Feynman Simplified

3A: Quantum Mechanics

Part One

Everyone's Guide to the Feynman Lectures

Robert L. Piccioni, Ph.D.

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Everyone's Guide
to the
Feynman Lectures on Physics

by
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Everyone's Guide to the Feynman Lectures on Physics

Feynman Simplified gives mere mortals access to the fabled ***Feynman Lectures on Physics***.

Quantum mechanics is rarely taught well in introductory physics courses, largely because this challenging subject was not well taught to many of today's instructors. Few had the opportunity to learn quantum mechanics from some who understood it profoundly; almost none learned it from one of its creators. Here more than anywhere else, Feynman excels. Here more than anywhere else, ***Feynman Simplified*** can help you learn from the very best, but at a humane pace.

This Book

Feynman Simplified: 3A covers the first half of Volume 3 and chapters 37 and 38 of Volume 1 of The Feynman Lectures on Physics. The topics we explore include:

- Why the Micro-World is Different.
- Quantization and Particle-Wave Duality
- Indeterminism and the Uncertainty Principle
- Probabilities and Amplitudes
- Identical Particle Phenomena
- Bosons and Spin One
- Fermions and Spin One-Half
- Time Evolution and the Hamiltonian Operator
- The Two-State Ammonia Maser

Readers will greatly benefit from a prior understanding of the material in ***Feynman Simplified 1A, 1B*** and ***1C***. A familiarity with elementary calculus is assumed.

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Chapter 1

What Is Quantum Mechanics?

Much of the material in this chapter supplements the ***Feynman Lectures***.

Quantum mechanics is the physical theory of elementary particles, how they interact with one another, and how they form atoms and larger structures.

It is often said that quantum mechanics is strange, "unnatural", and bizarrely contrary to our innate sense of how things "really" are. For example, quantum mechanics says objects can be in different places at the same time, and can be simultaneously right-side-up and upside-down. It says particles are both everywhere and nowhere, until we look at them.

To that last assertion, Einstein scoffed: "Would the Moon disappear if we did not look at it?" Moon no, but electrons yes.

Other eminent physicists have also found quantum mechanics astonishing, including two Nobel Laureates honored for developing quantum mechanics:

"If quantum mechanics hasn't profoundly shocked you,
you haven't understood it yet"

— Niels Bohr

"I think that I can safely say that
no one understands quantum mechanics"

— Richard Feynman

Quantum mechanics is strange, but true.

Quantum mechanics correctly describes nature's fundamental processes, the nuts-and-bolts of reality at its core. This strange theory is one of the most extensively tested and precisely confirmed creations of the human mind. The everyday world we perceive is a hazy, superficial, diluted version of the tempestuous reality at nature's core. Perhaps, it is **our** perception of reality that is unnatural.

How can Feynman say "no one understands quantum mechanics" when he and many others have filled library shelves with books explaining it? Let me address that with an analogy. Many people say they understand computers because they can email and surf the web. But few dive inside the box. And even fewer comprehend the internal structure of all those gray plastic centipedes that populate its

guts. We know how to use quantum mechanics, but its gray centipedes are still bewildering.

Quantum mechanics is truly how the world works: it is the heart of physics.

Although many conclusions of quantum mechanics defy our intuition, physicists believe we now know all its rules. We can solve all its equations, even if we must laugh at some of the answers. In this sense, quantum mechanics is a mystery that we have solved but not fully digested.

Key Principles of Quantum Mechanics

The greatest physical theories blossom from just a few remarkable but simple-sounding ideas. Galileo's Principle of Relativity has one idea: only relative velocities are physically meaningful. Einstein's Special Theory of Relativity has one: the speed of light never changes. Einstein's General Theory of Relativity has one: locally, gravity is equivalent to acceleration.

I say quantum mechanics has two key principles. These two ideas are so interrelated that they could be combined into one idea. Perhaps, but it is easier to learn them separately, and put them together later. The two principles are:

1. **Quantization**

2. **Particle-Wave Duality**

We will first describe what quantization means, then examine the development of particle-wave duality, and ultimately discover how duality leads to quantization.

Quantization

Quantization is the simple notion that some things in nature are countable — they come in integer quantities.

Money is *quantized*. In the U.S., the amount of money in any transaction is an integer multiple of 1¢. In Japan it's 1¥. Particles and people are also quantized: there is no such thing as 1.37 electrons or π people. Conversely, at least on a human scale, water, air, space, and time *appear* to be continuous. As our understanding has advanced, we have discovered that more and more entities that seem continuous are actually quantized. Perhaps we will ultimately discover that everything really is quantized.

The staircase and ramp in Figure 1-1 illustrate the difference between quantized and continuous. On a ramp, every elevation is possible; just slide along to the right spot.

Figure 1-1 Staircase and Ramp

On a staircase, only a few discrete elevations are possible. One can be as high as the second step, or the third step, but never as high as the 2.7th step, because no such step exists. On a staircase, elevation changes abruptly and substantially.

The micro-world of atoms and particles is replete with significant staircases — the steps are large and dramatically impact natural processes in this realm. We, however, live on a much larger scale, billions of times larger. As one's perception zooms out from the atomic scale toward the human scale, the steps in nature's staircases appear ever smaller and ever more numerous, as depicted in Figure 1-2.

Figure 1-2 Staircase Viewed On Ever-Larger Scales

Eventually, the steps become too small to be seen individually, and we observe smooth ramps instead of staircases.

Nature does not have one set of laws for the atomic scale and another set of laws for the human scale. Nature's laws are universal; these steps exist always and everywhere. But at our level, the steps are so small that they almost never make a discernible difference.

In our macro-world, planets can orbit stars at any distance, baseballs can have any speed, and nothing is ever in two different places at the same time.

In the micro-world, electrons circle nuclei only in specific orbits, only with specific energies, and are everywhere simultaneously, until the macro-world intervenes.

Quantum mechanics is all about understanding what happens when the staircase steps are important, when quantization dominates.

The Beginning of Quantum Theory

Feynman Simplified 1B Chapter 20 discusses how and why the first glimmers of quantum mechanics emerged in 1900, when Max Planck "solved" the *Ultraviolet Catastrophe* in the theory of thermal radiation.

Thermal radiation is the light (often infrared light) that objects emit due to their heat energy, the random motion of "hot" atoms. Recall that classical physics predicts thermal radiation has the same

wave amplitude at each frequency. Since frequency f has an unlimited range, when one integrates to infinite f, the total radiation becomes infinite. Lighting a match should cremate the entire universe — clearly that is ridiculous. For theorists it was a catastrophe at high frequency, ultraviolet and beyond.

To fix this, Planck postulated that thermal radiation is quantized. He said energy is emitted only in integer multiples of hf: he said the emitted energy E must equal nhf, where n is an integer and h is a constant named in Planck's honor. For a high enough frequency f, the available energy is less than 1•hf. Allowing only integer multiples of hf precludes any n•hf except 0•hf. This truncates high frequency emission and makes the integral finite. Planck offered no physical rationale for quantizing thermal emission; he viewed it as simply a mathematical formalism that worked.

Truly, this was a solution without an explanation. But a profound explanation came five years later.

Einstein & The Photoelectric Effect

In 1887, Heinrich Hertz observed that when light strikes a metal, an electric current is produced *sometimes*. Careful experiments determined this *photoelectric effect* is due to light knocking electrons out of the atoms of the metal. In 1839, A. Edmond Becquerel discovered the closely related *photovoltaic effect* in which light *sometimes* elevates atomic electrons to higher energy states. Both originate from the same basic physics. But mysteriously, both effects only happen *sometimes***.**

Knocking an electron away from a positively charged nucleus requires energy. Since light carries energy, any beam of light of sufficient intensity should eject electrons. But here's the mysterious part: blue light ejects electrons but red light doesn't. Even extremely intense beams of red light fail to eject electrons. Conversely, even low intensity beams of blue light eject a few electrons.

What's wrong with red?

Einstein solved this mystery in 1905 by proclaiming that light is **both** a particle and a wave. This was heresy ─ every other physicist was certain that waves and particles were entirely distinct and incompatible. Yet, Einstein claimed these two very different phenomena are actually two aspects of a more fundamental entity.

Einstein said light beams are composed of vast numbers of individual particles, which we now call photons. He said each photon's energy E is proportional to its frequency f, according to: E=hf. Blue light has twice the frequency of red light, hence a blue photon has twice the energy of a red photon. When a beam of photons strikes a metal surface, Einstein explained, the fundamental interaction is one photon hitting one electron ─ there's no double-teaming. It takes one good whack to eject an electron; a thousand little nudges wouldn't do the trick. An electron is ejected only if a single photon has sufficient energy. A blue photon does have enough energy, but a red photon doesn't. That's why blue works and red doesn't.

Einstein realized his concept of light being individual particles fit perfectly with Planck's quantization of thermal radiation. Since particles always come in integer quantities, it is *evident* that the energy of radiation must be quantized, an integer multiple of the energy of one photon, hf. (Recall

that Feynman said: "The real *glory* of science is that we can *find a way of thinking* such that the law is *evident*.")

Let's go back to 1900 and understand why Einstein's claim that light is both particle and wave was so revolutionary.

Particles Versus Waves

Physicists in 1900 were universally convinced that there were two completely separate and incompatible entities in nature: particles and waves. Particles were simpler, just very small versions of golf balls. Waves were much more complex. Their essential differences are tabulated below:

Property	Particles	Waves
Quantized	YES	NO
Localized	YES	NO
Diffraction	NO	YES
Interference	NO	YES
Needs Medium	NO	YES
Speed	no restrictions	set by medium

Particles always come in integer numbers and are precisely localized. They require no medium to travel through and can move at any speed. Conversely, waves can have any amplitude; with waves it is how much, not how many. Waves spread throughout all available space, and exhibit complex diffraction and interference effects that particles never do. Waves are the organized motion of a medium, such as air molecules for sound waves. Waves cannot exist without a medium, and their speed is determined by the properties of their medium, not by their source.

We cover the basics of particle motion in **1** chapter, ***Feynman Simplified 1A*** Chapter 6, and the behavior of light waves in **9** chapters, ***Feynman Simplified 1C*** Chapters 30 through 38.

If all that doesn't convince you that waves and particles are completely different, consider their force-free equations of motion, with A being amplitude and ***a*** being acceleration:

Particle: $0 = \boldsymbol{a}$
Wave: $0 = d^2A/dx^2 + d^2A/dy^2 + d^2A/dz^2 - d^2A/dt^2$

It was obvious to everyone that particles and waves had nothing in common. Obvious to everyone, that is, except Einstein.

One might say Einstein thought "outside the box." But, it may be more accurate to say that he never even noticed that there were boxes.

Particle–Wave Duality

Einstein said light is both a wave and a particle. Louis de Broglie expanded on this idea and said particles are also waves. In particular, de Broglie said every particle has a wavelength λ given by $\lambda=h/p$, where p is the particle's momentum. This remarkable combination, now called particle-wave duality, became the essential foundation of quantum mechanics. It also ultimately led to conclusions that Einstein could never accept.

Einstein opened Pandora's box and was never able to squeeze Uncertainty back into the box.

We now know "particle" and "wave" are really labels for the opposite ends of a continuous spectrum, similar to the labels "black" and "white." Everything in our universe is really a shade of gray. In our macro-world, everything is almost completely black or white. But in the micro-world, gray rules. Sometimes particle-waves are more particle-like and sometimes more wave-like, but fundamentally, everything is always really both.

Two-Slit With Classical Particles

Let's examine the impact of particles having wavelengths using the iconic experiment of quantum mechanics: the two-slit experiment discussed in ***Feynman Simplified 1C*** Chapter 31.

To establish a baseline, first imagine running the experiment with classical particles: tennis balls. A ball throwing machine S shoots tennis balls in random directions toward a net that is broken in two places. Some tennis balls pass through the holes, but are randomly deflected. These eventually reach the backboard F, shown in Figure 1-3, where we tally each ball's impact point. Let the figure's vertical axis be y and let y=0 be the horizontal midline, which is the axis of symmetry of our apparatus.

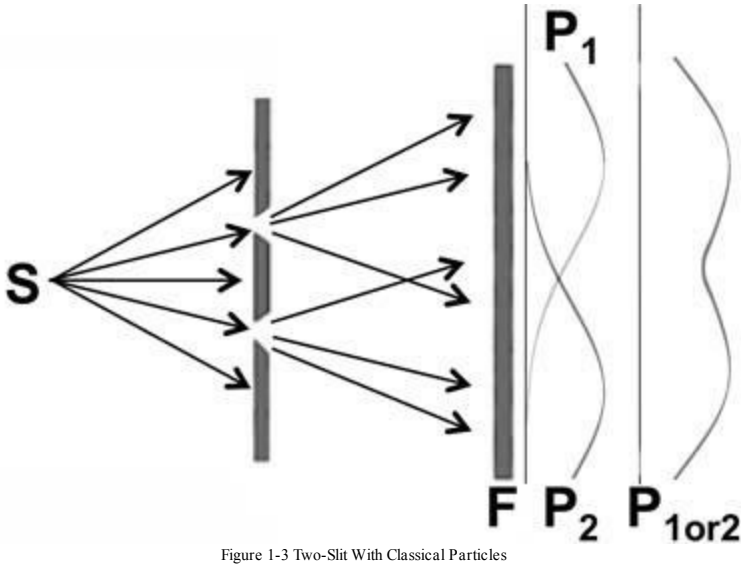

Figure 1-3 Two-Slit With Classical Particles

Call the upper hole #1 and the lower #2. After thousands of balls have reached F, we plot a probability distribution $P_1(y)$, the probability that a ball passing through hole #1 reaches F at vertical position y. We similarly plot $P_2(y)$ for balls passing through hole #2, and $P_{1+2}(y)$ for balls passing through either hole. In our experiment, $P_1(y)$ is quite broad but peaks along the line from S through hole #1. $P_2(y)$ is also broad and peaks along line from S through hole #2.

Clearly, $P_{1+2}(y) = P_1(y) + P_2(y)$. For tennis balls, and all classical particles, probabilities simply add; balls that reach F have gone through **either** one hole **or** the other. Additionally, each classical particle is a single entity that arrives at a single localized point.

None of this is surprising; it's what we expect particles to do.

Two-Slit With Classical Waves

Now, let's do a similar experiment with a classical wave. In Figure 1-4, waves from source S strike an opaque barrier with two small slits, producing two coherent wave sources with zero phase shift and equal amplitudes and frequencies.

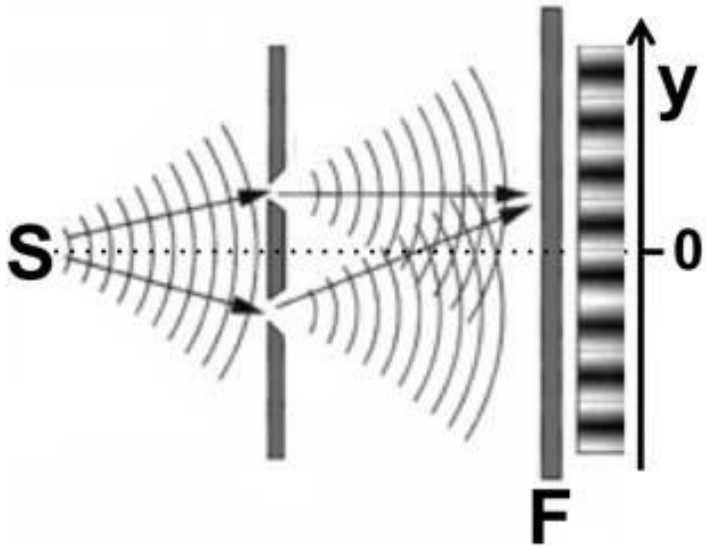

Figure 1-4 Two-Slit Experiment With Waves

The two waves interfere on a detection screen F. Define D to be the slit separation, L to be the distance between barrier and F, and as before, y is the vertical axis with y=0 on the midline. The apparatus is symmetric about the midline and L>>D.

If we close one slit, the wave intensity broadly and smoothly spreads across F, peaking along a line from S through the open hole. By symmetry, we get the same total intensity when either hole is open and the other is closed.

But with both holes open things become much more interesting: we get the interference fringe pattern shown at the right side of Figure 1-4. We mastered wave interference in ***Feynman Simplified 1C*** Chapter 31, but let's review it again so you don't have to switch back and forth between eBooks.

The waves passing through each slit are coherent with zero phase shift and equal amplitudes and

frequencies ω. As the two waves proceed to F, they may travel different distances, changing their relative phase shift.

For the wave passing through slit #1, the upper slit, its wave equation at point y on F can be written:

$A\cos(\omega t+z_1 k)$

where wave number k equals ω/c, z_1 is the distance from slit #1 to y, and A is the wave amplitude at F. We assume any path length differences are << L, so the 1/r amplitude dependence can be ignored.

Similarly for the wave from slit #2:

$A\cos(\omega t+z_2 k)$

Now, consider how these two waves combine on F.

Waves from slit #1 and slit #2 travel the same distance to reach y=0: $z_1=z_2$, by symmetry. Therefore their relative phase shift remains zero, as it was at the barrier. At y=0, the waves interfere entirely constructively. Their combined amplitude is 2A, making their combined intensity $4A^2\langle\cos^2(\omega t)\rangle = 2A^2$. Recall that < > means time average over a full cycle. The two waves produce a high-intensity white *fringe* surrounding y=0.

As y increases, z_1 decreases while z_2 increases. At some y value, the difference in slit distances, z_2-z_1, equals λ/2. There the lower wave arrives one-half cycle after the upper wave, resulting in entirely destructive interference: their combined amplitude is zero. The waves produce a zero-intensity black fringe.

At about twice that y value, z_2 becomes one full wavelength longer than z_1, $z_2-z_1=\lambda$. Since a full cycle delay is equivalent to zero phase shift, here interference is again entirely constructive, producing another white fringe. The pattern repeats up and down screen F, producing an alternating pattern of black and white fringes of nearly equal width.

We can calculate z_1 and z_2 for any y value on screen F, recalling that the slits are at vertical positions ±D/2, and assuming L >> D or y.

$z_1 = \sqrt{(L^2+(y-D/2)^2)} \approx L + (y-D/2)^2/2L$
$z_2 = \sqrt{(L^2+(y+D/2)^2)} \approx L + (y+D/2)^2/2L$
$\Delta z = z_2-z_1 \approx yD/L$

(The convention on Δq and dq is: Δq is a difference of any magnitude between two values of any variable q, whereas dq implies an infinitesimal change in variable q. The latter are of particular interest in the calculus of derivatives and integrals.)

For any y, the combined amplitude A_{1+2} and intensity I_{1+2} are:

$A_{1+2} = A\cos(\omega t+z_1 k) + A\cos(\omega t+z_2 k)$

$A_{1+2} = 2A \cos(\omega t+[z_1+z_2]k/2) \cos(\Delta z\, k/2)$
$I_{1+2} = \langle A_{1+2}^2 \rangle$
$I_{1+2} = 4A^2 (1/2) \cos^2[\Delta z\, k/2]$
$I_{1+2} = A^2 (1+\cos[\Delta z\, k])$

As we see, the intensity varies between $2A^2$ and zero, averaging A^2, which is the sum of the intensity of each wave separately. The combined intensity peaks at $y=\Delta z=0$, and drops to zero when $\cos[\Delta z\, k]=-1$, where:

$\pm\pi = (\Delta z)\ (k)$
$\pm\pi = (yD/L)\ (2\pi/\lambda)$
$y = \pm\ \lambda L / 2D$

The corresponding beam angles are:

$\theta \approx \tan\theta = y/L = \pm\ \lambda / 2D$

The angular span of a full cycle, one white and one black fringe, is $2\theta = \lambda/D$.

These are wave interference effects to which classical physicists were well accustomed.

Two-Slit With Real Electrons

In V1p37-4 and V3p1-4, Feynman suggested performing the two-slit experiment using electrons in place of classical particles or classical waves. Because of the extremely short wavelength of electrons, such an experiment is so technically challenging that no one, Feynman included, expected that it could actually be done in the foreseeable future. He invited his students to consider this as just a thought experiment.

However, nothing inspires experimental physicists more than the world's preeminent theorist saying an experiment is impossible. It's like waving a red flag at a bull.

An Italian group led by Guilio Pozzi achieved the first successful electron interference experiment in 1974. In 1989, Akira Tonomura of Hitachi followed with an improved experiment. Rather than trying to make slits of the required minuteness, both experiments used optical separators. In 2013, a U.S.-Canadian team led by Herman Batelaan published the grandest such experiment so far. They made two physical slits in gold foil that were 62 nm wide and 272 nm apart. Their source emitted electrons of energy 600 eV, corresponding to a wavelength of 0.008 nm. One of their images is shown in Figure 1-5. It clearly demonstrates interference fringes, proving electrons have wave properties.

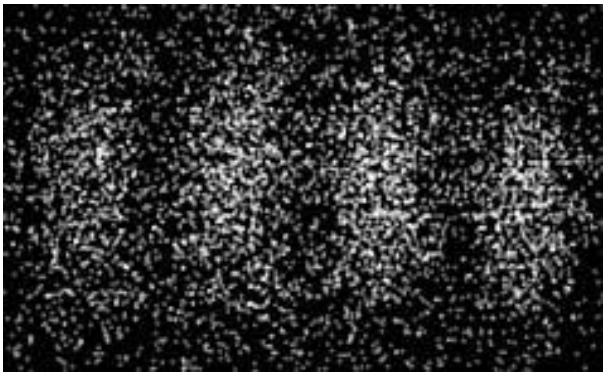

Figure 1-5 Two-Slit
Experiment With Electrons

Looking closely at the image, we see countless tiny discrete dots. Each dot corresponds to a single electron impact. Since waves impact broad areas not single points, the tiny dots prove electrons have particle properties. The combination of both effects proves electrons are simultaneously particles and waves.

Here is the most staggering conclusion. In this experiment, electrons were emitted at a rate of one per second, so only one electron at a time went through the apparatus. This means each electron must have gone through **both slits simultaneously** and **interfered with itself** at the detector. This is nothing like your grandfather's electron.

This two-slit experiment, suggested by Feynman, reveals the ultimate expression of particle-wave duality.

In V3p1-1, Feynman says the two-slit experiment "is impossible, *absolutely* impossible, to explain in any classical way, and … has in it the heart of quantum mechanics." He said this experiment encapsulates all the mysteries of quantum mechanics, and adds: "We cannot make the mystery go away by 'explaining' how it works.

To paraphrase Hermann Minkowski on spacetime: particles of themselves and waves of themselves will sink into mere shadows, and only a kind of union between them, particle-waves, shall survive. There is no longer any doubt: particle-wave duality is a fundamental property of nature.

We have lost both our classical particles and our classical waves, but as Feynman points out in V3p1-1, at least they have become the same entity: we now need only deal with particle-waves.

The Glory of Unification

Einstein unified two seemingly distinct phenomena — particles and waves — into a single entity. This was one of many occasions in which Einstein found the underlying unity of apparently disparate entities and thereby advanced our understanding of nature. Newton unified all gravitational effects in a single theory of universal gravity. Maxwell and Faraday unified electricity, magnetism, and light. Einstein unified particles and waves, then he unified space and time, then mass and energy, and finally spacetime and mass-energy. Each unifying advance brought a more profound understanding of our natural world.

Chapter 1 Review: Key Ideas

1. Quantum mechanics is the physical theory of elementary particles, how they interact with one another, and how they form atoms and larger structures.

2. Quantum mechanics is truly how the world works: it is the heart of physics.

3. Although many conclusions of quantum mechanics defy our intuition, physicists believe we now know all its rules. We can solve all its equations. In this sense, quantum mechanics is a mystery that we have solved but not fully digested.

4. The two key principles of quantum mechanics are:

    - Quantization
    - Particle-Wave Duality

5. Quantization is the notion that many things in nature come in integral numbers, like steps on a staircase. In the micro-world, staircases dominate; their steps are large and abrupt. In the macro-world, the steps are so small and so numerous that nature's staircases seem like ramps.

6. We now know "particle" and "wave" are really labels, like "black" and "white", for the opposite ends of a continuous spectrum. Everything in our universe is really a shade of gray. In our macro-world, almost everything is almost completely black or white. But in the micro-world, gray rules.

7. Every particle has a wavelength λ determined by its momentum p: $\lambda=h/p$.

8. Classical particles and classical waves do not exist. Both are now replaced with particle-waves.

Chapter 2

Particle-Wave Duality & Uncertainty

Particles Are Wave Packets

Much of this section supplements the ***Feynman Lectures***.

How can a particle, which seems localized, also be a wave that strives to spread across all space? Said another way, how can there be a wave-like entity inside a particle?

Particles can indeed be composed of waves in the form of *wave packets*, as illustrated in Figure 2-1. Here five waves of different frequencies combine to form a wave packet. Each of the five waves on the left spreads evenly throughout all space; they are completely un-localized. Yet their sum is somewhat localized, being confined within a horizontal distance Δx. It looks somewhat particle-like.

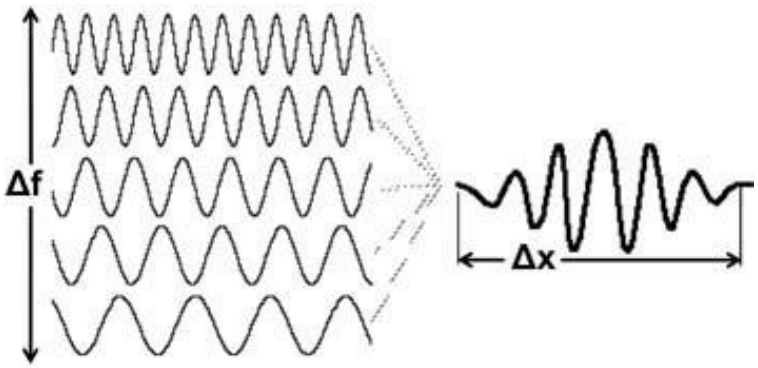

Figure 2-1 Waves Making a Wave Packet

But on the other hand, each of the five waves on the left has a definite frequency, which their sum does not. The wave packet is a composite of frequencies spanning a range Δf.

Particle-wave duality leads us to model all entities as wave-packets, each spanning a distance Δx and composed of frequencies spanning a range Δf. When an entity is more localized (small Δx) it is more particle-like. When an entity is spread broadly (large Δx) it is more wave-like with a better-defined frequency (small Δf).

Wave packets impose tradeoffs. Adding more different frequencies reduces a wave packet's width Δx, but at the expense of broadening its frequency spread Δf. Adding fewer different frequencies reduces Δf at the expense of broadening Δx. The wave packet-particle's position can become less uncertain (smaller Δx) only if its frequency becomes more uncertain (larger Δf), and vice versa.

The tradeoff is mathematically inescapable: Δf and Δx are inversely proportional. We can express

that relationship in terms of momentum p:

$\Delta x \sim 1/\Delta f$
$f \sim 1/\lambda \sim p$
$\Delta x \sim 1/\Delta p$
$\Delta x \bullet \Delta p = \text{constant}$

This is the basis of the Heisenberg Uncertainty Principle: particle-waves have an irreducible minimum product of (the uncertainty in position) × (the uncertainty in momentum).

When the uncertainties have normal Gaussian distributions, and the Δ's represent one standard deviation, the proper mathematical statement of Heisenberg's Uncertainty Principle is contained in four equations:

$\Delta x \bullet \Delta p_x \geq \hbar/2$
$\Delta y \bullet \Delta p_y \geq \hbar/2$
$\Delta z \bullet \Delta p_z \geq \hbar/2$
$\Delta t \bullet \Delta E \geq \hbar/2$

where $\hbar = h/2\pi$

The conditions described above are optimal. Uncertainties could be worse (more uncertain) than above, but they can never be better.

These uncertainties are the direct consequence of particles having wavelengths.

Phase Space

This section supplements the ***Feynman Lectures***.

One way to think about the uncertainty principle is in geometric terms. Einstein excelled at transforming complex dynamics into geometry, which is generally simpler and more intuitive.

Consider $\Delta x \bullet \Delta p_x \geq \hbar/2$. It says the product of two quantities has a minimum value. We can think of the product of two quantities as representing an area. In this case, an area that is Δx wide by Δp long. What the uncertainty principle is saying, in these terms, is that a particle-wave requires some minimum area — it demands an area of at least ħ/2. It is impossible, Heisenberg says, to squeeze a particle-wave into anything smaller.

Now, this minimum area can have almost any shape, as illustrated in Figure 2-2. It can be square, long and narrow, or even curved. But the area on a (x,p_x) plot cannot be smaller than ħ/2.

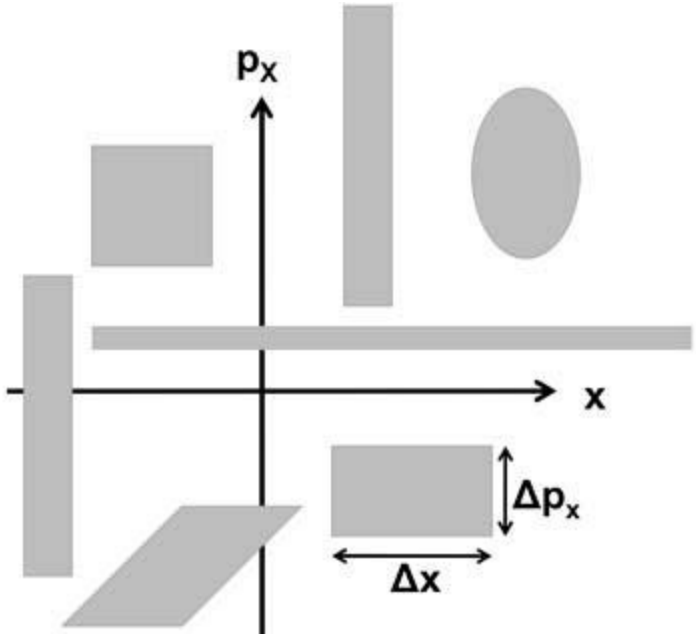

Figure 2-2 Phase Space for x,p_x

This (x,p_x) expanse is called *phase space*. Phase space is a mathematical tool, not a representation of any real physical object. In classical physics, each particle occupies a single point in phase space, with a definite position and momentum. And a wave occupies a single horizontal line (one momentum value, but spread across all x values). In quantum physics, particle-waves occupy an area. This is the essential distinction between the quantum and the classical viewpoints.

For the single spatial dimension x, the corresponding phase space is two-dimensional. Since the uncertainty principle applies equally and independently to all three spatial dimensions, the complete phase space representation is six dimensional: (x,y,z,p_x,p_y,p_z). In 6-D phase space, a particle-wave must occupy a volume of at least $\hbar^3/8$.

There is no need to add E, t and go to an 8-D phase space. Since $E^2=p^2c^2+m^2c^4$ for any particle-wave, knowing the three Δp's defines ΔE.

The only limitation to this geometric analogy is that one cannot trade x real estate for z real estate; the minimum area requirement applies separately for x, y, and z.

Can We Peek Under The Curtain?

That nature prevents us from knowing fundamental quantities with unlimited precision is anathema to physicists. Since the dawn of quantum mechanics over a century ago, physicists have tried hundreds of clever schemes to circumvent the uncertainty inherent in particle-wave duality. All such schemes have failed; exhaustive searching has found no chinks in quantum mechanics' armor.

In V3p1-6, Feynman examines the failure of one such scheme.

For the electrons that we know and love, going through two slits simultaneously is "utterly impossible" some say. Some suggest sneaking a spy into the two-slit experiment to reveal which slit each electron "really" goes through. As shown in Figure 2-3, a light source shines downward

covering both slit exits. After going through the slits, electrons traversing the spy light beam deflect the beam's photons. We collect the deflected photons with an optical system that focuses light from the upper slit at one point and from the lower slit at another point. This informs us which slit (or slits) each electron went through.

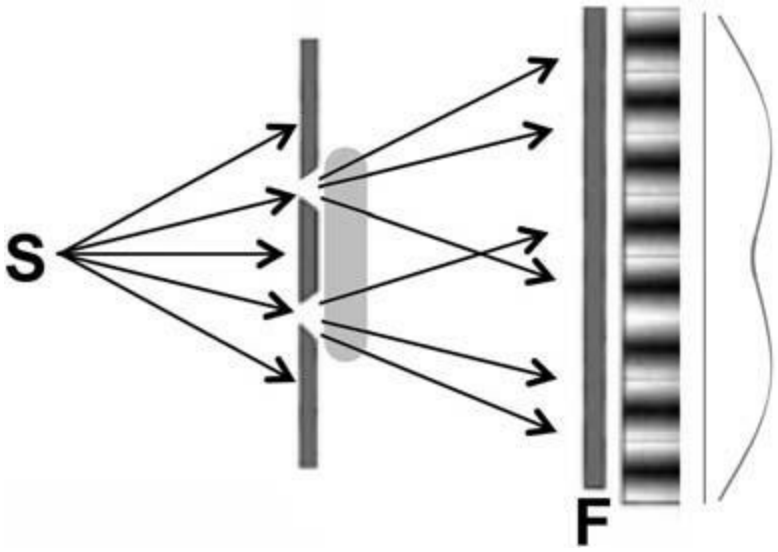

Figure 2-3 Spy Light Spans Slit Exits

Will we see light flashes from one slit **or** the other as each electron-particle goes by, or will we see simultaneous flashes from **both** slits as each electron-wave goes by? Will the intensity distribution be the broad smooth curve of a classical particle, like that on the extreme right? Or will we see the interference fringes of a classical wave, like the pattern next to F? Surely with this setup, we can now determine whether electrons are fish or fowl.

Let's see about that.

No one has yet done a two-slit experiment of this type. But innumerable experiments have **never** detected the same electron in two places simultaneously. Whenever measurements determine an electron's location, the result is one and only position. As we will discuss throughout this eBook, measurement requires a substantial interaction, and substantial interactions localize particles to specific positions. Thus we can say with confidence that our spy light will never detect simultaneous flashes from both slits.

This leaves three possible outcomes for each electron that strikes F:

1. We detect a light flash from slit #1
2. We detect a light flash from slit #2
3. We detect no flash

Our experiment can separately determine the probability distribution of impact points on F for each of the three outcomes. In the above order, the probability distributions actually observed are:

1. P_1, classical particle through slit #1
2. P_2, classical particle through slit #2

3. Interference, classical wave

When we identify which slit each electron passes through, the interference effect vanishes. If we turn off the spy light, the interference pattern returns.

Detecting where electrons go, changes where they go.

This is not completely unexpected if we consider what "detection" entails. To measure an object's location, momentum, or other characteristics, we must cause an interaction. Measurement is a contact sport, and the contact is more like boxing than dancing. In this case, we bombard electrons with photons. We detect an electron's position only when it substantially deflects a photon in a collision. Recalling Newton's third law, this happens only when the electron's momentum changes as much as the photon's momentum. Measuring an electron's position requires changing its momentum and possibly destroying the coherence of the two electron-waves exiting the slits.

Can we probe less invasively? Let's try reducing the intensity of our spy light.

Lower intensity means fewer photons. At very low intensity, not every electron is struck by a photon; some electrons travel from slit to screen without interaction. We then have two types of events: those with collisions, where we know which slit was traversed; and those without collision, where we don't. We find that events with collisions have the probability distribution of classical particles with no interference, while events without collisions have the probability distribution of classical waves with interference fringes.

We have gained no new information. This is because, while there are fewer photons, each has the same energy as before. Fewer collisions occur, but each collision that does occur entails the same substantial change of the electron's momentum.

Instead of reducing photon numbers, let's instead reduce each photon's energy — let's use redder light. We repeat the experiment many times, each time reducing the spy light photons' energy and increasing their wavelength.

At first, longer wavelengths make no difference. Electrons hit by our spy light do not produce interference fringes. But eventually, our photons are gentle enough not to destroy the electrons' coherence. At some spy light wavelength, call it λ_γ, the electrons produce a hybrid probability distribution; wave interference peaks and troughs start to breakup the broad smooth particle curve. When the spy light wavelength is several times λ_γ, the broad smooth curve is entirely replaced by interference fringes.

Astonishingly, the wavelength λ_γ at which interference returns is the wavelength at which we can no longer tell which slit each electron traversed. It seems like a conspiracy.

We measure an electron's position by bombarding it with photons, capturing deflected photons, and using their initial and final trajectories to locate the collision. The collision location, and hence the electron's location, is measured with a precision of one photon wavelength. Here's a crude but apt analogy. Imagine trying to find a glass vase in a dark room. We might throw golf balls in various

directions. The sound of shattering glass reveals where the vase was to within ±2cm. If one used basketballs, the position would be known to ±12cm. For beach balls, it might be ±75cm. For photons it's ± half a wavelength.

What is λ_γ in our spy light two-slit experiment? Define λ_e and p_e to be the electron's wavelength and momentum, and p_γ to be the spy photons' momentum. For each $\lambda p=h$. Recall from above that the interference fringe pattern repeats with an angular width of $\Delta\theta=\lambda_e/D$. If spy photons deflect electrons exiting the slits by an angle $2\Delta\theta$, the black and white interference fringes will be smeared out to a uniform gray. That deflection corresponds to a momentum change:

$\Delta p_e/p_e = \Delta\theta = \lambda_e/D$
$\Delta p_e = p_e\, \lambda_e/D = h/D$

The electron's momentum change equals the photon's momentum change (action=reaction). We detect deflected photons only if their trajectories change substantially. Assume as a rough approximation that this means the photon's momentum change equals its total momentum:

$\Delta p_e = \Delta p_\gamma \approx p_\gamma$
$h/D = h/\lambda_\gamma$
$\lambda_\gamma = D$

Interference effects are averaged out when λ_γ is less than the slit separation D, and they appear when $\lambda_\gamma>D$, when the spy photons' wavelengths are too large to determine which slit the electrons traverse.

Lest we bemoan a conspiracy to withhold "truth", this is nothing more than nature's inexorable logic. As often discussed in ***Feynman Simplified 1C*** Chapters 30 through 38, waves interfere when and only when they are coherent, having the same frequency and stable phase shifts. Substantial interactions, either due to random macroscopic chaos or to intentional measurements, destroy coherence and interference.

In V3p1-9, Feynman raises the question of why we don't see interference fringes with macroscopic particles. As Feynman says, interference is everywhere in nature, it's just a question of scale. For a 57-gram tennis ball served at 263 km/hr, momentum p = 4 kg m/s, corresponding to a wavelength of 1.6×10^{-34} m. That is impossible to detect.

Sliding Slits

In V3p1-11, Feynman examines another attempt to defeat the uncertainty principle. This time the barrier with two slits is mounted on rollers, allowing it to slide in the vertical direction, as shown in Figure 2-4. Again, D is the slit separation and L is the distance from barrier to screen F.

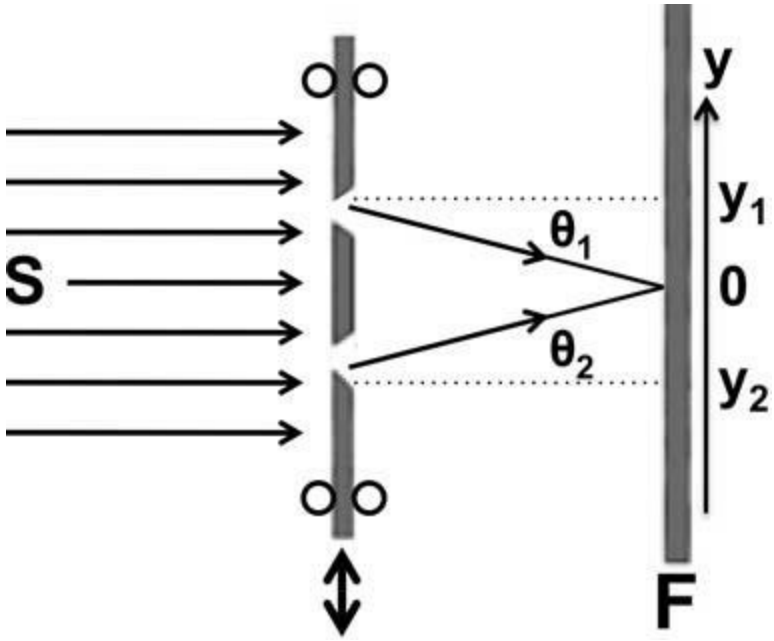

Figure 2-4 Two Slit with Barrier on Rollers

Consider two electrons traversing opposite slits and converging on F at y=0, as shown in the figure. The electron from the upper slit must be deflected downward by the barrier to reach y=0. This means the barrier must be deflected upward by the electron (action=reaction). Similarly, the electron from the lower slit must deflect the barrier downward to reach y=0. Thus by measuring the barrier's motion, we can determine which slit each electron traversed, without having to interact directly with the electron.

How clever. But not clever enough to defeat quantum mechanics.

It's best to use electrons from an extremely distant source S, with momentum p incident normal to the barrier. The momentum transfer required for an electron to reach point y on F is:

upper slit: $\Delta p_1/p = \theta_1 \approx \tan\theta_1 = (-D/2 - y)/L$
lower slit: $\Delta p_2/p = \theta_2 \approx \tan\theta_2 = (+D/2 - y)/L$
$\Delta p = \Delta p_2 - \Delta p_1 = pD/L$

Note that if the electron source was nearby, the electrons would have to bend more, and the total Δp would be larger. Our plan is to identify which slit electrons traverse by observing the barrier's recoil. As shown above, Δp does not depend on y; Δp has the same value for every position on the screen.

As Feynman explains, for this scheme to work, we must know, for each electron transit, both the barrier's vertical position and momentum change. However, everything in nature is a particle-wave; the barrier itself is subject to quantum mechanics. The uncertainty principle limits our ability to precisely know both the barrier position and its momentum. Measuring its momentum to a precision $\Delta p_в$ results in an uncertainty $\Delta y_в$ in the barrier's position of:

$\Delta y_в \approx h/\Delta p_в = h\,(L/pD) = (h/p)\,L/D = \lambda L/D$

In our analysis of the normal two-slit experiment, we found that the distance Δy on screen F at which

the interference pattern repeats is:

$\Delta y = \lambda L/D$

This means the fringe pattern width Δy is comparable to the barrier position uncertainty $\Delta y_{в}$.

The conclusion is: measuring the barrier momentum precisely enough for slit identification results in a barrier position uncertainty comparable to the width of one black and one white fringe. As the barrier jitters up and down, the interference pattern is averaged out to the smooth broad distribution of a classical particle.

A Brave New World of Uncertainty

In English, what Heisenberg announced in 1927 is universally called the "uncertainty" principle. Unfortunately, "uncertainty" connotes a human quandary, which is not the correct interpretation. **The uncertainty principle is not about our ignorance**. In the original German, Heisenberg said "ungenauigkeit", which is better translated as "unexactness" or "imprecision."

The uncertainty principle declares that nature is a bit fuzzy. There is an intrinsic imprecision in some fundamental quantities, including position, time, energy, and momentum. Even with the best imaginable instruments, we can never know with unlimited precision both the location and momentum of an electron. We can't know because **nature does not simultaneously establish both these quantities with unlimited precision**. Nature is inherently fuzzy, but in a very precise way.

In v1p1-10 Feynman says: "*We would like to emphasize a very important difference between classical physics and quantum mechanics*".

The laws of classical physics are completely deterministic. The entire future of every particle is predetermined with absolute precision, and has been since the Big Bang, if not before.

Conversely, imprecision / indetermination / uncertainty / fuzziness are inescapable in the quantum world. No particle's future is predetermined or can be predicted with certainty. By extension, this means no system of particles — atoms, humans, stars, galaxies, even the entire universe — has a precisely predetermined future.

Physicists were once sure we could measure anything and predict everything, with sufficient instruments. Now we must retreat to the goal of predicting probabilities.

For physicists, this is a traumatic setback, one that Einstein and others never accepted.

Einstein, Newton, and the little scientist in all our souls, believed the universe is a giant clockwork. Like a perfect clock, it ticks inexorably forward, governed by immutable laws, toward a predetermined future. They believed humanity could aspire to understand those laws and glimpse that future.

Quantum mechanics rudely wakens us from our utopian dreams to face the dreary reality that we are just playing craps. We can never again claim to predict **exactly** what electrons or anything else will do. We can only predict **exactly the probabilities** of various possible future outcomes.

Einstein bewailed: "God does not play with dice." Niels Bohr, his dear friend and colleague, replied: "Don't tell God what to do with His dice." As Feynman often admonishes, we must accept nature as it is, not as we wish it were.

Principles of Probabilistic Quantum Mechanics

In V3p1-9, Feynman lists principles governing probabilistic quantum mechanics. He defines an "ideal experiment" as one in which all the initial and final conditions are as precisely known as quantum mechanics allows. He defines an "event" as a specific set of these initial and final conditions. For example, the two-slit experiment is ideal, at least theoretically. One event might be: an electron leaves source S, arrives at position y on detector F, and no spy light flashes are observed. Here are the principles that countless experiments have confirmed:

1. In an ideal experiment, if event A can occur in **only one** way, the probability of that event, P(A) is:

$$P(A) = \phi\phi^* = |\phi|^2$$

where ø(A) is a complex number called the *probability amplitude* of A, and ø* is its complex conjugate. The probability of A is the square of the magnitude of the probability amplitude of A.

It is essential to understand the distinction between probability and probability amplitude. In quantum mechanics, "probability" has the same meaning as in mathematics: it describes the likelihood of some outcome. The term "probability amplitude" is unique to quantum mechanics; it has no analog in either our daily experience or in normal statistics. Being complex numbers, probability amplitudes are able to describe the interference effects of waves, which can combine either constructively or destructively. Probabilities always add, but probability amplitudes always interfere, either adding, subtracting, or somewhere in between.

