. Simplify and express your answer with positive exponents.

a. $\dfrac{x^{-2} - y^{-1}}{(xy^2)^{-1}}$ **b.** $(5x^{-2}y^{-3})^{-2}$

16. (Requires a calculator that can evaluate powers.) Calculators and spreadsheets use slightly different formats for scientific notation. For example, if you type in Avogadro's number either as 602,000,000,000,000,000,000,000 or as $6.02 \cdot 10^{23}$, the calculator or spreadsheet will display 6.02 E 23, where E stands for "exponent" or power of 10. Perform the following calculations using technology, then write the answer in standard scientific notation rounded to three places.

a. $\left(\dfrac{9}{5}\right)^{50}$ **d.** $\dfrac{7}{6^{15}}$

b. 2^{35} **e.** $(5)^{-10}(2)^{10}$

c. $\left(\dfrac{1}{3}\right)^7$ **f.** $(-4)^5\left(\dfrac{1}{(16)^{12}}\right)$

17. Describe at least three different methods for entering $5.23 \cdot 10^{-3}$ into a calculator or spreadsheet.

18. Using rules of exponents, show that $\dfrac{9^5}{27^{-7}} = 3^{31}$.

19. Using rules of exponents, show that $\dfrac{1}{x^{-n}} = x^n$.

20. Write an expression that displays the calculation(s) necessary to answer the question. Then use scientific notation and exponent rules to determine the answer.

a. Find the number of nickels in $500.00.

b. The circumference of Earth is about 40.2 million meters. Find the radius of Earth, in kilometers, using the formula $C = 2\pi r$.

21. According to the National Confectioners Association, in 2006 there were 35 million pounds (or 9 billion kernels) of candy corn made for Halloween. How many kernels are in a pound?

22. The robot spacecraft NEAR (Near Earth Asteroid Rendezvous) is on a four-year mission through the inner solar system to study asteroids. In February 2001 the spacecraft landed on Eros, a Manhattan-sized asteroid 160 million miles from Earth.

a. If radio messages travel at the speed of light, how long would it take for a message sent back from the NEAR spacecraft to reach the scientists?

b. The near-Earth asteroid Cruithne is now known to be a companion, and an unusual one, of Earth. This asteroid shares Earth's orbit, its motion "choreographed" in such a way as to remain stable and avoid colliding with our planet. At its closest approach Cruithne gets to within 0.1 astronomical unit of Earth (about 15 million kilometers). The asteroid in 2004 was about 0.3 astronomical unit (45 million kilometers) from Earth. If the NEAR spacecraft was in orbit around Cruithne at that time, how long would a radio signal transmitted from Earth take to reach the spacecraft?

4.4 *Converting Units*

Exploration 4.1 will help you understand the relative ages and sizes of objects in our universe and give you practice in scientific notation and unit conversion.

Problems in science constantly require converting back and forth between different units of measure. To do so, we need to be comfortable with the laws of exponents and the basic metric and English units (see Table 4.1 or a more complete table on the inside back cover). The following unit conversion examples describe a strategy based on *conversion factors.*

Converting Units within the Metric System

EXAMPLE 1 **Conversion Factors**

Light travels at a speed of approximately $3.00 \cdot 10^5$ kilometers per second (km/sec). Describe the speed of light in meters per second (m/sec).

SOLUTION The prefix *kilo* means thousand. One kilometer (km) is equal to 1000 or 10^3 meters (m):

$$1 \text{ km} = 10^3 \text{ m} \qquad (1)$$

Dividing both sides of Equation (1) by 1 km, we can rewrite it as

$$1 = \frac{10^3 \text{ m}}{1 \text{ km}}$$

If instead we divide both sides of Equation (1) by 10^3 m, we get

$$\frac{1 \text{ km}}{10^3 \text{ m}} = 1$$

The ratios $\frac{10^3 \text{ m}}{1 \text{ km}}$ and $\frac{1 \text{ km}}{10^3 \text{ m}}$ are called *conversion factors*, because we can use them to convert between kilometers and meters.

What is the right conversion factor? If units in $\frac{\text{km}}{\text{sec}}$ are multiplied by units in meters per kilometer, we have

$$\frac{\cancel{\text{km}}}{\text{sec}} \cdot \frac{\text{m}}{\cancel{\text{km}}}$$

and the result is in meters per second. So multiplying the speed of light in km/sec by a conversion factor in m/km will give us the correct units of m/sec.

A conversion factor always equals 1. So we will not change the value of the original quantity by multiplying it by a conversion factor. In this case, we use the conversion factor of $\frac{10^3 \text{ m}}{1 \text{ km}}$.

$$\begin{aligned} 3.00 \cdot 10^5 \text{ km/sec} &= 3.00 \cdot 10^5 \frac{\cancel{\text{km}}}{\text{sec}} \cdot \frac{10^3 \text{ m}}{1 \cancel{\text{km}}} \\ &= 3.00 \cdot 10^5 \cdot 10^3 \text{ m/sec} \\ &= 3.00 \cdot 10^8 \text{ m/sec} \end{aligned}$$

Hence light travels at approximately $3.00 \cdot 10^8$ m/sec.

EXAMPLE 2 Check your answer in Example 1 by converting $3.00 \cdot 10^8$ m/sec back to km/sec.

SOLUTION Here we use the same strategy, but now we need to use the other conversion factor. Multiplying $3.00 \cdot 10^8$ m/sec by (1 km)/(10^3 m) gives us

$$3.00 \cdot 10^8 \frac{\cancel{m}}{\text{sec}} \cdot \frac{1\text{ km}}{10^3\cancel{m}} = 3.00 \cdot \frac{10^8\text{ km}}{10^3\text{ sec}}$$
$$= 3.00 \cdot 10^5\text{ km/sec}$$

which was the original value given for the speed of light.

Converting between the Metric and English Systems

EXAMPLE 3 You're touring Canada, and you see a sign that says it is 130 kilometers to Toronto. How many miles is it to Toronto?

SOLUTION The crucial question is, "What conversion factor should be used?" From Table 4.1 we know that

$$1\text{ km} \approx 0.62\text{ mile}$$

This equation can be rewritten in two ways:

$$1 \approx \frac{0.62\text{ mile}}{1\text{ km}} \quad \text{or} \quad 1 \approx \frac{1\text{ km}}{0.62\text{ mile}}$$

It produces two possible conversion factors:

$$\frac{0.62\text{ mile}}{1\text{ km}} \quad \text{and} \quad \frac{1\text{ km}}{0.62\text{ mile}}$$

> **SOMETHING TO THINK ABOUT**
> ? *Why is the conversion factor 1 km/ 0.62 miles not helpful in solving this problem?*

Which one will convert kilometers to miles? We need one with kilometers in the denominator and miles in the numerator, namely $\frac{0.62\text{ mile}}{1\text{ km}}$, so that the km will cancel when we multiply by 130 km:

$$130\,\cancel{\text{km}} \cdot \frac{0.62\text{ mile}}{1\,\cancel{\text{km}}} = 80.6\text{ miles}$$

So it is a little over 80 miles to Toronto.

Using Multiple Conversion Factors

EXAMPLE 4 Light travels at $3.00 \cdot 10^5$ km/sec. How many kilometers does light travel in one *year*?

SOLUTION Here our strategy is to use more than one conversion factor to convert from seconds to years. Use your calculator to perform the following calculations:

$$3.00 \cdot 10^5 \frac{\text{km}}{\cancel{\text{sec}}} \cdot \frac{60\,\cancel{\text{sec}}}{1\,\cancel{\text{min}}} \cdot \frac{60\,\cancel{\text{min}}}{1\,\cancel{\text{hr}}} \cdot \frac{24\,\cancel{\text{hr}}}{1\,\cancel{\text{day}}} \cdot \frac{365\,\cancel{\text{days}}}{1\text{ year}} = 94{,}608{,}000 \cdot 10^5\text{ km/year}$$
$$\approx 9.46 \cdot 10^7 \cdot 10^5\text{ km/year}$$
$$= 9.46 \cdot 10^{12}\text{ km/year}$$

So a light year, the distance light travels in one year, is approximately equal to $9.46 \cdot 10^{12}$ kilometers.

Algebra Aerobics 4.4

On the back inside cover of the text there are tables containing metric prefixes and conversion facts. Round your answers to two decimal places.

1. Coca-Cola Classic comes in a 2-liter container. How many fluid ounces is that? (*Note:* There are 32 ounces in a quart.)
2. A child's height is 120 cm. How tall is she in inches?
3. Convert to the desired unit:
   a. 12 inches = __________ cm
   b. 100 yards = __________ meters
   c. 20 kilograms = __________ pounds
   d. $40,000 per year = $__________ per hour (assume a 40-hour work week for 52 weeks)
   e. 24 hr/day = __________ sec/day
   f. 1 gallon = __________ ml
   g. 1 mph = __________ ft/sec
4. The distance between the sun and the moon is $3.84 \cdot 10^8$ meters. Express this in kilometers.
5. The mean distance from our sun to Jupiter is $7.8 \cdot 10^8$ kilometers. Express this distance in meters.
6. A light year is about $5.88 \cdot 10^{12}$ miles. Verify that $9.46 \cdot 10^{12}$ kilometers $\approx 5.88 \cdot 10^{12}$ miles.
7. 1 angstrom (Å) $= 10^{-8}$ cm. Express 1 angstrom in meters.
8. If a road sign says the distance to Quebec is 218 km, what is the distance in miles?
9. The distance from Earth to the sun is about 93,000,000 miles. There are 5280 feet in a mile, and a dollar bill is approximately 6 inches long. Estimate how many dollar bills would have to be placed end to end to reach from Earth to the sun.
10. Fill in the missing parts of the following conversion.

$$\frac{2560 \text{ mi}}{4.2 \text{ hrs}} = \frac{2560 \text{ mi}}{4.2 \text{ hrs}} \cdot \frac{1.6 \text{ km}}{?} = \frac{?}{?}$$
$$= \frac{?}{?} \cdot \frac{?}{60 \text{ min}} = \frac{? \text{ km}}{? \text{ min}}$$

11. Anthrax spores, which were inhaled by postal workers, causing severe illness and death, are no larger than 5 microns in diameter. How much larger than a spore is the tip of a pencil that is 1 millimeter in diameter? (*Note:* A micron is the same as a micrometer, μm.)
12. Use the conversion factor of 1 light year $= 9.46 \cdot 10^{12}$ kilometers or $5.88 \cdot 10^{12}$ miles to determine the following.
    a. Alpha Centauri, the nearest star to our sun, is 4.3 light years away. What is the distance in kilometers? How many miles away is it?
    b. The radius of the Milky Way is 10^8 light years. How many meters is that?
    c. Deneb is a star 1600 light years from Earth. How far is that in feet?
13. If 1 angstrom, Å, $= 10^{-10}$ meter, determine the following values.
    a. The radius of a hydrogen atom is 0.5 angstrom. How many meters is the radius?
    b. The radius of a cell is 10^5 angstroms. How many meters is the cell's radius?
    c. A radius of a proton is 0.00001 angstrom. Express the proton's radius in meter.
14. The Harvard Bridge, which connects Cambridge to Boston along Massachusetts Avenue, is literally marked off in units called *Smoots*. A Smoot is equal to about 5.6 feet, the height of an M.I.T. fraternity pledge named Oliver Smoot. The bridge is approximately 364 Smoots long. How long is the bridge in feet? Show all units when doing your conversion.

Exercises for Section 4.4

Use the conversion table on the back cover of the text for problems in this section.

1. Change the following English units to the metric units indicated.
   a. 50 miles to kilometers
   b. 3 feet to meters
   c. 5 pounds to kilograms
   d. 12 inches to centimeters
   e. 60 feet to meters
   f. 4 quarts to liters
2. Change the following metric units to the English units indicated.
   a. 25 kilometers to miles
   b. 700 meters to yards
   c. 250 centimeters to inches
   d. 50 grams to ounces
   e. 10 kilograms to pounds
   f. 10,000 milliliters to quarts

3. For the following questions, make an estimate and then check your estimate using the conversion table on the inside back cover:

a. One foot is how many centimeters?

b. One foot is what part of a meter?

4. A football field is 100 yards long. How many meters is this? What part of a kilometer is this?

5. How many droplets of water are in a river that is 100 km long, 250 m wide, and 25 m deep? Assume a droplet is 1 milliliter. (*Note:* one liter = one cubic decimeter and 10 decimeters = l meter.)

6. a. A roll of aluminum foil claims to be 50 sq ft or 4.65 m^2. Show the conversion factors that would verify that these two measurements are equivalent.

b. One cm^3 of aluminum weighs 2.7 grams. If a sheet of aluminum foil is 0.003 8 cm thick, find the weight of the roll of aluminum foil in grams.

7. If a falling object accelerates at the rate of 9.8 meters per second every second, how many feet per second does it accelerate each second?

8. Convert the following to feet and express your answers in scientific notation.

a. The radius of the solar system is approximately 10^{12} meters.

b. The radius of a proton is approximately 10^{-15} meter.

9. The speed of light is approximately $1.86 \cdot 10^5$ miles/sec.

a. Write this number in decimal form and express your answer in words.

b. Convert the speed of light into meters per year. Show your work.

10. The average distance from Earth to the sun is about 150,000,000 km, and the average distance from the planet Venus to the sun is about 108,000,000 km.

a. Express these distances in scientific notation.

b. Divide the distance from Venus to the sun by the distance from Earth to the sun and express your answer in scientific notation.

c. The distance from Earth to the sun is called 1 astronomical unit (1 A.U.) How many astronomical units is Venus from the sun?

d. Pluto is 5,900,000,000 km from the sun. How many astronomical units is it from the sun?

11. The distance from Earth to the sun is approximately 150 million kilometers. If the speed of light is $3.00 \cdot 10^5$ km/sec, how long does it take light from the sun to reach Earth? If a solar flare occurs right now, how long would it take for us to see it?

12. Earth travels in an approximately circular orbit around the sun. The average radius of Earth's orbit around the sun is $9.3 \cdot 10^7$ miles. Earth takes one year, or 365 days, to complete one orbit.

a. Use the formula for the circumference of the circle to determine the distance the Earth travels in one year.

b. How many hours are in one year?

c. Speed is distance divided by time. Find the orbital speed of Earth in miles per hour.

13. A barrel of U.S. oil is 42 gallons. A barrel of British oil is 163.655 liters. Which barrel is larger and by how much?

