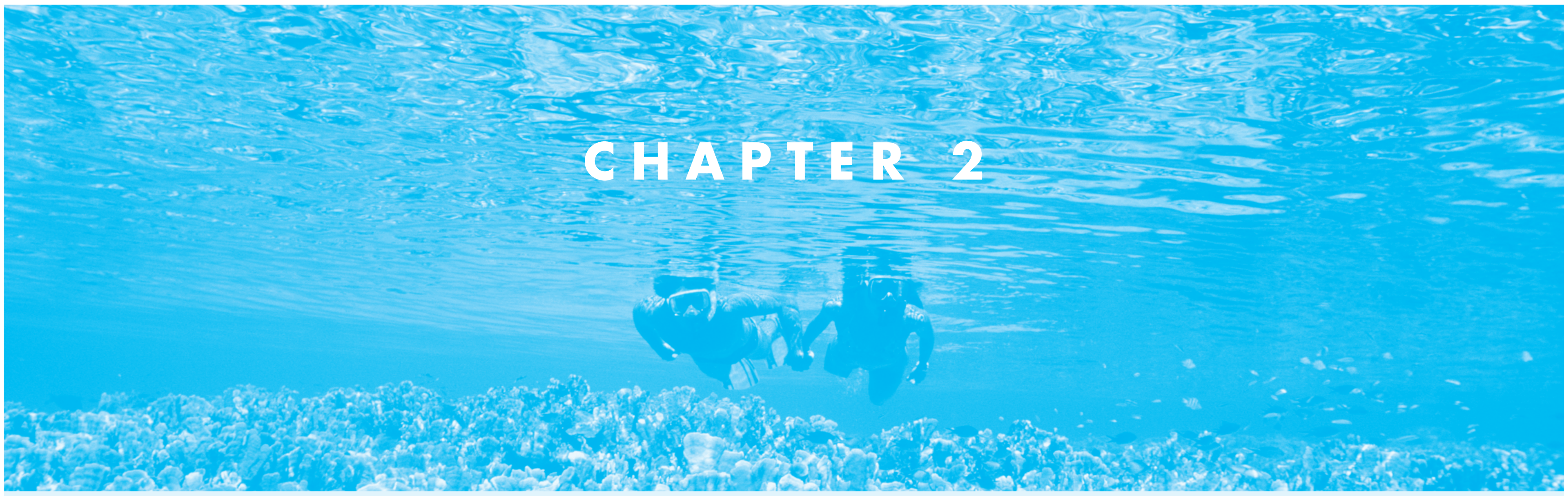

RATES OF CHANGE AND LINEAR FUNCTIONS

OVERVIEW

How does the U.S. population change over time? How do children's heights change as they age? Average rates of change provide a tool for measuring how change in one variable affects a second variable. When average rates of change are constant, the relationship is linear.

After reading this chapter you should be able to

- calculate and interpret average rates of change
- understand how representations of data can be biased
- recognize that a constant rate of change denotes a linear relationship
- construct a linear equation given a table, graph, or description
- derive by hand a linear model for a set of data

2.1 *Average Rates of Change*

In Chapter 1 we looked at how change in one variable could affect change in a second variable. In this section we'll examine how to measure that change.

Describing Change in the U.S. Population over Time

DATA
USPOP

We can think of the U.S. population as a function of time. Table 2.1 and Figure 2.1 are two representations of that function. They show the changes in the size of the U.S. population since 1790, the year the U.S. government conducted its first decennial census. Time, as usual, is the independent variable and population size is the dependent variable.

Change in population

Figure 2.1 clearly shows that the size of the U.S. population has been growing over the last two centuries, and growing at what looks like an increasingly rapid rate. How can the change in population over time be described quantitatively? One way is to pick two points on the graph of the data and calculate how much the population has changed during the time period between them.

Suppose we look at the change in the population between 1900 and 1990. In 1900 the population was 76.2 million; by 1990 the population had grown to 248.7 million. How much did the population increase?

$$\begin{aligned} \text{change in population} &= (248.7 - 76.2) \text{ million people} \\ &= 172.5 \text{ million people} \end{aligned}$$

This difference is portrayed graphically in Figure 2.2, at the top of the next page.

Change in time

Knowing that the population increased by 172.5 million tells us nothing about how rapid the change was; this change clearly represents much more dramatic growth if it happened over 20 years than if it happened over 200 years. In this case, the length of time over which the change in population occurred is

$$\begin{aligned} \text{change in years} &= (1990 - 1900) \text{ years} \\ &= 90 \text{ years} \end{aligned}$$

Population of the United States: 1790–2000

Year	Population in Millions
1790	3.9
1800	5.3
1810	7.2
1820	9.6
1830	12.9
1840	17.1
1850	23.2
1860	31.4
1870	39.8
1880	50.2
1890	63.0
1900	76.2
1910	92.2
1920	106.0
1930	123.2
1940	132.2
1950	151.3
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4

Table 2.1

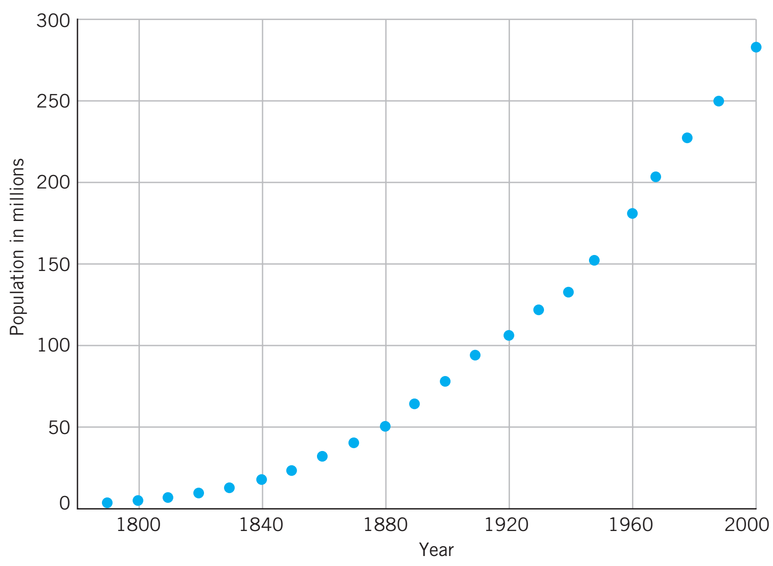

Figure 2.1 Population of the United States.
Source: U.S. Bureau of the Census, *Statistical Abstract of the United States: 2002.*

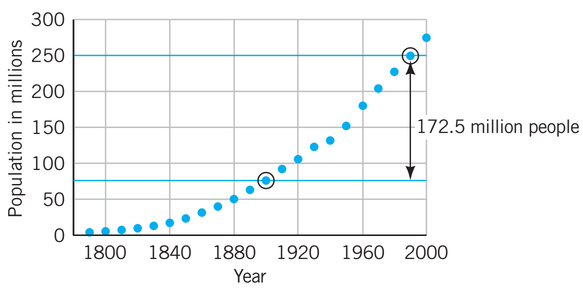

Figure 2.2 Population change: 172.5 million people.

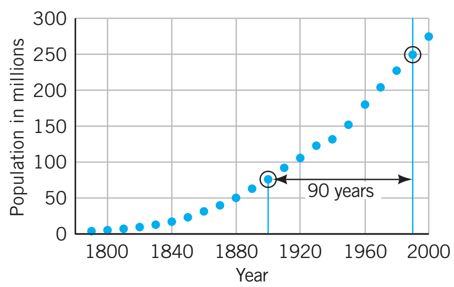

Figure 2.3 Time change: 90 years.

This interval is indicated in Figure 2.3 above.

Average rate of change

To find the *average rate of change* in population per year from 1900 to 1990, divide the change in the population by the change in years:

$$\begin{aligned}\text{average rate of change} &= \frac{\text{change in population}}{\text{change in years}}\\ &= \frac{172.5 \text{ million people}}{90 \text{ years}}\\ &\approx 1.92 \text{ million people/year}\end{aligned}$$

In the phrase "million people/year" the slash represents division and is read as "per." So our calculation shows that "on average," the population grew at a rate of 1.92 million people per year from 1900 to 1990. Figure 2.4 depicts the relationship between time and population increase.

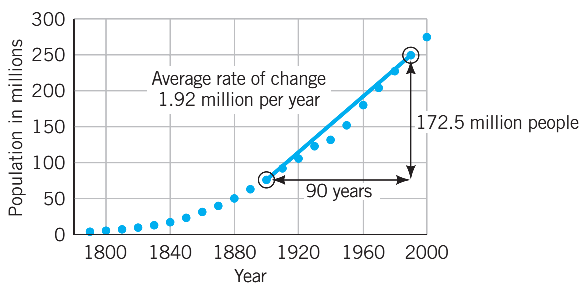

Figure 2.4 Average rate of change: 1900–1990.

Defining the Average Rate of Change

The notion of average rate of change can be used to describe the change in any variable with respect to another. If you have a graph that represents a plot of data points of the form (x, y), then the average rate of change between any two points is the change in the y value divided by the change in the x value.

> The *average rate of change* of y with respect to $x = \dfrac{\text{change in } y}{\text{change in } x}$

If the variables represent real-world quantities that have units of measure (e.g., millions of people or years), then the average rate of change should be represented in terms of the appropriate units:

$$\text{units of the average rate of change} = \frac{\text{units of } y}{\text{units of } x}$$

For example, the units might be dollars/year (read as "dollars per year") or pounds/person (read as "pounds per person").

EXAMPLE 1 Between 1850 and 1950 the median age in the United States rose from 18.9 to 30.2, but by 1970 it had dropped to 28.0.

a. Calculate the average rate of change in the median age between 1850 and 1950.

b. Compare your answer in part (a) to the average rate of change between 1950 and 1970.

SOLUTION **a.** Between 1850 and 1950,

$$\begin{aligned}\text{average rate of change} &= \frac{\text{change in median age}}{\text{change in years}}\\ &= \frac{(30.2 - 18.9)\text{ years}}{(1950 - 1850)\text{ years}} = \frac{11.3\text{ years}}{100\text{ years}}\\ &= 0.113\text{ years/year}\end{aligned}$$

The units are a little confusing. But the results mean that between 1850 and 1950 the median age increased an average of 0.113 years each calendar year.

b. Between 1950 and 1970,

$$\begin{aligned}\text{average rate of change} &= \frac{\text{change in median age}}{\text{change in years}}\\ &= \frac{(28.0 - 30.2)\text{ years}}{(1970 - 1950)\text{ years}} = \frac{-2.2\text{ years}}{20\text{ years}}\\ &= -0.110\text{ years/year}\end{aligned}$$

Note that since the median age dropped in value between 1950 and 1970, the average rate is negative. The median age decreased by 0.110 years/year between 1950 and 1970, whereas the median age increased by 0.113 years/year beween 1850 and 1950.

Limitations of the Average Rate of Change

The average rate of change is an average. Average rates of change have the same limitations as any average. Although the average rate of change of the U.S. population from 1900 to 1990 was 1.92 million people/year, it is highly unlikely that in each year the population grew by exactly 1.92 million. The number 1.92 million people/year is, as the name states, an average. Similarly, if the arithmetic average, or *mean*, height of students in your class is 67 inches, you wouldn't expect every student to be 67 inches tall. In fact, it may be the case that not even one student is 67 inches tall.

The average rate of change depends on the end points. If the data points do not all lie on a straight line, the average rate of change varies for different intervals. For instance, the average rate of change in population for the time interval 1840 to 1940 is 1.15 million people/year and from 1880 to 1980 is 1.76 million people/year. (See Table 2.2. *Note:* Here we abbreviate "million people" as "million.") You can see on the graphs that the line segment is much steeper from 1880 to 1980 than from 1840 to 1940 (Figures 2.5 and 2.6). Different intervals give different impressions of the rate of change in the U.S. population, so it is important to state which end points are used.

Time Interval	Change in Time	Change in Population	Average Rate of Change
1840–1940	100 yr	132.2 − 17.1 = 115.1 million	$\frac{115.1 \text{ million}}{100 \text{ yr}} \approx 1.15$ million/yr
1880–1980	100 yr	226.5 − 50.2 = 176.3 million	$\frac{176.3 \text{ million}}{100 \text{ yr}} \approx 1.76$ million/yr

Table 2.2

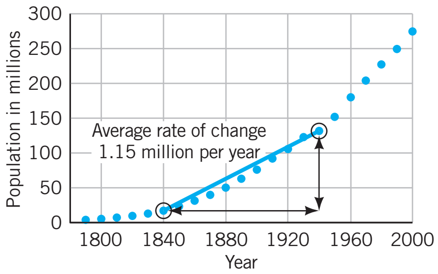

Figure 2.5 Average rate of change: 1840–1940.

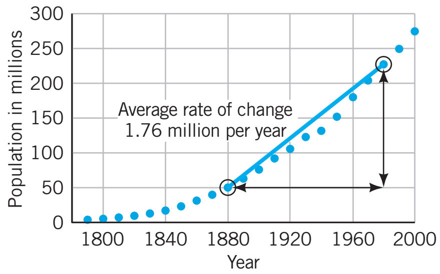

Figure 2.6 Average rate of change: 1880–1980.

The average rate of change does not reflect all the fluctuations in population size that may occur between the end points. For more specific information, the average rate of change can be calculated for smaller intervals.

Algebra Aerobics 2.1

1. Suppose your weight five years ago was 135 pounds and your weight today is 143 pounds. Find the average rate of change in your weight with respect to time.
2. Table 2.3 shows data on U.S. international trade as reported by the U.S. Bureau of the Census.

Year	U.S. Exports (billions of $)	U.S. Imports (billions of $)	U.S. Trade Balance = Exports − Imports (billions of $)
1990	537.2	618.4	−81.2
2006	820.2	1273.2	−453.0

Table 2.3

   a. What is the average rate of change between 1990 and 2006 for:
      i. Exports?
      ii. Imports?
      iii. The trade balance, the difference between what we sell abroad (exports) and buy from abroad (imports)?
   b. What do these numbers tell us?
3. Table 2.4 indicates the number of deaths in motor vehicle accidents in the United States as listed by the U.S. Bureau of the Census.

Annual Deaths in Motor Vehicle Accidents (thousands)

1980	1990	2000	2004
52.1	44.6	41.8	42.6

Table 2.4

   Find the average rate of change:
   a. From 1980 to 2000
   b. From 2000 to 2004
   Be sure to include units.
4. A car is advertised to go from 0 to 60 mph in 5 seconds. Find the average rate of change (i.e., the average acceleration) over that time.
5. According to the National Association of Insurance Commissioners, the average cost for automobile insurance has gone from $689 in 2000 to $867 in 2006. What is the average rate of change?
6. A football player runs for 1056 yards in 2002 and for 978 yards in 2006. Find the average rate of change in his performance.
7. The African elephant is an endangered species, largely because poachers (people who illegally hunt elephants) kill elephants to sell the ivory from their tusks. In the African country of Kenya in the last 10 years, the elephant population has dropped from 150,000 to 30,000. Calculate the average rate of change and describe what it means.

Exercises for Section 2.1

1. If r is measured in inches, s in pounds, and t in minutes, identify the units for the following average rates of change:

   a. $\frac{\text{change in } r}{\text{change in } s}$ b. $\frac{\text{change in } t}{\text{change in } r}$ c. $\frac{\text{change in } s}{\text{change in } r}$

2. Assume that R is measured in dollars, S in ounces, T in dollars per ounce, and V in ounces per dollar. Write a product of two of these terms whose resulting units will be:

   a. Dollars b. Ounces

3. Your car's gas tank is full and you take a trip. You travel 212 miles, then you fill your gas tank up again and it takes 10.8 gallons. If you designate your change in distance as 212 miles and your change in gallons as 10.8, what is the average rate of change of gasoline used, measured in miles per gallon?

4. The gas gauge on your car is broken, but you know that the car averages 22 miles per gallon. You fill your 15.5-gallon gas tank and tell your friend, "I can travel 300 miles before I need to fill up the tank again." Explain why this is true.

5. The consumption of margarine (in pounds per person) decreased from 9.4 in 1960 to 7.5 in 2000. What was the annual average rate of change? (*Source: www.census.gov*)

6. The percentage of people who own homes in the United States has gone from 65.5% in 1980 to 69.0% in 2005. What is the average rate of change in percentage points per year?

7. The accompanying table shows females' SAT scores in 2000 and 2005.

Year	Average Female Verbal SAT	Average Female Math SAT
2000	504	498
2005	505	504

Source: www.collegeboard.com.

   Find the average rate of change:

   a. In the math scores from 2000 to 2005

   b. In the verbal scores from 2000 to 2005

8. a. In 1992 the aerospace industry showed a net loss (negative profit) of \$1.84 billion. In 2002 the industry had a net profit of \$8.97 billion. Find the average annual rate of change in net profits from 1992 to 2002.

   b. In 2005, aerospace industry net profits were \$2.20 billion. Find the average rate of change in net profits:

      i. From 1992 to 2005

      ii. From 2002 to 2005

9. According to the U.S. Bureau of the Census, in elementary and secondary schools, in the academic year ending in 1985 there were about 630,000 computers being used for student instruction, or about 84.1 students per computer. In the academic year ending in 2005, there were about 13,600,000 computers being used, or about 4.0 students per computer. Find the average rate of change from 1985 to 2005 in:

   a. The number of computers being used

   b. The number of students per computer

10. According to the U.S. Bureau of the Census, the percentage of persons 25 years old and over completing 4 or more years of college was 4.6 in 1940 and 27.6 in 2005.

    a. Plot the data, labeling both axes and the coordinates of the points.

    b. Calculate the average rate of change in percentage points per year.

    c. Write a topic sentence summarizing what you think is the central idea to be drawn from these data.

11. Though reliable data about the number of African elephants are hard to come by, it is estimated that there were about 4,000,000 in 1930 and only 500,000 in 2000.

    a. What is the average annual rate of change in elephants over time? Interpret your result.

    b. During the 1980s it was estimated that 100,000 elephants were being killed each year due to systematic poaching for ivory. How does this compare with your answer in part (a)? What does this tell you about what was happening before or after the 1980s? (*Source: www.panda.org*)

12. a. According to the U.S. Bureau of the Census, between 1980 and 2004 domestic new car sales declined from 6581 thousand cars to 5357 thousand. (*Note:* This does not include trucks, vans, or SUVs.) Calculate the annual average rate of change.

    b. During the same period Japanese car sales in the United States dropped from 1906 thousand to 798 thousand. Calculate the average rate of change.

    c. What do the two rates suggest about car sales in the United States?

13. Use the information in the accompanying table to answer the following questions.

Percentage of Persons 25 Years Old and Over Who Have Completed 4 Years of High School or More

	1940	2004
All	24.5	85.2
White	26.1	85.8
Black	7.3	80.6
Asian/Pacific Islander	22.6	85.0

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States:* 2006.

a. What was the average rate of change (in percentage points per year) of completion of 4 years of high school from 1940 to 2004 for whites? For blacks? For Asian/Pacific Islanders? For all?

b. If these rates continue, what percentages of whites, of blacks, of Asian/Pacific Islanders, and of all will have finished 4 years of high school in the year 2007? Check the Internet to see if your predictions are accurate.

c. Write a 60-second summary describing the key elements in the high school completion data. Include rates of change and possible projections for the near future.

d. If these rates continue, in what year will 100% of whites have completed 4 years of high school or more? In what year 100% of blacks? In what year 100% of Asian/Pacific Islanders? Do these projections make sense?

14. The accompanying data show U.S. consumption and exports of cigarettes.

Year	U.S. Consumption (billions)	Exports (billions)
1960	484	20
1980	631	82
2000	430	148
2005	378	113

Source: U.S. Department of Agriculture.

a. Calculate the average rates of change in U.S. cigarette consumption from 1960 to 1980, from 1980 to 2005, and from 1960 to 2005.

b. Compute the average rate of change for cigarette exports from 1960 to 2005. Does this give an accurate image of cigarette exports?

c. The total number of cigarettes consumed in the United States in 1960 was 484 billion, very close to the number consumed in 1995, 487 billion. Does that mean smoking was as popular in 1995 as it was in 1960? Explain your answer.

d. Write a paragraph summarizing what the data tell you about the consumption and exports of cigarettes since 1960, including average rates of change.

15. Use the accompanying table on life expectancy to answer the following questions.

Average Number of Years of Life Expectancy in the United States by Race and Sex Since 1900

Life Expectancy at Birth by Year	White Males	White Females	Black Males	Black Females
1900	46.6	48.7	32.5	33.5
1950	66.5	72.2	58.9	62.7
2000	74.8	80.0	68.2	74.9
2005	75.4	81.1	69.9	76.8

Source: U.S. National Center for Health Statistics, *Statistical Abstract of the United States*, 2007.

a. What group had the highest life expectancy in 1900? In 2005? What group had the lowest life expectancy in 1900? In 2005?

b. Which group had the largest average rate of change in life expectancy between 1900 and 2005?

c. Write a short summary of the patterns in U.S. life expectancy from 1900 to 2005 using average rates of change to support your points.

16. The accompanying table gives the number of unmarried males and females over age 15 in the United States.

Marital Status of Population 15 Years Old and Older

Year	Number of unmarried males (in thousands)	Number of unmarried females (in thousands)
1950	17,735	19,525
1960	18,492	22,024
1970	23,450	29,618
1980	30,134	36,950
1990	36,121	43,040
2000	43,429	50,133
2004	41,214	47,616

Source: U.S. Bureau of the Census, *www.census.gov.*

a. Calculate the average rate of change in the number of unmarried males between 1950 and 2004. Interpret your results.

b. Calculate the average rate of change in the number of unmarried females between 1950 and 2004. Interpret your results.

c. Compare the two results.

d. What does this tell you, if anything, about the *percentages* of unmarried males and females?

2.2 *Change in the Average Rate of Change*

We can obtain an even better sense of patterns in the U.S. population if we look at how the average rate of change varies over time. One way to do this is to pick a fixed interval period for time and then calculate the average rate of change for each successive time period. Since we have the U.S. population data in 10-year intervals, we can calculate the average rate of change for each successive decade. The third column in Table 2.5 shows the results of these calculations. Each entry represents the average population growth *per year* (the average annual rate of change) during the previous decade. A few of these calculations are worked out in the last column of the table.

SOMETHING TO THINK ABOUT

What might be some reasons for the slowdown in population growth from the 1960s through the 1980s?

Average Annual Rates of Change of U.S. Population: 1790–2000

Year	Population (millions)	Average Annual Rate for Prior Decade (millions/yr)	Sample Calculations
1790	3.9	Data not available	
1800	5.3	0.14	0.14 = (5.3 − 3.9)/(1800 − 1790)
1810	7.2	0.19	
1820	9.6	0.24	
1830	12.9	0.33	
1840	17.1	0.42	0.42 = (17.1 − 12.9)/(1840 − 1830)
1850	23.2	0.61	
1860	31.4	0.82	
1870	39.8	0.84	
1880	50.2	1.04	
1890	63.0	1.28	
1900	76.2	1.32	
1910	92.2	1.60	
1920	106.0	1.38	
1930	123.2	1.72	
1940	132.2	0.90	0.90 = (132.2 − 123.2)/(1940 − 1930)
1950	151.3	1.91	
1960	179.3	2.80	
1970	203.3	2.40	
1980	226.5	2.32	
1990	248.7	2.22	
2000	281.4	3.27	

Table 2.5

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States: 2002.*

What is happening to the average rate of change over time?

Start at the top of the third column and scan down the numbers. Notice that until 1910 the average rate of change increases every year. Not only is the population growing every decade until 1910, but it is growing at an increasing rate. It's like a car that is not only moving forward but also accelerating. A feature that was not so obvious in the original data is now evident: In the intervals 1910 to 1920, 1930 to 1940, and 1960 to 1990 we see an increasing population but a decreasing rate of growth. It's like a car decelerating—it is still moving forward but it is slowing down.

The graph in Figure 2.7, with years on the horizontal axis and average rates of change on the vertical axis, shows more clearly how the average rate of change

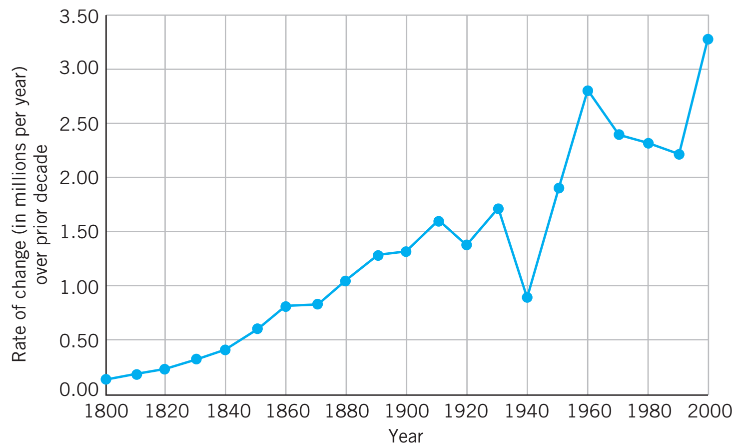

Figure 2.7 Average rates of change in the U.S. population by decade.

fluctuates over time. The first point, corresponding to the year 1800, shows an average rate of change of 0.14 million people/year for the decade 1790 to 1800. The rate 1.72, corresponding to the year 1930, means that from 1920 to 1930 the population was increasing at a rate of 1.72 million people/year.

What does this tell us about the U.S. population?

The pattern of growth was fairly steady up until about 1910. Why did it change? A possible explanation for the slowdown in the decade prior to 1920 might be World War I and the 1918 flu epidemic, which by 1920 had killed nearly 20,000,000 people, including about 500,000 Americans.

In Figure 2.7, the steepest decline in the average rate of change is between 1930 and 1940. One obvious suspect for the big slowdown in population growth in the 1930s is the Great Depression. Look back at Figure 2.1, the original graph that shows the overall growth in the U.S. population. The decrease in the average rate of change in the 1930s is large enough to show up in our original graph as a visible slowdown in population growth.

The average rate of change increases again between 1940 and 1960, then drops off from the 1960s through the 1980s. The rate increases once more in the 1990s. This latest surge in the growth rate is attributed partially to the "baby boom echo" (the result of baby boomers having children) and to a rise in birth rates and immigration.

Algebra Aerobics 2.2

1. Table 2.6 and Figure 2.8 show estimates for world population between 1800 and 2050.

a. Fill in the third column of the table by calculating the average annual rate of change.

b. Graph the annual average rate of change versus time.

World Population

Year	Total Population (millions)	Annual Average Rate of Change (over prior 50 years)
1800	980	n.a.
1850	1260	
1900	1650	
1950	2520	
2000	6090	
2050	9076 (est.)	

Table 2.6
Source: Population Division of the United Nations, *www.un.org/popin.*

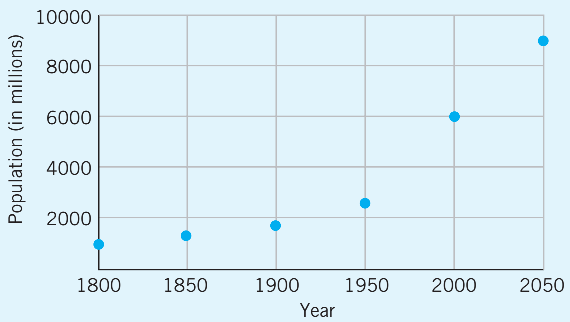

Figure 2.8 World population.

c. During what 50-year period was the average annual rate of change the largest?

d. Describe in general terms what happened (and is predicted to happen) to the world population and its average rate of change between 1800 and 2050.

2. A graph illustrating a corporation's profits indicates a positive average rate of change between 2003 and 2004, another positive rate of change between 2004 and 2005, a zero rate of change between 2005 and 2006, and a negative rate of change between 2006 and 2007. Describe the graph and the company's financial situation over the years 2003–2007.

3. Table 2.7 shows educational data collected on 18- to 24-year-olds between 1960 and 2004 by the National Center for Educational Statistics. The table shows the number of students who graduated from high school or completed a GED (a high school equivalency exam) during the indicated year.

High School Completers

Year	Number (thousands)	Average Rate of Change (thousands per year)
1960	1679	n.a.
1970	2757	
1980	3089	
1990	2355	
2000	2756	
2004	2752	

Table 2.7

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a. Fill in the blank cells with the appropriate average rates of change for high school completers.

b. Describe the pattern in the number of high school completers between 1960 and 2004.

c. What does it mean here when the average rate of change is positive? Give a specific example from your data.

d. What does it mean here when the average rate of change is negative? Give another specific example.

e. What does it mean when two adjacent average rates of change are positive, but the second one is smaller than the first?

Exercises for Section 2.2

Technology is optional for Exercises 4 and 11.

1. Calculate the average rate of change between adjacent points for the following function. (The first few are done for you.)

x	$f(x)$	Average Rate of Change
0	0	n.a.
1	1	1
2	8	7
3	27	
4	64	
5	125	

a. Is the function $f(x)$ increasing, decreasing, or constant throughout?

b. Is the average rate of change increasing, decreasing, or constant throughout?

2. Calculate the average rate of change between adjacent points for the following function. The first one is done for you.

x	$f(x)$	Average Rate of Change
0	0	n.a.
1	1	1
2	16	
3	81	
4	256	
5	625	

a. Is the function $f(x)$ increasing, decreasing, or constant throughout?

b. Is the average rate of change increasing, decreasing, or constant throughout?

3. The accompanying table shows the number of registered motor vehicles in the United States.

Year	Registered Motor Vehicles (millions)	Annual Average Rate of Change (over prior decade)
1960	74	n.a.
1970	108	
1980	156	
1990	189	
2000	218	

a. Fill in the third column in the table.

b. During which decade was the average rate of change the smallest?

c. During which decade was the average rate of change the largest?

d. Write a paragraph describing the change in registered motor vehicles between 1960 and 2000.

4. (Graphing program optional.) The accompanying table indicates the number of juvenile arrests (in thousands) in the United States for aggravated assault.

Year	Juvenile Arrests (thousands)	Annual Average Rate of Change over Prior 5 Years
1985	36.8	n.a.
1990	54.5	
1995	68.5	
2000	49.8	
2005	36.9	

a. Fill in the third column in the table by calculating the annual average rate of change.

b. Graph the annual average rate of change versus time.

c. During what 5-year period was the annual average rate of change the largest?

d. Describe the change in aggravated assault cases during these years by referring both to the number and to the annual average rate of change.

5. Calculate the average rate of change between adjacent points for the following functions and place the values in a third column in each table. (The first entry is "n.a.")

x	$f(x)$
0	5
10	25
20	45
30	65
40	85
50	105

x	$g(x)$
0	270
10	240
20	210
30	180
40	150
50	120

a. Are the functions $f(x)$ and $g(x)$ increasing, decreasing, or constant throughout?

b. Is the average rate of change of each function increasing, decreasing, or constant throughout?

6. Calculate the average rate of change between adjacent points for each of the functions in Tables A–D and place the values in a third column in each table. (The first entry is "n.a.") Then for each function decide which statement best describes it.

Table A

x	$f(x)$
0	1
1	3
2	9
3	27
4	81
5	243

Table C

x	$h(x)$
0	50
10	55
20	60
30	65
40	70
50	75

Table B

x	$g(x)$
0	200
15	155
30	110
45	65
60	20
75	−25

Table D

x	$k(x)$
0	40
1	31
2	24
3	19
4	16
5	15

a. As x increases, the function increases at a constant rate.

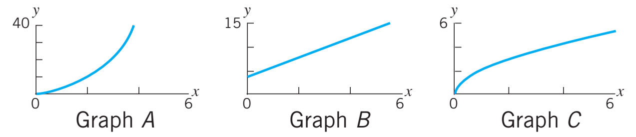
b. As x increases, the function increases at an increasing rate.

c. As x increases, the function decreases at a constant rate.

d. As x increases, the function decreases at an increasing rate.

7. Each of the following functions has a graph that is increasing. If you calculated the average rate of change between sequential equal-size intervals, which function can be said to have an average rate of change that is:

a. Constant? **b.** Increasing? **c.** Decreasing?



Graph *A* Graph *B* Graph *C*

8. Match the data table with its graph.

Table A

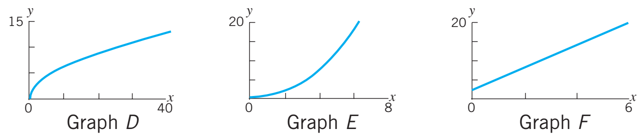
x	y
0	2
1	5
2	8
3	11
4	14
5	17
6	20

Table B

x	y
0	0
1	0.5
2	2
3	4.5
4	8
5	12.5
6	18

Table C

x	y
0	0
1	2
4	4
9	6
16	8
25	10
36	12



Graph *D* Graph *E* Graph *F*

9. Refer to the first two data tables (A and B) in Exercise 8. Insert a third column in each table and label the column "average rate of change."

a. Calculate the average rate of change over adjacent data points.

b. Identify whether the table represents an average rate of change that is constant, increasing, or decreasing.

c. Explain how you could tell this by looking at the corresponding graph in Exercise 8.