2. When state A can occur in N different, **undistinguished** ways, we have:

$$\phi = \phi_1 + \phi_2 + \ldots + \phi_N$$

$$P(A) = \phi\phi^* = |\phi|^2$$

The probability amplitudes are summed, properly accounting for interference among the terms, and then that sum is squared. Note all N ways must produce the *same* state A.

3. When event A can occur in N different, **distinguished** ways, we have:

$P(A) = |ø_1|^2 + |ø_2|^2 + \ldots + |ø_N|^2$

$P(A) = P_1 + P_2 + \ldots + P_N$

The probability amplitudes are squared first, eliminating the possibility of interference, and then those squares are summed. This summing of probabilities is the standard of classical physics.

Quantum mechanics teaches us how to calculate ø and how to determine what is distinguished.

In V3p1-10, Feynman says:

> "One might still like to ask: 'How does it work? What is the machinery behind the law?' No one has found any machinery behind the law. No one can 'explain' any more than we have just 'explained.' No one can give you any deeper representation of the situation. We have no ideas about a more basic mechanism from which these results can be deduced."

While this may not be a "deeper representation", I can say something about when we square the sum of amplitudes (#2 above) and when we sum the squares of amplitudes (#3 above). The determining factor is coherence, or lack thereof.

Everything has wave properties. When coherent waves combine, they interfere, and one must sum amplitudes before squaring to get intensity. Without coherence, one sums probabilities, the square of amplitudes, in the classical manner.

Initially coherent waves, traveling different paths, retain their coherence if they remain undisturbed. Any substantial interaction disrupts coherence. That interaction could arise randomly from the chaotic macro-world, or it could arise from human measurement that determines the wave's path. The dividing line between distinguished and undistinguished is defined by the magnitude of any disruptive interactions.

The key factor is not human intention or knowledge. It is the magnitude of the change in wave frequency and/or phase due to external interaction. This was demonstrated in our discussion of the spy light. Coherence, and therefore interference, is lost when and only when the spy light photons are energetic enough to distinguish which slit the electron traverses. Whether or not humans actually record spy light flashes is irrelevant.

Many have proposed various "hidden variable" theories to circumvent the uncertainty principle's destruction of the beloved deterministic, clockwork universe. These theories claim elementary particles have internal characteristics, of which we are not yet aware, and that these hidden characteristics control the particles' behavior according to completely deterministic laws.

Einstein and two junior associates, Boris Podolsky and Nathan Rosen, made the first such proposal in a brilliant paper in 1935. Combining the first letter of each author's name, this concept is called the

EPR paradox. Einstein claimed to show that the predictions of quantum mechanics were so absurd that it must be an incomplete theory — it must be missing something hidden. EPR, and the many variants that it spawned, claim particles are always in well-defined states, with positions and momenta with zero uncertainty. If we were able to observe particles' hidden variables, they claim, we would see that nature is deterministic. The uncertainty espoused by quantum mechanics, EPR says, arises from our ignorance of these hidden variables.

The EPR saga continued for the rest of the 20th century. It witnessed many brilliant theories and amazing experiments. Long after Einstein and Feynman had passed, this issue was definitively settled: every possible class of hidden variable theories has been refuted by observations that only quantum mechanics can explain.

I plan to fully address EPR, as this fascinating subject deserves, later in the ***Feynman Simplified*** series.

Chapter 2 Review: Key Ideas

1. Heisenberg's Uncertainty Principle is stated in four equations:

   $\Delta x \, \Delta p_x \geq \hbar/2$
   $\Delta y \, \Delta p_y \geq \hbar/2$
   $\Delta z \, \Delta p_z \geq \hbar/2$
   $\Delta t \, \Delta E \geq \hbar/2$

   where $\hbar = h/2\pi = 1.055 \times 10^{-34}$ joule-sec, and
   $h = 6.626 \times 10^{-34}$ joule-sec is Planck's constant

2. The Uncertainty Principle forces us to abandon the hope of **exactly predicting the future.** Instead, we must resign ourselves to the goal of **exactly predicting the probabilities** of all possible future outcomes.

3. Observation requires interaction. Substantial interactions alter the behavior of what is observed.

4. The probability P(A) of an event A that can occur in **only one** way is:

   $P(A) = \phi\phi^* = |\phi|^2$

   where $\phi(A)$ is a complex number called the *probability amplitude* of A.

   When event A can occur in N different, **undistinguished** ways, we have:

   $\phi = \phi_1 + \phi_2 + \ldots + \phi_N$
   $P(A) = \phi\phi^* = |\phi|^2$

Note, for the above, all N ways must produce the *same* state A.

When event A can occur in N different **distinguished** ways, we have:

$P(A) = |ø_1|^2 + |ø_2|^2 + \ldots + |ø_N|^2$
$P(A) = P_1 + P_2 + \ldots + P_N$

Chapter 3

Particles, Waves & Particle-Waves

This chapter is more qualitative than quantitative. Starting in V3p2-1, Feynman explores particle-wave behavior, delineating the still-useful and the now-outdated aspects of classical ideas of waves and particles. Subsequent chapters provide comprehensive quantitative analysis.

In the prior chapter, we learned that, in quantum theory, the probability of finding a particle at location x at time t is proportional to the magnitude-squared of a probability amplitude ø(x,t), which is a complex number.

One example is $ø = A\exp\{-i\omega t+i\boldsymbol{k}\bullet\boldsymbol{r}\}$, for frequency ω, wave number 3-vector $\boldsymbol{k}$, and position vector $\boldsymbol{r}$. We write exp{y} rather than e^y for better ereading clarity. Amplitude ø describes a classical wave that fills all of space; its magnitude-squared is A^2 everywhere. This could be the equation for a plane wave of light or sound. Because ø has a definite frequency and wave number, it has a definite energy and momentum:

$E = \hbar\omega$
$\boldsymbol{p} = \hbar\boldsymbol{k}$

Conversely, if a particle is confined within a region of width Δy, the probability of finding it outside Δy must be zero. This means, the magnitude of the probability amplitude, and hence the amplitude itself, must be zero outside Δy. Mathematically, a wave-packet of limited spatial extent does not have a definite frequency. In quantum mechanics, de Broglie's equation λ=h/p extends that indefiniteness to the wave-packet's momentum.

Let's gain some insight into the uncertainty principle by examining two attempts to simultaneously measure particle-wave location y and momentum p.

Y & P Measured by a Barrier With Aperture

Consider a parallel beam of particle-waves incident on a barrier with a small aperture that is followed by a detection screen, as shown in Figure 3-1. Let D be the aperture width, L be the distance between barrier and screen, and let y=0 on the horizontal axis of symmetry.

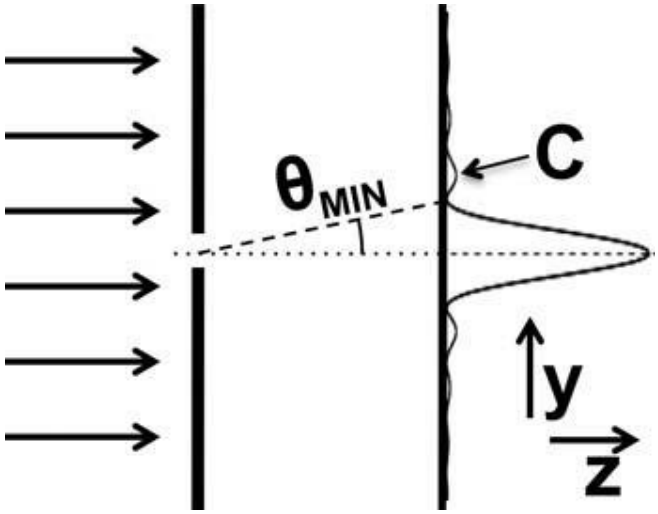

Figure 3-1 Plane Wave Diffracting
from Barrier with Aperture

Assume the incident particles have momenta $p_y=0$ and $p_z=p_0$. What do we know about the particles after they pass through the aperture?

Particles exiting the aperture must have y values between +D/2 and –D/2.

Feynman cautions: we might assume the particle's exiting y momentum is zero with zero uncertainty, "but that is wrong. We once knew the momentum was [entirely] horizontal, but we do not know it any more. Before the particles passed through the hole, we did not know their vertical positions. Now that we have found the vertical position … we have lost our information about the vertical momentum!"

According to particle-wave duality, these particles have a wavelength $\lambda=h/p_0$. We know waves diffract after passing through apertures (see ***Feynman Simplified 1C***, Chapter 31); so must these particle-waves.

At each point along the screen, such as point C indicated in the figure, there is some *probability* of detecting a *particle* hit. Unlike classical waves, each particle-wave hits the screen at one and only one point. But the probability of a particle-wave hit is distributed across the entire screen. A typical probability distribution is shown on the right side of Figure 3-1.

The uncertainty in y momentum can be can approximated by the angular width of the central peak of the probability distribution. As we discovered in the above-mentioned chapter, the angle of the first minimum intensity, θ_{MIN}, is:

$$\theta_{MIN} = \lambda/D$$

This means the spread in y-momentum is:

$$\Delta p_y = p_0 \lambda/D = h/D$$

The product of (the y position uncertainty) × (the y momentum uncertainty) is:

$$\Delta y \, \Delta p_y = (D/2)\,(h/D) = h/2$$

If we had been more precise in defining uncertainties, the constant on the right side would be smaller.

A brief mathematical side note: uncertainties should correspond to 1σ, one standard deviation. For example, with a probability distribution P(x) centered at x=0, $\sigma^2=\int x^2P(x)dx$. For P(x)=1/D for |x|<D/2 and P(x)=0 for |x|>D/2, $\sigma^2=(1/3)(D^2/4)$, so $\sigma=D/\sqrt{12}$, not D/2 as we used above.

Nonetheless, the essential point is clear. We can make Δy smaller by shrinking the aperture, which increases Δp_y. Making Δp_y smaller requires increasing Δy. We can trade one uncertainty for another but we cannot make both uncertainties arbitrarily small.

Feynman says some might protest that, after a particle hits the screen, we can precisely know both its position and momentum. We could, in principle, measure with great precision both the energy E and position y of the particle when it hits the screen. Combining that with the particle's y position at the slit (0±D/2), we can calculate its y momentum:

$$p_y = (y \pm D/2)/(Lc)\ \sqrt{(E^2 - m^2c^4)}$$

For sufficiently large L, the uncertainty Δp_y can be made arbitrarily small.

In V3p1-3, Feynman says:

> "It is quite true that we can receive a particle, and on reception determine what its position is and what its momentum would have had to have been to have gotten there. That is true, but that is not what the uncertainty relation [$\Delta y\ \Delta p_y \geq \hbar/2$] refers to. [That relation] refers to the *predictability* of a situation, not remarks about the *past*."

Feynman's point is that the uncertainty principle limits what we can know about what a particle's condition is before it is measured.

Y & P Measured by a Diffraction Grating

We next employ a diffraction grating to simultaneously measure the y position and y momentum of a particle-wave. A diffraction grating allows us to measure the wavelength of the particle-wave. We refer to the analysis of diffraction gratings in ***Feynman Simplified 1C***, Chapter 33.

Consider the grating shown in Figure 3-2, which features N=5 diffracting grooves, each separated by distance D.

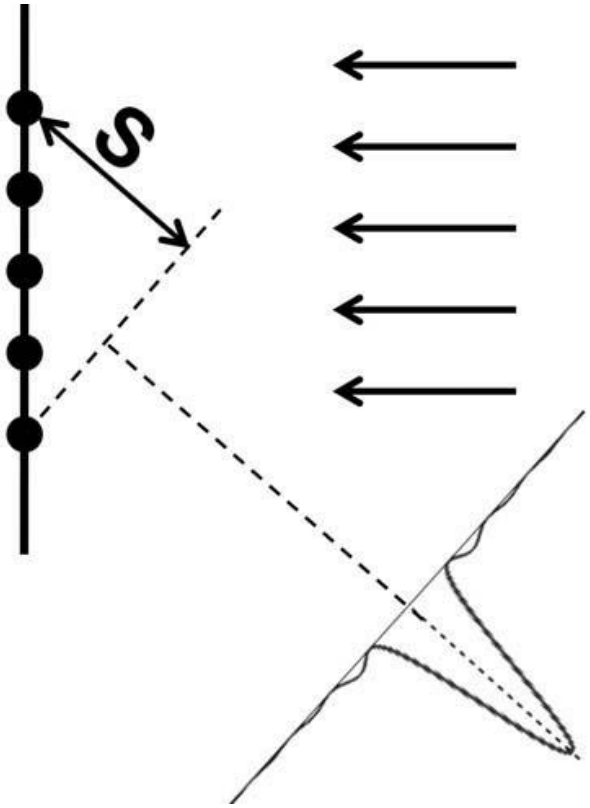

Figure 3-2 Diffraction from Grating

For a plane wave of definite wavelength λ, incident normal to the grating, an *mth order beam* is emitted at angle β, according to:

$\sin\beta = m\lambda/D$

To make the discussion simpler, take the case of m=1. Eager readers can work through the analysis for arbitrary m.

The emitted beam arrives at a remote screen, where we display its intensity profile, seen in the lower right portion of the figure. The m=1 beam requires that groove-to-screen path lengths increase by 1λ per groove. This means the top groove has a path length (N–1)λ longer than the path length of the bottom groove. In Figure 3-2, that extra path length is labeled S.

Also in ***Feynman Simplified 1C*** Chapter 33, a diffraction grating's resolving power is calculated, based on the Rayleigh criterion. A grating can distinguish two wavelengths that differ by Δλ when:

$\Delta\lambda/\lambda = \pm 1/Nm = \pm 1/N$ for m=1

That analysis assumed all grooves contributed to the m=1 order diffraction beam. Now let's examine more closely what that means. Imagine that the incident plane wave has just been switched on. At the detection screen, the first particles to arrive are those from the bottom groove, which is the closest. We will call that groove #1, and call the first arrival time t_1. Groove #2 is λ farther away, so its contribution arrives λ/v later, where v is the particle-wave velocity; call its arrival time t_2. Since λf=v for any wave, we can write $t_2=1/f$. The top groove's contribution arrives at $t_N=(N–1)/f$.

The key point here is that the full diffraction pattern shown in the figure, and the full resolving power of the grating, is achieved only when the incident wave lasts longer than t_N.

Imagine for example, that the plane wave has a duration equal to 2/f, two full wave cycles. At t_3, diffracted beams arrive at the screen from grooves #2 and #3. Groove #1 is no longer being excited by the incident wave and is no longer radiating. Contributions from grooves #4 and #5 haven't arrived yet. The groove #2 and #3 beams interfere, producing a diffraction pattern whose width and resolving power correspond to two grooves, not N grooves. At t_4, beams arrive from grooves #3 and #4. We again get a two-groove diffraction pattern, which is displaced from the pattern at t_3.

We never get the full N-groove pattern, unless the wave persists for at least N full cycles.

If the duration of the incident particle-wave is $\Delta t=J/f$, the best wavelength resolution possible is:

$$\Delta\lambda/\lambda = \pm 1/J$$

We define the particle-wave's direction of motion to be y and its momentum and energy to be p_y and E. We will approximate $\Delta t=J/f$ to be the uncertainty in the time of the particle-wave, and $\Delta y = v\ \Delta t$ to be the uncertainty in its position.

Note that:

$$\Delta E = h\ \Delta f$$
$$v = \lambda f, \text{ so } 0 = \lambda\ df + f\ d\lambda$$
$$df/f = -d\lambda/\lambda = 1/J$$
$$h = p\lambda, \text{ so } dp/p = -\ d\lambda/\lambda$$

Since we are dealing with uncertainties like $\pm\Delta\lambda$, the above minus signs are irrelevant. Combining these equations:

$$\Delta E\ \Delta t = (h\ \Delta f)\ \Delta t = h\ (f/J)\ (J/f)$$
$$\Delta E\ \Delta t = h$$

$$\Delta p_y\ \Delta y = (p_y\ \Delta\lambda/\lambda)\ (v\ \Delta t) = p_y\ (1/J)\ v\ (J\lambda/v)$$
$$\Delta p_y\ \Delta y = p_y\ \lambda = h$$

Feynman points out that all the above (except $\lambda=h/p$) result from the fundamental properties of classical waves, not from quantum mechanics. Recalling the wave packet of Figure 2-1, a wave packet confined to distance L has an uncertain number of wavelengths of "something like" ± 1. This corresponds to a wavelength uncertainty of $\pm 1/L$. Similarly, a wave packet confined to a time interval T has a frequency uncertainty of $\pm 1/T$.

In V3p2-4, Feynman says the classical notion of definite momentum fails when a particle-wave is confined to a small space, according to $\Delta p \sim h / \Delta x$. Similarly, the classical notion of definite energy fails for small time intervals, according to $\Delta E \sim h / \Delta t$.

Diffraction from a Crystal

Crystals are solids that are composed of repeating patterns of atoms. In V3p2-5, Feynman gives a 2-D analogy: wallpaper, whose repeating pattern, sometimes simple and sometimes more complex, repeats across the entire wall. The pattern that repeats is called the *unit cell*, and the manner in which the repetition occurs is called the *lattice type*.

X-rays are often used to study crystal structures, since their wavelengths are comparable to atomic spacings. Indeed, as discussed in ***Feynman Simplified 1C***, Chapter 33, x-ray crystallography is the most precise method of determining the atomic structure of anything that can be crystallized, from the simplest compounds to DNA containing billions of atoms. To get a strong reflected x-ray beam, we want constructive interference from each plane of atoms in the crystal. This is again a diffraction problem.

Figure 3-3 shows an incident x-ray beam at angle θ to the crystal plane that reflects, also at angle θ, and goes off to the upper right. We define d to be the spacing of adjacent crystal planes.

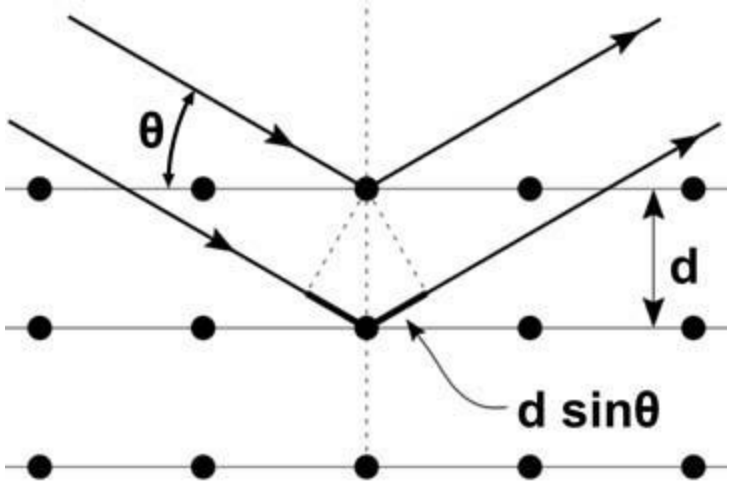

Figure 3-3 X-ray Diffraction from Crystal

Constructive interference requires the path lengths for reflections from adjacent planes to differ by an integer number of wavelengths. This means:

$2d \sin\theta = n\lambda$, for any integer $n>0$

If there were atoms halfway between the upper and middle planes of Figure 3-3, they would interfere completely destructively for the n=1 beam, since $2(d/2)\sin\theta$ would equal $\lambda/2$. This means the crystal orientation relative to the x-ray beam is critical.

Interestingly, the angles at which diffraction occurs, the angles with constructive interference, determine the lattice type, whereas the ratios of intensities at the various angles reveals the structure of the unit cell.

As with most scientific endeavors, getting the most out of x-ray crystallography requires a profound understanding of the underlying physics plus a good deal of creativity and craftsmanship — there's a reason outstanding performance is called the "state of the art."

Figure 3-4 shows an x-ray image of ZnS, zinc sulfide, taken by Max Laue, who received the 1914 Nobel Prize for discovering x-ray diffraction by crystals. This provided strong evidence that x-rays

are a form of light.

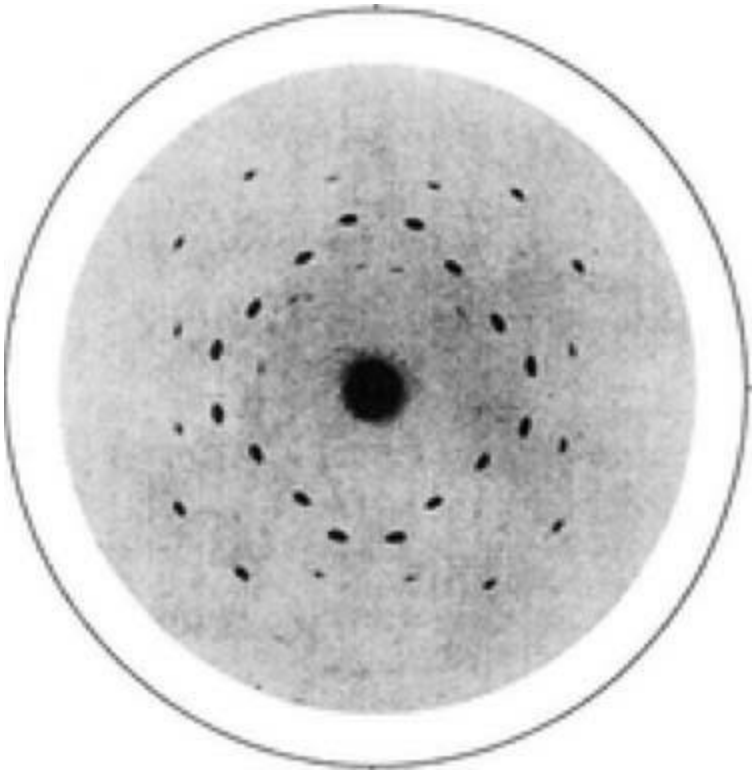

Figure 3-4 X-ray Crystallographic
Image of ZnS by Max Laue

Figure 3-5 shows an x-ray crystallographic image of an enzyme of the SARS virus. This highlights the crucial role of x-ray crystallography in biology and medicine.

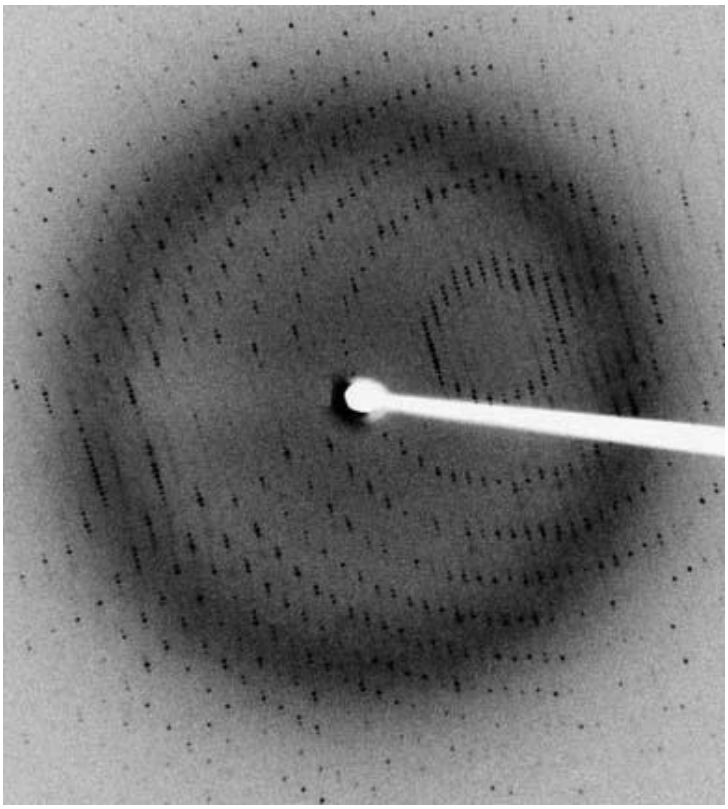

Figure 3-5 X-ray Image of SARS
Protease by Jeff Dahl

In V3p2-5, Feynman points out that for long wavelengths, particle-waves do not scatter coherently in a crystal. For $\lambda>2d$, the equation $2d\sin\theta=n\lambda$ has no solutions except $n=0$. A particle-wave incident normal to the crystal's exterior surface propagates in a straight line. In general some incoherent scattering and possibly other interactions will attenuate the incident beam, causing an exponential decrease in its amplitude. But that attenuation is sometimes very modest. As mentioned earlier in this course, visible light attenuation in glass is only 2% per km.

An interesting example of the absence of diffractive scattering of long wavelengths occurs in atomic reactors. Neutrons with low energies and $\lambda>2d$ traverse graphite without coherent scattering. But higher energy neutrons with $\lambda<2d$ are coherently scattered. This phenomenon allows the production of very low energy neutron beams. It also proves that neutrons have wave properties just as electrons do. And as Feynman says, neutrons "are obviously particles, for anyone's money!"

Uncertainty and the Size of Hydrogen

In V3p2—5, Feynman uses the uncertainty principle to calculate the size of the lowest-energy electron orbit in a hydrogen atom. This is an approximate analysis, enlightening but not rigorous.

Suppose an electron orbits the nucleus at radius r. If r were zero, we would know the electron's position with zero uncertainty and it would have to have infinite momentum, according to the uncertainty principle. For a position uncertainty $\approx \pm r$, the momentum uncertainty $\approx \pm\hbar/r$. Equating the momentum uncertainty with the electron's total momentum means the electron's kinetic energy is:

$$mv^2/2 = p^2/2m \approx \hbar^2/(2mr^2).$$

Also, at radius r, the electron's potential energy is $-e^2/r$ (see ***Feynman Simplified 1C*** Chapter 35). Hence the electron's total energy is:

$$E = \hbar^2/(2mr^2) - e^2/r$$

As r decreases, the first term above increases while the second decreases (becomes more negative). Nature always seeks the minimum energy, which is where $dE/dr=0$.

$$0 = dE/dr = -2\hbar^2/(2mr^3) + e^2/r^2$$
$$0 = -\hbar^2/m + e^2r$$
$$r = \hbar^2/(me^2), \text{ for hydrogen}$$
$$r = 0.529 \text{ angstroms} = 0.529\times10^{-10} \text{ m}$$

This is called the *Bohr radius*, denoted a_0. It provides a rough estimate of the size of a hydrogen atom. As we will soon discover, electron atomic states are more complex than simply "orbiting at radius r." Defining a "radius" is somewhat subjective; quoted measured values include 0.25 and 0.37 angstroms.

We can rewrite the equation for electron energy as:

$E = (\hbar c/r)^2/2mc^2 - (e^2/\hbar c)\,(\hbar c/r)$

Here, $\hbar c$ is a common combination whose value is 197.33 MeV fermi, in particle physics units. The expression $e^2/\hbar c$ is the famous *fine structure constant* α. Physicists usually quote its reciprocal: $1/\alpha$ equals 137.036. (In the early 20th century, when $1/\alpha$ was known to only 3 digits, Sir Arthur Eddington "proved" $1/\alpha$ was a prime number. More precise measurements refuted this notion. We now know α to better than 10 digits.)

With these two common constants, the electron energy at the Bohr radius is:

$\hbar c/r = (197.33 \text{ MeV fermi}) / (0.529\times10^{+5} \text{ fermi})$
$\hbar c/r = 3.73\times10^{-3} \text{ MeV}$

$(\hbar c/r)^2/2m = (3.73\times10^{-3} \text{ MeV})^2/(2\times0.511 \text{ MeV})$
$(\hbar c/r)^2/2m = +13.61 \text{ eV}$

$-e^2/r = -(3.73\times10^{-3} \text{ MeV}) / (137.036)$
$-e^2/r = -27.22 \text{ eV}$

$E = +13.61 - 27.22 = -13.61 \text{ eV}$

The total energy is negative because the electron is bound to the nucleus. Energy is required to move an electron away from the positive nucleus to a remote location where its potential is defined to be zero.

This energy, called the *Rydberg constant* in honor of Johannes Rydberg, is very precisely definable and is measured to 20 parts per billion. Since the electron's kinetic energy is 40,000 times less than its mass, our use of non-relativistic equations is entirely justified.

Another key point is that the electron's potential energy equals –2 times its kinetic energy. Known as the *virial theorem*, this is a general rule for systems subject to gravitational or electrostatic forces. It applies to the time-averaged kinetic and potential energies in stable systems, those that have reached equilibrium. The virial theorem applies to any force related to potential energy U by F=–grad(U).

The virial theorem leads to an intriguing conclusion about the potential energy released when very distant bodies are pulled together. Half of the released potential energy becomes kinetic energy, and the other half must be dissipated to achieve a stable orbit.

Why Atoms Are Stable

This section supplements the ***Feynman Lectures***.

Classical physics cannot explain why atoms are stable, but quantum mechanics can. As discussed extensively in ***Feynman Simplified 1C***, Maxwell's equations show that any accelerating electric charge radiates energy in the form of light. An orbiting body — Earth orbiting the Sun, or an electron

orbiting a nucleus — is accelerating because its velocity is continuously changing direction. For a circular orbit, $a=v^2/r$.

This means electrons orbiting nuclei must radiate energy. As their total energy, kinetic plus potential, decreases, their orbit must get smaller. Eventually, every electron in every atom must be confined within the nucleus of its atom. If you do the math, "eventually" means billionths of a second. If that did happen, atoms would be inert, unable to form molecules and compounds. They would also be trillions of times smaller. Earth would collapse to a tiny ball less than 200 meters wide and have a surface gravity of billions of g's.

Fortunately, classical physics is wrong.

Atoms are stable because electrons have wavelengths, as Niels Bohr first explained.

The following discussion is illustrative, but not fully realistic. The left side of Figure 3-6 depicts an electron orbiting a nucleus. For clarity, we split the electron's orbit at A and straighten it out, forming the rectangle shown in the upper right. The electron's wave amplitude, the black sinusoidal curve, is shown within the electron's orbit.

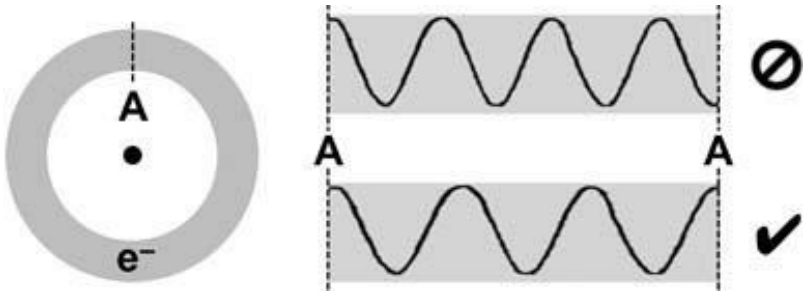

Figure 3-6 Bohr Model of an Atom
Circumference is Multiple of Wavelength

What we've done here is similar to making a flat map of Earth, something that is intrinsically spherical. Just as maps of Earth are often cut at the International Date Line, our map of the electron orbit is cut at A. Like Earth maps, our map shows A at both left and right ends. Both A's represent the same part of the electron orbit. What occurs at the left A must also occur at the right A, just as an island on the International Date Line must appear identically on both sides of a map of Earth.

We come now to the key physics. The wave amplitude at left A must equal the wave amplitude at right A, because they are the same point. An electron wave cannot have two different values at A, or anywhere else. This means the electron wave must complete an integer number of full cycles in one orbit, going from A around and back to A. The wave in the upper rectangle fails this requirement, while the wave in the lower rectangle passes.

We discover that an electron's orbit is **quantized**; its circumference must be an integer multiple of the electron's wavelength:

$$2\pi r = n\lambda$$

Since zero wavelength and zero wave cycles do not correspond to any real particle-waves, the smallest possible orbit occurs when n=1. Electrons can never have smaller orbits. Electrons in the

n=1 orbit cannot radiate, lose energy, and spiral into the nucleus.

Therefore atoms are stable.

The quantization of electron orbits in atoms is entirely contrary to our macro-world experience. Planets can orbit stars at any distance. Airplanes can fly at any height, and at any speed. But, electrons are permitted only a few possible orbits, corresponding to only a few possible energies.

It is impossible to overstate the importance of this one fact. Without it, no chemical or biological reactions would be possible. Even the entire evolution of the universe would have been very different, and vastly less interesting.

Electron Atomic Energy Levels

This section supplements the ***Feynman Lectures***.

For a single electron orbiting a nucleus with Z protons, the kinematic equations are:

$ma = F$
$mv^2/r = Ze^2/r^2$
$v^2 = Ze^2/mr$

Now add the Bohr requirement: $2\pi r = n\lambda$.

$2\pi r = n (h/mv)$
$mvr = n\hbar$

The last equation says electrons' angular momenta, mvr, are quantized in integral multiples of ħ. Quantizing angular momentum is mathematically equivalent to quantizing orbital circumference. Continuing:

$m^2(Ze^2/mr) r^2 = n^2\hbar^2$
$r = n^2\hbar^2 / Zme^2 = n^2 a_0$
$E_n = -Ze^2/2r = -13.61 \text{ eV } Z^2 / n^2$

The result we just derived based on particle wavelengths is the same as the result Feynman derived based on the uncertainty principle. This demonstrates the self-consistency of quantum mechanics.

As a result of the quantization of orbital size, energy levels are also quantized, although the steps are unequal, as shown in Figure 3-7.

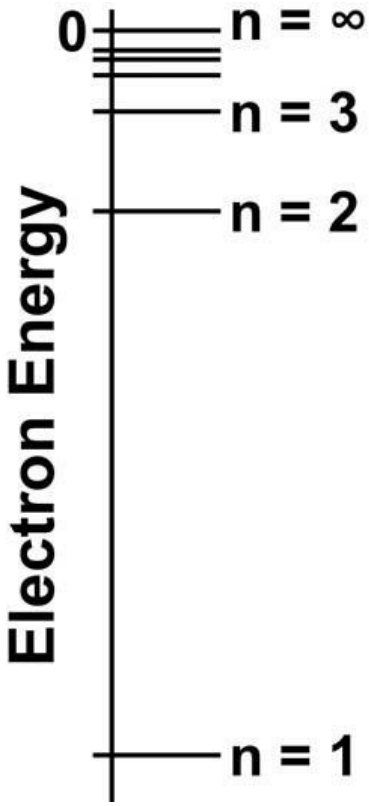

Figure 3-7 Electron Energy Levels in Hydrogen

Electrons in atoms are restricted to a very limited number of energy levels.

We still haven't explored the full complexity of electron orbits, but the most important physics is right here.

Feynman labels the lowest level E_0 to emphasize it is the ground state. While that is customary elsewhere, the energy levels in atoms are conventionally numbered E_1, E_2, …, where the index n is the *principal quantum number*, the electron shell number of the Periodic Table, as I've shown above. For the known elements, n ranges from 1 through 7. As we'll discover later, the nth orbit can contain up to $2n^2$ electrons.

For atoms with multiple electrons, the above equation must be revised due to electron-electron interactions. One can roughly say that inner electrons partially shield the nuclear charge, reducing the effective charge experienced by outer electrons, and requiring us to replace Z with a lesser value.

Electron Transitions & Light Frequencies

In V3p2-7, Feynman explains how atomic spectra result from the quantization of electron energies. Electrons can drop from a higher-energy orbit to a lower one by emitting a photon. If it drops from orbit m to orbit n (m>n), the change in electron energy and the photon energy are:

$$\Delta E_e = -(E_m - E_n) = (Z^2e^2/2a_0)\,(1/m^2 - 1/n^2) < 0$$
$$E_\gamma = \hbar\omega = -\Delta E > 0$$

The emitted photon carries off the energy that the electron loses. Since m and n are small integers, there are only a limited number of frequencies such photons can have. Each element has a different

number of protons in its nucleus, which leads to a different set of energy levels. This means each element has it own unique set of frequencies, its own *spectrum*, of light that it can emit.

The reverse process also occurs. An electron can jump from orbit n to orbit m by absorbing a photon of energy ħω, if and only if:

$$\hbar\omega = (E_m - E_n)$$

This means each atom absorbs light only at those frequencies at which it can emit light.

Figure 3-8 shows the spectra of hydrogen, helium, and carbon. In each row, the emission and absorption lines are shown, with low frequencies at the left and high frequencies at the right.

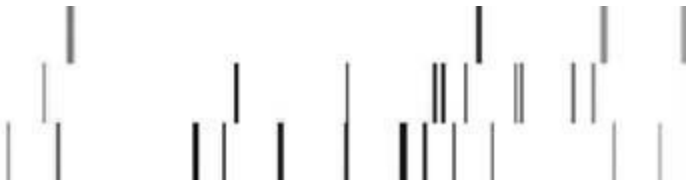

Figure 3-8 Emission & Absorption Lines for
Top to Bottom: Hydrogen, Helium & Carbon

Clearly, each element leaves a unique fingerprint in the light it emits and absorbs. Analyzing the spectra of the remotest galaxies reveals that they are made of the same elements that we are. At the core, everything in the universe is made of the same few parts.

Philosophical Implications

In V3p2-8, Feynman says that in developing quantum theory, physicists were forced to reevaluate what scientific theories can achieve. Quantum mechanics revealed that some quantities cannot be simultaneously measured, and that some quantities cannot be known until they are measured. Leading theorists came to believe that our models need not predict anything that is unmeasurable.

For example, if we do not measure through which slit an electron passed, our theories need not tell us that electron's path. Bohr said: "Nothing exists until it is measured." What happens to a particle between measurements, these theorists say, is not a question science can or should try to answer.

Feynman says:

> "The basis of a science is its ability to predict. To predict means to tell what will happen in an experiment that has never been done. How can we do that? By assuming that we know what is there…We must take our concepts and extend them to places where they have not yet been checked…We do not know where we are [wrong] until we 'stick our neck out'…the only way to find out that we are wrong is to find out *what* our predictions are. It is absolutely necessary to make constructs."

Feynman thought too much is made of the indeterminacy that quantum mechanics espouses. He said classical physics is not as deterministic as some assume. Classically, if we did know the **exact**

position and **exact** velocity of **every** particle in the universe, then yes we could predict the future precisely. But that is so unrealistic as to be meaningless.

Feynman says

> "Given an arbitrary accuracy, no matter how precise, we can find a time long enough that we cannot make predictions valid for that long a time. Now the point is that this length of time is not very large…it turns out that in only a very, very tiny time we lose all our information. If the accuracy is taken to be one part in billions and billions and billions…[then in less] time than it took to state the accuracy… we can no longer predict what is going to happen…Already in classical mechanics there was indeterminability from a practical point of view."

Indeed, in the years since his lectures, chaos theory, a branch of mathematics, has emerged. Chaos theory analyzes the rapidity with which tiny changes avalanche in large systems and randomizes their behaviors. They famously say a butterfly flapping its wings in the Amazon can cause a tornado in Kansas.

Chapter 3 Review: Key Ideas

1. The wave properties of particles preclude simultaneously measuring their position and momentum with unlimited precision.

2. The uncertainty principle provides a reasonable estimate of the radius of the lowest energy orbit of an electron in a hydrogen atom. Called the Bohr radius a_0, it is given by: $\hbar^2/(me^2) = 0.529\times10^{-10}$ m.

3. The Bohr model of atoms states electron orbits are **quantized**; the circumference of their orbits must be an integer multiple of their wavelength: $2\pi r = n\lambda$. In the smallest possible orbit, where n=1, electrons cannot radiate, lose energy, and spiral closer to the nucleus. This makes atoms stable, and enables chemical and biological reactions.

4. In the Bohr model of a single electron orbiting a nucleus with Z protons, the allowed orbital radii and energies are:

   $r_n = n^2 a_0 / Z$
   $E_n = -Ze^2/2r_n = -13.61 \text{ eV } Z^2 / n^2$

5. The virial theorem states that an orbiting body's potential energy equals –2 times its kinetic energy, in stable systems subject to gravitational or electrostatic forces.

6. Since electron energies are quantized, when they move from orbit m to orbit n, they must emit or absorb a photon whose energy exactly balances the change in electron energy: $\hbar\omega = (E_m - E_n)$. This means each element emits and absorbs a unique set of light frequencies, a unique spectrum.

Chapter 4

Probability Amplitudes

In V3p3-1, Feynman explains his plan to teach quantum mechanics as it had never been taught before. The traditional teaching approach is to follow the development of quantum theory through the different stages that physicists went through as their knowledge matured. Feynman elects to bypass all that and go straight to the most modern version of quantum theory.

Feynman takes beginning students right to the cutting edge.

My goal is to ensure that, for you, this isn't the bleeding edge.

Henceforth, for brevity, I will often say "particle" or "wave" instead of "particle-wave", since we now know that everything is actually a particle-wave. Also, I will often shorten "probability amplitude" to simply "amplitude."

In quantum mechanics, we frequently use the term *state* to describe the condition of a particle or system. A quantum state defines all the variable properties that entities can have, such as position, momentum, energy, and angular momentum. A quantum state does not define the intrinsic properties that each particle has, such as charge and mass. In general, different types of particles can be put into a given state, and can be moved from one state to another.

It is beneficial to think of a quantum state as a vector. It defines a location in the abstract space of all possible properties.

Often only some of a particle's variable properties are of interest. For example, we might be concerned with directions and energies of photons but not their polarizations or locations. In such cases, we often say photons are in the same *state* if their directions and energies are the same, regardless of differences in other variables.

Combining Amplitudes in Bra-Ket Notation

In this lecture, Feynman discusses more general principles of quantum mechanics, and assigns numbers to some of them. I will try to follow that format for your convenience, but you should know that there is no standard numbering scheme for these principles as there is for Newton's and Kepler's laws.

Recall the two-slit experiment: particles, emitted by a source S, encounter a barrier with two small slits. Those particles that transit the slits arrive at a detection screen F, where their impact point is recorded.

The **first general principle** of quantum mechanics is that the probability P(y) of a particle arriving at point y is proportional to the square of the magnitude of a probability amplitude ø(y), which is a complex number.

Paul A. M. Dirac developed the standard notation for probability amplitudes, called the *bra-ket* notation. The format of Dirac's notation is:

The amplitude that A results in (or goes to) B is written: $< B \mid A >$

Note that this notation is read from right to left.

Recalling our analogy of a quantum state as a vector in property space, <B|A> is analogous to the dot product of vectors A and B — <B|A> measures the degree to which A and B have the same properties.

For the two-slit experiment, we write:

ø(y) = <particle arrives at y | particle leaves S>

Or even more simply:

ø(y) = <y|S>

Here, for any specific y, <y|S> is a single complex number. More precisely, ø(y) is a *scalar function* of the variable y. That's a fancy way of saying: at every y, ø(y) is a single number.

The **second general principle** of quantum mechanics states that when an event y can occur in two or more ways that are **undistinguished**, ø(y) equals the sum of the probability amplitudes of each separate way. For example, if the amplitudes of three undistinguished possible paths to reach y are $ø_1$, $ø_2$, and $ø_3$, the total amplitude to reach y is:

$$ø(y) = ø_1 + ø_2 + ø_3$$

Note that this principle applies only if every undistinguished path reaches the *same* state y.

In Chapter 2, we discussed the specific meaning of "distinguished" in quantum mechanics. When waves combine after traveling different paths, they interfere **if** they haven't been substantially disturbed. A substantial disturbance alters a wave's frequency and/or phase sufficiently to randomize its phase angle relative to other waves. That destroys coherence and eliminates the possibility of interference.

As the two-slit experiment demonstrates, with no substantial disturbances, it is impossible to *distinguish* which paths particles travel to reach an event y — impossible for both man and nature. Paths are *distinguished* only when waves are substantially disturbed. What matters is the magnitude of the disturbance, not whether or not it is human-directed. But due to our scale, any human-directed interaction will almost certainly substantially disturb a quantum system.

For the two-slit experiment, if the paths through slits #1 and #2 are undistinguished, the amplitude to reach y is:

$\langle y|S\rangle = \langle y|1\rangle\langle 1|S\rangle + \langle y|2\rangle\langle 2|S\rangle$

We read the Dirac notation $\langle y|1\rangle\langle 1|S\rangle$ as: "(the amplitude of a particle from S passing through slit #1) multiplied by (the amplitude of a particle from slit #1 arriving at y)."

Here, we used what I will call the **fourth general principle** of quantum mechanics: the amplitude for a sequence of events equals the product of the amplitudes for each event separately.

The **third general principle** of quantum mechanics states that when an event y can occur in two (or more) ways that are **distinguished**, the probability of y, P(y), equals the sum of the squares of the amplitudes of event y to occur each way separately. In Dirac notation:

$|\langle y|S\rangle|^2 = |\langle y|1\rangle\langle 1|S\rangle|^2 + |\langle y|2\rangle\langle 2|S\rangle|^2$

Note that $|z|^2$ denote the magnitude squared of the complex number z, whereas the | between < and > means "results in." Again, unfortunately, we have multiple uses for the same symbols.

Let's now do some bra-ket calisthenics addressing a more interesting situation. Augment the two-slit experiment by adding a second barrier with three slits between the original barrier and screen F, as shown in Figure 4-1.