14. A barrel of wheat is 3.2812 bushels (U.S. dry) or 4.0833 cubic feet.

a. How many cubic feet are in a bushel of wheat?

b. How many cubic inches are in a barrel?

c. How many cubic centimeters are in a bushel?

15. In the United States, land is measured in acres and one acre is 43,560 sq ft.

a. If you buy a one-acre lot that is in the shape of a square, what would be the length of each side in feet?

b. A newspaper advertisement states that all lots in a new housing development will be a minimum of one and a half acres. Assuming the lot is rectangular and has 150 ft of frontage, how deep will the minimal-size lot be? If the new home owner wants to fence in the lot, how many yards of fencing would be needed?

c. The metric unit for measuring land is the square hectometer. (A hectometer is a length of 100 meters.) Find the size of a one-acre lot if it were measured in square hectometers.

d. A hectare is 100 acres. How many one-acre lots can fit in a square mile? How many hectares is that?

16. Estimate the number of heartbeats in a lifetime. Explain your method.

17. A nanosecond is 10^{-9} second. Modern computers can perform on the order of one operation every nanosecond. Approximately how many feet does an electrical signal moving at the speed of light travel in a computer in 1 nanosecond?

18. Since light takes time to travel, everything we see is from the past. When you look in the mirror, you see yourself not as you are, but as you were nanoseconds ago.

a. Suppose you look up tonight at the bright star Deneb. Deneb is 1600 light years away. When you look at Deneb, how old is the image you are seeing?

b. Even more disconcerting is the fact that what we see as simultaneous events do not necessarily occur simultaneously. Consider the two stars Betelgeuse and Rigel in the constellation Orion. Betelgeuse is 300 and Rigel 500 light years away. How many years apart were the images generated that we see simultaneously?

19. The world population in 2005 was approximately 6.45 billion people. During that year the Coca-Cola company claimed that 15,000 of their beverages were consumed every second. What was the worldwide consumption of their beverages per year per person in 2005?

20. A homeowner would like to spread shredded bark (mulch) over her flowerbeds. She has three flowerbeds measuring 25 ft by 3 ft, 15 ft by 4 ft, and 30 ft by 1.5 ft. The recommended depth for the mulch is 4 inches, and the shredded bark costs $27.00 per one cubic yard. How much will it cost to cover all of the flowerbeds with shredded bark? (*Note:* You cannot buy a portion of a cubic yard of mulch.)

21. A circular swimming pool is 18 ft in diameter and 4 ft deep.

a. Determine the volume of the pool in gallons if one gallon is 231 cubic inches.

b. The pool's filter pump can circulate 2500 gal per hour. How many hours do you need to run the filter in order to filter the number of gallons contained in the pool?

c. One pound of chlorine shock treatment can treat 10,000 gal. How much of the shock treatment should you use?

22. An angstrom, Å, is a metric unit of length equal to one ten billionth of a meter. It is useful in specifying wavelengths of electromagnetic radiation (e.g., visible light, ultraviolet light, X-rays, and gamma rays).

a. The visible-light spectrum extends from approximately 3900 angstroms (violet light) to 7700 angstroms (red light) Write this range in centimeters using scientific notation.

b. Some gamma rays have wavelengths of 0.0001 angstrom. Write this number in centimeters using scientific notation.

c. The nanometer (nm) is 10 times larger than the angstrom, so 1 nm is equal to how many meters?

23. The National Institutes of Health guidelines suggest that adults over 20 should have a body mass index, or BMI, under 25. This index is created according to the formula

$$\text{BMI} = \frac{\text{weight in kilograms}}{(\text{height in meters})^2}$$

a. Given that 1 kilogram = 2.2 pounds, and 1 meter = 39.37 inches, calculate the body mass index of President George W. Bush, who is 6 feet tall and in 2003 weighed 194 pounds. According to the guidelines, how would you describe his weight?

b. Most Americans don't use the metric system. So in order to make the BMI easier to use, convert the formula to an equivalent one using weight in pounds and height in inches. Check your new formula by using Bush's weight and height, and confirm that you get the same BMI.

c. The following excerpt from the article "America Fattens Up" (*The New York Times,* October 20, 1996) describes a very complicated process for determining your BMI:

To estimate your body mass index you first need to convert your weight into kilograms by multiplying your weight in pounds by 0.45. Next, find your height in inches. Multiply this number by 0.254 to get meters. Multiply that number by itself and then divide the result into your weight in kilograms. Too complicated? Internet users can get an exact calculation at http://141.106.68.17/bsa.acgl.

Can you do a better job of describing the process?

d. A letter to the editor from Brent Kigner, of Oneonta, N.Y., in response to the *New York Times* article says:

Math intimidates partly because it is often made unnecessarily daunting. Your article "American Fattens Up" convolutes the procedure for calculating the Body Mass Index so much that you suggest readers retreat to the Internet. In fact, the formula is simple: Multiply your weight in pounds by 703, then divide by the square of your height in inches. If the result is above 25, you weigh too much.

Is Brent Kigner right?

24. Computer technology refers to the storage capacity for information with its own special units. Each minuscule electrical switch is called a "bit" and can be off or on. As the information capacity of computers has increased, the industry has developed some much larger units based on the bit:

1 byte = 8 bits

1 kilobit = 2^{10} bits, or 1024 bits (a kilobit is sometimes abbreviated Kbit)

1 kilobyte = 2^{10} bytes, or 1024 bytes (a kilobyte is sometimes abbreviated Kbyte)

1 megabit = 2^{20} bits, or 1,048,576 bits

1 megabyte = 2^{20} bytes, or 1,048,576 bytes

1 gigabyte = 2^{30} bytes, or 1,073,741,824 bytes

a. How many kilobytes are there in a megabyte? Express your answer as a power of 2 and in scientific notation.

b. How many bits are there in a gigabyte? Express your answer as a power of 2 and in scientific notation.

25. The accompanying excerpt is from an article about Planck's length, which at 10^{-35} meter is believed to be the smallest length or size anything can be in the universe (from G. Johnson, "How Is the Universe Built? Grain by Grain," in the science section of the Dec. 7, 1999, *New York Times,* p. D1). Read the accompanying excerpt and then answer the following questions.

a. How many kilometers is Planck's length?

b. How many miles is Planck's length?

c. If light travels at $3 \cdot 10^8$ m/sec, how long will it take light to cross a distance equivalent to Planck's length?

Slightly smaller than what Americans quaintly insist on calling half an inch, a centimeter (one-hundredth of a meter) is easy enough to see. Divide this small length into 10 equal slices and you are looking, or probably squinting, at a millimeter (one-thousandth, or 10 to the minus 3 meters). By the time you divide one of these tiny units into a thousand minuscule micrometers, you have far exceeded the limits of the finest bifocals. But in the mind's eye, let the cutting continue, chopping the micrometer into a thousand nanometers and the nanometers into a thousand picometers, and those in steps of a thousandfold into femtometers, attometers, zeptameters, and yoctometers. At this point, 10 to the minus 24 meters, about one-billionth the radius of a proton, the roster of Greek names runs out. But go ahead and keep dividing, again and again until you reach a length only one hundred-billionth as large as that tiny amount: 10 to the minus 35 meters. . . . You have finally hit rock bottom: a span called the Planck length, the shortest anything can get. According to recent developments in the quest to devise the "theory of everything," space is not an infinitely divisible continuum. It is not smooth but granular, and the Planck length gives the size of the smallest possible grains.

The time it takes for a light beam to zip across this ridiculously tiny distance . . . is called Planck time, the shortest possible tick of an imaginary clock.

4.5 *Fractional Exponents*

So far we have derived rules for operating with expressions of the form a^n, where n is any integer. These rules can be extended to expressions of the form $a^{m/n}$, where the exponent is a fraction. We need first to consider what an expression such as $a^{m/n}$ means.

The expression m/n can also be written as $m \cdot (1/n)$ or $(1/n) \cdot m$. If the laws of exponents are consistent, then

$$a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m$$

For example, if we apply Rule 3 for exponents to the expression $(a^{1/3})^2$, then the following is true:

$$(a^{1/3})^2 = a^{(1/3)2} = a^{2/3}$$

Square Roots: Expressions of the Form $a^{1/2}$

The expression $a^{1/2}$ is called the *principal square root* (or just the *square root*) of a and is often written as $\sqrt{a}$. The symbol $\sqrt{}$ is called a *radical*. The principal square root of a is the *nonnegative* number b such that $b^2 = a$. Both the square of -2 and the square of 2 are equal to 4, but the notation $\sqrt{4}$ is defined as *only the positive root*. If both -2 and 2 are to be considered, we write $\pm\sqrt{4}$, which means "plus or minus the square root of 4."

So, $\sqrt{4} = 2$ and $\pm\sqrt{4} = 2$ and -2. When you solve $x^2 = 4$, the solution is 2 and -2.

In the real numbers, $\sqrt{a}$ is not defined when a is negative. For example, $\sqrt{-4}$ is undefined, since there is no real number b such that $b^2 = -4$.

> **The Square Root**
> For $a \geq 0$,
>
> $$a^{1/2} = a^{0.5} = \sqrt{a}$$
>
> where $\sqrt{a}$ is the nonnegative number b such that $b^2 = a$.
>
> For example, $25^{1/2} = \sqrt{25} = 5$, since $5^2 = 25$.

Estimating Square Roots. A number is called *a perfect square* if its square root is an integer. For example, 25 and 36 are both perfect squares since $25 = 5^2$ and $36 = 6^2$, so $\sqrt{25} = 5$ and $\sqrt{36} = 6$. If we don't know the square root of some number x and don't have a calculator handy, we can estimate the square root by bracketing it between two perfect squares, a and b, for which we do know the square roots. If $0 \leq a < x < b$, then $\sqrt{a} < \sqrt{x} < \sqrt{b}$. For example, to estimate $\sqrt{10}$,

we know	$9 < 10 < 16$	where 9 and 16 are perfect squares
so	$\sqrt{9} < \sqrt{10} < \sqrt{16}$	
and	$3 < \sqrt{10} < 4$	

Therefore $\sqrt{10}$ lies somewhere between 3 and 4, probably closer to 3 because 10 is closer to 9 than to 16. According to a calculator, $\sqrt{10} \approx 3.16$.

EXAMPLE 1 Estimate $\sqrt{27}$.

SOLUTION

We know $25 < 27 < 36$

therefore $\sqrt{25} < \sqrt{27} < \sqrt{36}$

and $5 < \sqrt{27} < 6$

So $\sqrt{27}$ lies somewhere between 5 and 6. Would you expect $\sqrt{27}$ to be closer to 5 or to 6? Check your answer with a calculator.

Using a calculator

Many calculators and spreadsheet programs have a square root function, often labeled $\sqrt{\ }$ or perhaps "SQRT." You can also calculate square roots by raising a number to the $\frac{1}{2}$ or 0.5 power using the ^ key, as in 4 ^ 0.5. Try using a calculator to find $\sqrt{4}$ and $\sqrt{9}$.

In any but the simplest cases where the square root is immediately obvious, you will probably use the calculator. For example, use your calculator to find

$$8^{1/2} = \sqrt{8} \approx 2.8284$$

Double-check the answer by verifying that $(2.8284)^2 \approx 8$.

EXAMPLE 2 **Calculating square roots**

The function $S = \sqrt{30d}$ describes the relationship between S, the speed of a car in miles per hour, and d, the distance in feet a car skids after applying the brakes on a dry tar road. Use a calculator to estimate the speed of a car that:

a. Leaves 40-foot-long skid marks on a dry tar road.

b. Leaves 150-foot-long skid marks.

SOLUTION

a. If $d = 40$ feet, then $S = \sqrt{30 \cdot 40} = \sqrt{1200} \approx 35$, so the car was traveling at about 35 miles per hour.

b. If $d = 150$ feet, then $S = \sqrt{30 \cdot 150} = \sqrt{4500} \approx 67$, so the car was traveling at almost 70 miles per hour.

EXAMPLE 3 Assuming that the surface area of Earth is approximately 200 million square miles, estimate the radius of Earth.

SOLUTION

Step 1. Find the formula for the radius of a sphere.

If we assume that Earth is roughly spherical, we can solve for the radius r in the formula for the surface of a sphere, $S = 4\pi r^2$. We get

$$r = \sqrt{\frac{S}{4\pi}} = \sqrt{\frac{1}{4}} \cdot \sqrt{\frac{S}{\pi}} = \frac{1}{2}\sqrt{\frac{S}{\pi}}$$

Step 2. Estimate the radius of Earth.

Given that Earth's surface area is approximately 200,000,000 square miles, we can use our derived formula to estimate Earth's radius. Substituting for S, we get

$$\begin{aligned} r &= \frac{1}{2}\sqrt{\frac{200{,}000{,}000 \text{ miles}^2}{\pi}} \\ &\approx \frac{1}{2}\sqrt{63{,}661{,}977 \text{ miles}^2} \\ &\approx \frac{1}{2} \cdot 7979 \text{ miles} \approx 3989 \text{ miles} \end{aligned}$$

So Earth has a radius of about 4000 miles.

*n*th Roots: Expressions of the Form $a^{1/n}$

The term $a^{1/n}$ denotes the *n*th root of *a*, often written as $\sqrt[n]{a}$. For $a \geq 0$, the *n*th root of *a* is the nonnegative number whose *n*th power is *a*.

$$8^{1/3} = \sqrt[3]{8} = 2 \qquad \text{since } 2^3 = 8 \text{ (we call 2 the third or cube root of 8)}$$
$$16^{1/4} = \sqrt[4]{16} = 2 \qquad \text{since } 2^4 = 16 \text{ (we call 2 the fourth root of 16)}$$

For $a < 0$, if *n* is odd, $\sqrt[n]{a}$ is the negative number whose *n*th power is *a*. Note that if *n* is even, then $\sqrt[n]{a}$ is not a real number when $a < 0$.

$$(-8)^{1/3} = \sqrt[3]{-8} = -2 \qquad \text{since } (-2)^3 = -8.$$
$$(-27)^{1/3} = \sqrt[3]{-27} = -3 \qquad \text{since } (-3)^3 = -27$$
$$(-16)^{1/4} = \sqrt[4]{-16} \text{ is not a real number}$$

The *n*th Root

If *a* is a real number and *n* is a positive integer,

$$a^{1/n} = \sqrt[n]{a}, \qquad \text{the } n\text{th root of } a$$

For $a \geq 0$,

$\sqrt[n]{a}$ is the nonnegative number *b* such that $b^n = a$.