10. Following are data on the U.S. population over the time period 1830–1930 (extracted from Table 2.1).

U.S. Population

Year	Population (in millions)	Average Rate of Change (millions/yr)
1830	12.9	n.a.
1850	23.2	
1870		0.83
1890	63.0	1.16
1910	92.2	
1930		1.55

Source: U.S. Bureau of the Census, *www.census.gov.*

a. Fill in the missing parts of the chart.

b. Which 20-year interval experienced the largest average rate of change in population?

c. Which 20-year interval experienced the smallest average rate of change in population?

11. (Technology recommended.) The accompanying data give a picture of the two major methods of news communication in the United States. (See also Excel or graph link files NEWPRINT and ONAIRTV.)

a. Use the U.S. population numbers from Table 2.1 (at the beginning of this chapter) to calculate and compare the number of copies of newspapers *per person* in 1920 and in 2000.

b. Create a table that displays the annual average rate of change in TV stations for each decade since 1950. Create a similar table that displays the annual average rate of change in newspapers published for the same period. Graph the results.

c. If new TV stations continue to come into existence at the same rate as from 1990 to 2000, how many will there be by the year 2010? Do you think this is likely to be a reasonable projection, or is it overly large or small judging from past rates of growth? Explain.

d. What trends do you see in the dissemination of news as reflected in these data?

Number of U.S. Newspapers

Year	Newspapers (thousands of copies printed)	Number of Newspapers Published
1920	27,791	2042
1930	39,589	1942
1940	41,132	1878
1950	53,829	1772
1960	58,882	1763
1970	62,108	1748
1980	62,202	1745
1990	62,324	1611
2000	55,800	1480

Source: U.S. Bureau of the Census, *www.census.gov.*

Number of U.S. Commercial TV Stations

Year	Number of Commercial TV Stations
1950	98
1960	515
1970	677
1980	734
1990	1092
2000	1248

Source: U.S. Bureau of the Census, *www.census.gov.*

2.3 *The Average Rate of Change Is a Slope*

Calculating Slopes

The reading "Slopes" describes many of the practical applications of slopes, from cowboy boots to handicap ramps.

On a graph, the average rate of change is the *slope* of the line connecting two points. The slope is an indicator of the steepness of the line.

If (x_1, y_1) and (x_2, y_2) are two points, then the change in y equals $y_2 - y_1$ (see Figure 2.9). This difference is often denoted by Δy, read as "delta y," where Δ is the Greek letter capital D (think of D as representing difference): $\Delta y = y_2 - y_1$. Similarly, the change in x (delta x) can be represented by $\Delta x = x_2 - x_1$. Then

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x}$$

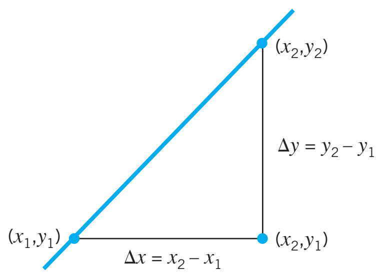


Figure 2.9 Slope $= \Delta y/\Delta x$.

> The average rate of change represents a *slope*. Given two points (x_1, y_1) and (x_2, y_2),
>
> $$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$$

When calculating a slope, it doesn't matter which point is first

Given two points, (x_1, y_1) and (x_2, y_2), it doesn't matter which one we use as the first point when we calculate the slope. In other words, we can calculate the slope between (x_1, y_1) and (x_2, y_2) as

$$\frac{y_2 - y_1}{x_2 - x_1} \quad \text{or as} \quad \frac{y_1 - y_2}{x_1 - x_2}$$

The two calculations result in the same value.

We can show that the two forms are equivalent.

Given	$\text{slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$
multiply by $\dfrac{-1}{-1}$	$= \dfrac{-1}{-1} \cdot \dfrac{y_2 - y_1}{x_2 - x_1}$
simplify	$= \dfrac{-y_2 + y_1}{-x_2 + x_1}$
rearrange terms	$= \dfrac{y_1 - y_2}{x_1 - x_2}$

In calculating the slope, we do need to be consistent in the order in which the coordinates appear in the numerator and the denominator. If y_1 is the first term in the numerator, then x_1 must be the first term in the denominator.

EXAMPLE 1 Plot the two points $(-2, -6)$ and $(7, 12)$ and calculate the slope of the line passing through them.

SOLUTION Treating $(-2, -6)$ as (x_1, y_1) and $(7, 12)$ as (x_2, y_2) (Figure 2.10), then

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12 - (-6)}{7 - (-2)} = \frac{18}{9} = 2$$

We could also have used -6 and -2 as the first terms in the numerator and denominator, respectively:

$$\text{slope} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{-6 - 12}{-2 - 7} = \frac{-18}{-9} = 2$$

Either way we obtain the same answer.

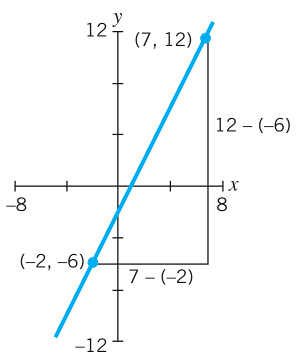


Figure 2.10

EXAMPLE 2 The percentage of the U.S. population living in rural areas decreased from 84.7% in 1850 to 21.0% in 2000. Plot the data, then calculate and interpret the average rate of change in the rural population over time.

SOLUTION If we treat year as the independent and percentage as the dependent variable, our given data can be represented by the points (1850, 84.7) and (2000, 21.0). See Figure 2.11.

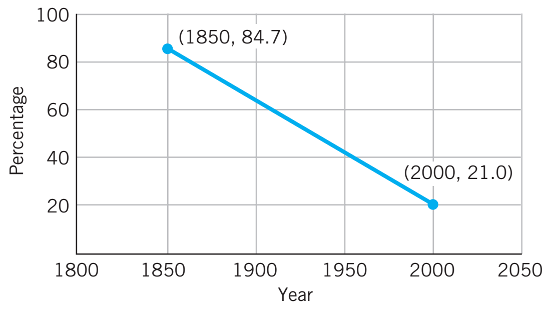


Figure 2.11 Percentage of the U.S. population living in rural areas.
Source: U.S. Bureau of the Census, *www.census.gov.*

$$\begin{aligned}\text{The average rate of change} &= \frac{\text{change in the percentage of rural population}}{\text{change in time}}\\ &= \frac{(21.0 - 84.7)\%}{(2000 - 1850)\text{ years}}\\ &= \frac{-63.7\%}{150 \text{ years}}\\ &\approx -0.42\% \text{ per year}\end{aligned}$$

SOMETHING TO THINK ABOUT
What kind of social and economic implications does a population shift of this magnitude have on society?

The sign of the average rate of change is negative since the percentage of people living in rural areas was decreasing. (The negative slope of the graph in Figure 2.11 confirms this.) The value tells us that, on average, the percentage living in rural areas decreased by 0.42 percentage points (or about one-half of 1%) each year between 1850 and 2000. The change per year may seem small, but in a century and a half the rural population went from being the overwhelming majority (84.7%) to about one-fifth (21%) of the population.

> If the slope between any two points (the average rate of change of y with respect to x) is *positive*, then the graph of the relationship rises when read from left to right. This means that as x increases in value, y also increases in value.
>
> If the slope is *negative*, the graph falls when read from left to right. As x increases, y decreases.
>
> If the slope is *zero*, the graph is flat. As x increases, there is no change in y.

EXAMPLE 3 Given Table 2.8, a listing of civil disturbances over time, plot and then connect the points, and (without doing any calculations) indicate on the graph when the average rate of change between adjacent data points is positive (+), negative (−), and zero (0).

SOLUTION The data are plotted in Figure 2.12. Each line segment is labeled +, −, or 0, indicating whether the average rate of change between adjacent points is positive, negative, or zero. The largest positive average rate of change, or steepest upward slope, seems to be

Civil Disturbances in U.S. Cities

Year	Period	Number of Disturbances
1968	Jan.–Mar.	6
	Apr.–June	46
	July–Sept.	25
	Oct.–Dec.	3
1969	Jan.–Mar.	5
	Apr.–June	27
	July–Sept.	19
	Oct.–Dec.	6
1970	Jan.–Mar.	26
	Apr.–June	24
	July–Sept.	20
	Oct.–Dec.	6
1971	Jan.–Mar.	12
	Apr.–June	21
	July–Sept.	5
	Oct.–Dec.	1
1972	Jan.–Mar.	3
	Apr.–June	8
	July–Sept.	5
	Oct.–Dec.	5

Table 2.8

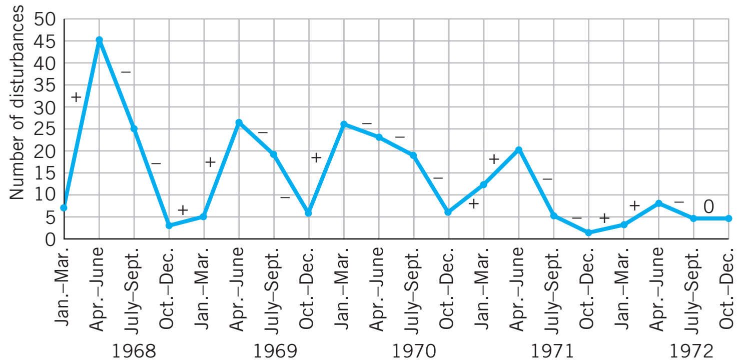


Figure 2.12 Civil disturbances: 1968–1972.
Source: D. S. Moore and G. P. McCabe, *Introduction to the Practice of Statistics*.

between the January-to-March and April-to-June counts in 1968. The largest negative average rate of change, or steepest downward slope, appears later in the same year (1968) between the July-to-September and October-to-December counts.

Civil disturbances between 1968 and 1972 occurred in cycles: The largest numbers generally occurred in the spring months and the smallest in the winter months. The peaks decrease over time. What was happening in America that might correlate with the peaks? This was a tumultuous period in our history. Many previously silent factions of society were finding their voices. Recall that in April 1968 Martin Luther King was assassinated and in January 1973 the last American troops were withdrawn from Vietnam.

Algebra Aerobics 2.3

1. **a.** Plot each pair of points and then calculate the slope of the line that passes through them.
   - **i.** (4, 1) and (8, 11)
   - **ii.** (−3, 6) and (2, 6)
   - **iii.** (0, −3) and (−5, −1)

   b. Recalculate the slopes in part (a), reversing the order of the points. Check that your answers are the same.
2. Specify the intervals on the graph in Figure 2.13 for which the average rate of change between adjacent data points is approximately zero.

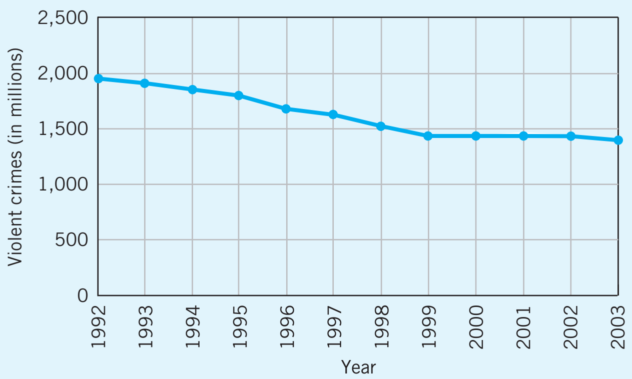


Figure 2.13 Violent crimes in the United States.

3. Specify the intervals on the graph in Figure 2.14 for which the average rate of change between adjacent data points appears positive, negative, or zero.

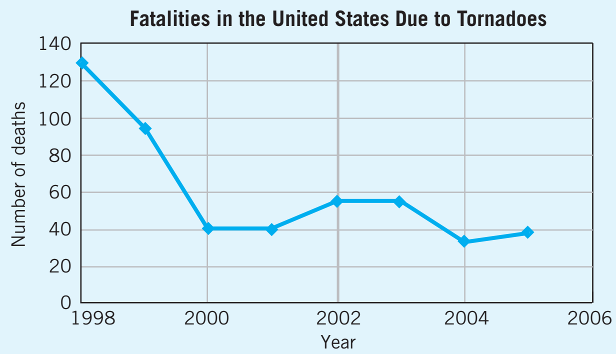


Figure 2.14 Deaths due to tornadoes: 1998–2005.

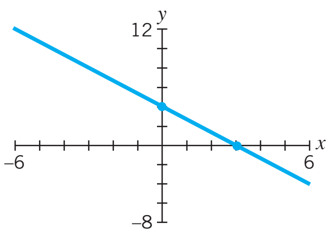
4. What is the missing y-coordinate that would produce a slope of 4, if a line were drawn through the points (3, −2) and (5, y)?
5. Find the slope of the line through the points (2, 9) and (2 + h, 9 + 2h).

6. Consider points $P_1 = (0, 0)$, $P_2 = (1, 1)$, $P_3 = (2, 4)$, and $P_4 = (3, 9)$.
   a. Verify that these four points lie on the graph of $y = x^2$.
   b. Find the slope of the line segments connecting P_1 and P_2, P_2 and P_3, and P_3 and P_4.
   c. What do these slopes suggest about the graph of the function within those intervals?

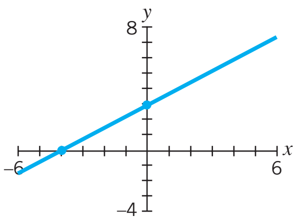
Exercises for Section 2.3

1. Find the slope of a straight line that goes through:
   a. $(-5, -6)$ and $(2, 3)$
   b. $(-5, 6)$ and $(2, -3)$

2. Find the slope of each line using the points where the graph intersects the x and y axes.

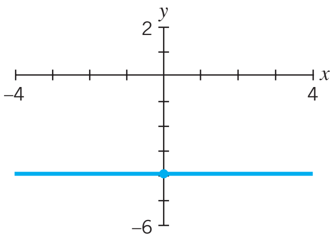


Graph *A*

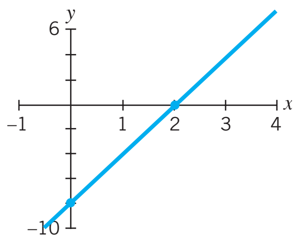


Graph *B*

3. Find the slope of each line using the points where the line crosses the x- or y-axis.

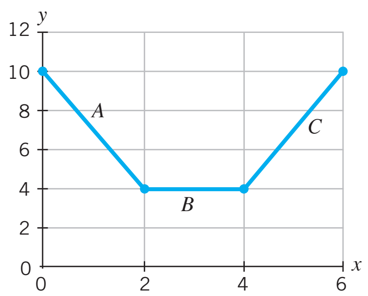


Graph *A*

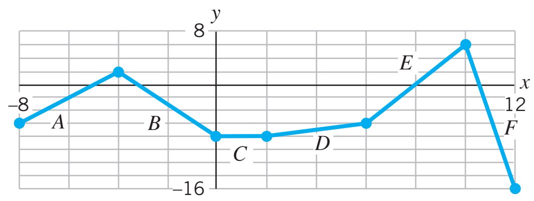


Graph *B*

4. Examine the line segments A, B, and C.
   a. Which line segment has a slope that is positive? That is negative? That is zero?
   b. Calculate the exact slope for each line segment A, B, and C.



5. Given the graph at the top of the next column:
   a. Estimate the slope for each line segment A–F.
   b. Which line segment is the steepest?
   c. Which line segment has a slope of zero?



6. Plot each pair of points and calculate the slope of the line that passes through them.
   a. $(3, 5)$ and $(8, 15)$
   b. $(-1, 4)$ and $(7, 0)$
   c. $(5, 9)$ and $(-5, 9)$
   d. $(-2, 6)$ and $(2, -6)$
   e. $(-4, -3)$ and $(2, -3)$

7. The following problems represent calculations of the slopes of different lines. Solve for the variable in each equation.
   a. $\dfrac{150 - 75}{20 - 10} = m$
   b. $\dfrac{70 - y}{0 - 8} = 0.5$
   c. $\dfrac{182 - 150}{28 - x} = 4$
   d. $\dfrac{6 - 0}{x - 10} = 0.6$

8. Find the slope m of the line through the points $(0, b)$ and (x, y), then solve the equation for y.

9. Find the value of t if m is the slope of the line that passes through the given points.
   a. $(3, t)$ and $(-2, 1)$, $m = -4$
   b. $(5, 6)$ and $(t, 9)$, $m = \frac{2}{3}$

10. a. Find the value of x so that the slope of the line through $(x, 5)$ and $(4, 2)$ is $\frac{1}{3}$.
    b. Find the value of y so that the slope of the line through $(1, -3)$ and $(-4, y)$ is -2.
    c. Find the value of y so that the slope of the line through $(-2, 3)$ and $(5, y)$ is 0.
    d. Find the value of x so that the slope of the line through $(-2, 2)$ and $(x, 10)$ is 2.
    e. Find the value of y so that the slope of the line through $(-100, 10)$ and $(0, y)$ is $-\frac{1}{10}$.
    f. Find at least one set of values for x and y so that the slope of line through $(5, 8)$ and (x, y) is 0.

11. Points that lie on the same straight line are said to be *collinear*. Determine if the following points are collinear.
    a. $(2, 3)$, $(4, 7)$, and $(8, 15)$
    b. $(-3, 1)$, $(2, 4)$, and $(7, 8)$

12. **a.** Find the slope of the line through each of the following pairs of points.

i. $(-1, 4)$ and $(-2, 4)$

ii. $(7, -3)$ and $(-7, -3)$

iii. $(-2, -6)$ and $(5, -6)$

b. Summarize your findings.

13. Graph a line through each pair of points and then calculate its slope.

a. The origin and $(6, -2)$ **b.** The origin and $(-4, 7)$

14. Find some possible values of the y-coordinates for the points $(-3, y_1)$ and $(6, y_2)$ such that the slope $m = 0$.

15. Calculate the slope of the line passing through each of the following pairs of points.

a. $(0, \sqrt{2})$ and $(\sqrt{2}, 0)$

b. $\left(0, -\frac{3}{2}\right)$ and $\left(-\frac{3}{2}, 0\right)$

c. $(0, b)$ and $(b, 0)$

d. What do these pairs of points and slopes all have in common?

16. Which pairs of points produce a line with a negative slope?

a. $(-5, -5)$ and $(-3, -3)$ **d.** $(4, 3)$ and $(12, 0)$

b. $(-2, 6)$ and $(-1, 4)$ **e.** $(0, 3)$ and $(4, -10)$

c. $(3, 7)$ and $(-3, -7)$ **f.** $(4, 2)$ and $(6, 2)$

17. In the previous exercise, which pairs of points produce a line with a positive slope?

18. A study on numerous streams examined the effects of a warming climate. It found an increase in water temperature of about 0.7°C for every 1°C increase in air temperature.

a. Find the rate of change in water temperature with respect to air temperature. What are the units?

b. If the air temperature increased by 5°C, by how much would you expect the stream temperature to increase?

19. Handicapped Vietnam veterans successfully lobbied for improvements in the architectural standards for wheelchair access to public buildings.

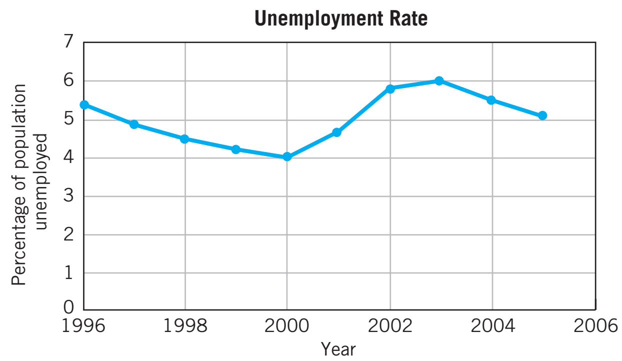
a. The old standard was a 1-foot rise for every 10 horizontal feet. What would the slope be for a ramp built under this standard?

b. The new standard is a 1-foot rise for every 12 horizontal feet. What would the slope of a ramp be under this standard?

c. If the front door is 3 feet above the ground, how long would the handicapped ramp be using the old standard? Using the new?

20. **a.** The accompanying graph gives information about the unemployment rate. Specify the intervals on the graph for which the slope of the line segment between adjacent data points appears positive. For which does it appear negative? For which zero?

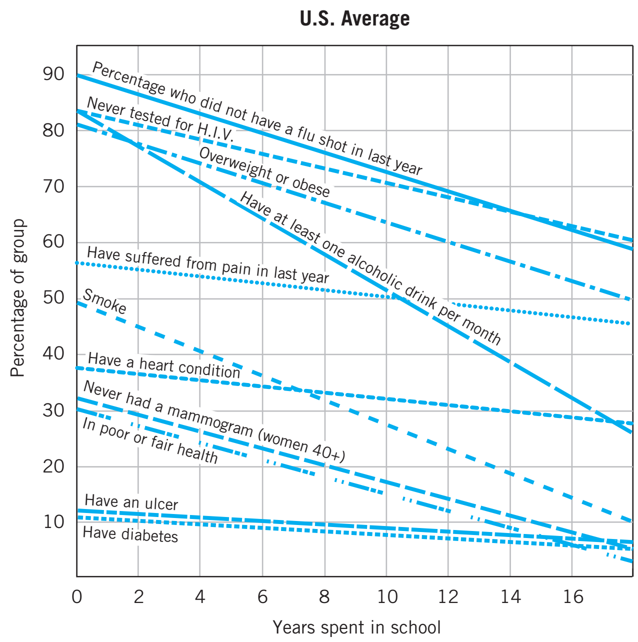
b. What might have caused the increase in the unemployment rate just after 2000?



Source: U.S. Department of Labor, Department of Labor Statistics, *www.bls.gov*

21. Read the Anthology article "Slopes" on the course website and then describe two practical applications of slopes, one of which is from your own experience.

22. The following graph shows the relationship between education and health in the United States.



Source: "A Surprising Secret to a Long Life, According to Studies: Stay in School," *New York Times,* January 3, 2007.

a. Estimate the percentages for those with 8 years of education and for those with 16 years of education (college graduates) who:

i. Are overweight or obese

ii. Have at least one alcoholic drink per month

iii. Smoke

b. For each category in part (a), calculate the average rate of change with respect to years of education. What do they all have in common?

c. What does the graph suggest about the link between education and health? Could there be other factors at play?

2.4 *Putting a Slant on Data*

Whenever anyone summarizes a set of data, choices are being made. One choice may not be more "correct" than another. But these choices can convey, either accidentally or on purpose, very different impressions.

Slanting the Slope: Choosing Different End Points

Within the same data set, one choice of end points may paint a rosy picture, while another choice may portray a more pessimistic outcome.

EXAMPLE 1 The data in Table 2.9 and the scatter plot in Figure 2.15 show the number of people below the poverty level in the United States from 1960 to 2005. How could we use the information to make the case that the poverty level has decreased? Has increased?

People in Poverty in the United States

Year	Number of People in Poverty (in thousands)
1960	39,851
1970	25,420
1980	29,272
1990	33,585
2000	31,581
2005	36,950

Table 2.9
Source: U.S. Bureau of the Census, *www.census.gov.*

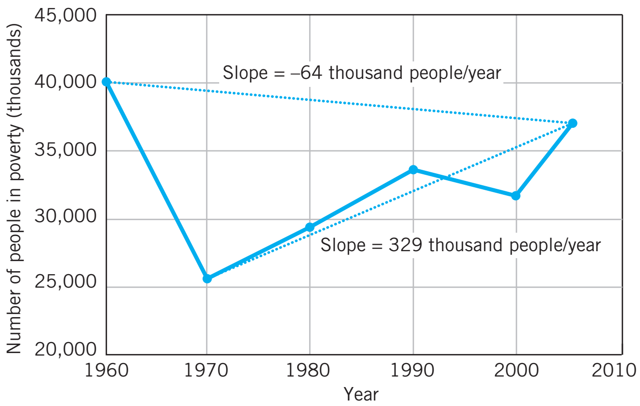


Figure 2.15 Number of people in poverty in the United States between 1960 and 2005.

SOLUTION *Optimistic case:* To make an upbeat case that poverty numbers have decreased, we could choose as end points (1960, 39851) and (2005, 36950). Then

$$\begin{aligned}\text{average rate of change} &= \frac{\text{change in no. of people in poverty (000s)}}{\text{change in years}}\\ &= \frac{36{,}950 - 39{,}851}{2005 - 1960}\\ &= \frac{-2901}{45}\\ &\approx -64 \text{ thousand people/year}\end{aligned}$$

So between 1960 and 2005 the number of impoverished individuals *decreased* on average by 64 thousand (or 64,000) each year. We can see this reflected in Figure 2.15 in the negative slope of the line connecting (1960, 39851) and (2005, 36950).

Pessimistic case: To make a depressing case that poverty numbers have risen, we could choose (1970, 25420) and (2005, 36950) as end points. Then

$$\begin{aligned}\text{average rate change of change} &= \frac{\text{change in no. of people in poverty (000s)}}{\text{change in years}}\\ &= \frac{36{,}950 - 25{,}420}{2005 - 1970}\end{aligned}$$

$$= \frac{11{,}530}{35}$$

$$\approx 329 \text{ thousand people/year}$$

So between 1970 and 2005 the number of impoverished individuals *increased* on average by 329 thousand (or 329,000) per year. This is reflected in Figure 2.15 in the positive slope of the line connecting (1970, 25420) and (2005, 36950). Both average rates of change are correct, but they give very different impressions of the changing number of people living in poverty in America.

Slanting the Data with Words and Graphs

See "C4: Distortion by Clipping and Scaling" in *Rates of Change.*

If we wrap data in suggestive vocabulary and shape graphs to support a particular viewpoint, we can influence the interpretation of information. In Washington, D.C., this would be referred to as "putting a spin on the data."

Take a close look at the following three examples. Each contains exactly the same underlying facts: the same average rate of change calculation and a graph with a plot of the same two data points (1990, 248.7) and (2000, 281.4), representing the U.S. population (in millions) in 1990 and in 2000.

The U.S. population increased by only 3.27 million/year between 1990 and 2000

Stretching the scale of the horizontal axis relative to the vertical axis makes the slope of the line look almost flat and hence minimizes the impression of change (Figure 2.16).

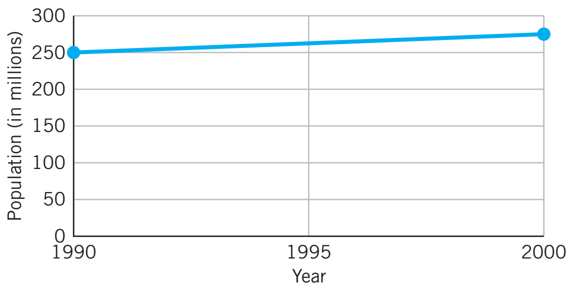


Figure 2.16 "Modest" growth in the U.S. population.

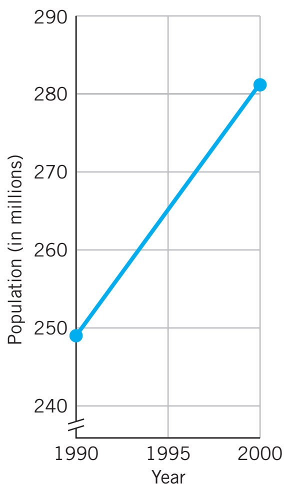


Figure 2.17 "Explosive" growth in the U.S. population.

The U.S. population had an explosive growth of over 3.27 million/year between 1990 and 2000

Cropping the vertical axis (which now starts at 240 instead of 0) and stretching the scale of the vertical axis relative to the horizontal axis makes the slope of the line look steeper and strengthens the impression of dramatic change (Figure 2.17).

The U.S. population grew at a reasonable rate of 3.27 million/year during the 1990s

Visually, the steepness of the line in Figure 2.18 seems to lie roughly halfway between the previous two graphs. *In fact, the slope of 3.27 million/year is precisely the same for all three graphs.*

How could you decide upon a "fair" interpretation of the data? You might try to put the data in context by asking, How does the growth between 1990 and 2000 compare with other decades in the history of the United States? How does it compare with growth in other countries at the same time? Was this rate of growth easily accommodated, or did it strain national resources and overload the infrastructure?

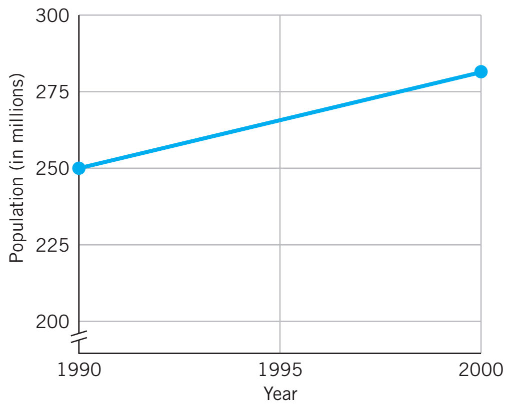


Figure 2.18 "Reasonable" growth in the U.S. population.

Exploration 2.1 gives you a chance to put your own "spin" on data.

A statistical claim is never completely free of bias. For every statistic that is quoted, others have been left out. This does not mean that you should discount all statistics. However, you will be best served by a thoughtful approach when interpreting the statistics to which you are exposed on a daily basis. By getting in the habit of asking questions and then coming to your own conclusions, you will develop good sense about the data you encounter.

Algebra Aerobics 2.4

1. Table 2.10 shows the *percentage* of the U.S. population in poverty between 1960 and 2005. In each case identify two end points you could use to make the case that poverty:

Year	% of Population in Poverty
1960	22.2
1970	12.6
1980	13.0
1990	13.5
2000	11.3
2005	12.6

Table 2.10
Source: U.S. Bureau of the Census, *www.census.gov.*

a. Has declined dramatically
b. Has remained stable
c. Has increased substantially

2. Assume you are the financial officer of a corporation whose stock earnings were $1.02 per share in 2005, and $1.12 per share in 2006, and $1.08 per share in 2007. How could you make a case for:

a. Dramatic growth?
b. Dramatic decline?

3. Sketch a graph and compose a few sentences to forcefully convey the views of the following persons.

a. You are an antiwar journalist reporting on American casualties during a war. In week 1 there were 17, in week 2 there were 29, and in week 3 there were 26.
b. You are the president's press secretary in charge of reporting war casualties [listed in part (a)] to the public.

4. The graph in Figure 2.19 appeared as part of an advertisement in the *Boston Globe* on June 27, 2003. Identify at least three strategies used to persuade you to buy gold.

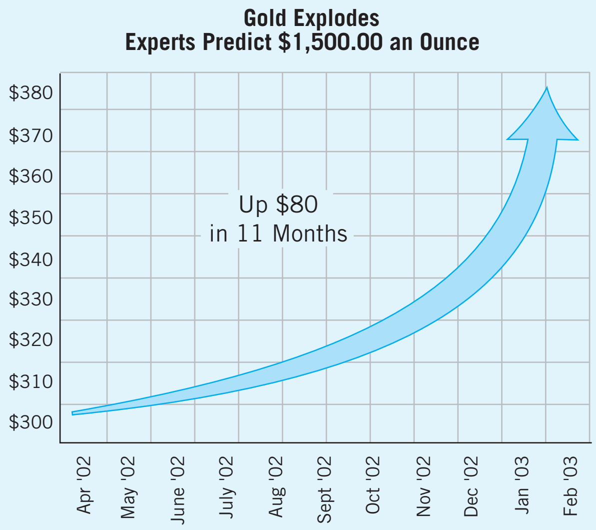
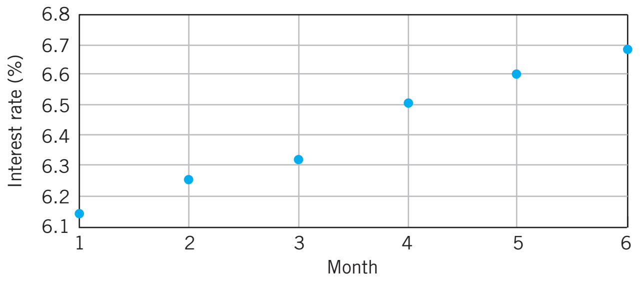


Figure 2.19 Prices for an ounce of gold.

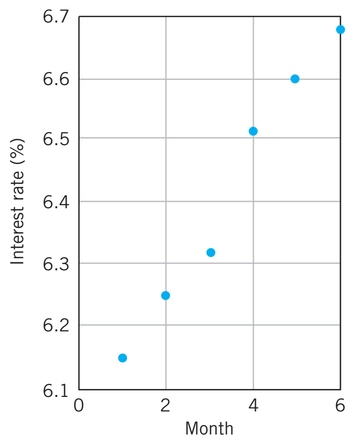
Exercises for Section 2.4

Graphing program optional for Exercise 10. Course software is required for Exercise 12.

