INTERDISCIPLINARY MATHEMATICS

VOLUME VI

TOPICS IN THE MATHEMATICS OF QUANTUM MECHANICS

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MATH SCI PRESS
53 JORDAN ROAD
BROOKLINE, MASSACHUSETTS

Second Printing, 1977

MATH SCI PRESS
53 JORDAN ROAD
BROOKLINE, MASSACHUSETTS

PRINTED IN THE UNITED STATES OF AMERICA

ISBN: 0915692058

PREFACE

As the title indicates, this volume proceeds into geometric material that is, in some sense, relevant to the study of quantum mechanics. I have already extensively presented material along these lines in my earlier books, but the subject is vast, and ever-growing! Note, for example, the connection to such topics in mathematics as Lie group representation theory, global analysis, functional analysis, symplectic manifolds (and via this route with number theory, automorphic function theory, Kähler manifolds, algebraic geometry), and the theory of partial differential equations!

A comprehensive treatise on the mathematics of quantum mechanics is obviously needed. One such is being written by P. Chernoff and J. Marsden. Right now, the book by Prugovecki [1] is a good standard reference. Thus, one of my main purposes in this work is to provide to the scientific world material which will feed into this goal. In addition, I will cover many topics which interest me personally, and some of my own research is included here.

PREFACE TO THE SECOND PRINTING

MARCH 1977

This volume contains a good deal of original material, which probably (at least judging by reviews) is not particularly well known even now. I am convinced that understanding quantum mechanics is not as simple and straightforward a matter as the physicists would have it--and differential quantum-Lie theoretic problems are a good deal closer to the real questions than much of the standard, functional-analysis oriented material. This concern for developing the differential geometric thread is the unifying thread through the diverse topics treated here! Most of them form an elaboration of certain material that is done only crudely or in special situations in the physics literature.

In particular, Chapter 4 on "current algebras" is probably the most significant for further development. I started work on this topic at the suggestion of Murray Gell-Mann. He thought that a direct analysis of the way these objects appeared in quantum mechanics would lead to interesting properties of elementary particles, just as the study of skew-Hermitian representations of finite dimensional Lie algebras leads to useful information in atomic and nuclear physics. Alas, this turned out to be too vain a hope. On the one hand, even the mathematical theory of these objects--which are in fact infinite dimensional Lie algebras--is very difficult (not too much non-trivial information is known, certainly nothing comparable to what is known about finite dimensional Lie algebras!) and, on the other hand, every attempt to explicate how these Lie algebras may appear in interesing physical situations led right back to quantum field theory and all its difficulties. However, there are certainly important objects--no doubt both the mathematical and physical world will be studying them some day--and I have written out for future reference a few facts about them. Note particularly the material in Sections 9-11: After constructing "groups" whose "Lie algebras" are the "current algebras", I believe it is clearly perceived to be possible to construct at least some useful unitary representation (which are needed for quantum mechanical purposes) by generalizing methods of the theory of induced representations-vector bundles.

I regret that there is still no even remotely adequate book on the mathematics of quantum mechanics. (Apparently, the Chernoff-Marsden effort mentioned in the original Preface did not materialize.) Quantum Mechanics is a marvelous subject, with tentacles reaching into many parts of mathematics. (The faithful readers of my books will, no doubt, be aware that I am not happy with recent attempts to develop it mathematically from a strict functional analysis point of view.) There are tremendous and fascinating possibilities (basically still unrealized) for rewarding collaboration between mathematicians and physicists.

Finally, I might mention that I have written a short introduction to some of the ideas of quantum mechanics as an Appendix to Nolan Wallach's Symplectic Geometry and Fourier Analysis, the fifth volume of the companion series, Lie Groups: History, Frontiers and Applications.

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Chapter I

POISSON BRACKET SPACES

1. INTRODUCTION

In this chapter we shall investigate a circle of questions relating classical and quantum mechanics, Lie group theory and differential geometry.

Recall from LAQM the following general setting for problems in classical mechanics. Let M be a manifold. A Poisson bracket structure on M is defined by giving the following data:

a) An algebra F of C^∞ real-valued functions on M.

b) A Lie algebra structure $\{,\}: F\times F \to F$ such that

$$\{f_1,f_2f_3\} = \{f_1,f_2\}f_3+f_2\{f_1,f_3\}$$

for $f_1,f_2,f_3 \in F$.

An automorphism of the structure is a linear map $A: F \to F$ such that

$$A(\{f_1,f_2\}) = \{A(f_1),A(f_2)\}$$
$$A(f_1f_2) = A(f_1)A(f_2)$$

An infinitesimal automorphism is a linear map $X: F \to F$ such that

$$X(\{f_1,f_2\}) = \{X(f_1),f_2\} + \{f_1,X(f_2)\}.$$
$$X(f_1f_2) = X(f_1)f_2 + f_1X(f_2).$$

An infinitesimal automorphism X is inner if there is an $h \in F$ such that

$$X(f) = \{h,f\} \qquad 1.1$$

for all $f \in F$.

A one-parameter group $t \to A_t$ of automorphisms has X as <u>infinitesimal generator</u> if:

$$\frac{\partial}{\partial t}(A_t(f)) = X(A_t(f)) \qquad 1.2$$

for all $f \in F$, $-\infty < t < \infty$.

If X is inner, then 1.2 takes the following form:

$$\frac{\partial}{\partial t}(A_t(f)) = \{h, A_t(f)\} \qquad 1.3$$

1.3 is then a general form of Hamilton's equations of classical mechanics, with h the "Hamiltonian".

A good deal of "mechanics" - both classical and quantum - is involved in the study of such structures. A standard way of defining a Poisson bracket structure on M is to be given a closed, 2 differential form ω on M of constant rank. As in LAQM, we can then let F be the space of functions f on M which are constant on the characteristic curves of ω. The Poisson bracket of two such functions is then defined as follows:

$$\{f_1, f_2\} = X_{f_1}(f_2), \qquad 1.4$$

where X_{f_1} is a vector field such that:

$$df_1 = X_{f_1} \lrcorner\, \omega \qquad 1.5$$

The automorphisms of the Poisson bracket structure are then obviously tied in with the automorphisms of the form ω. Accordingly, most of the work in this chapter shall be concerned with the study of groups of automorphisms (and Lie algebras of infinitesimal automorphisms) of closed differential forms, independently of the question of automorphisms of Poisson bracket structures. In turn, this will lead us to study such standard differential-geometric objects as "complex analytic manifolds" and "Kähler manifolds". Such material has great independent interest and importance in mathematics, of course. Accordingly, we shall take some space to describe their most important properties. We shall begin with a more leisurely exposition of the theory of complex manifolds than was presented in DGCV.

2. COMPLEX STRUCTURES ON THE REAL VECTOR SPACE

Let V be a vector space with the complex numbers as field of scalars. By restricting the field of scalars down to the real numbers, it can also be considered as a real vector space. How does one recognize in terms of this real vector space structure that the complex structure was originally present? There are essentially two ways of answering this question.

The first works only with V itself. Turning things around, we know that V is a real vector in which one knows in <u>addition</u> how to multiply a $v \in V$ by a complex number. In particular, let us denote the product of a $v \in V$ and the particular complex number $i = \sqrt{-1}$ by: $J(v)$. Then, the mapping $v \to J(v)$ is an <u>R-linear</u> mapping: $V \to V$ such that:

$$J^2 = (\text{identity}) \qquad 2.1$$

Conversely, if one is given a real vector space V, together with an R-linear map $J: V \to V$ satisfying 2.1, one can make V into a complex vector space by using the following formula:

$$(a+bi)(v) = av+bJ(v) \qquad 2.2$$

for $a,b \in R$.

<u>Exercise</u> Show that 2.2 defines V as a genuine complex vector space.

We can put these remarks together to give the following result.

<u>Theorem 2.1</u>. Given a real vector space V, there is a one-one correspondence between complex vector space structures on V which reduce - when the field of scalars is restricted from C to R - to the given real structure and R-linear maps $J: V \to V$ which satisfy 2.1.

Now for an alternate approach. Suppose that V is a real vector space. Construct its "complexification", $V' = V \otimes C$. (Here, "C" is regarded as a real vector space, so that the tensor product is considered as the tensor product of real vector spaces.) Now, V' can be regarded as a complex vector space, using the following formula:

$$c(v \otimes c_1) = v \otimes (cc_1) \qquad 2.3$$

$$\text{for } c, c_1 \in C,\ v \in V.$$

Since $C = R \oplus iR$, we can also write:

$$V' = V \oplus iV, \qquad 2.4$$

with the complex structure on V' defined by 2.3 given by the following more explicit formula:

$$(a+bi)(v_1 \oplus iv_2) = (av_1 - bv_2) \oplus i(bv_1 + av_2) \qquad 2.5$$

$$\text{for } a,b \in R,\ v_1, v_2 \in V$$

We can also define an R-linear map $v' \to v'^*$ on V', called <u>complex conjugation</u>, in the following way:

$$(v_1+iv_2)^* = v_1-iv_2 \qquad 2.6$$

for $v_1, v_2 \in V$.

Thus, V may - as a real vector space - be identified with the space of elements $v \in V'$ such that $v^* = v$.

Now, suppose that V _itself_ has a complex structure, defined, say, by an R-linear map $J: V \to V$ satisfying 2.1. Extend J to be a complex linear map: $V' \to V'$, using the following formula:

$$J(v_1+iv_2) = J(v_1) + iJ(v_2) \qquad 2.7$$

for $v_1, v_2 \in V$

Thus extended J again satisfies 2.1. Now, because of 2.1, the eigenvalues of J are $\pm i$. Let V_+ and V_- be the space of eigenvectors of J corresponding to the eigenvalues $+i$ and $-i$.

Exercise Show that:

$$V_+ = \{v-iJ(v): v \in V\}$$
$$V_- = \{v+iJ(v): v \in V\} \qquad 2.8$$

V' is a direct sum (as a complex vector space) of the complex vector spaces V_+ and V_-. 2.9

$$(V_+)^* = V_- \qquad 2.10$$

Conversely, suppose we do not assume that such a J exists, but that $V' = V \otimes C$ is decomposed into a direct sum of complex vector spaces V_+' and V' which are related via 2.10.

Exercise Show that there is a $J: V \to V$ satisfying 2.1 such that V_+, V_- take the form 2.8.

After these exercises are done, the reader will recognize that the following result has been proved.

Theorem 2.2. Given a real vector space V, there is a one-one correspondence between complex structures on V which reduce to the given real structure and direct sum decompositions $V' = V \otimes C = V_+ \oplus V_-$ of V into complex subspaces which go into each other via complex-conjugation, i.e. which satisfy 2.10.

So far, we have worked on a given vector space. Now, let us apply these simple algebraic remarks to the case of vector spaces which are the tangent spaces to a manifold.

3. COMPLEX STRUCTURES ON MANIFOLDS

Let M be a manifold. It has been explained in Chapter 32 of DGCV what is meant by saying that M has a structure of "complex analytic" manifold. We require that M be covered by an atlas of coordinate patches, in each of which there is a diffeomorphism with an open subset of C^n, such that in the overlap of two such coordinate patches the different ways of labelling points of M by complex numbers are related by complex - analytic functions in the usual sense. Now, an "almost complex manifold" only has this property "up to the first order". In other words, each point p of M is contained in a family of coordinate patches, in each of which there is a diffeomorphism with an open subset of C^n, such that the different ways of labelling p by complex numbers are related by transformations which are complex analytic only "up to the first order".

This, at any rate, is the intuitive idea of "almost complex structure". Luckily, the formalism of manifold theory enables us to give a more precise technical definition.

Definition. An almost complex structure on the manifold M is defined by a field $p \to J_p$ of linear transformations $J_p: M_p \to M_p$, one given for each $p \in M$, which varies smoothly (i.e. in a C^∞ way) with p, such that:

$$J_p^2 = (\text{identity}) \qquad 3.1$$

for each $p \in M$.

As explained in Section 2, such a structure enables us to define a structure of a complex vector space on the real tangent space M_p to each point $p \in M$. To see the relation to the "intuitive" idea of almost complex structure defined above, we may remark that, intuitively, the tangent space M_p to a point $p \in M$ can be used to represent points of M near p "up to the first order". A consistent way of introducing a complex structure in each such tangent space then enables up to regard points of M as labelled by complex-analytic parameters "up to the first order".

Such a field $p \to J_p$ of complex-structures for the tangent spaces to M can also be defined in terms of $F(M)$-module language.

Alternative definition. An almost complex structure for M is defined by an $F(M)$-linear map $J: V(M) \to V(M)$ such that:

$$J^2 = -(\text{identity}) \qquad 3.2$$

The relation between the two sorts of J's is given as follows:

$$J_p(X(p)) = J(X)(p) \qquad 3.3$$

for $p \in M$, $X \in V(M)$.

Because it is technically more convenient, we will work here with the F(M)-module version of J.

Now, the alternative definition of complex structure on a real vector space - as a decomposition of $V \otimes C$ into complex subspaces V_+, V_- - suggests an alternative way of regarding J. Set:

$$V_c(M) = V(M) \otimes C \qquad 3.4$$

Again, since $C = R \oplus iR$, $V_c(M)$ can be regarded as a direct sum:

$$V_c(M) = V(M) \oplus iV(M) \qquad 3.5$$

In turn, 3.5 suggests an interpretation of $V_c(M)$ as the "complex-valued vector fieldson M".

<u>Exercise</u>. Let $F_c(M)$ be the complex-valued, C^∞ functions on M. Show that $V_c(M)$ can be identified with the space of derivations of $F_c(M)$.

Given an F(M)-linear map $J: V(M) \to V(M)$ satisfying 3.2, extend it to an $F_c(M)$-linear map, $J: V_c(M) \to V_c(M)$, as follows:

$$J(X+iY) = JX+iJY \qquad 3.6$$

for $X,Y \in V(M)$

Set:

$$V_c^+(M) = \{Z \in V_c(M): J(Z) = iZ\} \qquad 3.7$$

$$V_c^-(M) = \{Z \in V_c(M): J(Z) = -iZ\} \qquad 3.8$$

<u>Exercise</u>. Show that:

$$V_c^{+}(M) = \{X - iJ(X) : X \in V(M)\} \qquad 3.9$$

$$V_c^{-}(M) = \{X + iJ(X) : X \in V(M)\} \qquad 3.10$$

Now, we want a way of recognizing when an "almost complex structure" arises from a "complex analytic structure". Technically, it is most convenient to proceed as follows:

Definition. An almost complex structure on M, defined by an $F(M)$-linear map $J: V(M) \to V(M)$, is said to be locally flat if each point $p \in M$ lies in a coordinate patch, with a real coordinate system $(x_1,\dots,x_n, y_1,\dots,y_n)$ of functions such that:

$$J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}$$
$$J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j} \qquad 3.11$$

for $1 \leq j \leq n$.

Exercise. Suppose that $(x_1',\dots,y_n')$ is another coordinate system valid in the patch, which also satisfies 2.11. Set:

$$z_j' = x_j' + iy_j' \in F_c(M)$$

$$j = 1,\dots,n,$$

Show that the $(z_1',\dots,z_n')$ are given in terms of complex analytic functions of the $(z_1,\dots,z_n)$, with $z_j = x_j + iy_j$.

Using this exercise, we can assign a complex analytic structure on M to each locally flat almost complex structure. Namely, to each coordinate system $(x_1,\dots,y_n)$ satisfying 2.11, define $z_j = x_j + iy_j \in F_c(M)$, and the diffeomorphism

$p \to (z_1(p),\ldots,z_n(p))$ of the patch with an open subset of C^n. The result of the exercise guarantees that, in the overlap of two such patches, the complex coordinates are related by complex-analytic functions. This is precisely what is required to define a "complex analytic structure" for M. Summing up, we have proved:

Theorem 3.1. To such a locally flat almost complex structure one can assign a structure of complex analytic manifold on M. Conversely, each structure of complex analytic manifold in M arises in this way from a unique almost complex structure on M.

Now, the following idea of "integrable" structure is, in principle, different from "locally flat".

Definition. The almost complex structure is integrable if:

$$[V_c^{+}(M),\ V_c^{+}(M)] \subset V_c^{+}(M). \qquad 3.12$$

The conditions 3.13 are called the integrability conditions. Using 3.9, they can be rewritten in a more convenient form, using only $V(M)$.

Theorem 3.2. Conditions 3.12 are satisfied if and only if:

$$[X,Y]+J[JX,Y]+J[X,JY]-[JX,JY] = 0 \qquad 3.13$$

$$\text{for } X,Y \in V(M).$$

Proof Define a map $\pi: V_c(M) \to V_c(M)$ as follows:

$$\pi(Z) = Z+iJ(Z).$$

Notice that:

$$\pi(Z) = 0 \text{ if and only if } Z \in V_c^{+}(M).$$

Using 2.9, 2.12 takes the following form:

$$\pi([X-iJX, Y-iJY]) = 0 \qquad 3.14$$

for $X,Y \in V(M)$.

But, the left hand side of 3.14 works out as follows:

$$\pi([X,Y]-[JX,JY]$$
$$-i[JX,Y]-i[X,JY])$$
$$= [X,Y]-[JX,JY]-i[JX,Y]-i[X,JY]$$
$$+ iJ[X,Y]-iJ[JX,JY]+J[JX,Y]+J[X,JY]$$

Setting this equal to zero, and equating the real and imaginary parts gives 3.13.

Exercise. Given an almost complex structure, defined by an $F(M)$-linear map $J\colon V(M) \to V(M)$, set:

$$T(X,Y) = [X,Y]-[JX,JY]+J[X,JY] + J[JX,Y] \qquad 3.15$$

Show that T is an $F(M)$-linear map: $V(M)\times V(M) \to V(M)$. By the general principles of differential geometry, such a map defines a tensor field on M. Its vanishing is then necessary and sufficient that the almost complex structure be integrable. It is called the integrability tensor.

Theorem 3.3. If an almost complex structure is locally flat, then it is integrable.

Proof. Since T defined by 3.14 is $F(M)$-linear, to prove that it vanishes it suffices to show that it vanishes for any choice of $F(M)$-module basis of $V(M)$. But, a coordinate system $(x_1,\dots,y_n)$ satisfying 3.11 provides such a basis, since, because of 3.11, each term on the right hand side of 3.15

vanishes for $X,Y = \frac{\partial}{\partial x_j}$ or $\frac{\partial}{\partial y_j}$.

The converse of this is much harder (for C^∞ structures) to prove, and was first given by Newlander and Nirenberg [1]. We shall state it without proof.

Theorem 3.4. An integrable almost complex structure is locally flat. Thus, in view of Theorem 3.1, the integrable structures correspond to the complex-analytic manifold structures on M.

Exercise. There is an alternate way of describing the integrability condition 3.13, in terms of differential forms instead of vector fields. Let $F_c^r(M) = F^r(M)\otimes C = F^r(M)\oplus iF^r(M)$ be the complex r-forms on M. The decomposition $V_c(M) = V_c^+(M)\oplus V_c^-(M)$ defined by 3.7, 3.8 defines a similiar decomposition

$$F_c^1(M) = F_c^{1,0}(M) \oplus F_c^{0,1}(M)$$

($F_c^{1,0}(M)$ is the space of complex-valued differential 1 forms ω such that

$$\omega(V_c^-(M)) = 0.)$$

This decomposition defines a decomposition

$$F_c^r(M) = \sum_{s_1+s_2=r} F_c^{s_1,s_2}(M)$$

of r-forms into forms of type (s_1,s_2). (For example, a 2-form is in $F_c^{1,1}(M)$ if it is the sum of exterior products of forms in $F_c^{0,1}(M)$ and $F_c^{(1,0)}(M)$). Show that there is an operator $\partial\colon F_c^{s_1,s_2}(M) \to F_c^{s_1+1,s_2}(M)$ such that:

$$d = \partial + \partial^*.$$

(∂^* is the operator $\omega \to (\partial\omega^*)^*$, where $*$ denotes complex-conjugation). Show also that the integrability conditions 3.13 are satisfied if and only if:

$$\partial^2 = 0$$

4. INVARIANT COMPLEX STRUCTURES ON MANIFOLDS

Suppose now that M is a coset space G/L of a connected Lie group G, with L a closed subgroup of G. In this section, we will investigate almost complex and complex structures on M with regard to which G acts as a group of automorphisms.

Let $\underset{\sim}{G}$ be the Lie algebra of G, considered as a Lie subalgebra of $V(M)$. Let $\underset{\sim}{L}$ be the subalgebra of $\underset{\sim}{G}$ corresponding to the subgroup L. Let J be the $F(M)$-linear map: $V(M) \to V(M)$ defining an almost complex structure on M.

Exercise. Show that G acts as a group of automorphisms of the almost complex structure determined by J (that is, each $g \in G$ leaves invariant the tensor field on M determined by J) if and only if:

$$[X,J(Y)] = J([X,Y]) \qquad 4.1$$

for all $Y \in V(M)$, all $X \in \underset{\sim}{G}$.

Now, let p be the identity coset of $M = G/L$. Then, the map $X \to X(p)$ identifies $\underset{\sim}{G}/\underset{\sim}{L}$ with M_p. The value of J at p maps M_p into itself, hence carries over to a linear map, that we again denote by J, of $\underset{\sim}{G}/\underset{\sim}{L} \to \underset{\sim}{G}/\underset{\sim}{L}$. Condition 3.1 carries over to the following condition

$$\text{Ad } X(J(v)) = J \text{ Ad } X(v) \qquad 4.2$$

for $X \in L$, $v \in \underset{\sim}{G}/\underset{\sim}{L}$.

Now, let us investigate the conditions that the almost complex structure be integrable. Let T be the integrability tensor, defined by 3.14. Since it too is obviously invariant under G, to prove that the structure is integrable it suffices to prove that T vanishes at p. To facilitate writing this condition, let us suppose that the map $J: \underset{\sim}{G}/\underset{\sim}{L} \to \underset{\sim}{G}/\underset{\sim}{L}$ arises from a map $J': \underset{\sim}{G} \to \underset{\sim}{G}$ such that:

$$J'(\underset{\sim}{L}) \subset \underset{\sim}{L} \qquad 4.3$$

$$(J'^2+1)(\underset{\sim}{G}) \subset \underset{\sim}{L} \qquad 4.4$$

(In other words, we require that J be the map induced on the quotient vector space by J'). Then, the condition that $T = 0$ can be written as follows:

$$[X,Y]-[J'X,J'Y]+J'[X,J'Y] + J'[J'X,Y] \in \underset{\sim}{L} \qquad 4.5$$

for $X,Y \in \underset{\sim}{G}$.

In summary, conditions 4.2-4.5 are the conditions that G/L admit on invariant complex structure.

We will not attempt a general classification of these conditions. However, we will discuss a few simple situations where one can indeed analyze these conditions in a relatively simple way.

First, let us discuss the case where G and L are "complex" Lie groups, i.e. where their Lie algebras have the structure of complex vector spaces, with respect to which the Lie bracket operation is complex bilinear. The conditions for this are readily derived following the linear algebra pattern discussed in Section 2.

Exercise. Let $\underset{\sim}{G}$ be a real Lie algebra. Show that the complex vector space structures on $\underset{\sim}{G}$ such that it becomes a Lie algebra over the complex numbers are in one-ont correspondence with the real linear maps j: $\underset{\sim}{G} \to \underset{\sim}{G}$ which satisfy the following conditions:

$$j^2 = -1$$

$$j([X,Y]) = [j(X),Y] \qquad 4.6$$

for $X,Y \in \underset{\sim}{G}$.

Given such a complex Lie algebra structure for $\underset{\sim}{G}$, show that a real subalgebra $\underset{\sim}{L}$ is also a complex subalgebra if and only if:

$$j(\underset{\sim}{L}) \subset \underset{\sim}{L}. \qquad 4.7$$

Now, let us suppose $M = G/L$ is such that $\underset{\sim}{G}$ is such a complex Lie algebra, with $\underset{\sim}{L}$ a complex subalgebra. Let j be the map: $\underset{\sim}{G} \to \underset{\sim}{G}$ satisfying 4.6-4.8. Set:

$$J' = j \qquad 4.8$$

Then, we see that conditions 4.4-4.5 are satisfied, i.e. the invariant almost complex structure defined on G/L by J (which is the quotient J' induces in $\underset{\sim}{G}/\underset{\sim}{L}$) is integrable. This proves the following rather trivial result:

Theorem 4.1. Let G be a connected, complex Lie group (i.e., a

Lie group whose Lie algebra has a structure of complex Lie algebra), and let L be a connected complex subgroup. Then, the coset space $M = G/L$ has a G-invariant complex manifold structure.