For $a < 0$,

If *n* is odd, $\sqrt[n]{a}$ is the negative number *b* such that $b^n = a$.
If *n* is even, $\sqrt[n]{a}$ is not a real number.

If the *n*th root exists, you can find its value on a calculator. For example, to determine a fifth root, raise the number to the $\frac{1}{5}$ or the 0.2 power. So

$$3125^{1/5} = \sqrt[5]{3125} = 5$$

Double-check your answer by verifying that $5^5 = 3125$.

EXAMPLE 4 Simplify:

a. $625^{1/4}$ **b.** $(-625)^{1/4}$ **c.** $125^{1/3}$ **d.** $(-125)^{1/3}$

SOLUTION

a. $625^{1/4} = 5$ since $5^4 = 625$

b. $(-625)^{1/4}$ does not have a real-number solution

c. $125^{1/3} = 5$ since $5^3 = 125$

d. $(-125)^{1/3} = -5$ since $(-5)^3 = -125$

EXAMPLE 5

a. The volume of a sphere is given by the equation $V = \frac{4}{3}\pi r^3$. Rewrite the formula, solving for the radius as a function of the volume.

b. If the volume of a sphere is 370 cubic inches, what is its radius? What common object might have that radius?

c. What are the dimensions of a cube that contains this volume?

SOLUTION

a. Given: $V = \frac{4}{3}\pi r^3$

multiply both sides by 3 $3V = 4\pi r^3$

divide by 4π $\quad \frac{3V}{4\pi} = r^3$

take the cube root and switch sides $\quad r = \sqrt[3]{\frac{3V}{4\pi}}$

b. Substituting 370 for V and 3.14 for π in our derived formula in part (a), we have

$$r \approx \sqrt[3]{\frac{3 \cdot 370}{4 \cdot 3.14}} \approx 4.45 \text{ inches}$$

A regulation-size soccer ball is basically a sphere with a radius of about 4.45 inches.

c. A cube has the same length on all three sides. If s is the side length, then the volume of the desired cube is $s^3 = 370$ cubic inches. So $s = \sqrt[3]{370} \approx 7.18$ inches. A cube of length 7.18 inches on each side would give a volume equivalent to a sphere with a radius of 4.45 inches.

Rules for Radicals

The following rules can help you compute with radicals. They represent extensions of the rules for integer exponents. In the following table we assume that m and n are positive integers and that $\sqrt[n]{a}$ and $\sqrt[n]{b}$ exist.

Rules for Radicals

	Example
1. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$	$\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$
2. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad b \neq 0$	$\frac{\sqrt[4]{125}}{\sqrt[4]{25}} = \sqrt[4]{\frac{125}{25}} = \sqrt[4]{5}$
3. $(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a \quad a > 0$	$(\sqrt{7})^2 = \sqrt{7^2} = 7$

EXAMPLE 6 **Simplifying radicals**

Simplify the following radical expressions. Assume all variables are nonnegative real numbers.

a. $\sqrt[3]{625x^4}$ **b.** $3\sqrt{48} - 5\sqrt{27}$

SOLUTION **a.** Factor 625 $\quad \sqrt[3]{625x^4} = \sqrt[3]{5^4 \cdot x^4}$

rewrite using perfect cube factors $\quad = \sqrt[3]{5^3 \cdot 5 \cdot x^3 \cdot x}$

use Rule 1 for radicals $\quad = \sqrt[3]{5^3x^3} \cdot \sqrt[3]{5x}$

extract the perfect cubes (Rule 3), leaving the remaining factors under the radical $\quad = 5x \cdot \sqrt[3]{5x}$

b. Find the largest perfect square factors $\quad 3\sqrt{48} - 5\sqrt{27} = 3\sqrt{16 \cdot 3} - 5\sqrt{9 \cdot 3}$

extract the perfect squares (Rule 3) $\quad = 3 \cdot 4\sqrt{3} - 5 \cdot 3\sqrt{3}$

multiply $\quad = 12\sqrt{3} - 15\sqrt{3}$

use distributive law $\quad = (12 - 15)\sqrt{3}$

$\quad = -3\sqrt{3}$

Algebra Aerobics 4.5a

1. Evaluate each of the following without a calculator.
   a. $81^{1/2}$ b. $144^{1/2}$ c. $36^{1/2}$ d. $-49^{1/2}$ e. $(-36)^{1/2}$
2. Assume that all variables represent nonnegative quantities. Then simplify and rewrite the following without radical signs. (Use fractional exponents if necessary.)
   a. $\sqrt{9x}$ b. $\sqrt{\frac{x^2}{25}}$ c. $\sqrt{36x^2}$ d. $\sqrt{\frac{9y^2}{25x^4}}$ e. $\sqrt{\frac{49}{x^2}}$ f. $\sqrt{\frac{4a}{169}}$
3. Use the formula in Example 2 in this section to estimate the following:
   a. The speed of a car that leaves 60-foot-long skid marks on a dry tar road.
   b. The speed of a car that leaves 200-foot-long skid marks on a dry tar road.
4. Without a calculator, find two consecutive integers between which the given square root lies.
   a. $\sqrt{29}$ b. $\sqrt{92}$ c. $\sqrt{117}$ d. $\sqrt{79}$ e. $\sqrt{39}$
5. Evaluate each of the following without a calculator:
   a. $27^{1/3}$ b. $16^{1/4}$ c. $8^{-1/3}$ d. $32^{1/5}$ e. $27^{-1/3}$ f. $25^{-1/2}$ g. $\left(\frac{8}{27}\right)^{-1/3}$ h. $\left(\frac{1}{16}\right)^{1/2}$
6. Evaluate:
   a. $\sqrt[3]{-27}$ b. $(-10{,}000)^{1/4}$ c. $(-1000)^{1/3}$ d. $-16^{1/4}$ e. $(-8)^{1/3}$ f. $\sqrt{2500}$
7. Estimate the radius of a spherical balloon with a volume of 2 cubic feet.
8. Simplify if possible.
   a. $\sqrt{9+16}$ b. $-\sqrt{49}$ c. $\sqrt[3]{-125}$ d. $\sqrt{45} - 3\sqrt{125}$
9. Change each radical expression into exponent form, then simplify. Assume all variables are nonnegative.
   a. $\sqrt{36}$ b. $\sqrt[3]{27x^6}$ c. $\sqrt[4]{81a^4b^{12}}$
10. Solve for the indicated variable. Assume all variables represent nonnegative quantities.
    a. $V = \pi r^2 h$ for r
    b. $V = \frac{1}{3}\pi r^2 h$ for r
    c. $V = s^3$ for s
    d. $c^2 = a^2 + b^2$ for a
    e. $S = 6x^2$ for x

You may want to do Exploration 4.2 on Kepler's laws of planetary motion after reading this section.

Fractional Powers: Expressions of the Form $a^{m/n}$

In the beginning of this section, we saw that we can write $a^{m/n}$ either as $(a^m)^{1/n}$ or $(a^{1/n})^m$. Writing it as $(a^m)^{1/n}$ means that we would first raise the base, a, to the mth power and then take the nth root of that. Writing it as $(a^{1/n})^m$ implies first finding the nth root of a and then raising that to the mth power. For example,

$$\begin{aligned} 2^{3/2} &= (2^3)^{1/2} \\ &= (8)^{1/2} \\ \text{using a calculator} \quad &\approx 2.8284 \end{aligned}$$

$$\begin{aligned} \text{Equivalently,} \quad 2^{3/2} &= (2^{1/2})^3 \\ &\approx (1.414)^3 \\ &\approx 2.8284 \end{aligned}$$

We could, of course, use a calculator to compute $2^{3/2}$ (or $2^{1.5}$) directly by raising 2 to the $\frac{3}{2}$ or 1.5 power.

If $a \geq 0$ and m and n are integers $(n \neq 0)$, then using radical notation,

$$\begin{aligned} a^{m/n} &= \left(\sqrt[n]{a}\right)^m \\ \text{or equivalently} \quad &= \sqrt[n]{a^m} \end{aligned}$$

Exponents expressed as ratios of the form m/n are called *rational exponents*. The set of laws for simplifying expressions with integer exponents also holds for real exponents, which includes rational exponents.

EXAMPLE 7 Find the product of $(\sqrt{5}) \cdot (\sqrt[3]{5})$, leaving the answer in exponent form.

SOLUTION

$$\left(\sqrt{5}\right) \cdot \left(\sqrt[3]{5}\right) = 5^{1/2} \cdot 5^{1/3} = 5^{(1/2)+(1/3)} = 5^{5/6}$$

EXAMPLE 8 According to McMahon and Bonner in *On Size and Life*,[4] common nails range from 1 to 6 inches in length. The weight varies even more, from 11 to 647 nails per pound. Longer nails are relatively thinner than shorter ones. A good approximation of the relationship between length and diameter is given by the equation

$$d = 0.07L^{2/3}$$

where d = diameter and L = length, both in inches. Estimate the diameters of nails that are 1, 3, and 6 inches long.

SOLUTION

When $L = 1$ inch, the diameter $d = 0.07 \cdot (1)^{2/3} = 0.07 \cdot 1 = 0.07$ inches.
When $L = 3$ inches, then $d = 0.07 \cdot (3)^{2/3} \approx 0.07 \cdot 2.08 \approx 0.15$ inches.
When $L = 6$ inches, then $d = 0.07 \cdot (6)^{2/3} \approx 0.07 \cdot 3.30 \approx 0.23$ inches.

Summary of Zero, Negative, and Fractional Exponents

If m and n are integers and $a \neq 0$, then

$$a^0 = 1$$
$$a^{-n} = \frac{1}{a^n}$$
$$a^{1/n} = \sqrt[n]{a}$$
$$a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m \qquad a > 0$$

Algebra Aerobics 4.5b

Assume all variables represent positive quantities.

1. Find the product expressed in exponent form:
   a. $\sqrt{2} \cdot \sqrt[3]{2}$ c. $\sqrt{3} \cdot \sqrt[3]{9}$ e. $\sqrt[4]{x^3} \cdot \sqrt{x}$
   b. $\sqrt{5} \cdot \sqrt[4]{5}$ d. $\sqrt[4]{x} \cdot \sqrt[3]{x}$ f. $\sqrt[3]{xy^2} \cdot \sqrt{xy}$
2. Find the quotient by representing the expression in exponent form. Leave the answer in positive exponent form.
   a. $\dfrac{\sqrt{2}}{\sqrt[3]{2}}$ b. $\dfrac{2}{\sqrt[4]{2}}$ c. $\dfrac{\sqrt[4]{5}}{\sqrt[3]{5}}$ d. $\dfrac{\sqrt{x}}{\sqrt[4]{x^3}}$ e. $\dfrac{\sqrt[3]{xy^2}}{\sqrt{xy}}$
3. McMahon and Bonner give the relationship between chest circumference and body weight of adult primates as

$$c = 17.1w^{3/8}$$

   where w = weight in kilograms and c = chest circumference in centimeters. Estimate the chest circumference of a:
   a. 0.25-kg tamarin b. 25-kg baboon
4. Simplify each expression by removing all possible factors from the radical.
   a. $\sqrt{20x^2}$ c. $\sqrt[3]{16x^3y^4}$
   b. $\sqrt{75a^3}$ d. $\dfrac{\sqrt[4]{32x^4y^6}}{\sqrt[4]{81x^8y^5}}$
5. Change each radical expression into a form with fractional exponents, then simplify.
   a. $\sqrt{4a^2b^6}$ c. $\sqrt[3]{8.0 \cdot 10^{-9}}$
   b. $\sqrt[4]{16x^4y^6}$ d. $\sqrt{8a^{-4}}$

[4] T. A. McMahon and J. Tyler Bonner, *On Size and Life* (New York: Scientific American Library, Scientific American Books, 1983).

Exercises for Section 4.5

1. Evaluate without a calculator:

a. $100^{1/2}$ **c.** $100^{-1/2}$ **e.** $-1000^{1/3}$

b. $-100^{1/2}$ **d.** $-100^{-1/2}$ **f.** $(-1000)^{1/3}$

2. Evaluate without a calculator:

a. $\sqrt{10{,}000}$ **c.** $625^{1/2}$ **e.** $\left(\frac{1}{9}\right)^{1/2}$

b. $\sqrt{-25}$ **d.** $100^{1/2}$ **f.** $\left(\frac{625}{100}\right)^{1/2}$

3. Assume that all variables represent positive quantities and simplify.

a. $\sqrt{\frac{a^2b^4}{c^6}}$ **b.** $\sqrt{36x^4y}$ **c.** $\sqrt{\frac{49x}{y^6}}$ **d.** $\sqrt{\frac{x^4y^2}{100z^6}}$

4. Simplify each expression by removing all possible factors from the radical, then combining any like terms.

a. $2\sqrt{50} + 12\sqrt{8}$ **c.** $10\sqrt{32} - 6\sqrt{18}$

b. $3\sqrt{27} - 2\sqrt{75}$ **d.** $2\sqrt[3]{16} + 4\sqrt[3]{54}$

5. Simplify by removing all possible factors for each radical. Assume all variable quantities are positive.

a. $\sqrt{125a}$ **c.** $\sqrt{8x^3y^2}$

b. $\sqrt{\frac{x^2}{4x^4y^6}}$ **d.** $\sqrt{64x^4y^5}$

6. Estimate the radius, r, of a circular region with an area, A, of 35 ft^2 (where $A = \pi r^2$).

7. Evaluate each expression without using a calculator.

a. $\sqrt{36 \cdot 10^6}$ **c.** $\sqrt[4]{625 \cdot 10^{20}}$

b. $\sqrt[3]{8 \cdot 10^9}$ **d.** $\sqrt{1.0 \cdot 10^{-4}}$

8. Calculate the following:

a. $4^{1/2}$ **c.** $27^{1/3}$ **e.** $8^{2/3}$ **g.** $16^{1/4}$

b. $-4^{1/2}$ **d.** $-27^{1/3}$ **f.** $-8^{2/3}$ **h.** $16^{3/4}$

9. Calculate:

a. $\left(\frac{1}{100}\right)^{1/2}$ **b.** $25^{-1/2}$ **c.** $\left(\frac{9}{16}\right)^{-1/2}$ **d.** $\left(\frac{1}{1000}\right)^{1/3}$