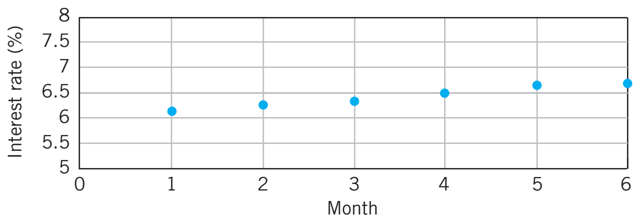
1. Examine the three graphs *A*, *B*, and *C*. All report the same data on 30-year mortgage interest rates for the first six months in 2006. Create a different title for each graph exaggerating their differences.



Graph *A*

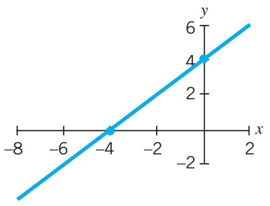


Graph *B*

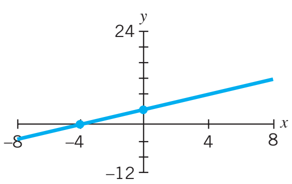


Graph *C*

2. Compare the accompanying graphs.
   - **a.** Which line appears to have the steeper slope?
   - **b.** Use the intercepts to calculate the slope. Which graph actually does have the steeper slope?

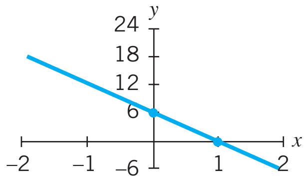


Graph *A*

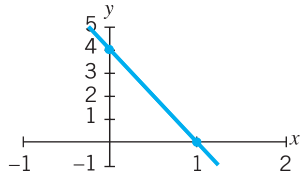


Graph *B*

3. Compare the accompanying graphs.
   - **a.** Which line appears to have the steeper slope?
   - **b.** Use the intercepts to calculate the slope. Which graph has the more negative (and hence steeper) slope?

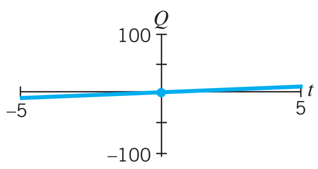


Graph *A*

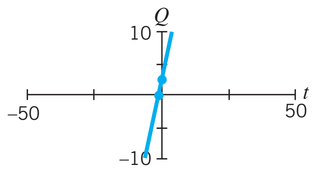


Graph *B*

4. The following graphs show the same function graphed on different scales.
   - **a.** In which graph does *Q* appear to be growing at a faster rate?
   - **b.** In which graph does *Q* appear to be growing at a near zero rate?
   - **c.** Explain why the graphs give different impressions.

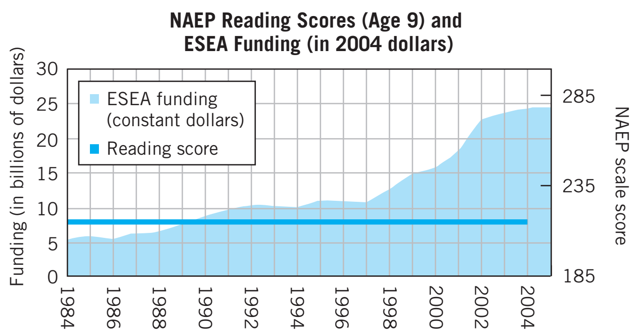


Graph *A*



Graph *B*

5. The following graph shows both federal spending on K–12 education under the Education and Secondary Education Act (ESEA) and National Assessment of Educational Progress (NAEP) reading scores (for age 9).



   - **a.** Summarize the message that this chart conveys concerning NAEP reading scores and federal spending on ESEA.
   - **b.** What strategy is used to make the reading scores seem lower?

6. Generate two graphs and on each draw a line through the points (0, 3) and (4, 6), choosing *x* and *y* scales such that:
   - **a.** The first line appears to have a slope of almost zero.
   - **b.** The second line appears to have a very large positive slope.

7. a. Generate a graph of a line through the points $(0, -2)$ and $(5, -2)$.

b. On a new grid, choose different scales so that the line through the same points appears to have a large positive slope.

c. What have you discovered?

8. What are three adjectives (like "explosive") that would imply rapid growth?

9. What are three adjectives (like "severe") that would imply rapid decline?

10. (Graphing program optional.) Examine the data given on women in the U.S. military forces.

Women in Uniform: Female Active-Duty Military Personnel

Year	Total	Army	Navy	Marine Corps	Air Force
1970	41,479	16,724	8,683	2,418	13,654
1980	171,418	69,338	34,980	6,706	60,394
1990	227,018	83,621	59,907	9,356	74,134
2000	204,498	72,021	53,920	9,742	68,815
2005	189,465	71,400	54,800	8,498	54,767

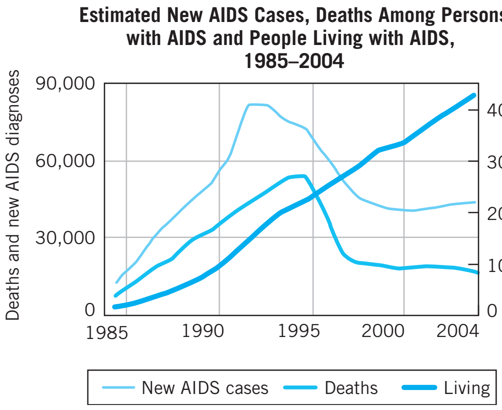
Source: U.S. Defense Department.

a. Make the case with graphs and numbers that women are a growing presence in the U.S. military.

b. Make the case with graphs and numbers that women are a declining presence in the U.S. military.

c. Write a paragraph that gives a balanced picture of the changing presence of women in the military using appropriate statistics to make your points. What additional data would be helpful?

11. The first case in the United States of what later came to be called AIDS appeared in June 1981. The accompanying graph shows the progress of AIDS cases in the United States as reported by the Centers for Disease Control and Prevention (CDC).



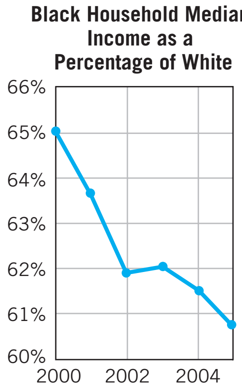
a. Find something encouraging to say about these data by using numerical evidence, including estimated average rates of change.

b. Find numerical support for something discouraging to say about the data.

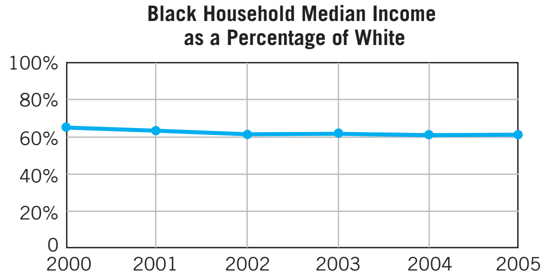
c. How might we explain the enormous increase in new AIDS cases reported from 1985 to a peak in 1992–1993, and the drop-off the following year?

12. (Course software required.) Open "L6: Changing Axis Scales" in *Linear Functions*. Generate a line in the upper left-hand box. The same line will appear graphed in the three other boxes but with the axes scaled differently. Describe how the axes are rescaled in order to create such different impressions.

13. The accompanying graphs show the same data on the median income of a household headed by a black person as a percentage of the median income of a household headed by a white person. Describe the impression each graph gives and how that was achieved. (Data from the U.S. Bureau of the Census.)

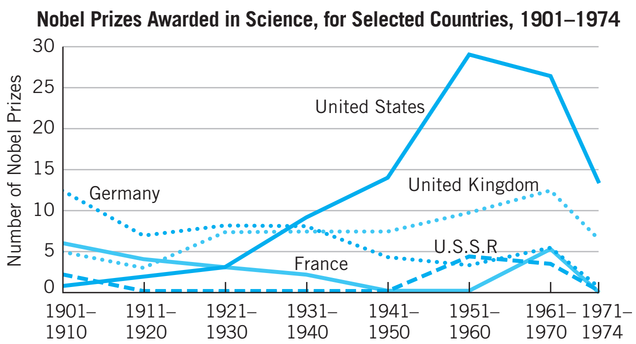


Graph *A*



Graph *B*

14. The accompanying graph shows the number of Nobel Prizes awarded in science for various countries between 1901 and 1974. It contains accurate information but gives the impression that the number of prize winners declined drastically in the 1970s, which was not the case. What flaw in the construction of the graph leads to this impression?



Source: E. R. Tufte, *The Visual Display of Quantitative Information* (Cheshire, Conn.: Graphics Press, 1983).

2.5 *Linear Functions: When Rates of Change Are Constant*

In many of our examples so far, the average rate of change has varied depending on the choice of end points. Now we will examine the special case when the average rate of change remains constant.

What If the U.S. Population Had Grown at a Constant Rate? A Hypothetical Example

In Section 2.2 we calculated the average rate of change in the population between 1790 and 1800 as 0.14 million people/year. We saw that the average rate of change was different for different decades. What if the average rate of change had remained constant? What if in every decade after 1790 the U.S. population had continued to grow at the same rate of 0.14 million people/year? That would mean that starting with a population estimated at 3.9 million in 1790, the population would have grown by 0.14 million each year. The slopes of all the little line segments connecting adjacent population data points would be identical, namely 0.14 million people/year. The graph would be a straight line, indicating a constant average rate of change.

Experiment with varying the average velocities and then setting them all constant in "C3: Average Velocity and Distance" in *Rates of Change.*

On the graph of actual U.S. population data, the slopes of the line segments connecting adjacent points are increasing, so the graph curves upward. Figure 2.20 compares the actual and hypothetical results.

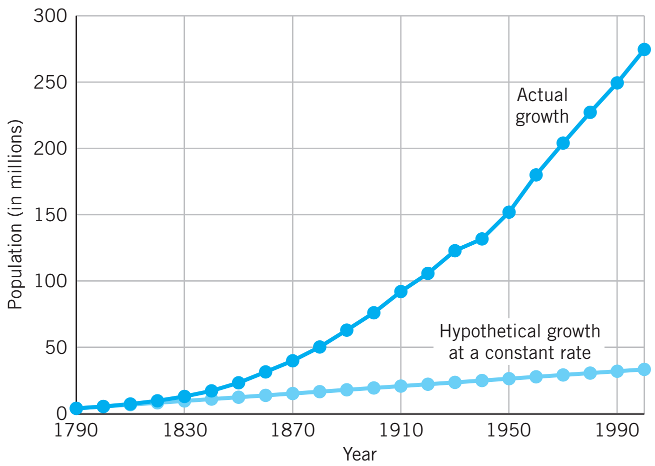


Figure 2.20 U.S. population: A hypothetical example.

Any function that has the same average rate of change on every interval has a graph that is a straight line. The function is called linear. This hypothetical example represents a linear function. When the average rate of change is constant, we can drop the word "average" and just say "rate of change."

> A linear function has a constant rate of change. Its graph is a straight line.

Real Examples of a Constant Rate of Change

EXAMPLE 1 According to the standardized growth and development charts used by many American pediatricians, the median weight for girls during their first six months of life increases at an almost constant rate. Starting at 7.0 pounds at birth, female median weight increases by approximately 1.5 pounds per month. If we assume that the median weight for females, W, is increasing at a constant rate of 1.5 pounds per month, then W is a linear function of age in months, A.

a. Generate a table that gives the median weight for females, W, for the first six months of life and create a graph of W as a function of A.

b. Find an equation for W as a function of A. What is an appropriate domain for this function?

c. Express the equation for part (b) using only units of measure.

SOLUTION

a. For female infants at birth $(A = 0 \text{ months})$, the median weight is 7.0 lb $(W = 7.0 \text{ lb})$. The rate of change, 1.5 pounds/month, means that as age increases by 1 month, weight increases by 1.5 pounds. See Table 2.11.

The dotted line in Figure 2.21 shows the trend in the data.

Median Weight for Girls

A, Age (months)	W, Weight (lb)
0	7.0
1	8.5
2	10.0
3	11.5
4	13.0
5	14.5
6	16.0

Table 2.11
Source: Data derived from the Ross Growth and Development Program, Ross Laboratories, Columbus, OH.

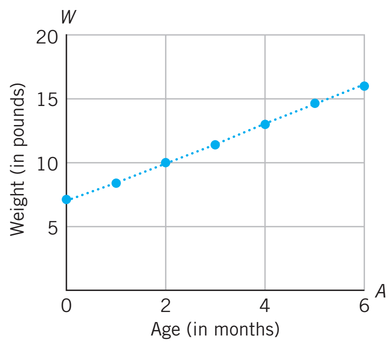


Figure 2.21 Median weight for girls.

b. To find a linear equation for W (median weight in pounds) as a function of A (age in months), we can study the table of values in Table 2.11.

$$\begin{aligned} W &= \text{initial weight} + \text{weight gained} \\ &= \text{initial weight} + \text{rate of growth} \cdot \text{number of months} \\ &= 7.0 \text{ lb} + 1.5 \text{ lb/month} \cdot \text{number of months} \end{aligned}$$

The equation would be

$$W = 7.0 + 1.5A$$

An appropriate domain for this function would be $0 \leq A \leq 6$.

c. Since our equation represents quantities in the real world, each term in the equation has units attached to it. The median weight W is in pounds (lb), rate of change is in lb/month, and age, A, is in months. If we display the equation

$$W = 7.0 + 1.5A$$

showing only the units, we have

$$\text{lb} = \text{lb} + \left(\frac{\text{lb}}{\text{month}}\right)\text{month}$$

The rules for canceling units are the same as the rules for canceling numbers in fractions. So,

$$\text{lb} = \text{lb} + \left(\frac{\text{lb}}{\cancel{\text{month}}}\right)\cancel{\text{month}}$$

$$\text{lb} = \text{lb} + \text{lb}$$

SOMETHING TO THINK ABOUT

If the median birth weight for baby boys is the same as for baby girls, but boys put on weight at a faster rate, which numbers in the model would change and which would stay the same? What would you expect to be different about the graph?

This equation makes sense in terms of the original problem since pounds (lb) added to pounds (lb) should give us pounds (lb).

EXAMPLE 2 You spend $1200 on a computer and for tax purposes choose to depreciate it (or assume it decreases in value) to $0 at a constant rate over a 5-year period.

a. Calculate the rate of change of the assumed value of the equipment over 5 years. What are the units?

b. Create a table and graph showing the value of the equipment over 5 years.

c. Find an equation for the value of the computer as a function of time in years. Why is this a linear function?

d. What is an appropriate domain for this function? What is the range?

SOLUTION **a.** After 5 years, your computer is worth $0. If V is the value of your computer in dollars and t is the number of years you own the computer, then the rate of change of V from $t = 0$ to $t = 5$ is

$$\text{rate of change} = \frac{\text{change in value}}{\text{change in time}} = \frac{\Delta V}{\Delta t}$$

$$= -\frac{\$1200}{5 \text{ years}} = -\$240/\text{year}$$

Thus, the worth of your computer drops at a constant rate of $240 per year. The rate of change in V is negative because the worth of the computer decreases over time. The units for the rate of change are dollars per year.

Value of Computer Depreciated over 5 Years

Numbers of Years	Value of Computer ($)
0	1200
1	960
2	720
3	480
4	240
5	0

Table 2.12

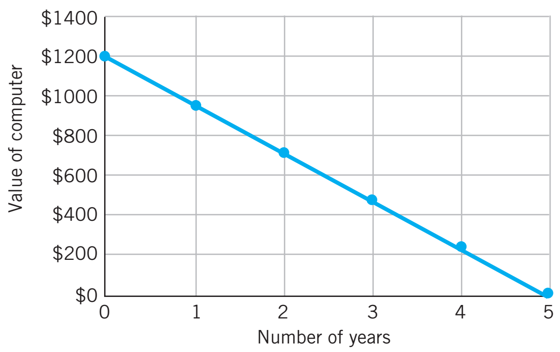


Figure 2.22 Value of computer over 5 years.

b. Table 2.12 and Figure 2.22 show the depreciated value of the computer.

c. To find a linear equation for V as a function of t, think about how we found the table of values.

$$\text{value of computer} = \text{initial value} + (\text{rate of decline}) \cdot (\text{number of years})$$

$$V = \$1200 + (-\$240/\text{year}) \cdot t$$

$$V = 1200 - 240t$$

This equation describes V as a function of t because for every value of t, there is one and only one value of V. It is a linear function because the rate of change is constant.

d. The domain is $0 \le t \le 5$ and the range is $0 \le V \le 1200$.

The General Equation for a Linear Function

Explorations 2.2A and 2.2B (along with "L1: *m* & *b* Sliders" in *Linear Functions*) allow you to examine the effects of *m* and *b* on the graph of a linear function.

The equations in Examples 1 and 2 can be rewritten in terms of the output (dependent variable) and the input (independent variable).

$$\text{weight} = \text{initial weight} + \text{rate of growth} \cdot \text{number of months}$$
$$\text{value of computer} = \text{initial value} + \text{rate of decline} \cdot \text{number of years}$$
$$\underbrace{\text{output}}_{y} = \underbrace{\text{initial value}}_{b} + \underbrace{\text{rate of change}}_{m} \cdot \underbrace{\text{input}}_{x}$$

Thus, the general linear equation can be written in the form

$$y = b + mx$$

where we use the traditional mathematical choices of y for the output (dependent variable) and x for the input (independent variable). We let m stand for the rate of change; thus

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$
$$= \text{slope of the graph of the line}$$

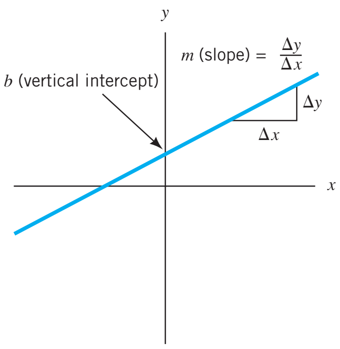


Figure 2.23 Graph of $y = b + mx$.

In our general equation, we let b stand for the initial value. Why is b called the initial value? The number b is the value of y when $x = 0$. If we let $x = 0$, then

given $\quad y = b + mx$

$$y = b + m \cdot 0$$
$$y = b$$

The point $(0, b)$ satisfies the equation and lies on the y-axis. The point $(0, b)$ is technically the vertical intercept. However, since the coordinate b tells us where the line crosses the y-axis, we often just refer to b as the *vertical* or *y-intercept* (see Figure 2.23).

A Linear Function

A function $y = f(x)$ is called *linear* if it can be represented by an equation of the form

$$y = b + mx$$

Its graph is a straight line where m is the *slope*, the rate of change of y with respect to x, so

$$m = \frac{\Delta y}{\Delta x}$$

b is the *vertical* or *y-intercept* and is the value of y when $x = 0$.

The equation $y = b + mx$ could, of course, be written in the equivalent form $y = mx + b$. In linear mathematical models, b is often the initial or starting value of the output, so it is useful to place it first in the equation.

EXAMPLE 3 For each of the following equations, identify the value of b and the value of m.

a. $y = -4 + 3.25x$ **c.** $y = -4x + 3.25$

b. $y = 3.25x - 4$ **d.** $y = 3.25 - 4x$

SOLUTION

a. $b = -4$ and $m = 3.25$

b. $b = -4$ and $m = 3.25$

c. $b = 3.25$ and $m = -4$

d. $b = 3.25$ and $m = -4$

EXAMPLE 4 In the following equations, L represents the legal fees (in dollars) charged by four different law firms and h represents the number of hours of legal advice.

$$L_1 = 500 + 200h \qquad L_3 = 800 + 350h$$
$$L_2 = 1000 + 150h \qquad L_4 = 500h$$

a. Which initial fee is the highest?
b. Which rate per hour is the highest?
c. If you need 5 hours of legal advice, which legal fee will be the highest?

SOLUTION

a. L_2 has the highest initial fee of $1000.
b. L_4 has the highest rate of $500 per hour.
c. Evaluate each equation for $h = 5$ hours.

$$\begin{aligned} L_1 &= 500 + 200h \\ &= 500 + 200(5) \\ &= \$1500 \end{aligned} \qquad \begin{aligned} L_3 &= 800 + 350h \\ &= 800 + 350(5) \\ &= \$2550 \end{aligned}$$

$$\begin{aligned} L_2 &= 1000 + 150h \\ &= 1000 + 150(5) \\ &= \$1750 \end{aligned} \qquad \begin{aligned} L_4 &= 500h \\ &= 500(5) \\ &= \$2500 \end{aligned}$$

For 5 hours of legal advice, L_3 has the highest legal fee.

Algebra Aerobics 2.5

1. From Figure 2.21, estimate the weight W of a baby girl who is 4.5 months old. Then use the equation $W = 7.0 + 1.5A$ to calculate the corresponding value for W. How close is your estimate?
2. From the same graph, estimate the age of a baby girl who weighs 11 pounds. Then use the equation to calculate the value for A.
3. **a.** If $C = 15P + 10$ describes the relationship between the number of persons (P) in a dining party and the total cost in dollars (C) of their meals, what is the unit of measure for 15? For 10?

   b. The equation $W = 7.0 + 1.5A$ (modeling weight as a function of age) expressed in units of measure only is

   $$\text{lb} = \text{lb} + \left(\frac{\text{lb}}{\text{month}}\right)\text{months}$$

   Express $C = 15P + 10$ from part (a) in units of measure only.
4. (True story.) A teenager travels to Alaska with his parents and wins $1200 in a rubber ducky race. (The race releases 5000 yellow rubber ducks marked with successive integers from one bridge over a river and collects them at the next bridge.) Upon returning home he opens up a "Rubber Ducky Savings Account," deposits his winnings, and continues to deposit $50 each month. If D = amount in the account and M = months since the creation of the account, then $D = 1200 + 50M$ describes the amount in the account after M months.

   a. What are the units for 1200? For 50?

   b. Express the equation using only units of measure.
5. Assume $S = 0.8Y + 19$ describes the projected relationship between S, sales of a company (in millions of dollars), for Y years from today.

   a. What are the units of 0.8 and what does it represent?

   b. What are the units of 19 and what does it represent?

   c. What would be the projected company sales in three years?

   d. Express the equation using only units of measure.
6. Assume $C = 0.45N$ represents the total cost C (in dollars) of operating a car for N miles.

   a. What does 0.45 represent and what are its units?

   b. Find the total cost to operate a car that has been driven 25,000 miles.

   c. Express the equation using only units of measure.
7. The relationship between the balance B (in dollars) left on a mortgage loan and N, the number of monthly payments, is given by $B = 302{,}400 - 840N$.

   a. What is the monthly mortgage payment?

   b. What does 302,400 represent?

   c. What is the balance on the mortgage after 10 years? 20 years? 30 years? (*Hint:* Remember there are 12 months in a year.)

8. Identify the slope, m, and the vertical intercept, b, of the line with the given equation.

a. $y = 5x + 3$
b. $y = 5 + 3x$
c. $y = 5x$
d. $y = 3$
e. $f(x) = 7.0 - x$
f. $h(x) = -11x + 10$
g. $y = 1 - \frac{2}{3}x$
h. $2y + 6 = 10x$

9. If $f(x) = 50 - 25x$:

a. Why does $f(x)$ describe a linear function?
b. Evaluate $f(0)$ and $f(2)$.
c. Use your answers in part (b) to verify that the slope is -25.

10. Identify the functions that are linear. For each linear function, identify the slope and the vertical intercept.

a. $f(x) = 3x + 5$
b. $f(x) = x$
c. $f(x) = 3x^2 + 2$
d. $f(x) = 4 - \frac{2}{3}x$

11. Write an equation for the line in the form $y = mx + b$ for the indicated values.

a. $m = 3$ and $b = 4$
b. $m = -1$ and passes through the origin
c. $m = 0$ and $b = -3$

12. Write the equation of the graph of each line in Figure 2.24 in $y = mx + b$ form. Use the y-intercept and the slope indicated on each graph.

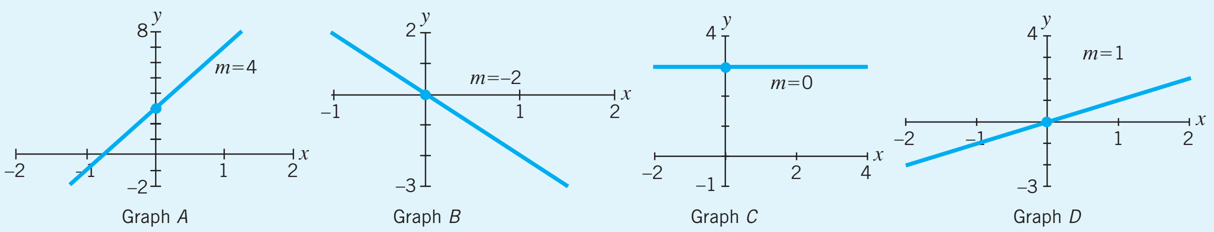


Figure 2.24 Four linear graphs.

Exercises for Section 2.5

You might wish to hone your algebraic mechanical skills with three programs in the course software; *Linear Functions:* "L1: Finding m & b", "L3: Finding a Line through 2 Points" and "L4: Finding 2 Points on a Line." They offer practice in predicting values for m and b, generating linear equations, and finding corresponding solutions.

1. Consider the equation $E = 5000 + 100n$.

a. Find the value of E for $n = 0, 1, 20$.
b. Express your answers to part (a) as points with coordinates (n, E).

2. Consider the equation $G = 12{,}000 + 800n$.

a. Find the value of G for $n = 0, 1, 20$.
b. Express your answers to part (a) as points with coordinates (n, G).

3. Determine if any of the following points satisfy one or both of the equations in Exercises 1 and 2.

a. (5000, 0) **b.** (15, 24000) **c.** (35, 40000)

4. Suppose during a 5-year period the profit P (in billions of dollars) for a large corporation was given by $P = 7 + 2Y$, where Y represents the year.

a. Fill in the chart.

Y	0	1	2	3	4
P					

b. What are the units of P?
c. What does the 2 in the equation represent, and what are its units?
d. What was the initial profit?

5. Consider the equation $D = 3.40 + 0.11n$

a. Find the values of D for $n = 0, 1, 2, 3, 4$.
b. If D represents the average consumer debt, in thousands of dollars, over n years, what does 0.11 represent? What are its units?
c. What does 3.40 represent? What are its units?

6. Suppose the equations $E = 5000 + 1000n$ and $G = 12{,}000 + 800n$ give the total cost of operating an electrical (E) versus a gas (G) heating/cooling system in a home for n years.

a. Find the cost of heating a home using electricity for 10 years.
b. Find the cost of heating a home using gas for 10 years.
c. Find the initial (or installation) cost for each system.
d. Determine how many years it will take before $40,000 has been spent in heating/cooling a home that uses:

i. Electricity **ii.** Gas

7. If the equation $E = 5000 + 1000n$ gives the total cost of heating/cooling a home after n years, rewrite the equation using only units of measure.

8. Over a 5-month period at Acadia National Park in Maine, the average night temperature increased on average 5 degrees Fahrenheit per month. If the initial temperature is 25 degrees, create a formula for the night temperature N for month t, where $0 \le t \le 4$.

9. A residential customer in the Midwest purchases gas from a utility company that charges according to the formula $C(g) = 11 + 10.50(g)$, where $C(g)$ is the cost, in dollars, for g thousand cubic feet of gas.

a. Find $C(0)$, $C(5)$, and $C(10)$.

b. What is the cost if the customer uses no gas?

c. What is the rate per thousand cubic feet charged for using the gas?

d. How much would it cost if the customer uses 96 thousand cubic feet of gas (the amount an average Midwest household consumes during the winter months)?

10. Create the formula for converting degrees centigrade, C, to degrees Fahrenheit, F, if for every increase of 5 degrees centigrade the Fahrenheit temperature increases by 9 degrees, with an initial point of $(C, F) = (0, 32)$.

11. Determine the vertical intercept and the rate of change for each of these formulas:

a. $P = 4s$

b. $C = \pi d$

c. $C = 2\pi r$

d. $C = \frac{5}{9}F - 17.78$

12. A hiker can walk 2 miles in 45 minutes.

a. What is his average speed in miles per hour?

b. What formula can be used to find the distance traveled, d, in t hours?

13. The cost $C(x)$ of producing x items is determined by adding fixed costs to the product of the cost per item and the number of items produced, x. Below are three possible cost functions $C(x)$, measured in dollars. Match each description [in parts (e) − (g)] with the most likely cost function and explain why.

a. $C(x) = 125{,}000 + 42.50x$

b. $C(x) = 400{,}000 + 0.30x$

c. $C(x) = 250{,}000 + 800x$

e. The cost of producing a computer

f. The cost of producing a college algebra text

g. The cost of producing a CD

14. A new \$25,000 car depreciates in value by \$5000 per year. Construct a linear function for the value V of the car after t years.

15. The state of Pennsylvania has a 6% sales tax on taxable items. (*Note:* Clothes, food, and certain pharmaceuticals are not taxed in Pennsylvania.)

a. Create a formula for the total cost (in dollars) of an item $C(p)$ with a price tag p. (Be sure to include the sales tax.)

b. Find $C(9.50)$, $C(115.25)$, and $C(1899)$. What are the units?

16. a. If $S(x) = 20{,}000 + 1000x$ describes the annual salary in dollars for a person who has worked for x years for the Acme Corporation, what is the unit of measure for 20,000? For 1000?

b. Rewrite $S(x)$ as an equation using only units of measure.

c. Evaluate $S(x)$ for x values of 0, 5, and 10 years.

d. How many years will it take for a person to earn an annual salary of \$43,000?

17. The following represent linear equations written using only units of measure. In each case supply the missing unit.

a. inches = inches + (inches/hour) · (?)

b. miles = miles + (?) · (gallons)

c. calories = calories + (?) · (grams of fat)

18. Identify the slopes and the vertical intercepts of the lines with the given equations.

a. $y = 3 + 5x$

b. $f(t) = -t$

c. $y = 4$

d. $Q = 35t - 10$

e. $f(E) = 10{,}000 + 3000E$

19. For each of the following, find the slope and the vertical intercept, then sketch the graph. (*Hint*: Find two points on the line.)

a. $y = 0.4x - 20$ **b.** $P = 4000 - 200C$

20. Construct an equation and sketch the graph of its line with the given slope, m, and vertical intercept, b. (*Hint*: Find two points on the line.)

a. $m = 2, b = -3$

b. $m = -\frac{3}{4}, b = 1$

c. $m = 0, b = 50$

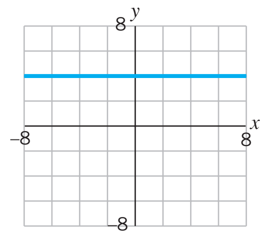
In Exercises 21 to 23 find an equation, generate a small table of solutions, and sketch the graph of a line with the indicated attributes.

21. A line that has a vertical intercept of −2 and a slope of 3.

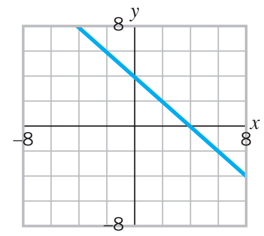
22. A line that crosses the vertical axis at 3.0 and has a rate of change of −2.5.

23. A line that has a vertical intercept of 1.5 and a slope of 0.

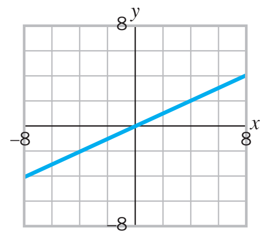
24. Estimate b (the y-intercept) and m (the slope) for each of the accompanying graphs. Then, for each graph, write the corresponding linear function.



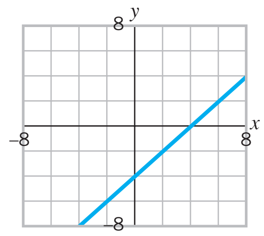
Graph *A*



Graph *C*



Graph *B*



Graph *D*

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2.6 *Visualizing Linear Functions*

The values for b and m in the general form of the linear equation, $y = b + mx$, tell us about the graph of the function. The value for b tells us where to anchor the line on the y-axis. The value for m tells us whether the line climbs or falls and how steep it is.

The Effect of b

In the equation $y = b + mx$, the value b is the vertical intercept, so it anchors the line at the point $(0, b)$ (see Figure 2.25).

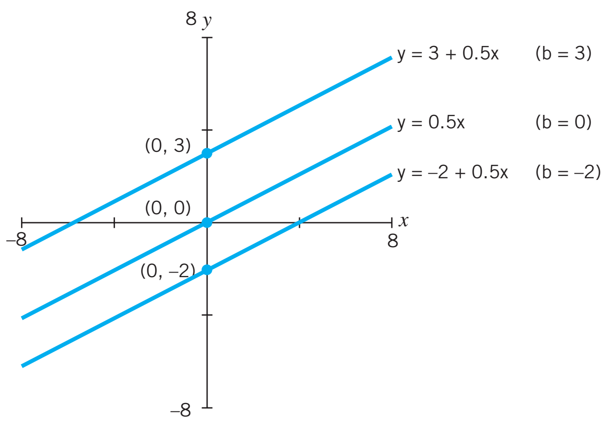


Figure 2.25 The effect of b, the vertical intercept.