Now, it is less trivial that there are Lie groups G which are not complex groups, but which have subgroups L such that G/L has a G-invariant complex structure. We will only investigate the conditions for this in case the map J' satisfies a further condition, namely:

$$J' \text{ is a derivation of } \underset{\sim}{G}, \text{ i.e.}$$

$$J'([X,Y] = [J'X,Y]+[X,JY] \qquad 4.9$$

$$\text{for } X,Y \in \underset{\sim}{G}.$$

<u>Theorem 4.2</u>. Suppose that $\underset{\sim}{G}$ is a direct sum (as a vector space) of $\underset{\sim}{L}$ and a linear subspace $\underset{\sim}{M}$, and that J' is a derivation: $\underset{\sim}{G} \to \underset{\sim}{G}$ such that:

$$J'(\underset{\sim}{L}) = 0 \qquad 4.10$$

$$J'^2(X) = X \quad \text{for} \quad X \in \underset{\sim}{M} \qquad 4.11$$

Suppose also that:

$$[J'X,J'Y]-[X,Y] \in \underset{\sim}{L} \qquad 4.12$$

$$\text{for } X,Y \in \underset{\sim}{M}.$$

Let $M = G/L$, with L a connected subgroup of G where Lie algebra is $\underset{\sim}{L}$. Then J' defines a G-invariant complex structure on M.

For the proof, one has only to notice that 4.5, 4.9, and 4.11 combine to give condition 4.12.

This result is most useful in case $\underset{\sim}{L}$ is a symmetric subalgebra of $\underset{\sim}{G}$, i.e. $\underset{\sim}{M}$ can be chosen so that:

$$[\underset{\sim}{M},\underset{\sim}{M}] \subset \underset{\sim}{L} \tag{4.13}$$

$$J'(\underset{\sim}{M}) \subset \underset{\sim}{M} \tag{4.14}$$

In this case, 4.12 is obviously satisfied. These symmetric spaces $M = G/L$ which admit invariant complex structures - which were first studied by E. Cartan - are very important in various branches of mathematics, e.g. algebraic geometry and the theory of automorphic functions. Further discussion can be found in the book by Helgason [1].

5. KÄHLER MANIFOLDS

In order to present the idea of a "Kähler manifold", let us return to the study of a general manifold M, without assumptions about possible homogenity of M. Suppose given a Riemannian metric on M, i.e. an $F(M)$-linear map $\beta: V(M)\times V(M) \to F(M)$ which is non-degenerate and symmetric, i.e.

$$\beta(X,V(M)) = 0 \quad \text{implies} \quad X = 0,$$

Recall that a <u>complex structure</u> on M is defined by an $F(M)$-linear map $J: V(M) \to V(M)$ which satisfies: $J^2 = -1$, and the integrability condition 3.13.

<u>Definition</u>. A <u>Kähler structure</u> on the manifold M is a pair (β,J) of a Riemannian and complex structure on M, which satisfy the following conditions:

$$\beta(X,JY) = -\beta(JX,Y) \tag{5.1}$$

$$\text{for} \quad X,Y \in V(M)$$

$$\nabla_X(JY) = J(\nabla_X Y) \qquad 5.2$$

for $X,Y \in V(M)$

In 5.2, ∇ denotes the torsion-free affine connection associated with the metric β. (See DGCV, p. 273. This connection is called the Riemannian connection). Condition 5.2 means that the covariant derivative of the J-tensor with respect to this affine connection is zero.

Now, conditions 5.1 and 5.2 are independent of each other. In order to study the interrelation between these two conditions, it is convenient to give a separate name to the structures which only satisfy 5.1.

Definition. A Hermitian structure on a manifold M is a pair (β,J) of Riemannian and complex structures which satisfy condition 5.1 only.

Given such a Hermitian structure, one can define a 2-differential form ω by the following formula:

$$\omega(X,Y) = \beta(X,JY) \qquad 5.3$$

for $X,Y \in V(M)$.

ω is called the fundamental 2-form of the Hermitian structure. Notice that ω is a 2-form of maximal rank, i.e. $\omega(X,V(M)) = 0$ implies $X = 0$. However, ω is not necessarily a closed form. In fact, a basic result in Kähler manifold theory, which we will prove below, is that ω is closed if and only if the structure (β,J) is Kähler. One of the reasons that Kähler manifolds are of interest in physics is that the fundamental 2-form ω (which we shall see in a moment is closed) defines a Poisson bracket structure which is of interest in various problems of mechanics.

Theorem 4.1. Given a Hermitian structure (β, J), there is an affine connection $\nabla' : V(M) \times V(M) \to V(M)$ such that:

$$\nabla_X{}'Y = \frac{1}{2}(\nabla_X Y - J\nabla_X J(Y)) \qquad 5.4$$

for $X,Y \in V(M)$

$$\nabla_X{}'(JY) = J\nabla_X{}'Y \qquad 5.5$$

$$\nabla_X{}'(\beta(Y,Z)) = \beta(\nabla_X{}'Y,Z) + \beta(Y,\nabla_X{}'Z) \qquad 5.6$$

for $X,Y,Z \in V(M)$

This connection is called the Hermitian affine connection.

Proof. Let us define ∇' by formula 5.4. We must first show that ∇' is a genuine affine connection. As explained in DGCV, Chapter 19, to do this we must show that ∇' satisfies the following conditions:

$$\nabla'_{fX}Y = f\nabla_X{}'Y$$
$$\nabla_X{}'(fY) = X(f)Y + f\nabla_X{}'(Y) \qquad 5.7$$

for $X,Y \in V(M)$, $f \in F(M)$.

However, 5.7 readily follows from 5.4.

Let us now prove 5.5

$$\nabla_X{}'(JY) = \frac{1}{2}(\nabla_X JY - J\nabla_X J(JY))$$
$$= \frac{1}{2}(\nabla_X JY + J\nabla_X Y)$$

$$J\nabla_X{}'JY = \frac{1}{2}(J\nabla_X JY - \nabla_X Y)$$
$$= -\nabla_X{}'Y$$

which proves 5.5. 5.6 is proved similarly.

We now introduce the torsion tensor T, of the affine connection ∇'. (See DGCV, p. 266).

$$T(X,Y) = \nabla_X'Y - \nabla_Y'X - [X,Y] \tag{5.8}$$

for $X,Y \in V(M)$.

We can, of course, calculate T using 5.4 and the fact that the torsion tensor of the Riemannian affine connection ∇ is zero:

$$\begin{aligned} T(X,Y) &= \frac{1}{2}(\nabla_X Y - J\nabla_X JY \\ &\quad - \nabla_Y X + J\nabla_Y JX) - [X,Y] \\ &= \frac{1}{2}([X,Y] - J\nabla_X JY + J\nabla_Y JX) \end{aligned} \tag{5.9}$$

Theorem 5.2. The torsion tensor T vanishes if and only if the Hermitian structure (β,J) is Kahler.

Proof. Suppose first that the Hermitian structure is Kahler. Then, $\nabla_X JY = J\nabla_X Y$, hence, using 5.4, $\nabla_X Y = \nabla_{X'} Y$, i.e. the Hermitian connection equals the Riemannian connection. But, the latter connection is torsion-free, hence $T = 0$.

Conversely, suppose $T = 0$. Then, by 5.6, the covariant derivative of the β-tensor with respect to the ∇'-tensor vanishes. As was shown in DGCV, p. 273, there is but one connection satisfying these two conditions, hence: $\nabla = \nabla'$. 5.5 then shows that the covariant derivative of the J-tensor with respect to ∇ is zero, i.e. the structure (β,J) is Kahler.

We can now "extend the ground field" from R to C, i.e. introduce $V_c(M) = V(M) \otimes C$, and extend tensor-fields (such as

β, T and J) that were defined initially as F(M)-linear objects associated with V(M) to F_c(M)-linear objects associated with $V_c(M)$. Thus, $V_c(M)$ splits as a direct sum:

$$V_c(M) = V_c^{+}(M) \oplus V_c^{-}(M), \qquad 5.10$$

with:

$$J(X) = iX \quad \text{or} \quad +iX \qquad 5.11$$

$$\text{for} \quad X \in V_c^{+}(M) \quad \text{or} \quad V_c^{-}(M).$$

Then, also:

$$0 = \beta(V_c^{+}(M), V_c^{+}(M)) = \beta(V_c^{-}(M), V_c^{-}(M)). \qquad 5.12$$

<u>Proof</u>. Given $X, Y \in V_c^{+}(M)$,

$$\beta(JX, JY) = -\beta(J^2X, Y) = \beta(X,Y)$$

$$= i^2\beta(X,Y) = -\beta(X,Y), \text{ which implies } \beta(X,Y) = 0.$$

Now, let (z_j), $1 \le j, k \le n$, be a system of complex analytic coordinates valid in a coordinate patch of M. Then, the complex vector fields $\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_j^*}\right)$ form a basis for $V_c(M)$. Further,

$$\frac{\partial}{\partial z_j} \in V_c^{+}(M).$$

Set:

$$g_{jk} = \beta\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k^*}\right) \qquad 5.13$$

Then,

$$g_{jk} = g_{kj}^*, \qquad 5.14$$

i.e. (g_{jk}) is Hermitian symmetric

$$\beta = g_{jk} dz_j \cdot dz_k{}^* \qquad 5.15$$

$$\omega = -i g_{jk} dz_j \wedge dz_k{}^* \qquad 5.16$$

Proof

$$\omega\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k{}^*}\right) = \beta\left(\frac{\partial}{\partial z_j}, J \frac{\partial}{\partial z_k{}^*}\right)$$

$$= -i\beta\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k{}^*}\right) = -i g_{jk}$$

5.16 follows from this relation.

Lemma 5.3. $d\omega = 0$ if and only if:

$$\frac{\partial g_{jk}}{\partial z_\ell} = \frac{\partial g_{\ell k}}{\partial z_j} \qquad 5.17$$

Proof. This follows from 5.15

$$d\omega = -i(dg_{jk}) \quad dz_j \wedge dz_k{}^*$$

$$= -i\left(\frac{\partial g_{jk}}{\partial z_\ell} \; dz_\ell + \frac{\partial g_{jk}}{\partial z_\ell} dz_\ell{}^*\right) \quad dz_j \wedge dz_k{}^*$$

Equating to zero the term containing only one factor $dz_j{}^*$ gives 5.17. The other term vanishes also by Hermitian symmetry, i.e. 5.14.

Lemma 5.4. T, the torsion tensor associated with the affine connection ∇', vanishes if and only if:

$$\nabla_{\frac{\partial}{\partial z_j}} \left(\frac{\partial}{\partial z_k{}^*}\right) + \nabla_{\frac{\partial}{\partial z_k{}^*}} \left(\frac{\partial}{\partial z_j}\right) = 0 \qquad 5.18$$

Proof. Let us use formula 5.9 for T to prove this. Note first that 5.9 implies that:

$$T\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = 0 = T\left(\frac{\partial}{\partial z_j{}^*}, \frac{\partial}{\partial z_k{}^*}\right) \qquad 5.19$$

Hence, $T = 0$ if and only if:

$$T\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k{}^*}\right) = 0.$$

However, 5.9 is used to show that 5.17 is the condition needed for this:

Now, let us work out the conditions that 5.17 hold, in terms of the metric tensor (g_{jk}) defined by 5.12. To do this, let us use the following formula, which is proved (in the case of "real" vector fields) on p. 273 of DGCV:

$$\begin{aligned} 2\,\beta(\nabla_Z X, Y> &= Z(\beta(X,Y)) - \beta(X,[Z,Y]) \\ &- Y(\beta(X,Z)) + \beta(Z,[Y,X] \\ &+ X(\beta(Y,Z)) - \beta(Y,[X,Z]) \end{aligned} \qquad 5.19$$

Using this,

$$\begin{aligned} &2\beta\left(\nabla_{\frac{\partial}{\partial z_j}} \frac{\partial}{\partial z_k}, \frac{\partial}{\partial z_\ell}\right) \\ &= \frac{\partial}{\partial z_j}(g_{\ell k}) - \frac{\partial}{\partial z_\ell}(g_{jk}) \end{aligned} \qquad 5.20$$

$$\begin{aligned} &2\beta\left(\nabla_{\frac{\partial}{\partial z_k^*}} \frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_\ell}\right) \\ &= -\frac{\partial}{\partial z_\ell}(g_{jk}) + \frac{\partial}{\partial z_j}(g_{\ell k}) \end{aligned} \qquad 5.21$$

Notice that 5.16 implies the vanishing of 5.20 and 5.21.

<u>Theorem 5.5</u> A Hermitian manifold is Kähler if and only if its fundamental two form is closed.

<u>Proof</u> Suppose first that the metric is Kahler. Then, $\nabla = \nabla'$,

hence the covariant derivative of the β and J tensors with respect to ∇ is zero. Now, ω is constructed from β and J, using formula 5.3. This implies that its covariant derivative is zero. The proof that $d\omega = 0$ will follow from a general fact about torsion-free affine connections, which we leave as an exercise.

Exercise Suppose that ∇ is a torsion-free affine connection on a manifold M, and that θ is an r-differential form such that: $\nabla_X(\theta) = 0$ for all $X \in V(M)$. Show that $d\theta = 0$.

Conversely, suppose that $d\omega = 0$. Then, 5.16 holds. This implies that 5.20 and 5.21 vanish, which in turn proves that 5.17 holds, which implies that $T = 0$.

Problem In the above proof, we had to use local complex analytic coordinate systems in order to express the condition: $d\omega = 0$ in a simple form. Is there a proof that $d\omega = 0$ implies $\nabla = \nabla'$ that is more "intrinsic", i.e. does not use such local coordinates, but that uses the integrability conditions for J directly?

6. INVARIANT CLOSED 2-FORMS ON HOMOGENEOUS SPACES

As we have just seen, a Kähler metric on a manifold M leads to a closed 2-form. If a transformation group G leaves invariant the Kähler metric, it also leaves invariant the form. We shall now investigate the conditions this imposes on G if it acts transitively on M. In fact, we shall work in considerably greater generality than the "Kähler" situation, and investigate the existence of a closed 2-form on a homogeneous space G/L, without assuming that the form is the fundamental form of a Kähler metric.

Let M be a manifold, let $\underset{\sim}{G}$ be a Lie algebra of vector fields on M, and let ω be a closed 2-form on M which is invariant under $\underset{\sim}{G}$, i.e. is such that:

$$X(\omega) = 0 \quad \text{for all} \quad X \in \underset{\sim}{G}. \tag{6.1}$$

(We do not necessarily assume that ω is of maximal rank, hence the work in this section will cover a large amount of material that is involved in mechanics).

Let us work out the algebraic consequences of 6.1, and the condition that $d\omega = 0$. Let p_0 be a point of M, and let $\underset{\sim}{L}$ be the isotropy subalgebra of $\underset{\sim}{G}$ at p_0, i.e. the set of $X \in \underset{\sim}{G}$ such that:

$$X(p_0) = 0. \tag{6.2}$$

For $X,Y \in \underset{\sim}{G}$, set:

$$\omega_0(X,Y) = \omega(X,Y)(p_0). \tag{6.3}$$

Thus, ω_0 is a skew-symmetric, bilinear form: $\underset{\sim}{G} \times \underset{\sim}{G} \to R$. Note that 6.2 implies that:

$$\omega_0(\underset{\sim}{L},\underset{\sim}{G}) = 0 \tag{6.4}$$

<u>Exercise</u> Show that 6.1 implies the following condition:

$$\omega_0([X,Y],Z)+\omega_0(Y,[X,Z]) = 0 \tag{6.5}$$

$$\text{for} \quad X \in \underset{\sim}{L};\ Y,Z \in \underset{\sim}{G}.$$

The algebraic conditions 6.4 and 6.5 do not involve the condition: $d\omega = 0$. To understand this condition, note that: For $X \in \underset{\sim}{G}$,

$$0 = X \lrcorner d\omega = X(\omega) + d(X \lrcorner \omega)$$
$$= d(X \lrcorner \omega)$$

Hence, for $X, Y, Z \in \underset{\sim}{G}$,

$$\begin{aligned} 0 &= d(X \lrcorner \omega)(Y,Z) \\ &= Y((X \lrcorner \omega)(Z)) - Z((X \lrcorner \omega)(Y)) \\ &\qquad - (X \lrcorner \omega)([Y,Z]) \\ &= Y(\omega(X,Z)) - Z(\omega(X,Y)) \\ &\qquad - \omega(X,[Y,Z]) \\ &= \omega([Y,X],Z) + \omega(X,[Y,Z]) \\ &\quad - \omega([Z,X],Y) - \omega(X,[Z,Y]) \\ &\quad - \omega(X,[Y,Z]) \\ &= \omega([Y,X],Z) - \omega([Z,X],Y) \\ &\quad - \omega(X,[Z,Y]) \end{aligned} \tag{6.6}$$

Let us sum up as follows:

Theorem 6.1 Let ω_0 be the form on $\underset{\sim}{G} \times \underset{\sim}{G}$, defined via 6.3 - as the value of ω at p_0. Then, 6.4-6.5 are satisfied. Further,

$$\begin{aligned} &\omega_0([Y,X],Z) - \omega_0([Z,X],Y) \\ &\quad - \omega_0(X,[Z,Y]) = 0 \\ &\quad \text{for } X,Y,Z \in \underset{\sim}{G}. \end{aligned} \tag{6.7}$$

Remark Condition 6.7 is readily understood in terms of Lie algebra cohomology. ω_0 defines a 2-cochain of $\underset{\sim}{G}$, with coefficients in R defined by the "zero" representation of $\underset{\sim}{G}$. Then, 6.7 means that the coboundary of ω_0 is zero, i.e. that ω_0 is a cocycle.

Conditions 6.4-6.5 and 6.7 are also sufficient to determine ω in case $\underset{\sim}{G}$ acts transitively on M.

Problem Suppose that M is the coset space G/L, where L is a connected, closed subgroup of the connected Lie group G. Let p_0 be the point of M represented by the identity coset. Suppose that ω_0 is a skew-symmetric, bilinear form on $\underset{\sim}{G}$ which satisfies 6.4-6.5 and 6.7. Show that there is a G-invariant, closed 2-form ω on M whose value at p_0 is ω_0.

Now we shall investigate methods to determine all possible forms ω_0 which satisfy 6.7.

Theorem 6.2 Suppose that ω_0 is a skew-symmetric, bilinear form: $\underset{\sim}{G} \times \underset{\sim}{G} \to R$. Let $B: \underset{\sim}{G} \times \underset{\sim}{G} \to R$ be a symmetric, non-degenerate form such that:

$$B([X,Y],Z)+B(Y,[X,Z]) = 0 \quad \text{for } X,Y,Z \in \underset{\sim}{G}. \tag{6.8}$$

Let A be a linear transformation: $\underset{\sim}{G} \to \underset{\sim}{G}$ such that:

$$\omega_0(X,Y) = B(AX,Y) \quad \text{for } X,Y \in \underset{\sim}{G}. \tag{6.9}$$

Then, ω_0 satisfies 6.7 if and only if A is a derivation of $\underset{\sim}{G}$, i.e.

$$A([X,Y]) = [AX,Y]+[X,AY] \quad \text{for } X,Y \in \underset{\sim}{G}. \tag{6.10}$$

Proof Using 6.9, the left hand side of 6.7 is:

$$B(A([Y,X]),Z)+B([Z,X],AY)$$
$$-B(AX,[Z,Y])$$

= , using 6.8,

$$B(A([Y,X]),Z)-B(Z,[AY,X]) + B([AX,Y],Z)$$

We see that this is zero if and only if 6.10 is satisfied.

Remark: If $\underset{\sim}{G}$ is the direct sum of an abelian and a semisimple Lie algebra, then a B satisfying 6.8 exists. If, further, $\underset{\sim}{G}$ is semisimple, then B may be taken as the Killing form of $\underset{\sim}{G}$, and it is known that the derivation A is inner, i.e. there is a $W \in \underset{\sim}{G}$ such that:

$$A(X) = [W,X] \quad \text{for all } X \in \underset{\sim}{G}. \qquad 6.11$$

In terms of this W, we have then:

$$\begin{aligned} \omega_0(X,Y) &= B(AX,Y) \\ &= B([W,X],Y) \\ &= -B(W,[X,Y]) \\ &= -\theta([X,Y]), \qquad 6.12 \end{aligned}$$

where:

$$\theta(X) = B(W,X) \qquad 6.13$$

Condition 6.12 is well-known in the theory of cohomology of Lie algebras. It expresses the fact that the coboundary of the 1-cochain θ of $\underset{\sim}{G}$ (with coefficients in R) is ω_0. In the next section we shall investigate the consequences of this "cobounding" condition.

Exercise Suppose that ω_0 is given in terms of B and the derivation A by formula 6.9. Show that ω_0 satisfies 6.4 and 6.5 if and only if the following conditions is satisfied:

$$A(\underset{\sim}{L}) = 0 \tag{6.14}$$

In particular, if A is an inner derivation, given in terms of 6.11 by an element $W \in \underset{\sim}{G}$, then 6.14 is satisfied if and only if:

$$[W,\underset{\sim}{L}] = 0, \tag{6.16}$$

i.e. if $\underset{\sim}{L}$ lies in the centralizer of W in $\underset{\sim}{G}$.

7. INVARIANT CLOSED 2-FORMS DETERMINED BY ELEMENTS OF THE DUAL SPACE OF THE LIE ALGEBRA

Let G be a connected Lie group, L a closed subgroup. Denote by $\underset{\sim}{G}$ the Lie algebra of G, and by $\underset{\sim}{G}^d$ the dual space of the vector space $\underset{\sim}{G}$, i.e. the space of real linear maps $\theta: \underset{\sim}{G} \to R$.

Let $g \to Ad(g): \underset{\sim}{G} \to \underset{\sim}{G}$ denote the usual "adjoint" representation of G by linear transformations on $\underset{\sim}{G}$. Thus, if $X \in \underset{\sim}{G}$, and if $t \to \exp(tX)$ is the one parameter subgroup of G determined by X, then

$$\exp(t\, Ad\, g(X)) = g \exp(tX) g^{-1} \tag{7.1}$$

The representation $g \to Ad\, g$ of G by linear transformations on $\underset{\sim}{G}$ "dualizes" to define a representation $g \to Ad^d(g)$ of G by linear transformations on $\underset{\sim}{G}$, by means of the following formula:

$$Ad^d(g)(\theta)(X) = \theta(Ad(g^{-1})X) \tag{7.2}$$

for $\theta \in \underset{\sim}{G}^d$, $X \in \underset{\sim}{G}$.

Suppose now that θ is an element of $\underset{\sim}{G}^d$ such that:

$$\mathrm{Ad}^d(\ell)(\theta) = \theta \quad \text{for all} \quad \ell \in L. \tag{7.3}$$

Define a skew-symmetric bilinear transformation $\omega_0 \colon \underset{\sim}{G} \times \underset{\sim}{G} \to R$ by means of the following formula:

$$\omega_0(X,Y) = -\theta([X,Y]) \tag{7.4}$$

Remark: In terms of Lie algebra cohomology theory of $\underset{\sim}{G}$ (with coefficients in R, determined by the trivial representation of $\underset{\sim}{G}$) 7.4 expresses the fact that the 2-cochain ω_0 is the coboundary of the 1-cochain θ.

Exercises With ω_0 defined by 7.4, show that 6.4-6.5 and 6.7 are satisfied. In particular, show that ω_0 determines a closed 2-form ω on the coset space $M = G/L$ which is invariant under the action of G, such that:

$$\omega(X,Y)(p_0) = \omega_0(X,Y) \quad \text{for} \quad X,Y \in \underset{\sim}{G}, \tag{7.5}$$

where p_0 denotes the identity coset of G/L. In particular, if the second cohomology group of $\underset{\sim}{G}$ with coefficients in R vanishes, then every G-invariant, closed 2-form on G/L is determined in this way by an element of $\underset{\sim}{G}^d$ which is invariant under $\mathrm{Ad}^d(L)$.

These results are due to B. Kostant [1]. He emphasizes a slightly different point of view; namely, consider the orbits of $\mathrm{Ad}^d(G)$ on $\underset{\sim}{G}^d$. If θ lies on such an orbit, let L denote the isotropy subgroup of G at θ. Then, formula 7.4

determines a G-invariant, closed 2-form ω on G/L. Further, ω has no characteristic vectors, i.e. ω is a non-degenerate 2-form, hence determines a Poisson-bracket structure on the functions on G/L. In turn, this geometric structure (which we call a canonical structure) provides methods for constructing various infinite-dimensional unitary representations on Hilbert spaces, which have been extensively studied by Kirillov and Kostant. However, our main interest in these structures is that they seem to be associated with the mechanical systems which appear in Nature in maximally simple and "solvable" form, such as the harmonic oscillator and hydrogen atom. We shall now study one class of such structures.

8. INVARIANT CANONICAL STRUCTURES ON COSET SPACES OF SEMISIMPLE LIE GROUPS

With the ideas developed in the last two sections, we are in position to tackle a number of interesting and important problems concerning the classification of various invariant geometric structures one can impose on a given coset space G/L. We shall now discuss one of the simplest of such problems, the classification of invariant canonical structures. For the reader's convenience, I will now recapitulate the ideas in a form suited to treating this problem.