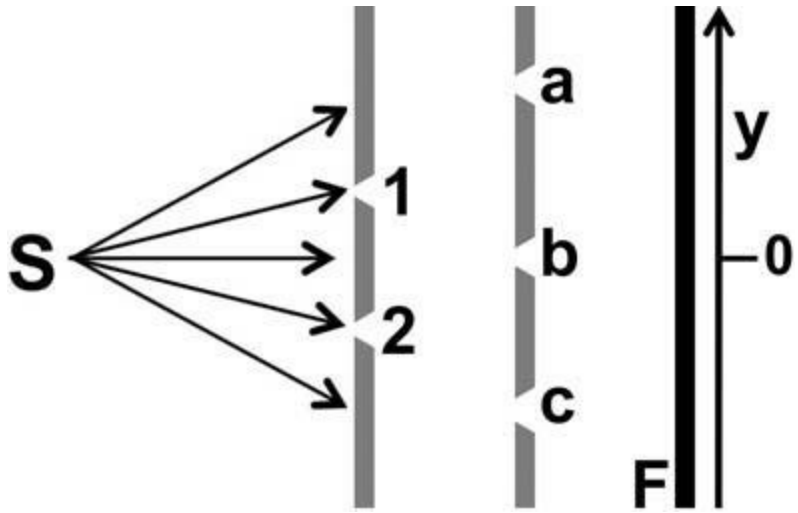

Figure 4-1 Experiment with Two Barriers & Five Slits

There are six ways for a particle to go from S to a point y on F. One way passes through slit #1 and then slit #b, for which the amplitude is:

$\langle y|b\rangle\langle b|1\rangle\langle 1|S\rangle$

The total amplitude for a particle to leave S and arrive at y is the sum of the amplitudes of all six ways:

$\phi(y)=\langle y|a\rangle\langle a|2\rangle\langle 2|S\rangle+\langle y|a\rangle\langle a|1\rangle\langle 1|S\rangle$

$+ \langle y|b\rangle\langle b|2\rangle\langle 2|S\rangle + \langle y|b\rangle\langle b|1\rangle\langle 1|S\rangle$
$+ \langle y|c\rangle\langle c|2\rangle\langle 2|S\rangle + \langle y|c\rangle\langle c|1\rangle\langle 1|S\rangle$

We can write that more compactly as the sum over q=a,b,c, and m=1,2:

$\phi(y) = \Sigma_{qm} \langle y|q\rangle\langle q|m\rangle\langle m|S\rangle$

Calculating Amplitudes

Next we need to know how to calculate a probability amplitude for each specific alternative.

Take the simplest case first: a free particle traveling through empty space with no forces acting on it. We recall the wave equation:

$\text{Wave height}(t,\boldsymbol{r}) = A \exp\{-i(\omega t - \boldsymbol{k}\bullet\boldsymbol{r})\}$

Since $E=\hbar\omega$ and $\boldsymbol{p}=\hbar\boldsymbol{k}$, we could equally well write:

$\text{Wave height}(t,\boldsymbol{r}) = A \exp\{(Et - \boldsymbol{p}\bullet\boldsymbol{r})/i\hbar\}$

In V3p3-4, Feynman says the amplitude μ for this wave to propagate from event $(t_1,\boldsymbol{r}_1)$ to event $(t_2,\boldsymbol{r}_2)$ is:

$\mu = \langle t_2,\boldsymbol{r}_2|t_1,\boldsymbol{r}_1\rangle = A \exp\{(E[t_2-t_1] - \boldsymbol{p}\bullet[\boldsymbol{r}_2-\boldsymbol{r}_1])/i\hbar\}$

While this looks like a familiar wave equation, Feynman cautions that the *wave function* that is a solution to this wave equation is a mathematical construction not directly associated with any real physical entity. Indeed, the wave function of a two-particle system is a function of eight variables, the four coordinates of each of the two particles.

The only connection to physical reality that a solution of these wave equations has is that the square of its magnitude is the probability of a real event.

For multiple particles, we have a **fifth general principle**, similar to the fourth for multiple actions of one particle. If ϕ_1 is the amplitude of particle 1 going from S_1 to F_1, and ϕ_2 is the amplitude of particle 2 going from S_2 to F_2, then the amplitude for both events to occur is $\phi_1 \times \phi_2$.

Feynman emphasizes another point. In discussing the original two-slit experiment, we wrote the amplitude for a particle to go from source S through slit #1 and arrive at point y on screen F as:

$\langle y|1\rangle\langle 1|S\rangle$

Everything we really need to know about the particle's initial condition is $\langle 1|S\rangle$. We don't need to know what source S is or how it works. Just one complex number is all we need. Someone could completely change how particles are emitted. None of those details concern us, as long as we know

<1|S>.

Two-Slit with Spy Light in Bra-Ket Notation

Let's reexamine our two-slit with spy light experiment (from Chapter 2) using < | >'s. The setup is repeated here in Figure 4-2. Electrons from source S strike a barrier with two slits. Those electrons passing through each slit may deflect photons from a spy light shining down into the screen of your ereader. An optical system collects deflected photons and focuses them on two light detectors; photons from slit #1 are focused on detector #1 and those from slit #2 are focused on detector #2. The slits are separated by distance D and are a distance L from screen F. As before, L>>D.

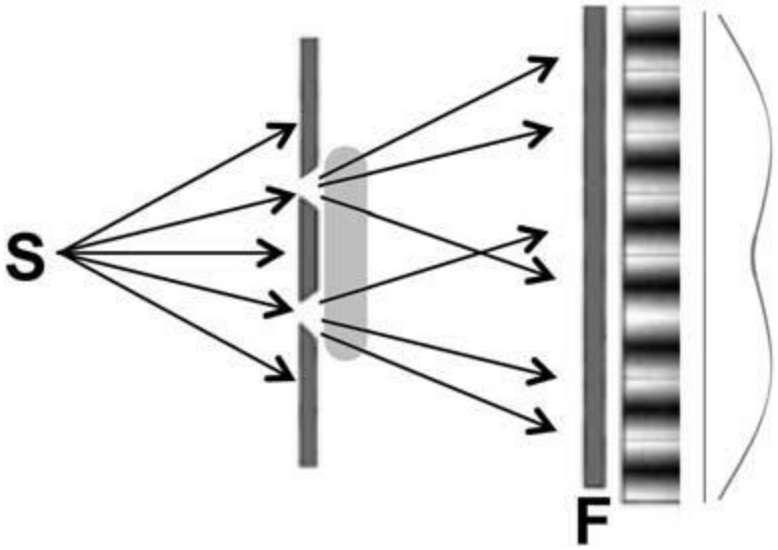

Figure 4-2 Two-Slit with Spy Light

Recall the result we found in Chapter 2. With the spy light off, waves from the two slits maintain coherence and produce the interference fringe pattern shown next to F. With the spy light on and set to a short wavelength, λ<D, electrons that deflect photons lose coherence and produce the smooth broad distribution on the far right of the figure. With the spy light on and set to a long wavelength, λ>D, electrons that deflect photons are not substantially disturbed and they produce interference fringes. For all spy light conditions, electrons that avoid deflecting photons maintain coherence and produce interference fringes. When the spy light wavelength is greater than D, we cannot tell which slit electrons traverse — no optical system can separate the images of slit #1 photons versus slit #2 photons, when λ>D.

Now let's define amplitudes for various events:

$ø_1$ = <y|1><1|S>, S thru slit #1 to y
$ø_2$ = <y|2><2|S>, S thru slit #2 to y
$ε_{JK}$ = photon from slit #J hits light detector #K
$ε_0$ = no photon detected from either slit

Ideally, $ε_{11}=ε_{22}=1$ and $ε_0=ε_{12}=ε_{21}=0$, which means photons from slit #1 always hit detector #1, photons from slit #2 always hit detector #2, and no electrons go undetected. We would then be 100% certain from which slit each photon originated. Very high intensity spy light with λ<<D approaches ideal

discrimination, while completely disrupting the coherence of every electron. At reduced spy light intensity, some electrons avoid photon collisions, decreasing the correct identification magnitudes ε_{11} and ε_{22}, and increasing ε_0. At long spy light wavelengths, the misidentification amplitudes ε_{12} and ε_{21} increase in magnitude.

Recall that no experiment ever finds the same electron in two places at once, so we never detect photons in both detectors from a single electron.

We will include all εterms in our analysis to obtain a completely general result.

Using the standard symbols e for electron and γ for photon, the amplitudes we want are:

e in slit #1 and ϒ in detector #1 = $\varnothing_1\ \varepsilon_{11}$
e in slit #1 and ϒ in detector #2 = $\varnothing_1\ \varepsilon_{12}$
e in slit #2 and ϒ in detector #1 = $\varnothing_2\ \varepsilon_{21}$
e in slit #2 and ϒ in detector #2 = $\varnothing_2\ \varepsilon_{22}$
e in slit #1 and no ϒ detected = $\varnothing 1\ \varepsilon_0$
e in slit #2 and no ϒ detected = $\varnothing 2\ \varepsilon_0$

Now, the amplitude for an electron to arrive at y and a photon to hit detector#1 is the sum of two terms:

$$\varnothing_1\ \varepsilon_{11} + \varnothing_2\ \varepsilon_{21}$$

The first term is for the electron to traverse slit #1 and deflect a photon that hits detector #1. The second term is for the electron to traverse slit #2 and deflect a photon that hits detector #1. Similar expressions apply to the other combinations of events.

Let's examine the probability P(y,1) of an electron arriving at y and a photon hitting detector #1.

$$P(y,1) = |\varnothing_1\ \varepsilon_{11} + \varnothing_2\ \varepsilon_{21}|^2$$

Similarly, the probability P(y,2) of an electron arriving at y and a photon hitting detector #2 is:

$$P(y,2) = |\varnothing_1\ \varepsilon_{12} + \varnothing_2\ \varepsilon_{22}|^2$$

Next, consider this important case: P(y,1or2), the probability of an electron at y with a photon hitting either detector, when spy light wavelength λ<D and we can distinguish which slit was traversed.

$$P(y,1or2) = P(y,1) + P(y,2)$$
$$P(y,1or2) = |\varnothing_1\ \varepsilon_{11} + \varnothing_2\ \varepsilon_{21}|^2 + |\varnothing_1\ \varepsilon_{12} + \varnothing_2\ \varepsilon_{22}|^2$$
NOT: $P(y,1or2) = |\varnothing_1\ \varepsilon_{11}+\varnothing_2\ \varepsilon_{21}+\varnothing_1\ \varepsilon_{12}+\varnothing_2\ \varepsilon_{22}|^2$

Feynman strongly advises avoiding the common mistake in the last line. Events (y,1) and (y,2) are two distinct, distinguishable final states, requiring we add their probabilities not their amplitudes. Although arguing with Feynman is a real long shot, someone might claim that they don't care which

detector the photon hits, that they choose not to distinguish which slit was traversed, that they won't even bother to record detector hits, and so amplitudes should be added before squaring.

That is wrong! Nature doesn't care whether or not we examine data. Nature's actions can't depend on decisions that we could make after the action is finished. Fringe patterns can't suddenly come and go depending on what we decide after the fact.

The critical point is the occurrence of substantial interactions that disrupt coherence. For a photon to hit a detector, the photon-electron collision must substantially change the momentum of both particles.

Never add amplitudes when the final states are different and distinguishable, even if the experimenter chooses not to distinguish them.

If our optical system had zero misidentification ($\varepsilon_{12}=\varepsilon_{21}=0$), our result would simply be the classical result for a particle without wave properties: the probability of an electron reaching y through slit#1 times the probability of detecting a deflected photon.

$P(y,1) = |\phi_1|^2 \, |\varepsilon_{11}|^2$

Conversely, if the spy light wavelength $\lambda >> D$ and any deflected photon is equally likely to hit either detector ($\varepsilon_{11}=\varepsilon_{21}$), then:

$P(y,1) = |\phi_1 + \phi_2|^2 \, |\varepsilon_{11}|^2$

This is the result for a classical wave producing interference fringes, with the added requirement of detecting a photon in either detector.

For spy light wavelength $\lambda \sim D$, P(y,1) is a mixture of partial coherence and partial incoherence.

Spin

Throughout the remainder of this chapter, and indeed our entire course on quantum mechanics, we will deal with *spin,* an iconic quantum property. Spin is a form of angular momentum that is intrinsic to each elementary particle.

By analogy a rotating top has spin. Like all forms of momentum, spin is a vector. The direction of the spin vector is along the axis of rotation, and the length of the spin vector, called the *total spin*, equals the amount of angular momentum. Like all vectors, we can define the *component of spin* along with any selected coordinate direction.

But since nothing in the micro-world is exactly the same as anything in the macro-world, quantum spin is different from that of a child's spinning top. Most importantly, quantum spin is quantized.

The primary fermions — quarks, leptons, protons, neutrons, and electrons — are all *spin one-half*: s = 1/2. Their component of spin along any chosen axis can only be $+\hbar/2$ or $-\hbar/2$. Physicists call these

two options *spin up* and *spin down*. Other fermions also have half-integral spin: s = 1/2, 3/2, …

Bosons, the force carrying particles, have integral spin: s = 0, 1, or 2. Their component of spin along any chosen axis can only be: –sħ, …, 0, …+sħ. Photons have spin 1, but since they move only at speed c, they never have spin component s=0, only s=+1 or –1. Gluons and the weak force bosons, Z and W, have spin 1. The Higgs boson has spin zero. If the graviton exists, it must have spin 2.

Above we discussed the components of spin along some axis. Spin is a vector that can point in any direction. If an electron has spin +1/2 along the z-axis, that does not mean its spin components are zero along other axes. Indeed, the length of an electron's spin vector is more than 1/2. That length is rarely referred to, but the spin vector's length is called *total spin.* A particle with spin s has total spin $S = \hbar\sqrt{[s(s+1)]}$; $S = \hbar\sqrt{(3/4)}$ for spin 1/2 particles, and $S= \hbar\sqrt{2}$ for spin 1 particles.

Note that any particle's component of spin along any axis can change only by integer multiples of ħ.

Neutron Scattering from a Crystal

In V3p3-7, Feynman presents a remarkable lesson in what is and what isn't distinguishable.

We previously discussed x-ray crystallography, the scattering of x-rays from a solid material whose atoms are arranged in a continually repeating pattern. We discovered that x-rays are coherently scattered at angles at which the equally spaced emitters interfere constructively.

In this study, we consider the scattering of neutrons by a crystal. Since neutrons have zero net charge, they do not interact with electric charges. Neutrons interact with, and can be scattered by, the nuclei of crystal atoms.

We learned in the last chapter that very low energy neutrons are not scattered coherently by crystals. If their wavelengths are greater than twice the crystal layer spacing: (λ>2d), constructive interference is impossible. Conversely, high energy neutrons can knock atoms out of the crystal lattice, which is a more violent collision than we wish to study here. We will therefore use neutrons of moderate energy that can scatter coherently but not disrupt the crystal structure.

Detecting neutrons is much more difficult than detecting protons, electrons, and other charged particles that leave in their wake long trails of ionized atoms. Neutron detection is accomplished primarily by observing charged particles that are released in neutron-nucleus interactions.

Let's imagine placing a neutron detector array around a crystal to measure the angle θ at which neutrons scatter from the crystal. We will assume all nuclei act identically, and number them 1, 2, … N. (N is going to be a very big number, but not to worry, there are more integers than nuclei in the observable universe.) The amplitude for a neutron from source S to be scattered at angle θ by crystal nucleus K is:

$\phi(\theta,K) = \langle\theta|K\rangle\, \sigma(\theta)\, \langle K|S\rangle$

Reading from right to left, the overall amplitude ø equals (the amplitude for a neutron to go from source S to nucleus K) × (the amplitude of a neutron to scatter at angle θ) × (the amplitude for a neutron to go from nucleus K to a detector at angle θ). This is almost like assembling Legos — we can snap parts together to make anything we can imagine.

A moderate energy neutron cannot dislodge an atom from the crystal; the recoil momentum is effectively transferred to the entire crystal. This means it is impossible to know from which nucleus the scattering occurred. There are N undistinguished ways for a neutron to scatter; we therefore sum the amplitudes before squaring.

$\Phi(\theta) = \Sigma_K \, ø(\theta,K) = \Sigma_K \, \langle\theta|K\rangle \, \sigma(\theta) \, \langle K|S\rangle$

Since the scattering sources are at different distances from the detector, there will be interference effects and we expect coherent scattering at specific angles, as illustrated in Figure 4-2a.

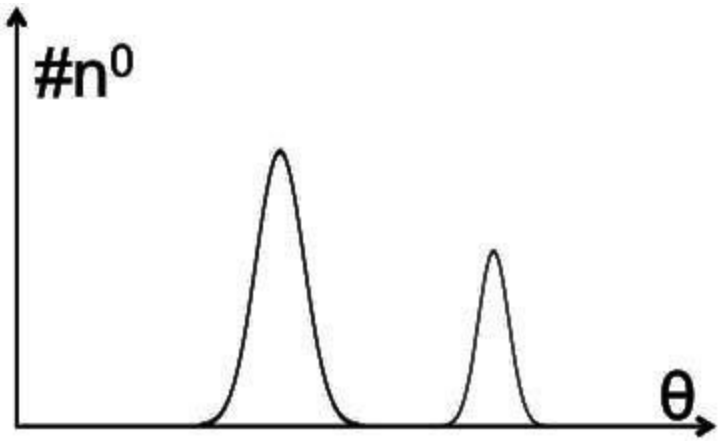

Figure 4-2a Spin-less Neutron Scattering vs. Angle

In some crystals, neutron-nucleus interactions are more complex due to spin.

Like all spin 1/2 particles, a neutron is always either spin up or spin down. But nuclei can have a variety of spins.

A nucleus has spin zero if equal numbers of protons and neutrons have their spins pointed in opposite directions. Some nuclei, however, have an odd number of fermions, resulting in an overall spin of 1/2; they are either spin up or spin down.

The neutron's spin orientation is irrelevant when it interacts with a spin zero nucleus. But spin matters when a neutron interacts with a spin 1/2 nucleus. In the latter case, there are two possibilities: (1) the neutron and nuclear spins don't change; or (2) if the spins were initially in opposite directions, they can both flip. Angular momentum conservation prohibits only one spin flipping. If no spins change, spin is not a factor and scattering proceeds exactly as if spins didn't exist. The same coherent interference effects are observed as seen in Figure 4-2a.

Now comes the most interesting alternative: spin flipping. Imagine source S produces neutrons that are all spin up, and imagine we arrange for our crystal nuclei to be all spin down. If the neutron and the nucleus it hits both flip spin, we could in principle know that the neutron spin changed and also know which atom it scattered from, namely the only atom that is now spin up. (The immense difficulty

of finding which one of a trillion atoms has changed spin is "only" a technical problem that is irrelevant to nature; nature's determining factor is whether or not that one atom is *distinguishable in principle*.)

Since we can know which atom scattered the neutron, the N alternative scattering sources are no longer undistinguished. Now we add probabilities, not amplitudes. The amplitude for spin-flip scattering from nucleus K and the probability of spin-flip scattering from all nuclei are:

$$\phi_{FLIP}(\theta,K) = \langle\theta|K\rangle\, \sigma_{FLIP}(\theta)\, \langle K|S\rangle$$
$$P_{FLIP}(\theta) = \Sigma_K\, |\phi_{FLIP}(\theta,K)|^2$$
$$P_{FLIP}(\theta) = N\, |\phi_{FLIP}(\theta,K)|^2$$

By excluding interference effects, spin-flip scattering is incoherent with the broad smooth distribution shown in Figure 4-2b. Whether or not I choose to measure the scattered neutron's spin, and whether or not I choose to identify which nucleus has flipped spin, the fact that the spins have changed makes spin-flip events distinguishable. The final states are substantially different, which nature recognizes even if I don't. It isn't about me.

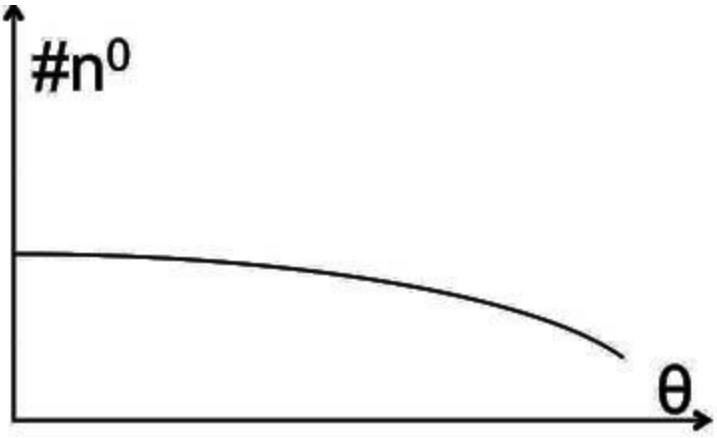

Figure 4-2b Spin-Flip Neutron Scattering vs. Angle

If we combine all events, those with spin flips and those without, the scattering distribution becomes a mixture of coherent and incoherent scattering, as shown in Figure 4-2c.

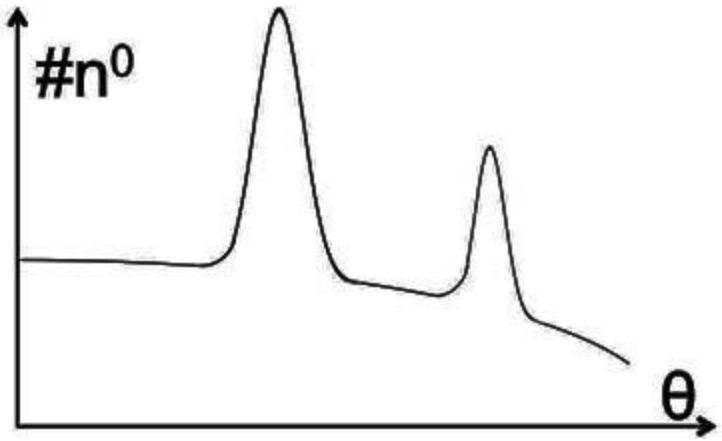

Figure 4-2c Any Spin Neutron Scattering vs. Angle

To review the key physics:

If alternative **final states** are **distinguishable, even in principle** if not in practice, we sum the probabilities, the square of amplitudes, of each alternative final state.

If alternative final states are **indistinguishable, even in principle**, we sum the amplitudes of each alternative final state, then we square that sum to obtain the probability.

Feynman cautions us again not to strive to connect probability amplitudes with any physical reality. No one knows how to incorporate spin into a classical wave concept. These quantum mechanical concepts are mathematical tools that we learn to manipulate to predict outcomes.

Identical Particle Interactions

Special circumstances arise in quantum situations involving identical particles, because they are indistinguishable.

We will consider scattering events between various types of particles. For simplicity, we examine these in their *center of mass frame*, the reference frame in which the total momentum is zero. Less commonly but more precisely, this is also called the *center of momentum frame*. By labeling it the *CM frame*, we can have it both ways. Collisions are almost always easier to analyze in the CM frame.

Let's start with two non-identical particles: an oxygen-16 nucleus and a helium-4 nucleus. The latter is also called an *alpha particle*. The collisions will be at energies low enough to ensure simple scattering, rather than anything more exciting such as nuclear reactions. The nuclei scatter because they are both positively charged. From a classical prospective, they repel one another with the electric force.

Scattering might well occur with different probabilities at different angles. We will measure this dependence by surrounding the interaction point with detectors, as is done at colliding-beam particle accelerators, including the LHC. Let $f(\theta)$ be the amplitude that each nucleus scatters at angle θ, in the CM frame. Figure 4-3 shows two scattering events. On the left, each nucleus is deflected by angle θ, while on the right, each is deflected by angle $\pi-\theta$. Particles are seen exiting at the same angles in both events. The difference between these events is: on the left, the helium nucleus goes up, and on the right it goes down.

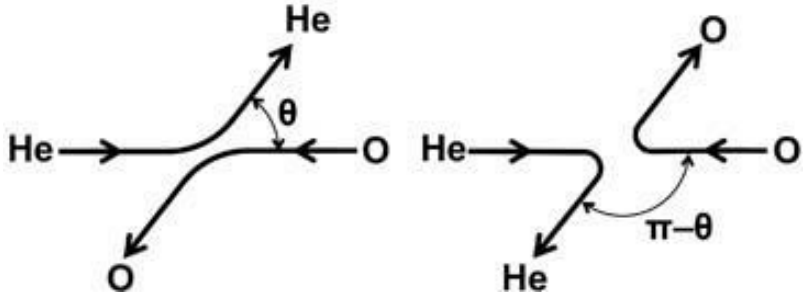

Figure 4-3 Scattering of Helium and Oxygen Nuclei
At Angles θ (Left) and π–θ (Right)

If our detectors discriminate between helium and oxygen nuclei, we can count only helium nuclei that go up and thereby directly measure $|f(\theta)|^2$, the probability of scattering at angle θ.

Conversely, if our detectors respond equally to either nucleus, we will measure a combined distribution:

$P(\text{either nucleus at } \theta) = |f(\theta\})|^2 + |f(\pi-\theta\})|^2$

Do you know why we add probabilities and not amplitudes? Check your answer at the end of this chapter.

The above result, adding probabilities, is correct for helium-4 nuclei colliding with any other nucleus — except one. Can you guess which one? Again the answer is at the end of this chapter.

Now let's try something more interesting: scattering two electrons. In the first case, we choose two electrons that are both spin up, as shown in Figure 4-4.

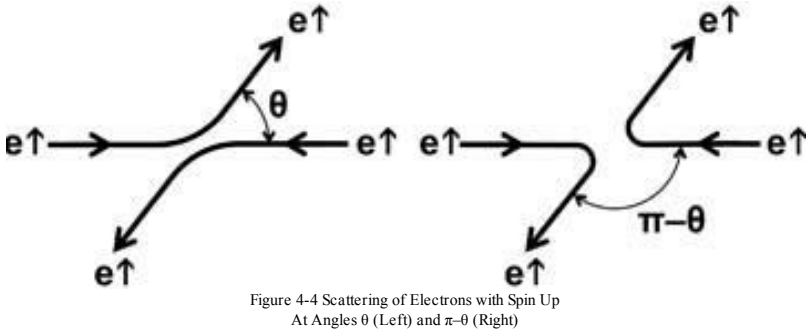

Figure 4-4 Scattering of Electrons with Spin Up
At Angles θ (Left) and π–θ (Right)

Since these are identical particles, we cannot distinguish between each electron scattering at angle θ (left side of figure) and each electron scattering at angle π–θ (right side). We therefore expect to add amplitudes $f(\theta)+f(\pi-\theta)$ and then square, exactly as we have so diligently learned.

But that's not nature's way. Here we discover a new principle: the Exclusion Principle, espoused by Wolfgang Pauli. We can call this the **sixth general principle** of quantum mechanics.

Like many physicists, Pauli had an unusual sense of humor. He famously berated another physicist's theory by saying: "This isn't right. This isn't even wrong." Perhaps he meant that clever people can learn from ideas that are wrong but that nonetheless illuminate essential points. But some ideas are so ridiculous that they provide no insight whatsoever.

In any case, Pauli's exclusion principle says fermions are anti-social — two identical fermions will **never** share the same quantum state. Bosons, conversely, are gregarious — the more bosons are in one quantum state, the more likely it is that others will join them. The exclusion principle means that **identical fermions interfere with a minus sign**. The correct probability $P(\theta)$ for identical electron scattering at angle θ is:

$P(\theta) = |f(\theta) - f(\pi-\theta)|^2$

An essential key to the minus sign is that both electrons have spin up. If one were up and the other down, the final states would be distinguishable, not identical.

If we collide electrons with opposite spins, as in Figure 4-5, we can distinguish, in principle,

whether the particles scattered by angle θ or by angle π–θ.

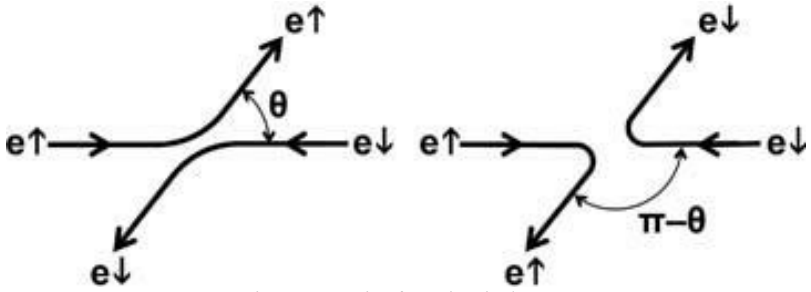

Figure 4-5 Scattering of Opposite Spin Electrons
At Angles θ (Left) and π–θ (Right)

Therefore the left and right side processes are incoherent, do not interfere, and we add their probabilities:

$$P(\theta) = |f(\theta)|^2 + |f(\pi-\theta)|^2$$

Finally, consider scattering electrons of random spins, and count all electrons that reach our detectors equally, regardless of their spin. We expect each incident electron to have a 50% probability of being spin up and a 50% probability of being spin down.

Half the collisions will be between electrons with the same spin. These interfere with a minus sign; we subtract amplitudes, and square that sum to obtain probabilities.

The other half of the collisions will be between electrons of opposite spin. Those cannot interfere; we sum the squares of amplitudes to obtain probabilities.

The total of all that is:

$$P(\theta) = 1/2 \{ |f(\theta) - f(\pi-\theta)|^2 + |f(\theta)|^2 + |f(\pi-\theta)|^2 \}$$

Promised Answers

The first question was: if our detectors respond equally to either helium or oxygen nuclei, should we add amplitudes or probabilities? Oxygen is clearly distinguishable from helium; just try breathing them. Therefore the final states are not identical, and we add probabilities:

$$P(\theta) = |f(\theta)|^2 + |f(\pi-\theta)|^2$$

The second question was: if we replace oxygen nuclei with different nuclei, for which nuclei is the above equation wrong? If both nuclei are helium-4, we can no longer distinguish which one went up and which went down. We must, therefore, add amplitudes before squaring. Note helium-3 yields the same result as oxygen; neither is identical to helium-4.

Chapter 4 Review: Key Ideas

A quantum state defines all the variable properties that entities can have, such as position, momentum, energy, spin, and angular momentum. A quantum state does not define the intrinsic properties that each particle has, such as charge and mass. In general, different types of particles can be put into a given state, and can be moved from one state to another. It is beneficial to think of a quantum state as a vector; it defines a location in the space of all possible properties.

The standard notation for probability amplitudes, due to Paul Dirac, is:

<B| is called a **bra**
|A> is called a **ket**
<B|A> is a **bra-ket**

<B|A> is the amplitude that A results in, or goes to, B. <B|A> is analogous to the dot product of vectors A and B.

In bra-ket notation, **the general principles** of quantum mechanics are:

First: the probability P(y) of event y is proportional to the square of the magnitude of amplitude ø(y), which is a complex number. For the two-slit experiment, "event y" is a particle arriving at y. We write:

ø(y) = <particle arrives at y | particle leaves S>

Or: ø(y) = <y|S>, and $P(y) = |ø(y)|^2$

Second: when event y can occur in N undistinguished ways, the amplitude of y equals the sum of the amplitudes for each separate way:

$\langle y|S\rangle = \Sigma_K \langle y|K\rangle\langle K|S\rangle$

Note that this applies only if all N ways result in the *same* event y.

Third: when event y can occur in N distinguished ways, the probability of y equals the sum of the probabilities for each separate way:

$P(y) = \Sigma_K |\langle y|K\rangle\langle K|S\rangle|^2$

Fourth: the amplitude for a sequence of events equals the product of the amplitudes for each event separately. The amplitude for a particle to go from S through slit #1 and from there to y is: <y|slit#1> <slit#1|S>.

Fifth: For multiple particles, if $ø_1$ is the amplitude of particle 1 going from S_1 to F_1, and $ø_2$ is the amplitude of particle 2 going from S_2 to F_2, then the amplitude for both events to occur is $ø_1 \times ø_2$.

Sixth: Identical particle statistics. When two identical particles can enter, exit, or be in the same state, their amplitudes interfere. If the particles are bosons governed by *Bose-Einstein statistics*, their

amplitudes add. If the particles are fermions governed by *Fermi-Dirac statistics*, their amplitudes subtract. There is no third alternative. For two identical particles 1 and 2, and any two states A and B, the combined amplitude is:

Fermions: $\langle 1|A\rangle\langle 2|B\rangle - \langle 1|B\rangle\langle 2|A\rangle$

Bosons : $\langle 1|A\rangle\langle 2|B\rangle + \langle 1|B\rangle\langle 2|A\rangle$

Fermions with different spins are not identical.

Spin is a form of angular momentum that is intrinsic to each elementary particle. Particle spins are quantized. The primary fermions have spin s = 1/2. Their component of spin along any chosen axis can only be $+\hbar/2$ or $-\hbar/2$, called *spin up* and *spin down*. Bosons have integral spin: s = 0, 1, or 2. Their component of spin along any chosen axis must be: $-s\hbar$, …, 0, …$+s\hbar$. Photons have spin 1, but cannot have a spin component of zero along any axis. The full length of the spin vector is called total spin S; $S = \hbar\sqrt{[s(s+1)]}$. Any particle's component of spin along any axis can change only by integer multiples of $\hbar$.

Chapter 5

Identical Particles

In the last chapter, we encountered some special circumstances associated with identical particles. We are all familiar with macroscopic things that are "identical", including identical twins, and identical postage stamps. But in truth, no two macroscopic objects are ever absolutely identical. Each new dime from the U.S. Mint in Philadelphia contains over a trillion, trillion neutrons, protons, and electrons. While new dimes may appear identical, no one would claim each has **exactly** the same number of neutrons.

But in the micro-world, all fundamental particles of each type are **exactly** identical. I'm not saying physicists can't see differences; it's much more than that. We are sure nature can't detect any intrinsic differences between one electron and another. They may be in different places and have different energies and spin orientations, but their intrinsic properties are **absolutely** identical in a way that has no macro-world analogy.

We know this is true because of the unique quantum behavior of identical particles.

When two particles 1 and 2 are in two different states A and B, one particle in each state, two alternatives are possible: 1 in A and 2 in B (<1|A><2|B>); or <1|B><2|A>. If the particles are truly identical, the two alternatives are indistinguishable and their amplitudes interfere. If the particles are in any way distinct from one another, the alternative states are not identical and their amplitudes do not interfere.

We observe interference effects with pairs of electrons, proving that nature cannot distinguish one electron from another. The same interference effects are observed with pairs of protons, pairs of neutrons, and also pairs of fundamental particles of each type. All fundamental particles of each type are intrinsically absolutely identical.

Bosons and Fermions

Let's reexamine the two-particle collision, viewed in the CM frame, discussed in the last chapter and shown again in Figure 5-1. The colliding particles will now be identical, which requires that their spin states be identical.

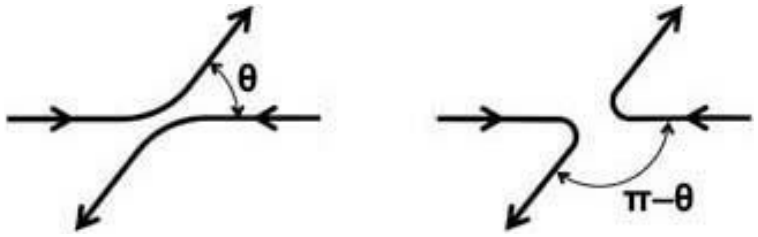

Figure 5-1 CM collision of Two Identical Particles

On the left side, each particle scatters by angle θ, for which we assign the amplitude f(θ). On the right side the same particles scatter by angle π–θ, with amplitude f(π–θ). Since the left-side and right-side final states are the same, the probabilities of the two alternatives must be equal. Thus:

$|f(\theta)|^2 = |f(\pi-\theta)|^2$
$f(\pi-\theta) = f(\theta)\exp\{i\phi\}$

Equal probabilities ensure equal magnitudes, but the amplitudes can differ by phase angle ø. This means, if we start with the left side of Figure 5-1, where the left particle goes up, the amplitude is f(θ). If we next exchange the two identical particles, we have the right-side collision with amplitude $f(\theta)\exp\{i\phi\}$. And, if we exchange them again, we multiply by $\exp\{i\phi\}$ once again, and go back to the left-side collision with amplitude $f(\theta)\exp\{i2\phi\}$. But since two exchanges return us to the original state:

$f(\theta)\exp\{i2\phi\} = f(\theta)$
$\exp\{i2\phi\} = 1$
$\exp\{i\phi\} = \pm 1$

Which sign does nature pick? **Both**.

For bosons: $\exp\{i\phi\} = +1$
For fermions: $\exp\{i\phi\} = -1$

Physicists describe this distinction as being two types of *statistics*. Nature is divided into bosons governed by *Bose-Einstein statistics* (the + sign), and fermions governed by *Fermi-Dirac statistics* (the – sign). Both statistics have momentous consequences.

Next, consider the scattering of two composite particles. Protons and neutrons, collectively called *nucleons*, are each composed of three quarks. Protons have two up quarks and one down quark, while neutrons have two downs and one up. If we identify the black-centered circles in Figure 5-2 as up quarks, and the white-centered circles as down quarks, these collisions involve two protons.

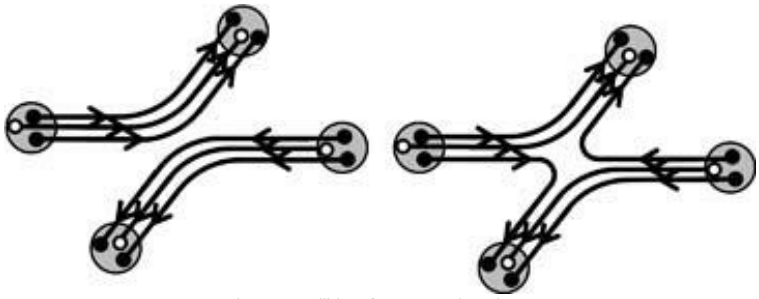
Figure 5-2 Collision of Two 3-Quark Nucleons

All quarks are spin 1/2 fermions, and all quarks of each type are identical. Hence, the left and right final states are identical; neither we nor nature can tell whether or not the protons swapped an up quark.

Two positively-charged protons repel one another. In low-energy collisions, that repulsion prevents the protons from getting close to one another. This is because their electrical potential energy is $+e^2/r$. As r, the distance between them, reduces, their kinetic energy must convert into potential energy. If their kinetic energy is initially low, it runs out before r becomes very small.

Since the range of the strong nuclear force is only about the diameter of a proton, quark-exchange is possible only if r is about 1 fermi. But in high-energy collisions, quark-exchange is appreciable. There is an amplitude for scattering without quark-exchange and an amplitude for scattering with quark-exchange. We must add the amplitudes of these indistinguishable alternatives.

Recall that spin is a form of angular momentum. Three spin 1/2 quarks combine to make a spin 1/2 proton. Experiments show that the quarks within protons also have orbital angular momentum. The orbital angular momenta combine with the three spins in some way, not fully determined, to produce a spin 1/2 proton.

Two protons and two neutrons, all spin 1/2 particles, combine to make an alpha particle, the nucleus of helium-4, which is a spin zero boson. Adding another proton and two more neutrons yields the nucleus of lithium-7, which is a spin 3/2 fermion.

Whatever an object's composition, it is ultimately either a fermion if its overall spin is half-integral, or a boson if its overall spin is integral. There is no middle ground.

Due to the sign with which the amplitudes of identical particles combine, bosons and fermions have dramatically different behaviors.

Let's next consider the behavior of groups of bosons, focusing on groups of photons.

Two-Boson States

We begin with two bosons that are either emitted into or scattered into similar final states, as illustrated in Figure 5-3. For now, it doesn't matter how the particles get there, our interest is in the final states.

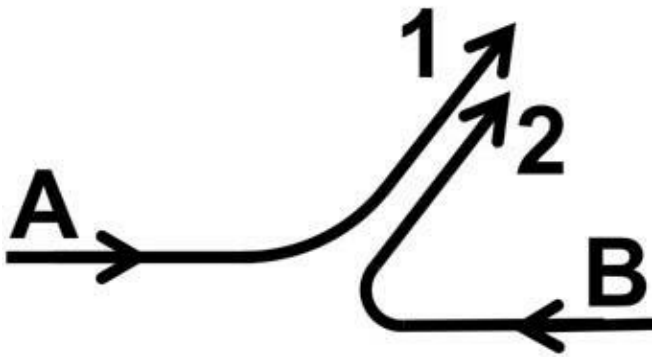

Figure 5-3 Two Bosons Enter Similar State

Here particle A enters state 1 and particle B enters a very similar state 2. (Feynman will shortly make states 1 and 2 identical.) Considering particle A by itself, the amplitude that it goes to state 1 is: $\langle 1|A\rangle$. The amplitude for B to go to state 2 is: $\langle 2|B\rangle$. Hence, according to our fifth general principle, the amplitude that A goes to 1 and B goes to 2 is:

$$\langle 1|A\rangle\langle 2|B\rangle$$

with probability

$$P(1A,2B) = |\langle 1|A\rangle\langle 2|B\rangle|^2 = |\langle 1|A\rangle|^2 \, |\langle 2|B\rangle|^2$$

To reduce clutter, define $a_1 = \langle 1|A\rangle$ and $b_2 = \langle 2|B\rangle$.

Going back to Figure 5-3, it might happen that particle A goes to state 2 while particle B goes to state 1. The amplitude for that is:

$$\langle 2|A\rangle\langle 1|B\rangle = a_2 \, b_1$$

with probability $P(2A,1B) = |a_2|^2 \, |b_1|^2$

Assuming the particles or final states are not identical, the probability that either particle goes to state 1 and the other particle goes to state 2 is:

$$P(AB,12) = |a_1|^2 \, |b_2|^2 + |a_2|^2 \, |b_1|^2$$

Now Feynman takes the limit as state 2 becomes identical to state 1: the two A amplitudes must become equal, as must the two B amplitudes. Namely:

a_2 goes to a_1 , which we will call a, and
b_2 goes to b_1 , which we will call b

So that $P(AB,12) = 2 \, |a|^2 \, |b|^2$

Next, Feynman makes particles A and B identical bosons, so that event (1A,2B) is indistinguishable from event (2A,1B). We now add amplitudes before squaring.

$$P(AB,12) = |a_1 \, b_2 + a_2 \, b_1|^2$$

$$P(AB,12) = |2ab|^2 = 4 \, |a|^2 \, |b|^2$$

This means two identical bosons are twice as likely to be in the same state as two non-identical particles.

Next, Feynman becomes more precise about defining scattering amplitudes. Imagine that state 1 is defined as those particles that hit detector D_1 whose cross sectional area is σ_1. Redefine the amplitude for particle A, assuming it was by itself, as:

probability of A hitting D_1 per unit area = $|a_1|^2$

To get the probability of A hitting D_1 anywhere, we integrate over the detector's cross section, which we denote: $\int_A |a_1|^2$. We assume area σ_1 is so small that the scattering amplitude doesn't vary across its surface. This is written:

probability of A hitting D_1 anywhere = $\int_A |a_1|^2$
probability of A hitting D_1 anywhere = $|a_1|^2 \sigma_1$

Similarly, state 2 is defined as particle B hitting detector D_2 whose cross sectional area is σ_2. For particle B, assuming it was by itself, we get:

probability of B hitting D_2 per unit area = $|b_2|^2$
probability of B hitting D_2 anywhere = $\int_B |b_2|^2$
probability of B hitting D_2 anywhere = $|b_2|^2 \sigma_2$

The joint probability that A hits D_1 and B hits D_2 is as before, but with the σ factors:

$P(1A,2B) = |a_1|^2 |b_2|^2 \int_A \int_B$
$P(1A,2B) = |a_1|^2 |b_2|^2 \sigma_1 \sigma_2$

This is the probability for two independent non-identical particles. If we now require that both non-identical particles hit D_1, the probability is:

$P(1A,1B) = |a_1|^2 |b_1|^2 \sigma_1^2$

Now go back to two detectors and make A and B identical bosons, the events (1A,2B) and (2A,1B) are indistinguishable. We therefore add amplitudes before squaring:

$P(AB,12) = |a_1 b_2 + a_2 b_1|^2 \sigma_1 \sigma_2$

Next, we will combine detectors D_1 and D_2 into a single detector D with cross section σ. It turns out that properly counting combinations suddenly becomes surprisingly tricky. During his lecture, Feynman's explanation was quite brief, even by Caltech standards. A terse footnote, added on V3p4-5, helps some, but still leaves me scratching my head and reaching for paper and pen. Let's understand this using some graphics.

To calculate P(AB,12), two identical bosons in two different detectors, we integrate over both D_1 and D_2. The upper half of Figure 5-4 shows one outcome: A in the upper part of D_1 and B in the lower part of D_2. The lower half of the figure shows a different outcome, in which A and B are exchanged. The two outcomes are different but indistinguishable; we therefore add the two amplitudes.

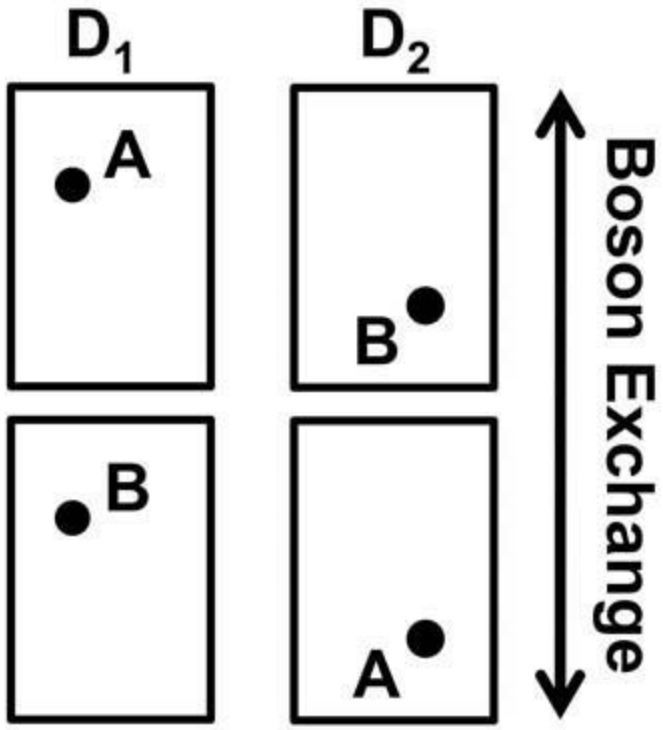

Figure 5-4 Two Identical Bosons
in Two Detectors

When we combine two detectors into one, we get four combinations. In the upper half of Figure 5-5, two outcomes are shown: (1) A in the upper part of detector D and B in the lower part of D; and (2) A in the lower part of D and B in the upper part. Assuming D fires in the same way regardless of where it is hit, these outcomes are indistinguishable. Here, the integral $\int_A$ over the cross section of D is represented by the black-centered circle, and the integral $\int_B$ is represented by the white-centered circle.

