Advanced Quantum Mechanics

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Chapter 1

Introduction and Review

The business of mechanics, both classical and quantum, is to predict the future, i.e. given a system with some set of initial conditions, describe its subsequent behavior. Quantum mechanics gives us a deceptively simple-looking solution to this problem; it's called Schrodingers' equation.

$$\psi(t) = e^{-i\hat{H}(t-t')}\psi(t') \tag{1.1}$$

The equation is not very useful because $\hat{H} = \hat{T} + \hat{V}$ is the sum of two non-commuting operators.[1] There is the trivial case in which the state is an eigenstate of the complete Hamiltonian. If

$$\hat{H}\psi_n(x,t) = E_n\psi(x,t) \tag{1.2}$$

then

$$\psi_n(t) = e^{-iE_n(t-t')}\psi_n(t') \tag{1.3}$$

These are called "stationary stattes" because there is no time dependence except for the unobservable phase factor. If everything were that simple there would be nothing to live for. The fact that such states exist, however, suggests a way of calculating time dependence. We simply expand the wave function in a series of energy eigenstates, each one of winch has trivial time dependence. The following calculation is done in every introductory quantum text. We define energy eigenstates (with no time dependence) as

$$\hat{H}\phi_n(x) = E_n\phi(x) \tag{1.4}$$

[1] I will use "hats" on symbols when I want to call attention to the fact that they are operators.

Now for some $t' < t$,

$$\psi(x,t) = \sum_n c_n \phi(x) e^{-iE_n t} \tag{1.5}$$

$$\int dx' \phi_m^*(x')\ \psi(x',t') = c_m e^{-iE_m t'} \tag{1.6}$$

I have assumed that

$$\int dx \phi_m^*(x)\phi_n(x) = \delta_{mn} \tag{1.7}$$

Now reinsert (6) into (7)

$$\psi(x,t) = \int dx' \left[\sum_n \phi_n^*(x')\phi_n(x) e^{-iE_n(t-t')}\right] \psi(x',t') \tag{1.8}$$

The quantity in brackets is called the *propagator*.

$$K(x,t;x',t') \equiv \sum_n \phi_n^*(x')\phi_n(x) e^{-iE_n(t-t')} \tag{1.9}$$

$$\psi(x,t) = \int dx' K(x,t;x',t')\psi(x',t') \tag{1.10}$$

Obviously, the propagator has the ability to "move the wave function along in time." Equation (9) is a complete in-principle solution to the problem of time development, but it's still not very useful. For one thing, there are only a few potintials for which we can find exact eigenfunctions, and these are usually exotic transcendental functions. The worst part of it is that there is no way to make systematic approximations. Nontheless, the notion of a propagator as defined implicitly by (10) is very powerful. Everything we will do in the next few chapters has to do ket in some way with finding propagators.

In order to keep the discussion on familiar ground (for a little while), I will redo the calculation above with a more abstract notation. This would be a good time to review the difference between the Schrodinger picture and the Heisenberg picture in elementary quantum mechanics. When we write an equation like,

$$|\psi(t)\rangle_S = e^{-iHt}|\psi(0)\rangle \tag{1.11}$$

we are implicitly working the Schrodinger picture. The state vectors evolve in time, whereas the operators do not. I have emphasized this by putting the subscript S on the ket on the left. (There is no subscript on the other ket

for reasons that will soon be clear.) Operators in the Schrodinger picture usually don't depend on time, but of course, their expectation values do.

$$\langle Q(t)\rangle = \langle \psi_S(t)|\hat{Q}_S|\psi_S(t)\rangle = \langle \psi(0)|e^{i\hat{H}t}\hat{Q}_S e^{-i\hat{H}t}|\psi(0)\rangle \tag{1.12}$$

Define

$$\hat{Q}_H(t) = e^{i\hat{H}t}\hat{Q}_S e^{-i\hat{H}t}, \tag{1.13}$$

so that

$$\langle Q(t)\rangle = \langle \psi(0)|\hat{Q}_H(t)|\psi(0)\rangle \tag{1.14}$$

All time dependence is now gathered into the operator. Since the two terms on the right of (12) are identical for $t = 0$, we make no distinction between Schrodinger states and Heisenberg states at this time. For this reason we often write $|\psi(0)\rangle = |\psi\rangle$. The question of exactly what time it was when $t = 0$ (last week, last year?) just doesn't come up.

If the Hamiltonian itself doesn't depend on time, then the Heisenberg operators satisfy their own version of Schrodinger's equation.

$$\frac{d\hat{Q}_H(t)}{dt} = i\left[\hat{H}, \hat{Q}_H(t)\right] \tag{1.15}$$

Now look at an eigenstate of $\hat{Q}$.

$$\hat{Q}_S|q\rangle_S = q|q\rangle_S \tag{1.16}$$

$$e^{i\hat{H}t}\hat{Q}_S e^{-i\hat{H}t}e^{i\hat{H}t}|q\rangle_S = qe^{i\hat{H}t}|q\rangle_S \tag{1.17}$$

Define

$$e^{i\hat{H}t}|q\rangle_S \equiv |q,t\rangle_H \tag{1.18}$$

so that

$$\hat{Q}_H|q,t\rangle_H = q|q,t\rangle \tag{1.19}$$

These are the "Heisenberg eigenstates." There is something paradoxical about them. We are used to saying the Heisenberg states do not change in time, but in fact, the eigenstates of Heisenberg operators transform "backwards" in time.

The Creed of Quantum Mechanics says, among other things, that to every observable quantity there corresponds a Hermitian operator. The eigenvalue spectrum of this operator corresponds to all possible results of measurements of the observable, and the eigenstates themselves constitute a complete set of states in terms of which any arbitrary state vector can be expanded. This is not something we can prove mathematically. We believe

in creeds, because life wouldn't make sense otherwise. Certainly the position of a particle is something that can be measured, so there must be a position operator with eigenstates.

$$\hat{X}|x\rangle = x|x\rangle \tag{1.20}$$

The eigenvalue spectrum is continuous, so we can take advantage of completeness by writing

$$\hat{I} = \int dx|x\rangle\langle x| \tag{1.21}$$

Students often find this troubling, and for good reason. The variable x, in addition to being an observable quantity, is also, along with time, one of the independent variables in Schrodinger's equation. And yet – we never think of time as an observable, and the notion of a "time eigenstate" is bizarre! The fact that position has this ambiguous dual role is one of the drawbacks of Schrodinger theory. The problem largely disappears in the Heisenberg picture, however, and for this and other reasons having to do with the role of space and time, we will use the Heisenberg picture almost exclusively in what follows.

Position eigenstates can be thought of as a notational device for switching back and forth between the elementary formulation of quantum mechanics in terms of wave functions and the more abstract formulation in terms of bras and kets. For example

$$\psi(x,t) = \langle x,t|\psi\rangle_H = |x\rangle\psi(t)\rangle_S \tag{1.22}$$

Notice that wave functions are really scalar products and as such do not change between the Heisenberg and Schrodinger pictures. We can reverse the procedure as follows:

$$|\psi(t)\rangle_S = \int dx\psi(x,t)|x\rangle_S \tag{1.23}$$

Now we can get back to the business of calculating propagators. Define energy eigenstates as

$$\hat{H}|n\rangle = E_n|n\rangle \tag{1.24}$$

The symbol n stands for all the discrete quantum numbers required to specify the eigenstate. We'll assume that the states are complete and normalized so that

$$\hat{I} = \sum_n |n\rangle\langle n| \tag{1.25}$$

$$e^{-iH(t-t')} = e^{-iH(t-t')}\sum_n |n\rangle\langle n| = \sum e^{-iE_n(t-t')}|n\rangle\langle n| \tag{1.26}$$

This is sometimes called the "spectral representation" of the operator $e^{-i\hat{H}(t-t')}$. This gives us an alternative derivation of (8).

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|e^{-iH(t-t')}|\psi(t')\rangle = \sum_n e^{-iE_n(t-t')}\langle x|n\rangle\langle n|\psi(t')\rangle$$

$$= \int dx' \sum_n e^{-iE_n(t-t')}\langle x|n\rangle\langle n|x'\rangle\psi(x',t') \tag{1.27}$$

This is identical to (8) if we remember that $\langle x|n\rangle = \phi_n(x)$.

In order to take the next step toward the path integral formulation of quantum mechanics, it will be necessary to calculate the propagator without using energy eigenstates. So one last time $\cdots$

$$\langle x|\psi(t)\rangle = \int dx' \langle x|e^{-iH(t-t')}|x'\rangle\langle x'|\psi(t')\rangle \tag{1.28}$$

Compare this with (10). We see the propagator lurking inside the integral.

$$K(x,t;x't') = \langle x|e^{-iH(t-t')}|x'\rangle = \left(\langle x|e^{-iHt}\rangle\right)\left(e^{iHt'}|x'\rangle\right) = \langle x,t|x',t'\rangle \tag{1.29}$$

This is the starting point for the path integral formulation in the next section. Before going on, we need one all-important result.

$$\hat{I} = \int dx|x\rangle\langle x| = \int dx\; e^{iHt}|x\rangle\langle x|e^{-iHt} = \int dx\langle x,t|x,t\rangle \tag{1.30}$$

Insert this in (29).

$$K(x,t;x',t') = \int dx''\langle x,t|x'',t''\rangle\langle x'',t''|x',t'\rangle = \int dx'' K(x,t;x'',t'')K(x'',t'';x',t') \tag{1.31}$$

This is the "composition property" of propagators. It implies that we can always insert a complete set of Heisenberg position eigenstates using (30). Physically it means that the probability for a particle to propagate from (x',t') to (x,t) is equal to the probability of propagating from (x',t') to (x'',t'') time the probability of propagating from (x'',t'') to (x,t) integrated over all x''.

Chapter 2

The Path Integral Formulation

2.1 Introduction

In the usual formulation of quantum mechanics, one tries to calculate the time dependence of the wave functions directly using Schrodinger's equation. The path integral formulation seeks to calculate the propagator $K(x_f, t_f; x_\circ, t_\circ)$. The procedure follows:

1. Draw all *causal* paths in the x-t plane connecting $(x_\circ, t_\circ)$ with (x_f, t_f). By "causal" I mean that the paths must not loop back in time. There are no other restrictions. The paths can be wildly unphysical.

2. Find the classical action $S[x(t)]$ for each path $x(t)$.

3. Perform the following sum.

$$K(x_f, t_f; x_\circ, t_\circ) = A \sum_{\text{paths}} e^{iS[x(t)]/\hbar} \tag{2.1}$$

The constant A is a normalization factor, more about this later. I have put in the $\hbar$ to illustrate the classical limit. The real question is how to do the sum over paths. This will require learning some new math, functional integration. First let's do a brief review of classical mechanics.

The classical equations of motion can be derived from Hamilton's principle. We start with the Lagrangian $L = T - V$, the kinetic energy minus

the potential energy. Calculate the action.

$$S[q,\dot{q}] = \int_{t_\circ}^{t_f} L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n) dt \tag{2.2}$$

The q_i's are the generalized coordinates. They are functions of time, and these functions $q_i = q_i(t)$ collectively describe the path taken by the system. Each choice of functions will yield a *numerical* value for the action. We say that it is a *functional* of the coordinates and their derivatives. Hamilton's principle says that the actual path followed by the system is the one that makes the action an extremum, i.e. a maximum or a minimum. Your favorite classical mechanics text will now prove that this statement is equivalent to a set of n Euler-Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.3}$$

We will be much concerned with the propagation of free particles. Let's see how that works classically in one dimension We know the answer ahead of time

$$x(t) = x_\circ + \frac{(x_f - x_\circ)}{(t_f - t_\circ)}(t - t_\circ) \tag{2.4}$$

$$S = \int_{t_\circ}^{t_f} \frac{1}{2} m\dot{x}^2 dt = \frac{m}{2}\int_{t_\circ}^{t_f} \left(\frac{x_f - x_\circ}{t_f - t_\circ}\right)^2 dt = \frac{m}{2}\frac{(x_f - x_\circ)^2}{(t_f - t_\circ)} \tag{2.5}$$

In classical mechanics, the particle follows just this one path. In quantum mechanics, the particle somehow follows all paths (that don't involve propagation backwards in time), but as can be seen from (1), those with action much different from the classical action will tend to cancel one another through destructive interference. Terms close to the classical path add coherently. Let's put in some numbers. Suppose a classical particle travels 1 cm in 1 second. If it starts at the origin and follows the "right" path $x = t$ in the appropriate units

$$S = \int_0^1 \frac{1}{2} m(1)^2 dt = \frac{1}{2}m. \tag{2.6}$$

However, if the particle follows an unphysical path, say $x = t^2$ then

$$S = \int_0^1 \frac{1}{2} m(2t)^2 dt = \frac{2}{3}m. \tag{2.7}$$

The units of S must be erg-seconds. If the mass were one gram, the "right" and "wrong" action would differ by only 1/6 erg-sec., but remember that in

(1) the action is divided by $\hbar = 10^{-27}$ erg-sec. Consequently the wrong path is out of phase by 10^{26} radians! On the other hand, if we are talking about an electron with a mass of 9.11×10^{-28} grams, then the wrong path is out of phase by only 8.7 degrees. Clearly the classical particle must follow the straight and narrow way, and how narrow the path must be is determined by the extremely small size of $\hbar$. To put it in less biblical terms, $S \gg \hbar$ gives the classical limit.

The postulate (1) is usually attributed to Feynman[1] who claimed that he discovered it in a moment if inspiration during a high school physics class. What Feynman could intuit, the rest of us can prove, though the proof will take many pages. The key idea is to discretize time. We will divide the interval between $t_\circ$ and t_f into $n+1$ equal slices bounded by $t_\circ < t_1 < t_2 < \cdots < t_{n-1} < t_f$. We represent the path of the particle as a set of $n+1$ straight-line segments connecting the space-time points, i.e.

$$(x_\circ, t_\circ) \to (x_1, t_1) \to (x_2, t_2) \to \cdots \to (x_n, t_n) \to (x_f, t_f)$$

We now use the composition property (1.31) $n+1$ times.

$$\langle x_f, t_f | x_\circ, t_\circ \rangle = \int dx_n dx_{n-1} \cdots dx_1 \langle x_f, t_f | x_n, t_n \rangle \langle x_n, t_n | x_{n-1}, t_{n-1} \rangle \cdots \langle x_1, t_1 | x_\circ, t_\circ \rangle \tag{2.8}$$

We will use the more compact notation,

$$\langle x_f, t_f | x_\circ, t_\circ \rangle = \int \prod_{i=1}^{n} dx_i \prod_{j=0}^{n} \langle x_{j+1}, t_{j+1} | x_j, t_j \rangle, \tag{2.9}$$

but remember that $x_0 = x_\circ$ and $x_{n+1} = x_f$. We will eventually take the limit $n \to \infty$ to represent all smooth paths, but this is a strange limit in which there are an infinite number of integrations! We now work on the short-time propagator,

$$\langle x_{j+1}, t_{j+1} | x_j, t_j \rangle = \langle x_{j+1} | e^{-iH\tau} | x_j \rangle \approx \langle x_{j+1} | (1 - iH\tau | x_j \rangle \tag{2.10}$$

The first step makes use of (1.29). The second step takes advantage of the fact that $\tau = t_{j+1} - t_j$ is very small. At this point I would like to insert a complete set of momentum eigenstates, but there are a few technicalities regarding normalization and boundary conditions that will repay a brief degression.

[1] Actually, most of these ideas were published earlier by Dirac. I don't know if Feynman was aware of his work.

The momentum eigenvalue equation in the Schrodinger picture is

$$\hat{P}\xi_p(x) = p\,\xi(x) \tag{2.11}$$

$$\xi_p \propto e^{ipx}$$

We often make the eigenvalue spectrum discrete by imposing periodic boundary conditions, $\xi(x) = \xi(x+L)$. in this case

$$\xi_n = \frac{1}{\sqrt{L}} e^{ip_n x} \qquad p_n = \frac{2\pi n}{L} \tag{2.12}$$

The normalization is chosen so that

$$\int_0^L dx\; \xi_n^* \xi_{n'} = \delta_{nn'} \tag{2.13}$$

If we don't use periodic boundary conditions, the eigenvalue spectrum is continuous. You can go back and forth between these two representations with the following "cookbook" recipe.

$$\xi_n = \frac{e^{ip_n x}}{\sqrt{L}} \leftrightarrow \xi_p = e^{ipx} \qquad \sum_n \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} dp \tag{2.14}$$

In the language of bras and kets

$$\hat{P}|p\rangle = p|p\rangle \tag{2.15}$$

but we have to pay some attention to the normalization of $|p\rangle$. Let's choose

$$\langle x|p\rangle = \xi_p(x) \qquad \langle x|p_n\rangle = \xi_n(x) \tag{2.16}$$

We would also like $\langle x'|x\rangle = \delta(x-x')$. In terms of discrete states this requires

$$\langle x'|x\rangle = \sum_n \langle x'|p_n\rangle\langle p_n|x\rangle = \sum_n \xi_n(x')\xi_n^*(x) = \delta(x-x') \tag{2.17}$$

So far, so good, When we make the transition to continuum states according to (14) this becomes

$$\langle x'|x\rangle = \frac{1}{2\pi}\int\; dp\langle x'|p\rangle\langle p|x\rangle = \delta(x-x'). \tag{2.18}$$

The point is that whereas

$$\int dx|x\rangle\langle x| = \hat{I},$$

to be consistent we must have

$$\frac{1}{2\pi}\int dp|p\rangle\langle p| = \hat{I} \tag{2.19}$$

It is this last expression that we need for the short-time propagator.

The Hamiltonian in (10) can be written

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(x) \tag{2.20}$$

Our strategy will be to use position and momentum eigenstates to replace the momentum and position operators with their eigenvalues. When we have done that, *there will no longer be operators in our formalism.* They will have been replaced by classical or c-number variables. Let's do the kinetic energy term first.

$$\langle x_{j+1}|\frac{\hat{P}^2}{2m}|x_j\rangle = \frac{1}{2\pi}\int dp_j\langle x_{j+1}|\frac{p_j^2}{2m}|p_j\rangle\langle p_j|x_j\rangle \tag{2.21}$$

$$= \frac{1}{2\pi}\int dp_j e^{ip_j(x_{j+1}-x_j)}\frac{p_j^2}{2m}$$

The potential energy term is simpler.

$$\langle x_{j+1}|V(x)|x_j\rangle = V(x_j)\langle x_{j+1}|x_j\rangle = V(x_j)\frac{1}{2\pi}\int dp_j e^{ip_j(x_{j+1}-x_j)} \tag{2.22}$$

Combine (10), (21) and (22).

$$\langle x_{j+1},t_{j+1}|x_j,t_j\rangle = \frac{1}{2\pi}\int dp_j e^{ip_j(x_{j+1}-x_j)}\left[1-i\tau H(p_j,x_j)\right] \tag{2.23}$$

Two points to notice:

- The Hamiltonian contains no operators. It is the *classical* Hamiltonian.
- The symbol p_j does not imply discrete momentum values. The subscript means that $|p_j\rangle$ is being used to help do the integrals associated with the t_j time slice.

Use the fact that τ is small so that $[1-i\tau H] \approx e^{-i\tau H}$.

$$\langle x_{j+1},t_{j+1}|x_j,t_j\rangle = \frac{1}{2\pi}\int dp_j \exp\left\{i\left[p_j(x_{j+1}-x_j)-\tau H(p_j,x_j)\right]\right\} \tag{2.24}$$

Substitute (24) into (9).

$$\langle x_f, t_f | x_\circ, t_\circ \rangle = \int \left(\prod_{i=1}^{n} dx_i \right) \left(\prod_{j=0}^{n} \frac{1}{2\pi} \int dp_j \exp\left\{ i \left[p_l (x_{j+1} - x_j) - \tau H(p_j, x_j) \right] \right\} \right)$$
$$= \int \left(\prod_{i=1}^{n} dx_i \right) \left(\prod_{j=0}^{n} \frac{1}{2\pi} \int dp_j \right) \exp\left\{ i \sum_{l=0}^{n} \left[p_l (x_{l+1} - x_l) - \tau H(p_l, x_l) \right] \right\} \tag{2.25}$$

The notation in the last line is potentially confusing. I have used three independent indices, i, j, and l, since there are three indexed operations, two products and a sum. The intent is to describe an equation with n factors, each consisting of an integral over one of the x's and one of the p's.

In order to calculate a propagator, one evaluates (25) term by term. The final result will be an expression that depends on n. One then takes the limit $n \to \infty$ in the hope that the limit is well-defined. Unfortunately, there are only a few simple potentials for which this agenda can be carried through exactly, so we are usually working with approximations. To this end, note that something remarkable happens when we take the limit $n \to \infty$. Concentrate on the sum inside the exponential.

$$\lim_{n\to\infty} \sum_{l=0}^{n} \tau \left[p_l \left(\frac{x_{l+1} - x_l}{t_{l+1} - t_l} \right) - H(p_l, x_l) \right] = \int_{t_\circ}^{t_f} dt (p\dot{x} - H) = \int_{t_\circ}^{t_f} L = S \tag{2.26}$$

Yes! L is the classical Lagrangian and S is the classical action. Equation (25) can now be collapsed as follows:

$$\lim_{n\to\infty} \prod_{j=0}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_j \equiv \int D[p] \tag{2.27}$$

$$\lim_{n\to\infty} \prod_{i=1}^{n} \int dx_i \equiv \int D[x] \tag{2.28}$$

$$\langle x_f, t_f | x_\circ, t_\circ \rangle = \int D[x] D[p] e^{iS} \tag{2.29}$$

Equations (25) through (29) not only prove Feynman's conjecture (1), they also tell us what it means to sum over paths. They are the most general formulas for evaluating path integrals.

The mathematics before the limit $n \to \infty$ is conventional math. Equations like (25) can be done with ordinary calculus without much regard to rigor. After the limit, the equations are in the new arena of functional integration. Eventually we will do functional integrals just as we do ordinary integrals – by following a set of rules we have memorized. In order to derive the rules we must go back to (25), do what needs to be done, and then retake the limit. Our first example of this is the Feynman propagator, which we will do in the next section.

2.2 The Feyman Path Integral

Most nonrelativistic Hamiltonians consist of a kinetic energy term, which is some simple quadratic function of the momenta, and a potential energy term, which is only a function of the coordinates. In the simplest case, $H = p^2/2m + V(x)$. When this is true the momentum integrations in (25) or (27) can be done once and for all. The resulting expression is called the Feynman propagator.

Let's go back to the integral in (24). In order to simplify the notation, I will drop all the subscripts.

$$\int dp \exp\left\{ i \left[p\Delta x - \tau \frac{p^2}{2m} - \tau V \right] \right\} \tag{2.30}$$

The gaussian integral comes in many different forms. This is one of them. The formula we need here is

$$\int_{-\infty}^{\infty} dp \ \exp\left[i(-ap^2 + bp + c) \right] = \sqrt{\frac{\pi}{ia}} \exp\left[ib^2/4a + ic \right] \tag{2.31}$$

We use this to evaluate (30) and substitute the result back into (25),

$$\langle x_f, t_f | x_i, t_i \rangle = \int \left(\prod_{i=1}^{n} dx_i \right) \left(\frac{m}{2\pi i \tau} \right)^{\frac{n+1}{2}} \exp\left\{ i\tau \sum_{l=0}^{n} \left[\frac{m}{2} \left(\frac{\Delta x_l}{\tau} \right)^2 - V(x_l) \right] \right\}, \tag{2.32}$$

where $\Delta x_l = x_{l+1} - x_l$. In the limit that $n \to \infty$ the multiplicative constant is

$$N = \lim_{n \to \infty} \left(\frac{m}{2\pi i \tau} \right)^{\frac{n+1}{2}} \tag{2.33}$$

This diverges like ∞^{∞}, but no matter, you have to get used to this sort of thing. As you will see when we calculate things that are physically relevant, this constant always factor out or disappears in some miraculous way. All

this is to say the the limit $n \to \infty$ is tricky. If you are trying to be rigorous, the limit should be the *last* step in the calculation after all the troublesome factors have eaten each other.

There is another bit of math that we have ignored so far. The integral (31) doesn't really converge to anything, because of the oscillating exponential. This, as the engineers would say, is not a bug but a feature. We could make it converge unambiguously either by deforming the contour on integration slightly into the complex plane or by adding a small convergence factor in the exponential. The way we choose to do this will determine whether the particle propagates forwards in time or backwards in time or some combination of the two We will return to this issue presently.

So with all these caveats, let's take the limit $n \to \infty$ with our fingers firmly crossed behind our backs.

$$\langle x_f, t_f | x_i, t_i \rangle = N \int D[x] e^{iS} \tag{2.34}$$

This is the Feynman path integral.

2.3 The free particle propagator

So far as I know, there are only two problems for which this last integral can be done exactly, the free particle and the harmonic oscillator. Everything else is approximation, but the approximations are all based on these two cases. We need to study them carefully for this reason. Let's start with the free particle. Go back to (32) and set $V = 0$. The crux of the matter is this integral,

$$\int dx_1 \cdots dx_n \ \exp\left\{ \frac{im}{2\tau} \left[(x_f - x_n)^2 + (x_n - x_{n-1})^2 + \cdots + (x_1 - x_i)^2 \right] \right\} \tag{2.35}$$

This is another example of the famous gaussian integral. Watch what happens when we integrate over dx_1 using (31).

$$\int dx_1 \exp\left\{ \frac{im}{2\tau} \left[(x_2 - x_1)^2 + (x_1 - x_i)^2 \right] \right\} = \sqrt{\frac{i\pi\tau}{m}} \exp\left[\frac{im}{4\tau} (x_2 - x_i)^2 \right] \tag{2.36}$$

After you have done n of these integral you will be left with

$$\langle x_f, t_f | x_i, t_i \rangle = \sqrt{\frac{m}{2\pi i (t_f - t_i)}} \exp\left[\frac{im(x_f - x_i)^2}{2(t_f - t_i)} \right] \tag{2.37}$$

Exercise: Do the calculation and confirm this result.

There are several points here to contemplate.

- We never had to take the limit $n \to \infty$, because by the time we had done all the integrals, *there were no n's left.*
- By the same token, there are no divergent factors.
- Look at our example (5). The argument of the exponential is just the *classical* action, so
$$\langle x_f, t_f | x_i, T_i \rangle = Ce^{iS} \tag{2.38}$$
- We could have gotten the same result directly from (1.6). The point is, I guess, that for free particles, energy eigenstates are also momentum eigenstates, so the sum in (1.6) it trivial.
- The propagator (37) seems to allow propagation backwards in time. This is not surprising, since we could have replaced $t \to -t$ in any of our derivations and gotten more of less the same results. As hinted previously, this ambiguity in the direction of time is related to the ambiguity associated with the convergence of integrals with oscillating phase factors. This phenomenon is not peculiar to path integrals or even to quantum mechanics. One encounters the same problem calculating Greens' functions for the propagation of electromagnetic waves. The free particle propagator is just simple enough to allow us to investigate these issues carefully in the next section.

2.4 Causality and the propagator

Equation (1.9) gives a simple formula for the free-particle propagator in terms of the energy eigenfunctions. In our current notation,

$$\langle x_f, t_f | x_i, t_i \rangle = \sum_n \phi_n^*(x_i)\phi_n(x_f)e^{-iE_n(t_f - t_i)} \tag{2.39}$$

In the case of a free particle, energy eigenfunctions are also momentum eigenfunctions, since

$$\hat{H}\phi_n = \frac{p_n^2}{2m}\phi_n. \tag{2.40}$$

We can use the momentum eigenfunctions defined in (12)and (14) for our ϕ's,

$$\phi_n(x) = \frac{1}{\sqrt{L}}e^{ip_n x} \to e^{ipx}, \tag{2.41}$$

where the arrow represents the continuum limit. Replace the sum in (39) with an integral as in (14)

$$\langle x_f, t_f | x_i, t_i \rangle = \frac{1}{2\pi} \int dp \; e^{i(x_f - x_i)} e^{-i\frac{p^2}{2m}(t_f - t_i)} \tag{2.42}$$

This can be integrated immediately using (31) to get (37), but for the purposes of discussing causality, it is better to get rid of the time and space variables by means of a Fourier transform.

Propagators are closely related to Green's functions. For example, we can define a "retarded" Green's function as

$$G_R(x_f, t_f; x_i, t_i) \equiv -i\theta(t_f - t_i)\langle x_f, t_f | x_i, t_i \rangle \tag{2.43}$$

$$= -\frac{i}{2\pi} \int dp \; \theta(t_f - t_i) \exp\left\{ i \left[-\frac{p^2}{2m}(t_f - t_i) + p(x_f - x_i) \right] \right\}$$

I put in the step function "by hand" so that the Green's function is zero for all $t_i > t_f$. The $-i$ factor is a convention, but there is a good reason for it as you will see eventually. In order to save ink, I will set x_i and t_i equal to zero and abbreviate

$$G_R(x_f, t_f; 0, 0) \equiv G(x, t) \tag{2.44}$$

Eliminate the x variable with a Fourier transform.

$$\int dx \; e^{-ip'x} G(x, t) = -i\theta(t) e^{-iE_{p'}t} \equiv G(p', t) \tag{2.45}$$

Finally, take the Fourier transform with respect to time.

$$\int dt \; e^{i\omega t} G(p, t) = -i \int_0^\infty e^{-i(E-\omega)t} dt \equiv G(p, \omega) \tag{2.46}$$

The last integral is not well defined. There are several strategies for dealing with it. The simplest is to add a small negative imaginary part to the energy, $E \to E - i\delta$. Then

$$G(p, \omega) = \frac{1}{\omega - E + i\delta} \tag{2.47}$$

To see the significance of the "fudge factor," Fourier transform back into configuration space.

$$G(x, t) = \frac{1}{(2\pi)^2} \int dp \int d\omega \frac{e^{i(px - \omega t)}}{\omega - E_p + i\delta} \tag{2.48}$$

The ω integration can be done as a complex contour integral. For $t < 0$, the integration contour closes in the upper half ω plane and the integral is zero. When $t > 0$, the contour closes in the lower half plane and the integral is just $-2\pi i$ times the residue of the pole at $\omega = E - i\delta$. This displacing the pole slightly below the real axis gives the retarded Green's function, i.e. the one that vanishes when $t < 0$. Obviously, we could get the advanced Green's function by displacing the pole upward.

Here is an assortment of points that you should absorb from the preceding derivation:

- Compare the formula for the propagator in configuration space, equation (37), with the Fourier transform of the Green's function, (47). Not only is (47) much simpler mathematically, its causality properties are unambiguous and obvious. (The $-i$ factor in (43) was inserted to make (47) as simple as possible.) This is a recurring theme in field theory. Equations of motion and path integrals always start out in configuration space, but the results are simpler in momentum space.

- Integrals over momentum like (42) tend to be *mathematically* ambiguous because the integrand oscillates infinitely rapidly as $p \to \infty$. They are *physically* ambiguous because they could describe propagation backward in time just as well as propagation forward in time. We can remove this ambiguity in several ways. In (43) we simply put in a theta function. This makes causality explicit, but is clumsy to work with. A better scheme is to give energies a small negative imaginary part, or alternatively, just remember that the propagator in momentum space always has the form (47). This apparently ad hoc scheme removes the physical ambiguity and the mathematical ambiguity simultaneously, because they are really two manifestations of the same problem.

- In non-relativistic quantum mechanics based on Schrodinger's equation, cause and effect always propagate "forward" in time. The point is that the equation is second order in space and first order in time, so that if you replace $t \to -t$ in Schrodinger's equation, *you get a different equation.* I emphasize this because relativistic theories are profoundly different; space and time coordinates (except for that damnable minus sign) must be handled in the same way. The Kline-Gordon equation, for example, is second order in space and time. The Dirac equation is first order in both variables. Such theories do not have a well-defined sense of past and future. The propagators must therefore account for

both forward and backward propagation. I'll have much more to say about this later.

- If you replace $t \to -it$ in Schrodinger's equation, you get something that looks like the equation for heat flow,

$$\nabla^2\phi - a\frac{\partial\phi}{\partial t} = 0 \tag{2.49}$$

This is significant for two reasons. First, we all know that heat flows from somewhere hot to somewhere colder. The second law of thermodynamics defines an unambiguous direction of time. Schrodinger's equation has this kind of causality built into it. Second, the transformation $t \to -it$ is called a *Wick rotation.* It is a profound and puzzling fact that the Wick rotation always turns quantum mechanics into thermodynamics. You have just seen one example of this. There are more to follow.

- You doubtless learned in Middle School that the Green's function is the field due to a unit impulse function. You should convince yourself that $G(x,t)$ is no different, i.e.

$$\left(\hat{H} - i\frac{\partial}{\partial t}\right) G(x,t) = -\delta(x)\delta(t) \tag{2.50}$$

2.5 Generalizing the one-particle propagator

We have been working on the propagator

$$\langle x_f, t_f | x_i, t_i\rangle = \langle x_f | e^{-iH(t_f - t_i)} | x_i\rangle \tag{2.51}$$

Remember that the states on the left are time-dependent position eigenstates in the Heisenberg picture. The states on the right are time-independent Schrodinger states in which there is one particle that happens to be at x_i or x_f. Let's think through a series of generalizations. First, we replace x with a generalized coordinate q. We can measure this coordinate, so by the Central Dogma of Quantum Mechanics, there must be some corresponding operator, eigenvalues and eigenstates.

$$\hat{Q}|q\rangle = q|q\rangle \tag{2.52}$$

We could repeat all our derivations with q replacing x and get

$$\langle q_f, t_f | q_i, t_i \rangle = \langle q_f | e^{-iH(t_f - t_i)} | q_i \rangle \tag{2.53}$$

Now that we have introduced one generalized coordinate, there is no reason why we can't have N of them. The classical Hamiltonian is

$$H = K(p_1 \cdots p_N) + V(q_1 \cdots q_N) \tag{2.54}$$

where

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \tag{2.55}$$

Again, repeat all the derivations. This time replace the single integrals with multiple integrals

$$\int dp_j \to \int \cdots \int dp_{j1} \cdots dp_{jN} \tag{2.56}$$

$$\int dq_j \to \int \cdots \int dq_{j1} \cdots dq_{jN}$$

This does not introduce any additional problems.

In many respects we seem to be doing classical Langrangian mechanics with N degrees of freedom. In order to do *field* theory, however, we must pass to the limit of continuum mechanics, which is to say, mechanics with an infinite number of degrees of freedom. In this case the *coordinates* q_j are replaced with displacement *fields* $\phi(x)$. The motivations for doing this and the mathematics involved are the subjects of the next chapter.

2.6 Correlation functions

It turns out that many of the quantities that will be of interest to us have the form

$$\langle F | T \left[\hat{Q}(t_1) \hat{Q}(t_2) \cdots \hat{Q}(t_n) \right] | I \rangle \tag{2.57}$$

The $\hat{Q}$ operators were explained in the previous section. The states $|I\rangle$ and $\langle F|$ are shorthand notation for the initial and final states. The notation $T[\cdots]$ introduces the *time-ordered product*. It means that the operators in the brackets must be arranged so that operators evaluated at earlier time

always stand to the right of operators evaluated at later times. for example, in the case of a single particle,

$$\langle F|T[\hat{Q}(t_1)\hat{Q}(t_2)]|I\rangle = \langle q_f,t_f|\hat{Q}(t_1)\hat{Q}(t_2)|q_i,t_i\rangle\theta(t_1-t_2) \tag{2.58}$$

$$\pm\langle q_f,t_f|\hat{Q}(t_2)\hat{Q}(t_1)|q_i,t_i\rangle\theta(t_2-t_1)$$

The $\pm$ sign anticipates some arcane math that will be necessary when we deal with fermion operators, which anticommute. For the time being, this will be a plus sign. As a mnemonic for remembering (58), recall that quantum mechanics reads like Hebrew, from right to left, so that $t_f > t_2 \geq t_1 > t_i$.

I will anticipate the next section by saying that we will usually evaluate (57) in the limits, $t_i \to -\infty$ and $t_f \to +\infty$. This has the effect of projecting out the ground state or lowest-energy state of the system, so that we are really concerned with

$$\langle\Omega|T\left[\hat{Q}(t_1)\hat{Q}(t_2)\cdots\hat{Q}(t_n)\right]|\Omega\rangle, \tag{2.59}$$

where $|\Omega\rangle$ and $\langle\Omega|$ are the initial and final ground states. I would prefer to do this one step at a time. First I will show how to evaluate (57) , and in a later section, I'll show the significance of the limits $t_i \to -\infty$ and $t_f \to +\infty$.

We start by evaluating (57) with a single operator. Transforming it into the Schrodinger picture gives

$$\langle q_f,t_f|\hat{Q}(t)|q_i,t_i\rangle = \langle q_f|e^{-iH(t_f-t)}\hat{Q}e^{-iH(t-t_i)}|q_i\rangle. \tag{2.60}$$

Now insert a complete set of states just to the right of $\hat{Q}$ and take advantage of the fact that $\hat{Q}|q\rangle = q|q\rangle$.

$$= \int dq\ q\ \langle q_f|e^{-iH(t_f-t)}|q\rangle\langle q|e^{-iH(t-t_i)}|q_i\rangle = \int dq\ q(t)\ \langle q_f,t_f|q,t\rangle\langle q,t|q_i,t_i\rangle$$

In this last expression we have two ordinary propagators $\langle q_f,t_f|q,t\rangle$ and $\langle q,t|q_i,t_i\rangle$. Each one could be written as in (32) or (34). This is just as if we were calculating $\langle q_f,t_f|q_i,t_i\rangle$ and stuck in one "extra" time slice at t between t_i and t_f. Repeating the steps that led to (34) we have

$$\langle q_f,t_f|\hat{Q}(t)|q_i,t_i\rangle = \int D[q]q(t)e^{iS} \tag{2.61}$$

Exercise: Verify this. Pay careful attention to the normalization factor.

This procedure can be repeated as many times as we like, so

$$\langle q_f, t_f | T\left[\hat{Q}(t_1)\hat{Q}(t_2)\cdots\hat{Q}(t_n)\right]|q_i, t_i\rangle = \int D[q]q(t_1)q(t_2)\cdots q(t_n)e^{iS} \tag{2.62}$$

Notice that there is a time-ordering operator on the left side of the equation and none on the right. One way to look at this is that the q's commute, so the order in which they are written makes no difference. A better way to look at it is that when we insert sets of complete sets as in (60), we will always put them in in such a way that $t_f > t_n > \cdots > t_2 > t_1 > t_i$ (in order to prevent backwards propagation in time). Thus the $q(t_i)$'s are automatically time ordered.

Path integrals are notoriously difficult to do, so what hope do we have of evaluating an integral like (62) with all those additional $q(t)$'s? The answer is that if we can evaluate it exactly without the q's, then we can do it with the q's almost trivially! If we can get an approximate answer without the q's, then to the same degree of approximation, we can evaluate it with the q's. In order to understand how this trick works, look at the simpler integral

$$\int_{-\infty}^{\infty} dq\; q^n e^{-\frac{1}{2}aq^2}$$

We know that

$$\int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}aq^2+Jq} = \left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{J^2/2a}$$

Differentiating this n times with respect to J,

$$\int_{-\infty}^{\infty} dq\; q^n e^{-\frac{1}{2}aq^2+Jq} = \frac{d^n}{dJ^n}\int_{-\infty}^{\infty} dq\; e^{-\frac{1}{2}aq^2+Jq} = \frac{d^n}{dJ^n}\left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{J^2/2a}$$

Finally

$$\int_{-\infty}^{\infty} dq\; q^n e^{-\frac{1}{2}aq^2} = \lim_{J\to 0}\frac{d^n}{dJ^n}\left(\frac{2\pi}{a}\right)^{\frac{1}{2}} e^{J^2/2a} \tag{2.63}$$

You see the point? Integration is hard. Differentiation is easy. Using the "J trick," we can do difficult integrals by differentiating!

Now go back to (???) and put in the J.

$$z[J] \equiv \langle q_f, t_f | q_i, t_i\rangle^J \equiv \int D[q]\exp\left\{i\int_{t_i}^{t_f} dt\,[L(t) + J(t)q(t)]\right\} \tag{2.64}$$

I can always get the propagator by taking the limit $J \to 0$, but in the meantime, I can put as many q's as I like in the integrand by differentiating

(64) with respect to J. $z[J]$ is called a *generating function* for this reason. This is more than just a trick as you will see later on In some theories J has a clear physical interpretation as a source of particles.

Before we can use (64) we must dispose of a technical difficulty. In (63), J is a parameter, In (64) it's a function of t buried inside a definite integral. This requires a new technique called functional differentiation, I have put this material in an appendix labelled Chapter 3, to which I will be referring from time to time. This would be a good time to study it, We can use (3.5) to get our final result:

$$\begin{aligned} &\langle q_f, t_f | T\left[\hat{Q}(t_1)\hat{Q}(t_2)\cdots\hat{Q}(t_n)\right]|q_i, t_i\rangle \\ &= \left(\frac{1}{i}\right)^n \frac{\delta^n z[J]}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0} \end{aligned} \tag{2.65}$$

Chapter 3

Mathematical Appendix

3.1 Functional Differentiation

Consider the expression

$$F(J(\tau))$$

By this we understand that τ is the independent variable and F is a function of a function of τ. The familiar rules of differentiation apply, so for example,

$$dF = \frac{\partial F}{\partial J}\frac{dJ}{d\tau}d\tau$$

These rules fail when there is buried inside F some definite integral in which τ appears as the variable of integration. This is the case in equation (2.64), which I repeat here for reference.

$$\langle q_f, t_f | q_i, t_i \rangle^J \equiv \int D[q] \exp\left\{ i \int_{t_i}^{t_f} dt \, [L(t) + J(t)q(t)] \right\} \tag{3.1}$$

We call a expression of this sort a *functional* of J and write it $F[J]$. We can't differentiate this with respect to J, but we can do something similar called *functional differentiation.*

Suppose, for example that the function $f = f(J(\tau))$ is a function of J, which itself is a function of τ. The function f is turn is buried inside an integral, so

$$F[J] \equiv \int f(J(\tau))d\tau \tag{3.2}$$

Define the *variation* of F as follows.[1] Let $\eta(\tau)$ be any old well-behaved

[1] This definition is taken from Goldstein, *Classical Mechanics,* 3'rd ed. Section 2.3

function of τ, and define

$$J(\tau,\alpha) = J(\tau) + \alpha\eta(\tau)$$

$$\frac{\partial[F(\alpha)]}{\partial\alpha} = \int \frac{\partial f}{\partial J}\frac{\partial J}{\partial\alpha}d\tau = \int \frac{\partial f}{\partial J}\eta d\tau \tag{3.3}$$

The variation is then

$$\delta F \equiv \left(\frac{\partial F}{\partial\alpha}\right)\delta\alpha = F[J+\delta\eta] - F[J] \tag{3.4}$$

Where $\delta\alpha$ is some very small value of α, and $\delta\eta = \eta\delta\alpha$. Combine (3.3) and (3.4)

$$\delta F = \int \frac{\partial f}{\partial J}\delta\eta\, d\tau \tag{3.5}$$

This is usually written

$$F[J+\delta\eta] - F[J] = \int \frac{\delta F[J]}{\delta J(\tau)}\delta\eta\, d\tau \tag{3.6}$$

Equation (3.6) implicitly defines the functional derivative $\frac{\delta F[J]}{\delta J(\tau)}$ and also gives us an algorithm for calculating it. As an example, take a simplified form of (2.54)

$$F[J] = \exp\left\{i\int_{\tau_i}^{\tau_f} d\tau[L + J(\tau)q(\tau)]\right\} \tag{3.7}$$

$$F[J+\delta\eta] = \exp\left\{i\int_{\tau_i}^{\tau_f} d\tau[L + J(\tau)q(\tau) + \delta\eta(\tau)q(\tau)]\right\} = F[J]e^{i\int_{\tau_i}^{\tau_f}\delta\eta(\tau)q(\tau)d\tau}$$

$$\delta F = iF[J]\int_{\tau_i}^{\tau_f} q(\tau)\delta\eta(\tau)d\tau$$

$$\frac{\delta F[J]}{\delta J(\tau)} = iq(\tau)F[J] \tag{3.8}$$

This is a good time to point out that the functional derivative of a functional is a function.

Now that you have the idea, you can take the shortcut. (This is sometimes used as a definition of the functional derivative.)

$$\frac{\delta F[J(\tau)]}{\delta J(\tau')} = \lim_{\epsilon\to 0}\frac{F[J(\tau)+\epsilon\delta(\tau-\tau')] - F[J(\tau)]}{\epsilon} \tag{3.9}$$

The notation $F[J(\tau)]$ means that τ is the variable of integration.

We can use the shortcut to evaluate a more difficult and important case from a later chapter. The following expression occurs naturally when we calculate correlation functions using the path integral formalism.

$$F[J] = -\frac{i}{2}\int d^4x\, d^4y J(x) D(x-y) J(y) \tag{3.10}$$

All you need to know about this is that x and y are 4-vectors and $D(x) = D(-x)$. We use functional differentiation to pull $D(x-y)$ out of the functional. To do this, we need to evaluate the expression,

$$\frac{\delta^2 F[J]}{\delta J(x')\delta J(y')} \tag{3.11}$$

The first derivative is found as follows:

$$\begin{aligned}\frac{\delta F}{\delta J(x')} = \lim_{\epsilon\to 0}\left(\frac{1}{\epsilon}\right)\Bigg\{-\frac{i}{2}\int d^4x d^4y [J(x)+\epsilon\delta^{(4)}(x-x')]D(x-y) \\ \times[J(y)+\epsilon\delta^{(4)}(y-x')] + \frac{i}{2}\int d^4x d^4y J(x)D(x-y)J(y)\Bigg\} \\ = -i\int d^4x D(x-x')J(x)\end{aligned}$$

Notice that both J's are incremented by the ϵ term. The last step is easier.

$$\frac{\delta^2 F}{\delta J(x')\delta J(y')} = -iD(x'-y')$$

The primes have done their job of distinguishing the independent variables from the variables of integration. We don't need them anymore, so let's write it

$$\frac{\delta^2 F}{\delta J(x)\delta J(y)} = -iD(x-y) \tag{3.12}$$

We differentiated with respect to $J(x)$ and $J(y)$, therefore the resulting function must be a function of x and y.

3.2 Functional Taylor series

"When in doubt, expand in a Taylor series." This is good advice with functionals as well as ordinary functions This is particularly true, since most path integrals cannot be evaluated exactly, and we need some way to make systematic approximations.

As in (3.2) we can write

$$F[J(\alpha)] = \int f[J(\tau) + \alpha\eta(\tau)]d\tau \tag{3.13}$$

This time we will expand $F[J(\alpha)]$ in an ordinary power series in α and set $\alpha = 1$ at the end of the calculation.

$$F[J(\alpha)] = F[J(0)] + \frac{d}{d\alpha}\, F[J(\alpha)]|_{\alpha=0} + \frac{1}{2}\frac{d^2}{d\alpha^2}\, F[J(\alpha)]|_{\alpha=0} + \cdots \tag{3.14}$$

Repeating the steps from (3.3) – (3.6)

$$\frac{d}{d\alpha}F[J(\alpha)]|_{\alpha=0} = \int \frac{\partial f}{\partial J}\eta d\tau = \int \frac{\delta F[J]}{\delta J(\tau)}\eta(\tau)d\tau \tag{3.15}$$

$$\frac{d^2}{d\alpha^2}F[J(\alpha)] = \int d\tau \frac{\delta}{J(\tau)}\eta(\tau') \int d\tau' \frac{\delta F[J]}{\delta J(\tau')}\eta(\tau') \tag{3.16}$$

Collecting these terms and setting $\alpha = 1$ gives

$$F[J+\eta] = F[J] + \int \frac{\delta F[J]}{\delta J(\tau)}\eta(\tau)d\tau + \frac{1}{2}\int\int d\tau d\tau' \eta(\tau)\eta(\tau')\frac{\delta^2 F[J]}{\delta J(\tau)\delta J(\tau')} + \cdots \tag{3.17}$$

Chapter 4

Introduction to Field Theory

The formalism I have employed so far tries to look as much as possible like classical Lagrangian mechanics with a finite number of degrees of freedom. Our strategy has been to get rid of operators and replace them with classical, c-number variables. These variables commute with one another, which as a great simplifying feature. They are embedded in the path integral formalism, which is anything but simple, but most of the time this formalism "lurks in the background." We can construct Lagrangians, calculate interactions, and discuss symmetries without actually doing path integrals. So long as we work with a finite number of degrees of freedom, however, we can never accommodate the creation and annihilation of particle. There is no way to change the number of q's in our Lagrangian. To do this, we need to replace the mechanics of discrete masses with the mechanics of continuous media. The motivation for this is explained nicely in Zee's book in the section about the mattress, which I will paraphrase.

Imagine that space is like a rubber sheet. If I put a bowling ball on the sheet, it will create a depression, and nearby objects will roll into it. This is an imperfect analogy for an attractive potential. We could describe the attraction in one of two ways: we could say that there is an attractive potential between any pair of point-like masses, or we could introduce a continuous variable, $\phi(x, y)$ which describes the displacement of the sheet as a function of position. Such a continuous displacement variable is a *field* in the strict mathematical sense: it assigns a numerical value (or set of values) to each point in space. The quantum mechanics of such fields is called quantum field theory. Now suppose that instead of using a bowling ball I jump up and down on the sheet. The sheet will oscillate in response. My activity becomes a *source* of energy, which propagates outward in the form

of waves. This is the rubber-sheet analogy to the propagation of particles.

This analogy can easily be misleading. For one thing, I don't want you to think we are doing general relativity. The rubber sheet is not intended as an analogy for ordinary space-time as it is often used in explaining general relativity. The field $\phi(x, y)$ describes a displacement, and I know you want to ask, "Displacement of what?"

The same question comes up in classical electromagnetic theory. When an electromagnet wave is propagating through space, what is waving? Folks in the 19'th century thought it must be some sort of mechanical medium, which they called the ether. According to the textbooks, Michaelson and Morley proved that wrong with their famous interferometer. But just saying that the ether does't exist doesn't answer the question, it just makes it impossible to answer! Let's bite the bullet and agree for the purposes of this course that space is pervaded by a medium, which for lack of a better name, we will call the ether. Well, actually the *ethers*. Each species of particle corresponds to a set of vibrations in it's own specific ether. Electrons are all vibrations in the electron ether, etc. Space-time points in the ether can be labelled with Lorentz four-vectors or $(\boldsymbol{x}, t)$ as usual, and these points obey the usual rules for Lorentz transformations. This much is required by the M-M experiment. Ordinary bulk media have elastic properties that are described by two parameters, the density and Young's modulus. These parameters are not themselves relevant to our formalism, but their ratio gives the velocity of propagation, which is what we really care about.

I am fond of saying, "When correctly viewed, everything is a harmonic oscillator." Now you see that this is profoundly true. Each point on the rubber sheet or ether acts like a harmonic oscillator! *Quantum field theory is a theory about harmonic oscillators.*

Well – I have to modify that slightly. If each point on the sheet behaved like a *simple* harmonic oscillator with a quadratic potential, the waves propagating on the sheet would never interact. The principle of linear superposition would hold everywhere. This is a theory of free particles. If our theory is to describe interactions, then we must modify the potential so that it becomes anharmonic. Unfortunately, the anharmonic oscillator cannot be solve exactly in quantum mechanics. (If you think of a way to do it, tell me and I'll be famous.) We have to resort to approximations, and here is where the path integral formalism really becomes a help rather than a complication. It makes possible a systematic and understandable perturbation theory that, with enough patience, can be carried out to any degree of accuracy desired.

There is an alternative way of dealing with the creation and annihilation

of particles. It is the older way, sometimes called canonical quantization or second quantization. The path integral formalism, seeks to banish all operators from the theory. Second quantization goes in the other direction. It turns the wave functions themselves into operators by imbedding creation and annihilation operators into them; *but they are the raising and lowering operators of the harmonic oscillator!* The universe, according to second quantization, is an infinity of harmonic oscillators. This approach is complementary to path integrals in other ways as well. We need to master both.

Continuum mechanics is not covered in most graduate mechanics classes. There is a good discussion in the last chapter of Goldstein, but we never make it that far. What follows is a brief introduction.

4.1 Introduction to Continuum Mechanics

The rules of continuum mechanics are derived by starting with a system with a finite number of degrees of freedom and then passing to the limit in which the number becomes infinite. Let's do this with the simplest possible system, a long chain of masses connected by springs. It's a one-dimensional problem. The masses can only oscillate along the chain. We will use φ_i, the displacement of the i-th particle from its equilibrium position, as the generalized coordinate. The Lagrangian is constructed in the obvious way.

$$T = \frac{1}{2}\sum_i m\ddot{\varphi}_i^2 \tag{4.1}$$

$$V = \frac{1}{2}\sum_i k(\varphi_{i+1} - \varphi_i)^2 \tag{4.2}$$

$$L = T - V = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\varphi}_i^2 - ka\left(\frac{\varphi_{i+1} - \varphi_i}{a}\right)^2\right] \tag{4.3}$$

The equilibrium separation between masses is a. The spring constant is k. The Euler-Lagrange equations of motion are obtained from

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_i} - \frac{\partial L}{\partial \varphi_i} = 0 \tag{4.4}$$

If there are N masses, then there are N coupled equation of this sort. They look like

$$\frac{m}{a}\ddot{\varphi}_i - ka\left(\frac{\varphi_{i+1} - \varphi_i}{a^2}\right) + ka\left(\frac{\varphi_i - \varphi_{i-1}}{a^2}\right) = 0 \tag{4.5}$$

We need different parameters to describe the continuum limit:

$$m/a \to \mu \qquad \text{mass per unit length}$$

$$ka \to Y \qquad \text{Young's modulus}$$

The index i points to the i-th mass, and φ_i gives its displacement. In the continuum limit, the index is replaced by the coordinate x. In elementary mechanics, x would be the displacement of a particle. Here $\varphi(x)$ is the displacement of the string *at the point* x. In the continuum limit

$$\frac{\varphi_{i+1} - \varphi_i}{a} \to \frac{\varphi(x+a) - \varphi(x)}{a} \to \frac{d\varphi}{dx}$$

$$L \to \frac{1}{2}\int dx \left[\mu\dot{\varphi}^2 - Y\left(\frac{d\varphi}{dx}\right)^2\right] \equiv \int dx \mathcal{L}(\varphi, \dot{\varphi}) \tag{4.6}$$

The last integral implicitly defines the *Lagrangian density* . The continuum version of the Euler-Lagrange equation is[1]

$$\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\right] + \frac{d}{dx}\left[\frac{\delta \mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right] - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{4.7}$$

Use the Lagrangian density from (6) in (7)

$$\frac{\partial^2 \varphi}{\partial x^2} = \left(\frac{\mu}{Y}\right)\frac{d^2\varphi}{dt^2} \tag{4.8}$$

(4) and (5) represent a set of N coupled equations for N degrees of freedom. (7) is one equation for an infinite number of degrees of freedom. In this sense, continuum mechanics is much easier that discrete mechanics.

(8) should remind you of the equation for the propagation of electromagnetic waves.

$$\left(\frac{\partial^2 \varphi}{\partial x^2}\right) + \left(\frac{\partial^2 \varphi}{\partial y^2}\right) + \left(\frac{\partial^2 \varphi}{\partial z^2}\right) = \frac{1}{c^2}\left(\frac{\partial^2 \varphi}{\partial t^2}\right)$$

As you know, photons are *massless* particles. Notice that a string of *massive* particles yields a wave equation that when quantized describes the propagation of *massless* particles. (With a different velocity, of course.) This is worth a brief digression.

[1] See Goldstein for a derivation of this important equation.

What does it mean to say that a wave function describes the propagation of a particle of a particular mass? The wave function $\psi = e^{i(kx-\omega t)}$ might describe a wave in classical E&M, or a massive particle in non-relativistic or relativistic quantum mechanics. The question is, what is the relation between k and ω? The relationship between the two is called a *dispersion relation*. It contains a great deal of information. In the case of EM waves in vacuum, $k = \omega/c$. Frequency and wave number are simply proportional. This is the hallmark of a massless field. The velocity is the constant of proportionality, so there can only be one velocity. In Schrodinger theory

$$\frac{\hbar^2 k^2}{2m} = \hbar\omega \tag{4.9}$$

The relationship is quadratic. The relativistic wave equation for a spin-zero particle is called the Kline-Gordon equation.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi - \frac{m^2c^2}{\hbar^2}\varphi = 0 \tag{4.10}$$

The dispersion relation is

$$(c\hbar k)^2 + m^2c^4 = (\hbar\omega)^2, \tag{4.11}$$

or in other words, $p^2c^2 + m^2c^4 = E^2$. All these equations can be obtained from (7) with the appropriate Lagrangian density. They are all three-dimensional variations of our "waves on a rubber sheet" model. What does this have to do with the particle's mass? It's useful to plot (9) and (11), i.e. plot ω versus k for small values of k. In both cases the curves are parabolas. This means that in the limit of small k, the group velocity,

$$v_{\text{group}} = \frac{d\omega}{dk} \approx \frac{\hbar k}{m} \tag{4.12}$$

In other words, the group velocity is equal to the classical velocity for a massive particle $v = p/m$. All the wave equations I know of fall in one of these two categories; either ω is proportional to k, in which case the particle is massless and its velocity $v = \omega/k$, or the relationship is quadratic, in which case

$$m = \lim_{k\to 0}\left(\hbar k\frac{dk}{d\omega}\right). \tag{4.13}$$

So long as we are talking about wave-particle duality, *this is what mass means.*

One of the advantages of using Lagrangians rather than Hamiltonians is that Lagrangians have simple transformation properties under Lorentz transformations. To see this, let's rewrite (7) in relativistic notation. Construct the contravariant and covariant four-vectors

$$x^\mu \equiv (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{4.14}$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \tag{4.15}$$

and the corresponding contravariant and covariant derivatives

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{4.16}$$

This puts the Euler-Lagrange equation in tidy form

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{4.17}$$

This is slightly amazing. Equation (7) was derived without reference to Lorentz transformations, and yet (17) has the correct form for a scalar wave equation. We get relativity for free! If we can manage to make $\mathcal{L}$ a Lorentz scalar, then (17) will have the same form in all Lorentz frames. Better yet, the action

$$S = \int dt\, L = \int dt \int d^3x\, \mathcal{L} = \frac{1}{c} \int d^4x\, \mathcal{L} \tag{4.18}$$

is also a Lorentz scalar. We can do relativity with path integrals without having to "fix" anything.

Here's an example. Rewrite (6) in 3-d

$$\mathcal{L} = \frac{1}{2} \left\{ \mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - Y \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \right\} \tag{4.19}$$

This would be the Lagrangian density for oscillations in a huge block of rubber. Take

$$\frac{\mu}{Y} = \frac{1}{c^2}. \tag{4.20}$$

Obviously $\mathcal{L}$ can be multiplied by any constant without changing the equations of motion. Rescale it so that it becomes

$$\mathcal{L} = \frac{1}{2} \left\{ \left(\frac{\partial \varphi}{\partial t} \right)^2 - c^2 \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \right\} \tag{4.21}$$

Substituting (21) into (17) yields the usual equation for EM waves, $\Box\varphi = 0$.

Notice how the Lagrangian for oscillations a block of rubber (19) turns into the Lagrangian for oscillations in the ether (21). We don't have to worry about the mechanical properties of the ether, because μ and Y are scaled away. Despite what you may have been told, the Michelson-Morley experiment proves the *existence* of the ether. When correctly viewed, everything is a bunch of harmonic oscillators, even the vacuum!

Using Einstein's neat notation, we can collapse (21) into one term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) \equiv \frac{1}{2}(\partial\varphi)^2 \tag{4.22}$$

The last piece of notation $(\partial\varphi)^2$, is used to save ink. The fact that we can write $\mathcal{L}$ like this is proof that it is a Lorentz scalar. This is an important point; we can deduce the symmetry properties of a theory by glancing at $\mathcal{L}$.

Now you can make up your own field theories. All you have to do is add scalar terms to (22). Try it. Name the theory after yourself. Here's a theory that already has a name. It's the Kline-Gordon theory.

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\phi^2\right] \tag{4.23}$$

(I have set $c = 1$ and $\hbar = 1$.) Using our new notation, the equation of motion is

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0 \tag{4.24}$$

If we assume that $\varphi(x)$ (x is a 4-vector in this notation.) is a one-component Lorentz scalar, then this describes a spinless particle with mass m propagating without interactions. Spin can be included by adding more components to φ. More about this later.

It is plausible that we can adopt all the path integral results from the previous sections. Simply replace the displacement q_i with the displacement *field* $\varphi(x)$ and the momentum p_i with the canonical momentum $\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}$. Let's look at this more closely. We have been calculating the propagator,

$$\langle q_f, t_f | q_i, t_i \rangle_H = \langle q_f | e^{-iH(t_f - t_i)} | q_f \rangle_S = \int D[x]D[p]e^{iS}. \tag{4.25}$$

(Equation (29) in Chapter 2) The natural generalization of this is

$$\langle \varphi_f(\boldsymbol{x}) | e^{-iH(t_f - t_i)} | \varphi_i(\boldsymbol{x}) \rangle = \int D[\varphi]D[\pi] \exp\left[i\int_{t_i}^{t_f} d^4x\mathcal{L}(\varphi)\right] \tag{4.26}$$

Where the functions $\varphi(x)$ over which we integrate are constrained to the specific configurations $\varphi_i(\boldsymbol{x})$ at $x_0 = t_i$, and $\varphi_f(\boldsymbol{x})$ at $t_0 = t_f$.

In the previous chapter we were able to derive (2.29) directly from Feynman's sum over paths hypothesis. Here here that derivation breaks down. The trouble is that the field operators $\hat{\varphi}$ do not have complete sets of eignstates. The derivation can be completed using the notion of coherent states,[2] but I propose a different approach. To make the notation a bit more compact I will write the propagator,

$$\langle \varphi_f, t_f | \varphi_i, t_i \rangle = U(\varphi_i, \varphi_f; T), \tag{4.27}$$

where $T = t_f - t_i$. Now it is possible to show[3] without using the completeness hypothesis that U satisfies Schrodinger's equation

$$i\hbar \frac{\partial}{\partial T} U = HU \tag{4.28}$$

The Hamiltonian in this equation is obtained from the Lagrangian in (26) by the usual procedure. We argue that (26) is correct, even if we are unable to derive it from first principles. Let's just take it as a definition of U.

Since the exponent in (26) is quadratic in π, we can complete the square and evaluate the $D[\pi]$ integral to obtain

$$\langle \varphi_f(\boldsymbol{x}) | e^{-iH(t_f - t_i)} | \varphi_i(\boldsymbol{x}) \rangle = \int D[\varphi] \exp\left[i \int_{t_i}^{t_f} d^4x \, \mathcal{L} \right] \tag{4.29}$$

You will recall from (2.32) and (2.33) that this integration produces a badly behaved normalization constant. Back then, I simply called it N and reassured you that it would go away. Here, this is true *a fortiori*. I have no idea what the constant is and only a clue what it means. I do know that all physically interesting quantities are quotients of two terms, each containing the same constant. Since it always cancels, I have not even bothered to write it in (29).