Let M be a manifold. A closed 2-form ω on M which has no non-zero characteristic vectors (i.e. which induces a non-degenerate, skew-symmetric, bilinear form on each tangent space to M) defines a <u>canonical structure for M.</u> A diffeomorphism $\phi: M \to M$ is a <u>canonical automorphism</u> if:

$$\phi^*(\omega) = \omega.$$

Such a canonical structure defines a Poisson-bracket operation $(f_1,f_2) \to \{f_1,f_2\}$ on functions on M. It is defined as follows:

$$\{f_1,f_2\} = -X_{f_1}(f_2) \quad \text{for } f_1,f_2 \in F(M), \tag{8.1}$$

where X_{f_1} is the vector field on M such that:

$$df = X_{f_1} \lrcorner \ \omega \tag{8.2}$$

If M is a coset space G/L, a canonical structure is said to be <u>G-invariant</u> if the natural geometric action of G on G/L defines G as a group of canonical automorphisms. In this section, we shall suppose that this is satisfied, and that in addition $\underset{\sim}{G}$ is semisimple. By the results of Section 6, there is then an element $Z \in \underset{\sim}{G}$ such that:

$$\omega(X,Y)(p_0) = B([Z,X],Y) \quad \text{for all } X,Y \in \underset{\sim}{G}, \tag{8.3}$$

where p_0 is the identity coset of $M = G/L$, and where $B(\ ,\ )$ is the Killing form on the semisimple algebra $\underset{\sim}{G}$. (Thus, $B(X,Y) = \text{trace}(\text{Ad } X \text{ Ad } Y)$. See Jacobson [1], LGP and VB, vol. II, Chapter 2 for material about the Killing form. In particular recall Cartan's theorem: $\underset{\sim}{G}$ is semisimple if and only if the Killing form is non-degenerate.)

So far, ω could be any closed, G-invariant 2-form on M. The condition that ω define a canonical structure is that it have no non-zero characteristic vectors, which - in view of the

G-invariance - is equivalent to the following condition:

$$\omega(X,\underset{\sim}{G})(p_0) = 0 \quad \text{implies} \quad X \in \underset{\sim}{L} \qquad 8.4$$

Combining 8.3 and 8.4 then proves the following result:

Theorem 8.1. Let G be a semisimple Lie group, $M = G/L$ a coset space of G which admits a G-invariant canonical structure. Then, there is an element $Z \in \underset{\sim}{G}$ such that:

$$L = \text{centralizer of } Z \text{ in } \underset{\sim}{G}, \quad \text{i.e. the set of } X \in \underset{\sim}{G} \text{ such that } [X,Z] = 0. \qquad 8.5$$

Further, the form ω determining the canonical structure is determined in terms of Z by formula 8.3.

Of course, the determination of the invariant canonical structure in terms of the element $Z \in \underset{\sim}{G}$ is dependent on the choice of the point $p_0 \in M$ to evaluate the form ω. Changing this point to a point $p_1 = g(p_0) \in M$, with $g \in G$, amounts to changing Z to $\text{Ad}\, g(Z)$. Thus, the G-invariant canonical structures are parametrized by points of the orbit space

$$\text{Ad}\, G \backslash \underset{\sim}{G} \qquad 8.6$$

If G is a compact, semisimple Lie group, the enumeration of the orbit space 8.6 is essentially known by the work of E. Cartan (See LGP and Helgason [1]). IF $\underset{\sim}{G}$ is non-compact, a complete description of the orbit spaces is not known, save in certain special cases and situations. However, a considerable amount of partial information is known, since the algebraic problems involved here are very similar to those involved in the study of infinite-dimensional representations of Lie groups. In the next few sections we shall describe certain of these special situations.

9. COSET SPACES OF SEMISIMPLE LIE GROUPS WHICH ADMIT INVARIANT KÄHLER METRICS

Let $M = G/L$ be a coset space of a semisimple, connected Lie group which admits an invariant, positive Kähler metric. Also, suppose that G acts effectively on G/L, i.e. no element of G except the identity acts as the identity on G/L.

Exercise. Show that this is so if and only if no invariant subgroup of L is also an invariant subgroup of G.

Let Z be the element of $\underset{\sim}{G}$ satisfying 8.5, i.e. which determines $\underset{\sim}{L}$ as the centralizer of Z in $\underset{\sim}{G}$.

Exercise. Show that the condition that G/L admit an invariant, positive Kahler metric forces L to be compact.

By the "conjugacy of maximal subgroups theorem" of E. Cartan (see Helgason [1]), there is a maximal, connected, compact subgroup K of G such that:

$$L \subset K \subset G. \tag{9.1}$$

This triple of groups determines a fiber space $G/L \to G/K$, with fiber K/L.

Theorem 9.1. If G/L admits an invariant, positive Kahler metric, then:

$$\text{rank } G = \text{rank } K, \tag{9.2}$$

where K is the maximal compact subgroup of G.

Proof. By Cartan's theorem, $\underset{\sim}{K}$ is a symmetric subalgebra of $\underset{\sim}{G}$, i.e. there is a linear subspace $\underset{\sim}{P} \subset \underset{\sim}{G}$ such that:

$$\underset{\sim}{G} = \underset{\sim}{K} \oplus \underset{\sim}{P}$$
$$[\underset{\sim}{K},\underset{\sim}{P}] \subset \underset{\sim}{P};\ [\underset{\sim}{P},\underset{\sim}{P}] \subset \underset{\sim}{K}. \qquad 9.3$$

Further, $\underset{\sim}{G}_\mu = \underset{\sim}{K} + i\underset{\sim}{P}$ is a "compact real form" of $\underset{\sim}{G}$, i.e. it is the Lie algebra of a compact Lie group, whose complexification is the same as the complexification of G. We see that:

$$\underset{\sim}{L} = \text{centralizer of } Z \text{ in } \underset{\sim}{G}_\mu. \qquad 9.4$$

This implies 9.2, since Z belongs to a Cartan subalgebra of $\underset{\sim}{G}_\mu$ (Recall that the "rank" of a Lie group is the common dimension of all its Cartan subalgebras. Thus, rank $\underset{\sim}{G}$ = rank $\underset{\sim}{G}_\mu$, and it is known that all Cartan subalgebras of the compact semisimple Lie algebra $\underset{\sim}{G}_\mu$ are abelian and mutually conjugate.)

Theorem 9.2. Suppose conversely that G is a connected, semisimple Lie group, that K is a maximal connected compact subgroup of G, that L is a connected subgroup of K such that $\underset{\sim}{L}$ is the centralizer in $\underset{\sim}{G}$ of an element $Z \in \underset{\sim}{K}$. Then, M = G/L admits a G-invariant, positive Kahler metric.

Proof. Let p_0 be the identity coset of G/L. Let B(,) again denote the Killing form on $\underset{\sim}{G}$. Let $\underset{\sim}{G} = \underset{\sim}{K} \oplus \underset{\sim}{P}$ be the Cartan decomposition of $\underset{\sim}{G}$, satisfying 9.3. Let ω be the G-invariant, closed 2-form on M defined by formula 8.3. We must show that ω is the fundamental 2-form of a G-invariant Kahler metric. To do this, we shall find the relevant G-invariant complex structure on M.

Let $\underset{\sim}{G}_c = \underset{\sim}{G} + i\underset{\sim}{G}$ be the complexification of $\underset{\sim}{G}$. Since Z lies in a compact subalgebra of $\underset{\sim}{G}$, the eigenvalues of $\text{Ad } Z$ acting in $\underset{\sim}{G}_c$ are pure imaginary. Let $\underset{\sim}{G}_+(Z)$, denote the space of elements $X \in \underset{\sim}{G}_c$ such that:

$$[Z,X] = i\lambda X, \qquad 9.4$$

$$\text{with } \lambda \in R \text{ and } \lambda \geq 0.$$

<u>Exercise</u>. Show that $\underset{\sim}{G}_+(Z)$ is a complex Lie subalgebra of $\underset{\sim}{G}_c$. Further, show that:

$$\text{real dimension } \underset{\sim}{G}_c/\underset{\sim}{G}_+(Z) = \text{dimension } \underset{\sim}{G}/\underset{\sim}{L} \qquad 9.5$$

Now, $\underset{\sim}{G}$ is a real subspace of $\underset{\sim}{G}_c$, and L is a real subspace of $\underset{\sim}{G}_+(Z)$. Thus, using 9.5, we see that $\underset{\sim}{G}/\underset{\sim}{L}$ is isomorphic - under the injection map $\underset{\sim}{G} \to \underset{\sim}{G}_c$ - to $\underset{\sim}{G}_c/\underset{\sim}{G}_+(Z)$. Further, $\underset{\sim}{G}_c$ and $\underset{\sim}{G}_+(Z)$ are complex vector spaces. This then gives a complex vector space structure to the vector space $\underset{\sim}{G}/\underset{\sim}{L}$, which in turn is identified with the tangent space to $M = G/L$ at the identity coset point p_0.

<u>Problem</u>. Show that M has a unique G-invariant complex manifold structure, such that the complex vector space structure induced on $M_{p_0} = \underset{\sim}{G}/\underset{\sim}{L}$ agrees with that defined via the isomorphism between $\underset{\sim}{G}/\underset{\sim}{L}$ and $\underset{\sim}{G}_c/\underset{\sim}{G}_+(Z)$ defined above.

<u>Problem</u>. Complete the proof of Theorem 9.2 by showing that there is a G-invariant positive Hermitian (with respect to the complex structure defined in the previous exercise) metric on M whose fundamental 2-form ω is given by formula 8.3. Deduce that the Hermitian metric is Kähler.

Remark: The class of G-invariant positive Kähler manifolds constructed using these results are very important for various applications to pure mathematics. First, we may consider the case where G itself is compact, semisimple. The class of manifolds obtained in this way includes the various types of projective Riemannian and "flag" manifolds. The simplest type of such space important in physics is the 2-sphere $SU(2)/U(1)$. Another is $M = SU(3)/U(2)$, the 2 (complex) dimensional projective space. As shown in LGP, when one decomposes the action of $G = SU(3)$ on C^∞ functions on M, one obtains the representations of $SU(3)$ modulo its center that seem to appear in the "Eightfold way" classification of elementary particles.

Another important class of spaces is that where:

G = simple, connected, non-compact

$K = L$ = maximal compact subgroup.

Exercise. Show that $M = G/K$ admits an invariant Kahler metric if and only if the center of K is one-dimensional.

Finally, we might mention that, as a consequence of theorems proved by E. Cartan and Harish-Chandra, this class of spaces admits geometric realizations as "symmetric bounded domains" in complex Euclidean space. These spaces are important in the theory of functions of several complex variables, particularly the theory of automorphic functions.

10. CANONICAL COSET SPACES OF THE POINCARÉ GROUP

Let G be the connected Poincare group, which is the semidirect product of $SO^{+}(1,3)$ (the connected Lorentz group) and R^4, the space-time translation group. Now, it is well-known that the irreducible unitary representationsof G determine the relativistically invariant, "free" quantum mechanical particles. It is then interesting to see that the canonical coset spaces G/L of G are related to the free, relativistically invariant "classical mechanics" particles. (This has been particularly emphasized in work by R. Arens [1]). We shall briefly investigate this point of view in this section.

<u>Theorem 10.1</u>. The second cohomology group $H^2(\underset{\sim}{G},R)$ of the Poincare Lie algebra, with coefficients in R determined by the trivial representation of $\underset{\sim}{G}$, is zero.

<u>Proof</u>. I shall use the ideas concerning Lie algebra cohomology theory described in my series papers "Analytic continuation of group representations. (A brief explanation is also given in VB, vol. II, Chap. 2). Let $C^2(\underset{\sim}{G},R)$ be the 2-cochains, i.e. the skew-symmetric, bilinear maps

$$\omega: \underset{\sim}{G} \times \underset{\sim}{G} \to R.$$

Let $\underset{\sim}{G}$ act - via a sort of "Lie derivative"-in $C^2(\underset{\sim}{G},R)$ as follows:

$$X(\omega)(Y_1,Y_2) = \omega([X,Y_1],Y_2) + \omega(Y_1,[X,Y_2])$$
$$\text{for } X,Y_1,Y_2 \in \underset{\sim}{G}. \qquad 10.1$$

Let $Z^2(\underset{\sim}{G},R)$ be the cocycles in $C^2(\underset{\sim}{G},R)$. Let $B^2(\underset{\sim}{G},R)$ be the coboundaries, i.e. the image of $C^1(\underset{\sim}{G},R)$ under the coboundary operator. Then,

$$B^2(\underset{\sim}{G},R) \subset Z^2(\underset{\sim}{G},R)$$
$$H^2(\underset{\sim}{G},R) = Z^2(\underset{\sim}{G},R)/B^2(\underset{\sim}{G},R). \qquad 10.2$$

Now, via the action 10.1 of $\underset{\sim}{G}$, one readily verifies that $\underset{\sim}{G}$ leaves the subspaces $B^2(\underset{\sim}{G},R)$ and $Z^2(\underset{\sim}{G},R)$ invariant. $\underset{\sim}{G}$ is the direct sum

$$\underset{\sim}{L} \oplus \underset{\sim}{T}$$

of a semisimple subalgebra $\underset{\sim}{L}$, an abelian subalgebra $\underset{\sim}{T}$, such that:

$$[\underset{\sim}{L},\underset{\sim}{T}] \subset \underset{\sim}{T}$$

By the theorem of "complete reducibility" of finite dimensional representations of a semisimple Lie algebra (see Jacobson [1] or VB, vol. II, Chap. 2) there is a linear subspace $\underset{\sim}{A} \subset Z^2(\underset{\sim}{G},R)$ such that:

$$Z^2(\underset{\sim}{G},R) = \underset{\sim}{A} \oplus B^2(\underset{\sim}{G},R) \qquad 10.3$$

$$\underset{\sim}{L}(\underset{\sim}{A}) \subset \underset{\sim}{A} \tag{10.4}$$

<u>Problem</u>. Show that 10.1, 10.3 and 10.4 implies that:

$$\underset{\sim}{L}(\underset{\sim}{A}) = 0 \tag{10.5}$$

Using 10.5, we shall now prove that $\underset{\sim}{A} = 0$ which (in view of 10.3) would finish the proof. Suppose that $\omega \in \underset{\sim}{A}$. Consider ω restricted to $\underset{\sim}{T}$. It is a skew-symmetric bilinear map: $\underset{\sim}{T} \times \underset{\sim}{T} \to R$. Now, T is isomorphic to R^4. 10.5 means that ω is invariant under the action of the Lorentz group on R^4.

<u>Exercise</u>. Show that there is no non-zero bilinear, skew-symmetric map: $R^4 \times R^4 \to R$ which is invariant under the action of the Lorentz group. Similiarly, show that there is no bilinear, skew-symmetric map: $\underset{\sim}{L} \times \underset{\sim}{L} \to R$ which is invariant under the adjoint representation of L.

We deduce the following facts from this exercise:

$$\omega(\underset{\sim}{T},\underset{\sim}{T}) = 0 = \omega(\underset{\sim}{L},\underset{\sim}{L}) \tag{10.6}$$

Finally, we must also have:

$$\omega(\underset{\sim}{T},\underset{\sim}{L}) = 0, \tag{10.7}$$

for otherwise the representations of L on R^4 and on $\underset{\sim}{L}$ itself would be dual to each other, which they are not. (<u>Exercise</u>: Prove this).

10.6 and 10.7 then imply that $\underset{\sim}{A} = 0$, which completes the proof of Theorem 10.1.

<u>Problem</u>. Refine the above argument to give sufficient conditions that a general semidirect product algebra $\underset{\sim}{G} = \underset{\sim}{L} \oplus \underset{\sim}{T}$, with $\underset{\sim}{L}$

semisimple, $\underline{T}$ abelian, have vanishing second cohomology group $H^2(\underline{G},R)$. Can you construct an example of such a $\underline{G}$ for which $H^2(\underline{G},R)$ is not equal to zero?

<u>Remark</u>: Recall that the second cohomology group $H^2(\underline{G},R)$ plays a key role in another mathematical topic important for physics, namely the classification of central extensions and ray-representations of $\underline{G}$. See VB, vol. II.

Return to the case where G is the connected Poincare group. Let $M = G/L$ be a coset space of G which admits an invariant canonical structure defined by a closed 2-form ω. By Theorem 10.1, there is then an element $\theta \in \underline{G}^d$ such that:

$$\omega(X,Y)(p_0) = \theta([X,Y]) \qquad 10.8$$

$$\text{for } X,Y \in \underline{G}$$

$$\theta([\underline{L},\underline{G}]) = 0 \qquad 10.9$$

For $X \notin \underline{L}$, there is a $Y \in \underline{G}$ such that:

$$\theta([X,Y]) \neq 0 \qquad 10.10$$

11. LAGRANGIAN SUBMANIFOLDS AND FIBRATIONS

Let M be a manifold, with a canonical structure defined by a closed 2-form ω. Let p be a point of M. A linear subspace γ M_p is said to be a <u>Lagrangian subspace</u> of M_p if:

$$\omega(\gamma,\gamma) = 0, \qquad 11.1$$

and γ is contained in no larger subspace with property 11.1.

Exercise. Show that a linear subspace $\gamma \subset M_p$ is a Lagrangian subspace of M_p if and only if 11.1 is satisfied, and:

$$2 \dim \gamma = \dim M.$$

A submanifold $N \subset M$ is called a Lagrangian submanifold of the form ω if, for each $p \in N$, the tangent space $N_p \subset M_p$ is a Lagrangian subspace of M_p.

A fiber space map $\pi\colon M \to B$ is called a Lagrangian fibration if, for each $b \in B$, the fiber $\pi^{-1}(b)$ is a Lagrangian submanifold of the form ω.

Let G be a group of canonical automorphisms acting on M. A fiber space map $\pi\colon M \to B$ is said to be a G-covariant Lagrangian fibration if G also acts on B as a transformation group, and if π intertwines the action of G on both of these spaces.

The simplest case of a Lagrangian fibration is that where B is an arbitrary manifold, M is the cotangent bundle to B, and the form ω on M is $d\theta$, where θ is the "contact 1-form" on M. (See VB, vol. II, Chap. 1). This is also the situation occurring in classical mechanics, of course. For example, if (q_i), $1 \leq i, j \leq n$, are coordinates of the "configuration space" B, let (q_i, p_i) denote coordinates of $M = T^d(B)$ defined as follows:

$$q_i(b,\beta) = q_i(b) \qquad 11.2$$

$$p_i(b,\beta) = \beta\left(\frac{\partial}{\partial q_i}\right) \qquad 11.3$$

$$\text{for } b \in B,\ \beta \in B_b{}^d.$$

($B_b{}^d$ denotes the dual space to the tangent space B_b, i.e. the space of 1-covectors to B at b). Thus, the contact one-form θ

on $M = T^d(B)$ is defined by the following formula:

$$\theta = p_i dq_i \qquad 11.4$$

(Summation convention is in force on the indices $i, j, \ldots$).
Then, the canonical structure on M is defined by the following closed 2-form :

$$\omega = d\theta = dp_i \wedge dq_i \qquad 11.5$$

A submanifold N of M is then a Lagrangian submanifold if the following conditions are satisfied:

$$\begin{aligned} &\dim N = n \\ &\omega \text{ restricted to } N \text{ is zero.} \end{aligned} \qquad 11.6$$

In particular, we see that the submanifolds defined by setting the p_i = constant are Lagrangian. Using 11.3, we see that they are just the fibers of the map $M = T^d(B) \to B$, i.e. the spaces of cotangent vectors to points of B.

In this case, we can construct examples of G-covariant Lagrangian fibrations in the following way: Let G be an arbitrary group of diffeomorphisms of B. Each $g \in G$ then acts on covectors to B in the following way:

$$\begin{aligned} &g(\beta)(v) = (g_*^{-1}(v)) \\ &\quad \text{for } v \in T(B),\ \beta \in T^d(B). \end{aligned} \qquad 11.7$$

This formula then defines a transformation group action of G on $T^d(B) = M$.

<u>Exercise</u>. Show that G, defined via 11.7 as acting on $T^d(B)$, acts as a group of canonical automorphisms, with the map $\pi: T^d(B) \to B$

intertwining the action of G. In other words, $(T^d(B), B, \pi)$ defines a G-covariant Lagrangian fibration of $M = T^d(B)$.

12. A GROUP-THEORETIC CONSTRUCTION OF LAGRANGIAN SUBMANIFOLDS

In this section we shall give a useful general construction of Lagrangian submanifolds. Let G be a connected Lie group. Let $\underset{\sim}{G}$ be its Lie algebra. Let Z be an element of $\underset{\sim}{G}$ which satisfies the following three conditions:

$$\text{Ad } Z \text{ acting in } \underset{\sim}{G} \text{ is completely reducible} \qquad 12.1$$

There is a non-degenerate, symmetric, bilinear form $B: \underset{\sim}{G} \times \underset{\sim}{G} \to R$ such that:

$$B([Z,X],Y)+B(X,[Z,Y]) = 0 \quad \text{for } X,Y \in \underset{\sim}{G}. \qquad 12.2$$

$$\text{Ad } Z \text{ acting in } \underset{\sim}{G} \text{ has only real eigenvalues.} \qquad 12.3$$

With conditions 12.1 and 12.3, we can split up $\underset{\sim}{G}$ into the direct sum (as a vector space) of three subalgebras, $\underset{\sim}{G}_0$, $\underset{\sim}{G}_+$, $\underset{\sim}{G}_-$:

$$\underset{\sim}{G}_0 = \{X \in \underset{\sim}{G}: [X,Z] = 0\} = \text{centralizer of } Z \text{ in } \underset{\sim}{G}. \qquad 12.4$$

$$\underset{\sim}{G}_+ = \text{subspace of } \underset{\sim}{G} \text{ spanned by the eigenvectors of Ad } Z \text{ for positive eigenvalues.} \qquad 12.5$$

$$\underset{\sim}{G}_- = \text{subspace of } \underset{\sim}{G} \text{ spanned by the eigenvectors of } \operatorname{Ad} Z \text{ for negative eigenvalues.} \qquad 12.6$$

Exercise. Show that $\underset{\sim}{G}_+$, $\underset{\sim}{G}_-$ are indeed Lie subalgebras of $\underset{\sim}{G}$, and that $\underset{\sim}{G}$ is decomposed, as a vector space, as this direct sum. (They are not ideals of $\underset{\sim}{G}$, so that this is not a "direct sum" in the sense of Lie algebra theory).

Let G_0, G_+, G_- be the connected subgroups of G generated by the subalgebras $\underset{\sim}{G}_0$, $\underset{\sim}{G}_+$, $\underset{\sim}{G}_-$. Let M be the coset space G/G_0. From the work of previous sections, we see that M has a G-invariant, closed 2-form ω, which defines a canonical structure on M, such that:

$$\omega(X,Y)(p_0) = B(Z,[X,Y]) \quad \text{for all } X,Y \in \underset{\sim}{G}, \qquad 12.7$$

where p_0 is the identity coset element of M.

Lemma 12.1. $B(Z,\underset{\sim}{G}_+) = 0 = B(Z,\underset{\sim}{G}_-)$ 12.8

Proof. We deal with $\underset{\sim}{G}_+$. The proof for $\underset{\sim}{G}_-$ is similiar. Now, $\underset{\sim}{G}_+$ is spanned by elements $X \in \underset{\sim}{G}$ such that:

$$[Z,X] = \lambda X, \text{ with } \lambda > 0.$$

Hence, using 12.2,

$$0 = B([Z,Z],X) + B(Z,[Z,X]) = \lambda B(Z,X),$$

which forces: $B(Z,X) = 0$, which implies 12.8.

Lemma 12.2. ω is zero on the orbit submanifolds G_+p_0, G_-p_0 of M.

Proof. First, note that 12.7 and 12.8 imply that ω restricted to the orbits vanishes at p_0. Then, the fact that ω is invariant under the action of G on M implies that ω vanishes at all points of the orbits.

If G is semisimple, we can show that these orbit submanifolds are Lagrangian.

Theorem 12.3. Suppose in addition that G is semisimple, and that B is the Killing form of $\underset{\sim}{G}$. Then,

$$\dim \underset{\sim}{G}_+ = \dim \underset{\sim}{G}_- = \frac{1}{2} \dim M \qquad 12.8$$

In particular, the orbits G_+p, G_-p are Lagrangian submanifolds of M.