EXAMPLE 1 Explain how the graph of $y = 4 + 5x$ differs from the graphs of the following functions:

a. $y = 8 + 5x$

b. $y = 2 + 5x$

c. $y = 5x + 4$

SOLUTION Although all of the graphs are straight lines with a slope of 5, they each have a different vertical intercept. The graph of $y = 4 + 5x$ has a vertical intercept at 4.

a. The graph of $y = 8 + 5x$ intersects the y-axis at 8, four units above the graph of $y = 4 + 5x$.

b. The graph of $y = 2 + 5x$ intersects the y-axis at 2, two units below the graph of $y = 4 + 5x$.

c. Since $y = 5x + 4$ and $y = 4 + 5x$ are equivalent equations, they have the same graph.

The Effect of *m*

The sign of m

The sign of m in the equation $y = b + mx$ determines whether the line climbs (slopes up) or falls (slopes down) as we move left to right on the graph. If m is positive, the line climbs from left to right (as x increases, y increases). If m is negative, the line falls from left to right (as x increases, y decreases) (see Figure 2.26).

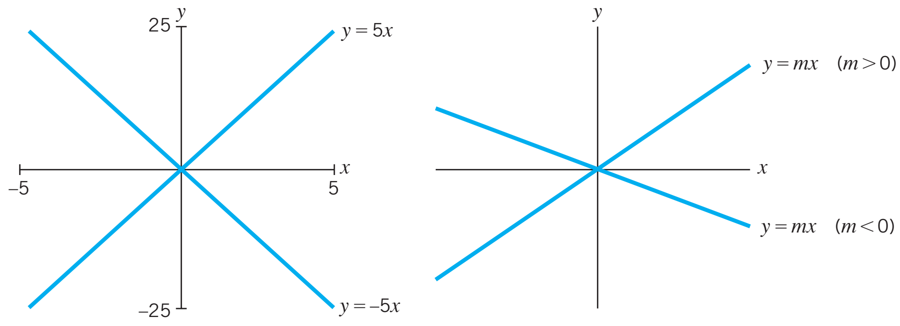


Figure 2.26 The effect of the sign of *m*.

EXAMPLE 2 Match the following functions to the lines in Figure 2.27.

$$f(x) = 3 - 2x$$
$$g(x) = 3 + 2x$$
$$h(x) = -5 - 2x$$
$$j(x) = -5 + 2x$$

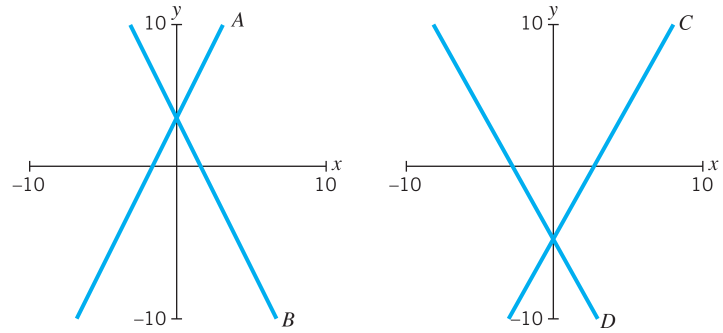


Figure 2.27 Matching graphs.

SOLUTION A is the graph of $g(x)$.

B is the graph of $f(x)$.

C is the graph of $j(x)$.

D is the graph of $h(x)$.

The magnitude of m

SOMETHING TO THINK ABOUT

Describe the graph of a line where m = 0.

The magnitude (absolute value) of m determines the steepness of the line. Recall that the absolute value of m is the value of m stripped of its sign; for example, $|-3| = 3$. The greater the magnitude $|m|$, the steeper the line. This makes sense since m is the slope or the rate of change of y with respect to x. Notice how the steepness of each line in Figure 2.28 (next page) increases as the magnitude of m increases. For example, the slope ($m = 5$) of $y = 7 + 5x$ is steeper than the slope ($m = 3$) of $y = 7 + 3x$.

In Figure 2.29 we can see that the slope ($m = -5$) of $y = 7 - 5x$ is steeper than the slope ($m = -3$) of $y = 7 - 3x$ since $|-5| = 5 > |-3| = 3$. The lines $y = 7 - 5x$ and $y = 7 + 5x$ have the same steepness of 5 since $|-5| = |5| = 5$.

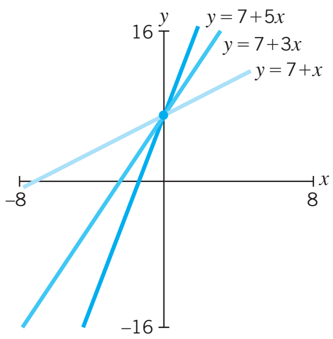


Figure 2.28 Graphs with positive values for *m*.

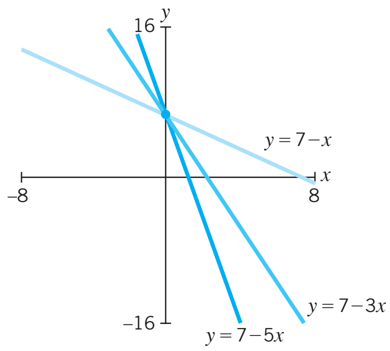


Figure 2.29 Graphs with negative values for *m*.

EXAMPLE 3 Pair each graph in Figure 2.30 with a matching equation.

$f(x) = 9 - 0.4x$
$g(x) = -4 + x$
$h(x) = 9 - 0.2x$
$i(x) = -4 + 0.25x$
$j(x) = 9 + 0.125x$
$k(x) = 4 - 0.25x$

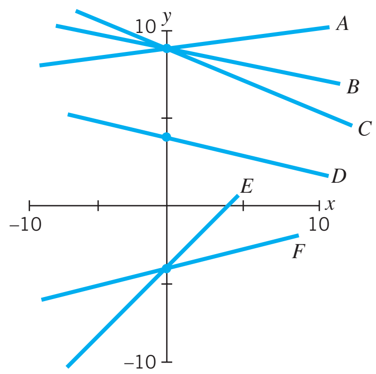


Figure 2.30 Graphs of multiple linear functions.

SOLUTION *A* is the graph of $j(x)$.
B is the graph of $h(x)$.
C is the graph of $f(x)$.
D is the graph of $k(x)$.
E is the graph of $g(x)$.
F is the graph of $i(x)$.

EXAMPLE 4 Without graphing the following functions, how can you tell which graph will have the steepest slope?

a. $f(x) = 5 - 2x$ **b.** $g(x) = 5 + 4x$ **c.** $h(x) = 3 - 6x$

SOLUTION The graph of the function h will be steeper than the graphs of the functions f and g since the magnitude of m is greater for h than for f or g. The greater the magnitude of m, the steeper the graph of the line. For $f(x)$, the magnitude of the slope is $|-2| = 2$. For $g(x)$, the magnitude of the slope is $|4| = 4$. For $h(x)$, the magnitude of the slope is $|-6| = 6$.

EXAMPLE 5 Which of the graphs in Figure 2.31 has the steeper slope?

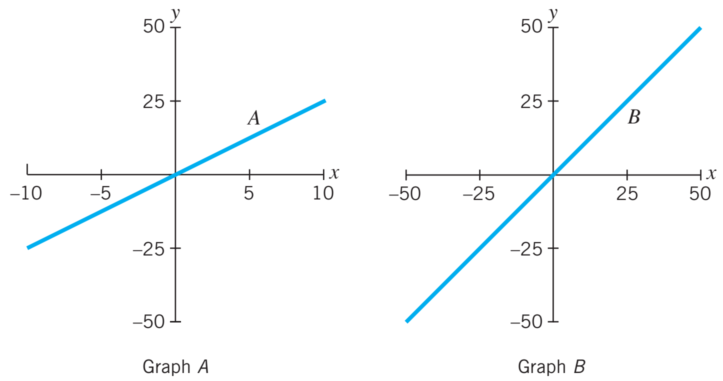


Figure 2.31 Comparing slopes.

SOLUTION Remember from Section 2.4 that the steepness of a linear graph is not related to its visual impression, but to the numerical magnitude of the slope. The scales of the horizontal axes are different for the two graphs in Figure 2.31, so the impression of relative steepness is deceiving. Line A passes through (0, 0) and (10, 25), so its slope is $(25 - 0)/(10 - 0) = 2.5$. Line B passes through (0, 0) and (50, 50) so its slope is $(50 - 0)/(50 - 0) = 1$. So line *A* has a steeper slope than line *B*.

The Graph of a Linear Equation

Given the general linear equation, $y = b + mx$, whose graph is a straight line:

The y-intercept, b, tells us where the line crosses the y-axis.

The slope, m, tells us how fast the line is climbing or falling. The larger the magnitude (or absolute value) of m, the steeper the graph.

If the slope, m, is positive, the line climbs from left to right. If m is negative, the line falls from left to right.

Algebra Aerobics 2.6

1. Place these numbers in order from smallest to largest.

$$|-12|, |-7|, |-3|, |-1|, 0, 4, 9$$

2. Without graphing the function, explain how the graph $y = 6x - 2$ differs from the graph of
   - **a.** $y = 6x$
   - **b.** $y = 2 + 6x$
   - **c.** $y = -2 + 3x$
   - **d.** $y = -2 - 2x$
3. Without graphing, order the graphs of the functions from least steep to steepest.
   - **a.** $y = 100 - 2x$
   - **b.** $y = 1 - x$
   - **c.** $y = -3x - 5$
   - **d.** $y = 3 - 5x$
4. On an x-y coordinate system, draw a line with a positive slope and label it $f(x)$.
   - **a.** Draw a line $g(x)$ with the same slope but a y-intercept three units above $f(x)$.
   - **b.** Draw a line $h(x)$ with the same slope but a y-intercept four units below $f(x)$.
   - **c.** Draw a line $k(x)$ with the same steepness as $f(x)$ but with a negative slope.
5. Which function has the steepest slope? Create a table of values for each function and graph the function to show that this is true.
   - **a.** $f(x) = 3x - 5$
   - **b.** $g(x) = 7 - 8x$
6. Create three functions with a y-intercept of 4 and three different negative slopes. Indicate which function has the steepest slope.
7. Match the four graphs in Figure 2.32 with the given functions.
   - **a.** $f(x) = 2 + 3x$
   - **b.** $g(x) = -2 - 3x$
   - **c.** $h(x) = \frac{1}{2}x - 2$
   - **d.** $k(x) = 2 - 3x$

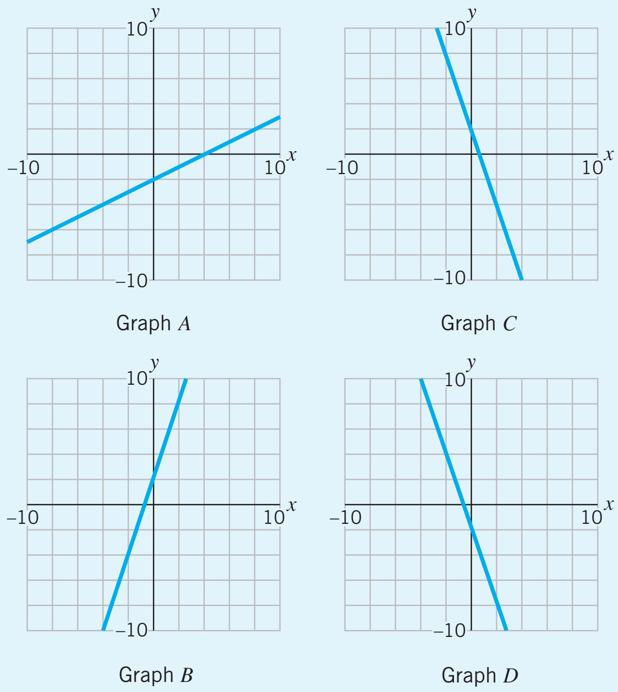
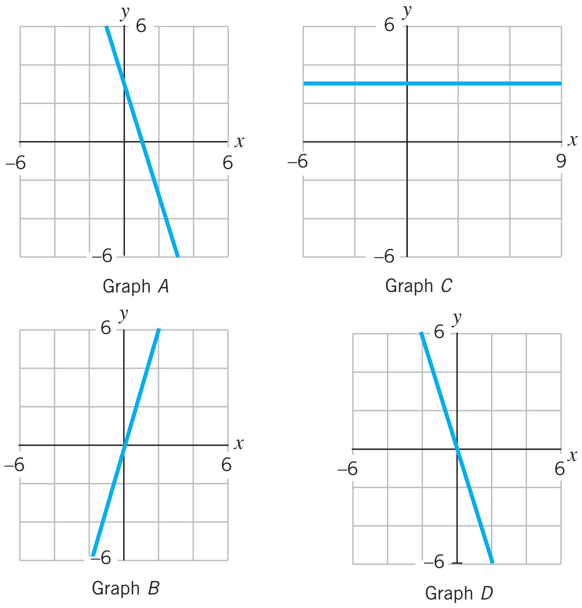


Figure 2.32 Four graphs of linear functions.

Exercises for Section 2.6

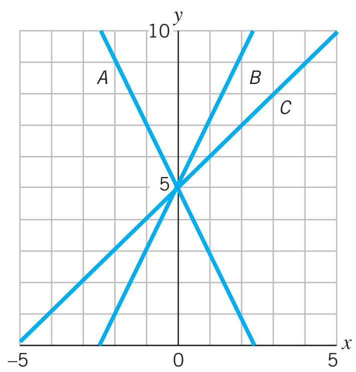
1. Assuming m is the slope, identify the graph(s) where:

a. $m = 3$ **b.** $|m| = 3$ **c.** $m = 0$ **d.** $m = -3$



2. Which line(s), if any, have a slope m such that:

a. $|m| = 2$? **b.** $m = \frac{1}{2}$?



3. For each set of conditions, construct a linear equation and draw its graph.

a. A slope of zero and a y-intercept of -3

b. A positive slope and a vertical intercept of -3

c. A slope of -3 and a vertical intercept that is positive

4. Construct a linear equation for each of the following conditions.

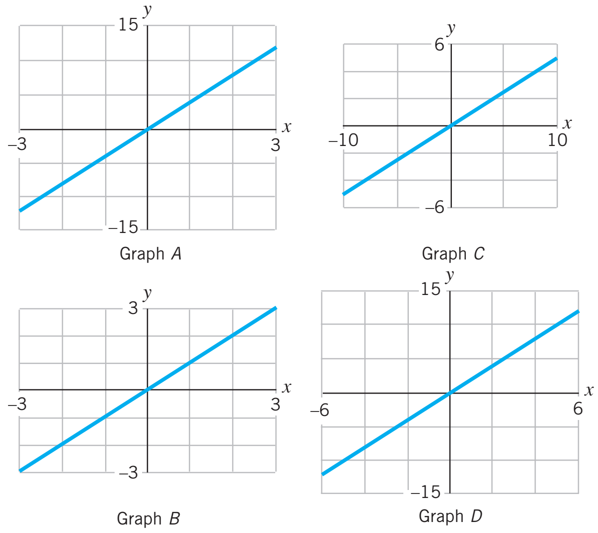
a. A negative slope and a positive y-intercept

b. A positive slope and a vertical intercept of -10.3

c. A constant rate of change of $1300/year

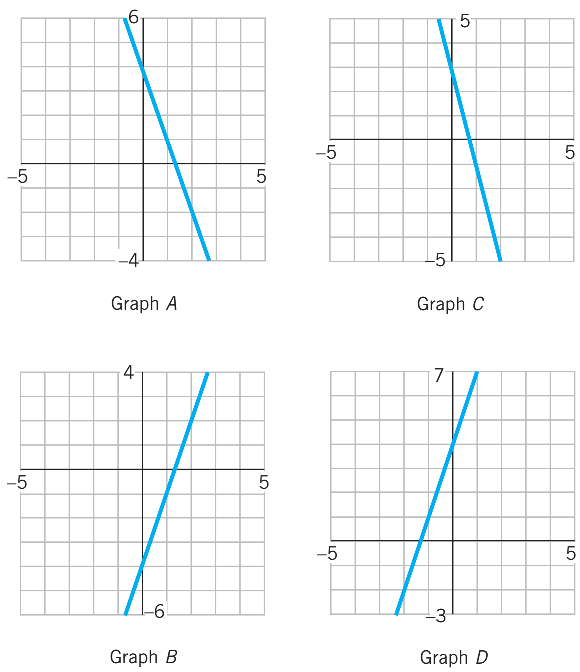
5. Match the graph with the correct equation.

a. $y = x$ **b.** $y = 2x$ **c.** $y = \frac{x}{2}$ **d.** $y = 4x$



6. Match the function with its graph. (*Note:* There is one graph that has no match.)

a. $y = -4 + 3x$ **b.** $y = -3x + 4$ **c.** $y = 4 + 3x$



7. In each part construct three different linear equations that all have:

a. The same slope

b. The same vertical intercept

8. Which equation has the steepest slope?

a. $y = 2 - 7x$ **b.** $y = 2x + 7$ **c.** $y = -2 + 7x$

9. Given the function $Q(t) = 13 - 5t$, construct a related function whose graph:

a. Lies five units above the graph of $Q(t)$

b. Lies three units below the graph of $Q(t)$

c. Has the same vertical intercept

d. Has the same slope

e. Has the same steepness, but the slope is positive

10. Given the equation $C(n) = 30 + 15n$, construct a related equation whose graph:

a. Is steeper

b. Is flatter (less steep)

c. Has the same steepness, but the slope is negative

11. On the same axes, graph (and label with the correct equation) three lines that go through the point (0, 2) and have the following slopes:

a. $m = \frac{1}{2}$ **b.** $m = 2$ **c.** $m = \frac{5}{6}$

12. On the same axes, graph (and label with the correct equation) three lines that go through the point (0, 2) and have the following slopes:

a. $m = -\frac{1}{2}$ **b.** $m = -2$ **c.** $m = -\frac{5}{6}$

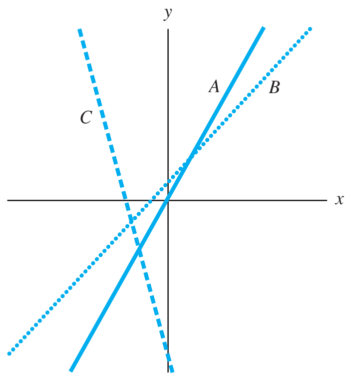
13. Given the following graphs of three straight lines:

a. Which has the steepest slope?

b. Which has the flattest slope?

c. If the slope of the lines *A, B,* and *C* are m_1, m_2, and m_3, respectively, list them in increasing numerical order.

d. In this example, why does the line with the steepest slope have the smallest numerical value?



2.7 *Finding Graphs and Equations of Linear Functions*

Finding the Graph

EXAMPLE 1 **Given the equation**

A forestry study measured the diameter of the trunk of a red oak tree over 5 years. The scientists created a linear model $D = 1 + 0.13Y$, where D = diameter in inches and Y = number of years from the beginning of the study.

a. What do the numbers 1 and 0.13 represent in this context?

b. Sketch a graph of the function model.

SOLUTION

a. The number 1 represents a starting diameter of 1 inch. The number 0.13 represents the annual growth rate of the oak's diameter (change in diameter/change in time), 0.13 inches per year.

b. The linear equation $D = 1 + 0.13Y$ tells us that 1 is the vertical intercept, so the point (0, 1) lies on the graph. The graph represents solutions to the equation. So to find a second point, we can evaluate D for any other value of Y. If we set $Y = 1$, then $D = 1 + (0.13 \cdot 1) = 1 + 0.13 = 1.13$. So (1, 1.13) is another point on the line. Since two points determine a line, we can sketch our line through (0, 1) and (1, 1.13) (see Figure 2.33).

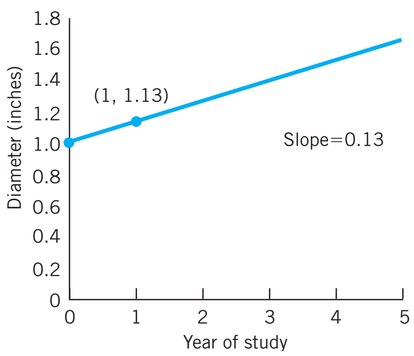


Figure 2.33 The diameter of a red oak over time.

The program "L4: Finding 2 Points on a Line" in *Linear Functions* will give you practice in finding solutions to equations.

EXAMPLE 2 **Given a point off the *y*-axis and the slope**

Given a point (2, 3) and a slope of $m = -5/4$, describe at least two ways you could find a second point to plot a line with these characteristics without constructing the equation.

SOLUTION Plot the point (2, 3).

a. If we write the slope as $(-5)/4$, then a change of 4 in x corresponds to a change of -5 in y. So starting at (2, 3), moving horizontally four units to the right (adding 4 to the x-coordinate), and then moving vertically five units down (subtracting 5 from the y-coordinate) gives us a second point on the line at $(2 + 4, 3 - 5) = (6, -2)$. Now we can plot our second point $(6, -2)$ and draw the line through it and our original point, (2, 3) (see Figure 2.34).

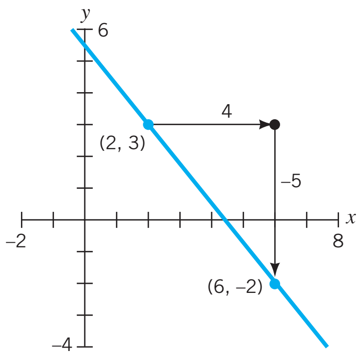


Figure 2.34 Graph of the line through (2, 3) with slope $-5/4$.

b. If we are modeling real data, we are more likely to convert the slope to decimal form. In this case $m = (-5)/4 = -1.25$. We can generate a new point on the line by starting at (2, 3) and moving 1 unit to the right and down (since m is negative) -1.25 units to get the point $(2 + 1, 3 - 1.25) = (3, 1.75)$.

EXAMPLE 3 **Given a general description**

A recent study reporting on the number of smokers showed:

a. A linear increase in Georgia

b. A linear decrease in Utah

c. A nonlinear decrease in Hawaii

d. A nonlinear increase in Oklahoma

Generate four rough sketches that could represent these situations.

SOLUTION See Figure 2.35.

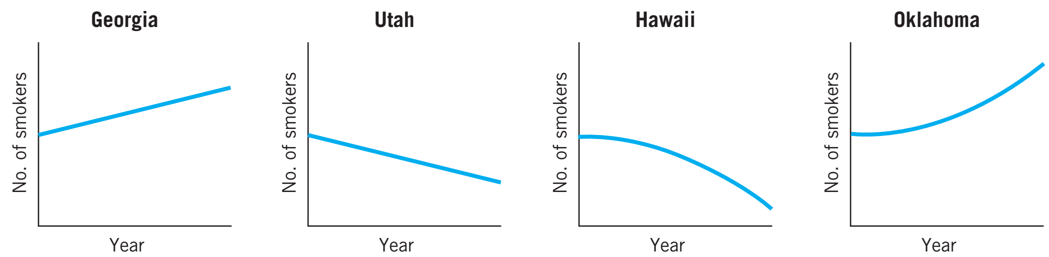


Figure 2.35 The change in the number of smokers over time in four states.

Finding the Equation

The program "L.3: Finding a Line Through 2 Points" in *Linear Functions* will give you practice in this skill.

To determine the equation of any particular linear function $y = b + mx$, we only need to find the specific values for m and b.

EXAMPLE 4 **Given two points**

Find the equation of the line through the two points $(-3, -5)$ and (4, 9).

SOLUTION If we think of these points as being in the form (x, y), then the slope m of the line connecting them is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{9 - (-5)}{4 - (-3)} = \frac{14}{7} = 2$$

So the equation is of the form $y = b + 2x$. To find b, the y-intercept, we can substitute either point into the equation. Substituting in (4, 9) we get $9 = b + (2 \cdot 4) \Rightarrow 9 = b + 8 \Rightarrow b = 1$. So the equation is $y = 1 + 2x$.

EXAMPLE 5 **From a graph**

Find the equation of the linear function graphed in Figure 2.36.

SOLUTION We can use any two points on the graph to calculate m, the slope. If, for example, we take $(-3, 0)$ and $(3, -2)$, then

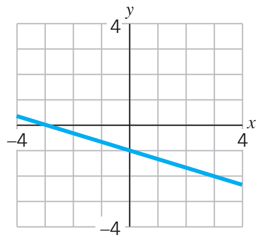


Figure 2.36 Graph of a linear function.

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{-2 - 0}{3 - (-3)} = \frac{-2}{6} = \frac{-1}{3}$$

From the graph we can estimate the y-intercept as -1. So $b = -1$.

Hence the equation is $y = -1 - \frac{1}{3}x$.

EXAMPLE 6 **From a verbal description**

In 2006 AT&T's One Rate plan charged a monthly base fee of \$3.95 plus \$0.07 per minute for long-distance calls. Construct an equation to model a monthly phone bill.

SOLUTION In making the transition from words to an equation, it's important to first identify which is the independent and which the dependent variable. We usually think of the phone bill, B, as a function of the number of minutes you talk, N. If you haven't used any phone minutes, then $N = 0$ and your bill $B = \$3.95$. So \$3.95 is the vertical intercept. The number \$0.07 is the rate of change of the phone bill with respect to number of minutes talked. The rate of change is constant, making the relationship linear. So the slope is \$0.07/minute and the equation is

$$B = 3.95 + 0.07N$$

EXAMPLE 7 The top speed a snowplow can travel on dry pavement is 40 miles per hour, which decreases by 0.8 miles per hour with each inch of snow on the highway.

a. Construct an equation describing the relationship between snowplow speed and snow depth.

b. Determine a reasonable domain and then graph the function.

SOLUTION

a. If we think of the snow depth, D, as determining the snowplow speed, P, then we need an equation of the form $P = b + mD$. If there is no snow, then the snowplow can travel at its maximum speed of 40 mph; that is, when $D = 0$, then $P = 40$. So the point (0, 40) lies on the line, making the vertical intercept $b = 40$. The rate of change, m, (change in snowplow speed)/(change in snow depth) $= -0.8$ mph per inch of snow. So the desired equation is

$$P = 40 - 0.8D$$

b. Consider the snowplow as only going forward (i.e., not backing up). Then the snowplow speed does not go below 0 mph. So if we let $P = 0$ and solve for D, we have

$$0 = 40 - 0.8D$$
$$0.8D = 40$$
$$D = 50$$

So when the snow depth reaches 50 inches, the plow is no longer able to move. A reasonable domain then would be $0 \le D \le 50$. (See Figure 2.37.)

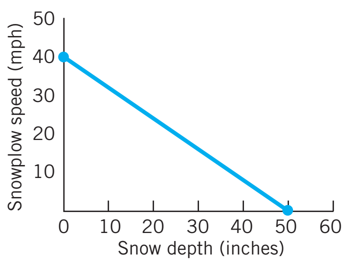


Figure 2.37 Snowplow speed versus snow depth.

EXAMPLE 8 Pediatric growth charts suggest a linear relationship between age (in years) and median height (in inches) for children between 2 and 12 years. Two-year-olds have a median height of 35 inches, and 12-year-olds have a median height of 60 inches.

a. Generate the average rate of change of height with respect to age. (Be sure to include units.) Interpret your result in context.

b. Generate an equation to describe height as a function of age. What is an appropriate domain?

c. What would this model predict as the median height of 8-year-olds?

SOLUTION **a.** Average rate of change $= \dfrac{\text{change in height}}{\text{change in age}} = \dfrac{60 - 35}{12 - 2} = \dfrac{25}{10} = 2.5$ inches/year

The chart suggests that, on average, children between the ages of 2 and 12 grow 2.5 inches each year.

b. If we think of height, H (in inches), depending on age, A (in years), then we want an equation of the form $H = b + m \cdot A$. From part (a) we know $m = 2.5$, so our equation is $H = b + 2.5A$. To find b, we can substitute the values for any known point into the equation. When $A = 2$, then $H = 35$. Substituting in, we get

$$H = b + 2.5A$$
$$35 = b + (2.5 \cdot 2)$$
$$35 = b + 5$$
$$30 = b$$

So the final form of our equation is

$$H = 30 + 2.5A$$

where the domain is $2 \leq A \leq 12$.

c. When $A = 8$ years, our model predicts that the median height is $H = 30 + (2.5 \cdot 8) = 30 + 20 = 50$ inches.

EXAMPLE 9 **From a table**

a. Determine if the data in Table 2.13 represent a linear relationship between values of blood alcohol concentration and number of drinks consumed for a 160-pound person. (One drink is defined as 5 oz of wine, 1.25 oz of 80-proof liquor, or 12 oz of beer.)

D, Number of Drinks	A, Blood Alcohol Concentration
2	0.047
4	0.094
6	0.141
10	0.235

Table 2.13

Note that federal law requires states to have 0.08 as the legal BAC limit for driving drunk.

b. If the relationship is linear, determine the corresponding equation.

SOLUTION **a.** We can generate a third column in the table that represents the average rate of change between consecutive points (see Table 2.14). Since the average rate of change of A with respect to D remains constant at 0.0235, these data represent a linear relationship.

D	A	Average Rate of Change
2	0.047	n.a.
4	0.094	$\frac{0.094 - 0.047}{4 - 2} = \frac{0.047}{2} = 0.0235$
6	0.141	$\frac{0.141 - 0.094}{6 - 4} = \frac{0.047}{2} = 0.0235$
10	0.235	$\frac{0.235 - 0.141}{10 - 6} = \frac{0.094}{4} = 0.0235$

Table 2.14

b. The rate of change is the slope, so the corresponding linear equation will be of the form

$$A = b + 0.0235D \qquad (1)$$

To find b, we can substitute any of the original paired values for D and A, for example, (4, 0.094), into Equation (1) to get

$$\begin{aligned} 0.094 &= b + (0.0235 \cdot 4) \\ 0.094 &= b + 0.094 \\ 0.094 - 0.094 &= b \\ 0 &= b \end{aligned}$$

So the final equation is

$$A = 0 + 0.0235D$$

or just $$A = 0.0235D$$

So when D, the number of drinks, is 0, A, the blood alcohol concentration, is 0, which makes sense.

Algebra Aerobics 2.7

For Problems 1 and 2, find an equation, make an appropriate table, and sketch the graph of:

1. A line with slope 1.2 and vertical intercept −4.
2. A line with slope −400 and vertical intercept 300 (be sure to think about scales on both axes).
3. Write an equation for the line graphed in Figure 2.38.

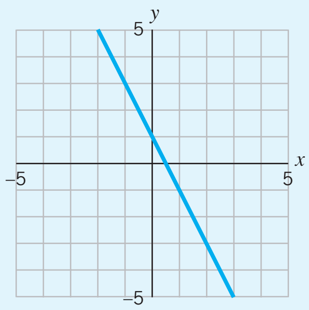


Figure 2.38 Graph of a linear function.

4. **a.** Find an equation to represent the current salary after x years of employment if the starting salary is \$12,000 with annual increases of \$3000.
   b. Create a small table of values and sketch a graph.
5. **a.** Plot the data in Table 2.15.

Years of Education	Hourly Wage
8	\$5.30
10	\$8.50
13	\$13.30

Table 2.15

   b. Is the relationship between hourly wage and years of education linear? Why or why not?
   c. If it is linear, construct a linear equation to model it.
6. Complete this statement regarding the graph of the line with equation $y = 6.2 + 3x$: Beginning with any point on the graph of the line, we could find another point by

moving up ___ units for each unit that we move horizontally to the right.

7. Given the equation $y = 8 - 4x$, complete the following statements.

a. Beginning with the vertical intercept, if we move one unit horizontally to the right, then we need to move down ___ units vertically to stay on the line and arrive at point (__, __).

b. Beginning with the vertical intercept, if we move one unit horizontally to the left, then we need to move up ___ units vertically to stay on the line and arrive at point (__, __).

8. The relationship between the number of payments *P* made and the balance *B* (in dollars) of a $10,800 car loan can be represented by Table 2.16.

P, Number of Monthly Payments	*B*, Amount of Loan Balance ($)
0	10,800
1	10,500
2	10,200
3	9,900
4	9,600
5	9,300
6	9,000

Table 2.16

a. Based on the table, develop a linear equation for the amount of the car loan balance *B* as a function of the number of monthly payments *P*.

b. What is the monthly car payment?

c. What is the balance after 24 payments?

d. How many months are needed to produce a balance of zero?

9. The relationship between the number of tickets purchased for a movie and the revenue generated from that movie is indicated in Table 2.17.

Number of Tickets Purchased, *T*	Revenue, *R* ($)
0	0
10	75
20	150
30	225
40	300

Table 2.17

a. Based on this table, construct a linear equation for the relationship between revenue, *R*, and the number of tickets purchased, *T*.

b. What is the cost per ticket?

c. Find the revenue generated by 120 ticket purchases.

10. For each graph in Figure 2.39, identify two points on each line, determine the slope, then write an equation for the line.

