On the gravitational redshift

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Abstract

The purpose of this paper is twofold - to demonstrate that in the gravitational redshift it is the frequency a photon that is constant, and to describe the mechanism responsible for the change of its wavelength.

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It is usually assumed that both frequency and wavelength of a photon in the gravitational redshift change whereas its velocity remains constant. In this note we shall show that it is the frequency of a photon that does not change whereas its velocity and wavelength change. It will be also shown that it is the change of the coordinate velocity of the photon along its path that leads to a change in its wavelength.

Three things should be kept in mind when dealing with the gravitational redshift:

1. If two observers at different points A and B in a gravitational field determine the characteristics of a photon emitted from identical atoms placed at A and B, each observer will find that the photon characteristics - frequency, wavelength and local velocity - will have the same numerical values.

2. In a *parallel* gravitational field coordinate and proper distances coincide $dx = dx_A = dx_B$ [1] and therefore the wavelength of a photon *at a point* is the same for all observers - $\lambda_A = \lambda_B = \lambda$.

3. The local velocity of a photon *at a point* is different for different observers (it is c only for an observer at that point).

Consider a non-inertial frame N^g at rest in a *parallel* gravitational field of strength $\mathbf{g}$. If the z-axis is anti-parallel to the acceleration $\mathbf{g}$ the spacetime metric in N^g has the form [2]

$$ds^2 = \left(1 + \frac{2gz}{c^2}\right) c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

from where the coordinate velocity of light at a point z in a parallel gravitational field is immediately obtained (for $ds^2 = 0$)

$$c^g = c\left(1 + \frac{gz}{c^2}\right). \tag{2}$$

Notice that (1) is the standard spacetime interval in a *parallel* gravitational field [2], which does not coincide with the expression for the spacetime interval in a spherically symmetric gravitational field (i.e. the Schwarzschild metric expressed here in Cartesian coordinates) [3, p. 395]

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 + \frac{2GM}{c^2 r}\right)\left(dx^2 + dy^2 + dz^2\right). \tag{3}$$

The metric (1) can be written in a form similar to (3) if we choose $r = r_0 + z$ where r_0 is a constant

$$ds^2 = \left(1 - \frac{2GM}{c^2 (r_0 + z)}\right) c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right). \tag{4}$$

As $g = GM/r_0^2$ and for $z/\ r_0 < 1$ we can write

$$ds^2 = \left(1 - \frac{2GM}{c^2 r_0} + \frac{2gz}{c^2}\right) c^2 dt^2 - \left(dx^2 + dy^2 + dz^2\right). \quad (5)$$

As the gravitational potential is undetermined to within an additive constant we can choose $GM/r_0 = 0$ in (5); more precisely, when calculating the gravitational potential we can set the constant of integration to be equal to $-GM/r_0$. With this choice of the integration constant (5) coincides with (1). Although similar (4) and (3) have different values for g_{ii} $(i = 1, 2, 3)$: $g_{ii} = -1$ in (4), whereas $g_{ii} = -\left(1 + 2GM/c^2 r\right)$ in (3). This reflects the fact that in a parallel gravitational field proper and coordinate times do not coincide (except for the proper time of an observer at infinity) whereas proper and coordinate distances are the same [1].

Consider an atom stationary at a point B in a parallel gravitational field. The atom emits a photon - a B-photon - which is observed at a point A at a distance h above B. As seen at B the B-photon is emitted with a frequency $f_B = (d\tau_B)^{-1}$, where $d\tau_B$ is the proper period. As seen from A, however, the B-photon's period is $d\tau_A$ and therefore its frequency is $f_B^A = (d\tau_A)^{-1}$. Notice that if an identical atom at A emits a photon its frequency at A will be $f_A = (d\tau_A)^{-1} = f_B$, which means that the corresponding periods will be (numerically) equal: $d\tau_A = d\tau_B$. In the case of the redshift experiment, however, when a B-photon is measured at A, $d\tau_A$ and $d\tau_B$ are different - $d\tau_B$ is the proper period (measured at B) whereas $d\tau_A$ is the *observed* period as measured at A. $d\tau_A$ and $d\tau_B$ are the proper times at A and B that correspond to the *same* coordinate time, i.e. the same coordinate period dt:

$$d\tau_A = \left(1 + \frac{gz_A}{c^2}\right) dt$$

and

$$d\tau_B = \left(1 + \frac{gz_B}{c^2}\right) dt.$$

As $z_A = z_B + h$ it follows from (1) that the ratio between $d\tau_A$ and $d\tau_B$ is

$$\frac{d\tau_A}{d\tau_B} = \frac{\left(1 + gz_A/c^2\right)}{\left(1 + gz_B/c^2\right)} \approx 1 + \frac{gh}{c^2}.$$