Proof. Consider the bilinear map $\alpha\colon \underset{\sim}{G}_+ \times \underset{\sim}{G}_- \to R$ defined as follows:

$$\alpha(X,Y) = \omega(X,Y)(p_0) \quad \text{for } X \in \underset{\sim}{G}_+,\ Y \in \underset{\sim}{G}_- \qquad 12.9$$

Exercise. Show that relation 12.8 is equivalent to showing that the map α is non-degenerate, i.e. that the following conditions are satisfied:

$$\alpha(X,\underset{\sim}{G}_-) = 0 \text{ implies } X = 0 \qquad 12.10$$

$$\alpha(\underset{\sim}{G}_+,Y) = 0 \text{ implies } Y = 0 \qquad 12.11$$

$$\text{for } X \in \underset{\sim}{G}_+,\ Y \in \underset{\sim}{G}_-.$$

We shall now prove 12.10.

$$\alpha(X,Y) = B(Z,[X,Y])$$

$$= B([Z,X],Y)$$

Thus, if $X \in \underset{\sim}{G}_+$ and $\alpha(X,\underset{\sim}{G}_-) = 0$, then $[Z,X] = 0$, which implies $X \in \underset{\sim}{G}_0$, which forces $X = 0$. The proof of 12.1 is similiar, which proves 12.8.

Let $\underset{\sim}{G}_1$ = subalgebra of G spanned by eigenvectors of $\mathrm{Ad}\, Z$ with non-negative eigenvalues. Let G_1 be the connected subgroup of G generated by $\underset{\sim}{G}_1$.

<u>Exercise</u>. Show that G_+ is an invariant subgroup of G_1.

Then, we have:

$$G_0 \subset G_1 \subset G.$$

This induces a fibration.

$$\pi\colon\ M = G/G_0 \to G/G_1. \qquad 12.12$$

The fibers are the translaters under G of the orbits $G_1 p_0$. Now, $G_1 p_0 = G_+ p_0$. This proves the following main result:

<u>Theorem 12.4</u>. Suppose that $\underset{\sim}{G}$ is semisimple. Then, the fibers of the map π indicated by 12.11 are Lagrangian submanifolds of the form ω. In particular, the fibration is a Lagrangian fibration of the form ω.

<u>Remark</u>: The coset space G/G_1 plays a basic role in the representation theory of semisimple Lie groups. "Induced" representations defined by cross-sections of homogeneous vector bundles on this space (see LGP) are the basic objects, which often define irreducible unitary representations of G. B. Kostant has remarked [1] that these seem to be close relations between these induced representations and representations defined by the means of the canonical structure on G/G_0.

Chapter II

THE GEOMETRY OF COMPLEX PROJECTIVE SPACES AND THE STATE SPACE OF QUANTUM MECHANICS

1. INTRODUCTION

It is known that a "Poisson bracket" structure for spaces of observables is associated with a geometric structure - namely, a "symplectic" structure - for the state space. (See LAQM and VB.)

From the point of view of differential geometry, the "Kähler manifolds" form one important class of symplectic manifolds. The simplest sort of non-flat Kähler manifolds are the complex projective spaces. This suggests that the complex projective spaces are interesting spaces to study.

However, in addition to this "deduction" of a reason to study the finite dimensional complex projective spaces at this point, there is the more important fact that they are related to quantum mechanics. (Recall that the complex projective space associated with an infinite dimensional complex Hilbert space is the state space of a quantum mechanical system when it is looked at in the usual "Schrodinger picture" way.)

Now, the "symplectic manifolds" of most interest to physics are of "infinite dimensional" type. The theory of these structures is not yet in as definitive a form as one would like, from the point of view of these applications. Accordingly, I shall not attempt to discuss the infinite dimensional situation systematically, but shall deal with the finite dimensional ones in a way that I believe will give clues as to how to deal with the infinite dimensional situation. When appropriate, I shall also insert direct comments about how one might deal with the infinite dimensional generalizattions.

Mathematically, the work in this chapter will also be on two levels of the concreteness-abstraction scale. Although we are dealing with a specific sort of space, which may be as readily studied with classical methods, I believe it is valuable to look at it from certain general differential-geometric points of view. Accordingly, I will insert some sections which explain the general background of the ideas, as they are encountered.

2. THE PROJECTIVE SPACE OF A FINITE DIMENSIONAL VECTOR SPACE

Let K be a "field" in the sense of algebra. (Thus, K has the algebraic properties of the real or complex numbers. The reader who does not want to worry about this level of abstraction can think of K as the real or complex numbers.)

Let H be a finite dimensional vector space, over K as the field of scalars. Denote typical elements of H by the letter ψ.

Let P(H) be the space whose "points" are the one-dimensional, linear subspaces of H. Thus, each non-zero element $\psi \epsilon H$ is contained in precisely one such subspace, namely the space of vectors ψ' of H of the form:

$$\psi' = k\psi,$$

$$\text{for } k \epsilon K.$$

This is an element of P(H), which we denote by (ψ). Let us define a map $\pi: H-(0) \rightarrow P(H)$ by setting

$$\pi(\psi) = (\psi)$$

$$\text{for } \psi \epsilon H - (0).$$

P(H) is called the <u>projective space</u> associated with the vector space H. π is called the <u>projection map.</u>

<u>Remarks</u>: One might encounter vector spaces with two or more fields of scalars. For example, a complex vector space of dimension n is a vector space with respect to the real numbers (since R is a subfield of C) of dimension 2n, and an infinite dimensional vector space over the field Q of rational numbers. A notation for dealing with this situation might be: $P_K(H)$: meaning "the projective space defined by regarding H as a vector space over K."

One might hope to examine geometric situations in P(H) by referring them back, via π, to H. In the language of classical projective geometry, this is called "introducing homogeneous coordinates."

What are classically called "non-homogeneous coordinates" for P(H) are related to the topic we now discuss, namely a manifold-like structure for P(H).

Let H' be a linear subspace of H of codimension one. (Recall that the <u>codimension</u> of a linear subspace H' of H is the dimension of the quotient space H/H'. Everything is, for the rest of this section at least, defined relative to the field K of scalars, which is fixed.) Each one dimensional subspace of H' is obviously a subspace of H; hence P(H') may be identified with a subset of P(H). Such a subset of P(H) is called a <u>hyperplane of P(H)</u>

We shall now show that P(H) - P(H') may be identified with H' itself. To this end, let ψ_0 be any element in H-H'.

Assign to $\psi' \varepsilon H'$ the element:

$$p(\psi') = (\psi' + \psi_0) \qquad 2.1$$

of P(H).

Exercise. Show that the assignment 2.1, $\psi' \to p(\psi')$, is a one-one, onto mapping from H' to P(H) - P(H').

Remark: In classical language, P(H') is called a hyperplane at infinity of P(H). The assignment to a point $p \varepsilon P(H) - P(H')$ of the element $\psi' \varepsilon H'$ such that: $p = p(\psi')$, is called the assignment of inhomogeneous coordinates to points P(H) - P(H'). A subspace of P(H) of the form P(H) - P(H'), for some subspace H', is called an affine subspace of P(H).

In case K = R or C, these assignments can be used to define a manifold structure for P(H). Suppose that:

$$\dim H = n+1.$$

Then dimension H' = n.

Exercise. Show that there are linear subspaces $H_1', \ldots, H_{n+1}'$ of codimension one such that each point of P(H) belongs to one of the subsets $P(H) - P(H_1'), \ldots, P(H) - P(H_{n+1}')$.

Show also that there is a manifold structure for P(H) such that each of the "inhomogeneous coordinate" maps: $P(H) - P(H') \to H'$ defined by formula 2.1 are diffeomorphisms, with H' carrying the manifold structure which is customary for finite dimension vector spaces. If $K = C$, show further that $P(H)$ can be given a complex manifold structure, so that the coordinate maps are complex analytic mappings.

Remark. One might now proceed to develop ideas of algebraic geometry in this framework; in the case where K is a general field, one would mimic standard ideas of manifold theory. Unfortunately, the accessible textbooks in algebraic geometry develop a separate way of looking at these ideas, which often masks their geometric origin.

3. THE TANGENT VECTOR BUNDLE FOR A PROJECTIVE SPACE

Let H continue to be a finite dimensional vector space, with K a field of scalars. For a point $p \in P(H)$, consider an element $\psi \in H-(0)$ such that:

$$p = (\psi)$$

Consider the quotient vector space

$$H/(\psi).$$

Definition. A tangent vector to P(H) at the point (ψ) is a K-linear map $v:(\psi) \to H/(\psi)$. The collection of such maps form a K-vector space (under addition), called the tangent vector space to P(H) at the point (ψ), denoted by $P(H)_{(\psi)}$. (In the case that K = R or C, we shall soon see why the name "tangent space" is justified.) Notice that the dimension of $P(H)_{(\psi)}$ is equal to:

$$\dim H - 1.$$

$P(H)_{(\psi)}$ could be identified with $H/(\psi)$ itself, if a specific **representative** ψ_0 for (ψ) were chosen, i.e. a $v \in P(H)_{(\psi)}$ could be identified with the image

$$v(\psi_0) \text{ in } H/(\psi).$$

However, if one wants to maintain complete "covariance," the distinction between $P(H)_{(\psi)}$ and $H/(\psi)$ as vector spaces should be maintained.

<u>Definition</u>. <u>The tangent bundle to P(H)</u>, denoted by T(P(H)), is the space of ordered pairs (p,v), where:

$$p \in P(H); \quad v \in P(H)_p.$$

Thus, T(P(H)) is a vector bundle one P(H) with the projection map (p,v) $\to$ p), with the fiber the K-vector spaces formed by the tangent spaces.

Let A be a K-linear transformation: H $\to$ H. Associate with A a map X_A: P(H) $\to$ T(P(H)), $X_A((\psi)) = v \in P(H)_{(\psi)}$, where v is defined as follows:

$$v(\psi') = \text{image of } A(\psi') \text{ in } H/(\psi), \quad \text{for } \psi' \in (\psi). \qquad 3.2$$

Notice that X_A is a cross-section map: P(H) $\to$ T(P(H)). It will play an important role in quantum mechanics. Further, we have

<u>Theorem 3.1.</u> $X_A((\psi)) = 0$ if and only if

$$A(\psi) = \lambda\psi \text{ for some } \lambda \in K, \qquad 3.3$$

i.e., ψ is an eigenvector of A.

One could proceed further in this direction to define precisely algebraically the objects which, when K = R or C, reduce to the familiar differential-geometric objects. This is a "game" which the algebraic geometers enjoy playing. Perhaps I might pursue it further later on in this treatise.

4. THE TANGENT BUNDLE TO REAL OR COMPLEX PROJECTIVE SPACE

In section 3 we have defined an algebraic object which we have called the "tangent bundle." Now, we look at the construction analytically, in case K = C or R.

First, let us be more explicit about the manifold structure for P(H). Suppose

$$\text{dimension } H = n + 1. \tag{4.1}$$

Let p be a point of P(H), and let H' be a linear subspace of H of codimension 1 such that:

$$H' \cap p = (0) \tag{4.2}$$

We shall coordinatize the affine space P(H) - P(H'), ie exhibit a correspondence between it and K^n, such that p goes into the zero element of K^n. These mappings are, as we mentioned in Section 2, the coordinate charts defining the manifold structure for P(H).

Let $\psi_0, \psi_1, \ldots, \psi_n$ be a basis for H, such that:

$$(\psi_0) = p \tag{4.3}$$

$$(\psi_1, \ldots, \psi_n) \text{ are a basis for } H' \tag{4.4}$$

Each point $\psi \epsilon H - H'$ then has a unique expansion of the following form:

$$k_0\psi = \psi_0 + k_1\psi_1 + \dots + k_n\psi_n, \qquad 4.5$$

with $k_0 \neq 0$; $k_0, k_1, \dots, \epsilon K$. Assign to $(\psi)\epsilon P(H)-P(H')$ the element $(k_1,\dots,k_n)\epsilon K^n$. By 4.3, p goes into $(0,\dots,0)$.

Exercise. Show that the assignment

$$(\psi) \rightarrow (k_1,\dots,k_n)\epsilon K^n$$

defines a one-one, onto map between $P(H)-P(H')$ and K^n. Calculate explicitly the transition map $K^n \rightarrow K^n$ resulting from coordinatization using two different bases of H.

With the results of this exercise, one shows readily that P(H) has a real-analytic manifold structure for K = R, and a complex analytic manifold structure for K = C.

Let us now see why the vector bundle T(P(H)) constructed purely algebraically in Section 3 is really the "manifold" tangent bundle to P(H). Let ψ be a point of H - (0), with (ψ) the projection in P(H). Let $t \rightarrow p(t)$, $-1<t<1$, be a one parameter family of one dimensional subspaces of H, i.e. a curve in P(H), such that:

$$p(0) = (\psi).$$

Suppose that $t \rightarrow \psi(t)$ is a curve in H - (0) such that:

$$p(t) = (\psi(t)).$$

Define $v\epsilon P(H)_{(\psi)}$ so that:

$$v(\psi(0)) = \text{projection in } H/(\psi) \text{ of } \frac{d\psi}{dt}(0) \qquad 4.6$$

Let us regard v defined by 4.6 as the "tangent vector" to the curve $t \to p(t)$ if $t = 0$.

<u>Problem</u>. Show that the curves $t \to p(t)$, $p_1(t)$ in $P(H)$ which equal (ψ) at $t = 0$ have the same tangent vector at $t = 0$ (relative to the manifold structure defined already) if and only if $v = v_1$, where v_1 is the element of $P(H)_{(\psi)}$ defined analogously to 4.6.

This identification then permits us to identify the tangent bundle to P(H) in the sense of manifold theory with the tangent bundle defined algebraically in Section 3.

In case K = C, one sees that the affine space coordinate charts: $P(H) - P(H') \to C^n$ constructed above define a complex manifold structure for P(H). We should then be able to define an R-linear operator J: $T(P(H)) \to T(P(H))$ such that: $J^2 = -(\text{identity})$, defining the complex structure on P(H). The definition of this J can be described as follows:

<u>Exercise</u>. Suppose that H is a complex vector space. For $(\psi)\in P(H)$, $v\in P(H)_{(\psi)}$ set:

$$J(v) = iv \qquad 4.6$$

Show that $J:T(P(H)) \to T(P(H))$ defines a complex manifold structure for P(H) which agrees with the complex manifold structure defined via the affine space coordinate charts.

We shall now see that this way of looking at the manifold structure of P(H) - identifying the tangent bundle to P(H) defined via manifold theory with the "algebraic" bundle described in Section 2 - is very convenient for describing other geometric properties of complex projective spaces.

5. THE KÄHLER STRUCTURE FOR COMPLEX PROJECTIVE SPACES

Let H continue to be a finite dimensional complex vector space. In addition, let us suppose that H has a Hilbert space structure, defined by an R-bilinear map $(\psi_1,\psi_2) \to \langle\psi_1/\psi_2\rangle$ of $H\times H \to C$ such that:

$$\langle\psi_1/\psi_2\rangle^* = \langle\psi_2/\psi_1\rangle$$

$$\langle\psi_1/c\psi_2\rangle = c\langle\psi_1/\psi_2\rangle = \langle c^*\psi_1/\psi_2\rangle \qquad 5.1$$

for $\psi_1,\psi_2 \epsilon H$; $c\epsilon C$.

$$\langle\psi/\psi\rangle > 0 \quad \text{for} \quad \psi\epsilon H - (0).$$

(Notice that we use physicist's notations for Hilbert spaces; * denotes complex conjugate of a complex number.

Recall that the <u>orthogonal complement</u> of an element $\psi\epsilon H$ is defined as the space of $\theta\epsilon H$ such that:

$$\langle\theta/\psi\rangle = 0, \qquad 5.2$$

and denoted by $\psi^\perp$.

Thus, $H/(\psi)$ can be identified (because the form 5.1 is positive) with $\psi^\perp$. $P(H)_{(\psi)}$ may then be defined as the space of C-linear maps

$$v:(\psi) \to \psi^\perp$$

Define a Hilbert space structure on $P(H)_{(\psi)}$ - also denoted by $\langle\ /\ \rangle$ -via the following formula:

$$\langle v_1/v_2 \rangle = \frac{\langle v_1(\psi)/v_2(\psi) \rangle}{\langle \psi/\psi \rangle} \qquad 5.3$$

for $v_1, v_2 \epsilon P(H)_{(\psi)}$. (Note that formula 5.3 is independent of the ψ chosen to generate the one-dimensional linear subspace (ψ).)

Define an R-bilinear, symmetric map

$$\beta : P(H)_{(\psi)} \times P(H)_{(\psi)} \to R$$

by the following formula:

$$\beta(v_1, v_2) = \frac{1}{2} (\langle v_1/v_2 \rangle + \langle v_2/v_1 \rangle) \qquad 5.4$$

for $v_1, v_2 \epsilon P(H)_{(\psi)}$.

Define an R-bilinear, skew-symmetric map $\omega = P(H)_{(\psi)} \times P(H)_{(\psi)} \to R$ by the following formula:

$$\omega(v_1, v_2) = \frac{1}{2i} (\langle v_1/v_2 \rangle - \langle v_2/v_1 \rangle) \qquad 5.5$$

As (ψ) varies over P(H), β defines a Riemannian metric for P(H). ω defines a symplectic structure which is non-degenerate, i.e. a "canonical" structure for P(H). (Exercise. Show that $d\omega = 0$, so that ω really does define a symplectic structure.)

Let $J : P(H)_{(\psi)} \to P(H)_{(\psi)}$ be defined as follows:

$$J(v) = iv \text{ for } v \epsilon P(H)_{(\psi)} \qquad 5.6$$

Then, ω,β and J are interrelated as follows:

$$\omega(v_1,v_2) = \beta(Jv_1,v_2) \qquad 5.7$$

for $v_1,v_2 \in T(P(H))$.

As explained in Chapter II, relation 5.7, together with the integrability of the almost complex structure defined by J and the condition $d\omega = 0$, amount to saying that the triple (β,ω,J) define a Kähler structure for P(H).

Let us now compute the distance between two points of P(H) defined by the metric β.

Exercise. Let $t \to \psi(t)$, $a \leq t \geq b$, be a C^∞ curve in H-(0). Set:

$$v(t)(\psi(t)) = \frac{d\psi}{dt} - \langle\psi/ \frac{d\psi}{dt}\rangle \frac{\psi(t)}{\langle\psi(t)/\psi(t)\rangle^{1/2}},$$

regarding v(t) as a C-linear map: $(\psi(t)) \to \psi(t)^\perp$. Show that v(t) may be identified with the tangent vector to the curve $t \to (\psi(t))$. Show, as a consequence, that the length of the curve $t \to (\psi(t))$ with respect to the Riemannian metric defined above is equal to:

$$\int_a^b \langle v(t)/v(t)\rangle^{1/2} dt.$$

Let ψ_1,ψ_2 be elements of H-(0). To study their projections in P(H), we might as well suppose that they are of unit length, i.e.,

$$||\psi_1|| = ||\psi_2|| = 1 \qquad 5.8$$

(Recall that $|| \quad ||$ is the standard notation in Hilbert space theory for the norm, i.e.

$$||\psi|| = \langle\psi/\psi\rangle^{1/2}).$$

Consider the inner product $\langle\psi_1/\psi_2\rangle$ as a complex number:

$$\langle\psi_1/\psi_2\rangle = re^{i\alpha}, \qquad 5.9$$

with r and α real. Thus,

$$\langle\psi_1/e^{-i\alpha}\psi_2\rangle = r \qquad 5.10$$

Hence, in studying the image of ψ_1,ψ_2 in P(H), we may suppose that:

$$\langle\psi_1/\psi_2\rangle \text{ is a real number} \qquad 5.11$$

Let θ be the angle between ψ_1 and ψ_2, such that: $0 \leq \theta \leq \pi/2$, i.e.

$$\cos\theta = |\langle\psi_1/\psi_2\rangle| \qquad 5.11$$

<u>Exercise</u>. Show that there is an element $\psi_3 \epsilon H$ such that:

$$\langle \psi_1 / \psi_3 \rangle = 0 \tag{5.12}$$

$$||\psi_3|| = 1 \tag{5.13}$$

$$\psi_2 = \cos\theta\ \psi_1 + \sin\theta\ \psi_3 \tag{5.14}$$

Now set:

$$\psi(t) = \cos t\ \psi_1 + \sin t\ \psi_3 \tag{5.1}$$

Thus, $\psi(0) = \psi_1$, $\psi(\theta) = \psi_2$.

$$\langle \psi(t)) / \frac{d\psi}{dt} \rangle = 0 \tag{5.16}$$

Consider the curve $t \to (\psi(t))$, $0 \leq t \leq \theta$, in $H-(0)$. It goes from (ψ_1) to (ψ_2). Its length in the β-metric is: (because of 5.16)

$$\int^{\theta} \beta(\psi'(t), \psi'(t))^{1/2} dt$$

$$= \int^{\theta} \langle \frac{d\psi}{dt} / \frac{d\psi}{dt} \rangle^{1/2} dt$$

$$= \theta \tag{5.17}$$

Exercise. Show that the curve $t \to (\psi(t))$, $0 \leq t \leq \theta$, given by 5.18, going from (ψ_1) to (ψ_2) is a geodesic of the β metric. Hence the distance from (ψ_1) to (ψ_2) with respect to the β-metric is θ, where θ is the unique number between 0 and $\pi/2$ such that:

$$\cos\theta = \frac{|\langle\psi_1|\psi_2\rangle|}{||\psi_1||\,||\psi_2||} \qquad 5.18$$

Exercise. Define a function $d: P(H)\times P(H) \to R$ by the following rules:

$$0 \leq d((\psi_1),(\psi_2)) \leq \pi/2 \qquad 5.19$$

for $(\psi_1),(\psi_2)\in P(H)$

$$\cos d((\psi_1),(\psi_2)) = \frac{|\langle\psi_1/\psi_2\rangle|}{||\psi_1||\,||\psi_2||} \qquad 5.20$$

for $\psi_1,\psi_2 \in H-(\theta)$

Show (independently of Riemannian geometry or finite dimensionality of H) that d is a metric space function for P(H).

Remark. In case H is infinite dimensional, this result will provide a convenient way to define a topology for P(H).

This is important for quantum mechanics, where P(H) plays the role of "state space."

We now turn to the study of the relation between R-linear transformation on H and distance-preserving transformations on P(H).

6. LINEAR AND ANTI-LINEAR AUTOMORPHISMS OF COMPLEX PROJECTIVE SPACE

Let H be a complex vector space, (not necessarily finite dimensional) with a Hilbert space inner product $(\psi_1,\psi_2) \to \langle\psi_1/\psi_2\rangle$. Let P(H) denote the projective space of one dimensional complex-linear subspaces of H. Let d (,) be the metric distance function on P(H), defined by formula 5.20. Let $A:H \to H$ be a one-one, R-linear map. We shall first find the conditions that A pass to the quotient to define a map: $P(H) \to P(H)$, and then investigate the conditions that this map preserves the distance function d.

Let us suppose that A does pass to the quotient. Fix a $\psi\varepsilon H$. Then, $A(\psi)$ and $A(i\psi)$ must project into the same element of P(H); i.e. there exists a complex number $\lambda(\psi)$, such that:

$$A(i\psi) = \lambda(\psi)A(\psi) \tag{6.1}$$

Also,

$$-A(\psi) = A(ii\psi) = \lambda(i\psi)A(i\psi)$$

$$= -A(i\psi)/\lambda(\psi)$$

Hence,

$$\lambda(i\psi)\lambda(\psi) = -1 \tag{6.2}$$

<u>Lemma 6.1.</u> $\lambda(\psi)$ is not real, for all $\psi\varepsilon H-(0)$.

<u>Proof</u>. Suppose otherwise. Since A is R-linear, we have, from 6.1 ,

$$A(i\psi) = A(\lambda(\psi)\psi),$$

or

$$A(i\psi - \lambda(\psi)\psi) = 0,$$

or

$$i\psi = \lambda(\psi)\psi,$$

since A is one-one. This is the desired contradiction.

Lemma 6.2. If ψ_1, ψ_2 are linearly independent over the complex numbers, so are $A(\psi_1)$, $A(\psi_2)$.

Proof. Otherwise, there would be a relation of the following form, with a complex number c.

$$A(\psi_1) + cA(\psi_2) = 0 \qquad 6.2$$

Set:

$$c = a + bi$$

$$\lambda(\psi_2) = a'+b'i,$$

with $a, b, a', b' \in R$.

By Lemma 6.1, $b' \neq 0$. 6.2 now takes the following form:

$$A(\psi_1) + A(a\psi_2) + iA(b\psi_2) = 0 \qquad 6.3$$

Now,

$$A(i\psi_2) = \lambda(\psi_2)A(\psi_2) = (a'+b'i)A(\psi_2)$$

$$= A(a'\psi_2) + iA(b'\psi_2).$$

Hence,

$$iA(\psi_2) = \frac{A(i\psi_2) - A(a'\psi_2)}{b'} \qquad 6.4$$

Combining 6.3 and 6.4:

$$A(\psi_1) + A(a\psi_2) + A\left(\frac{b}{b'}(i\psi_2 - a'\psi_2)\right) = 0$$

Since A is one-one, this forces a complex linear dependence relation between ψ_1 and ψ_2, which is a contradiction.