4.2 Correlation functions

The propagator in (25) is just the ticket for ordinary quantum mechanics. After all, the q's are measurable quantities. Quantum fields, on the other hand, are not measurable, so U as it stands is not of much use. It turns out

[2] This is worked out in Prof. Wasserman's lecture notes.

[3] See Peskin and Schroeder, Section 9.1

that the quantities of interest are the correlation functions defined in Section 2.6, equation (59), but with fields replacing the $\hat{Q}$ operators. In other words we are looking for

$$G(x_1, x_2, \cdots, x_n) \equiv \langle \Omega | T\left[\hat{\varphi}(x_1)\hat{\varphi}(x_2)\cdots\hat{\varphi}(x_n)\right] | \Omega \rangle \tag{4.30}$$

This is sometimes called the n-point function. The most important instance of this is the two-point correlation function, which we will now investigate. Consider the following object:[4]

$$\langle \varphi_b | e^{-iHT} T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] e^{-iHT} | \varphi_a \rangle \tag{4.31}$$

The $\hat{\varphi}$'s are field operators in the Heisenberg picture. The times are arranged so that both x_1^0 and x_2^0 fall in the interval between $-T$ and T. φ_a represents the field evaluated at time $-T$, $\varphi_a = \varphi(-T, \boldsymbol{x})$ and $\varphi_b = \varphi(T, \boldsymbol{x})$. The construction $T[\cdots]$ refers to the time ordering operator, as in Section 2.6. Let's assume to start with that $x_2^0 > x_1^0$. We can convert the Heisenberg fields to Schrodinger fields, e.g.

$$\hat{\varphi}(x_2) = e^{iHx_2^0}\, \hat{\varphi}_S(\boldsymbol{x}_2) e^{-iHx_2^0}$$

Now (31) can be written

$$\langle \varphi_b | e^{-iH(T-x_2^0)} \hat{\varphi}_S(\boldsymbol{x}_2) e^{-iH(x_2^0 - x_1^0)} \hat{\varphi}_S(\boldsymbol{x}_1) e^{-iH(x_1^0+T)} | \varphi_a \rangle \tag{4.32}$$

At this point in Section 2.6 we inserted complete sets of position eigenstates and replaced the operators with their eigenvalues. As mentioned previously, this step is not immediately justified, because the field operators don't have eigenstates. I don't want to get into the technicalities of coherent states, so let me just reassure you that we won't make any mistakes, if we assume that we can replace field operators with c-number fields

$$\hat{\varphi}_S(\boldsymbol{x}_1)|\varphi_1\rangle = \varphi_1(\boldsymbol{x}_1)|\varphi_1\rangle,$$

and that the completeness relation

$$\int D[\varphi_1]\ |\varphi_1\rangle\langle\varphi_1| = \hat{1},$$

[4] I am using the notation from Section 9.2 of Peskin and Schroeder. I hope no one will be so symbol-minded as to confuse the T in the exponentials (meaning some fixed time) and the T in front of the fields (the time ordering operator).

makes sense. Now insert $\hat{1}$ after each of the fields in (32). The result is

$$\int D[\varphi_1(\boldsymbol{x})] \int D[\varphi_2(\boldsymbol{x})]\ \varphi_1(\boldsymbol{x}_1)\varphi_2(\boldsymbol{x}_2)\langle\varphi_b|e^{-iH(T-x_2^0)}|\varphi_2\rangle \tag{4.33}$$

$$\times\langle\varphi_2|e^{-iH(x_2^0-x_1^0)}|\varphi_1\rangle\langle\varphi_1|e^{-iH(x_1^0+T)}|\varphi_a\rangle$$

This is a product of three propagators that jolly along the field from $-T \to x_1^0 \to x_2^0 \to T$. For example,

$$\langle\varphi_b|e^{-iH(T-x_2^0)}|\varphi_2\rangle = \int D[\varphi]\exp\left[i\int_{x_2^0}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

combining these three factors yields the final result

$$\langle\varphi_b|e^{-iHT}T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)]e^{-iHT}|\varphi_a\rangle \tag{4.34}$$

$$= \int D[\varphi]\varphi(x_1)\varphi(x_2)\exp\left[i\int_{-T}^{T} d^4x\ \mathcal{L}(\varphi)\right]$$

where the boundary conditions on the path integral are $\varphi(-T,\boldsymbol{x}) = \varphi_a(\boldsymbol{x})$ and $\varphi(T,\boldsymbol{x}) = \varphi_b(\boldsymbol{x})$ for some φ_a and φ_b. Notice that if we had started with the assumption that $x_1^0 > x_2^0$, we would have arrived at the same result. We get the time ordering for free.

I have set up the correlation function with the e^{-iHT}'s in (31) because I want to discuss the limit $T \to \infty$. This has the effect of projecting out the ground state, but exactly what does this mean and what is the ground state anyhow? We are accustomed to dividing the Hamiltonian into two parts, the free-particle and the potential or interaction term, $H = H_0 + H_I$. If we set $H_I = 0$, the particle has energy eigenstates and eigenenergies, and clearly, the minimum energy is $E_0 = 0$. This makes sense mathematically, but not physically: nature doesn't come equipped with switches to do this. This is a highly non-trivial problem, because a single particle can interact with itself via intermediate particles. Worse yet, even the vacuum can interact with itself through the spontaneous creation of particle-antiparticle pairs. We call this minimum-energy vacuum state $|\Omega\rangle$.

Define the eigenstates of H as follows:

$$H|n\rangle = E_n|n\rangle$$

Imagine starting with $|\varphi_a\rangle$, and evolving through time with H.

$$e^{-iHT}|\varphi_a\rangle = \sum_n e^{-iE_nT}|n\rangle\langle n|\varphi_a\rangle$$

$$= e^{-iE_0T}|\Omega\rangle\langle\Omega|\varphi_a\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|\varphi_a\rangle$$

Give a small negative imaginary part to T and look at what happens when $T \to \infty(1 - i\epsilon)$. Each term tends toward zero, but since $E_0 < E_n$ for all $n \neq 0$, the first term approaches zero more slowly. Put it another way: if we divide the equation through by $e^{-iE_0T}\langle\Omega|\varphi_a\rangle$, the term containing the sum will vanish.

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} (e^{-iE_0T}\langle\Omega|\varphi_a\rangle)^{-1} e^{-iHT}|\varphi_a\rangle. \tag{4.35}$$

Insert (35) into (34).

$$\langle\Omega|T[\varphi(\hat{x}_1)\varphi(\hat{x}_2)]|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int D[\varphi]\varphi(x_1)\varphi(x_2)\exp\left[i\int_{-T}^{T} d^4x\mathcal{L}\right]}{\int D[\varphi]\exp\left[i\int_{-T}^{T} d^4x\mathcal{L}\right]} \tag{4.36}$$

I remarked previously that the measure $\int D[\varphi]$ produced weird normalization constants. Here as always, these constants cancel between the numerator and denominator. Often in doing perturbation theory, we are concerned with the correlation function for two free particles. In this case we replace $|\Omega\rangle$ with $|0\rangle$. The denominator is then proportional to $\langle 0|0\rangle = 1$, so we can ignore it altogether.

Back at the end of Chapter 2, I introduced the notion of a generating function. The key equations here are (2.64) and (2.65), which I repeat here for reference.

$$z[J] \equiv \langle q_f, t_f|q_i, t_i\rangle^J \equiv \int D[q]\exp\left\{i\int_{t_i}^{t_f} dt\,[L(t) + J(t)q(t)]\right\} \tag{4.37}$$

$$\langle q_f, t_f|T\left[\hat{Q}(t_1)\hat{Q}(t_2)\cdots\hat{Q}(t_n)\right]|q_i, t_i\rangle = \left(\frac{1}{i}\right)^n \frac{\delta^n z[J]}{\delta J(t_1)\delta J(t_2)\cdots\delta J(t_n)}\bigg|_{J=0} \tag{4.38}$$

We take advantage of recent developments by upgrading these formulas in two ways: (1) replacing the Q_i's with the fields, $\varphi(x)$, and (2) taking the limit $T \to \infty(1 - i\epsilon)$. It is customary to use the capital Z for this new generating functional.

$$Z[J] \equiv \int D[\varphi]\exp\left\{i\int_{-\infty}^{\infty} d^4x\,[\mathcal{L}(\varphi) + J(x)\varphi(x)]\right\} \tag{4.39}$$

We can use this technology to rewrite (36) as follows:

$$\langle\Omega|T[\varphi(\hat{x}_1)\varphi(\hat{x}_2)]|\Omega\rangle = \frac{1}{Z[0]}\left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \tag{4.40}$$

The generalization to higher-order correlation functions is obvious. We sometimes use the notation

$$\langle\Omega|T[\varphi(\hat{x}_1)\varphi(\hat{x}_2)\cdots\hat{\varphi}(x_n)]|\Omega\rangle = G(x_1, x_2, \cdots, x_n) \tag{4.41}$$

The function $G(x_1, x_2, \cdots, x_n)$ is called the "n-point function."

4.3 The Feyman Propagator

I will now show how this formalism works with the free Klein-Gordon field. Use the Lagrangian (23) in (39)

$$Z[J] = \int D[\varphi] \exp\left\{ i \int d^4x \left[\frac{1}{2}\left((\partial\varphi)^2 - m^2\varphi^2 + i\epsilon\varphi^2 \right) + J\varphi \right] \right\} \tag{4.42}$$

I have inserted the $i\epsilon\varphi^2$ to ensure that the exponential integrals converge as explained previously. The key to doing the $\int d^4x$ integral is to Fourier transform the integrand to momentum space.

$$\varphi(x) = \int \frac{d^4p}{(2\pi)^2} e^{-ipx} \tilde{\varphi}(p) \tag{4.43}$$

Substituting (43) into (42) gives the following:

$$\int d^4x \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2 + i\epsilon\varphi^2 \right] \tag{4.44}$$

$$= \int \frac{d^4p_1 d^4p_2}{2(2\pi)^4} = \left[-p_1p_2 - m^2 + i\epsilon \right] \tilde{\varphi}(p_1)\tilde{\varphi}(p_2) \int d^4x e^{-i(p_1+p_2)}$$

$$= \frac{1}{2} \int d^4p \; \tilde{\varphi}(-p)\tilde{\varphi}(p)(p^2 - m^2 + i\epsilon)$$

On the other hand

$$\int d^4x J(x)\varphi(x) = \int d^4p \tilde{J}(-p)\tilde{\varphi}(p) \qquad \text{or} \qquad \int d^4p \tilde{J}(p)\tilde{\varphi}(-p) \tag{4.45}$$

$$\text{or (in fact)} \qquad \frac{1}{2}\int d^4p \left[\varphi(\tilde{P})\tilde{J}(-p) + \tilde{\varphi}(-p)\tilde{J}(p) \right]$$

The reason for the two terms will be clear soon. In order to make the notation a bit more compact, I will define

$$\Delta^{-1} \equiv p^2 - m^2 + i\epsilon \tag{4.46}$$

The complete action becomes

$$\begin{aligned} S &= \int d^4x \, (\mathcal{L} + J\hat{\varphi}) \\ &= \frac{1}{2}\int d^4p \left[\tilde{\varphi}(-p)\tilde{\varphi}(p)\Delta^{-1} + \tilde{J}(p)\tilde{\varphi}(-p) + \tilde{J}(-p)\tilde{\varphi}(p)\right] \end{aligned} \tag{4.47}$$

Define a new field

$$\tilde{\varphi}(\pm p) = \tilde{\varphi}_0(\pm p) - \tilde{J}(\pm p)\Delta \tag{4.48}$$

$$S = \frac{1}{2}\int d^4p \left[\tilde{\varphi}_0(-p)\tilde{\varphi}_0(p)\Delta^{-1} - \tilde{J}(p)\tilde{J}(-p)\Delta\right] \tag{4.49}$$

You see the point of (49)? It break up the action into two pieces, one depending only on the fields and one depending only on the sources. The peculiar sum in (45) was necessary to make this happen. Substituting into (42) yields

$$Z[J] = \int D[\varphi_0] \exp\left\{-\frac{i}{2}\int d^4p \; \tilde{\varphi}_0(p)\tilde{\varphi}_0(-p)\Delta^{-1}\right\} \tag{4.50}$$

$$\times \exp\left\{\frac{i}{2}\int d^4p \; \tilde{J}(p)\tilde{J}(-p)\Delta\right\}$$

$$= Z[0]e^{iW[J]} \tag{4.51}$$

where

$$W[J] \equiv -\frac{1}{2}\int d^4p \; \tilde{J}(p)\tilde{J}(-p)\Delta(p) \tag{4.52}$$

Most path integrals are impossible to do, we just don't have to do them. We don't have a snowball's chance of calculating $Z[0]$, but (40) guarantees that it will always cancel out!

At this point it is useful to transform back to configuration space.

$$\tilde{J}(\pm p) = \int \frac{d^4x}{(2\pi)^2} e^{\pm ipx} J(x) \tag{4.53}$$

$$\Delta(p) = \int d^4x \; e^{ipx} D(x)$$

Our final result is

$$W = -\frac{1}{2}\int d^4x \; d^4y \; J(x)J(y)D(x-y) \tag{4.54}$$

where

$$D(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon} \tag{4.55}$$

This is the Feynman propagator.[5] It is the relativistic generalization of the one-particle Green's function (2.48). As such it represents the amplitude for a particle at the space time point x to propagate to y. The boundary conditions are different, however, since $D(x-y) = D(y-x)$. Relativity forces this on us, as I mentioned before.

The derivation of $Z[J]$ assumed that sources were turned on and off at finite times before and after the limit $T \to \infty$. Let's see how this idea plays out in (50). First expand $Z[J]/Z[0]$ in an exponential series. (See equation (51).)

$$\frac{Z[J]}{Z[0]} = 1 + iW + \frac{i^2}{2}W^2 + \cdots \tag{4.56}$$

The second term is easy to understand.

$$iW = -\frac{i}{2}\int\int d^4x\, d^4y\, J(x)J(y)D(x-y) \tag{4.57}$$

Suppose our source consists of two local centers of activity, so that $J(x) = J_0[\delta(x-x_1) + \delta(x-x_2)]$. Then

$$iW = -iJ_0^2[D(x_1-x_2) + D(0)] \tag{4.58}$$

The $D(0)$ is meaningless. It's an artifact of our delta function sources. The other term is simply the probability of a particle to be created by the source at x_1 (when the source was turned on) and reabsorbed at x_2 (when the source was turned off) *and vice versa.*

Now calculate the two-point function corresponding to iW without specifying the nature of the sources. The following calculations use equations

[5] To be consistent I suppose, we should use $\tilde{D}(p)$ instead of $\Delta(p)$, but this is not customary notation.

(40), (41), (56) and (57).

$$\begin{aligned} G(x_1, x_2) &= \langle \Omega | T[\hat{\varphi}(x_1)\hat{\varphi}(x_2)] | \Omega \rangle \\ &= \frac{1}{Z[0]} \left(\frac{1}{i}\right)^2 \frac{\delta^2 Z[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\ &= \left(\frac{1}{i}\right)^2 \frac{\delta^2 i W[J]}{\delta J(x_1)\delta J(x_2)}\bigg|_{J=0} \\ &= \left(\frac{1}{i}\right)^2 \frac{\delta^2}{\delta J(x_1)\delta J(x_2)} \left[-\frac{i}{2}\int\int d^4x\, d^4y\, J(x)J(y)D(x-y)\right]_{J=0} \end{aligned} \tag{4.59}$$

This functional derivative was worked out as an example in Chapter 3. The relevant equations are (3.10) and (3.12). The effect of the functional derivatives has been to "peel off" the sources. The final result is

$$G(x_1, x_2) = iD(x_1 - x_2). \tag{4.60}$$

This is in fact the exact amplitude for a free particle to propagate from x_1 to x_2. The higher order terms in (56) vanish when we set $J = 0$. It is easy to see that the n-th term of the expansion (56) contains $2n$ factors of J and that all the n-point functions vanish except for $G(x_1, x_2, \cdot, x_{2n})$. Your should work out the 4-point function as an exercise. It consists or a sum of three terms corresponding to all possible ways in which two particles can propagate between four points. Now that you have the idea, you can represent any n-point function as a set of (2n-1)!! diagrams. Just draw n points labelled $x_1, x_2, \cdots, x_{2n}$, and draw lines connecting them in pairs, These are Feynman diagrams.

The theory so far has been trivial in the sense that the particles don't interact with one another. We started with the Lagrangian based on the simple harmonic oscillator, i.e. with a quadratic potential, and in such a model all the normal modes are linearly independent. This is well known from continuum mechanics, and it's true in quantum mechanics as well. Interacting field theories have additional terms in the Lagrangian. We'll look at a simple example in the next section.

4.4 Interacting Fields

In order to make out fields interact, we must add some additional term to the Lagrangian. Many texts use the interaction term $-(\lambda/4!)\varphi^4(x)$, which is as

good as any. As you will see, this describes identical particles interacting in groups of four. An example is the scattering process in which two particles come together at a point and two particles emerge from that point. Another is the decay process in which one particle decays into three clones of itself. The parameter λ is called a "coupling constant." It determines the strength of the interaction. You probably don't know any particles that act like this, but it turns out to be an important ingredient in the non-Abelian gauge theories we will study much later in the course. More familiar theories all involve two or more different particle species and spin, but "Sufficient unto the day is the evil thereof."

The simplest non-trivial process involving these hypothetical particles is the two-body scattering process described above, It turns out that everything we can know about this process is contained in the four-point Green's function. We need to calculate the four-field version of (40),

$$\frac{1}{Z[0]}\left(\frac{1}{i}\right)^4 \frac{\delta^4 Z[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} = G(x_1,x_2,x_3,x_4) \qquad (4.61)$$

with our new $Z[J]$,

$$Z[J] = \int D[\varphi]\exp\left\{i\int d^4x\left[\frac{1}{2}\left((\partial\varphi)^2 - m^2\varphi^2 + i\epsilon\varphi^2\right) + J\varphi - (\lambda/4!)\varphi^4\right]\right\} \qquad (4.62)$$

The tricks we used in arriving at (50) no longer work, and we are forced to make approximations. As I am fond of saying, "When in doubt, expand in a power series." Feynman diagrams are a kind of perturbation theory in which the coupling constant is assumed to be "small" so that perturbation series is really an expansion in powers of the coupling constant, in this case λ. Let's start with the first-order term.

$$Z_1[J] = \frac{-i\lambda}{4!}\int d\omega \int D[\varphi]\exp\left\{i\int d^4x\ \left[-\frac{1}{2}\varphi(\partial^2 + m^2 - i\epsilon)\varphi + J\varphi\right]\right\}\varphi^4(\omega) \qquad (4.63)$$

You will notice that I replaced $(\partial\varphi)^2$ in the exponential with $-\varphi\partial^2\varphi$. This

is just integration by parts. Now substitute (4.63) into (4.61)

$$\begin{aligned} G(x_1,x_2,x_3,x_4) =& \frac{1}{Z[0]}\left(\frac{1}{i}\right)^4 \frac{\delta^4 Z_1[J]}{\delta J(x_1)\delta J(x_2)\delta J(x_3)\delta J(x_4)}\bigg|_{J=0} \\ =& \frac{-i\lambda}{4!}\frac{1}{Z[0]}\int d^4\omega \int D[\varphi]\exp\left\{i\int d^4x\,\left[-\frac{1}{2}\varphi(\partial^2+m^2-i\epsilon)\varphi+J\varphi\right]\right\}\bigg|_{J=0} \\ & \times\varphi^4(\omega)\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4) \end{aligned} \tag{4.64}$$

You see that long string of φ's on the last line of (4.64)? That is the paradigmatic integral. If you can do that, you can calculate any Green's function to any order in λ. In fact, there is a simple, intuitive "cookbook" procedure for evaluating all such expressions effortlessly. The trick is yet another application of our basic gaussian integral.[6]

$$\int_{-\infty}^{\infty} dx_1 dx_2\cdots dx_N\; e^{-\frac{i}{2}x\cdot A\cdot x+iJ\cdot x} = \sqrt{\frac{(2\pi i)^N}{\det[iA]}}e^{\frac{i}{2}J\cdot A^{-1}\cdot J} \tag{4.65}$$

The notation $x\cdot A\cdot x$ means $\sum_{ij}x_iA_{ij}x_j$, etc. The matrix A is assumed to be real and symmetric. Let's define

$$\langle x_ix_j\cdots x_kx_l\rangle \equiv \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1dx_2\cdots dx_Ne^{-\frac{i}{2}x\cdot A\cdot x}x_ix_j\cdots x_kx_l}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1dx_2\cdots dx_Ne^{-\frac{i}{2}x\cdot A\cdot x}} \tag{4.66}$$

By repeated differentiation of (4.65) we can derive

$$\langle x_ix_j\cdots x_kx_l\rangle = \sum_{\text{Wick}}(A^{-1}/i)_{ab}\cdots(A^{-1}/i)_{cd} \tag{4.67}$$

The subscripts ab and cd in (4.67) stand for any pair of the indices $i,j,\cdots k,l$. A sum over "Wick contractions" consists of all possible ways of connecting the indices in pairs. If there is only one pair,

$$\langle x_ix_k\rangle = (A^{-1}/i)_{ik} \tag{4.68}$$

There is a useful analogy between (4.65) and (4.66) and the last two lines of (4.64). Clearly J_i is analogous to $J(x)$, and x_i is analogous to $\varphi(x)$. We've discussed this before. So far so good. The term $\frac{i}{2}J\cdot A^{-1}\cdot J$ is reminiscent

[6]There is a brief proof of this in Appendix 1 of Zee's book. I will ask you to derive it in more detail for homework.

of (54), so it's a good guess that A^{-1} is analogous to $-D(x-y)$, but if this is true, A must be analogous to $(\partial^2 + m^2 - i\epsilon)$. *Quel horreur!* Worse yet, what does it mean to take the determinant of it?

The answer to the last question is clear: we don't care, since the determinant cancels in equation (4.66). In fact, come to think of it, the denominator in (4.66) must be analogous to $Z[0]$. The other point is almost clear, since

$$(\partial^2 + m^2 - i\epsilon)D(x-y) = -\delta^{(4)}(x-y) \tag{4.69}$$

The proof of (4.69) is easily established using the definition (55). $D(x-y)$ is, after all, a Green's function, and Green's functions are by definition, inverses of differential operators. That's what they do for a living. In summary then, I claim that we can use (4.65) and (4.66) for fields rather than discrete variables by making the following replacements:

$$\begin{aligned}
x_i &\rightarrow \varphi(x_i) \\
J_i &\rightarrow J(x_i) \\
A^{-1} &\rightarrow -D(x-y) \\
\frac{i}{2}J\cdot A^{-1}\cdot J &\rightarrow -\frac{i}{2}\int d^4x d^4y J(y)D(x-y)J(y) \\
A &\rightarrow (\partial^2 + m^2 - i\epsilon) \\
-\frac{i}{2}x\cdot A\cdot x &\rightarrow -\frac{i}{2}\int d^4x \varphi(x)(\partial^2 + m^2 - i\epsilon)\varphi(x) \\
\int_{-\infty}^{\infty} dx_1 dx_2 \cdots dx_n &\rightarrow \int D[\varphi]
\end{aligned}$$

The proof that my claim is valid is sketched out in Ryder,[7] but in fact, we have implicitly proved it, at least for expressions of the form (4.64), as you will see. Anyhow, if my claim is true then the correct generalization of (4.66) is

$$\begin{aligned}
&\langle\varphi(x_i)\varphi(x_j)\cdots\varphi(x_k)\varphi(x_l)\rangle \\
&= \frac{\int D[\varphi]\exp\left\{i\int d^4x\left[-\frac{1}{2}\varphi(\partial^2 + m^2 - i\epsilon)\varphi\right]\right\}\varphi(x_i)\varphi(x_j)\cdots\varphi(x_k)\varphi(x_l)}{\int D[\varphi]\exp\left\{i\int d^4x\left[-\frac{1}{2}\varphi(\partial^2 + m^2 - i\epsilon)\varphi\right]\right\}}
\end{aligned} \tag{4.70}$$

The denominator is just $Z_0[0]$, ie. equation (4.62) with $J = \lambda = 0$.

[7] Lewis Ryder, *Quantum Field Theory*, Section 6.2

To see how this works, let's evaluate (4.70) in the case of two fields.

$$\begin{aligned} G(x_1,x_2) &= \frac{1}{Z_0[0]}\left(\frac{1}{i}\right)^2 \left.\frac{\delta^2 Z_0[J]}{\delta J(x_1)\delta J(x_2)}\right|_{J=0} \\ &= \frac{\int D[\varphi]\exp\left\{i\int d^4x\ \left[-\frac{1}{2}\varphi(\partial^2+m^2-i\epsilon)\varphi\right]\right\}\varphi(x_1)\varphi(x_2)}{\int D[\varphi]\exp\left\{i\int d^4x\ \left[-\frac{1}{2}\varphi(\partial^2+m^2-i\epsilon)\varphi\right]\right\}} \end{aligned} \tag{4.71}$$

The first line follows from (59), and the second line uses (4.62) with $\lambda = 0$. A glance at (4.70) reveals that what we have just calculated is $\langle\varphi(x_1)\varphi(x_2)\rangle$.

Let's pause for a moment to see how all the pieces fit together. $\langle\varphi(x_1)\varphi(x_2)\rangle$ as defined by (4.70) equals $G(x_1,x_2)$. That proof required nothing more than functional differentiation. Reasoning by analogy from (4.68) led us to hope that $\langle\varphi(x_1)\varphi(x_2)\rangle$ would be equal to $-D(x_1-x_2)/i$. But we already know from (60) that $G(x_1,x_2) = iD(x_1-x_2)$, so our cookbook analogy has given us the right answer for the Feynman propagator without any of the clever calculations in Section 4.3. Put it another way: we have proved that the analogy works correctly in the case of (71). The analogy to (4.67) is

$$\langle\varphi(x_i)\varphi(x_j)\cdots\varphi(x_k)\varphi(x_l)\rangle = \sum_{\text{Wick}} G(x_a,x_b)\cdots G(x_c,x_d) \tag{4.72}$$

So here's the procedure: make a list of all possible ways of pairing up the indices $i,j,\cdots,k,l$. Each entry in the list corresponds to one term in the sum. That's all there is to it. I'm willing to believe this without further proof.

Now you can evaluate (4.64) using (4.70).

$$G(x_i,x_2,x_3,x_4) = -\frac{i\lambda}{4!}\int d^4\omega\langle\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\varphi^4(\omega)\rangle \tag{4.73}$$

The Wick sum contains in principle $(N-1)!! = 7!! = 105$ terms. It is probably the case, however, that most of these terms are of no interest. Draw some pictures to find the interesting ones. Label five points 1,2, 3, 4, and ω. Each term in the expansion corresponds to a way of connecting these points. Here are the rules:

- There are eight fields in (4.73), therefore there must be four lines connecting points. Each line corresponds to one of the $G(x_a,x_b)$'s in (4.72).
- The points 1, 2, 3, and 4, must be attached to one and only one line. Since there are four factors of $\varphi(\omega)$ in (4.73), there must be four "line ends" at the ω point, *ie.* it can be connected to itself by one or two lines.

The points 1, 2, 3, and 4 are connected to external sources, or to put it another way, $J(x_1)$, $J(x_2)$, $J(x_3)$, and $J(x_4)$ were used to define $G(x_1, x_2, x_3, x_4)$. The ω point is not. If you want all four of these particles to interact, they must each be contracted with one of the $\varphi(\omega)$'s. There are 4! terms like this in the Wick sum. The final result is

$$G(x_1, x_2, x_3, x_4) = -i\lambda \int d^4\omega G(x_1, \omega)G(x_2, \omega)G(x_3, \omega)G(x_4, \omega) \qquad (4.74)$$

Forgive me if I gloat a bit. A few paragraphs ago we had some algebra, which, if done correctly, would have yielded 105 terms. Now we draw a few pictures, and the answer is obvious by inspection. These pictures are called Feynman diagrams.

You have noticed that we have ignored 105-4!=81 terms, These are all pathological in the sense that the ω point is connected to itself by at least one line. It is a general rule that all Feynman diagrams with loops give rise to divergent integrals. This is ultimately related to the fact that there are parameters such as the mass and coupling constant (or charge) of a particle that are fundamentally mysterious. If these integrals converged, then we might be able to calculate them. These divergences are nature's way of slamming the door in our face. That's the big bad news. The little bit of good news is that we can go on doing field theory calculating everything else, if we treat these divergences *very carefully*. This comes under the general heading of renormalization, to which we will return later.

4.5 Momentum Space

Our derivations have for the most part used configuration space. Momentum space was only used occasionally as a tool for evaluating certain integrals. Momentum space is more natural for many quantum processes, however. Scattering is a good example. The particles in the beam have a well-defined momentum. The target particle is at rest. The momenta of the final state particles are determined by our apparatus. The rules for Feynman diagrams discussed briefly above also have a simpler form in momentum space.

Define the momentum space Green's functions in the obvious way.

$$\tilde{G}(p_1, p_2, \cdots, p_N) = \int d^4x_1 d^4x_2 \cdots d^4x_N e^{i(\pm p_1x_1 \pm p_2x_2 \pm \cdots \pm p_Nx_N)} G(x_1, x_2, \cdots, x_N) \qquad (4.75)$$

I call your attention to the $\pm$ signs in the exponential. To some extent the sign doesn't matter, since $G(x) = G(-x)$ and $\Delta(p) = \Delta(-p)$. Moreover,

when x or p appear as integration variables, one is always free to replace $x \to -x$ or $p \to -p$. In one respect, however, the signs are all-important. Take the Fourier transform of (4.74) using (4.75) and (4.53).

$$\tilde{G}(p_1, p_2, p_3, p_4) = -i\lambda(2\pi)^4\delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm p_4)\Delta(p_1)\Delta(p_2)\Delta(p_3)\Delta(p_4) \tag{4.76}$$

The δ-function is there to enforce momentum conservation. For example, if the scattering process is one in which p_1 and p_2 are the incident momentum and p_3 and p_4 are the outgoing momentum, then the δ-function must be $\delta^{(4)}(p_1 + p_2 - p_3 - p_4)$ or $\delta^{(4)}(p_3 + p_4 - p_1 - p_2)$. Let's agree on a convention: outgoing momenta are positive and incoming momenta are negative. That way we will always get the right signs in (4.75).

Equation (4.76) illustrates some of the rules associated with Feynman diagrams. The momentum space Green's function corresponding to any one diagram consists of a product of the following factors.

- A factor $-i\lambda$ for each vertex.
- A factor $i\Delta(p_i)$ for each external line.
- An energy-momentum conserving δ-function $(2\pi)^4\delta^{(4)}(\pm p_1 \pm p_2 \pm p_3 \pm p_4)$

There is one more important item that we have not yet encountered. What happens to internal lines, *ie.* those that do not terminate on external sources. To see that, let's look at one of the pathological diagrams in which one of the lines connected to the ω loops back on itself. One such term in the Wick expansion is[8]

$$G(x_1, x_2, x_3, x_4) = -\frac{i\lambda}{4!}\int d^4\omega\; G(x_1, x_2)G(x_3, \omega)G(x_4, \omega)G(\omega, \omega) \tag{4.77}$$

The momentum space Green's function is

$$G(p_1, p_2, p_3, p_4) = -\frac{i\lambda}{4!}(2\pi)^4\delta^{(4)}(p_1 - p_2)(2\pi)^4\delta^{(4)}(p_3 - p_4) \times \Delta(p_1)\Delta(p_3)\Delta(p_4)\int \frac{d^4p_4'}{(2\pi)^4}\Delta(p_4') \tag{4.78}$$

In this case, the particle that starts out with momentum p_1 does not interact with the one with momentum p_3, therefore their momenta are conserved

[8] We have not yet come to terms with indistinguishable particles. For the moment, assume that the particles have little numbers on them like billiard balls.

separately. There are two δ-function The new feature here is an internal line, this one labelled by p_4'. Since this momentum can have any value, there is nothing to constrain the $\int d^4p_4'$ integral. *Such integrals always diverge.* To see that this is so, look at

$$\int \frac{d^4p_4'}{p_4'^2 - m^2 + i\epsilon} \tag{4.79}$$

The numerator contains four powers of the momentum, whereas the denominator contains only two. There is no oscillating phase factor, so the integral diverges quadratically. Such integrals can be dealt with rigorously with a variety of different techniques. For example, one can put a cutoff on the range on integration, and then investigate analytically how the result depends on the cutoff. For a certain privileged class of theories, one can show that the only cutoff-dependent quantities are the particle's mass and coupling constant and the normalization of the wave function. Since we know these things already, we can simply ignore the cutoff-dependent parts. Such theories are said to be *renormalizable*. The fact that we *have* to do this is a sign that our knowledge and theory are incomplete. The fact that we *can* do it is also significant. All theories we know that describe the real world, at least approximately, are renormalizable. If you give me a theory that is not renormalizable, *I know that it's wrong!*

4.6 Virtual Particles

We have encountered the free-particle Feynman propagator, first in momentum variables (46) and later in space-time variables (55). When we evaluate out Green's functions they appear attached to the external sources as well as to points inside the region where the particles interact with themselves. In both cases they have a peculiar feature: in some sense they represent particles that are not "on the mass shell." For free particles, after all, $p^2 = p^\mu p_\mu = m^2$, so it seems that the denominators of (46) and (55) should vanish. They don't of course, because p^μ is not a physical momentum, but a variable of integration. There is more to it than that, however. We cannot specify space and momentum coordinates with arbitrary accuracy. The propagator $D(x-y)$ describes a situation in which x and y are microscopically close together, so there is some "fuzziness" in the momentum as well. Since we cannot specify momentum and energy with arbitrary precision, we can't insist that they are perfectly conserved. The propagator takes this into account in a way that is consistent with relativity. p^2 is a Lorentz scalar,

it's just not equal to m^2. We say that the propagator describes "virtual particles."

The propagator $\Delta(p_4')$ in (4.78) describes particles that are virtual because they are emitted and absorbed in a very short distance. The other propagators in (4.77) and (4.78) describe particles that are virtual for a different reason: the sources $J(x_i)$ that we built into the theory are also quantum mechanical objects. They are not external sources like x-ray tubes or cyclotrons. That should be evident from the discussion in Section 4.2. The sources were turned on sometime after $-T$ and turned off again before T, but the limits $-T \to -\infty$ and $+T \to +\infty$ that were used to project out the ground states refer to the macroscopic past and future where we live and move and have our being. Put it another way, we live in the ground state. The action takes place after the sources turn on , and it stops when the sources turn off.

We often use the momentum space Green's functions to calculate scattering reactions. Indeed, that is their principal use. But particles in scattering experiments are produced by cyclotrons and other macroscopic sources. We can use our Green's functions for this by the simple expedient of removing the Δ's from the external legs. This is sometimes called "amputating the legs of the diagram."[9] This procedure then becomes part of the rules for Feynman diagrams.

Leaving aside the very technical issue of renormalization, you now have everything you need to do perturbation theory calculations with scalar field theory to any order. There is a complete description of the rules in Zee's book, pages 53 and 54. I will restate them here for reference.

To calculate a momentum space Green's function corresponding to a Feynman diagram, do the following:

- Draw the diagram and label each line with a momentum.
- Associate each line carrying momentum p with a propagator $i/(p^2 - m^2 + i\epsilon)$.
- Associate with each interaction vertex the coupling $-i\lambda$ and $(2\pi)^4\delta^{(4)}(\sum_i p_i - \sum_j p_j)$, forcing the sum of momenta $\sum_i p_i$ going out of the vertex to be equal to the sum of momenta $\sum_j p_j$ going out of the vertex.
- Momenta associated with internal lines are to be integrated over with the measure $d^4p/(2\pi)^4$.
- Remove those propagators associated with external lines.

[9] Don't let PETA find out about this.

4.7 A Word About Renormalization

I have said repeatedly that all diagrams with complete loops lead to divergent integrals. That's bad news enough, but what is worse, these divergent integrals contribute real measurable effects to the physical processes described by the Green's functions. The challenge is to divide these integrals into two parts; one part that is infinite and one part that is finite and contains all measurable effects. This is really a huge subject, but I would like to give you a few of the key ideas. I will do this in the context of the simplest possible divergent integral. It arises when one tries to do calculations in the so-called "phi-third theory."

$$\mathcal{L}_{\text{int}} = -i\frac{\lambda}{3!}\varphi^3 \tag{4.80}$$

When one calculates the two-point function $G(x_1, x_2)$ to second order in perturbation theory, one comes up against the integral

$$G_2(x_1, x_2) = (-i\lambda)^2 \int d^4w\; d^4z\; G(x, w)G(w, z)G(w, z)G(z, x_2) \tag{4.81}$$

This will diverge because of the loop. It represents the second-order correction to the propagator.

First take the Fourier transform

$$\begin{aligned} G_2(p_1, p_2) = \int d^4x_1\; d^4x_2\; d^4w\; d^4z\; e^{i(p_2x_2 - p_1x_1)} \int \frac{d^4p'_1 \cdots d^4p'_4}{(2\pi)^{16}} \\ \times\, e^{i(x_1-w)}e^{i(w-z)}e^{i(w-z)}e^{i(z-x_2)}i\Delta(p'_1)i\Delta(p'_2)i\Delta(p'_3)i\Delta(p'_4) \end{aligned} \tag{4.82}$$

Do the integrals over $\int d^4x_1\; d^4x_2\; d^4w\; d^4z$, then do three momentum integrals $\int d^4p'_1\; d^4p'_3\; d^4p'_4$. The result when all is said and done is

$$G_2(p_1, p_2) = (-i\lambda)^2(2\pi)^4\delta^{(4)}(p_1 - p_2)\Delta(p_1)\Delta(p_2) \int \frac{d^4k}{(2\pi)^4}\Delta(k)\Delta(p_1 - k) \tag{4.83}$$

The integration variable p'_2 has be renamed k, since there are no other primed variables left.

That last integral is going to give us trouble. Let's concentrate on it by defining

$$F(p) \equiv \int \frac{d^4k}{(2\pi)^4}\Delta(k)\Delta(p - k) \tag{4.84}$$

The integral is not only difficult, it's divergent. We will deal with these two problems with two amazing tricks; the first due to Feynman. Feynman

argued that a four-fold integral is hard to do, so let's make it a five-fold integral! The basic formula is this:

$$\frac{1}{A_1 A_2} = \int_0^1 dz[A_1 z + A_2(1-z)]^{-2} \equiv \int_0^1 \frac{dz}{D(z)^2} \tag{4.85}$$

The key idea is that the product $A_1 A_2$ is replaced by the single factor D^2. This, as it turns out, is enough to make the d^4k integrable doable. In our case

$$D^2 = B^2 + Q^2 - k'^2 - i\epsilon = C^2 - k'^2 - i\epsilon \tag{4.86}$$

where

$$Q = p(1-z) \qquad B^2 = m^2 - p^2(1-z) \qquad k = k' + Q \qquad c^2 = B^2 + Q^2$$

Dropping the prime, we have

$$F(p) = \int_0^1 dz \int \frac{d^4k}{(2\pi)^4} \frac{1}{[C^2 - k^2 - i\epsilon]^2} \tag{4.87}$$

A bit of power counting shows that there are four powers of k in the numerator and four in the denominator. The integral diverges logarithmically, and here is where the next great idea comes in. Note that the integral

$$\int_1^\infty \frac{dx}{x}$$

diverges, but

$$\int_1^\infty \frac{dx}{x^{1+\epsilon}} = \frac{1}{\epsilon}$$

ie. increasing the exponent in the denominator by an infinitesimal amount makes the integral convergent. We would like to integrate over $1/D^{2+\epsilon}$ but this makes no sense whatsoever (D is a four vector). Suppose we could integrate over slightly less that four dimensions! In the context of the trivial example above, suppose we could integrate

$$\int_1^\infty \frac{d^{(1-\epsilon)}x}{x}$$

Here's the idea

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(C^2 - k^2 - i\epsilon)^2} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \left(\frac{1}{C^2}\right)^{2-d/2} \tag{4.88}$$

This was derived for integer d (don't ask me how), but the Γ functions and everything else on the right side of this equation are perfectly well-defined continuous functions of d. We use (4.88) as a *definition* of what it means to integrate over a non-integer number of dimensions. We will use (4.88) to integrate (4.87) and then see what happens in the limit $d \to 4$. Incidentally,

$$\Gamma(x) = \int_0^\infty dt\; t^{x-1} e^{-t}, \tag{4.89}$$

Also $\Gamma(n) = (n-1)!$ for integer n, and $\Gamma(x+1) = x\Gamma(x)$ in general.

After some algebra we find

$$F(p) = \frac{i}{(4\pi)^{d/2}} \frac{2\Gamma(1+\epsilon/2)}{\epsilon} \int_0^1 \frac{dz}{[m^2 - p^2 z(1-z)]^{\epsilon/2}} \tag{4.90}$$

where $\epsilon \equiv 4 - d$. Well $\Gamma(1) = 1$, so the integral still diverges in the limit $\epsilon \to 0$. We expected that, but I have one more trick up my sleeve. Look at the expansion

$$A^{-\epsilon/2} = 1 - \frac{\epsilon}{2} \ln A + \frac{1}{8}\epsilon^2 (\ln A)^2 + \cdots \tag{4.91}$$

Insert (4.91) into (4.90).

$$\lim_{\epsilon \to 0} F(p) = \infty - \frac{i}{(2\pi)^2} \int_0^1 dz\; \ln\left[m^2 - p^2 z(1-z)\right] \tag{4.92}$$

There are two points to notice here.

- The remaining integral is straightforward, since it involves only one real variable of integration.
- The term "∞" in (4.92) is really

$$\lim_{\epsilon \to 0} \frac{2i}{(4\pi)^2} \frac{1}{\epsilon} \tag{4.93}$$

  The point is that this is a pure number; *It does not depend on p!* We have cleanly separated the measurable physics in the second term from the meaningless constant in (4.93). To used an old cliche, we don't have to worry about throwing the baby(s) out with the bath water, *because we have gotten all the kids out of the tub!*

Well, that's not quite true. Recall that $F(p)$ is multiplied by λ^2 in (4.81). It's as if we were working with an infinite coupling constant. Rather than

"throwing out an infinite term," let's say that the λ that appears in (4.80) is not physically measurable. It's sometimes called the "bare" coupling constant λ_0. What we are doing is replacing $\lambda_0 \infty \to \lambda$. We have "renormalized" the coupling constant. If we were to calculate $G(p_1, p_2)$ to higher orders of perturbation theory, we would find that these divergent integrals also have the effect of renormalizing the mass. We use the same argument; the "bare" mass m_0 is not measurable, but we can always replace $m_0 \infty \to m$, where m is the physical mass. It is remarkable fact that all the infinities that occur to all orders of perturbation theory can be disposed of using these two maneuvers. A theory in which all infinities can be sequestered in a finite number of physical parameters is said to be "renormalizible." We have learned from decades of experience that *all theories that are not renormalizible are wrong* in the sense that they make predictions that are falsified by experiments. The bad news is that we can't calculate m and λ. The good news is that we can calculate everything else. This raises a profound and unanswered question. Are the masses and coupling constants of elementary particles really fundamental constants, or can they be calculated on the basis of something else, and if so, what is that something else?

Chapter 5

Identical Particles and Many-Particle States

Ordinary matter consists of vast numbers of identical particles at low energy. Quantum mechanics requires that the wave functions for such ensembles be totally symmetric or totally antisymmetric, and this in turn is responsible for many important properties, including the fact that atoms don't simply collapse into their nuclei! Ordinary quantum mechanics is capable of describing such systems so long as the number of particles remains constant. In a sense this is always true; there is never enough energy in a semiconductor, for example, to create a single electron; but it is convenient to regard holes in the Fermi-Dirac sea as real particles. Thus the formation of a hole entails the creation of two particles, a positive "particle," the hole, and a negative particle, the excited electron. There are equivalent processes in Boson condensates as we shall see. Second quantization is ideally suited to describe such processes, but first we need to develop the ordinary quantum mechanics of many-particle systems.

5.1 The Quantum Mechanics of Identical Particles

Let's write a many-particle wave function as follows:

$$\psi = \psi(1, 2, \ldots, N) \tag{5.1}$$

The notation 1, for example, means $\mathbf{x_1}, \sigma_\mathbf{1}$. Suppose the particles had numbers on them like billiard balls. They don't of course, that's the whole point, but suppose they did. Then $\psi(1, 2)$ means that the particle numbered 1 was at the point $\mathbf{x_1}$ and it's z-component of spin was σ_1. The wave function

$\psi(2,1)$ means that the number-one ball has components $\mathbf{x_2}$ and σ_2. Our job is to construct a theory in which the sequence of numbers in ψ has no observable consequences. That is what we mean by indistinguishable particles.

Define the permutation operator P_{ij}

$$P_{ij}\psi(\ldots,i,\ldots,j,\ldots) = \psi(\ldots,j,\ldots,i,\ldots) \tag{5.2}$$

Here are a few simple theorems about permutations. The reader is invited to supply the simple proofs.

1. For every symmetric operator $S(1,\ldots,N)$ we have

$$[P,S] = 0 \tag{5.3}$$

   An important example of a symmetric operator is the Hamiltonian. This should be obvious, since a low-energy Hamiltonian will have the form

$$H = \sum_i \frac{p_i^2}{2m} + \sum_{i,j} V_{ij}$$

   where V_{ij} is the interaction potential between particles i and j. From Newton's third law, we expect $V_{ij} = V_{ji}$.

2.
$$\langle P\varphi|P\psi\rangle = \langle\varphi|\psi\rangle, \tag{5.4}$$

   which follows by renaming the integration variables.

3. Permutations have no effect on the expectation values of symmetric operators.

$$\langle P\varphi|S|P\psi\rangle = \langle\varphi|P^\dagger SP|\psi\rangle = \langle\varphi|S|\psi\rangle \tag{5.5}$$

4. If $\langle\varphi|O|\psi\rangle = \langle P\varphi|O|P\psi\rangle$ for all φ and ψ, then O must be symmetric. We believe that all physical observables must be invariant under all permutations of the particles, so all operators corresponding to observable quantities must be symmetric.

It is well known that wave functions describing identical bosons must be symmetric with respect to the exchange of any pair of particles. Functions describing identical fermions must be antisymmetric in the same sense. There is a vast amount of experimental evidence to corroborate this. There is also a deep result known as the *spin-statistics theorem*, which shows that

it is virtually impossible to construct a covariant theory that does not have this property.

One way to make completely symmetric or antisymmetric states is simply to multiply single-particle states in all possible combinations. We'll call the basic ingredient $|i\rangle_\alpha$. By this I mean that the ball that wears the number α is in the quantum state given by i. We assume these are orthonormal, $\langle i|j\rangle = \delta_{ij}$. We can write an N-particle state

$$|i_1, i_2, \ldots, i_N\rangle = |1\rangle_1 |2\rangle_2 \cdots |i_N\rangle_N \tag{5.6}$$

The symmetrized and antisymmetrized basis states are can be written

$$S_\pm |i_1, i_2, \ldots, i_N\rangle \equiv \frac{1}{\sqrt{N!}} \sum_P (\pm)^P P |i_1, i_2, \ldots, i_N\rangle \tag{5.7}$$

The sum goes over all of the $N!$ possible permutations of N objects. Equation (5.6) defines the symmetric- and antisymmetric-making operators $S_\pm$. The symbol $(\pm)^P$ requires further explanation. All the $N!$ permutations of N objects can be obtained by exchanging successive pairs of particles. The P in $(\pm)^P$ stands for the number of such exchanges required to achieve the corresponding permutation. Of course, the upper sign refers to bosons and the lower, to fermions. It is a fact that no permutation can be reached by an even *and* an odd number of exchanges. Put it another way, even and odd permutations constitute two disjoint sets.

5.2 Boson States

We must allow for the possibility that there are several particles in one quantum state. If, for example, there are n_i particles in the i-th state, there will be $n_i!$ permutations that leave the N-particle state unchanged. In this case (5.6) will not be normalized to unity. A properly normalized state can be constructed as follows:

$$|n_1, n_2, \ldots\rangle = S_+ |i_1, i_2, \ldots, i_N\rangle \frac{1}{\sqrt{n_1! n_2! \cdots}} \tag{5.8}$$

The numbers n_1, n_2, etc. are called *occupation numbers*. The sum of all occupation numbers must equal to the total number of particles:

$$\sum_i n_i = N \tag{5.9}$$

All this assumes that there are exactly N particles. We are interested in precisely the situation in which the total number of particles is not fixed. We allow for this by taking the basis states to be the direct sum of the space with no particles, the space with one particle, the space with two particles, etc. A typical basis element is written $|n_1, n_2, \ldots\rangle$. There in no constraint on the sum of the n_i. The normalization is

$$\langle n_1, n_2, \ldots | n_1', n_2', \ldots\rangle = \delta_{n_1,n_1'}\delta_{n_2,n_2'}\cdots \tag{5.10}$$

and the completeness relation

$$\sum_{n_1,n_2,\ldots} |n_1, n_2, \ldots\rangle\langle n_1, n_2, \ldots| = \mathbf{1} \tag{5.11}$$

Since there are physical processes that change the number of particles in a system, it is necessary to define operators that have this action. The basic operators for so doing are the *creation and annihilation operators.* As you will see, they are formally equivalent to the raising and lowering operators associated with the harmonic oscillator. For example, suppose a state has n_i particles in the i'th eigenstate. We introduce the creation operator $a_i^\dagger$ by

$$a_i^\dagger|\ldots, n_i, \ldots\rangle = \sqrt{n_i+1}|\ldots, n_i+1, \ldots\rangle, \tag{5.12}$$

ie. $a_i^\dagger$ increases by one the number of particles in the i'th eigenstate. The adjoint of this operator reduces by one the number of particles. This can be seen as follows: Take the adjoint of (5.12) and multiply on the right by $|\ldots, n_i+1, \ldots\rangle$.

$$\begin{aligned}&\langle\ldots, n_i, \ldots|a_i|\ldots, n_i+1, \ldots\rangle \\ &\qquad = \sqrt{n_i+1}\langle\ldots, n_i+1, \ldots|\ldots, n_i+1, \ldots\rangle = \sqrt{n_i+1}\end{aligned}$$

Now replace n_i everywhere by $n_i - 1$.

$$\begin{aligned}&\langle\ldots, n_i-1, \ldots|a_i|\ldots, n_i, \ldots\rangle \\ &\qquad = \sqrt{n_i}\langle\ldots, n_i, \ldots|\ldots, n_i, \ldots\rangle = \sqrt{n_i}\end{aligned} \tag{5.13}$$

The effect of a_i on the state $|\ldots, n_i, \ldots\rangle$ has been to produce a state in which the number of particles in the i'th state has been reduced by one. Eqn. (5.13) also tells us what the normalization must be. In short

$$a_i|\ldots, n_i, \ldots\rangle = \sqrt{n_i}|\ldots, n_i-1, \ldots\rangle \text{ for } n_i \geq 1 \tag{5.14}$$

Of course if $n_i = 0$, the result is identically zero.

$$a_i|\ldots, n_i = 0, \ldots\rangle = 0$$

The commutation relations are important. Clearly all the a_i's commute among one another, since it makes no difference in what order the different states are depopulated, and by the same argument, the $a_i^\dagger$'s commute as well. a_i and $a_j^\dagger$ also commute if $i \neq j$, since

$$\begin{aligned} a_i a_j^\dagger|\ldots, n_i, \ldots, n_j, \ldots\rangle \\ &= \sqrt{n_1}\sqrt{n_j+1}|\ldots, n_i - 1, \ldots, n_j + 1, \ldots\rangle \\ &= a_j^\dagger a_i|\ldots, n_i, \ldots, n_j, \ldots\rangle \end{aligned}$$

Finally

$$\begin{aligned} \left(a_i a_i^\dagger - a_i \dagger a_i\right)|\ldots, n_i, \ldots, n_j, \ldots\rangle \\ &= \left(\sqrt{n_i+1}\sqrt{n_i+1} - \sqrt{n_i}\sqrt{n_i}\right)|\ldots, n_i, \ldots, n_j, \ldots\rangle \end{aligned}$$

In summary

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \qquad [a_i, a_j^\dagger] = \delta_{ij} \tag{5.15}$$

If it were not for the i and j subscripts, (5.15) would be the commutation relations for the harmonic oscillator, $[a, a] = [a^\dagger, a^\dagger] = 0$, $[a, a^\dagger] = 1$. In this context they are called *ladder operators* or *raising and lowering operators.* This is the essence of second quantization. Try to imagine a quantum system as an infinite forest of ladders, each one corresponding to one quantum state labelled by an index i. The rungs of the i'th ladder are labelled by the integer n_i. The entire state of the system is uniquely specified by these occupation numbers. The effect of a_i and $a_i^\dagger$ is to bump the system down or up one rung of the i'th ladder. There are several important results from harmonic oscillator theory that can be taken over to second quantization. One is that we can build up many-particle states using the $a_i^\dagger$'s. Starting with the *vacuum state* $|0\rangle$ with no particles, we can construct single-particle states, $a_i^\dagger|0\rangle$, two-particle states

$$\frac{1}{\sqrt{2!}}\left(a_i^\dagger\right)^2|0\rangle \qquad \text{or} \qquad a_i^\dagger a_j^\dagger|0\rangle,$$

or in general, many-particles states.

$$|n_1, n_2, \ldots\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots}}\left(a_1^\dagger\right)^{n_1}\left(a_2^\dagger\right)^{n_2}\cdots|0\rangle \tag{5.16}$$

Another familiar result is that the number operator

$$N_i = a_i^\dagger a_i \tag{5.17}$$

is a Hermitian operator whose eigenvalue is the number of particles in the i'th quantum state.

$$N_i|\ldots, n_i, \ldots\rangle = n_i|\ldots, n_i, \ldots\rangle \tag{5.18}$$

Here is a useful result that you can prove by brute force or induction.

$$\left[a_i, \left(a_i^\dagger\right)^n\right] = n\left(a_i^\dagger\right)^{n-1}$$

Use this to do the following exercises.

- Show that (5.16) has the normalization required by (5.10).
- Prove (5.18).
- Show that the mysterious factor of $\sqrt{n_i + 1}$ in (5.12) is in fact required by (5.16).

5.3 More General Operators

The matrix element of a single-particle operator[1] in a one-particle state is just

$$O_{ij} = \langle i|\hat{O}|j\rangle. \tag{5.19}$$

If the $|i\rangle$'s represent a complete set of states, we can represent the operator as

$$\hat{O} = \sum_{i,j} O_{ij}|i\rangle\langle j| \tag{5.20}$$

In a many-particle system, we must distinguish between operators that work on specific individual particles and those that operate on all the particles in the system. One way to do this is to simply sum over one-particle states and generalize (5.20) as follows:

$$\hat{O} = \sum_{i,j} O_{ij} \sum_{\alpha=1}^{N} |i\rangle_\alpha\langle j|_\alpha \tag{5.21}$$

[1] I will sometimes use "hats" to emphasize that some particular symbol represents an operator. I will do this only occasionally, however, and I reserve the right to be inconsistent.

The index α points to the individual particles, but the sum over α is awkward; we would like to express it in terms of creation and annihilation operators. Note that when $\sum |i\rangle_\alpha \langle j|_\alpha$ operates on an arbitrary N-particle state, it produces n_j terms in which $|j\rangle$ is replaced by $|i\rangle$, one for each particle in the state, but these n_j terms are still normalized as if they had n_j particles in the j'th state and n_i particles in the i'th state. Taking this into account gives

$$\begin{aligned}
&\sum_{\alpha=1}^{n} |i\rangle_\alpha \langle j|_\alpha |\ldots, n_i, \ldots, n_j, \ldots\rangle \\
&\qquad = n_j |\ldots, n_i + 1, \ldots, n_j - 1, \ldots\rangle \frac{\sqrt{n_i + 1}}{\sqrt{n_j}} \\
&\qquad = \sqrt{n_j}\sqrt{n_i + 1} |\ldots, n_i + 1, \ldots, n_j - 1, \ldots\rangle \\
&\qquad\qquad = a_i^\dagger a_j |\ldots, n_i, \ldots, n_j, \ldots\rangle \\
&\qquad\qquad\qquad \equiv N_{ij} |\ldots, n_i, \ldots, n_j, \ldots\rangle \qquad (5.22)
\end{aligned}$$

You see that $N_{ij} = a_i^\dagger a_j$ is a generalization of (5.17). Evidentally, N_{ii} as defined by (5.22) is the same as N_i defined by (5.17). Finally

$$\hat{O} = \sum_{ij} O_{ij} N_{ij} \qquad (5.23)$$

From here it's only a hop, skip, and jump to defining two-particle operators.

$$\hat{O}^{(2)} = \frac{1}{2} \sum_{i,j,k,m} \langle i, j|\hat{O}^{(2)}|k, m\rangle a_i^\dagger a_j^\dagger a_k a_m, \qquad (5.24)$$

where

$$\langle i, j|\hat{O}^{(2)}|k, m\rangle = \int dx \int dy\ \varphi_i^*(x)\varphi_j^*(y)\hat{O}^{(2)}(x, y)\varphi_k(x)\varphi_m(y). \qquad (5.25)$$

The factor $\frac{1}{2}$ in (5.25) insures that each interaction is counted only once, since for identical particles $\hat{O}^{(2)}(x, y) = \hat{O}^{(2)}(y, x)$.

5.4 Field Operators

I have used the symbol $|i\rangle$ to indicate a one particle "quantum state." For example, if one were dealing with hydrogen, i would stand for the discrete eigenvalues of some complete set of commuting operators, in this case n,

l, m, and m_s. The creation operators $a_i^\dagger$ create particles in such a state. Whatever these operators might be, however, **none** of them is the position operator. An important question is what the creation operator formalism looks like in terms of the position operator. First consider two basis systems based on two alternative sets of commuting observables. Use i to index one set and λ to index the other. Then

$$|\lambda\rangle = \sum_i |i\rangle\langle i|\lambda\rangle. \tag{5.26}$$

Since this is true for states, it must also be true of creation operators.

$$a_\lambda^\dagger = \sum_i \langle i|\lambda\rangle a_i^\dagger \tag{5.27}$$

Now remember that what we call a wave function in elementary quantum mechanics is really a scalar product on Hilbert space of the corresponding state and eigenstates of the position operator, i.e.

$$\langle \mathbf{x}|i\rangle = \varphi_i(\mathbf{x}). \tag{5.28}$$

We assume that the φ_i are a complete set of orthonormal functions, so that

$$\int d^3x\ \varphi_i^*(\mathbf{x})\varphi_j(\mathbf{x}) = \delta_{ij} \tag{5.29}$$

and

$$\sum_i \varphi_i^*(\mathbf{x})\varphi_i(\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{5.30}$$

Combining (5.27) and (5.28) gives

$$\psi^\dagger(\mathbf{x}) = \sum_i \varphi_i^*(\mathbf{x}) a_i^\dagger \tag{5.31}$$

and its adjoint

$$\psi(\mathbf{x}) = \sum_i \varphi_i(\mathbf{x}) a_i$$

$\psi^\dagger(\mathbf{x})$ is the creation operator in the basis defined by the position operator. It creates a particle at $\mathbf{x}$. $\psi^\dagger$ and ψ are called *field operators.* Their commutation relations are important.

$$[\psi(\mathbf{x}), \psi(\mathbf{x}')]_\pm = [\psi^\dagger(\mathbf{x}), \psi^\dagger(\mathbf{x}')]_\pm = 0 \tag{5.32}$$

$$[\psi(\mathbf{x}), \psi^\dagger(\mathbf{x}')]_\pm = \delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

I have used the symbol $[\cdots,\cdots]_\pm$ to allow for both boson (commutation) and fermion (anticommutation) rules. The first line of (5.32) is more or less obvious. The second line follows from (5.30)

Quantum mechanics according to Heisenberg was obtained by replacing classical quantities like **P** and **L** with noncommuting operators. Our formalism takes the further step of making the wave functions themselves into operators via (5.32). This, I suppose, is the reason why it's called "second quantization." Many texts in fact, start with (5.32) and use it to infer the existence of the creation and annihilation operators. Let's see what ordinary quantum mechanics looks like in this formalism and then see what new aspects appear. Start with the kinetic energy. Eqn. (5.19) gives the matrix element

$$T_{ij} = \int d^3x\ \varphi_i^*(\mathbf{x})\left(-\frac{\hbar^2}{2m}\nabla^2\right)\varphi_j(\mathbf{x}) = \frac{\hbar^2}{2m}\int d^3x\ \nabla\varphi_i^*(\mathbf{x})\cdot\nabla\varphi_{\mathbf{j}}(\mathbf{x}). \quad (5.33)$$

The last step involves an integration by parts. Eqn.(5.23) becomes

$$\hat{T} = \frac{\hbar^2}{2m}\sum_{ij}\int d^3x\ a_i^\dagger\nabla\varphi_i^*(\mathbf{x})\cdot\nabla\varphi_j a_j(\mathbf{x}) = \frac{\hbar^2}{2m}\int d^3x\ \nabla\psi^\dagger(\mathbf{x})\cdot\nabla\psi(\mathbf{x}) \quad (5.34)$$

The potential operator is similar.

$$\hat{U} = \int d^3x\ \psi^\dagger(\mathbf{x})U(\mathbf{x})\psi(\mathbf{x}) \quad (5.35)$$

Equations (5.24) and (5.25) give us to form of any two-particle operator,

$$\hat{V} = \frac{1}{2}\sum_{i,j,k,m}\int d^3x\int d^3y\ a_i^\dagger a_j^\dagger\varphi_i^*(\mathbf{x})\varphi_j^*(\mathbf{y})V(\mathbf{x},\mathbf{y})\varphi_k(\mathbf{x})\varphi_m(\mathbf{y})a_k a_m \quad (5.36)$$

$$= \frac{1}{2}\int d^3x\int d^3y\ \psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})V(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})$$

The total Hamiltonian operator is then

$$\hat{H} = \hat{T} + \hat{U} + \hat{V} \quad (5.37)$$

From (5.17) $N = \sum_i a_i^\dagger a_i$ is the total number of particles operator. It follows that $\rho(\mathbf{x},t) = \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t)$ is the spatial density operator. This can be seen as follows. Inverting (5.31) gives

$$a_i^\dagger(t) = \int d^3x\ \psi^\dagger(\mathbf{x},t)\varphi_i^*(\mathbf{x})$$

$$a_i(t) = \int d^3x\ \varphi_i(\mathbf{x})\psi(\mathbf{x})$$

$$N = \sum_i a_i^\dagger a_i = \sum_i \left[\int d^3x\ \psi^\dagger(\mathbf{x},t)\varphi_i^*(\mathbf{x},t)\right]\left[\int d^3x'\ \varphi_i(\mathbf{x}')\psi(\mathbf{x}',t)\right]$$

$$= \int d^3x \int d^3x' \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x}',t)\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$

$$= \int d^3x\ \psi^\dagger(\mathbf{x},t)\psi(\mathbf{x},t) = \int d^3x\ \rho(\mathbf{x},t)$$

Notice that

$$\rho(\mathbf{x}',t)\psi^\dagger(\mathbf{x},t)|0\rangle = \psi^\dagger(\mathbf{x}',t)\psi(\mathbf{x}',t)\psi^\dagger(\mathbf{x},t)|0\rangle$$

$$= \psi^\dagger(\mathbf{x}',t)\left[\psi^\dagger(\mathbf{x},t)\psi(\mathbf{x}',t) + \delta^{(3)}(\mathbf{x}-\mathbf{x}')\right]|0\rangle$$

$$= \delta^{(3)}(\mathbf{x}-\mathbf{x}')\psi^\dagger(\mathbf{x}',t)|0\rangle = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\psi^\dagger(\mathbf{x},t)|0\rangle$$

The point of this is that $\psi^\dagger(\mathbf{x},t)|0\rangle$ is an eigenstate of ρ with eigenvalue $\delta^{(3)}(\mathbf{x}-\mathbf{x}')$. This is consistent both with our interpretation of $\psi^\dagger$ as a creation operator and with our interpretation of ρ as a particle density operator. $\psi^\dagger(\mathbf{x},t)$ does indeed create a point-like particle at $\mathbf{x}$ and time t.