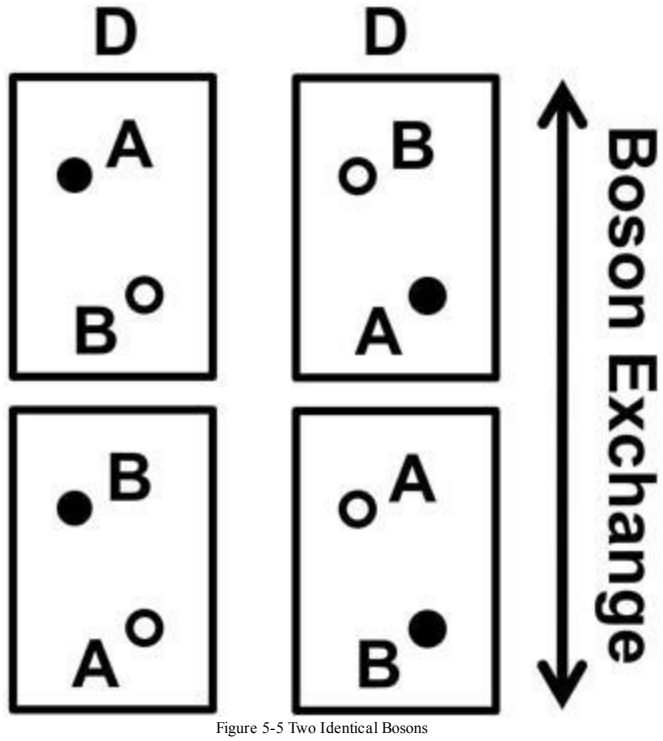

Figure 5-5 Two Identical Bosons
in One Detector

In the lower half of the figure are two more outcomes that result from exchanging A and B in the prior outcomes. The trouble is, the four outcomes are not all distinct. The upper left outcome and the lower right outcome are the same — the same impact points are hit by the same particles. This is just one

outcome, but the double integral picks up the same outcome twice. The upper right and lower left are also the same outcome that is counted twice.

The double integral double counts outcomes.

For two identical particles, the double integral overcounts outcomes by a factor of two. We therefore divide our result by 2 to compensate.

$P(AB,D) = |a_1 b_2 + a_2 b_1|^2 \sigma_1 \sigma_2 / 2$
$P(AB,D) = |a\ b + a\ b|^2 \sigma^2 / 2$
$P(AB,D) = 2\ |a|^2\ |b|^2\ \sigma^2$

This is twice the probability of two non-identical particles hitting detector D.

N-Boson States

Now let's up our game to N bosons in N similar states, all headed toward one remote detector D, as shown in Figure 5-6. (We will soon get to N identical bosons in one state, such as in a laser beam.)

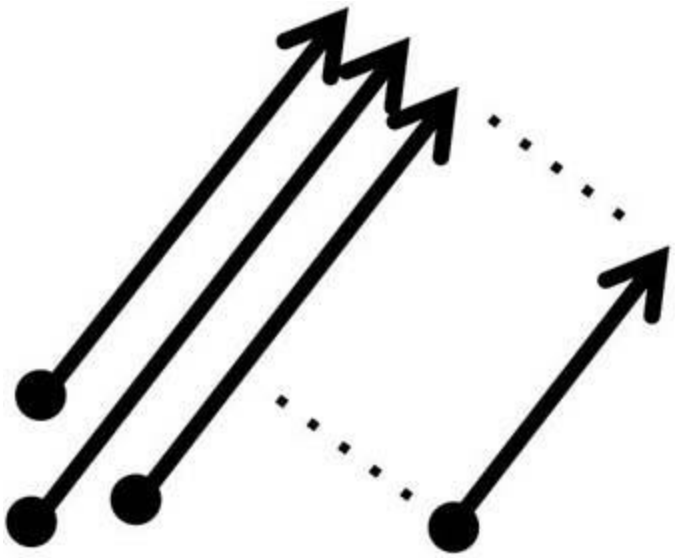

Figure 5-6 N Bosons in Similar States

We label the bosons A, B, C, … and the final states 1, 2, 3, …N. As before, we normalize the amplitudes so that a single, lone boson has probability P of hitting detector D whose cross sectional area is σ, according to:

$P(A,D) = |a|^2\ \sigma$

If the particles are all different, they act independently, and the joint probability that all N hit the detector is the product of their independent probabilities:

$P(\text{N different bosons, D}) = \{|a|^2\ |b|^2\ |c|^2\ \ldots\}\ \sigma^N$

If, however, these are all identical bosons, photons for example, then the combinations multiply. Generalizing the results of the last section, the number of *permutations* (the number of distinct re-

orderings) of N objects is N! = 1×2×3…×N. Three identical bosons A, B, and C have 6 permutations:

ABC, ACB, BAC, BCA, CAB, CBA

Each permutation is a different but indistinguishable outcome, and each is obtainable from another by the exchange of two bosons. We therefore need to add the amplitudes for N permutations.

But as we found above, the N-fold integral overcounts the number of distinct outcomes. For N=2, the double integral overcounts by a factor of 2. What correction do we need for N bosons?

Consider the case of three bosons, N=3. Before we get to exchanging bosons, let's take one specific set of A, B, and C impact points, and see how many outcomes the triple integral counts. In Figure 5-7, specific integration points in the three integrals are marked with a black-centered circle, a white-centered circle, and a heart, respectively.

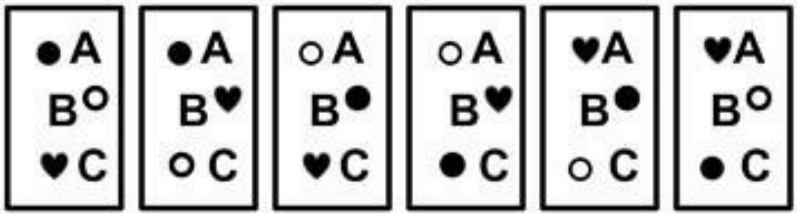

Figure 5-7 Six Outcomes from Triple Integral

One real event, the impact of A at one point, B at a second point, and C at a third point, is counted 6 times by the triple integral. If we added a fourth boson, the 4-fold integral would count 4 outcomes for each of the 6 outcomes in the figure, resulting in 24 outcomes counted for one real event for N=4. Evidently, the number of outcomes counted per real event by an N-fold integral is N! Another way to say this is: for each real event, the N-fold integral will count N! permutations.

We have not yet included identical particle exchange. As we saw above, each of the 6 outcomes in Figure 5-7 has 6 particle-exchange permutations, yielding 36 total outcomes for N=3. Evidently, that would be $N!^2$ counted outcomes for N identical bosons. The true number of distinct permutations is actually N!, which means our N-fold integral overcounts by a factor of N! The correct result is:

$P(N) = |\text{ all N! permutations of abc}\ldots|^2\ \sigma^N / N!$
$P(N) = N!\ |\text{ abc}\ldots|^2\ \sigma^N$

This is larger than the probability for non-identical particles by a factor of N!. That factor can be phenomenally large: 59! is about 10^{80}, equal to the number of nucleons in our observable universe. For many situations, N, the number of elementary particles can exceed millions.

In V3p4-7, Feynman calculates the probability of a boson joining a group of N other bosons in a common state, when all are identical particles. Call the new boson W and let w be the amplitude for it to be in this state when no other particles are present. We expand the above equation from N to N+1, and obtain:

$P(N+1) = (N+1)!\ |\text{ abc}\ldots w|^2\ \sigma^{N+1}$
$P(N+1) = (N+1)\ |w|^2\ \sigma\ \{N!\ |\text{ abc}\ldots|^2\ \sigma^N\}$

$P(N+1) = (N+1)\,|w|^2\,\sigma\,\{P(N)\}$

Hence, the probability of W entering a state occupied by N identical bosons is N+1 times larger than it would be if the state were empty. Bosons are groupies: the bigger the party, the greater the attraction.

Chapter 5 Review: Key Ideas

1. In the micro-world, all fundamental particles of each type are intrinsically **exactly** identical. Additionally, it seems there is only one way to combine three quarks to make a proton, and only one other way to combine three quarks to make a neutron. While not fundamental, all protons are exactly identical as are all neutrons. Even nature cannot distinguish between identical particles, as proven by interference effects.

2. When two identical particles can enter, exit, or be in the same state, their amplitudes interfere. If the particles are bosons governed by *Bose-Einstein statistics*, the amplitudes **add**. If the particles are fermions governed by *Fermi-Dirac statistics*, the amplitudes **subtract**. There is no third alternative.

3. The probability that N identical bosons are in a common state is larger by a factor of N! than the probability of N non-identical particles being in that state.

4. The probability of one more boson entering a state occupied by N identical bosons is N+1 times greater than it would be if the state were empty. Bosons are groupies: the bigger the party, the greater the attraction.

Chapter 6

Impact of Identical Particles

In the last chapter, we discovered the remarkable behaviors of identical particles. This chapter examines some consequences of those uniquely quantum mechanical behaviors. Our topics include lasers, black body radiation, superconductivity, and the structure of atoms that enables chemistry and life.

Light Emission & Absorption

The prior chapter demonstrated that the probability that an atom will emit a photon into state ø is enhanced by a factor of N+1 if there are already N photons in that state. This means of course that the amplitude is enhanced by $\sqrt{(N+1)}$.

Feynman introduces here a principle of quantum mechanics that is proven in the next chapter. This principle allows us to more profoundly examine Einstein's laws of radiation and Planck's equation for black body radiation. This new principle is:

$\langle \xi | ø \rangle = \langle ø | \xi \rangle^*$

This says the amplitude to go from state ø to state ξ equals the complex conjugate of the amplitude to go from ξ to ø. The forward and reverse amplitudes are complex conjugates of one another.

Let's apply this to atoms radiating and absorbing light.

Define $|N\rangle$ to be a state containing N photons, and define A to be the amplitude that a photon enters this state when it is empty. The last chapter showed:

$\langle N+1|N\rangle = \sqrt{(N+1)}\ A$

Here $\langle N+1|N\rangle$ is the amplitude to add a photon to a state already containing N photons.

With our new principle, the amplitude for a photon to exit a state with N+1 photons, leaving N photons behind, is:

$\langle N|N+1\rangle = \sqrt{(N+1)}\ A^*$

Here A* is the amplitude to absorb a photon from state ø if it contains one photon.

Let's restate the prior equation for one less photon: the amplitude for a photon to exit a state with N

photons, leaving N–1 behind, is:

$\langle N-1|N\rangle = (\sqrt{N})\ A^*$

Now, imagine a box whose interior surfaces are ideal mirrors. Let the box contain N photons and one atom. We assume all N photons are in the same state ø; all have the same frequency, same velocity, and the same polarization. The probability that the atom will absorb a photon from state ø, leaving N–1 photons behind, is:

$P_{ABSORB} = N\ |A^*|^2 = N\ |A|^2$

The last equation says the probability of absorption is proportional to the intensity of light, not a surprise.

The probability the atom will emit a photon into state ø is:

$P_{EMIT} = (N+1)\ |A|^2$

This equation was first derived by Einstein, as discussed in ***Feynman Simplified 1B*** Chapter 20. Einstein said the emission process has two parts: *spontaneous* and *stimulated*. An isolated atom spontaneously emits photons at a rate determined by $|A|^2$. When N other photons are already in the state that the emitted photon will enter, the atom's emission rate is stimulated, increased by $N\ |A|^2$, which is proportional to the intensity of photons already in the final state.

As we describe in the above referenced chapter, this is the basis for the operation of lasers. In brief, large numbers of atoms placed in excited states stimulate one another to emit photons with the same frequency and velocity, forming a coherent beam of photons — a laser beam.

Planck's Black Body Spectrum

Feynman Simplified 1B Chapter 20 also discusses Planck's development of the quantum theory of black body radiation. Let's re-examine it in terms of what we now know about the quantum behavior of bosons.

Begin with a box containing photons and atoms in thermal equilibrium. In V3p4-8, Feynman supposes that, in each frequency band $\omega \pm d\omega/2$, there are N atoms with two electron energy levels separated by $\Delta E = \hbar\omega$. Call the lower energy level the *ground state*, and the upper energy level the *excited state*. As shown in Figure 6-1, electrons can absorb a photon of energy $\hbar\omega$ and rise from the ground state to the excited state (left side of image). Electrons can also emit a photon of energy $\hbar\omega$ and drop from the excited state to the ground state (right side of image).

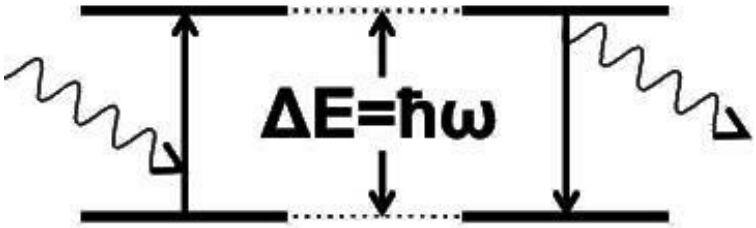

Figure 6-1 Atomic Absorption & Emission of Light

Let N_G and N_E be the average number of atoms in the ground and excited states, respectively. Recalling Boltzmann's law from ***Feynman Simplified 1B*** Chapter 20, with k being Boltzmann's constant and T being temperature, the population ratio of such states at equilibrium is:

$$N_E / N_G = \exp(-\hbar\omega/kT)$$

Define N_γ to be the number of photons in any specific state in the frequency band $\omega \pm d\omega/2$. At equilibrium, the photon absorption rate equals the photon emission rate at that frequency.

$$\text{Emission Rate} = N_E(N_\gamma+1)\,|A|^2$$
$$\text{Absorption Rate} = N_G\,N_\gamma\,|A^*|^2$$
$$\text{Thus: } N_E(N_\gamma+1) = N_G\,N_\gamma$$

Combining this with the population ratio:

$$(N_\gamma+1)/N_\gamma = N_G / N_E = \exp(+\hbar\omega/kT)$$
$$N_\gamma + 1 = N_\gamma \exp(+\hbar\omega/kT)$$
$$N_\gamma = 1 / \{\exp(+\hbar\omega/kT) - 1\}$$

We aren't there yet. This expression tells us the number distribution of photons in a specific state with frequency ω. We need to multiply that by the number of states with that frequency. These are called the number of *modes*.

Let's examine the modes in one dimension along a line of length L. As Figure 6-2 shows, just like a violin string, the amplitude of each allowed mode must be zero at both ends of the line.

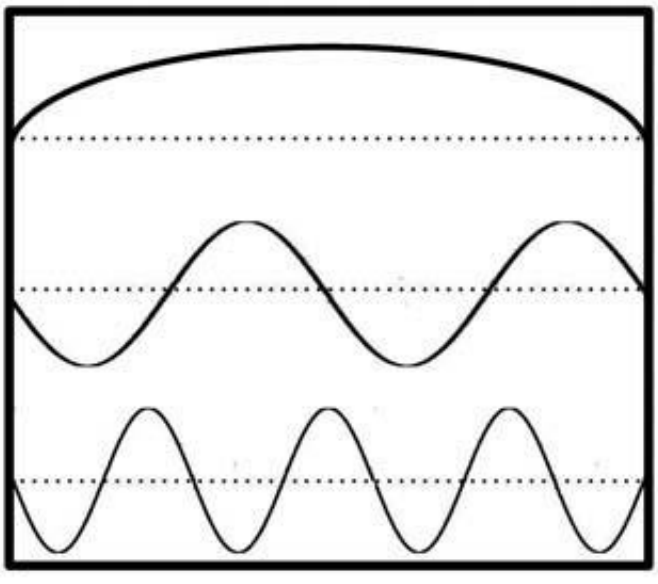

Figure 6-2 Some Allowed
Modes For Length L

The allowed wavelengths are integer submultiples of 2L: $\lambda_J=2L/J$, for any integer $J>0$. Each value of J is one mode. In terms of the wave number k:

$k_J = 2\pi/\lambda_J = J\pi/L$

Define: $\Delta k = \Delta J\ \pi/L$

For a line much longer than the wavelength of any light of interest, $L>>\lambda$, the ratio $\Delta J/\Delta k$ approaches the derivative dJ/dk. We can then calculate dM, the number of allowed modes in an interval $k\pm dk/2$:

$dM = (\Delta J/\Delta k)\ dk = (L/\pi)\ dk$

We now have dM/dk in one dimension. The next step is to expand this to three dimensions. Wave number k becomes a vector $\boldsymbol{k}$ with components (k_x, k_y, k_z), and the line of length L becomes a cube of volume $V=L^3$.

$dM = V/\pi^3\ dk_x\ dk_y\ dk_z$

The three components of $\boldsymbol{k}$ range from 0 to +K. K is a very large number, but not infinite. Experiments and Planck's quantization "trick" show that the intensity of black body radiation approaches zero at high frequencies.

The expression $dk_x\ dk_y\ dk_z$ is an infinitesimal volume in a 3-dimensional k-space, illustrated in Figure 6-3. We need to find the number of modes between k and k+dk, where k is the length of the wave number 3-vector. We know how to do this in normal 3-D space.

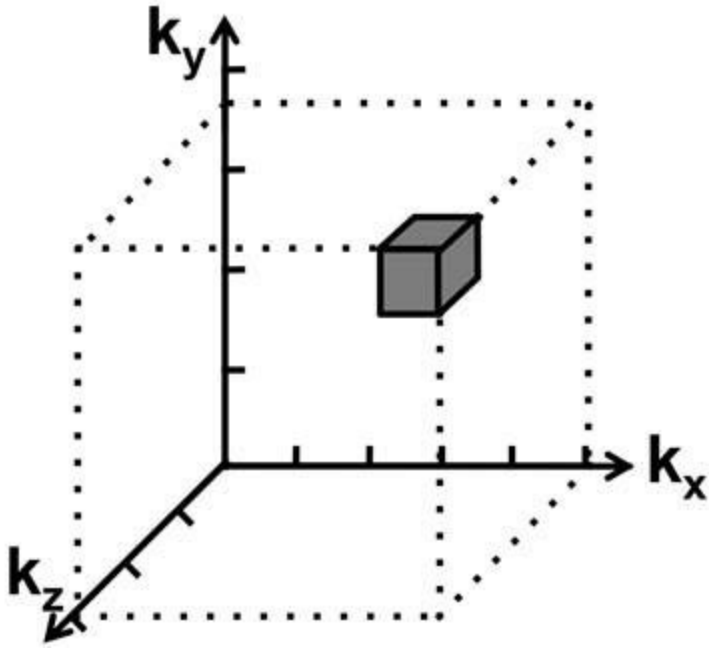

Figure 6-3 Infinitesimal Volume in k-Space

In real 3-D space, dx dy dz is the equivalent infinitesimal volume element. Expressed in polar coordinates this volume element is:

$r^2 \sin\theta \, dr \, d\theta \, d\phi$

Space is spherically symmetric, and waves travel through space in the same way, at the same velocity, in all directions. To find the total volume between r and r+dr, we integrate over θ (0 to π) and over ø (0 to 2π). That yields the volume of a shell of radius r and thickness dr:

$4\pi r^2 dr$

This 3-D physical space volume is equivalent to the volume we seek in k-space, except that the three ***k*** components range from 0 to +K. The volume they enclose in 3-D k-space is 1/8 of a full sphere.

Replacing $dk_x \, dk_y \, dk_z$ with $4\pi k^2 dk/8$ yields:

$dM = V/(2\pi^2) \, k^2 \, dk$

Feynman prefers having the three k components range from –K to +K, which requires replacing V/π^3 by $V/(2\pi)^3$. We get the same result either way. In V3p4-11, Feynman recommends memorizing either the above expression for the number of modes in 3-D k-space, or the equivalent: $dM = V/(2\pi)^3 \, d^3k$.

Up to this point, our mode analysis is correct for any type of wave. Now, let's make it specific to light. We use $k=\omega/c$, and multiply the number of modes by 2 for the two polarization states that photons can have.

$dM = V/(\pi^2 c^3) \, \omega^2 \, d\omega$

We next calculate the energy dE in the frequency band $\omega \pm d\omega/2$.

dE = (energy/photon)×(#modes)×(#photons/mode)

Putting in all the pieces, and noting that k is once again Boltzmann's constant and no longer the wave number:

$dE = (\hbar\omega) \, (V\omega^2/\pi^2 c^3) \, (1 \, / \, \{\exp(+\hbar\omega/kT) - 1\}) \, d\omega$

Light intensity *I* is energy per unit area per second. The volume V swept out by a light beam per second equals the beam's cross sectional area times light's speed c. Hence:

$I(\omega)d\omega = dE \, c/V$
$I(\omega) = (\hbar\omega^3/\pi^2 c^2)/\{\exp(+\hbar\omega/kT) - 1\}$

This is Planck's black body spectrum.

Feynman reminds us that we derived the same equation by assuming atoms were quantized harmonic oscillators (see ***Feynman Simplified 1B*** Chapter 20). In V3p4-9, Feynman says:

> "That is one of the marvelous miracles of quantum mechanics…There is no way to make up your mind whether the electromagnetic field is really to be described as a quantized harmonic

oscillator or by giving how many photons there are in each condition. The two views turn out to be mathematically identical."

Liquid Helium

At very low temperatures, helium has several remarkable properties that arise from the fact that it is a spin zero boson. One of these phenomena is *superfluidity*, the ability of a liquid to flow with zero viscosity, the total absence of friction or turbulence.

In V3p4-12, Feynman explains that at low enough temperatures, the Boltzmann factor exp{–E/kT} effectively "freezes" out all but the lowest energy state, the ground state. At the helium-4 superfluidity transition temperature, 2.17K (–271°C, –456°F), kT = 1.9×10^{-4} eV. This is much less than the energy needed to elevate an atom to an excited state, making it extremely unlikely that a helium atom can be bumped up to a higher energy state, even if jostled by its neighbors.

Unable to either gain or lose energy, helium atoms can only go with the flow. The transition to superfluidity occurs suddenly as the temperature is reduced.

Feynman said, as everyone believed at the time, that superfluidity could not occur in helium-3, because it is a spin 1/2 fermion.

One of my classmates apparently wasn't entirely convinced. In 1971, Doug Osheroff and colleagues discovered superfluidity in helium-3 at 0.0025K, for which they were awarded the 1996 Nobel Prize. It seems helium-3 can form the equivalent of *Cooper pairs*, in which two electrons with anti-parallel spins combine to make a spin zero boson, the process that enables superconductivity.

This confirms what Feynman himself often said, scientific truth is not determined by the proclamations of our Prophets, however esteemed, but by observing and learning from nature.

Pauli Exclusion Principle

We now turn from bosons to fermions, which as Feynman says: "act in a completely different way."

In Figure 6-4, two identical fermions A and B enter two similar states.

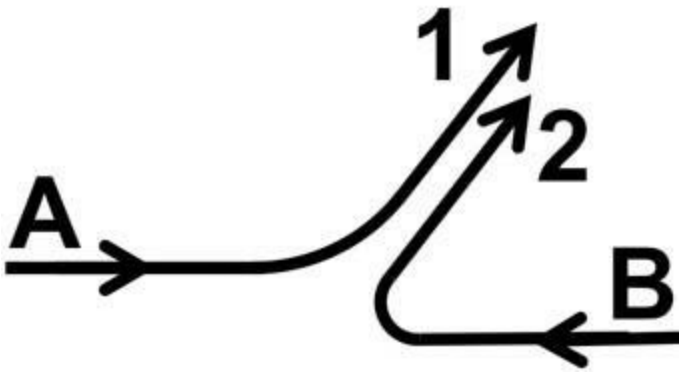

Figure 6-4 Two Identical Fermions Entering Similar States

The amplitude for A to go into state 1 is a_1, and the amplitude for B to go into state 2 is b_2. The amplitude that both events occur is:

$a_1 b_2 = \langle 1|A\rangle \langle 2|B\rangle$

Conversely, the amplitude of A going to state 2 and B going to state 1 is:

$a_2 b_1 = \langle 2|A\rangle \langle 1|B\rangle$

Since A and B are identical fermions, we must **subtract** amplitudes to obtain the amplitude that A goes to either state and B goes to the other state.

$a_1 b_2 - a_2 b_1 = \langle 1|A\rangle \langle 2|B\rangle - \langle 2|A\rangle \langle 1|B\rangle$

In the limit that states 1 and 2 become identical, with the same spin and direction of motion, a_1 becomes a_2 and b_1 becomes b_2. This means the above amplitude becomes zero. Combining fermions is much simpler than combining bosons (zero is simple). A fermion will never exist in the same state as another identical fermion. (Two fermions are not in the same state if their spins are different.)

This antisocial behavior is called the *Pauli Exclusion Principle*.

As a consequence, two electrons cannot have the same atomic orbit unless they have different spins. Since all of the Big Three particles of matter — electrons, protons, and neutrons — are spin 1/2 fermions, their spins along any selected axis can only be +1/2 or –1/2, which are more commonly called spin up and spin down.

Each electron atomic orbit, defined by its distribution in space and angular momentum, can have only two electrons, one with spin up and the other with spin down.

In V3p4-13, Feynman muses about what the universe might be like if fermions behaved like bosons. If so, any number of electrons could occupy the lowest energy orbit closest to the nucleus. Indeed it would be much more likely than not that they all would occupy the same state. This also applies to protons and neutrons inside the nucleus. All particles would occupy the lowest energy states, with their spins aligned in the same direction. No chemical reactions would occur, because all electrons would already be in the lowest possible energy level. We carbon-based life forms wouldn't need to worry about any of the other possible consequences of repealing the exclusion principle.

Fortunately, fermions are antisocial and atoms are much more interesting as a result.

Hydrogen has only one proton and one electron, so the exclusion principle does not apply.

Helium has two protons, two neutrons, and two electrons. All occupy the lowest energy level, denoted n=1, that is available to their species. To do that, all pairs have their spins anti-parallel. As we said earlier, helium-4 is a spin 0 boson. In helium the n=1 *shell* is full. All orbits in the n=1 shell are spherically symmetric, are denoted "1s" orbits, and are as close to the atom's center as the uncertainty principle and Bohr quantization allow.

Beyond helium, additional particles enter the n=2 shell, which has the second lowest energy orbits. This shell is more diverse. The n=2 electron orbits are of two types: a single "2s" spherically symmetric orbit; and three asymmetric orbits called "2p". Each of these four n=2 orbits can accommodate two electrons, one with spin up and the other with spin down, for a total of 8 electrons.

Figure 6-5 illustrates some of the more common electron orbits in atoms. The 1s orbit is a simple ball. The 2p orbits are asymmetric, with the electron wave at each end having opposite phases, represented in the figure by different shadings. Mathematically, one side is proportional to $\cos(\omega t)$ while the opposite side is proportional to $\cos(\omega t+\pi)$. The p orbits have one symmetry axis, as do American footballs. The axes of the three p orbits are mutually orthogonal in each shell.

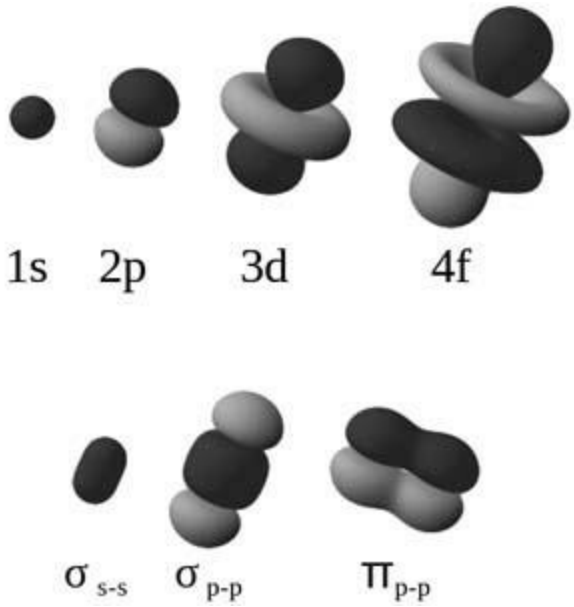

Figure 6-5 Some Electron Orbits

The n=3 shell is even more diverse, with one "3s", three "3p", and five "3d" orbits. Again each orbit accommodates two electrons with opposite spins, for a total of 18 electrons. The 3d orbital looks like a barbell of one wave phase surrounded by a donut of the opposite phase.

The n=4 shell has one "4s", three "4p", five "4d", and seven "4f" orbits, and accommodates a total of 32 electrons. The higher-numbered shells are progressively more complex.

One general rule is: shell n accommodates up to $2n^2$ electrons.

Electrons are less tightly bound in outer orbits than inner orbits. For example, 25 volts is required to pull an electron out of a 1s orbit in helium, while only 5 volts extracts an electron from a 2s orbit in lithium. An atom's outermost, most loosely bound electrons, are most easily transferred to or shared with other atoms, enabling chemistry.

Antisocial fermions also make matter rigid and stable. Any attempt to squeeze together two objects requires forcing their electrons into less space. Since electrons are already in the lowest energy states that the exclusion principle allows, the only "space" they can be pushed into is higher energy states. At several volts per electron, and nearly a trillion electrons per cubic centimeter, crushing even a tiny piece of solid matter requires hundreds of thousands of joules.

Two-Nucleon Systems

Feynman ends this lecture discussing two-nucleon states.

A proton and a neutron can combine to form the stable nucleus of deuterium, hydrogen-2. But neither two neutrons nor two protons form stable pairs. We know two protons repel one another electrically, but the strong force is strong enough to overcome that. Helium-3, two protons and one neutron, attests to that. The key issue, Feynman says, involves the exclusion principle.

Experiments show that the strong force is stronger between nucleons whose spins are parallel than between nucleons with opposite spin. Two protons (or two neutrons) cannot exist in their lowest energy state (1s) if they have the same spin, and the next higher energy state isn't bound.

The only bound two-nucleon system is the spin 1 deuteron, the nucleus of deuterium, one proton and one neutron with parallel spins. Parallel spins enhance the strong force attraction, and are permitted by the exclusion principle because the particles are non-identical.

Chapter 6 Review: Key Ideas

1. The gregarious behavior of bosons underlies Einstein's laws of radiation. The probability that an atom will absorb a photon from a state containing N photons is proportional to N. The probability that an atom will emit a photon into a state containing N photons is proportional to N+1.

2. Feynman recommends memorizing the number of modes, M, in 3-D k-space: $dM = V/(2\pi)^3\ dk_x\ dk_y\ dk_z = V/(2\pi)^3\ d^3k$, where the k's, the wave numbers in each dimension, can be positive or negative.

3. Gregarious bosons and Einstein's laws of radiation explain Planck's black body equation for the light intensity, the energy per unit area per second, emitted by a body of temperature T:

   $$I(\omega) = (\hbar\omega^3/\pi^2c^2)/\{\exp(+\hbar\omega/kT) - 1\}$$

4. Fermions are antisocial; they never exist in the same state as another identical fermion. This is called the *Pauli Exclusion Principle*. Two fermions are not in the same state if their spins are different.

5. Due to the exclusion principle, each electron orbit in an atom, defined by its distribution in space and angular momentum, can have only two electrons, one with spin up and the other with spin down. This leads to the rich structure of atoms that enables chemical processes.

6. Nature has room for both groupies and individualists.

Chapter 7

Spin One

In V3p5-1, Feynman declares that he will henceforth describe quantum mechanical phenomena in purely quantum terms, with no attempts at classical analogs or explanations.

He starts with the quantum phenomena of particles with *spin one*. We previously described spin as an intrinsic form of quantized angular momentum that elementary particles possess. Two key points to stress are: (1) a body with spin one has a spin component of either +ħ, 0, or –ħ along any selected axis; and (2) measuring a body's spin along one axis nullifies any knowledge of its spin along all other axes. Physicists generally say the spin is +1, 0, or –1, leaving the ħ unmentioned but understood.

Feynman says spin one is "sufficiently complicated that it can stand as a prototype which can be generalized for the description of all quantum mechanical phenomena."

What we learn in this chapter about the interplay of quantum states will provide a foundation for exploring many other quantum phenomena.

Stern-Gerlach Device for Spin Separation

In 1922 Otto Stern and Walther Gerlach developed a device that sorts atoms according to their magnetic moments along any selected axis. Any object with spin has a magnetic moment; along any axis, the magnetic moment is proportional the component of spin along that axis. Stern and Gerlach proved that quantum entities cannot spin in any direction, like a child's top. They showed that quantum spin is quantized, as we have previously described. Stern was awarded the 1943 Nobel Prize.

In V2p35-3, Feynman gives a detailed explanation of how a Stern-Gerlach device works, which I will briefly summarize here. Between the pole faces of a normal magnet, the magnetic field is uniform. But in a Stern-Gerlach device, one pole face is tapered, as shown in Figure 7-1, resulting in a magnetic field that varies across the pole gap, being highest at the tip of the taper.

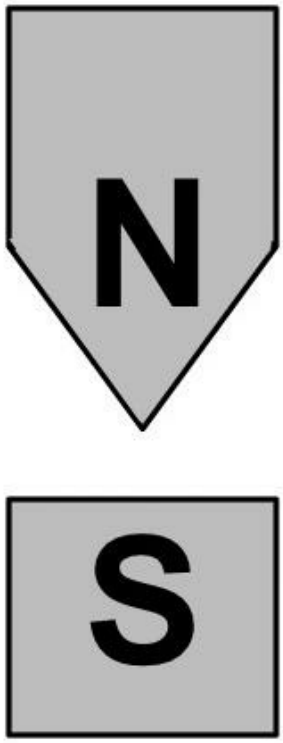

Figure 7-1 Tapered Magnet
in Stern-Gerlach Device

This field gradient exerts a force on magnetic dipoles that is parallel to the pole gap (vertical in the above image). Particles with magnetic dipole moments deflect toward one pole or the other, depending on their spin orientation. Stern-Gerlach devices are most effective with beams of electrically neutral particles, since charged particles will experience a much greater force from the qv×B term of the Lorentz force.

In V3p5-2, Feynman imagines an enhanced version of the basic Stern-Gerlach device that is illustrated in Figure 7-2. Atoms (or particles) enter from point A, pass through a collimator, traverse three magnets, pass through a final collimator, and arrive at point B. The collimators define a narrow beam. In this 2-D image, the tapering of the magnet pole faces isn't visible; the upper poles of each magnet are tapered in the third dimension, in and out of the screen.

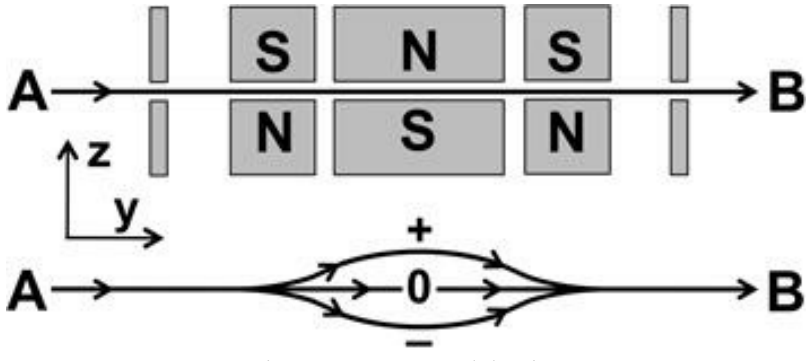

Figure 7-2 Feynman's Stern-Gerlach Device

The lower half of Figure 7-2 is a magnified view of beam deflections inside the device. It shows that atoms with positive magnetic moments in the z-direction deflect upward, those with negative moments deflect downward, and those with zero moments in the z-direction are not deflected.

The three-magnet arrangement first separates the beams, and then recombines them. If all three

pathways are open, as in Figure 7-2, the net effect of the entire apparatus is to leave the atoms completely unaltered.

But we have options. Absorbers can be placed midway through the device to block one or more of the three pathways. This enables selection of those atoms (or particles) with desired spin components along the z-axis. For example, if we block the "0" and "–" beams, the beam at B will contain only atoms with spin +1 along the z-axis. Such a beam is variously described as *polarized*, *filtered*, or *in a definite spin state*.

To eliminate needless complications, Feynman stipulates that atoms are at rest at points A and B. Unspecified mechanisms accelerate the atoms at A and decelerate them at B.

We will employ a shorthand notation to describe various Stern-Gerlach device configurations. While different from Feynman's notation, ours is more suitable for ereaders. The following symbols denote: a beam enters from the left, traverses a Stern-Gerlach device labeled S that passes only those atoms with spin +1 along the axis of device S, after which the beam exits to the right.

–> {S:+} –>

The above device passes only the spin + beam because it contains absorbers that block the spin –1 and spin 0 beams. As Stern-Gerlach devices may be oriented in any direction, their magnetic fields can select spin along any axis. We define |+S> to denote a pure state of spin +1 with respect to the axis of device S.

Examples of this notation are:

–> {A:+} –> device A passes only spin |+A>

–> {B:**0**} –> device B passes only spin |0B>

–> {C:+–} –> device C passes |+C> and |–C>

–> {D:–} –> device D passes only spin |–D>

–> {E:**none**} –> device E passes no atoms

–> {F:**all**} –> device D passes all spins

We can put two devices, S and T, in a row by writing:

–> {S:–} –> {T:**all**} –>

Here, atoms go through S first, and subsequently go through T. After S, the beam is in a pure spin |–S> state, and passes through T unaltered because T passes all spins.

Conversely, in the following configuration, no atoms pass through the second device:

–> {S:–} –> {S:**0**} –>

Recall our prior notation: <U|V> is the probability amplitude that an atom in state V goes into state U. Here we can write:

<+S|+S> = 1, 100% likely that +S goes to +S
<0S|+S> = 0, it never happens

In V3p5-5, Feynman lays out a *matrix* listing all nine combinations of <To|From>:

	From		
To	+S	0S	–S
+S	1	0	0
0S	0	1	0
–S	0	0	1

Matrices are used to describe quantum phenomena. The more interesting phenomena have non-zero off-diagonal components.

Two Devices with Rotation

Now let's get more elaborate. What happens if we send atoms through two sequential Stern-Gerlach devices when the second is rotated? Let T, our second device, be rotated about the y-axis (the beam axis) by angle θ relative to S, the first device. Figure 7-3 shows the view looking down the beam axis from starting point A.

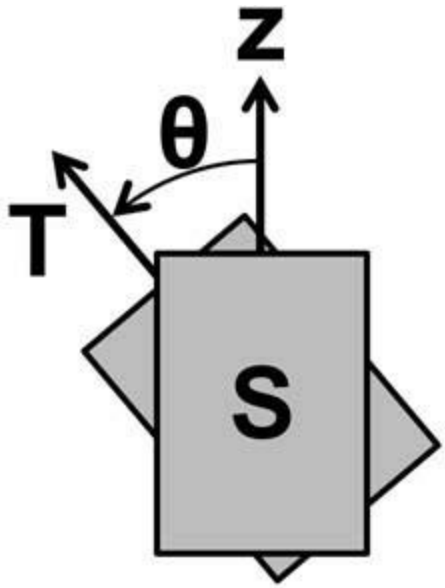

Figure 7-3 Device S Followed by Device T Rotated by θ

Our devices are configured:

–> {S:+} –> {T:+} –>

How many atoms pass through device T? Unless θ=0, the answer is less than 100% of the atoms that pass through S. This is because the atoms exiting device S are in a pure |+S> state, which is different from |+T>. This is similar to expressing a vector in two different coordinate systems. It's the same vector, but its components are different in different coordinate systems.

In our present case, state |+S> is represented in the rotated system by various amounts of |+T>, |0T>, and |–T>. Those amounts are written: <+T|+S>, <0T|+S>, and <–T|+S>. Similarly for states |0S> and |–S>. In total, there are nine coefficients for transforming 3 S states into 3 T states. These nine coefficients form a 3×3 *transformation matrix*.

In the next chapter, we will derive some transformation matrices for various rotations.

If an atom in the |+S> state enters T it must end up in some combination of T states with total probability 1; it has to go somewhere. Mathematically, we write:

$$|\langle +T|+S\rangle|^2 + |\langle 0T|+S\rangle|^2 + |\langle -T|+S\rangle|^2 = 1$$

Three Sequential Devices

Next, imagine three sequential Stern-Gerlach devices: S, T, and U. Let S and T be as above, with T rotated about the y-axis by angle θ, and let U be parallel to S. In our notation, this configuration is:

–> {S:+} –> {T:**0**} –> {U:+} –>

Feynman asks: will all the atoms that pass through T also pass through U? Do the atoms that previously passed through S retain any knowledge of once being in state |+S>? The answer is No.

Every measurement forces the observed entity into a definite quantum state, a state with an allowed value for that measurement. In this case, only three values are allowed: +1, 0 and –1 along the measurement axis.

Quantum states do not record prior history. Particles and atoms have no clocks and no diaries. They exist entirely in the Now.

When an atom exits T in |0T>, nothing additional can be known about its spin. It makes no difference whether T is preceded by one S device, 79 S devices, or none. Since T is rotated relative to U, state |0T> is a combination of the three U states, with amplitudes that depend on the rotation angle. If, for example, 1/3 of the atoms exiting T also exit U, then no matter what precedes T, that 1/3 ratio will never change.

Consider an example using two experiments:

Expt. “00+”: –> {S:**0**} –> {T:**0**} –> {U:+} –>

S passes spin 0 along the z-axis, then T passes spin 0 along its rotated axis, and finally U passes spin +1 along the z-axis.

Expt. “000”: –> {S:**0**} –> {T:**0**} –> {U:**0**} –>

S passes spin 0 along the z-axis, then T passes spin 0 along its rotated axis, and finally U passes spin 0 along the z-axis

For these two experiments, the amplitudes that an atom exiting S passes through both T and U are:

Expt. “00+”: <+U|0T> <0T|0S>
Expt. “000”: <0U|0T> <0T|0S>

The ratio of amplitudes of these two experiments is:

<+U|0T> <0T|0S> / <0U|0T> <0T|0S>
= <+U|0T> / <0U|0T>

This result depends only on the configuration of T and U, not on S. This is true even if U is not parallel to S. That should be clear since the last result shows that we can remove S without changing this ratio.

Basis Vectors & Basis States

In normal 3-D space, a vector $\boldsymbol{v}$ is represented by an ordered triplet of numbers (v_x, v_y, v_z) that are the vector’s components along each axis of a chosen coordinate system. Define *basis vector* $\boldsymbol{e}_x$ to be a unit vector in the x-direction, and similarly for $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$. To conveniently represent any vector in 3-D, our coordinate axes, and therefore the three basis vectors, must be mutually orthogonal. In this case, we have these relationships:

$\boldsymbol{e}_j \bullet \boldsymbol{e}_j = 1$ for j = x, y, and z, and
$\boldsymbol{e}_j \bullet \boldsymbol{e}_k = 0$ for $j \neq k$ = x, y, and z

The first equation says the e vectors are unit vectors, and the second says they are orthogonal.

This can be written more compactly by introducing the *Kronecker delta* δ_{jk}, which equals 1 if j=k and equals 0 if j≠k. Then:

$$\boldsymbol{e}_j \bullet \boldsymbol{e}_k = \delta_{jk}$$

Any vector in 3-D space can be written as a linear combination of basis vectors. In particular:

$$\boldsymbol{v} = v_x \boldsymbol{e}_x + v_y \boldsymbol{e}_y + v_z \boldsymbol{e}_z$$

where $v_j = \boldsymbol{v} \bullet \boldsymbol{e}_j$, for j = x, y, z

In the same manner, in quantum mechanics, we can represent any state as a linear combination of *basis states*. An essential step in addressing any quantum situation is identifying a set of basis states that are mutually orthogonal and that span the entire range of possible states. The latter requirement ensures that every possible state is some linear combination of the basis states.

(The terms *basis vectors* and *basis states* are more commonly used than Feynman's terms, base vector and base state. But both have the same meaning.)

Just as with basis vectors, generally there are a great many, if not an uncountable number, of possible sets of basis states. All are equally valid. Referring back to the last section, in our idealized thought experiment, two equally valid choices of basis states are:

|+S>, |0S>, |–S>; and
|+T>, |0T>, |–T>

Let's go back to the experiment with S and T illustrated in Figure 7-3, and set the absorbers to this configuration:

–> {S:+} –> {T:**0**} –>

Feynman reminds us that atoms exiting T are in pure |0T> state and have no memory of once being in state |+S>. Feynman says:

> "Some people would say that in filtering by T we have 'lost the information' about the previous state |+S> because we have 'disturbed' the atoms when we separated them into three beams in the apparatus T. But that is not true. The past information is not lost by the *separation* into three beams, but by the [*absorbers*]."

He demonstrates this with the following sequence of thought experiments.

Firstly, arrange three devices, S, T, and U, with U parallel to S. In the first experiment, call it "+0+", S and U are identical, both passing |+S>, while T is rotated at an unspecified angle θ and passes |0T>. Define N to be the number of atoms exiting S. Some fraction α of those will pass through T; the number exiting T is αN. Of those, some other fraction β will pass through U; the number exiting U is βαN. Thus we have:

Expt. "+0+": –> {S:+} N –> {T:**0**} αN –> {U:+} βαN –>

After each device, we added above the number of atoms that pass through that device.

In the next experiment, call it "+00", reset U to pass only |0U>. Of the αN atoms exiting T, a fraction ε pass through the re-configured U.

Expt. "+00": –> {S:+} N –> {T:**0**} αN –> {U:**0**} εαN –>

We next remove all absorbers from T and repeat each of the last two experiments, which we now label "+all+" and "+all0", with "all" representing a completely unblocked device.

Expt. "+all+": –> {S:+} N –> {T:**all**} N –> {U:+} N –>

Expt. "+all0": –> {S:+} N –> {T:**all**} N –> {U:**0**} 0 –>

When S = U, opening T increases the pass rate:
  βαN becomes N

When S ≠ U, opening T decreases the pass rate:
  βαN becomes 0

Feynman says it may seem surprising that *removing* absorbers *drops* a pass rate. But, he adds, we have seen such surprises before. In the two-slit experiment, opening a second slit reduces the intensity at half the points on the detection screen, due to destructive interference.