With these two lemmas, we can now determine more precisely the structure of the function $\psi \to \lambda(\psi)$ from $H \to C$. Let ψ_1, ψ_2 be two complex linearly independent elements of H. Then,

$$A(i(\psi_1+\psi_2)) = \lambda(\psi_1+\psi_2)A(\psi_1+\psi_2) = \lambda(\psi_1+\psi_2)(A(\psi_1) + A(\psi_2))$$

$$= \lambda(\psi_1)A(\psi_1) + \lambda(\psi_2)A(\psi_2)$$

Since $A(\psi_1)$, $A(\psi_2)$ are complex linearly independent, we have:

$$\lambda(\psi_1) = \lambda(\psi_1+\psi_2) = \lambda(\psi_2). \qquad 6.5$$

Let us apply this to ψ_1 and $i\psi_2$. We then have:

$$\lambda(i\psi_2) = \lambda(\psi_1) = \lambda(\psi_2).$$

Let us set $\psi_2 = \psi$ in relation 6.2:

$$\lambda(\psi)^2 = -1.$$

In particular, we have proved:

$$\lambda(\psi) = \pm i \qquad 6.6$$

for all $\psi \epsilon H$.

The $+i$ possibility corresponds to the case where A is a complex-linear map. The $-i$ possibility corresponds to the "anti-linear" case, according to the following definition:

Definition. A map $A:H \to H'$ between complex vector spaces is anti-linear if it is R-linear, and if:

$$A(c\psi) = c^*A(\psi)$$

for $\psi \epsilon H$, $c \epsilon C$

We can now sum up as follows.

Theorem 6.3. Let H be a complex vector space, and let $A:H \to H$ be a one-one, R-linear map. Then, A passes to the quotient to define a map $P(H) \to P(H)$ if and only if A is linear or anti-linear.

Remark. Notice that the proof given above indeed only requires that A be one-one, and that the Hilbert space structure for H was not used.

Now, let us reimpose the Hilbert space structure on H, and use it to construct it to construct the Riemannian metric, and the corresponding distance function d(,), as explained in Section 2. Let $A:H \to H$ be a one-one, complex linear or anti-linear map. We shall now find the conditions that the map A induces on P(H) preserves the distance function d, i.e. that it be an isometry.

Now, by its definition

$$\cos d((\psi_1),(\psi_2)) = \frac{|<\psi_1/\psi_2>|}{||\psi_1||\,||\psi_2||} \qquad 6.7$$

Hence, the condition that A act as isometry on P(H) is that:

$$\frac{|<A\psi_1/A\psi_2>|}{||A\psi_1||\,||A\psi_2||} = \frac{|<\psi_1/\psi_2>|}{||\psi_1||\,||\psi_2||} \qquad 6.8$$

for $\psi_1,\psi_2 \in H$.

In particular, note the following fact:

If ψ_1,ψ_2 are perpendicular and non-zero, so are $A(\psi_1),A(\psi_2)$. 6.9

Suppose now that ψ_1,ψ_2 are two elements of H such that:

$$<\psi_1/\psi_2> = 0; \quad ||\psi_1|| = 1 = ||\psi_2||. \qquad 6.10$$

Notice that 6.9 gives us no direct information about $||A(\psi_1)||$. Such information can be obtained by applying a suitable "polarization" process to 6.9. Namely, set:

$$\psi_1' = a\psi_1 + b\psi_2$$

$$\psi_2' = -b\psi_1 + a\psi_2, \tag{6.11}$$

with a,b real numbers.

Then, $<\psi_1'/\psi_2'> = 0$, hence also, by 6.9,

$$0 = <A\psi_1'/A\psi_2'>$$

$$= <aA(\psi_1) + bA\psi_2/ - bA(\psi_1) + aA\psi_2>$$

$$= -ab||A\psi_1||^2 + ab||A\psi_2||^2$$

This identity forces - since a, b are real numbers -

$$||A\psi_1||^2 = ||A\psi_2||^2 \tag{6.12}$$

<u>Theorem 6.4.</u> If A is on R-linear map: $H \to H$ which acts on P(H) preserving the distance function, then there is a positive real number such that:

$$\rho = \frac{||A\psi||}{||\psi||} \tag{6.13}$$

for all $\psi \varepsilon H$

Proof. Let ψ_1, ψ be arbitrary elements of H, with $||\psi_1|| = 1$. We can then construct (by the Gram-Schmidt orthogonalization process, for example) an element $\psi_2 \varepsilon H$ such that:

$$||\psi_2|| = 1; \quad \langle\psi_1/\psi_2\rangle = 0.$$

$$\psi = a\psi_1 + b\psi_2, \text{ with } a,b \text{ real.}$$

Set: $\rho = ||A\psi_1|| =$, by 3.12, $||A\psi_2||$. Then, $||\psi||^2 = a^2+b^2$;

$$||A\psi||^2 = a^2||A\psi_1||^2 + b^2||A\psi_2||^2$$

$$= \rho^2(a^2+b^2),$$

which prove 6.13.

At most changing A to $A' = \frac{A}{\sqrt{\rho}}$, we may suppose without loss in generality that:

$$||A\psi|| = ||\psi|| \qquad 6.14$$

Quantum mechanically, 6.14 means that the transformation $\psi \to A(\psi)$ "preserves probabilities." Such transformation were first studied systematically by Wigner [1], who also proved the basic fact that such probability preserving transformations (not assumed, a-priori, to be linear) arose from a unitary or anti-unitary transformation on H.

<u>Definition</u>. A transformation $A:H \to H$ is said to be an <u>isometry of the Hilbert space structure</u> if it is complex linear, and if:

$$\langle A\psi_1/A\psi_2\rangle = \langle\psi_1/\psi_2\rangle \qquad 6.16$$

for $\psi_1,\psi_2 \epsilon H$.

A is said to be an <u>anti-isometry</u> if it is anti-linear and if

$$\langle A\psi_1/A\psi_2\rangle = \langle\psi_2/\psi_1\rangle \qquad 6.17$$

for $\psi_1,\psi_2 \epsilon H$

A is said to be <u>unitary</u> if it is an isometry and A^{-1} exists; A is <u>anti-unitary</u> if A is an anti-isometry and A^{-1} exists.

<u>Exercise</u>. If $A:H \to H$ is complex linear and satisfies condition 6.14, show that A is an isometry. If A is anti-linear and satisfies 6.14, show that it is an anti-isometry.

<u>Problem</u>. If $A:P(H) \to P(H)$ is a distance preserving map, show that is arises in this way from an isometry or anti-isometry of H.(This is, roughly, equivalent to Wigner's theorem.)

<u>Problem</u>. Describe, in abstract groups theoretical language, the relation between the group of R-linear isomorphisms: $H \to H$ which are either unitory or anti-unitory and the group of distance-preserving isomorphisms: $P(H) \to P(H)$. Discuss the interesting subgroups and invariant subgroups of these groups.

7. QUANTUM OBSERVABLES AS FUNCTIONS ON COMPLEX PROJECTIVE SPACE

The complex projective space associated with a Hilbert space is important for quantum mechanics, since it plays the role of the "state" space, similiarly to the role the cotangent bundle in configuration space plays in classical mechanics.

Let us briefly recall this connection, using the ideas developed in LMP, Vol. II. In the standard approach to quantum mechanics one is given a Hilbert space H. (In the simplest examples of particle quantum mechanics, H is the space of "Schrodinger wave functions," with the Hilbert space inner product $< / >$ related to the "probabilistic" features of quantum mechanics.)

Recall also that an "observable" is a Hermitian operator: $H \to H$. Given such an operator A, one can define a real-valued function $f_A : P(H) \to P(H)$ by the following formula:

$$f_A((\psi)) = \frac{<\psi/A\psi>}{<\psi/\psi>} \qquad 7.1$$

for $\psi \in H$

Quantum mechanically, 7.1 is the "expectation value" of the "observable A" in the "state (ψ)." From the mathematical point of view it is interesting as a realization of the "observables" as real-valued functions on the space of "states." (Recall that such a realization is typical of mechanics, whether of "classical" or "quantum" type.)

Now, let us suppose that H is finite dimensional, so that P(H) is a manifold in the standard sense. We shall compute df_A and the integral of f_A over P(H) with respect to the volume element defined by the Riesmannian metric on P(H) defined in Section 6.

Since A is Hermitian, iA is skew-Hermitian. Set:

$$g(t) = \exp(itA), \qquad 7.2$$

$$-\infty < t < \infty$$

Then, $t \to g(t)$ is a one parameter group of unitary operators on H. Each g(t) permutes the one-dimensional subspace of H, hence each g(t) acts as a diffeomorphism of P(H). It is readily verified that this diffeomorphism is an automorphism of the Kähler structure on P(H). Thus, $t \to g(t)$ acts on P(H) as a one-parameter group of Kähler metric automorphisms.

Let X be the vector field on P(H) which is the infintisimal generator of the one parameter group $t \to g(t)$. Recall that, we have defined in Section 3, X_B as a vector field on P(H), for any linear transformation $B:H \to H$, via the following formula:

$$X_B((\psi))(\psi') = \text{image of } B(\psi)$$

$$\text{in } H/(\psi) = \psi^{\perp}, \qquad 7.3$$

$$\text{for all } \psi' \varepsilon (\psi).$$

Problem. Show that X, the infinitesimal generator of the group $t \to g(t)$ given by 7.2, is given by the following formula:

$$X = X_{iA} \qquad 7.4$$

Show also that:

$$df_A = X \lrcorner \omega, \qquad 7.5$$

where ω is the 2-form of the symplectic structure defined by the Kähler metric on P(H).

Exercise. Show that $df_A = 0$ at a point $(\psi)\varepsilon P(H)$ if and only if ψ is an eigenvector of A.

Remark: These results are literally only true for the case of finite dimensional H, because it is only for these that we have a properly defined manifold structure. However, in the infinite dimensional case, they remain true in some "symbolic" sense, thus giving us a way to think of quantum mechanics in a way that is more geometric than customary.

Having discussed the derivatives of the f_A, let us consider the integrals:

Theorem 7.1. Suppose H is a finite dimensional complex vector space. There is a constant c such that i for each Hermitian operator $A:H \to H$,

$$\int_{P(H)} f_A(p)\,dp = c \text{ trace } (A), \qquad 7.6$$

where "dp" denotes the volume element differential form on P(H) defined by the Riemannian metric β.

Proof. Let G be the group of unitary transformations of the vector space H. G passes to the quotient to act on P(H).

Exercise. Show that G acts transitively on P(H). Identify the Lie group which is the isotropy subgroup.

G is a compact Lie group. Let dg be its bi-invariant volume element differential form, normalized so that the total volume of G is equal to one.

Now, the compact G acts transitively on P(H), and the volume element dp is invariant under G. Up to a constant multiple, there is but one volume element invariant under G; it is related to the biinvariant volume element dg on G, as indicated in the following exercise.

Let ψ_1 be a fixed element of H such that: $||\psi_1|| = 1$.

Exercise. Show that there is a constant c' such that:

$$\int_{P(H)} f(p)dp = c' \int_G f(g(\psi_1))dg \qquad 7.7$$

for all continuous functions f on P(H).

We shall compute the integral of f_A over P(H) using 7.7, and some standard orthogonality relations about integrals over G of matrix elements of irreducible representations. (For this, see LMP, Vol. II or Wallach [1]).

Let $\psi_1, \ldots, \psi_n$ be an orthonomial basis for H_1 with ψ_1 the fixed element involved in 7.7. Choose indices i,j running from 1 to n, with the summation convention, and let $a_{ij}(g)$ be the matrix element function on G such that:

$$g(\psi_i) = a_{ji}(g)\psi_j \qquad 7.8$$

The orthogonality relations are now:

$$\int_G a_{ji}(g)^* a_{j'i'}(g)dg = n\sigma_{jj'}\sigma_{ii'} \qquad 7.9$$

Hence, using 7.7-7.8, we have:

$$\int_{P(H)} f(p)dp = c' \int_G \langle g\psi_1/Ag\psi_1\rangle dg$$

$$= c' \int_G a_{i1}(g)^* a_{j1}(g) \langle \psi_i/A\psi_j\rangle dg$$

= , using 7.9,

$$c'n \langle \psi_i/A\psi_i\rangle$$

$$= c'n \text{ trace } (A).$$

This finishes the proof of formula 7.6, hence also of Theorem 7.1.

Remarks: The unknown constant c in 7.6 may be put in another form. In 7.6, set A = (identity operator). Then,

$$\text{trace } A = n = \dim H.$$

f_A is of course identically one. Hence, the left hand side of 7.6 is equal to the volume V(n) of P(H) with respect to the Riemannian volume element dp. Thus, we have:

$$V = cn, \quad \text{or}$$

$$c = \frac{V(n)}{n} \tag{7.10}$$

Now, the volume V is not uniquely determined, since the metric on P(H) is determined only up to a constant multiple. However, the matric can be normalized using the curvature.

Exercise. Suppose that the Kähler metric on P(H) is normalized so that the maximum value of its sectional curvatures is equal to one. Compute V as a function of n.

So far, we have been working with finite dimensional vector spaces H, which are only indirectly of interest for quantum mechanics. However, recall one of our basic themes - we are studying finite dimensional situations in part because of clues we can get for possible generalizations to infinite dimensional situations. We can apply this here, and make certain speculative suggestions about what happens in the infinite dimensional situations.

Let us combine 7.6 and 7.10 as follows:

$$\int_{P(H)} f_A dp = \frac{V(n)}{n} \text{ trace } (A). \qquad 7.11$$

Suppose H is an infinite dimensional Hilbert space. Let

$$H_1, H_2, \ldots$$

be a sequence of finite dimensional subspaces of H such that:

$$H_1 \subset H_2 \subset \ldots \qquad 7.12$$

$$\dim (H_n) = d(n) \qquad 7.13$$

The union of the subspaces H_n is dense in H. 7.14

One might call a structure $(H, H_n,\ n = 1, \ldots)$ satisfying 7.12-7.14 a <u>filtered Hilbert space structure</u> for H. (In PALG I have presented some material related to this concept.)

Then, $P(H_n)$ is a subspace of P(H), for all n. A given Hilbert space structure on H restricts to define a Hilbert space structure on H_n, hence also a metric on $P(H_n)$. Suppose that:

$$\lim_{n\to\infty} \frac{V(n)}{d(n)} = V. \tag{7.15}$$

Let $A_n : H_n \to H_n$ be the operator such that:

$$A_n(\psi) = \text{orthogonal projection of } A(\psi) \text{ on } H_n, \text{ for } \psi \varepsilon H_n \tag{7.16}$$

Suppose that A is a "trace-class" or "nuclear" operator. (See Gelfand-Vilenkin [1] or Yosida [1]). Then "trace A" is defined as a real number, and it has the property that:

$$\text{trace } A = \lim_{n\to\infty} \text{trace } (A_n) \tag{7.17}$$

Let $f : P(H) \to R$ be a real valued function. Let us define the integral over P(H) as the following limit, if it exists:

$$\int_{P(H)} f\, dp = \lim_{n\to\infty} \int_{P(H_n)} f(p_n) dp_n \tag{7.18}$$

Putting all this together, we have the following formula, for a trace class Hermitian operator A:

$$\int_{P(H)} f(p) dp = V \text{ trace } (A) \tag{7.19}$$

Ramifications of these ideas should be investigated further, since they seem to be important for understanding, on the one hand, infinite dimensional "integral geometry," and on the other hand they should be very useful in quantum mechanics and statistical mechanics. I hope to go into this myself at a later point of this treatise.

Chapter III

THE GEOMETRY OF SCATTERING THEORY

1. INTRODUCTION

In FA, LMP, vol. II, and PALG I have already presented some of the fascinating mathematical ideas implicit in the area of physics called "scattering theory". In this chapter, I want to delve into the material more deeply from the point of view of differential geometry and Lie group theory.

As an introduction, I shall briefly define the basic mathematical objects needed to describe the scattering of two relativistic, spin-zero particles, in the form usually to be found in the physics literature.

Let p denote an element of R^4. Physically, p denotes the relativistic 4-momentum of particles. Consider p as an ordered pair $(E, \vec{p})$ of a real number E (the energy of the particle) and a vector $\vec{p} \in R^3$ (the momentum of the particle in the sense of ordinary, 3-dimensional mechanics). Define the Lorentz inner product as follows:

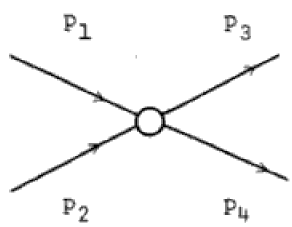
If $p_1 = (E_1, \vec{p}_1)$, $p_2 = (E_2, \vec{p}_2)$, then

$$p_1 \cdot p_2 = \frac{E_1 E_2}{c^2} - \vec{p}_1 \cdot \vec{p}_2 \qquad 1.1$$

(c = velocity of light)

In 1.1, $\vec{p}_1 \cdot \vec{p}_2$ denotes the usual Euclidean, positive-definite inner product for vectors in R^3.

Now, consider a scattering situation, consisting of two spinless particles of 4-momentum $p_1, p_2 \in R^4$ coming in, and two spinless particles p_3, p_4 going out, as symbolized by the following diagram:



1.2

Suppose the particles are of mass m_1, m_2, m_3, m_4 i.e.,

$$m_i^2 = p_i \cdot p_i; \quad i = 1,2,3,4. \tag{1.3}$$

<u>Definition</u> The <u>kinematic space</u>, denoted by M, for this scattering situation consists of the 4-tuples of vectors (p_1, p_2, p_3, p_4) such that:

$$p_1 + p_2 = p_3 + p_4 \tag{1.4}$$

$$p_i \cdot p_i = m_i^2, \; i = 1,2,3,4. \tag{1.5}$$

Let s, t be the functions: $M \to R$ defined as follows:

$$s = (p_1 + p_2)^2 \tag{1.6}$$

$$t = (p_1 - p_3)^2 \tag{1.7}$$

(Physically, s is called the <u>center of mass energy</u>, t is called the <u>momentum transfer</u>).

The functions s, t are the invariants of the action of the Lorentz group L on M. A <u>scattering amplitude function</u> is a function $f: M \to C$ which is invariant under the action of L. We shall suppose, for simplicity - or at least until another assumption seems reasonable on either mathematical or physical grounds - that f is infinitely differentiable, i.e. of differentiability class C^∞.

We can now indicate how such a scattering amplitude function gives rise to a "scattering operator". Let M_i, $i = 1,2,3,4$, denote the hyperboloid of R^4, i.e., the set of vectors $p_i \in R^4$ such that:

$$p_i^2 = m_i^2 \tag{1.8}$$

1.8

(In the physics literature, this is called the <u>mass-shell</u>.)

Let dp_i denote the Lorentz-invariant volume element on M_i:

<u>Exercise</u> If $p_i = (E_i, \vec{p}_i)$, show that

$$dp_i = \frac{d\vec{p}_i}{|E_i|}, \tag{1.9}$$

where $d\vec{p}_i$ is the Euclidean volume element form on R^3.

Let H_i denote the Hilbert space of C^∞, complex valued, rapidly decreasing functions $\psi: M_i \to C$, with the Hilbert inner product defined as follows:

$$\langle\psi|\psi\rangle = \int_{M_i} \psi(p_i)^*\psi(p_i)dp_i \tag{1.10}$$

The Lorentz group, $SO(1,3)$, acts unitarily on H_i, via its geometric action on M_i.

Let H be the tensor product Hilbert space $H_1 \otimes H_2$. Elements of H represent the Hilbert space of "incoming" states, of the scattering situation. Also, $H' = H_3 \otimes H_4$ are the "outgoing" states. Geometrically, an element $\psi \in H$ is a function $\psi(p_1,p_2)$ on $M_1 \times M_2$. Similiarly, an element $\psi' \in H'$ is a function $\psi'(p_3,p_4)$ on $M_3 \times M_4$.

Definition Let S be a linear operator: $H \to H'$. Then, a function f on M is said to be a scattering amplitude function associated to the operator S if:

$$\langle\psi'|S|\psi\rangle = \int_M \psi'(p_3,p_4)^* f(p_1,p_2,p_3,p_4)\psi(p_1,p_2)dp \qquad 1.11$$

for $\psi \in H$, $\psi' \in H'$,

where "dp" denotes a suitable Lorentz-invariant volume element on M. (In fact, "dp" can be chosen as the volume element associated with the Riemannian metric on M given by its natural imbedding into R^{16}.)

The physicists use the Dirac notations to symbolize relation 1.11:

$$\langle p_3,p_4|S|p_1,p_2\rangle = \delta(p_1+p_2-p_3-p_4)f(p_1,p_2,p_3,p_4) \qquad 1.12$$

We see that 1.11 is just the form 1.12 takes when one integrates over test-functions $\psi \in H$, $\psi' \in H'$.

Exercise Show that relation 1.11, plus invariance of f under the action of the Lorentz groups $(\ell,(p_1,p_2,p_3,p_4)) \to (\ell p_1,\ldots,\ell p_4)$ on M, implies that S intertwines the action of the Poincare group on H and H'.

Such a scattering amplitude function f is a function of s and t, since s and t serve to parametrize the oribt space of the action of L on M. However, there are other ways of presenting such f's. We shall consider the one most relevant to the study of the "partial wave expansion" of a scattering amplitude.

For each positive real number b, let N_b denote the set of all vectors $(p_3,p_4) \in R^4 \times R^4$ such that:

$$p_3^2 = m_3^2; \; p_4^2 = m_4^2. \qquad 1.13$$

$$p_3+p_4 = (\sqrt{b},0)$$

Let us parametrize N_b explicitly. Set:

$$p_3 = (E_3,\vec{p}_3); \; p_4 = (E_4,\vec{p}_4)$$

Then, condition 1.13 means that:

$$E_3+E_4 = b; \; \vec{p}_3 = -\vec{p}_4. \qquad 1.14$$

$$\frac{E_3^2}{c^2} - \vec{p}_3^{\,2} = m_3^2 \qquad 1.15$$

$$\frac{E_4^2}{c^2} - \vec{p}_4^{\,2} = m_4^2 \qquad 1.16$$

We shall now eliminate E_3 and E_4 from these equations, and obtain an equation for $\vec{p}_3$ alone:

$$E_4 = c\sqrt{\vec{p}_4^{\,2}+m_4^2} = c\sqrt{\vec{p}_3^{\,2}+m_4^2}$$

$$E_3 = \sqrt{b} - c\sqrt{\vec{p}_3^{\,2}+m_4^2},$$

$$\vec{p}_3^{\,2} = \frac{1}{c^2}\left(b-2\sqrt{a}\; c\sqrt{\vec{p}_3^{\,2}+m_4^2} + c^2(\vec{p}_3^{\,2}+m_4^2)\right)-m_3^2$$

$$= \frac{b}{c^2} - \frac{2\sqrt{b}}{c}\sqrt{\vec{p}_3^{\,2}+m_4^2} + \vec{p}_3^{\,2}+m_4^2-m_3^2,$$

or

$$\frac{2\sqrt{b}}{c}\sqrt{\vec{p}_3{}^2+m_4{}^2} = \frac{b}{c^2} + m_4{}^2 - m_3{}^2,$$

$$\frac{4b}{c^2}(\vec{p}_3{}^2+m_4{}^2) = \frac{(b+c^2(m_4{}^2-m_3{}^2))^2}{c^4}$$

$$\vec{p}_3{}^2 = \frac{(b+c^2(m_4{}^2-m_3{}^2))^2}{4bc^2} - m_4{}^2 \qquad 1.17$$

Notice that this relation shows that $\vec{p}_3$ lies on a sphere in R^3 whose radius is given by the right hand side of 1.17.

<u>Exercise</u> Show that the map $(p_3,p_4) \to \vec{p}_3$ defines a diffeomorphism of N_b with the sphere in R^3, of radius λ, where λ^2 is the right hand side of 1.17.

To understand the geometric meaning of "partial wave analysis", for each positive real number b and each vector $\vec{p}_0 \in R^3$ define a map

$$\phi_{b,\vec{p}_0}: N_a \to M$$

by the following formulas:

$$\phi_{b,\vec{p}_0}(p_3,p_4) = ((c\sqrt{m_1{}^2+\vec{p}_0{}^2},\vec{p}_0),(b-c\sqrt{m_1{}^2+\vec{p}_0{}^2}),p_3,p_4) \qquad 1.18$$

(Physically, $\vec{p}_0$ is then the 3-momentum of the first particle). If $f: M \to C$ is a scattering amplitude function, it can be pulled back under $\phi_{b,\vec{p}_0}$ to define a function

$$f_{b,\vec{p}_0} = \phi_{b,\vec{p}_0}{}^*(f) \quad \text{on } N_b.$$

Usually the dependence on $\vec{p}_0$ is suppressed, and this function is denoted simply by f_b. Further, since $(p_3+p_4)^2 \equiv s(\phi_{b,\vec{p}_0}(p_3,p_4)) = b$, the label "s" is essentially just the same as the label "b", and the physicists usually make no notational distinction between them.