5.5 Field Equations

The field operators so far have had no time dependence. We have been working in the Schrodinger picture. Field theory calculations are typically done in the Heisenberg picture, however, so we must derive an equation of motion. The basic relation is

$$\psi(\mathbf{x},t) = e^{iHt/\hbar}\psi(\mathbf{x},0)e^{-iHt/\hbar} \tag{5.38}$$

Where H is the operator given by (5.37). The Heisenberg equation of motion is obtained by differentiating (5.38),

$$i\hbar\frac{\partial}{\partial t}\psi((x),t) = -[H,\psi(\mathbf{x},t)] = -e^{iHt/\hbar}[H,\psi(\mathbf{x},0)]e^{-iHt/\hbar} \tag{5.39}$$

The commutator is something of a challenge. We'll do it one term at a time.

$$[T,\psi(\mathbf{x})] = \left(-\frac{\hbar^2}{2m}\right)\left[\sum_{ij}\int d^3x'\varphi_i^*(\mathbf{x}')\nabla'^2\varphi_j(\mathbf{x}')a_i^\dagger a_j\ ,\ \sum_k \varphi_k(\mathbf{x})a_k\right]$$

I have used (5.33) for T and (5.31) for ψ. At this point we use the commutation relations, either (5.15) for bosons or (5.93) for fermions, to reduce the product of three a_i's.

$$[a_i^\dagger a_j, a_k] = -a_j \delta_{ki}$$

$$[T, \psi(\mathbf{x})] = \left(\frac{\hbar^2}{2m}\right) \int d^3x' \sum_i \varphi_i(\mathbf{x}) \varphi_i^*(\mathbf{x}') \nabla'^2 \sum_j \varphi_j(\mathbf{x}') a_j$$

$$= \left(\frac{\hbar^2}{2m}\right) \int d^3x' \delta^{(3)}(\mathbf{x} - \mathbf{x}') \nabla'^2 \psi(\mathbf{x}') = \left(\frac{\hbar^2}{2m}\right) \nabla^2 \psi(\mathbf{x}) \tag{5.40}$$

The potential energy is rather easier.

$$[U, \psi(\mathbf{x})] = \int d^3x U(\mathbf{x}') [\psi^\dagger(\mathbf{x}') \psi(\mathbf{x}'), \psi(\mathbf{x})] = -U(\mathbf{x}) \psi(\mathbf{x}) \tag{5.41}$$

Finally, the interaction term. A straightforward application of (5.32) and (5.36) gives

$$[V, \psi(\mathbf{x})] = \frac{1}{2} \int d^3x' d^3x'' [\psi^\dagger(\mathbf{x}') \psi^\dagger(\mathbf{x}'') V(\mathbf{x}', \mathbf{x}'') \psi(\mathbf{x}'') \psi(\mathbf{x}'), \psi(\mathbf{x})]$$
$$= -\int d^3x' \psi(\mathbf{x}') V(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}') \psi(\mathbf{x}) \tag{5.42}$$

Incidentally, these results are valid for both boson and fermion fields so long as one keeps the fields ordered in exactly the way shown above. (The reader is invited to work through these derivations to see how this comes about. Finally combining (5.37), (5.39), (5.40), (5.41), and (5.42) gives the equation of motion

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{x}, t) = \left(-\frac{\hbar^2}{2m} \nabla^2 + U(\mathbf{x})\right) \psi(\mathbf{x}, t) +$$
$$\int d^3x' \psi^\dagger(\mathbf{x}', t) V(\mathbf{x}, \mathbf{x}') \psi(\mathbf{x}', t) \psi(\mathbf{x}, t) \tag{5.43}$$

The first line looks like the conventional Schrodinger equation but with field operators rather than the usual wave functions. The last term is nonlinear and has no counterpart in single-particle systems.

5.6 Momentum Representation

As we have seen before, it's usually easier to formulate a theory in position space and easier to interpret it in momentum space. In this case we work exclusively in a finite volume with discrete momentum eigenvalues. The basic eigenfunctions are

$$\varphi_{\mathbf{k}} = \frac{1}{\sqrt{V}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.44}$$

We assume the usual periodic boundary conditions force the momentum eigenvalues to be

$$\mathbf{k} = 2\pi \left(\frac{n_x}{L_x}, \frac{n_y}{L_y}, \frac{n_z}{L_z} \right) \tag{5.45}$$

Where each of the n's can take the values $0, \pm 1, \pm 2, \cdots$ independently. With this normalization, the eigenfunctions are orthonormal,

$$\int_V d^3x \; \varphi^*_{\mathbf{k}}(\mathbf{x}) \varphi_{\mathbf{k}'}(\mathbf{x}) = \delta_{\mathbf{k},\mathbf{k}'} \tag{5.46}$$

We can now calculate matrix elements of the Hamiltonian. Eqn. (5.33) gives

$$\langle \mathbf{k}|\hat{T}|\mathbf{k}'\rangle = \int d^3x \; \varphi^*_{\mathbf{k}}(\mathbf{x}) \left(-\frac{\hbar^2}{2m}\nabla^2 \right) \varphi_{\mathbf{k}'}(\mathbf{x}) = \frac{\hbar^2 k^2}{2m} \delta_{\mathbf{k},\mathbf{k}'} \tag{5.47}$$

For the potential energy

$$\langle \mathbf{k}|\hat{U}|\mathbf{k}'\rangle = \int d^3x \; \varphi^*_{\mathbf{k}}(\mathbf{x}) U(\mathbf{x}) \varphi_{\mathbf{k}'}(\mathbf{x}) = \frac{1}{V} \int d^3x \; e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} U(\mathbf{x}) \tag{5.48}$$

$$\equiv \frac{1}{V} U_{\boldsymbol{k}-\boldsymbol{k}'}$$

This is the Fourier transform of $U(\mathbf{x})$ with respect to the momentum $\mathbf{k} - \mathbf{k}'$. The two-particle potential in (5.36) should only be a function of the difference of the two coordinates, ie. $V(\mathbf{x}, \mathbf{y}) = V(|\mathbf{x} - \mathbf{y}|)$. As with $U(\mathbf{x})$ we can take the Fourier transform[2]

$$V_{\mathbf{q}} \equiv \int d^3x \; e^{-i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \tag{5.49}$$

$$V(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{q}} V_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}$$

[2]Anyone who confuses the volume V with the potential V will be taken out and tied to a tree.

The matrix element of V is then

$$\langle \mathbf{p}', \mathbf{k}' | V(\mathbf{k} - \mathbf{k}') | \mathbf{p}, \mathbf{k} \rangle = \frac{1}{V} \sum_{\mathbf{q}} V_{\mathbf{q}} \, \delta_{\mathbf{q},\mathbf{p}'-\mathbf{p}} \, \delta_{\mathbf{q},\mathbf{k}-\mathbf{k}'} \tag{5.50}$$

The complete Hamiltonian is then

$$H = \sum_{\mathbf{k}} \frac{(\hbar \mathbf{k})^2}{2m} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}',\mathbf{k}} U_{\mathbf{k}'-\mathbf{k}} a^\dagger_{\mathbf{k}'} a_{\mathbf{k}} + \frac{1}{2V} \sum_{\Delta} V_{\mathbf{k}-\mathbf{k}'} a^\dagger_{\mathbf{k}'} a^\dagger_{\mathbf{p}'} a_{\mathbf{k}} a_{\mathbf{p}} \tag{5.51}$$

The meaning of the sum over Δ is that one sums over all values of $\mathbf{k}, \mathbf{p}, \mathbf{k}', \mathbf{p}'$ such that $\mathbf{k} + \mathbf{p} = \mathbf{k}' + \mathbf{p}'$. This is easy to interpret. Two particles with momenta $\mathbf{k}$ and $\mathbf{p}$ interact in such a way that there is a momentum $\mathbf{k} - \mathbf{k}' = \mathbf{p} - \mathbf{p}'$ transferred between them. After this scattering process their new momenta are $\mathbf{k}'$ and $\mathbf{p}'$.

5.7 Superfluids

^{4}He atoms have zero angular moment and as such are, to a good approximation, Bose "particles". Helium is unique in that it does not crystalize even in the limit of zero temperature (except under extreme high pressure). The reason is that it is so light that its endpoint oscillations have sufficient energy to prevent the formation of a lattice. Helium does liquify at $T \approx 4$K and at the lambda point $T_\lambda = 2.18$K it enters the superfluid state with some remarkable properties. One of the most striking is the ability to flow through narrow channels without friction. Another is the existence of quantized vortices. The occurrence of frictionless flow can be described in a rough qualitative way by the two-fluid model. The two fluids – the normal and the superfluid components – are interpenetrating, and their densities depend on temperature. At very low temperatures the density of the normal component vanishes, while the density of the superfluid component approaches the total density of the liquid. Near the transition temperature to the normal state, the situations is reversed: the superfluid density tends towards zero while the normal density approaches the density of the liquid.

At a more microscopic level, superfluidity is explained in terms of elementary excitations. In an ideal gas an elementary excitation corresponds to the addition of a single quasiparticle in a momentum eigenstate. For small momenta the excitations in liquid ^{4}He are sound waves or *phonons*. The phonons have a linear dispersion relation $E \propto k$, which is to say that they are massless particles. This point is crucial to the phenomena of superfluidity as can be seen from the following argument. Imagine helium flowing

Coordinate System	S	S'
Ground State	$E = E_0$	$E' = E_0 + M\mathbf{v}^2/2$
	$\mathbf{P} = 0$	$\mathbf{P}' = -M\mathbf{v}$
GS + One Quasiparticle	$E = E_0 + \epsilon(\mathbf{p})$	$E' = E_0 + \epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v} + M\mathbf{v}^2/2$
	$\mathbf{P} = \mathbf{p}$	$\mathbf{P}' = \mathbf{p} - M\mathbf{v}$
Excitation Energy	$\Delta E = \epsilon(\mathbf{p})$	$\Delta E' = \epsilon(\mathbf{p}) - \mathbf{p}\cdot\mathbf{v}$

Table 5.1: Kinematics for Quasiparticle Formation

through a tube at constant velocity. We can describe this in terms of two different coordinate systems. In the system S the helium is at rest and the tube is moving with velocity $\mathbf{v}$, and in system S' the tube is at rest and the helium has velocity $-\mathbf{v}$. Evidentally S' is moving relative to S with velocity $\mathbf{v}$. Momentum and energy are related by the Galilean transformation,

$$\mathbf{P}' = \mathbf{P} - M\mathbf{v} \tag{5.52}$$

$$E' = E - \mathbf{P}\cdot\mathbf{v} + M\mathbf{v}^2/2$$

If the fluid is ordinary, ie. not a superfluid, there will be friction between the fluid and the walls of the tube. In S it would appear that the tube was dragging a thin layer of fluid along with the it. In S' the tube is decelerating this same layer. In the limit $T = 0$, this deceleration comes about through the creation of quasiparticles, and this is only kinematically possible if it lowers the total energy of the fluid. Let's try to create our first quasiparticle in S with energy $\epsilon(\mathbf{p})$ and momentum $\mathbf{p}$. Call the ground state energy of the fluid E_0 and its momentum, $\mathbf{P}$. Table 1 shows the relevant kinematics. Only when $\Delta E' < 0$ does the flowing fluid lose energy. This means there is a minimum velocity

$$v_{\min} = \frac{\epsilon}{p}. \tag{5.53}$$

This is called the Landau criterion. If the flow velocity is smaller that this, no quasiparticles are excited and the fluid flows unimpeded and loss-free through the tube. If the quasiparticles had finite mass so that $\epsilon(\mathbf{v}) = \mathbf{p}^2/2m$, then $v_{\min} = 0$ in the limit $\mathbf{p} = 0$, and there could be no superfluidity. It is essential that ϵ/p pass through the origin with a finite positive slope.

It is possible to calculate the dispersion relation for low-energy quasiparticles using the field theory formalism we have just developed. This was done originally by N.N. Bogoliubov,[3] and is one of the crown jewels of many-body

[3]N. N. Bogoliubov, J. Phys. U.S.S.R. **11**, 23 (1947)

theory. Though this was not realized in 1947, this calculation introduced into physics some profound ideas about symmetry breaking that are important in theories of fundamental interactions. I will take you through the Bogoliubov calculation now and come back to the fundamental issues later.

5.7.1 Bogoliubov Theory

We have derived the relevant Hamiltonians, (5.33) through (5.37) in coordinate space and (5.51) in momentum space. Since we are not concerned with the effect of external potentials, we set $U = 0$. Notice that (5.37) in invariant under the simple gauge transformation,

$$\psi \to \psi' = e^{-i\alpha}\psi. \tag{5.54}$$

Moreover (5.51) obeys a simple conservation law; is conserves the number of particles in the system. Each term has the same number of creation and annihilation operators. For every particle created, one is annihilated and vice versa. Though it's by no means obvious, these two properties are closely related. For *every* continuous symmetry transformation that leaves a Hamiltonian invariant there is a corresponding conservation law. This deep result is called Noether's theorem. I will have much more to say about it later.

We know from statistical mechanics, that if the particles do not interact with one another, then in the limit of zero temperature, they are all in the ground state. If we introduce a weak two-particle interaction via V in (5.51), then it is plausible that *most* of the particles remain in the ground state. The crucial approximation consists in ignoring interactions between excited particles and keeping only those terms corresponding to particles in the ground state interacting with one another and terms in which one excited particle interacts with one particle in the ground state. That means keeping in (5.51) only those terms that have at least two of the ground state creation and annihilation operators a_i and $a_i^\dagger$. We are left with

$$H \approx \sum_{\mathbf{k}} \frac{(\hbar\mathbf{k})^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2V} V_0 a_0^\dagger a_0^\dagger a_0 a_0 + \frac{1}{V} \sum_{\mathbf{k}\neq\mathbf{0}} (V_0 + V_{\mathbf{k}}) a_0^\dagger a_0 a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$$

$$+ \frac{1}{2V} \sum_{\mathbf{k}\neq\mathbf{0}} V_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger a_0 a_0 + a_0^\dagger a_0^\dagger a_{\mathbf{k}} a_{-\mathbf{k}}) \tag{5.55}$$

We have ignored those terms with three a_0's, because they would not conserve momentum. Bogoliubov argued that the a_0 and $a_0^\dagger$ operators in (5.55)

can be replaced with c-numbers, $a_0 \to \sqrt{N_0}$, $a_0^\dagger \to \sqrt{N_0}$ where N_0 is the number of particles in the ground state. For one thing, the system is in an eigenstate of $a_0^\dagger a_0$ (with eigenvalue N_0) and of $a_0 a_0^\dagger$ (with eigenvalue N_0+1). It figures that since N_0 is a number on the order of 10^{20} or 10^{22}, we can neglect the 1 in $N_0 + 1$. To this extent a_0 and $a_0^\dagger$ commute! Another point the that the a_0's and $a_0^\dagger$'s are all divided by the volume V.

$$\frac{1}{V}[a_0, a_0^\dagger]|N_0\rangle = \frac{1}{V}|N_0\rangle \tag{5.56}$$

The left side of (5.56) consist of two terms, both equal to the particle density. This right side goes to zero as $V \to \infty$. With that replacement (5.55) becomes

$$H \approx \sum_{\mathbf{k}} \frac{(\hbar\mathbf{k})^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2V} N_0^2 V_0$$

$$+\frac{N_0}{V} \sum_{\mathbf{k}\neq\mathbf{0}} \left[(V_0 + V_{\mathbf{k}}) a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{1}{2} V_{\mathbf{k}} (a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger + a_{\mathbf{k}} a_{-\mathbf{k}}) \right] \tag{5.57}$$

The $a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ operator has three contributions. The first is the kinetic energy. The second, $N_0 V_0/V$, is called the Hartree energy , which comes from the direct interaction of a particle in the state $\mathbf{k}$ with the N_0 atoms in the zero-momentum state. The third is the exchange , or Fock term, in which an atom in the state $\mathbf{k}$ is scattered into the zero-momentum state, while a second atom is simultaneously scattered from the condensate to the state $\mathbf{k}$.

We can get some insight about the approximations made so far by noticing the following. We obtained (5.51) from (5.36) by replacing

$$\varphi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.58}$$

If we had made the replacement

$$\varphi(\mathbf{x}) \to \sqrt{\rho_0} + \frac{1}{\sqrt{V}} \sum_{\mathbf{k}\neq 0} a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \tag{5.59}$$

where $\rho_0 = N_0/V$, we would have gotten (5.57) directly. You see that (5.59) differs from (5.58) in two related ways: the classical density ρ_0 and the $\mathbf{k} = 0$ exclusion in the summation. The meaning is clear. We are treating the ground state like a classical fluid and all the other particles as a dilute quantum fluid. I should mention a related matter. The original Hamiltonian conserved particle number: for every creation operator there

was a corresponding annihilation operator. Eqn. (5.57) no longer does. The particles simply appear out of or disappear into the ground state where they are "invisible," since we are treating the ground state classically. We will have much more to say about this later on in the course when we discuss spontaneous symmetry breaking.

Eqn. (5.57) refers to N_0, the number of particles in the ground state, but this is not something we know ahead of time. We do know the total number of particles, that's something that can be measured. The two can be related as follows:

$$N = N_0 + \sum_{\mathbf{k}\neq\mathbf{0}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \tag{5.60}$$

You see the similarity between (5.60) and (5.59). N_0 is a c-number, but the number of excited particles is obtained via an operator operating on the state function. Substituting (5.60) into (5.57) gives

$$H \approx \sum_{\mathbf{k}\neq\mathbf{0}} E_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{N^2}{2V} V_0 + \frac{N}{2V} \sum_{\mathbf{k}\neq\mathbf{0}} V_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}}) \tag{5.61}$$

$$E_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m} + \frac{N}{V} V_{\mathbf{k}} \tag{5.62}$$

From (5.49), $V_{\mathbf{k}} = V_{-\mathbf{k}}$, so that (5.61) can be written in a more symmetric fashion,

$$H \approx \frac{N^2}{2V} V_0 + \sum_{\mathbf{k}>\mathbf{0}} \left[E_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + a^\dagger_{-\mathbf{k}} a_{-\mathbf{k}}) + \frac{N}{V} V_{\mathbf{k}} (a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}}) \right] \tag{5.63}$$

5.7.2 The Bogoliubov Transformation

Physically realizable states must be eigenstates of the Hamiltonian, but systems with a definite number of particles in the various energy levels are *not* eigenstates of (5.63), since it does not conserve particle number. In order to study the physical states, we must diagonalize H. The procedure for doing this was originally invented by Bogoliubov in this context, and the technique as since been applied to many different problems. In order to work this out in a generic fashion (and also to keep the algebra under control), I will pose the problem like this.[4] Each term in the sum in (5.63) has the form

$$H = \epsilon_0 (a^\dagger a + b^\dagger b) + \epsilon_1 (a^\dagger b^\dagger + ba) \tag{5.64}$$

[4] I am following the development in *Bose-Einstein Condensation in Dilute Gasses*, C. J. Pethick & H. Smith, Cambridge (2002)

In this particular instance, $a = a_{\mathbf{k}}$ and $b = a_{-\mathbf{k}}$, but the final result will hold with the generality of (5.64). ϵ_0 and ϵ_1 are arbitrary numbers with units of energy, or since $\hbar = 1$, of frequency. Of course

$$[a, a^\dagger] = [b, b^\dagger] = 1, \text{ and } [a, b] = [a, b^\dagger] = [a^\dagger, b] = [a^\dagger, b^\dagger] = 0 \tag{5.65}$$

We would like to find linear combinations of these

$$\alpha = ua + vb^\dagger, \text{ and } \beta = ub + va^\dagger \tag{5.66}$$

such that

- $$[\alpha, \alpha^\dagger] = [\beta, \beta^\dagger] = 1, \text{ and } [\alpha, \beta^\dagger] = [\beta, \alpha^\dagger] = 0 \tag{5.67}$$
- The only operators appearing in the Hamiltonian are $\alpha^\dagger\alpha$ and $\beta^\dagger\beta$.

In other words, we will diagonalize the Hamiltonian, and resulting states, whatever they may be, will be the observable, physical states. Insert (5.66) into (5.67) and use (5.65). Assuming u and v to be real we get,

$$u^2 - v^2 = 1. \tag{5.68}$$

The inverse of (5.66) is

$$a = u\alpha - v\beta^\dagger, \qquad b = u\beta - v\alpha^\dagger. \tag{5.69}$$

Now substitute (5.69) into (5.64). After a bit of algebra we get,

$$\begin{aligned} H = {} & 2v^2\epsilon_0 - 2uv\epsilon_1 + [\epsilon_0(u^2 + v^2) - 2uv\epsilon_1](\alpha^\dagger\alpha + \beta^\dagger\beta) \\ & + [\epsilon_1(u^2 + v^2) - 2uv\epsilon_0](\alpha\beta + \beta^\dagger\alpha^\dagger) \end{aligned} \tag{5.70}$$

We would like the last term to vanish, i.e.

$$\epsilon_1(u^2 + v^2) - 2uv\epsilon_0 = 0 \tag{5.71}$$

Combining (5.68) and (5.71) gives

$$u^2 = \frac{1}{2}\left(\frac{\epsilon_0}{\epsilon} + 1\right) \text{ and } v^2 = \frac{1}{2}\left(\frac{\epsilon_0}{\epsilon} - 1\right), \tag{5.72}$$

where

$$\epsilon = \sqrt{\epsilon_0^2 - \epsilon_1^2} \tag{5.73}$$

It is necessary to choose the positive branch of the square root so that u and v are real. Substituting these into (5.70) gives

$$H = \epsilon(\alpha^\dagger\alpha + \beta^\dagger\beta) + \epsilon - \epsilon_0 \tag{5.74}$$

It's time to make contact with our original Hamiltonian, (5.61) or (5.63). The parameter we have been calling ϵ is best thought of as a frequency. Where convenient I will use $n \equiv N/V$.

$$\epsilon_0 \to \frac{\mathbf{k}^2}{2m} + nV_{\mathbf{k}} \qquad \epsilon_1 \to nV_{\mathbf{k}}$$

$$\epsilon \to \omega_{\mathbf{k}} \equiv \sqrt{\left(\frac{\mathbf{k}^2}{2m}\right)^2 + \frac{n\mathbf{k}^2V_{\mathbf{k}}}{m}} \tag{5.75}$$

$$\begin{aligned} H &\approx \frac{N^2}{2V}V_0 + \sum_{\mathbf{k}>0}\left[\omega_{\mathbf{k}}(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}) + \omega_{\mathbf{k}} - \left(\frac{\mathbf{k}^2}{2m} + nV_{\mathbf{k}}\right)\right] \\ &= \frac{N^2}{2V}V_0 - \frac{1}{2}\sum_{\mathbf{k}\neq 0}\left(\frac{\mathbf{k}^2}{2m} + nV_{\mathbf{k}} - \omega_{\mathbf{k}}\right) + \sum_{\mathbf{k}\neq\mathbf{0}}\omega_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} \end{aligned} \tag{5.76}$$

The Hamiltonian consists of the ground-state energy and a sum of oscillators of energy $\omega_{\mathbf{k}}$. Our new creation and destruction operators $\alpha^\dagger_{\mathbf{k}}$ and $\alpha_{\mathbf{k}}$ refer to the quasiparticles discussed earlier. The ground state of the system is that in which there are no quasiparticles.

$$\alpha_{\mathbf{k}}|\Omega\rangle = 0 \text{ for all } \mathbf{k}. \tag{5.77}$$

This new ground state plays the same role for the $\alpha_{\mathbf{k}}$'s as the quantum vacuum $|0\rangle$ plays for the $a_{\mathbf{k}}$'s. It is in some sense, *the new vacuum.* I'm sure this was not realized back in 1947 when this work was originally published, but we are standing on the threshold of a profound new idea. The point is that in making the transition from the Hamiltonian (5.51) and the field operator (5.31) to The Hamiltonian (5.63) and field (5.59), we automatically make a transition to a new vacuum with some unusual properties. I will get back to that in a minute.

This solution makes an interesting point about perturbation theory. Let's calculate the number of real particles, helium atoms say, that are excited out of the ground state.

$$N_{\text{ex}} = \langle\Omega|\sum_{\mathbf{k}\neq\mathbf{0}} a^\dagger_{\mathbf{k}}a_{\mathbf{k}}|\Omega\rangle = \langle\Omega|\sum_{\mathbf{k}\neq\mathbf{0}} v^2_{\mathbf{k}}\alpha_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}}|\Omega\rangle = \sum_{\mathbf{k}\neq\mathbf{0}} v^2_{\mathbf{k}} \tag{5.78}$$

To make my point, I will take the simplest possible potential, $V(\mathbf{x}) = \lambda\delta(\mathbf{x})$, so that $V_{\mathbf{k}} = \lambda$. If we were doing ordinary perturbation theory, we would pray that λ was "small" and expand the solution in powers of $n\lambda$. In this case we can get a solution which is "exact" in the sense that we don't need this approximation. We can replace the sum in (5.78) with an integral. After a horrendous integration we get the simple result

$$n_{\rm ex} \equiv \frac{N_{\rm ex}}{V} = \frac{m^{3/2}}{3\pi^2}(\lambda n)^{3/2} \tag{5.79}$$

Now $\lambda^{3/2}$ considered as a complex function of λ is non-analytic at $\lambda = 0$. It has a branch point. You can't make a power series expansion around a branch point; it just doesn't make sense. In this problem perturbation theory, in the usual sense, *would always get the wrong answer.* It's a sobering thought that in scattering theory we always use perturbation theory. We aren't smart enough to do anything else. *And it could all be wrong.*

It's clear that $\omega_{\mathbf{k}}$ from Eqn. (5.75) is the energy of a single quasiparticle. In the limit of small k we have

$$\omega_{\mathbf{k}} = vk \text{ with } v = \sqrt{\frac{nV_0}{m}} \tag{5.80}$$

The linear dispersion relation is the hallmark of massless particles. If there were no potential or the potential somehow vanished at low energy, the dispersion relation would be

$$\omega_{\mathbf{k}} = \frac{\mathbf{k}^2}{2m}.$$

This is the relation for an ordinary particle of mass m, *i.e.* the potential creates massless particles. We could say that (5.75) interpolates between phonon-like behavior at small k and massive particle-like behavior at large k.

Now let's get back to the ground state. Since the $\alpha_{\mathbf{k}}$'s and $a_{\mathbf{k}}$'s are related by a canonical transformation, there must exist an unitary operator U such that

$$\alpha_{\mathbf{k}} = U a_{\mathbf{k}} U^{-1} \qquad a_{\mathbf{k}} = U^{-1}\alpha_{\mathbf{k}} U. \tag{5.81}$$

It is possible to find an exact form for this operator,[5] but for our purposes it is enough to know that it exists. Since $a_{\mathbf{k}}|0\rangle = 0$, it must be that $\alpha_{\mathbf{k}} U|0\rangle = 0$. Look at (5.77). It must be that

$$|\Omega\rangle = U|0\rangle. \tag{5.82}$$

[5]This is assigned as Problem 17.4 in *Gauge Theories in Particle Physics, Vol. II*, I.J.R. Aitchison and A. J. G. Hey, Institute of Physics Publishing, 2004

The exact form of U shows that $|\Omega\rangle$ is a linear superposition of states with different numbers of particles, therefor it is not an eigenstate of the number operator (5.17). We had an intimation of this, since the Hamiltonian (5.57) does not conserve particle number. It will turn out that the conservation of particle number is related to gauge transformations. Our original Hamiltonian is invariant under the simple $U(1)$ symmetry transformation, $\varphi(\mathbf{x}) \to e^{-i\alpha}\varphi(\mathbf{x})$, whereas the modified Hamiltonian is not. It turns out that the breaking of this symmetry is directly responsible for the existence of the massless quasiparticles. I will prove this assertion when we get around to the Goldstone theorem.

The expectation value of the field $\varphi(\mathbf{x})$ from (5.58) for any state with a definite number of particles will always vanish. Let $|N\rangle$ be any state with N noninteracting particles. Then

$$\langle N|\varphi(\mathbf{x})|N\rangle = 0,$$

but this is not true of our new field (5.59).

$$\langle N|\tilde{\varphi}(\mathbf{x})|N\rangle = \rho_0^{1/2}. \tag{5.83}$$

We can rewrite (5.59) in terms of the $\alpha_{\mathbf{k}}$'s. Using (5.69) and (5.75) we get

$$\tilde{\varphi}(\mathbf{x}) = \rho_0^{1/2} + \frac{1}{\sqrt{V}}\sum_{\mathbf{k}}(u_{\mathbf{k}}\alpha_{\mathbf{k}} - v_{\mathbf{k}}\alpha_{-\mathbf{k}})e^{i\mathbf{k}\cdot\mathbf{x}}$$

Then from (5.77),

$$\langle\Omega|\tilde{\varphi}(\mathbf{x})|\Omega\rangle = \rho_0^{1/2} \tag{5.84}$$

All this was obtained as the end result of a number of approximations. Suppose we could do the theory exactly (whatever that means), and find the "true" field and vacuum state. We postulate that something of (5.84) would survive, that in fact

$$\langle\Omega|\tilde{\varphi}(\mathbf{x})|\Omega\rangle \neq 0. \tag{5.85}$$

While we are speculating, let's guess that because we are not making any approximations, the Hamiltonian would preserve the $U(1)$ symmetry discussed in connection with (5.54). Then we should not change any physical results by replacing

$$\tilde{\varphi}(\mathbf{x}) \to \tilde{\varphi}'(\mathbf{x}) = e^{-i\alpha}\tilde{\varphi}(\mathbf{x}) \tag{5.86}$$

This transformation can be brought about by a unitary operator U_α, $\tilde{\varphi}' = U_\alpha\tilde{\varphi}U_\alpha^{-1}$. Then

$$\langle\Omega|U_\alpha\tilde{\varphi}U_\alpha^{-1}|\Omega\rangle = e^{-i\alpha}\langle\Omega|\tilde{\varphi}|\Omega\rangle \tag{5.87}$$

I choose to interpret (5.87) as follows. I regard $U_\alpha^{-1}|\Omega\rangle \equiv |\Omega,\alpha\rangle$ as an alternative ground state. Then (5.87) becomes

$$\langle\Omega,\alpha|\tilde{\varphi}|\Omega,\alpha\rangle = e^{-i\alpha}\langle\Omega|\tilde{\varphi}|\Omega\rangle \tag{5.88}$$

Obviously there are an infinite number of these ground states, (5.84) corresponds to the special case $\alpha = 0$, but the choice of α can have no observable physical consequences.

It's time to summarize the plot. It is postulated that our exact Hamiltonian is invariant under the $U(1)$ transformation (5.54). The Hamiltonian gives rise to an infinite number of alternative ground states, however, and in choosing one of them *we break the symmetry.* This is called "spontaneous symmetry breaking," and I will have much more to say about it later. In the meantime, there is another example of spontaneous symmetry breaking that you're all familiar with, the ferromagnet. In this case the broken symmetry is rotation invariance. Surely the Hamiltonian for the interactions of atoms and electrons is invariant under rotation, and so an infinite mass of molten iron is also, but when the iron is cooled below the Curie point, the iron spontaneously (spontaneously – get it?) "chooses" one direction to line up all the magnetic moments, resulting in a ground state that has a preferred direction. Thus rotation invariance is broken. It is widely speculated that the "Hamiltonian of the universe" has many symmetries of which we are unaware, but we are living in or close to the ground state of this Hamiltonian, and much of the physical world is the end result of this sort of symmetry breaking.

5.8 Fermions

The fact that fermion wave functions are antisymmetric introduces a few small complications regarding the creation and annihilation operators. They are easy to explain looking at two-particle states. When I write $|i_1, i_2\rangle$, I mean that particle 1 is in state i_1, which is to say that the left-most entry in the ket refers to particle 1, the second entry on the left refers to particle 2, etc. Antisymmetry then decrees that $|i_1, i_2\rangle = -|i_2, i_1\rangle$. If both particles were in the same state $|i_1, i_1\rangle = -|i_1, i_1\rangle$, so double occupancy is impossible. If I describe this state in terms of occupation numbers $|n_1, n_2\rangle$, the left-most entry refers to the first quantum state (rather than the first particle), but which state is the first state? You have to decide on some convention for ordering states and then be consistent about it.

These states are constructed with creation and annihilation operators as in the case of bosons, but now we must be more careful about ordering. Consider the following two expressions.

$$a^\dagger_{i_1} a^\dagger_{i_2} |0\rangle = |i_1, i_2\rangle$$

$$a^\dagger_{i_2} a^\dagger_{i_1} |0\rangle = |i_2, i_1\rangle = -|i_1, i_2\rangle$$

I have decreed, and again this is a convention, that the first operator on the left creates particle 1, etc. Obviously

$$a^\dagger_{i_1} a^\dagger_{i_2} + a^\dagger_{i_2} a^\dagger_{i_1} = 0 \tag{5.89}$$

We say that fermion operators *anticommute*. The usual notation is

$$[A, B]_+ = \{A, B\} = AB + BA \tag{5.90}$$

Of course, fermions don't have numbers painted on them any more than bosons do, so we must use occupation numbers. Here the convention is

$$|n_1, n_2, \cdots\rangle = \left(a^\dagger_i\right)^{n_1} \left(a^\dagger_2\right)^{n_2} \cdots |0\rangle \qquad n_i = 0, 1 \tag{5.91}$$

The effect of the operator $a^\dagger_i$ must be

$$a^\dagger_i |\ldots, n_i, \ldots\rangle = \eta |\ldots, n_i + 1, \ldots\rangle, \tag{5.92}$$

where $\eta = 0$ of $n_i = 1$ and otherwise $\eta = +/-$, depending on the number of anticommutations necessary to bring the $a^\dagger_i$ to the position i. The commutation relations of the a_i's and $a^\dagger_i$'s are obtained by arguments similar to those leading to (5.15). The results are

$$\{a_i, a_j\} = \{a^\dagger_i, a^\dagger_j\} = 0 \qquad \{a_i, a^\dagger_j\} = \delta_{ij} \tag{5.93}$$

One can construct one- and two-particle operators just as we did for bosons. Formulas (5.23) and (5.24) are correct as they stand, but one must be careful to keep the creation and annihilation operators in the exact order in which they appear in these equations.[6]

[6]Strictly speaking, I should prove this. If you want to see all the excruciating details, consult F. Schabel *Advanced Quantum Mechanics*, Section 1.4.2

5.8.1 Inclusion of Spin

Spin is just "tacked on" to nonrelativistic quantum mechanics. The wave function for a spin-1/2 particle is just

$$\psi_\sigma(\mathbf{r},t) = \psi(\mathbf{r},t)\mathbf{u}_\sigma(t). \tag{5.94}$$

The spin wave function $\mathbf{u}_\sigma$ is a two-component column matrix. The subscript σ is an index that takes on the values $\pm 1/2$ corresponding to spin "up" or "down" (with respect to some axis of quantization). The space wave function is $\psi(\mathbf{r},t)$. The point is that spin and space degrees of freedom are completely decoupled. (This is not true in relativistic theory.) Sometime we need to specify that a particle is in a specific spin state. This can be done in cartoon fashion by writing $\psi_\uparrow$ or $\psi_\downarrow$.

It's easy to incorporate spin into Fock-state formulation. For example, the operator $a^\dagger_{\mathbf{k}\sigma}$ creates a particle in a momentum and spin eigenstate with momentum $\mathbf{k}$ and z-projection of spin σ. If we say that there are n_i particles in the i'th eigenstate, we must remember that a complete specification of the i'th eigenstate must include the spin projection σ_i.

5.8.2 The Fermi Sphere and the Debye Cutoff

The ground state of a system of bosons is one in which all the particles have zero energy. Because of the exclusion principle, this is not possible with fermions. In the case of fermions the ground state is that in which all the energies are as small as possible. Let's try to make that more precise. Since the kinetic energy is a function of k^2, $T = \hbar^2 k^2/2m$, there will be a certain momentum associated with the highest occupied energy level. This is called the Fermi momentum $\hbar k_F$. The ground state can be constructed in momentum space as

$$|\Omega\rangle = \prod_{\substack{\mathbf{k}\\ |\mathbf{k}|<k_F}} \prod_\sigma a^\dagger_{\mathbf{k}\sigma}|0\rangle \tag{5.95}$$

The set of all wave numbers $|\mathbf{k}| < k_F$ is called the *Fermi sphere*. Suppose the system is contained in a cube of length L on a side. If we impose periodic boundary conditions along x axis for example, the wave numbers are limited by the condition $e^{ik_x x} = e^{ik_x(x+L)}$, or $k_x L = 2\pi n_x$, where n_x is an integer.

$$dn_x = \frac{L}{2\pi} dk_x, \tag{5.96}$$

where dn_x is the number of allowed momentum states in the interval between k_x and $k_x + dk_x$. The number of states in the Fermi sphere is then

$$N = 2V \int_0^{k_f} \frac{d^3k}{(2\pi)^3} = \frac{Vk_F^3}{3\pi^2} \tag{5.97}$$

The factor of 2 in front of the integral arises because there are two allowed spin states for every one momentum state. It follows that

$$k_F^3 = 3\pi^2 n, \tag{5.98}$$

where $n = N/V$ is the average particle density. The Fermi energy is defined by $\epsilon_F = (\hbar k_F)^2/(2m)$. Typically it is on the order of a few eV's.

At absolute zero temperature all the electrons would be inside the Fermi sphere. The simplest excitation of this system is obtained by promoting an electron from inside the sphere to an energy level outside. In terms of creation and annihilation operators, such a state is written

$$|\mathbf{k}_2\sigma_2, \mathbf{k}_1\sigma_1\rangle = a^\dagger_{\mathbf{k}_2\sigma_2} a_{\mathbf{k}_1\sigma_1} |\Omega\rangle \tag{5.99}$$

One can think of this as the creation of an electron-hole pair. We can define hole creation and annihilation operators as $b_{\mathbf{k}\sigma} \equiv a^\dagger_{-\mathbf{k},-\sigma}$ and $b^\dagger_{\mathbf{k}\sigma} \equiv a_{-\mathbf{k},-\sigma}$. Then (5.98) becomes

$$|\mathbf{k}_2\sigma_2, \mathbf{k}_1\sigma_1\rangle = a^\dagger_{\mathbf{k}_2\sigma_2} b^\dagger_{-\mathbf{k}_1,-\sigma_1} |\Omega\rangle \tag{5.100}$$

Bosons do not obey the exclusion principle, so there is nothing exactly like the Fermi sphere for integer-spin particles. There is something similar for low-energy phonons in crystalline solids, however, called the *Debye cutoff.* Phonons in crystals are sound waves in which the crystal planes oscillate longitudinally along the direction of propagation. The fact that the oscillators are discrete imposes limitations on the wave numbers that can exist. Consider a line of length L consisting of $N + 1$ atoms spaced at a distance a. Assume that the first and last atoms don't move because they are at the surface of the crystal. The normal modes will be standing waves in which there are an integer number of half-wavelengths in the length L. Consequently, the allowed values are

$$k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \cdots, \frac{(N-1)\pi}{L} \tag{5.101}$$

Notice that if $k = N\pi/L$, there would be a node at the site of each atom, so that *no atom could move!* It doesn't make sense to have half-wavelengths

smaller than the interatomic separation, so the number of allowed modes is one less that the number of atoms – just those given in (5.101). We can now calculate the number of modes with wave number less than k. The calculation is identical to (5.97) (without the factor of 2),

$$N = V \int_0^k \frac{d^3k'}{(2\pi)^3} = \frac{Vk^3}{6\pi^2}. \tag{5.102}$$

We know that the number of modes is just equal to the number of atoms that are oscillating. If we call that number N also, (5.102) gives us a formula for the maximum allowed k.

$$k_D = (6\pi^2 N/V)^{1/3} \tag{5.103}$$

In the long-wavelength limit, the energy and frequency are simply proportional.

$$\omega = vk \tag{5.104}$$

where v is the velocity of sound. This allows us to define a cutoff frequency[7]

$$\omega_D = (6\pi^2 v^3 N/V)^{1/3}. \tag{5.105}$$

It will be relevant to our discussion of superconductors that the Debye energy $\hbar\omega_D$ is usually several orders of magnitude smaller than the Fermi energy.

5.9 Superconductors

Some metals when cooled to a temperature close to absolute zero enter a distinct thermodynamic state in which current flows without resistance. Metals in this state have many other remarkable properties as well. For example, they are perfectly diamagnetic. Magnetic fields are excluded except for a small penetration depth at the surface. This is called the Meissner effect. This rich phenomenology will be covered in the spring quarter of your solid state physics class. Here I want to concentrate on the basic physics of the superconducting state.

Our basic ideas about superconductors come from a classic paper by Bardeen, Cooper, and Schrieffer.[8] They were awarded the Noble Prize for

[7]Unlike (5.103), (5.105) is not an exact result. The point is that (5.104) holds in the long-wavelength limit, whereas the Debye cutoff argument is based on short-wavelength behavior. Nonetheless, these ideas are at the heart of several useful results for solids at low temperature. See C. Kittel, *Introduction to Solid State Physics.*

[8]J. Bardeen, L. N. Cooper, J. R. Schrieffer, *Phys. Rev.* **106**, 162 (1957); **108**, 1175 (1957)

this, and the body of theory has come to be called "BCS theory." Electrons of course repel one another, and even with the screening effect of the lattice, the net coulomb force is still repulsive. They can interact in another way, however, through the exchange of phonons. It was L. N. Cooper who first realized that this interaction produces a weak attractive force over a narrow range of electron energies. This force is sufficient under some circumstances to produce a weakly bound spin-zero state of two electrons called a "Cooper pair." Since the energy of an electron in a pair is less than the Fermi energy, the ground state becomes unstable, and electrons rush out of the Fermi sea to make pairs. The pairs act as bosons, so they can "condense" much like the helium atoms in a superfluid. The crucial difference is that their wave functions overlap, so there is a strong coherence among the pairs. For this reason they are able to spontaneously lock into phase with one another with a macroscopic coherence length. The pairs move, not as individual particles, but as a fluid with a rigid phase. It's not easy to scatter one of the pairs, because it's locked "in step" with all the others. It is this that accounts for the lack of electrical resistance.

This is the barest outline of an enormously rich and elegant body of theory. I will concentrate on the central ideas and make numerous idealizations and approximations along the way.

5.9.1 Effective Electron-Electron Interaction

Phonons in this context are acoustic waves in crystals. They arise because the crystal lattice can "flex" slightly. The phonon waves are purely longitudinal, they have no polarization. In this respect they are a three-dimensional generalization of compression waves on a rubber band. Since we will be dealing with phonons in the long-wavelength approximation, the classical analogy is apt. In both cases, the frequency and energy are proportional. Phonons are massless particles. If we think of phonons as particles with definite momentum, we must allow for them to scatter from electrons. The Hamiltonian for this interaction is derived in every advanced text on solid state physics.

$$H' = \sum_{\mathbf{k},\mathbf{q}} D_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} (a_{\mathbf{q}} + a^\dagger_{-\mathbf{q}}) \tag{5.106}$$

Here $c_{\mathbf{k}}$ is the annihilation operator for electrons, and $a_{\mathbf{q}}$ is the corresponding operator for phonons. $D_{\mathbf{q}}$ is the coupling "constant." It actually has some momentum dependence, which we will ignore. The derivation of this constant involves some assumptions about the mechanical properties of the lattice. This is really a topic for a solid state class, so I will just take (5.106)

as "given." The significance of the terms is clear enough. This first term corresponds to an electron of momentum $\mathbf{k}$ absorbing a phonon of momentum $\mathbf{q}$ and going on its way with momentum $\mathbf{q}+\mathbf{k}$. The second term gives the emission of a phonon of momentum $\mathbf{q}$ leaving an electron with momentum $\mathbf{k}-\mathbf{q}$. This is the Hamiltonian we would guess, even if we had no theory to find $D_{\mathbf{q}}$. In addition to the interaction term, there is the free-particle Hamiltonian,

$$H_0 = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}} a^\dagger_{\mathbf{q}} a_{\mathbf{q}} + \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}}. \tag{5.107}$$

The exchange of a phonon between two electrons is at least a two-step process, so we need to do something like second-order perturbation theory to combine two factors of (5.106) to get an effective electron-electron Hamiltonian. There is an elegant trick for doing this without any special formalism. Suppose you have a Hamiltonian of the form

$$H = H_0 + \lambda H'. \tag{5.108}$$

The parameter λ is supposed to be "small." We will eventually set it equal to one. It's really just a bookkeeping device. Our strategy is to find a transformed Hamiltonian

$$\tilde{H} \equiv e^{-S} H e^{S} = H + [H,S] + \frac{1}{2}[[H,S],S] + \cdots, \tag{5.109}$$

such that all first-order terms in λ are transformed away.[9] This can be accomplished by

$$\lambda H' + [H_0, S] = 0 \tag{5.110}$$

Substitute (5.110) into (5.109) bearing in mind that S is itself first order in λ.

$$\tilde{H} = H_0 + \frac{1}{2}[\lambda H', S] + O(\lambda^3) \tag{5.111}$$

That commutator will be our second-order interaction Hamiltonian. We can make an educated guess at the form of S (with $\lambda = 1$).

$$S = \sum_{\mathbf{k},\mathbf{q}} (A a^\dagger_{-\mathbf{q}} + B a_{\mathbf{q}}) D_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \tag{5.112}$$

[9] The solid state community calls this the Schrieffer-Wolff transformation.

Here A and B are functions that we are about to determine. A long tedious calculation gives

$$[H_0, S] = \sum_{\mathbf{k},\mathbf{q}} D_{\mathbf{q}} a^\dagger_{-\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} (\hbar\omega_{\mathbf{q}} + \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}) A + \sum_{\mathbf{k},\mathbf{q}} D_{\mathbf{q}} a_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} (\epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} - \hbar\omega_{\mathbf{q}}) B \quad (5.113)$$

I have assumed that $\omega_{\mathbf{q}} = \omega_{-\mathbf{q}}$. Now all this is supposed to equal $-H'$ from (5.106). It must be that

$$\begin{aligned} (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} - \hbar\omega_{\mathbf{q}}) A &= 1 \\ (\epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} + \hbar\omega_{\mathbf{q}}) B &= 1 \end{aligned} \quad (5.114)$$

Use the notation $\Delta_\pm(\mathbf{k},\mathbf{q}) = \epsilon_{\mathbf{k}} - \epsilon_{\mathbf{k}+\mathbf{q}} \pm \hbar\omega_{\mathbf{q}}$. Our formula for S is then.

$$S = \sum_{\mathbf{k},\mathbf{q}} \left[\frac{a^\dagger_{-\mathbf{q}}}{\Delta_-(\mathbf{k},\mathbf{q})} + \frac{a_{\mathbf{q}}}{\Delta_+(\mathbf{k},\mathbf{q})} \right] D_{\mathbf{q}} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} \quad (5.115)$$

Some comments about (5.115) are in order.

- It's possible to show that S is antihermitian, i.e. $S^\dagger = -S$. This is necessary so that (5.109) is a unitary transformation. In some sense we have "diagonalized" the Hamiltonian. I'll explain what I mean by that below.

- The energy denominators are typical for nonrelativistic perturbation theory. They vanish on the energy shell. For Δ_+ this corresponds to the absorption of a phonon of momentum $\mathbf{q}$, and Δ_- to the emission of a phonon of momentum $-\mathbf{q}$.

- Our new second-order interaction Hamiltonian from (5.111) is $[H', S]/2$. In order to calculate this we will need commutators like

  $$[(a^\dagger_{-\mathbf{q}} + a_{\mathbf{q}}) c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}}, a^\dagger_{-\mathbf{q}'} c^\dagger_{\mathbf{k}'+\mathbf{q}'} c_{\mathbf{k}'}] = \delta_{\mathbf{q},-\mathbf{q}'} c^\dagger_{\mathbf{k}+\mathbf{q}} c_{\mathbf{k}} c_{\mathbf{k}'+\mathbf{q}'} c_{\mathbf{k}'} + \cdots \quad (5.116)$$

  The $\cdots$ in (5.116) refers to terms with one pair of boson operators and one pair of electron operators. These are second order all right, but they are the product of two *single-particle interactions* and as such do not interest us. Now you see the sense in which (5.109) diagonalizes the Hamiltonian. Our new Hamiltonian connects two-electron states with two-electron states, and states with one electron and one boson with similar states with one boson and one electron.

It's instructive to combine the two terms of (5.115) over a common denominator. That gives the final form for our effective, second-order, two electron Hamiltonian as

$$\frac{1}{2}[H', S] = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} D_{\mathbf{q}}^2 c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}'-\mathbf{q}}^\dagger c_{\mathbf{k}'} c_{\mathbf{k}} \frac{\hbar\omega}{(\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}'-\mathbf{q}})^2 - (\hbar\omega)^2} \tag{5.117}$$

The electron energies $\epsilon_{\mathbf{k}}$, as I pointed out in Section 8.2, are much larger than phonon frequencies, $\omega_{\mathbf{q}}$, but there is a narrow window around $|\epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}'-\mathbf{q}}| \approx \hbar\omega_D$ in which the potential changes sign. Outside the window the potential is repulsive (just like a positive Coulomb potential), but inside *it is attractive.* This is the famous "energy gap" upon which all superconducting phenomena depend. Electrons with just this range of energy form Cooper pairs.

At this point it is customary to make several approximations based partly on mathematical convenience and partly on an understanding of the physics of superconductors. First it is known that most of the phonons in a low-temperature solid have energies close to the Debye limit. Since the spread in energies is so small, we do not sum over $\mathbf{q}$ in (5.117) but gather all the numerical factors into one phenomonological (positive) coupling constant V. Second, for a variety of reasons, electron pairs tend to have zero total momentum and zero total spin. Thus if one electron of a pair has momentum $\mathbf{k}$ and $s_z = 1/2$, the other will have $-\mathbf{k}$ and $s_z = -1/2$. We therefore adopt the convention that a state written explicitly as $\mathbf{k}$ has spin $\uparrow$, and one written as $-\mathbf{k}$ has spin $\downarrow$. The complete electron Hamiltonian with these assumptions is

$$H_{\text{BCS}} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}(c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + c_{-\mathbf{k}}^\dagger c_{-\mathbf{k}}) - V\sum_{\mathbf{k},\mathbf{k}'} c_{\mathbf{k}'}^\dagger c_{-\mathbf{k}'}^\dagger c_{-\mathbf{k}} c_{\mathbf{k}} \tag{5.118}$$

This is known as the BCS reduced Hamiltonian. It operates only on the subspace consisting of the Cooper pairs that satisfy the above assumptions.

Notice that the interaction part of this Hamiltonian cannot be written as a product of number operators. Put it another way, a state described in terms of occupation numbers is *not* an eigenstate of H_{BCS}. We have not completely succeeded in our goal of diagonalizing it. You will recall that this was the situation with the superfluid Hamiltonian. In fact, (5.118) apart from the double sum is formally equivalent to to (5.63). It won't be a surprise then that the Bogoliubov transformation is the key to diagonalizing it.

In 1956 Cooper[10] proved a profound result that was the prelude to the full BCS theory. He showed that two electrons directly below the Fermi

[10]L. N. Cooper, Phys. Rev. **104**, 1189 (1956)

surface can lower their energy by being excited into a Cooper pair with momentum $(\mathbf{k}, -\mathbf{k})$ just above the Fermi surface, provided that an attractive interaction of the form (5.118) exists. This is known as the *Cooper instability.*[11] He also showed that a small center of mass momentum has the effect of much reducing the binding energy. This is one reason why we only consider $(\mathbf{k}, -\mathbf{k})$ pairs. It is also the reason why superconductivity only exists at low temperature. Now suppose we could start with a free electron gas close to $T = 0$, so that the Fermi sphere is completely filled up to a radius ϵ_F. As we "turn on" the interaction (5.118), electrons in a thin shell, $\epsilon_F - \omega_D < \epsilon < \epsilon_F + \omega_D$ form Cooper pairs. The one-electron states in $\mathbf{k}$-space in this shell are used in forming the BCS ground state, i.e. the minimum-energy of the Hamiltonian (5.118). Part of our job will be to construct this state explicitly. For the time being I will simply call it $|\Omega\rangle$.

We'll start by deriving equations of motion for the $c_{\mathbf{k}}$'s.

$$\begin{aligned} i\dot{c}_{\mathbf{k}} &= \epsilon_{\mathbf{k}} c_{\mathbf{k}} - c^\dagger_{-\mathbf{k}} V \sum_{\mathbf{k}'} c_{-\mathbf{k}'} c_{\mathbf{k}'} \\ i\dot{c}^\dagger_{-\mathbf{k}} &= -\epsilon_{\mathbf{k}} c^\dagger_{-\mathbf{k}} - c_{\mathbf{k}} V \sum_{\mathbf{k}'} c^\dagger_{\mathbf{k}'} c^\dagger_{-\mathbf{k}'} \end{aligned} \tag{5.119}$$

The key to solving the superfluid problem was to linearize the equations of motion by replacing the product of operators $a^\dagger_k a_k$ with their expectation value in the ground state, $|\Omega\rangle$. Here the "particles" we are concerned with are pairs, so the corresponding approximation is

$$\begin{aligned} V \sum_{\mathbf{k}} c_{-\mathbf{k}} c_{\mathbf{k}} &\to V \langle\Omega| \sum_{\mathbf{k}} c_{-\mathbf{k}} c_{\mathbf{k}} |\Omega\rangle \equiv \Delta_{\mathbf{k}} \\ V \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} &\to V \langle\Omega| \sum_{\mathbf{k}} c^\dagger_{\mathbf{k}} c^\dagger_{-\mathbf{k}} |\Omega\rangle \equiv \Delta^*_{\mathbf{k}} \end{aligned} \tag{5.120}$$

Substituting (5.120) back into (5.119) puts us back in the realm of familiar mathematics. We look for solutions of the form $a_{\mathbf{k}}(t) = a_{\mathbf{k}}(0) e^{-i\omega_{\mathbf{k}} t}$.

$$\begin{aligned} \omega_{\mathbf{k}} c_{\mathbf{k}}(0) &= \epsilon_{\mathbf{k}} c_{\mathbf{k}}(0) - \Delta_{\mathbf{k}} c^\dagger_{-\mathbf{k}}(0) \\ \omega_{\mathbf{k}} c^\dagger_{-\mathbf{k}}(0) &= -\epsilon_{\mathbf{k}} c^\dagger_{-\mathbf{k}}(0) - \Delta^*_{\mathbf{k}} c_{\mathbf{k}}(0) \end{aligned} \tag{5.121}$$

These equations are consistent if

$$\begin{vmatrix} \omega_{\mathbf{k}} - \epsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta^*_{\mathbf{k}} & \omega_{\mathbf{k}} + \epsilon_{\mathbf{k}} \end{vmatrix} = 0 \tag{5.122}$$

[11] See, P. Philips, *Advanced Solid State Physics* for an accessible proof.

Or

$$\omega_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \tag{5.123}$$

This is an important result. It means that all excited states are separated from $\epsilon_{\mathbf{k}}$ by a finite amount, the gap parameter Δ.

The eigenvectors of (5.121) are in fact the Bogoliubov transformations from Section 7.2. The analog to (5.66) is

$$\beta_{\mathbf{k}} = u_k c_{\mathbf{k}} - v_k c_{-\mathbf{k}}^\dagger \qquad \beta_{-\mathbf{k}} = u_k c_{-\mathbf{k}} - v_k c_{\mathbf{k}}^\dagger \tag{5.124}$$

with inverses

$$c_{\mathbf{k}} = u_{\mathbf{k}} \beta_{\mathbf{k}} + v_{\mathbf{k}} \beta_{-\mathbf{k}}^\dagger \qquad c_{-\mathbf{k}} = u_{\mathbf{k}} \beta_{-\mathbf{k}} - v_{\mathbf{k}} \beta_{\mathbf{k}}^\dagger \tag{5.125}$$

Both u_k and v_k are real numbers; u_k is an even function of k, v_k is odd. We would like the β's to satisfy fermion anticommutation relations.

$$\{\beta_{\mathbf{k}}, \beta_{\mathbf{k}'}\} = u_{\mathbf{k}} u_{\mathbf{k}'} \{c_{\mathbf{k}}, c_{\mathbf{k}'}\} = v_{\mathbf{k}} v_{\mathbf{k}'} \{c_{-\mathbf{k}}^\dagger, c_{-\mathbf{k}'}\} = \delta_{\mathbf{k}\mathbf{k}'}(u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2)$$

This is the first condition on $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$.

$$u_{\mathbf{k}}^2 + v_{\mathbf{k}}^2 = 1 \tag{5.126}$$

The natural parameterization is

$$u_{\mathbf{k}} = \cos\theta_{\mathbf{k}} \qquad v_{\mathbf{k}} = \sin\theta_{\mathbf{k}} \tag{5.127}$$

Substituting (5.125) into the first of (5.121) gives

$$\omega_{\mathbf{k}} u_{\mathbf{k}} = \epsilon_{\mathbf{k}} u_{\mathbf{k}} + \Delta u_{\mathbf{k}} v_{\mathbf{k}}. \tag{5.128}$$

Square and substitute (5.123)

$$\Delta^2 (u_{\mathbf{k}}^2 - v_{\mathbf{k}}^2) = 2\epsilon_{\mathbf{k}} \Delta u_{\mathbf{k}} v_{\mathbf{k}} \tag{5.129}$$

from which we conclude

$$\tan 2\theta_{\mathbf{k}} = \Delta/\epsilon_{\mathbf{k}}$$

$$\cos^2\theta_{\mathbf{k}} = \frac{1}{2}\left(1 + \frac{\epsilon_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \qquad \sin^2\theta_{\mathbf{k}} = \frac{1}{2}\left(1 - \frac{\epsilon_{\mathbf{k}}}{\omega_{\mathbf{k}}}\right) \tag{5.130}$$

So far I have said nothing about the phase of Δ. There is an important issue lurking here. The BCS Hamiltonian (5.118) is invariant under the global gauge transformation $c_{\mathbf{k}} \to c'_{\mathbf{k}} = e^{-i\alpha} c_{\mathbf{k}}$ for all $\mathbf{k}$. (It is clear that if H_{BCS} did not observe this symmetry, it would also not conserve fermion

number.) It is speculated that the ground state observes this symmetry as well so that any choice of the phase of Δ is equally valid. We take Δ real. The magnitude of Δ can be estimated as follows. Substitute (5.125) into (5.120) and use the fermion anticommutation relations.

$$\begin{aligned}\Delta &= V\langle\Omega|\sum_{\mathbf{k}}(\cos\theta_{\mathbf{k}}\beta_{-\mathbf{k}}+\sin\theta_{\mathbf{k}}\beta^\dagger_{\mathbf{k}})(\cos\theta_{\mathbf{k}}\beta_{\mathbf{k}}+\sin\theta_{\mathbf{k}}\beta^\dagger_{-\mathbf{k}})|\Omega\rangle \\ &= V\langle\Omega|\sum_{\mathbf{k}}\cos\theta_{\mathbf{k}}\sin\theta_{\mathbf{k}}\beta_{-\mathbf{k}}\beta^\dagger_{-\mathbf{k}}|\Omega\rangle \\ &= V\sum_{\mathbf{k}}\frac{\Delta}{2\left[\epsilon^2_{\mathbf{k}}+\Delta^2\right]^{1/2}} \qquad (5.131)\end{aligned}$$

I have used (5.130) in the last step. This can be converted into an integral equation,

$$1 = \frac{VN_F}{2}\int_{-\omega_D}^{\omega_D}\frac{d\epsilon}{\left[\epsilon^2+\Delta^2\right]^{1/2}} = VN_F\sinh^{-1}(\omega_D/\Delta) \qquad (5.132)$$

The limits of the integral are chosen in light of the fact that the Hamiltonian only works on the states $\epsilon_{\mathbf{k}}$ where $\epsilon_F-\omega_D<\epsilon_{\mathbf{k}}<\epsilon_F+\omega_{\mathbf{k}}$. N_F is the density of states at the Fermi level. Eqn. (5.132) can be inverted to yield

$$\Delta = \frac{\omega_D}{\sinh(1/VN_F)} \qquad (5.133)$$

This was one of the original discoveries of the BCS collaborators. I understand that it is in reasonable agreement with experiment. Remember that ordinary perturbation theory requires that the interaction be in some sense "small." Here the magnitude of the interaction is set by the quantity VN_F. In the limit that this is small, the denominator looks like the hyperbolic sine of infinity! This is not something you will ever get from a perturbation theory.

Finally as promised, we construct the explicit form of the BCS ground-state wave function. I claim that apart from normalization

$$|\Omega\rangle = \prod_{\mathbf{k}}\beta_{-\mathbf{k}}\beta_{\mathbf{k}}|0\rangle \qquad (5.134)$$

Substituting (5.124) we get,

$$|\Omega\rangle = \prod_{\mathbf{k}}(-v_{\mathbf{k}})(u_{\mathbf{k}}+v_{\mathbf{k}}c^\dagger_{\mathbf{k}}c^\dagger_{-\mathbf{k}}) \qquad (5.135)$$

It turns out that the right normalization is obtained by dropping the factors of $(-v_{\mathbf{k}})$. This can be seen as follows:

$$\begin{aligned}\langle\Omega|\Omega\rangle &= \langle 0|\prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}}c_{-\mathbf{k}}c_{\mathbf{k}})(u_{\mathbf{k}} + v_{\mathbf{k}}c^{\dagger}_{\mathbf{k}}c^{\dagger}_{-\mathbf{k}}|0\rangle) \\ &= \prod_{\mathbf{k}}(u^2_{\mathbf{k}} + v^2_{\mathbf{k}})\langle 0|0\rangle = 1 \end{aligned} \tag{5.136}$$

Thus the normalized ground state is

$$|\Omega\rangle = \prod_{\mathbf{k}}(u_{\mathbf{k}} + v_{\mathbf{k}}c^{\dagger}_{\mathbf{k}}c^{\dagger}_{-\mathbf{k}}) \tag{5.137}$$

Just to make sure, let's verify that $\beta|\Omega\rangle = 0$.

$$\beta_{\mathbf{k}'}|\Omega\rangle = \beta_{\mathbf{k}'}\beta_{-\mathbf{k}'}\beta_{\mathbf{k}'}\prod_{\mathbf{k}}{}' \beta_{-\mathbf{k}}\beta_{\mathbf{k}}|0\rangle = 0$$

The prime on the product indicates that the $\mathbf{k} = \mathbf{k}'$ term has been factored out. The expression is zero because $\beta_{\mathbf{k}'}\beta_{\mathbf{k}'} = 0$ for a fermion operator. Eqn. (5.137) represents a coherent superposition of correlated pairs, with no restraint on the particle number.