The complementary effect is explored in ***Feynman Simplified 1C*** Chapter 36, where adding a third Polaroid filter allows light transmission through an otherwise opaque pair of orthogonal filters. There the third filter was oriented at 45º and placed between the orthogonal pair.

Feynman says, in V3p5-10, that the effect in our current case is due to interfering amplitudes. It's not interference in the same sense as in two-slit experiments, diffraction gratings, and optics. Those situations require careful analysis of varying path lengths and changing phase angles. Here, the situation is much easier: pathways are simply being switched on and off.

Let's see the exact effect of switching pathways on and off.

First, consider the pair of experiments with both S and U passing |+S>, one with T passing |0T> and the other with T passing all spins. We said states can be represented using any basis states. We can represent |+S> using the T states as a basis. For some a, b, and c:

$$|+S\rangle = a|+T\rangle + b|0T\rangle + c|-T\rangle$$

If T passes only |0T>, the state entering U is: b|0T>. We can next represent that state in a basis of S states, which are identical to U states. For some d, e, f:

$$|0T\rangle = d|+S\rangle + e|0S\rangle + f|-S\rangle$$

Hence the state exiting U is bd|+S>, when T passes |0T>.

But if T passes all spins, the state exiting T is unaltered by T; it remains:

$$a|+T\rangle + b|0T\rangle + c|-T\rangle = |+S\rangle$$

which U passes in its entirety. As Feynman says, T separates the beam into three parts and then recombines them exactly — "Humpty Dumpty has been put back together again."

Next, consider the pair of experiments with different S and U: S passing |+S> and U passing |0S>. When T passes only |0T>, the state entering U is: b|0T>, as above. The final state is b times the part of |0T> that U passes: b•e|0S>. While when T passes all spins, the state entering U is |+S>, which U totally blocks.

Rules for Basis States

Feynman does the same analysis slightly differently. For the case of identical S and U, and T passing all spin states, he says the amplitude of passing U is the sum of the amplitudes for three different but undistinguished paths. The three paths are those of the three beams passing through T. The amplitude for the |+T> path is:

<+S|+T><+T|+S>

This is (the amplitude of going from |+S> to |+T>) × (the amplitude of going from |+T> to |+S>).

We can write the sum of three such amplitudes more compactly as:

$Q = \Sigma_J$ <+S|J><J|+S>

where J is summed over all T states.

When T passes all spins, T really has no effect — nothing would change if T were removed. In this case, we can remove T from the above equations and rewrite them as:

Σ_J <+S|J><J|+S> = <+S|+S> = 1
Σ_J <0S|J><J|+S> = <0S|+S> = 0
Σ_J <–S|J><J|+S> = <–S|+S> = 0

All this results from the fact that S states and T states are basis sets.

If we now rotate U to some other angle ø, the amplitude would become:

Σ_J <0U|J><J|+S> = <0U|+S>

Indeed, Feynman says this is a general rule that applies to any initial and final state. Replacing the initial and final states with more generic symbols, we can state the rule in a general form:

Σ_J <ψ|J><J|φ> = <ψ|φ>

This rule holds provided the |J> states are a complete orthonormal basis. Orthonormal means <J|K> = δ_{JK}, and complete means every possible state |φ> is a linear combination of |J> states.

Setting ψ=φ, we get:

Σ_J <φ|J><J|φ> = <φ|φ> = 1

Let's next derive another important rule by examining the representation of state $|\varphi\rangle$ as a linear combination of three basis states $|J\rangle$:

$$|\varphi\rangle = a_1 |1\rangle + a_2 |2\rangle + a_3 |3\rangle$$

where $a_J = \langle J|\varphi\rangle$, or more compactly

$$|\varphi\rangle = \Sigma_J \langle J|\varphi\rangle |J\rangle$$

It is essential that the sum of the squares of the a_J's equals 1: the sum of the probabilities of the particle being in some combination of basis states must be 100%. Hence:

$$1 = \Sigma_J |\langle\varphi|J\rangle|^2 = \Sigma_J \langle\varphi|J\rangle\langle\varphi|J\rangle^*$$

Combining that with an earlier result:

$$1 = \Sigma_J \langle\varphi|J\rangle\langle J|\varphi\rangle = \Sigma_J \langle\varphi|J\rangle\langle\varphi|J\rangle^*$$
$$0 = \Sigma_J \langle\varphi|J\rangle \{\langle J|\varphi\rangle - \langle\varphi|J\rangle^*\}$$

This must be true for any state φ and any set of basis states J. This requires:

$$\langle J|\varphi\rangle = \langle\varphi|J\rangle^*$$

This important relationship is valid for any J and φ.

The Power of Basis States

In V3p5-12, Feynman demonstrates the utility of employing basis states.

Suppose, he says, we have a three-state system such as atoms with spin one. We send those atoms through device S to obtain a pure state, and then send those atoms through a very complex apparatus A, and finally filter the atoms exiting A with device U, which need not be identical to S. All this is symbolized by:

–> {S:+} –> {A:**???**} –> {U:+} –>

Feynman says:

> "By A we mean any complicated arrangement of Stern-Gerlach apparatuses with mask [absorber] or half-mask, oriented at peculiar angles, with odd electric and magnetic fields… almost anything you want to put. (It's nice to do thought experiments — you don't actually have to go to all the trouble of actually building the apparatus!)"

The standard notation for the amplitude to go from state $|+S\rangle$ go through A and end in state $|0U\rangle$ is:

$$\langle 0U | A | +S \rangle$$

which we read right to left as:
<end | through | start >.

Recalling our analogy of a quantum state as a vector, A is an *operator* that transforms vectors in some way, such as a rotation, and $\langle\psi|A|\varphi\rangle$ is analogous to the dot product of ψ with the transformed vector $A\varphi$.

If A does nothing to alter the particle-wave states, we can write either of the following equivalent expressions:

$\langle 0U|A|+S\rangle = \langle 0U|1|+S\rangle = \langle 0U|+S\rangle$

Since there are limitless possible initial and final states, understanding $\langle 0U|A|+S\rangle$ might seem hopeless. But using the rules of the last section, we can represent any initial and any final state using any basis states we wish. We choose three states denoted |1>, |2>, and |3>. We then have (with J and K summed over 1, 2, and 3):

$\langle 0U| = \Sigma_K \langle 0U|K\rangle \langle K|$
$|+S\rangle = \Sigma_J \langle J|+S\rangle |J\rangle$
$\langle 0U|A|+S\rangle = \Sigma_{KJ} \langle 0U|K\rangle \langle K|A|J\rangle \langle J|+S\rangle$

For any initial state φ and final state ψ, this is:

$\langle\psi|A|\varphi\rangle = \Sigma_{KJ} \langle\psi|K\rangle \langle K|A|J\rangle \langle J|\varphi\rangle$

This means any device A, regardless of its complexity, is completely characterized by just nine complex numbers, the nine amplitudes:

$\langle K|A|J\rangle$

In V3p5-14, Feynman says:

> "This then is the machinery of quantum mechanics for a spin one particle. Every *state* is described by three numbers which are the amplitudes to be in each of some selected set of [basis] states. Every apparatus is described by nine numbers which are the amplitudes to go from one [basis] state to another in the apparatus. From these numbers anything can be calculated."

Feynman's statements can be generalized to systems with any number of basis states.

The nine amplitudes of a spin one apparatus are often written in matrix form:

$$\begin{vmatrix} \langle +T|A|+T\rangle & \langle +T|A|0T\rangle & \langle +T|A|-T\rangle \\ \langle 0T|A|+T\rangle & \langle 0T|A|0T\rangle & \langle 0T|A|-T\rangle \\ \langle -T|A|+T\rangle & \langle -T|A|0T\rangle & \langle -T|A|-T\rangle \end{vmatrix}$$

Components of matrix A are identified by indices: $A_{KJ} = \langle K|A|J\rangle$.

Now imagine that apparatus A actually consists of two independent devices B and C. What we previously described as "atoms enter A in state |J> and exit A in state |K>" is now replaced by "atoms enter B in state |J>, exit B and enter C in state |M>, and exit C in state |K>." The amplitude is now written:

$$\langle\psi|CB|\varphi\rangle = \Sigma_{KMJ} \langle\psi|K\rangle\langle K|C|M\rangle\langle M|B|J\rangle\langle J|\varphi\rangle$$

As before, we can represent B and C by 3×3 matrices. The matrix A is the product of the matrix C times the matrix B: A=CB. Since matrix multiplication is not commutative, one must multiply matrices in the proper order: CB≠BC in general. By convention, the proper order puts the first device to act on the right, the next device to act to its immediate left, and so forth. In this case, atoms enter B before C, so matrix B is on the right.

The rule for matrix multiplication can be read from the last equation:

$$A = C\,B$$
$$A_{KJ} = \Sigma_M\, C_{KM}\, B_{MJ}$$
$$A_{KJ} = \Sigma_M \langle K|C|M\rangle \langle M|B|J\rangle$$

Unpolarized Beams

Often the original source of beams of atoms is unpolarized. A solid or liquid is heated; the most energetic atoms escape into the vapor phase, and a portion thereof form the source for Stern-Gerlach and other experiments. Since heat energy has no preferred direction, atoms exiting a furnace have randomly distributed spin orientations.

This means we expect the probability of an atom being in state |+S> to be 1/3, and the same for |0S> and |–S>, for any set of basis states S. In V3p5-17, Feynman stresses that equal probabilities **do not** mean equal probability amplitudes. The atom's initial state |ψ> is **not** known to have equal amplitudes for each basis state:

$$|\psi\rangle \neq (1/\sqrt{3})|{+S}\rangle + (1/\sqrt{3})|0S\rangle + (1/\sqrt{3})|{-S}\rangle$$

The above state has specific phase relationships and interference possibilities that unpolarized states do not have. We only know that:

$$1/3 = |\langle\psi|{+S}\rangle|^2 = |\langle\psi|0S\rangle|^2 = |\langle\psi|{-S}\rangle|^2$$

The probability that such an atom initially happens to be in state |+S>, goes through device A, and ends up in state |φ> is: $|\langle\varphi|A|{+S}\rangle|^2/3$. The probability that an atom from an unpolarized beam goes through A and ends up in state |φ> is the sum of probabilities for each initial spin state:

$$|\langle\varphi|A|{+S}\rangle|^2/3 + |\langle\varphi|A|0S\rangle|^2/3 + |\langle\varphi|A|{-S}\rangle|^2/3$$

The choice of basis states S is arbitrary. If we instead use T states as a basis set, with:

$|JS\rangle = \Sigma_K \langle KT|JS\rangle |KT\rangle$

we would get the same result, for unpolarized beams.

$\Sigma_J |\langle\varphi|A|JS\rangle|^2 = \Sigma_{JK} |\langle\varphi|A|KT\rangle \langle KT|JS\rangle|^2$
$\Sigma_J |\langle\varphi|A|JS\rangle|^2 = \Sigma_K \{|\langle\varphi|A|KT\rangle|^2 (\Sigma_J |\langle KT|JS\rangle|^2)\}$
$\Sigma_J |\langle\varphi|A|JS\rangle|^2 = \Sigma_K |\langle\varphi|A|KT\rangle|^2$ (1)

Chapter 7 Review: Key Ideas

1. Stern-Gerlach devices separate atoms (or particles) into beams of pure spin states.

2. Every measurement forces the observed entity into a definite state, one with an allowed value for that measurement. Particles have no memory of their prior history. If an atom's spin is measured to be +1 along the z-axis, nothing additional can be known about its spin.

3. Any quantum state can be represented as a linear combination of *basis states*, which are mutually orthogonal and span the entire range of possible states. Generally there are many possible sets of equally valid basis states. For a set of N basis states $|J\rangle$, J=1…N, these relationships hold for any states ψ and φ, and with J and K summed from 1 to N:

   $\langle J|K\rangle = \delta_{JK} = 1$ if J=K, and = 0 if J≠K

   $|\varphi\rangle = a_1 |1\rangle + a_2 |2\rangle + \ldots + a_N |N\rangle$
   for $a_J = \langle J|\varphi\rangle$, $|\varphi\rangle = \Sigma_J \langle J|\varphi\rangle |J\rangle$

   $1 = \Sigma_J |\langle\varphi|J\rangle|^2 = \Sigma_J \langle\varphi|J\rangle\langle\varphi|J\rangle^*$
   $1 = \Sigma_J \langle\varphi|J\rangle\langle J|\varphi\rangle = \langle\varphi|\varphi\rangle$

   $\langle\psi|\varphi\rangle = \Sigma_J \langle\psi|J\rangle\langle J|\varphi\rangle$
   $\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$

4. The standard notation for the amplitude to go from state $|\varphi\rangle$, through apparatus A, and end in state $|\psi\rangle$ is:

   $\langle \psi | A | \varphi \rangle$

5. For N state systems, every *state* is completely described by N complex numbers: the amplitudes to be in each of the N states of any selected basis set. Every apparatus A is completely described by N×N complex numbers: the amplitudes for A to transform one basis state into another. "From these numbers anything can be calculated." Apparatus A is represented by a quantum *operator* A with N×N amplitudes that are often arrayed as a matrix: $A_{KJ} = \langle K|A|J\rangle$.

6. Two sequential apparatuses B and C, with B preceding C, which are represented by matrices B_{KJ} and C_{KJ}, are equivalent to a single apparatus A whose matrix is the product of matrix C times matrix B. The matrix equations are:

   $A = C\,B$
   $A_{KJ} = \Sigma_M\, C_{KM}\, B_{MJ}$
   $A_{KJ} = \Sigma_M \langle K|C|M\rangle \langle M|B|J\rangle$

   Matrix multiplication is not commutative: $CB \neq BC$ in general.

Chapter 8

Rotations for Spin ½

This chapter is entirely devoted to mathematically deriving transformation matrices for rotations of basis states, using the specific example of spin 1/2 particles.

In V3p6-2, Feynman suggests some might wish to skip the math and just use the results, which I provide in the review section at the end of this chapter.

Even if you skip everything else in this chapter, be sure to read the last section.

While this chapter is more math-heavy than most, what we learn here about the properties of spin 1/2 particles explains the fundamental behavior of all material objects, because all matter is composed entirely of spin 1/2 particles.

Feynman describes this lecture as:

> "a sort of cultural excursion. …it is intended to show that the principles of quantum mechanics are not only interesting, but are so deep that by adding only a few extra hypotheses about the structure of space, we can deduce a great many properties of physical systems."
>
> "…as long as our laws of physics are incomplete — as we know they are — it is interesting to find out whether the places where our theories fail to agree with experiment is where our logic is the best or the where our logic is the worst. Until now, it appears that where our logic is the most abstract it always gives the correct results — it agrees with experiment. Only when we try to make specific models of the internal machinery of the fundamental particles and their interactions are we unable to find a theory that agrees with experiment."

In the last 50 years, great advances have been achieved in the theory of fundamental particles. The Standard Model of particle physics effectively models *almost* all particle interactions. The number of unanswerable questions is much smaller today than it was when Feynman gave these lectures.

Feynman adds:

> "We are going to derive all the coefficients for the transformation from one representation to another by pure reasoning — plus a few assumptions. *Some* assumptions are always necessary in order to use 'pure' reasoning."

Transforming Basis States

Let's review how transformations fit into our quantum description of nature.

In the last chapter, we discovered that any quantum state can be represented as a linear sum of basis states. For example state |U> can be represented as:

$$|U\rangle = \Sigma_J \langle J|U\rangle |J\rangle$$

where J is summed over all basis states |J>, and <J|U> is the amplitude for something in state |U> to be found in state |J>. We can think of <J|U> as measuring the overlap between the two states; it is the dot product of vectors J and U within the space of all states.

In any quantum situation, there can be many (even infinitely many) sets of equally acceptable basis states. For convenience, we choose basis states that are orthonormal and complete. Orthonormal means for any two basis states |J> and |K>:

$$\langle J|K\rangle = \delta_{JK} = 1 \text{ if } J=K, = 0 \text{ if } J\neq K$$

A basis set is complete if every possible state is some linear combination of basis states.

We also discussed transforming from one set of basis states to another. If we know <J|U> for all |J>, we know the complete representation of |U> in basis states |J>. We can transform to basis states |L> according to:

$$|U\rangle = \Sigma_{LJ} \langle L|J\rangle \langle J|U\rangle |L\rangle$$

When a basis set has N states, there are N^2 transformation amplitudes <L|J> that allow us to transform any state in the J basis to the corresponding state in the L basis. The <L|J> amplitudes can be arranged into an N×N matrix.

The objective of this chapter is to derive a complete set of transformation matrices for rotations, for the case of one basis set rotated relative to another.

Since any quantum state can be represented using basis states, each state is mathematically equivalent to a vector in N dimensions, where N is the number of linearly independent basis states. We can therefore employ the mathematics of N-dimensional rotational transformations. This mathematics was fully developed long before quantum mechanics. We will not add much here to the theory of rotations.

The essential concept is: a vector V defined in one basis is transformed to a rotated basis by multiplying it by one or more matrices. If we rotate once, by angle θ about the z-axis, the transformed vector U is given by:

$$U_j = \Sigma_k R_{jk} V_k$$

where the components of R depend on θ and z, and the indices j and k range over all N basis states or basis vectors. To rotate multiple times, we repeat the procedure. For three rotations, we could write:

$$U_j = \Sigma_{kmn} T_{jn} S_{nm} R_{mk} V_k$$

Here R, S, and T each perform a rotation by some angle about some axis. Equivalently, we could calculate matrix A, the product of matrices R, S, and T, and use A to perform the same transformation in one step:

$$A_{jk} = \Sigma_{mn}\, T_{jn}\, S_{nm}\, R_{mk}$$

$$U_j = \Sigma_k\, A_{jk}\, V_k$$

It is easier to derive the transformations individually, and later decide whether to employ them serially or in combination.

For spin 1/2, there are only two states: +1/2 and –1/2, which we will further abbreviate to + and –. In a Stern-Gerlach device, spin 1/2 particles separate into two beams, one less than the three beams of spin 1 particles in the last chapter. For spin 1/2, a basis set needs only two states; N=2, and rotation matrices have only 4 components. We choose to start with the simplest case.

Feynman discusses at some length the fact that the components of a rotation matrix are not uniquely defined. As we know from the study of wave interference, only phase angle differences have physical significance. If we add a constant phase angle to every wave, no element of physical reality changes. Similar situations arise repeatedly with the rotation of quantum states; multiplying every state by $\exp\{i\phi\}$ has no physical consequences. We will, therefore, arbitrarily assign overall phase angles as is convenient and without exhaustive discussion. We also adopt the standard sign conventions: the right-hand rule, and the determinant of all transformation matrices must be +1.

Rotational & Translational Invariance

The first point Feynman makes is that absolute angles have no physical significance: rotation matrices depend not on angles, but only on angle *differences*. This principle, called rotational invariance, states the laws of nature are the same in all directions. If the entire universe were rotated by 10 degrees, nothing would change. Rotational invariance is directly related to the conservation of angular momentum; angular momentum would not be conserved if the laws of nature changed as we turned.

Feynman doesn't mention this, but associated with rotational invariance is translational invariance, which states the laws of nature are the same here, as they are over there. This is directly related to the conservation of linear momentum.

All this is interesting and important, but not particularly quantum.

Rotations About the Z-Axis

Consider two devices, S and T, which pass all spins (no absorbers) and are represented by dashed rectangles in Figure 8-1. The magnetic fields of both S and T are in the +z-direction, pointing toward you, perpendicular to the screen. A particle with spin + deflects up out of the screen in each device,

while spin – particles deflect downward. Recall that our Stern-Gerlach devices recombine all beams after deflection.

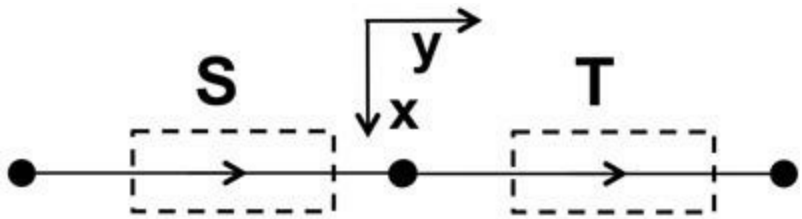

Figure 8-1 Two Devices with Zero Rotation

S and T lie along a straight line parallel to the y-axis. In this orientation, S and T have zero rotation relative to one another. The rotation matrix for transforming S states into T states is the identity matrix:

| 1 0 |

| 0 1 |

The probability of a spin + particle being in state |+S> is 100%, as it is for being in |+T>. Similarly for spin –, the probabilities are 100% for both |–S> and |–T>.

In Figure 8-2 shows the same devices, with T rotated by 90º about the z-axis.

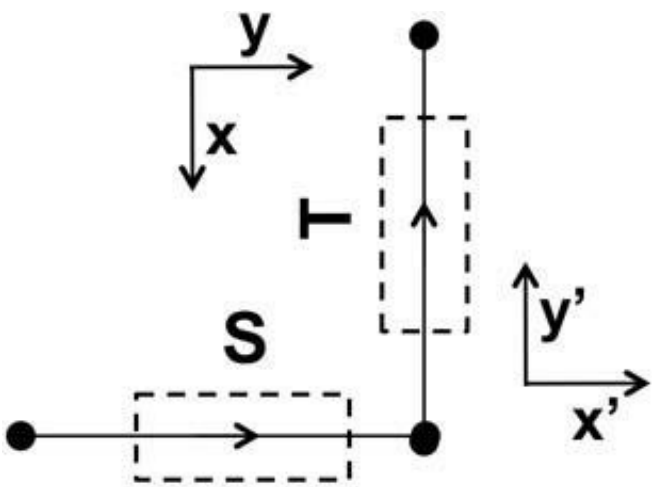

Figure 8-2 Two Devices with
90º Rotation about Z-Axis

For any T rotation angle about the z-axis, the probabilities remain 100% that a spin + particle is in state |+T> and a spin – particle is in |–T>. This means the amplitudes must have magnitude 1, but they can and do have different phases, as we show next.

Suppose that particles entering S are in a pure +x spin state. S separates particles according to their z-spin components, but then seamlessly recombines them into one beam. As we learned in the last chapter, a device that passes all spins makes no measurement and does not alter particles' states. Hence, the beam entering T remains in a pure +x spin state. But, since T is rotated by 90º, the +x-direction corresponds to T's –y'-direction. As represented in a T state basis, the beam's spin states are changed by this 90º rotation.

Since we showed that the magnitudes of the T states don't change, the states can change only by a

phase angle. We also know that the + and – spins must change by different phase angles (both changing by the same angle is equivalent to no change). Since only phase differences matter, we assign a positive phase shift to spin + and a negative phase shift to spin –:

$|+T\rangle = \exp\{+i\lambda\}\ |+S\rangle$
$|-T\rangle = \exp\{-i\lambda\}\ |-S\rangle$

Now what is λ? Since rotations are related to sine and cosine functions that are continuous, an infinitesimal rotation should result in an infinitesimal phase shift λ. For sufficiently small angles, assume λ is proportional to the rotation angle ε: $\lambda=m\varepsilon$, where m is an unknown constant.

Let T be rotated relative to S by a very small angle ε, and add another device U after T that is rotated relative to T by the same angle ε also about the z-axis. The phase change for the second rotation must equal that of the first rotation (absolute angles don't matter; only angle differences matter). Hence:

$|+U\rangle = \exp\{+im\varepsilon\}|+T\rangle = \exp\{+2im\varepsilon\}\ |+S\rangle$
$|-U\rangle = \exp\{-im\varepsilon\}|-T\rangle = \exp\{-2im\varepsilon\}\ |-S\rangle$

We can continue repeating this indefinitely, eventually building up to a large rotation. Evidently, for any angle ε, regardless of size, $\lambda=m\varepsilon$ and $|+T\rangle = \exp\{+im\varepsilon\}\ |+S\rangle$. Now what is m?

Let's try rotating T by 360º, all the way around to its starting position, on a straight line with S. We might rush to say that $|+T\rangle$ must equal $|+S\rangle$, so $\exp\{im2\pi\}=1$ and m=1. But Feynman says: *"This argument is wrong!"*

To see the problem, try ε=180º. This flips T, which is certainly a physically significant change. For ε=180º, the rotated states are:

$|+T\rangle = -1\ |+S\rangle$
$|-T\rangle = -1\ |-S\rangle$

These rotated T states are equivalent to the S states; multiplying both states by –1 makes no difference. But the physical state is different: what S sees as +x spin, the flipped T sees as –x spin. We return to the same physical situation only when ε = 360º. For 360º to be the smallest angle that produces states equivalent to the original states, m must be 1/2. For $m\varepsilon=360º/2$, $|+T\rangle$ and $|-T\rangle$ are both multiplied by –1.

Therefore, to transform from basis state S to basis states T, which are rotated about the z-axis by angle ø, the transformation equations are:

$|+T\rangle = \exp\{+i\text{ø}/2\}\ |+S\rangle$
$|-T\rangle = \exp\{-i\text{ø}/2\}\ |-S\rangle$

Rotations About the Y-Axis

Next we derive the rotation equations for two rotations about the y-axis: for angles 180º and 90º.

Consider the configuration shown in Figure 8-3, with device S preceding device T, with T rotated by 180º about the y-axis, which is the beam axis.

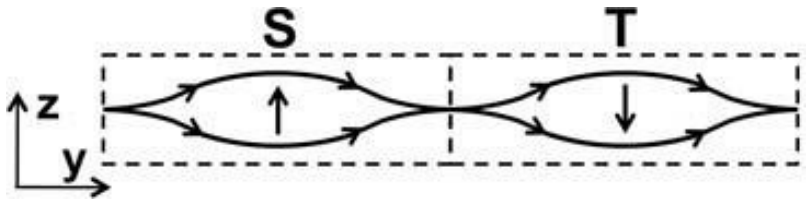

Figure 8-3 Two Devices with
180º Rotation About Y-axis

The magnetic field direction is indicated by the arrow at the center of each device. Note that the field is inverted in T.

Particles with spin + have a 100% probability of being in the |+S> state and 100% probability of being in the |–T> state. Spin – particles are in the opposite states with 100% probability. Hence the states are switched, possibly with phase angles β and γ:

$$|+T\rangle = \exp\{i\beta\}\ |-S\rangle$$
$$|-T\rangle = \exp\{i\gamma\}\ |+S\rangle$$

Rotating by 360º leads to the same issues as before, so we conclude that a 360º rotation results in each state being multiplied by –1. We accomplish a 360º rotation with two consecutive 180º rotations. Call the 180º-rotated states T and the 360º-rotated states U. Using the above expressions twice, we get:

$$|+U\rangle = \exp\{i\beta\}\ |-T\rangle = \exp\{i\beta\}\ \exp\{i\gamma\}\ |+S\rangle$$
$$|-U\rangle = \exp\{i\gamma\}\ |+T\rangle = \exp\{i\gamma\}\ \exp\{i\beta\}\ |-S\rangle$$

$$|+U\rangle = -\ |+S\rangle = \exp\{i\beta\}\ \exp\{i\gamma\}\ |+S\rangle$$
$$|-U\rangle = -\ |-S\rangle = \exp\{i\gamma\}\ \exp\{i\beta\}\ |-S\rangle$$

$$\exp\{i\beta\}\ \exp\{i\gamma\} = -1$$
$$\exp\{i\gamma\} = -\exp\{-i\beta\}$$

Again the overall phase angle is arbitrary. By convention, we chose β=0, which yields for a 180º rotation about y-axis:

$$|+T\rangle = +\ |-S\rangle$$
$$|-T\rangle = -\ |+S\rangle$$

We next turn to a 90º rotation about the y-axis. Since we know how to do 180º rotations, let's see what equating that with two 90º rotations yields. The most general form for any spin 1/2 rotation is:

$$|+T\rangle = a\ |+S\rangle + b\ |-S\rangle$$
$$|-T\rangle = c\ |+S\rangle + d\ |-S\rangle$$

Assume the above is the transformation for one 90º rotation. Now perform a second 90º rotation about the same axis, going from T to U:

$|+U\rangle = a\,|+T\rangle + b\,|-T\rangle$
$|-U\rangle = c\,|+T\rangle + d\,|-T\rangle$

$|+U\rangle = (a^2+bc)\,|+S\rangle + (ab+bd)\,|-S\rangle$
$|-U\rangle = (ca+cd)\,|+S\rangle + (cb+d^2)\,|-S\rangle$

Now require this result to match that for a single 180º rotation:

$+\,|-S\rangle = (a^2+bc)\,|+S\rangle + (ab+bd)\,|-S\rangle$
$-\,|+S\rangle = (ca+cd)\,|+S\rangle + (cb+d^2)\,|-S\rangle$

$a^2 + bc = 0$
$ab + bd = +1$
$ca + cd = -1$
$cb + d^2 = 0$

From the first and last equation, we see $a^2=d^2$. This means a=d or a=–d. However, a=–d would make the second equation: a(b–b)=1, which cannot be right. So a=d. The second and third equations now reduce to:

$2ab = +1$, so $b = +1/2a$
$2ca = -1$, so $c = -1/2a$

Putting all this into the first equation:

$a^2 + (1/2a)\,(-1/2a) = 0$
$a^4 = 1/4$
$a = 1/\sqrt{2}$

Our final result for a 90º rotation about the y-axis is:

$|+T\rangle = (1/\sqrt{2})\,(+|+S\rangle + |-S\rangle)$
$|-T\rangle = (1/\sqrt{2})\,(-|+S\rangle + |-S\rangle)$

Feynman also provides the equations for a –90º rotation about the y-axis:

$|+T\rangle = (1/\sqrt{2})\,(+|+S\rangle - |-S\rangle)$
$|-T\rangle = (1/\sqrt{2})\,(+|+S\rangle + |-S\rangle)$

Rotations About the X-axis

Before you panic, as I did, know that we now have all we need for any rotation about any axis, at least for spin 1/2 in three dimensions.

Consider for example rotating about the x-axis. Rotating 90º about the y-axis moves the z-axis into the orientation of the original x-axis. We then rotate by any angle θ about the new z-axis (old x-axis), which we learned how to do above. Finally, Rotating –90º about the new y-axis brings the x-axis back to its starting position. Try it with any 3-D object; it works.

In V3p6-11, Feynman remarks on how hard it is for us to intuitively grasp combined rotations: "It is rather strange, because we live in three dimensions, but it is hard to for us to appreciate what happens if we turn this way and then that way. Perhaps, if we were fish or birds…"

To calculate an x rotation, use the procedure discussed at the start of this chapter: multiply the three matrices for (–90º y-rotation) × (rotation by angle θ about z-axis) × (+90º y-rotation). The result is:

$|+T\rangle = \cos(\theta/2)\,|+S\rangle + i\sin(\theta/2)\,|-S\rangle$
$|-T\rangle = i\sin(\theta/2)\,|+S\rangle + \cos(\theta/2)\,|-S\rangle$

Rotation by Euler Angles

Arbitrary rotations — any angles about any axes — the pièce de résistance of 3-D rotation, are defined using Euler angles, which are shown in Figure 8-4. The transformation equations are given in the chapter review section.

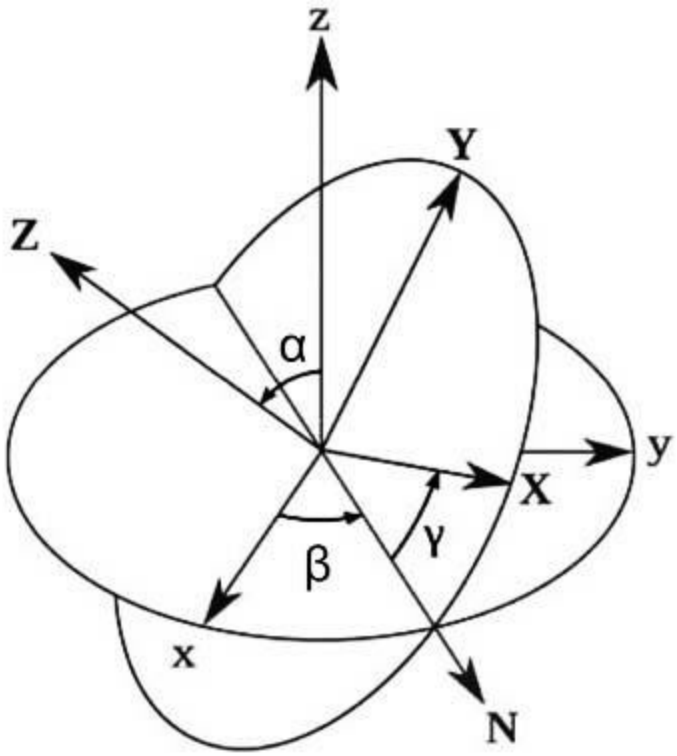

Figure 8-4 Euler Angles α, β, γ

Chapter 8 Review: Tables of Rotations

We tabulate here the equations for several rotation transformations between a set S of basis states and a rotated set T of basis states. In all cases, the y-axis is the beam axis and the z axis is the axis of separation of spins in a Stern-Gerlach device.

For **spin 1/2** particles, each basis set has two linearly independent states, denoted + and –.

To rotate about the **z-axis by angle ø**:

$|+T\rangle = \exp\{+i\text{ø}/2\}\ |+S\rangle$
$|-T\rangle = \exp\{-i\text{ø}/2\}\ |-S\rangle$

To rotate about the **y-axis by 180º**:

$|+T\rangle = +\ |-S\rangle$
$|-T\rangle = -\ |+S\rangle$

To rotate about the **y-axis by +90º**:

$|+T\rangle = (1/\sqrt{2})\ (+|+S\rangle + |-S\rangle)$
$|-T\rangle = (1/\sqrt{2})\ (-|+S\rangle + |-S\rangle)$

To rotate about the **y-axis by –90º**:

$|+T\rangle = (1/\sqrt{2})\ (+|+S\rangle - |-S\rangle)$
$|-T\rangle = (1/\sqrt{2})\ (+|+S\rangle + |-S\rangle)$

To rotate about the **x-axis by angle θ**:

$|+T\rangle = \cos(\theta/2)\ |+S\rangle + i\sin(\theta/2)\ |-S\rangle$
$|-T\rangle = i\sin(\theta/2)\ |+S\rangle + \cos(\theta/2)\ |-S\rangle$

For **Euler angles**:
rotate first about the **z-axis by angle β**,
then rotate about the **new x-axis by angle α**,
then rotate about the **new z-axis by angle γ**:

$\langle+T|+S\rangle = \cos(\alpha/2)\ \exp\{+i(\beta+\gamma)/2\}$
$\langle+T|-S\rangle = i\sin(\alpha/2)\ \exp\{-i(\beta-\gamma)/2\}$
$\langle-T|+S\rangle = i\sin(\alpha/2)\ \exp\{+i(\beta-\gamma)/2\}$
$\langle-T|-S\rangle = \cos(\alpha/2)\ \exp\{-i(\beta+\gamma)/2\}$

For **spin 1 particles**, each basis set has three linearly independent states, denoted +, 0, –.

To rotate about the **y-axis by angle θ**:

$\langle +T|+S\rangle = (1+\cos\theta)/2$
$\langle 0T|+S\rangle = -(\sin\theta)/\sqrt{2}$
$\langle -T|+S\rangle = (1-\cos\theta)/2$

$\langle +T|0S\rangle = +(\sin\theta)/\sqrt{2}$
$\langle 0T|0S\rangle = +\cos\theta$
$\langle -T|0S\rangle = -(\sin\theta)/\sqrt{2}$

$\langle +T|-S\rangle = (1-\cos\theta)/2$
$\langle 0T|-S\rangle = +(\sin\theta)/\sqrt{2}$
$\langle -T|-S\rangle = (1+\cos\theta)/2$

To rotate about the **z-axis by angle θ**:

$\langle +T|+S\rangle = \exp\{+i\theta\}$
$\langle 0T|0S\rangle = 1$
$\langle -T|-S\rangle = \exp\{-i\theta\}$

The other six amplitudes are zero.

Spins & Rotations

Compare the equations for rotations by angle θ about the z-axis, the direction of the spin-separating magnetic field. For spin 1/2 particles, the allowed z-axis spin components are s = +1/2 and –1/2. For spin 1 particles the allowed spin components are s = +1, 0, –1. For both spin types and all spin components, we have **a remarkable general rule**:

$\langle sT|sS\rangle = \exp\{is\theta\}$

In V3p4-3, Feynman says there is no simple explanation for the Pauli Exclusion Principle, adding that its proof requires relativistic quantum field theory. However, the following explanation seems simple. If you see anything wrong with it, let me know.

When two particles, ϕ and θ on the left side of Figure 8-5, are rotated 180º around the dotted axis, the result is what is shown on the right side of the figure. Similarly, rotating the right side produces the left side.

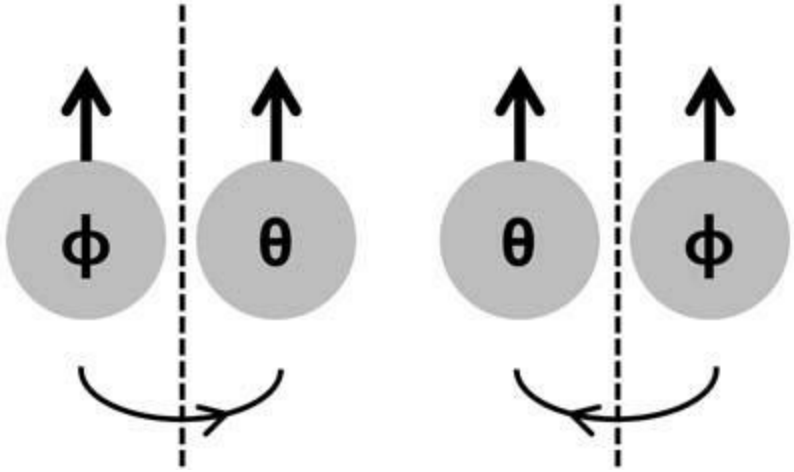

Figure 8-5 Swapping Identical Particles

From above, we found that rotating a spin s particle by 180º around the axis parallel to its spin multiplied its amplitude by $\exp\{is\pi\}$. The combined state of the two spin s particles on the left side of Figure 8-5 is:

$|A\rangle = \langle left|\phi\rangle \langle right|\theta\rangle$

After rotating both particles by 180º, amplitude A changes to:

$|B\rangle = [\langle left|\phi\rangle \exp\{is\pi\}] [\langle right|\theta\rangle \exp\{is\pi\}]$
$|B\rangle = \exp\{i2s\pi\} \langle left|\phi\rangle \langle right|\theta\rangle$
$|B\rangle = \exp\{i2s\pi\} |A\rangle$

This rotation is equivalent to swapping particles ϕ and θ, matching the right side of the figure. If ϕ and θ are identical particles, we find:

For spin 1/2, $\exp\{i2s\pi\} = -1$, and $|B\rangle = -|A\rangle$.

For spin 1, $\exp\{i2s\pi\} = +1$, and $|B\rangle = +|A\rangle$.

This demonstrates that swapping fermions flips the sign of the amplitude, as Pauli claimed, while swapping bosons leaves the sign unchanged.

If ϕ and θ are moved together into the same state, $|A\rangle$ and $|B\rangle$ become indistinguishable, which requires we combine their amplitudes.

For spin 1/2, the combined amplitude is:

$|A\rangle + |B\rangle = |A\rangle - |A\rangle = 0$

This yields Fermi-Dirac statistics and explains why identical fermions are precluded from occupying the same state.

For spin 1, the combined amplitude is:

$|A\rangle + |B\rangle = |A\rangle + |A\rangle = 2|A\rangle$

This yields Bose-Einstein statistics and explains the enhanced probability of identical bosons occupying the same state.

Chapter 9

Time Dependence of Amplitudes

This chapter focuses on the evolution of probability amplitudes over time. To illuminate the essential physics, we consider a very simple situation, indeed one that is greatly over-simplified. (Physicists sometimes use "toy models." These are very simple models that we know are quite incomplete, but that we can push around to see how far they can take us. The aim is not providing realism but rather promoting understanding.)

In V3p7-1, Feynman compares entities that have a definite energy with those that do not.

A lone, stationary electron has a definite energy, its rest mass. Physicists have measured that value to 10 digits. But because electrons have existed for nearly 14 billion years (maybe longer), nature knows their mass to at least 40 digits — that's quite definite. Let's see why.

In atoms, an electron in a stable orbit with the lowest possible energy has a definite total energy. Electrons can also briefly occupy excited states of higher energy. In the latter case, their energy is not as definite. An electron in an excited state doesn't stay there long; it quickly emits a photon and drops to a lower energy state. If Δt is the mean time an electron remains in an excited state, the mean lifetime of that state, the uncertainty principle says the energy of that excited state is uncertain to $\Delta E = \hbar/(2\Delta t)$. Physicists don't always mention that when quoting energy levels, but it is understood.

Some particles are stable, but others spontaneously decay, each with a specific mean lifetime. Such particles do not have completely definite masses. For example, K*-mesons have a broad mass distribution. Their *average* mass is 891.7 MeV, but the mass of an individual K* can vary considerably. With a mean lifetime of 1.3×10^{-23} sec, the uncertainty principle spreads K* masses across a resonance-like distribution with a width of 51 MeV.

Stationary States

Now, consider something with definite energy, such as an isolated atom at rest. Let E be its energy, by which we mean c^2 times its rest mass m. This mass includes the masses of its component particles and their binding energies. If the atom was in an excited state, it would have less negative binding energy, and its mass would be greater than in the ground state. When it emits a photon of energy $\hbar\omega$ and drops to the ground state, the atom's mass decreases by $\hbar\omega/c^2$.

For our toy model atom — at rest, isolated, and in its ground state — the quantum mechanical amplitude to find that atom at position $\boldsymbol{r}$ is the same for all $\boldsymbol{r}$: the atom is everywhere with equal

probability. (Not our usual notion of atoms.) Indeed, not just the probability but also the amplitude is exactly the same everywhere.

The amplitude is the same at every point in space, but not at every moment in time. The amplitude for the atom to be at location (x,y,z) at time t is:

$$ø(t,x,y,z) = A \exp\{Et/iħ\} = A \exp\{-i\omega t\}$$

where $ħω = E = mc^2$, and A is a normalization constant. E, m, and ω are three equally valid ways to describe the atom's energy.

While it is strange to think of a particle being spread throughout all space, this results directly from the uncertainty principle: $\Delta E=0$ implies $\Delta p=0$ implies $\Delta x=\infty$.

If we measure any internal property of an atom, such as the states of its electrons, and if the probability of each outcome never changes, we say the atom is in a *stationary state*.

Let's contrast that with a non-stationary state. Imagine the atom might or might not absorb a photon, elevating one of its electrons to an excited state. If the ground state and the excited state are not distinguished, the outcome of any measurement would depend on the sum of two amplitudes, the amplitude to be in the excited state plus the amplitude to be in the ground state. This summing of amplitudes leads to interference effects.

The ground and excited states have slightly different energies, E and E*, resulting in different time dependencies. For example, the following amplitude ø is a linear combination of ground and excited states with different energies:

$$ø = A \exp\{Et/iħ\} + B \exp\{E^*t/iħ\}$$
$$ø = A \exp\{Et/iħ\} \; [1 + (B/A) \exp\{(E^*-E)t/iħ\}]$$

Assuming $(E^*-E) << E$, ø oscillates rapidly at frequency E/ħ due to the first exponential, while slowly waxing and waning at frequency $(E^*-E)/ħ$, as the term in [ ]'s cycles between 1+B/A and 1–B/A. The rate of waxing and waning is called a *beat frequency*.

One more point: shifting the energy scale has no impact on physical reality. If we add Δ to every energy level, every amplitude is multiplied by a common factor: ø becomes $øexp\{i\Delta t/ħ\}$. This added factor has no effect on $|ø|^2$. If we add two amplitudes, both will contain this added factor, which again disappears when calculating probabilities. We are therefore free to chose any zero-point for our energy scale. In specifying an atom's energy, we can include the rest masses of all its component particles, or not. We can define the ground state energy to be zero, or we can define zero to be the energy that all its pieces would have if they were separated by great distances. The latter is the most common choice.

Uniform Motion

Enough rest; let's get moving.

An atom at rest in one reference frame may be moving in another reference frame. In the atom's rest frame, its amplitude is:

$$ø(t,x,y,z) = A \exp\{Et/iħ\}$$

The real part of ø, Re{ø}, crests along lines of constant time ($t=n2\pi ħ/E=n/f$ for any integer n). The crests lie along the horizontal dashed lines in Figure 9-1, which is a plot of rest frame time t vs. position x.

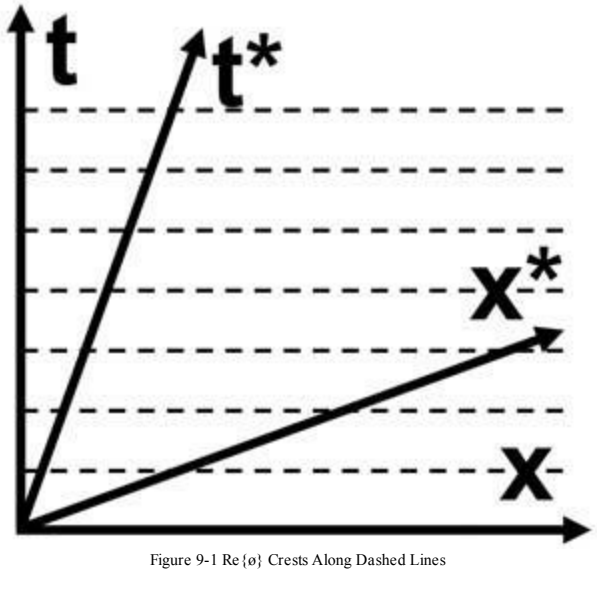

Figure 9-1 Re{ø} Crests Along Dashed Lines

Also plotted in Figure 9-1 are the moving frame time t* and position x*. The moving frame axes both rotate toward the x=t line at 45º (see ***Feynman Simplified 1C*** Chapter 27). The crests of Re{ø} have a different spacing along the t*-axis than along the t-axis. Also note that the dashed lines, which are parallel to the x-axis, intersect the x*-axis. This means ø varies with x*; the amplitude is no longer identical throughout all of space.