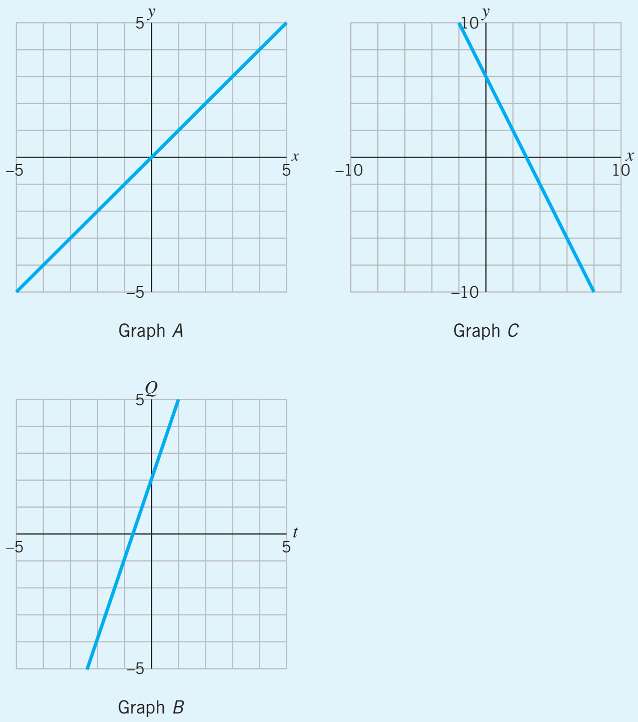


Figure 2.39

11. The revenue from one season of a college baseball team is the sum of allotted funds from alumni gifts and ticket sales from home games. In 2007 a college baseball team began the season with $15,000 in allotted funds. Tickets are sold at an average price of $12 each.

a. Write an equation for the relationship between tickets sold at home games, *T*, and revenue, *R* (in dollars).

b. Find the revenue for the team if 40,000 home game tickets are sold over the entire season.

12. Write the equations of three lines each with vertical intercept of 6.

13. Write the equations of three lines with slope -3.

14. Write an equation for the line that passes through:

a. The point $(-2, 1)$ and has a slope of 4

b. The point $(3, 5)$ and has a slope of $-2/3$

c. The point $(-1, 3)$ and has a slope of -10

d. The point $(1.2, 4.5)$ and has a slope of 2

15. Write an equation for the line:

a. Through (2, 5) and (4, 11)

b. Through (−3, 2) and (6, 1)

c. Through (4, −1) and (−2, −7)

16. Write each of the following in the form $y = mx + b$.

a. $3x + 4y = -12$

b. $7x - y = 5$

c. $2x + 8y = 1$

d. $x - 2y = 0$

e. $y - 2 = 3(x + 1)$

f. $y + 4 = -5(x - 2)$

17. Sketch the graph of the line:

a. With slope 2 and vertical intercept 5

b. With slope −1/2 and vertical intercept 6

c. With slope −3/4 and passing through the point (−4, 2)

d. With slope 3 and passing through the point (−5, −6)

Exercises for Section 2.7

Graphing program optional in Exercises 8, 10 and 23.

1. Write an equation for the line through (−2, 3) that has slope:

a. 5 **b.** $-\frac{3}{4}$ **c.** 0

2. Write an equation for the line through (0, 50) that has slope:

a. −20 **b.** 5.1 **c.** 0

3. Calculate the slope and write an equation for the linear function represented by each of the given tables.

a.

x	y
2	7.6
4	5.1

b.

A	W
5	12
7	16

4. Determine which of the following tables represents a linear function. If it is linear, write the equation for the linear function.

a.

x	y
0	3
1	8
2	13
3	18
4	23

b.

q	R
0	0.0
1	2.5
2	5.0
3	7.5
4	10.0

c.

x	$g(x)$
0	0
1	1
2	4
3	9
4	16

d.

t	r
10	5.00
20	2.50
30	1.67
40	1.25
50	1.00

e.

x	$h(x)$
20	20
40	−60
60	−140
80	−220
100	−300

f.

p	T
5	0.25
10	0.50
15	0.75
20	1.00
25	1.25

5. Plot each pair of points, then determine the equation of the line that goes through the points.

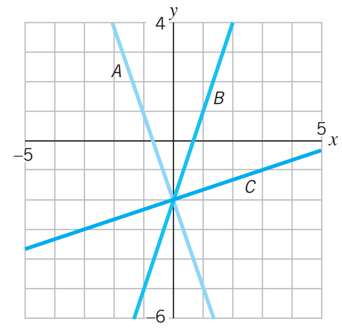
a. (2, 3), (4, 0)

b. (−2, 3), (2, 1)

c. (2, 0), (0, 2)

d. (4, 2), (−5, 2)

6. Find the equation for each of the lines *A*–*C* on the accompanying graph.



7. Put the following equations in $y = mx + b$ form, then identify the slope and the vertical intercept.

a. $2x - 3y = 6$

b. $3x + 2y = 6$

c. $\frac{1}{3}x + \frac{1}{2}y = 6$

d. $2y - 3x = 0$

e. $6y - 9x = 0$

f. $\frac{1}{2}x - \frac{2}{3}y = -\frac{1}{6}$

8. (Graphing program optional.) Solve each equation for y in terms of x, then identify the slope and the y-intercept. Graph each line by hand. Verify your answers with a graphing utility if available.

a. $-4y - x - 8 = 0$

b. $\frac{1}{2}x - \frac{1}{4}y = 3$

c. $-4x - 3y = 9$

d. $6x - 5y = 15$

9. Complete the table for each of the linear functions, and then sketch a graph of each function. Make sure to choose an appropriate scale and label the axes.

a.

x	$f(x) = 0.10x + 10$
−100	
0	
100	

b.

x	$h(x) = 50x + 100$
−0.5	
0	
0.5	

10. (Graphing program optional.) The equation $K = 4F - 160$ models the relationship between F, the temperature in degrees Fahrenheit, and K, the number of chirps per minute for the snow tree cricket.

a. Assuming F is the independent variable and K is the dependent variable, identify the slope and vertical intercept in the given equation.

b. Identify the units for K, 4, F, and -160.

c. What is a reasonable domain for this model?

d. Generate a small table of points that satisfy the equation. Be sure to choose realistic values for F from the domain of your model.

e. Calculate the slope directly from two data points. Is this value what you expected? Why?

f. Graph the equation, indicating the domain.

11. Find an equation to represent the cost of attending college classes if application and registration fees are $150 and classes cost $120 per credit.

12. a. Write an equation that describes the total cost to produce x items if the startup cost is $200,000 and the production cost per item is $15.

b. Why is the total average cost per item less if the item is produced in large quantities?

13. Your bank charges you a $2.50 monthly maintenance fee on your checking account and an additional $0.10 for each check you cash. Write an equation to describe your monthly checking account costs.

14. If a town starts with a population of 63,500 that declines by 700 people each year, construct an equation to model its population size over time. How long would it take for the population to drop to 53,000?

15. A teacher's union has negotiated a uniform salary increase for each year of service up to 20 years. If a teacher started at $26,000 and 4 years later had a salary of $32,000:

a. What was the annual increase?

b. What function would describe the teacher's salary over time?

c. What would be the domain for the function?

16. Your favorite aunt put money in a savings account for you. The account earns simple interest; that is, it increases by a fixed amount each year. After 2 years your account has $8250 in it and after 5 years it has $9375.

a. Construct an equation to model the amount of money in your account.

b. How much did your aunt put in initially?

c. How much will your account have after 10 years?

17. You read in the newspaper that the river is polluted with 285 parts per million (ppm) of a toxic substance, and local officials estimate they can reduce the pollution by 15 ppm each year.

a. Derive an equation that represents the amount of pollution, P, as a function of time, t.

b. The article states the river will not be safe for swimming until pollution is reduced to 40 ppm. If the cleanup proceeds as estimated, in how many years will it be safe to swim in the river?

18. The women's recommended weight formula from Harvard Pilgrim Healthcare says, "Give yourself 100 lb for the first 5 ft plus 5 lb for every inch over 5 ft tall."

a. Find a mathematical model for this relationship. Be sure you clearly identify your variables.

b. Specify a reasonable domain for the function and then graph the function.

c. Use your model to calculate the recommended weight for a woman 5 feet, 4 inches tall; and for one 5 feet, 8 inches tall.

19. In 1977 a math professor bought her condominium in Cambridge, Massachusetts, for $70,000. The value of the condo has risen steadily so that in 2007 real estate agents tell her the condo is now worth $850,000.

a. Find a formula to represent these facts about the value of the condo $V(t)$, as a function of time, t.

b. If she retires in 2010, what does your formula predict her condo will be worth then?

20. The y-axis, the x-axis, the line $x = 6$, and the line $y = 12$ determine the four sides of a 6-by-12 rectangle in the first quadrant (where $x > 0$ and $y > 0$) of the xy plane. Imagine that this rectangle is a pool table. There are pockets at the four corners and at the points (0, 6) and (6, 6) in the middle of each of the longer sides. When a ball bounces off one of the sides of the table, it obeys the "pool rule": The slope of the path after the bounce is the negative of the slope before the bounce. (*Hint:* It helps to sketch the pool table on a piece of graph paper first.)

a. Your pool ball is at (3, 8). You hit it toward the y-axis, along the line with slope 2.

i. Where does it hit the y-axis?

ii. If the ball is hit hard enough, where does it hit the side of the table next? And after that? And after that?

iii. Show that the ball ultimately returns to (3, 8). Would it do this if the slope had been different from 2? What is special about the slope 2 for this table?

b. A ball at (3, 8) is hit toward the y-axis and bounces off it at $\left(0, \frac{16}{3}\right)$. Does it end up in one of the pockets? If so, what are the coordinates of that pocket?

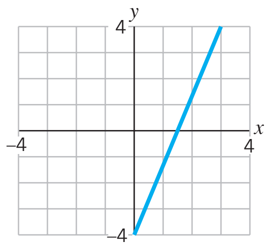
c. Your pool ball is at (2, 9). You want to shoot it into the pocket at (6, 0). Unfortunately, there is another ball at (4, 4.5) that may be in the way.

i. Can you shoot directly into the pocket at (6, 0)?

ii. You want to get around the other ball by bouncing yours off the y-axis. If you hit the y-axis at (0, 7), do you end up in the pocket? Where do you hit the line $x = 6$?

iii. If bouncing off the y-axis at (0, 7) didn't work, perhaps there is some point $(0, b)$ on the y-axis from which the ball would bounce into the pocket at (6, 0). Try to find that point.

21. Find the equation of the line shown on the accompanying graph. Use this equation to create two new graphs, taking care to label the scales on your new axes. For one of your graphs, choose scales that make the line appear steeper than in the original graph. For your second graph, choose scales that make the line appear less steep than in the original graph.



22. The exchange rate that a bank gave for euros in October 2006 was 0.79 euros for $1 U.S. They also charged a constant fee of $5 per transaction. The bank's exchange rate from euros to British pounds was 0.66 pounds for 1 euro, with a transaction fee of 4.1 euros.

a. Write a general equation for how many euros you got when changing dollars. Use E for euros and D for dollars being exchanged. Draw a graph of E versus D.

b. Would it have made any sense to exchange $10 for euros?

c. Find a general expression for the percentage of the total euros converted from dollars that the bank kept for the transaction fee.

d. Write a general equation for how many pounds you would get when changing euros. Use P for British pounds and E for the euros being exchanged. Draw a graph of P versus E.

23. (Graphing program optional.) Suppose that:

- For 8 years of education, the mean annual earnings for women working full-time are approximately $19,190.
- For 12 years of education, the mean annual earnings for women working full-time are approximately $31,190.
- For 16 years of education, the mean annual earnings for women working full-time are approximately $43,190.

a. Plot this information on a graph.

b. What sort of relationship does this information suggest between earnings and education for women? Justify your answer.

c. Generate an equation that could be used to model the data from the limited information given (letting E = years of education and M = mean earnings). Show your work.

24. a. Create a third column in Tables A and B, and insert values for the average rate of change. (The first entry will be "n.a.".)

t	d
0	400
1	370
2	340
3	310
4	280
5	250

Table A

t	d
0	1.2
1	2.1
2	3.2
3	4.1
4	5.2
5	6.1

Table B

b. In either table, is d a linear function of t? If so, construct a linear equation relating d and t for that table.

25. Adding minerals or organic compounds to water lowers its freezing point. Antifreeze for car radiators contains glycol (an organic compound) for this purpose. The accompanying table shows the effect of salinity (dissolved salts) on the freezing point of water. Salinity is measured in the number of grams of salts dissolved in 1000 grams of water. So our units for salinity are in parts per thousand, abbreviated ppt. Is the relationship between the freezing point and salinity linear? If so, construct an equation that models the relationship. If not, explain why.

Relationship between Salinity and Freezing Point

Salinity (ppt)	Freezing Point (°C)
0	0.00
5	−0.27
10	−0.54
15	−0.81
20	−1.08
25	−1.35

Source: Data adapted from P.R. Pinel, *Oceanography: An Introduction to the Planet Oceanus* (St. Paul, MN: West, 1992), p. 522.

26. The accompanying data show rounded average values for blood alcohol concentration (BAC) for people of different weights, according to how many drinks (5 oz wine, 1.25 oz 80-proof liquor, or 12 oz beer) they have consumed.

Blood Alcohol Concentration for Selected Weights

Number of Drinks	100 lb	140 lb	180 lb
2	0.075	0.054	0.042
4	0.150	0.107	0.083
6	0.225	0.161	0.125
8	0.300	0.214	0.167
10	0.375	0.268	0.208

a. Examine the data on BAC for a 100-pound person. Are the data linear? If so, find a formula to express blood alcohol concentration, A, as a function of the number of drinks, D, for a 100-pound person.

b. Examine the data on BAC for a 140-pound person. Are the data linear? If they're not precisely linear, what might be a reasonable estimate for the average rate of change of blood alcohol concentration, A, with respect to number of drinks, D? Find a formula to estimate blood alcohol concentration, A, as a function of number of drinks, D, for a 140-pound person. Can you make any general conclusions about BAC as a function of number of drinks for all of the weight categories?

c. Examine the data on BAC for people who consume two drinks. Are the data linear? If so, find a formula to express blood alcohol concentration, A, as a function of weight, W, for people who consume two drinks. Can you make any general conclusions about BAC as a function of weight for any particular number of drinks?

2.8 Special Cases

Direct Proportionality

The simplest relationship between two variables is when one variable is equal to a constant multiple of the other. For instance, in the previous example $A = 0.0235D$; blood alcohol concentration A equals a constant, 0.0235, times D, the number of drinks. We say that *A is directly proportional to D.*

How to recognize direct proportionality

Linear functions of the form

$$y = mx \qquad (m \neq 0)$$

describe a relationship where y is directly proportional to x. If two variables are directly proportional to each other, the graph will be a straight line that passes through the point (0, 0), the origin. Figure 2.40 shows the graphs of two relationships in which y is directly proportional to x, namely $y = 2x$ and $y = -x$.

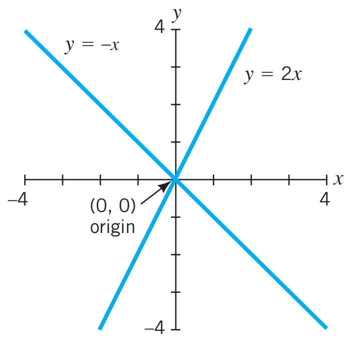


Figure 2.40 Graphs of two relationships in which y is directly proportional to x. Note that both graphs are lines that go through the origin.

> **SOMETHING TO THINK ABOUT**
> *If y is directly proportional to x, is x directly proportional to y?*

> **Direct Proportionality**
>
> In a linear equation of the form
>
> $$y = mx \qquad (m \neq 0)$$
>
> we say that
>
> y is *directly proportional to x.*
>
> Its graph will go through the origin.

EXAMPLE 1 Braille is a code, based on six-dot cells, that allows blind people to read. One page of regular print translates into 2.5 Braille pages. Construct a function describing this relationship. Does it represent direct proportionality? What happens if the number of print pages doubles? Triples?

SOLUTION If P = number of regular pages and B = number of Braille pages, then $B = 2.5P$. So B is directly proportional to P. If the number of print pages doubles, the number of Braille pages doubles. If the number of print pages triples, the number of Braille pages will triple.

EXAMPLE 2

You are traveling to Canada and need to exchange American dollars for Canadian dollars. On that day the exchange rate is approximately 1 American dollar for 1.13 Canadian dollars.

a. Construct an equation for converting American to Canadian dollars. Does it represent direct proportionality?

b. Suppose the Exchange Bureau charges a \$2 flat fee to change money. Alter your equation from part (a) to include the service fee. Does the new equation represent direct proportionality?

SOLUTION

a. If we let A = American dollars and C = Canadian dollars, then the equation

$$C = 1.13A$$

describes the conversion from American (the input) to Canadian (the output). The amount of Canadian money you receive is directly proportional to the amount of American money you exchange.

b. If there is a \$2 service fee, you would have to subtract \$2 from the American money you have before converting to Canadian. The new equation is

$$C = 1.13(A - 2)$$

or equivalently,

$$C = 1.13A - 2.26$$

where 2.26 is the service fee in Canadian dollars. Then C is no longer directly proportional to A.

EXAMPLE 3

A prominent midwestern university decided to change its tuition cost. Previously there was a ceiling on tuition (which included fees). Currently the university uses what it calls a linear model, charging \$106 per credit hour for in-state students and \$369 per credit hour for out-of-state students.

a. Is it correct to call this pricing scheme a linear model for in-state students? For out-of-state students? Why?

b. Generate equations and graphs for the cost of tuition for both in-state and out-of-state students. If we limit costs to one semester during which the usual maximum credit hours is 15, what would be a reasonable domain?

c. In each case is the tuition directly proportional to the number of credit hours?

SOLUTION

a. Yes, both relationships are linear since the rate of change is constant in each case: \$106 per credit hour for in-state students and \$369 per credit hour for out-of-state students.

b. Let N = number of credit hours, C_i = cost for an in-state student, and C_o = cost for an out-of-state student. In each case if the number of credit hours is zero ($N = 0$), then the cost would be zero ($C_i = 0 = C_o$). Hence both lines would pass through the origin (0, 0), making the vertical intercept 0 for both equations. So the results would be of the form

$$C_i = 106N \quad \text{and} \quad C_o = 369N$$

which are graphed in Figure 2.41. A reasonable domain would be $0 \le N \le 15$.

c. In both cases the tuition is directly proportional to the number of credit hours. The graphs verify this since both are straight lines going through the origin.

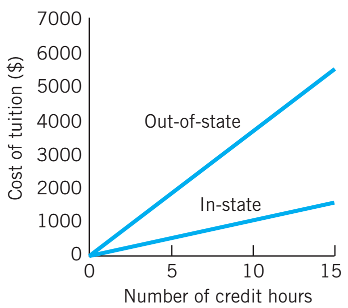


Figure 2.41 Tuition for in-state and out-of-state students at a midwestern university.

Algebra Aerobics 2.8a

1. Construct an equation and draw the graph of the line that passes through the origin and has the given slope.

a. $m = -1$ **b.** $m = 0.5$

2. For each of Tables 2.18 and 2.19, determine whether x and y are directly proportional to each other. Represent each relationship with an equation.

a.

x	y
−2	6
−1	3
0	0
1	−3
2	−6

Table 2.18

b.

x	y
0	5
1	8
2	11
3	14
4	17

Table 2.19

3. In September 2006 the exchange rate was \$1.00 U.S. to 0.79 euros, the common European currency.

a. Find a linear function that converts U.S. dollars to euros.

b. Find a linear function that converts U.S. dollars to euros with a service fee of \$2.50.

c. Which function represents a directly proportional relationship and why?

4. Suppose you go on a road trip, driving at a constant speed of 60 miles per hour. Create an equation relating distance d in miles and time traveled t in hours. Does it represent direct proportionality? What happens to d if the value for t doubles? If t triples?

5. The total cost C for football tickets is directly proportional to the number of tickets purchased, N. If two tickets cost \$50, construct the formula relating C and N. What would the total cost of 10 tickets be?

6. Write a formula to describe each situation.

a. y is directly proportional to x, and y is 4 when x is 12.

b. d is directly proportional to t, and d is 300 when t is 50.

7. Write a formula to describe the following:

a. The diameter, d, of a circle is directly proportional to the circumference, C.

b. The amount of income tax paid, T, is directly proportional to income, I.

c. The tip amount t, is directly proportional to the cost of the meal, c.

8. Assume that a is directly proportional to b. When $a = 10$, $b = 15$.

a. Find a if b is 6.

b. Find b if a is 4.

Horizontal and Vertical Lines

The slope, m, of any horizontal line is 0. So the general form for the equation of a horizontal line is

$$y = b + 0x$$

or just

$$y = b$$

For example, Table 2.20 and Figure 2.42 show points that satisfy the equation of the horizontal line $y = 1$. If we calculate the slope between any two points in the table—for example, $(-2, 1)$ and $(2, 1)$—we get

$$\text{slope} = \frac{1-1}{-2-2} = \frac{0}{-4} = 0$$

x	y
−4	1
−2	1
0	1
2	1
4	1

Table 2.20

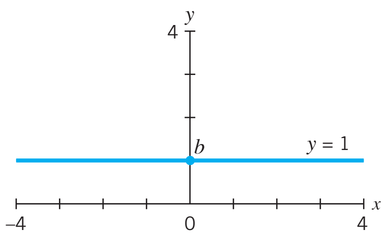


Figure 2.42 Graph of the horizontal line $y = 1$.

For a vertical line the slope, m, is undefined, so we can't use the standard $y = b + mx$ format. The graph of a vertical line (as in Figure 2.43) fails the vertical line test, so y is not a function of x. However, every point on a vertical line does have the same horizontal coordinate, which equals the coordinate of the horizontal intercept. Therefore, the general equation for a vertical line is of the form

$$x = c \quad \text{where } c \text{ is a constant (the horizontal intercept)}$$

For example, Table 2.21 and Figure 2.43 show points that satisfy the equation of the vertical line $x = 1$.

x	y
1	−4
1	−2
1	0
1	2
1	4

Table 2.21

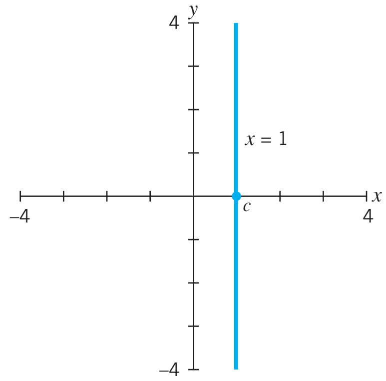


Figure 2.43 Graph of the vertical line $x = 1$.

Note that if we tried to calculate the slope between two points, say $(1, -4)$ and $(1, 2)$, on the vertical line $x = 1$ we would get

$$\text{slope} = \frac{-4 - 2}{1 - 1} = \frac{-6}{0} \quad \text{which is undefined.}$$

The general equation of a *horizontal line* is

$$y = b$$

where b is a constant (the vertical intercept) and the slope is 0.

The general equation of a *vertical line* is

$$x = c$$

where c is a constant (the horizontal intercept) and the slope is undefined.

EXAMPLE 4 Find the equation for each line in Figure 2.44.

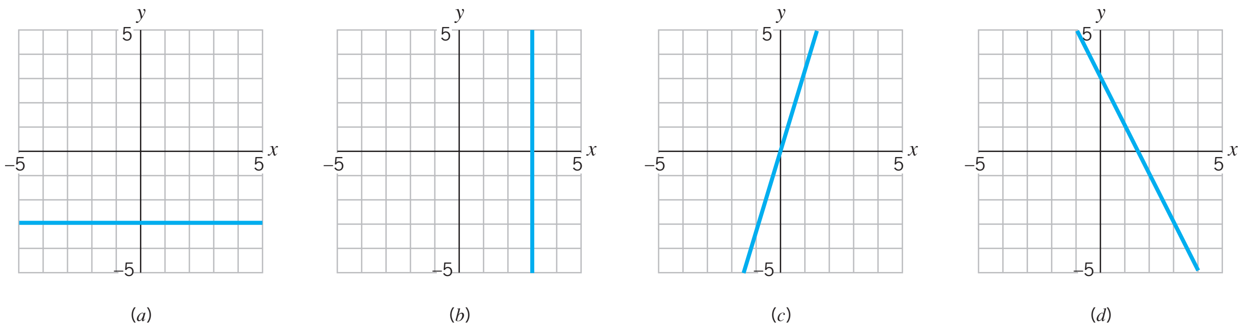


Figure 2.44 Four linear graphs.

SOLUTION

a. $y = -3$, a horizontal line
b. $x = 3$, a vertical line
c. $y = 3x$, a direct proportion, slope $= 3$
d. $y = -2x + 3$, a line with slope -2 and y-intercept 3

Parallel and Perpendicular Lines

Parallel lines have the same slope. So if the two equations $y = b_1 + m_1x$ and $y = b_2 + m_2x$ describe two parallel lines, then $m_1 = m_2$. For example, the two lines $y = 2.0 - 0.5x$ and $y = -1.0 - 0.5x$ each have a slope of -0.5 and thus are parallel (see Figure 2.45).

SOMETHING TO THINK ABOUT

Describe the equation for any line perpendicular to the horizontal line $y = b$.

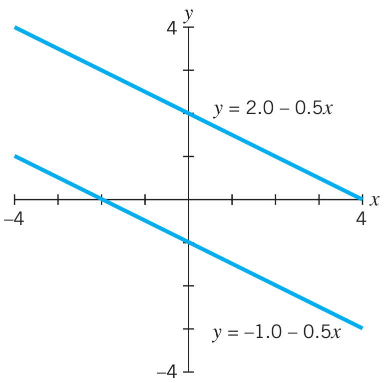


Figure 2.45 Two parallel lines have the same slope.

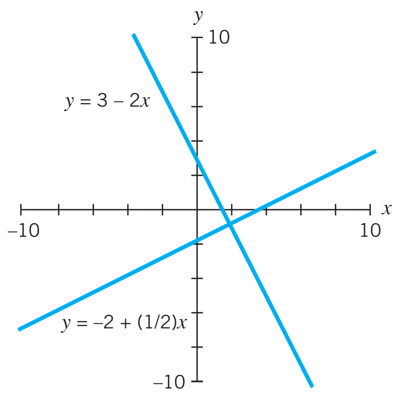


Figure 2.46 Two perpendicular lines have slopes that are negative reciprocals.

Two lines are perpendicular if their slopes are negative reciprocals. If $y = b_1 + m_1x$ and $y = b_2 + m_2x$ describe two perpendicular lines, then $m_1 = -1/m_2$. For example, in Figure 2.46 the two lines $y = 3 - 2x$ and $y = -2 + \frac{1}{2}x$ have slopes of -2 and $\frac{1}{2}$, respectively. Since -2 is the negative reciprocal of $\frac{1}{2}$ (i.e., $-\frac{1}{(\frac{1}{2})} = -1 \div \frac{1}{2} = -1 \cdot \frac{2}{1} = -2$), the two lines are perpendicular.

Why does this relationship hold for perpendicular lines?

Consider a line whose slope is given by v/h. Now imagine rotating the line 90 degrees clockwise to generate a second line perpendicular to the first (Figure 2.47). What would the slope of this new line be?

The positive vertical change, v, becomes a positive horizontal change. The positive horizontal change, h, becomes a negative vertical change. The slope of the original line is v/h, and the slope of the line rotated 90 degrees clockwise is $-h/v$. Note that $-h/v = -1/(v/h)$, which is the original slope inverted and multiplied by -1.

In general, the slope of a perpendicular line is the negative reciprocal of the slope of the original line. If the slope of a line is m_1, then the slope, m_2, of a line perpendicular to it is $-1/m_1$.

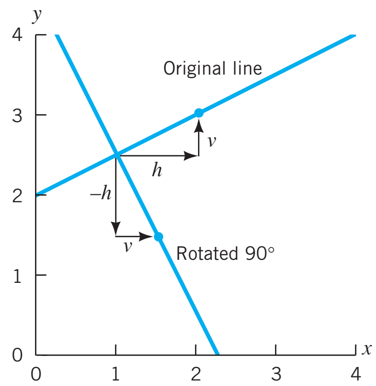


Figure 2.47 Perpendicular lines $m_2 = -1/m_1$.

This is true for any pair of perpendicular lines for which slopes exist. It does not work for horizontal and vertical lines since vertical lines have undefined slopes.

> Parallel lines have the same slope.
>
> Perpendicular lines have slopes that are negative reciprocals of each other.

EXAMPLE 5 Determine from the equations which pairs of lines are parallel, perpendicular, or neither.

a. $y = 2 + 7x$ and $y = 7x + 3$

b. $y = 6 - x$ and $y = 6 + x$

c. $y = 5 + 3x$ and $y = 5 - 3x$

d. $y = 3x + 13$ and $3y + x = 2$

SOLUTION **a.** The two lines are parallel since they share the same slope, 7.

b. The two lines are perpendicular since the negative reciprocal of -1 (the slope of first line) equals $-(1/(-1)) = -(-1) = 1$, the slope of the second line.

c. The lines are neither parallel nor perpendicular.

d. The lines are perpendicular. The slope of the first line is 3. If we solve the second equation for y, we get

$$3y + x = 2$$
$$3y = 2 - x$$
$$y = 2/3 - (1/3)x$$

So the slope of the second line is $-(1/3)$, the negative reciprocal of 3.

Algebra Aerobics 2.8b

1. In each case, find an equation for the horizontal line that passes through the given point.
   a. $(3, -5)$ **b.** $(5, -3)$ **c.** $(-3, 5)$
2. In each case, find an equation for the vertical line that passes through the given point.
   a. $(3, -5)$ **b.** $(5, -3)$ **c.** $(-3, 5)$
3. Construct the equation of the line that passes through the points.
   a. $(0, -7)$, $(3, -7)$, and $(350, -7)$
   b. $(-4.3, 0)$, $(-4.3, 8)$ and $(-4.3, -1000)$
4. Find the equation of the line that is parallel to $y = 4 - x$ and that passes through the origin.
5. Find the equation of the line that is parallel to $W = 360C + 2500$ and passes through the point where $C = 4$ and $W = 1000$.
6. Find the slope of a line perpendicular to each of the following.
   a. $y = 4 - 3x$ **c.** $y = 3.1x - 5.8$
   b. $y = x$ **d.** $y = -\frac{3}{5}x + 1$
7. **a.** Find an equation for the line that is perpendicular to $y = 2x - 4$ and passes through $(3, -5)$.
   b. Find the equations of two other lines that are perpendicular to $y = 2x - 4$ but do not pass through the point $(3, -5)$.
   c. How do the three lines from parts (a) and (b) that are perpendicular to $y = 2x - 4$ relate to each other?
   d. Check your answers by graphing the equations.
8. Find the slope of the line $Ax + By = C$ assuming that y is a function of x. (*Hint:* Solve the equation for y.)
9. Use the result of the previous exercise to determine the slope of each line described by the following linear equations (again assuming y is a function of x).
   a. $2x + 3y = 5$ **d.** $x = -5$
   b. $3x - 4y = 12$ **e.** $x - 3y = 5$
   c. $2x - y = 4$ **f.** $y = 4$
10. Solve the equation $2x + 3y = 5$ for y, identify the slope, then find an equation for the line that is parallel to the line $2x + 3y = 5$ and passes through the point $(0, 4)$.

11. Solve the equation $3x + 4y = -7$ for y, identify the slope, then find an equation for the line that is perpendicular to the line of $3x + 4y = -7$ and passes through (0, 3).

12. Solve the equation $4x - y = 6$ for y, identify the slope, then find an equation for the line that is perpendicular to the line $4x - y = 6$ and passes through $(2, -3)$.

13. Determine whether each equation could represent a vertical line, a horizontal line, or neither.

a. $x + 1.5 = 0$ **b.** $2x + 3y = 0$ **c.** $y - 5 = 0$

14. Write an equation for the line perpendicular to $2x + 3y = 6$:

a. That has vertical intercept of 5

b. That passes through the point $(-6, 1)$

15. Write an equation for the line parallel to $2x - y = 7$:

a. That has a vertical intercept of 9

b. That passes through (4, 3)

Piecewise Linear Functions

Some functions are not linear throughout but are made up of linear segments. They are called *piecewise linear functions*. For example, we could define a function $f(x)$ where:

$$f(x) = 2 + x \quad \text{for } x \le 1 \qquad f(x) = 3 \quad \text{for } x > 1 \qquad \text{or, more compactly,} \qquad f(x) = \begin{cases} 2 + x & \text{for } x \le 1 \\ 3 & \text{for } x > 1 \end{cases}$$

The graph of $f(x)$ in Figure 2.48 clearly shows the two distinct linear segments.

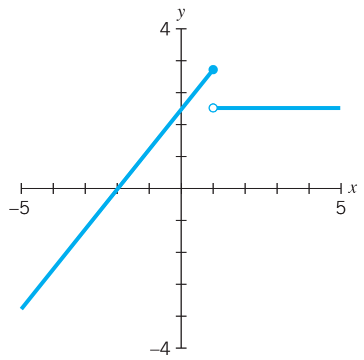


Figure 2.48 Graph of a piecewise linear function.