10. Estimate the length of a side, s, of a cube with volume, V, of 6 cm^3 (where $V = s^3$).

11. Evaluate when $x = 2$:

a. $(-x)^2$ **c.** $x^{1/2}$ **e.** $x^{3/2}$

b. $-x^2$ **d.** $(-x)^{1/2}$ **f.** x^0

12. Determine if the following statements are true or false.

a. $\sqrt[4]{(3x^2)^4} = 3x^2$

b. $\sqrt[3]{(x+1)^4} = (x+1)\left(\sqrt[3]{x+1}\right)$

c. $\sqrt[3]{\frac{9}{25}} = \frac{\sqrt[3]{45}}{5}$

d. $\sqrt{15} - \sqrt{3} = \sqrt{12}$

13. Use $<$, $>$, or $=$ to make each statement true.

a. $\sqrt{3} + \sqrt{7}$ _?_ $\sqrt{3+7}$

b. $\sqrt{3^2 + 2^2}$ _?_ 5

c. $\sqrt{5^2 - 4^2}$ _?_ 2

14. Fill in the missing forms in the table.

Radical Form	Rational Exponent Form
a. $\sqrt[3]{64} = 4$	
b.	$-(144)^{1/2} = -12$
c. $\left(\sqrt[4]{81x}\right)^3 = 27 \cdot \sqrt[4]{x^3}$	
d.	$(-243)^{1/5} = -3$
e.	$16^{5/4} = 32$

15. Evaluate:

a. $27^{2/3}$ **b.** $16^{-3/4}$ **c.** $25^{-3/2}$ **d.** $81^{-3/4}$

16. Without using a calculator, find two *consecutive* integers such that one is smaller and one is larger than each of the following (for example, $3 < \sqrt{11} < 4$). Show your reasoning.

a. $\sqrt{13}$ **b.** $\sqrt{22}$ **c.** $\sqrt{40}$

17. Estimate the radius of a spherical balloon that has a volume of 4 ft^3.

18. *Constellation.* Reduce each of the following expressions to the form $u^a \cdot m^b$; then plot the exponents as points with coordinates (a, b) on graph paper. Do you recognize the constellation?

a. $\frac{(u^2)^2 \cdot m}{u^2 \cdot m^{-4}}$ **e.** $\frac{u^{-3/2} \cdot u^{-7/2} \cdot m^1 \cdot (m^3)^3}{(um)^2}$

b. $\frac{u^{-9/5} \cdot m^3}{(umu^2)^1 \cdot m^{-1}}$ **f.** $\frac{1}{u^{12} \cdot m^{-9}}$

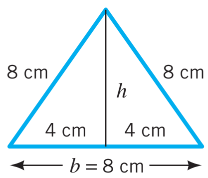
c. $\frac{u^2 \cdot u^{-4}}{u^3 \cdot (m^{-2})^3}$ **g.** $\frac{(mu)^0 \cdot (u^{10})^{-1} \cdot m^{1/4}}{(m^{-3} \cdot u^{-1/3})^3}$

d. $\frac{(um^2)^3 \cdot u^2}{(um)^4}$

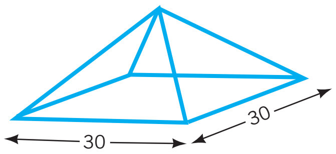
19. An equilateral triangle has sides of length 8 cm.

a. Find the height of the triangle. (*Hint:* Use the Pythagorean theorem on the inside back cover.)

b. Find the area A of the triangle if $A = \frac{1}{2}bh$.



20. An Egyptian pyramid consists of a square base and four triangular sides. A model of a pyramid is constructed using four equilateral triangles each with a side length of 30 inches. Find the surface area of the pyramid model, including the base. (*Note:* Surface area is the sum of the areas of the four triangular sides and the rectangular base. The previous exercise gives the formula for finding the area of a triangle.)

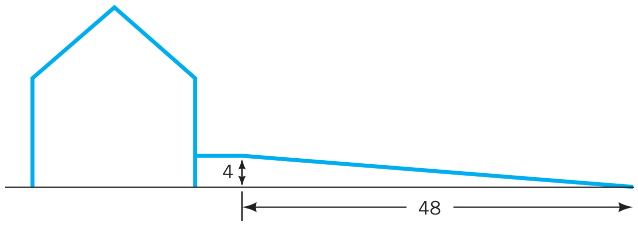


21. The time it takes for one complete swing of a pendulum is called the *period* of its motion. The period T (in seconds) of a swinging pendulum is found using the formula $T = 2\pi\sqrt{\frac{L}{32}}$, where L is the length of the pendulum in feet and 32 is the acceleration of gravity in feet per second.^{2}

a. Find the period of a pendulum whose length is 2 ft 8 in.

b. How long would a pendulum have to be to have a period of 2 seconds?

22. (Requires the use of a calculator that can evaluate powers.) A wheelchair ramp is constructed at the end of a porch, which is 4 ft off the ground. The base of the ramp is 48 ft from the porch. How long is the ramp? (*Hint:* Use the Pythagorean theorem on the inside back cover.)



23. (Requires the use of a calculator that can evaluate powers.) The breaking strength S (in pounds) of a three-strand manila rope is a function of its diameter, D (in inches). The relationship can be described by the equation $S = 1700D^{1.9}$. Calculate the breaking strength when D equals:

a. 1.5 in **b.** 2.0 in

24. (Requires the use of a calculator that can evaluate powers.) If a rope is wound around a wooden pole, the number of pounds of frictional force, F, between the pole and the rope is a function of the number of turns, N, according to the equation $F = 14 \cdot 10^{0.70N}$. What is the frictional force when the number of turns is:

a. 0.5 **b.** 1 **c.** 3

4.6 *Orders of Magnitude*

Comparing Numbers of Widely Differing Sizes

We have seen that a useful method of comparing two objects of widely different sizes is to calculate the ratio rather than the difference of the sizes. The ratio can be estimated by computing *orders of magnitude,* the number of times we would have to multiply or divide by 10 to convert one size to the other. Each factor of 10 represents one order of magnitude.

For example, the radius of the observable universe is approximately 10^{26} meters and the radius of our solar system is approximately 10^{12} meters. To compare the radius of the observable universe to the radius of our solar system, calculate the ratio

$$\frac{\text{radius of the universe}}{\text{radius of our solar system}} \approx \frac{10^{26} \text{ meters}}{10^{12} \text{ meters}} \approx 10^{26-12} \approx 10^{14}$$

Orders of Magnitude

The radius of the universe is roughly 10^{14} times larger than the radius of the solar system; that is, we would have to multiply the radius of our solar system by 10 fourteen times in order to obtain the radius of the universe. Since each factor of 10 is counted as a single order of magnitude, the radius of the universe is *fourteen orders of magnitude larger* than the radius of our solar system. Equivalently, we could say that the radius of our solar system is *fourteen orders of magnitude smaller* than the radius of the universe.

When something is one order of magnitude larger than a *reference object,* it is 10 times larger. You *multiply* the *reference size* by 10 to get the other size. If the object is two orders of magnitude larger, it is 100 or 10^2 times larger, so you would multiply the reference size by 100. If it is one order of magnitude smaller, it is 10 times smaller, so you would *divide* the reference size by 10. Two orders of magnitude smaller means the reference size is divided by 100 or 10^2.

EXAMPLE 1

The radius of the sun (10^9 meters) is how much larger than the radius of a hydrogen atom (10^{-11} meter)?

SOLUTION

$$\frac{\text{radius of sun}}{\text{radius of the hydrogen atom}} \approx \frac{10^9 \text{ meters}}{10^{-11} \text{ meters}}$$
$$\approx 10^{9-(-11)}$$
$$\approx 10^{20}$$

So the radius of the sun is 10^{20} times, or twenty orders of magnitude, larger than the radius of the hydrogen atom.

EXAMPLE 2

Compare the length of an unwound DNA strand (10^{-2} meter) with the size of a living cell (radius of 10^{-5} meter).

SOLUTION

$$\frac{\text{length of DNA strand}}{\text{radius of the living cell}} \approx \frac{10^{-2} \text{ meter}}{10^{-5} \text{ meter}}$$
$$\approx 10^{-2-(-5)}$$
$$\approx 10^{-2+5}$$
$$\approx 10^3$$

Surprisingly enough, the average width of a living cell is approximately three orders of magnitude *smaller* than one of the single strands of DNA it contains, if the DNA is uncoiled and measured lengthwise.

The Richter Scale

The reading "Earthquake Magnitude Determination" describes how earthquake tremors are measured.

The *Richter scale,* designed by the American Charles Richter in 1935, allows us to compare the magnitudes of earthquakes throughout the world. The Richter scale measures the maximum ground movement (tremors) as recorded on an instrument called a seismograph. Earthquakes vary widely in severity, so Richter designed the scale to measure order-of-magnitude differences. The scale ranges from less than 1 to over 8. Each increase of one unit on the Richter scale represents an increase of ten times, or one order of magnitude, in the maximum tremor size of the earthquake. So an increase from 2.5 to 3.5 indicates a 10-fold increase in maximum tremor size. An increase of two units from 2.5 to 4.5 indicates an increase in maximum tremor size by a factor of 10^2 or 100.

Description of the Richter Scale

Richter Scale Magnitude	Description
2.5	Generally not felt, but recorded on seismographs
3.5	Felt by many people locally
4.5	Felt by all locally; slight local damage may occur
6	Considerable damage in ordinary buildings; a destructive earthquake
7	"Major" earthquake; most masonry and frame structures destroyed; ground badly cracked
8 and above	"Great" earthquake; a few per decade worldwide; total or almost total destruction; bridges collapse, major openings in ground, tremors visible

Table 4.6

Table 4.6 contains some typical values on the Richter scale along with a description of how humans near the center (called the *epicenter*) of an earthquake perceive its effects. There is no theoretical upper limit on the Richter scale. The U.S. Geological Survey reports that the largest measured earthquake in the United States was in Prince William Sound, Alaska, in 1964 (magnitude 9.2), and the largest in the world was in Chile in 1960 (magnitude 9.5).[5]

Graphing Numbers of Widely Differing Sizes: Log Scales

Exploration 4.1 asks you to construct a graph using logarithmic scales on both axes.

If the sizes of various objects in our solar system are plotted on a standard linear axis, we get the uninformative picture shown in Figure 4.1. The largest value stands alone, and all the others are so small when measured in terameters that they all appear to be zero. When objects of widely different orders of magnitude are compared on a linear scale, the effect is similar to pointing out an ant in a picture of a baseball stadium.

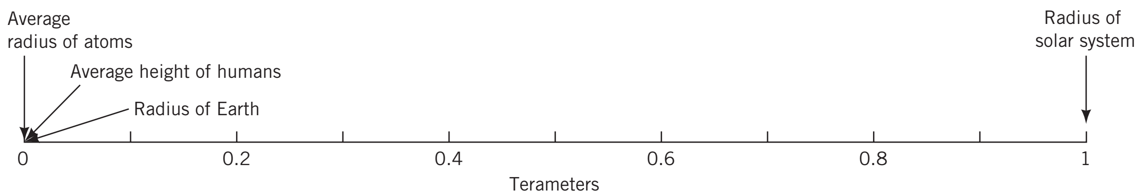


Figure 4.1 Sizes of various objects in the universe on a linear scale.
(*Note:* One terameter = 10^{12} meters.)

A more effective way of plotting sizes with different orders of magnitude is to use an axis that has powers of 10 evenly spaced along it. This is called a *logarithmic* or *log scale*. The plot of the previous data graphed on a logarithmic scale is much more informative (see Figure 4.2).

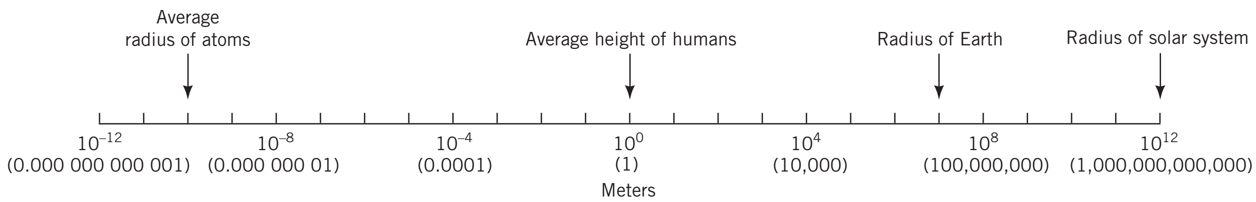


Figure 4.2 Sizes of various objects in the universe on an order-of-magnitude (logarithmic) scale.

Reading Log Scales

Graphing sizes on a log scale can be very useful, but we need to read the scales carefully. When we use a linear scale, each move of one unit to the right is equivalent to *adding* one unit to the number, and each move of k units to the right is equivalent to *adding* k units to the number (Figure 4.3).

[5]See the National Earthquake Information Center website at *http://earthquake.usgs.gov/eqcenter/historic_eqs.php.*

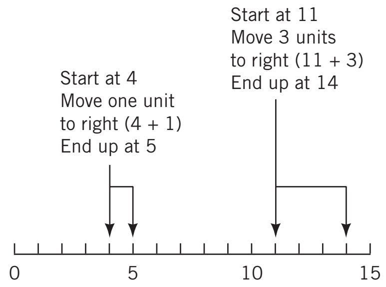


Figure 4.3 Linear scale.

When we use a log scale (see Figure 4.4), we need to remember that one unit of length now represents a change of one order of magnitude. Moving one unit to the right is equivalent to *multiplying by 10*. So moving from 10^4 to 10^5 is equivalent to multiplying 10^4 by 10. Moving three units to the right is equivalent to *multiplying* the starting number by 10^3, or 1000. In effect, a linear scale is an "additive" scale and a logarithmic scale is a "multiplicative" scale.

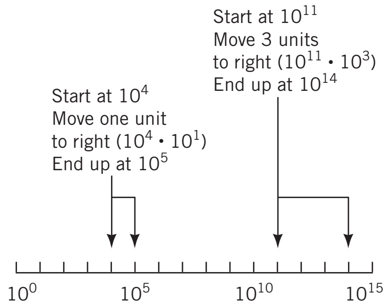


Figure 4.4 Order-of-magnitude (logarithmic) scale.