Recall the *Lorentz transformation* for time coordinates (see ***Feynman Simplified 1C*** Chapter 25):

$$t = (t^*–x^*\beta/c)\gamma$$

where βc = the relative velocity of the two frames, and $\gamma=1/\sqrt{(1–\beta^2)}$.

In the moving frame, the amplitude is:

$$ø(t^*,x^*) = A \exp\{E(t^*\gamma–\beta x^*\gamma/c)/iħ\}$$

Using $E^*=E\gamma$ and $p^*=\beta E^*/c$ for the atom's energy and momentum in the moving frame, we can rewrite this equation as:

$$ø(t^*,x^*) = A \exp\{(E^*t^*–p^*x^*)/iħ\}$$

Also recall the expressions for the *position 4-vector*, *momentum* 4-vector, and their *invariant*

product:

$x^*_\mu = (ct^*, x^*, y^*, z^*)$
$p^*_\mu = (E^*/c, p^*_x, p^*_y, p^*_z)$
$x^*_\mu\, p^*_\mu = E^*t^* - x^*p^*_x + y^*p^*_y + z^*p^*_z$

Being invariant, $(x^*_\mu\, p^*_\mu)$ is the same in all frames. In the atom's rest frame $x_\mu\, p_\mu = Et$.

The wave equation can also be written:

$\phi = A \exp\{-i(\omega t - \boldsymbol{k}\bullet\boldsymbol{x})\}$

We know $E = \hbar\omega$, $\boldsymbol{p}=\hbar\boldsymbol{k}$, and $k=2\pi/\lambda$ (see ***Feynman Simplified 1C*** Chapter 37). Hence:

$p = \hbar k = \hbar\,(2\pi/\lambda)$
$p = h/\lambda$

$\lambda = h/p$

This is the famous de Broglie equation, and Feynman says this is how he derived it.

Note that:

$|A \exp\{Et/i\hbar\}|^2 = |A \exp\{(E^*t^* - p^*x^*)/i\hbar\}|^2 = |A|^2$

This means the probability of finding the atom hasn't changed; it still has the same probability of being everywhere. The phase of the amplitude changes in the moving frame, but not its magnitude and not the corresponding probability.

To localize a particle and recognize its motion, we must make a wave packet. We do so by summing waves of different frequencies.

The *group velocity* of a wave packet is explained in V1p48-7; it equals:

$v_g = d\omega/dk$

In the moving frame:

$v_g = d(E^*/\hbar) \,/\, d(p^*/\hbar) = dE^*/dp^*$
$E^{*2} = p^{*2}c^2 + m^2 c^4$
$2E^*\, dE^*/dp^* = 2p^*c^2$
$dE^*/dp^* = p^*c^2/E^*$
$v_g = p^*c^2/(cp^*/\beta)$
$v_g = \beta c = v$

This means a wave packet composed of many frequencies near E^*/h has a group velocity equal to the velocity v of a classical particle.

Effect of Potential Energy

We next address the impact of potential energy on the evolution of probability amplitudes.

The simplest case is a constant potential. Feynman suggests an electrically conductive box attached to a battery. The electrical potential everywhere within the box is Φ, so that a particle of charge q has a potential energy $V = q\Phi$.

Let m be the rest mass, which may include internal energies, and let E remain the sum of rest mass and kinetic energy. Since quantum mechanics is employed mostly to describe atomic phenomena, where energies are much less than rest masses, we can generally use non-relativistic kinematics in quantum mechanics. Thus, $E=p^2/2m+mc^2$. The particle's total energy equals E+V, and the amplitude at time t at location $\boldsymbol{r}$ is:

$$\text{ø} = A \exp\{ [(E+V)t - \boldsymbol{p}\bullet\boldsymbol{r}] /i\hbar\}$$

In V3p7-6, Feynman says it is a general principle that the coefficient of t in the wave equation is always the total energy.

Recall there is some arbitrariness about the zero point on the energy scale, as that only involves a universal phase shift. For example, if V is constant in time and the same at every point inside the box, it merely adds a common phase angle that has no impact on physical reality.

Now consider a potential that is constant in time but varies spatially. Two electrically conductive boxes, attached to different batteries, are connected by an electrically insulating tunnel, as shown in Figure 9-2.

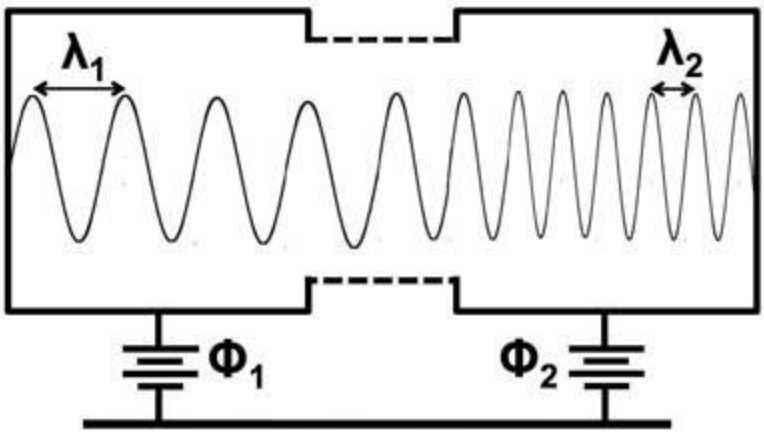

Figure 9-2 Two Boxes at Different Potentials

The potential is Φ_1 in the left box, is Φ_2 in the right box, and gradually transitions between them in the tunnel.

The amplitude in each box is:

Left : $\exp\{(E_1+V_1)t/i\hbar - \boldsymbol{p}_1\bullet\boldsymbol{x}/i\hbar\}$, with $V_1 = q\Phi_1$

Right: $\exp\{(E_2+V_2)t/i\hbar - \boldsymbol{p}_2 \bullet \boldsymbol{x}/i\hbar\}$, with $V_2 = q\Phi_2$

In V3p7-7, Feynman stresses a key point: due to energy conservation, the amplitude oscillates at the same frequency throughout the apparatus, even though its wavelength changes. If all parts of a wave do not oscillate at the same frequency, its waveform is torn apart.

Recall a related phenomenon: light refraction, which is explored in detail in ***Feynman Simplified 1C*** Chapter 36. Light entering a dense medium refracts, changing direction and wavelength. Light accelerates electrons in the medium, causing them to radiate. This secondary radiation cancels the incident wave within the medium along light's original direction of motion. This cancelling occurs only because all field oscillations have the same frequency despite having different wavelengths.

In our current example, conservation of energy requires the total energy, E+V, to be the same throughout. Both E and V change but their sum cannot. This is why the frequency is the same throughout:

$$\hbar\omega_1 = E_1+V_1 = E_2+V_2 = \hbar\omega_2$$

The momenta and wavelengths in the two boxes are:

$$p_1 = \sqrt{\{2m(E_1-mc^2)\}}$$
$$p_2 = \sqrt{\{2m(E_2-mc^2)\}}$$

$$\lambda_1 = h/p_1$$
$$\lambda_2 = h/p_2$$

Barrier Penetration

In V3p7-8, Feynman examines a remarkable phenomenon: negative kinetic energy. That is impossible in classical physics, but in quantum mechanics, negative kinetic energy has real and important consequences.

Consider water in a lake created by a dam. The water level is much higher on the upstream side of the dam than on the downstream side. Yet, water molecules cannot reach the downstream side because they don't have enough kinetic energy to rise up to the dam's rim and flow over. To rise a distance h, a molecule's potential energy would have to increase by mgh and its kinetic energy would have to decrease by the same amount. If the molecule's initial kinetic energy were almost zero, rising up to the rim would make its kinetic energy negative. Classically this is impossible. Our macro-world experience supports that — 50,000 dams worldwide can't be wrong.

But in the micro-world, particles can overcome barriers that we would normally consider too high to scale. An electron with a kinetic energy of 1 electron volt (1 eV) has a chance of passing through a thin strip that has a potential of –2 volts. Classically, the electron is forbidden to enter this strip because its kinetic energy would be –1 eV. But in quantum mechanics, it's possible — not likely perhaps, but possible.

Indeed, this important phenomenon, called *barrier penetration*, is employed in most modern microelectronics. It is also the essential physics in radioactive decay and in the fusion reactions that power the stars. Let's see how this bizarre and uniquely quantum phenomenon works.

Imagine a particle-wave traveling in the +x direction in a space that has two distinct regions. Let the x=0 plane be the boundary between these regions: $x<0$ is region #1, and $x\geq 0$ is region #2. Using the notation of the last section, in region #1 the wave's potential energy is V_1 and its kinetic plus rest mass energy is E_1. In region #2, the wave's potential energy is V_2 and its kinetic plus rest mass energy is E_2.

From the conservation of energy, we calculate the kinetic energy in region #2:

$$E_1+V_1 = E_2+V_2$$
$$E_2 = E_1+ V_1 – V_2$$
$$E_2 – mc^2 = E_1 – mc^2 + V_1 – V_2$$

What happens when $V_2 > E_1–mc^2+V_1$, and the kinetic energy in region #2, $E_2–mc^2$, is negative? Define $\varepsilon = –(E_2 – mc^2)$; in region #2, ε is positive.

$$–\varepsilon = E_2 – mc^2 = p_2^2/2m$$
$$p_2^2 = 2m\,(–\varepsilon)$$
$$p_2 = i\sqrt{(2m\varepsilon)}$$

Now define $\Lambda = i\hbar/p_2$; in region #2, Λ is real and positive.

$$\Lambda = i\hbar \,/\, [i\sqrt{(2m\varepsilon)}]$$
$$\Lambda = \hbar \,/\, \sqrt{(2m\varepsilon)}$$

For $x\geq 0$, in the classically forbidden zone, the amplitude is:

$$ø = A \exp\{[(E+V)t–p_2x]/i\hbar\}$$
$$ø = A \exp\{(E+V)t/i\hbar\}\ \exp\{–(i\hbar/\Lambda)\ x/i\hbar\}$$
$$ø = A \exp\{(E+V)t/i\hbar\}\ \exp\{–x/\Lambda\}$$

The most interesting part is the last term with the negative real exponent. Amplitude ø decreases exponentially with increasing distance x into the classically forbidden zone, where the kinetic energy K.E. is negative. This is illustrated in Figure 9-3, which plots the real part of ø vertically and x horizontally.

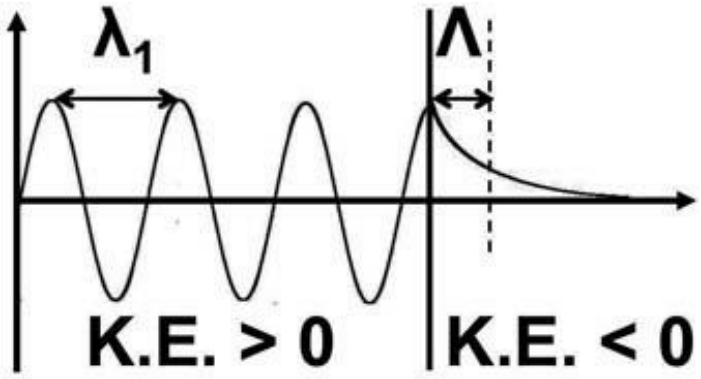

Figure 9-3 Re{ø} vs. x. Exponential Decline Within Barrier

As the figure shows, there is some chance of the particle being inside the forbidden zone within a region close to the boundary. The rate of exponential decline is inversely proportional to Λ; as the kinetic energy becomes more negative, Λ decreases, and the amplitude declines more rapidly.

If the barrier is thin enough, a particle can pass completely through the barrier and emerge on the opposite side, as shown in Figure 9-4.

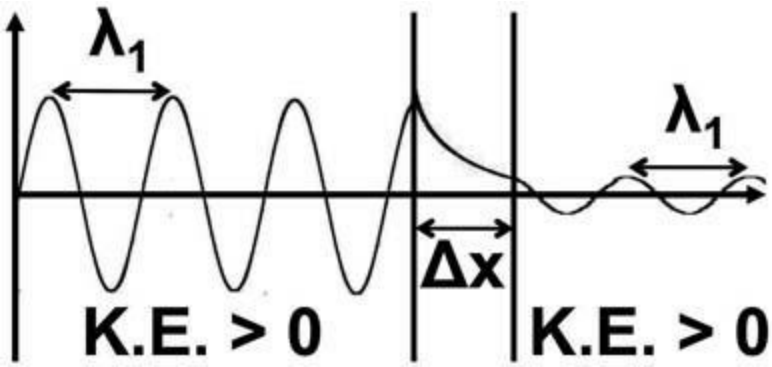

Figure 9-4 Penetration of Barrier where Kinetic Energy<0

Here again the real part of the amplitude is plotted vertically versus the particle's horizontal position. For barrier thickness Δx and Λ defined as above, the attenuation of the amplitude in crossing the barrier is:

$\exp\{-\Delta x/\Lambda\}$

Beyond the barrier, the amplitude oscillates as before, with the same frequency and wavelength, but with the attenuated amplitude.

A very important point is not mentioned in this lecture. The amplitude is non-zero within the classical forbidden zone, Δx, above. But experiments will **never** detect a negative energy particle. The particle may be found to the left of the barrier, or to the right of the barrier, but never within the barrier itself. Quantum mechanics stipulates that when an object is measured, its state changes instantaneously, becoming a state with an allowed value for that measurement. The result of every measurement must be consistent with energy conservation, momentum conservation, and other fundamental principles. Negative kinetic energy is not an allowed measurement outcome.

Between measurements, negative energy and other bizarre states are allowed. This is because, between measurements, it is impossible to know what a particle's energy, momentum, position, and

spin "really" are. Between measurements, those quantities **are not real**. This is a bit like: "Don't ask; don't tell."

In the prior two paragraphs, "measurement" means any substantial interaction that disrupts coherence. Due to our macroscopic size, human-directed interactions disrupt quantum coherence. But human involvement is not the essential factor. Even non-human natural macroscopic interactions disrupt coherence. Negative energy and other bizarre states exist only in isolated havens on the quantum scale.

We completely understand the rules of barrier penetration. We can calculate the answer to any question. But no one can honestly say they are comfortable with the quantum "explanation" of this phenomenon. We will return to such issues many more times.

Some simply refuse to accept the "unnatural" pronouncements of quantum mechanics, all of which have been extensively and precisely validated by countless experiments. It's best not to fight quantum mechanics; resistance is futile.

Radioactivity & Nuclear Fusion

Inside nuclei, nucleons (protons and neutrons) occupy discrete energy levels, similar to electron energy levels in atoms. In very heavy nuclei, such as uranium, protons in the highest energy levels have negative binding energies — energy is released if they are removed to a great distance from the nucleus.

Normally when two objects attract one another, potential energy is released as they move closer. *Binding energy* is the amount of energy released as objects move from infinitely far away to their adjoined states. The same amount of energy must be expended to pull the objects apart.

When a single attractive force dominates, binding energy is always positive.

Nuclei, however, are a bit more complex. The potential energy of protons near a nucleus results from two opposing forces: (1) the strong force that pulls nucleons together; and (2) the electric force that pushes positive charges apart. The strong force contributes a large negative potential well with a very small radius; call that r_N. The electric force contributes a more modest positive potential proportional to $1/r^2$.

The resulting potential, the heavy black line in the upper portion of Figure 9-5, resembles a volcanic caldera. To escape to a lower energy state far from the nucleus, a proton or alpha particle (the nucleus of helium-4) must first scale the caldera's rim.

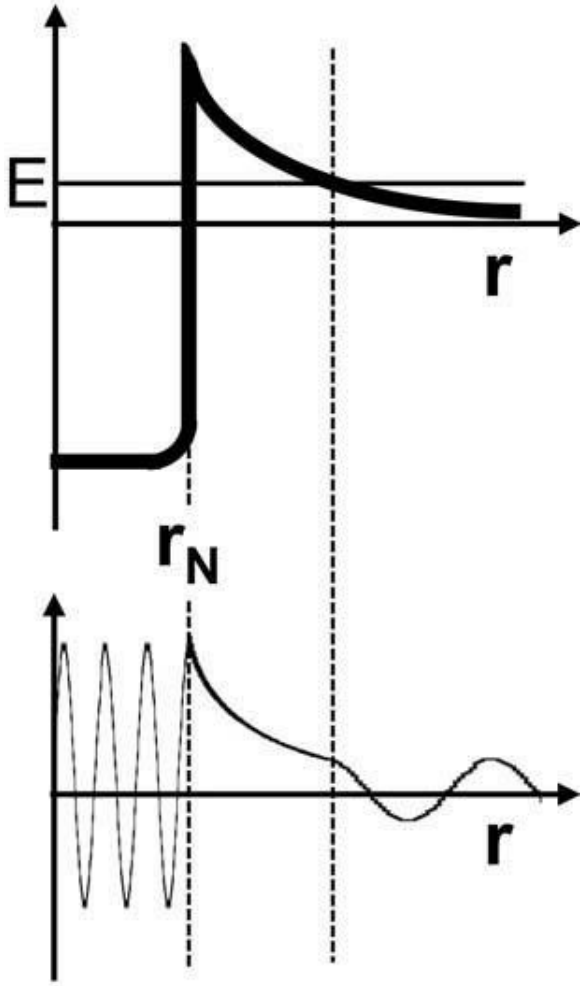

Figure 9-5 Top: Potential of Heavy Nucleus
Bottom: Real Part of Amplitude of Nucleon

In the upper portion of Figure 9-5, the energy level of the highest proton is labeled E. Without barrier penetration, the proton would be unable to escape and this nucleus would remain intact forever. With barrier penetration, a proton's probability amplitude penetrates into the negative kinetic energy zone. The amplitude attenuates exponentially at a rate determined by the barrier's height and width. If the attenuation is not excessive, there is a meaningful probability of the particle appearing outside the caldera and escaping the nucleus with positive kinetic energy.

Individual protons or alpha particles can therefore escape a nucleus if they occupy energy levels with negative binding energy. This process is one example of *radioactive nuclear decay*.

Due to its exponential character, the decay rates of different nuclei span an incredible range. Mean decay times are as short as 20 trillionths of a trillionth of a second for hyrodgen-7, and as long as 3 trillion, trillion years for tellurium-128. During one *mean decay time* the number of surviving nuclei drops by a factor of e (2.71828…). A closely related term is *half-life*; during one half-life the number of survivors is cut in half. The conversion factor is: (one half-life) = (one mean decay time) × ln2.

Nuclear fusion, the process that powers stars, has similar characteristics, but operates in the opposite direction. Fusion is the merging of two smaller nuclei to produce one larger nucleus. Fusion is possible only if two positively charged nuclei can overcome their mutual electrical repulsion.

Due to enormous temperatures, nuclei in stellar cores have high kinetic energies. A miniscule fraction of these nuclei have enough energy to defy Coulomb repulsion and come within range of the strong

force. In our caldera analogy, extremely energetic smaller nuclei are able to roll up the caldera's flanks and drop into the central pit, where they form a larger, stable nucleus. As they drop into the deep potential well, these nuclei release fabulous quantities of energy — as much as 40 million times more energy per kg of reacting matter than typical chemical reactions. This is the vast power source inside all stars.

Well almost. The success rate is abysmally small. For our Sun, only about one in 30,000 trillion nuclei fuse per second. And almost none of those actually make it to the caldera's rim. Instead, most tunnel through the upper flanks employing barrier penetration. They race toward the goal, get close, and finish with a Hail Mary; their attenuated amplitudes sneak in for the score.

Without barrier penetration, nuclear fusion rates and stellar energy production would be vastly diminished. Only the most mammoth stars would be able to initiate nuclear fusion. Our Sun isn't massive enough; it would not be a true star, and we would all be extremely cold.

Forces as a Classical Limit

In V3p7-9, Feynman demonstrates how classical concepts of *force* translate into the quantum world.

Consider the motion of a particle traversing an electric field, as illustrated in Figure 9-6. Let p, v, and m be the particle's original momentum, velocity, and mass. Let the x-axis be the particle's original direction of motion, and let y be the direction of the applied force F.

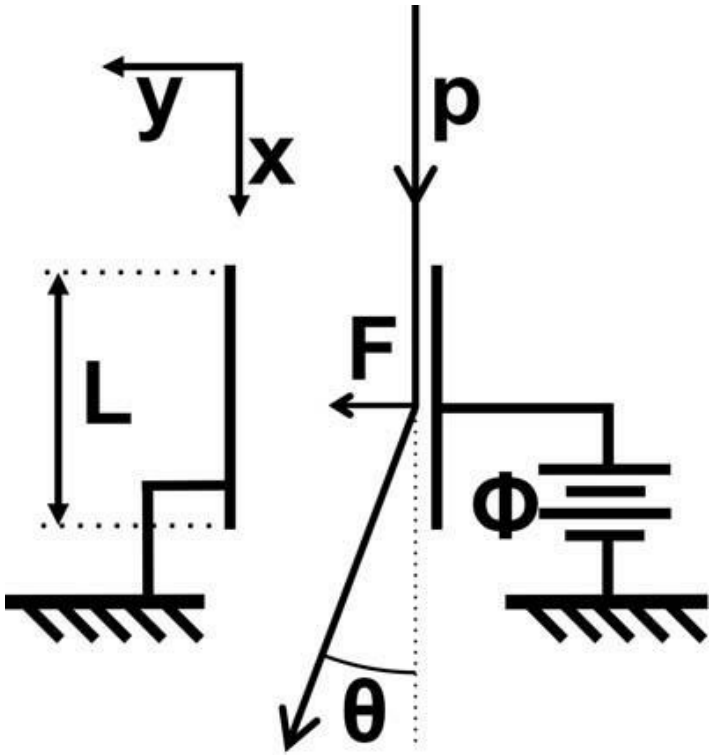

Figure 9-6 Particle Deflecting in Electric Field

The particle passes between two electrodes of length L, whose different potentials establish an electric potential gradient along the y-axis. The particle's potential energy V equals its charge q times the electric potential. We will assume $\phi>0$ and $q>0$. The force exerted on the particle is:

$F_y = -dV/dy$

In our setup, dV/dy is negative, making $F_y>0$, as Figure 9-6 shows. This force acts while the particle is between the electrodes, a time interval of $\Delta t=L/v$. For a small deflection, the particle acquires a component of momentum in the y-direction of:

$p_y = F_y \Delta t$
$p_y = (-dV/dy)\ (L/v) > 0$

Since there are a lot of signs coming and going, I will periodically indicate the overall polarity of various expressions.

The deflection angle θ is:

$\theta = p_y / p$
$\theta = -(L/pv)\ dV/dy > 0$

Now, let's examine the same situation using quantum wave concepts. We will assume the wavelength $\lambda=h/p$ is much smaller than any of the dimensions of our apparatus. In Figure 9-7, the particle is replaced by a wave, with dotted lines marking wave crests. For clarity, the wave crest separations are greatly exaggerated.

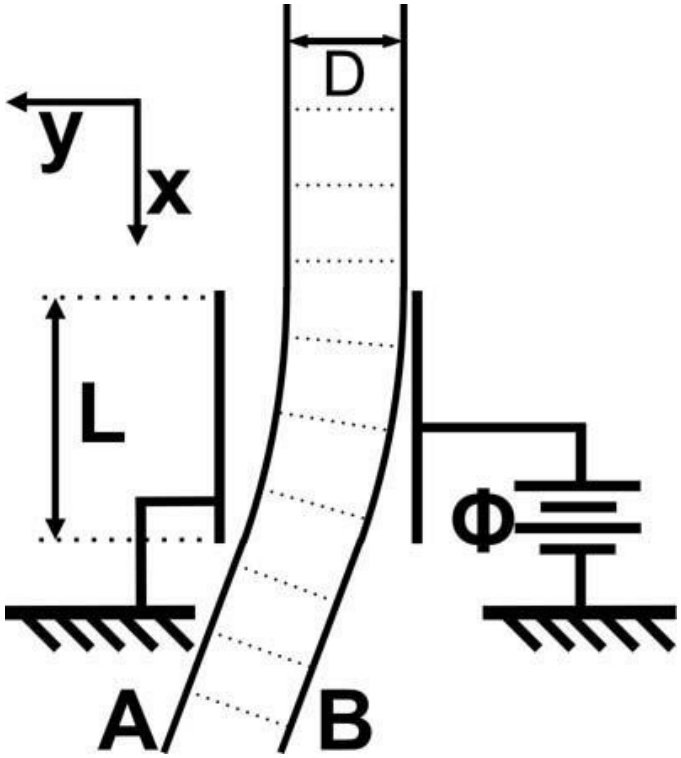

Figure 9-7 Wave Turning in Electric Field

We show two paths A and B on opposite sides of the wave front. Paths A and B are separated by distance D. The probability amplitude is proportional to:

$ø \sim \exp\{[Ut-\boldsymbol{p}\bullet\boldsymbol{r}]/i\hbar\}$, with
$U = V+p^2/2m+mc^2$

Energy conservation requires that U be constant. This determines how changes in V affect changes in p:

$\Delta V + 2p\ \Delta p\ /2m = \Delta U = 0$
$\Delta V = -\ \Delta p\ p/m$

Where V is larger, p is smaller, and the wavelength $\lambda=h/p$ is larger. As Figure 9-7 shows, the potential is higher at path B than at path A. Thus between electrodes, the wavelength is longer along Path B than Path A. Since the wave oscillates at the same frequency everywhere, the wave moves faster on path B than path A, resulting in the wave turning toward A.

We next calculate the differences that arise between paths A and B as the wave traverses the electrodes. The potential energy difference is:

$V_B - V_A = \Delta V = (y_B - y_A)\ dV/dy$
$V_B - V_A = \Delta V = -\ D\ dV/dy > 0$

This leads to a momentum difference of:

$p_B - p_A = \Delta p = +\ (m/p)\ (D\ dV/dy) < 0$

We want to compare where wave crests occur along paths A and B. At a fixed time, crests occur at certain values of $\boldsymbol{p \bullet r}$. Since path B has a smaller momentum p, it must travel a longer distance r to attain the same $\boldsymbol{p \bullet r}$ at the same time as path A.

$d(\boldsymbol{p \bullet r}) = 0 = r\ dp + p\ dr$
$r_B - r_A = dr = -dp\ (r/p) > 0$

The cumulative path length difference after traversing distance r=L is:

$\Delta r = -\ (m/p)\ (D\ dV/dy)\ (L/p) > 0$

Since wave crests on path B are ahead of crests on path A by distance Δr, the wave turns toward path A by angle θ.

$\theta = \Delta r\ /\ D$
$\theta = -\ (mL/p^2)\ dV/dy > 0$
$\theta_{QUANTUM} = -\ (L/pv)\ dV/dy > 0$

This matches exactly with our calculation for a classical particle:

$\theta_{CLASSICAL} = -\ (L/pv)\ dV/dy$

This demonstrates that the same result is obtained from classical equations based on F=ma as from the quantum formalism based on phase angle changes due to energy differences (dV t/ħ).

As Feynman says in V3p7-10: *"In the classical limit, the quantum mechanics will agree with*

Newtonian mechanics." We found something similar when exploring special relativity: in the classical limit, special relativity agrees with Newtonian mechanics. Newtonian mechanics is well confirmed for modest velocities and macroscopic energies. Quantum mechanics and special relativity must the wealth of evidence in the classical realm.

Precession of Spin ½ Particles

In the last section, the only assumption we made about force F was that it equals the negative gradient of a potential, which is equivalent to saying F is a conservative force, as defined in ***Feynman Simplified 1A*** Chapter 10. Recall that all fundamental forces are conservative.

A particle with magnetic moment $\boldsymbol{\mu}$ in a magnetic field $\boldsymbol{B}$ has potential energy $V= -\boldsymbol{\mu}\bullet\boldsymbol{B}$. In a Stern-Gerlach device, $\boldsymbol{B}$ has a gradient, which means V has a gradient. This exerts a force on particles with magnetic moments that depends on their spin orientation. With what we now know, we can describe this quantum mechanically: waves with spins in different directions experience different energy potentials, resulting in different phase shifts that steer the waves in different directions.

We now switch to a uniform, constant magnetic field along the z-axis. A particle with magnetic moment μ_z along the z-axis has an amplitude proportional to:

$$ø \sim \exp\{(-\mu_z B)t/i\hbar\}$$

For a spin 1/2 particle, it z-component of spin must be either up or down, hence its magnetic moment along z is either $+\mu$ or $-\mu$. We therefore have two possible amplitudes:

$$ø_+ \sim \exp\{-\mu Bt/i\hbar\}$$
$$ø_- \sim \exp\{+\mu Bt/i\hbar\}$$

In V3p7-11, Feynman considers the specific case of a muon decaying to an electron, a neutrino, and an anti-neutrino, all of which are spin 1/2 elementary fermions.

μ decays to $e + \nu + \nu$

Energy conservation dictates that in any particle decay the rest mass of the original particle must be greater than the sum of the rest masses of all the decay products. Since the muon mass is 105.7 MeV, the electron mass is 0.511 MeV, and neutrinos masses are less than 0.2 eV, more than 105 MeV of mass energy is converted into kinetic energy in each muon decay. Since that is much more than the rest masses of any of the decay products, each can have a velocity comparable to the speed of light.

The vast majority of electrons from muon decay can have kinetic energies well above 0.511 MeV, making them relativistic. It turns out that these relativistic electrons are preferentially emitted in the opposite direction of the muon's spin. This enables physicists to identify the spin direction of muons at the instant they decay.

You should know that muon experiments are a proud Piccioni family tradition. My thesis experiment

employed muonic decays of neutral K-mesons to demonstrate an asymmetry between matter and anti-matter. My father, Oreste Piccioni, used muonic decays to establish the distinction between strongly and weakly interacting particles. His was the first particle physics experiment that went beyond mere observation of naturally occurring events. Science historian Robert Crease selected my father's experiment as one of the twenty most beautiful in the history of all branches of science.

Figure 9-8 schematically illustrates a muon decay experiment. Muons enter from the left, and come to rest in absorber A in the presence of uniform magnetic field B. Detector D counts electrons emanating from muon decay. Neutrinos from muon decay, indicated by the dotted arrows, escape detection.

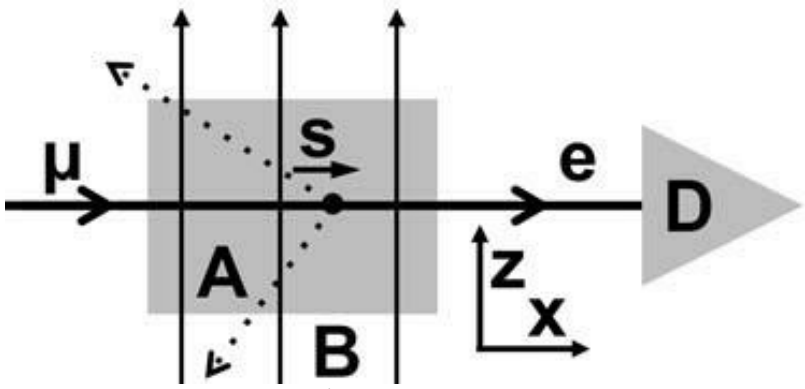

Figure 9-8 Muon Capture
and Decay in Magnetic Field

In this experiment, the incident muons are fully polarized, with spin +x, as indicated by the arrow labeled "s" in the figure. We wish to determine whether muons' spins change over time due to the magnetic field. By producing a muon beam from a particle accelerator, we can very precisely know when muons arrive at A. We can also precisely know when their decay electrons arrive at D. The time difference is how long the muon persisted in the magnetic field before decaying.

The number of decaying particles declines exponentially with time, according to:

Number of decays at time t ~ $\exp\{-t/\tau\}$

where τ is the mean decay time. For muons, τ is a rather long 2.2 microseconds. Nature thus provides us with muons whose time in the magnetic field spans a wide range: from zero to 10 or more microseconds. With this experiment, we can measure the evolution of the amplitude for muons to be in spin state +x after being in the magnetic field for time t.

Above, we derived the evolution of spin states that are parallel to the magnetic field ($ø_+$ and $ø_-$). How do we relate that to spin states perpendicular to the field?

The end of Chapter 8 tabulates equations for transforming x and z basis states. A –90º rotation about the y-axis, turns +x into +z; use $|+S\rangle = |+x\rangle$ and $|+T\rangle = |+z\rangle$.

$\langle +z|+x\rangle = 1/\sqrt{2} = \langle +x|+z\rangle^*$
$\langle -z|+x\rangle = 1/\sqrt{2} = \langle +x|-z\rangle^*$

Thus the muons' initial spin state is:

$\phi(t=0) = (|+z\rangle + |-z\rangle) /\sqrt{2}$

Applying the time evolution terms to the z states, we obtain the amplitude at time t:

$\phi(t) = (|+z\rangle \exp\{-i\omega t\} + |-z\rangle \exp\{+i\omega t\}) /\sqrt{2}$

where $\omega = \mu B/\hbar$

We can now transform back to the x-basis representation. Define $A_{+x}(t)$ to be the amplitude to be in spin state +x at time t.

$A_{+x}(t) = \langle +x|+z\rangle \langle +z|\phi(t)\rangle + \langle +x|-z\rangle \langle -z|\phi(t)\rangle$
$A_{+x}(t) = (1/\sqrt{2})(\exp\{-i\omega t\}/\sqrt{2}) + (1/\sqrt{2})(\exp\{+i\omega t\}/\sqrt{2})$
$A_{+x}(t) = (\exp\{-i\omega t\} + \exp\{+i\omega t\})/2$
$A_{+x}(t) = \cos(\omega t)$

The probability of a muon having spin +x at time t is:

$P_{+x}(t) = \cos^2(\omega t)$

As shown in Figure 9-9, this function equals 1 whenever $\omega t = \mu Bt/\hbar = n\pi$, for any integer n, and equals zero when $\mu Bt/\hbar = (n+1/2)\pi$. $P_{+x}(t)$ is a periodic function that repeats at frequency 2ω.

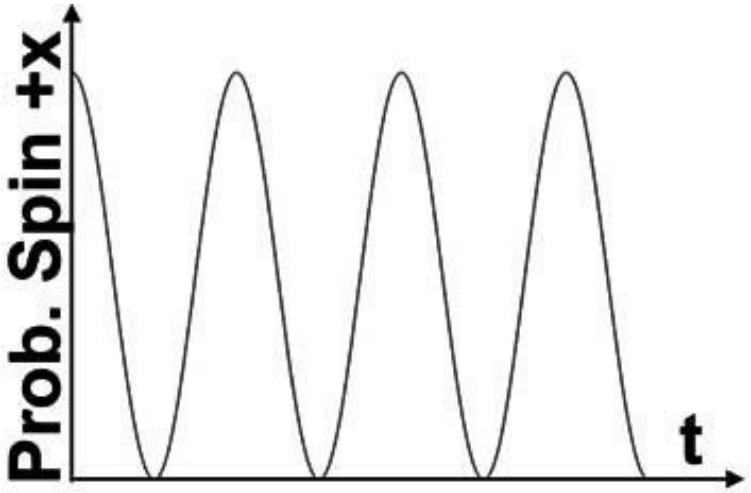

Figure 9-9 Probability vs. Time of Muon in Spin State +x

Feynman states that the probability of a muon being in the +y spin state equals $\cos^2(\omega t - \pi/4)$, which oscillates at the same frequency but with a one-quarter-cycle lag. Feynman says this shows that muon spins rotate in the xy-plane. This is not hard to confirm.

Define a uvw coordinate system, with the u-axis parallel to the z-axis. A time t=0, let the v-axis be parallel to the x-axis and let the w-axis be parallel to the y-axis. Rotate the uvw-axes at frequency 2ω about the u-axis.

According to the rotation transformations in Chapter 8, for rotation of spin 1/2 states about the z-axis by angle ø:

$\langle +u|+z\rangle = \exp\{+i\phi/2\} = \exp\{+i\omega t\}$
$\langle -u|-z\rangle = \exp\{-i\phi/2\} = \exp\{-i\omega t\}$
$\langle -u|+z\rangle = \langle +u|-z\rangle = 0$

The muon spin amplitudes in the u spin states are:

$\langle +u|\phi(t)\rangle = (\langle +u|+z\rangle\exp\{-i\omega t\}+\langle +u|-z\rangle\exp\{+i\omega t\}) /\sqrt{2}$
$\langle +u|\phi(t)\rangle = (\exp\{+i\omega t\}\ \exp\{-i\omega t\} + [0]\ \exp\{+i\omega t\}) /\sqrt{2}$
$\langle +u|\phi(t)\rangle = (1 + 0) /\sqrt{2} = 1/\sqrt{2}$

$\langle -u|\phi(t)\rangle = (\langle -u|+z\rangle\exp\{-i\omega t\}+\langle -u|-z\rangle\exp\{+i\omega t\}) /\sqrt{2}$
$\langle -u|\phi(t)\rangle = ([0]\ \exp\{-i\omega t\} + \exp\{-i\omega t\}\ \exp\{+i\omega t\}) /\sqrt{2}$
$\langle -u|\phi(t)\rangle = (0 + 1) /\sqrt{2} = 1/\sqrt{2}$

Now rotate about the w-axis by +90º, turning u into the xy-plane. With $|+S\rangle = |+u\rangle$ and $|+T\rangle = |+v\rangle$, the transformation amplitudes are:

$\langle +v|+u\rangle = +1/\sqrt{2}$
$\langle -v|+u\rangle = -1/\sqrt{2}$
$\langle +v|-u\rangle = +1/\sqrt{2}$
$\langle -v|-u\rangle = +1/\sqrt{2}$

The muon spin amplitudes in the v spin states are:

$\langle +v|\phi(t)\rangle = \langle +v|+u\rangle\langle +u|\phi(t)\rangle + \langle +v|-u\rangle\langle -u|\phi(t)\rangle$
$\langle +v|\phi(t)\rangle = (+1/\sqrt{2})(1/\sqrt{2}) + (+1/\sqrt{2})(1/\sqrt{2})$
$\langle +v|\phi(t)\rangle = 1/2 + 1/2 = 1$

$\langle -v|\phi(t)\rangle = \langle -v|+u\rangle\langle +u|\phi(t)\rangle + \langle -v|-u\rangle\langle -u|\phi(t)\rangle$
$\langle -v|\phi(t)\rangle = (-1/\sqrt{2})(1/\sqrt{2}) + (+1/\sqrt{2})(1/\sqrt{2})$
$\langle -v|\phi(t)\rangle = -1/2 + 1/2 = 0$

These equations show that the muon spin is always entirely in the +v-direction. Recall that +v was the same as +x at t=0, and that the v-axis rotates within the xy-plane at frequency $+2\omega$. This confirms that the muon spin *precesses* about the magnetic field direction, the z-axis, at frequency 2ω.

Radiation & Entropy

As an aside, Feynman comments on why atoms are so much more likely to emit photons and drop to ground states than to absorb photons and rise to excited states. It's all about *entropy*. As Feynman describes: "When energy is in the electromagnetic field, there are so many different ways it can be — so many different places it can wander — that if we look for the equilibrium condition, we find that in the most probable situation the field is excited and the atom is de-excited. It takes a very long time for the photon to come back and find that it can knock the atom back up again."

Entropy can be understood in terms of phase space, which is described in Chapter 2. For a given amount of energy, nothing has greater momentum or traverses more space than light. This means that transferring energy from an electron to light increases the occupied phase space and thus the amount of entropy.

Chapter 9 Review: Key Ideas

1. If an atom had a definite energy, one specific value with zero uncertainty, the probability of finding it would be the same everywhere throughout all of space. Its amplitude to be at location (x,y,z) at time t would be:

   $\phi(t,x,y,z) \sim \exp\{Et/i\hbar\} = \exp\{-i\omega t\}$

2. The amplitude for a particle of momentum p and energy E (kinetic plus mass), with potential energy V, is proportional to:

   $\exp\{(E+V)t/i\hbar - \boldsymbol{p}\bullet\boldsymbol{r}/i\hbar\}$

3. **Barrier Penetration**. In a classically forbidden region, where a particle's kinetic energy T is negative, its amplitude is proportional to:

   $\phi(x) \sim \exp\{-x/\Lambda\}$

   where x is the distance into the forbidden zone, and $\Lambda=\hbar/\sqrt{\{-2mT\}}$.

4. Barrier penetration enables radioactive decay and nuclear fusion, and is employed in most modern microelectronics.

5. Feynman says: *"In the classical limit, the quantum mechanics will agree with Newtonian mechanics."* Planck's constant h sets the scale at which quantum mechanics becomes important, just as the speed of light sets the scale at which special relativity becomes important.

6. A spin 1/2 particle with spin in the xy-plane, in a magnetic field along +z, precesses according to: $P_{+x}(t) = \cos^2(\mu Bt/\hbar)$; $P_{+y}(t) = \cos^2(\mu Bt/\hbar - \pi/4)$.

Chapter 10

The Hamiltonian

We have so far introduced many rather abstract concepts and notations. The first third of this chapter is a comprehensive review of that material.

Our description of quantum mechanics has been very abstract. One might ask: "Can't you just explain that in English?" Actually, No. Quantum mechanics is a highly abstract theory because it describes phenomena with which none of us has any direct experience. No human language has words that adequately describe these phenomena. The only language that can address the micro-world is mathematics.

I have 100 trillion, trillion electrons in my hand, but I have never held an electron in my hand and examined it in detail. All I can tell you is that an electron is $|\psi> = \exp\{Et/i\hbar\ldots$. Unfortunately, my description is entirely abstract. Its only virtue is that it works. We can use these abstract descriptions to explain the behavior of atoms, fundamental physics, chemistry, and biology.

Quantum mechanics describes nature extremely well. Its predictions are confirmed by thousands of very precise measurements. No flaw has ever been found in quantum mechanics. For the micro-world, no alternative theory is even worth discussing.

On V3p8-1, Feynman begins this lecture stressing the similarities between the algebra of quantum states and the algebra of traditional vectors. We stressed that analogy in earlier chapters, but it is so beneficial that it merits repeating. Those who find this too repetitious might skip a section or two.

Similarity of Vectors & States

In normal 3-D space, a vector $\boldsymbol{v}$ specifies any selected point within space.

In quantum mechanics, $|S>$ specifies a *quantum state*, any selected set of particle properties, which might include energy, momentum, position, and spin.

In 3-D, we choose a *complete* set of three *orthonormal basis vectors* $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$.

In QM, we choose a complete set of orthonormal *basis states* $|J>$, J=1,…,N.

In 3-D, an "orthonormal" basis means for each basis state J and K:

$\boldsymbol{e}_J \bullet \boldsymbol{e}_K = \delta_{JK}$

where δ_{JK} is the *Kronecker delta*, which equals 1 if J=K and equals 0 if J≠K.

In QM, orthonormal means for each J and K:

$\langle K|J\rangle = \delta_{JK}$

In 3-D, a basis is "complete" if every vector in 3-D space is some linear combination of basis vectors — for any vector v:

$\boldsymbol{v} = \Sigma_J \, v_J \, \boldsymbol{e}_J = v_x \, \boldsymbol{e}_x + v_y \, \boldsymbol{e}_y + v_z \, \boldsymbol{e}_z$

$\boldsymbol{v} = \Sigma_J \, (\boldsymbol{v} \bullet \boldsymbol{e}_J) \, \boldsymbol{e}_J$

where $v_J = \boldsymbol{v} \bullet \boldsymbol{e}_J$, for J = x, y, z. The v_J are *real* numbers. The set of all v_J completely defines $\boldsymbol{v}$.

In QM, a basis is complete if every possible state is some linear combination of basis states — for any state $|\varphi\rangle$:

$|\varphi\rangle = \Sigma_J \, a_J \, |J\rangle = a_1 \, |1\rangle + a_2 \, |2\rangle + \ldots \, a_N \, |N\rangle$

$|\varphi\rangle = \Sigma_J \, \langle J|\varphi\rangle \, |J\rangle$

where $a_J = \langle J|\varphi\rangle$ are *complex* numbers. The set of all a_J completely defines $|\varphi\rangle$.

In 3-D, the *dot product* of two vectors A and B is:

$\boldsymbol{A} \bullet \boldsymbol{B} = \Sigma_J \, A_J B_J = A_x B_x + A_y B_y + A_z B_z$

$\boldsymbol{A} \bullet \boldsymbol{B} = \Sigma_J \, (\boldsymbol{A} \bullet \boldsymbol{e}_j) \, (\boldsymbol{e}_j \bullet \boldsymbol{B})$

In QM, the *product* of two states φ and ψ is:

$\langle \varphi|\psi\rangle = \Sigma_J \, \langle \varphi|J\rangle \, \langle J|\psi\rangle$

Here: $\langle \ldots|$ is called a *bra*; $|\ldots\rangle$ is called a *ket*; and $\langle \ldots|\ldots\rangle$ is called a *bra-ket*, following Dirac.

Differences Between Vectors & States

In 3-D, the dot product is commutative:

$\boldsymbol{A \bullet B = B \bullet A}$.

But, in QM the product of two states is **not** commutative:

$\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$

where $\langle\psi|\varphi\rangle^*$ is the complex conjugate of $\langle\psi|\varphi\rangle$. The complex conjugate of a complex quantity is obtained by replacing i with –i. In general $\langle\psi|\varphi\rangle \neq \langle\varphi|\psi\rangle$.

In 3-D, three orthogonal basis vectors span the entire space, forming a complete basis set.