Therefore, the *initial* frequency of the B-photon at B as seen from A will be

$$f_A = \frac{1}{d\tau_A} = \frac{1}{d\tau_B\left(1 + gh/c^2\right)} \approx f_B\left(1 - \frac{gh}{c^2}\right). \quad (6)$$

As seen from (6) for an observer at A the B-photon is emitted with a reduced initial frequency $f_A < f_B$. This demonstrates that the frequency of the B-photon does not change during its journey from A to B since its final frequency at A should be also (6).

The same expression for the initial frequency of the B-photon at B as seen from A can be obtained if one makes use of the fact that in a parallel gravitational field proper and coordinate distance coincide. This means that the initial wavelength λ_A of the B-photon at B as seen from A is equal to the initial wavelength λ_B as measured at B - $\lambda_A = \lambda_B = \lambda$. The initial velocity of the B-photon at B as seen from A can be easily calculated

$$c_A = \frac{dz_B}{d\tau_A} = \frac{dz_B}{dt}\frac{dt}{d\tau_A}$$

where and dz_B/dt is the coordinate velocity at point B

$$c' = c\left(1 + \frac{gz_B}{c^2}\right)$$

and

$$dt = \left(1 - \frac{gz_A}{c^2}\right) d\tau_A.$$

As $z_A = z_B + h$ we can write

$$c_A = c\left(1 - \frac{gh}{c^2}\right) \tag{7}$$

Hence, the frequency of the B-photon at B as seen from A is

$$f_A = \frac{c_A}{\lambda} = f_B\left(1 - \frac{gh}{c^2}\right)$$

where $f_B = c/\lambda$.

The fact that the B-photon's frequency does not change demonstrate that its energy is constant - an indication that the photon is not losing energy while moving against the gravitational field. Inversely, if an A-photon is observed at B its constant energy will indicate that it is not gaining energy and therefore is not falling in the gravitational field (if it were falling its average downward speed would be greater than its upward average speed which is not the case).

We have seen that it is the frequency that is constant - a conclusion also pointed out by Okun, Selivanov, and Telegdi [5]. What changes as the B-photon travels toward the observation point A , as seen from A, is its velocity and wavelength. The initial velocity of the B-photon at B , as seen from A, is given by (7); its final velocity at A , as seen from A, should be obviously c. The change of the photon's velocity on its way toward A also explains the mechanism responsible for the change of its wavelength. As seen from A any wavefront moving away from the gravitational field (toward A) acquires a greater velocity as compared to the velocity of the next wavefront that follows it. Due to the speeding up of the first wavefront the spacing between the two wavefronts increases for one period $d\tau_A$ (as seen by A) by a fraction $\delta\lambda = \delta c\ d\tau_A$ where

$$\delta c = c\left[1 + \frac{g\,(z + dz)}{c^2}\right] - c\left(1 + \frac{gz}{c^2}\right) = c\frac{gdz}{c^2}$$

is the change of the coordinate velocity over the distance dz.

Then the total increase of the wavelength from B to A is

$$\Delta\lambda = \int_0^h \delta c\ d\tau_A = c\frac{gd\tau_A}{c^2}\int_0^h dz = c\frac{gh}{c^2}d\tau_A.$$

As

$$d\tau_A = d\tau_B\left(1 + \frac{gh}{c^2}\right)$$

we can write for $\Delta\lambda$ by keeping only the terms proportional to c^{-2}

$$\Delta\lambda = c\frac{gh}{c^2}d\tau_B = \lambda\frac{gh}{c^2}$$

where $c\ d\tau_B = \lambda$ is the initial wavelength as determined at B. The final (measured) wavelength of the B-photon at A is then

$$\lambda_A = \lambda + \Delta\lambda = \lambda\left(1 + \frac{gh}{c^2}\right).$$

Therefore, in the gravitational redshift it is the velocity and wavelength of a photon that change whereas its frequency does not change.

References

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