Algebra Aerobics 4.6

1. In 1987 Los Angeles had an earthquake that measured 5.9 on the Richter scale. In 1988 Armenia had an earthquake that measured 6.9 on the Richter scale. Compare the sizes of the two earthquakes using orders of magnitude.
2. On July 15, 2003, Little Rock, Arkansas, had an earthquake that measured 6.5 on the Richter scale. Compare the size of this earthquake to the largest ever recorded, 9.5 in Chile in 1960.
3. If my salary is \$100,000 per year and you make an order of magnitude more, what is your salary? If Henry makes two orders of magnitude less money than I do, what is his salary?
4. For each of the following pairs, determine the order-of-magnitude difference:
   a. The radius of the sun (10^9 meters) and the radius of the Milky Way (10^{21} meters)
   b. The radius of a hydrogen atom (10^{-11} meter) and the radius of a proton (10^{-15} meter)
5. Joe wants to move from Wyoming to California, but he has been advised that houses in California cost an order of magnitude more than houses in Wyoming.
   a. If Joe's house in Wyoming is worth \$400,000, how much would a similar house cost in California?
   b. If a house in California sells for \$650,000, how much would it cost in Wyoming?
6. How many orders of magnitude greater is a kilometer than a meter? Than a millimeter?
7. By rounding the number to the nearest power of 10, find the approximate location of each of the following on the logarithmic scale in Figure 4.2 on page 244.
   a. The radius of the sun, at approximately 1 billion meters
   b. The radius of a virus, at 0.000 000 7 meter
   c. An object whose radius is two orders of magnitude smaller than that of Earth

Exercises for Section 4.6

1. What is the order-of-magnitude difference between the following units? (Refer to table on inside back cover.)

a. A millimeter and a gigameter

b. A second and a day

c. A square centimeter and an acre ($1 \text{ acre} = 43{,}560 \text{ ft}^2$)

d. A microfarad and a picofarad

2. Fill in the blanks to make each of the following statements true.

a. Attaching the prefix "micro" to a unit __________ the size by __________ orders of magnitude.

b. Attaching the prefix "kilo" to a unit __________ the size by __________ orders of magnitude.

c. Scientists and engineers have designated prefix multipliers from septillionths (10^{-24}) to septillions (10^{24}), a span of __________ orders of magnitude.

3. Compare the following numbers using orders of magnitude.

a. 5.261 and 52.61 **c.** $5.261 \cdot 10^6$ and 526.1

b. 5261 and 5.261

4. An ant is roughly 10^{-3} meter in length and the average human roughly one meter. How many times longer is a human than an ant?

5. Refer to the chart in Exploration 4.1.

a. How many orders of magnitude larger is the Milky Way than the first living organism on Earth?

b. How many orders of magnitude older is the Pleiades (a cluster of stars) than the first *Homo sapiens*?

6. Water boils (changes from a liquid to a gas) at 373 kelvins. The temperature of the core of the sun is 20 million kelvins. By how many orders of magnitude is the sun's core hotter than the boiling temperature of water?

7. An electron weighs about 10^{-27} gram, and a raindrop weighs about 10^{-3} gram. How many times heavier is a raindrop than an electron? How many times lighter is an electron than a raindrop? What is the order-of-magnitude difference?

8. On Nov. 20, 2001, *The New York Times* reported that FBI scientists had found a sealed plastic bag with a letter addressed to Senator Patrick Leahy that was highly contaminated with anthrax. The article said that a sample taken from the bag "showed the presence of 23,000 anthrax spores. This, the scientists said, was roughly three orders of magnitude more spores than found in samples from any of the other 600 bags of mail the bureau examined."

Estimate the number of spores found in any of the 600 other bags of mail.

9. In the December 1999 issue of the journal *Science,* two Harvard scientists describe a pair of "nanotweezers" they created that are capable of manipulating objects as small as one-50,000th of an inch in width. The scientists used the tweezers to grab and pull clusters of polystyrene molecules, which are of the same size as structures inside cells. A future use of these nanotweezers may be to grab and move components of biological cells.

a. Express one-50,000th of an inch in scientific notation.

b. Express the size of objects the tweezers are able to manipulate in meters.

c. The prefix "nano" refers to nine subdivisions by 10, or a multiple of 10^{-9}. So a nanometer would be 10^{-9} meters. Is the name for the tweezers given by the inventors appropriate?

d. If not, how many orders of magnitude larger or smaller would the tweezers' ability to manipulate small objects have to be in order to grasp things of nanometer size?

10. Determine the order-of-magnitude difference in the sizes of the radii for:

a. The solar system (10^{12} meters) compared with Earth (10^7 meters)

b. Protons (10^{-15} meter) compared with the Milky Way (10^{21} meters)

c. Atoms (10^{-10} meter) compared with neutrons (10^{-15} meter)

11. To compare the sizes of different objects, we need to use the same unit of measure.

a. Convert each of these to meters:

i. The radius of the moon is approximately 1,922,400 yards.

ii. The radius of Earth is approximately 6400 km.

iii. The radius of the sun is approximately 432,000 miles.

b. Determine the order-of-magnitude difference between:

i. The surface areas of the moon and Earth

ii. The volumes of the sun and the moon

12. The pH scale measures the hydrogen ion concentration in a liquid, which determines whether the substance is acidic or alkaline. A strong acid solution has a hydrogen ion concentration of 10^{-1} M. One M equals $6.02 \cdot 10^{23}$ particles, such as atoms, ions, molecules, etc., per liter, or 1 mole per liter.[6] A strong alkali solution has a hydrogen ion concentration of 10^{-14} M. Pure water, with a concentration of 10^{-7} M, is neutral. The pH value is the power without the minus sign, so pure water has a pH of 7, acidic substances have a pH less than 7, and alkaline substances have a pH greater than 7.

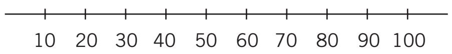
a. Tap water has a pH of 5.8. Before the industrial age, rain water commonly had a pH of about 5. With the spread of modern industry, rain in the northeastern United States and parts of Europe now has a pH of about 4, and in extreme cases the pH is about 2. Lemon juice has a pH of 2.1. If acid rain with a pH of 3 is discovered in an area, how much more acidic is it than preindustrial rain?

[6] You may recall from Algebra Aerobics 4.1 that $6.02 \cdot 10^{23}$ is called Avogadro's number. A mole of a substance is defined as Avogadro's number of particles of that substance. M is called a molar unit.

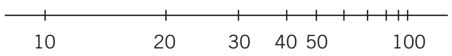
b. Blood has a pH of 7.4; wine has a pH of about 3.4. By how many orders of magnitude is wine more acidic than blood?

13. Which is an additive scale? Explain why. Which is a multiplicative or logarithmic scale? Explain why.

a.



b.



14. Graph the following on a power-of-10 (logarithmic) scale. (See sample log scale at the end of the exercises.)

a. 1 meter
b. 10 meters
c. 1 hectometer
d. 1000 kilometers
e. 10 gigameters

15. (Refer to the chart in Exploration 4.1.) Plot on the logarithmic scale in Figure 4.2 an object whose radius is:

a. Five orders of magnitude larger than the radius of the first atoms

b. Twenty orders of magnitude smaller than the radius of the sun

16. a. Read the chapter entitled "The Cosmic Calendar" from Carl Sagan's book *The Dragons of Eden*.

b. Carl Sagan tried to give meaning to the cosmic chronology by imagining the almost 15 billion–year lifetime of the universe compressed into the span of one calendar year. To get a more personal perspective, consider your date of birth as the time at which the Big Bang took place. Map the following five cosmic events onto your own life span:

i. The Big Bang
ii. Creation of Earth
iii. First life on Earth
iv. First *Homo sapiens*
v. American Revolution

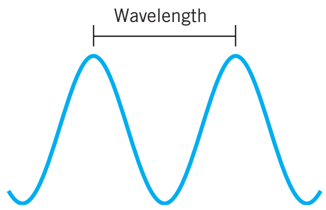
Once you have done the necessary mathematical calculations and placed your results on either a chart or a timeline, form a topic sentence and write a playful paragraph about what you were supposedly doing when these cosmic events took place. Hand in your calculations along with your writing.

17. Graph the following on a power-of-10 (logarithmic) scale. (See sample log scale at the end of the exercises.)

a. 1 watt
b. 10 kilowatts
c. 100 billion kilowatts
d. 1000 terawatts

18. Radio waves, sent from a broadcast station and picked up by the antenna of your radio, are a form of electromagnetic (EM) radiation, as are microwaves, X-rays, and visible, infrared, and ultraviolet light. They all travel at the speed of light. Electromagnetic radiation can be thought of as oscillations like the vibrating strings of a violin or guitar or like ocean swells that have crests and troughs. The distance between the crest or peak of one wave and the next is called the wavelength. The number of times a wave crests per minute, or per second for fast-oscillating waves, is called its frequency. Wavelength and frequency are inversely proportional: the longer the wavelength, the lower the frequency, and vice versa—the faster the oscillation, the shorter the wavelength. For radio waves and other EM, the number of oscillations per second of a wave is measured in hertz, after the German scientist who first demonstrated that electrical waves could transmit information across space. One cycle or oscillation per second equals 1 hertz (Hz).

For the following exercise you may want to find an old radio or look on a stereo tuner at the AM and FM radio bands. You may see the notation kHz beside the AM band and MHz beside the FM band. AM radio waves oscillate at frequencies measured in the kilohertz range, and FM radio waves oscillate at frequencies measured in the megahertz range.



a. The Boston FM rock station WBCN transmits at 104.1 MHz. Write its frequency in hertz using scientific notation.

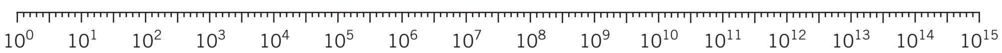
b. The Boston AM radio news station WBZ broadcasts at 1030 kHz. Write its frequency in hertz using scientific notation.

The wavelength λ (Greek lambda) in meters and frequency μ (Greek mu) in oscillations per second are related by the formula $\lambda = \frac{c}{\mu}$ where c is the speed of light in meters per second.

c. Estimate the wavelength of the WBCN FM radio transmission.

d. Estimate the wavelength of the WBZ AM radio transmission.

e. Compare your answers in parts (c) and (d), using orders of magnitude, with the length of a football field (approximately 100 meters).



Sample log Scale

4.7 Logarithms Base 10

In Section 4.6 we used a logarithmic scale to graph numbers of widely disparate sizes. We labeled the axis with powers of 10, so it was easy to plot numbers such as $1000 = 10^3$ or $100{,}000 = 10^5$ that are integer powers of 10. But how would we plot a number such as $4{,}600{,}000{,}000 = 4.6 \cdot 10^9$, the approximate age of Earth in years? To do that we need to understand logarithms.

Finding the Logarithms of Powers of 10

For handling very large or very small numbers, it is often easier to write the number using powers of 10. For example,

$$100{,}000 = 10^5$$

We say that

100,000 equals the base 10 to the fifth power

But we could rephrase this as

5 is the exponent of the base 10 that is needed to produce 100,000

The more technical way to say this is

5 is the *logarithm* base 10 of 100,000

In symbols we write

$$5 = \log_{10} 100{,}000$$

So the expressions

$$100{,}000 = 10^5 \qquad \text{and} \qquad 5 = \log_{10} 100{,}000$$

are two ways of saying the same thing. The key point to remember is that a logarithm is an exponent.

Definition of Logarithm

The *logarithm base 10 of x* is the exponent of 10 needed to produce x:

$$\log_{10} x = c \qquad \text{means} \qquad 10^c = x$$

So to find the logarithm of a number, write it as 10 to some power. The power is the logarithm of the original number.

EXAMPLE 1 Find log (1,000,000,000) without using a calculator.

SOLUTION Since $\quad 1{,}000{,}000{,}000 = 10^9$

then $\quad \log_{10} 1{,}000{,}000{,}000 = 9$

and we say that the logarithm base 10 of 1,000,000,000 is 9. The logarithm of a number tells us the exponent of the number when written as a power of 10. Here the logarithm is 9, so that means that when we write 1,000,000,000 as a power of 10, the exponent is 9.

EXAMPLE 2 Find log 1 without using a calculator.

SOLUTION Since $\quad 1 = 10^0$

then $\quad \log_{10} 1 = 0$

and we say that the logarithm base 10 of 1 is 0. Since logarithms represent exponents, this says that when we write 1 as a power of 10, the exponent is 0.

EXAMPLE 3 How do we calculate the logarithm base 10 of decimals such as 0.000 01?

SOLUTION Since $0.000\,01 = 10^{-5}$

then $\log_{10} 0.000\,01 = -5$

and we say that the logarithm base 10 of 0.000 01 is -5.

In the previous example, we found that the log (short for "logarithm") of a number can be negative. This makes sense if we think of logarithms as exponents, since exponents can be any real number. But we cannot take the log of a negative number or zero; that is, $\log_{10} x$ is not defined when $x \leq 0$. Why? If $\log_{10} x = c$, where $x \leq 0$, then $10^c = x$ (a number ≤ 0). But 10 to any power will never produce a number that is negative or zero, so $\log_{10} x$ is not defined if $x \leq 0$.

$\log_{10} x$ is not defined when $x \leq 0$.

Table 4.7 gives a sample set of values for x and their associated logarithms base 10. To find the logarithm base 10 of x, we write x as a power of 10, and the logarithm is just the exponent.

Most scientific calculators and spreadsheet programs have a LOG function that calculates logarithms base 10. Try using technology to double-check some of the numbers in Table 4.7.

Logarithms of Powers of 10

x	Exponential Notation	$\log_{10} x$
0.0001	10^{-4}	-4
0.001	10^{-3}	-3
0.01	10^{-2}	-2
0.1	10^{-1}	-1
1	10^{0}	0
10	10^{1}	1
100	10^{2}	2
1000	10^{3}	3
10,000	10^{4}	4

Table 4.7

Logarithms base 10 are used frequently in our base 10 number system and are called *common logarithms*. We write $\log_{10} x$ as $\log x$.

Common Logarithms

Logarithms base 10 are called *common logarithms.*

$\log_{10} x$ is written as $\log x$.