In QM, a complete set of basis states may be much larger, even infinitely large.

Since $\langle\varphi|\psi\rangle = \Sigma_J \langle\varphi|J\rangle \langle J|\psi\rangle$ for all states φ and ψ, Feynman says the following is the "great law of quantum mechanics":

$| = \Sigma_J |J\rangle \langle J|$

Feynman calls this an *open equation*; it is valid whenever the equation is multiplied by any bra and/or by any ket.

Feynman then chases the relationship $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$ around a full circle, seemingly just for fun.

$\langle\varphi|\psi\rangle = \Sigma_J \langle\varphi|J\rangle \langle J|\psi\rangle$

Feynman says that since this is true for all ψ, we can simply drop ψ and write:

$\langle\varphi| = \Sigma_J \langle\varphi|J\rangle \langle J|$

Apply the universal rule $\langle A|B\rangle = \langle B|A\rangle^*$ to the last equation:

$\langle\varphi| = \Sigma_J \langle J|\varphi\rangle^* \langle J|$

As we said above, we can represent any state by a linear combination of basis states. We do that for ψ:

$|\psi\rangle = \Sigma_K |K\rangle \langle K|\psi\rangle$

Combining the last equations for $\langle\varphi|$ and $|\psi\rangle$, we obtain:

$\langle\varphi|\psi\rangle = \{ \Sigma_J \langle J|\varphi\rangle^* \langle J| \} \{ \Sigma_K |K\rangle \langle K|\psi\rangle \}$
$\langle\varphi|\psi\rangle = \Sigma_{JK} \langle J|\varphi\rangle^* \langle J|K\rangle \langle K|\psi\rangle$
$\langle\varphi|\psi\rangle = \Sigma_{JK} \langle J|\varphi\rangle^* \delta_{JK} \langle K|\psi\rangle$
$\langle\varphi|\psi\rangle = \Sigma_J \langle J|\varphi\rangle^* \langle J|\psi\rangle = \Sigma_J \langle\varphi|J\rangle \langle J|\psi\rangle$

Thus, we return to the equation we started with.

Other Comments on Notation

Bra and ket quantities are essential intermediate steps in solving problems in quantum mechanics, but they are not end results by themselves. The measurable quantities that we seek to predict are the probabilities of various outcomes, which have the form:

Probability = $|\langle\varphi|\psi\rangle|^2$

Calculations must therefore not end with expressions having unmatched bras or kets.

Bra and ket are complex conjugates: $\langle\psi| = |\psi\rangle^*$. This means $\langle\psi|$ and $|\psi\rangle$ are different, in general. Either could properly represent a quantum state. Feynman says he always uses kets $|\psi\rangle$ to identify states, for consistency.

Chapter 8 introduced the concept of and notation for *operators*, which transform one state vector into another. We previously discussed rotation operators, which have the effect of rotating basis states by a specified angle about a specified axis. As we will soon discover, operators are also used to represent the passage of time and all measurements of particle properties, such as momentum or spin. Any action that changes particle states is represented in quantum mechanics by an operator.

As an example:

$\langle\chi|A|\psi\rangle$

denotes the product of state χ with the vector resulting from A operating on state ψ. The above expression can be expanded in basis states as:

$\langle\chi|A|\psi\rangle = \Sigma_{JK} \langle\chi|K\rangle \langle K|A|J\rangle \langle J|\psi\rangle$

Knowing $\langle K|A|J\rangle$ for all basis states J and K, completely determines the properties of A; those amplitudes contain all the knowledge of A that exists. With the $\langle K|A|J\rangle$ we can calculate what A will do to any state.

One can also define φ as the state resulting from A operating on ψ.

$|\varphi\rangle = A|\psi\rangle = \Sigma_K |K\rangle \langle K|A|\psi\rangle$
or
$|\varphi\rangle = \Sigma_{JK} |K\rangle \langle K|A|J\rangle \langle J|\psi\rangle$

$\langle\chi|A|\psi\rangle = \langle\chi|\varphi\rangle$
or
$\langle\chi|A|\psi\rangle = \Sigma_K \langle\chi|K\rangle \langle K|\varphi\rangle$

One can even write an open equation for A:

$A = \Sigma_{JK} |K\rangle \langle K|A|J\rangle \langle J|$

This expression is valid whenever it is multiplied by any bra and/or by any ket.

Operators are often represented as N×N matrices, where N is the number of basis states. Matrix component A_{KJ} equals $\langle K|A|J\rangle$. If multiple operators act on a particle sequentially, we express that using the product of those operators. Matrix products are not commutative, so the order of factors is essential. The first operator to act holds the right-most position, with the second on its immediate left, and continuing on to the last operator to act in the left-most position. For two operators, A followed by B, the expression is:

$\langle\chi|BA|\psi\rangle = \Sigma_{JKL} \langle\chi|L\rangle \langle L|B|K\rangle \langle K|A|J\rangle \langle J|\psi\rangle$

The World's Basis States

In V3p8-5, Feynman examines at length the choice of basis states.

He states again the machinery through which quantum mechanics solves problems: identify a complete set of basis states; then determine how the physics of the situation transforms each basis state. This machinery works for any initial state, because any state $|A\rangle$ is a linear combination of basis states $|J\rangle$. The final state of $|A\rangle$ is the same linear combination of the final states of each $|J\rangle$.

But which basis states should one choose? Typically there are many, if not an infinite number, of choices. For example, spin 1/2 has the property that any measurement will find it either up or down along any selected axis. Which of the limitless number of axes should we choose? Sometimes it makes no difference: for an isolated hydrogen atom, all directions are equivalent. Sometimes circumstances dictate a clear choice: for Stern-Gerlach devices, the magnetic field gradient defines the most convenient axis.

When is a set of basis states complete? The short answer is probably never, except in extremely simplified situations. One might treat a hydrogen atom as a single entity with basis states covering its momentum and position possibilities. For some conditions, such as very low energies, this could be adequate. But, we know hydrogen atoms have internal parts, which play important roles in many phenomena. To analyze those phenomena, we must add basis states for the proton's momentum, position, and spin, and also for the electron's momentum, position, and spin. For helium, there are 4 more particles, each with their own basis states.

The conclusion is: we will never have a complete basis set covering every possibility of each of the 10^{80} particles in the observable universe. We must exercise judgment and identify what is most important and most useful.

Time Evolution of Basis States

We discussed operators that transform states, such as rotation operators. Another essential operator is time. What happens to our basis states with the passage of time?

Even isolated particles can change over time. Particles also change when exposed to external fields or other particles. Recall the example of spin precession of muons in a magnetic field.

Physicists define the operator $U(t_2,t_1)$ to represent time advancing from t_1 to t_2. The amplitude for state φ to change into state ψ during the time interval from t_1 to t_2 is:

$$\langle\psi|U(t_2,t_1)|\varphi\rangle = \Sigma_{JK} \langle\psi|K\rangle \langle K|U(t_2,t_1)|J\rangle \langle J|\varphi\rangle$$

Knowing all the amplitudes $\langle K|U(t_2,t_1)|J\rangle$ completely defines $U(t_2,t_1)$.

In V3p8-8, Feynman discusses a key application, saying:

> "The high-class theoretical physicist working in high-energy physics considers problems of the following general nature (because it's the way experiments are usually done). He starts with a couple of particles, like a proton and a proton, coming together from infinity…The things go crash and out come, say two K-mesons, six π-mesons, and two neutrons in certain directions with certain momenta. What's the amplitude for this to happen?"

The mathematical procedure for dealing with such complex phenomena is: (1) define state φ specifying all initial particle momenta and spins; (2) define state ψ specifying all the measured properties of the final particles; and (3) take the limit as t_1 goes to $-\infty$ and t_2 goes to $+\infty$. (We take these limits since experiments can never actually see what happens "inside"; we only know what goes in and what comes out.)

The limiting case of $U(+\infty,-\infty)$ is so common that it has a special name: the S matrix, where S stands for scattering. Almost all high-energy physics experiments are scattering experiments.

For high energies, evaluating S requires relativistic quantum mechanics, which is beyond the scope of this course (and you thought this was hard). In non-relativistic situations, we can evaluate S incrementally. We divide the total time into a series of small time intervals, and then integrate.

For three times, t_1, t_2, and t_3, we can define two U's: $U(t_2,t_1)$ and $U(t_3,t_2)$. Each U is an operator, so the sequence of $U(t_2,t_1)$ followed by $U(t_3,t_2)$ is represented by the product of those operators:

$$U(t_3,t_1) = U(t_3,t_2) \times U(t_2,t_1)$$

Let's now evaluate $U(t_2,t_1)$ in the limit that the time interval $dt=t_2-t_1$ goes to zero. Define $|\Psi(t)\rangle$ to be some state at time t. It evolves according to:

$$|\Psi(t+dt)\rangle = U(t+dt,t)\ |\Psi(t)\rangle$$

Expanding in basis states, this is:

$\langle J|\Psi(t+dt)\rangle = \Sigma_K \langle J|U(t+dt,t)|K\rangle \langle K|\Psi(t)\rangle$

Before we all drown in notation, let's break this down into more manageable pieces.

The last piece, $\langle K|\Psi(t)\rangle$, is the amplitude for state $|\Psi(t)\rangle$ to be in basis state $|K\rangle$ at time t; $\langle K|\Psi(t)\rangle$ is a complex number that changes over time. We can think of $|\psi\rangle$ as a vector whose tip is moving through the space of all states.

The complex number $\langle K|\Psi(t)\rangle$ changes because U operates on it; U drives that change.

We express how U drives change by writing:

$\langle J|U(t+dt,t)|K\rangle$

This defines the change in each basis state $|K\rangle$. U changes $|K\rangle$ into something else. We express that something else as a combination of basis states $|J\rangle$: the amount by which U changes $|K\rangle$ into $|J\rangle$ is $\langle J|U(t+dt,t)|K\rangle$.

Knowing how U changes each basis state enables us to calculate the change in any other state, like $|\psi\rangle$.

We understand the evolution of $|\Psi(t)\rangle$ by understanding how its amplitude to be in $|K\rangle$ changes into its amplitude to be in $|J\rangle$, for all $|K\rangle$ and $|J\rangle$. Everything we need to know is given by:

$\langle J|U(t+dt,t)|K\rangle$

Hope that helps.

The Hamiltonian

In general U(t+dt,t) can be quite complicated, but there are a few simple things we can say about it.

Firstly, if no time passes (dt=0), nothing can change; every state must remain as it is. This means:

$\langle J|U(t,t)|K\rangle = U_{JK}(t,t) = \delta_{JK} = 1$ if J=K, =0 if J≠K

We can also reasonably assume that for a very small time interval Δt, the change in U is proportional to Δt. This is just the property of a derivative: $\Delta U = dU/dt\ \Delta t$. In general each component U_{JK} of U can change differently during time interval Δt. Let M_{JK} be the proportionality constant for component U_{JK}:

$U_{JK}(t+\Delta t,t) = U_{JK}(t,t) + M_{JK}\ \Delta t$
$U_{JK}(t+\Delta t,t) - U_{JK}(t,t) = M_{JK}\ \Delta t$

$\langle J|U(t+\Delta t,t)|K\rangle - \langle J|U(t,t)|K\rangle = \Delta t \langle J|M|K\rangle$

Or, as an open equation, for any initial and final states:

$U(t+\Delta t,t) - U(t,t) = \Delta t\, M$

Now multiply this by $|\Psi(t)\rangle$, and let Δt become the infinitesimally small dt.

$U(t+dt,t)\ |\Psi(t)\rangle - U(t,t)\ |\Psi(t)\rangle = dt\, M\ |\Psi(t)\rangle$

Recall our earlier equation: $|\Psi(t+dt)\rangle = U(t+dt,t)\ |\Psi(t)\rangle$.

$|\Psi(t+dt)\rangle - |\Psi(t)\rangle = dt\, M\ |\Psi(t)\rangle$
$\{|\Psi(t+dt)\rangle - |\Psi(t)\rangle\} / dt = M\ |\Psi(t)\rangle$
$d\ |\Psi(t)\rangle / dt = M\ |\Psi(t)\rangle$

By convention, and for historical reasons, this equation is written:

$i\hbar\, d|\Psi(t)\rangle/dt = H\ |\Psi(t)\rangle$

Here H is the *Hamiltonian operator*. In V3p8-10, Feynman says this equation is: "the quantum mechanical law for the dynamics of the world." The Hamiltonian matrix contains all the physics that causes states to evolve over time. In general, H is a function of time and position.

Feynman is bemused that the most important operator in quantum mechanics is named after William Rowan Hamilton, a mathematical physicist who worked in the 1830's. This really isn't unusual; 25 centuries ago, Pythagoras had a great idea that we still frequently use and is still named in his honor. I believe that for many centuries, the Feynman diagrams and the Feynman path integral formulation will be taught using Feynman's name.

Expanding the last equation in basis states yields:

$i\hbar \langle K|\, d\Psi(t)/dt \rangle = \Sigma_J \langle K|H|J\rangle \langle J|\Psi(t)\rangle$

We can write that in a more vector-like form:

$i\hbar\, d(\Psi_K)/dt = \Sigma_J\, H_{KJ}\, \Psi_J$

If we knew the Hamiltonian for all phenomena, quantum physics would be complete — job done, over and out. Not to worry: we know the Hamiltonian for only some processes; employment opportunities in physics still exist.

One general rule governing all Hamiltonians is:

$H_{JK} = H_{KJ}^*$

This could be described as the principle of conservation of probability. More formally, this is called the *Unitarity Principle*. This requirement ensures that any state operated on by H must become some combination of final states, with a total probability of 100%. If H operates on a lone electron, exactly one electron must remain, somewhere.

In the simplest case, with just one basis state, H has just one component. If H is constant in time, we have:

$$i\hbar \, d\Psi/dt = H \Psi$$

$$\Psi = A \exp\{Ht/i\hbar\}$$

This means Ψ has a definite energy H, and has the same probability $|A|^2$ of being everywhere in space.

The prior equation should remind us of the wave equation $\exp\{Et/i\hbar\}$. Feynman says the coefficient of t in this equation is always energy. This means in a one-state system, H=E. For this reason the Hamiltonian is often called the *energy matrix*.

Two-State Ammonia Molecule

The next simplest systems have two basis states. Consider a system with two states that we imaginatively name: |1> and |2>. We can express any state $|\psi>$ as:

$$|\Psi> = C_1 |1> + C_2 |2>$$

We then have two independent Hamiltonian equations:

$$i\hbar \, d(C_1)/dt = H_{11} C_1 + H_{12} C_2$$
$$i\hbar \, d(C_2)/dt = H_{21} C_1 + H_{22} C_2$$

In V3p8-11, Feynman demonstrates the utility of these equations by picking: “an interesting but simple example in which, by making some reasonable guesses about the Hamiltonian, we can work out some important — and even practical — results.” (Some “practical results”; imagine that.)

The example is the ammonia molecule, illustrated in Figure 10-1. Ammonia consists of four atoms arranged in a tetrahedral pyramid. Three hydrogen atoms, shown as black dots, form the pyramid’s base, with one nitrogen atom, shown as an open circle labeled N, at the pyramid’s other corner.

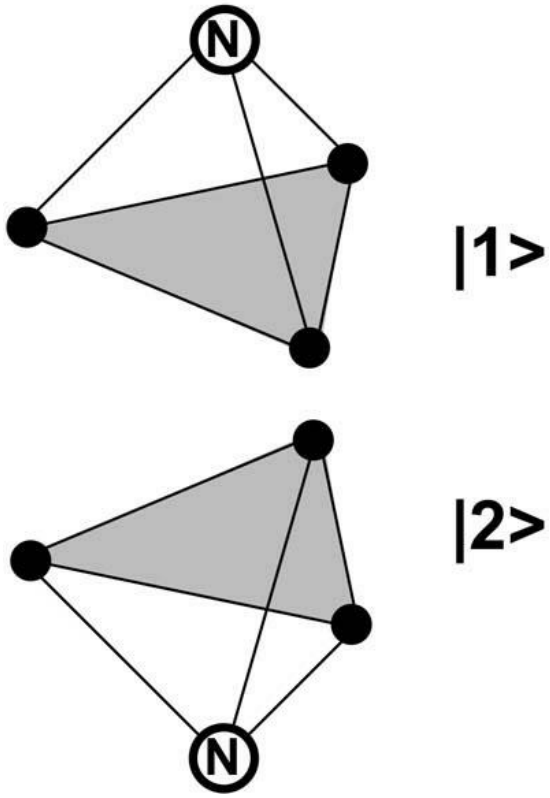

Figure 10-1 Two States of Ammonia Molecule

In the image, the plane containing the three hydrogen atoms is shown in gray for emphasis.

Although this molecule has many rotational and vibrational degrees of freedom, we will assume no changes in any of those states. We will focus on just two states. These are: |1> nitrogen above plane of hydrogen atoms; and |2> nitrogen below that plane.

State |2> is just state |1> flipped upside down. Together these form a complete set of basis states.

At each time t, the state $|\Psi(t)\rangle$ of the ammonia molecule will be some linear combination of the basis states:

$$|\Psi(t)\rangle = C_1(t)\,|1\rangle + C_2(t)\,|2\rangle$$

In general, $C_1(t)$ and $C_2(t)$ are functions of time. The Hamiltonian equations are as above:

$$i\hbar\, d(C_1)/dt = H_{11}\, C_1 + H_{12}\, C_2$$
$$i\hbar\, d(C_2)/dt = H_{21}\, C_1 + H_{22}\, C_2$$

A two-state system has four Hamiltonian components: H_{JK} is the amplitude to go from state |K> to state |J>. The four amplitudes are:

H_{11} = amplitude state |1> goes to state |1> = E_1
H_{12} = amplitude state |2> goes to state |1>
H_{21} = amplitude state |1> goes to state |2>
H_{22} = amplitude state |2> goes to state |2> = E_2

H_{12} and H_{21} are amplitudes to change states. H_{11} equals the energy of state |1>: E_1, while H_{22} equals the

energy of state |2>: E_2.

At high temperatures, ammonia molecules dissociate; the atoms become energetic enough to separate. At lower temperatures, the nitrogen atom does not have enough energy to move through the plane of hydrogen atoms. Classically, for the nitrogen atom, the plane is an impenetrable barrier, a forbidden zone. This would preclude transitions between |1> and |2>, meaning $H_{12}=H_{21}=0$.

But as we know, in the quantum realm, barrier penetration allows transitions through negative kinetic energy zones. This means the nitrogen atom can tunnel through the hydrogen plane and "switch" sides. Hence, quantum mechanics allows non-zero H_{12} and H_{21}.

By the way: when a nitrogen atom "switches" sides, all four atoms move. The molecule's center of mass cannot move without an external force. Nitrogen has atomic mass 14 and three hydrogen atoms have a total atomic mass of only 3. Hence when the nitrogen atom "switches" to the other side, the hydrogen atoms actually move almost 5 times farther than the nitrogen.

States |1> and |2> are identical except for orientation. Since nature doesn't distinguish "up" from "down", by symmetry, both states must have the same energy and other basic properties. This means $H_{11}=H_{22}$ and $H_{12}=H_{21}$.

Define $E_0 = H_{11} = H_{22}$ and $-A = H_{12} = H_{21}$. The differential equations become:

$$i\hbar\, d(C_1)/dt = E_0\, C_1 - A\, C_2$$
$$i\hbar\, d(C_2)/dt = E_0\, C_2 - A\, C_1$$

The ability to solve multiple coupled differential equations is a survival skill for theoretical physicists. This one is much easier than most. Notice what happens when we add the two equations:

$$i\hbar\, d(C_1+C_2)/dt = E_0(C_1+C_2) - A\,(C_1+C_2)$$
$$i\hbar\, d(C_1+C_2)/dt = (E_0-A)\,(C_1+C_2)$$
$$(C_1+C_2) = 2\alpha \exp\{(E_0-A)t/i\hbar\}$$

We choose the arbitrary integration constant to be 2α.

Now subtract the Hamiltonian equations:

$$i\hbar\, d(C_1-C_2)/dt = E_0(C_1-C_2) + A\,(C_1-C_2)$$
$$i\hbar\, d(C_1-C_2)/dt = (E_0+A)\,(C_1-C_2)$$
$$(C_1-C_2) = 2\beta \exp\{(E_0+A)t/i\hbar\}$$

Taking sums and differences of the two equations for (C_1+C_2) and (C_1-C_2), yields:

$$C_1(t) = \alpha \exp\{(E_0-A)t/i\hbar\} + \beta \exp\{(E_0+A)t/i\hbar\}$$
$$C_2(t) = \alpha \exp\{(E_0-A)t/i\hbar\} - \beta \exp\{(E_0+A)t/i\hbar\}$$

In V3p8-13, Feynman says: "We have the solutions; now what do they mean? (The trouble with

quantum mechanics is not only in solving the equations but in understanding what the solutions mean!)"

We have found two independent solutions to two linear differential equations, which is as many as there can be. Any linear combination of these two solutions must also be a solution. (See ***Feynman Simplified*** 1B Chapter 14 for a discussion of linear differential equations.) We can use this fact to match any initial conditions.

If, for example, we knew an ammonia molecule started in state $|1\rangle$, we would select $\alpha=\beta=1/2$.

Conversely, if we knew an ammonia molecule started in state $|2\rangle$, we would select $\alpha=-\beta=1/2$.

That takes care of initial conditions. Let's next examine the time evolution of C_1 and C_2:

For $\alpha=\beta=1/2$, we can rewrite these as:

$$C_1(t) = \exp\{E_0 t/i\hbar\}\ [\exp\{At/i\hbar\} + \exp\{At/i\hbar\}]\ /2$$
$$C_2(t) = \exp\{E_0 t/i\hbar\}\ [\exp\{At/i\hbar\} - \exp\{At/i\hbar\}]\ /2$$

$$C_1(t) = \exp\{E_0 t/i\hbar\}\ [+\cos(At/\hbar)]$$
$$C_2(t) = \exp\{E_0 t/i\hbar\}\ [-\sin(At/\hbar)]$$

The probabilities of being in each state are:

$$P_1(t) = |C_1(t)|^2 = \cos^2(At/\hbar)$$
$$P_2(t) = |C_2(t)|^2 = \sin^2(At/\hbar)$$

Note that $P_1(t) + P_2(t) = 1$, as it must.

As Feynman says, $|1\rangle$ and $|2\rangle$ are not stationary states; the probability of finding the ammonia molecule "sloshes back and forth" between the two states. With $\alpha=\beta=1/2$, the sloshing begins in C_1 at $t=0$. We could just as well chose $\alpha=1/2$ and $\beta=-1/2$, which would start the sloshing in C_2.

Now let's find the stationary states. The most general solution is $|\psi(t)\rangle$ defined by:

$$C_1(t) = \alpha \exp\{(E_0-A)t/i\hbar\} + \beta \exp\{(E_0+A)t/i\hbar\}$$
$$C_2(t) = \alpha \exp\{(E_0-A)t/i\hbar\} - \beta \exp\{(E_0+A)t/i\hbar\}$$

$$|\psi(t)\rangle = C_1(t)\ |1\rangle + C_2(t)\ |2\rangle$$

Choose $\alpha=1/\sqrt{2}$ and $\beta=0$, and call the resulting state $|\psi-\rangle$.

$$|\psi-\rangle = \exp\{(E_0-A)t/i\hbar\}\ (|1\rangle + |2\rangle)/\sqrt{2}$$

$|\psi-\rangle$ is a stationary state of definite energy $E_-=E_0-A$. Here the amplitudes of states $|1\rangle$ and $|2\rangle$ have the same phase; they oscillate together. If the system started in this state, it would remain there

forever, absent any other interactions.

Now chose $\alpha=0$ and $\beta=1/\sqrt{2}$, and call the resulting state $|\psi+\rangle$.

$$|\psi+\rangle = \exp\{(E_0+A)t/i\hbar\}\ (|1\rangle - |2\rangle)/\sqrt{2}$$

$|\psi+\rangle$ is a stationary state of definite energy $E_+=E_0+A$. Here the amplitudes of states $|1\rangle$ and $|2\rangle$ have opposite phases; they oscillate at the same frequency but with opposite polarity. If the system started in this state, it would remain there forever, absent any other interactions.

In V3p8-14, Feynman notes that the ammonia molecule is like a pair of coupled pendulums. There is one state of definite frequency, with energy E_0-A, where C_1 and C_2 oscillate together. There is another state of a different definite frequency, with energy E_0+A, where C_1 and C_2 oscillate oppositely.

Chapter 10 Review: Key Ideas

See the beginning of this chapter for a concise review of basis state concepts, notations, and rules.

1. Since $\langle\varphi|\psi\rangle = \Sigma_J \langle\varphi|J\rangle \langle J|\psi\rangle$ for all states φ and ψ, Feynman says the "great law of quantum mechanics" is expressed by an *open equation*:

   $$| = \Sigma_J |J\rangle\langle J|$$

   This expression is valid whenever multiplied by any bra and/or any ket.

2. Bra and ket quantities are essential intermediate steps, but not end results. The measurable quantities that we seek to predict are probabilities. Calculations must not end with unmatched bras or kets.

3. Bra and ket are complex conjugates: $\langle\psi| = |\psi\rangle^*$. This means $\langle\psi|$ and $|\psi\rangle$ are different, in general. Either could properly represent a quantum state. Feynman says he always uses kets $|\psi\rangle$ to identify states, for consistency.

4. The time evolution of quantum states is govern by the Hamiltonian equation:

   $$i\hbar\, d|\Psi(t)\rangle/dt = H\,|\Psi(t)\rangle$$

5. For a two-state system, with basis states $|1\rangle$ and $|2\rangle$, any state $|\psi\rangle$ can be represented as:

   $$|\Psi(t)\rangle = C_1(t)\,|1\rangle + C_2(t)\,|2\rangle$$

   The Hamiltonian equation yields two independent differential equations:

$i\hbar\, d(C_1)/dt = H_{11}\, C_1 + H_{12}\, C_2$
$i\hbar\, d(C_2)/dt = H_{21}\, C_1 + H_{22}\, C_2$

With $E_0 = H_{11} = H_{22}$ and $-A = H_{12} = H_{21}$, the solutions are:

$C_1(t) = \exp\{E_0 t/i\hbar\}\ [+\cos(At/\hbar)]$
$C_2(t) = \exp\{E_0 t/i\hbar\}\ [-\sin(At/\hbar)]$

$|\psi-\rangle = \exp\{(E_0 - A)t/i\hbar\}\ (|1\rangle + |2\rangle)/\sqrt{2}$
$|\psi+\rangle = \exp\{(E_0 + A)t/i\hbar\}\ (|1\rangle - |2\rangle)/\sqrt{2}$

$|\psi-\rangle$ is a stationary state of definite energy $E_- = E_0 - A$.
$|\psi+\rangle$ is a stationary state of definite energy $E_+ = E_0 + A$.

$C_1(t)$ and $C_2(t)$ are not stationary states; they are coupled, with probabilities "sloshing back and forth" between them.

Chapter 11

Ammonia Maser

In the last chapter, we analyzed the ammonia molecule, modeling it as a two-state system. Let's briefly review.

The structure of the ammonia molecule is a tetrahedral pyramid, with three hydrogen atoms forming a plane and one nitrogen atom at the other corner. The two states are defined as:

|1> Nitrogen above plane of hydrogen atoms

|2> Nitrogen below plane of hydrogen atoms

These states form a complete basis set. Any state $|\psi>$ can be represented as a linear combination of states |1> and |2>:

$$|\Psi(t)> = C_1(t)\,|1> + C_2(t)\,|2>$$

The Hamiltonian equation:

$$i\hbar\, d\Psi/dt = H\,\Psi$$

yields two independent differential equations:

$$i\hbar\, d(C_1)/dt = H_{11}\,C_1 + H_{12}\,C_2$$

$$i\hbar\, d(C_2)/dt = H_{21}\,C_1 + H_{22}\,C_2$$

From the symmetry of the physical states, we expect $H_{11}=H_{22}$ and $H_{12}=H_{21}$. Defining $E_0=H_{11}=H_{22}$ and $-A=H_{12}=H_{21}$, the differential equations become:

$$i\hbar\, d(C_1)/dt = E_0\,C_1 - A\,C_2$$
$$i\hbar\, d(C_2)/dt = E_0\,C_2 - A\,C_1$$

The solutions, including arbitrary constants α and β, are:

$$C_1(t) = \alpha \exp\{(E_0-A)t/i\hbar\} + \beta \exp\{(E_0+A)t/i\hbar\}$$
$$C_2(t) = \alpha \exp\{(E_0-A)t/i\hbar\} - \beta \exp\{(E_0+A)t/i\hbar\}$$

Choosing $\alpha=1/\sqrt{2}$ and $\beta=0$ yields state $|\psi->$, and $\beta=1/\sqrt{2}$ and $\alpha=0$ yields state $|\psi+>$.

$$|\psi-> = \exp\{(E_0-A)t/i\hbar\}\ (|1> + |2>)/\sqrt{2}$$

$|\psi+\rangle = \exp\{(E_0+A)t/i\hbar\}\ (|1\rangle - |2\rangle)/\sqrt{2}$

$|\psi-\rangle$ is a stationary state of definite energy $E_- = E_0 - A$.
$|\psi+\rangle$ is a stationary state of definite energy $E_+ = E_0 + A$.

All that is from the last chapter.

Let's define a new pair of basis states:

$|-\rangle = (|1\rangle + |2\rangle)/\sqrt{2}$
$|+\rangle = (|1\rangle - |2\rangle)/\sqrt{2}$

We can show these states are orthonormal.

$\langle -|-\rangle = \{\langle 1| + \langle 2|\}\ \{|1\rangle + |2\rangle\}/2$
$\langle -|-\rangle = \{\langle 1|1\rangle + \langle 2|1\rangle + \langle 1|2\rangle + \langle 2|2\rangle\}/2$
$\langle -|-\rangle = \{1 + 0 + 0 + 1\}/2$
$\langle -|-\rangle = 1$

$\langle +|+\rangle = \{\langle 1| - \langle 2|\}\ \{|1\rangle - |2\rangle\}/2$
$\langle +|+\rangle = \{\langle 1|1\rangle - \langle 2|1\rangle - \langle 1|2\rangle + \langle 2|2\rangle\}/2$
$\langle +|+\rangle = \{1 - 0 - 0 + 1\}/2$
$\langle +|+\rangle = 1$

$\langle -|+\rangle = \{\langle 1| + \langle 2|\}\ \{|1\rangle - |2\rangle\}/2$
$\langle -|+\rangle = \{\langle 1|1\rangle + \langle 2|1\rangle - \langle 1|2\rangle - \langle 2|2\rangle\}/2$
$\langle -|+\rangle = \{1 + 0 - 0 - 1\}/2$
$\langle -|+\rangle = 0$

Looking ahead, we included the $\sqrt{2}$'s to provide the proper normalization. This means $|+\rangle$ and $|-\rangle$ also form a complete set of orthonormal basis states for the two-state ammonia molecule.

Repeating the logic of the last chapter, this time using basis states $|+\rangle$ and $|-\rangle$, any state $|\psi\rangle$ can be represented as:

$|\Psi(t)\rangle = C_+(t)\ |+\rangle + C_-(t)\ |-\rangle$

The Hamiltonian equation yields two new independent differential equations:

$i\hbar\ d(C_+)/dt = E_0\ C_+ + A\ C_+ = E_+\ C_+$
$i\hbar\ d(C_-)/dt = E_0\ C_- - A\ C_- = E_-\ C_-$

The normalized solutions are:

$C_+ = \exp\{E_+ t/i\hbar\}$
$C_- = \exp\{E_- t/i\hbar\}$

$|\psi\text{–}\rangle = \exp\{(E_0\text{–}A)t/i\hbar\}\ |\text{–}\rangle$
$|\psi\text{+}\rangle = \exp\{(E_0\text{+}A)t/i\hbar\}\ |\text{+}\rangle$

Transforming the Hamiltonian to the |+> and |–> basis states, the H matrix becomes diagonal:

$$\begin{vmatrix} E_+ & 0 \\ 0 & E_- \end{vmatrix}$$

Feynman doesn't mention this here, but I think it's interesting that the difference between E_+ and E_- is related to symmetry properties. In stationary state $|\psi\text{–}\rangle$ with energy E_0–A, the amplitudes of states |1> and |2> oscillate together. Conversely, in stationary state $|\psi\text{+}\rangle$ with energy E_0+A, the amplitudes of states |1> and |2> oscillate opposite to one another. Figure 11-1 schematically illustrates how basis states |+> and |–> compare with basis states |1> and |2>.

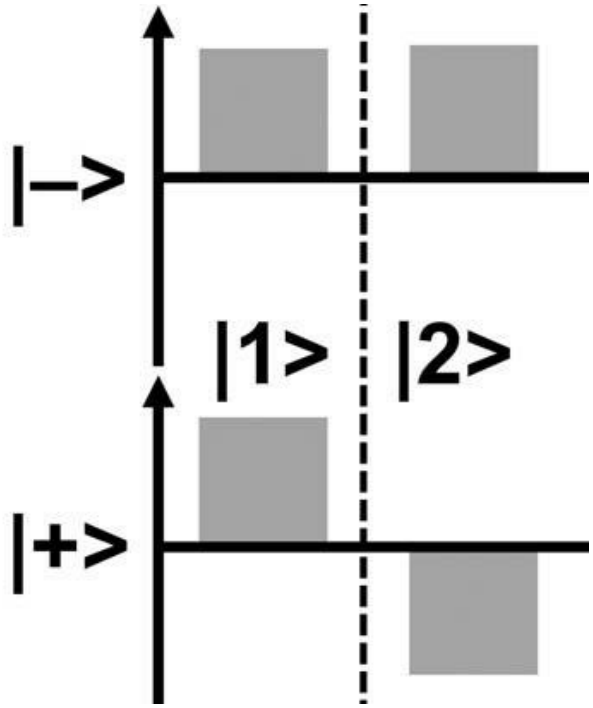

Figure 11-1 Comparing Basis |–>, |+> with Basis |1>, |2>

State |–> is symmetric in the sense that it has the same amplitudes in |1> and |2>. Conversely, state |+> is asymmetric with opposite amplitudes in |1> and |2>. The asymmetric distribution of |+> corresponds to a higher frequency and therefore a higher energy.

In V3p9-2, Feynman compares the energy levels of different types of excited states. He notes that raising an electron in an atom to a higher orbital shell requires 10 eV or more, corresponding to visible or ultra-violet photons. Atomic vibrational degrees of freedom require only about 0.1 eV, and can be excited by infrared photons. Rotational modes are even less energetic, around 0.01 eV, and can be excited by far-infrared photons. The energy gap between the two states of ammonia is a mere 10^{-4} eV, corresponding to a photon frequency of only f = 24 GHz, which is in the microwave range with a wavelength of 12mm.

Ammonia in an Electric Field

The states |+> and |–> are stationary; molecules in these states would remain there forever, absent other interactions. The higher energy state |+> could transition to the lower energy state |–> by emitting a photon. Or, the lower energy state could transition to the higher energy state by absorbing a photon.

Let's discover the impact of immersing ammonia molecules in an electric field.

The first topic is the electric dipole moment of ammonia. Nitrogen has a higher electronegativity than hydrogen. This means nitrogen attracts electrons more forcefully than does hydrogen. When ammonia forms, the three hydrogen electrons reduce their energies by moving closer to the nitrogen nucleus than to the hydrogen nuclei. This gives the nitrogen atom a net negative electric charge, –0.344e, and gives each hydrogen atom a net positive charge, +0.115e, where e is the charge of one proton.

Electric dipole moments are defined in terms of the product of charge multiplied by distance. Normally, the orbits of electrons in atoms are centered on the nucleus. If an electron's orbit is displaced from that center by a distance d, a dipole moment μ results with μ=ed. The dipole moment is a vector that points from the net negative charge toward the net positive charge.

Dipole moments are quoted in many different units, most commonly in electron-angstroms and in Debye, named in honor of Peter J. W. Debye. One electron-angstrom corresponds to an electron displaced by one angstrom, which equals 4.8 Debye.

The dipole moment of ammonia, μ, is 1.46 Debye or 0.30 electron-angstroms. This corresponds to each hydrogen electron being displaced by 0.10 angstroms toward the nitrogen nucleus.

Figure 11-2 shows the orientation of the dipole moment for both ammonia states |1> and |2>.

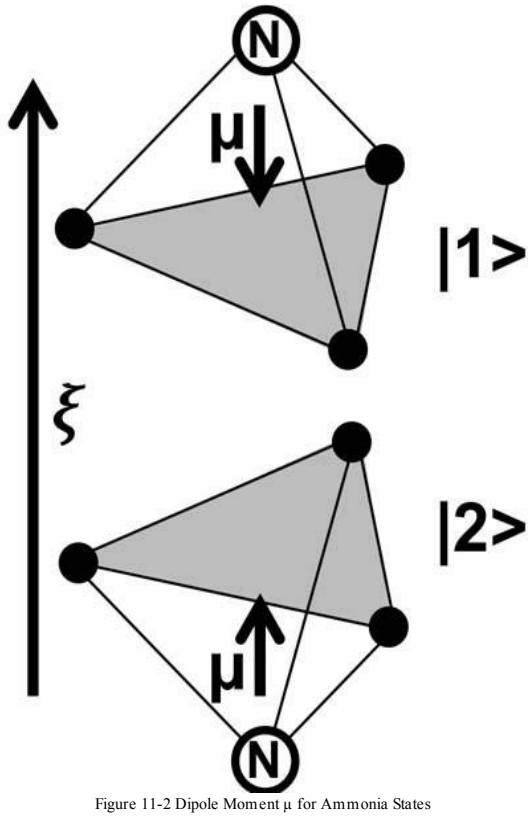

Figure 11-2 Dipole Moment μ for Ammonia States

In an electric field $\boldsymbol{\xi}$, an ammonia molecule has a lower energy if its dipole moment $\boldsymbol{\mu}$ is parallel to $\boldsymbol{\xi}$. The molecule's potential energy is decreased by $V=\mu\xi$ when $\boldsymbol{\mu}$ is parallel to $\boldsymbol{\xi}$, as for state |2> in Figure 11-2. The potential energy is increased by V when $\boldsymbol{\mu}$ is anti-parallel to $\boldsymbol{\xi}$, as for state |1> in the figure.

Unfortunately, physicists conventionally use E for both energy and electric field. Here, energy is E, the electric field vector is $\boldsymbol{\xi}$, and the magnitude of $\boldsymbol{\xi}$ is ξ.

In V3p9-5, Feynman notes regretfully that even with a complete knowledge of quantum mechanics, no one has been able to calculate all the properties of the ammonia molecule, and other systems with similar or greater complexity. We would like to be able to calculate ammonia's μ and A, but we can't. With four nuclei and ten electrons, the equations are just too complex to solve. All the properties of this important molecule are well known, but only through experiment and measurement.

We will assume that we can ignore all other degrees of freedom and address only two states: nitrogen up and nitrogen down. On that basis, we can do a reasonable job of calculating what happens to the molecule in an electric field.

Go back to the Hamiltonian, represented in basis states |1> and |2>.

We found that H_{12} is the amplitude for state |2> to transition to state |1>, which we previously called –A. Furthermore, H_{21} is the amplitude for state |1> to transition to state |2>, which was also –A. If the electric field is not strong enough to distort the molecule, we have no reason to expect these transition amplitudes to change; we leave them equal to –A.

$H_{12} = -A$
$H_{21} = -A$

We also found that H_{11} is the energy of state |1>, and H_{22} is the energy of state |2>. These do change: H_{11} increases by V, and H_{22} decreases by V, with the orientation of ξ in Figure 11-2.

$H_{11} = E_0 + V$
$H_{22} = E_0 - V$

Since we will need solutions to two-state Hamiltonian problems often in the future, Feynman chooses to solve this problem for any combination of H_{JK}, and plug in the above values at the end. This makes the current problem harder, but it will pay off later.

For arbitrary H_{JK}, the differential equations are:

$i\hbar\, dC_1/dt = H_{11}\, C_1 + H_{12}\, C_2$
$i\hbar\, dC_2/dt = H_{21}\, C_1 + H_{22}\, C_2$

As we discovered in ***Feynman Simplified 1B*** Chapter 12 through 14, any set of linear differential equations with constant coefficients can be solved with exponentials. Try exponentials for C_1 and C_2 that have the same energy E:

$C_1 = \alpha \exp\{Et/i\hbar\}$
$C_2 = \beta \exp\{Et/i\hbar\}$

Plugging these into the differential equations, and cancelling the common exponentials in each term, yields:

$i\hbar\, (1/i\hbar)\, E\, \alpha = H_{11}\, \alpha + H_{12}\, \beta$
$i\hbar\, (1/i\hbar)\, E\, \beta = H_{21}\, \alpha + H_{22}\, \beta$

$(E - H_{11})\, \alpha = H_{12}\, \beta$
$(E - H_{22})\, \beta = H_{21}\, \alpha$

Multiply these two equations:

$(E - H_{22})\, (E - H_{11})\, \alpha\beta = H_{12}\, H_{21}\, \alpha\beta$
$E^2 - E\, (H_{11} + H_{22}) + H_{11}\, H_{22} - H_{12}\, H_{21} = 0$

Recall the quadratic equation and its solution:

$ax^2 + bx + c = 0$
$x = \{-b \pm \sqrt{[b^2 - 4ac]}\} / 2$

Let's define the square root to equal 2E*, and simplify it slightly:

$2E^* = \sqrt{[\, (H_{11}+H_{22})^2 - 4\, H_{11}\, H_{22} + 4\, H_{12}\, H_{21}\,]}$
$2E^* = \sqrt{[\, (H_{11}-H_{22})^2 + 4\, H_{12}\, H_{21}\,]}$
$E^* = \sqrt{[\, (H_{11}-H_{22})^2/4 + H_{12}\, H_{21}\,]}$

The solution to our quadratic equation is:

$E = (H_{11}+H_{22})/2 \pm E^*$

In V3p9-7, Feynman suggests checking our result for some simple limiting cases — always a good idea at the end of a long calculation.

For A = 0 ($H_{12} = H_{21} = 0$):
$E = (H_{11}+H_{22})/2 \pm (H_{11}-H_{22})/2$
$E = H_{11}$ or H_{22}

That is correct. Without transitions between states |1> and |2>, both are stationary states with energies E_0+V and E_0-V respectively.

Now try V = 0 ($H_{11} = H_{22} = E_0$):

$E = E_0 \pm \sqrt{(-A)(-A)}$
$E = E_0 \pm A$

This confirms our result without the electric field. These checks don't mean our calculations are correct, but they would have spotted some gross errors.

Plugging in the values of H_{JK} for our current problem:

$E^* = \sqrt{[(2V)^2/4 + A^2]}$
$E^* = \sqrt{[V^2 + A^2]}$
$E = E_0 \pm E^*$

As before, the stationary basis states are:

$|+\rangle = \{|1\rangle - |2\rangle\} /\sqrt{2}$
$|-\rangle = \{|1\rangle + |2\rangle\} /\sqrt{2}$

And the stationary states of ammonia atoms are:

$|\psi+\rangle = |+\rangle \exp\{E_+t/i\hbar\}$, with $E_+ = E_0+E^*$
$|\psi-\rangle = |-\rangle \exp\{E_-t/i\hbar\}$, with $E_- = E_0-E^*$

In Figure 11-3, we plot the energy of the two stationary states versus the electric potential $V=\boldsymbol{\mu}\bullet\boldsymbol{\xi}$. The two dotted lines mark $E=\pm V$, and the two solid curves plot $E = E_0 \pm E^* = E_0 \pm \sqrt{[V^2+A^2]}$.

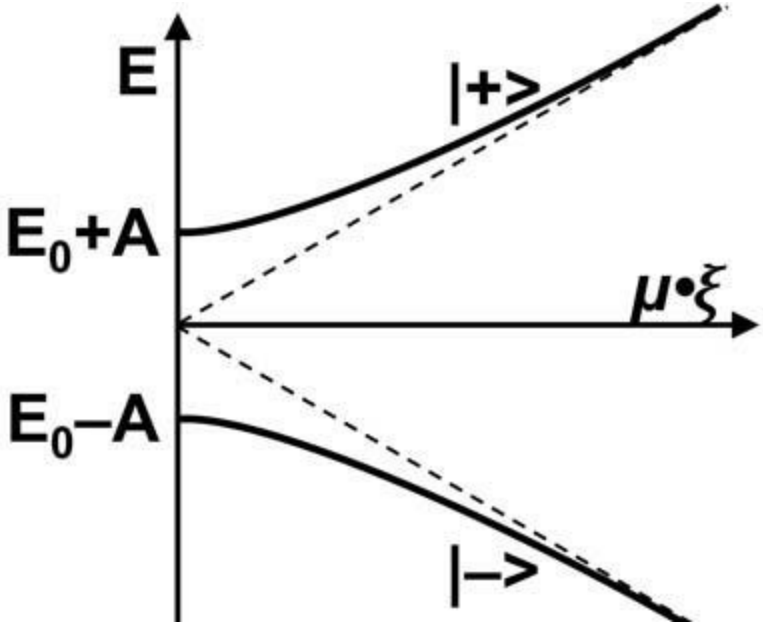

Figure 11-3 Ammonia Molecule
Energy vs. Field Potential

We see that as the electric field increases, the splitting between the energies of the two stationary states increases. For very large electric fields, the splitting is dominated by the dipole alignment, and the possibility of nitrogen atoms "switching" sides become much less likely.

In V3p9-8, Feynman says: "This is an interesting point that we will come back to later."

The Ammonia Maser

Before getting to all the nuts and bolts, here is an overview of an ammonia *maser*.