Chapter 6

Second Quantization and Relativistic Fields

In Chapter 5 I introduced the concept of the field operators $\psi(\mathbf{x})$ and $\psi^\dagger(\mathbf{x})$ (5.31). I explained that $\psi^\dagger(\mathbf{x})$ is the operator that creates a particle at the position $\mathbf{x}$. In this context it seemed like a technical device for introducing interactions in which particles are exchanged so that the number of particles is not a constant of the motion. All this was done to "soup up" Schrodinger's equation. We didn't worry too much about the physical interpretation. There are other ways of regarding quantum field operators, however, that are important when we are exploring other wave equations, particularly those arising in relativistic theories.[1]

6.1 Introduction to Field Theory

Imagine that space is like a rubber sheet. If I put a bowling ball on the sheet, it will create a depression, and nearby objects will roll into it. This is an imperfect analogy for an attractive potential. We could describe the attraction in one of two ways: we could say that there is an attractive potential between any pair of point-like masses, or we could introduce a continuous variable, $\phi(x, y)$ which describes the displacement of the sheet as a function of position. Such a continuous displacement variable is a *field* in the strict mathematical sense: it assigns a numerical value (or set of values) to each point in space. The quantum mechanics of such fields is called quantum field theory. Now suppose that instead of using a bowling ball I

[1] This material also appears at the beginning of Chapter 4. It is included here to make the next few chapters self-contained.

jump up and down on the sheet. The sheet will oscillate in response. My activity becomes a *source* of energy, which propagates outward in the form of waves. This is the rubber-sheet analogy to the propagation of particles.

This analogy can easily be misleading. For one thing, I don't want you to think we are doing general relativity. The rubber sheet is not intended as an analogy for ordinary space-time as it is often used in explaining general relativity. The field $\phi(x, y)$ describes a displacement, and I know you want to ask, "Displacement of what?"

The same question comes up in classical electromagnetic theory. When an electromagnet wave is propagating through space, what is waving? Folks in the 19'th century thought it must be some sort of mechanical medium, which they called the ether. According to the textbooks, Michaelson and Morley proved that wrong with their famous interferometer. But just saying that the ether does't exist doesn't answer the question, it just makes it impossible to answer! Let's bite the bullet and agree for the purposes of this course that space is pervaded by a medium, which for lack of a better name, we will call the ether. Well, actually the *ethers*. Each species of particle corresponds to a set of vibrations in it's own specific ether. Electrons are all vibrations in the electron ether, etc. Space-time points in the ether can be labelled with Lorentz four-vectors or $(\boldsymbol{x}, t)$ as usual, and these points obey the usual rules for Lorentz transformations. This much is required by the M-M experiment. Ordinary bulk media have elastic properties that are described by two parameters, the density and Young's modulus. These parameters are not themselves relevant to our formalism, but their ratio gives the velocity of propagation, which is what we really care about.

I am fond of saying, "When correctly viewed, everything is a harmonic oscillator." Now you see that this is profoundly true. Each point on the rubber sheet or ether acts like a harmonic oscillator! *Quantum field theory is a theory about harmonic oscillators.*

Well – I have to modify that slightly. If each point on the sheet behaved like a *simple* harmonic oscillator with a quadratic potential, the waves propagating on the sheet would never interact. The principle of linear superposition would hold everywhere. This is a theory of free particles. If our theory is to describe interactions, then we must modify the potential so that it becomes anharmonic. Unfortunately, the anharmonic oscillator cannot be solve exactly in quantum mechanics. (If you think of a way to do it, tell me and I'll be famous.) We have to resort to approximations. The generic name for these approximations is perturbation theory. The path integral formalism in Chapters 1-4 comes equipped with its own natural way of doing perturbation theory, which you understand by now if you have plowed

through this material.

There is an alternative way of dealing with interaction involving the creation and annihilation of particles. It is the older way, sometimes called canonical quantization or second quantization. The path integral formalism, seeks to banish all operators from the theory. Second quantization goes in the other direction. It turns the wave functions themselves into operators by imbedding creation and annihilation operators into them; *but they are the raising and lowering operators of the harmonic oscillator!* The universe, according to second quantization, is an infinity of harmonic oscillators. This approach is complementary to path integrals in other ways as well. We need to master both.

Continuum mechanics is not covered in most graduate mechanics classes. There is a good discussion in the last chapter of Goldstein, but we never make it that far. What follows is a brief introduction.

6.2 Introduction to Continuum Mechanics

The rules of continuum mechanics are derived by starting with a system with a finite number of degrees of freedom and then passing to the limit in which the number becomes infinite. Let's do this with the simplest possible system, a long chain of masses connected by springs. It's a one-dimensional problem. The masses can only oscillate along the chain. We will use φ_i, the displacement of the i-th particle from its equilibrium position, as the generalized coordinate. The Lagrangian is constructed in the obvious way.

$$T = \frac{1}{2}\sum_i m\ddot{\varphi}_i^2 \tag{6.1}$$

$$V = \frac{1}{2}\sum_i k(\varphi_{i+1} - \varphi_i)^2 \tag{6.2}$$

$$L = T - V = \frac{1}{2}\sum_i a\left[\frac{m}{a}\dot{\varphi}_i^2 - ka\left(\frac{\varphi_{i+1} - \varphi_i}{a}\right)^2\right] \tag{6.3}$$

The equilibrium separation between masses is a. The spring constant is k. The Euler-Lagrange equations of motion are obtained from

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}_i} - \frac{\partial L}{\partial \varphi_i} = 0 \tag{6.4}$$

If there are N masses, then there are N coupled equation of this sort. They look like

$$\frac{m}{a}\ddot{\varphi}_i - ka\left(\frac{\varphi_{i+1}-\varphi_i}{a^2}\right) + ka\left(\frac{\varphi_i-\varphi_{i-1}}{a^2}\right) = 0 \tag{6.5}$$

We need different parameters to describe the continuum limit:

$$m/a \to \mu \qquad \text{mass per unit length}$$

$$ka \to Y \qquad \text{Young's modulus}$$

The index i points to the i-th mass, and φ_i gives its displacement. In the continuum limit, the index is replaced by the coordinate x. In elementary mechanics, x would be the displacement of a particle. Here $\varphi(x)$ is the displacement of the string *at the point* x. In the continuum limit

$$\frac{\varphi_{i+1}-\varphi_i}{a} \to \frac{\varphi(x+a)-\varphi(x)}{a} \to \frac{d\varphi}{dx}$$

$$L \to \frac{1}{2}\int dx\left[\mu\dot{\varphi}^2 - Y\left(\frac{d\varphi}{dx}\right)^2\right] \equiv \int dx\mathcal{L}(\varphi,\dot{\varphi}) \tag{6.6}$$

The last integral implicitly defines the *Lagrangian density* . The continuum version of the Euler-Lagrange equation is[2]

$$\frac{d}{dt}\left[\frac{\partial\mathcal{L}}{\partial\left(\frac{d\varphi}{dt}\right)}\right] + \frac{d}{dx}\left[\frac{\delta\mathcal{L}}{\partial\left(\frac{d\varphi}{dx}\right)}\right] - \frac{\partial\mathcal{L}}{\partial\varphi} = 0 \tag{6.7}$$

Use the Lagrangian density from (6.6) in (6.7).

$$\frac{\partial^2\varphi}{\partial x^2} = \left(\frac{\mu}{Y}\right)\frac{d^2\varphi}{dt^2} \tag{6.8}$$

(6.4) and (6.5) represent a set of N coupled equations for N degrees of freedom. (6.7) is one equation for an infinite number of degrees of freedom. In this sense, continuum mechanics is much easier that discrete mechanics.

Equation (6.8) should remind you of the equation for the propagation of electromagnetic waves.

$$\left(\frac{\partial^2\varphi}{\partial x^2}\right) + \left(\frac{\partial^2\varphi}{\partial y^2}\right) + \left(\frac{\partial^2\varphi}{\partial z^2}\right) = \frac{1}{c^2}\left(\frac{\partial^2\varphi}{\partial t^2}\right)$$

[2]See Goldstein for a derivation of this important equation.

As you know, photons are *massless* particles. Notice that a string of *massive* particles yields a wave equation that when quantized describes the propagation of *massless* particles. (With a different velocity, of course.) This is worth a brief digression.

What does it mean to say that a wave function describes the propagation of a particle of a particular mass? The wave function $\psi = e^{i(kx-\omega t)}$ might describe a wave in classical E&M, or a massive particle in non-relativistic or relativistic quantum mechanics. The question is, what is the relation between k and ω? The relationship between the two is called a *dispersion relation*. It contains a great deal of information. In the case of EM waves in vacuum, $k = \omega/c$. Frequency and wave number are simply proportional. This is the hallmark of a massless field. The velocity is the constant of proportionality, so there can only be one velocity. In Schrodinger theory

$$\frac{\hbar^2 k^2}{2m} = \hbar\omega \tag{6.9}$$

The relationship is quadratic. The relativistic wave equation for a spin-zero particle is called the Klein-Gordon equation.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\varphi - \frac{m^2c^2}{\hbar^2}\varphi = 0 \tag{6.10}$$

The dispersion relation is

$$(c\hbar k)^2 + m^2c^4 = (\hbar\omega)^2, \tag{6.11}$$

or in other words, $p^2c^2 + m^2c^4 = E^2$. All these equations can be obtained from (6.7) with the appropriate Lagrangian density. They are all three-dimensional variations of our "waves on a rubber sheet" model. What does this have to do with the particle's mass? It's useful to plot (6.9) and (6.11), i.e. plot ω versus k for small values of k. In both cases the curves are parabolas. This means that in the limit of small k, the group velocity,

$$v_{\text{group}} = \frac{d\omega}{dk} \approx \frac{\hbar k}{m} \tag{6.12}$$

In other words, the group velocity is equal to the classical velocity for a massive particle $v = p/m$. All the wave equations I know of fall in one of these two categories; either ω is proportional to k, in which case the particle is massless and its velocity $v = \omega/k$, or the relationship is quadratic, in which case

$$m = \lim_{k\to 0}\left(\hbar k\frac{dk}{d\omega}\right). \tag{6.13}$$

So long as we are talking about wave-particle duality, *this is what mass means.*

One of the advantages of using Lagrangians rather than Hamiltonians is that Lagrangians have simple transformation properties under Lorentz transformations. To see this, let's rewrite (6.7) in relativistic notation. Construct the contravariant and covariant four-vectors

$$x^\mu \equiv (x^0, x^1, x^2, x^3) = (ct, x, y, z) \tag{6.14}$$

$$x_\mu = (x_0, x_1, x_2, x_3) = (ct, -x, -y, -z) \tag{6.15}$$

and the corresponding contravariant and covariant derivatives

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} \qquad \partial_\mu \equiv \frac{\partial}{\partial x^\mu}. \tag{6.16}$$

This puts the Euler-Lagrange equation in tidy form

$$\partial^\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{6.17}$$

This is slightly amazing. Equation (6.7) was derived without reference to Lorentz transformations, and yet (6.17) has the correct form for a scalar wave equation. We get relativity for free! If we can manage to make $\mathcal{L}$ a Lorentz scalar, then (6.17)) will have the same form in all Lorentz frames. Better yet, the action

$$S = \int dt\, L = \int dt \int d^3x\, \mathcal{L} = \frac{1}{c} \int d^4x\, \mathcal{L} \tag{6.18}$$

is also a Lorentz scalar. We can do relativistic quantum mechanics using the canonical formalism of classical mechanics.

Here's an example. Rewrite (6.6) in 3-d

$$\mathcal{L} = \frac{1}{2} \left\{ \mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - Y \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \right\} \tag{6.19}$$

This would be the Lagrangian density for oscillations in a huge block of rubber. Take

$$\frac{\mu}{Y} = \frac{1}{c^2}. \tag{6.20}$$

Obviously $\mathcal{L}$ can be multiplied by any constant without changing the equations of motion. Rescale it so that it becomes

$$\mathcal{L} = \frac{1}{2} \left\{ \left(\frac{\partial \varphi}{\partial t} \right)^2 - c^2 \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] \right\} \tag{6.21}$$

Substituting (6.21) into (6.17) yields the usual equation for EM waves, $\Box\varphi = 0$.

Notice how the Lagrangian for oscillations a block of rubber (6.19) turns into the Lagrangian for oscillations in the ether (6.21). We don't have to worry about the mechanical properties of the ether, because μ and Y are scaled away. Despite what you may have been told, the Michelson-Morley experiment proves the *existence* of the ether. When correctly viewed, everything is a bunch of harmonic oscillators, even the vacuum!

Using Einstein's neat notation, we can collapse (6.21) into one term

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) \equiv \frac{1}{2}(\partial\varphi)^2 \tag{6.22}$$

The last piece of notation $(\partial\varphi)^2$, is used to save ink. The fact that we can write $\mathcal{L}$ like this is proof that it is a Lorentz scalar. This is an important point; we can deduce the symmetry properties of a theory by glancing at $\mathcal{L}$.

Now you can make up your own field theories. All you have to do is add scalar terms to (6.22). Try it. Name the theory after yourself. Here's a theory that already has a name. It's the Klein-Gordon theory.

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\phi^2\right] \tag{6.23}$$

(I have set $c = 1$ and $\hbar = 1$.) Using our new notation, the equation of motion is

$$(\partial_\mu\partial^\mu + m^2)\varphi = 0 \tag{6.24}$$

If we assume that $\varphi(x)$ (x is a 4-vector in this notation.) is a one-component Lorentz scalar, then this describes a spinless particle with mass m propagating without interactions. Spin can be included by adding more components to φ. More about this later.

6.3 Introduction to Second Quantization

So far this has all been "classical field theory," ie. the fields are ordinary functions, c-numbers as they are called, and not operators. We have not yet introduced second quantization. In the previous chapter I introduced the field operator

$$\hat{\psi}(\mathbf{x}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}} \tag{6.25}$$

I said that $\hat{a}^\dagger_{\mathbf{k}}$ and $\hat{a}_{\mathbf{k}}$ were creation and annihilation operators and that all this was necessary to treat systems in which the number of particles was

not constant. In this section I would like to examine the motivations behind (6.25) more carefully and also investigate the many subtleties that arise when we apply these ideas to relativistic wave equations. We will eventually derive a completely relativistic generalization of (6.25), which will be our starting point for doing relativistic field theory.

We have encountered so far three quantum mechanical wave equations, the Schrodinger equation, the Klein-Gordon equation, and the equation for massless scalar particles, which is just the K-G equation with $m = 0$. I will write the free-particle versions of them in a peculiar way to emphasize the complementary roles of time and energy.

$$i\frac{\partial}{\partial t}\psi(\mathbf{x},t) = \frac{\hat{\mathbf{k}}^2}{2m}\psi(\mathbf{x},t) \tag{6.26}$$

$$\left(i\frac{\partial}{\partial t}\right)^2\psi(\mathbf{x},t) = (\hat{\mathbf{k}}^2 + m^2)\psi(\mathbf{x},t) \tag{6.27}$$

$$\left(i\frac{\partial}{\partial t}\right)^2\psi(\mathbf{x},t) = \hat{\mathbf{k}}^2\psi(\mathbf{x},t) \tag{6.28}$$

I have used $\hat{\mathbf{k}} = -i\nabla_{\mathbf{x}}$ and $\hbar = c = 1$. The operator on the right side of (6.26) is the kinetic energy. Einstein's equation $E^2 = \mathbf{k}^2 + m^2$ suggests that the operators on the right side of (6.27) and (6.28) are the total energy squared. Suppose that the ψ's are eigenfunctions of these operators with eigenvalue $\omega(\mathbf{k})$. (Each of these three equations will define a different functional relation between $\mathbf{k}$ and ω, of course.) The equations become

$$i\frac{\partial}{\partial t}\psi_\omega(\mathbf{x},t) = \omega(\mathbf{k})\psi_\omega(\mathbf{x},t) \tag{6.29}$$

$$\left(i\frac{\partial}{\partial t}\right)^2\psi_\omega(\mathbf{x},t) = \omega^2(\mathbf{k})\psi_\omega(\mathbf{x},t) \tag{6.30}$$

$$\left(i\frac{\partial}{\partial t}\right)^2\psi_\omega(\mathbf{x},t) = \omega^2(\mathbf{k})\psi_\omega(\mathbf{x},t) \tag{6.31}$$

Although we don't usually use this language, we could think of $i\partial/\partial t$ as a kind of energy operator whose eigengvalues are the total energy of the particle. Suppose now that the ψ_ω's are also momentum eigenstates so that $\hat{\mathbf{k}}\psi_\omega = \mathbf{k}\psi_\omega$. The simplest solutions of (6.26) and (6.29) with $\omega = \mathbf{k}^2/2m$ are

$$\psi_{\mathbf{k}}(\mathbf{x},t) = \frac{1}{\sqrt{V}}e^{i(\pm\mathbf{k}\cdot\mathbf{x}-\omega t)} \tag{6.32}$$

whereas the simplest solutions of (6.27) and (6.30) with $\omega^2 = \mathbf{k}^2 + m^2$ or (6.28) and (6.31) with $\omega^2 = \mathbf{k}^2$ are

$$\psi_{\mathbf{k}}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} e^{i(\pm\mathbf{k}\cdot\mathbf{x}\mp\omega t)} \tag{6.33}$$

(The $1/\sqrt{V}$ is a normalization factor put in for later convenience.) Evidentally the solutions to (6.30) and (6.31) comprise a larger family than those of (6.29), and it is this larger family that I want to investigate.

To avoid ambiguity, I will assume that the *symbol* ω refers to a positive quantity. Then

$$i\frac{\partial}{\partial t} e^{\mp i\omega t} = \pm\omega e^{\mp i\omega t} \tag{6.34}$$

Since $\hbar = 1$, ω has the units of energy. Schrodinger's equation does not have negative energy solutions. This is the clue that the upper sign in (6.33) and (6.34) gives positive-energy solutions and the lower sign gives negative-energy solutions *whatever that may mean!* What about the other sign ambiguity? Think about the solution $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$. Pick out some point on the wave where the phase of the exponential is ϕ. As time goes by, this point will move so that $\mathbf{k}\cdot\mathbf{x} - \omega t = \phi$, or

$$\mathbf{k}\cdot\mathbf{x} = \omega t + \phi.$$

This point is moving in the general direction of $\mathbf{k}$. We call this a *positive-frequency* solution. If ω and $\mathbf{k}\cdot\mathbf{x}$ have opposite signs, the solution has positive frequency (in the sense above). If the signs are the same, one gets the negative-frequency solution.

Now take an arbitrary time-independent wave function and expand it in a Fourier series. Assume periodic boundary conditions so that $\mathbf{k}$ is discretized as in (5.45).

$$\psi(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}} \tag{6.35}$$

At this point $a_{\mathbf{k}}$ is a Fourier coefficient and nothing more. We can make ψ time dependent by building the time dependence into the $a_{\mathbf{k}}$'s, $a_{\mathbf{k}} \to a_{\mathbf{k}}(t)$. In order that (6.30) and (6.31) be satisfied, the $a_{\mathbf{k}}$'s should satisfy

$$\ddot{a}_{\mathbf{k}} + \omega_{\mathbf{k}}^2 a_{\mathbf{k}} = 0 \tag{6.36}$$

This is the differential equation for the harmonic oscillator, except for two peculiar features. First, the $a_{\mathbf{k}}$'s are complex functions, and second, the frequency (and hence the "spring constant") is a function of $\boldsymbol{k}$. In some

sense, each term in (6.35) has a harmonic oscillator associated with it. We can tie into the usual harmonic oscillator formalism and avoid the complex coordinates at the same time by defining the real generalized coordinate,

$$q_{\mathbf{k}}(t) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}} \left[a_{\mathbf{k}}(t) + a^*_{\mathbf{k}}(t) \right]. \tag{6.37}$$

The conjugate momentum is given by $\mathbf{p}(t) = \dot{\mathbf{q}}(t)$, but before we take the derivative, we must decide on whether we are dealing with the positive- or negative-energy solution. In order that each term in (6.35) has the form (6.33), $\dot{a}_{\mathbf{k}}(t) = \mp i\omega a_{\mathbf{k}}$. For the time being take positive energy (upper sign)

$$p_{\mathbf{k}}(t) = -i\sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[a_{\mathbf{k}}(t) - a^*_{\mathbf{k}}(t) \right] \tag{6.38}$$

These are real variables oscillating with frequency ω. We know that the Hamiltonian for simple harmonic motion is

$$H_{\mathbf{k}} = \frac{1}{2} \left[p^2_{\mathbf{k}} + \omega^2_{\mathbf{k}} q^2_{\mathbf{k}} \right]. \tag{6.39}$$

You can verify with the definitions (6.37) and (6.38) that $H_{\mathbf{k}}$ is time-independent, that $p_{\mathbf{k}}$ is canonically conjugate to $q_{\mathbf{k}}$, and that Hamilton's equations of motion are satisfied. We can turn (6.39) into a quantum-mechanical Hamiltonian by simply making $a_{\mathbf{k}}$, $a^*_{\mathbf{k}}$, $q_{\mathbf{k}}$ and $p_{\mathbf{k}}$ into operators. We know from our work in Chapter 5 that $\hat{a}_{\mathbf{k}}$ must be an annihilation operator with the commutation relations (5.15). The operators in (5.15), however, are time-independent, Schrodinger-picture operators as is the field operator (6.25). We will want to work in the Heisenberg representation, so we must be careful about the time dependence. The natural assumption is

$$a_{\mathbf{k}}(t) \to \hat{a}_{\mathbf{k}} e^{-i\omega t} \qquad a^*_{\mathbf{k}} \to \hat{a}^\dagger_{\mathbf{k}} e^{+i\omega t} \tag{6.40}$$

In (6.40) $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$ are time-independent, Schrodinger-picture operators. I'll argue presently that these is a consistent and reasonable assumption. The commutation relations are then,

$$[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \qquad [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}] = 0 \tag{6.41}$$

Since $\hat{p}_{\mathbf{k}}$ and $\hat{q}_{\mathbf{k}}$ don't have this simple time dependence, the commutation relations must be taken at equal times.

$$[\hat{q}_{\mathbf{k}}(t), \hat{p}_{\mathbf{k}'}(t)] = i\delta_{\mathbf{k},\mathbf{k}'} \qquad [\hat{q}_{\mathbf{k}}(t), \hat{q}_{\mathbf{k}'}(t)] = [\hat{p}_{\mathbf{k}}(t), \hat{p}_{\mathbf{k}'}(t)] = 0 \tag{6.42}$$

With this substitution (6.39) becomes

$$\hat{H}_{\mathbf{k}} = \frac{1}{2}\omega_{\mathbf{k}}\left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right] = \omega_{\mathbf{k}}\left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\right] \tag{6.43}$$

The same replacement turns (6.35) into (6.25).

The last factor of 1/2 in (6.43) presents something of a dilemma. This $\hat{H}_{\mathbf{k}}$ is just the Hamiltonian for a single $\mathbf{k}$ value. The complete Hamiltonian is a sum over all values.

$$\hat{H} = \sum_{\mathbf{k}} \hat{H}_{\mathbf{k}} \tag{6.44}$$

An infinite number of 1/2's is still infinity. It is customary to discard the constant with some weak argument to the effect that in defining energy, additive constants are meaningless. Since this problem will appear again and again in different contexts, it is useful to have some formal procedure for sweeping it under the rug. To this end we introduce the concept of "normal ordering." We will say that an operator has been normal ordered if *all creation operators are placed to the left of all annihilation operators.* The usual symbol to indicate that an operator has been normal ordered is to place it between colons, so for example,

$$:\hat{H}_{\mathbf{k}}: = \omega_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} \tag{6.45}$$

To put it another way, (6.45) was obtained from (6.43) by commuting the $\hat{a}_{\mathbf{k}}$ past the $\hat{a}^\dagger_{\mathbf{k}}$ in the second term *and discarding the commutator.* Whenever we use a Hamiltonian in a practical calculation, we will assume that it has been normal ordered.

We can check that this Hamiltonian is consistent with the time dependence assumed in (6.40) First note that $[\hat{a}_{\mathbf{k}}, \hat{H}] = \omega_{\mathbf{k}}\hat{a}_{\mathbf{k}}$, so

$$\hat{H}\hat{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}(\hat{H} - \omega_{\mathbf{k}}) \tag{6.46}$$

hence

$$\hat{H}^n\hat{a}_{\mathbf{k}} = \hat{a}_{\mathbf{k}}(\hat{H} - \omega_{\mathbf{k}})^n \tag{6.47}$$

as a consequence

$$\begin{aligned} \hat{a}_{\mathbf{k}}(t) &= e^{i\hat{H}t}\hat{a}_{\mathbf{k}}e^{-i\hat{H}t} = \hat{a}_{\mathbf{k}}e^{-i\omega_{\mathbf{k}}t} \\ \hat{a}^\dagger_{\mathbf{k}}(t) &= e^{i\hat{H}t}\hat{a}^\dagger_{\mathbf{k}}e^{-i\hat{H}t} = \hat{a}^\dagger_{\mathbf{k}}e^{i\omega_{\mathbf{k}}t} \end{aligned} \tag{6.48}$$

The deeper question is why this quantization procedure makes any sense at all. The justification lies in the canonical quantization procedure from

elementary quantum mechanics. It uses the Poisson bracket formalism of classical mechanics and then replaces the Poisson brackets with commutator brackets to get the corresponding quantum expression A Poisson bracket is defined as

$$\{F, G\} \equiv \sum_{k=1}^{N} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \tag{6.49}$$

where q_k and p_k are any pair of conjugate variables, and F and G are any two arbitrary functions of q_k and p_k. The sum runs over the complete set of generalized coordinates. Obviously

$$\begin{aligned} \{q_n, p_m\} &= \delta_{mn} \\ \{q_n, q_m\} = \{p_n, p_m\} &= 0 \end{aligned} \tag{6.50}$$

This looks like the uncertainty relation in Quantum Mechanics, $[x, p] = i\hbar$. We get the quantum expression from the classical expression by the replacement

$$\{F, G\} \to [\hat{F}, \hat{G}]/i\hbar, \tag{6.51}$$

where $\hat{F}$ and $\hat{G}$ are the quantum mechanical operators corresponding to the classical quantities F and G, and $[\hat{F}, \hat{G}] = \hat{F}\hat{G} - \hat{G}\hat{F}$. In the case where $\hat{F}$ and $\hat{G}$ depend on time, the commutator is taken at equal times. This seems like a leap of faith, but it is valid for all the familiar operators in quantum mechanics.[3] Now inverting (6.38) and (6.39) gives

$$a_{\mathbf{k}} = \frac{ip_{\mathbf{k}} + \omega_{\mathbf{k}} q_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}} \qquad a^*_{\mathbf{k}} = \frac{-ip_{\mathbf{k}} + \omega_{\mathbf{k}} q_{\mathbf{k}}}{\sqrt{2\omega_{\mathbf{k}}}} \tag{6.52}$$

Substituting (6.52) into (6.49) gives $\{a_{\mathbf{k}}, a^*_{\mathbf{k}'}\} = -i\delta_{\mathbf{k},\mathbf{k}'}$ and $\{a_{\mathbf{k}}, a_{\mathbf{k}'}\} = \{a^\dagger_{\mathbf{k}}, a^*_{\mathbf{k}'}\} = 0$, so that

$$\begin{aligned} [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}] &= \delta_{\mathbf{k},\mathbf{k}'} \\ [\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}}] = [\hat{a}^\dagger_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}}] &= 0 \end{aligned} \tag{6.53}$$

(with $\hbar = 1$).

The value of the Poisson bracket $\{F, G\}$ is independent of the choice of canonical variables. That is a fundamental theorem. Since (6.37) and (6.38) are together a canonical transformation, (6.42) and (6.53) are identical. *Any* choice of variables will do so long as they are related to $q_{\mathbf{k}}$ and $p_{\mathbf{k}}$

[3]Much of the formal structure of quantum mechanics appears as a close copy of the Poisson bracket formulation of classical mechanics. See Goldstein, Poole and Safko, *Classical Mechanics* Third Ed., Sec. 9.7

by a canonical transformation. We simply chose $q_{\mathbf{k}}$ so that it was real and had convenient units. The rest followed automatically. The fact that the resultant Hamiltonian is that of harmonic oscillators is simply a consequence of the fact that we choose to expand $\psi(\mathbf{x}, t)$ in a Fourier series.

I can now write Equation (6.25) as

$$\hat{\psi}^{(+)}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} \hat{a}_{\mathbf{k}} \tag{6.54}$$

The superscript (+) means the positive energy solution. The functional form of $\omega(\mathbf{k})$ is determined by the wave equation it represents, but I want to concentrate on solutions of the Klein-Gordon equation. Suppose we had chosen the negative energy solution.

$$\hat{\psi}^{(-)}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{i(\mathbf{k}\cdot\mathbf{x} + \omega(\mathbf{k})t)} \hat{c}_{\mathbf{k}} \tag{6.55}$$

(This automatically becomes a negative frequency solution as well.) The operators $\hat{c}_{\mathbf{k}}$ and $\hat{c}^{\dagger}_{\mathbf{k}}$ annihilate and create these new negative energy particles. Everything goes through as before except that $\hat{p}_{\mathbf{k}}(t) = \dot{\hat{q}}_{\mathbf{k}}(t) = +i\omega\hat{q}_{\mathbf{k}}(t)$ changes sign, so that (6.38) becomes

$$\hat{p}_{\mathbf{k}} = i\sqrt{\frac{\omega(\mathbf{k})}{2}} \left[\hat{c}_{\mathbf{k}} - \hat{c}^{\dagger}_{\mathbf{k}}\right]. \tag{6.56}$$

The counterpart of (6.41) is

$$[\hat{c}^{\dagger}_{\mathbf{k}}, \hat{c}_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'} \tag{6.57}$$

It seems that the new creation operator $c^{\dagger}_{\mathbf{k}}$ stands in the place of the old annihilation operator $\hat{a}_{\mathbf{k}}$. This is not just a mathematical accident. It points to an important piece of physics. To see this we define another pair of creation and annihilation operators.

$$\hat{d}_{\mathbf{k}} = \hat{c}^{\dagger}_{-\mathbf{k}} \qquad \hat{d}^{\dagger}_{\mathbf{k}} = \hat{c}_{-\mathbf{k}} \tag{6.58}$$

Substituting this in (6.55) and changing the sign of the summation variable from $\mathbf{k}$ to $-\mathbf{k}$ gives

$$\hat{\psi}^{(-)}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} e^{-i(\mathbf{k}\cdot\mathbf{x} - \omega(\mathbf{k})t)} \hat{d}^{\dagger}_{\mathbf{k}} \tag{6.59}$$

What is it that the $\hat{d}$'s are creating and destroying, and what is the significance of the change in the sign of the momentum? In Chapter 5 we considered the Fermi sea to be a set of low-lying energy levels all occupied by fermions. Removing one particle from the sea leaves a "hole," which behaves in some ways like a real particle. If the hole moves in the positive $\mathbf{k}$ direction, a real particle must move in the $-\mathbf{k}$ direction to backfill the hole. Dirac proposed this mechanism to explain the negative energy states that appear in relativistic electron theory. The correspondence between holes moving forward and real particles moving backward is a good way of visualizing the significance of (6.58). Unfortunately, the Klein-Gordon equation describes bosons, so there is no Fermi sea.[4] Nowadays, we regard these negative-energy solutions as representing real positive-energy antiparticles. There are two lines of evidence for this. For one thing, the states created by $\hat{d}_{\mathbf{k}}$ have momentum $\mathbf{k}$ (rather than $-\mathbf{k}$). This can be proved formally, but it is almost obvious from (6.59), which is a positive-frequency solution. Later on when we discuss the interaction of the electromagnetic field with bosons, we will show that $\hat{d}^{\dagger}_{\mathbf{k}}$ creates a particle of the *opposite* charge to that created by $\hat{a}^{\dagger}_{\mathbf{k}}$. The complete operator-valued wave function is the sum of (6.54) and (6.59).

$$\hat{\psi}(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left[e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \hat{a}_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \hat{d}^{\dagger}_{\mathbf{k}} \right] \tag{6.60}$$

There are several profound reasons why the positive- and negative-energy solutions must be added in just this way. These will emerge as we go along.

Let's note in passing that there are several neutral spin-zero particles such as the π^0 that have no non-zero additive quantum numbers. Such particles are thereby identical to their antiparticles. If $\hat{a}^{\dagger}_{\mathbf{k}}$ creates a π^0, then $\hat{d}_{\mathbf{k}}$ destroys the *same* particle. In this case there is no point in distinguishing between $\hat{a}_{\mathbf{k}}$ and $\hat{d}_{\mathbf{k}}$. We write (6.60)

$$\hat{\psi}(\mathbf{x},t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \left[e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} a_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)} \hat{a}^{\dagger}_{\mathbf{k}} \right] \tag{6.61}$$

Fields corresponding to neutral particles are Hermitian. Those corresponding to charged particles are not.

In some ways (6.60) and (6.61) are relics of our nonrelativistic fields from Chapter 5. Because they are based on discrete $\mathbf{k}$ values and periodic boundary conditions they behave under Lorentz transformations in a most

[4]The idea is only relevant to low-temperature conductors anyway.

awkward fashion. We are accustomed to passing to the continuum limit through the replacement

$$\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^3k}{(2\pi)^3},$$

but this may be a bad idea for relativistic fields. The trouble is that the integration measure d^3k does not transform like a scalar under Lorentz transformations. A better choice might be

$$\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^4k}{(2\pi)^3}\,\delta(k^2 - m^2), \tag{6.62}$$

which is clearly invariant. (The symbol k here refers to the usual four-vector $k^\mu \to (k^0, \mathbf{k})$.) The dk^0 integration is done as follows.

$$\int dk^0\,\delta(k^2 - m^2) = \int d(k^0)^2 \left(\frac{dk^0}{d(k^0)^2}\right)\delta((k^0)^2 - \omega_k^2)$$

$$= \int \frac{d(k^0)^2}{2k^0}\delta((k^0)^2 - \omega_k^2) = \frac{1}{2\omega_k}$$

Equation (6.62) becomes

$$\frac{1}{V}\sum_{\mathbf{k}} \to \int \frac{d^3k}{(2\pi)^3 2\omega_k} \tag{6.63}$$

Although this convention is sometimes used, it is somewhat simpler to use the following

$$\boxed{\hat{\varphi}(\boldsymbol{x}, t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left[\hat{a}(\boldsymbol{k})e^{-ikx} + \hat{d}^\dagger(\boldsymbol{k})e^{ikx}\right]} \tag{6.64}$$

where $kx = \omega t - \mathbf{k}\cdot\mathbf{x}$. The point is that we must also consider the transformation properties of the creation and annihilation operators. The natural generalization of (6.41) is

$$\boxed{[\hat{a}(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = \delta^{(3)}(\mathbf{k} - \mathbf{k}') \qquad [\hat{a}(\mathbf{k}), \hat{a}(\mathbf{k}')] = [\hat{a}^\dagger(\mathbf{k}), \hat{a}^\dagger(\mathbf{k}')] = 0} \tag{6.65}$$

and and similarly for the $\hat{d}$'s. Although $\delta^{(3)}(\mathbf{k} - \mathbf{k}')$ by itself is not a Lorentz scalar, the field definition (6.64) together with (6.65) does have the right transformation properties. This will be apparent once we have calculated the propagator.

6.4 Field Theory and the Klein-Gordon Equation

The Klein-Gordon equation and its associated Lagrangian were discussed briefly in Section 6.2. The Lagrangian density (6.54) is[5]

$$\mathcal{L} = \frac{1}{2}\left[(\partial\varphi)^2 - m^2\varphi^2\right] \tag{6.66}$$

Suppose we regard φ as a generalized "coordinate" perhaps referring to the displacement of some hypothetical continuous medium. The conjugate field is

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\dot{\varphi}} = \partial_0\varphi(x) \tag{6.67}$$

We can use the Poisson bracket approach to quantize these fields just as we quantized the $a_{\mathbf{k}}$'s in the previous section. Classically,[6]

$$\{\varphi(\mathbf{x},t), \pi(\mathbf{x}',t)\} = \delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{6.68}$$

Our quantized version of this is

$$\boxed{[\hat{\varphi}(\mathbf{x},t), \hat{\pi}(\mathbf{x}',t)] = i\delta^{(3)}(\mathbf{x}-\mathbf{x}')} \tag{6.69}$$

It is easy to verify that (6.64) satisfies (6.69) so long as the creation operators are normalized according to (6.65). It's crucial that the two fields in (6.69) are evaluated at the same time. As such they are called *equal time commutation relations.* Many texts in fact consider (6.69) to be the fundamental postulate of field theory and use it to derive the properties of the creation and annihilation operators.

Before we leave the subject of commutation relations, there is an issue that has been hanging fire since (6.60). In that equation the positive- and negative-energy solutions appear with equal magnitude. Nothing I have said so far requires the presence of the negative-energy term, let alone that it have the same normalization as the positive-energy part. In fact, we have no choice in the matter. The argument goes like this. Consider two space-like separated points $(\mathbf{x},t)$ and $(\mathbf{x}',t')$. There will always be a Lorentz frame such that $t = t'$. If $\mathbf{x}$ and $\mathbf{x}'$ are distinct points, then a signal can propagate from one to the other only by travelling at infinite velocity. We believe this to be impossible, so our theory must not allow it, even in principle. We

[5] I am using φ in this section rather than ψ to refer specifically to solutions of the Klein-Gordon equation.

[6] See J.V. Jose and E. J. Saletan, *Classical dynamics: a contemporary approach*, Sec 9.3.1 for a derivation of this rather non-trivial result.

call this the requirement of *causality*. It can be argued that a necessary requirement is that the fields $\hat{\varphi}(x)$ and $\hat{\varphi}^\dagger(x')$ also commute at equal times.[7] Let's rewrite (6.64) with an additional parameter α that we can tweak at our pleasure.

$$\hat{\varphi}(\boldsymbol{x},t) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \left[\hat{a}(\boldsymbol{k})e^{-ikx} + \alpha\hat{a}^\dagger(\boldsymbol{k})e^{ikx}\right] \tag{6.70}$$

A simple calculation now gives

$$[\hat{\varphi}(\mathbf{x},t), \hat{\varphi}^\dagger(\mathbf{x}',t)] = \int \frac{d^3k}{2(2\pi)^3\omega_k}(1-|\alpha|^2)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \tag{6.71}$$

This is not zero (because of the ω in the denominator) unless $|\alpha| = 1$. Relativity and causality together require an equal admixture of negative- and positive-energy states. This argument takes on additional significance when spin is taken into account. It can be shown that the requirement proved above only holds for *integer* spin. In the case of half-odd integer spin, the result only holds if the corresponding creation and annihilation operators *anticommute*.

6.5 The Propagator

At this point I must anticipate some developments from the next few chapters. It will turn out that one of the key ingredients of any perturbation calculation is the Feynman propagator defined by

$$G(x,y) = iD(x-y) = \langle 0|T[\hat{\phi}(x), \hat{\phi(y)}]|0\rangle \tag{6.72}$$

Where $T[\ ,\ ]$ is the "time-ordered product" defined by

$$T[\hat{\varphi}(x)\hat{\varphi}(y)] = \theta(x^0-y^0)\hat{\varphi}(x)\hat{\varphi}(y) + \theta(y^0-x^0)\hat{\varphi}(y)\hat{\varphi}(x) \tag{6.73}$$

In a time-ordered product the time-dependent operators are ordered so that later times stand to the left of earlier times. Time-ordered products read like Hebrew, right to left. There are several ways of interpreting $D(x-y)$. From a mathematical point of view, it is the Green's function of the Klein-Gordon equation, i.e.

$$(\partial_\mu\partial^\mu - m^2)D(x-y) = \delta^{(4)}(x-y) \tag{6.74}$$

[7] See Paul Teller, *An Interpretive Introduction to Quantum Field Theory*, Chapter 4, for a careful discussion of this point.

From a physical point of view, it is the probability amplitude for a particle to propagate from y to x. I need to prove the central result that

$$\boxed{iD(x-y) = i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x-y)}}{k^2-m^2+i\epsilon}} \tag{6.75}$$

Each kind of particle has its own propagator, but the $+i\epsilon$ term in the denominator is ubiquitous. The ϵ stands for an infinitesimal *positive* quantity. It's job is to get the boundary conditions right as you will see in the derivation.

We know that this will be a function of $x-y$, so we can make the algebra a bit simpler by setting $y=0$. Just remember to replace $x \to x-y$ at the end of the calculation. Substituting the fields from (6.70) into (6.72) and taking the vacuum expectation value gives

$$\begin{aligned} iD(x) &= \langle 0|T[\hat{\varphi}(\boldsymbol{x},t)\hat{\varphi}(0,0)]|0\rangle \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left[\theta(t)e^{-i(\omega_k t-\boldsymbol{k}\cdot\boldsymbol{x})}+\theta(-t)e^{i(\omega_k t-\boldsymbol{k}\cdot\boldsymbol{x})}\right] \end{aligned} \tag{6.76}$$

Equations (6.75) and (6.76) are really the same result, though this is far from obvious. In principle, we could derive either form from the other, but it's probably easier to start from (6.75).

$$iD(x) = i\int \frac{d^3k}{(2\pi)^4}\, e^{i\mathbf{k}\cdot\mathbf{x}}\int dk_0 \frac{e^{-ik_0 t}}{(k_0-\omega_k+i\epsilon)(k_0+\omega_k-i\epsilon)} \tag{6.77}$$

Notice how the denominator is factored. Multiplying the two factors and making the replacements, $2i\omega_k\epsilon \to i\epsilon$ and $\epsilon^2 \to 0$, gives the same denominator as (6.75). The dk_0 integration is now performed as a contour integration in the complex k_0 plane. For $t<0$ the contour is completed in the upper half-pane enclosing the point $k_0 = -\omega_k + i\epsilon$, and for $t>0$ the contour is completed in the lower half-plane enclosing the point $k_0 = \omega - i\epsilon$. The result is identical to (6.76). You see how the $i\epsilon$ in the denominator displaces the poles so as to pick up the right integrand depending on whether t is positive or negative. Notice finally that (6.73) is a Lorentz scalar since kx, k^2 and d^4k are all scalar quantities. You will see how $D(x-y)$ becomes a key player in perturbation theory via the interaction picture in the next chapter.

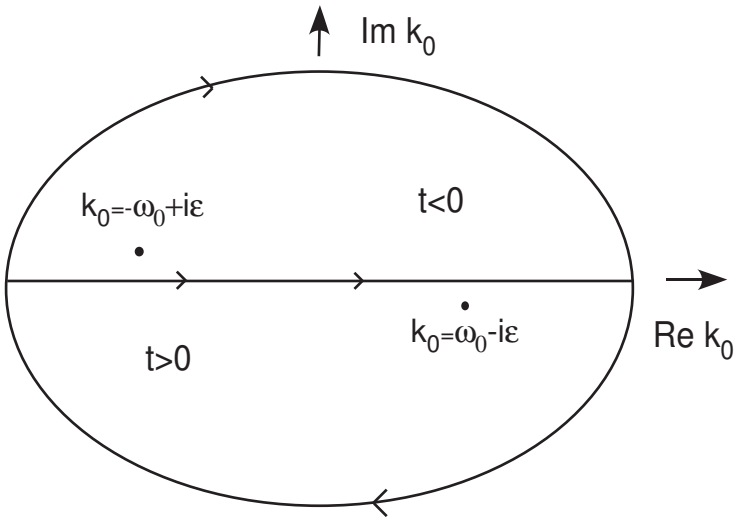


Figure 6.1: The complex k_0 plane

Chapter 7

The Interaction Picture and the S-Matrix

Most of what we know about subatomic physics comes from two kinds of experiments: decay experiments and scattering experiments. In a decay experiment, one starts with some system such as an atom or nucleus or "elementary" particle and observes the spontaneous transitions that it undergoes. One can determine the lifetime of the system, the identity of the decay products, the relative frequency of the various decay modes, and the distribution of momentum and energy among the resulting particles. In a scattering experiment, one starts with a stable system and bombards it with another particle. One measures the distribution of momenta among the various particles produced by the reaction and determines the probability that the scattering will lead to a particular final state. One common feature of both these experiments is that the particles are detected when they are not interacting with one another. They are detected on a scale of distance that is many orders of magnitude larger than the characteristic de Broglie wavelength and at times that are vastly longer than the time of interaction. In non-relativistic quantum mechanics these processes are calculated with first-order, time dependent perturbation theory using a bit of hocus pocus called "Fermi's golden rule." This is not altogether wrong, but it is inadequate for several reasons: it can't accommodate the creation of new particles, it's hopelessly non-relativistic, and it only works to first order.

Real scattering theory is difficult. There are many subtle issues involved. Much of the material in advanced quantum books relates to scattering in one way or another. I say this because it's easy to lose sight of the goal amidst all the technical difficulties. Roughly speaking, there are two basic issues: how

do the quantum fields $\varphi(x)$ evolve in time, and given this information, how can we calculate the results of these experiments in terms of the momenta of particles measured in the asymptotic regime as explained above? The first question is answered neatly by the interaction picture formalism first developed by Freeman Dyson. The second question is much more difficult and will require the remainder of this chapter.

7.1 The Interaction Picture

Path integrals use the Lagrangian; the interaction picture uses the Hamiltonian; so I should say a few words about Hamiltonians in general. In classical mechanics, the two are related by

$$H = p\dot{q} - L \tag{7.1}$$

In our variety of continuum mechanics the Hamiltonian density becomes

$$\mathcal{H} = \pi\dot{\varphi} - \mathcal{L} \tag{7.2}$$

For the free Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}\left[(\partial_\mu\varphi)(\partial^\mu\varphi) - m^2\varphi^2\right] = \frac{1}{2}\left[\dot{\varphi}^2 - (\nabla\varphi)^2 - m^2\varphi^2\right] \tag{7.3}$$

$$\mathcal{H} = \frac{1}{2}\left[\dot{\varphi}^2 + (\nabla\varphi)^2 + m^2\varphi^2\right] \tag{7.4}$$

We get the Hamiltonian by integrating (7.4) over all space.

$$H = \int d^3x\mathcal{H} \tag{7.5}$$

We assume that the Hamiltonian can be split up into two pieces: $H = H_0 + H_{\text{int}}$. H_0 is the Hamiltonian of the free field, and H_{int} is everything left over. We assume the H_{int} is "small" in some sense; so that the perturbation series converges. We also have to assume that H_{int} is a polynomial of the fields and their derivatives. The reason for this technical assumption will appear later.

You will recall the relation between the Schrodinger and Heisenberg pictures is

$$\begin{aligned} |\psi(t)\rangle_S &= e^{-iHt}|\psi\rangle_H \\ Q_H(t) &= e^{iHt}Q_S e^{-iHt} \end{aligned} \tag{7.6}$$

This is the usual convention at any rate. $|\psi\rangle_H$, which does not depend on time, is a "snapshot" of $|\psi(t)\rangle_S$ taken at $t = 0$. This is not necessarily the most convenient time to take the picture as we will see.

We define the interaction state by

$$|\psi(t)\rangle_I \equiv e^{iH_0t}|\psi(t)\rangle_S \tag{7.7}$$

$$Q_I(t) = e^{iH_0t}Q_S\, e^{-iH_0t}$$

Some comments:

- It's clear from (7.6) the Hamiltonian H is the same in the Schrodinger and Heisenberg pictures, and equally clear from (7.7) that H_0 is the same in the interaction picture as well. This is not true of the interaction piece; since in general, $[H_{\rm int}, H_0] \neq 0$. I will write

  $$H^I_{\rm int}(t) \equiv e^{iH_0t}H^S_{\rm int}e^{-iH_0t} \tag{7.8}$$

  Naturally, $H^S_{\rm int}$ is the interaction Hamiltonian in the Schrodinger picture. Notice that $H^I_{\rm int}(t)$ *depends on time.*

- If there were no interactions, $|\psi(t)\rangle_I = |\psi\rangle_H$, so the interaction picture state would be stationary. Now think of a scattering experiment. When the particles are far apart, there is no interaction between them. In this regime, the interaction picture and Heisenberg pictures are identical. As the particles begin to interact, $H_{\rm int}$ "turns on" and $|\psi(t)\rangle_I$ begins to change. Eventually, after the particles are separated it settles down to a new Heisenberg state.

  Actually, this business of "turning on" and " turning off" skirts a profound and difficult problem. The trouble is that $H_{\rm int}$ never really turns off. Even when the particles are not interacting with one another, they are still interacting with themselves by emitting and reabsorbing virtual particles. Worse yet, even the vacuum interacts with itself by creating particle-antiparticle pairs. This is a difficult problem, but one that can be dealt with rigorously.

Define the time evolution operator in the interaction picture, $U(t, t_0)$, by

$$|\psi(t)\rangle_I = U(t, t_0)|\psi(t_0)\rangle_I. \tag{7.9}$$

Since H is Hermitian and the norm of $|\psi\rangle_I$ is conserved, we feel entitled by the rules of quantum mechanics to the following basic relations:

$$U^\dagger(t, t_0)U(t, t_0) = \mathbf{1} \tag{7.10}$$

$$U(t,t) = \mathbf{1} \tag{7.11}$$

$$U(t,t_0) = U^{-1}(t_0,t) \tag{7.12}$$

$$U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0) \tag{7.13}$$

We can derive an equation of motion for U by starting with Schrodinger's equation

$$i\frac{\partial}{\partial t}|\psi(t)\rangle_S = H|\psi(t)\rangle_S. \tag{7.14}$$

A quick calculation with (7.7) and (7.9) yields

$$i\frac{\partial}{\partial t}U(t,t_0) = H^I_{\rm int}(t)U(t,t_0). \tag{7.15}$$

We know how to solve equations like this.

$$U(t,t_0) = 1 - i\int_{t_0}^{t} dt'\ H^I_{\rm int}(t')U(t',t_0) \tag{7.16}$$

Well – maybe solve is too strong a word, since U appears on both sides of (7.16). We would like to claim that

$$U(t,t_0) = (?)\exp\left\{-i\int_{t_0}^{t} dt'\ H^I_{\rm int}(t')\right\}$$

This would be the obvious solution if U and H were not operators. The flaw in the reasoning here is that factors of $H_{\rm int}$ don't commute at different times, $[H_{\rm int}(t), H_{\rm int}(t')] \neq 0$. We can come up with a valid solution by iterating (7.16) paying careful attention to the time dependence.

$$\begin{aligned} U(t,t_0) &= 1 - i\int_{t_0}^{t} dt_1 H^I_{\rm int}(t_1) \\ &+(-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H^I_{\rm int}(t_1)H^I_{\rm int}(t_2) + \cdots \end{aligned} \tag{7.17}$$

The entire series can be summarized by

$$U(t,t_0) = \sum_{n=0}^{\infty}(-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2\cdots\int_{t_0}^{t_{n-1}} dt_n H^I_{\rm int}(t_1)H^I_{\rm int}(t_2)\cdots H^I_{\rm int}(t_n) \tag{7.18}$$

The series (7.18) is more difficult than it looks. Since the Hamiltonians don't commute, we must be meticulously careful to keep **later times to**

the left of earlier times. This is called *time ordering.* We need some machinery to do this for us. Define the time ordering operator,

$$T[H(t_1)H(t_2)] \equiv H(t_1)H(t_2)\theta(t_1 - t_2) + H(t_2)H(t_1)\theta(t_2 - t_1) \tag{7.19}$$

The generalization to three or more Hamiltonians is obvious. You should convince yourself that the following things are true:

$$\begin{aligned}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H(t_1)H(t_2) &= \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 H(t_2)H(t_1) \\ &= \frac{1}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 T[H(t_1)H(t_2)],\end{aligned}$$

and in general

$$\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t_{n-1}} dt_n H(t_1)\cdots H(t_n) = \frac{1}{n!}\int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n T[H(t_1)\cdots H(t_n)]$$

So our final result for the U operator is

$$U(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!}\int_{t_0}^{t}\cdots\int_{t_0}^{t} dt_1 dt_2 \cdots dt_n T[H^I_{\text{int}}(t_1)H^I_{\text{int}}(t_2)\cdots H^I_{\text{int}}(t_n)], \tag{7.20}$$

which can be written in shorthand notation as

$$\boxed{U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} dt' H^I_{\text{int}}(t')\right\}} \tag{7.21}$$

The reason that this is a workable approximation is that the fields that make up H^I_{int} are interaction-picture fields, which according to (7.8), transform as free fields. Equation (7.20) is a "recipe." You just insert the formulas for the free fields and do the algebra.

7.2 The S Matrix

We can imagine quantum scattering as taking place in three phases. In the limit $t \to -\infty$ the particles are separated and not interacting *with one another.* As the particles approach one another, the interaction between them "turns on" as they exchange virtual particles. At some later time $t \to +\infty$, this interaction turns off and they again become free particles. This is difficult to treat theoretically, since the interaction Hamiltonian itself in no way

turns on or off. So far as theoretical difficulties are concerned, this is only the tip of the iceberg. As soon as we allow particles to interact with other particles, they inevitably interact with themselves. When this happens, the notions of free particles, charge, and vacuum become profoundly subtle.

From the point of view of theory, a free particle is a boiling stream of self-interactions. From the point of view of experiment, every electron in the universe is an unremarkable little thing with a well-defined mass, spin, charge, and magnetic moment. These two views can exist simultaneously because the self-interactions are virtual processes, and as such they are contained within a region roughly the size of the Compton wavelength. So long as one does not probe inside this region, the electron has it's usual physical properties, and probing inside the region means doing a high-energy scattering experiment.

The same sort of ambiguities also complicate our understanding of the vacuum state. According to perturbation theory, virtual particle-antiparticle pairs are spontaneously emitted from and reabsorbed into empty space. This fact is established experimentally (to some extent) by an experiment measuring the so-called Casimir effect,[1] but apart from this, these vacuum fluctuations have very little consequence, and at any rate, are not well understood.[2]

Finally consider the charge of an electron (or more generally, the coupling constant for any sort of interaction). The charge is the measure of the strength with which photons couple to electrons, but this can only be measured by doing a scattering experiment, and the experiment must be interpreted in light of the virtual processes involved.

We believe that all these problems can be addressed, at least for quantum electrodynamics, but this is a long story, which will occupy us for the next few chapters. For the time being, let us say that the theory recognizes two kinds of mass and two kinds of charge. There are the physical or "dressed" masses and coupling constants – those that are measured in the lab – and the "bare" masses and coupling constants, which are the parameters that go into the interaction Lagrangians and Hamiltonians. Somehow the effect of all virtual processes formulated in terms of the bare particles is to produce the physical particles. What then are the numerical values of these bare masses and charges? Alas, we do not know. There is no way to measure them directly. They could be calculated in principle, *but every such calculation yields infinity!* Despite this, we are able to calculate everything that can be

[1] Two charged plates suspended parallel to one another in vacuum experience a very small repulsive force because of virtual electron-positron pairs created between them.

[2] One can calculate the gravitational mass of all the virtual particle-antiparticle pairs in the universe. The result come out too large by many orders of magnitude.

measured (except mass and charge, of course) using only the physical mass and charge. The fact that we have learned to skirt infinities so violent as to give sci-fi fans nightmares, is one of the ironic triumphs of modern physics.

In view of this, the interaction picture formulation in the previous section is a bit too naive. What masses are we to put in $H_{\rm int}$, the bare or the physical mass? It seems that either way we are stuck. We need to back up and look at the basic description of scattering processes.

Consider the state of a scattering experiment before the particles have interacted. For mathematical purposes we say this corresponds to a time "$t \to -\infty$" (The actual time for a relativistic particle could be as small as $t \to -10^{-23}$ seconds.) At this time the particles are free in the sense explained above. It is customary to call such states "in states," and write them $|\alpha, {\rm in}\rangle$. The symbol α stands for all the quantum numbers required to completely specify the state. We will be particularly concerned with the momentum. If there are n particles in the initial state, we can write $|k_1, k_2, \ldots, k_n, {\rm in}\rangle$. The field operators introduced in the previous chapter can be written

$$\varphi_{\rm in}(x) = \int \frac{d^3x}{\sqrt{2E_k(2\pi)^3}} \left[a_{\rm in}(k)e^{-ikx} + a^\dagger_{\rm in}(k)e^{ikx}\right] \tag{7.22}$$

It is understood that in calculating with this expression, one uses the physical mass of the particle. In the same way, one defines "out states" $\langle\beta, {\rm out}|$, "out fields" $\varphi_{\rm out}(x)$, and "out" creation operators $a^\dagger_{\rm out}(k)$ corresponding to the limit $t \to \infty$. A scattering experiment can be idealized as follows. The initial target and beam particles are prepared so that they are in a unique eigenstate of all the relevant operators. We call this state $|\alpha, {\rm in}\rangle$. The detectors of the scattered particles are so refined that they can specify that the final state was in the exact eigenstate $\langle\beta, {\rm out}|$. The object of the experiment is to find what fraction of particles in $|\alpha, {\rm in}\rangle$ make the transition to $\langle\beta, {\rm out}|$. In other words, we are measuring the probability

$$P_{\beta\alpha} = |S_{\beta\alpha}|^2 \tag{7.23}$$

where the so-called S matrix is defined by

$$S_{\beta\alpha} = \langle\beta, {\rm out}|\alpha, {\rm in}\rangle \tag{7.24}$$

If we assume that the in states and out states are both complete sets, there must be some operator S that transforms in states into out states.

$$\langle\beta, {\rm out}| = \langle\beta, {\rm in}|S \tag{7.25}$$

$$S_{\beta\alpha} = \langle \beta, \text{in}|S|\alpha, \text{in}\rangle \tag{7.26}$$

The goal of scattering theory is to calculate the S matrix and the corresponding S operator.

Here are some common-sense properties of S.

1. Nothing in – nothing out, i.e. $|S_{00}|^2 = 1$. Put it another way

$$\langle 0, \text{in}|S = \langle 0, \text{out}| = e^{i\theta}\langle 0, \text{in}| \tag{7.27}$$

   The vacuum is stable, but we must allow the possibility that phase of the vacuum state evolves with time.

2. One-particle states are also stable.

$$\langle p, \text{out}|p, \text{in}\rangle = \langle p, \text{in}|S|p, \text{in}\rangle = \langle p, \text{in}|p, \text{in}\rangle = 1 \tag{7.28}$$

3. $$\varphi_{\text{in}} = S\varphi_{\text{out}}S^{-1} \tag{7.29}$$

   Proof:

$$\langle \beta, \text{out}|\varphi_{\text{out}}|\alpha, \text{in}\rangle = \langle \beta, \text{in}|S\varphi_{\text{out}}|\alpha, \text{in}\rangle$$

   But $\langle \beta, \text{in}|\varphi_{\text{out}}$ is itself an in state, so we can write

$$\langle \beta, \text{in}|\varphi_{\text{out}}|\alpha, \text{in}\rangle = \langle \beta, \text{out}|\varphi_{\text{in}}S|\alpha, \text{in}\rangle$$

   Comparing these two equations gives

$$S\varphi_{\text{out}} = \varphi_{\text{in}}S$$

4. S is unitary.

   Proof:

$$\langle \beta, \text{in}|S = \langle \beta, \text{out}| \qquad S^\dagger|\alpha, \text{in}\rangle = |\alpha, \text{out}\rangle$$

$$\langle \beta, \text{in}|SS^\dagger|\alpha, \text{in}\rangle = \langle \alpha, \text{out}|\beta, \text{out}\rangle = S_{\beta\alpha}$$

7.3 The LSZ Reduction Scheme

What is the relation between the fully-interacting field φ and φ_{in}? The natural guess is

$$\lim_{t\to-\infty} \varphi(x) = \varphi_{\text{in}}(x)$$

$$\lim_{t\to\infty} \varphi(x) = \varphi_{\rm out}(x)$$

On second thought though, this can't be right. The complete field $\varphi(x)$ is capable of creating states out of the vacuum with two or more virtual particles. A more careful analysis suggests

$$\lim_{t\to-\infty} \varphi(x) = \sqrt{Z}\ \varphi_{\rm in}$$

$$\lim_{t\to\infty} \varphi(x) = \sqrt{Z}\ \varphi_{\rm out} \tag{7.30}$$

Z is understood as the probability for $\varphi(x)$ to produce a single-particle state out of the vacuum. Even this is not quite correct as an operator equation for some subtle mathematical reasons that are discussed in the original article.[3] It *is* correct in terms of matrix elements, for example

$$\lim_{t\to-\infty} \langle\alpha|\varphi(x)|\beta\rangle = \sqrt{Z}\langle\alpha|\varphi_{\rm in}(x)|\beta\rangle$$

where $\langle\alpha|$ and $|\beta\rangle$ are arbitrary states. Since we are only interested in matrix elements in the long run, we will assume (7.30) without any further comment. Furthermore, the processes in which $\varphi(x)$ produces multiparticle states out of the vacuum only occur in higher orders of perturbation theory. When we are working to lowest non-trivial order we can (and will) set $Z = 1$.

Now we have the apparatus at hand for defining and studying the S matrix defined in (7.24). We start from an initial state of a system with n noninteracting physical particles, denoted by

$$|k_1 \cdots k_n \text{ in}\rangle = |\alpha \text{ in}\rangle, \tag{7.31}$$

and a final state in which m particles emerge denoted by

$$\langle p_1 \cdots p_m \text{ out}| = \langle\beta \text{ out}| \tag{7.32}$$

Our agenda is to convert (7.24) into an expression involving vacuum expectation values of interacting fields. This is the LSZ reduction technique.

Let us write the in-field in a way analogous to (6.64)

$$\varphi_{\rm in}(x) = \int d^3k \left[f_k(x) a_{\rm in}(k) + f_k^*(x) a_{\rm in}^\dagger(k) \right] \tag{7.33}$$

where

$$f_k(x) = \frac{e^{-ikx}}{\sqrt{(2\pi)^3 2\omega_k}} \tag{7.34}$$

[3] See also *Quantum Theory of Point Particles and Strings*, Brian Hatfield, Chapter 7.