Algebra Aerobics 4.7a

1. Without using a calculator, find the logarithm base 10 of:
   a. 10,000,000 **c.** 10,000 **e.** 1000 **g.** 1
   b. 0.000 000 1 **d.** 0.0001 **f.** 0.001
2. Rewrite the following expressions in an equivalent form using powers of 10:
   a. $\log 100{,}000 = 5$ **c.** $\log 10 = 1$
   b. $\log 0.000\,000\,01 = -8$ **d.** $\log 0.01 = -2$
3. Evaluate without using a calculator. Find a number if its log is:
   a. 3 **c.** 6 **e.** -2
   b. -1 **d.** 0
4. Find c and then rewrite as a logarithm:
   a. $10^c = 1000$ **d.** $10^c = 0.000\,01$
   b. $10^c = 0.001$ **e.** $10^c = 1{,}000{,}000$
   c. $10^c = 100{,}000$ **f.** $10^c = 0.000\,001$
5. Find the value of x that makes the statement true.
   a. $10^{x-3} = 10^2$ **c.** $\log (x - 2) = 1$
   b. $10^{2x-1} = 10^4$ **d.** $\log 5x = -1$

Finding the Logarithm of Any Positive Number

Scientific calculators have a LOG function that will calculate the log of any positive number. However, it's easy to make errors typing in numbers, so it's important not to rely solely on technology-generated answers. To verify that the calculated number is the right order of magnitude, you should estimate the answer without using technology.

EXAMPLE 4 **Estimating, then using technology to calculate logs**

a. Estimate the size of log (2000) and log (0.07).

b. Use a calculator to find the logarithm of 2000 and 0.07.

SOLUTION **a. i.** If we place 2000 between the two closest integer powers of 10, we have

$$1000 < 2000 < 10{,}000$$

Rewriting 1000 and 10,000 as powers of 10 gives

$$10^3 < 2000 < 10^4$$

Taking the log of each term preserves the inequality, so we would expect

$$3 < \log 2000 < 4$$

ii. If we place 0.07 between the two closest integer powers of 10, we have

$$0.01 < 0.07 < 0.10$$

Rewriting 0.01 and 0.10 as powers of 10 gives

$$10^{-2} < 0.07 < 10^{-1}$$

Taking the log of each term preserves the inequality, so we would expect

$$-2 < \log 0.07 < -1$$

b. Using a calculator, we have

i. $\log 2000 \approx 3.301$ **ii.** $\log (0.07) \approx -1.155$.

So our estimates were correct.

EXAMPLE 5 **Calculating logs of very large or small numbers**

Find the logarithm of

a. 3.7 trillion

b. A Planck length of 0.000 000 000 000 000 000 000 000 000 000 000 016 meter

SOLUTION Our strategy in each case is to

- Write the number in scientific notation.
- Then convert the number into a single power of 10.

The resulting exponent is the desired log.

a. In scientific notation 3.7 trillion is $3.7 \cdot 10^{12}$. To convert the entire expression into a single power of 10, we need to first convert the coefficient 3.7 to a power of 10. Using a calculator, we have

	$\log 3.7 \approx 0.568$
so	$3.7 \approx 10^{0.568}$
If we substitute for 3.7,	$3.7 \cdot 10^{12} \approx 10^{0.568} \cdot 10^{12}$
and use rules for exponents,	$= 10^{0.568+12}$
we have	$= 10^{12.568}$

So the exponent 12.568 is the desired logarithm.

b. In scientific notation a Planck length is $1.6 \cdot 10^{-35}$ meter. We need to convert 1.6 to a power of 10. Using a calculator, we have $\log(1.6) \approx 0.204$, so $1.6 = 10^{0.204}$. If we

substitute for 1.6	$1.6 \cdot 10^{-35} \approx 10^{0.204} \cdot 10^{-35}$
use rules for exponents	$= 10^{0.204-35}$
and subtract, we get	$= 10^{-34.796}$

So the exponent -34.796 is the desired log.

So far we have dealt with finding the logarithm of a given number. Logarithms can, of course, occur in expressions involving variables.

EXAMPLE 6 **Finding the number given the log**

Rewrite the following expressions using exponents, and then solve for x without using a calculator.

a. $\log x = 3$ **b.** $\log x = 0$ **c.** $\log x = -2$

SOLUTION

a. If $\log x = 3$, then $10^3 = x$, so $x = 1000$.

b. If $\log x = 0$, then $10^0 = x$, so $x = 1$.

c. If $\log x = -2$, then $10^{-2} = x$, so $x = 1/10^2 = 1/100 = 0.01$.

EXAMPLE 7 Rewrite the following expressions using logarithms and then solve for x using a calculator. Round off to three decimal places.

a. $10^x = 11$ **b.** $10^x = 0.5$ **c.** $10^x = 0$

SOLUTION

a. If $10^x = 11$, then $\log 11 = x$. Using a calculator gives $x \approx 1.041$.

b. If $10^x = 0.5$, then $\log 0.5 = x$. Using a calculator gives $x \approx -0.301$.

c. There is no power of 10 that equals 0. Hence there is no solution for x.

Plotting Numbers on a Logarithmic Scale

We are finally prepared to answer the question posed at the very beginning of this section: How can we plot on a logarithmic (or order-of-magnitude) scale a number such as 4.6 million years, the estimated age of Earth?

A Strategy for Plotting Numbers on a Logarithmic Scale

- Write the number in scientific notation.
- Then convert the number into a single power of 10.
- Use the exponent to help plot the number.

In scientific notation the age of Earth equals 4.6 billion = 4,600,000,000 = $4.6 \cdot 10^9$ years. To plot this number on a logarithmic scale, we need to convert 4.6 into a power of 10. Using a calculator, we have $\log 4.6 \approx 0.663$, so $4.6 \approx 10^{0.663}$. If we

substitute for 4.6	$4.6 \cdot 10^9 \approx 10^{0.663} \cdot 10^9$
and use rules of exponents	$= 10^{0.663+9}$
we have	$= 10^{9.663}$

The power of 10 seems reasonable since $10^9 < 4.6 \cdot 10^9 < 10^{10}$.

Having converted our original number $4.6 \cdot 10^9$ into $10^{9.663}$, we can plot it on an order-of-magnitude graph between 10^9 and 10^{10} (see Figure 4.5).

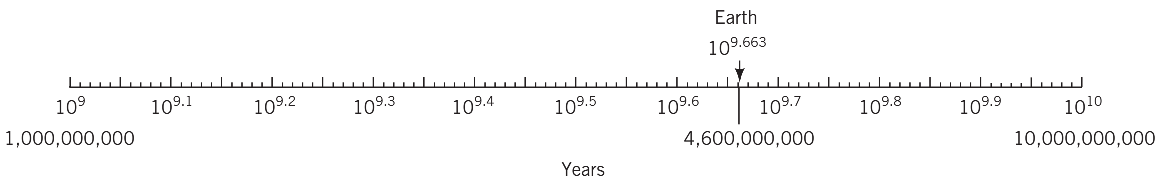


Figure 4.5 Age of Earth plotted on an order-of-magnitude (or logarithmic) scale.

EXAMPLE 8 Plot the numbers 100, 200, 300, 400, 500, 600, 700, 800, 900, and 1000 on a logarithmic scale.

SOLUTION Using our log plotting strategy, we first convert each number to a single power of 10. We have

$100 = 10^2$	$600 \approx 10^{2.778}$
$200 \approx 10^{2.301}$	$700 \approx 10^{2.845}$
$300 \approx 10^{2.477}$	$800 \approx 10^{2.903}$
$400 \approx 10^{2.602}$	$900 \approx 10^{2.954}$
$500 \approx 10^{2.699}$	$1000 = 10^3$

We can now use the exponents of each power of 10 to plot the numbers directly onto a logarithmic scale (see Figure 4.6).

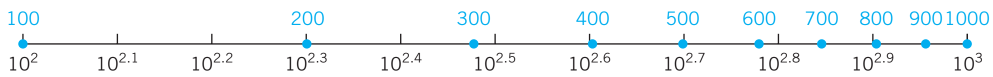


Figure 4.6 A logarithmic plot of the numbers 100, 200, 300, . . . , 1000.

Note that on the logarithmic scale in Figure 4.6 the point halfway between 10^2 and 10^3 is at $10^{2.5} = 316$.

When are numbers evenly spaced on a logarithmic scale?

On a linear (additive) scale the numbers 100, 200, . . . , 1000 would be evenly spaced, since you *add* a constant amount to move from one number to the next. On a logarithmic (multiplicative) scale, numbers that are evenly spaced are generated by *multiplying* by a constant amount to get from one number to the next. For example, the integer powers of 10 are all evenly spaced on a log plot since you multiply each number by 10 to get the next number in the sequence. Similarly, the numbers 100, 200, 400, and 800 are evenly spaced in Figure 4.6 since you multiply by the constant 2 to get from one number in the sequence to the next. The sequence 100, 200, 300, . . . , 1000 is not evenly spaced, since there is not a constant factor that you could multiply one number by to get to the next. In this last sequence, since the multiplication factor needed to move from one number to the next decreases as the numbers approach 1000, the distance between points decreases.

Labeling using only the exponent

Instead of labeling the axis using powers of 10, we can label it using just the exponents of 10 as in Figure 4.7. Remember that exponents are logarithms, which is why we call the scale logarithmic.

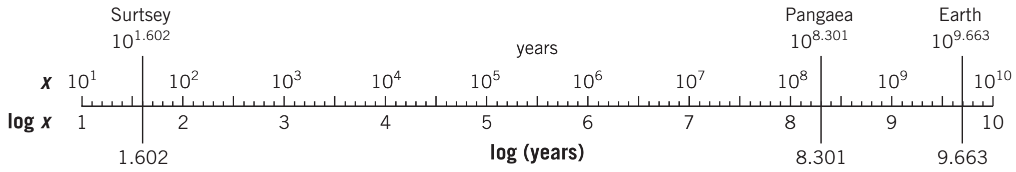


Figure 4.7 The age of Surtsey, Pangaea, and Earth plotted using an order-of-magnitude or logarithmic scale.

EXAMPLE 9 Add on to the logarithmic plot in Figure 4.7 the following numbers:

a. 200 million, the number of years since all of Earth's continents collided to form one giant land mass called Pangaea

b. 40, the number of years since the volcanic island of Surtsey, Earth's newest land mass, emerged near Iceland

SOLUTION **a.** In scientific notation 200 million $= 200{,}000{,}000 = 2.0 \cdot 10^8$. Using a calculator, we have $\log 2.0 \approx 0.301$, so $2.0 \approx 10^{0.301}$. Hence $2.0 \cdot 10^8 \approx 10^{0.301} \cdot 10^8 = 10^{8.301}$ is the age of Pangaea in a form easily plotted on a logarithmic scale.

b. In scientific notation $40 = 4.0 \cdot 10^1$. Using a calculator, we have $\log 4.0 \approx 0.602$, so $4.0 \approx 10^{0.602}$. Therefore $4.0 \cdot 10^1 \approx 10^{0.602} \cdot 10^1 = 10^{1.602}$ is the age of Surtsey in a form easily plotted on a log scale.

The two numbers are plotted in Figure 4.7.

Algebra Aerobics 4.7b

Most of these problems require a calculator that can evaluate logs.

1. Use a calculator to estimate each of the following:

   a. log 3 **b.** log 6 **c.** log 6.37

2. Use the answers from Problem 1 to estimate values for:

   a. log 3,000,000 **b.** log 0.006

   Then use a calculator to check your answers.

3. Write each of the following as a power of 10:

   a. 0.000 000 7 m (the radius of a virus)

   b. 780,000,000 km (the mean distance from our sun to Jupiter)

   c. 0.0042

   d. 5,400,000,000

4. Rewrite the following equations using exponents instead of logarithms. Estimate the solution for x. Check your estimate with a calculator. Round the value of x to the nearest integer.

   a. $\log x = 4.125$

   b. $\log x = 5.125$

   c. $\log x = 2.125$

5. Rewrite the following equations using logs instead of exponents. Estimate a solution for x and check your estimate with a calculator. Round the value of x to three decimal places.

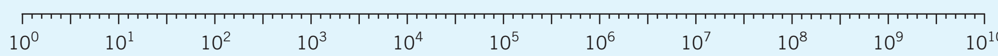
   a. $10^x = 250$ **c.** $10^x = 0.075$

   b. $10^x = 250{,}000$ **d.** $10^x = 0.000\ 075$

6. Write each number as a power of 10 and then plot them all on the logarithmic scale below.

   a. 57 **c.** 25,000

   b. 182 **d.** 7,200,000,000



Exercises for Section 4.7

Many of the problems in this section require the use of a calculator that can evaluate logs.

1. Rewrite in an equivalent form using logarithms:

a. $10^4 = 10{,}000$ **c.** $10^0 = 1$
b. $10^{-2} = 0.01$ **d.** $10^{-5} = 0.000\,01$

2. Use your calculator to evaluate to two decimal places:

a. $10^{0.4}$ **c.** $10^{0.6}$ **e.** $10^{0.8}$
b. $10^{0.5}$ **d.** $10^{0.7}$ **f.** $10^{0.9}$

3. Express the number 375 in the form 10^x.

4. Estimate the value of each of the following:

a. log 4000 **b.** log 5,000,000 **c.** log 0.0008

5. Rewrite the following statements using logs:

a. $10^2 = 100$ **b.** $10^7 = 10{,}000{,}000$ **c.** $10^{-3} = 0.001$

Rewrite the following statements using exponents:

d. $\log 10 = 1$ **e.** $\log 10{,}000 = 4$ **f.** $\log 0.0001 = -4$

6. Evaluate the following without a calculator.

a. Find the following values:

i. log 100 **ii.** log 1000 **iii.** log 10,000,000

What is happening to the values of $\log x$ as x gets larger?

b. Find the following values:

i. log 0.1 **ii.** log 0.001 **iii.** log 0.000 01

What is happening to the values of $\log x$ as x gets closer to 0?

c. What is log 0?

d. What is $\log(-10)$? What do you know about $\log x$ when x is any negative number?