EXAMPLE 6 Consider the amount of gas in your car during a road trip. You start out with 20 gallons and drive for 3 hours, leaving you with 14 gallons in the tank. You stop for lunch for an hour and then drive for 4 more hours, leaving you with 6 gallons.

a. Construct a piecewise linear function for the amount of gas in the tank as a function of time in hours.

b. Graph the results.

SOLUTION **a.** Let t = time (in hours). For $0 \le t < 3$, the average rate of change in gasoline over time is (14–20) gallons/(3 hr) = (−6 gallons)/(3 hr) = −2 gallons/hr; that is, you are consuming 2 gallons per hour. The initial amount of gas is 20 gallons, so $G(t)$, the amount of gas in the tank at time t, is given by

$$G(t) = 20 - 2t \quad \text{for } 0 \le t < 3$$

While you are at lunch for an hour, you consume no gasoline, so the amount of gas stays constant at 14 gallons. So

$$G(t) = 14 \quad \text{for } 3 \le t < 4$$

At the end of lunch, $t = 4$ and $G(t) = 14$. You continue to drive for 4 more hours, ending up with 6 gallons. You are still consuming 2 gallons per hour since (6–14) gallons/4 hr $= -2$ gallons/hr. So your equation will be of the form $G(t) = b - 2t$. Substituting in $t = 4$ and $G(t) = 14$, we have $14 = b - 2 \cdot 4 \Rightarrow b = 22$. So

$$G(t) = 22 - 2t \quad \text{for } 4 \le t < 8$$

Writing $G(t)$ more compactly, we have

$$G(t) = \begin{cases} 20 - 2t & \text{if } 0 \le t < 3 \\ 14 & \text{if } 3 \le t < 4 \\ 22 - 2t & \text{if } 4 \le t < 8 \end{cases}$$

b. The graph of $G(t)$ is shown in Figure 2.49.

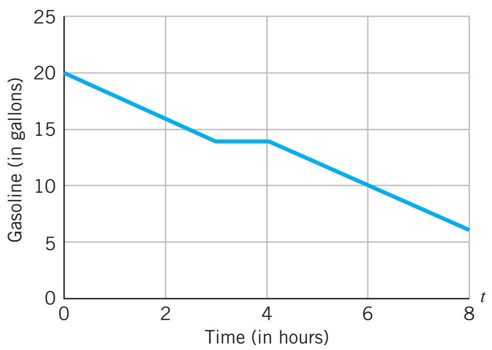


Figure 2.49 Number of gallons left in the car's tank.

The absolute value function

The absolute value of x, written as $|x|$, strips x of its sign. That means we consider only the magnitude of x, so $|x|$ is never negative.[1]

- If x is positive (or 0), then $|x| = x$.
- If x is negative, then $|x| = -x$.

For example, $|-5| = -(-5) = 5$.

We can construct the absolute value function using piecewise notation.

> **The absolute value function**
>
> If $f(x) = |x|$, then $f(x) = \begin{cases} x & \text{for } x \ge 0 \\ -x & \text{for } x < 0 \end{cases}$

EXAMPLE 7 If $f(x) = |x|$, then:

a. What is $f(6)$? $f(0)$? $f(-6)$?

b. Graph the function $f(x)$ for x between -6 and 6.

c. What is the slope of the line segment when $x > 0$? When $x < 0$?

[1]Graphing calculators and spreadsheet programs usually have an absolute value function. It is often named *abs* where *abs* $(x) = |x|$. So *abs*$(-3) = |3| = 3$.

SOLUTION

a. $f(6) = 6; f(0) = 0; f(-6) = |-6| = 6$

b. See Figure 2.50.

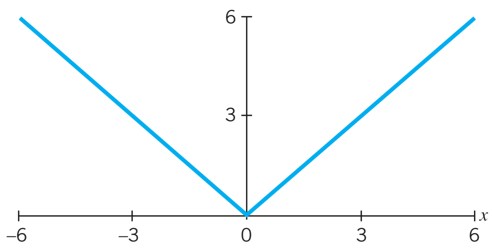


Figure 2.50 Graph of the absolute value function $f(x) = |x|$.

c. When $x > 0$, the slope is 1; when $x < 0$, the slope is -1.

Absolute value inequalities (in one variable) are frequently used in describing an allowable range above or below a certain amount.

EXAMPLE 8 Range in values of poll results

Poll figures are often given with a margin of error. For example, in January 2007 a CNN poll said that 63% of Americans felt that the economy was in good condition, with a sample error of ±3 points. Construct an absolute value inequality that describes the range of percentages P that are possible within this poll. Restate this condition without using absolute values, and display it on a number line.

SOLUTION

$|P - 63| \leq 3$; that is, the poll takers are confident that the difference between the estimated percentage, 63%, and the actual percentage, P, is less than or equal to 3%.

Equivalently we could write $60 \leq P \leq 66$; that is, the actual percentage P is somewhere between 60% and 66% (see Figure 2.51).

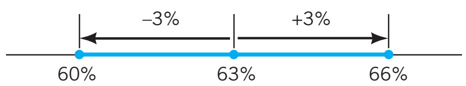


Figure 2.51 Range of error around 63% is ±3 percentage points.

Absolute value functions are useful in describing distances between objects.

EXAMPLE 9 Distance between a cell phone and cell tower

You are a passenger in a car, talking on a cell phone. The car is traveling at 60 mph along a straight highway and the nearest cell phone tower is 6 miles away.

a. How long will it take you to reach the cell phone tower (assuming it is right by the road)?

b. Construct a linear function $D(t)$ that describes your distance *to* the cell tower (in miles ≥ 0) or *from* the cell tower (in miles < 0), where t is the number of hours traveled.

c. Graph your function using a reasonable domain for t.

d. What does $|D(t)|$ represent? Graph $|D(t)|$ on a separate grid and compare it with the graph of $D(t)$.

SOLUTION

a. Traveling at 60 miles per hour is equivalent traveling at 1 mile per minute. So traveling 6 miles from the start will take you 6 minutes or 0.1 hr to reach the cell tower.

b. At the starting time $t = 0$ hours, the distance to the nearest cell phone tower is 6 miles, so $D(0) = 6$ miles. Thus, the vertical intercept is at (0, 6). After $t = 0.1$ hr you are at the cell tower, so the distance between you and the cell tower is 0. Thus, $D(0.1) = 0$ miles and hence the horizontal intercept is (0.1, 0). The slope of the line is $(0 - 6)/(0.1 - 0) = -60$. So the distance function is $D(t) = 6 - 60t$, where t is in hours and $D(t)$ is in miles (to or from the cell tower).

c. A reasonable domain might be $0 \leq t \leq 0.2$ hours. See Figure 2.52.

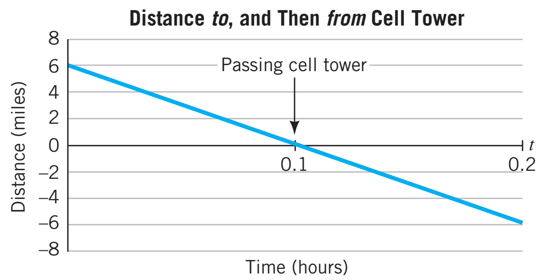


Figure 2.52 Graph of $D(t) = 6 - 60t$.

d. $|D(t)|$ describes the absolute value of the distance between you and the cell tower, indicating that the direction of travel no longer matters. Whether you are driving toward or away from the tower, the absolute value of the distance is always positive (or 0). For example, $|D(0.2)| = |6 - 60 \cdot 0.2| = |6 - 12| = |-6| = 6$ miles, which means that after 0.2 hours (or 12 minutes) you are 6 miles away from the tower. See Figure 2.53.

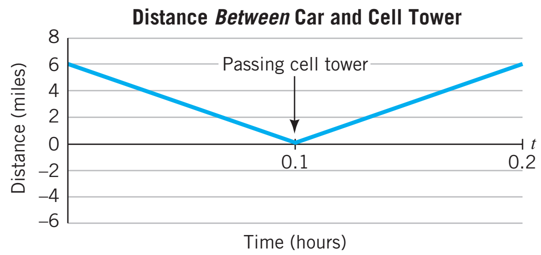


Figure 2.53 Graph of $|D(t)| = |5 - 25t|$.

Step functions

Some piecewise linear functions are called *step functions* because their graphs look like the steps of a staircase. Each "step" is part of a horizontal line.

EXAMPLE 10

A step function: Minimum wages

The federal government establishes a national minimum wage per hour. Table 2.22 shows the value of the minimum wage over the years 1990 to 2007.

Federal Minimum Wage for 1990–2007

Year New Minimum Wage Set	Minimum Wage (per hour)
1990	\$3.80
1991	\$4.25
1996	\$4.75
1997	\$5.15
2007	\$5.85

Table 2.22

a. Construct a step function $M(x)$, where $M(x)$ is the minimum wage at year x. What is the domain?

b. What is $M(1992)$? $M(2006)$?

c. Graph the step function.

d. Why do you think there was a lot of discussion in 2006 about raising the minimum wage? (*Note:* Individual states can set a higher minimum for their workers.)

SOLUTION

a. $$M(x) = \begin{cases} 3.80 & \text{for } 1990 \le x < 1991 \\ 4.25 & \text{for } 1991 \le x < 1996 \\ 4.75 & \text{for } 1996 \le x < 1997 \\ 5.15 & \text{for } 1997 \le x < 2007 \\ 5.85 & \text{for } 2007 \le x < 2008 \end{cases}$$

The domain is $1990 \le x < 2008$.

b. $M(1992) = \$4.25/\text{hr}$; $M(2006) = \$5.15/\text{hr}$

c. See Figure 2.54.

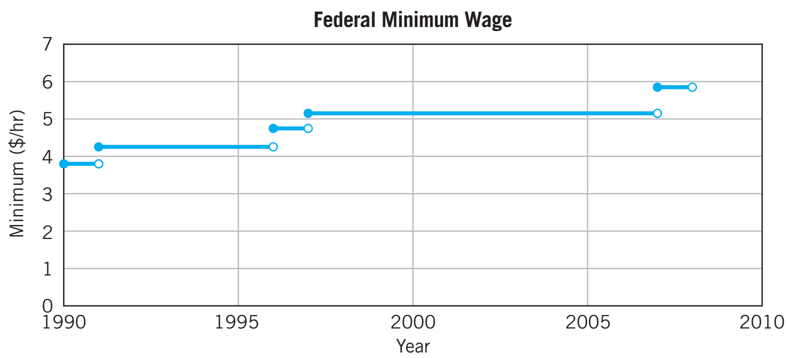


Figure 2.54 Step function for federal minimum wage between 1990 and 2007.

d. By the end of 2006 the federal minimum wage had stayed the same ($5.15/hr) for 9 years. Inflation always erodes the purchasing power of the dollar, so $5.15 in 2006 bought a lot less than $5.15 in 1997. So many felt an increase in the minimum wage was way overdue.

Algebra Aerobics 2.8c

1. Construct the graphs of the following piecewise linear functions. (Be sure to indicate whether each endpoint is included on or excluded from the graph.)

   a. $$f(x) = \begin{cases} 1 & \text{for } 0 < x \le 1 \\ 0 & \text{for } 1 < x \le 2 \\ -1 & \text{for } 2 < x \le 3 \end{cases}$$

   b. $$g(x) = \begin{cases} x + 3 & \text{for } -4 \le x < 0 \\ 2 - x & \text{for } \quad 0 \le x \le 4 \end{cases}$$

2. Construct piecewise linear functions $Q(t)$ and $C(r)$ to describe the graphs in Figures 2.55 and 2.56.

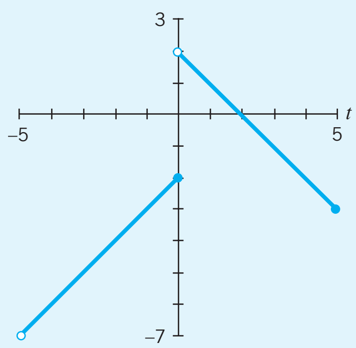


Figure 2.55 Graph of $Q(t)$.

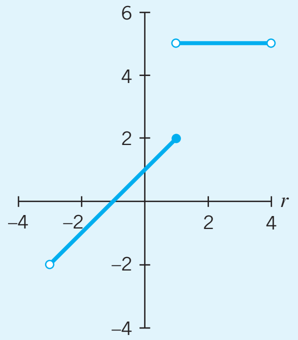


Figure 2.56 Graph of $C(r)$.

3. Evaluate the following:

a. $|-2|$ **c.** $|3-5|$ **e.** $-|3| \cdot |-5|$

b. $|6|$ **d.** $|3| - |5|$

4. Given the function $g(x) = |x - 3|$:

a. What is $g(-3)$? $g(0)$? $g(3)$? $g(6)$?

b. Sketch the graph of $g(x) = |x - 3|$ for $-6 \leq x \leq 6$.

c. Compare the graph of $g(x) = |x - 3|$ with the graph of $f(x) = |x|$.

d. Write $g(x)$ using piecewise linear notation.

5. Rewrite the following expressions without using an absolute value sign, and then describe in words the result.

a. $|t - 5| \leq 2$ **b.** $|Q - 75| < 6$

6. The optimal water temperature for trout is 55°F, but they can survive water temperatures that are 20° above or below that. Write an absolute value inequality that describes the temperature values, T, that lie within the trout survival temperature range. Then write an equivalent expression without the absolute value sign.

7. a. Sketch a piecewise linear graph of the *total* distance you would travel if you walked at a constant speed from home to a coffee shop, stopped for a cup, and then walked home at a faster pace.

b. For the scenario in part (a), sketch a piecewise linear graph that shows the distance *between* you and the coffee shop over time.

8. The federal funds rate is the short-term interest rate charged by the Federal Reserve for overnight loans to other federal banks. This is one of the major tools the Federal Reserve Board uses to stimulate the economy (with a rate decrease) or control inflation (with a rate increase). Table 2.23 shows the week of each rate change during 2006.

Federal Funds Rates During 2006

Week in 2006 When Rate Was Changed	Federal Funds Rate (%)
Week 1	4.25
Week 5	4.50
Week 13	4.75
Week 19	5.00
Week 26	5.25

Table 2.23

a. What was the federal funds rate in week 4? In week 52? What was the longest period in 2006 over which the federal funds rate remained the same?

b. What appears to be the typical percentage increase used by the Board? Do the increases occur at regular intervals?

c. During 2006 was the Federal Reserve Board more concerned about stimulating economic growth or curbing inflation?

d. Construct a piecewise linear function to describe the federal funds rate during 2006.

e. Sketch a graph of your function.

Exercises for Section 2.8

1. Using the general formula $y = mx$ that describes direct proportionality, find the value of m if:

a. y is directly proportional to x and $y = 2$ when $x = 10$.

b. y is directly proportional to x and $y = 0.1$ when $x = 0.2$.

c. y is directly proportional to x and $y = 1$ when $x = \frac{1}{4}$.

2. For each part, construct an equation and then use it to solve the problem.

a. Pressure P is directly proportional to temperature T, and P is 20 lb per square inch when T is 60 degrees Kelvin. What is the pressure when the temperature is 80 degrees Kelvin?

b. Earnings E are directly proportional to the time T worked, and E is \$46 when T is 2 hours. How long has a person worked if she earned \$471.50?

c. The number of centimeters of water depth W produced by melting snow is directly proportional to the number of centimeters of snow depth S. If W is 15.9 cm when S is 150 cm, then how many centimeters of water depth are produced by a 100-cm depth of melting snow?

3. In the accompanying table y is directly proportional to x.

Number of CDs purchased (x)	3	4	5
Cost of CDs (y)	42.69	56.92	

a. Find the formula relating y and x, then determine the missing value in the table.

b. Interpret the coefficient of x in this situation.

4. The electrical resistance R (in ohms) of a wire is directly proportional to its length l (in feet).

a. If 250 feet of wire has a resistance of 1.2 ohms, find the resistance for 150 ft of wire.

b. Interpret the coefficient of l in this context.

5. For each of the following linear functions, determine the independent and dependent variables and then construct an equation for each function.

a. Sales tax is 6.5% of the purchase price.

b. The height of a tree is directly proportional to the amount of sunlight it receives.

c. The average salary for full-time employees of American domestic industries has been growing at an annual rate of \$1300/year since 1985, when the average salary was \$25,000.

6. On the scale of a map 1 inch represents a distance of 35 miles.

a. What is the distance between two places that are 4.5 inches apart on the map?

b. Construct an equation that converts inches on the map to miles in the real world.

7. Find a function that represents the relationship between distance, d, and time, t, of a moving object using the data in the accompanying table. Is d directly proportional to t? Which is a more likely choice for the object, a person jogging or a moving car?

t (hours)	d (miles)
0	0
1	5
2	10
3	15
4	20

8. Determine which (if any) of the following variables (w, y, or z) is directly proportional to x:

x	w	y	z
0	1	0.0	0
1	2	2.5	$-\frac{1}{3}$
2	5	5.0	$-\frac{2}{3}$
3	10	7.5	-1
4	17	10.0	$-\frac{4}{3}$

9. Find the slope of the line through the pair of points, then determine the equation.

a. (2, 3) and (5, 3)

b. (−4, −7) and (12, −7)

c. (−3, 8) and (−3, 4)

d. (2, −3) and (2, −1)

10. Describe the graphs of the following equations.

a. $y = -2$

b. $x = -2$

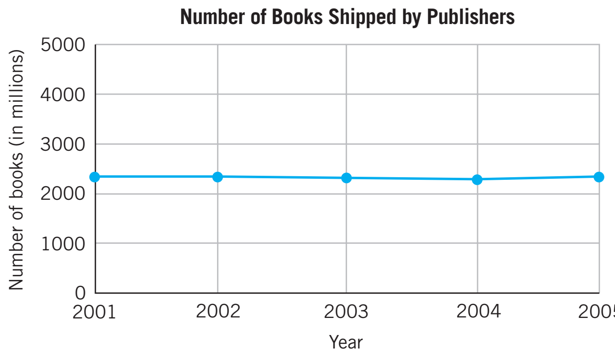
c. $x = \frac{2}{5}$

d. $y = \frac{x}{4}$

e. $y = 324$

f. $y = \frac{2}{3}$

11. The accompanying figure shows the quantity of books (in millions) shipped by publishers in the United States between 2001 and 2005. Construct the equation of a horizontal line that would be a reasonable model for these data.



Source: U.S. Bureau of the Census. *Statistical Abstract of the United States*, 2006.

12. An employee for an aeronautical corporation had a starting salary of \$25,000/year. After working there for 10 years and not receiving any raises, he decides to seek employment elsewhere. Graph the employee's salary as a function of time for the time he was employed with this corporation. What is the domain? What is the range?

13. For each of the given points write equations for three lines that all pass through the point such that one of the three lines is horizontal, one is vertical, and one has slope 2.

a. (1, −4) **b.** (2, 0)

14. Consider the function $f(x) = 4$.

a. What is $f(0)$? $f(30)$? $f(-12.6)$?

b. Describe the graph of this function.

c. Describe the slope of this function's graph.

15. A football player who weighs 175 pounds is instructed at the end of spring training that he has to put on 30 pounds before reporting for fall training.

a. If fall training begins 3 months later, at what (monthly) rate must he gain weight?

b. Suppose that he eats a lot and takes several nutritional supplements to gain weight, but due to his metabolism he still weighs 175 pounds throughout the summer and at the beginning of fall training. Sketch a graph of his weight versus time for those 3 months.

16. a. Write an equation for the line parallel to $y = 2 + 4x$ that passes through the point (3, 7).

b. Find an equation for the line perpendicular to $y = 2 + 4x$ that passes through the point (3, 7).

17. a. Write an equation for the line parallel to $y = 4 - x$ that passes through the point (3, 7).

b. Find an equation for the line perpendicular to $y = 4 - x$ that passes through the point (3, 7).

18. Construct the equation of a line that goes through the origin and is parallel to the graph of given equation.

a. $y = 6$ **b.** $x = -3$ **c.** $y = -x + 3$

19. Construct the equation of a line that goes through the origin and is perpendicular to the given equation.

a. $y = 6$ **b.** $x = -3$ **c.** $y = -x + 3$

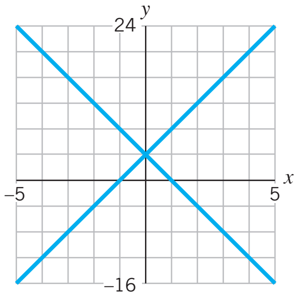
20. Which lines are parallel to each other? Which lines are perpendicular to each other?

a. $y = \frac{1}{3}x + 2$ **c.** $y = -2x + 10$ **e.** $2y + 4x = -12$

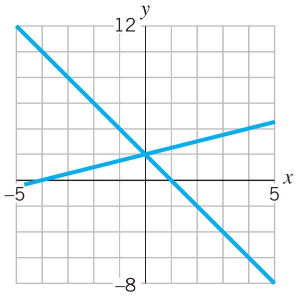
b. $y = 3x - 4$ **d.** $y = -3x - 2$ **f.** $y - 3x = 7$

21. Because different scales may be used on the horizontal and vertical axes, it is often difficult to tell if two lines are perpendicular to each other. In parts (a) and (b), determine the equations of each pair of lines and show whether or not the paired lines are perpendicular to each other.

a.



b.



22. In each part construct the equations of two lines that:

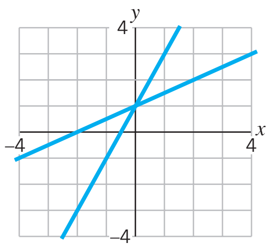
a. Are parallel to each other

b. Intersect at the same point on the y-axis

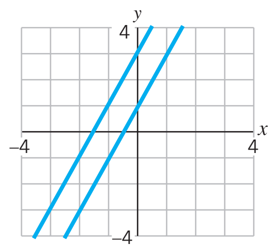
c. Both go through the origin

d. Are perpendicular to each other

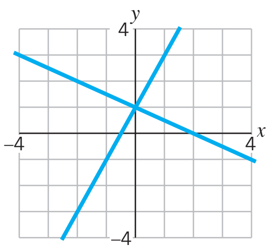
23. For each of the accompanying graphs you don't need to do any calculations or determine the actual equations. Rather, using just the graphs, determine if the slopes for each pair of lines are the same. Are the slopes both positive or both negative, or is one negative and one positive? Do the lines have the same y-intercept?



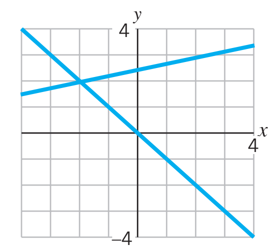
Graph *A*



Graph *C*



Graph *B*



Graph *D*

24. Find the equation of the line in the form $y = mx + b$ for each of the following sets of conditions. Show your work.

a. Slope is $1400/year and line passes through the point (10 yr, $12,000).

b. Line is parallel to $2y - 7x = y + 4$ and passes through the point $(-1, 2)$.

c. Equation is $1.48x - 2.00y + 4.36 = 0$.

d. Line is horizontal and passes through (1.0, 7.2).

e. Line is vertical and passes through (275, 1029).

f. Line is perpendicular to $y = -2x + 7$ and passes through (5, 2).

25. In the equation $Ax + By = C$:

a. Solve for y so as to rewrite the equation in the form $y = mx + b$.

b. Identify the slope.

c. What is the slope of any line parallel to $Ax + By = C$?

d. What is the slope of any line perpendicular to $Ax + By = C$?

26. Use the results of Exercise 25, parts (c) and (d), to find the slope of any line that is parallel and then one that is perpendicular to the given lines.

a. $5x + 8y = 37$ **b.** $7x + 16y = -14$ **c.** $30x + 47y = 0$

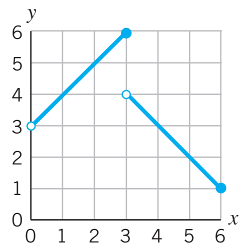
27. Construct the graphs of the following piecewise linear functions. Be sure to indicate whether an endpoint is included in or excluded from the graph.

a. $f(x) = \begin{cases} 2 & \text{for } -3 < x \le 0 \\ 1 & \text{for } \ \ 0 < x \le 3 \end{cases}$

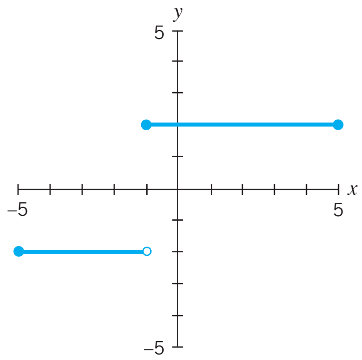
b. $g(x) = \begin{cases} x + 3 & \text{for } -4 < x \le 0 \\ 2 - x & \text{for } \ \ 0 < x \le 4 \end{cases}$

28. Construct piecewise linear functions for the following graphs.

a. Graph of $f(x)$



b. Graph of $g(x)$

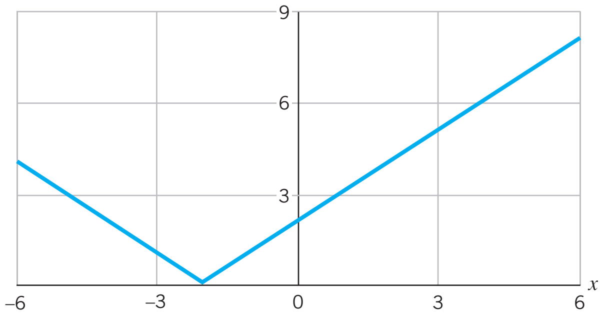


29. Given the following graph of $g(x)$:

a. Construct a piecewise linear description of $g(x)$.

b. Construct another description using absolute values.

c. Describe the relationship between this graph and the graph of $f(x) = |x|$.



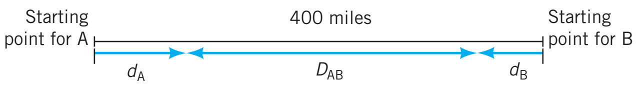
30. a. Normal human body temperature is often cited as 98.6°F. However, any temperature that is within 1°F more or less than that is still considered normal. Construct an absolute value inequality that describes normal body temperatures T that lie within that range. Then rewrite the expression without the absolute value sign.

b. The speed limit is set at 65 mph on a highway, but police do not normally ticket you if you go less than 5 miles above or below that limit. Construct an absolute value inequality that describes the speeds S at which you can safely travel without getting a ticket. Rewrite the expression without using the absolute value sign.

31. Assume two individuals, A and B, are traveling by car and initially are 400 miles apart. They travel toward each other, pass and then continue on.

a. If A is traveling at 60 miles per hour, and B is traveling at 40 mph, write two functions, $d_A(t)$ and $d_B(t)$, that describe the distance (in miles) that A and B each has traveled over time t (in hours).

b. Now construct a function for the distance $D_{AB}(t)$ *between* A and B at time t (in hours). Graph the function for $0 \le t \le 8$ hours.



c. At what time will A and B cross paths? At that point, how many miles has each traveled?

d. What is the distance between them one hour before they meet? An hour after they meet? Interpret both values in context. (*Hint:* If they are traveling toward each other, the distance between them is considered positive. Once they have met and are traveling away from each other, the distance between them is considered negative.)

e. Now construct an absolute value function that describes the (positive) distance between A and B at any point, and graph your result for $0 \le t \le 8$ hours.

32. The greatest integer function $y = [x]$ is defined as the greatest integer $\le x$ (i.e., it rounds x down to the nearest integer below it).

a. What is [2] ? [2.5] ? [2.9999999] ?

b. Sketch a graph of the greatest integer function for $0 \le x < 5$. Be sure to indicate whether each endpoint is included or excluded.

[*Note:* A bank employee embezzled hundreds of thousands of dollars by inserting software to round down transactions (such as generating interest on an account) to the nearest cent, and siphoning the round-off differences into his account. He was eventually caught.]

33. The following table shows U.S. first-class stamp prices (per ounce) over time.

Year	Price for First-Class Stamp
2001	34 cents
2002	37 cents
2006	39 cents

a. Construct a step function describing stamp prices for 2001–2006.

b. Graph the function. Be sure to specify whether each of the endpoints is included or excluded.

c. In 2007 the price of a first-class stamp was raised to 41 cents. How would the function domain and the graph change?

34. Sketch a graph for each of the following situations.

a. The amount in your savings account over a month, where you direct-deposit your paycheck each week and make one withdrawal during the month.

b. The amount of money in an ATM machine over one day, where the ATM is stocked with dollars at the beginning of the day, and then ATM withdrawals of various sizes are made.

2.9 *Constructing Linear Models of Data*

According to Edward Tufte in *Data Analysis of Politics and Policy,* "Fitting lines to relationships is the major tool of data analysis." Of course, when we work with actual data searching for an underlying linear relationship, the data points will rarely fall exactly in a straight line. However, we can model the trends in the data with a linear equation.

Linear relationships are of particular importance not because most relationships are linear, but because straight lines are easily drawn and analyzed. A human can fit a straight line by eye to a scatter plot almost as well as a computer. This paramount convenience of linear equations as well as their relative ease of manipulation and interpretation means that lines are often used as first approximations to patterns in data.

Fitting a Line to Data: The Kalama Study

DATA KALAMA

Children's heights were measured monthly over several years as a part of a study of nutrition in developing countries. Table 2.24 and Figure 2.57 show data collected on the mean heights of 161 children in Kalama, Egypt.

Mean Heights of Kalama Children

Age (months)	Height (cm)
18	76.1
19	77.0
20	78.1
21	78.2
22	78.8
23	79.7
24	79.9
25	81.1
26	81.2
27	81.8
28	82.8
29	83.5

Table 2.24
Source: D. S. Moore and G. P. McCabe, *Introduction to the Practice of Statistics.* Copyright © 1989 by W. H. Freeman and Company. Used with permission.

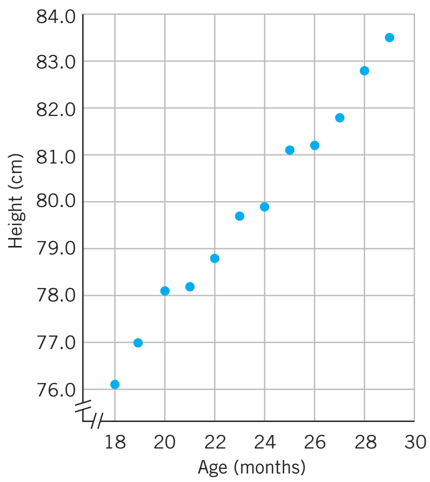


Figure 2.57 Mean heights of children in Kalama, Egypt.

Sketching a line through the data

Although the data points do not lie exactly on a straight line, the overall pattern seems clearly linear. Rather than generating a line through two of the data points, try eyeballing a line that approximates all the points. A ruler or a piece of black thread laid down through the dots will give you a pretty accurate fit.

In the Extended Exploration on education and earnings following this chapter, we will use technology to find a "best-fit" line. Figure 2.58 on the next page shows a line sketched that approximates the data points. This line does not necessarily pass through any of the original points.

Finding the slope

Estimating the coordinates of two points *on the line,* say (20, 77.5) and (26, 81.5), we can calculate the slope, m, or rate of change, as

$$\begin{aligned} m &= \frac{(81.5 - 77.5)\text{ cm}}{(26 - 20)\text{ months}} \\ &= \frac{4.0\text{ cm}}{6\text{ months}} \\ &\approx 0.67\text{ cm/month} \end{aligned}$$

So our model predicts that for each additional month an "average" Kalama child will grow about 0.67 centimeter.

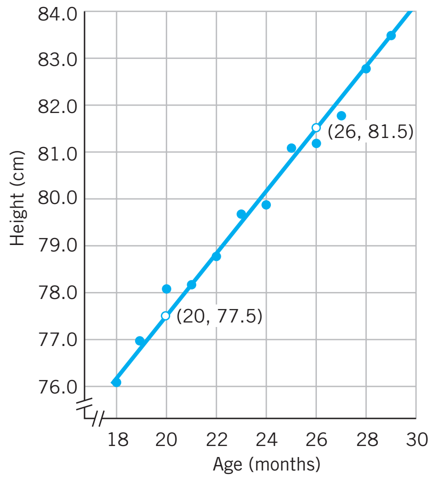


Figure 2.58 Estimated coordinates of two points on the line.

Constructing the equation

Since the slope of our linear model is 0.67 cm/month, then our equation is of the form

$$H = b + 0.67A \qquad (1)$$

where A = age in months and H = mean height in centimeters.

How can we find b, the vertical intercept? We have to resist the temptation to estimate b directly from the graph. As is frequently the case in social science graphs, both the horizontal and the vertical axes are cropped. Because the horizontal axis is cropped, we can't read the vertical intercept off the graph. We'll have to calculate it.

Since the line passes through (20, 77.5) we can

substitute (20, 77.5) in Equation (1)	$77.5 = b + (0.67)(20)$
simplify	$77.5 = b + 13.4$
solve for b	$b = 64.1$

Having found b, we complete the linear model:

$$H = 64.1 + 0.67A$$

where A = age in months and H = height in centimeters. It offers a compact summary of the data.