Equation (7.33) can be inverted to yield

$$a_{\rm in}(k) = i\int d^3x\; f_k^*(x)\overset{\leftrightarrow}{\partial_0}\varphi_{\rm in}(x)$$

$$a_{\rm in}^\dagger(k) = -i\int d^3x\; f_k(x)\overset{\leftrightarrow}{\partial_0}\varphi_{\rm in}(x) \tag{7.35}$$

The notation $a\overset{\leftrightarrow}{\partial_0}b$ means $a\partial_0 b-(\partial_0 a)b$. Since the in-state in (**??**) $|\alpha\text{ in}\rangle$ is made up with a product of creation operators as in (**??**), we can write

$$S_{\beta\alpha} = \langle\beta\text{ out}|a_{\rm in}^\dagger(k)|\alpha-k\text{ in}\rangle \tag{7.36}$$

where $|\alpha-k\text{ in}\rangle$ represents the initial assemblage of particles represented by α with one particle of momentum k removed. The procedure is now to use (7.35) to replace $a_{\rm in}^\dagger(k)$ with an integral over $\varphi_{\rm in}(x)$. The procedure is then repeated until all the particles have been removed from the in- and out- states leaving the vacuum expectation value of the fields.

$$S_{\beta\alpha} = \langle\beta\text{ out}|a_{\rm out}^\dagger(k)|\alpha-k\text{ in}\rangle \tag{7.37}$$

$$+\langle\beta\text{ out}|\left[a_{\rm in}^\dagger(k)-a_{\rm out}^\dagger(k)\right]|\alpha-k\text{ in}\rangle$$

$$=\langle\beta{-}k\text{ out}|\alpha{-}k\text{ in}\rangle{-}i\langle\beta\text{ out}|\left[\int d^3x\; f_k(x)\overset{\leftrightarrow}{\partial_0}(\varphi_{\rm in}(x)-\varphi_{\rm out}(x))\right]|\alpha-k\text{ in}\rangle$$

I have added and subtracted $a_{\rm out}^\dagger(k)$ and used (7.35) to eliminate $a_{\rm in}^\dagger$ and $a_{\rm out}^\dagger$. The first term in the last line vanishes unless the initial and final states are identical. This is the unscattered wave, which we will neglect for the time being. Now use (7.30) to replace the in- and out- fields with the complete interacting field. The scattered part can be written as follows:

$$S_{\beta\alpha}\sim\frac{i}{\sqrt{Z}}\left(\lim_{x_0\to\infty}-\lim_{x_0\to-\infty}\right)\int d^3x\; f_k(x)\overset{\leftrightarrow}{\partial_0}\langle\beta\text{ out}|\varphi(x)|\alpha-k\text{ in}\rangle \tag{7.38}$$

(The symbol $\sim$ means that I have temporarily neglected the forward scattering term.) The limits have a nasty non-covariant look about them. I will clean up the expression with a series of elegant mathematical tricks. For any two functions such as $f_k(x)$ and $\varphi(x)$,

$$\left(\lim_{x_0\to\infty}-\lim_{x_0\to-\infty}\right)\int d^3x\; f_k\overset{\leftrightarrow}{\partial_0}\varphi=\int_{-\infty}^{\infty}d^4x\;\partial_0\left[f_k\overset{\leftrightarrow}{\partial_0}\varphi\right]$$

$$= \int_{-\infty}^{\infty} d^4x \left[f_k \frac{\partial^2}{\partial x_0^2}\varphi - \varphi \frac{\partial^2}{\partial x_0^2} f_k \right] \tag{7.39}$$

The fact that f_k satisfies the Klein-Gordon equation allows us to write

$$\varphi \frac{\partial^2}{\partial x_0^2} f = \varphi(\nabla^2 - m^2) f \to f(\nabla^2 - m^2)\varphi \tag{7.40}$$

The sense of the right arrow is that the subsequent expression is obtained by integrating by parts twice and discarding the surface terms. Finally substituting (7.39) and (7.40) into (7.38) gives the final result

$$S_{\beta\alpha} = \langle \beta \text{ out}|\text{in } \alpha\rangle = \langle \beta - k \text{ out}|\alpha - k \text{ in}\rangle \tag{7.41}$$
$$+\frac{i}{\sqrt{Z}} \int d^4x \; f_k(x)(\Box + m^2)\langle \beta \text{ out}|\varphi(x)|\alpha - k \text{ in}\rangle$$

You see the pattern? We have "reduced" an asymptotic particle from the in-state and replaced it with an interacting field $\varphi(x)$. The price we pay for this is an integral, a wave function $f_k(x)$, and a differential operator $(\Box + m^2)$. We will eventually Fourier transform this expression, whereupon all these complications will disappear leaving only the inverse of the momentum space propagator $\Delta(k)$.

As an exercise (I am getting tired of typing) you should take the next step and reduce a particle of momentum p out of the out-state. Convince yourself that the result is

$$S_{\beta\alpha} = \langle \beta - p \text{ out}|\alpha - k \text{ in}\rangle \tag{7.42}$$
$$\sim \left(\frac{i}{\sqrt{z}}\right)^2 \int d^4x \int d^4y \; f_k(x) f_p^\dagger(y)(\Box_x + m^2)(\Box_y + m^2)$$
$$\times \langle \beta - p, \text{out}|T[\varphi(x)\varphi(y)]|\alpha - k, \text{in}\rangle$$

The $\sim$ sign again means that we have dropped the forward scattering terms. The new thing here is the time ordered product. You should do the calculation carefully to see how this comes about. Now suppose there are m particles in the initial state and n particles in the final state. Just repeat this procedure $n + m$ times to get the following result:

$$\boxed{\begin{aligned} S_{\beta\alpha} =& \langle p_1 \cdots p_n \text{ out}|k_1 \cdots k_m \text{ in}\rangle \\ \sim \left(\frac{i}{\sqrt{Z}}\right)^{m+n} & \prod_{i=1}^{m} \int d^4x_i f_{k_i}(x_i)(\Box_{x_i} + m^2) \prod_{j=1}^{n} d^4y_j f_{p_j}^*(y_j)(\Box_{y_j} + m^2) \\ & \times \langle 0|T[\varphi(y_1)\cdots\varphi(y_n)\varphi(x_1)\cdots\varphi(x_m)]|0\rangle \end{aligned}} \tag{7.43}$$

Equation (7.43), to quote Bjorken and Dell, "serves as a cornerstone for all calculation of scattering amplitudes in modern quantum field theory." (This was before the days of path integrals.)

Feynman rules are usually expressed in momentum space, so introduce the Fourier transform,

$$G(x_1 \cdots x_m y_1 \cdots y_n) = \prod_{i=1}^{m} \int d^4k_i \, \frac{e^{ik_i x_i}}{(2\pi)^4} \prod_{j=1}^{n} \int d^4p_j \, \frac{e^{-ip_j y_j}}{(2\pi)^4} \, \tilde{G}(k_1 \cdots k_m p_1 \cdots p_n) \tag{7.44}$$

There is a possible confusion about this equation. Since

$$G(x_1 \cdots x_m y_1 \cdots y_n) = \langle 0|T[\varphi(y_1) \cdots \varphi(y_n)\varphi(x_1) \cdots \varphi(x_m)]|0\rangle \tag{7.45}$$

There is no distinction between x's and y's, but in (7.44), x's are clearly associated with incoming particles, and y's are associated with outgoing particles. The point is that $G(x_1 \cdots x_m y_1 \cdots y_n)$ is a "general purpose" tool that can be used in many situations including those on a quantum-mechanical time scale in which there is no clear-cut distinction between past and future. In scattering experiments, on the other hand, past and future are unambiguous. We therefore associate some coordinates with in-states and some with out-states. We do this by assigning them to k's (for incoming particles) or with p's (for outgoing particles). In this simple theory with self-interacting scalar fields, the only difference is the $+$ or $-$ signs in the exponentials in (7.44), but this is necessary to get the conservation of momentum right. When there are several different kinds of particles involved, this choice becomes even more significant. For example, if we were looking at the process $e^+ + e^- \to \gamma + \gamma$, (positron annihilation) we would need to assign x's to the electron fields and y's to the photon fields. After making this assignment, substitute (7.44) into (7.43) and do the necessary integrals.

$$\begin{aligned} S_{\beta\alpha} =& \langle p_1 \cdots p_n \text{ out}|k_1 \cdots k_m \text{ in}\rangle \\ &\sim \left(\frac{i}{\sqrt{Z}}\right)^{m+n} \prod_{i=1}^{m} \frac{1}{\sqrt{(2\pi)^3 2E_{k_i}}}(k_i^2 - m^2) \\ &\times \prod_{j=1}^{n} \frac{1}{\sqrt{(2\pi)^3 2E_{p_j}}}(p_j^2 - m^2)\tilde{G}(k_1 \cdots k_m p_1 \cdots p_n) \end{aligned} \tag{7.46}$$

7.4 Correlation Functions

The heart of (7.43) is the vacuum correlation function

$$G(x_1 \cdots x_n) = \langle 0|T[\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)]|0\rangle \tag{7.47}$$

The fields $\varphi(x)$ are the fully interacting fields in the Heisenberg picture. If we are to calculate anything in this formalism, we must somehow relate them to the "in" fields defined in (7.30). Let's hope there exists an operator $U(t)$ such that

$$\varphi(x) = U^{-1}(t)\varphi_{\text{in}}(x)U(t) \qquad \text{with } t = x^0 \tag{7.48}$$

I happen to know that the necessary operator is given by

$$U(t) = \lim_{t_0 \to -\infty} U(t, t_0) \tag{7.49}$$

where $U(t, t_0)$ is time time-development operator given by (7.21). This is plausible, but it need to be proved.

Both φ and φ_{in} satisfy Heisenberg equations of motion.

$$\frac{\partial}{\partial t}\varphi(t) = i[H(\varphi), \varphi(t)] \tag{7.50}$$

$$\frac{\partial}{\partial t}\varphi_{\text{in}}(t) = i[H_0(\varphi_{\text{in}}), \varphi_{\text{in}}(t)] \tag{7.51}$$

I have written $H(\varphi)$ and $H_0(\varphi_{\text{in}})$ to emphasize that the Hamiltonians are functionals of their respective fields. I have also written $\varphi(t)$. I realize the φ is really a function of the four vector x^μ, but I am specifically concerned with the time development here, and $x_0 = t$. Note that

$$\frac{d}{dt}\left[UU^{-1}\right] = \frac{dU}{dt}U^{-1} + U\frac{dU^{-1}}{dt} = 0$$

In a condensed notation this is

$$U\dot{U}^{-1} = -\dot{U}U^{-1} \tag{7.52}$$

Now substitute (7.48) into (7.51) and use (7.50) to eliminate $\dot{\varphi}$.

$$\frac{\partial}{\partial t}\varphi_{\text{in}} = \frac{\partial}{\partial t}\left[U\varphi U^{-1}\right] = \dot{U}\varphi U^{-1} + U\dot{\varphi}U^{-1} + U\varphi\dot{U}^{-1}$$

$$= \dot{U}(U^{-1}\varphi_{\text{in}}U)U^{-1} + Ui\left[H(\varphi), \varphi\right]U^{-1} + U(U^{-1}\varphi_{\text{in}}U)\dot{U}^{-1}$$

$$= \dot{U}U^{-1}\varphi_{\rm in} - \varphi_{\rm in}\dot{U}U^{-1} + i\left[(UH(\varphi)U^{-1}), U\varphi U^{-1}\right]$$
$$= \left[\dot{U}U^{-1}, \varphi_{\rm in}\right] + i\left[H(\varphi_{\rm in}), \varphi_{\rm in}\right] = i\left[H_0(\varphi_{\rm in}), \varphi_{\rm in}\right] \tag{7.53}$$

I have used (7.52) to simplify the second line of (7.53). I have also used the fact that Hamiltonians are polynomials of φ (and possibly its derivatives), so for example, $UH(\varphi)U^{-1} = H(\varphi_{\rm in})$. Therefore

$$\left[\left\{\dot{U}U^{-1} + iH_{\rm int}(\varphi_{\rm in})\right\}, \varphi_{\rm in}\right] = 0$$

The operator in curly brackets commutes with all $\varphi_{\rm in}$. It must therefore be a c number. We could think of this constant as a number added to the Hamiltonian. We have had several occasions to discard such numbers before and will do so now. We conclude that (to within a meaningless additive constant)

$$i\frac{dU(t)}{dt} = H_{\rm int}(\varphi_{\rm in})U(t) \tag{7.54}$$

This is equation (7.15) with the important clarification that $H_{\rm int}$ is constructed with in fields, i.e. fields using the physical masses of the particles involved. We can take $U(t, t_0)$ to be given by (7.21) and $U(t)$ by (7.49). Using (7.12) and (7.13) we can argue that

$$U(t_2, t_1) = \lim_{t\to-\infty} U(t_2, t_0)U(t_0, t_1)$$
$$= \lim_{t\to-\infty} U(t_2, t_0)U^{-1}(t_1, t_0) = U(t_2)U^{-1}(t_1) \tag{7.55}$$

Now consider a set of time-ordered points $x_1, \cdots, x_n$ satisfying $x_0^1 > x_2^0 > \cdots > x_n^0$. Using (7.55) we can write

$$G(x_1, \cdots, x_n) = \langle 0|\varphi(x_1)\cdots\varphi(x_n)|0\rangle$$
$$= \langle 0|U(t_0, t_1)\varphi_{\rm in}(x_1)U(t_1, t_2)\varphi_{\rm in}(x_2)\cdots U(t_{n-1}, t_n)\varphi_{\rm in}(x_n)U(t_n, t_0)|0\rangle \tag{7.56}$$

The sequence of times reading from left to right runs from $t_0 \to -\infty$ to t_1 and back again to $-\infty$. We can deform the contour by writing $U(t_0, t_1) = U(t_0, -t_0)U(-t_0, t_1)$. An operator of the form $U(-t_0, t_0)$ runs from $-\infty$ to $+\infty$. It has the effect of transforming asymptotic initial states to asymptotic final states. This is just the role of the S operator defined in (7.50). We will simply call $U(t_0, -t_0) = S^{-1}$. With this change, our correlation function becomes

$$G(x_1, \cdots, x_n)$$
$$= \langle 0|S^{-1}U(-t_0, t_1)\varphi_{\rm in}(x_1)U(t_1, t_2)\cdots U(t_{n-1}, t_n)\varphi_{\rm in}(x_n)U(t_n, t_0)|0\rangle \tag{7.57}$$

According to (7.27), the vacuum state is an eigenstate of S with a complex phase. We can write $\langle 0|S^{-1} = e^{-i\theta}\langle 0|$ and find the phase as follows.

$$\langle 0|S|0\rangle\langle 0|S^{-1} = \langle 0|$$

The point is that since the vacuum is stable, $|0\rangle\langle 0|$ constitutes a complete set of states. Consequently

$$\langle 0|S^{-1} = \frac{\langle 0|}{\langle 0|S|0\rangle} \tag{7.58}$$

Equation (7.57) becomes

$$G(x_1, \cdots, x_n) \tag{7.59}$$
$$= \frac{\langle 0|U(-t_0, t_1)\varphi_{\rm in}(x_1)U(t_1, t_2)\cdots U(t_{n-1}, t_n)\varphi_{\rm in}(x_n)U(t_n, t_0)|0\rangle}{\langle 0|S|0\rangle}$$

Consider an operator like $U(t_{i-1}, t_i)$. It contains an infinite sum of products of fields evaluated at various times, but every term in the sum contains fields that are time-ordered between t_{i-1} and t_i. The entire numerator of (7.59) consists of a vast array of fields, *but they are all time ordered.* Of course, we started out with the assumption that the sequence of coordinates $x_1, x_2, \cdots x_n$ was already time ordered, so this makes the "scaffolding" around which the U's are time ordered. The original definition of G, equation (7.47) already contains the time ordering operator, so we need not have made that assumption. We just need to write it

$$G(x_1, \cdots, x_n) \tag{7.60}$$
$$= \frac{\langle 0|T\left[U(-t_0, t_1)\varphi_{\rm in}(x_1)U(t_1, t_2)\cdots U(t_{n-1}, t_n)\varphi_{\rm in}(x_n)U(t_n, t_0)\right]|0\rangle}{\langle 0|S|0\rangle}$$

With the time-ordering operator in place, we are free to move the factors around in the numerator to suit our convenience. In particular, we can gather all the U's together.

$$U(-t_0, t_1)U(t_1, t_2)\cdots U(t_n, t_0) = U(-t_0, t_0) = S$$

Our final result for the correlation function function is

$$\boxed{G(x_1, \cdots, x_n) = \frac{\langle 0|T\left[\varphi_{\rm in}(x_1)\varphi_{\rm in}(x_2)\cdots\varphi_{\rm in}(x_n)S\right]|0\rangle}{\langle 0|S|0\rangle}} \tag{7.61}$$

Itzykson and Zuber[4] in a masterstroke of French understatement, call this derivation "heuristic." The fact is that we just don't know how to do rigorous mathematics with such badly-behaved operator expansions as (7.20), and there are many subtle physical questions as well. You might call this "experimental mathematics." We use this formula and it works. It is also true that equivalent formulas can be derived using the path integral formalism. This also goes way beyond the limits of conventional mathematics, but the same result is obtained in the end.

7.5 The Wick Expansion

If we can calculate $S_{\beta\alpha}$ in (7.46), we can calculate all possible decay rates and cross sections. In order to do this we need the Fourier transformed Green's function given (implicitly) by (7.44).

$$\tilde{G}(k_1 \cdots k_m p_1 \cdots p_n) = \prod_{i=1}^{m} \int d^4x_i \; e^{-ik_i x_i} \prod_{j=1}^{n} \int d^4y_j \; e^{ip_j y_j} \; G(x_1 \cdots x_m y_1 \cdots y_n) \tag{7.62}$$

Now we need the configuration space Green's function $G(x_1 \cdots x_m y_1 \cdots y_n)$. The formula for this is (7.61)

$$\boxed{\begin{aligned} G(x_1, x_2, \cdots, x_n) &= \langle 0|T[\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)]|0\rangle \\ &= \frac{\langle 0|T[\varphi_I(t_1)\varphi_I(t_2)\cdots\varphi_I(t_n)S]|0\rangle}{\langle 0|S|0\rangle}, \end{aligned}} \tag{7.63}$$

By S in (7.63) I mean the operator $U(-t_0, t_0)$ in the limit $t_0 \to -\infty$. Then the equation for S is (7.20) with infinite limits on the integrals.

$$\boxed{S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 d^4x_2 \cdots d^4x_n T[\mathcal{H}^I_{\rm int}(x_1)\mathcal{H}^I_{\rm int}(x_2)\cdots\mathcal{H}^I_{\rm int}(x_n)],} \tag{7.64}$$

where it is understood that the limits on the time integrations are $-\infty < x^0 < \infty$ and that $H^I_{\rm int}$ is constructed with $\varphi_{\rm in}$ given by (7.33).

This seems (and is) an excruciatingly complicated set of equations to wade through. The fact is that when you finally get the results in momentum space, they are often embarrassingly simple. This led Feynman to formulate a set of rules that give the right answer in momentum space without having

[4] *Quantum Field Theory*, C. Itzykson and J.-B. Zuber, McGraw-Hill 1980

to calculate anything. I could just tell you the rules now, but they would be devoid of any intellectual content. Let's derive them step by step.

Take a look at the derivation of the propagator at the end of Chapter 6. There we were dealing with free fields so we should rewrite (6.72)

$$\langle 0|T[\varphi_I(x_1)\varphi_I(x_2)]|0\rangle = iD(x_1 - x_2) \tag{7.65}$$

I am going to describe the steps that led from (6.72) to (6.75) in rather formal terms that make it easy to generalize to situations in which there are more than two fields. In order to keep the notation under control, I will write $\varphi(x) = \varphi^+(x) + \varphi^-(x)$, where φ^+ is the positive-energy piece containing an annihilation operator, and φ^- is the negative-energy term with the creation operator.

- Multiply the fields together with the time-ordering θ functions. This gives eight terms, each one containing a product of two of the creation and/or annihilation operators.

$$\begin{aligned} T[\varphi(x_1)\varphi(x_2)] = (\varphi^+(x_1) + \varphi^-(x_1))(\varphi^+(x_2) + \varphi^-(x_2))\theta(t_1 - t_2) \\ + (\varphi^+(x_2) + \varphi^-(x_2))(\varphi^+(x_1) + \varphi^-(x_1))\theta(t_2 - t_1) \end{aligned}$$

- Use the commutation relations (6.65) to rewrite each term (if necessary) in such a way that **all creation operators stand to the left of all annihilation operators.** This is called *normal ordering.* The usual notation for this is to enclose the expression between colons, i.e. $: \varphi(x_1)\varphi(x_2) :$ means that the terms between the colons have already been placed in normal order. In the course of normal ordering there will appear terms without operators since, for example

$$\varphi^+(x_1)\varphi^-(x_2) =: \varphi^-(x_2)\varphi^+(x_1) : + \int \frac{d^3k}{(2\pi)^3 2\omega} e^{-ik(x_1 - x_2)}$$

  The first term on the right is normal ordered. The second term is a c-number.

- Take advantage of the fact that the vacuum expectation value of any normal ordered product is zero. As a consequence

$$\langle 0|T[\varphi(x_1)\varphi(x_2)]|0\rangle = \langle 0| : \varphi(x_1)\varphi(x_2) : |0\rangle + iD(x_1 - x_2) \tag{7.66}$$

  Of course, the first term on the right is zero. We say that $iD(x_1 - x_2)$ is the *contraction* of $\varphi(x_1)$ and $\varphi(x_2)$. I will use the overbrace to indicate contraction. In this notation (7.66) is

$$T[\varphi(x_1)\varphi(x_2)] =: \varphi(x_1)\varphi(x_2) : + \overbrace{\varphi(x_1)\varphi(x_2)} \tag{7.67}$$

I am now in a position to state Wick's theorem. The time ordered product of n fields $T[\varphi(x_1)\varphi(x_2)\cdots\varphi(x_n)]$ can be written as a sum of terms as follows:

- A single term in which all n fields are normal-ordered.
- A sum of terms, each one of which consists of one pair of contracted φ's. The $n-2$ remaining fields are normal-ordered. There is one term in the sum for each possible pairing of two of the fields.
- A sum of terms, each one of which consists of two pairs of contracted φ's. The remaining $n-4$ remaining fields are normal-ordered. There is one term in the sum for each way in which two distinct pairs can be made out of n fields.
- You see the pattern? One term with no contractions plus all possible terms with one contraction plus all possible terms with two contractions plus all possible terms with three contractions and so forth. If n is even, repeat this procedure $n/2$ times. The last group of terms will consist of $n/2$ contractions. There will be one term in this group for each way of choosing $n/2$ pairs out of n objects. If n is odd, repeat this procedure $(n-1)/2$ times. The last terms will have one normal-ordered field left over

This is all much easier to understand that to explain. The following example with $n=4$ should make this clear.

$$\begin{aligned}
T[\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)] =& :\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4): \\
&+\overbrace{\varphi(x_1)\varphi(x_2)}:\varphi(x_3)\varphi(x_4): +\overbrace{\varphi(x_1)\varphi(x_3)}:\varphi(x_2)\varphi(x_4): \\
&+\overbrace{\varphi(x_1)\varphi(x_4)}:\varphi(x_2)\varphi(x_3): +\overbrace{\varphi(x_2)\varphi(x_3)}:\varphi(x_1)\varphi(x_4): \\
&+\overbrace{\varphi(x_2)\varphi(x_4)}:\varphi(x_1)\varphi(x_3): +\overbrace{\varphi(x_3)\varphi(x_4)}:\varphi(x_1)\varphi(x_2): \\
&+\overbrace{\varphi(x_1)\varphi(x_2)}\overbrace{\varphi(x_3)\varphi(x_4)} +\overbrace{\varphi(x_1)\varphi(x_3)}\overbrace{\varphi(x_2)\varphi(x_4)} \\
&+\overbrace{\varphi(x_1)\varphi(x_4)}\overbrace{\varphi(x_2)\varphi(x_3)}
\end{aligned} \tag{7.68}$$

You should try working out one of these time ordered products by brute force starting with (6.64) and (6.65). You will be convinced that the theorem is true. The proof is more work that it's worth.[5]

[5]If you want a proof see J. D. Bjorken and S. D. Drell, *Relativistic quantum Fields*, Section 17.4

This would be a good time to go back to the free field Hamiltonian, Eq. (6.43).

$$H = \frac{1}{2}\sum_{n=-\infty}^{n=\infty} \hbar\omega_n(a_n a_n^\dagger + a_n^\dagger a_n) = \sum_{n=-\infty}^{n=\infty} \hbar\omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right)$$

We had to discard the $\hbar\omega/2$ term to avoid an infinite sum. This is a common problem whenever we make Hamiltonians our of field operators. In order to insure that this never happens, we will agree that all Hamiltonians come already normally ordered. Accordingly (7.64) should be written

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 d^4x_2 \cdots d^4x_n T[: \mathcal{H}_{\rm int}^I(x_1) :: \mathcal{H}_{\rm int}^I(x_2) : \cdots : \mathcal{H}_{\rm int}^I(x_n) :]. \tag{7.69}$$

When we use Wick's theorem on this expression, we only make contractions of pairs of fields from *different* Hamiltonians.

7.6 An Example

Let's take the interaction Hamiltonian $\mathcal{H}_{\rm int} = (\lambda/4!)\varphi^4(x)$ that we studied in Sections 4.3 and 4.4, and calculate the four-point Green's function $G(x_1, x_2, x_3, x_4)$ to first order in λ. The S-operator (7.64) to first order is

$$S = 1 - \frac{i\lambda}{4!}\int d^4x : \varphi^4(x) : + \cdots \tag{7.70}$$

Note that to first order $\langle 0|S|0\rangle = 1$, so (7.63) becomes

$$\begin{aligned} G(x_1, x_2, x_3, x_4) &= \langle 0|T[\varphi_I(x_1)\varphi_I(x_2)\varphi_I(x_3)\varphi_I(x_4)]|0\rangle \\ &- \frac{i\lambda}{4!}\langle 0|T\left[\varphi_I(x_1)\varphi_I(x_2)\varphi_I(x_3)\varphi_I(x_4)\int d^4x : \varphi_I^4(x) :\right]|0\rangle + \cdots \end{aligned} \tag{7.71}$$

The zeroth-order term can be read off of (7.68)

$$G(x_1, x_2)G(x_3, x_4) + G(x_1, x_3)G(x_2, x_4) + G(x_1, x_4)G(x_2, x_3) \tag{7.72}$$

This is the sum of all possible ways two particles can propagate between two pairs of points. As you would expect in this zeroth-order term, the particles do not interact .

The first-order term has eight fields each with two terms. If you were to simply multiply them out there would be $2^8 = 256$ terms, so here's where

Mr. G. C. Wick earns his keep. First notice that the vacuum expectation value of any normal-ordered product of fields vanishes. We need only be concerned with terms in which all eight fields are contracted. Because of the normal-ordered Hamiltonian, we don't contract the four $\varphi(x)$'s with one another. The only surviving terms are those in which each of $\varphi(x_1)$, $\varphi(x_2)$, $\varphi(x_3)$, and $\varphi(x_4)$ is contracted with one of the $\varphi(x)$'s. There are, of course, 4! ways of doing this all leading to the same expression for the first-order contribution to $G(x_1, x_2, x_3, x_4)$.

$$G(x_1, x_2, x_3, x_4) = -i\lambda \int d^4x G(x_1, x)G(x_2, x)G(x_3, x)G(x_4, x), \qquad (7.73)$$

which is exactly (4.74).

Now suppose you want to calculate the S matrix element for the scattering of two particles $k_1 + k_2 \to p_1 + p_2$. There are two ways of doing this.

- First do the d^4x integration in (7.73). Then substitute $G(x_1, x_2, x_3, x_4)$ into (7.62) and find the Fourier transform $\tilde{G}(k_1, k_2, p_1, p_2)$. Finally substitute $\tilde{G}$ into (7.46) to get $S_{\beta\alpha}$.

- G is a Green's function after all, so

$$(\Box_x + m^2)G(x, x') = i(\Box_x + m^2)D(x - x') = -i\delta^{(4)}(x - x') \qquad (7.74)$$

  You see, all the $(\Box_{x_i} + m^2)$ operators in (7.43) produce factors of $-i\delta^{(4)}(x_i - x)$ when they operate on $G(x_i, x)$. Now all the integrals are trivial.

$$\begin{aligned} S_{\beta\alpha} =& \langle p_1 p_2 \text{ out} | k_1 k_2 \text{ in} \rangle \\ =& \left(\frac{i}{\sqrt{Z}}\right)^4 (-i\lambda)(2\pi)^4 \delta^{(4)}(p_1 + p_2 - k_1 - k_2) \\ & \times \prod_{i=1}^{2} \frac{1}{\sqrt{(2\pi)^3 2E_{k_i}}} \prod_{j=1}^{2} \frac{1}{\sqrt{(2\pi)^3 2E_{p_j}}} \end{aligned} \qquad (7.75)$$

This second technique, though less obvious, saves a factor of two in scratch paper.

This simple calculation illustrates most of the rules for calculating more complicated S-matrix elements in the φ^4 theory. To start with draw some diagrams with the incoming and outgoing particles labelled with their momenta. The diagrams will consist of straight lines and vertices where four

lines meet. The number of vertices must be less than or equal to the corresponding order of perturbation theory, i.e. if you working at the n-th order approximation, you must draw all possible diagrams in which there are n or fewer vertices. In the example, there was only one vertex corresponding to first-order perturbation theory. There will be many diagrams in which one or more of the particles don't interact with the others. One way to say this is that these diagrams can be separated into two or more pieces without cutting any lines. These are called "disconnected" diagrams. These will fall into two categories: "vacuum fluctuation" diagrams in which there is a piece that is disconnected from all the external particles and "self interaction" diagrams in which one of the external particles emits and reabsorbs virtual particles without interacting with the other external particles. Discard all these. It can be shown that the vacuum fluctuations only contribute to the $\langle 0|S|0\rangle$ term, which cancelled in the formula for the S-matrix. The self interaction diagrams represents particles that *don't* scatter, and we are presumably not interested in the probability of things not happening. Each of the remaining diagrams contributes a term to the perturbation expansion. Each term will have the following factors.

- A factor $1/\sqrt{(2\pi)^3 2E_k}$ for each external particle with momentum k.
- A factor $(-i\lambda)$ for each vertex.
- A factor $(2\pi)^4\delta^{(4)}(k_1+k_2-p_1-p_2-\cdots)$ to enforce all-over momentum conservation.

These factors are all evident in (7.75). There are several new features that are only present in higher-order terms.

- There will be internal lines, i.e. lines that begin and end in the diagram. Each one of these gets a propagator

$$\frac{i}{p^2 - m^2 + i\epsilon}$$

  where p is the momentum of the internal particle.
- Impose momentum conservation at each vertex.
- Integrate over each undetermined loop momentum:

$$\int \frac{d^4p}{(2\pi)^4}$$

- Divide by the symmetry factor.

This last issue, the symmetry factor, has to do with redundant diagrams. It's too complicated to explain in the text. I'll explain it in class when I can draw lots of diagrams on the board.

There's still one more step on the road. How do you turn an S-matrix into a cross section. That is the subject of the next chapter.

Chapter 8

Cross Sections and Decay Rates

The last step in the road from Lagrangians to cross sections is this: take the S-matrix computed in Chapter 7 and convert it to a scattering cross section or decay rate. There are two easy ways of doing this and one right way. One easy way is to assume that the initial and final state particles can be described by plane waves. Plane waves occupy infinite volume, unfortunately, so several key integrals diverge. This problem can be ducked with some judicious hand waving. The other easy way is to quantize the wave functions in a box of finite size. This avoids the infinite integrals, but of course, real experiments aren't done in a box. It is best to acknowledge that the beam and target particles should be represented by wave packets, but this makes for a long tedious derivation. We'll do it the ssecond easy way (at least in this edition).

8.1 Classical Scattering

From the point of view of classical mechanics, the cross section of a particle is just its projected area. If the particle is a sphere of radius R, the cross section $\sigma_T = \pi R^2$. This is usually called the "total" cross section. If the particle were very small, we might contrive to measure its cross section by means of a scattering experiment. First we prepare a target with n_t of these particles per unit volume, and bombard this target with N_i incident point-like particles. We assume that the target is so thin and the target particles so small that no one of them gets in front of another. We then count the number N of incident particles that are deflected as a result of a

collision. The fraction of particles that are scattered, N/N_i must be equal to the fraction of the target area covered by the projected area of the target particles, $n_t L\sigma_T$, where L is the target thickness. The cross section is then given by

$$\sigma_T = \frac{N}{N_i n_t L}$$

The idea works even if σ_T is arbitrarily small. We just have to make N_i and n_t large enough.

We can also determine the shape with a scattering experiment. Imagine that the target particles have concentric circles painted on them like an archery target. If one of the incident particles hits a particular area $d\sigma$ it will scatter into a solid angle $d\Omega$. OK – so your're not as good with a bow and arrow as Legolas. Just fire a gazillion arrows at random an observe the distribution of scattering angles. Your favorite theorist should then be able to infer the shape of the target particle from the distribution. The relevant equation is

$$\frac{d\sigma}{d\Omega} = \frac{1}{N_i n_t L}\frac{dN}{d\Omega} \tag{8.1}$$

If you are an experimentalist, you measure $dN/d\Omega$ in your lab, compute $d\sigma/d\Omega$, and pass it along to the theorist. This computed quantity is called the "differential cross section."

The quantities N_i, n_t, and L in (8.1) are directly measurable, but for purposes of comparing with theory, it is more convenient to express them in terms of *flux.* If the beam is moving at velocity v toward a stationary target, then the number of particles in the beam N_i is equal to the density of the beam n_i times the volume. If the beam is a pulse that is turned on for T seconds, then the volume of the beam is vTA, where A is the cross-sectional area of the beam (assumed smaller than the target.) Therefore $N_i = n_i vTA$. The cross section can be written as:

$$\begin{aligned}\frac{d\sigma}{d\Omega} =& \frac{1}{(n_i vTA)n_t L}\frac{dN}{d\Omega} \\ =& \frac{1}{I}\frac{dR}{d\Omega},\end{aligned} \tag{8.2}$$

where the transition rate R is the number of scattering events per unit time per unit target volume, $V = AL$.

$$R = N/VT, \tag{8.3}$$

and the flux I is

$$I = n_i n_t v \tag{8.4}$$

8.2 Quantum Scattering

In the quantum mechanics regime, quantities like size and shape don't have any direct meaning, since any putative size is bound to be smaller that the corresponding deBroglie wavelength; but the experiment can still be done exactly as I have described it, because all the measurements are macroscopic and asymptotic. The differential cross section (DCS for short) can't be interpreted directly in terms of shape, but it does contain most of the information we are entitled to have about particles and other subatomic systems.

The description of scattering in terms of archery targets is appropriate to low-energy potential scattering, but at higher energies new particles can be created and the total momentum shared among them. It requires more than a scattering angle to describe the final state. Rather we have to specify the momenta of each of the particles: $\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n$. Not all values of the momenta are allowed of course, because of the conservation of momentum and energy. We can imagine the momenta as occupying some complicated-shaped region in a $3n$-dimensional space whose coordinates are the components of the individual momenta. This construct is called "phase space."[1] A typical multiparticle scattering experiment measures the transition probability into some region of phase space. We still use the differential cross section notation $d\sigma$, but now $d\sigma$ might be an $3n$-fold differential, depending on the details of the experiment. I will also use the notation $\Delta\Omega$ to indicate some small region of phase space (not just a region of solid angle).

Now let's put on our theorist's hats. We know how to calculate the S-matrix elements. How do we now find the cross sections? You will recall from the end of Chapter 7 that the first term of the perturbation theory expansion of the S-matrix describes particles that don't scatter. Feynman's rules decree that all other terms contain the factor $(2\pi)^4\delta^{(4)}(P_f - P_i)$, where P_i and P_f are the total momentum four-vectors of the initial and final state particles. It is customary to define a transition matrix $\mathcal{T}_{fi}$ as follows:

$$S_{fi} = \delta_{fi} - i(2\pi)^4\delta^{(4)}(P_f - P_i)\mathcal{T}_{fi} \tag{8.5}$$

The δ_{fi} just represents the particles that don't scatter. $\mathcal{T}$ is the non-trivial part of the S-matrix with the delta function "peeled off." If should be clear from (6.25) and (6.35) that the S *operator* is unitary. The S-matrix in the current notation is $S_{fi} = \langle f|S|i\rangle$, where $|i\rangle$ and $\langle f|$ represent complete

[1] Note that phase space in thermodynamics is $6n$-dimensional, since it includes space as well as momentum coordinates.

sets of states. As such we expect the S-matrix to be unitary as well. Very schematically, this means that

$$\sum_f S_{fi}^* S_{fk} = \delta_{ik} \tag{8.6}$$

I say "very schematically" because it's not at all clear what it means to sum over one of these indices. I'll get back to that.

Since S_{fi} is a probability amplitude, we should get the corresponding probability by squaring it. So for example,

$$P_{fi} = |S_{fi}|^2 \tag{8.7}$$

is the probability that the initial state $|i\rangle$ will scatter into the final state $|f\rangle$. Unfortunately this means that we have to square the delta function in (8.5), and this is bad news, since $\delta(P_f - P_i)\delta(P_f - P_i) = \delta(P_f - P_i)\delta(0)$ and $\delta(0)$ is infinite.

I'll deal with this last problem first. Squaring (8.5) gives

$$P_{fi} = (2\pi)^4\delta^{(4)}(0)(2\pi)^4\delta^{(4)}(P_f - P_i)|\mathcal{T}_{fi}|^2 \tag{8.8}$$

(I have ignored the non-scattering term.) Here's the kludge. The factor $(2\pi)^4\delta^{(4)}(0)$ is interpreted as a volume of space-time, since in the limit $P \to 0$

$$(2\pi)^4\delta^{(4)}(P) = \int d^4x\, e^{iPx} \to \int d^4x = VT. \tag{8.9}$$

But the time and volume of what? We can use the derivation of (8.2) as a model. Let V be a small arbitrary volume inside the interaction region. Assume that it is small enough to fit inside the beam and target but very large compared the typical quantum-mechanical dimensions. T is the time required for a beam particle to pass through V. The analogy to (8.3) is

$$\begin{aligned} R_{fi} = \text{transition rate per unit volume } &= \frac{P_{fi}}{VT} \\ &= \frac{(2\pi)^4 VT\delta^{(4)}(P_f - P_i)|\mathcal{T}_{fi}|^2}{VT} \\ &= (2\pi)^4\delta^{(4)}(P_f - P_i)|\mathcal{T}_{fi}|^2 \end{aligned} \tag{8.10}$$

This is the first of several instances in which the arbitrary V's and T's cancel.

Now to the question of counting and normalizing states. This is not explained clearly in any book I know of, so I hope you will indulge me while

I explain this boring technical detail in excruciating detail. Someone has to do it.

First of all, Klein-Gordon wave functions have a conserved norm, but it isn't $\int dV|\varphi|^2$. In order to have the right behavior under Lorentz transformations, the norm must transform like the time-like component of a conserved current. That is, there must be a probability density ρ and a probability current $\boldsymbol{j}$ such that

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla \cdot j} = 0 \tag{8.11}$$

The reason is that the rate of decrease of particles in a given volume is equal to the total flux of particles out of that volume, that is

$$-\frac{\partial}{\partial t}\int_V \rho \, dV = \int_S \mathbf{j}\cdot\hat{\mathbf{n}}\, dS = \int_V \nabla\cdot\mathbf{j}\, dV$$

The procedure for finding this current should be familiar from nonrelativistic quantum mechanics. Write the Klein-Gordon equation as

$$-\frac{\partial^2\varphi}{\partial t^2} + \nabla^2\varphi = m^2\varphi$$

Multiply this equation by $-i\varphi^*$ and the complex conjugate equation by $-i\varphi$ and subtract.

$$\frac{\partial}{\partial t}\underbrace{\left[i\left(\varphi^*\frac{\partial\varphi}{\partial t} - \varphi\frac{\partial\varphi^*}{\partial t}\right)\right]}_{\rho} + \nabla\cdot\underbrace{[-i(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*)]}_{\mathbf{j}} = 0 \tag{8.12}$$

As a consequence the norm is

$$\rho = i\int d^3x\, \varphi^*(x)\overleftrightarrow{\partial_0}\varphi(x) \tag{8.13}$$

By $\varphi(x)$ I mean the wave function, not the quantized field, and

$$\varphi^*\overleftrightarrow{\partial_0}\varphi \equiv \varphi^*\partial_0\varphi - (\partial_0\varphi^*)\varphi.$$

We have been using the quantized field operator

$$\hat{\varphi} = \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}}\left[\hat{a}_k e^{-ikx} + \hat{d}_k^\dagger e^{ikx}\right]$$

from (6.64). We obtain the classical wave function φ from the quantized field operator $\hat{\varphi}$ as follows

$$\begin{aligned}\varphi(x) = \langle 0|\hat{\varphi}(x)|\mathbf{p}\rangle &= \int \frac{d^3k}{\sqrt{(2\pi)^3 2E_k}} e^{-ikx}\langle 0|\hat{a}_k\hat{a}_p^\dagger|0\rangle \\ &= \frac{e^{-ipx}}{\sqrt{(2\pi)^3 2E_p}}\end{aligned} \tag{8.14}$$

This result depends on three normalization conventions that are to some extent arbitrary. First is the factor $1/\sqrt{(2\pi)^3 2E}$ in the field operator (6.64). Second is the commutation relations for the creation and annihilation operators (6.65). Finally, we have agreed to construct single-particle states as $|\mathbf{p}\rangle = \hat{a}_\mathbf{p}^\dagger|0\rangle$. Only the equal-time commutation relations (6.69) are nonnegotiable. We will finally arrive at formulas for the cross sections and decay rates that must be independent of all conventions. This will happen, but no two books do it the same way!

Our choice of normalization for the field operator has the advantage of making (6.65) as simple as possible, but it results in a rather odd-looking norm.

$$\begin{aligned}\rho = i(\varphi^* \overleftrightarrow{\partial_0} \varphi) = 2E\varphi^*\varphi = (2\pi)^{-3} \\ \int_V \rho \, dV = (2\pi)^{-3} V\end{aligned} \tag{8.15}$$

Yes that's right – there are $(2\pi)^{-3}$ particles per unit volume!

Equation (8.10) refers to scattering into a single final state $|f\rangle$. What does this mean? Since we already are committed to the imaginary box, we can impose periodic boundary condition to discretize the momenta. For a single particle, quantum theory restricts the number of final states in a volume V with momenta in element d^3p to be $Vd^3p/(2\pi)^3$. Our V has $(2\pi)^{-3}V$ particles, however, so

$$\text{No. of final states/particle} = \frac{Vd^3p}{(2\pi)^3(2\pi)^{-3}V} = d^3p \tag{8.16}$$

In the general case in which two particles go into N particles with initial and final states given by

$$\begin{aligned}|i\rangle &= |\boldsymbol{k}_2, \boldsymbol{k}_2\rangle \\ |f\rangle &= |\boldsymbol{p}_1, \boldsymbol{p}_2, \cdots, \boldsymbol{p}_N\rangle\end{aligned} \tag{8.17}$$

the "density of states" is

$$\text{No. of available states} = \prod_{j=1}^{N} d^3 p_j \tag{8.18}$$

Now turn to the incident flux (8.4). With our norm $n_i = n_t = (2\pi)^{-3}$, and v is the relative velocity of the colliding particles.

$$v = \left| \frac{\boldsymbol{k}_1}{E_1} - \frac{\boldsymbol{k}_2}{E_2} \right| = \frac{1}{E_1 E_2} \sqrt{(k_1 \cdot k_2)^2 - (m_1 m_2)^2} \tag{8.19}$$

The generalization of (8.4) is

$$d\sigma = \frac{R}{I} \times \text{density of states}$$

Using (8.10), (8.18) and (8.7) we get

$$\begin{aligned} d\sigma &= \frac{(2\pi)^4 \delta(P_f - P_i) |\mathcal{T}_{fi}|^2}{(2\pi)^{-6}\, v} \prod_{j=1}^{N} d^3 p_j \\ &= \frac{1}{v} (2\pi)^{10} \delta^{(4)}(P_f - P_i) |\mathcal{T}_{fi}|^2 \prod_{j=1}^{N} d^3 p_j . \end{aligned} \tag{8.20}$$

Finally, we use this formalism to compute the probability of the decay of a single particle. The differential decay rate $d\Gamma$ of an unstable particle is defined by

$$\begin{aligned} d\Gamma &= \frac{R}{n_t} \times \text{density of states} \\ &= (2\pi)^7 \delta^{(4)}(P_f - P_i) |\mathcal{T}_{fi}|^2 \prod_{j=1}^{N} d^3 p_j \end{aligned} \tag{8.21}$$

In this case n_t is the number of decaying particles per unit volume, which as usual is equal to $(2\pi)^{-3}$.

8.3 Phase Space

The outcome of a scattering experiment as described by (8.20) depends, generally speaking, on three kinds of things. First there are the fundamental dynamics, which are encoded in the Hamiltonian of the system. Second,

there are the details of the experiment itself, e.g. the number of particles in the beam and target, the construction of the detectors, etc. There is a third category of influences that arise because momentum and energy are conserved and because the experiment must be describable in a relativistic invariant way. These last considerations are incorporated in the notion of phase space. Phase space in this context is the set of momentum states available to the particles after the interaction. Imagine leaving the parking lot in front of a movie theater after the show is over. If there is only one narrow road leading out of the lot, then it will take you a long time to escape. If there are freeways leading out in all directions, the lot will empty quickly. This is an analogy based on coordinate space. Try to imagine something like this in momentum space and you will have a feeling for what phase space means. If there are many momentum states available to the final-state particles, the scattering cross section will be large or the decay will happen quickly. I will explain this again in the next paragraph using a semi-quantitative argument based on the uncertainty principle and finally present a version of the argument that is correct relativistically.

In classical statistic mechanics the state of motion of a particle is represented as a point in a six-dimensional manifold called phase space consisting of three position and three momentum coordinates, i.e. (x, y, z, p_x, p_y, p_z). Classical mechanics places no restrictions on the density of these points, since $\mathbf{x}$ and $\mathbf{p}$ are supposedly measurable with complete precision. Quantum mechanics on the other hand limits the precision by the uncertainty principle,

$$\Delta x_j \Delta p_j \geq 2\pi\hbar.$$

Phase space can be thought of as being subdivided into elementary cells of size $(2\pi\hbar)^3$. The particle's coordinates cannot be localized to any smaller region. The number of states available to one particle will therefore be equal to the total volume of phase space divided by the size of the elementary cell. Assuming that the particle is contained in a volume V, the number of available states is

$$N = \frac{1}{(2\pi\hbar)^3} \int d^3x \; d^3p = \frac{V}{(2\pi\hbar)^3} \int d^3p.$$

The limits of integration come about from the conservation of mass and energy. Since $p_\mu p^\mu = m^2$, the momentum integration is bounded by a sphere of radius $|\mathbf{p}|^2 = E^2 - m^2$, so that N depends on m and E. The factor $V/(2\pi\hbar)^2$ is part of the normalization discussed in the previous section. I am concerned here with the integral over d^3p.

Now suppose there are N such particles in the final state of a scattering experiment.

$$N_N = \int \prod_{j=1}^{N-1} d^3p_j \tag{8.22}$$

There are only $N-1$ integrals in (8.22), because the N $\mathbf{p}_j$'s are not all independent,

$$\sum_{j=1}^{N} \mathbf{p}_j \equiv \mathbf{P}_f = \mathbf{P}_i.$$

We can incorporate this constraint in a more flexible way by introducing a δ-function.

$$\int d^3p_N \; \delta^{(3)} \left(\mathbf{p}_N + \sum_{k=1}^{N-1} \mathbf{p}_k - \mathbf{P}_i \right) = 1$$

$$N_N = \int \prod_{j=1}^{N-1} d^3p_j = \int \prod_{j=1}^{N} d^3p_j \; \delta^{(3)} \left(\sum_{k=1}^{N} \mathbf{p}_k - \mathbf{P}_i \right)$$

Energy is also conserved, and this constraint can be included with a similar trick.

$$\int dE_i \; \delta(E_f - E_i) = 1$$

$$N_N = \int dE_i \prod_{j=1}^{N} d^3p_j \; \delta^{(3)} \left(\sum_{k=1}^{N} \mathbf{p}_k - \mathbf{P}_i \right) \delta \left(\sum_{k=1}^{N} E_k - E_i \right)$$

$$\frac{dN_N}{dE_i} = \int \prod_{j=1}^{N} d^3p_j \; \delta^{(4)}(P_f - P_i) \tag{8.23}$$

These are just the kinematic factors appearing in (8.20). It appears from this heuristic argument, that the reaction rate is proportional to the density of states dN_N/dE_i.[2] This is not the whole story, because (8.23) as it stands is not Lorentz invariant, but according to the Feynman rules from Section

[2] You will recall from elementary quantum mechanics that the transition probability rate for a system to make a transition from an energy E_l to E_k is given by "Fermi's golden rule,"

$$w_{lk} = \frac{2\pi}{\hbar} g(E_k) |H_{kl}|^2$$

where $g(E_k)$ is the density of states whose energy is close to E_k. See R. L. Liboff, *Introductory Quantum Mechanics*.

7.5, the square of the scattering amplitude, $|\langle f|\mathcal{T}|i\rangle|^2$ in (8.20), contains a factor $1/2E_i(2\pi)^3$ for each particle in the final state. This is good, since

$$\int d^4p\ \delta(p^2 - m^2) = \int \frac{d^3p}{2E}$$

is clearly a Lorentz scalar. The point is that the factors

$$\left(\prod_{j=1}^{N} \int d^3p_j\right) (2\pi)^4 \delta^{(4)}(P_i - P_f)$$

from (8.20) together with the factors $1/2E_i$ from $|\langle f|\mathcal{T}|i\rangle|^2$ can be combined into a Lorentz-invariant quantity

$$\begin{aligned} \mathcal{P} &= \int \prod_{j=1}^{N} \frac{d^3p_j}{2E_j}\ \delta^{(4)}(P_f - P_i) \\ &= \int \prod_{j=1}^{N} d^4p_j\ \delta(p_j^2 - m^2)\ \delta^{(4)}(P_f - P_i) \end{aligned} \tag{8.24}$$

The quantity $\mathcal{P}$ is called the Lorentz-invariant phase space. In general we are more interested in integrating some of the momenta in (8.24) over some portion of their allowed kinematic range, in which case

$$d\mathcal{P} = \int_{p\in\Delta} \prod_{j=1}^{N} d^4p_j \delta(p_j^2 - m^2) \delta^{(4)}(P_f - P_i). \tag{8.25}$$

You might say that (8.24) and (8.25) lie on the interface between theory and experiment. For example, if you were using (8.20) to predict the results of an experiment, the limits on the integrals would be set to represent the details of the experimental apparatus. If the apparatus did not detect particle j (it might be a neutral particle), then the corresponding momentum p_j would be integrated out. If the momentum of the k'th particle was measured, then the corresponding integral would not be done, and $\mathcal{P}$ would be left as a function of p_k.

For many-particle reactions, $\mathcal{P}$ will have a complicated dependence on the momentum variables.[3] It is often the case that the data from an experiment are determined mostly by $\mathcal{P}$ regardless of the dynamics. Such an experiment is a failure almost by definition. The general purpose of an experiment is always to find some deviation from phase space.

[3]There are entire books written on the subject, e.g. *Kinematics and Multiparticle Systems*, M. Nikolic, Gordon and Breach (1968)

8.4 Two-Particle Scattering

To make this more concrete, let's calculate the cross section for the elastic scattering of two particles $k_1+k_2 \to p_1+p_2$ from the example in the previous chapter. Comparing (7.75) (with $Z=1$) with (8.5) gives

$$\mathcal{T}_{fi} = \frac{\lambda}{(2\pi)^6\sqrt{2E_{k_1}2E_{k_2}2E_{p_1}2E_{p_2}}} \tag{8.26}$$

Then (8.20) gives

$$\begin{aligned} d\sigma =& \frac{\lambda^2}{4(2\pi)^2 v E_{k_1}E_{k_2}} \int \frac{d^3p_1}{2E_{p_1}}\frac{d^3p_2}{2E_{p_2}}\delta^{(4)}(P_f - P_i) \\ =& \frac{\lambda^2}{4(2\pi)^2 v E_{k_1}E_{k_2}} \int d\mathcal{P} \end{aligned} \tag{8.27}$$

The transition matrix $\mathcal{T}_{fi}$ is especially simple in this example. This is why there is nothing inside the integral except $d\mathcal{P}$. In general there will be some complicated function that we may or may not want to integrate.

I will complete the calculation in the center of mass system assuming equal-mass particles. In this case $|\mathbf{k_1}| = |\mathbf{k_2}| = |\mathbf{p_1}| = |\mathbf{p_2}|$, $\mathbf{k_1} = -\mathbf{k_2}$, $\mathbf{p}_1 = -\mathbf{p}_2$, $E_{p_2} = E_{p_1} \equiv E_1$, and $E_{k_1} + E_{k_2} \equiv E_i$. For the remainder of this section only I will use the notation $|\mathbf{p}_1| = p_1$. Integrating out d^3p_2 gives

$$\mathcal{P} = \int \frac{1}{4E_1^2}\, p_1^2 dp_1 \delta(2E_{p_1} - E_i) d\Omega_1$$

It's convenient to change the variable of integration from p_1 to E_1.

$$p_1 dp_1 = E_1 dE_1 \qquad \delta(2E_1 - E_i) = \frac{1}{2}\delta(E_1 - E_i/2)$$

$$\mathcal{P} = \int \frac{p_1}{8E_1} dE_1 \delta(E_1 - E_i/2) d\Omega_1$$

There is still one δ-function, which will enforce energy conservation and hence fix the value of p_1.

$$\mathcal{P} = \int \frac{p_1}{8E_1}\, d\Omega_1 = \frac{\pi p_1}{4E_1}. \tag{8.28}$$

Notice that symmetry together with the conservation of momentum and energy largely predetermine the outcome of such a scattering experiment. If the particles are spinless there can be no dependence on the azimuthal

angle. Only the scattering angle θ_1 is unconstrained, and the only way the underlying dynamics could manifest itself is by affecting the differential cross section as a function of scattering angle. Our example is trivial in the sense that there is no θ dependence, but let's go ahead and calculate the differential cross section anyhow. For this purpose it is best to leave (8.28) in the form of a differential

$$d\mathcal{P} = \frac{p_1}{8E_1}\, d\Omega_1 \tag{8.29}$$

We also need v from (8.19), which in this simple case gives

$$v = \frac{2p_1}{E_1} \tag{8.30}$$

Equation (8.27) becomes

$$\begin{aligned} d\sigma =& \frac{\lambda^2}{4(2\pi)^2 E_1^2}\, \frac{E_1}{2p_1}\, \frac{p_1}{8E_1} d\Omega_1 \\ \frac{d\sigma}{d\Omega_1} =& \frac{\lambda^2}{(16\pi E_1)^2} \end{aligned} \tag{8.31}$$

As anticipated, the scattering is isotropic. There is no θ_1 dependence.

8.5 The General Case

Here is a cookbook procedure for calculating scattering cross sections and decay rates for the general case in which there are N particles in the final state.

- Use Feynman's rules from Section 7.5 to calculate the S matrix.
- Find the transition matrix $\mathcal{T}_{fi}$ from (8.5).
- Find the relative velocity v from (8.19).
- The differential cross sections and decay rates are calculated from (8.20) and (8.21) integrated over the appropriate regions of phase space.

$$d\sigma = \frac{(2\pi)^{10}}{v} \int_{p\in\Delta\Omega} \prod_{j=1}^{N} d^3p_j \delta^{(4)}(P_f - P_i) |\mathcal{T}_{fi}|^2 \tag{8.32}$$

$$d\Gamma = (2\pi)^7 \int_{p\in\Delta\Omega} \prod_{j=1}^{N} d^3p_j \delta^{(4)}(P_f - P_i)|\mathcal{T}_{fi}|^2 \tag{8.33}$$

When there are only two particles in the final state, one integrates over all the momentum components that are constrained by the delta function and leaves the remaining two angles in the form of a differential. When there are more than two particles in the final state, the decision regarding which variables to integrate and which to leave in the differential depends on the experiment that the theory is describing. That is the meaning of the notation $p \in \Delta\Omega$ that specifies the limits of the integrals

Because of our normalization conventions, $|\mathcal{T}_{fi}|^2$ will contain one factor of $1/(2\pi)^3 2E$ for each external particle in the reaction. The factors of 2π as well as the energies of the *initial-state* particles are constants that are gathered together outside the integrals. The energies of the *final-state* particles are functions of the variables of integration. It is best to think of them as part of the phase space defined by (8.24). Needless to say, these rules are tailored to the Feynman's rules from Section 7.5. If you combine formulas from different books, you are likely to get the wrong answer.

The rules so far only hold for spinless particles. The Dirac wave functions used for spin-$\frac{1}{2}$ scattering are normalized differently. I'll return to that point at the end of Chapter 10.

Chapter 9

The Dirac Equation

There is a story to the effect that Neils Bohr once asked Dirac what he was working on. Dirac, who was famous for not saying much, replied, "I'm trying to take the square root of the Klein-Gordon equation." The background to the story is this: the Klein-Gordon equation is invariant under Lorentz transformations, but it was known to have a strange pathology, it has negative energy solutions. This is inevitable with a second-order equation. The Schrodinger equation, on the other hand, is hopeless so far as relativity is concerned, but because it has only a first derivative with respect to time, it has only positive-energy solutions. (I'm talking about free-particle equations here. Bound state problems usually have negative total energy.) Dirac thought that he could combine the two equations in a way that would preserve the best features of each. The result is called the Dirac equation. Ironically, it has negative-energy solutions anyway, but it does provide a formalism for treating spin 1/2 particles in a way that is consistent with relativity.

9.1 The Equation

Let's follow in the footsteps of our taciturn master and see what we can come up with. Start with the Schrodinger equation.

$$i\frac{\partial\psi}{\partial t} = H\psi$$

Can we come up with a Hamiltonian that (1) is first order in space derivatives, (2) is Hermitian, and (3) leads to a covariant equation of motion? The

answer is *yes if ψ is a matrix.* Try

$$i\frac{\partial\psi}{\partial t} = H\psi = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)\psi \tag{9.1}$$

here $\boldsymbol{\alpha}$ and β are Hermitian matrices and $\boldsymbol{p} = -i\boldsymbol{\nabla}$. We will also use the replacement $E = i\partial/\partial t$. If E is the total energy of special relativity, it must satisfy $E^2 = p^2 + m^2$. Combine this with equation (9.1)

$$E^2\psi = \left(i\frac{\partial}{\partial t}\right)^2\psi = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)\left(i\frac{\partial\psi}{\partial t}\right) = (\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta m)^2\psi$$

We would like this to equal $(\boldsymbol{p}^2 + m^2)\psi$. This will be possible if the matrices have some special properties.

$$(\alpha_i p^i + \beta m)^2 = \beta^2 m^2 + (\alpha_i)^2(p^i)^2 + \{\alpha_i, \beta\}\, mp^i + \frac{1}{2}\{\alpha_i, \alpha_j\}_{i\neq j}\, p^i p^j \tag{9.2}$$

I have used the convention that repeated indices are summed. The curly brackets are endemic to Diracology. They represent the anticommutator.

$$\{A, B\} \equiv AB + BA$$

We would like (9.2) to boil down to $p^i p_i + m^2$. This will be possible if these matrices obey the following constraints:

$$\beta^2 = (\alpha_i)^2 = 1 \text{ for each } i \tag{9.3}$$

$$\{\alpha_i, \beta\} = \{\alpha_i, \alpha_j\}_{i\neq j} = 0 \tag{9.4}$$

We are trying to construct a theory of spin 1/2 particles, and we know that in non-relativistic QM Pauli spinors have two components. We expect therefore that α and β will be 2×2 matrices. Alas, it is not to be. They must be at least 4×4. Here is the argument:

1. $\beta\alpha_i + \alpha_i\beta = 0$ so $\beta\alpha_i\beta = -\alpha_i\beta^2 = -\alpha_i$. Take the trace of this equation. $Tr[\beta\alpha_i\beta] = Tr[\alpha_i\beta^2] = Tr[\alpha_i] = -Tr[\alpha_i]$. So α_i is traceless. In the same way we could prove that $Tr[\beta] = 0$.

2. We want α and β to be Hermitian. This together with (9.3) is sufficient to prove that their eigenvalues are ± 1

3. Arguments 1. and 2. lead us to conclude that the matrices must have even dimension.

4. There are only three independent traceless Hermitian 2×2 matrices, and you know what they are: the Pauli spin matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.5}$$

Unfortunately, we need four such matrices, so 2×2 won't do. We must have 4×4. The traditional choice is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{9.6}$$

All this has a nasty, non-relativistic look about it. We therefore abandon Dirac's original notation for the following:[1]

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{9.7}$$

Now (9.4) becomes

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{9.8}$$

and (9.1) is

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{9.9}$$

This is Dirac's equation. Of course ψ is a four-component column matrix, the so-called Dirac spinor. $g^{\mu\nu}$ is the usual metric tensor. There is one other convenient bit of notation.

$$\gamma_\mu B^\mu = \gamma^\mu B_\mu \equiv \not{B} \tag{9.10}$$

Dirac's equation finally is $(i\not{\partial} - m)\psi = 0$. It looks simple. It's not. It is however, covariant. This is far from obvious, but I will not prove it at this time.

Wave functions are supposed to have something to do with probability. This is true of ψ as well, but to see the connection, we need to derive the formula for a conserved current. Starting with (9.1) and taking the Hermitian conjugate, we have

$$-i\frac{\partial \psi^\dagger}{\partial t} = i\frac{\partial \psi^\dagger}{\partial x_i}\alpha_i + \psi^\dagger \beta m \tag{9.11}$$

[1] Some authors use different conventions. This is consistent with Zee, Gross, and many other texts.

Multiply on the right by $\beta^2 = 1$ and define

$$\overline{\psi} \equiv \psi^\dagger \beta = \psi^\dagger \gamma^0 \tag{9.12}$$

This object $\overline{\psi}$ is called the Dirac conjugate. It often appears in bilinear expressions in place of the usual Hermitian conjugate. Anyhow, the resulting equation is

$$-i\frac{\partial \overline{\psi}}{\partial t}\gamma^0 = -i\frac{\partial \overline{\psi}}{\partial x_i}\gamma_i + \overline{\psi} m$$

or

$$\overline{\psi}(i \overleftarrow{\not\partial} + m) = 0 \tag{9.13}$$

The arrow indicates that the derivative acts to the left. To find the current, multiply (9.9) on the left by $\overline{\psi}$ and (9.13) on the right by ψ and add the two. The result is

$$\partial_\mu \overline{\psi}\gamma^\mu \psi + \overline{\psi}\gamma^\mu \partial_\mu \psi = 0 = \partial_\mu(\overline{\psi}\gamma^\mu \psi) \tag{9.14}$$

So the conserved current is

$$j^\mu = \overline{\psi}\gamma^\mu \psi \tag{9.15}$$

The current density $j^0 = \psi^\dagger \psi$ is positive and has the same form $|\psi|^2$ as the probability density in non-relativistic QM.