There is a further symmetry to exploit; f is invariant under the action of the Lorentz group L on M. Now, an $\ell \in L$ acts on M via the "diagonal" action:

$$\ell(p_1,p_2,p_3,p_4) = (\ell p_1,\ell p_2,\ell p_3,\ell p_4).$$

f is invariant under the action, i.e.

$$f(\ell(p_1,p_2,p_3,p_4)) = f(p_1,p_2,p_3,p_4) \qquad 1.19$$

$$\text{for } (p_1,p_2,p_3,p_4) \in M,\ \ell \in L.$$

(The generalization to allow particles with "spin" involves allowing f to transform under L as one of a number $f_1,\ldots,f_N$ of functions which transform under L via a finite dimensional representation of L. In the physics literature, these are called "covariant amplitudes", see Barut [1]).

The action of $\ell \in L$ on $p \in R^4$ is just the usual action of $L = SO(1,3)$ by its 4-dimensional, real representation. Let K be the subgroup of $k \in L$ such that:

$$k(E,\vec{p}) = (E,k(\vec{p})), \qquad 1.20$$

where $\vec{p} \to k\vec{p}$ is a linear transformation $R^3 \to R^3$ which preserves the inner product. Then K is isomorphic to SO(3,R), the notation group of R^3. Let K_0 be the subgroup of $k \in K$ such that:

$$k(\vec{p}_0) = \vec{p}_0.$$

Then, K_0 is isomorphic to SO(2,R). Now, the invariance 1.19 of f under K_0 pulls-back via the map $\phi_{b,\vec{p}_0}$ to give invariance of $f_{b,\vec{p}_0}$ under K_0.

In summary, $f_{b,\vec{p}_0}$ is a function on the sphere of radius λ in R^3 (where λ^2 is the right hand side of 1.17), which is invariant under the action of a one parameter group of rotations, K_0. It is then appropriate to expand $f_{b,\vec{p}_0}$ in terms of "spherical functions", i.e. eigenvalues of the Laplace-Beltrami operator on the sphere which are invariant under the action of K_0. The parameter λ is given as a function of b, i.e. s, when c, m_4, m_3 are held fixed (as they will be in any specific physical situation, of course). Now, in appropriate coordinates for the sphere, these spherical functions are just the Legendre functions $P_n(\ )$. Thus, the "partial wave expansion" of f will take the following form:

$$f_{b,\vec{p}_0} = \sum_n (2n+1)a_n(b)P_n \qquad 1.21$$

Now, the physicists make no notational distinction between $f_{b,\vec{p}_0}$ and f and between b and s. Thus, in their notation, the expansion 1.21 would take the following form:

$$f = \sum_n (2n+1)a_n(s)P_n \qquad 1.22$$

For example, in the simplest case where all the masses are equal, $m = m_1 = m_2 = m_3 = m_4$, and where $c = 1$, the expansion 1.22 take the more explicit form:

$$f(s,t) = \sum_n (2n+1)a_n(s)P_n\left(1 + \frac{2t}{s-4m^2}\right) \qquad 1.23$$

(See Collins and Squires [1], p. 21).

We now see an interesting deformation problem: Describe what happens to the expansion 1.21 as $b \to 0$ or ∞, with all the other parameters held fixed. From the relation defining $N_{b,\vec{p}_0}$, namely:

$$\vec{p}_3{}^2 = \lambda^2;\ \lambda^2 = \frac{\left(b+c^2(m_4{}^2-m_3{}^2)\right)^2}{4bc^2} - m_4{}^2, \qquad 1.24$$

wee see that this deformation problem corresponds geometrically to a problem of expansion over a sphere whose radius goes to infinity or zero.

Remark: There are even more interesting deformation problems, where one allows all the free parameters to approach limiting values, and/or extends the vectors and functions to complex values. Of course, a good deal of the emphasis of physicsts concerned with "S-matrix theory" is the study of these problems. One of my aims in this book is to develop systematic and general mathematical methods for their study.

2. THE KINEMATIC SPACES

Now we shall translate the material in Section 1 (which is more-or-less the standard material in physics books) into a more general context.

Let V be a real, finite dimensional vector space, with a symmetric, bilinear, non-degenerate form: $V \times V \to R$ given on V. We shall denote the image of $(v_1, v_2) \in V \times V$ under this form by:

$$v_1 \cdot v_2$$

For $v \in V$, set:

$$v^2 = v \cdot v.$$

Let r be an integer greater or equal to one, and let $m_1^2, \ldots, m_r^2$ be real numbers. Let M be the space of r-tuples $(v_1, \ldots, v_r)$ of elements of V, such that:

$$\text{a)} \quad v_1^2 = m_1^2, \ldots, v_r^2 = m_r^2 \qquad 2.1$$

$$\text{b)} \quad v_1 + \ldots + v_r = 0$$

M will be called a <u>kinematic space</u>. Denote a typical point of M by p. Thus, M is the subset of V^r, the Cartesian product of r-copies of V with itself, defined by conditions 2.1.

<u>Definition</u> A point $p = (v_1, \ldots, v_r) \in M$ is in <u>general position</u> if the dimension of the linear subspace of V spanned by the vectors $v_1, \ldots, v_r$ of V is equal to:

$$\begin{aligned} &(r-1) \quad \text{if} \quad (r-1) \leq \dim V \\ &\dim V \quad \text{if} \quad (r-1) \geq \dim V., \end{aligned} \qquad 2.2$$

i.e. if the dimension of this linear subspace is as large as possible, consistent with the constraint 2.1(b) and the fact that the vectors spanning it lie in a finite dimensional subspace V.

Let G be the group of linear transformations $g: V \to V$ which leave the form invariant, i.e. such that:

$$gv_1 \cdot gv_2 = v_1 \cdot v_2 \qquad \text{for } v_1, v_2 \in V \tag{2.3}$$

G acts on M as a transformation group, in the following way:

$$\text{If } p = (v_1, \ldots, v_r), \qquad gp = (gv_1, \ldots, gv_r) \tag{2.4}$$

If $p = (v_1, \ldots, v_r)$, let $U(p)$ be the subspace of V spanned by the vectors $(v_1, \ldots, v_r)$. Let G^p be the isotropy subgroups of G at p, i.e. the set of $g \in G$ such that:

$$g(v_1) = v_1, \ldots, g(v_r) = v_r \tag{2.5}$$

Then, g acts as the identity in $U(p)$, hence g passes to the quotient to define a linear transformation in the vector space $V/U(p)$. Let $U(p)^{\perp}$ denote the orthogonal complement to $U(p)$ in V, i.e. the space of vectors $v \in V$ such that:

$$v \cdot U(p) = 0 \tag{2.6}$$

Let $\text{rad}(U(p))$ be the <u>radical</u> of $U(p)$, i.e. the vectors $v \in U(p)$ such that 2.6 satisfied. In other words,

$$\text{rad } U(p) = U(p) \cap U(p)^{\perp} \tag{2.7}$$

Consider the quotient projection map:

$$U(p)^{\perp} \to V/U(p) \tag{2.8}$$

The kernel of this map is obviously the radical of $U(p)$. G^p acts on $U(p)$ and $V/U(p)$. The map 2.8 intertwines the action of G^p

on $U(p)^{\perp}$ and $V/U(p)$.

Now, let us suppose that:

$$\text{rad } U(p) = (0), \qquad 2.9$$

i.e. that the form restricted to $U(p)$ is non-singular. Then, one sees readily that,

$$V = U(p) \oplus U(p)^{\perp} \qquad 2.10$$

2.10 shows that $U(p)$ can be identified with the quotient space $V/U(p)$. Let G' be the group of all linear transformations mapping $U(p)^{\perp}$ into itself that preserve the form. Then, the above remarks define a group homomorphism: $G^p \to G'$.

<u>Exercise</u> Show that this homomorphism is an isomorphism between G^p and G', if 2.9 is satisfied.

<u>Definition</u> Let M^o consist of the points $p \in M$ such that:

p is in general position, i.e. satisfies 2.2 — 2.11

The form restricted to $U(p)$ is non-singular,

i.e. $\text{rad } U(p) = (0)$. — 2.12

<u>Exercise</u> Show that M^o is an open, dense subset of M. (In fact, $M-M^o$ is an "algebraic subset" of M, i.e. a union of spaces defined by algebraic conditions).

Also, note that:

$$gM^o \subset M^o \text{ for } g \in G. \qquad 2.13$$

Note that:

$$\dim G^p = \text{constant for } p \in M^o \qquad 2.14$$

Thus, the orbit structure G/M^o is rather "nice". The troubles one encounters near the boundary of M^o in M are what the physicists call "kinematic singularities".

We can now parameterize $G\backslash M$, using the inner product form. Choose indices $1 \leq a,b \leq r-1$. Given $p = (v_1,\ldots,v_r) \in M$, set

$$s_{ab}(p) = (v_a+v_b)^2 \qquad 2.15$$

In 2.15, suppose that:

$$1 \leq a < b \leq r-1. \qquad 2.16$$

Then, one can regard (s_{ab}) as an element of $R^{(r-1)(r-2)/2}$. Then, 2.15 defines a map:

$$\phi\colon M \to R^{(r-1)(r-2)/2}, \qquad 2.17$$

Explicitly, the map ϕ sends $p = (v_1,\ldots,v_r) \in M$ into the element:

$$\phi(p) = (s_{ab}(p)), \qquad 2.18$$

with the matrix $(s_{ab}(p))$ given by the right hand side of 2.15.

Let us now study the fibers of the map ϕ defined by 2.17-2.18. First, the following fact follows from the condition that the action of G on V leaves invariant the inner product:

> The orbits of G on M are contained in the fibers of the map ϕ. In particular, ϕ passes to the quotient to define a map of the orbit space $G\backslash M \to R^{(r-1)(r-2)/2}$. 2.19

Let us now examine the extent of the validity of the converse of 2.16. Suppose then that:

$$p = (v_1,\ldots,v_r)$$
$$p' = v_1',\ldots,v_r') \qquad 2.20$$

are the points of M which lie on the same fiber of ϕ, i.e. which satisfy the following conditions:

$$(v_a+v_b)^2 = (v_a'+v_b')^2 \qquad 2.21$$

On working out both sides of 2.16, and using the conditions 2.1, we have the following conditions:

$$v_a \cdot v_b = v_a' \cdot v_b' \qquad 2.22$$

Let $U(p)$ and $U(p')$ be the linear subspaces of V spanned by the vectors on the right hand side of 2.17. If $\dim U(p) = \dim U(p')$, one can set up a linear isomorphism: $U(p) \to U(p')$ which preserves the form β. The Witt Theorem (See LMP, vol. II, p. 422) then implies that this isomorphism can be extended to an isomorphism: $V \to V$ which preserves the form , i.e. which belongs to G. We can now sum up what we have proved as follows:

Theorem 2.1. If $\dim U(p) = \dim U(p')$, then there is a $g \in G$ such that

$$gp = p'.$$

In other words, two points of M lie on the same orbit of G if and only if they lie on the same fiber of ϕ, and span a subspace of V of the same dimension.

Let us now examine the image $\phi(M) \subset R^{(r-1)(r-2)/2}$ of M. Because of 2.1 (b), it is obviously determined by a single, algebraic equation in $R^{(r-1)(r-2)/2}$, namely:

$$m_r^2 = \sum_{a,b} v_a \cdot v_b$$

$$= 2 \sum_{a<b} v_a \cdot v_b + \sum_a m_a^2$$

$$= \sum_{a<b} (v_a+v_b)^2-m_a^2-m_b^2) + \sum_a m_a^2 \qquad 2.23$$

$$= \sum_{a<b} (s_{ab}(p)-m_a^2-m_b^2) + \sum_a m_a^2$$

These constraints plus the inequality constraints imposed by the conditions that the vectors lie in the "physical regions" (in the case $V = R^4$, with the inner product given by the Lorentz form) - are extensively treated in the physics literature. We shall not emphasize the explicit, calculational side of the theory, since it is our aim to try to discern general geometric features of the situation.

3. FIBRATIONS OF THE KINEMATIC SPACES, AND PARTIAL WAVE ANALYSIS

$M, V, r, \cdot, G$ be as in Section 2. Recall that each point p of M is an r-tuple $(v_1, \ldots, v_r)$ of elements of V such that:

$$\text{a) } v_1^2 = m_1^2, \ldots, v_r^2 = m_r^2 \qquad \text{b) } v_1 + \ldots + v_r = 0. \tag{3.1}$$

Now, let s be an integer, $0 < s < r$. Let V^s be the Cartesian product of s copies of V. Let π be the map: $M \to V^s$ defined by the following formula:

$$\pi(p) = (v_1, \ldots, v_s) \tag{3.2}$$

If $q = (v_1, \ldots, v_s) \in V^s$, set:

$$M(q) = \pi^{-1}(q).$$

Thus, $M(q)$ is identified with the $(r-s)$-tuples $(v_{s+1}, \ldots, v_r)$ such that:

$$v_{s+1} + \ldots + v_r = v, \tag{3.3}$$

where

$$v = v_1 + \ldots + v_s \tag{3.4}$$

Thus, $M(q)$ is a subset of V^{r-s}. A space of this form will be called the kinematic space with inhomogeneous constraints.

Now, π is not quite a "fiber space" in the technical sense, at least of the "local product" type. It is, in fact, a sort of "fiber space with singularities". The "singularities", i.e. the points where the linear map π induces on tangent vectors degenerates in rank, are related to what the physicists call "kinematic singularities". However, one will not get involved in the technicalities of the theory of singularities of mappings, but will proceed rather carelessly, at least in the use of terminology.

Now, a "geometric" approach to "partial wave analysis" of the scattering amplitude has been suggested in Section 3 of Chapter V of FA. Namely, given a "scattering amplitude" function f on M, restrict f to each fiber $M(q)$, and expand f in terms of eigenfunctions of the Laplace-Beltrami operator of the Riemannian metric induced on $M(q)$ via its imbedding in V^{r-s} and the flat Riemannian metric on V induced by the form β on V. For example, suppose that each fiber of π is a compact manifold, with the metric on each $M(q)$ positive. Let $\psi_{1,q}, \psi_{2,q}, \ldots$ be the appropriately normalized eigenfunctions of this Laplace-Beltrami operator. Then, we have an expansion of the following general type:

$$f(p) = \sum_j a_j(q)\psi_{j,q}(p), \qquad 3.5$$

for $p \in M(q)$. Then, the "partial wave amplitude" functions a_j are functions on $\pi(M)$.

The map π intertwines the action of G on the spaces M and V^s. For $g \in V^s$, let $G(q)$ be the group of all $g \in G$ such that:

$$g(v_1+\ldots+v_s) = v_1+\ldots+v_s \qquad 3.6$$

Then, $G(q)$ acts on each space $M(q)$ in the following way:

$$g(v_{r+1},\ldots,v_s) = (gv_{r+1},\ldots,gv_{r+s}) \quad \text{for } (v_{r+1},\ldots,v_s) \in V^{s-r} \qquad 3.7$$

The action 3.7 of this group transforms the partial wave expansion 3.5 in an obvious "covariant" manner.

For example, if $V = R^4$, with β the Lorentz metric, and with $r = 4$, $m_1{}^2,\ldots,m_4{}^2 > 0$, the situation reduces to that which physically describes two-particle scattering. In this case, it is known that:

$G(q) = SO(3,R)$, and that $M(q)$ is a 2-sphere. The functions $\psi_1, \ldots$ are the usual "spherical harmonies", and 3.5 reduces to the usual partial wave expansion in terms of Legendre functions. To prove this in a general manner, without explicit calculation, we will show, in Section 4, that the metric on $M(q)$ is, in fact, of constant curvature. Of course, what will be important is not a general proof of something that is relatively easy to do by explicit calculation, but the embedding of the partial wave expansion in the two particle case in a general scheme which may be generalized to many particles.

4. RIEMANNIAN GEOMETRY OF THE KINEMATIC SPACES

Let us suppose that G as defined in Chapter III. Now, $M(v^o)$ is not a "random" space, but has special and important and beautiful geometric properties. Unfortunately, most of the physics treatises are so intent on working out the detailed formulas that they ignore these features. Above all, each of the "kinematic" spaces described in Section 3 inherits a Riemannian metric, since it may be imbedded as a subspace of a Cartesian product of copies of R^4. This metric is "natural", in the sense that the transformation group induced by the action of the Lorentz group acts as a group of isometries. In this section we will describe certain features of these Riemannian metrics. For the concepts of manifold theory and Riemannian geometry used here, refer to DGCV and Chapter I.

The Riemannian metric on the mass-shell hyperboloids in terms of sterographic projection.

Adopt the following notation: V is a real, finite dimensional vector space, with a symmetric, bilinear, non-singular form $(v_1,v_2) \to v_1 \cdot v_2$. This form induces a flat Riemannian metric on V and each of its linear subspaces. Let M be the submanifold of V consisting of the points $v \in V$ such that:

$$v \cdot v = 1 \qquad 4.1$$

Then, as for any submanifold of a Riemannian manifold, M inherits a Riemannian metric from the flat Riemannian metric on V.

Exercise Show that this Riemannian metric on M has constant curvature.

Suppose that v_0 is a fixed point of V. Let U be the orthogonal compliment of v_0 in V. U is a linear subspace of V, with V the direct orthogonal sum of U and the one dimensional subspace spanned by v_0. Let M' be the subset of M consisting of the vectors $v \in M$ such that:

$$v \cdot v_0 \neq 1 \tag{4.2}$$

Define a map ϕ: $M' \to U$ as follows:

$$\phi(v) = \frac{v-(v \cdot v_0)v_0}{1-v_0 \cdot v} \tag{4.3}$$

Definition The map ϕ defined by 4.3 is called the stereographic projection of M on the linear subspace U, with the point v_0 as the projecting point.

Exercise To justify the name "sterographic projection", show that, in case $V = R^3$, $v_0 = (0,0,1)$, M = unit sphere in R^3, ϕ is the usual sterographic projection of the unit sphere minus the north pole onto R^2.

The main property that interests us is that ϕ gives rise to a simple formula for the Riemannian matric on M induced from the imbedding of $M \to V$, with the flat Riemannian metric on V induced by the form $(v_1,v_2) \to v_1 \cdot v_2$. Namely, we have:

Theorem 4.1 ϕ is a conformal transformation between the Riemannian metric on M and the flat metric on the linear subspace U.

Proof. We will verify this by a direct calculation based on 4.3. This will also give us a useful formula for the Riemannian metric on M.

Suppose that $t \to v(t)$, $0 \leq t \leq 1$, is a curve in M. Set:

$$v'(t) = \frac{d}{dt} v(t)$$

$$u(t) = v(t)-(v(t) \cdot v_0)v_0$$

$$f(t) = 1-v_0 \cdot v(t)$$

Then,

$$f(t)^2 \frac{d}{dt} \phi(v(t)) = f(t) \frac{du}{dt} - u \frac{df}{dt}$$

$$= f(t)(v'-(v'\cdot v_0)v_0) + (v-(v\cdot v_0)v_0)(v_0\cdot v')$$

$$= v' - (v'\cdot v_0)v_0 - (v_0\cdot v)v' + (v_0\cdot v)(v'\cdot v_0)v_0$$

$$+ v(v_0\cdot v') - (v\cdot v_0)(v_0\cdot v')v_0$$

$$= f(t)v' - (v'\cdot v_0)v_0 + v(v_0\cdot v') \qquad 4.4$$

Now, since the curve $t \to v(t)$ satisfies 4.1, we have:

$$v\cdot v' = 0 \qquad 4.5$$

Squaring 4.4, and using 4.5, we have:

$$f(t)^4 \frac{d}{dt} \phi(v) \cdot \frac{d}{dt} \phi(v)$$

$$= f^2 v'\cdot v' + (v'\cdot v_0)^2 + (v_0\cdot v')^2$$

$$-2(v'\cdot v_0)^2 v\cdot v_0$$

$$= f^2 v'\cdot v'.$$

Thus, we have:

$$f(t)^2\left(\frac{d}{dt} \phi(v) \cdot \frac{d}{dt} \phi(v)\right) = \left(\frac{d}{dt} v(t) \cdot \frac{d}{dt} v(t)\right) \qquad 4.6$$

4.6 shows that ϕ preserves the angles between tangent vectors, i.e. is a conformal transformation.

<u>Corollary</u> Let f be the function

$$v \to 1-v\cdot v_0 \qquad 4.7$$

on M'. Let ds^2 be the flat Riemannian metric on U. Then,

$$f^2\phi^*(ds^2) \qquad 4.8$$

is the Riemannian metric on M induced from the flat metric on V.

Let us now derive the formula for the inverse map $\phi^{-1}: U \to M$. Suppose $u \in U$, and

$$\phi^{-1}(u) = u_1 + av_0 = v,$$

with $u_1 \in U$, $a \in R$

Using 2.3,

$$u = \frac{u_1+av_0-av_0}{1-a} = \frac{u_1}{1-a}$$

Also, $v \cdot v = 2$, or

$$u_1 \cdot u_1 + a^2 = 1.$$

$$u \cdot u = \frac{u_1 \cdot u_1}{(1-a)^2} = \frac{1-a^2}{(1-a)^2} = \frac{1+a}{1-a},$$

or

$$a = \frac{u \cdot u - 1}{1 + u \cdot u}$$

$$1-a = \frac{2}{1+u \cdot u},$$

$$u_1 = \frac{2u}{1+u \cdot u}$$

$$\phi^{-1}(u) = \frac{2u+(u \cdot u-1)v_0}{1+u \cdot u} \qquad 4.9$$

In summary, formula 4.9 gives a very convenient way to parametrize the mass-shell hyperboloid, and compute the Riemannian metric. In fact, note, from 4.7 that:

$$\phi^{-1*}(f) = 1 - \frac{(u \cdot u-1)}{(u \cdot u+1)}$$

$$= \frac{2}{u \cdot u+1} \qquad 4.10$$

Thus, if ds^2 is the Euclidean metric on U, the metric obtained by pulling back the metric on M by the map ϕ^{-1} is given by the following formula:

$$\frac{4}{(u \cdot u+1)^2} \, ds^2 \qquad 4.11$$

For example, suppose that V is R^4, with the metric on V given by the Lorentz metric. Choose:

$$v_0 = (1,0,0,0).$$

Suppose that:

$$u = (0,k_1,k_2,k_3)$$

$$\vec{k} = (k_1,k_2,k_3);\ \vec{k}^2 = k_1^2+k_2^2+k_3^2.$$

Then, 2.10 takes the following explicit form:

$$\frac{4}{(\vec{k}^2-1)^2}(dk_1^2+dk_2^2+dk_3^2) \qquad 4.12$$

<u>Exercise</u> Show that, in this case, the map ϕ^{-1} is a diffeomorphism between the unit ball $\vec{k}^2 < 1$ of R^3 and the upper sheet of the hyperboloid M, with the exterior $\vec{k}^2 > 1$ going into the lower sheet by a diffeomorphism.

<u>Exercise</u> Identifying $\vec{k} = (k_1,k_2,k_3)$ with an element of R^3, let B be the unit ball $\vec{k}^2 < 1$ of R^3, with the metric defined by 4.11. Compute directly that this metric has constant negative curvature. This space goes under the name of "Poincare's realization of the Lobochewski space".

<u>The metric on the inhomogeneous kinematic manifolds.</u>

Continue with V as above. Let r be an integer, $m_1^2,\ldots,m_r^2$ be positive real numbers, and let v_0 be a fixed element with $v_0 \cdot v_0 = 1$, and let λ be a real number of V. Let $M(\lambda v_0)$ be the set of r-tuples $(v_1,\ldots,v_r)$ of elements of V such that:

a) $v_1+\ldots+v_r = \lambda v_0$

b) $v_1^2 = m_1^2,\ldots,v_r^2 = m_r^2.$ 4.13

For the sake of simplicity, we will suppose, in this section, that the $m_1,\dots,m_r$ are equal to 1. Let U be the orthogonal complement subspace to v_0. Let U^r be the Cartesian product

$$U\times\dots\times U$$

of r copies of U.

Define a map $\alpha\colon U^r \to V^r$ as follows:

$$\alpha(u_1,\dots,u_r)$$
$$= \left(\frac{2u_1+(u_2\cdot u_1-1)v_0}{1+u_1\cdot u_1}\ ,\ \dots\ ,\ \frac{2u_r+(u_r\ u_r-1)v_0}{1+u_r\cdot u_r}\right) \qquad 4.14$$

(Thus, if $r = 1$, α reduces to the map ϕ^{-1} defined by 4.9). As we have seen above, the image of α satisfies conditions 4.13 b).