What is the domain of this model? In other words, for what inputs does our model apply? The data were collected on children age 18 to 29 months. We don't know its predictive value outside these ages, so

the domain consists of all values of A for which $18 \le A \le 29$

The vertical intercept may not be in the domain

Although the H-intercept is necessary to write the equation for the line, it lies outside of the domain.

Compare Figure 2.58 with Figure 2.59. They show graphs of the same equation, $H = 64.1 + 0.67A$. In Figure 2.58 both axes are cropped, while Figure 2.59 includes the origin (0, 0). In Figure 2.59 the vertical intercept is visible, and the shaded area between the dotted lines indicates the region that applies to our model. So a word of warning when reading graphs: Always look carefully to see if the axes have been cropped.

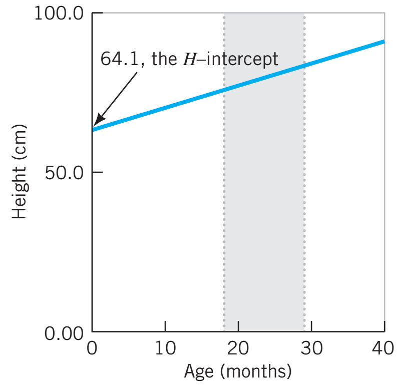


Figure 2.59 Graph of $H = 64.1 + 0.67A$ that includes the origin (0, 0). Shaded area shows the region that models the Kalama data.

Reinitializing the Independent Variable

When we model real data, it often makes sense to reinitialize the independent variable in order to have a reasonable vertical intercept. This is especially true for time series, as shown in the following example, where the independent variable is the year.

EXAMPLE 1 **Time series**

How can we find an equation that models the trend in smoking in the United States?

SOLUTION The American Lung Association website provided the data reproduced in Table 2.25 and graphed in Figure 2.60. Although in some states smoking has increased, the overall trend is a steady decline in the percentage of adult smokers in the United States between 1965 and 2005.

Year	Percentage of Adults Who Smoke
1965	41.9
1974	37.0
1979	33.3
1983	31.9
1985	29.9
1990	25.3
1995	24.6
2000	23.1
2005	20.9

Table 2.25

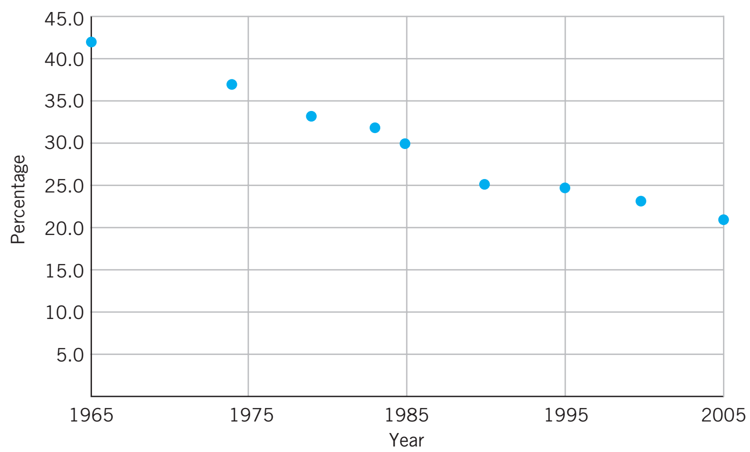


Figure 2.60 Percentage of adults who smoke.

The relationship appears reasonably linear. So the equation of a best-fit line could provide a fairly accurate description of the data. Since the horizontal axis is cropped, starting at the year 1965, the real vertical intercept would occur 1965 units to the left, at 0 A.D.! If you drew a big enough graph, you'd find that the vertical intercept would occur at approximately (0, 1220). This nonsensical extension of the model outside its known values would say that in 0 A.D., 1220% of the adult population smoked. A better strategy would be to define the independent variable as the number of years *since* 1965. Table 2.26 shows the reinitialized values for the independent variable, and Figure 2.61 gives a sketched-in best-fit line.

Year	No. of Years Since 1965	% of Adults Who Smoke
1965	0	41.9
1974	9	37.0
1979	14	33.3
1983	18	31.9
1985	20	29.9
1990	25	25.3
1995	30	24.6
2000	35	23.1
2005	40	20.9

Table 2.26

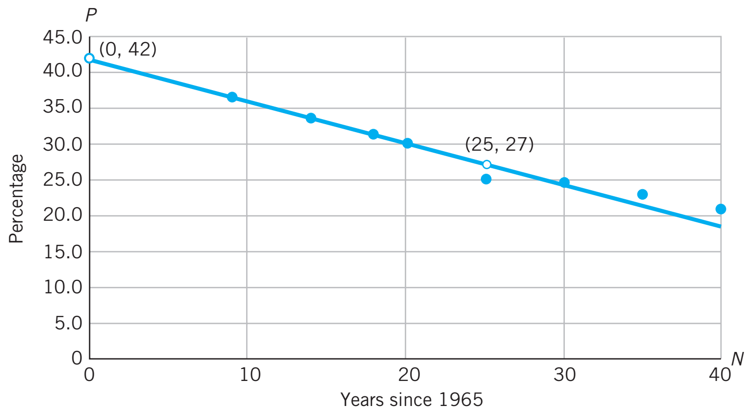


Figure 2.61 Percentage of adult smokers since 1965 with estimated best-fit line.

We can estimate the coordinates of two points, (0, 42) and (25, 27), on our best-fit line. Using them, we have

$$\text{vertical intercept} = 42 \quad \text{and} \quad \text{slope} = \frac{42 - 27}{0 - 25} = \frac{15}{-25} = -0.6$$

If we let N = the number of years since 1965 and P = percentage of adult smokers, then the equation for our best-fit line is

$$P = 42 - 0.6N$$

where the domain is $0 \leq N \leq 35$ (see Figure 2.61). This model says that starting in 1965, when about 42% of U.S. adults smoked, the percentage of the adult smokers has declined on average by 0.6 percentage points a year for 35 years.

What this model doesn't tell us is that (according to the U.S. Bureau of the Census) the total number of smokers during this time has remained fairly constant, at 50 million.

Interpolation and Extrapolation: Making Predictions

We can use this linear model on smokers to make predictions. We can *interpolate* or estimate new values between known ones. For example, in our smoking example we have no data for the year 1970. Using our equation we can estimate that in 1970 (when $N = 5$), $P = 42 - (0.6 \cdot 5) = 39\%$ of adults smoked. Like any other point on the best-fit line, this prediction is only an estimate and may, of course, be different from the actual percentage of smokers (see Figure 2.62).

We can also use our model to *extrapolate* or to predict beyond known values. For example, our model predicts that in 2010 (when $N = 45$), $P = 42 - (0.6 \cdot 45) = 15\%$ of adults will smoke. Extrapolation much beyond known values is risky. For 2035 (where $N = 70$) our model predicts that 0% will smoke, which seems unlikely. After 2035 our model would give the impossible answer that a negative percentage of adults will smoke.

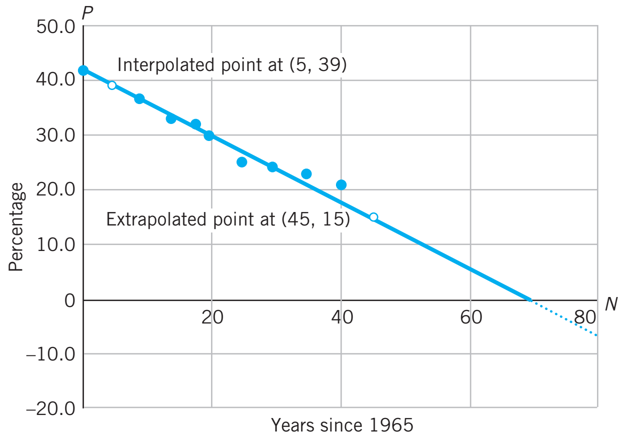


Figure 2.62 Interpolation and extrapolation of percentage of smokers.

Algebra Aerobics 2.9

1. Figure 2.63 shows the total number of U.S. college graduates (age 25 or older) between 1960 and 2005.

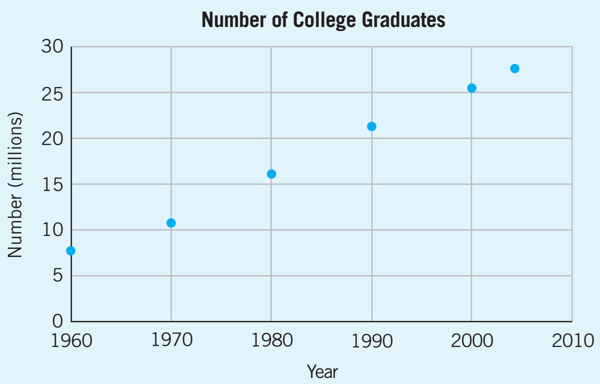


Figure 2.63 Total number of U.S. college graduates over time.

Source: U.S. National Center for Education Statistics, *Digest of Education Statistics,* annual.

a. Estimate from the scatter plot the number of college graduates in 1960 and in 2005.

b. Since the data look fairly linear, sketch a line that would model the growth in U.S. college graduates.

c. Estimate the coordinates of two points *on your line* and use them to calculate the slope.

d. If x = number of years *since* 1960 and y = total number (in millions) of U.S. college graduates, what would the coordinates of your two points in part (c) be in terms of x and y?

e. Construct a linear equation using the x and y values defined in part (d).

f. What does your model tell you about the number of college graduates in the United States?

2. Figure 2.64 shows the percentage of adults who (according to the U.S. Census Bureau) had access to the Internet either at home or at work between 1997 and 2003.

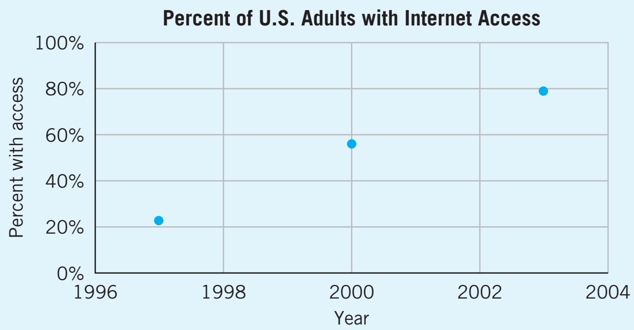


Figure 2.64 U.S. adults with Internet access.

a. Since the data appear roughly linear, sketch a best-fit line. (This line need not pass through any of the three data points.)

b. Reinitialize the years so that 1996 becomes year 0. Then identify the coordinates of any two points that lie on the line that you drew. Use these coordinates to find the slope of the line. What does this tell you about the percentage of adults with Internet access?

c. Give the approximate vertical intercept of the line that you drew (using the reinitialized value for the year).

d. Write an equation for your line.

e. Use your equation to estimate the number of adults with Internet access in 1998 and in 2002.

f. What would you expect to happen to the percentage after 2003? Do you think your linear model will be a good predictor after 2003?

Exercises for Section 2.9

"Extended Exploration: Looking for Links between Education and Earnings," which follows this chapter, has many additional exercises that involve finding best-fit lines using technology.

Graphing program recommended (or optional) for Exercises 3, 4, 13, and 15.

1. Match each equation with the appropriate table.

a. $y = 3x + 2$ **b.** $y = \frac{1}{2}x + 2$ **c.** $y = 1.5x + 2$

A.

x	y
0	2
2	3
4	4
6	5
8	6

B.

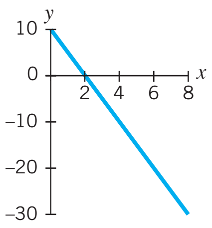
x	y
0	2
2	5
4	8
6	11
8	14

C.

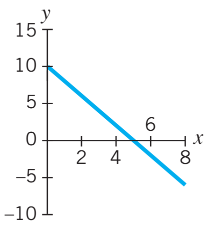
x	y
0	2
2	8
4	14
6	20
8	26

2. Match each of the equations with the appropriate graph.

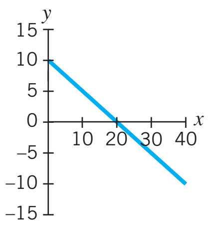
a. $y = 10 - 2x$ **b.** $y = 10 - 5x$ **c.** $y = 10 - 0.5x$



Graph *A*



Graph *B*



Graph *C*

3. (Graphing program recommended.) Identify which of the following data tables represent exact and which approximate linear relationships. For the one(s) that are exactly linear, construct the corresponding equation(s). For the one(s) that are approximately linear, generate the equation of a best-fit line; that is, plot the points, draw in a line approximating the data, pick two points on the line (not necessarily from your data) to generate the slope, and then construct the equation.

a.

x	−2	−1	0	1	2	3
y	−6.5	−5.0	−3.5	−2.0	−0.5	1.0

b.

t	−1	0	1	2	3	4
Q	8.5	6.5	3.0	1.2	−1.5	−2

c.

N	0	15	23	45	56	79
P	35	80	104	170	203	272

4. (Graphing program recommended.) Plot the data in each of the following data tables. Determine which data are exactly linear and which are approximately linear. For those that are approximately linear, sketch a line that looks like a best fit to the data. In each case generate the equation of a line that you think would best model the data.

a. The number of cocaine-related emergency room episodes.

Year	1994	1996	1998	2000
Cocaine-related emergency room episodes (in thousands)	143	152	172	174

Source: Centers for Disease Control and Prevention, National Center for Health-Related Statistics, 2005.

b. The amount of tax owed on a purchase price.

Price	\$2.00	\$5.00	\$10.00	\$12.00
Tax	\$0.12	\$0.30	\$0.60	\$0.72

c. The number of pounds in a given number of kilograms.

Kilograms	1	5	10	20
Pounds	2.2	11	22	44

Use the linear equations found in parts (a), (b), and (c) to approximate the values for:

d. The number of cocaine-related emergency room episodes predicted for 2004. How does your prediction compare with the actual value in 2004 of 131 thousand cases?

e. The amount of tax owed on \$7.79 and \$25.75 purchase prices.

f. The number of pounds in 15 kg and 150 kg.

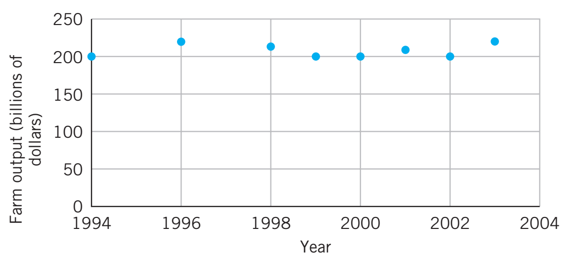
5. Determine which data represent exactly linear and which approximately linear relationships. For the approximately linear data, sketch a line that looks like a best fit to the data. In each case generate the equation of a line that you think would best model the data.

a. The number of solar energy units consumed (in quadrillions of British thermal units, called Btus)

Year	1998	1999	2000	2001
Solar consumption (in quadrillions of Btus)	0.07	0.07	0.07	0.07

Source: U.S. Bureau of the Census, *Statistical Abstract.*

b. Gross farm output (in billions of dollars)



Source: U.S. Census Bureau, *Statistical Abstract.*

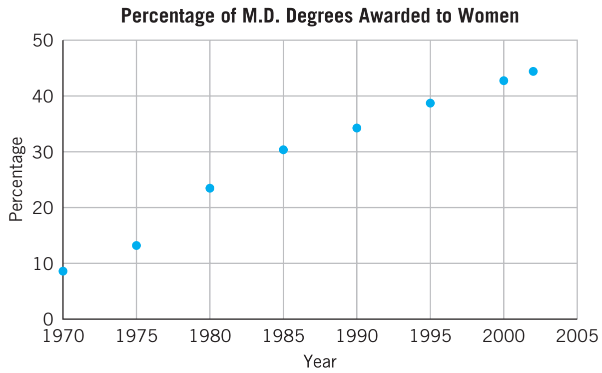
6. In 1995 the United States consumed 464 million gallons of wine and in 2004, about 667 million gallons.

a. Assuming the growth was linear, create a function that could model the trend in wine consumption.

b. Estimate the amount of wine consumed in 2005. The actual amount was about 703 million gallons worth of wine. How accurate was your approximation?

(*Source*: U.S. Department of Agriculture, Economic Research Service, 2007.)

7. The percentage of medical degrees awarded to women in the United States between 1970 and 2002 is shown in the accompanying graph.



Source: U.S. National Center for Education Statistics, "Digest of Education Statistics," annual; *Statistical Abstract of the United States.*

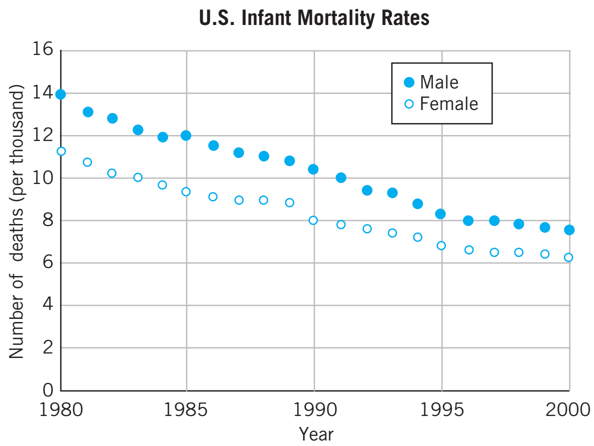
The data show that since 1970 the percentage of female doctors has been rising.

a. Sketch a line that best represents the data points. Use your line to estimate the rate of change of the percentage of M.D. degrees awarded to women.

b. If you extrapolate your line, estimate when 100% of doctors' degrees will be awarded to women.

c. It seems extremely unlikely that 100% of medical degrees will ever be granted to women. Comment on what is likely to happen to the rate of growth of women's degrees in medicine; sketch a likely graph for the continuation of the data into this century.

8. The accompanying graph shows the mortality rates (in deaths per 1000) for male and female infants in the United States from 1980 to 2000.



Source: Centers for Disease Control and Prevention, *www.cdc.gov.*

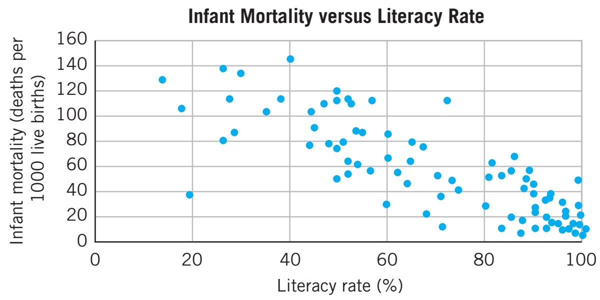
a. Sketch a line through the graph of the data that best represents female infant mortality rates. Does the line seem to be a reasonable model for the data? What is the approximate slope of the line through these points? Show your work.

b. Sketch a similar line through the male mortality rates. Is it a reasonable approximation? Estimate its slope. Show your work.

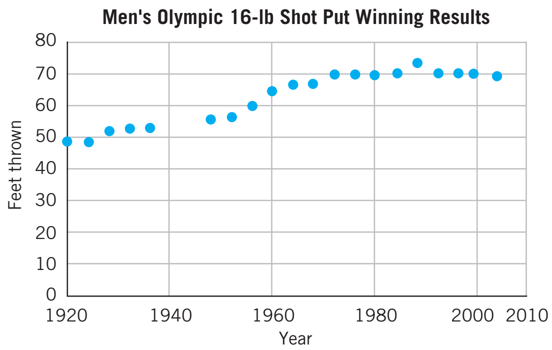
c. List at least two important conclusions from the graph.

9. The accompanying scatter plot shows the relationship between literacy rate (the percentage of the population who can read and write) and infant mortality rate (infant deaths per 1000 live births) for 91 countries. The raw data are contained in the Excel or graph link file NATIONS and are described at the end of the Excel file. (You might wish to identify the outlier, the country with about a 20% literacy rate and a low infant mortality rate of about 40 per 1000 live births.) Construct a linear model. Show all your work and clearly identify the variables and units. Interpret your results.

DATA



10. The accompanying graph shows data for the men's Olympic 16-pound shot put.



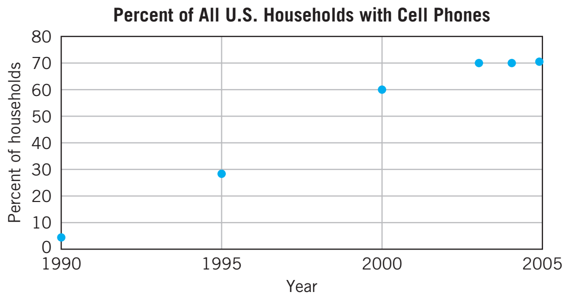
Source: 2006 World Almanac and Book of Facts.
Note: There were no Olympics in 1940 and 1944 due to World War II.

a. The shot put results are roughly linear between 1920 and 1972. Sketch a best-fit line for those years. Estimate the coordinates of two points on the line to calculate the slope. Interpret the slope in this context.

b. What is happening to the winning shot put results after 1972? Estimate the slope of the best-fit line for the years after 1972.

c. Letting x = years since 1920, construct a piecewise linear function $S(x)$ that describes the winning Olympic shot put results (in feet thrown) between 1920 and 2004.

11. The accompanying chart shows the percentage of U.S. households that have at least one cell phone.



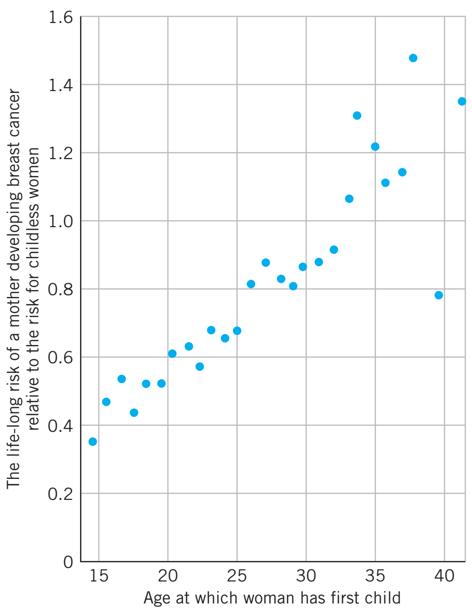
Source: 2006 World Almanac and Book of Facts Forrester Reports.

a. A linear model seems reasonable between 1990 and 2003. Sketch a best-fit line for those dates, then pick two points on your line and calculate the slope. Interpret the slope in this context.

b. What appears seems to be happening after 2003? What might be the explanation?

c. Letting x = years since 1990, construct a piecewise linear function $P(x)$ that would describe the percentage of households with cell phones between 1990 and 2005.

12. The accompanying graph shows the relationship between the age of a woman when she has her first child and her lifetime risk of getting breast cancer relative to a childless woman.



Source: J. Cairns, *Cancer: Science and Society* (San Francisco: W. H. Freeman, 1978), p. 49.

a. If a woman has her first child at age 18, approximately what is her risk of developing cancer relative to a woman who has never borne a child?

b. At roughly what age are the chances the same that a woman will develop breast cancer whether or not she has a child?

c. If a first-time mother is beyond the age you specified in part (b), is she more or less likely to develop breast cancer than a childless woman?

d. Sketch a line that looks like a best fit to the data, estimate the coordinates of two points on the line, and use them to calculate the slope.

e. Interpret the slope in this context.

f. Construct a linear model for these data, identifying your independent and dependent variables.

13. (Graphing program recommended.) The given data show that health care is becoming more expensive and is taking a bigger share of the U.S. gross domestic product (GDP). The GDP is the market value of all goods and services that have been bought for final use.

Year	1960	1970	1980	1990	2000	2003
U.S. health care costs as a percentage of GDP	5.1	7.1	8.9	12.2	13.3	15.3
Cost per person, $	141	341	1052	2691	4675	5671

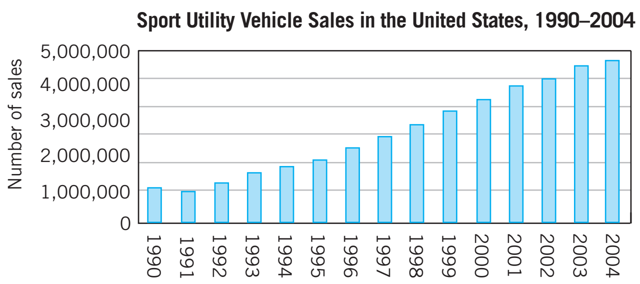
Source: U.S. Health Care Financing Administration and Centers for Medicare and Medicaid Services.

a. Graph health care costs as a percentage of GDP versus year, with time on the horizontal axis. Measure time in years since 1960. Draw a straight line by eye that appears to be the closest fit to the data. Figure out the slope of your line and create a function $H(t)$ for health care's percentage of the GDP as a function of t, years since 1960.

b. What does your formula predict for health care as a percentage of GDP for the year 2010?

c. Why do you think the health care cost per person has gone up so much more dramatically than the health care percentage of the GDP?

14. a. From the accompanying chart showing sport utility vehicle (SUV) sales, estimate what the rate of increase in sales has been from 1990 to 2004. (*Hint:* Convert the chart into an equivalent scatter plot.)



Source: Ward's Communications.

b. Estimate a linear formula to represent sales in years since 1990.

c. Do you think the popularity of SUVs will continue to grow at the same rate? Why or why not?

15. (Graphing program optional.) The Gas Guzzler Tax is imposed on manufacturers on the sale of new-model cars (*not* minivans, sport utility vehicles, or pickup trucks) whose fuel economy fails to meet certain statutory regulations, to discourage the production of fuel-inefficient vehicles. The tax is collected by the IRS and paid by the manufacturer. The table shows the amount of tax that the manufacturer must pay for a vehicle's miles per gallon fuel efficiency.

Gas Guzzler Tax

MPG	Tax per Car
12.5	$6400
13.5	$5400
14.5	$4500
15.5	$3700
16.5	$3700
17.5	$2600
18.5	$2100
19.5	$1700
20.5	$1300
21.5	$1000
22.5	$0

Source: http://www.epa.gov/otaq/cert/factshts/fefact 0.1.pdf.

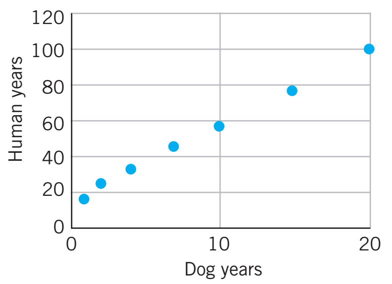
a. Plot the data, verify that they are roughly linear, and add a line of best fit.

b. Choose two points on the line, find the slope, and then form a linear equation with x as the fuel efficiency in mpg and y as the tax in dollars.

c. What is the rate of change of the amount of tax imposed on fuel-inefficient vehicles? Interpret the units.

16. A veterinarian's office displayed the following table comparing dog age (in dog years) to human age (in human years). The chart shows that the relationship is fairly linear.

Comparative Ages of Dogs and Humans



a. Draw a line that looks like a best fit to the data.

b. Estimate the coordinates and label two points on the line. Use them to find the slope. Interpret the slope in this context.

c. Using H for human age and D for dog age, identify which you are using as the independent and which you are using as the dependent variable.

d. Generate the equation of your line.

e. Use the linear model to determine the "human age" of a dog that is 17 dog years old.

f. Middle age in humans is 45–59 years. Use your model equation to find the corresponding middle age in dog years.

g. What is the domain for your model?

CHAPTER SUMMARY

The Average Rate of Change

The average rate of change of y with respect to x is

$$\frac{\text{change in } y}{\text{change in } x}$$

$$\text{The units of the average rate of change} = \frac{\text{units of } y}{\text{units of } x}$$

For example, the units might be dollars/year (read as "dollars per year") or pounds/person (read as "pounds per person").

The average rate of change between two points is the slope of the straight line connecting the points. Given two points (x_1, y_1) and (x_2, y_2),

$$\text{average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$$= \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \text{slope}$$

If the slope, or average rate of change, of y with respect to x is *positive,* then the graph of the relationship rises when read from left to right. This means that as x increases in value, y increases in value.

If the slope is *negative,* the graph falls when read from left to right. As x increases, y decreases.

If the slope is *zero,* the graph is flat. As x increases, there is no change in y.

Linear Functions

A *linear function* has a constant average rate of change. It can be described by an equation of the form

$$\underbrace{\text{output}}_{y} = \underbrace{\text{initial value}}_{b} + \underbrace{\text{rate of change}}_{m} \cdot \underbrace{\text{input}}_{x}$$

or

$$y = b + mx$$

where b is the vertical intercept and m is the slope, or rate of change of y with respect to x.

The Graph of a Linear Function

The graph of the linear function $y = b + mx$ is a straight line.

The y-intercept, b, tells us where the line crosses the y-axis.

The slope, m, tells us how fast the line is climbing or falling. The larger the magnitude (or absolute value) of m, the steeper the graph.

If the slope, m, is positive, then the line climbs from left to right. If m is negative, the line falls from left to right.

Special Cases of Linear Functions

Direct proportionality: y is *directly proportional to (or varies directly with)* x if

$$y = mx \quad \text{where the constant } m \neq 0$$

This equation represents a linear function in which the y-intercept is 0, so the graph passes through (0, 0), the origin.

Horizontal line: A line of the form $y = b$, with slope 0.

Vertical line: A line of the form $x = c$, with slope undefined.

Parallel lines: Two lines that have the same slope.

Perpendicular lines: Two lines whose slopes are negative reciprocals.

Piecewise linear function: A function constructed from different linear segments. Some examples are the *absolute value function,* $f(x) = |x|$, and *step functions,* whose linear segments are all horizontal.

Fitting Lines to Data

We can visually position a line to fit data whose graph exhibits a linear pattern. The equation of a best-fit line offers an approximate but compact description of the data. Lines are of special importance in describing patterns in data because they are easily drawn and manipulated and give a quick first approximation of trends.

CHECK YOUR UNDERSTANDING

I. Is each of the statements in Problems 1–30 true or false? Give an explanation for your answer.

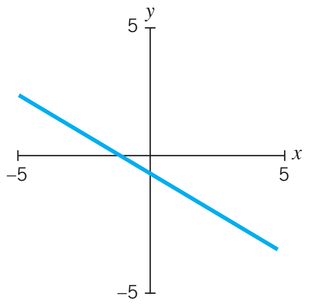
In Problems 1–4 assume that y is a function of x.

1. The graph of $y - 5x = 5$ is decreasing.

2. The graph of $2x + 3y = -12$ has a negative slope and negative vertical intercept.

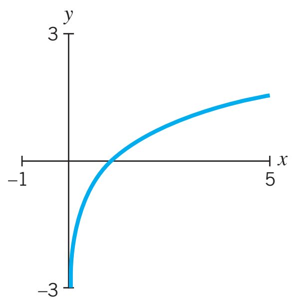
3. The graph of $x - 3y + 9 = 0$ is steeper than the graph of $3x - y + 9 = 0$.

4. The accompanying figure is the graph of $5x - 3y = -2$.



5. Health care costs between the years 1990 and 2007 would most likely show a positive average rate of change.

6. If we choose any two points on the graph in the accompanying figure, the average rate of change between them would be positive.



7. The average rate of change between two points (t_1, Q_1) and (t_2, Q_2) is the same as the slope of the line joining these two points.

8. The average rate of change of a variable M between the years 1990 and 2000 is the slope of the line joining two points of the form $(M_1, 1990)$ and $(M_2, 2000)$.

9. To calculate the average rate of change of a variable over an interval, you must have two distinct data points.

10. A set of data points of the form (x, y) that do not fall on a straight line will generate varying average rates of change depending on the choice of endpoints.

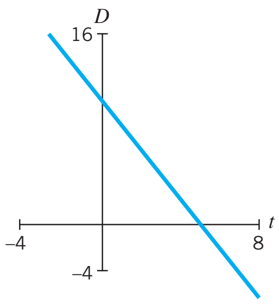
11. If the average rate of change of women's salaries from 2003 to 2007 is \$1000/year, then women's salaries increased by exactly \$1000 between 2003 and 2007.

12. If the average rate of change is positive, the acceleration (or rate of change of the average rate of change) may be positive, negative, or zero.

13. If the average rate of change is constant, then the acceleration (or rate of change of the average rate of change) is zero.

14. The average rate of change between (W_1, D_1) and (W_2, D_2) can be written as either $(W_1 - W_2)/(D_1 - D_2)$ or $(W_2 - W_1)/(D_2 - D_1)$.

15. If we choose any two distinct points on the line in the accompanying figure, the average rate of change between them would be the same negative number.



16. The average rate of change between (t_1, Q_1) and (t_2, Q_2) can be written as either $(Q_1 - Q_2)/(t_1 - t_2)$ or $(Q_2 - Q_1)/(t_2 - t_1)$.

17. On a linear graph, it does not matter which two distinct points on the line you use to calculate the slope.

18. If the distance a sprinter runs (measured in meters) is a function of the time (measured in minutes), then the units of the average rate of change are minutes per meter.

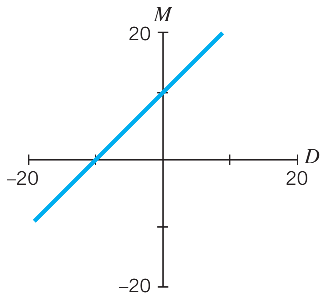
19. Every linear function crosses the horizontal axis exactly one time.