9.2 Plane Wave Solutions

We know that electrons have two spin components. Why then are there four components to the Dirac spinor? The answer is that despite Dirac's best intentions, there are still negative energy solutions. This in turn is related to the existence of antiparticles. We will get to the particle interpretation later. For the time being we will just reconcile ourselves to negative energies. We start with

$$\psi^\pm = e^{\mp ipx} \begin{pmatrix} \chi \\ \eta \end{pmatrix} \tag{9.16}$$

As a convention, the symbol $E = p^0$ will always refer to a positive number. χ and η are two-component spinors. They are not independent; the Dirac equation imposes a relationship between the two. Our job is to find it. Start with the upper sign (positive energy) solution. Substitute (9.16) into (9.1).

$$\begin{pmatrix} E - m & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ -\boldsymbol{\sigma}\cdot\boldsymbol{p} & E + m \end{pmatrix} \begin{pmatrix} \chi \\ \eta \end{pmatrix} = 0$$

This will only have a solution if the determinant is equal to zero, *i.e.* $E^2 - m^2 - (\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = 0$. Remember that

$$\sigma_i^2 = I \qquad \{\sigma_i, \sigma_j\}_{i\neq j} = 0 \tag{9.17}$$

so that

$$(\boldsymbol{\sigma}\cdot\boldsymbol{p})^2 = \boldsymbol{p}^2, \tag{9.18}$$

which is a useful identity to have at your disposal. We conclude that $E^2 = \boldsymbol{p}^2 + m^2$, which we knew anyhow. Unfortunately, this means that $E = \pm\sqrt{\boldsymbol{p}^2 + m^2}$, so we are stuck with negative energy solutions. At any rate

$$\eta = \left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\right)\chi$$

We can make χ into a Pauli spinor that satisfies $\chi^\dagger\chi = 1$.

$$\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{9.19}$$

The complete positive energy solution to the Dirac equation is written

$$\psi^+(x,s) = u(\boldsymbol{p},s)e^{-ipx} \tag{9.20}$$

where

$$u(\boldsymbol{p},s) = C\begin{bmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{bmatrix} \qquad s = 1,2 \tag{9.21}$$

Notice that in the non-relativistic limit, the third and fourth components become negligible.

The normalization constant C has to be chosen according to some convention. We will use $\overline{u}(\boldsymbol{p},s)u(\boldsymbol{p},s) = 1$. In order to appreciate this, it is necessary to use one of these group theory results that I have not yet proved. It is a fact that combinations like $\overline{\psi}\psi$ and $\overline{u}u$ are Lorentz scalars, so that if we choose the normalization in the rest frame of the particle, it will have the same value in all other frames. A quick calculation shows that

$$\overline{u}(\boldsymbol{p},s)u(\boldsymbol{p},s) = C^2\frac{2m}{E+m},$$

so our normalized spinor is

$$u(\boldsymbol{p},s) = \sqrt{\frac{E+m}{2m}}\begin{bmatrix} \chi^{(s)} \\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s)} \end{bmatrix} \qquad s = 1,2 \tag{9.22}$$

so that in the rest frame

$$u(0,1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u(0,2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{9.23}$$

Now back to the negative energy solutions, which I will provisionally call $u(\boldsymbol{p},3)$ and $u(\boldsymbol{p},4)$. Repeating the above arguments with the lower sign in (9.16) brings us to

$$\chi = -\left(\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\right)\eta$$

and

$$u(\boldsymbol{p},s) = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} -\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E+m}\chi^{(s-2)} \\ \\ \chi^{(s-2)} \end{bmatrix} \qquad s = 3,4 \tag{9.24}$$

It is easy to see that in the rest frame (and so everywhere else) (9.24) is normalized so that $\overline{u}(\boldsymbol{p},s)u(\boldsymbol{p},s) = -1$. The minus sign is forced on us. The complete negative-energy solution is

$$\psi^-(x,s) = u(\boldsymbol{p},s)e^{ipx} \qquad s = 3,4 \tag{9.25}$$

9.3 Charge Conjugation and Antiparticles

The appearance of negative-energy states was both an embarrassment and a disappointment; a disappointment because this wasn't supposed to happen with a linear wave equation and an embarrassment because of its physical implications. Let us consider an atomic electron. An excited electron will normally lose energy by emitting photons until it drops down to the lowest-energy or ground state. In the original version of Dirac theory there is no lowest-energy state since there exists an continuum of negative-energy states from $-mc^2$ to $-\infty$. The electron would fall into a bottomless pit emitting an infinite amount of energy in the process! To sidestep this difficulty, Dirac proposed that under normal circumstances, *all negative-energy states are completely filled.* This agglomeration of filled states is called the Dirac sea. Like all of our theories I suppose, it's a half truth. Unfortunately, the other half is very confusing. For one thing, it depends on the Pauli exclusion principle. The states are full because no two Fermions can occupy a single state. This is not true for integer spin particles, and the Klein-Gordon

equation also has negative energy states. Obviously the Dirac sea can't be the whole answer, but it contains a surprising amount of physical insight.[2]

Suppose one of the negative-energy electrons were to absorb a photon of energy $> 2mc^2$ and become an $E > 0$ state. As a result a "hole" is created in the Dirac sea. The *observable* energy of the Dirac sea is now the energy of the vacuum *minus* the *negative* energy of the vacated state, hence a *positive* quantity. In this way we expect that the absence of a negative-energy electron appears as the presence of a positive-energy particle. The same argument works with charge. Let Q_0 represent the charge of the vacuum and e the negative charge of an electron in the sea, then the total charge of the Dirac sea is

$$Q = Q_0 - e = Q_0 - (-|e|) = Q_0 + |e| \tag{9.26}$$

The observable charge is then $Q_{obs} = Q - Q_0 = |e|$. This is interpreted as a particle with positive energy and charge. This process actually happens. In the laboratory it looks like $\gamma \to e^- + e^+$. (It can't happen in empty space because of energy and momentum conservation. It does happen whenever high energy photons pass through matter.) When Dirac first proposed this in 1930, however, no such particle was known. For this reason it was assumed that the theory couldn't be right. The positron was discovered soon thereafter.

According to this theory, the total energy of the vacuum as well as the total charge is negative infinity. We have to take it on faith that these infinities are not observable. What is observable are the deviations from infinity corresponding to the energy and charge of a single electron! This is an unsatisfactory aspect of the theory of course, but it will continue to devil us long after we have abandoned the Dirac sea.

The argument made above regarding charge and energy also works for momentum and spin. The absence of momentum $\boldsymbol{p}$ in the Dirac sea appears as the presence of momentum $-\boldsymbol{p}$. Similarly the absence of a spin-up $E < 0$ electron is to be interpreted as the presence of a spin-down $E > 0$ positron.

This is all well and good, but we must face the fact that the Dirac sea is really just a metaphor. We need to describe antiparticles in a way that does not invoke negative energies. I once heard Edward Teller explain antiparticles like this: "Take a particle in one hand and an antiparticle in the other. Put them together, and what you have is – nothing"! Put less cryptically, all the additive quantum numbers sum to zero. What are the

[2]The best discussion of this entire subject is still the classic, *Advanced Quantum Mechanics*, by J. J. Sakurai, Addison-Wesley 1967, Sections 3.9-3.10

additive quantum numbers? Well, charge, baryon number, three kinds of lepton number, and strangeness for starters. Electrons have zero baryon number and strangeness anyhow. I will come to the issue of lepton number presently. Let's concentrate on the matter of charge. We can define a transformation called "charge conjugation." It has two effects. First, it replaces all particles with their antiparticles, and second, it changes the wave function in such a way that it correctly describes the new situation.

It's easy to modify the Dirac equation to include the electromagnetic potential. The procedure is called "minimal substitution." The rule is everywhere you see the derivative ∂_μ simply replace it with $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ where q is the charge of the particle. This rule is based on the idea of gauge invariance, which we will discuss in the next chapter. For the time being – just believe it. With this replacement the Dirac equation for an electron with charge $-e$ ($e > 0$) becomes

$$(i\not{\partial} + e\ \not{A} - m)\psi = 0 \tag{9.27}$$

Charge conjugation must change the charge of the electron so that Teller's dictum comes true. The transformed equation is

$$(i\not{\partial} - e\ \not{A} - m)\psi_c = 0. \tag{9.28}$$

Here ψ_c is the new "charge conjugated" wave function. Take the complex conjugate of (9.28) and multiply by C_0, a 4×4 matrix that operates on the space of the γ's.

$$\begin{aligned} &C_0(i\gamma^{\mu *}\partial_\mu + e\gamma^{\mu *}A_\mu + m)\psi_c^* \\ &= \left[(C_0\gamma^{\mu *}C_0^{-1})(i\partial_\mu + eA_\mu) + m\right] C_0\psi_c^* = 0 \end{aligned} \tag{9.29}$$

At this point we don't know what C_0 is. The challenge is to get (9.28) looking like (9.27). For this we will need $C_0\gamma^{\mu *}C_0^{-1} = -\gamma^\mu$ and $C_0\psi_c^* = \psi$. It doesn't take much trial and error to realize that all the γ's are pure real except γ^2. Try $C_0 = i\gamma^2$. (The i is necessary to make C_0 Hermitian.)[3] You can check that $C_0^2 = I$ and $C_0\gamma^{\mu *}C_0^{-1} = -\gamma^\mu$ as promised. The bottom line is

$$\psi_c = C_0\psi^* = i\gamma^2\psi^* \tag{9.30}$$

Let us see what this operation does to our prefab spinors. The following identities are useful.

$$\boldsymbol{\sigma}\cdot\boldsymbol{p} = \begin{pmatrix} p_z & p_- \\ p_+ & -p_z \end{pmatrix} \tag{9.31}$$

[3]There are several different representations of the gamma matrices that are useful in other contexts. Equation (9.30) is valid only in the the standard or Dirac-Pauli representation we are using.

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{9.32}$$

$$\psi^+(x,1) = u(\boldsymbol{p},1)e^{-ipx} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ p_z/(E+m) \\ p_+/(E+m) \end{bmatrix} e^{-ipx} \tag{9.33}$$

$$\psi^+(x,2) = u(\boldsymbol{p},2)e^{-ipx} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 0 \\ 1 \\ p_-/(E+m) \\ -p_z/(E+m) \end{bmatrix} e^{-ipx} \tag{9.34}$$

$$\psi^-(x,3) = u(\boldsymbol{p},3)e^{ipx} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} -p_z/(+m) \\ -p_+/(E+m) \\ 1 \\ 0 \end{bmatrix} e^{+ipx} \tag{9.35}$$

$$\psi^-(x,4) = u(\boldsymbol{p},4)e^{ipx} = \sqrt{\frac{E+m}{2m}} \begin{bmatrix} -p_-/(E+m) \\ p_z/(E+m) \\ 0 \\ 1 \end{bmatrix} e^{+ipx} \tag{9.36}$$

Let's find the charge-conjugated version of $\psi^+(x,1)$. Using (9.30), (9.32), and (9.33) gives

$$\begin{aligned} \psi_c^+(x,1) &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \sqrt{\frac{E+m}{2m}} \begin{bmatrix} 1 \\ 0 \\ p_z/(E+m) \\ p_+/(E+m) \end{bmatrix}^* e^{+ipx} \\ &= \sqrt{\frac{E+m}{2m}} \begin{bmatrix} -p_-/(E+m) \\ p_z/(E+m) \\ 0 \\ -1 \end{bmatrix} e^{+ipx} = -u(-\boldsymbol{p},4)e^{ipx} \end{aligned} \tag{9.37}$$

In the same way one can prove that

$$\psi_c^+(x,2) = u(-\boldsymbol{p},3)e^{ipx} \tag{9.38}$$

Thus the charge-conjugate wave function obtained from the positive-energy plane-wave solution is the wave function for a negative-energy plane wave

with the same $|E|$ and opposite momentum. The spin direction is also reversed since the spin-up spinor with $s = 1$ is changed into the spin-down spinor with $s = 4$. According to Dirac's hypothesis, the *observable* behavior of the charge-conjugate wave function should be a positron with $E > 0$ and the *same* momentum and spin. This is consequence of the *absence* of an electron with negative charge and energy and reversed momentum and spin.

If you buy into this bizarre notion of the Dirac sea, the rest is logically consistent, but expectation values of the charge-conjugated wave functions do not give the results you would like in terms of antiparticles. Try for example, to calculate the expectation value of the charge of a positron. The electromagnetic current should be proportional to the zeroth component of the probability current (9.15).

$$Q = e \int j^0 d^3x = e \int \psi^\dagger \psi d^3x \tag{9.39}$$

The charge-conjugated wave functions give the same value for Q, since

$$Q_c = e \int \psi_c^\dagger \psi_c d^3x = e \int (C_0 \psi^*)^\dagger (C_0 \psi^*) d^3x = e \int (\psi^\dagger \psi)^* d^3x = Q \tag{9.40}$$

where I have used the fact that C_0 is Hermitian, $C_0^2 = I$, and currents are real functions. It's trivial to show that the expectation values of the momentum and spin do change sign. The reason is that ψ_c is not the positron wave function, it's the wave function that is *interpreted* as the positron wave function. Under the skin, it's still the wave function of a fictitious negative energy particle.[4] In order to get sensible expectation values we need to introduce second quantization with creation operators that create real positrons. In order to do that, it is customary to define spinors for real positrons as follows.

$$\begin{aligned} v(\mathbf{p}, 1) &\equiv -u(-\mathbf{p}, 4) \\ v(\mathbf{p}, 2) &\equiv +u(-\mathbf{p}, 3) \end{aligned} \tag{9.41}$$

With this definition

$$\begin{aligned} u_c(\mathbf{p}, s) &= C_0 u(\mathbf{p}, s)^* = v(\mathbf{p}, s) \qquad s = 1, 2 \\ v_c(\mathbf{p}, s) &= C_0 v(\mathbf{p}, s)^* = u(\mathbf{p}, s) \qquad s = 1, 2 \end{aligned} \tag{9.42}$$

$u(\mathbf{p}, 3)$ and $u(\mathbf{p}, 4)$ are properly called negative-energy spinors, while $v(\mathbf{p}, 1)$ and $v(\mathbf{p}, 2)$ are positron spinors.

[4] Again – thanks to Sakurai for finally making me understand this point.

Here is a summary of the important equations involving u and v:

$$\psi^+(x) = u(\boldsymbol{p}, s)e^{-ipx} \tag{9.43}$$

$$\psi^-(x) = v(\boldsymbol{p}, s)e^{+ipx} \tag{9.44}$$

$$\overline{u}(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = -\overline{v}(\boldsymbol{p}, s)v(\boldsymbol{p}, s') = \delta_{ss'} \tag{9.45}$$

$$u^\dagger(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = v^\dagger(\boldsymbol{p}, s)v(\boldsymbol{p}, s') = \frac{E}{m}\delta_{ss'} \tag{9.46}$$

$$\overline{u}(\boldsymbol{p}, s)v(\boldsymbol{p}, s') = \overline{v}(\boldsymbol{p}, s)u(\boldsymbol{p}, s') = 0 \tag{9.47}$$

$$u^\dagger(-\boldsymbol{p}, s)v(\boldsymbol{p}, s') = v^\dagger(\boldsymbol{p}, s)u(-\boldsymbol{p}, s') = 0 \tag{9.48}$$

$$\sum_s u(\boldsymbol{p}, s)\overline{u}(\boldsymbol{p}, s) = \left(\frac{\not{p} + m}{2m}\right) \tag{9.49}$$

$$\sum_s v(\boldsymbol{p}, s)\overline{v}(\boldsymbol{p}, s) = \left(\frac{\not{p} - m}{2m}\right) \tag{9.50}$$

These equations will all be useful later on. You can verify them by substituting the representations (9.22), (9.24), and (9.7).

9.4 Quantizing the Field

We repeat the procedure from Chapter 5 used to quantize the Kline-Gordon field. First we'll need a Lagrangian, which can be taken as

$$\mathcal{L} = \overline{\psi}(x)(i\ \not{\partial} - m)\psi(x) \tag{9.51}$$

The conjugate momentum is

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\psi}} = i\psi^\dagger \tag{9.52}$$

and the Hamiltonian is

$$H = \int d^3x\ (\pi\dot{\psi} - \mathcal{L}) = \int d^3x\ \overline{\psi} i\gamma^0 \partial_0 \psi \tag{9.53}$$

Next we will need the plane wave solutions of the Dirac equation. These are $u(\mathbf{p}, s)e^{-ipx}$ and $v(\mathbf{p}, s)e^{ipx}$. We then expand $\psi(x)$ and $\pi(x)$ in a Fourier series in which creation and annihilation operators appear as Fourier coefficients. This should look like (6.60) with the following differences:

1. ψ is a four-component spinor rather than a scalar like φ.
2. $u(\mathbf{p}, x)$ and $v(\mathbf{p}, s)$ depend on spin, so in addition to integrating over all momentum components, we must also sum over the spin components.
3. We need two kinds of creation and annihilation operators, one set for electrons and one for positrons. In the usual notation

$$\begin{array}{ll} b(\mathbf{p}, s) & \text{Annihilates a positive energy electron} \\ b^\dagger(\mathbf{p}, s) & \text{Creates a positive energy electron} \\ d(\mathbf{p}, s) & \text{Annihilates a positive energy positron} \\ d^\dagger_(\mathbf{p}, s) & \text{Creates a positive energy positron} \end{array} \tag{9.54}$$

The analogs to (6.60) are

$$\begin{aligned} \psi(x) &= \int \sqrt{\frac{m}{E}} \frac{d^3p}{\sqrt{(2\pi)^3}} \sum_s [b(\mathbf{p}, s) u(\mathbf{p}, s) e^{-ipx} + d^\dagger(\mathbf{p}, s) v(\mathbf{p}, s) e^{ipx}] \\ \overline{\psi}(x) &= \int \sqrt{\frac{m}{E}} \frac{d^3p}{\sqrt{(2\pi)^3}} \sum_s [b^\dagger(\mathbf{p}, s) \overline{u}(\mathbf{p}, s) e^{ipx} + d(\mathbf{p}, s) \overline{v}(\mathbf{p}, s) e^{-ipx}] \end{aligned} \tag{9.55}$$

In analogy with (6.69) we would expect the equal-time commutation relations to be

$$[\psi_\alpha(\boldsymbol{x}, t), \psi^\dagger_\beta(\boldsymbol{y}, t)] = \delta_{\alpha\beta} \delta^3(\boldsymbol{x} - \boldsymbol{y})$$

$$[\psi_\alpha(\boldsymbol{x}, t), \psi_\beta(\boldsymbol{y}, t)] = [\psi^\dagger_\alpha(\boldsymbol{x}, t), \psi^\dagger_\beta(\boldsymbol{y}, t)] = 0.$$

Unfortunately, this leads to some wildly unphysical consequences. The origin of the problem is that electrons are fermions, *i.e.* particles with half-odd integer spin. Such particles obey the Fermi exclusion principle, no two particles can occupy the same quantum state. Mathematically, this means that creation and annihilation operators anticommute, and so as a consequence, must the fields themselves. We must have

$$\begin{aligned} \left\{ \psi_\alpha(\boldsymbol{x}, t), \psi^\dagger_\beta(\boldsymbol{y}, t) \right\} &= \delta_{\alpha\beta} \delta^3(\boldsymbol{x} - \boldsymbol{y}) \\ \{ \psi_\alpha(\boldsymbol{x}, t), \psi_\beta(\boldsymbol{y}, t) \} &= \left\{ \psi^\dagger_\alpha(\boldsymbol{x}, t), \psi^\dagger_\beta(\boldsymbol{y}, t) \right\} = 0. \end{aligned} \tag{9.56}$$

Pay careful attention to the notation. $\psi\psi^\dagger$ is a 4×4 matrix, whereas $\psi^\dagger\psi$ is a 1×1 matrix, so $[\psi, \psi^\dagger]$ doesn't make any sense. The commutators in (9.56) refer to the α-th *component* of one ψ matrix with the β-th *component* of another. Put it another way, each commutator is a 4×4 matrix with rows and columns indexed by α and β. Be alert to this ambiguity in other contexts.

You should verify that (9.55) is consistent with (9.56) assuming the following anticommutation relations

$$\left\{b(\mathbf{p},s), b^\dagger(\mathbf{p}',s')\right\} = \delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}')$$

$$\left\{d(\mathbf{p},s), d^\dagger(\mathbf{p}'s')\right\} = \delta_{ss'}\delta^3(\boldsymbol{p}-\boldsymbol{p}') \tag{9.57}$$

All other combinations of b and d anticommute. The factor of $\sqrt{m/E}$ in (9.55) was chosen so that (9.56) would come out right.

> There are at least three places where we have to make a somewhat arbitrary choice of normalization. One is the normalization of u and v, equations (9.22) and (9.24). The second is the abovementioned factor in the Fourier expansion of ψ. The third is the normalization of the creation and annihilation operators, (9.57). Only (9.56) is non-negotiable. It is only a slight exaggeration to say that no two books do it the same way. At least Zee, Michio Kaku (*Quantum Field Theory, A Modern Introduction*), and I are consistent.

It's easy to generalize charge conjugation (9.30) for the quantized fields. Define the charge conjugation *operator* $\hat{C}$ by

$$\hat{\psi}_c \equiv \hat{C}\hat{\psi}\hat{C}^{-1} = C_0\hat{\psi}^{\dagger T} = i\gamma^2\hat{\psi}^{\dagger T} \tag{9.58}$$

The point is that complex conjugation is not defined for the creation and annihilation operators. The combination $^{\dagger T}$ has the same effect as * on matrices and the same effect as † on $\hat{b}$ and $\hat{d}$. (I have put the hats back on the operators for emphasis. They are implied in the other operator equations in this section.) Now using (9.29), (9.55) becomes

$$\begin{aligned}
\psi(x) &= \int \sqrt{\frac{m}{E}}\frac{d^3p}{\sqrt{(2\pi)^3}}\sum_s [d(\mathbf{p},s)u(\mathbf{p},s)e^{-ipx} + b^\dagger(\mathbf{p},s)v(\mathbf{p},s)e^{ipx}] \\
\overline{\psi}(x) &= \int \sqrt{\frac{m}{E}}\frac{d^3p}{\sqrt{(2\pi)^3}}\sum_s [d^\dagger(\mathbf{p},s)\overline{u}(\mathbf{p},s)e^{ipx} + b(\mathbf{p},s)\overline{v}(\mathbf{p},s)e^{-ipx}]
\end{aligned} \tag{9.59}$$

The only change is that particle operators have been replaced by antiparticles operators and vice versa, $b(\mathbf{p},s) \leftrightarrow d(\mathbf{p},s)$ and $b^\dagger(\mathbf{p},s) \leftrightarrow d^\dagger(\mathbf{p},s)$ with *no change in spin*. This point will be central importance in the case of the weak interactions.

Now insert the Fourier decomposition back into the expression for the Hamiltonian, equation (9.53). Again – as an exercise in Diracology, you should verify that the following is obtained.

$$H = \int d^3pE \sum_s [b^\dagger(\mathbf{p},s)b(\mathbf{p},s) - d(\mathbf{p},s)d^\dagger(\mathbf{p},s)] \tag{9.60}$$

Here is the first of many places where commutation relations lead to unphysical results. If d and $d^\dagger$ commuted, the last term could be written $-d^\dagger(p,s)d(p,s)$, so by creating many particles with $d^\dagger$ we could create states of arbitrarily large negative energy. With the anticommutation rules we have

$$:H: = \int d^3pE \sum_s [b^\dagger(\mathbf{p},s)b(\mathbf{p},s) + d^\dagger(\mathbf{p},s)d(\mathbf{p},s)] \tag{9.61}$$

As in (6.45), the Hamiltonian must be normal ordered to avoid the infinite constant. When dealing with fermions, it is necessary to include the minus sign arising from the anticommutators. For example, if a, b, c, and d are fermion annihilation operators,

$$:(a+b^\dagger)(c+d^\dagger): = ac - d^\dagger a + b^\dagger c + b^\dagger d^\dagger \tag{9.62}$$

Take another look at the total charge calculated in (9.39)

$$Q = e\int d^3x\ \psi^\dagger(x)\psi(x)$$

where e is the electron charge, a negative number. The usual Dirac business gives us

$$Q = e\int d^3p \sum_s [b^\dagger(\mathbf{p},s)b(\mathbf{p},s) + d(\mathbf{p},s)d^\dagger(\mathbf{p},s)] \tag{9.63}$$

This operator also has to be normal ordered.

$$:Q: = e\int d^3p \sum_s [N^-(\mathbf{p},s) - N^+(\mathbf{p},s)] \tag{9.64}$$

Where N^+ and N^- are the number operators for positive and negative energy electrons respectively.

9.5 The Lorentz Group

We can't get much further with the Dirac equation without using some group theory. Here is a brief review of the Lorentz group.

The basic equation of Lorentz transformation is

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{9.65}$$

$\Lambda^{\mu}{}_{\nu}$ is any 4×4 real matrix that satisfies

$$g_{\mu\nu} = \Lambda^{\sigma}{}_{\mu} g_{\sigma\gamma} \Lambda^{\gamma}{}_{\nu}, \tag{9.66}$$

or in matrix notation

$$g = \Lambda^T g \Lambda. \tag{9.67}$$

It is easy to show that this implies $\det \Lambda = \pm 1$. The identity transformation is given by $\Lambda = I$. Any transformation that can be obtained from this by a continuous variation of the transformation parameters (β for example) is called a *proper* transformation. There are three ways that a transformation might be not proper.

1. Try $\Lambda^0{}_0 = -1$, $\Lambda^i{}_i = 1$, and all off-diagonal terms equal to zero. This simply makes time run backwards, and so it is called the *time reversal transformation* or T for short. It is easy to show from (9.66) that this is improper.

2. $\Lambda^0{}_0 = 1$, $\Lambda^i{}_i = -1$, and all off-diagonal terms equal to zero. This reverses the spatial coordinates. We call it a *parity transformation* and refer to it as P.

3. We could have a product of the two, TP=PT.

These are examples of discrete transformations, and $T^2 = P^2 = I$. The most general Lorentz transformation consists of a product of a proper transformation and P, T, or PT.[5]

All this is preliminary to analyzing the proper Lorentz group and its associated Lie algebra.[6] Consider the basic Lorentz transformation

$$x' = \gamma(x - vt) \qquad y' = y \qquad t' = \gamma(t - vx/c^2) \tag{9.68}$$

[5]Later on we will use P and T to represent more general operators that have the effect of reversing $\boldsymbol{x}$ or t.

[6]I am following a particularly fine exposition in *Quantum Field Theory* by Michio Kaku.

If we make the standard replacement

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \cosh\zeta_1 \qquad \beta\gamma = \sinh\zeta_1 \qquad \beta = v/c \tag{9.69}$$

then this transformation can be written as:

$$\begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \cosh\zeta_1 & -\sinh\zeta_1 & 0 & 0 \\ -\sinh\zeta_1 & \cosh\zeta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{9.70}$$

Transformations of this sort are called *boosts* and the angle ζ_1 that parameterizes the boost in (9.69) and (9.70) is called the *rapidity*. The 4×4 matrix above represents the tensor $\Lambda^\mu{}_\nu$ in (9.65). In matrix notation $x' = \Lambda x$.

According to the doctrine of Lie groups, we can calculate the group generators by differentiating Λ with respect to its parameters and then setting the parameters equal to zero.

$$K^1 \equiv \frac{\partial}{\partial\zeta_1}\Lambda(\zeta_1)\bigg|_{\zeta_1=0} \qquad \text{(provisional definition)}$$

The K's obtained in this way are constant matrices called the "generators" of the transformation. The group can then be "reconstituted" by exponentiating the generators.

$$\Lambda(\zeta_1) = e^{K^1\zeta_1}$$

At this point physicists often muddy the water by inserting i's in ways that appal the mathematicians. We usually write the above as

$$K^1 \equiv -i\,\frac{\partial}{\partial\zeta_1}\Lambda(\zeta_1)\bigg|_{\zeta_1=0} \tag{9.71}$$

and

$$\Lambda(\zeta_1) = e^{iK^1\zeta_1} \tag{9.72}$$

The reason for putting in the i and then taking it out will appear presently. With this convention

$$K^1 = -i\begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.73}$$

The generators of boosts in the y and z directions are

$$K^2 = i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 = i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{9.74}$$

Rotations are also Lorentz transformations, or to put it another way, the rotation group is a subgroup of the Lorentz group. The usual parameters are a set of three angles. For example, a rotation through an angle θ_1 around the 1 or x axis would yield

$$\Lambda(\theta_1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & 0 & -\sin\theta_1 & \cos\theta_1 \end{pmatrix} \tag{9.75}$$

The corresponding generators are

$$J^1 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad J^2 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{9.76}$$

$$J^3 = i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{9.77}$$

Notice that the J^i's are antisymmetric and because of the i's, Hermitian. The corresponding rotation matrices

$$\Gamma(\boldsymbol{\theta}) = e^{i\boldsymbol{J}\cdot\boldsymbol{\theta}} \tag{9.78}$$

are unitary. This is why physicists (who are fond of unitarity) put in the i's and mathematicians (who would like to keep the structure constants real) do not. The boost generators by contrast, are symmetric and boost matrices are not unitary. The generators have the following commutation relations:[7]

$$\begin{aligned} [K^i, K^j] &= -i\epsilon^{ijk}J^k \\ [J^i, J^j] &= i\epsilon^{ijk}J^k \\ [J^i, K^j] &= i\epsilon^{ijk}K^k \end{aligned} \tag{9.79}$$

[7]This is also worked out in Jackson, Sec. 11.7

The matrices above constitute the defining representations of the group and the algebra. That is to say, they operate on four-vectors. We are looking for a representation that transforms Dirac spinors, however, and for this purpose the commutation relations (9.79) must be put in a more convenient form. To this end we start by proving a simple theorem about the symmetry properties of the generators.

Consider infinitesimal Lorentz transformations of the form

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}_{\nu} + \epsilon M^{\mu}{}_{\nu} \tag{9.80}$$

with $M^{\mu}{}_{\nu}$ a constant matrix, *i.e.* a member of the algebra, and ϵ infinitesimal. Substitute (9.80) into (9.66) and discard the second-order terms.

$$M_{\mu\nu} + M_{\nu\mu} = 0 \tag{9.81}$$

This is equivalent in matrix notation to the statement that M (with both indices down) is an antisymmetric matrix. It is easy to check that the maximum number of independent 4×4 antisymmetric matrices is six and that the commutator of two antisymmetric matrices is itself antisymmetric. Thus this set of matrices constitutes a closed Lie algebra, the algebra of the Lorentz group. A particularly useful way of choosing such matrices is

$$(M^{\mu\nu})_{\alpha\beta} = i(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\mu}_{\beta}\delta^{\nu}_{\alpha}) \tag{9.82}$$

Just so there's no confusion, $\mu\nu$ is the *name* of the matrix and α and β index it's elements. It's easy to remember. The matrix named $M^{\mu\nu}$ contains all zeros except for two elements. The element that sits on the μ'th row and ν'th column is i and the element at the ν'th row and μ'th column is $-i$. One can derive the following commutation relations:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(g^{\nu\rho}M^{\mu\sigma} - g^{\mu\rho}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\rho} + g^{\mu\sigma}M^{\nu\rho}) \tag{9.83}$$

The M's defined by (9.82) constitute a matrix representation, but any set of objects that satisfy the commutation relations (9.83) also constitutes a valid representation of the Lie algebra. There are two other representations that are important. One is the differential operator

$$M^{\mu\nu} = i(x^{\mu}\partial^{\nu} - x^{\nu}\partial^{\mu}). \tag{9.84}$$

The other is the spinor representation, which I will come to shortly.

Referring again to (9.82), there are twelve non-zero 4×4 antisymmetric matrices related in a simple way to the K's and J's defined previously. It's

easy to verify that $gM^{01} = K^1$, $gM^{12} = J^3$, etc. Infinitesimal transformations are written

$$\Lambda^{\alpha}{}_{\beta}(\omega) = \delta^{\alpha}_{\beta} - \frac{i}{2}\omega_{\mu\nu}(M^{\mu\nu})^{\alpha}{}_{\beta}, \tag{9.85}$$

where $\omega_{\mu\nu}$ is an infinitesimal antisymmetric tensor. We can iterate (9.85) to build up finite transformations as usual.

$$\Lambda(\omega) = \exp\left\{-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}\right\} \tag{9.86}$$

9.6 Spinor Representations

The anticommutation relations (9.8) define what is called a Clifford algebra. (This works in any even dimension.) These algebras can always be used to define a new sort of representation. Define

$$\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^{\mu}, \gamma^{\nu}] \tag{9.87}$$

It is straightforward to show using (9.8) that the object $\frac{1}{2}\sigma^{\mu\nu}$ satisfies the commutation relations (9.83) and hence constitute a representation of the Lorentz group. The operator

$$S(\omega) \equiv \exp\left\{-\frac{i}{4}\omega_{\mu\nu}\sigma^{\mu\nu}\right\} \tag{9.88}$$

is exactly analogous to (9.86). This can also be written $S(\Lambda)$ to remind ourselves that given the parameters $\omega_{\mu\nu}$, we can calculate the defining representation $\Lambda^{\mu}{}_{\nu}$ as well as the spinor representation S. It is also straightforward to show that

$$[\gamma^{\mu}, \frac{1}{2}\sigma^{\rho\sigma}] = (M^{\rho\sigma})^{\mu}{}_{\nu}\gamma^{\nu} \tag{9.89}$$

where $M^{\rho\sigma}$ is defined by (9.82).

Before we apply all this to the Dirac equation, we should think about what it means to transform a vector field. The Dirac spinor $\psi(x)$, like the EM vector potential $A^{\mu}(x)$, describes some multi-component property (of the ether, if you like) at the space-time point x. I take the passive point of view, that a Lorentz transformed field describes the same phenomena at the same space-time point, but both the point and the components of the field are referred to a different set of axis, so

$$S(\Lambda)\psi(x) = \psi'(x'). \tag{9.90}$$

$x' = \Lambda x$ as usual, and the new field components ψ' will be determined by $S(\Lambda)$. If the Dirac equation is to make any sense, it must remain form invariant under this transformation, so that we must have both

$$(\gamma^\mu p_\mu - m)\psi(x) = 0 \tag{9.91}$$

and

$$(\gamma^\mu p'_\mu - m)\psi'(x') = 0 \tag{9.92}$$

(Here $p_\mu = -i\partial_\mu$.) This requirement leads to an important equation for $S(\Lambda)$. Substitute (9.90) and $p'_\mu = \Lambda_\mu{}^\nu p_\nu$ into (9.92).

$$S^{-1}(\Lambda)(\gamma^\mu \Lambda_\mu{}^\nu p_\nu - m)S(\Lambda)\psi(x) = 0$$

This will be identical to (9.91) if

$$S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu \tag{9.93}$$

This is interesting. The left side is a similarity transformation. This is how one transforms a matrix like γ^μ. The right side treats γ^μ like a four-vector. So is γ^μ a matrix like $S(\Lambda)$ or a four-vector like p^μ? The answer of course, is yes!

Or at least we hope so. The covariance of the Dirac equation stands or falls depending on (9.93). Here its proof: The first step is to prove that (9.93) is true for infinitesimal transformations

$$\begin{aligned}(1 + \frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma})\gamma^\mu(1 - \frac{i}{4}\omega_{\rho\sigma}\sigma^{\rho\sigma}) &= (\gamma^\mu - \frac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, \frac{1}{2}\sigma^{\rho\sigma}]) \\ &= (1 - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma})^\mu{}_\nu \gamma^\nu\end{aligned} \tag{9.94}$$

The last line makes use of (9.89). The left side of (9.94) is the infinitesimal form of (9.93) by construction. A glance at (9.86) shows that the last line of (9.94) is also the infinitesimal form of the right side of (9.93). Thus (9.93) is valid for infinitesimal transformations. It can be proved for finite transformations by iterating infinitesimal transformations. It's not a new argument, but it's very tedious to write out. I'll leave it as an exercise.

We are now in a position to prove the claim made earlier that $\overline{\psi}\psi$ is a Lorentz scalar. First take the Hermitian conjugate of (9.90) and right multiply by γ^0.

$$\psi'^\dagger(x')\gamma^0 = \psi^\dagger(x)\gamma^0\gamma^0 S^\dagger(\Lambda)\gamma^0$$

Remember that $(\gamma^0)^2 = 1$ and $(\gamma^0)^\dagger = \gamma^0$.

$$\overline{\psi}'(x') = \overline{\psi}(x)\gamma^0 S(\Lambda)^\dagger \gamma^0$$

$$= \overline{\psi}(x)S^{-1}(\Lambda) \tag{9.95}$$

The last line follows from (9.87) and (9.88). Evidentally, $\overline{\psi}'(x')\psi'(x') = \overline{\psi}(x)\psi(x)$. By the same token, $\overline{\psi}\gamma^\mu\psi$ is a genuine vector.

$$\overline{\psi}'(x')\gamma^\mu\psi'(x') = \overline{\psi}(x)S^{-1}\gamma^\mu S\psi = \overline{\psi}(x)\Lambda^\mu{}_\nu\gamma^\nu\psi(x) \tag{9.96}$$

The last step used (9.93).

Expressions like $\overline{\psi}\psi$ and $\overline{\psi}\gamma^\mu\psi$ are called Dirac bilinear forms. Notice that unlike ψ and γ^μ, they are not matrices but ordinary tensors. We can also show that $\overline{\psi}\sigma^{\mu\nu}\psi$ is a second-rank antisymmetric tensor. Since these forms are essential in describing interactions, we should enumerate the various possibilities. We need to define a new gamma matrix:

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{9.97}$$

The following properties can be verified with the help of the anticommutation relations.

$$(\gamma^5)^\dagger = \gamma^5 \tag{9.98}$$

$$(\gamma^5)^2 = 1 \tag{9.99}$$

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{9.100}$$

With our conventions

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{9.101}$$

There are sixteen numbers in a 4×4 matrix, so there should be sixteen linearly independent 4×4 matrices with well-defined transformation properties. Here they are,

1	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu}$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1

for a total of sixteen. The terms *pseudo-tensor* and *pseudo-scalar* are "pseudo" in the sense that they change sign under parity conjugation. I will return to this important subject later. One can show that these sixteen matrices are indeed linearly independent. Any product of gamma matrices can be reduced to a linear combination of these terms.

9.7 The Dirac Propagator

Consider a physical process in which an electron is created out of the vacuum at the point x and is subsequently reabsorbed at x'. This emission and reabsorption might take place because of sources, which we discussed in connection with path integrals, or it might come about because the electron interacted with other particles at x and x'. In any case, the amplitude for this process is called the propagator. It is the basic ingredient in all perturbation theory calculations. I have argued before that the field $\overline{\psi}(x)$ creates an electron at x and ψ destroys it, so the product $\langle 0|\psi(x')\overline{\psi}(x)|0\rangle$ must have something to do with the propagator. This can't be the whole story, however, because we cannot annihilate the electron before it has been created. True, simultaneity is relative, but it is logically impossible in any reference frame to make something disappear that doesn't exist.[8] Therefore, we should write out propagator as $\langle 0|\psi(x')\overline{\psi}(x)|0\rangle\theta(t'-t)$. This looks non-covariant, because or the $\theta(t'-t)$. The argument can be made in a completely covariant way, but this is just pedantry. We get a covariant result in the end anyway.

This is still not the complete propagator, because there is a distinct process that is physically equivalent. $\overline{\psi}$ doesn't just create electrons, *it reduces the charge*. So $\langle 0|\psi(x')\overline{\psi}(x)|0\rangle$ also include the process in which negative energy electrons travel backwards in time from x to x', or equivalently, positrons travel from x' to x. We must still insist that the positron be created before it is destroyed, so finally

$$iS_F(x',x) = \langle 0|\psi(x')\overline{\psi}(x)|0\rangle\theta(t'-t) - \langle 0|\overline{\psi}(x)\psi(x')|0\rangle\theta(t-t') \qquad (9.102)$$

So why subtract the two terms rather than adding them? The glib answer is that the spinor fields anticommute. We will get more insight into this presently. Just as we did in the case of the Kline-Gordon equation, eq. (5.33), we introduce the time ordering operator

$$T[\psi(x')\overline{\psi}(x)] = \begin{cases} \psi(x')\overline{\psi}(x) & t' > t \\ -\overline{\psi}(x)\psi(x') & t > t' \end{cases} \qquad (9.103)$$

Finally

$$i(S_F)_{\alpha\beta}(x',x) = \langle 0|T[\psi_\alpha(x')\overline{\psi}_\beta(x)]|0\rangle \qquad (9.104)$$

S_F is a 4×4 matrix indexed with α and β. It's easy to calculate. Substitute (9.55) into (9.102) and use (9.49) and (9.50) to do the spin sums. Zee

[8] "Yesterday upon a stair, / I met a man who wasn't there. / He wasn't there again today. / I wish that man would go away."

goes through this calculation in detail on pages 108-109. He arrives at the following:

$$iS(x) = i \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \frac{\not{p} + m}{p^2 - m^2 + i\epsilon} \tag{9.105}$$

It turns out that $(\not{p} - m)(\not{p} + m) = p^2 - m^2$. We could say that the factors of $(\not{p} + m)$ in the numerator and denominator "cancel" leaving the rather whimsical equation,[9]

$$iS(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ipx} \frac{i}{\not{p} - m + i\epsilon}. \tag{9.106}$$

In momentum space this is

$$iS(p) = \frac{i}{\not{p} - m + i\epsilon} \tag{9.107}$$

Causality requires that the propagator vanishes outside the light cone. You can show that (9.105) does have that property. It also reproduces the minus sign in (9.102). It can also be shown[10] that without that minus sign, the propagator *violates* causality. This is an illustration of a deep result, called the spin-statistics theorem. It says that in order to preserve causality, Fermion fields must anticommute, and boson fields must commute.

At this point in our treatment of the scalar field, we added some interactions and did perturbation theory, but electrons don't interact with themselves like our hypothetical scalar particle. They do interact with the electromagnetic vector potential and with the potential that carries the weak interactions. (The're the same field actually, but that's a story for another time.) Our next order of business is the electromagnetic field, to which we turn in the next chapter.

[9] Your instructor will become homicidal if you divide by a matrix under any other circumstances.

[10] Peskin and Schroeder, *An Introduction to Quantum Field Theory* page 56

Chapter 10

The Photon Field

10.1 Maxwell's Equations

In some sense Maxwell discovered both quantum electrodynamics and relativity. I mean by this odd remark that Maxwell's equations are consistent with both. Nothing about them needs to be changed. Here they are:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \boldsymbol{E} &= \rho \\ \boldsymbol{\nabla} \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} &= \boldsymbol{j} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \\ \boldsymbol{\nabla} \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} &= 0 \end{aligned} \tag{10.1}$$

You may never have seen them in this simple form. It arises because: (1) We use Heaviside-Lorentz units. (See the appendix in Jackson regarding units.) In this system, all the factors of 4π are absent from Maxwell's equation. (They retaliate by showing up in Coulomb's law, but we don't care about that.) There are no dielectrics or magnetic materials, so $\epsilon = \mu = 1$. (3) We set $c = 1$ as usual. The electromagnetic current is conserved.

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \tag{10.2}$$

The vector fields can be derived from potentials.

$$\boldsymbol{E} = -\boldsymbol{\nabla} A^0 - \frac{\partial \boldsymbol{A}}{\partial t} \qquad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{10.3}$$

This can all be put in covariant notation by defining the following tensors:

$$\begin{aligned} A^\mu &= (A^0, \boldsymbol{A}) \\ j^\mu &= (\rho, \boldsymbol{j}) \\ F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{10.4}$$

Then current conservation is just

$$\partial_\mu j^\mu = 0 \tag{10.5}$$

and Maxwell's equations can be summarized as:

$$\begin{aligned} \partial_\mu F^{\mu\nu} &= j^\nu \\ \partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} &= 0 \end{aligned} \tag{10.6}$$

The fact that these can be written in covariant form indicates that relativity is "built in" to Maxwell's equations.

We need a Lagrangian to do field theory, Jackson has some obscure things to say about Lagrangians in general, but if you know the answer ahead of time, it's really very easy.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j_\mu A^\mu \tag{10.7}$$

To show that this does indeed generate the right equations of motion, simplify the expression as follows:

$$\begin{aligned} -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} &= -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= -\frac{1}{2} [(\partial_\mu A_\nu)(\partial^\mu A^\nu) - (\partial_\mu A_\nu)(\partial^\nu A^\mu)] \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\partial^\mu A^\nu + \partial^\nu A^\mu = -F^{\mu\nu} \end{aligned} \tag{10.8}$$

The Euler-Lagrange equations are

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = -\partial_\mu F^{\mu\nu} + j^\nu = 0,$$

which is exactly (10.6).

The potential contains some unphysical information in the sense that Maxwell's equations are invariant under gauge transformations. The familiar form for gauge transformation is

$$\mathbf{A} \to \mathbf{A} + \nabla \Lambda$$

$$\phi \to \phi - \frac{\partial \Lambda}{\partial t}$$

where $\Lambda(x)$ is any arbitrary function of space and time. We can write this in covariant form as follows.

$$A^\mu(x) \to A^\mu(x) - \partial^\mu \Lambda(x) \tag{10.9}$$

This leaves the field tensor invariant $F^{\mu\nu}(x) \to F^{\mu\nu}(x)$, and hence preserves Maxwell's equations. We can use this symmetry to make A^μ satisfy certain relations.[1] For example, we can choose Λ so that $A^0(x) = 0$. Imposing constraints like this is called "gauge fixing." This particular constraint defines what is called temporal gauge. It is possible to impose an additional constraint

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0 \tag{10.10}$$

This is known as Coulomb gauge or radiation gauge. Another choice is Lorentz gauge.

$$\partial_\mu A^\mu = 0 \tag{10.11}$$

This has the advantage of being covariant, but it does not fully fix the gauge. This poses a dilemma: we can't fully fix the gauge without making the equations non-covariant. We can't make the theory covariant without building into the theory unphysical degrees of freedom known as Fadeev-Popov ghosts.[2] The subject is so difficult and convoluted that one distinguished text[3] simply refuses to discuss it!

Components of A^μ that can be set equal to zero by a gauge transformation cannot have any physical significance. It's a well-known result from classical field theory that any vector can be decomposed into a longitudinal and a transverse part.

$$\boldsymbol{A} = \boldsymbol{A}_T + \boldsymbol{A}_L$$

where by definition

$$\boldsymbol{\nabla} \times \boldsymbol{A}_L = 0 \qquad \boldsymbol{\nabla} \cdot \boldsymbol{A}_T = 0$$

Furthermore, the decomposition is unique up to an additive constant. Since F^{ij} is basically the curl of A, it cannot depend on the longitudinal components. Only the transverse components are meaningful. There are only two

[1] This subject is discussed extensively in Jackson and our Ph632 class. Here I am quoting the main results without proof.

[2] Where is Buffy now that we need her?

[3] Peskin and Schroder

physical degrees of freedom, which correspond to the two polarization states of an electromagnetic wave.

Perhaps the easiest way out of this mess is to quantize in Coulomb gauge, though even this is not straightforward. This is the approach taken in the previous generation of texts.[4] Modern texts do this with path integrals. It's ironic that the photon propagator, which is all you need to do perturbation theory in QED, is very simple. This is my excuse for spending minimal effort on this vexatious subject.

10.2 Quantization in the Coulomb Gauge

We have two goals.[5] First, to find an expansion of the photon field in terms of creation and annihilation operators, and second, find the photon propagator. The Lagrangian density for the free fields is

$$\mathcal{L}(x) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = \frac{1}{2}(\boldsymbol{E}^2 - \boldsymbol{B}^2). \tag{10.12}$$

The conjugate momenta to A^μ are,

$$\pi_0(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^0)} = 0$$

$$\pi_i(x) = \frac{\partial\mathcal{L}}{\partial(\partial_0 A^i)} = -E_i \tag{10.13}$$

and the Hamiltonian density is

$$\mathcal{H} = \pi_\mu \dot{A}^\mu - \mathcal{L} = \frac{1}{2}(\boldsymbol{E}^2 + \boldsymbol{B}^2) + \boldsymbol{E}\cdot\boldsymbol{\nabla}A^0 \tag{10.14}$$

The next step is to impose the equal-time commutation relations. (Compare with (73) in the previous chapter.)

$$[A_\alpha(\boldsymbol{x},t), \pi_\beta(\boldsymbol{y},t)] = i\delta_{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) \tag{10.15}$$

$$[A_\alpha(\boldsymbol{x},t), A_\beta(\boldsymbol{y},t)] = [\pi_\alpha(\boldsymbol{x},t), \pi_\beta(\boldsymbol{y},t)] = 0$$

We are missing one relation, since $\pi_0 = 0$. This is an invitation to duck into coulomb gauge and set $A^0 = \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. We are not out of the woods yet,

[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics and Relativistic Quantum Fields* is the Gold Standard.

[5] I am following B. Hatfield, *Quantum Field Theory of Point Particles and Strings.*

however. Eq. (10.3) gives $\boldsymbol{\nabla}\cdot\boldsymbol{E}=0$. This is Gauss's law; so far so good. If we take the divergence of (10.15)

$$[A_i(\boldsymbol{x},t),E_j(\boldsymbol{y},t)]=-i\delta_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{10.16}$$

we get the odd result

$$0=-i\partial_i\delta(\boldsymbol{x}-\boldsymbol{y})$$

This is time for some creative thinking. Suppose we modify (10.15) slightly so that we avoid this contradiction. Lets replace the δ-function with a function $f(\boldsymbol{x}-\boldsymbol{y})$ such that $\partial_i f(\boldsymbol{x}-\boldsymbol{y})=0$. The usual choice is the transverse δ-function,

$$\delta^{\rm tr}_{ij}(\boldsymbol{x}-\boldsymbol{y})\equiv\int\frac{d^4k}{(2\pi)^3}e^{i\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y})}\left(\delta_{ij}-\frac{k_ik_j}{\boldsymbol{k}^2}\right). \tag{10.17}$$

$\delta^{\rm tr}_{ij}(\boldsymbol{x})$ has the desired property: $\partial_i\delta^{\rm tr}_{ij}(\boldsymbol{x})=0$. We modify (16) to read

$$[A_i(\boldsymbol{x},t),E_j(\boldsymbol{y},t)]=-i\delta^{\rm tr}_{ij}\delta(\boldsymbol{x}-\boldsymbol{y}), \tag{10.18}$$

The following comments are in order:

- $\delta^{\rm tr}_{ij}(\boldsymbol{x})\neq 0$ so A_i and E_i don't commute at space-like separations. This in itself is OK since $\boldsymbol{A}$ is not measurable.
- Electric and magnetic fields calculated using (10.18) do commute at space-like separations, and in fact, we get the same commutators for $\boldsymbol{E}$ and $\boldsymbol{B}$ regardless whether we use the transverse or ordinary δ-function in (10.18).
- The underlying idea here is that there are only two degrees of freedom, but (10.16) tries to quantize three. The transverse δ-function remedies this problem.

Now we are home safe. It remains only to expand $\boldsymbol{A}$ in plane waves and calculate the propagator.

$$\boldsymbol{A}(x)=\int\frac{d^3k}{\sqrt{(2\pi)^3 2k_0}}\sum_{\lambda=1}^{2}\boldsymbol{\varepsilon}^\lambda(k)\left[a^\lambda(k)e^{-ikx}+a^{\lambda\dagger}(k)e^{ikx}\right] \tag{10.19}$$

Since there is no antiphoton (or more correctly, the photon is its own antiparticle) there is only one set of creation and destruction operators. Since we are working in transverse gauge, $\boldsymbol{A}$ must be perpendicular to the direction

of propagation. This is accomplished by defining the photon polarization vectors $\boldsymbol{\varepsilon}^\lambda$ such that

$$\boldsymbol{\varepsilon}^\lambda \cdot \boldsymbol{k} = 0 \tag{10.20}$$

We also require that they be orthonormal.

$$\boldsymbol{\varepsilon}^\lambda(k) \cdot \boldsymbol{\varepsilon}^{\lambda'}(k) = \delta^{\lambda\lambda'} \tag{10.21}$$

Equation (10.18) will be satisfied if

$$[a^\lambda(k), a^{\lambda\dagger}(k')] = \delta^{\lambda\lambda'}\delta^{(3)}(\boldsymbol{k} - \boldsymbol{k'}) \tag{10.22}$$

Our last mission is to calculate the propagator. The process should be quite familiar by now.

$$\begin{aligned} iD_F(x'-x)_{\mu\nu} &= \langle 0|T[A_\mu(x')A_\nu(x)]|0\rangle \\ &= i\int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x'-x)}}{k^2+i\epsilon}\sum_{\lambda=1}^{2}\boldsymbol{\varepsilon}_\mu^\lambda(k)\boldsymbol{\varepsilon}_\nu^\lambda(k) \end{aligned} \tag{10.23}$$

The polarization tensors are not in a convenient form, and since they refer to a particular frame of reference, they are not covariant. That is the price of working in Coulomb gauge. Here is an argument that addresses both issues. We construct a set of orthonormal vectors based on the coordinate system in which the polarization vectors are defined. In this frame they are perpendicular to $\boldsymbol{k}$, and since they have no time component $\varepsilon_\mu k^\mu = 0$ (regardless of the time component of k). We introduce a time-like unit vector $\eta = (\ 1 \quad 0 \quad 0 \quad 0\ )$, $\eta^\mu\varepsilon_\mu = 0$, but $\eta_\mu k^\mu \neq 0$. We complete our set of four unit vectors by projecting out the part of k^μ along η^μ.

$$\hat{k}^\mu = \frac{k^\mu - (k\cdot\eta)\eta^\mu}{\sqrt{(k\cdot\eta)^2 - k^2}} \tag{10.24}$$

The four unit vectors then are η, ε^1, ε^2, and $\hat{k}$. If you think of them as column matrices,

$$\eta\eta^\dagger - \varepsilon^1\varepsilon^{1\dagger} - \varepsilon^2\varepsilon^{2\dagger} - \hat{k}\hat{k}^\dagger = g,$$

where g is the usual metric tensor. Writing this as a tensor equation gives,

$$\sum_{\lambda=1}^{2}\varepsilon_\mu^\lambda(k)\varepsilon_\nu^\lambda(k) = -g_{\mu\nu} + \eta_\mu\eta_\nu - \hat{k}_\mu\hat{k}_\nu. \tag{10.25}$$

The usual four-vector argument goes like this: equation (10.25) is valid in one particular reference frame. $g_{\mu\nu}$ and $\hat{k}_\mu\hat{k}_\nu$ are legitimate four-tensors. *If*

$\eta_\mu\eta_\nu$ were also, then (24) would be valid in all reference frames. Unfortunately, it isn't, so it's not. Let's go ahead and substitute (10.24) into (10.25) anyhow.

$$\sum_{\lambda=1}^{2}\varepsilon_\mu^\lambda(k)\varepsilon_\nu^\lambda(k) = -g_{\mu\nu} - \frac{k_\mu k_\nu}{(k\cdot\eta)^2 - k^2} + \frac{(k\cdot\eta)(k_\mu\eta_\nu + k_\nu\eta_\mu)}{(k\cdot\eta)^2 - k^2} - \frac{k^2\eta_\mu\eta_\nu}{(k\cdot\eta)^2 - k^2} \tag{10.26}$$

This is not covariant, but when the photon propagator is used in a actual calculation it is always coupled to a conserved current. Eq. (10.5) written in momentum space is $k_\mu j^\mu = 0$. All the terms above with uncontracted k's vanish! The last term is something of an embarrassment. When substituted into (10.23) it gives the term

$$-\eta_\mu\eta_\nu\frac{\delta(t-t')}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}$$

This is the instantaneous Coulomb interaction. It is a well known artifact of Coulomb gauge in classical E&M.[6] It can't be right, of course. Even the Coulomb interaction doesn't propagate instantaneously through space. In this case it is cancelled by another spurious term in the Hamiltonian.[7] The remaining term in (10.26) gives the correct propagator,

$$D_F(x'-x)_{\mu\nu} = g_{\mu\nu}\int\frac{d^4k}{(2\pi)^4}\frac{e^{-ik(x'-x)}}{k^2+i\epsilon}. \tag{10.27}$$

In momentum space, this is simply,

$$D(k) = \frac{g_{\mu\nu}}{k^2+i\epsilon} \tag{10.28}$$

This argument guarantees that so long as we use (10.27) or (10.28) in our calculations, the resulting S matrix elements and Green's functions will have the correct covariant behavior, even though the quantization was done in Coulomb gauge.

[6] See Jackson, Sec. 6.5 and references therein.

[7] See M. Kaku, *Quantum Field Theory, Section 4.3*

Chapter 11

Quantum Electrodynamics

11.1 Gauge Invariance

Before we can do any realistic calculations, we must figure out how the electromagnetic field couples to electrons. We could guess this as follows: we know that the classical electromagnetic Lagrangian has the term $-j^\mu A_\mu$, so we assume that the interaction Hamiltonian must be of the form $j^\mu A_\mu$. Furthermore, the electron probability current is $j^\mu = \bar{\psi}\gamma^\mu\psi$, so the charge current must be $e\bar{\psi}\gamma^\mu\psi$. We conclude that $\mathcal{H}_{\text{int}} = e\bar{\psi}\gamma^\mu\psi A_\mu$. This is true, but there is a profound principle here that we should linger over.

The free-particle Dirac Lagrangian is $\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$. If we make the transformation $\psi \to e^{iq\theta}\psi$, then $\mathcal{L} \to \mathcal{L}$. This is called a "global gauge transformation." It's not what we usually think of as a gauge transformation in E&M, but they are related as you will see. Try an infinitesimal transformation $\delta\psi = iq\delta\theta\psi$. If the Lagrangian is to remain invariant under this transformation we must have

$$\delta\mathcal{L} = 0 = \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta(\partial_\mu\psi) \tag{11.1}$$

$$= iq\delta\theta\left[\frac{\delta\mathcal{L}}{\delta\psi}\psi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\partial_\mu\psi\right]$$

Here θ is an infinitesimal and q is something like a charge. The Euler-Lagrange equation of motion is

$$\frac{\delta\mathcal{L}}{\delta\psi} = \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)} \tag{11.2}$$

Substituting (11.2) into (11.1) gives

$$iq\delta\theta\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\psi\right]=0 \tag{11.3}$$

Evidentally, the quantity in brackets is a conserved current.

$$j^\mu\propto\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\psi \tag{11.4}$$

and $\partial_\mu j^\mu=0$. Applying this formula to the Dirac Lagrangian gives $j_\mu=\bar{\psi}\gamma^\mu\psi$, which we knew before.

There is a stronger statement to be made if we make a different transformation at each point in space, *i.e.* if we make a local transformation.

$$\psi'=e^{iq\theta(x)}\psi. \tag{11.5}$$

The function $\theta(x)$ is arbitrary. Now

$$\partial_\mu\psi'=e^{iq\theta}\partial_\mu\psi+iq(\partial_\mu\theta)e^{iq\theta}\psi$$

Now $\mathcal{L}$ is not invariant by itself, but it can be made invariant by incorporating an additional field with special transformation properties. Let's insist that we replace $\partial_\mu\psi$ everywhere with $(\partial_\mu-ieqA_\mu)\psi$. If the following statement is true, the electron Lagrangian will be invariant:

$$(\partial_\mu-ieqA'_\mu)\psi'=e^{iq\theta}(\partial_\mu-ieqA_\mu)\psi,$$

but

$$(\partial_\mu-ieqA'_\mu)\psi'=e^{iq\theta}\partial_\mu\psi+iq(\partial_\mu\theta)e^{iq\theta}\psi-ieqA'_\mu e^{iq\theta}\psi$$

We conclude that

$$A'_\mu=\frac{1}{e}\partial_\mu\theta+A_\mu. \tag{11.6}$$

In other words, the electron Lagrangian will be invariant under the transformation (11.5) if our new fields transform according to (11.6), but these new fields must also appear in our Lagrangian, and this Lagrangian must be invariant under the transformation

$$A_\mu\rightarrow A_\mu+\frac{1}{e}\,\partial_\mu\theta(x). \tag{11.7}$$

Equation (11.7) is of course, the usual statement of gauge invariance in classical E&M, but now we see it in a larger context; quantum gauge invariance, (11.5), *requires the existence on an additional vector field.*

We know that the Lagrangian for classical E&M (excluding sources) is

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Transformation (11.7) does leave this invariant, but if this were the Lagrangian for a massive field, it would contain an additional term

$$m^2 A^{\dagger\mu} A_\mu$$

which would not be invariant. We conclude that local gauge invariance requires a *massless* vector field.

You will notice that the "charge" q cancels in the above calculation. As a consequence the sign and magnitude of e are undetermined. As a convention we usually write

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - e\gamma^\mu A^\mu - m)\psi \tag{11.8}$$

so that

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = e\bar{\psi}\gamma^\mu \psi = j^\mu A_\mu \tag{11.9}$$

11.2 Noether's Theorem

Zee calls Noether's theorem, "one of the most profound observations in theoretical physics." The theorem simply states that every continuous symmetry transformation is connected with a conserved current and hence, with some sort of conserved "charge." The converse is also true, for every conserved current, there is a corresponding symmetry group. The proof is a simple generalization to that given above. Given the Lagrangian density composed of N fields φ_i and their derivatives, $\mathcal{L} = \mathcal{L}(\varphi_i, \partial_\mu \varphi_i)$. Consider an infinitesimal change, $\delta\varphi_i$, brought about by some small parameter or parameters $\delta\epsilon_j$, and calculate the action.

$$\delta S = \sum_{i=1}^{N} \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \varphi_i} \, \delta\varphi_i + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_i)} \delta(\partial_\mu \varphi_i) \right)$$

$$= \sum_{i=1}^{N} \int d^4x \left(\frac{\delta \mathcal{L}}{\delta \varphi_i} \, \delta\varphi_i + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_i)} \partial_\mu \delta\varphi_i \right) \tag{11.10}$$

(The repeated μ indices are summed.) We can use the N E-L equations of motion

$$0 = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_i)} \right] - \frac{\delta \mathcal{L}}{\delta \varphi_i}$$

to combine the two terms in (11.10).

$$\delta S = \sum_{i=1}^{N} \int d^4x \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_i)} \frac{\partial \varphi_i}{\partial \epsilon_j} \delta \epsilon_j \right) \tag{11.11}$$

If the action is unchanged by this variation, we can define a current

$$J_j^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi_i)} \frac{\partial \varphi_i}{\partial \epsilon_j} \tag{11.12}$$

Equation (11.11) says that $\partial_\mu J_j^\mu = 0$. From this conserved current, we can also create a conserved charge by integrating over the time component of the current:

$$Q_j \equiv \int d^3x \; J_j^0 \tag{11.13}$$

Now integrate the conservation equation:

$$0 = \int d^3x \; \partial_\mu J_j^\mu = \int d^3x \; \partial_0 J_j^0 + \int d^3x \; \nabla \cdot \mathbf{J}_j$$

$$= \frac{d}{dt} Q_j + \int_s d\mathbf{S} \cdot \mathbf{J}_j$$

In the last term I have used the divergence theorem to convert the volume integral into a surface integral. Assume that the fields vanish at infinity so the last term can be neglected. Then:

$$\frac{d}{dt} Q_j(t) = 0 \tag{11.14}$$

In summary: a continuous symmetry implies a conserved current, which in turn implies a conserved charge. It's important to understand that the "charge" so defined usually has no connection with electrical charge. Most of the conserved quantities in particle physics, such as strangeness, charm, lepton number, *etc.* come about in this way. Many theories in particle physics came about because the conserved quantities were first discovered experimentally. Theorists then reasoned "backwards" from this discovery to construct the underlying conserved currents.