Let h be the map: $V^r \to V$ defined as follows:

$$h(v_1,\dots,v_r) = v_1+\dots+v_r \qquad 4.15$$

Then, the inverse image $\alpha^{-1}(M(\lambda v_0))$ is determined by the following conditions:

$$\alpha^*(h) = \lambda v_0 \qquad 4.16$$

where:

$$\alpha^*(h)(u_1,\dots,u_r)$$
$$= h(\alpha(u_1,\dots,u_r)$$

combining 4.14 and 4.15,

$$\frac{2u_1+(u_1\cdot u_1-1)v_0}{1+u_1\cdot u_1} +\dots+ \frac{2u_r+(u_r\cdot u_r-1)v_0}{1+u_r\cdot u_r} \qquad 4.17$$

Thus, using 4.16, $\alpha^{-1}(u(\lambda v_0))$ is determined by the following conditions:

$$\frac{u_1}{1+u_1 \cdot u_1} + \ldots + \frac{u_r}{1+u_r \cdot u_r} = 0. \qquad 4.18$$

$$\frac{u_1 \cdot u_1 - 1}{1+u_1 \cdot u_1} + \ldots + \frac{u_r \cdot u_r - 1}{1+u_r \cdot u_r} = \lambda \qquad 4.19$$

Consider the following Riemannian metric on U^r:

$$ds^2 = \frac{4}{(u_1 \cdot u_1 + 1)^2} \, ds_1{}^2 + \ldots + \frac{4}{(u_r \cdot u_r + 1)^2} \, ds_r{}^2, \qquad 4.20$$

where $ds_1{}^2, \ldots, ds_r{}^2$ are the metrics on the copies of U defined by the quadratic form $u \to u \cdot u$.

Putting these remarks together, proves the following result:

Theorem 4.2 The induced Riemannian metric on $\alpha^{-1}(M(v_0))$ is the metric 4.20, restricted to the submanifold of U^r determined by conditions 4.19 and 4.20.

Notice now that the constraints 4.18-4.19 involve rational functions of the variables $(u_1, \ldots, u_r)$. This feature shows that our method of parametrizing the kinematic sets has advantages over the usual methods to be found in the physics treatises which usually involve square roots in the constraints. For example, let us work out the case: $r = 2$.

Then, 4.18 takes the following form:

$$\frac{u_1}{1+u_1 \cdot u_1} = - \frac{u_2}{1+u_2 \cdot u_2} \qquad 4.21$$

Squaring 4.21 gives the relation

$$\frac{u_1{}^2}{(1+u_1{}^2)^2} = \frac{u_2{}^2}{(1+u_2{}^2)^2} \qquad 4.22$$

Set: $a = u_1^2$, $b = u_2^2$. Then, conditions 4.21 and 4.22 take the following form:

$$\frac{a}{(1+a)^2} = \frac{b}{(1+b)^2}$$

$$\frac{a-1}{1+a} + \frac{b-1}{1+b} = \lambda \qquad 4.23$$

Set:

$$c = \frac{\lambda(1+b)-b+1}{1+b} \qquad 4.24$$

Then, $a = \frac{c-1}{c+1}$

$$1+a = \frac{2c}{1+c}$$

$$\frac{a}{(1+a)^2} = \frac{(c-1)(c+1)}{c^2} = \frac{c^2-1}{4c^2} \qquad 4.25$$

Thus, the equations determining the values of a and b permissible from 4.23 are:

$$(c^2-1)(1+b)^2 = 4c^2b \qquad 4.26$$

Now, the equation 4.26 is a quartic in b. All we really need to know is that the values are discrete. Thus, we have proved the following general result.

Theorem 4.3 For $r = 2$, the inhomogeneous kinematic space $M(\lambda v_0)$ is identified with the quadrics $u \cdot u = b$, where b are the solutions of 4.26.

Remarks In the case of interest in relativistic physics, i.e. $V = R^4$, $U = R^3$, $u \cdot u < 0$ for $u \in U$, one can show that b has but one solution in the region $u \cdot u > 1$, hence the manifold $M(\lambda v_0)$ is a 2-sphere. This is well-known but it is nice to see it recaptured so readily in this formalism.

We can also now readily compute the metric ds^2 on the manifold $M(\lambda v_0)$. In fact, from 3.8 and 3.9, we have:

$$ds_1{}^2 = \left(\frac{1+a}{1+b}\right)^2 ds_2{}^2,$$

hence

$$ds^2 = 4/(1+b)^2\, ds_2{}^2 \qquad 4.27$$

In particular, this proves the following general result:

Theorem 4.4 The metric induced on $M(\lambda v_0)$, in the case $r = 2$, is the constant curvature metric.

This general result explains why "spherical functions" always appear in partial wave expansions of the S-matrix, for two particle scattering processes. In accordance with general principles the functions in which one expands the scattering amplitude are chosen as eigenfunctions of the Laplace-Beltrami operator on the non-homogeneous constraint manifolds.

Chapter IV

TENSOR PRODUCTS AND CURRENT ALGEBRA

1. INTRODUCTION

In this chapter we shall gather together certain ideas concerning the theory of tensor products of Hilbert spaces that seem to play an important role in quantum mechanics and quantum field theory. I shall emphasize certain natural geometric ideas here. (See Guichardet [1] for an exposition from a more precise functional analysis point of view.)

The main application that I hope to make of this material is the study of representations of the "current algebras" that appear in elementary particle physics and quantum field theory. As I explained in my paper "Infinite dimensional Lie algebras and current algebras", these "current algebras" form a certain class of infinite dimensional Lie algebras. Now, a main unsolved problem in this theory is to construct a supply of linear representations of these algebras that is adequate for the potential physical applications. It seems likely that the theory of tensor products of Hilbert spaces will ultimately provide a way of constructing such representations. In this chapter, I will provide a few further comments about this program, as well as provide some further geometric information about the infinite dimensional Lie algebras and groups which occur in current algebra theory.

2. THE FREE VECTOR SPACE GENERATED BY A SET

In this work, I have usually assumed that the reader is familiar with the elementary ideas of linear algebra, and in particular, the idea of "tensor product". However, in the theory of representations of infinite dimensional Lie groups and algebras - such as "current algebras" - certain confusing and interesting complications are involved. (For example, one must deal with tensor products of various sorts of infinite families of vector spaces.) Accordingly, it is worthwhile going over even very elementary material from first principles.

In this section, let us consider the definition of a "free vector space defined by a set". Let M be an abstract space of points, without any particular further structure. Suppose fixed a field of scalars; for simplicity we shall suppose that it is just the real numbers, R. Let $F_a(M)$ denote the space of real-valued functions on M, i.e. an element $f \in F_a(M)$ is a map $f:M \to R$. (Warning: Do not confuse $F_a(M)$ with $F(M)$, the real-valued C^∞ functions defined when M is a manifold. The subscript "a" stands for "all real-valued functions").

$F_a(M)$ is a real vector space, since two such functions can be added in the unreal way, and multiplied by scalars. Accordingly, we may form its dual space $F_a(M)^d$, consisting of the R-linear maps of $F_a(M)$ onto R itself. It too is a real vector space.

There is a natural imbedding of M as a subset of $F_a(M)^d$. Namely, to each $p \in M$, associate the element

$$f \to f(p) \equiv p(f)$$

of $F_a(M)^d$.

Definition The free vector space of formal linear combinations of elements of M is the smallest linear subspace of $F_a(M)^d$ containing M. We denote this vector space by FV(M).

Exercise 2.1 Let $p_1,\ldots,p_n$ be a finite number of distinct elements of M. Show that, as elements of $F_a(M)^d$, they are linearly independent.

Using Exercise 2.1, we see that an element $u \in FV(M)$ may be identified with a sum (in $F_a(M)^d$) of the form

$$u = c_1p_1 + \ldots + c_np_n, \qquad 2.1$$

where n is some integer, $p_1,\ldots,p_n$ are distinct elements of M $F_a(M)^d$, and where $c_1,\ldots,c_n$ are non-zero real scalars that are uniquely determined by u. Thus, FV(M) may loosely be described as the "formal linear combinations of elements of M, with real numbers as coefficients." For example, if M has m elements, FV(M) is naturally identified with R^m.

We now use this construction to define the notion of "tensor product" of two vector spaces.

3. THE TENSOR PRODUCT OF TWO VECTOR SPACES

Now, suppose that V_1,V_2 are two real vector spaces. In this book - and my previous books as well - I have been assuming that the reader was familiar with the elementary notions of linear and multilinear algebra, such as the "tensor product" notion. At this stage, it is desirable to define it precisely in terms of the "free vector space" notion of Section 2, in order to prepare the reader for the generalization to be considered later of the tensor product of infinite families of vector spaces.

Let us apply the construction of Section 2 to the Cartesian product set

$$M = V_1 \times V_2 .$$

Thus, an element of $FV(V_1 \times V_2)$ is a linear combination of the form:

$$c(v_1,v_2) + c'(v_1',v_2') + \dots$$

$$c,c',\dots \in R,\ v_1,v_1',\dots \in V_1,$$

with

$$v_2,v_2',\dots \in V_2 .$$

In $FV(V_1 x V_2)$, let us consider the linear subspace spanned by all elements of the form:

$$(v_1,v_2) + (v_1',v_2) - (v_1+v_1',v_2)$$

$$(v_1,v_2) + (v_1,v_2') - (v_1,v_2+v_2')$$

$$c(v_1,v_2) - (cv_1,v_2) \qquad 3.1$$

$$c(v_1,v_2) - (v_1,cv_2)$$

where $v_1,v_1' \in V_1, v_2,v_2' \in V_2, c \in R$.

Definition. The tensor product of the vector spaces V_1, V_2, denoted by $V_1 \otimes V_2$, is the quotient vector space of $FV(V_1 x V_2)$ by the linear subspace spanned by all elements of the form 3.1. It is denoted by $V_1 \otimes V_2$. Given $v_1 \in V_1$, $v_2 \in V_2$, the image of (v_1,v_2) in this quotient space is denoted by: $v_1 \otimes v_2$.

Exercise 3.1 Consider the map $\alpha: V_1 \times V_2 \to V_1 \otimes V_2$ which assigns to $(v_1,v_2) \in V_1 \times V_2$ its image $v_1 \otimes v_2$ in the vector space $V_1 \otimes V_2$.

Show that α is an R-bilinear map, i.e.

$$\alpha(v_1+v_1',v_2) = \alpha(v_1,v_2) + \alpha(v_1',v_2)$$

$$\alpha(v_1,v_2+v_2') = \alpha(v_1,v_2) + \alpha(v_1,v_2') \qquad 3.2$$

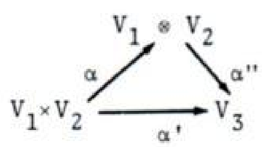
$$\alpha(cv_1,v_2) = c\alpha(v_1,v_2) = \alpha(v_1,cv_2)$$

for $v_1,v_1' \in V_1$; $v_2,v_2' \in V_2$; $c \in R$.

<u>Exercise 3.2.</u> Suppose that V_1,V_2,V_3 are vector spaces and that $\alpha':V_1 \times V_2 \to V_3$ is an R-bilinear map, i.e. α' satisfies 3.2, (with α' replacing α, of course.) Show that there is a <u>linear</u> map $\alpha'':V_1 \otimes V_2 \to V_3$ such that:

$$\alpha' = \alpha''\alpha \qquad 3.3$$

This relation is most readily kept in mind by writing down the following commutative diagram of maps:



<u>Remark</u> This is the "universality" property of the tensor product construction. It is the one emphasized in the Bourbaki approach to algebra. Despite its overall elegance, I believe that the more explicit definition given above is best for the physical applications.

<u>Exercise 3.3</u> Suppose V_1,V_2 are finite dimensional vector spaces, with $\dim V_1 = n_1$, $\dim V_2 = n_2$. Suppose (v_1^i), $i = 1,\ldots,n_1$; (v_2^a), $a = 1,\ldots,n_2$, are bases of V_1 and V_2. Show that the

elements

$$v^{ia} = v_1^i \otimes v_2{}^a \qquad 3.4$$

from a basis for $V_1 \otimes V_2$. In particular, deduce that:

$$\dim(V_1 \otimes V_2) = (\dim V_1)\dim(V_2) \qquad 3.5$$

Now, let us consider the action of tensor product relative to linear transformations. Suppose that V_1, V_2, W_1, W_2 are vector spaces, with:

$$A:V_1 \to W_1$$

$$B:V_2 \to W_2$$

linear transformations between them.

Exercise 3.4 Show that there is a unique linear transformation, denoted by $A \otimes B: V_1 \otimes V_2 \to W_1 \otimes W_2$, such that:

$$(A\otimes B)(v_1\otimes v_2) = (Av_1 \otimes Bv_2) \qquad 3.6$$

$$\text{for } v_1 \in V_1, v_2 \in V_2.$$

$A \otimes B$ is called the tensor or product of A and B. (An older terminology that is sometimes used is "Kronecker Product" or "outer product".)

Remark It is here that the reader needs to sort out in his own mind the difference between the vector space $V_1 \otimes V_2$ and the direct sum vector space $V_1 \oplus V_2$. Of course, as a set of points, $V_1 \oplus V_2$ is the Cartesian product $V_1 \times V_2$. The map

$$(v_1,v_2) \to (Av_1,Bv_2)$$

then is identified the direct sum $A \oplus B$ of the linear transformations. However, $A \otimes B$ and $A \oplus B$ have completely different properties, of course.

Exercise 3.5 With respect to given bases of V_1, V_2, W_1, W_2, compute the matrices of the linear transformations $A \oplus B$ and $A \otimes B$.

The tensor product construction plays a key role in group representation theory. Suppose that G_1, G_2 are groups, V_1, V_2 vector spaces, and that $\rho_1(G_1)$, $\rho_2(G_2)$ are representations of G_1 and G_2 by linear transformations on V_1 and V_2, respectively. Given $(g_1, g_2) \in G_1 \times G_2$, assign to it the linear transformation:

$$(\rho_1 \otimes \rho_2)(g_1, g_2) \equiv \rho_1(g_1) \otimes \rho(g_2) \qquad 3.7$$

Exercise 3.6 Show that 3.6 defines a linear representation $(g_1, g_2) \to (\rho_1 \otimes \rho_2)(g_1, g_2)$ of the direct product group $G_1 \times G_2$ by linear transformations on $V_1 \otimes V_2$, $\rho_1 \otimes \rho_2$ is called the tensor product of ρ_1 and ρ_2.

Remarks Again, one should keep in mind the difference between the tensor product $\rho_1 \otimes \rho_2$ and the direct sum $\rho_1 \oplus \rho_2$ representation of the group $G_1 \times G$ on the direct sumvector space $V_1 \oplus V_2$:

$$(\rho_1 \ \rho_2)(g_1, g_2) = \rho_1(g_1) \oplus \rho_2(g_2).$$

For example, one way of keeping in mind the difference is to note that $\rho_1 \oplus \rho_2$ is never irreducible. (The subspaces $V_1 \oplus (o), (o) \oplus V_2$), for example, are left invariant by it.) One can prove that $\rho_1 \otimes \rho_2$ is an irreducible representation of $G_1 \otimes G_2$ if $\rho_1(G_1)$ and $\rho_2(G_2)$ act irreducibly on V_1 and V_2 and if, for example, V_1 and V_2 are finite dimensional

Another confusing point to keep in mind is the difference between $\rho_1 \otimes \rho_2$ as a representation of $G_1 \times G_2$ and it as a representation of the "diagonal" subgroup, in case $G_1 = G_2 = G$, i.e. the subgroup of elements of $G \times G$ of the form (g,g). This representation is not irreducible; in fact, reducing it to its irreducible pieces is an important question for physics, involving what the physicists call "Clebsch-Gordon coefficients". The reader will find an extensive discussion of much of this material in LMP, vol. II, at least for the groups of most immediate use in physics.

Another important point is the tensor-product-of-representation notion at the Lie algebra level. Let $\underset{\sim}{G}_1$, $\underset{\sim}{G}_2$ be real Lie algebras, with $\rho_1(\underset{\sim}{G}_1)$, $\rho_2(\underset{\sim}{G}_2)$ linear representations of these algebras by operators on V_1 and V_2. ("Representation" means that Lie brackets go over into operator commutation, of course.) Then, the <u>tensor product representation</u> $\rho_1 \otimes \rho_2$ is a linear representation of the <u>direct sum</u> Lie algebra $G_1 \oplus G_2$ by linear transformations on $V_1 \times V_2$, defined as follows:

$$(\rho_1 \otimes \rho_2)(X_1 \oplus X_2) \cdot (v_1 \otimes v_2) = \rho_1(X_1)(v_1) \otimes v_2 + v_1 \otimes \rho_2(X_2)(v_2) \qquad 3.8$$

<u>Exercise 3.7</u> Show that formula 3.8 really defines a representation of the Lie algebra $G_1 \oplus G_2$.

<u>Remarks</u> One should note that the assignment $(X_1,X_2) \to \rho_1(X_1) \otimes \rho_2(X_2)$ is not a Lie algebra representation, a fact that sometimes confuses physicists. In fact, it is a linear mapping between the vector space $\underset{\sim}{G}_1 \otimes \underset{\sim}{G}_2$ and the space of all operators on $V_1 \otimes V_2$, but

the vector space $\underset{\sim}{G}_1 \otimes \underset{\sim}{G}_2$ is not in a natural way a Lie algebra. One may, of course, consider the smallest Lie algebra of linear transformations containing the $\rho_1(X_1) \otimes \rho_2(X_2)$. This was first introduced by Wigner in his SU(4)-theory of "spin-isospin" symmetry in nuclear physics. Work by Gursey, Radicati, and Sakita in 1964, and then by many other physicists in the next few years after that, attempted to generalize this to a "spin-unitary spin" SU(6) symmetry in relativistic elementary particle physics, but there was only limited success in either physical results or deep understanding of the underlying physical and/or mathematical mechanism.

Finally, we may indicate a "geometric" way of looking at tensor products that is important when one considers tensor products of infinite families of vector spaces. Suppose that M_1, M_2 are sets, and V_1, V_2 are <u>given</u> as linear subspaces of $F_a(M_1), F_a(M_2)$.

<u>Exercise 3.8</u> Show that $V_1 \otimes V_2$ may be identified with a subspace of $F_a(M_1 \times M_2)$ in such a way that:

$$(v_1 \otimes v_2)(p_1, p_2) = v_1(p_1)v_2(p_2) \qquad 3.9$$

$$\text{for } v_1 \in V_1,\ v_2 \in V_2,\ p_1 \in M_1, p_2 \in M_2$$

Further, if M_1, M_2 are finite sets of points, and if $V_1 = F_a(M_1)$, $V_2 = F_2(M_2)$, show that:

$$V_1 \otimes V_2 = F_a(M_1 \times M_2) \qquad 3.10$$

Of course, a relation as precise as 3.10 does not hold if M_1, M_2 are infinite sets. It is at this point that functional analysis subtleties enter in. For example, let us consider the case where M_1, M_2 are locally compact topological spaces, and that V_1, V_2 consist of the compact-support, continuous functions. (Or, C^∞ functions perhaps, if M_1, M_2 are manifolds). Then, $V_1 \otimes V_2$ may be identified, via 3.9, with a space of compact support continuous functions on $M_1 \times M_2$. Typically, this space of functions is dense in the space of all continuous, compact support functions on $M_1 \times M_2$, when various natural topologies are introduced for these function spaces. Frequently, the tensor product provides "enough" functions on $M_1 \times M_2$ to carry an analysis.

Another point of interest in connection with this "geometric" interpretation of the tensor product is in connection with the idea of "interaction" in quantum mechanics. To look at this in its simplest form, consider two particles moving in R^3. Introduce 3-vectors $\vec{x} = (x_i)$, $\vec{y} = (y_i) \in R^3$, $1 \leq i, j \leq 3$. A "Schrodinger wave function" for the particle whose position is labelled $\vec{x}$ is a complex-valued function $\vec{x} \to \psi_1(\vec{x})$ such that:

$$\int |\psi_1(\vec{x})|^2 d\vec{x} \equiv ||\psi_1||^2 = 1. \qquad 3.11$$

(Refer to LMP, vol. II, for the notions to be discussed now.) Let V_1 be the complex vector space of such functions ψ_1 such that

$$||\psi_1|| < \infty ,$$

i.e. which are "square integrable".

Similarly, introduce wave functions $\vec{y} \to \psi_2(\vec{y})$ for the second particle, and the vector space V_2 of such ψ_2's such that $||\psi_2|| < \infty$.

Now, a wave function for the composite system of the two particles is a complex-valued function $\psi(\vec{x}, \vec{y})$ on $R^3 \times R^3$, such that:

$$||\psi||^2 \equiv \iint |\psi(\vec{x},\vec{y})|^2 d\vec{x}d\vec{y} = 1. \qquad 3.12$$

The physical interpretation of ψ may be described as follows:

If D is a domain in $R^3 \times R^2$,

$$\iint_D |\psi(\vec{x},\vec{y})|^2 d\vec{x}d\vec{y}$$

is the "probability" that the joint position $(\vec{x},\vec{y})$ of the particles lies in $R^3 \times R^3$.

Introduce V as the space of functions $\psi: R^3 \times R^3 \to C$ for which $||\psi|| < \infty$. Then, we may imbed $V_1 \otimes V_2$ as a subspace of V as follows:

For $\psi_1 \in V$, $\psi_2 \in V_2$, identify $\psi_1 \otimes \psi_2$ with the function $(\vec{x}_1\vec{y}) \to \psi_1(\vec{x})\psi_2(\vec{y})$

(Of course, $V_1 \otimes V_2$ means here the tensor product in the sense of complex vector spaces. Notice that all we have done up to now in describing the algebraic properties of tensor products carries over to an arbitrary field of scalars replacing the real numbers e.g. the complex numbers.) We then obviously have the following relation between the square-integral norms defined by 3.11 and 3.12

$$||\psi_1 \otimes \psi_2|| = ||\psi_1|| \; ||\psi_2|| \qquad 3.13$$

In fact, relation 3.13, which here just follows from the explicit definition of the norm and the Fubini theorem for product integrals, is taken over in the abstract theory of Hilbert spaces in order to define tensor products as Hilbert spaces.

Relation 3.13 has a physical meaning also. Suppose D is of the form $D_1 \times D_2$, where D_1, D_2 are domains in $R^3 \times R^3$. Then,

$$\iint_D |(\psi_1 \otimes \psi_2)(\vec{x},\vec{y})|^2 d\vec{x}d\vec{y} = \left(\int_{D_1} |\psi_1(\vec{x})|^2 d\vec{x}\right)\left(\int_{D_2} |\psi_2(\vec{y})|^2 d\vec{y}\right) \qquad 3.14$$

Thus, 3.14 means that the joint probability that the first particle be in D_1, the second in D_2, is the product probability. This is the appropriate law that the particles are to be regarded as acting "independently" of each other.

An observable for the first particle is a Hermitian operator $A: V_1 \to V_1$. Similarly, an observable for the second is an Hermitian operator $B: V_2 \to V_2$. Then, the operator $A \otimes 1 + 1 \otimes B$ is the appropriate one for the particles "without interaction".

For example, the "energy" operator for the particles might take the following typical Schrödinger form:

$$A(\psi_1)A = \frac{1}{2m_1} \frac{\partial^2 \psi_1}{\partial x_i \partial x_i} + V_1 \psi_1,$$

$$B(\psi_2) = \frac{1}{2m_2} \frac{\partial^2 \psi_2}{\partial y_i \partial y_i} + V_2 \psi_2 \quad , \qquad 3.15$$

where m_1, m_2 are the "masses" of the particles, $\vec{x} \to V_1(\vec{x}), \vec{y} \to V_2(\vec{y})$ are "the potentials". Then, the operator $A \otimes 1 + 1 \otimes B$ acting on $V_1 \otimes V_2$ may be extended to V, as follows:

$$\psi(\vec{x},\vec{y}) \to \frac{1}{2m_1} \frac{\partial^2 \psi}{\partial x_i \partial x_i} + \frac{1}{2m_2} \frac{\partial^2 \psi}{\partial y_i y_i} \qquad 3.16$$

$$+ V_1(\vec{x})\psi(\vec{x}_1\vec{y}) + V_2(\vec{y})\psi(\vec{x}_1\vec{y})$$

This is the appropriate energy operator for the particles put together, acting without interaction. An "interaction" might typically be defined as an additional term acting on the right hand side of 3.16 that is "local" and that does not preserve the subspace $V_1 \otimes V_2$ of V_1 i.e. that introduces "correlations" between the joint probabilities. For example, one simple form for such an operator might be:

$$\psi(\vec{x}_1\vec{y}) \to V(\vec{x}-\vec{y})\ \psi\ (\vec{x},\vec{y}),$$

where V is an "interaction potential" function.

Now, let us generalize this procedure to the case of the tensor product of two Hilbert spaces. (Later on, we shall attempt to extend the idea to cover infinite families of Hilbert spaces.)