7. Rewrite the following equations using exponents instead of logs. Estimate a solution for x and then check your estimate with a calculator. Round the value of x to the nearest integer.

a. $\log x = 1.255$ **c.** $\log x = 4.23$
b. $\log x = 3.51$ **d.** $\log x = 7.65$

8. Rewrite the following equations using exponents instead of logs. Estimate a solution for x and then check your estimate with a calculator. Round the value of x to the nearest integer.

a. $\log x = 1.079$ **c.** $\log x = 2.1$
b. $\log x = 0.699$ **d.** $\log x = 3.1$

9. Rewrite the following equations using logs instead of exponents. Estimate a solution for x and then check your estimate with a calculator. Round the value of x to three decimal places.

a. $10^x = 12{,}500$ **c.** $10^x = 597$
b. $10^x = 3{,}526{,}000$ **d.** $10^x = 756{,}821$

10. Rewrite the following equations using logs instead of exponents. Estimate a solution for x and then check your estimate with a calculator. Round the value of x to three decimal places.

a. $10^x = 153$ **c.** $10^x = 0.125$
b. $10^x = 153{,}000$ **d.** $10^x = 0.001\,25$

11. Solve for x. (*Hint:* Rewrite each expression so that you can use a calculator to solve for x.)

a. $\log x = 0.82$ **c.** $\log x = 0.33$
b. $10^x = 0.012$ **d.** $10^x = 0.25$

12. Without using a calculator, show how you can solve for x.

a. $10^{x-2} = 100$ **c.** $10^{2x-3} = 1000$
b. $\log(x - 4) = 1$ **d.** $\log(6 - x) = -2$

13. Without using a calculator show how you can solve for x.

a. $10^{x-5} = 1000$ **c.** $10^{3x-1} = 0.0001$
b. $\log(2x + 10) = 2$ **d.** $\log(500 - 25x) = 3$

14. Find the value of x that makes the equation true.

a. $\log x = -2$ **b.** $\log x = -3$ **c.** $\log x = -4$

15. Without using a calculator, for each number in the form $\log x$, find some integers a and b such that $a < \log x < b$. Justify your answer. Then verify your answers with a calculator.

a. log 11 **b.** log 12,000 **c.** log 0.125

16. Use a calculator to determine the following logs. Double-check each answer by writing down the equivalent expression using exponents, and then verify this equivalence using a calculator.

a. log 15 **b.** log 15,000 **c.** log 1.5

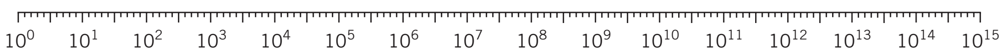
17. On a logarithmic scale, what would correspond to moving over to the right:

a. 0.001 unit **b.** $\frac{1}{2}$ unit **c.** 2 units **d.** 10 units

18. The difference in the noise levels of two sounds is measured in decibels, where decibels $= 10 \log\left(\frac{I_2}{I_1}\right)$ and I_1 and I_2 are the intensities of the two sounds. Compare noise levels when $I_1 = 10^{-15}$ watts/cm^2 and $I_2 = 10^{-8}$ watts/cm^2.

19. The concentration of hydrogen ions in a water solution typically ranges from 10 M to 10^{-15} M. (One M equals $6.02 \cdot 10^{23}$ particles, such as atoms, ions, molecules, etc., per liter or 1 mole per liter.) Because of this wide range, chemists use a logarithmic scale, called the pH scale, to measure the concentration (see Exercise 12 of Section 4.6). The formal definition of pH is pH $= -\log[H^+]$, where $[H^+]$ denotes the concentration of hydrogen ions. Chemists use the symbol H^+ for hydrogen ions, and the brackets [ ] mean "the concentration of."

a. Pure water at 25ºC has a hydrogen ion concentration of 10^{-7} M. What is the pH?



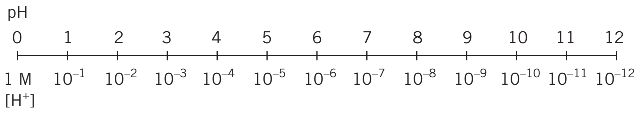
Sample log Scale

b. In orange juice, $[H^+] \approx 1.4 \cdot 10^{-3}$ M. What is the pH?

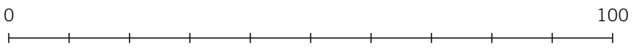
c. Household ammonia has a pH of about 11.5. What is its $[H^+]$?

d. Does a higher pH indicate a lower or a higher concentration of hydrogen ions?

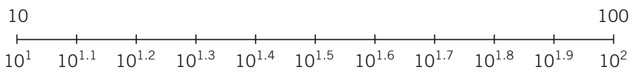
e. A solution with a pH > 7 is called basic, one with a pH $= 7$ is called neutral, and one with a pH < 7 is called acidic. Identify pure water, orange juice, and household ammonia as either acidic, neutral, or basic. Then plot their positions on the accompanying scale, which shows both the pH and the hydrogen ion concentration.



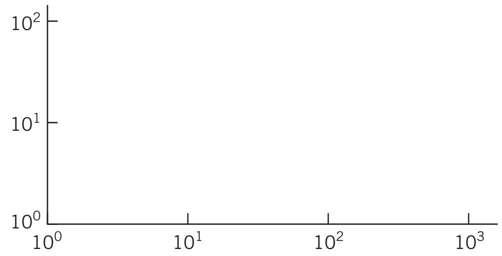
20. a. Place the number 50 on the *additive* scale below.



b. Place the number 50 on the *multiplicative* scale below.



21. The coordinate system below uses multiplicative or log scales on both axes. Position the point whose coordinates are (708, 25).



22. Change each number to a power of 10, then plot the numbers on a power-of-10 scale. (See sample log scale on page 254.)

a. 125 **b.** 372 **c.** 694 **d.** 840

23. Compare the times listed below by plotting them on the same order-of-magnitude scale. (*Hint:* Start by converting all the times to seconds.)

a. The time of one heartbeat (1 second)

b. Time to walk from one class to another (10 minutes)

c. Time to drive across the country (7 days)

d. One year (365 days)

e. Time for light to travel to the center of the Milky Way (38,000 years)

f. Time for light to travel to Andromeda, the nearest large galaxy (2.2 million years)

CHAPTER SUMMARY

Powers of 10

If n is a positive integer, we define

$$10^n = \underbrace{10 \cdot 10 \cdot 10 \cdot \cdots \cdot 10}_{n \text{ factors}}$$

$$10^0 = 1$$

$$10^{-n} = \frac{1}{10^n}$$

Scientific Notation

A number is in scientific notation if it is in the form

$$N \cdot 10^n$$

where N is called the *coefficient,* $1 \le |N| < 10$, and n is an integer.

Example: In scientific notation 67,000,000 is written as $6.7 \cdot 10^7$ and $-0.000\ 000\ 000\ 008\ 1$ is written as $-8.1 \cdot 10^{-12}$.

Powers of *a*

In the expression a^n, a is called the *base* and n is called the *exponent* or *power.*

If a is nonzero real and n is a positive integer, then

$$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}}$$

$$a^0 = 1$$

$$a^{-n} = \frac{1}{a^n}$$

If m and n are positive integers and the base, a, is restricted to values for which the power is defined, then

$$a^{1/2} = \sqrt{a}$$

$$a^{1/n} = \sqrt[n]{a}$$

$$a^{m/n} = (a^m)^{1/n} = (a^{1/n})^m = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$$

Rules of Exponents

If a and b are nonzero, then

1. $a^m \cdot a^n = a^{(m+n)}$

2. $\dfrac{a^n}{a^m} = a^{(n-m)}$

3. $(a^m)^n = a^{(m \cdot n)}$

4. $(ab)^n = a^n b^n$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Orders of Magnitude

We use *orders of magnitude* when we compare objects of widely different sizes. Each *factor* of 10 is counted as a single order of magnitude.

Example: The radius of the universe is 10^{14} times or fourteen orders of magnitude larger than the radius of the solar system. And vice versa: The radius of the solar system is fourteen orders of magnitude smaller than the radius of the universe.

Logarithms

The *logarithm base 10 of x* is the exponent of 10 needed to produce x. So

$$\log_{10} x = c \qquad \text{means} \qquad 10^c = x$$

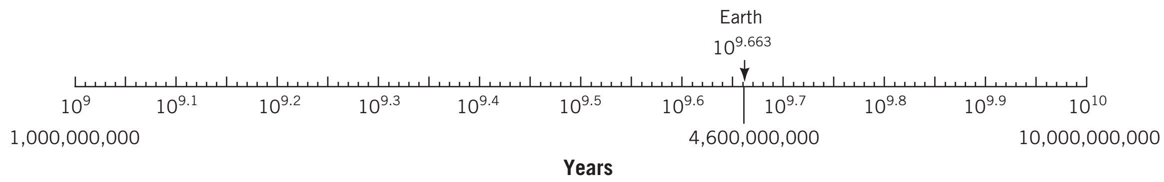
We say that c is the logarithm base 10 of x.

Example: $\log_{10} 6{,}370{,}000 \approx 6.804$ means that $10^{6.804} \approx 6{,}370{,}000$.

Logarithms base 10 are called *common logarithms*. We usually write $\log_{10} x$ as $\log x$. When $x \le 0$, $\log x$ is not defined.

Plotting Numbers on a Logarithmic Scale

Logarithmic or powers-of-10 scales are used to graph objects of widely differing sizes. We can plot a number on a log scale by converting the number to a power of 10.



Age of Earth plotted on a logarithmic scale.

CHECK YOUR UNDERSTANDING

I. Are the statements in Problems 1–26 true or false? Give an explanation for your answer.

1. A distance of 10 miles is longer than a distance of 10 kilometers.

2. There are 39 centimeters in 1 inch.

3. 10^{15} is 10 followed by fifteen zeros.

4. $10^0 = 0$.

5. $\dfrac{1}{10^{-m}} = 10^m$.

6. $-0.000\,005\,62 = -5.62 \cdot 10^{-6}$.

7. $15 \cdot 10^4$ is correct scientific notation for the number 150,000.

8. The age of the universe ($1.37 \cdot 10^{10}$ years) is about three times the age of Earth ($4.6 \cdot 10^9$ years).

9. In July 2004, the population of the world (about 6,377,642,000) was approximately three orders of magnitude larger than the population of the United States (about 293,028,000).

10. $-8^2 = (-8)^2$.

11. $\left(\dfrac{5}{3}\right)^{-3} = \dfrac{-15}{-9}$.

12. $10^2 + 10^1 + 10^5 = 10^{2+1+5} = 10^8$.

13. To convert a distance D in kilometers to miles, you could multiply D by $\dfrac{1 \text{ km}}{0.62 \text{ mile}}$.

14. The units of $300 \dfrac{\text{km}}{\text{hr}} \cdot \dfrac{1 \text{ hr}}{60 \text{ min}} \cdot \dfrac{1 \text{ min}}{60 \text{ sec}} \cdot \dfrac{10^3 \text{ m}}{1 \text{ km}}$ are meters per second.

15. $-9 < -\sqrt{75} < -8$.

16. $8^{1/2} = 8^{0.5}$.

17. $\log_{10} 0.0001 = 10^{-4}$.

18. $\log_{10} 1821$ is not defined since 1821 is not a power of 10.

19. $(81)^{1/2} = \pm 9$ because $(9)^2 = 81$ and $(-9)^2 = 81$.

20. $\log 0 = 1$.

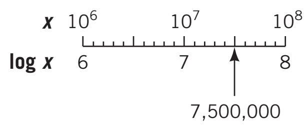
21. $\log(-3) = -\log(3)$.

22. $-4 < \log 0.00015 < -3$.

23. $\log 0.143 \approx -0.845$ means that $10^{-0.845} \approx 0.143$.

24. If $P > 0$, $\log P = Q$ means that $10^P = Q$.

25. The following figure illustrates the number 7,500,000 plotted correctly on a logarithmic scale.



26. If $10^x = 36$, then $x \approx 1.556$.

II. In Problems 27–32, give examples with the specified properties.

27. Populations of two cities A and B, where the population of city A is two orders of magnitude larger than that of city B.

28. A number x such that $\log x$ lies between 8 and 9.

29. A number x such that $\log x$ is a negative number.

30. A positive number b such that $\sqrt{b} > b$.

31. A non-zero number b such that $b^m = b^n$ for any numbers m and n.

32. A number b such that $|b| = -b$.

III. Are the statements in Problems 33–42 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

33. If one quantity is four orders of magnitude larger than a second quantity, it is four times as large as the second quantity.

34. $|c| = c$ for any real number c.

35. $\log x$ is defined only for numbers $x > 0$.

36. Raising a number to the $\frac{1}{3}$ power is the same as taking the cube root of that number.

37. $b^m \cdot b^m = b^{m^2}$

38. $(b^p)^q = b^{p+q}$

39. $(b + c)^m = b^m + c^m$

40. $(-b)^q = b^q$

41. $b^m \cdot c^n = (b \cdot c)^{m+n}$

42. If n is odd, $\sqrt[n]{b}$ can be positive, negative, or zero depending on the value of b.

CHAPTER 4 REVIEW: PUTTING IT ALL TOGETHER

1. Evaluate each of the following without a calculator.

a. $4.2 \cdot 10^3$ **b.** $(-5)2^3$ **c.** -4^2 **d.** $100^{-1/2}$ **e.** $\dfrac{3^5}{3^2}$

2. Use the rules of exponents to simplify the following. Express your answer with positive exponents.

a. x^4x^3 **b.** $\dfrac{10x^2y^4}{5xy^3}$ **c.** $(-2xy^2)^3$ **d.** $(x^{-1/2})^2$ **e.** $(x^{-3}y)(x^2y^{-1/2})$

3. For what integer values of x will the following statement be true?

$(-10)^x = -10^x$

4. a. Show with an example why the following is not a true statement for all values of x: $x^3 + x^5 = x^8$.

b. For what value of x is the above statement true?

5. Elephant seals can weigh as much as 5000 lb (for males) and 2000 lb (for females). On land, these seals can travel short distances quite quickly, as much as 20 feet in 3 seconds. How many miles per hour is this?

6. An NFL regulation playing field for football is 120 yd (110 m) long including the end zones, and 53 yd 1 ft (48.8 m) wide. An acre is 4840 square yards, and 1 yard = 3 feet.

a. Which is larger, a football field (including the end zones) or an acre? By how much?

b. If you bought a house on a square lot that measured half an acre, what would the dimensions of the lot be in feet?

7. In 2006, Tiger Woods was the highest paid athlete in the world (taking into account on and off the field earnings), making \$11.9 million in salary and \$100 million in endorsements for a total of \$111.9 million. By what order of magnitude is his salary greater than that of a minimum-wage worker in the same state making \$6.40/hr working 40 hours/week for 52 weeks/year? How many years would the minimum-wage worker have to work to earn what Tiger Woods made in 1 year?

8. The Yangtze River (China) is 6380 km long. The Colorado River is 1400 miles long.

a. Which river is longer?

b. Compare the lengths of these rivers using orders of magnitude.