11.3 Feynman's Rules for QED

The derivation of Feynman's rules for electrons and photons proceeds along the same lines as the derivation for scalar particles in Chapter 7. Because fermion fields anticommute, there are some additional minus signs to keep track of. The time-ordered product was defined in (9.103)

$$T[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)] = \psi_\alpha(x_1)\overline{\psi}_\beta(x_2)\theta(x_1^0 - x_2^0) - \overline{\psi}(x_2)_\beta\psi(x_1)_\alpha\theta(x_2^0 - x_1^0) \tag{11.15}$$

Decompose the field into positive- and negative-energy parts. $\psi = \psi^+ + \psi^-$. Remember that ψ^+ contains the electron annihilation operator $\hat{b}_k$ and ψ^+ contains the positron creation operator $\hat{d}_k^\dagger$. Rewrite (11.15) in such a way that all creation operators stand to the right of all annihilation operators.

$$\begin{aligned} &T[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)] \\ &= \{\psi_\alpha^+(x_1), \overline{\psi}_\beta^+(x_2)\}\theta(x_1^0 - x_2^0) - \{\overline{\psi}_\beta^-(x_2), \psi_\alpha^-(x_1)\}\theta(x_2^0 - x_1^0) \\ &-\overline{\psi}_\beta^+(x_2)\psi_\alpha^+(x_2) + \psi_\alpha^+(x_1)\overline{\psi}_\beta^-(x_2) + \psi_\alpha^-(x_1)\overline{\psi}_\beta^+(x_2) + \psi_\alpha^-(x_1)\overline{\psi}_\beta^-(x_2) \end{aligned} \tag{11.16}$$

You see that the second line of this equation is a c number and the third, a q number. We call the c-number part the "contraction" of the two fields. The q-number part is "normal ordered" in the sense that all creation operators stand to the left of all annihilation operators, but because of the additional minus sign we must amend the definition of normal ordering somewhat.

Definition: The fields $: \psi(x_1)\psi(x_2)\cdots\psi(x_n) :$ are normal ordered if in each term, all creation operators stand to the left of all annihilation operators. Each term is positive unless it required on odd number of permutations to bring the various creation and annihilation operators into that order.

In a way analogous to (7.67), we introduce the notion of the contraction of two fields.

$$\overbrace{\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)} \equiv T[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)] - : \psi_\alpha(x_1)\overline{\psi}_\beta(x_2) : \tag{11.17}$$

Since the vacuum expectation value of normal-ordered fields vanishes, we have

$$\overbrace{\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)} = \langle 0|T[\psi_\alpha(x_1)\overline{\psi}_\beta(x_2)]|0\rangle = i(S_F)_{\alpha\beta}(x_1 - x_2), \tag{11.18}$$

where I have borrowed the definition of the propagator from (9.104). It's clear that

$$\overbrace{\psi(x_1)\psi(x_2)} = \overbrace{\overline{\psi}(x_1)\overline{\psi}(x_2)} = 0 \tag{11.19}$$

An additional complication arises when there are more than two fields to time order. In this case we must include a minus sign for each operator interchange. For example, (compare with (7.68))

$$\begin{aligned} T[\psi(x_1)\psi(x_2)\overline{\psi}(x_3)\overline{\psi}(x_4)] =& :\psi(x_1)\psi(x_2)\overline{\psi}(x_3)\overline{\psi}(x_4): \\ &- \overbrace{\psi(x_1)\overline{\psi}(x_3)}:\psi(x_2)\overline{\psi}(x_4): + \overbrace{\psi(x_1)\overline{\psi}(x_4)}:\psi(x_2)\overline{\psi}(x_3): \\ &+ \overbrace{\psi(x_2)\overline{\psi}(x_3)}:\psi(x_1)\overline{\psi}(x_4): - \overbrace{\psi(x_2)\overline{\psi}(x_4)}:\psi(x_1)\overline{\psi}(x_3): \\ &- \overbrace{\psi(x_1)\overline{\psi}(x_3)}\,\overbrace{\psi(x_2)\overline{\psi}(x_4)} + \overbrace{\psi(x_1)\overline{\psi}(x_4)}\,\overbrace{\psi(x_2)\overline{\psi}(x_3)} \end{aligned} \tag{11.20}$$

Wick's theorem enables us to calculate the vacuum expectation value of any product of boson and fermion fields. The next step toward calculating the S matrix requires a fermion version of the LSZ reduction formulas developed in Section 7.3. The derivation is similar to that leading to (7.43). There are no new ideas, just a lot of niggling details regarding minus signs. I will simply state the results and refer the reader to the classic exposition in Bjorken and Drell.[1]

In analogy with (7.30) we define the fermion in-field

$$\lim_{t\to-\infty}\psi(x) = \sqrt{Z_2}\,\psi_{\text{in}}(x) \tag{11.21}$$

The constant Z_2 is called the electron wave function renormalization constant. There are three such constants in QED. One for the electron wave function, one for the photon, and one for the electric charge. If it were not for the fact that all Feynman diagrams with closed loops give divergent results, we would be able to calculate them. As it is, we must be sure that they do not appear in any final results. As it turns out, the photon renormalization constant can be set equal to unity and the other two can be gathered into the definition of the physical mass and charge of the electron. This is to say that QED is *renormalizable.* One of the key ideas to come out of this is that renormalizability is a very special property possessed by only a few theories, and non-renormalizable theories are wrong!

Define the plane-wave solutions of Dirac's equation as follows.

$$\begin{aligned} U_{ks}(x) &= \sqrt{\frac{m}{(2\pi)^3 E_k}}\,u(k,s)e^{-ikx} \\ V_{ks}(x) &= \sqrt{\frac{m}{(2\pi)^3 E_k}}\,v(k,s)e^{ikx} \end{aligned} \tag{11.22}$$

[1] *Relativistic Quantum Fields*, J. D. Bjorken and S. D. Drell, McGraw-Hill 1965, Section 16.9

In this notation the second-quantized Dirac field is

$$\psi_{\text{in}}(x) = \int d^3k \sum_s [b_{\text{in}}(k,s)U_{ks}(x) + d^\dagger_{\text{in}}(k,s)V_{ks}] \tag{11.23}$$

There are several formulas analogous to (7.41) depending on whether we remove a particle or antiparticle from the in-state or out-state. If we remove a particle from the in-state (refer to Section 7.3 for the signification of the notation)

$$\begin{aligned} S_{\beta\alpha} &= \langle \beta \text{ out}|\alpha \text{ in}\rangle \\ &\sim -\frac{i}{\sqrt{Z_2}} \int d^4x \, \langle \beta \text{ out}|\overline{\psi}(x)|\alpha - k \text{ in}\rangle \overleftarrow{(-i\partial\!\!\!/ - m)} U_{ks}(x) \end{aligned} \tag{11.24}$$

Removing an antiparticle from the in-state leads to

$$\sim \frac{i}{\sqrt{Z_2}} \int d^4x \, \overline{V}_{ks} \overrightarrow{(i\partial\!\!\!/ - m)} \langle \beta \text{ out}|\overline{\psi}(x)|\alpha - k \text{ in}\rangle \tag{11.25}$$

Removing a particle from the out-state leads to

$$\sim -\frac{i}{\sqrt{Z_2}} \int d^4x \, \overline{U}_{ks} \overrightarrow{(i\partial\!\!\!/ - m)} \langle \beta - k \text{ out}|\overline{\psi}(x)|\alpha \text{ in}\rangle U_{ks}(x) \tag{11.26}$$

Removing an antiparticle from the out-state gives

$$\sim \frac{i}{\sqrt{Z_2}} \int d^4x \, \langle \beta - k \text{ out}|\overline{\psi}(x)|\alpha \text{ in}\rangle \overleftarrow{(-i\partial\!\!\!/ - m)} V_{ks}(x) \tag{11.27}$$

After two or more particles have been "reduced," the fields inside the bra-ket combination are time ordered.

Finally, we need to be able to reduce photon fields. In this case the asymptotic condition is

$$\lim_{t\to-\infty} \mathbf{A}(\mathbf{x},t) = \sqrt{Z_3}\mathbf{A}_{\text{in}}(\mathbf{x},t) \tag{11.28}$$

The constant Z_3 will turn out to renormalize the electron charge. The wave function is

$$\mathbf{A}_{k\lambda} = \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{-ikx} \epsilon(k,\lambda) \tag{11.29}$$

so that the Fourier expansion of $\mathbf{A}_{\text{in}}(x)$ is

$$\mathbf{A}_{\text{in}} = \int d^3k \sum_\lambda [a_{\text{in}}(k,\lambda)\mathbf{A}_{k\lambda}(x) + a^\dagger_{\text{in}}(k,\lambda)\mathbf{A}^*_{k\lambda}(x)] \tag{11.30}$$

The reduction formulas are virtually identical to (7.43) (with $m = 0$ of course) except for one minus sign.

$$\begin{aligned} S_{\beta\alpha} &= \langle \beta \text{ out} | \alpha \text{ in} \rangle \\ &\sim \frac{-i}{\sqrt{Z_3}} \int d^4x \, \langle \beta \text{ out} | A_\mu(x) | \alpha - (k,\lambda) \rangle \overleftarrow{\Box} A^{\mu *}_{k\lambda}(x) \end{aligned} \tag{11.31}$$

The additional minus sign in (11.31) comes from the space-like nature of the polarization unit vector

$$\epsilon_\mu \epsilon^\mu = -\boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon} = -1$$

This would be a good point to summarize the results so far. According to (8.5) the S-matrix can be rewritten

$$S_{fi} = \delta_{fi} - i(2\pi)^4 \delta^{(4)}(P_f - P - i)\mathcal{T}_{fi} \tag{11.32}$$

The δ_{fi} stands for all those terms appearing in the reduction procedure in which the two initial particles don't scatter.[2] The $(2\pi)^4\delta(P_f - P_i)$ appears when we do the last integral over coordinate space. The interesting part is $\mathcal{T}_{fi}$, which is the crucial ingredient in (8.20) and (8.21) to calculate cross sections and decay rates. To calculate the S-matrix, we start with the basic QED Hamiltonian (11.9) and substitute it into the formula for the S-operator (7.69). This is an infinite series. We keep only those terms corresponding to the desired degree of approximation. These terms are substituted into (**??**)

$$S_{\beta\alpha} = \langle \beta \text{ out} | S | \alpha \text{ in} \rangle. \tag{11.33}$$

In this context, α stands for the specific initial state you have set up in your experiment, and β represents the particular final state whose probability you wish to calculate. We then use (11.24), (11.25), (11.26), (11.27), and (11.31) to reduce out the various electron and photon fields. Finally, do all the implied differentiation and integration.

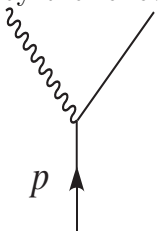
This is a "cookbook" procedure. No problem-solving ability is required. You should be able to do it while in a coma. You have to have patience though, even a modest calculation can involve, implicitly at least, tens of thousands of terms. After you have done a few such calculations, you will realize that you are doing the same few things over and over. These things can be boiled down to a few simple rules known as Feynman's rules.

[2] In a real experiment, the vast majority of incident beam particles don't scatter. All this business about calculating cross sections has to do with the infinitesimal fraction that does.

1. Draw all possible connected, topologically distinct diagrams, including loops, with n external legs. Ignore vacuum-to-vacuum graphs. Each vertex must have a continuous electron or positron line and a photon line terminating on the fermion.

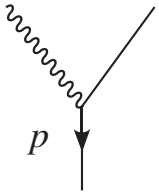
2. There is a factor given by the following rules for each external line.

Incoming fermion:



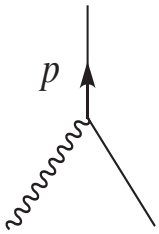
$$= \frac{u(p,s)}{\sqrt{Z_2(2\pi)^3 2E_p}}$$

Incoming antifermion:



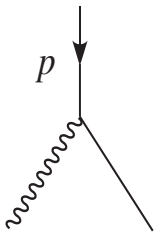
$$= \frac{\overline{v}(p,s)}{\sqrt{Z_2(2\pi)^3 2E_p}}$$

Outgoing fermion:



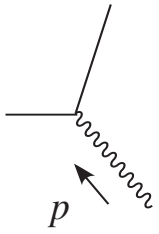
$$= \frac{\overline{u}(p,s)}{\sqrt{Z_2(2\pi)^3 2E_p}}$$

Outgoing antifermion:

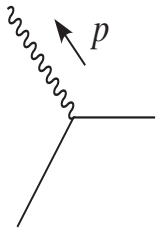


$$= \frac{v(p,s)}{\sqrt{Z_2(2\pi)^3 2E_p}}$$

Incoming photon:



$$= \frac{\epsilon^{\mu}(p)}{\sqrt{Z_3(2\pi)^3 2E_p}}$$

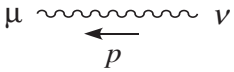
Outgoing photon: 

$$= \frac{\epsilon^{\mu *}(p)}{\sqrt{Z_3 (2\pi)^3 2E_p}}$$

3. For each internal line there is a propagator

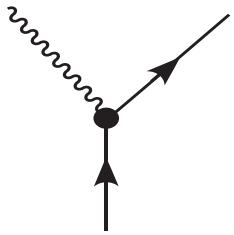
Electron propagator: 

$$= \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$$

Photon propagator: 

$$= \frac{-ig_{\mu\nu}}{p^2 + i\epsilon}$$

4. At each vertex, place a factor of $-ie\gamma^\mu$.

Electron-photon vertex: 
$$= -ie\gamma^\mu$$

5. Insert an additional factor of -1 for each closed fermion loop.

6. For each internal loop, integrate over:

$$\int \frac{d^4 q}{(2\pi)^4} \tag{11.34}$$

7. A relative factor -1 appears between graphs that differ from each other by an interchange of two identical external fermion lines.

8. Internal fermion lines appear with arrows in both clockwise and counterclockwise directions. However. diagrams that are topologically equivalent are counted only once.

Remember that the electron-photon coupling has the form $e\overline{\psi}\gamma^\mu \psi A_\mu$. As a consequence, the spinors associated with any vertex must appear in the order $\overline{u}\gamma^\mu u$ (with v replacing u as needed).

With this in mind, let's do some examples.

11.4 The Reaction $e^- + e^+ \to \mu^- + \mu^+$

Consider the reaction $e^- + e^+ \to \mu^- + \mu^+$. In terms of momentum, that is $p + p' \to k + k'$, and in terms of spin indices we have s and $s' \to r$ and r'. The momentum of the exchanged photon is $q = p + p' = k + k'$. The muons are almost identical to electrons, *i.e.* they have the same electromagnetic interactions, but they are heavier by a factor of 200. This simplifies the problem in several ways. For one thing, we are usually entitled to neglect the mass of the electron. For another, there are no identical particles in this example. Feyman's rules give

$$iM = \frac{ie^2}{q^2} \left(\overline{v}_e(p')\gamma^\mu u_e(p)\right) \left(\overline{u}_\mu(k)\gamma_\mu v_\mu(k')\right) \tag{11.35}$$

I have added subscripts e and μ to the spinors to indicate that they contain different masses. The spin indices are implicit. I will get back to spins in a minute.

In order to obtain a differential cross section from M we must do three difficult things: square it, sum over final spins, and average over initial spins. In order to make the notation more compact, I will write (11.35) as

$$iM = \frac{ie^2}{q^2} a^\mu b_\mu \tag{11.36}$$

In order to calculate $|M|^2$, we will need to figure out

$$|ab|^2 = a^\mu a^{\nu\dagger} b_\mu b_\nu^\dagger \tag{11.37}$$

Notice that a and b are 1×1 matrices in spin space and four-vectors in Minkowski space. There are two useful consequences; they all commute and complex conjugation and Hermitian conjugation have the same effect. This allows us to write

$$a^{\nu\dagger} = u^\dagger \gamma^{\nu\dagger} \gamma^{0\dagger} v = u^\dagger \gamma^0 \gamma^\nu v = \overline{u}\gamma^\nu v \tag{11.38}$$

With that insight (11.37) becomes

$$|M|^2 = \frac{e^4}{q^4} \left(\overline{v}(p')\gamma^\mu u(p)\overline{u}(p)\gamma^\nu v(p')\right) \left(\overline{u}(k)\gamma_\mu v(k')\overline{v}(k')\gamma_\nu u(k)\right) \tag{11.39}$$

Now make use of the spin information. The differential cross section will be proportional to

$$\frac{1}{2}\sum_s \frac{1}{2}\sum_{s'} \sum_r \sum_{r'} |M(s, s', r, r')|^2 \tag{11.40}$$

Remember from a previous chapter

$$\sum_s u(p,s)\overline{u}(p,s) = \frac{\not{p}+m}{2m} \tag{11.41}$$

$$\sum_s v(p,s)\overline{v}(p,s) = \frac{\not{p}-m}{2m}$$

Let's do the spin sums on the first term of (11.39)

$$\sum_{s,s'} \overline{v}(p',s')\gamma^\mu u(p,s)\overline{u}(p,s)\gamma^\nu v(p',s') \tag{11.42}$$

$$= \frac{1}{2m}\sum_{s'} \overline{v}_a(p',s')\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}v_d(p',s')$$

I apologize for the notational overload. An object like γ^μ is a 4×4 matrix in spin space. I have turned it in to a tensor by adding the matrix indices a and b. Tensors commute as you know, so (11.42) can be rewritten

$$= \frac{1}{2m}\sum_{s'} v_d(p',s')\overline{v}_a(p',s')\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}$$

Now v and $\overline{v}$ are in the right relation to one another so that we can use (11.41)

$$= \left(\frac{1}{2m}\right)^2 (\not{p}'-m)_{da}\gamma^\mu_{ab}(\not{p}+m)_{bc}\gamma^\nu_{cd}$$

Now this is recognizable as the trace of a product of matrices.

$$= \left(\frac{1}{2m}\right)^2 \mathrm{Tr}\left[(\not{p}'-m)\gamma^\mu(\not{p}+m)\gamma^\nu\right]$$

Finally combining (11.39), (11.40), and (11.42), we get

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|M|^2 = \frac{1}{4}\left(\frac{e}{2mq}\right)^4 &\mathrm{Tr}\left[(\not{p}'-m_e)\gamma^\mu(\not{p}+m_e)\gamma^\nu\right] \\ &\times \mathrm{Tr}\left[(\not{p}'-m_\mu)\gamma_\mu(\not{p}+m_\mu)\gamma_\nu\right]\end{aligned} \tag{11.43}$$

There you have it. QED calculations eventually come down to taking traces.

11.4.1 Trace Technology

The things we are taking traces of are all 4×4 matrices, even when they don't look like that. This results in some paradoxical formulas like Tr $1 = 4$ and Tr $m = 4m$. You can check from the representation of the gamma matrices that $\mathrm{Tr}\gamma^\mu = 0$. Traces of an even number of γ's are generally not zero. Here is a simple result.

$$\mathrm{Tr}(\not{a}\ \not{b}) = a_\mu b_\nu \mathrm{Tr}(\gamma^\mu\gamma^\nu) = \frac{1}{2}a_\mu b_\nu \mathrm{Tr}\{\gamma^\mu, \gamma^\nu\} = a_\mu b_\nu \mathrm{Tr}(g^{\mu\nu}) = 4a \cdot b$$

I have used the important fact that the trace of a product of matrices such as $\mathrm{Tr}(abc\cdots z)$ is invariant under any cyclic permutation of the matrices. Thus in the equation above $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = \mathrm{Tr}(\gamma^\nu\gamma^\mu)$. Our result can be summarized

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu} \tag{11.44}$$

The trace of an odd number of γ's always vanishes. The proof requires a trick. Remember the γ^5 matrix. It has the properties

$$\mathrm{Tr}(\gamma^5) = 0 \qquad (\gamma^5)^2 = 1 \qquad \{\gamma^5, \gamma^\mu\} = 0$$

Now the line of argument goes

$$\mathrm{Tr}(\not{a}_1\ \not{a}_2 \cdots\ \not{a}_n) = \mathrm{Tr}(\not{a}_1\ \not{a}_2 \cdots\ \not{a}_n\gamma^5\gamma^5) = \mathrm{Tr}(\gamma^5\ \not{a}_1\ \not{a}_2 \cdots\ \not{a}_n\gamma^5)$$
$$= (-1)^n \mathrm{Tr}(\not{a}_1\ \not{a}_2 \cdots\ \not{a}_n\gamma^5\gamma^5) = (-1)^n \mathrm{Tr}(\not{a}_1\ \not{a}_2 \cdots\ \not{a}_n)$$

The trace is zero for n odd.

There are many other identities that are easily proved. In a moment I will use

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) \tag{11.45}$$

which is obtained from repeated application of (11.44). Others can be found in any standard text. At the moment I am looking at Peskin and Schroder, *Introduction to Quantum Field Theory* Appendix A3.

11.4.2 Kinematics

A short calculation using (11.45) yields

$$\mathrm{Tr}\left[(\not{p}' - m)\gamma^\mu(\not{p} + m)\gamma^\nu\right] = 4[p'^\mu p^\nu + p'^\nu p^\mu - g^{\mu\nu}(p \cdot p' + m_e^2)] \tag{11.46}$$

We can set $m_e^2 = 0$. Then (11.43) can be evaluated by using (11.46) twice.

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^4}[(p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) + m_\mu^2(p \cdot p')] \tag{11.47}$$

Equation (11.47) is valid in any Lorentz frame. This is fine as far as it goes, but in order to compare the result with an experiment, we need to specialize to a definite frame of reference. Such an experiment is typically done with a colliding beam machine for which the laboratory is the CM frame. Let us assume then that $k = (E, \boldsymbol{k})$, $k' = (E, -\boldsymbol{k})$, $p = (E, E\hat{\boldsymbol{z}})$ and $p' = (E, -E\hat{\boldsymbol{z}})$. The angle between $\boldsymbol{k}$ and the z axis is θ. The following kinematic identities follow with $m_e = 0$.

$$q^2 = (p + p')^2 = 4E^2 \qquad p \cdot p' = 2E^2$$

$$p \cdot k = p' \cdot k' = E^2 - E|\mathbf{k}|\cos\theta \qquad p' \cdot k = p \cdot k' = E^2 + E|\mathbf{k}|\cos\theta$$

Eqn. (11.47) can be rewritten

$$\frac{1}{4}\sum_{\text{spins}} |M|^2 = e^4 \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta \right] \tag{11.48}$$

Back in Chapter 8 I derived the relationship between M and the differential cross section. Since then we have introduced the Dirac spinors, which must be normalized differently from the Kline-Gordon fields. I will therefore state (without proof) the appropriate version of ?? is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{2E_A E_B |\mathbf{v}_A - \mathbf{v}_B|} \frac{|\mathbf{p}_1|}{(2\pi)^2 4E_{CM}} |M(p_A + p_B \to p_1 + p_2)|^2 \tag{11.49}$$

This holds in any coordinate system in which generic particles A and B scatter into a final state consisting of two particles with momenta p_1 and p_2. In our case particles A and B are collinear and assumed massless. In this case

$$|\mathbf{v}_A - \mathbf{v}_B| = \left| \frac{\mathbf{k}_A}{E_A} - \frac{\mathbf{k}_B}{E_B} \right| = 2 \tag{11.50}$$

In our example the M in (11.49) is replaced by the spin-averaged M of (11.48).

11.5 Introduction to Renormalization

Every Feynman diagram with one or more closed loops yields a divergent integral.[3] One of the most important discoveries of twentieth-century physics

[3] There are some exceptions in the case of spin-zero particles, but it's universally true in qed.

is that these divergent terms contain real physical content that is part of the "right answer" to any higher-order calculation. The program of extracting useful information from meaningless infinities is called *renormalization.* This is an extremely technical and complicated subject, but I would like to expose you to some of the basic ideas. To do this, I will concentrate on a subset of divergent diagrams that go under the heading of electron self-energy.

Consider the quantum process in which a single photon of momentum k is emitted and reabsorbed by a virtual electron with initial and final momentum p. In the interim between emitting and reabsorbing the photon the electron has momentum $q = p - k$. According to Feynman's rules, Section 11.3, the amplitude for this process is

$$\begin{aligned} & iS_F(p) \int \frac{d^4k}{(2\pi)^4}(-ie\gamma^\mu)\left(\frac{-ig_{\mu\nu}}{k^2+i\epsilon}\right)\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon}(-ie\gamma^\nu)iS_F(p) \\ & = iS_F(p)\left[-ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\gamma^\mu(m+\not{p}-\not{k})\gamma_\mu}{(-k^2-i\epsilon)(m^2-(p-k)^2-i\epsilon)}\right]S_F(p) \\ & \equiv iS_F(p)\Sigma(p)S_F(p) \end{aligned} \tag{11.51}$$

The last equality implicitly defines the divergent quantity Σ, know as a "self-energy insertion." It is clearly divergent, since there are five powers of k in the numerator and only four in the denominator. We will eventually evaluate it, but for the time being I would like to investigate some of its general properties. Such as term would arise in perturbation theory as a second-order correction to the electron propagator. We could define a "more nearly exact" propagator

$$iS'_F(p) \approx iS_F(p) + iS_F(p)\Sigma(p)S_F(p)$$

I say "more nearly exact" meaning that $iS'_F(p)$ contains all the second-order corrections that would arise in the usual perturbation expansion of the propagator. We could make an even better approximation by including another self-energy insertion.

$$iS'_F \approx iS_F + iS_F\Sigma S_F + iS_F\Sigma S_F\Sigma S_F$$

This is not quite correct, even to fourth order, since I have ignored the diagram in which the electron emits another photon before reabsorbing the first. Diagrams of this sort are called "overlapping divergences." Such diagrams present a huge additional complication, but "sufficient unto the day is the

evil thereof." I will ignore them. My conclusions will still be qualitatively correct.

I can go on adding diagrams like this *ad infinitum*. The sum would look like this.

$$iS[1 + \Sigma S + \Sigma S \Sigma S + (\Sigma S)^3 + \cdots] \tag{11.52}$$

(I have suppressed the F subscript and well as the (p) to make the equations easier to read.) This looks like the familiar geometric series $(1-x)^{-1} = 1 + x^2 + x^3 + \cdots$, so I am tempted to write

$$iS' = iS\frac{1}{1 - \Sigma S} \tag{11.53}$$

The question is, what does it mean to divide by a 4×4 matrix? You can regard (11.53) as notational shorthand for (11.52). Here are some further observations.

- It's clear from (11.52) that Σ has the form

  $$\Sigma = mA(p^2) + \not{p}B(p^2) \tag{11.54}$$

  It's true that A and B are infinite, but let's ignore that for the time being. Furthermore

  $$S = \frac{\not{p} + m}{p^2 - m^2}$$

  (I will ignore the $i\epsilon$'s for the time being also. I don't want to distract you with details. I promise I haven't forgotten them and will put them back when necessary.) It follows that $[S, \Sigma] = 0$. Therefore the following forms are equivalent.

  $$S' = S\frac{1}{1 - \Sigma S} = \frac{1}{1 - \Sigma S}S = \frac{S}{1 - \Sigma S} \tag{11.55}$$

- Since $\not{p}\,\not{p} = p^2$ it makes sense to write in the same spirit

  $$S = \frac{\not{p} + m}{p^2 - m^2} = \frac{\not{p} + m}{(\not{p} - m)(\not{p} + m)} = \frac{1}{\not{p} - m}, \tag{11.56}$$

  and finally

  $$S^{-1} = \not{p} - m. \tag{11.57}$$

  The propagator can now be rewritten

  $$iS' = \frac{iS}{1 - \Sigma S}\frac{S^{-1}}{S^{-1}} = \frac{i}{\not{p} - m - \Sigma} \tag{11.58}$$

Leaving aside the fact that Σ is infinite, there is still a strange pathology in (11.57). Every propagator we have encountered so far has had a pole at the mass of the particle. It is almost axiomatic that a propagator is the probability amplitude for the particle to propagate from x to y *such that* $p^2 = m^2$ on the mass shell. It must be that the series of self-energy terms we have summed *has modified the mass of the particle.* We have to confess that at least in the context of perturbation theory, the m that appears in the Hamiltonian is not the physical mass of the particle. We call the m in the Hamiltonian the "bare" mass and the mass that marks the pole of the exact propagator, the "dressed" or physical mass, which I will call $\overline{m}$. It seems that given the bare mass, we should be able to calculate the physical mass or vice versa. The fact that Σ is infinite makes this permanently impossible. We have not faced up to this problem so far, because we have always worked at the lowest non-trivial order of perturbation theory. To that order we can assume the the bare and physical masses are identical, and everything works out OK. As soon as we include the self-energy terms, we must keep track of both m and $\overline{m}$. Worse yet, the same sort of thing happens when we calculate higher-order corrections to the electron-photon vertex. There we find that the charge e put into the Hamiltonian is not the physical charge of the electron but some "bare" charge, and again we must keep track of the e's and $\overline{e}$'s.

The agenda of keeping track of these corrections goes by the name "renormalization." It is a remarkable fact that all the infinite corrections brought about by all loop diagrams to all orders of perturbation theory can swept under the rug of these two constants. A theory for which this is possible is said to be "renormalizable." It is an article of faith that no non-renormalizable theory can be completely right, however useful it might be phenomenologically. Fermi's theory of weak interactions, which we will study in the next chapter, is a good example. It explains nuclear beta decay quite nicely, but it would require an infinite number of infinite constants to take care of all the loops! It was eventually modified by the addition of the intermediate vector bosons. The resulting theory is called the "standard model." It is renormalizable and right! It is also possible to do quantum field theory with gravitons. The theory at least predicts Newton's law of gravitation. To this extent it is correct. It is non-renormalizable, however, and we have no idea what the true theory might be.

It is necessary to rewrite (11.58) in such a way that the physical mass appears in the denominator. To this end we expand the self energy term as

follows.

$$\Sigma(\not{p}) = \Sigma(\overline{m}) + (\not{p} - \overline{m})\Sigma'(\overline{m}) + (\not{p} - \overline{m})^2 R(p^2) \tag{11.59}$$

This is the key equation, and it deserves some comment.

- It looks like a power series expansion about the point $\not{p} = m$, but in fact, $\not{p}$ can never equal m, since m is diagonal and $\not{p}$ never is. Rather, (11.59) defines implicitly what is meant by $R(p^2)$. I will sometimes write it

$$\Sigma(\not{p}) = \Sigma(\overline{m}) + (\not{p} - \overline{m})\Sigma'(\overline{m}) + \Sigma_R(\not{p}) \tag{11.60}$$

  In which case it defines Σ_R. By construction $\Sigma_R(\overline{m}) = 0$.

- It appears that in the term Σ' we have differentiated with respect to a matrix! That is not so strange as it seems. The point is that $p^2 = \not{p}^2$. For example, the $A(p^2)$ term in (11.54) can be differentiated as follows.

$$A'(\overline{m}) = \left.\frac{dA(p^2)}{d\ \not{p}}\right|_{\not{p}=\overline{m}} = \left.\frac{dA(p^2)}{dp^2}\frac{d\ \not{p}^2}{d\ \not{p}}\right|_{\not{p}=\overline{m}} = 2\overline{m}\left.\frac{dA(p^2)}{dp^2}\right|_{p^2=\overline{m}^2} \tag{11.61}$$

Strictly speaking, $\Sigma(\overline{m})$ and $\Sigma'(\overline{m})$ are both infinite, so the operations in (11.60) and (11.61) are not well defined. Our strategy for giving them a precise meaning works as follows. I will introduce an additional parameter ϵ into (11.51) in such a way the the integrals converge for $\epsilon > 0$. It will turn out that $A(p^2)$ and $B(p^2)$ in (11.54) can each be split into two parts. One part will depend on ϵ. It becomes infinite as $\epsilon \to 0$, but for $\epsilon \neq 0$ it is a well-behaved mathematical expression. The other part will depend on p^2 but not on ϵ. It is finite, calculable, and important. I am about to show that all the epsilon-dependent terms can be lumped together with the bare mass to give the physical mass $\overline{m}$ and a new epsilon-dependent term Z_2 called the wave function renormalization factor, which cancels out in any complete calculation of a Feynman diagram. The point is that all these calculations are done with finite (though epsilon-dependent) terms, and when the time comes to calculate some physical quantity such as a cross section or reaction rate, *there are no epsilon-dependent terms remaining.*

Now substitute (11.60) into (11.58)

$$iS' = \frac{-i}{m + \Sigma(\overline{m}) - \not{p} + (\not{p} - \overline{m})\Sigma'(\overline{m}) + \Sigma_R} \tag{11.62}$$

I looks like the "real" mass is $\overline{m} = m + \Sigma(\overline{m})$. We need one further constant.[4]

$$Z_2^{-1} \equiv 1 - \Sigma'(\overline{m}) \tag{11.63}$$

Equation (11.62) can be rewritten

$$iS' = \frac{i(\not{p} + \overline{m})}{p^2 - \overline{m}^2} \frac{Z_2}{1 + (\overline{m} - \not{p}) Z_2 R(p^2)} \tag{11.64}$$

The looks like the "bare" propagator iS (with m replaced by $\overline{m}$) multiplied by a factor, which, if Z_2 were not epsilon-dependent, would be finite and calculable. Before we set $\epsilon = 0$, however, Z_2 is finite, and we can show why it does not appear in the final result of any complete calculation of a Feynman diagram. First consider the Z_2 in the denominator of (11.64). You can see from (11.51) that Σ is proportional to e^2. We must confess again the the e that appears in the Hamiltonian is not the measured charge of the electron, but rather a "bare" charge, which is modified by all the loop diagrams to all orders of perturbation theory. Part of this modification is due to Z_2. Let's tentatively define a "dressed" charge, $\overline{e} = \sqrt{Z_2} e$. Then Z_2 disappears from the denominator because it is absorbed into Σ. The Z_2 disappears from the numerator when we realize that our virtual electron will not be observable until it couples to a photon at each end. (Draw some Feynman diagrams to convince yourself this is true.) Each electron-photon vertex comes with a factor of e, so the entire diagram is proportional to $e^2 Z_2 = \overline{e}^2$.

Now I will evaluate the very difficult integral (11.51). Notice that the denominator is the product of two factors that can be labelled

$$A = m^2 - (p - k)^2 - i\epsilon \qquad B = -k^2 - i\epsilon$$

Feyman realized that a four-dimensional integral can be simplified by making it five-dimensional. The trick is

$$\frac{1}{AB} = \int_0^1 \frac{dz}{[Az + B(1 - z)]^2} \equiv \int_0^1 \frac{dz}{D^2} \tag{11.65}$$

The term D can be simplified by completing the square. Define $k^\mu \equiv k'^\mu + zp^\mu$. Then

$$D = C^2 - k'^2 \qquad \text{where} \qquad C^2 \equiv z[m^2 - p^2(1 - z)]$$

[4] This is standard notation. There are other Z's, but since we are only dealing with the electron self-energy part, these other constants don't arise.

With these substitutions, the numerator of (11.51) becomes

$$N(p,z) = \gamma^\mu[m + \not{p}(1-z) - \not{k}']\gamma_\mu$$

I can delete the $\not{k}'$ term using the familiar argument that an odd function integrated over even limits yields zero. Using the fact that $\gamma^\mu\gamma_\mu = 4$, gives us the final form of the numerator function.

$$N = 2[2m - \not{p}(1-z)] \tag{11.66}$$

With all these definitions the self-energy term becomes

$$\Sigma(p) = -i\bar{e}^2 \int_0^1 dz N(p,z) \int \frac{d^4k}{(2\pi)^4} \frac{1}{(C^2 - k^2 - i\epsilon)^2} \tag{11.67}$$

The integral can be done for integer dimension d using the remarkable formula[5]

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(C^2 - k^2 - i\epsilon)^n} = \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{C^2}\right)^{n-d/2} \tag{11.68}$$

Here Γ is the Euler gamma function defined by

$$\Gamma(\alpha) \equiv \int_0^\infty dt\, t^{\alpha-1} e^{-t} \tag{11.69}$$

If α is an integer, say n, $\Gamma(n) = (n-1)!$, and in general $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$. It's useful to get Maple to plot this function for you. You will see that Γ interpolates smoothly between the integer values of α. You will also see that $\Gamma(0) = \infty$. We expected that. The integral (11.67) has to diverge. Now of course, the integral on the left side of (11.68) only makes sense when d is an integer, but the expression on the right *interpolates smoothly between integer values of d*. Even if d is only infinitesimally less than four, it is still finite. Now define $\epsilon \equiv 4 - d$. For our purposes, $n = 2$.

$$\frac{\Gamma(n - d/2)}{\Gamma(n)} = \frac{\Gamma(3 - d/2)}{2 - d/2} = \frac{2}{\epsilon}$$

$$\Sigma(p) = \frac{2\bar{e}^2}{(4\pi)^2} \int_0^1 dz N \frac{C^{-\epsilon}}{\epsilon} \tag{11.70}$$

Insert the expansion

$$C^{-\epsilon} = 1 - \epsilon \ln C + O(\epsilon^2)$$

[5]This is proved in Gross, *Relativistic Quantum Mechanics and Field Theory*, page 344

$$\Sigma(p) = \frac{2\overline{e}^2}{(4\pi)^2}\int_0^1 dz N(\not{p}, z)\left(\frac{1}{\epsilon} - \ln C(p^2, z)\right) + O(\epsilon) \tag{11.71}$$

The integrals in (11.71) are easy enough to do, although the resulting expressions are not very illuminating. The important point is that $\Sigma(\not{p})$ can be divided up as follows.

$$\Sigma(\not{p}) = (mB_1 + \not{p}B_2)/\epsilon + \tilde{\Sigma}(\not{p}) \tag{11.72}$$

where $\tilde{\Sigma}(\not{p})$ does not contain ϵ and so remains finite in the limit $\epsilon \to 0$. B_1 and B_2 are simple constants. When we expand $\Sigma(\not{p})$ using (11.60), $\Sigma_R(\not{p})$ is as promised, finite. Thus we can absorb all the infinite terms into the two constants $\overline{m}$ and Z_2, and calculate $\Sigma_R(\not{p})$and $R(p^2)$ which are, again as promised, finite and important.

Chapter 12

Weak Interactions

The field of nuclear physics began with the study of radioactive decay. There were three identifiable processes called α- β- and γ-decay corresponding to the emission of alpha particles, electrons, and photons respectively. In retrospect, they manifest the strong, weak, and electromagnetic forces, the three forces that operate at the level of nuclei and particles. Gravity is too weak to have any effect, and there has never been any convincing evidence of a "fifth force."

Consider some generic nucleus that undergoes β-decay, say ${}^N X_Z$. By 1930 the process was believed to be[1] ${}^N X_Z \to {}^N X_{Z+1} + e^-$. Unfortunately, the energy spectrum of the emitted electron looks something like the sketch in Fig. 1. If there were only two particles in the final state, all the electrons should have the energy marked $E_{\max}$. Several wild hypotheses were put forward to explain this including the suggestion that energy and/or momentum were not conserved in these reactions. In 1930 W. Pauli suggested that β-decay was in fact a three-body reaction, ${}^N X_Z \to {}^N X_{Z+1} + e^- + \bar{\nu}$. The $\bar{\nu}$ was a hitherto unknown particle, which Pauli called the "neutron." The neutron as we know it is a constituent of nuclei, so the name was later changed to "neutrino." The particle would have to have zero charge and spin $\hbar/2$ to conserve charge and angular momentum. So far as one could tell from the endpoint energy, it had to have a very small mass. Since it has proved very difficult to detect, it must interact weakly with ordinary matter.[2] Now we know that there are actually three kinds of neutrinos all

[1] There is a similar decay in which a positron is emitted. This is harder to recognize, however, because the positron immediately captures an electron forming positronium. This decays quickly into two or three photons.

[2] I have written $\bar{\nu}$ rather than ν, because in this case, the particle is actually an antineutrino. The distinction between neutrinos and antineutrinos is a subtle matter, to

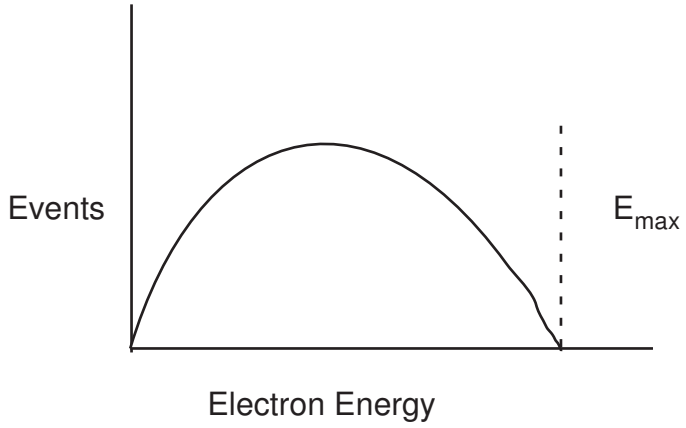


Figure 12.1: The β-decay energy spectrum

with very small masses, but for the time being I will consider only one kind, the so-called electron neutrino and make the very good approximation that it is massless.[3]

It is ironic that whereas the radioactive decay of nuclei was the first evidence of the operation of nuclear forces, it is also so very complicated that no first-principles calculation could ever reproduce the lifetimes and branching ratios of these decays. Nuclei are very complicated objects, and the strong, electromagnetic, and weak forces are inextricably entangled in all of these processes. The "scientific method" dictates that we should understand the simplest instances first. These turn out to be the purely leptonic decays such as $\mu \to e + \nu + \bar{\nu}$ where the strong interactions play no role. I would like to postpone that reaction for a bit and follow a more historical line of development. We know that once outside of a nucleus, a neutron will decay, $n \to p + e^- + \bar{\nu}$ in about 13 minutes. This process is complicated slightly by the effect of the strong interactions, but no nuclear physics is involved, and this was the first decay to be studied quantitatively.

By the mid 1950's it was believed that neutron decay as well as all the other weak interactions known at the time could be described by the interaction Hamiltonian

$$\mathcal{H}_{int} = \frac{G}{\sqrt{2}} J^{\alpha}(x) J^{\dagger}_{\alpha}(x) \tag{12.1}$$

which I will return later.

[3] "Sufficient unto the day is the evil thereof."

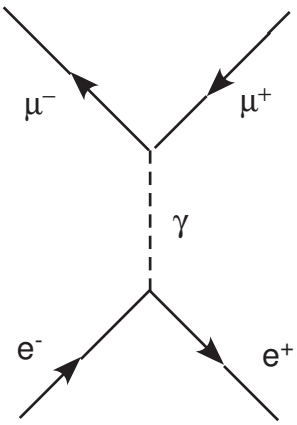


Figure 12.2: $e^+ + e^- \to \mu^+ + \mu^-$ via one photon exchange

where

$$G = 1.166 \times 10^{-5} \text{GeV}^{-2} \tag{12.2}$$

is the universal weak coupling constant or Fermi constant. This is modelled after the electromagnetic interactions that we have already studied. For example the process $e^+ + e^- \to \mu^+ + \mu^-$ shown in Fig. 2 can be described by the matrix element

$$\mathcal{M}_{int} = -ie(\bar{v}_e \gamma^\alpha u_e) D_{\alpha\beta} (\bar{u}_\mu \gamma^\beta v_\mu) = -ie J_e^\alpha D_{\alpha\beta} J_\mu^\beta \tag{12.3}$$

where $D_{\alpha\beta}$ is the photon propagator. Fermi suggested that protons and neutrons (collectively called nucleons) and electrons and neutrinos (collectively called leptons) could also constitute currents just as the electrons and muons do in (12.3). The weak current J^α consists of a leptonic part l^α and a hadronic part h^α,

$$J^\alpha(x) = l^\alpha(x) + h^\alpha(x). \tag{12.4}$$

The are three different leptons, the e, μ and τ, so l^α consists of a sum of three separate currents. Likewise there are many hadrons that undergo weak decay, so in principle h^α should be a sum of of all the corresponding currents. In this way (12.1) describes many different processes. There will be purely leptonic process such as muon decay, semileptonic processes like neutron decay, and many purely hadronic processes such as $\Lambda \to p + \pi$ and $K \to \pi + \pi$. It is now known that the weak interactions are mediated by the exchange of vector bosons just as the electromagnetic interactions are mediated by the exchange of the photon. The bosons, however, are

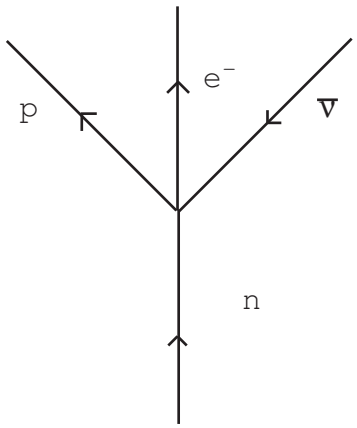


Figure 12.3: Feynman diagram for neutron decay

extremely heavy, so for ordinary nuclear processes, the propagator collapses into a single constant that is absorbed into G. Weak decays therefore are essentially point-like interactions. Neutron decay, for example, is described by the Feynman diagram Fig 13.3.

Equation (12.1), the so-called "current-current interaction," had its successes, but there is much more to the story. For one thing, (12.1) does not allow the nucleon's spin to flip, at least not in the nonrelativistic limit. The spin frequently does flip in nuclear β-decay, so other interaction terms must be considered. It is also non-renormalizable so it is at best a low-energy approximation.

12.1 Parity Violation

By far the most profound modification of (12.1), however, has to do with parity. Until 1956 it was assumed that parity was conserved in all interactions. Roughly speaking, this means that the mirror image of any allowed process is also an allowed process with all the same reaction rates. In more mathematical terms, there is an operation called parity conjugation, which has two effects: first it replaces x^μ with $x'^\mu = (ct, -\boldsymbol{x})$, and second, it replaces every wave function or quantum field $\psi(x)$ with a new function $\psi'(x')$ that describes the particle or system in the mirror-reflected world. The statement that parity is conserved is equivalent to saying that this transformation leaves all the equations of motion form invariant. In 1956

Lee and Yang pointed out that there was no evidence that this was true for the weak interactions. It was shown shortly thereafter that weak interactions violate this principle *maximally.*[4] This was done by polarizing ^{60}Co nuclei in a strong magnetic field. ^{60}Co β-decays emitting an electron. It was found that the intensity distribution of the emitted electrons could be described by the formula

$$I(\theta) = 1 - \langle \boldsymbol{J} \rangle \cdot \boldsymbol{p}/E = 1 - v\cos\theta \tag{12.5}$$

Here $\boldsymbol{J}$ is the nuclear spin and $\boldsymbol{p}$ the momentum of the electron. The important point about (12.5) is that the distribution is not symmetric between the forward (with respect to the nuclear spin or magnetic field) and backward direction. This in itself is proof that parity is not conserved. To see how this follows, imagine viewing the experiment in a mirror held perpendicular to the nuclear spin. Since angular momentum has the form $\boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{p}$, it does not change sign under parity conjugation. Its mirror image is identical to the real thing. It is an example of a pseudo- or axial vector. The electron distribution, however, does change. In this world it is peaked in the direction opposite to $\boldsymbol{J}$. In the mirror world it is peaked along $\boldsymbol{J}$. Ergo, parity is not conserved. Parenthetically, ^{60}Co also undergoes γ-decay, and the photon distribution is symmetric between the forward and backward directions. This is because electromagnetic interactions do conserve parity.

That was a simple argument, but incorporating parity into the theory of relativistic electrons requires a some formalism. Our postulate of form invariance requires that the Dirac equation in the mirror reflected world should be

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0 \tag{12.6}$$

Multiply the ordinary Dirac equation on the left by γ^0.

$$\gamma^0(i\gamma^\mu \partial_\mu - m)\psi(x) = (i\gamma^\mu \partial'_\mu - m)\gamma^0\psi(x) = 0 \tag{12.7}$$

In order for (12.6) to be true, it must be that

$$\psi'(x') = \gamma^0\psi(x) \tag{12.8}$$

In Chapter 9.4 we touched briefly on the five bilinear forms that can be constructed with Dirac spinors. I showed that $\bar{\psi}\psi$ transforms like a scalar under Lorentz transformations and that $\bar{\psi}\gamma^\mu\psi$ transforms like a vector. I also introduced the γ^5 matrix and made some vague remarks to the effect

[4] I will explain eventually what I mean by this

that it had something to do with parity. For easy reference, here are its definition and properties.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{12.9}$$

$$(\gamma^5)^\dagger = \gamma^5 \tag{12.10}$$

$$(\gamma^5)^2 = 1 \tag{12.11}$$

$$\{\gamma^5, \gamma^5\} = 0 \tag{12.12}$$

Let's take a look at $\bar{\psi}\gamma^5\psi$. Under parity conjugation,

$$\bar{\psi}(x)\gamma^5\psi(x) \to \bar{\psi}'(x')\gamma^5\psi'(x') = \bar{\psi}(x)\gamma^0\gamma^5\gamma^0\psi(x) = -\bar{\psi}(x)\gamma^5\psi(x) \tag{12.13}$$

It is, as claimed, a pseudo scalar. In the same way it can be shown that $\bar{\psi}\gamma^\mu\gamma^5\psi$ is a pseudo- or axial vector.

How are we to modify (12.1) to accommodate parity violation? At first sight one could replace γ^α by $\gamma^\alpha\gamma^5$ making the currents into pseudovectors. But since the Hamiltonian is a product of two such currents, the minus signs cancel, and parity conjugation has no effect. In principle one could use all five of the bilinear forms in arbitrary linear combinations, and in the case of some nuclear β-decay, that possibility must be taken seriously. In order to understand the fundamental nature of weak interactions, however, it is useful to look at reactions in which there are no strong interactions to complicate things. There are a few purely weak decay processes, and the most accessible is ordinary muon decay, $\mu \to e + \nu + \bar{\nu}$. After much experimental work it is clear that the way to construct purely leptonic weak currents is as follows.

$$l^\mu = \bar{u}_e\gamma^\mu\tfrac{1}{2}(1-\gamma^5)v_\nu \tag{12.14}$$

This is called the V-A or vector minus axial vector interaction. This precise form is tied up with the properties of the neutrino to which we now turn.

12.2 Neutrinos

We know that there are three kinds of neutrinos called the electron neutrino, the muon neutrino, and the tau neutrino. There is considerable indirect evidence that they all have some small mass, which from a theoretical point of view is both a complication and an embarrassment. The neutrino that appears in (12.14) is the electron neutrino whose mass is a tiny fraction of an electron volt. Except for those experiments involving neutrinos from the sun or from outer space, this mass is completely negligible, and it will simplify our work considerably if we ignore it. In this limit, half of the neutrino

spinor is "missing," and the role of the $\frac{1}{2}(1-\gamma^5)$ in (12.14) is to "delete" it. In order to see what this means, it is useful to reconstruct the Dirac spinors according to a different plan.

The gamma matrices are defined by their anticommutation relations, but there are several different matrix representations that have the same relations. The choice given in Equations 9.6 and 9.7 is called the Dirac or standard representation. When spinors are constructed with this choice, the third and fourth components are proportional to $\boldsymbol{p}/(E+m)$ and so vanish in the low-energy limit. There is another important representation called the Weyl or chiral representation.[5]

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{12.15}$$

The important difference is that γ^0 is diagonal in the standard representation, and γ^5 is diagonal in the chiral representation.

The Dirac equation in the limit $m = 0$ and in momentum space is simply $\not{p}u = 0$. Using the representation (12.15) this is

$$\begin{bmatrix} 0 & E - \boldsymbol{\sigma}\cdot\boldsymbol{p} \\ E + \boldsymbol{\sigma}\cdot\boldsymbol{p} & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = 0. \tag{12.16}$$

The determinant of the coefficient matrix must vanish, and so $E = \pm|\boldsymbol{p}|$. Concentrate on the positive-energy solution for the time being. We are left with

$$\begin{aligned} \chi &= \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi \\ \phi &= -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\phi \end{aligned} \tag{12.17}$$

In this limit the upper and lower components decouple. The operator $\sigma \cdot \mathbf{p}$ represents the component of spin along the particle's direction of motion. Its eigenvalues are called *helicity*. According to (12.17), massless particles are helicity eigenstates. The spin is either aligned parallel to the direction of motion (positive helicity, right-handed particles) or antiparallel (negative helicity, left-handed particles).

Let us refine our notation somewhat and use $\lambda = \pm 1$ to represent the helicity. Introduce the helicity eigenstates defined by

$$\begin{aligned} \boldsymbol{\sigma}\cdot\boldsymbol{p}\,\phi_\lambda &= \lambda|\boldsymbol{p}|\phi_\lambda \\ \boldsymbol{\sigma}\cdot\boldsymbol{p}\,\chi_\lambda &= \lambda|\boldsymbol{p}|\chi_\lambda \end{aligned} \tag{12.18}$$

[5] Beware. There are several different sign conventions used by other authors.

Massless positive-energy spinors are then

$$u(\mathbf{p}, \lambda = 1) = \begin{pmatrix} 0 \\ \phi_+ \end{pmatrix} \qquad u(\mathbf{p}, \lambda = -1) = \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \tag{12.19}$$

The matrix γ^5 is called the *chirality* operator.[6] In the chiral representation the operator $\frac{1}{2}(1 \pm \gamma^5)$ has a simple structure.

$$\tfrac{1}{2}(1 + \gamma^5)u = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = \begin{bmatrix} 0 \\ \chi \end{bmatrix} \tag{12.20}$$

$$\tfrac{1}{2}(1 - \gamma^5)u = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi \\ \chi \end{bmatrix} = \begin{bmatrix} \phi \\ 0 \end{bmatrix}$$

Clearly the $\frac{1}{2}(1 \pm \gamma^5)$'s are acting like projection operators. Let's be brief and write $P_\pm \equiv \frac{1}{2}(1 \pm \gamma^5)$. These operate like any God-fearing projection operators. For example, $P_+^2 = P_+$, $P_+P_- = 0$, and $P_+ + P_- = 1$. The fact that u_ν in (12.14) is always multiplied by P_- means that only left-handed neutrinos interact.

$$P_- u_\nu(\lambda = 1) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad P_+ u_\nu(\lambda = -1) = \begin{pmatrix} \phi_- \\ 0 \end{pmatrix} \tag{12.21}$$

So far as one can tell from the form of the interaction, it might be that neutrinos come in two helicity states, but somehow only the left-handed states are allowed to participate. In fact, neutrinos *only* interact weakly and *always* with the $\frac{1}{2}(1 - \gamma^5)$ in front of the spinor, so whether or not there are positive helicity states is something of a metaphysical question.[7] It is these "missing" or non-interacting helicity states that are directly responsible for parity violation. If we take the neutrino spinor to be

$$P_- u_\nu = \begin{bmatrix} \phi_- \\ 0 \end{bmatrix} \tag{12.22}$$

then the parity conjugation operation (12.8) forces us to conclude that

$$P_- u'_\nu = P_- \gamma^0 u_\nu = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{12.23}$$

[6] The word "chirality" comes from the Greek meaning "hand." Nonetheless, the terms right- and left-handed refer to helicity not chirality.

[7] The fact that the left-handed rather than right-handed states interact was established in a classic experiment by Goldhaber et al (1958) described in Perkins, *Introduction to High Energy Physics*, Addison-Wesley 1987

When I said that parity was violated maximally, I meant that the right side of (12.23) is zero without any remaining small components.[8]

12.3 Chirality, Helicity and Electrons

This simple association of chirality with helicity in (12.19) does not hold for massive particles. The Dirac equation in our new basis including the mass term is

$$\begin{bmatrix} -m & E-\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ E+\boldsymbol{\sigma}\cdot\boldsymbol{p} & -m \end{bmatrix}\begin{bmatrix} \phi \\ \chi \end{bmatrix} = 0 \tag{12.24}$$

$$(E+\boldsymbol{\sigma}\cdot\boldsymbol{p})\phi = m\chi$$

$$(E-\boldsymbol{\sigma}\cdot\boldsymbol{p})\chi = m\phi$$

It's convenient to take ϕ and χ to be helicity eigenstates. Equations (12.24) become

$$\frac{1}{m}(E+\lambda|\boldsymbol{p}|)\phi_\lambda = \chi_\lambda \tag{12.25}$$

The complete spinors are

$$u(\lambda) = C\begin{bmatrix} \phi_\lambda \\ \frac{E+\lambda|\boldsymbol{p}|}{m}\,\phi_\lambda \end{bmatrix} = \frac{1}{\sqrt{2m}}\begin{bmatrix} \sqrt{E-\lambda|\boldsymbol{p}|}\,\phi_\lambda \\ \sqrt{E+\lambda|\boldsymbol{p}|}\,\phi_\lambda \end{bmatrix} \tag{12.26}$$

The normalization constant C is chosen according to our convention that $\bar{u}u = 1$. Now look at the relative sizes of the upper and lower components.

$$\frac{\sqrt{E-|\boldsymbol{p}|}}{\sqrt{E+|\boldsymbol{p}|}} = \frac{\sqrt{E^2-|\boldsymbol{p}|^2}}{E+|\boldsymbol{p}|} = \frac{m}{E+|\boldsymbol{p}|} \approx \frac{m}{2E} \tag{12.27}$$

which vanishes in the limit $m \to 0$ or equivalently the limit $|\boldsymbol{p}| \gg m$.[9] In this limit

$$P_- u(\lambda=1) = \frac{1}{\sqrt{2m}}\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{12.28}$$

$$P_- u(\lambda=-1) = \frac{1}{\sqrt{2m}}\begin{pmatrix} \sqrt{2E}\,\phi_- \\ 0 \end{pmatrix}$$

[8] Now you see why the neutrino mass is such a problem. The statement is not *quite* correct.

[9] It is still meaningful to take the limit $m \to 0$ despite the factor of $1/\sqrt{2m}$. The point is that all physical results depend on bilinear products like $\bar{u}u$. Because of our normalization condition, all the m's cancel.

The point is that for $|\boldsymbol{p}| \gg m$, the P_- in the interaction suppresses the positive helicity components by the factor $m/2E$. It is easy to show that if u_1 and u_2 are any two spinors,

$$\bar{u}_1\gamma^\mu P_- u_2 = P_-\bar{u}_1\gamma^\mu u_2 = P_-\bar{u}_1\gamma^\mu P_- u_2 \tag{12.29}$$

so that in this limit, only the left-handed components of all fermions participate in the interaction.

One can construct the spinors for antiparticles by repeating the procedure with E replaced by $-E$ in (12.16) and (12.24). Equation (12.17) becomes

$$\phi = -\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\phi \tag{12.30}$$
$$\chi = \boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\,\chi$$

The signs have changed. Massless righthanded antiparticles have negative chirality. *Only right-handed antineutrinos participate in weak interactions.* Equation (12.26) becomes

$$v(\lambda) = C\begin{bmatrix} -\frac{1}{m}(E+\lambda|\boldsymbol{p}|)\chi_\lambda \\ \\ \chi_\lambda \end{bmatrix} = \frac{1}{\sqrt{2m}}\begin{bmatrix} -\sqrt{E+\lambda|\boldsymbol{p}|}\chi_\lambda \\ \\ \sqrt{E-\lambda|\boldsymbol{p}|}\chi_\lambda \end{bmatrix}. \tag{12.31}$$

(Remember that $\bar{v}v = -1$.) For $|\boldsymbol{p}| \gg m$, the negative helicity components are suppressed by $m/2E$.

The missing helicity state in the neutrino wave function also causes charge conjugation symmetry to be violated. According to (12.20), neutrinos have negative helicity. Equation (9.59) shows that the charge-conjugate state must also have negative helicity, but we have just showed that the helicities of massless antiparticles are opposite to those of particles. Negative helicity antineutrinos don't exist, or if they do, they don't participate. The combination of parity conjugation and charge conjugation is a different story. The helicity operator $\sigma\cdot\mathbf{p}$ is odd under parity conjugation, so the helicity of a parity-conjugated state is opposite to that of the original state. Two minuses make a plus, right? Weak interactions are invariant under CP.

Well – almost. There is a "superweak" component to the weak interactions that does violate CP. It is so weak that it only manifests itself in the decays of the neutral K and B mesons, and then only because of a complicated set of fortuitus circumstances. CP violation is one of the most profound discoveries of particle physics in recent years. It is believed to be the origin of the preponderance of matter over antimatter in the universe.

12.4 Lepton Number

I mentioned (without explanation) that ordinary muon decay $\mu^- \to e^- + \nu + \bar{\nu}$, requires a neutrino and an antineutrino in the final state. I also stated (again without explanation) (12.14) that the lepton current should be $l^\mu = \bar{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5) v_\nu$ rather than say $l^\mu = \bar{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5) u_\mu$, both of which are allowed by Feynman's rules. How is one to know about these things?

The following reactions and decays all conserve energy and charge, but none have ever been observed.

$$\begin{aligned} \pi^0 &\not\to \mu^+ + e^- \\ \mu^- &\not\to e^- + \gamma \\ \mu^- &\not\to e^+ + e^- + e^- \\ \mu^- + p &\not\to p + e^- \\ \bar{\nu} + n &\not\to p + e^- \end{aligned} \tag{12.32}$$

The last reaction seems impossible to realize, but there's a trick. Park a swimming pool full of cleaning fluid in front of a nuclear reactor. Reactors produce mostly antineutrinos, so one looks for the reaction

$$\bar{\nu} + {}^{37}Cl \to {}^{37}A + e^-$$

This can be identified since ${}^{37}Ar$ is a radioactive gas that can be separated from the liquid and its radioactivity measured. This is a heroic experiment, but it's been done, and the reaction just doesn't occur. What is "wrong" with these reactions? The (almost) obvious guess is that these leptons carry some sort of additive quantum numbers that in (12.32) just don't add up. First of all, muons can't decay into electrons, so there must be some measure of "muonness" and "electronness" that must be conserved. As a convention, we say that negative electrons have an electron number = +1, and positrons have electron number = −1. Similarly, negative muons have muon number = +1, and positive muons have muon number = −1. (There is also a tau lepton, so it must carry a tau number assigned in the same way.) Reactions like $e^+ + e^- \to \gamma + \gamma$ are allowed, since γ's are not leptons, and so all their lepton numbers are zero. What are we to make of $\mu^- \to e^- + \bar{\nu} + \nu$? We have to acknowledge that neutrinos also carry lepton number and that all neutrinos are not created identical. There must be electron neutrinos that carry electron number just as electrons do, as well as muon neutrinos and tau neutrinos. The complete set of quantum numbers is given in Table 13.1. The correct way to specify muon decay is $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$. Now it's clear

	e^-	ν_e	μ^-	ν_μ	τ^-	ν_τ
n_e	1	1	0	0	0	0
n_μ	0	0	1	1	0	0
n_τ	0	0	0	0	1	1

Table 12.1: Standard lepton number assignments. Antiparticle lepton numbers have the opposite sign. All other particles have zero lepton number.