Masers are essentially lasers operating in a different frequency domain. "Maser" is an acronym for Microwave Amplification by Stimulated Emission of Radiation. "Laser" is the same acronym with "Microwave" replaced by "Light." Since microwaves are a form of light, the distinction is not fundamental.

In an ammonia maser, a beam of ammonia molecules is separated into two states: a higher energy state |+>, and a lower energy state |–>. The |+> molecules are directed through a cavity with an oscillating electric field that has a frequency of f = 24 GHz, which matches the transition energy between the |+> and |–> states. The electric field stimulates the |+> state molecules to emit radiation at the transition energy frequency, forming a coherent beam of microwave photons.

The basic physics is identical to that of lasers, which is discussed in detail in ***Feynman Simplified 1B*** Chapter 20.

The apparatus used to prepare a pure |+> state beam is similar to a Stern-Gerlach device. The Stern-Gerlach device separates spin states by deflecting magnetic dipoles in a magnetic field with a large gradient. Here we deflect electric dipoles in a device with a large electric field gradient.

Figure 11-4 illustrates this process. Ammonia molecules enter from the left, and pass through two

collimating slits, thereby forming a narrow beam. The beam is then exposed to an electric field that increases rapidly in the direction of the downward arrow indicated in the figure.

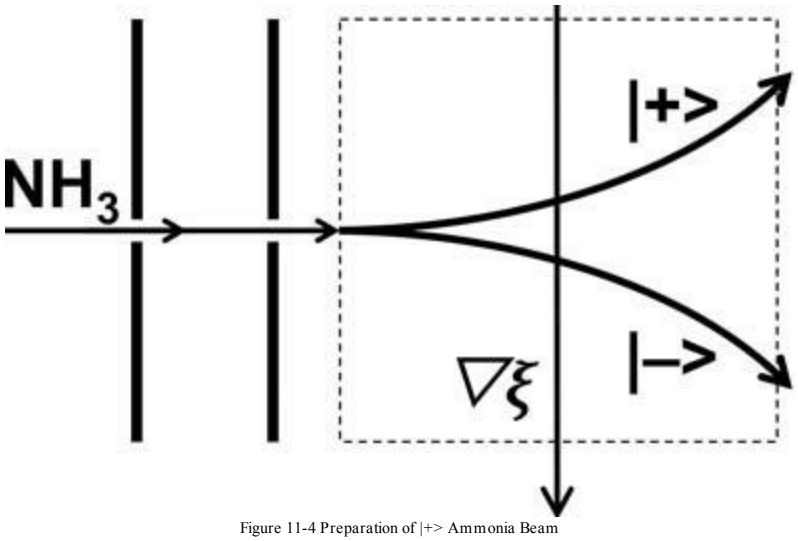

Figure 11-4 Preparation of |+> Ammonia Beam

Due to ammonia's electric dipole moment, the electric field decreases the potential energy of |–> state molecules and increases the potential energy of |+> state molecules. Molecules in the |–> state deflect downward toward the stronger field to maximize their potential energy decrease. Molecules in the |+> state deflect upward toward the weaker field to minimize their potential energy increase.

Practical considerations limit the magnitude of artificial electric fields. As a result, in these devices, V is always small compared with A. We can therefore simplify the energy equation by approximating the square root.

$E = E_0 \pm E^*$
$E = E_0 \pm \sqrt{[V^2+A^2]}$
$E \approx E_0 \pm \{A + V^2/2A\}$

State |+> has energy $E_+ = E_0 + \{A+V^2/2A\}$
State |–> has energy $E_- = E_0 - \{A+V^2/2A\}$

Note that the potential energy is proportional to V^2.

The deflecting force is proportional to minus the gradient of the potential energy. For an electric field parallel to the y-axis:

$F_y = - dV^2/dy /2A$

The beam of ammonia molecules in a pure |+> state next enters a cavity with an electric field that oscillates at microwave frequencies — basically a microwave oven. This is illustrated in Figure 11-5.

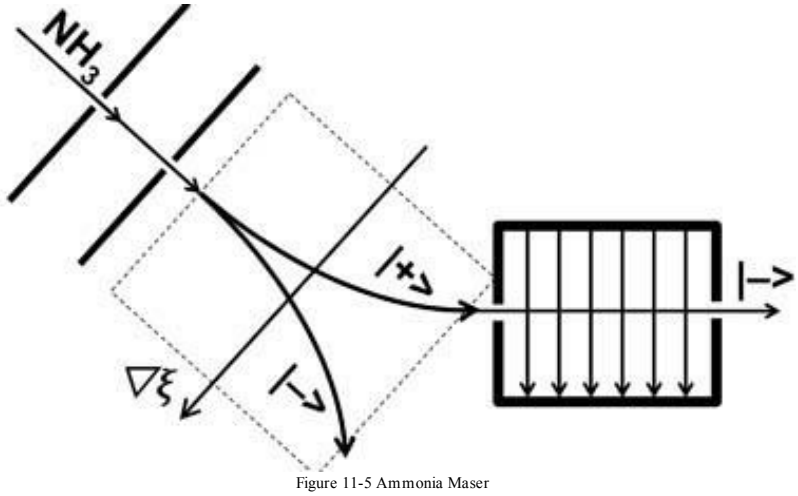

Figure 11-5 Ammonia Maser
Operating on Prepared Beam

By matching the cavity frequency to the transition frequency of ammonia, molecules are driven to emit photons of the same frequency. The transition frequency of ammonia is $\Omega=2A/\hbar$, where 2A is the energy difference between the two states |+> and |–>.

Time-Dependent Field

Since masers employ time-varying electric fields, our next step is to solve the Hamiltonian equation for a time-varying potential. We previously assumed H was constant, so this is something new. The two differential equations are:

$i\hbar\, dC_1/dt = (E_0+V)\, C_1 - A\, C_2$
$i\hbar\, dC_2/dt = -A\, C_1 + (E_0-V)\, C_2$

As before, we add and subtract these two equations to yield two other equations in the stationary state basis:

$C- = (C1+C2)/\sqrt{2}$
$C+ = (C1-C2)/\sqrt{2}$

$i\hbar\, dC_-/dt = (E_0-A)\, C_- + V\, C_+$
$i\hbar\, dC_+/dt = (E_0+A)\, C_+ + V\, C_-$

For a general electric field, even a sinusoidal one, these equations are not solvable with normal analytic methods.

Feynman was famous for solving complex problems. He was a genius at seeing hidden simplifications and making astute approximations. Every successful theoretical physicist has a "toolbox" full of mathematical tricks, learned mostly from the university of fire and brimstone. We have an opportunity here to observe master craftsmanship and add some wrenches to our toolbox.

We know that V is small compared with A, which at only 10^{-4} eV is small compared with E_0. Feynman therefore suggests using the trial solutions that follow. (As discussed more extensively in ***Feynman Simplified 1B*** Chapters 12 through 14, solving differential equations requires perception and

perspiration. There is no harm in guessing; if you're wrong, the equations will certainly let you know.) Let's try:

$C_+ = \Gamma_+ \exp\{E_+t/i\hbar\}$, with $E_+ = E_0+A$
$C_- = \Gamma_- \exp\{E_-t/i\hbar\}$, with $E_- = E_0-A$

In V3p9-10, Feynman explains that if the electric field were absent, Γ_+ and Γ_- would be complex constants that determine the probabilities of being in stationary states $|+\rangle$ and $|-\rangle$, as we have seen before. If a molecule began in $|-\rangle$, it would remain there forever, absent other interactions. Feynman says:

> "Now the idea of writing our equations in [this form] is that if [V] is small in comparison to A, the solutions can still be written this way, but Γ_+ and Γ_- become slowly varying functions of time — where by "slowly varying" we mean slowly in comparison with the exponential functions [$\exp\{E_\pm t/i\hbar\}$]. That is the trick."

We now plug our trial solution for C_+ into its differential equation, recognizing the time-dependence of Γ.

Try $C_+ = \Gamma_+ \exp\{E_+t/i\hbar\}$
$dC_+/dt = (E_+/i\hbar)\ \Gamma_+ \exp\{E_+t/i\hbar\} + d\Gamma_+/dt \exp\{E_+t/i\hbar\}$

$i\hbar\, dC_+/dt = E_+ C_+ + V C_-$

$[E_+ \Gamma_+ + i\hbar\, d\Gamma_+/dt] \exp\{E_+t/i\hbar\} =$
$E_+ \Gamma_+ \exp\{E_+t/i\hbar\} + V \Gamma_- \exp\{E_-t/i\hbar\}$

$[E_+ \Gamma_+ + i\hbar\, d\Gamma_+/dt] = E_+ \Gamma_+ + V \Gamma_- \exp\{(E_- - E_+)t/i\hbar\}$
$i\hbar\, d\Gamma_+/dt = V \Gamma_- \exp\{-i(E_- - E_+)t/\hbar\}$

Similarly, skipping intermediate steps, we have for C_-, Γ_-:

$i\hbar\, dC_-/dt = E_- C_- + V C_+$
$i\hbar\, d\Gamma_-/dt = V \Gamma_+ \exp\{-i(E_+ - E_-)t/\hbar\}$

Recall that $E_+ - E_- = 2A$. Define $\Omega = 2A/\hbar$. Since $\Omega\hbar = 2A$, Ω is the transition frequency between the two stationary states of an isolated ammonia molecule.

Our equations are then:

$i\hbar\, d\Gamma_+/dt = V \Gamma_- \exp\{+i\Omega t\}$
$i\hbar\, d\Gamma_-/dt = V \Gamma_+ \exp\{-i\Omega t\}$

We next focus on the time-varying electric field. For specificity, Feynman chooses a sinusoidally oscillating field given by:

$\xi = \xi_0 \cos(\omega t) = \xi_0 (\exp\{i\omega t\} + \exp\{-i\omega t\})/2$

We define $V_0 = \mu\xi_0/2$. With that definition, the equations become:

$i\hbar\, d\Gamma_+/dt = V_0\, \Gamma_- (\exp\{i(\omega+\Omega)t\} + \exp\{-i(\omega-\Omega)t\})$
$i\hbar\, d\Gamma_-/dt = V_0\, \Gamma_+ (\exp\{i(\omega-\Omega)t\} + \exp\{-i(\omega+\Omega)t\})$

Until this point, we haven't made any approximations regarding the electric field or the Γ's. Now is the time.

Observe that $d\Gamma/dt \sim V_0\, \Gamma$. For small V_0, this means the time derivatives of both Γ's will be small on a percentage basis. The Γ's change much more slowly than the $\exp\{i(\omega+\Omega)t\}$ terms. The latter will not contribute to $d\Gamma/dt$ because they will rapidly average out to their mean values: zero. We therefore drop these terms from the equations.

$i\hbar\, d\Gamma_+/dt = V_0\, \Gamma_- \exp\{-i(\omega-\Omega)t\}$
$i\hbar\, d\Gamma_-/dt = V_0\, \Gamma_+ \exp\{+i(\omega-\Omega)t\}$

Since $\exp\{i(\omega-\Omega)t\}$ has magnitude 1, the Γ's cannot vary more rapidly than $\exp\{V_0 t/i\hbar\}$. This is because:

$i\hbar\, d\Gamma/dt = V_0\, \Gamma$ implies $\Gamma = \exp\{V_0 t/i\hbar\}$

If $V_0/\hbar$ is much less than $(\omega-\Omega)$, $\exp\{i(\omega-\Omega)t\}$ will run through several full cycles for each full cycle of $\exp\{V_0 t/i\hbar\}$. That would make $d\Gamma/dt$ virtually zero. The only circumstance in which $d\Gamma/dt$ can be appreciable is when $\omega \approx \Omega$.

Feynman therefore makes the approximation that $\omega = \Omega$, which yields:

$i\hbar\, d\Gamma_+/dt = V_0\, \Gamma_-$
$i\hbar\, d\Gamma_-/dt = V_0\, \Gamma_+$

Rearranging the second equation, and differentiating it yields:

$\Gamma_+ = (i\hbar/V_0)\, d\Gamma_-/dt$
$d\Gamma_+/dt = (i\hbar/V_0)\, d^2\Gamma_-/dt^2$

Now use the equation for $d\Gamma_+/dt$:

$V_0\, \Gamma_- = i\hbar\, d\Gamma_+/dt = i\hbar\, (i\hbar/V_0)\, d^2\Gamma_-/dt^2$
$d^2\Gamma_-/dt^2 = -(V_0/\hbar)^2\, \Gamma_-$

The solutions are any combination of complex constants α and β, in:

$\Gamma_- = \alpha \exp\{-V_0 t/i\hbar\} + \beta \exp\{+V_0 t/i\hbar\}$

Plugging this into the equation for Γ_+:

$\Gamma_+ = (i\hbar/V_0)\ (V_0/i\hbar)\ [-\alpha\exp\{-V_0t/i\hbar\} + \beta\exp\{V_0t/i\hbar\}]$
$\Gamma_+ = -\alpha\ \exp\{-V_0t/i\hbar\} + \beta\ \exp\{V_0t/i\hbar\}$

As one example, for C_+ choose $-\alpha=\beta=1/2$, and for C_- choose $-\alpha=\beta=i/2$:

$C_+ = \cos(V_0t/i\hbar)\ \exp\{E_+t/i\hbar\}$, with $E_+ = E_0+A$
$C_- = \sin(V_0t/i\hbar)\ \exp\{E_-t/i\hbar\}$, with $E_- = E_0-A$

The corresponding probabilities are:

$P_+ = \cos^2(V_0t/\hbar)$
$P_- = \sin^2(V_0t/\hbar)$

Recall, this assumes E_0 is small compared with A, and that the derivatives of Γ are dominated by the resonance at frequency $\Omega=2A/\hbar$.

The beam entering the cavity of the ammonia maser in Figure 11-5 is a pure |+> state, corresponding to C_+ above. After time T in the cavity, when $V_0T/\hbar=\pi/2$, all those molecules transition to the |–> state. The molecules change states by emitting photons of frequency Ω, which is in the a microwave part of the light spectrum.

If ammonia molecules remain in the cavity for a time interval greater than T, they begin to transition back to the higher energy state |+> by absorbing photons from the cavity's electric field. For maximum efficiency, the cavity length L and the average molecule velocity v should be matched to the transition time T, such that vT=L. Since such matching is never perfect, masers are never 100% efficient.

Non-Resonant Transitions

Now consider transitions when the cavity frequency is not exactly equal to the transition frequency Ω.

Let's just take the simple case of weak electric field ***E*** and short time interval T, which together make the transition rate small. Since we start with a pure state of |+>, we expect almost all molecules to remain in that state and only a few to transition to the lower energy state |–>. This means $|\Gamma_+|^2$ remains close to 1 and $|\Gamma_-|^2$ remains close to zero. We will set $\Gamma_+ = 1$; if it has a phase angle, that will shortly become irrelevant.

Recall a prior equation:

$i\hbar\ d\Gamma_-/dt = V_0\ \Gamma_+\ \exp\{+i(\omega-\Omega)t\}$
$d\Gamma_-/dt \approx -i(V_0/\hbar)\ \exp\{+i(\omega-\Omega)t\}$
$\Gamma_-(T) \approx \int (d\Gamma_-/dt)\ dt$, from t=0 to t=T
$\Gamma_-(T) \approx -(V_0/\hbar)\ [\exp\{+i(\omega-\Omega)T\} -1]\ /\ (\omega-\Omega)$

Define $\phi = 2(\omega - \Omega)T$.

$[\exp\{+i2\phi\} - 1] = \exp\{+i\phi\}\ [\exp\{+i\phi\} - \exp\{-i\phi\}]$
$\Gamma_{-}(T) \approx -(V_0/\hbar)\ \exp\{+i\phi\}\ [2i \sin\phi]\ (2T/\phi)$
$\Gamma_{-}(T) \approx (V_0 T/i\hbar)\ \exp\{+i\phi\}\ [\sin\phi]\ /\phi$

$P(+ \text{ to } -) = |\Gamma_{-}(T)|^2 \approx (V_0 T/\hbar)^2\ \text{sinc}^2\phi$

Here $\text{sinc}(x) = \sin(x)/x$, and $P(+ \text{ to } -)$ is the probability to transition from state $|+\rangle$ to state $|-\rangle$ during time T.

The $\text{sinc}^2\phi$ function is very sharply peaked, being essentially zero when $|\phi| > \pi$. The peak width is $\Delta\phi = 2\pi$. Taking some representative numbers (T = 1 millisecond, and f = 24 GHz), this peak's Q-factor, height divided by width, is about 100 million. This justifies our assumption in the prior section of setting $\omega = \Omega$.

Light Absorption

In V3p9-14, Feynman notes that equations derived in this chapter are applicable to many other circumstances. While we spoke about ammonia, microwave frequencies, and fields inside cavities, none of our calculations depended on those particulars. Our equations apply equally well to any atom or molecule exposed to any form of light.

We can translate the electric field's magnitude into light energy intensity ***I*** (see ***Feynman Simplified*** 1C Chapter 34, and recall that $\langle\ \rangle$ means time-averaged value):

$\boldsymbol{I} = \varepsilon_0 c \langle \xi^2 \rangle = \varepsilon_0 c\ (\max \xi)^2 / 2$
$\boldsymbol{I} = \varepsilon_0 c \xi_0^{\,2}/2$
$P_{\text{ABSORB}} = \mu^2/(2\varepsilon_0 c\hbar^2)\ \boldsymbol{I}\ T^2\ \text{sinc}^2\phi$

For broad spectrum light, we must integrate the above equation over light's frequency range, using $\boldsymbol{I}(\omega)$ for the intensity at frequency ω.

$\int \boldsymbol{I}(\omega)\ \text{sinc}^2\phi\ d\omega = \int \boldsymbol{I}(\omega)\ \text{sinc}^2\phi\ (d\phi/2T)$

where $\phi = 2(\omega - \Omega)T$.

Here again, the sharp resonance greatly simplifies the integral. Figure 11-6 compares $\text{sinc}^2\phi$ with a typical light spectrum $\boldsymbol{I}(\omega)$.

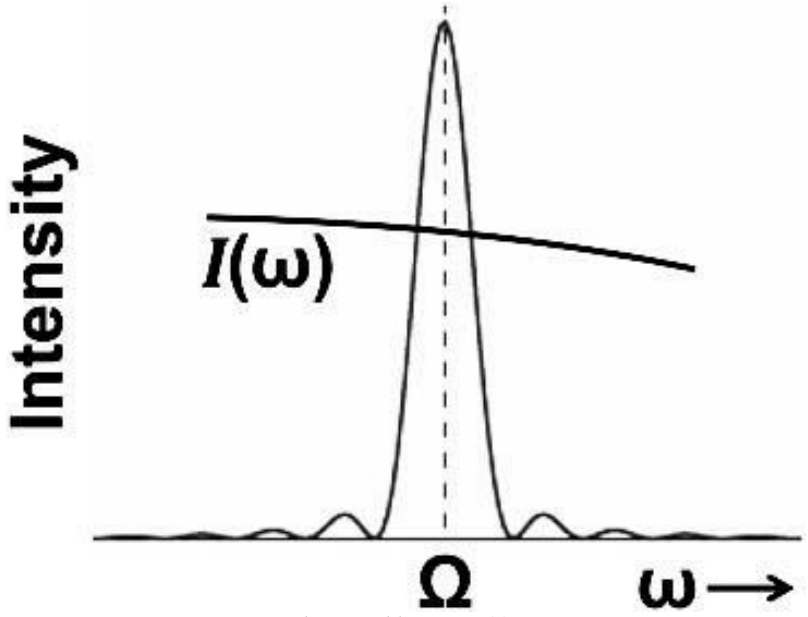

Figure 11-6 Light Spectrum $\boldsymbol{I}(\omega)$ Compared With $\text{Sinc}^2\text{ø}$

The figure shows that $\boldsymbol{I}(\omega)$ changes little across the narrow sinc² peak. In reality, the effect is much more dramatic. For clarity, the sinc² peak shown is much broader than the real peak, whose height to width ratio is 100 million to 1.

This reduces the integration range to an extremely narrow range near Ω:

$$\int \boldsymbol{I}(\omega)\ \text{sinc}^2\text{ø}\ d\omega = \boldsymbol{I}(\Omega)\ (2\pi/T)$$

$$P_{ABSORB} = \pi\mu^2/(\varepsilon_0 c\hbar^2)\ \boldsymbol{I}(\Omega)\ T$$

In V3p9-15, Feynman says this equation is the "*general theory of absorption of light by any molecular or atomic system*." It is valid, he says, regardless of which state, |+> or |–>, has the lower energy.

The transition probability is proportional to: time; light intensity; and the square of the coupling factor between the two states, |+> and |–>. Here that factor is $\boldsymbol{\mu \bullet \xi}$. In general, the coupling factor equals <–|H|+> and is called the *perturbation* term.

Feynman Simplified 1B Chapter 20 explores Einstein's laws of radiation, which include absorption, spontaneous emission, and stimulated emission. P(+ to _) above corresponds exactly, Feynman says, to B_{JK}, the coefficient of absorption and spontaneous emission.

Feynman concludes this portion of Volume 3 saying:

> "Our study of a simple two-state system has thus led us to an understanding of the general problem of the absorption of light."

Chapter 11 Review: Key Ideas

1. An isolated ammonia molecule (NH_3) has two stationary states: $|+\rangle$ with definite energy E_0+A; and $|-\rangle$ with definite energy E_0-A. Here, E_0 is the average energy and $-A$ is the amplitude for the nitrogen atom to "switch" sides of the plane of hydrogen atoms. The transition frequency between the stationary states of an isolated ammonia molecule is $\Omega=2A/\hbar$.

2. In the presence of a constant external electric field ξ, the energy gap between stationary states widens. The energies become E_0+E^* and E_0-E^*, where $E^*=\sqrt{[V^2+A^2]}$, $V=\mu\xi$, and μ is ammonia's electric dipole moment.

3. If an external electric field oscillates with frequency ω and magnitude ξ_0, and $V_0=\mu\xi_0/2$, the two solutions are:

   $C_+ = \Gamma_+ \exp\{E_+t/i\hbar\}$, with $E_+ = E_0+A$
   $C_- = \Gamma_- \exp\{E_-t/i\hbar\}$, with $E_- = E_0-A$

   $\Gamma_+ = -\alpha \exp\{-V_0t/i\hbar\} + \beta \exp\{V_0t/i\hbar\}$
   $\Gamma_- = +\alpha \exp\{-V_0t/i\hbar\} + \beta \exp\{V_0t/i\hbar\}$

   where α and β are arbitrary complex constants. We can choose solutions whose probabilities are:

   $P_+ = \cos^2(V_0t/\hbar)$
   $P_- = \sin^2(V_0t/\hbar)$

4. An ammonia maser operates by matching its cavity frequency ω to the ammonia transition frequency Ω. This drives molecules to emit photons, all with frequency Ω.

5. When a system (an atom or molecule) is exposed to light of frequency ω, with ω near a resonant frequency Ω of that system, the probability of absorption during time interval T is:

   $P_{ABSORB} = \sin^2\text{ø}\ (V_0T/\text{ø}\hbar)^2$

   where $\text{ø} = 2(\omega-\Omega)T$. For light intensity $\boldsymbol{I}(\omega)$ this equals:

   $P_{ABSORB} = \pi\mu^2/(\varepsilon_0c\hbar^2)\ \boldsymbol{I}(\Omega)\ T$

   Feynman says this equation is the "*general theory of absorption of light by any molecular or atomic system.*" It is valid, he says, regardless of which state, $|+\rangle$ or $|-\rangle$, has the lower energy.

   P_{ABSORB} is proportional to the square of the coupling factor between the two states, $|+\rangle$ and $|-\rangle$. Here that factor is $\boldsymbol{\mu\bullet\xi}$. In general, the coupling factor equals $\langle-|H|+\rangle$ and is called the *perturbation* term.

Chapter 12

Review of Quantum Mechanics Part One

Quantum mechanics is the physical theory of elementary particles, how they interact with one another, and how they form atoms and larger structures. Quantum mechanics is truly how the world works: it is the heart of physics. Although many conclusions of quantum mechanics defy our intuition, we now believe we know all its rules and can solve all its equations. In this sense, quantum mechanics is a mystery that we have solved but not fully digested.

Feynman says: *"In the classical limit, the quantum mechanics will agree with Newtonian mechanics."* Planck's constant h sets the scale at which quantum mechanics becomes important, just as the speed of light sets the scale at which special relativity becomes important.

The two key foundational principles of quantum mechanics are:

- Quantization
- Particle-Wave Duality

Quantization is the notion that many things in nature come in integral numbers, like steps on a staircase. In the micro-world, staircases dominate; their steps are large and abrupt. In the macro-world, the steps are so small and so numerous that nature's staircases seem like ramps.

Particle-wave duality states that "particle" and "wave" are just labels for opposite ends of a continuous spectrum, like "black" and "white." Everything in our universe is really a shade of gray. In our macro-world, everything is almost completely black or white. But in the micro-world, gray rules. Classical particles and classical waves do not exist. Both are now replaced with particle-waves.

Particle have wavelengths determined by their momenta: $\lambda = h/p$.

The wave properties of particles preclude simultaneously measuring their position and momentum with unlimited precision. Heisenberg's **Uncertainty Principle** quantifies this in four equations:

$\Delta x \, \Delta p_x \geq \hbar/2$
$\Delta y \, \Delta p_y \geq \hbar/2$
$\Delta z \, \Delta p_z \geq \hbar/2$
$\Delta t \, \Delta E \geq \hbar/2$

where
$\hbar = h/2\pi = 1.055 \times 10^{-34}$ joule-sec, and
$h = 6.626 \times 10^{-34}$ joule-sec is Planck's constant

The original German word "ungenauigkeit" is better translated as "unexactness" or "imprecision", rather than "uncertainty." **The uncertainty principle is not about our ignorance**, but rather about the indefiniteness of nature. Nature is fuzzy, but in a very precise way.

The Uncertainty Principle forces us to abandon the hope of **exactly predicting the future.** Instead, we must resign ourselves to the goal of **exactly predicting the probabilities** of all possible future outcomes.

A **quantum state** defines all the variable properties that entities can have, such as position, momentum, energy, spin, and angular momentum. A quantum state does not define the intrinsic properties that each particle has, such as charge and mass. In general, different types of particles can be put into a given state, and can be moved from one state to another. It is beneficial to think of a quantum state as a vector; it defines a location in the space of all possible properties.

Spin is a form of angular momentum that is intrinsic to each elementary particle. Particle spins are quantized. The primary fermions have spin s = 1/2. Their component of spin along any chosen axis can only be $+\hbar/2$ or $-\hbar/2$, called *spin up* and *spin down*. Bosons have integral spin: s = 0, 1, or 2. Their component of spin along any chosen axis must be: $-s\hbar$, …, 0, …$+s\hbar$. Photons have spin 1, but cannot have a spin component of zero along any axis. Any particle's component of spin along any axis can change only by integer multiples of $\hbar$.

Measurement is a contact sport, more like boxing than dancing. Observation requires interaction. Substantial interactions alter the behavior of what is observed. Every measurement forces the observed entity into a definite state, one with an allowed value for that measurement. Particles have no memory of their prior history. If an atom's spin is measured to be +1 along the z-axis, nothing additional can be known about its spin.

The **Bohr model** of atoms states electron orbits are quantized; the circumference of their orbits must be an integer multiple of their wavelength: $2\pi\ r = n\lambda$. In the smallest orbit, where n=1, electrons cannot radiate, lose energy, and spiral inward. This makes atoms stable, and enables chemical and biological reactions. For n=1, the Bohr radius is: $a_0 = \hbar^2/(me^2) = 0.529 \times 10^{-10}$ m. For a single electron orbiting a nucleus with Z protons, the allowed orbital radii and energies are:

$r_n = n^2\ a_0 / Z$
$E_n = -Ze^2/2r_n = -13.61\ \text{eV}\ Z^2 / n^2$

Since electron energies are quantized, when electrons move from orbit m to orbit n, they must emit or absorb a photon whose energy exactly balances the change in electron energy: $\hbar\omega = (E_m - E_n)$. This means each element emits and absorbs a **unique set of light frequencies**, a spectrum that is more unique than a human fingerprint.

In stable systems subject to gravitational or electrostatic forces, the **virial theorem** states that an

orbiting body's potential energy equals –2 times its kinetic energy.

Feynman enumerates these **general principles** of probabilistic quantum mechanics:

First: the probability P(y) of event y is proportional to the square of the magnitude of probability amplitude ø(y), which is a complex number. If event y can occur in only one way, P(y) is:

$P(y) = øø^* = |ø|^2$

The standard notation for probability amplitudes, due to Paul Dirac, is called the **bra-ket** notation. The format is:

The amplitude that A results in, or goes to, B is <B|A>. This is analogous to the dot product of normal vectors A and B. For the two-slit experiment:

ø(y) = <particle arrives at y | particle leaves source S>
ø(y) = <y|S>

Second: when event y can occur in N undistinguished ways, the amplitude of y equals the sum of the amplitudes for each separate way:

$\langle y|S\rangle = \Sigma_K \langle y|K\rangle\langle K|S\rangle$, sum K=1…N

We sum amplitudes only when each separate path results in exactly the same final state.

In quantum mechanics "distinguish" has a specific meaning. When waves combine after traveling different paths, they interfere if they haven't been substantially disturbed. Substantial disturbances alter a wave's frequency and/or phase sufficiently to randomize its phase angle relative to other waves. That destroys coherence and eliminates the possibility of interference.

As the two-slit experiment demonstrates, with no substantial disturbances, it is impossible — for both man and nature — to *distinguish* which paths particles travel to reach an event y. Paths are *distinguished* only when waves are substantially disturbed. What matters is the magnitude of the disturbance, not whether or not it is human-directed. But due to our scale, any human-directed interaction will almost certainly substantially disturb a quantum system.

Third: when event y can occur in N distinguished ways, the probability of y equals the sum of the probabilities for each separate way:

$P(y) = \Sigma_K |\langle y|K\rangle\langle K|S\rangle|^2$, sum K=1…N

Fourth: the amplitude for a sequence of events equals the product of the amplitudes for each event separately. The amplitude for a particle to go from S to x and from x to y is: <y|x> <x|S>.

Fifth: For multiple particles, if $ø_1$ is the amplitude of particle 1 going from S_1 to F_1, and $ø_2$ is the amplitude of particle 2 going from S_2 to F_2, then the amplitude for both events to occur is $ø_1 \times ø_2$.

Sixth: Identical particle statistics. When two identical particles can enter, exit, or be in the same state, their amplitudes interfere. If the particles are bosons governed by *Bose-Einstein statistics*, their amplitudes add. If the particles are fermions governed by *Fermi-Dirac statistics*, their amplitudes subtract. There is no third alternative. For two identical particles 1 and 2, and any two states A and B, the combined amplitude is:

Fermions: $\langle 1|A\rangle\langle 2|B\rangle - \langle 1|B\rangle\langle 2|A\rangle$

Bosons : $\langle 1|A\rangle\langle 2|B\rangle + \langle 1|B\rangle\langle 2|A\rangle$

Fermions with different spins are not identical.

In the micro-world, all fundamental particles of each type are **intrinsically exactly identical**. Additionally, it seems there is only one way to combine three quarks to make a proton, and only one other way to combine three quarks to make a neutron. While not fundamental, all protons are exactly identical as are all neutrons. Even nature cannot distinguish between identical particles, as proven by interference effects.

The class of fermions includes protons, electrons, neutrons, neutrinos, and quarks. All particles that form material objects are fermions with spin 1/2.

The class of bosons includes photons, gluons, $W^{\pm}$, Z^0, and the Higgs boson. All force-carriers are bosons. Photons are the force exchange particles of electromagnetism. Gluons mediate the strong force. The $W^{\pm}$ and Z^0 *intermediate vector bosons* mediate the weak force. The Higgs boson is credited with providing various amounts of mass to the other elementary particles. (See *Higgs & Bosons & Fermions....Oh My!* to further explore elementary particle physics.)

The probability that N identical bosons are in a common state is larger by a factor of N! than the probability of N non-identical particles being in that state.

The probability of one more boson entering a state occupied by N identical bosons is N+1 times greater than it would be if the state were empty. Bosons are groupies: the bigger the party, the greater the attraction.

The gregarious behavior of bosons underlies Einstein's laws of radiation and the operation of lasers. Gregarious bosons and Einstein's laws of radiation explain Planck's equation for the light intensity, the energy per unit area per second, emitted by a black body of temperature T:

$$I(\omega) = (\hbar\omega^3/\pi^2c^2)/\{\exp(+\hbar\omega/kT) - 1\}.$$

Fermions are antisocial; they never exist in the same state as another identical fermion. This is called the *Pauli Exclusion Principle*. Two fermions are not in the same state if their spins are different. Two electrons cannot have the same atomic orbit unless they have opposite spins. This leads to the rich structure of atoms that enables chemical processes.

Nature has room for both groupies and individualists.

Similarity of Vectors & States

In normal 3-D space, a vector $\boldsymbol{v}$ specifies any selected point within the space.
In quantum mechanics, $|S\rangle$ specifies a *state*, any selected set of particle properties.

In 3-D, we choose a *complete* set of *orthonormal basis vectors* $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$.
In QM, we choose a complete set of orthonormal *basis states* $|J\rangle$, J=1,…,N.

In 3-D, an "orthonormal" basis means for each J and K: $\boldsymbol{e}_J \bullet \boldsymbol{e}_K = \delta_{JK}$.
In QM, orthonormal means for each basis state J and K: $\langle K|J\rangle = \delta_{JK}$. Here δ_{JK} is the *Kronecker delta*, which equals 1 if J=K and equals 0 if J≠K.

In 3-D, a basis is "complete" if every vector in 3-D space is some linear combination of the basis vectors — for any vector v:

$$\boldsymbol{v} = \Sigma_J \, v_J \, \boldsymbol{e}_J = v_x \, \boldsymbol{e}_x + v_y \, \boldsymbol{e}_y + v_z \, \boldsymbol{e}_z$$
$$\boldsymbol{v} = \Sigma_J \, (\boldsymbol{v} \bullet \boldsymbol{e}_J) \, \boldsymbol{e}_J$$

where $v_J = \boldsymbol{v} \bullet \boldsymbol{e}_J$, for J = x, y, z. The v_J are *real* numbers. The set of all v_J completely defines $\boldsymbol{v}$.

In QM, a basis is "complete" if every possible state is some linear combination of basis states — for any state $|\varphi\rangle$:

$$|\varphi\rangle = \Sigma_J \, a_J \, |J\rangle = a_1 \, |1\rangle + a_2 \, |2\rangle + \ldots \, a_N \, |N\rangle$$
$$|\varphi\rangle = \Sigma_J \, \langle J|\varphi\rangle \, |J\rangle$$

where $a_J = \langle J|\varphi\rangle$ are *complex* numbers. The set of all a_J completely defines $|\varphi\rangle$.

Generally there are many possible sets of equally valid basis vectors and basis states. Our choice of basis is often driven by symmetries or by convenience, but is sometimes arbitrary.

In 3-D, the *dot product* of two vectors A and B is:

$$\boldsymbol{A} \bullet \boldsymbol{B} = \Sigma_J \, A_J B_J = A_x B_x + A_y B_y + A_z B_z$$
$$\boldsymbol{A} \bullet \boldsymbol{B} = \Sigma_J \, (\boldsymbol{A} \bullet \boldsymbol{e}_j) \, (\boldsymbol{e}_j \bullet \boldsymbol{B})$$

In QM, the *product* of two states φ and ψ is:

$$\langle \varphi|\psi\rangle = \Sigma_J \, \langle \varphi|J\rangle \, \langle J|\psi\rangle$$

Here:

$\langle \ldots|$ is called a *bra*;
$|\ldots\rangle$ is called a *ket*; and

<…|…> is called a *bra-ket*, following Dirac.

Differences Between Vectors & States

In 3-D, the dot product is commutative: $\boldsymbol{A}\bullet\boldsymbol{B} = \boldsymbol{B}\bullet\boldsymbol{A}$.
But, in QM the product of two states is **not** commutative: $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$, which is the complex conjugate of $\langle\psi|\varphi\rangle$. The complex conjugate of a complex quantity is obtained by replacing i with –i. In general $\langle\psi|\varphi\rangle \neq \langle\varphi|\psi\rangle$.

In 3-D, three orthogonal basis vectors span the entire space, forming a complete basis set.
In QM, a complete set of basis states may be much larger, even infinitely large.

Since $\langle\varphi|\psi\rangle = \Sigma_J \langle\varphi|J\rangle \langle J|\psi\rangle$ for all states φ and ψ, Feynman says the following is the "great law of quantum mechanics":

$| = \Sigma_J |J\rangle \langle J|$

Feynman calls this an *open equation*; it is valid whenever the equation is multiplied by any bra and/or by any ket.

Other Bra-Ket Notes.

Bras and kets are essential intermediate steps in solving problems, but they are not end results by themselves. Calculations of measurable quantities must not end with expressions having unmatched bras or kets.

Bra and ket are complex conjugates: $\langle\psi| = |\psi\rangle^*$. This means $\langle\psi|$ and $|\psi\rangle$ are different, in general. Either could properly represent a quantum state. Feynman says he always uses kets $|\psi\rangle$ to identify states, for consistency.

Operators transform one state vector into another. Any action that changes particle states is represented in quantum mechanics by an operator. As an example:

$\langle\chi|A|\psi\rangle = \Sigma_{JK} \langle\chi|K\rangle \langle K|A|J\rangle \langle J|\psi\rangle$

denotes the product of state χ with the vector resulting from A operating on state ψ, which is shown expanded in basis states. Knowing $\langle K|A|J\rangle$ for all basis states J and K completely determines A.

Operators are often represented as N×N matrices, where N is the number of basis states: matrix component A_{KJ} equals $\langle K|A|J\rangle$. If multiple operators act on a particle sequentially, we express that using the product of those operators. Matrix products are not commutative, so the order of factors is essential. The first operator to act holds the right-most position, with the second immediately on its left, and continuing on to the last operator to act in the left-most position. For two operators, A followed by B, the expression is:

$\langle\chi|BA|\psi\rangle = \Sigma_{JKL} \langle\chi|L\rangle \langle L|B|K\rangle \langle K|A|J\rangle \langle J|\psi\rangle$

For both spin 1/2 and spin 1, and for all spin components, the **general rule for rotations** about the spin axis from basis states S to basis states T is:

$\langle sT|sS\rangle = \exp\{is\theta\}$

Potential Energy. The probability amplitude for a particle of momentum p and energy E (kinetic plus mass), with potential energy V, is proportional to:

$\exp\{(E+V)t/i\hbar - \boldsymbol{p}\bullet\boldsymbol{r}/i\hbar\}$

Barrier Penetration. In a classically forbidden region, where a particle's kinetic energy T is negative, its amplitude is proportional to:

$\phi(x) \sim \exp\{-x/\Lambda\}$

where x is the distance into the forbidden zone, and $\Lambda=\hbar/\sqrt{\{-2mT\}}$.

Barrier penetration enables radioactive decay and nuclear fusion, and is employed in most modern microelectronics.

The **Hamiltonian equation** governs the time evolution of states:

$i\hbar\, d|\Psi(t)\rangle/dt = H\,|\Psi(t)\rangle$

For a system with two states |1> and |2>, the Hamiltonian equation yields two independent differential equations:

$i\hbar\, d(C_1)/dt = H_{11}\, C_1 + H_{12}\, C_2$
$i\hbar\, d(C_2)/dt = H_{21}\, C_1 + H_{22}\, C_2$

For a symmetric system with $E_0=H_{11}=H_{22}$ and $-A=H_{12}=H_{21}$, the solutions are:

$C_1(t) = \exp\{E_0 t/i\hbar\}\ [+\cos(At/\hbar)]$
$C_2(t) = \exp\{E_0 t/i\hbar\}\ [-\sin(At/\hbar)]$

$|\psi-\rangle = \exp\{(E_0-A)t/i\hbar\}\ (|1\rangle + |2\rangle)/\sqrt{2}$
$|\psi+\rangle = \exp\{(E_0+A)t/i\hbar\}\ (|1\rangle - |2\rangle)/\sqrt{2}$

$|\psi-\rangle$ is a stationary state of definite energy $E_-=E_0-A$.
$|\psi+\rangle$ is a stationary state of definite energy $E_+=E_0+A$.

C_1 and C_2 are not stationary states; they are coupled, with probabilities "sloshing back and forth" between them.

An isolated ammonia molecule, NH_3, has two stationary states: $|+\rangle$ with definite energy E_0+A; and $|-\rangle$

with definite energy E_0–A. Here, E_0 is the average energy and –A is the amplitude for the nitrogen atom to "switch" sides of the plane of hydrogen atoms. The transition frequency between the stationary states of an isolated ammonia molecule is $\Omega=2A/\hbar$.

In the presence of a constant external electric field $\boldsymbol{\xi}$, the energy gap between stationary states widens. The two energies become E_0+E^* and E_0-E^*, where $E^*=\sqrt{[V^2+A^2]}$, $\boldsymbol{V=\mu\bullet\xi}$, and $\boldsymbol{\mu}$ is ammonia's electric dipole moment.

If an external electric field oscillates with frequency ω and magnitude ξ_0, and $V_0=\mu\xi_0/2$, the two solutions are:

$C_+ = \Gamma_+ \exp\{E_+t/i\hbar\}$, with $E_+ = E_0+A$
$C_- = \Gamma_- \exp\{E_-t/i\hbar\}$, with $E_- = E_0-A$, with

$\Gamma_+ = -\alpha \exp\{-V_0t/i\hbar\} + \beta \exp\{+V_0t/i\hbar\}$
$\Gamma_- = +\alpha \exp\{-V_0t/i\hbar\} + \beta \exp\{+V_0t/i\hbar\}$

where α and β are arbitrary complex constants. We can choose solutions whose probabilities are:

$P_+ = \cos^2(V_0t/\hbar)$
$P_- = \sin^2(V_0t/\hbar)$

An ammonia maser matches its cavity frequency ω to the ammonia transition frequency Ω, thereby driving molecules to emit photons, all with frequency Ω.

When a system (atom or molecule) is exposed to light whose frequency ω is near a resonant frequency Ω of that system, the probability of absorption during time interval T is:

$P_{ABSORB} = \sin^2\text{ø}\ (V_0T/\text{ø}\hbar)^2$

where $\text{ø} = 2(\omega-\Omega)T$.

For light intensity $\boldsymbol{I}(\Omega)$ this equals:

$P_{ABSORB} = \pi\mu^2/(\varepsilon_0c\hbar^2)\ \boldsymbol{I}(\Omega)\ T$

Feynman says this equation is the "*general theory of absorption of light by any molecular or atomic system.*" It is valid, he says, regardless of which state, |+> or |–>, has the lower energy.

P_{ABSORB} is proportional to the square of the coupling factor between states |+> and |–>. Here that factor is $\boldsymbol{\mu\bullet\xi}$. In general, the coupling factor equals <–|H|+> and is called the *perturbation* term.

Meet The Author

Congratulations and thank you for reading my book. I know your time is valuable, and I sincerely hope you enjoyed this experience.

I'd like to tell you something about myself and share some stories.

First, the obligatory bio (as if 3 "tweets"-worth can define anyone): I have a B.S. in physics from Caltech, a Ph.D. in high-energy particle physics from Stanford University, and was on the faculty of Harvard University. Now "retired," I teach at the Osher Institutes at UCLA and CSUCI, where students honored me as "Teacher of the Year." In between, I ran eight high-tech companies and hold patents in medical, semiconductor, and energy technologies.

My goal is to help more people appreciate and enjoy science. We all know one doesn't have to be a world-class musician to appreciate great music — all of us can do that. I believe the same is true for science — everyone can enjoy the exciting discoveries and intriguing mysteries of our universe.

I've given 400+ presentations to general audiences of all ages and backgrounds, and have written 3 printed books and 29 eBooks. My books have won national and international competitions, and are among the highest rated physics books on Amazon.com. I'm delighted that two of these recently became the 2nd and 3rd best sellers in their fields.

Richard Feynman was a friend and colleague of my father, Oreste Piccioni, so I knew him well before entering Caltech. On several occasions, Feynman drove from Pasadena to San Diego to sail on our small boat and have dinner at our home. Feynman, my father, my brother and I once went to the movies to see *"Dr. Strangelove or: How I Learned to Stop Worrying and Love the Bomb."* It was particularly poignant watching this movie next to one of the Manhattan Project's key physicists.

At Caltech I was privileged to learn physics directly from this greatest scientist of our age. I absorbed all I could. His style and enthusiasm were as important as the facts and equations. Top professors typically teach only upper-level graduate classes. But Feynman realized traditional introductory physics didn't well prepare students for modern physics. He thought even beginners should be exposed to relativity, quantum mechanics, and particles physics. So he created a whole new curriculum and personally taught freshman and sophomore physics in the academic years 1961-62 and 1962-63.

The best students thrived on a cornucopia of exciting frontier science, but many others did not. Although Caltech may be the world's most selective science school, about half its elite and eager students drowned in Feynman's class. Even a classmate, who decades later received the Nobel Prize in Physics, struggled in this class. Feynman once told me that students sometimes gave him the "stink eye" — he added: "Me thinks he didn't understand angular momentum."

Some mundane factors made the class very tough: Feynman's book wasn't written yet; class notes

came out many weeks late; and traditional helpers (teaching assistants and upper classmen) didn't understand physics the way Feynman taught it.

But the biggest problem was that so much challenging material flew by so quickly. Like most elite scientists, Feynman's teaching mission was to inspire the one or two students who might become leading physicists of the next generation. He said in his preface that he was surprised and delighted that 10% of the class did very well.

My goal is to reach the other 90%.

It's a great shame that so many had so much difficulty with the original course — there is so much great science to enjoy. I hope to help change that and bring Feynman's genius to a wider audience.

Please let me know how I can make ***Feynman Simplified*** even better — contact me through my WEBSITE.

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