9. Use the accompanying table to answer the following questions.

Country	Area
Russia	17,075,200 km^2
Chile	290,125 mi^2
Canada	3,830,840 mi^2
South Africa	1,184,825 km^2
Norway	323,895 km^2
Monaco	0.5 mi^2

a. Which country has the largest area? The smallest?

b. Using scientific notation, arrange the countries from largest area to smallest area.

c. What is the order-of-magnitude difference between the country with the largest area and the country with the smallest area?

10. The Energy Information Administration of the U.S. Department of Energy estimates that in 2010 the world energy use will be 470.8 quadrillion Btu (British thermal units), where 1 Btu = 0.000 293 1 kWh (kilowatt-hours.)

a. Express 470.8 quadrillion Btu and 0.000 293 1 kWh in scientific notation.

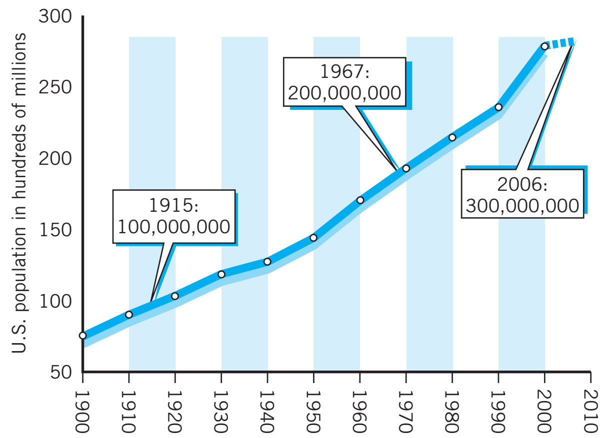
b. How many kilowatt-hours are there in 470.8 quadrillion Btu? Give your answer in scientific notation.

11. Is the following statement true or false? "An increase in one order of magnitude is the same as an increase of 100%." If true, explain why. If false, revise the statement to make it true.

12. On October 15, 2006, the *San Francisco Chronicle* published the accompanying graph and table derived from U.S. Bureau of the Census data on the growing size of the U.S. population. Compare the changes from 1915 to 2006 in two different categories, using at least one rate-of-change calculation and one order-of-magnitude comparison. Show your work.

A New Milestone for U.S. Population

The number of residents has tripled since 1915—and much has changed.



President	1915 Woodrow Wilson	1967 Lyndon B. Johnson	2006 George W. Bush
Price of new home	$3,200	$24,600	$290,600
Cost of gallon of regular gas	25¢	33¢	$2.25
Cost of a first-class stamp	2¢	5¢	39¢
Average household size	4.5 people	3.3 people	2.6 people
Number of people age 65 and older	4.5 million	19.1 million	36.8 million
Most popular baby names for boys and girls	John and Mary	Michael and Lisa	Jacob and Emily

13. The U.S. Census Bureau estimates that a baby is born somewhere in the country every 7 seconds, a new immigrant arrives every 31 seconds, and someone dies every 13 seconds, for a net average gain of one resident every 10 seconds.

a. On average, how many babies are born in the country per day?

b. On average, how many new immigrants arrive per day?

c. On average, how many people die per day?

14. Temperature can affect the speed of sound. The speed of sound, S (in feet/second), at an air temperature of T (in degrees Celsius) is

$$S = \frac{1087(273 + T)^{0.5}}{16.52}$$

a. Express T in terms of S.

b. The speed of sound is often given as 1120 feet/second. At what temperature in degrees Celsius would that be? At what temperature in degrees Fahrenheit would that be? (Recall that degrees Fahrenheit = 1.8(degrees Celsius) + 32.)

15. The radius of Earth is about $6.3 \cdot 10^6$ m and its mass is approximately $5.97 \cdot 10^{24}$ kg. Find its density in kg/m^3 (density = mass/volume).

16. Objects that are less dense than water will float; those that are more dense than water will sink. The density of water is 1.0 g/cm^3. A brick has a mass of 2268 g and a volume of 1230 cm^3. Show that the brick will sink in water (recall density = mass/volume).

17. An adult patient weighs 130 lb. The prescription for a drug is 5 mg per kg of the patient's weight per day. This drug comes in 100-mg tablets. What daily dosage should be prescribed?

18. On March 2, 2007, the *Boston Globe* reported the following:

An exabyte is 1 quintillion bytes. In 2006 alone, the human race generated 161 exabytes of digital information. So? Well, that's about 3 million times the information in all the books ever written or the equivalent of 12 stacks of books, each extending more than 93 million miles from Earth to the sun.

a. Use scientific notation to represent the amount of digital information generated in 2006. (One quintillion is 1 followed by eighteen zeros.)

b. Compare the amount of digital information generated in 2006 with the amount of information in all the books ever written, using orders of magnitude.

c. Estimate how many miles of books are needed to hold the equivalent information in 161 exabytes. Express your answer in scientific notation.

19. Find the logarithm of each of the following numbers:

a. 1 **b.** 1 billion **c.** 0.000 001

20. Estimate the following by placing the log between the two closest integer powers of 10.

a. log 3000 **b.** log 150,000 **c.** log 0.05

Item	Value	Value in Scientific Notation
Mass-energy of electron	0.000 000 000 000 051 J	
The kinetic energy of a flying mosquito	0.000 000 160 2 J	
An average person swinging a baseball bat	80 J	
Energy received from the sun at Earth's orbit on one square meter in one second	1,360 J	
Energy released by one gram of TNT	4,184 J	
Energy released by metabolism of one gram of fat	38,000 J	
Approximate annual power usage of a standard clothes dryer	320,000,000 J	

Source: http://en.wikipedia.org.

21. One way of defining the energy unit *joule* (J) is the amount of the energy required to lift a small apple weighing 102 grams one meter above Earth's surface. The accompanying table lists the estimated energy in joules for different situations.

Use the accompanying table to answer the following questions.

a. Write each value in scientific notation.

b. A year's use of a clothes dryer requires how many times the energy of swinging a baseball bat once?

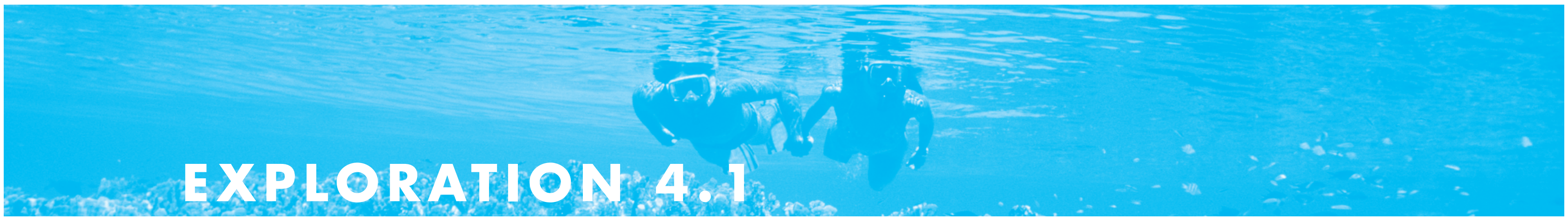
c. Metabolizing one gram of fat releases how many times the kinetic energy of a flying mosquito?

22. Rewrite each number as a power of 10, then create a logarithmic scale and estimate the location of the number on that scale. (*Hint:* $\log 2 = 0.301$.)

a. 10 **b.** 100 **c.** 200 **d.** 20,000

23. (Requires a scientific calculator.) Some drugs are prescribed in dosages that depend on a patient's BSA, or body surface area, an indicator of metabolic mass. One formula for calculating BSA is $BSA = 71.84W^{0.425}H^{0.725}$, where BSA is measured in square centimeters, W is weight in kilograms, and H is height in centimeters.

A patient weighs 180 lb and is 6 feet tall. His dosage of a particular drug is 15 mg/m^2/day (that is, 15 mg per square meter of body surface area per day). What is his daily dosage in mg? (*Source:* DuBois & DuBois, 1916, from *http://en.wikipedia.org/wiki/Body_surface_area# Calculation.*)



The Scale and the Tale of the Universe

Objective

- gain an understanding of the relative sizes and relative ages of objects in the universe using scientific notation and unit conversions

Materials/Equipment

- tape, pins, paper, and string to generate a large wall graph (optional)
- enclosed worksheet and conversion table on inside back cover

Related Readings/Videos

Powers of Ten and “The Cosmic Calendar” from *The Dragons of Eden*
Videos: *Powers of Ten* and *The Cosmic Calendar* in the PBS series *Cosmos*

Related Software

“E1: Tale and Scale of the Universe” in *Exponential & Log Functions*

Procedure

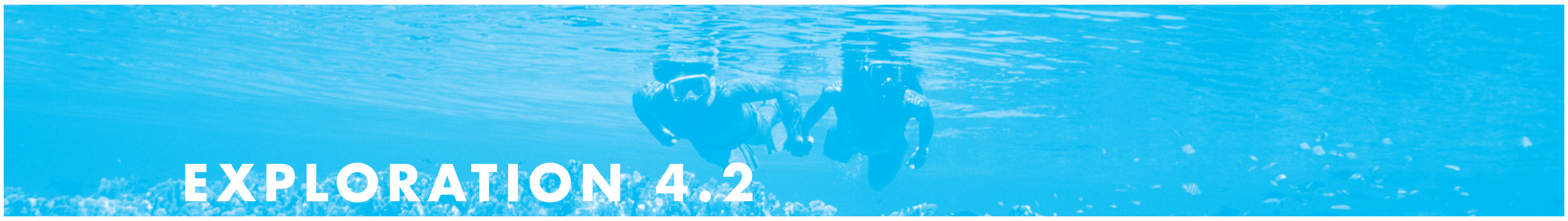
Work in small groups. Each group should work on a separate subset of objects on the accompanying worksheet.

1. Convert the ages and sizes of objects so they can be compared. You can refer to the conversion table that shows equivalences between English and metric units (see inside back cover). In addition, 1 light year $\approx 9.46 \cdot 10^{12}$ km.
2. Generate on the blackboard or on the wall (with string) a blank graph whose axes are marked off in orders of magnitude (integer powers of 10), with the units on the vertical axis representing age of object, ranging from 10^0 to 10^{11} years, and the units on the horizontal axis representing size of object, ranging from 10^{-12} to 10^{27} meters.
3. Each small group should plot the approximate coordinates of their selected objects (size in meters, age in years) on the graph. You might want to draw and label a small picture of your object to plot on your graph.

Discussion/Analysis

- Scan the plotted objects from left to right, looking only at relative sizes. Now scan the plotted objects from top to bottom, considering only relative ages. Does your graph make sense in terms of what you know about the relative sizes and ages of these objects?
- Describe the scale and the tale of the universe.

Object	Age (in years)	Size (of radius)	In Scientific Notation: Age (in years)	In Scientific Notation: Size (in meters)
Observable universe	13.7 billion	10^{26} meters		
Surtsey (Earth's newest land mass)	40 years	0.5 mile		
Pleiades (a galactic cluster)	100 million	32.6 light years		
First living organisms on Earth	4.6 billion	0.000 05 meter		
Pangaea (Earth's prehistoric supercontinent)	200 million	4500 miles		
First *Homo sapiens sapiens*	100 thousand	100 centimeters		
First *Tyrannosaurus rex*	200 million	20 feet		
Eukaryotes (first cells with nuclei)	2 billion	0.000 05 meter		
Earth	4.6 billion	6400 kilometers		
Milky Way galaxy	14 billion	50,000 light years		
First atoms	13.7 billion	0.000 000 0001 meter		
Our sun	5 billion	1 gigameter		
Our solar system	5 billion	1 terameter		



EXPLORATION 4.2

Patterns in the Positions and Motions of the Planets

Objective

- explore patterns in the positions and motions of the planets and discover Kepler's Law

Introduction and Procedure

Four hundred years ago, before Newton's laws of mechanics, Johannes Kepler discovered a law that relates the periods of planets with their average distances from the sun. (A period of a planet is the time it takes the planet to complete one orbit of the sun.) Kepler's strong belief that the solar system was governed by harmonious laws drove him to try to discover hidden patterns and correlations among the positions and motions of the planets. He used the trial-and-error method and continued his search for years.

At the time of his work, Kepler did not know the distance from the sun to each planet in terms of measures of distance such as the kilometer. But he was able to determine the distance from each planet to the sun in terms of the distance from Earth to the sun, now called the astronomical unit, or A.U. for short. One A.U. is the distance from Earth to the sun. The first column in the table below gives the average distance from the sun to each of the planets in astronomical units.

Patterns in the Positions and Motions of the Planets: Kepler's Discovery

Fill in the following table and look for the relationship that Kepler found.

Kepler's Third Law: The First Planet Table (Inner Planetary System)

Planet	Average Distance from Sun (A.U.)*	Cube of the Distance ($A.U.^3$)	Orbital Period (years)	Square of the Orbital Period ($years^2$)
Mercury	0.3870		0.2408	
Venus	0.7232		0.6151	
Earth	1.0000		1.0000	
Mars	1.5233		1.8807	
Jupiter	5.2025		11.8619	
Saturn	9.5387		29.4557	

*1 A.U. $\approx 149.6 \cdot 10^6$ km; 1 year $\approx$ 365.26 days.

Source: Data from S. Parker and J. Pasachoff, *Encyclopedia of Astronomy,* 2nd ed. (New York: McGraw-Hill, 1993), Table 1, Elements of Planetary Orbits. Copyright © 1993 by McGraw-Hill, Inc. Reprinted with permission.

The planets Uranus and Neptune were discovered after Kepler made his discovery. Check to see whether the relationship you found above holds true for these two planets.

The Second Planet Table (Outer Planetary System)

Planet	Average Distance from Sun (A.U.)	Cube of the Distance ($A.U.^3$)	Orbital Period (years)	Square of the Orbital Period ($years^2$)
Uranus	19.1911		84.0086	
Neptune	30.0601		164.7839	

Source: Data from S. Parker and J. Pasachoff, *Encyclopedia of Astronomy,* 2nd ed. (New York: McGraw-Hill, 1993), Table 1, Elements of Planetary Orbits. Copyright © 1993 by McGraw-Hill, Inc. Reprinted with permission.

Note: Pluto is no longer classified as a planet. It is now called a "dwarf planet."

Summary

- Express your results in words.
- Construct an equation showing the relationship between distance from the sun and orbital period. Solve the equation for distance from the sun. Then solve the equation for orbital period.
- Do your conclusions hold for all of the planets?