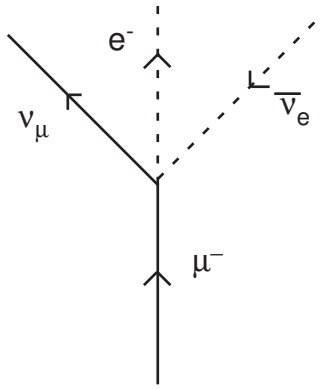


Figure 12.4: Muon decay $\mu^- \to e^- + \nu_\mu + \bar{\nu}_e$. The solid lines represent the flow of muon number current. The dotted lines represent electron number current.

that both sides of the reaction have $n_\mu = 1$ and $n_e = 0$.

Lepton numbers also resolve the question raised above regarding the construction of lepton currents. You will recall that the conservation of charge was a key requirement in constructing currents for electromagnetic currents. Take for example, the term $j_e^\alpha = \bar{v}_e \gamma^\alpha u_e$ in (12.5) illustrated in Figure 13.2. It represents an electron current that "flows" up the electron leg of the diagram and down the positron leg. Conservation of charge means that you can trace the current along this path following the arrows without lifting your pencil from the page. In the same way lepton number represents a kind of "charge" that must be conserved. In Figure 13.4 I have illustrated muon decay using solid lines to represent the flow of muon number and dotted lines to represent electron number current. The complete matrix

element for muon decay is then

$$\mathcal{M} = \frac{4G}{\sqrt{2}}\, \bar{u}_\nu \gamma^\alpha \tfrac{1}{2}(1-\gamma^5) u_\mu\, \bar{u}_e \gamma_\alpha \tfrac{1}{2}(1-\gamma^5) u_\mu \tag{12.33}$$

Chapter 13

The Electroweak Theory

13.1 Weak interaction of quarks

From the point of view of quarks, ordinary neutron decay, $n \to p + e^- + \bar{\nu}$, amounts to a process in which a down quark beta decays into an up quark and a lepton pair, $d \to u + e^- + \bar{\nu}$. When allowance is made for the differences in mass and phase space, the strength of this reaction is *almost* the same as the purely leptonic process $\mu \to e + \nu + \bar{\nu}$. We can construct weak V-A currents for these processes as follows,

$$j^\mu(\text{lepton}) \propto \bar{u}(\nu)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(e) \tag{13.1}$$

$$\propto \bar{\nu}_L \gamma^\mu e_L$$

$$j^\mu(\text{quark}) \propto \bar{u}(u)\gamma^\mu \tfrac{1}{2}(1-\gamma^5)u(d) \tag{13.2}$$

$$\propto \bar{u}_L \gamma^\mu d_L$$

Notice the condensed notation: $\bar{u}_L$ is the left-handed projection of the up quark. At this point I am not concerned with whether this is a u-type or a v-type spinor. Of course that is important when doing an exact calculation, but you have Feynman's rules to figure that out. Here I am trying to present an idea with a minimum of notational complications. By the same token, $\propto$ means "proportional to." The constant of proportionality will be all-important, but not now. Finally, notice that the projection operator, $\frac{1}{2}(1-\gamma^5)$ projects the left-handed components of both spinors as shown in class.

The currents in (13.1) and (13.2) are both "charged currents," in the sense that the currents change charge in the process of the interaction. Put it another way, they couple to charged intermediate vector bosons. Until the 1970's it was believed that there were no neutral current interactions,

since processes like $K_0 \to \mu^+\mu_-$ seem to be forbidden. The K_0 carries one unit of strangeness, however, so this is an example of a strangeness-changing interaction, and these are forbidden. With the advent of high-energy neutrino beams it became possible to see reactions such as $\nu_\mu + N \to \nu_\mu + X$, where N is some heavy nucleus, and X stands for some hadronic "stuff." In terms of quarks, these are examples of the reaction $\nu + q \to \nu + q$. (q stands for any quark.) There is no reason to believe that these interactions have the pure V-A form, so the correspondinq currents are

$$j^\mu(\text{leptonic neutral current}) \propto \bar{\nu}\gamma^\mu \tfrac{1}{2}(1-\gamma^5)\nu \tag{13.3}$$

$$j^\mu(\text{neutral hadronic current}) \propto \bar{q}\gamma^\mu \tfrac{1}{2}(c_V - c_A\gamma^5)q \tag{13.4}$$

The constants c_V and c_A have been determined experimentally with inelastic neutrino scattering. As a consequence of (13.4), weak neutral currents have both right- and left-handed components.

13.2 Weak Isospin

The electroweak theory per se does not involve quarks, nonetheless, the following facts established in the previous section are foundational.

- Weak leptonic currents and weak charged hadronic currents have only left-handed components.
- The weak, neutral, hadronic currents that they couple to have both right- and left-handed components.

Electrons, of course, also have both right- and left-handed components, but only the left-handed components participate in the weak interactions. *Or so it seems.* The underlying idea of the electroweak theory is that the weak and electromagnetic interactions are closely related, and the relationship is cemented by the marriage of the right-handed piece of the electron current and the right-handed piece of the weak, neutral, hadronic current.

To bring this marriage about I will define

$$J^+_\mu \equiv \bar{\nu}_L\gamma_\mu e_L \tag{13.5}$$

$$J^-_\mu = (J^+_\mu)^\dagger = \bar{e}_L\gamma_\mu\nu_L \tag{13.6}$$

The notation $J^+(J^-)$ means that the initial lepton gains (loses) one unit of charge in the course of the interaction. It does this by coupling to a negative

(positive) intermediate vector boson. It is customary to use the formalism of spin or isospin to combine these various currents. To this end, define the "spinor,"

$$\chi_{\rm L} = \begin{pmatrix} \nu \\ e^- \end{pmatrix}_L , \tag{13.7}$$

and raising and lowering operators,

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm \sigma_2) \tag{13.8}$$

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The σ_i's are the usual Pauli spin matrices. Then

$$J^i_\mu \equiv \bar\chi_{\rm L}\gamma_\mu \tfrac{1}{2}\sigma_i \chi_{\rm L} \qquad J^\pm_\mu \equiv \bar\chi_{\rm L}\gamma_\mu \sigma_\pm \chi_{\rm L} \tag{13.9}$$

This is familiar angular momentum physics in an unfamiliar context. Think of $\chi_{\rm L}$ as a Pauli spinor. We can combine two spin-1/2 doublets to make a spin-1 triplet and a spin-0 singlet. In this case, the underlying quantity is not spin but something conventionally called 'weak isospin." The currents $J^\pm_\mu$ are two components of the triplet. The third member is

$$J^3_\mu \equiv \bar\chi_{\rm L}\gamma_\mu \tfrac{1}{2}\sigma_3 \chi_{\rm L} = \tfrac{1}{2}\bar\nu_{\rm L}\gamma_\mu\nu_{\rm L} - \tfrac{1}{2}\bar e_{\rm L}\gamma_\mu e_{\rm L} \tag{13.10}$$

This J^3_μ is a weak neutral current, but it can't be the whole story, because the complete neutral weak current has right-handed components. The key idea is to split up the electron current into right- and left-handed pieces.

$$j^{\rm em}_\mu = -\bar e\gamma_\mu e = -\bar e\gamma_\mu \tfrac{1}{2}(1-\gamma^5)e - \bar e\gamma_\mu\tfrac{1}{2}(1+\gamma^5)e \tag{13.11}$$

$$= -\bar e_{\rm R}\gamma_\mu e_{\rm R} - \bar e_{\rm L}\gamma_\mu e_{\rm L}$$

Despite the fact that it contains electron spinors, $j^{\rm em}_\mu$ is a *neutral* current for reasons explained previously. Notice in this particular definition I have not included the electron charge. In (13.11) e is just the electron spinor. Suppose the "missing" right-handed components of the weak neutral current actually come from the right-handed components of the electromagnetic current. This is the idea, but it plays out in ways that are rather subtle.

We assume that in addition to charge and the third component of isotopic spin, there is a third "good quantum number" called *hypercharge* or Y. The three are related by

$$Q = T^3 + Y/2 \tag{13.12}$$

lepton	T	T^3	Q	Y
ν_e	$\frac{1}{2}$	$\frac{1}{2}$	0	-1
$e^-_{\rm L}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
$e^-_{\rm R}$	0	0	-1	-2

Table 13.1: Weak quantum number assignments

According to Noether's theorem, for every conserved charge there is a conserved current. The currents in this case must be related by

$$j^{\rm em}_\mu = J^3_\mu + \tfrac{1}{2} j^Y_\mu \tag{13.13}$$

The new current j^Y_μ is called the weak hypercharge current.[1] The quantum numbers are given in Table 12.1. Similar assignments can be made for the quarks.

13.3 The Intermediate Vector Bosons

In conventional QED, the interactions are mediated by the photon field A_μ and the basic interaction vertex is $-ieJ_\mu$. In electroweak theory there are four currents, the three components of J^i_μ and the hypercharge current j^Y_μ. In the conventional notation they couple to vector boson fields as follows

$$-ig\left(J^i\right)^\mu W^i_\mu - i\frac{g'}{2}\left(j^Y\right)^\mu B_\mu \tag{13.14}$$

The two new coupling constants g and g' will be related to the Fermi coupling constant and the electric charge later on. The fields W^i and B are assumed to be massless. They acquire mass through a mechanism called spontaneous symmetry breaking, which will be explained in a subsequent chapter. The so-called Weinberg angle θ_W, is another constant that must be determined from experiment. The particles that are detected in the laboratory are the photon which is massless, two heavy charged bosons, the $W^\pm$'s, and the even heavier neutral vector boson, the Z^0. Their fields are related to those in (13.15) as follows

$$W^\pm_\mu = \sqrt{\tfrac{1}{2}}(W^1_\mu \mp iW^2_\mu) \tag{13.15}$$

$$A_\mu = B_\mu \cos\theta_W + W^3_\mu \sin\theta_W \tag{13.16}$$

$$Z_\mu = -B_\mu \sin\theta_W + W^3_\mu \cos\theta_W$$

[1] There is a strong hypercharge also.

The neutral current piece of (13.14) is

$$-ig\left(J^3\right)^\mu W_\mu^3 - i\frac{g'}{2}\left(j^Y\right)^\mu B_\mu \tag{13.17}$$

$$= -i(g\sin\theta_W J_\mu^3 + g'\cos\theta_W j_\mu^Y/2)A_\mu - i(g\cos\theta_W J_\mu^3 - g'\sin\theta_W j_\mu^Y/2)Z^\mu$$

Since A_μ is the actual photon field, the current coupled to it must be $j_\mu^{\rm em}$ from (13.13). This can only be so if

$$g'\cos\theta_W = g\sin\theta_W = e \tag{13.18}$$

Whatever it is that couples to the Z^0 must be the complete neutral weak current, which I will call $J_\mu^{\rm NC}$. The last term in (13.17) can be rewritten

$$-ig\left[\cos\theta_W J_\mu^3 - \frac{\sin^2\theta_W}{\cos^2\theta_W}(j_\mu^{\rm em} - J_\mu^3)\right]Z^\mu = -i\frac{g}{\cos\theta_W}J_\mu^{\rm NC}Z^\mu \tag{13.19}$$

where

$$J_\mu^{\rm NC} \equiv J_\mu^3 - \sin^2\theta_W j_\mu^{\rm em} \tag{13.20}$$

As promised $J_\mu^{\rm NC}$ inherits its right-handed components from $j_\mu^{\rm em}$ and its left-handed components form J_μ^3 and $j_\mu^{\rm em}$.

13.4 The Feynman Rules

Now that the smoke has cleared, we are left with the following interactions.

- Two charged IVB's, the $W^\pm$'s, coupling to the weak charged current. The interaction term in (13.14) can be rewritten using (13.6), (13.9), and (13.15) as

$$-ig\left[(J^1)^\mu W_\mu^1 + (J^2)^\mu W_\mu^2\right] = -i\frac{g}{\sqrt{2}}(J^\mu W_\mu^+ + J^{\mu\dagger}W_\mu^-) \tag{13.21}$$

- One neutral IVB, the Z^0, coupled to $J^{\rm NC}$ with coupling constant $g/\cos\theta_W$. Typically we write the vertex function

$$-i\frac{g}{\cos\theta_W}\gamma^\mu\tfrac{1}{2}(c_V - c_A\gamma^5) \tag{13.22}$$

  The constants c_V and c_A can be determined from based on the charge and isotopic spin of the particles that make up $J^{\rm NC}$.

- The photon coupling to the electromagnetic current with the usual constant e.

The Feynman rules from Section 11.3 require the following additions.

1. At each vertex at which a $W^{\pm}$ couples to a charged current, insert the vertex factor
$$-i\frac{g}{\sqrt{2}}\gamma^{\mu}\tfrac{1}{2}(1-\gamma^{5}) \tag{13.23}$$
At each vertex at which a Z^0 couples to a neutral current, insert the vertex factor
$$-i\frac{g}{\cos\theta_W}\gamma^{\mu}\tfrac{1}{2}(c_V - c_A\gamma^{5}) \tag{13.24}$$

2. For each virtual W or Z insert the propagator
$$\frac{i(-g^{\mu\nu}+p^{\mu}p^{\nu}/M^2)}{p^2-M^2} \tag{13.25}$$

3. For each free W or Z insert the spin polarization tensor $\varepsilon_{\mu}(\lambda)$. When summing over the polarization state λ use the completeness relation
$$\sum_{\lambda}\varepsilon_{\mu}^{(\lambda)*}(p)\varepsilon_{\nu}^{(\lambda)}(p) = -g^{\mu\nu}+\frac{p^{\mu}p^{\nu}}{M^2} \tag{13.26}$$

The other rules are unchanged.

Chapter 14

Symmetry Breaking

Consider an atom with a spherically symmetric potential. Because of the symmetry, the energy eigenstates are independent of the m and m_s quantum numbers. Put it another way, the $SU(2)$ symmetry of the Hamiltonian causes these states to be degenerate. If we put the atom in an external magnetic field, the degeneracy is removed and the energies become functions of both m and m_s. The system is still symmetric about the direction of the field, so the imposition of the external field has broken the symmetry from $SU(2)$ down to $U(1)$. This sort of symmetry breaking is called the *Wigner Mode*.

There are several other ways to break symmetry that are more relevant to this chapter. Try standing a straight flexible rod upright and press down on the top of it with a force F. Let's assume that the rod remains straight for some minimal force, but beyond some threshold force F_c the rod "buckles," and assumes a curved shape. The symmetry is broken, the rod is no longer straight, but there are an infinite number of equivalent states corresponding to the possible orientations of the rod. We can summarize this as follows:

1. A parameter (in this case F) assumes some critical value. Beyond that value,
2. the symmetric situation becomes unstable, and
3. the ground state is degenerate.

The second example is the ferromagnet. The atoms in a ferromagnet interact through a spin-spin interaction

$$H = -\sum_{i,j} J_{ij} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

which is a scalar and therefore invariant under rotations. The *ground state* however, is one in which all the spins are aligned, and this is clearly not rotationally invariant. The direction of spontaneous magnetisation is random, and all the degenerate ground states may be reached from a given one by rotation. The spontaneous magnetisation disappears at high temperature, when the ground state becomes symmetric.

It is clear that the general situation here is the same as in our first example, the relevant parameter here being T. These two examples exhibit what is known as "spontaneous symmetry breaking. In both cases the system possesses a symmetry (rotation symmetry) but the ground state in not invariant under that symmetry; rather, it changes into one of the other (degenerate) ground states.

14.1 A Simple Example

Consider a classical self-interacting real scalar field ϕ with a Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 - \frac{1}{4}\lambda\phi^4 \tag{14.1}$$

We assume $\lambda > 0$ so that there is a finite minimum energy. This Lagrangian is invariant under the discrete transformation

$$\phi \to -\phi. \tag{14.2}$$

There are two quite different cases depending on the sign of μ^2. The potential for $\mu^2 > 0$ has a single minimum, $V(\phi) = 0$ at $\phi = 0$. This point corresponds to the ground state, so in quantum mechanical terms

$$\langle\phi\rangle_0 = \langle 0|\phi|0\rangle = 0$$

Expanding the Lagrangian about $\langle\phi\rangle_0$ to second order gives

$$\mathcal{L} \approx \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}\mu^2\phi^2 \tag{14.3}$$

which is the Lagrangian density of a free scalar field of mass μ. The perturbations induced by the ϕ^4 term can be thought of as oscillations about the origin.

Now consider the case $\mu^2 < 0$. Now the potential has minima at

$$\langle\phi\rangle_0 = \pm\sqrt{\frac{-\mu^2}{\lambda}} \equiv \pm v. \tag{14.4}$$

Now there are two degenerate vacuum states. The minima at $\langle\phi\rangle^0 = \pm v$ are equivalent and either may be chosen as the classical ground (vacuum) state of the system. Once this choice has been made, the vacuum is no longer invariant under the transformation (2). This is a new form of symmetry breaking: the Lagrangian is invariant but the vacuum state is not. This is called the *Goldstone mode*, spontaneous symmetry breaking, or hidden symmetry.

Let's choose

$$\langle\phi\rangle_0 = +v. \tag{14.5}$$

We can no longer do perturbation theory by expanding the Lagrangian about $\langle\phi\rangle_0 = 0$, since the slightest perturbation about this point would send the system plunging down into one of the two minima. It makes more sense to shift the field by defining

$$\xi(x) \equiv \phi(x) - \langle\phi\rangle_0 = \phi(x) - v. \tag{14.6}$$

In terms of this new variable the vacuum state is $\langle\xi\rangle_0 = 0$, and the Lagrangian density is (neglecting constant terms)

$$\mathcal{L} = \mathcal{L}_0 - \lambda v\xi^3 - \frac{1}{4}\lambda\xi^4 \tag{14.7}$$

We can think of the ξ^3 and ξ^4 terms as perturbations of the free-field Lagrangian

$$\mathcal{L}_0 = \frac{1}{2}(\partial\xi)^2 - \lambda v^2\xi^2 \tag{14.8}$$

which is the Lagrangian for a free scalar field with mass $m_\xi = \sqrt{-2\mu^2}$. (Remember that $\mu^2 < 0$.)

This is a toy model, of course, but it illustrates many of the features of spontaneous symmetry breaking as it manifests itself in quantum mechanics.[1]

1. There is a nonzero expectation value of some field in the vacuum state.
2. The resulting classical theory has a degenerate vacuum, with the choice among the equivalent vacuum states completely arbitrary.
3. The transition from a symmetric vacuum to a degenerate vacuum typically occurs as a phase transition as some order parameter (μ^2 in the example) is varied.

[1] I am copying almost verbatim from Mike Guidry, *Gauge Field Theories* John Wiley & Sons (1991)

4. The chosen vacuum state does not possess the same symmetry as the Lagrangian.

5. After we have expanded the Lagrangian around the chosen vacuum, the original symmetry is no longer apparent. The degenerate vacua are related to each other by symmetry operations, which tells us that the symmetry is still there, but it is not manifest; it is hidden.

6. The masses of the particles appearing in the theory with and without the spontaneous symmetry breaking may differ substantially. We say that the masses have been acquired spontaneously in the latter case.

7. Once the theory develops degenerate vacua the origin becomes an unstable point. Thus the symmetry may be "broken spontaneously" in the absence of external intervention.

14.2 Goldstone Bosons

The transformation (2) was discrete. The next generalization makes use of a continuous global gauge transformation. For this we will need a complex or non-Hermitian field. In previous chapters we used Hermitian scalar fields for spin zero particles. In terms of second quantization, this means that the particle is its own antiparticle. If the particle caries charge or some other quantum number that is different for particles and antiparticles, then the field must be non-Hermitian (as is the case with electron fields). Take the Lagrangian to be

$$\mathcal{L} = (\partial_\mu)^\dagger(\partial^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2 \tag{14.9}$$

The λ terms represents self interaction. Under normal circumstances, we would regard μ as a mass. Here μ^2 is just a parameter, since we are going to make $\mu^2 < 0$. This Lagrangian is invariant under the group of global gauge transformations,

$$\phi(x) \to \phi'(x) = e^{i\Lambda}\phi(x). \tag{14.10}$$

Using polar coordinates $\rho^2 = \phi^\dagger\phi$, we may identify a potential

$$V(\rho) = \mu^2\rho^2 + \lambda\rho^4. \tag{14.11}$$

As before, set $\mu^2 < 0$. The minima now occur in the complex ϕ plane on a circle of radius

$$|\phi|^2 = \frac{-\mu^2}{2\lambda} \equiv a^2 \tag{14.12}$$

The shape of the potential has been likened to a Mexican hat or perhaps the bottom of a champagne bottle. There is now an infinity of degenerate ground states, corresponding ro different positions on the ring of minima in the complex ϕ plane, and the symmetry operation (10) relates one to another. In the quantum theory, where ϕ becomes an operator, this condition refers to the vacuum expectation value of ϕ.

$$|\langle 0|\phi|0\rangle|^2 = a^2. \tag{14.13}$$

We could choose any point in the minimum and expand the Lagrangian about that point. For simplicity, we choose the real axis $Re(\phi) = a$, and expand around it to investigate the spectrum. It is convenient to work in polar "coordinates"

$$\phi(x) = \rho(x)e^{i\theta(x)}, \tag{14.14}$$

so the complex field ϕ is expressed in terms of two real scalar fields ρ and θ. In quantum language, we choose the vacuum state

$$\langle 0|\phi|0\rangle = a \tag{14.15}$$

where a is real; then

$$\langle 0|\rho|0\rangle = a, \qquad \langle 0|\theta|0\rangle = 0 \tag{14.16}$$

Now let us put

$$\phi(x) = [\rho'(x) + a]e^{i\theta(x)} \tag{14.17}$$

We regard ρ' and θ as "small" displacements from the point $\phi = a$. Substitute (17) into (9)and keeping only the quadratic terms gives

$$\mathcal{L} = (\partial_\mu\rho')(\partial^\mu\rho')+(\rho'+a)^2(\partial_\mu\theta)(\partial^\mu\theta)-4\lambda a^2\rho'^2+O(\rho'^3)+ \text{ constants} \tag{14.18}$$

The coefficient of the square of the field in a free-field Lagrangian is the mass squared of the corresponding particle. In this case the ρ' field has a mass given by

$$m^2_{\rho'} = 4\lambda a^2 \tag{14.19}$$

There is no term in θ^2, so θ is a massless field. As a result of spontaneous symmetry breaking, what would otherwise be two massive fields (the real and imaginary parts of ϕ), become one massive and one massless field. The difference between the two coordinates is easy to visualize. It requires no energy to displace a particle in the θ direction, but it does cost energy to displace ρ' against the restoring force of the potential. The θ particle is known

as a Goldstone boson. This phenomenon is general: spontaneous breaking of a continuous symmetry entails the existence of a massless particle. This is the Goldstone theorem.

Try the same thing in cartesian coordinates. Introduce two real fields ϕ_1 and ϕ_2.

$$\phi = a + \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \tag{14.20}$$

so that the vacuum expectation values of both fields are zero. Then

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - 2\lambda a^2 \phi_1^2 - \sqrt{2}\lambda \phi_1(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 \tag{14.21}$$

The ϕ_2 field is massless, but the ϕ_1 has a mass squared of $4\lambda a^2$, the same as (19).

There is an analogy here to spin waves in a ferromagnet. Spin waves are periodic variations in M. Because the forces in a ferromagnet are of short range, it costs very little energy excite such a wave. Such a wave could carry zero energy in the limit of infinite wavelength, *i.e.* it's a massless excitation.

14.3 A More Abstract Look

There is a profound result regarding symmetries known as Noether's theorem, to the effect that there is a conserved current associated with each generator of a continuous symmetry. The proof is simple: consider a Lagrangian composed of several fields, which we call ϕ_a. Our symmetry transformation changes each field ay an infinitesimal $\delta\phi_a$. since $\mathcal{L}$ does not change, we have

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\phi_a}\delta\phi_a + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}\delta(\partial_\mu\phi_a)$$

The equations of motion are

$$\frac{\delta\mathcal{L}}{\delta\phi_a} = \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}$$

so we can combine the two and get

$$0 = \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}\delta\phi_a \right) \tag{14.22}$$

If we define

$$J^\mu \equiv \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_a)}\delta\phi_a \tag{14.23}$$

Then (22) says that $\partial_\mu J^\mu = 0$. (We are also assuming that repeated a's are summed.) (We have already used this argument to get the electron current in Dirac theory.)

To make this more concrete, imagine that the N fields make up a multiplet,

$$\boldsymbol{\phi} \equiv \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_N \end{bmatrix},$$

and the Lagrangian contains terms like $(\partial\boldsymbol{\phi})^2 = (\partial_\mu\phi_a)(\partial^\mu\phi_a)$, and $\boldsymbol{\phi}^2 = \phi_a\phi_a$, arranged in the form

$$\mathcal{L} = \frac{1}{2}\left[(\partial\boldsymbol{\phi})^2 - m^2\boldsymbol{\phi}^2\right] - c_2(\boldsymbol{\phi}^2)^2 - c_3(\boldsymbol{\phi}^2)^3 - \cdots \tag{14.24}$$

(Unless otherwise stated, repeated indices are summed.) This Lagrangian will be invariant under a "rotation," $\phi_a \to R_{ab}\phi_b$, where R is a member of the rotation group $SO(N)$. (This is sometimes called an "internal symmetry.") We can write $R = e^{i\boldsymbol{\theta}\cdot\boldsymbol{X}}$ where the $\boldsymbol{X}^i$ are the group generators. Under a infinitesimal transformation $\phi_a \to R_{ab}\phi_b \approx (1+\theta^i X^i)_{ab}\phi_b$, or in other words, we have the infinitesimal change $\delta\phi_a = \theta^i X^i_{ab}\phi_b$. Equation (23) assures us that the $N(N-1)/2$ currents

$$J^i_\mu = (\partial_\mu\phi_a)X^i_{ab}\phi_b \tag{14.25}$$

are conserved. So there is one conserved current, J^i, for each of the N generators of the symmetry group.

For every conserved current there is a conserved "charge"

$$Q \equiv \int d^3x J^0 = \int d^3x \frac{\delta\mathcal{L}}{\delta(\partial_0\phi_a)}\delta\phi_a \tag{14.26}$$

We notice that $\delta\mathcal{L}/\delta(\partial_0\phi_a) = \pi_a$, the canonically conjugate field. The equal-time commutation relation then gives the elegant result

$$i[Q, \phi_a] = \delta\phi_a \tag{14.27}$$

We are now in a position to prove the Goldstone theorem. The charges defined by (26) are conserved, they are constant in time. Therefore,

$$[H, Q] = 0 \tag{14.28}$$

Let the vacuum (or ground state in quantum mechanics) be denoted by $|0\rangle$. We can always write $H|0\rangle = 0$. (The vacuum must be an eigenstate of the Hamiltonian. If it should happen that $H|0\rangle = c$, we could always redefine the Hamiltonian $H \to H - c$.) Normally the vacuum is invariant under the symmetry transformation, $e^{i\theta Q}|0\rangle = |0\rangle$, or in other words $Q|0\rangle = 0$.

In the previous examples, we had to choose a specific vacuum that was not invariant under the symmetry transformation; *i.e.* $Q|0\rangle \neq 0$. So what is this state, $Q|0\rangle$? Clearly, it's a zero-energy eigenstate of the Hamiltonian.

$$HQ|0\rangle = [H, Q]|0\rangle = 0 \tag{14.29}$$

I claim it corresponds to a massless particle. Consider the state

$$|\psi(\boldsymbol{k})\rangle = \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}} J^0(\boldsymbol{x}, t)|0\rangle \tag{14.30}$$

We make use of the fact that the momentum operator $\boldsymbol{P}$ satisfies

$$[\boldsymbol{P}, \phi(x)] = -i\boldsymbol{\nabla}\phi(x), \tag{14.31}$$

where $\phi(x)$ could be any field operator including $J(x)$. Then

$$\boldsymbol{P}|\psi(\boldsymbol{k})\rangle = \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{P} J^0(\boldsymbol{x}, t)|0\rangle$$

$$= \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}} [\boldsymbol{P}, J^0(\boldsymbol{x}, t)]|0\rangle = -i \int d^3x e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \boldsymbol{\nabla} J^0(\boldsymbol{x}, t)|0\rangle$$

Integration by parts then gives

$$\boldsymbol{P}|\psi(\boldsymbol{k})\rangle = \boldsymbol{k}|\psi(\boldsymbol{k})\rangle \tag{14.32}$$

Evidentally $|\psi(\boldsymbol{k})\rangle$ describes a particle with momentum $\boldsymbol{k}$. In the limit that $\boldsymbol{k}$ goes to zero, (30) becomes

$$\lim_{\boldsymbol{k}\to 0} |\psi(\boldsymbol{k})\rangle = Q|0\rangle \tag{14.33}$$

We know from (29) that $Q|0\rangle$ is a state of zero energy. This relationship between momentum and energy is the signature of a massless particle.

In general there will be as many conserved charges as there are generators in the symmetry group. For each one of these charges that does not leave the vacuum invariant, *i.e.* those for which $Q^i|0\rangle \neq 0$, there will be one of these massless Goldstone bosons.

14.4 Non-Relativistic Field Theory

Field theory is necessary whenever particles are created and destroyed. This is bound to happen at relativistic energies, but it can happen at very low energy, if the "particles" represent some collective phenomena. Spontaneous symmetry breaking can occur in solids and liquids. In fact, superconductivity and superfluidity are outstanding examples of the mechanism. In order to study these examples, we will have to have a low-energy version of the ϕ^4 theory discussed previously. Start with the Lagrangian

$$\mathcal{L} = (\partial\Phi)^\dagger(\partial\Phi) - m^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2 \tag{14.34}$$

When $\lambda = 0$ this gives the Kline-Gordon equation

$$(\partial^2 + m^2)\Phi = 0. \tag{14.35}$$

We can turn this into Schrodinger's equation as follows. Define

$$\Phi = e^{-i\omega t}\varphi(\boldsymbol{x}, t) \tag{14.36}$$

$$\frac{\partial\Phi}{\partial t} = e^{-i\omega t}\left(-im + \frac{\partial}{\partial t}\right)\varphi$$

$$\frac{\partial^2\Phi}{\partial t^2} = e^{-i\omega t}\left(-im + \frac{\partial}{\partial t}\right)^2\varphi$$

The low-energy approximation consists in setting $\partial^2\varphi/\partial t^2 = 0$. Then (35) becomes

$$-\frac{\nabla^2}{2m}\varphi = i\frac{\partial\varphi}{\partial t}, \tag{14.37}$$

which is Schrodinger's equation.

The same technique works on the Lagrangian, but we must be careful about the normalization.

$$\Phi = \frac{1}{\sqrt{2m}}e^{-imt}\varphi(\boldsymbol{x}, t) \tag{14.38}$$

First note that

$$\frac{\partial\Phi^\dagger}{\partial t}\frac{\partial\Phi}{\partial t} - m^2\Phi^\dagger\Phi = \frac{1}{2m}\left[(im + \frac{\partial}{\partial t})\varphi^\dagger(-im + \frac{\partial}{\partial t})\varphi - m^2\varphi^\dagger\varphi\right] \tag{14.39}$$

$$\approx \frac{i}{2}\left(\varphi^\dagger\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^\dagger}{\partial t}\varphi\right) \to i\varphi^\dagger\partial_0\varphi$$

The last step required an integration by parts. The final result is

$$\mathcal{L} = i\varphi^\dagger \partial_0 \varphi - \frac{1}{2m}\,\partial_i\varphi^\dagger \partial_i \varphi - g^2(\varphi^\dagger\varphi)^2 \tag{14.40}$$

where $g^2 = \lambda/4m^2$. (Relativistic is beautiful. Non-relativistic is ugly.)

It's often useful to write the fields in terms of polar coordinates.

$$\varphi = \sqrt{\rho}e^{i\theta} \tag{14.41}$$

The Lagrangian is

$$\mathcal{L} = \frac{i}{2}\partial_0\rho - \rho\partial_0\theta - \frac{1}{2m}\left[\rho(\partial_i\theta)^2 + \frac{1}{4\rho}(\partial_i\rho)^2\right] - g^2\rho^2 \tag{14.42}$$

ρ and θ should be thought of as independent fields, just like φ and $\varphi^\dagger$. In this case the canonical momentum is

$$\frac{\delta\mathcal{L}}{\delta(\partial_0\theta)} = -\rho, \tag{14.43}$$

so the equal-time commutation relation is

$$[\rho(\boldsymbol{x},t), \theta(\boldsymbol{x}',t)] = i\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{14.44}$$

We can normalize the wave function so that

$$N = \int \rho(\boldsymbol{x},t)\; d\boldsymbol{x} = \text{ number of bosons} \tag{14.45}$$

then

$$[N,\theta] = i \tag{14.46}$$

In other words, number is conjugate to phase angle.

What sort of interactions do these Lagrangians describe? There is a hint in (42). The interaction Hamiltonian is

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}} = g^2\rho^2$$

Since ρ represents the density of bosons, we can see that packing bosons together costs energy, *i.e.* bosons repel regardless of the sign of λ. We can learn more about this with a complicated argument from Zee I.3. Let's use the generating functional formalism from Fall quarter to calculate the propagator for a Lagrangian such as (34). Look at

$$Z = \int D[\Phi] e^{i\int d^4x \mathcal{L}(\Phi)} \tag{14.47}$$

Last quarter you proved that

$$\prod_i \int dx_i \, e^{\frac{i}{2}\,\boldsymbol{x}\cdot\boldsymbol{A}\cdot\boldsymbol{x}+i\boldsymbol{J}\cdot\boldsymbol{x}} = \left(\frac{(2\pi i)^N}{\det[\boldsymbol{A}]}\right)^{1/2} e^{-\frac{i}{2}\,\boldsymbol{J}\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{J}} \tag{14.48}$$

We learned to generalize equations such as this by replacing the coordinates with fields.

$$\int D[\varphi] e^{i\int d^4x(\frac{1}{2}\boldsymbol{\varphi}\cdot\boldsymbol{A}\cdot\boldsymbol{\varphi}+\boldsymbol{J}\cdot\boldsymbol{\varphi})} \propto e^{-\frac{i}{2}\int d^4x\,\boldsymbol{J}\cdot\boldsymbol{A}^{-1}\cdot\boldsymbol{J}} \tag{14.49}$$

Our strategy will be to introduce a new "auxiliary" field called σ which has the effect of transmitting the repulsive between the bosons. To see how this works, make the following substitutions in (49):

$$J \to 2\Phi\Phi^\dagger \qquad \varphi \to \sigma \qquad A \to 2/\lambda$$

Then

$$\frac{1}{2}\varphi\cdot A\cdot\varphi + J\cdot\varphi \to \sigma^2/\lambda + 2\sigma\Phi\Phi^\dagger \qquad -\frac{1}{2}J\cdot A^{-1}\cdot J \to 2\sigma\Phi\Phi^\dagger$$

The identity (49) becomes

$$\int D[\sigma] e^{i\int d^4x[2\sigma\Phi\Phi^\dagger+\sigma^2/\lambda]} = e^{-i\int d^4x\lambda(\Phi\Phi^\dagger)^2} \tag{14.50}$$

The right side of (50) contains the boson interaction term $\lambda\Phi\Phi^\dagger$, the left side represents a σ "particle" coupling to the Φ. Using this in (47) gives

$$Z = \int D[\phi]D[\sigma] \exp\left\{i\int d^4x[(\partial\Phi)^2 - m^2\Phi^2 + 2\sigma\Phi\Phi^\dagger + \sigma^2/\lambda]\right\} \tag{14.51}$$

To put it another way, we have a new Lagrangian[2]

$$\mathcal{L} = (\partial\Phi)^2 - m^2\Phi^2 + 2\sigma\Phi\Phi^\dagger + \sigma^2/\lambda \tag{14.52}$$

It contains a scalar Φ particle obeying the Klein-Gordon equation. We know that the exchange of such a particle will produce a force of range $\sim m^{-1}$. It is coupled to a σ particle via the $2\sigma\Phi\Phi^\dagger$ term, so presumably the exchange of a σ will produce a force between two Φ's, although it's not clear at this

[2]Zee calls the transition from (47) to (52) the Hubbard-Strantonovic transformation. I will have to take his word for it.

point what the range and sign may be. To determine these things I will first review Zee's argument from Section I.3-4 regarding the Φ.

Introduce the Fourier transform

$$\Phi(x) = \int \frac{d^4p}{(2\pi)^2}\, e^{-ipx}\tilde{\Phi}(p). \tag{14.53}$$

Then

$$\int d^4x\, [(\partial\Phi)^2 - m^2\Phi^2] = \int d^4p\, \tilde{\Phi}^2(p)(p^2 - m^2) \tag{14.54}$$

We have learned to extract the propagator from this integral. The argument briefly is this:

$$Z[J] = Z[0]e^{iW[J]} = \langle 0|e^{-iH(T-T')}|0\rangle \tag{14.55}$$

$$W[J] = -\frac{1}{2}\int d^4x d^4y J(x)J(y)D(x-y) \tag{14.56}$$

$$D(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip(x-y)}}{p^2 - m^2 + i\epsilon} \tag{14.57}$$

Let's put two delta function sources in (56) and see how they affect one another.

$$J(x) = \delta^{(3)}(x - x_1) + \delta^{(3)}(x - x_2)$$

$W[J]$ will contain terms with $\delta^{(3)}(x-x_1)\delta^{(3)}(y-x_2)$ and $\delta^{(3)}(x-x_2)\delta^{(3)}(y-x_1)$ representing the interaction between the two sources. Substitute them into (56)

$$W = -\int dx_0 \int dy_0 \int \frac{dp_0}{2\pi} e^{ip^0(x-y)_0} \int \frac{d^3p}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_1-\boldsymbol{x}_2)}}{k^2 - m^2 + i\epsilon}$$

Integrate over y_0 to get $\delta(p_0)$, then integrate over p_0.

$$W = \int_{-T/2}^{T/2} dx_0 \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}}{\boldsymbol{k}^2 + m^2} \tag{14.58}$$

Have another look at (55). It seems that $\langle 0|e^{-iHT}|0\rangle$ ought to be something like e^{-iET} where E is the energy due to the presence of the two sources acting on one another. Since this is just a qualitative argument that should do for now. In this spirit then $-iRT = iW$. (58) gives us

$$E = -\int \frac{d^3k}{(2\pi)^3}\frac{e^{i\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1)}}{\boldsymbol{k}^2 + m^2} \tag{14.59}$$

This contains two great truths: (1) a negative potential means an attractive force, and (2) the range of the force is $\sim 1/m$. The integral can be done easily. The result is

$$E = -\frac{1}{4\pi r} e^{-mr}. \tag{14.60}$$

Now go back to the Lagrangian (52). We have just analyzed the free Φ field, and the interaction term speaks for itself, but what are we to make of the free σ field? There is nothing in the Lagrangian but σ^2/λ. Let's repeat the steps from (52) to (57)

$$\sigma(x) = \int \frac{d^4p}{(2\pi)^2} e^{-ipx} \tilde{\sigma}(p) \tag{14.61}$$

$$\int d^4x \frac{1}{\lambda} \sigma^2(x) = \frac{1}{\lambda} \int d^4p \tilde{\sigma}^2(p) \tag{14.62}$$

Compare this with (54). The momentum space propagator is simply $+\lambda$. The Fourier transform of this is the delta function. The force is repulsive and zero-range.

14.5 Spontaneous Breaking of Local Symmetry

The situation is quite different when the broken symmetry is a local or gauge symmetry. Let's revisit the model from equations (9) and (10), this time with the transformation

$$\phi \to e^{i\Lambda(x)} \phi. \tag{14.63}$$

We discussed this in the chapter on quantum electrodynamics. The Lagrangian will remain invariant only if there is an additional massless field A_μ. Moreover, this field must appear in the Lagrangian through the replacement

$$\partial_\mu \to \partial_\mu + ieA_\mu, \tag{14.64}$$

and in the kinetic energy term $-\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Substituting this into (9) gives

$$\mathcal{L} = [(\partial_\mu + ieA_\mu)\phi]^\dagger [(\partial^\mu - ieA^\mu)\phi] - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{14.65}$$

As before, we treat ϕ as a complex classical field and μ^2 as a parameter that can be positive or negative.

The equations of motion for the A_μ field can be found from the Euler-Lagrange equations

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu A_\nu)} - \frac{\delta \mathcal{L}}{\delta A_\nu} = 0 \tag{14.66}$$

The resulting equation is

$$\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) = j^\nu \tag{14.67}$$

where

$$j^\nu = iq\left[\phi^\dagger \partial^\nu \phi - (\partial^\nu \phi^\dagger)\phi\right] - 2q^2 \phi^\dagger \phi A^\nu \tag{14.68}$$

For the time being, I will use the Lorentz gauge condition $\partial_\mu A^\mu = 0$. If we take ϕ to be real with constant value

$$\phi = \sqrt{\bar{\rho}} \tag{14.69}$$

a simple result emerges.

$$\Box A^\mu = j^\mu = -2q^2 \bar{\rho} A^\mu, \tag{14.70}$$

which is the equation for a free vector particle of mass $M = \sqrt{2\bar{\rho}}q$. Consider for a moment the static case. (70) becomes

$$\nabla^2 \boldsymbol{B} = M^2 \boldsymbol{B} \tag{14.71}$$

In one dimension this would give

$$B = B_0 e^{-Mx}. \tag{14.72}$$

In other words, the magnetic field is screened out by the currents. We might say that the magnetic field has developed a "mass." This is the origin of the Meisner effect in superconductors. The boson field ϕ is a coherent many-body wave function consisting of Cooper pairs.

How could this be relevant to particle physics? What is there in free space that is like the Bose-Einstein condensate in a superconductor? Perhaps an analogy will help. Maxwell noticed that in some sense, something like current flows between the plates of a capacitor when it is charging or discharging. He therefore postulated a displacement current

$$\boldsymbol{J}_d = \frac{\partial \boldsymbol{E}}{\partial t}.$$

Is there really a current flowing between the plates? It depends on what you mean by "really." In the same sense we will postulate vacuum screening currents. They come about through the mechanism of spontaneous symmetry breaking.

In the second-quantized version of quantum field theory, we represent the field operators in terms of creation and annihilation operators.

$$\hat{\phi} = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} [e^{-ikx}\hat{a}(k) + e^{ikx}\hat{a}^\dagger(k)] \tag{14.73}$$

with the understanding that

$$\langle 0|\hat{a}(k)|0\rangle = \langle 0|\hat{a}^\dagger|0\rangle = 0 \tag{14.74}$$

i.e. the vacuum is the state in which there are no particles. (I am temporarily using "hats" to emphasize that we are dealing with operators rather than classical fields.) In the path integral formulation, the vacuum was regarded as the lowest energy state, cf. Section 2.4. The two are not necessarily the same thing. In superconductors, for example, the ground state is one in which all the Cooper pairs are in their lowest energy state. It would not make sense to represent this state with a field operator of the form (73). It is at least plausible under these circumstances that $\langle 0|\hat{\phi}|0\rangle \neq 0$. The generic classical ϕ^4 Lagrangian (65) we have been considering is a phenomonological model for such systems. If $-\mu^2$ is taken as a positive parameter, the minimum energy state appears at $|\phi| = a$, where

$$a = \sqrt{\frac{-\mu^2}{2\lambda}} \tag{14.75}$$

In polar form, the ground state wave function is

$$\hat{\phi} = |\hat{\phi}|e^{i\hat{\theta}(x)}. \tag{14.76}$$

Any phase is possible, the Lagrangian is still symmetric, but the system will choose some vacuum, and this will break or hide the symmetry. The most familiar example of this symmetry breaking is the ground state of a ferromagnet below the transition temperature. The spin-spin interaction is invariant under rotation, but once the spins have "decided" to align themselves in one particular direction, the symmetry is apparently broken.

Let us assume then along with the high-energy physics community, that there is some field $\hat{\phi}$ called the Higgs field that fills space and has a non-zero

vacuum expectation value. We choose the vacuum phase $\theta = 0$. In the vicinity of this point we can write

$$\hat{\phi} = a + \hat{h} \tag{14.77}$$

The implication is that $\langle 0|\hat{\phi}|0\rangle = a$ is a constant given by (75), and $\hat{h}$ is a quantum field. Inserting (77) into (67) (with $\partial_\mu \hat{A}^\mu = 0$) gives

$$\Box \hat{A}^\mu = -2q^2a^2\hat{A}^\mu + \text{terms depending on } h \tag{14.78}$$

Compare with (70), the photon has acquired a mass $M = \sqrt{2q^2a^2}$.

This argument seems to depend on the gauge condition $\partial_\mu \hat{A}^\mu = 0$, so we have lost gauge invariance. Can we have our mass and gauge invariance too? To answer this question, first combine (76) and (77) so that

$$\hat{\phi} = ae^{i\hat{\theta}(x)} + \text{ terms depending on } \hat{h} \tag{14.79}$$

and substitute into the operator forms of (67) and (68)(again ignoring $\hat{h}$).

$$\Box \hat{A}^\mu - \partial^\nu(\partial_\mu \hat{A}^\mu) = -2q^2a^2(\hat{A}^\nu + q^{-1}\partial^\nu\hat{\theta}) \tag{14.80}$$

Let's define

$$\hat{A}^\nu \to \hat{A}'^\nu = \hat{A}^\nu + q^{-1}\partial^\nu\hat{\theta} \tag{14.81}$$

This is just a gauge transformation after all, and we know that the left side of (80) is gauge invariant, so

$$(\Box - M)\hat{A}'^\nu - \partial^\nu(\partial_\mu \hat{A}'^\mu) = 0. \tag{14.82}$$

Differentiating with respect to ∂_ν reveals

$$\partial_\nu \hat{A}'^\nu = 0, \tag{14.83}$$

so we can have our proverbial cake and eat it too.

$$(\Box + M^2)\hat{A}'^\nu = 0 \tag{14.84}$$

Gauge transformations are customarily written

$$\hat{A}^\mu \to \hat{A}'^\mu = \hat{A}^\mu - \partial^\mu\hat{\chi}^\mu, \tag{14.85}$$

so we have simply chosen the gauge $\hat{\chi} = -\hat{\theta}/q$.

Conclusions:

- Setting $\theta = 0$ in (76) breaks the rotational symmetry of the theory but it does not break gauge invariance, because any choice of θ can be compensated by an equivalent gauge transformation of A^μ.

- By allowing the photon to interact with a scalar field with a non-zero vacuum expectation value, we enable the photon to acquire a mass.

So far, we have only looked at the free-particle component of the theory. We can get an idea of the interactions by substituting the complete form of (77) into (68).

$$(\Box + M^2)\hat{A}^\nu = -4q^2 a\hat{h}\hat{A}^\nu - 2q^2\hat{h}^2\hat{A}^\nu \tag{14.86}$$

There is no such thing as a relativistic superconductor, of course, this example is only intended to illustrate the ideas behind the Higgs mechanism. Equation (86) is intended to illustrate one final point. There is a new particle, the $\hat{h}$ or Higgs boson, and it couples to the photon field in a specific way. The mechanism not only produces the mass, it decides the form of the interaction.

Suppose we did not set $\hat{\theta} = 0$ but kept it around as an additional particle? This would change the equations of motion and hence the propagator. This is not necessarily a bad thing, because the renormalization properties of the theory are more transparent with other choices of the gauge function $\hat{\theta}$. This is the key to the remarkable fact that gauge theories are renormalizable. This a technical issue that will not be covered in these lectures.

In the absence of the gauge field, the vacuum is at

$$|\phi| = a = \left(\frac{-\mu^2}{2\lambda}\right) \tag{14.87}$$

As in (20) we introduce two real fields ϕ_1 and ϕ_2 and substitute into (36). Keeping only the quadratic terms, we have

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2$$

$$-2\lambda a^2\phi_1^2 + \sqrt{2}eaA^\mu\partial_\mu\phi_2 + \cdots \tag{14.88}$$

Now it appears from the second term that the photon has developed a mass $m^2 = e^2a^2$. The ϕ_1 field has a mass, and ϕ_2, the Goldstone boson, is massless, but it couples to the "photon" with a derivative coupling. This suggests that the photon could simply turn itself into a ϕ_2. Let's count the degrees of freedom. Originally we had

2	(complex scalar field)
+2	(transverse field for massless photon)
4	

After the spontaneous symmetry breaking we have

1	(ϕ_1, a massive scalar field)
1	(ϕ_2 a massless scalar field)
+3	(massive vector field A_μ)
5	

Something is wrong. A degree of freedom seems to have been gained in the spontaneous symmetry breaking. This is an illusion, as can be make obvious by an appropriate change of gauge. The ϕ_2 field, however, can be removed by means of a gauge transformation. Putting an infinitesimal Λ in (34) gives

$$\phi_1' = \phi_1 - \Lambda\phi_2$$

$$\phi_2' = \phi_2 + \Lambda\phi_1 + \sqrt{2}\Lambda a \tag{14.89}$$

We can always choose Λ so that $\phi_2' = 0$. This will affect the higher-order terms, of course, but for the moment we are only concerned with the quadratic terms. Simply set $\phi_2 = 0$ in (38)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2a^2A_\mu A^\mu + \frac{1}{2}(\partial_\mu\phi_1)^2 - 2\lambda a^2\phi_1^2 + \cdots \tag{14.90}$$

There are now only two fields. A massive spin-one "photon" and a massive scalar field, the ϕ_1. The Goldstone boson has disappeared. This is called the Higgs mechanism.

Chapter 15

LNAGS and the Standard Model

These notes are intended to summarize the last few lectures on local non-abelian gauge fields (LNAGS) and the standard model. They will not be a substitute for your own careful notes.

We have been considering the consequences of the transformation

$$\psi \rightarrow \psi' = e^{ig\boldsymbol{\tau}\cdot\boldsymbol{\alpha}/2}\psi \tag{15.1}$$

Here $\boldsymbol{\tau}$ represent the three Pauli spin matrices, here interpreted as isospin operators, and ψ represents some generic two-component spinor. The Lagrangians we have used all have the "kinetic energy" term

$$(\partial_\mu\psi)^\dagger(\partial^\mu\psi) \tag{15.2}$$

In order to keep (2) invariant under the transformation (1), desperate measures are called for. We need to introduce a triplet of massless spin-one fields

$$\boldsymbol{W}^\mu = (W_1^\mu, W_2^\mu, W_3^\mu), \tag{15.3}$$

and construct with them the "covariant derivative"

$$D^\mu = \partial^\mu + ig\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu/2 \tag{15.4}$$

These fields must satisfy additional constraints so that

$$D'^\mu\psi' = e^{ig\boldsymbol{\tau}\cdot\boldsymbol{\alpha}(x)/2}D^\mu\psi \tag{15.5}$$

One consequence of these constraints is that the $\boldsymbol{W}^\mu$ fields interact with themselves. As a practical measure, we can derive the correct Lagrangian

and equations of motion by simply replacing $\partial_u \to D_\mu$ wherever it appears. The Lagrangian also has to be modified by the addition of the "free-particle" energy[1]

$$\boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{W}_\nu - \partial_\nu \boldsymbol{W}_\mu - g\boldsymbol{W}_\mu \times \boldsymbol{W}_\nu \tag{15.6}$$

15.1 Spontaneously broken global $SU(2)$ symmetry

The W particles will eventually acquire a mass via the Higgs mechanism. Let us first see how the massless precursors of the W's arise as goldstone bosons from spontaneously broken local $SU(2)$ symmetry. We consider an $SU(2)$ doublet, not of fermions this time, but of bosons.

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} (\phi_1 + i\phi_2)/\sqrt{2} \\ (\phi_3 + i\phi_4)/\sqrt{2} \end{pmatrix} \tag{15.7}$$

If these are interpreted as particle creation and annihilation operators, ϕ^+ destroys a positive particle or creates a negative antiparticle, whereas ϕ^- destroys a neutral particle or creates a neutral antiparticle.

The Lagrangian we choose is the generic "phi to the fourth" interaction

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{4}(\phi^\dagger \phi)^2 \tag{15.8}$$

This has a classical minimum at

$$(\phi^\dagger \phi)_{\min} = 2\mu^2/\lambda \equiv v^2/2, \tag{15.9}$$

which we interpret as a condition on the vacuum expectation value of $\phi^\dagger \phi$

$$\langle 0|\phi^\dagger \phi|0\rangle = v^2/2 \tag{15.10}$$

Notice that (8) is invariant under the transformation (1) and also under a separate global $U(1)$ transformation

$$\phi \to \phi' = e^{-iy\alpha}\phi, \tag{15.11}$$

where α is separate from $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$. The constant y is included for later use. It will stand for the weak hypercharge. The full symmetry is then referred to as $SU(2) \times U(1)$.

[1] "Free-particle" is in quotes because the particles are still interacting with themselves.

There are infinitely many ways of implementing (10), all summarized by the equation,

$$\langle 0|\phi^\dagger\phi|0\rangle = \langle 0|(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)|0\rangle = v^2. \tag{15.12}$$

There is an important subtlety here. We have to choose $\langle 0|\phi_i|0\rangle \neq 0$ for at least one of the fields ϕ_i. Since there are four independent symmetry transformation that leave the Lagrangian invariant, we expect four massless fields. It is inevitable, however, that any choice we make will be invariant under one of these four transformations, and hence the corresponding particle will remain massive. We choose

$$\langle 0|\phi|0\rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}. \tag{15.13}$$

In this case the vacuum remains invariant under the combined transformation of the third component of the $SU(2)$ transformation (1) plus the $U(1)$ transformation (11)

$$(1/2+t_3)\langle 0|\phi|0\rangle = 0 \tag{15.14}$$

and hence

$$\langle 0|\phi|0\rangle \to \langle 0|\phi|0\rangle' = \exp[i\alpha(1/2+t_3)]\langle 0|\phi|0\rangle = \langle 0|\phi|0\rangle \tag{15.15}$$

As usual $t_3 = \tau_3/2$. I am using lower case t for weak isospin. The point is that the vacuum is invariant under (15) so the corresponding field will not acquire a mass.

We now need to consider oscillations about (10) in order to see the physical particle spectrum. We parameterize these as

$$\phi = \exp(-i\boldsymbol{\theta}(x)\cdot\boldsymbol{\tau}/2v)\begin{pmatrix} 0 \\ (v+H(x))/\sqrt{2} \end{pmatrix} \tag{15.16}$$

Now ϕ is invariant under the $SU(2)$ gauge transformations, and clearly setting $\boldsymbol{\theta}=0$ is such a transformation, so consider it done and write

$$\phi = \begin{pmatrix} 0 \\ (v+H(x))/\sqrt{2} \end{pmatrix} \tag{15.17}$$

As a consequence of the gauge symmetry (11), the Lagrangian will contain an additional massless vector field, which we call B^μ. Our Lagrangian now is

$$\mathcal{L} = (D_\mu\phi)^\dagger(D^\mu\phi) + \mu^2\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 - \frac{1}{4}\boldsymbol{F}_{\mu\nu}\cdot\boldsymbol{F}^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \tag{15.18}$$

where

$$D^\mu = \partial^\mu + ig\boldsymbol{\tau}\cdot\boldsymbol{W}^\mu/2 + ig'yB^\mu/2 \tag{15.19}$$

$$G^{\mu\nu} = \partial^\mu B^\nu - \partial^\nu B^\mu \tag{15.20}$$

and $\boldsymbol{F}_{\mu\nu}$ is given by (6). Substitute (19), (6), and (20) into (18) and keep only terms that are quadratic in the fields. We find

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial_\mu H)(\partial^\mu H) - \mu^2 H^2 \\ &-\frac{1}{4}(\partial_\mu W_{1\nu} - \partial_\nu W_{1\mu})(\partial^\mu W_1^\nu - \partial^\nu W_1^\mu) + \frac{1}{8}g^2v^2 W_{1\mu}W_1^\mu \\ &-\frac{1}{4}(\partial_\mu W_{2\nu} - \partial_\nu W_{2\mu})(\partial^\mu W_2^\nu - \partial^\nu W_2^\mu) + \frac{1}{8}g^2v^2 W_{2\mu}W_2^\mu \\ &-\frac{1}{4}(\partial_\mu W_{3\nu} - \partial_\nu W_{3\mu})(\partial^\mu W_3^\nu - \partial^\nu W_3^\mu) - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} \\ &+\frac{1}{8}v^2(gW_{3\mu} - g'B_\mu)(gW_3^\mu - g'B^\mu)\end{aligned} \tag{15.21}$$

The fields W_3 and B are mixed in a way that suggests that neither of them are real physical fields. They can be unmixed by introducing

$$Z^\mu = \cos\theta_W W_3^\mu - \sin\theta_W B^\mu \tag{15.22}$$

and

$$A^\mu = \sin\theta_W W_3^\mu + \cos\theta_W B^\mu \tag{15.23}$$

The resulting Langrangian is

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\partial_\mu H\partial^\mu H - \mu^2 H^2 \\ &-\frac{1}{4}(\partial_\mu W_{1\nu} - \partial_\nu W_{1\mu})(\partial^\mu W_1^\nu - \partial^\nu W_1^\mu) + \frac{1}{8}g^2v^2 W_{1\mu}W_1^\mu \\ &-\frac{1}{4}(\partial_\mu W_{2\nu} - \partial_\nu W_{2\mu})(\partial^\mu W_2^\nu - \partial^\nu W_2^\mu) + \frac{1}{8}g^2v^2 W_{2\mu}W_2^\mu \\ &-\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu) + \frac{1}{8}v^2(g^2+g'^2)Z_\mu Z^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{15.24}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{15.25}$$

and

$$\tan\theta_W = g'/g \tag{15.26}$$

This is the "free-particle" Lagrangian for the vector fields. Now that the smoke has cleared, we see the consequences of our various maneuvers. The three W particles that were initially massless have acquired masses, although W_3 has done so by mixing with the B particle. We are left with a single massless particle, the A, which we identify with the photon. The masses and coupling constants can be read off the Lagrangian. This is left as an exercise.

Anyhow, these are the "messenger particles." But whose messages do they carry? The answer, of course, is the leptons and quarks. How they fit in is the subject of the next section.

15.2 Quarks and Leptons

The Dirac Langrangian is

$$\mathcal{L} = \bar{\psi}(i\partial_\mu \gamma^\mu - m)\psi \tag{15.27}$$

Because of our gauge hypothesis we must replace $\partial_\mu \to D_\mu$ where D_μ is given by (19), but now ψ is a two component object containing the relevant fermion fields. We have discussed weak interaction processes in which leptons scattered from leptons and those in which leptons scattered from quarks. Our new Lagrangian should include both processes, so we must construct ψ's with both quarks and leptons. We have in hand several clues to help us do this.

- The ψ's should transform as doublets under the weak isospin group $SU(2)_W$.
- Leptonic transitions associated with charged currents change leptons into neutrinos and vice versa. For example, $\nu_e \leftrightarrow e$ or $\nu_\mu \leftrightarrow \mu$. This suggests that these should be doublets of the group.
- We also talked about the quark doublet,

$$q = \begin{pmatrix} u \\ d_c \end{pmatrix} = \begin{pmatrix} u \\ d\cos\theta_c + s\sin\theta_c \end{pmatrix}, \tag{15.28}$$

  where θ_c is the Cabibo angle.
- Consider a process like $e^+ + d \to \nu_e + u$ resulting from the exchange of a W^+. The current that raises the quark charge from down ($q = -1/3$)

to up ($q = 2/3$) is

$$J^\mu = \frac{g}{\sqrt{2}}\,\bar{q}\gamma_\mu\,\left(\frac{\tau^+}{2}\right)\,\frac{1}{2}(1-\gamma_5)q \tag{15.29}$$

$$= \frac{g}{\sqrt{2}}\,\bar{u}_L d_L \cos\theta_c + \frac{g}{\sqrt{2}}\,\bar{u}_L s_L \sin\theta_c$$

The $\frac{1}{2}(1-\gamma_5)$ projects out the left-handed parts of the quark wave functions

$$\frac{1}{2}(1-\gamma_5)q \equiv q_L \tag{15.30}$$

So far as we know, there is no process that connects right and left-handed states, so we can write

$$J^{+\mu} = \frac{g}{\sqrt{2}}\,\bar{q}_L\gamma_\mu\,\left(\frac{\tau^+}{2}\right)\,q_L \tag{15.31}$$

We can combine charge raising, charge lowering and neutral-current terms by defining the operators

$$\frac{\tau^\pm}{2} = \frac{\sigma_1 \pm i\sigma_2}{2} \qquad \frac{\tau^0}{2} = \frac{\sigma_3}{2} \tag{15.32}$$

With this, (31) can be written more compactly as

$$J^{\alpha\mu} = \frac{g}{\sqrt{2}}\,\bar{q}_L\gamma^\mu\frac{\tau^\alpha}{2}q_L, \tag{15.33}$$

which also includes the neutral-current contribution.

Here then is the family portrait.

$$\text{leptons} \qquad \begin{pmatrix}\nu_e\\ e^-\end{pmatrix}_L \qquad \begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}_L \qquad \begin{pmatrix}\nu_\tau\\ \tau^-\end{pmatrix}_L \tag{15.34}$$

$$\text{quarks} \qquad \begin{pmatrix}u\\ d_c\end{pmatrix}_L \qquad \begin{pmatrix}c\\ s_c\end{pmatrix}_L \qquad \begin{pmatrix}t\\ b_c\end{pmatrix}_L \tag{15.35}$$

The subscript "c" on the d, s, and b remind us that these are not pure quark states, but rather they are mixed according to a scheme that we haven't time to discuss. The symbols that appear in these arrays stand for the Dirac spinors u and v with the left-handed parts projected out. For example "e^-" stands for $\frac{1}{2}(1-\gamma_5)u_e(k,\lambda)$. ($\lambda$ is the spin index.)

The couplings of the gauge fields to these left-handed doublets is given simply by

$$\mathcal{L}_{\rm int} = i\bar{\psi}' D^\mu \gamma_\mu \psi \tag{15.36}$$

where ψ and ψ' stand for any of the doublets in (34) and (35), and D^μ is given by (19). There is a technical point regarding the W's however. W_1 and W_2 are real fields, and as such they don't represent the physical particles. The real charged particles are given by

$$W^\pm = \frac{W_1 \pm iW_2}{\sqrt{2}} \tag{15.37}$$

The $\sqrt{2}$ in the denominator is an indirect consequence of the normalization we chose in (7). As a consequence, the vertex factor that appears whenever a W couples to a quark or lepton is

$$-i\frac{g}{\sqrt{2}}\gamma^\mu \frac{1-\gamma_5}{2} \tag{15.38}$$

Still unaccounted for are the right-handed components of the fermion fields. There is at present no evidence for any weak interactions coupling the the right-handed field components and it is therefore natural that all 'R' components are singlets under the weak isospin group. The 'R' components do interact via the $U(1)$ field B^μ: it is this that allows electromagnetism to emerge free of parity violating γ_5 terms. Just for the record then, the right handed fields are e_R, μ_R, τ_R, u_R, d_R, s_L, c_R, t_R, and b_R, all with weak isospin, $t = 0$. We will not have time this quarter to consider any of the interactions of these fields.