20. If a linear function in the form $y = mx + b$ has slope m, then increasing the x value by one unit changes the y value by m units.

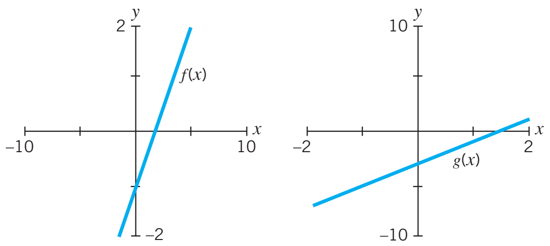
21. If the units of the dependent variable y are pounds and the units of the independent variable x are square feet, then the units of the slope are pounds per square foot.

22. If the average rate of change between any two data points is increasing as you move from left to right, then the function describing the data is linear and is increasing.

23. The function in the accompanying figure has a slope that is increasing as you move from left to right.



24. The slope of the function $f(x)$ is of greater magnitude than the slope of the function $g(x)$ in the accompanying figures.



25. You can calculate the slope of a line that goes through any two points on the plane.

26. If $f(x) = y$ is decreasing throughout, then the y values decrease as the x values decrease.

27. Having a slope of zero is the same as having an undefined slope.

28. Vertical lines are linear functions.

29. Two nonvertical lines that are perpendicular must have slopes of opposite sign.

30. All linear functions can be written in the form $y = b + mx$.

II. In Problems 31–40, give an example of a function or functions with the specified properties. Express your answer using equations, and specify the independent and dependent variables.

31. Linear and decreasing with positive vertical intercept

32. Linear and horizontal with vertical intercept 0

33. Linear and with positive horizontal intercept and negative vertical intercept

34. Linear and does not pass through the first quadrant (where $x > 0$ and $y > 0$)

35. Linear with average rate of change of 37 minutes/lap

36. Linear describing the value of a stock that is currently at \$19.25 per share and is increasing exactly \$0.25 per quarter

37. Two linear functions that are parallel, such that moving one of the functions horizontally to the right two units gives the graph of the other

38. Four distinct linear functions all passing through the point (0, 4)

39. Two linear functions where the slope of one is m and the slope of the second is $-1/m$, where m is a negative number

40. Five data points for which the average rates of change between consecutive points are positive and are increasing at a decreasing rate

III. Is each of the statements in Problems 41–53 true or false? If a statement is true, explain how you know. If a statement is false, give a counterexample.

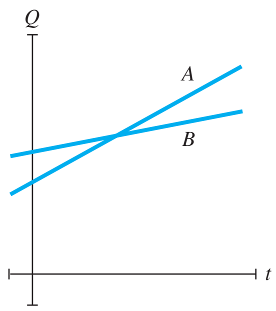
41. If the average rate of change between any two points of a data set is constant, then the data are linear.

42. If the slope of a linear function is negative, then the average rate of change decreases.

43. For any two distinct points, there is a linear function whose graph passes through them.

44. To write the equation of a specific linear function, one needs to know only the slope.

45. Function A in the accompanying figure is increasing at a faster rate than function B.



46. The graph of a linear function is a straight line.

47. Vertical lines are not linear functions because they cannot be written in the form $y = b + mx$, as they have undefined slope.

48. There exist linear functions that slant upward moving from left to right but have negative slope.

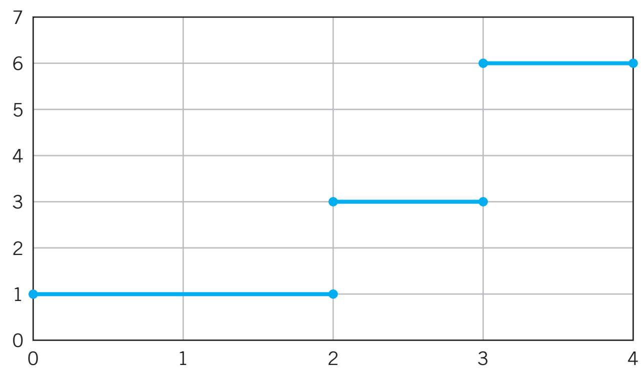
49. A constant average rate of change means that the slope of the graph of a function is zero.

50. All linear functions in x and y describe a relationship where y is directly proportional to x.

51. The function $h(t) = |t - 2|$ is always positive.

52. The function $h(t) = |t - 2|$ can be written as a piecewise linear function.

53. The graph to the right shows a step function.



CHAPTER 2 REVIEW: PUTTING IT ALL TOGETHER

1. According to the National Association of Realtors, the median price for a single-family home rose from \$147,300 in 2000 to \$217,300 in 2006. Describe the change in the following ways.

a. The absolute change in dollars

b. The percent increase

c. The annual average rate of change of the price with respect to year

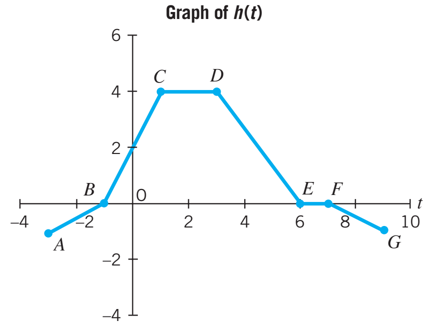
(*Note:* Be sure to include units in all your answers.)

2. a. According to Apple Computer, sales of its iPod (the world's best-selling digital audio player) soared from 304,000 in the third fiscal quarter of 2003 to 8,111,000 in the third fiscal quarter of 2006. What was the average rate of change in iPods sold per quarter? What was the average rate of change per month?

b. iPod sales reached an all-time high of 14,043,000 in Apple's first 2006 fiscal quarter (which included December 2005). What was the average rate of change in the number of iPods sold between the first and third fiscal quarters of 2006? What might be the reason for the decline?

3. Given the following graph of the function $h(t)$, identify any interval(s) over which:

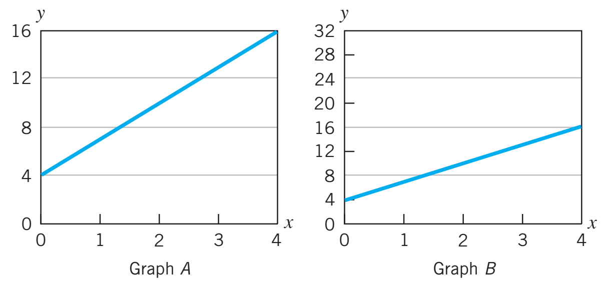
Graph of $h(t)$



a. The function is positive; is negative; is zero

b. The slope is positive; is negative; is zero

4. Which line has the steeper slope? Explain why.



5. You are traveling abroad and realize that American, British, and French clothing sizes are different. The accompanying table shows the correspondence among female clothing sizes in these different countries.

Women's Clothing Sizes

U.S.	Britain	France
4	10	38
6	12	40
8	14	42
10	16	44
12	18	46
14	20	48

a. Are the British and French sizes both linear functions of the U.S. sizes? Why or why not?

b. Write a sentence describing the relationship between British sizes and U.S. sizes.

c. Construct an equation that describes the relationship between French and U.S. sizes.

6. A bathtub that initially holds 50 gallons of water starts draining at 10 gallons per minute.

a. Construct a function $W(t)$ for the volume of water (in gallons) in the tub after t minutes.

b. Graph the function and label the axes. Evaluate $W(0)$ and $W(5)$ and describe what they represent in this context.

c. How would the original function and its graph change if the initial volume were 60 gallons? Call the new function $U(t)$.

d. How would the original function and graph change if the drain rate were 12 gallons per minute? Call the new function $V(t)$.

e. Add the graphs of $U(t)$ and $V(t)$ to the original graph of $W(t)$.

7. Cell phones have produced a seismic cultural shift. No other recent invention has incited so much praise—and criticism. In 2000 there were 109.4 million cell phone subscriptions in the United States; since then subscriptions steadily increased to reach 207.9 million in 2005. (*Note:* Some people had more than one subscription.) In 2005 about 66% of the U.S. population had a cell phone.

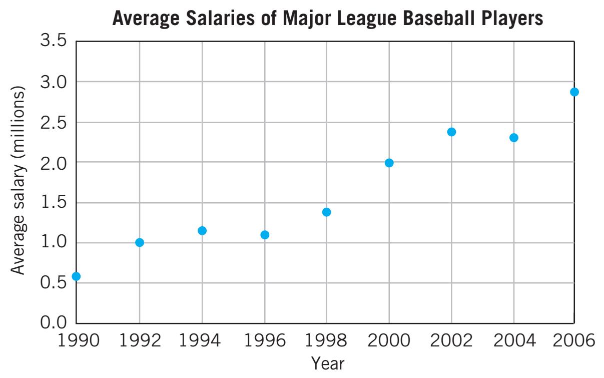
a. How many Americans were *without* a cell phone in 2005 (when the U.S. population was 296.4 million)?

b. What was the average rate of change in millions of cell phone subscriptions per year between 2000 and 2005?

c. Construct a linear function $C(t)$ for cell phone subscriptions (in millions) for t = years from 2000.

d. If U.S. cell phone subscriptions continue to increase at the same rate, how many will there be in 2010? Does your result sound plausible? (See Section 2.9, Exercise 11.)

8. The following graph shows average salaries for major league baseball players from 1990 to 2006.



a. On the graph, sketch a straight line that is an approximate mathematical model for the data.

b. Specify the coordinates of two points *on your line* and describe what they represent in terms of your model.

c. Calculate the slope and then interpret it in terms of the salary in millions of dollars and the time interval.

d. Let $S(x)$ be a linear approximation of the salaries where x = years since 1990. What is the slope of the graph of $S(x)$? The vertical intercept? What is the equation for $S(x)$?

9. Consider the following linear equations.

i. $3y + 2x - 15 = 0$

ii. $y + 4x = 1$

iii. $3x - y = 0$

iv. $8x - 5 = 2y$

Assuming x is graphed on the horizontal axis, which equation(s) have graphs that:

a. Are decreasing?

b. Have a positive vertical intercept?

c. Pass through the origin?

d. Is (are) the steepest?

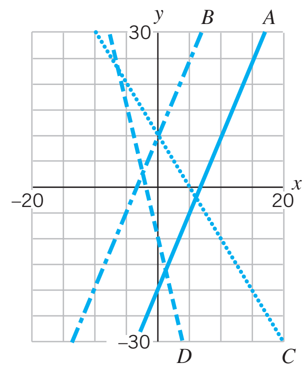
10. Match one or more of the following graphs with the following descriptions. Be sure to note the scale on each axis.

a. Represents a constant rate of change of 3

b. Has a vertical intercept of 10

c. Is parallel to the line $y = 5 - 2x$

d. Has a steeper slope than the line C



11. Gasoline prices spiked during the summer of 2006. The following table gives the weekly national average prices for regular gasoline as reported by the Department of Energy for three different weeks.

Regular Gas Prices in 2006

Week 1 (in January)	Week 31 (in August)	Week 43 (in October)
\$2.22/gal	\$3.04/gal	\$2.22/gal

a. Calculate the average rates of change for gasoline prices (in \$/gal/week) between weeks 1 and 31; between weeks 31 and 43; between weeks 1 and 43.

b. Write a 60-second summary about the price of gas during 2006.

(*Note:* Unfortunately in the summer of 2007, gas prices once again went over \$3.00 per gallon.)

12. You plan a trip to Toronto and find that on July 5, 2006, the exchange rate for Canadian money is CA\$1 = US\$0.899. Although this exchange rate favors the U.S. visitor to Canada, you are not delighted to discover that the Canadians have a 14% sales tax.

a. If a taxable item costs CA\$50, how much do you actually have to pay in Canadian dollars?

b. What is this worth in U.S. dollars?

c. Construct a function that describes the total cost C_{us} (in U.S.) of purchasing a taxable item with a price of P_{ca} (in Canadian dollars).

d. Is the cost in U.S. dollars, C_{us}, directly proportional to the Canadian price, P_{ca}?

13. Create a linear function for each condition listed below, using (at most) the numbers -2 and 3 and the variables Q and t. Assume t is the independent variable. The function's graph is:

a. Increasing

b. Decreasing

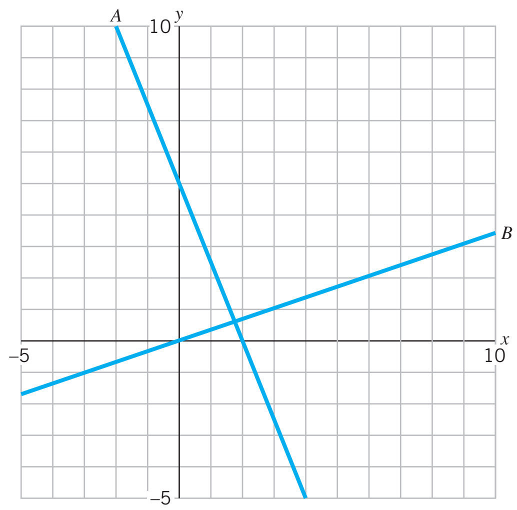
c. Horizontal

d. Steeper than that of $Q = 5 - t$

e. Parallel to $Q = 3t - 4$

f. Perpendicular to $Q = (1/2)t + 6$

14. Given the following graph:



a. Find the equations of line A and line B.

b. Show that line A is (or is not) perpendicular to line B.

15. According to the Environmental Protection Agency's "National Coastal Conditions Report II," more than half of the U.S. population live in the narrow coastal fringes. Increasing population in these areas contributes to degradation (by runoff, sewage spills, construction, and overfishing) of the same resources that make the coasts desirable. In 2003 about 153 million people lived in coastal counties. This coastal population is currently increasing by an average of 3600 people per day.

a. What is the average rate of change in millions of people per *year*?

b. Let x = years since 2003 and construct a linear function $P(x)$ for the coast population (in millions).

c. Graph the function $P(x)$ over a reasonable domain.

d. What does your model give for the projected population in 2008?

16. Generate the equation for a line under each of the following conditions.

a. The line goes through the points $(-1, -4)$ and $(4, 6)$.

b. The line is parallel to the line in part (a) and goes through the point $(0, 5)$.

c. The line is perpendicular to the line in part (a) and goes through the origin.

17. In 2005 a record 51.5% of paper consumed in the United States (51.3 million tons) was recycled. The American Forest and Paper Association states that its goal is to have 55% recovery by 2012.

a. Plot two data points that represent the percentage of paper recycled at t number of years since 2005. (You may want to crop the vertical axis to start at 50%.) Connect the two points with a line and calculate its slope.

b. Assuming linear growth, the line represents a model, $R(t)$, for the percentage of paper recycled at t years since 2005. Find the equation for $R(t)$.

c. Find $R(0)$, $R(5)$, and $R(20)$. What does $R(20)$ represent?

18. Sketch a graph that could represent each of the following situations. Be sure to label your axes.

a. The volume of water in a kettle being filled at a constant rate.

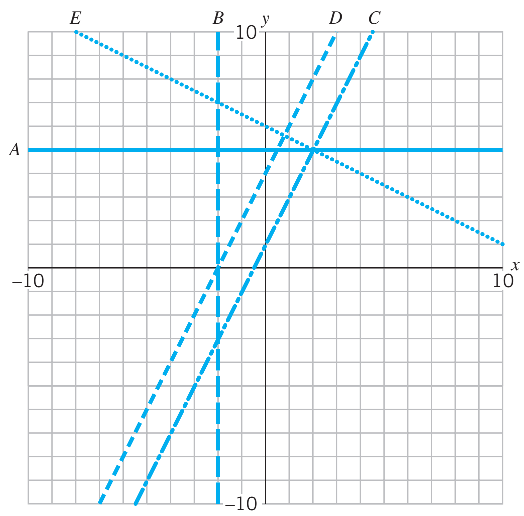
b. The height of an airplane flying at 30,000 feet for 3 hours.

c. The unlimited demand for a certain product, such that if the company produces 10,000 of them, they can sell them for virtually any price. (Use the convention of economists, placing price on the vertical axis and quantity on the horizontal.)

d. The price of subway tokens that periodically increase over time.

e. The distance from your destination over time when you travel three subway stops to get to your destination.

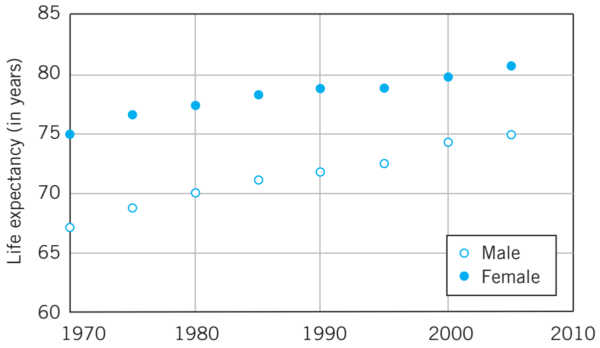
19. Generate an equation for each line A, B, C, D, and E shown in the accompanying graph.



20. U.S. life expectancy has been steadily increasing (see Exercise 15 in Section 2.1 and data file for LIFEXEC). The following graph shows the roughly linear trends for both males and females over more than three decades.

DATA

Trends in U.S. Life Expectancy by Sex, 1970–2005



Source: U.S. Bureau of the Census.

a. According to these data, what is the life expectancy of a female born in 1970? Of a male born in 2005?

b. Sketch a straight line to create a linear approximation of the data for your gender. Males and females will have different lines.

c. Identify two points *on your line* and calculate the slope. What does your slope mean in this context? The male slope is greater than the female slope. What does that imply?

d. Generate a linear function for your line. (*Hint:* Let the independent variable be the number of years since 1970.)

e. What would your model predict for your sex's life expectancy in 2010? How close does your model come to Census Bureau current predictions of life expectancies in 2010 of 75.6 years for males and 81.4 for females?

21. For the function $g(x) = |x| + 2$:

a. Generate a table of values for $g(x)$ using integer values of x between -3 and 3.

b. Use the table to sketch a graph of $g(x)$.

c. How does this graph compare with the graph of $f(x) = |x|$?

22. In Exercise 32 in Section 2.8 we met the greatest integer function $f(x) = [x]$, where $[x]$ is the greatest integer $\leq x$; (i.e., you round down to the nearest integer). So $[-1.5] = -2$ and $[3.99] = 3$. This function is sometimes written using the notation $\lfloor x \rfloor$ and called the *floor* function. There is a similar *ceiling* function, $g(x) = \lceil x \rceil$, where $\lceil x \rceil$ is the smallest integer $\geq x$; (i.e., here you round up to the nearest integer). So $\lceil -1.5 \rceil = -1$ and $\lceil 3.000001 \rceil = 4$.

a. Complete the following table.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
Floor $f(x) = \lfloor x \rfloor$									
Ceiling $g(x) = \lceil x \rceil$									

b. Plot the ceiling function and the floor function on two separate graphs, being sure to specify whether the endpoints of each line segment are included or excluded.

c. Why might a telephone company be interested in a ceiling function?

23. The table below gives historical data on voter turnout as a percentage of voting-age population (18 years and older) in U.S. presidential elections since 1960.

Year	Voting-Age Population	Turnout	%Turnout of Voting-Age Pop.
2004	221,256,931	122,294,978	55.3%
2000	205,815,000	105,586,274	51.3%
1996	196,511,000	96,456,345	49.1%
1992	189,529,000	104,405,155	55.1%
1988	182,778,000	91,594,693	50.1%
1984	174,466,000	92,652,680	53.1%
1980	164,597,000	86,515,210	52.6%
1976	152,309,190	81,555,789	53.5%
1972	140,776,000	77,718,554	55.2%
1968	120,328,186	73,211,875	60.8%
1964	114,090,000	70,644,592	61.9%
1960	109,159,000	68,838,204	63.1%

Using selected data from the table, graphs, and dramatically persuasive language, provide convincing arguments in a 60-second summary that during the years 1960 to 2004:

i. Voter turnout has plummeted.

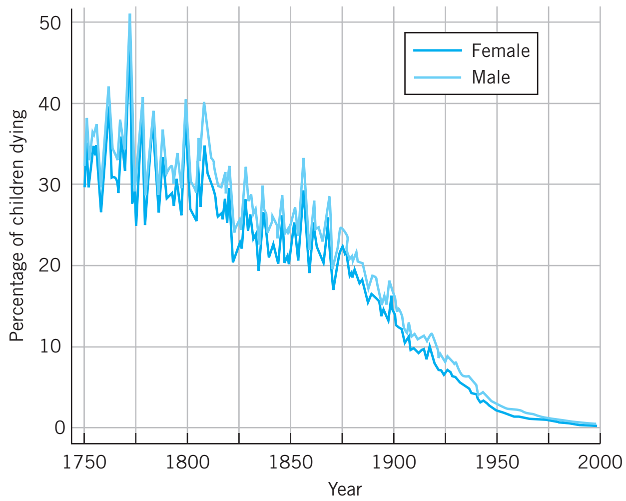
ii. Voter turnout has soared.

24. a. Construct the equation of a line y_1 whose graph is steeper than that of $8x + 2y = 6$ and does not pass through quadrant I (where both $x > 0$ and $y > 0$). Plot the lines for both equations on the same graph.

b. Construct another equation of a line y_2 that is perpendicular to the original equation, $8x + 2y = 6$, and *does* go through quadrant I. Add the plot of this line to the graph in part (a).

25. Sweden has kept meticulous records of its population for many years. The following graph shows the mortality rate (as a percentage) for female and male children in Sweden over a 250-year period.

250 Years of Mortality in Sweden for Children Between 0 and 5 Years



Source: Statistics Sweden.

a. Describe the overall trend, and the similarities and differences between the female and male graphs.

b. Assuming the graphs are roughly linear between 1800 and 1950, draw a line approximating the female childhood mortality rate during this time. What is the slope of the line and what does it tell you in this context?

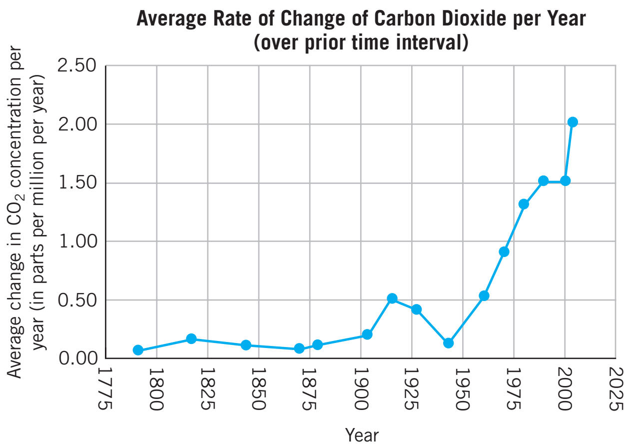
c. Construct a linear function to model the deaths over this period.

d. What is happening in the years after 1950?

Note: The large spike between 1750 and 1800 represents the more than 300,000 Swedish children (out of a total population of about 2 million) who died from smallpox, the most feared disease of that century.

26. According to the U.S. Environmental Protection Agency, carbon dioxide makes up 84.6% of the total U.S. greenhouse gas emissions. Carbon dioxide, or CO_2, arises from the combustion of coal, oil, and gas. Greenhouse gases trap heat on our planet, causing it to become warmer. Effects of this phenomenon are already being seen in increased melting of glaciers and permafrost. If this continues unchecked, water levels will rise, causing flooding in coastal communities, where the majority of the U.S. population lives. (See Exercise 15.)

The following graph shows the average rate of change in the concentration of atmospheric CO_2 over time.

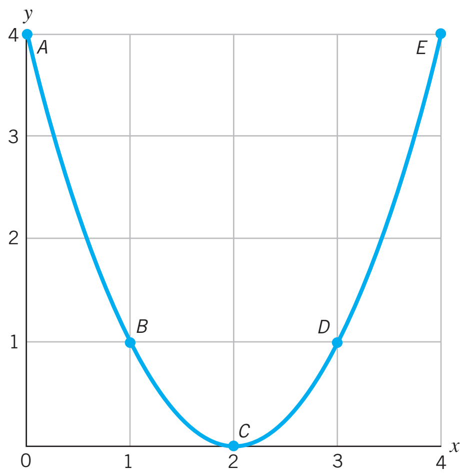


a. Is the average rate of change always positive? If so, does that imply that the amount of atmospheric carbon dioxide is always increasing? Explain your answer.

b. What does it mean when the average rate of change was decreasing (though not negative) between about 1915 and 1943? What was happening in the world during this period that might have caused carbon dioxide discharges to increase at a lower rate?

c. After 1943 the average rate of change steadily increases. What does that tell you about the increase of carbon dioxide?

27. Examine the following graph of a function $y = f(x)$.



a. Using the points *A*, *B*, *C*, *D*, and *E* labeled on the graph, fill in the rest of the accompanying table.

Point	x-Coordinate of Point	y-Coordinate of Point	Average Rate of Change Between Two Adjacent Points
A	0	4	n.a.
B	1	1	−3
C			
D			
E			

b. Over what x interval is the function increasing? What is happening to the average rate of change between the points in that interval?

c. Over what x interval is the function decreasing? What is happening to the absolute value of the average rate of change of the points in that interval? What does this mean? (Recall that the absolute value of a slope gives the steepness of the line.)

d. What is the concavity of the entire function? How do the changes in the steepness of the curve help confirm the concavity of the function graph?

EXPLORATION 2.1

Having It Your Way

Objective

- construct arguments supporting opposing points of view from the same data

Material/Equipment

- excerpts from the *Student Statistical Portrait* of the University of Massachusetts, Boston (in the Appendix) or from the equivalent for the student body at your institution
- computer with spreadsheet program and printer or graphing calculator with projection system (all optional)
- graph paper and/or overhead transparencies (and overhead projector)

Procedure

Working in Small Groups

Examine the data and graphs from the *Student Statistical Portrait* of the University of Massachusetts, Boston, or from your own institution. Explore how you would use the data to construct arguments that support at least two different points of view. Decide on the arguments you are going to make and divide up tasks among your team members.

Rules of the Game

- Your arguments needn't be lengthy, but you need to use graphs and numbers to support your position. You may use only legitimate numbers, but you are free to pick and choose those that support your case. If you construct your own graphs, you may, of course, use whatever scaling you wish on the axes.
- For any data that represent a time series, as part of your argument pick two appropriate endpoints and calculate the associated average rate of change.
- Use "loaded" vocabulary (e.g., "surged ahead," "declined drastically"). This is your chance to be outrageously biased, write absurdly flamboyant prose, and commit egregious sins of omission.
- Decide as a group how to present your results to the class. Some students enjoy realistic "role playing" in their presentations and have added creative touches such as mock protesters complete with picket signs.

Suggested Topics

Your instructor might ask your group to construct one or both sides of the arguments on one topic. If you're using data from your own institution, answer the questions provided by your instructor.

Using the *Student Statistical Portrait* from the University of Massachusetts, Boston (data located in the Appendix)

1. Use the table "Undergraduate Admissions Summary" to construct a persuasive case for each situation.

a. You are the Provost, the chief academic officer of the university, arguing in front of the Board of Trustees that the university is becoming more appealing to students.

b. You are a student activist arguing that the university is becoming less appealing to students.

2. Use the table "Trends in New Student Race/Ethnicity in the College of Liberal Arts" to make a convincing case for each of the following.

a. You are the Affirmative Action Officer arguing that her office has done a terrific job.

b. You are a reporter for the student newspaper criticizing the university for its neglect of minority students.

3. Use the table "SAT Scores of New Freshmen by College/Program" to "prove" each of the contradictory viewpoints.

a. You are the Dean of the College of Liberal Arts arguing that you have brighter freshmen than those in the College of Management.

b. You are the Dean of the College of Management arguing that your freshmen are superior.

Exploration-Linked Homework

With your partner or group prepare a short class presentation of your arguments, using, if possible, overhead transparencies or a projection panel. Then write individual 60-second summaries to hand in.

EXPLORATION 2.2A

Looking at Lines with the Course Software

Objective

- find patterns in the graphs of linear equations of the form $y = mx + b$

Equipment

- computer with course software "L1: *m* & *b* Sliders" in *Linear Functions*

Procedure

In each part try working first in pairs, comparing your observations and taking notes. Your instructor may then wish to bring the whole class back together to discuss everyone's results.

Part I: Exploring the Effect of m and b on the Graph of $y = mx + b$

Open the program *Linear Functions* and click on the button "L1: *m* & *b* Sliders."

1. What is the effect of *m* on the graph of the equation?

Fix a value for *b*. Construct four graphs with the same value for *b* but with different values for *m*. Continue to vary *m*, jotting down your observations about the effect on the line when *m* is positive, negative, or equal to zero. Do you think your conclusions work for values of *m* that are not on the slider?

Choose a new value for *b* and repeat your experiment. Are your observations still valid? Compare your observations with those of your partner.

2. What is the effect of *b* on the graph of the equation?

Fix a value for *m*. Construct four graphs with the same value for *m* but with different values for *b*. What is the effect on the graph of changing *b*? Record your observations. Would your conclusions still hold for values of *b* that are not on the slider?

Choose a new value for *m* and repeat your experiment. Are your observations still valid? Compare your observations with those of your partner.

3. Write a 60-second summary on the effect of *m* and *b* on the graph of $y = mx + b$.

Part II: Constructing Lines under Certain Constraints

1. Construct the following sets of lines still using "L1: *m* & *b* Sliders." Be sure to write down the equations for the lines you construct. What generalizations can you make about the lines in each case? Are the slopes of the lines related in some way? Are the vertical intercepts of the lines related?

Construct any line. Then construct another line that has a steeper slope, and then construct one that has a shallower slope.

Construct three *parallel lines.*

Construct three lines with the *same y-intercept*, the point where the line crosses the *y*-axis.

Construct a pair of lines that are *horizontal*.

Construct a pair of lines that go *through the origin*.

Construct a pair of lines that are *perpendicular* to each other.

2. Write a 60-second summary of what you have learned about the equations of lines.

EXPLORATION 2.2B

Looking at Lines with a Graphing Calculator

Objective

- find patterns in the graphs of linear equations of the form $y = mx + b$

Material/Equipment

- graphing calculator (instructions for the TI-83 and TI-84 families of calculators are available in the Graphing Calculator Manual)

Procedure

Getting Started

Set your calculator to the integer window setting. For the TI-83 or TI-84 do the following:

```
WINDOW FORMAT
Xmin=-47
Xmax=47
Xscl=10
Ymin=-31
Ymax=31
Yscl=10
Xres=1
```

1. Press ZOOM and select [6:ZStandard].
2. Press ZOOM and select [8:ZInteger], ENTER.
3. Press WINDOW to see whether the settings are the same as the duplicated screen image.

Working in Pairs

In each part try working first in pairs, comparing your observations and taking notes. Your instructor may then wish to bring the whole class back together to discuss everyone's results.

Part I: Exploring the Effect of m and b on the Graph of $y = mx + b$

1. What is the effect of m on the graph of the equation $y = mx$?

   a. Case 1: $m > 0$

   Enter the following functions into your calculator and then plot the graphs. To get started, try $m = 1, 2, 5$. Try a few other values of m where $m > 0$.

   $Y1 = x$
   $Y2 = 2x$
   $Y3 = 5x$
   $Y4 = \cdots$
   $Y5 =$
   $Y6 =$

   Compare your observations with those of your partner. In your notebook describe the effect of multiplying x by a positive value for m in the equation $y = mx$.

b. Case 2: $m < 0$

Begin by comparing the graphs of the lines when $m = 1$ and $m = -1$. Then experiment with other negative values for m and compare the graphs of the equations.

Y1 = x
Y2 = $-x$
Y3 = · · ·
Y4 =
Y5 =
Y6 =

Alter your description in part 1(a) to describe the effect of multiplying x by any real number m for $y = mx$ (remember to also explore what happens when $m = 0$).

2. What is the effect of b on the graph of an equation $y = mx + b$?

a. Enter the following into your calculator and then plot the graphs. To get started, try $m = 1$ and $b = 0, 20, -20$. Try other values for b as well.

Y1 = x
Y2 = $x + 20$
Y3 = $x - 20$
Y4 = · · ·
Y5 =
Y6 =

b. Discuss with your partner the effect of adding any number b to x for $y = x + b$. (*Hint:* Use "trace" to find where the graph crosses the y-axis.) Record your comments in your notebook.

c. Choose another value for m and repeat the exercise. Are your observations still valid?

3. Write a 60-second summary on the effect of m and b on the graph of $y = mx + b$.

Part II: Constructing Lines under Certain Constraints

1. Construct the following sets of lines using your graphing calculator. Be sure to write down the equations for the lines you construct. What generalizations can you make about the lines in each case? Are the slopes of the lines related in any way? Are the vertical intercepts related? Which graph is the steepest?

Construct three *parallel lines*.

Construct three lines with the *same y-intercept*.

Construct a pair of lines that are *horizontal.*

Construct a pair of lines that go *through the origin*.

Construct a pair of lines that are *perpendicular* to each other.

2. Write a 60-second summary of what you have learned about the equations of lines.