4. TENSOR PRODUCT OF TWO HILBERT SPACES

In quantum mechanics, the basic vector spaces have an additional "Hilbert space" structure. As explained in LMP, the structure differs from that used by the mathematicians, since one should not assume the "completeness" axiom for such spaces. In this section we shall show how the Hilbert space structure may be exploited to give a more elegant definition of the tensor product than was possible for general vector spaces.

Let H_1, H_2 be complex vector spaces, with Hermetian symmetric inner products $\langle | \rangle$ that make H_1, H_2 into Hilbert spaces. For example, $\langle | \rangle$ is an A-bilinear map

$$: H_1 \times H_1 \to C \quad \text{such that:}$$

$$\langle \psi_1 | \psi_1 \rangle > 0 \text{ if } \psi_1 \in H_1 \text{ is non-zero}$$

$$\langle c\psi_1 | \psi' \rangle = c^* \langle \psi_1 | \psi' \rangle = \langle \psi_1 | c^* \psi' \rangle$$

$$\text{for } \psi_1, \psi_1' \in H_1,\ c \in C.$$

Let M be the Cartesian product $H_1 \times H_2$ and let $FV(M)$ be the free vector space spanned by M. (Now, however, let us use the complex numbers, C, as the field of scalars.)

Let us define a mapping - again denoted by $<|>$ - of $M \times M \to C$ as follows:

$$<\psi_1 x \psi_2 | \psi_1' x \psi_2'> = <\psi_1|\psi_1'><\psi_2|\psi_2'> \qquad 4.1$$

for $\psi_1, \psi_1' \in H_1$; $\psi_2, \psi_2' \in H_2$.

(Of course, this formula is motivated by the "Schrödinger wave function" case considered in Section 3.)

Now, extend $<|>$ to a mapping of $FV(M) \times FV(M) \to C$ in the following way:

If $v = \sum_i c_i\, p_k$, $v' \in \sum_j c_j' p_j'$ are in $FV(M)$, with $c_i, c_j' \in C$, $p_i, p_j' \in M$, then

$$<v/v'> = \sum_{i,j} c_i^* c_j' <p_i|p_j> \qquad 4.2$$

Formula 4.2 now defines $<|>$ as a Hermitian symmetric, positive semi-definite form on the complex vector space $FV(M)$, i.e. we only have

$$<v/v> \geq 0 \text{ for } v \in FV(M).$$

Let V_o be the subspace of the $v_o \in FV(M)$ such that

$$<v_o/FV(M)> = 0, \qquad 4.3$$

i.e. V_o is the "radical" of the vector space $FV(M)$ with respect to the form $<|>$. Then, the quotient vector space $FV(M)/V_o$ inherits an inner product.

Definition The tensor product of the Hilbert spaces H_1, H_2, denoted by $H_1 \otimes H_2$, is the quotient vector space

$$FV(M)/V_o,$$

with the quotient inner product.

We shall now leave to the reader the details of the verification that this coincides with the purely algebraic definition of tensor product given in Section 3. This material is covered by the following exercises.

Exercise 4.1 Given $\psi_1, \psi_1', \psi_1'' \in H_1, \psi_2, \psi_2', \psi_2'' \in H_2'$, $c \in C$, show that the following elements of $FV(M)$ belong to V_o:

$$(\psi_1+\psi_2, \psi_1') - (\psi_1, \psi_1') - (\psi_2, \psi_1')$$

$$(\psi_1, \psi_2 + \psi_2') - (\psi_1, \psi_2) - (\psi_1, \psi_2')$$

$$c(\psi_1, \psi_1') - (c\psi_1, \psi_1')$$

$$c(\psi_1, \psi_1') - (\psi_1, c\psi_1')$$

Exercise 4.2 Let $H_1 \otimes H_2$ denote the tensor product of the complex vector spaces H_1, H_2, defined using the abstract procedure of Section 3. Show that the quotient map $H_1 \times H_2 = M \to FV(M)/V_o$ defines an isomorphism of $H_1 \otimes H_2$ with $FV(M)/V_o$.

Of course, this definition of $H_1 \otimes H_2$ as isomorphic to $FV(M)/V_o$ gives, as a bonus, a natural Hilbert space structure to $H_1 \otimes H_2$.

Exercise 4.3 Show that the Hilbert space structure defined in this way on $H_1 \otimes H_2$ is defined by the following formula:

$$<\psi_1 \otimes \psi_2 | \psi_1' \otimes \psi_2'>$$

$$= <\psi_1|\psi_1'><\psi_2|\psi_2'> \qquad 4.4$$

for $\psi_1 \psi_1' \in H_1$, ψ_2, $\psi_2' \in H_2$

Finally, we can show how these facts fit in with a more general picture. Let H_1, H_2 be Hilbert spaces. Let $F_a(H_1, C)$ denote the vector space of all maps: $H_1 \to C$. Then, given $\psi_1 \in H_1$, we can define an element $\alpha_1(\psi_1) \in F_a(H_1, C)$ as follows:

$$\alpha_1(\psi_1)(\psi_1') = <\psi_1'|\psi_1> \qquad 4.5$$

α_1 then defines a complex-linear map: $H_1 \to F_a(H_1, C)$.

Further, α_1 is one-one. In other words, the inner product exhibits H_1 as a vector space of complex valued funtions on another space, namely on H_1 itself. Similarly, define

$$\alpha_2 : H_2 \to F_a(H_2, C).$$

Given $\psi_1 \in H_1$, $\psi_2 \in H_2$, define

$$\alpha(\psi_1, \psi_2) \in F_a(H_1 \times H_2) \text{ as follows:}$$

$$\alpha(\psi_1,\psi_2)(\psi_1',\psi_2') = \alpha_1(\psi_1)(\psi_1')\alpha_2(\psi_2)(\psi_2') = <\psi_1'|\psi_1><\psi_2'|\psi_2> \qquad 4.6$$

Then, α is a map: $H_1 \times H_2 \to F_a(H_1 \times H_2, C)$

Exercise 4.4 Show that $H_1 \otimes H_2$ may be identified with the smallest linear subspace of $F_a(H_1 \times H_2)$, containing the functions

$$\alpha(H_1 \times H_2)$$

5. HEURISTIC IDEAS ABOUT CONTINUOUS TENSOR PRODUCTS

The theory of tensor products of continuous families of vector spaces is clearly necessary in order to treat the linear representations of such infinite dimensional Lie algebras as the "current algebras". However, this theory itself is not yet in a completely satisfactory form. Indeed, it has certain badly understood features which are probably closely related to the difficulties and complexities of quantum field theory. In the next few sections we shall present a number of ideas that should be present in some form or another in an ultimate theory.

Let H, M be topological spaces with $\pi : H \to M$ a map which defines H as a vector bundle over M. Thus, for $p \in M$, the fiber $\pi^{-1}(p) = H(p)$ is a vector space, say, over the complex numbers as a field of scalars. We assume also that the induced topology on $H(p)$ makes $H(p)$ a topological vector space.

Let E be another topological space, with π' a continuous, onto map:$E \to M$. For $p \in M$, let $E(p)$ denote the fiber $\pi'^{-1}(p)$ of E over p. For $p \in M$, let $H'(p)$ denote the complex vector space of continuous complex-valued functions $\psi : E(p) \to C$. Let H' be the space of ordered pairs (p,ψ), where $p \in M$, $\psi \in H'(p)$. Then, H' is a vector bundle over M also.

Let us suppose that we are given a map $\alpha : H \to H'$, such that

$$\alpha(H(p)) \subset H'(p) \quad \text{for} \quad p \in M,$$

and α maps $H(p)$ in a complex-linear way into $H'(p)$. (This is just a fancy way of saying that each fiber $H(p)$ is identified-via α-with a vector space of complex-valued functions on the space $E(p)$.)

Let $\Gamma(H)$ denote the space of cross-section maps: $M \to E$. What we want to do is to use this data to define $\Gamma(H)$ as a space of complex-valued functions on $\Gamma(E)$. The "infinite tensor product" $\bigotimes_p H(p)$ is then the smallest vector space of functions on $\Gamma(E)$ containing $\Gamma(H)$.

What we must do then is define the "value" of a $\gamma \in \Gamma(H)$ on a $\gamma' \in \Gamma(E)$, i.e. a real number that we denote by

$$\alpha(\gamma)(\gamma')$$

Now, the function $p \to \alpha(\gamma)(p)(\gamma'(p))$ a real valued function on M. Intuitively, we want to define $\alpha(\gamma)(\gamma')$ as something like the "product" $\prod_{p \in M} \alpha(\gamma)(p)(\gamma'(p))$ of the values of this function.

If M were a finite space, we could adopt this literally as the definition. However, the "continuous" case for M - which is the primary case in which we are interested - presents certain difficulties.

We shall discuss one approach to the problem. Let "dp" denote a measure on M. One might expect to define this infinite product by the following formula:

$$\alpha(\gamma)(\gamma') = \exp \int_M \ell og(\alpha(\gamma)(p)(\gamma'(p)))dp \qquad 5.1$$

Certainly, if M is a finite space, this is a sensible formula, which of course reduces (if the points of M have equal measure) to the product of the values of the function. However, in the continuous case, there are difficulties, since there are ambiguities in the law of the branch of the logarithm that is to be chosen for 5.1.

However, we shall suppose that there are subsets $\Gamma(H)',\Gamma(E)'$ of $\Gamma(H)$ and $\Gamma(E)$ for which formula 5.1 does make sense. Then, the "tensor product $\bigotimes_{p \in M} H(p)$" of the family $\{H(p): p \in M\}$ of vector spaces will be <u>defined</u> as the smallest complex vector space of complex-valued functions on $\Gamma(E)'$ containing the functions:

$$f_\gamma : \gamma' \to \alpha(\gamma)(\gamma') \text{ of } \Gamma(E)' \to C$$

defined by 5.1. Of course, how to choose the subspaces $\Gamma(H)'$ and $\Gamma(E)'$ and how to make sense of 5.1 must be studied at a more explicit, case-by-case level. Accordingly, in the next few sections we shall survey a few of the special cases that are most important for the physical applications.

6. TENSOR PRODUCTS OF CONTINUOUS FAMILIES OF HILBERT SPACES

Suppose, as in Section 5, that $\pi: H \to M$ is a vector bundle, with a complex vector space structure on each fiber $H(p) = \pi^{-1}(p)$. Let us suppose that there is also a Hilbert space inner product $<|>$ on each fiber $H(p)$. Let us then choose the fiber space E considered in Section 5 to be H itself. For $\gamma \in \Gamma(H)$, $\gamma' \in \Gamma(E) = \Gamma(H)$, $p \in M$, let the value $\alpha(\gamma)(p)$ on $\gamma'(p)$ be the following complex number:

$$\alpha(\gamma)(p)(\gamma'(p)) = <\gamma'(p)|\gamma(p)> \qquad 6.1$$

In this case, let us compress our notation, and define:

$$<\gamma_1,|\gamma_2> = \exp \int_M \log <\gamma_1(p)|\gamma_2(p)>\, dp \qquad 6.2$$

for $\gamma_1, \gamma_2 \in \Gamma(H)$.

(In other words, $\alpha(\gamma)(\gamma')$, defined by 5.1, is in this case set equal to $<\gamma'|\gamma>$.) Let us now proceed somewhat more formally.

Definition Let $H \to M$ be a vector bundle, whose fibers are complex Hilbert spaces. Let dp be a measure on M. Let Γ be a space-not necessarily even a linear space - of cross-section maps $\gamma: M \to H$ such that $<\gamma_1|\gamma_2>$ is well-defined by 6.2. Let $FV(\Gamma)$ be the complex free vector space spanned by 6.2, with the inner product $<|>$ extended to $FV(\Gamma)$ by bilinearity. Let V_o be the subspace of $v \in (FV(\Gamma)$ such that $<v|\Gamma> = 0$. Then, the quotient Hilbert space $FV(\Gamma)/V_o$ is defined to be the continuous tensor product (with respect to the measure dp) of the family $\{H(p): p \in M\}$ of Hilbert spaces.

Of course, the question of the proper interpretation of 6.2 is still open, since we have not given a precise definition of the right hand side of 6.2. Let us turn to a more specific situation where one can be more precise. Further this situation - involving tensor products of Fock spaces - as important for quantum field theory.

7. TENSOR PRODUCT OF BOSON FOCK SPACES

First, let us recall the definition of the Fock space. (For example, see VB, vol. I., Chap. IX). Let H be a Hilbert space, with elements of H denoted by ψ. Let $H^r, r = 0,1,2\ldots$ be the symmetric tensor product of r copies of H. Then, H^0 is a one dimensional vector space, spanned by an element that is labelled $|0\rangle$. (Physically, $|0\rangle$ is the "vacuum state".) H^1 is H itself. For $r \geq 2$, H^r is spanned by symmetric products $\psi_1 \circ \ldots \circ \psi_r$ of elements $\psi_1, \ldots, \psi_r$ of H. Let $B(H)$ be the direct sum $H^0 \oplus H^1 \oplus H^2 \oplus \ldots$ of these spaces. It is called the <u>boson Fock space</u>. (The "Fermion" Fock space corresponds to skew-symmetric tensors over H, but will not be considered here.)

$B(H)$ is made into a Hilbert space, using the following rules for the inner products:

$$\langle H^r | H^s \rangle = 0 \text{ for } r \neq s. \qquad 7.1$$

$$\langle \psi_1 \circ \ldots \circ \psi_r | \psi_1{}' \circ \ldots \circ \psi_r{}' \rangle = \frac{1}{r!} \sum \langle \psi_1 | \psi_{i_1}{}' \rangle \ldots \langle \psi_r | \psi_{i_r}{}' \rangle \qquad 7.2$$

for $\psi_1, \ldots, \psi_r, \psi_1{}', \ldots, \psi_r{}' \in H$.

On the right hand side of 7.2, the sum is over all permutations $(1,\ldots,r) \to (i_1,\ldots,i_r)$ of the numbers from one to r. (This accounts for the normalization $\frac{1}{r!}$, which is of course just the number of such permutations.)

In particular, note that:

$$\langle\psi_1^{\,r}|\psi_2^{\,r}\rangle = \langle\psi_1|\psi_2\rangle^r \qquad 7.3$$

for $\psi_1,\psi_2 \in H$, where $\psi_1^{\,r}$

denotes the product $\psi_1 \circ \ldots \circ \psi_1$

of r copies of ψ_1. If $r = 0$, set:

$$\psi^o = |0\rangle, \text{ the vacuum state} \qquad 7.4$$

Now, for $\psi \in H$, set:

$$e(\psi) = \sum_{j=o}^{\infty} \frac{\psi^j}{\sqrt{j!}} \qquad 7.5$$

Then, using 7.1 and 7.3, we have:

$$\langle e(\psi_1)|e(\psi_2)\rangle =$$

$$\sum_{j,k} \frac{\langle\psi_1^{\,j}|\psi_2^{\,k}\rangle}{\sqrt{j!k!}} \qquad \sum_j \frac{\langle\psi_1|\psi_2\rangle^j}{j!}$$

$$= \exp(\langle\psi_1|\psi_2\rangle) \qquad 7.6$$

In particular, we can define "log" for these elements by the following formula:

$$\log\langle e(\psi_1)|e(\psi_2)\rangle = \langle\psi_1|\psi_2\rangle \qquad 7.7$$

Now, let $\pi: H \to M$ be a vector bundle with Hilbert spaces for fibers, and with a measure dp on M. Let $B(H)$ be the vector bundle whose fiber $B(H)(p)$ over the point $p \in M$ is the boson Fock space $B(H(p))$ of the fiber $H(p)$. For $\gamma \in \Gamma(H)$, let $e(\gamma)$ be the cross-section of $B(H)$ such that:

$$e(\gamma)(p) = e(\gamma(p)) \text{ for } p \notin M.$$

Given $\gamma_1, \gamma_2 \in \Gamma(H)$, set:

$$<\gamma_1|\gamma_2> = \int_M <\gamma_1(p)|\gamma_2(p)>dp \qquad 7.8$$

Then, the inner product 7.8 is the appropriate one to make $\Gamma(H)$, with the usual definition of addition of cross-sections, into a Hilbert space. (With this structure, $\Gamma(H)$ should be considered as the generalized "direct sum" of the continuous family $\{H(p):p\in M\}$ of Hilbert spaces.)

Let us now attempt to make sense of the precription given in Section 6 for defining the continuous tensor product of the family $\{BH(p):p\in M\}$ of Hilbert spaces. To this end, let $e(\Gamma)$ consist of the set of cross-sections of $B(H)$ which are of the form $e(\gamma)$, where γ is a cross-section of H. Let us use 6.2 to define an inner product between two elements $e(\gamma_1)$, $e(\gamma_2)$ of $e(\Gamma)$, keeping in mind that 7.7 gives us a valid and consistent definition of "log".

$$<e(\gamma_1)|e(\gamma_2)> = \exp\int_M \log <e(\gamma_1)(p)|e(\gamma_2)(p)> dp$$

$$= \exp \int_M <\gamma_1(p)|\gamma_2(p) > dp$$

Then, using 7.8 as the definition of the "direct sum" inner product on $\Gamma(H)$, we have:

$$<e(\gamma_1)|e(\gamma_2)> = \exp <\gamma_1|\gamma_2> \qquad 7.9$$

This formula may be interpreted another way. Let Γ be a Hilbert space of cross-sections of H_1 with the inner product of the "direct sum" type 7.8. Let $B(\Gamma)$ be its boson Feck space. Then, for $\gamma \in \Gamma$, denote by $e'(\gamma)$ the element $(0) + \gamma + \frac{\gamma^2}{\sqrt{2!}} + \ldots$ in $B(\Gamma)$. As we have seen,

$$<e'(\gamma_1)|e'(\gamma_2)> = \exp <\gamma_1|\gamma_2> \qquad 7.10.$$

Then, the objects $e(\gamma)$ and $e'(\gamma)$ behave in exactly the same way as far as inner products go. This suggests that we may identify the continuous tensor product $\bigotimes_p BH(p)$ with the Fock space $B\Gamma$ of the Hilbert space Γ of cross-sections of H.

Exercise 7.1 Carry out the details of the identification of $B\Gamma$ with the continuous tensor product $FV(e(\Gamma))/V_o$.

Exercise 7.2 Let Γ^j, $j = 2,3,\ldots,$ be the space of symmetric tensors over the vector space Γ of cross-sections of H. Let H^2 be the vector bundle whose fiber $H^j(p)$ over the point p is the space of symmetric tensors of the vector space $H(p)$. Show that Γ^2 may be identified with the cross-sections $\Gamma(H^j)$ of H^j. Show that the Hilbert space inner products defined naturally on both spaces agree. Explain how these facts play a basic role in the identification - in Exercise 7.1 - of the tensor product $\bigotimes_{p \in M} BH(p)$ with $B\Gamma$.

Exercise 7.3 Suppose that the Hilbert space H is the direct sum of two orthogonal subspaces H_1,H_2. Show directly that $B(H)$ is isomorphic to the tensor product $B(H_1) \otimes B(H_2)$. Discuss why this result is the prototype of the ideas presented in this section.

Remarks. We have not been specific which space of cross-sections $\Gamma \subset \Gamma(H)$ to use. This might be left open, to be specified when the bundle is made more specific. For example, if M is a manifold and if dp is a measure defined by a volume element differential form on M, one appropriate choice might be to take Γ as the continuous cross-sections which vanish outside of a compact subset of M.

The ideas sketched here concerning infinite tensor products have been developed by Araki, Guichardet, Streater, Woods, and Wulfshon. We refer to the book by Guichardet [1] for a more systematic exposition, and references to further work. However, the reader should understand that the theory is still in an unfinished state. I have only aimed to give enough detail to suggest ideas that may be applicable to the problem of representing "current algebras". In fact, the formalism as it stands is not even applicable to the simplest form of current algebra, namely those that are gauge groups corresponding to a "charge" algebra that is a compact, semisimple Lie algebra. We shall now show how it may be modified in that direction.

8. HILBERT VECTOR BUNDLES WITH A DISTINGUISHED VACUUM STATE CROSS-SECTION

The key feature of Fock spaces that enables one to define their tensor product is that they come with a "vacuum" state. I shall now - following a suggestion to me by J. Feldman - develop the ideas only on the assumption that the Hilbert spaces have a distinguished cross-section.

Let $\pi: E \to M$ be a vector bundle, whose fibers $\{H(p)\}$ have Hilbert space structures, denoted by $\langle | \rangle$. Let γ_0 be a continuous cross-section map: $M \to H$ such that

$$\langle \gamma_0(p) | \gamma_0(p) \rangle = 1 \text{ for } p \in M. \qquad 8.1$$

Suppose that M is a locally compact topological space, and that dp is a measure on M such that the compact subsets of M are measurable, and have finite measure. (Then, the continuous, compact-support functions are integrable with respect to this measure.)

For $p \in M$, let $\gamma_o(p)^{\perp}$ denote the vectors $\psi \in H(p)$ that satisfy: $\langle\gamma_o(p)|\psi\rangle=0$. Let Γ denote the measurable cross-section map $\gamma:M \to H$ of the form

$$\gamma=\gamma_o+\gamma_1 , \qquad 8.2$$

where γ satisfies the following conditions:

γ_1, as a cross-section map: $M \to H$,
vanishes outside of   8.3
a compact subset of M

$\|\gamma_1(p)\| < 1$ for all $p \in M$ 8.4

$\gamma_1(p) \in \gamma_o(p)^{\perp}$ for all $p \in M$.   8.5

Suppose now that $\gamma,\gamma' \in \Gamma$, with

$$\gamma = \gamma_o + \gamma_1,\ \gamma' = \gamma_o + \gamma_1' ,$$

where γ_1,γ_1' satisfy 8.3-8.5. Then,

$$\langle\gamma(p)|\gamma'(p)\rangle = 1 + \langle\gamma_1(p)|\gamma_1'(p)\rangle.$$

Using the Schwarz inequality

$$|\langle\gamma(p)|\gamma'(p)\rangle-1| \leq \|\gamma_1(p)\|\ \|\gamma_1'(p)\| < 1 .$$

Hence, $\log(\langle\gamma(p)|\gamma'(p)\rangle)$ may be defined by using the standard power series of expansion. The function $p \to \log\langle\gamma(p)|\gamma'(p)\rangle$ is of compact support. Hence, we may define an inner product (of the "tensor product", not the "direct sum", type) in the following way:

$$\langle\gamma|\gamma'\rangle = \exp\int_M \log\langle\gamma(p)|\gamma'(p)\rangle dp \qquad 8.6 .$$

As before, the inner product may be extended to $FV(\Gamma)$, then defined on $FV(\Gamma)/V^o$, to define a Hilbert space structure $FV(\Gamma)/V^o$. We will call this Hilbert space the generalized Fock tensor product of the Hilbert spaces $\{H(p):p \in M\}$. It might be useful - for example in the problem of representative gauge groups - in cases where the individual Hilbert spaces $\{H(p)\}$ are not themselves Fock spaces.

9. GAUGE LIE ALGEBRAS AND GROUPS

The gauge Lie algebras are the simplest examples of current algebras. (They are given this name because they are the abstract Lie algebras responsible for what physicists call "gauge transformations".) In this section we shall briefly discuss some of their properties.

First, suppose that $\underset{\sim}{G}$ is a Lie algebra (over the real numbers, as field of scalars). Let F be an algebra over the real numbers, with the product law: $F \times F \to F$ denoted by $(f_1,f_2) \to f_1 f_2$.

Let $\underset{\sim}{G}_F = \underset{\sim}{G} \otimes F$ be the tensor product of these two real vector spaces. Denote the Lie bracket in $\underset{\sim}{G}$ by $[\ ,\ ]$. Define a bracket operation $[\ ,\ ]'$ as a bilinear map: $\underset{\sim}{G}_F \times \underset{\sim}{G}_F \to \underset{\sim}{G}_F$ by means of the following rule:

$$[X_1 \otimes f_1, X_2 \otimes f_2]' = [X_1, X_2] \otimes (f_1 f_2) \qquad 9.1$$

$$\text{for } X_1, X_2 \in \underset{\sim}{G}; f_1, f_2 \in F.$$

Theorem 9.1. Suppose that the given Lie algebra structure on $\underset{\sim}{G}$ is non-abelian. Then, the bracket $[\ ,\ ]'$ defined by 9.1 defines $\underset{\sim}{G}_F$ as a Lie algebra (over the real numbers) if and only if the given algebra structure on F is commutative and associative.

Proof. Let us first assume that 9.1 defines a Lie algebra structure on $\underset{\sim}{G}_F$. Then,

$$[X_1 \otimes f_1, X_2 \otimes f_2]' = -[X_2 \otimes f_2, X_1 \otimes f_1]'.$$

Hence, using 9.1,

$$[X_1, X_2] \otimes (f_1 f_2) = -[X_2, X_1] \quad (f_2 f_1). \qquad 9.2$$

But, the Lie algebra condition on the bracket [,] for $\underset{\sim}{G}$ implies that:

$$[X_1, X_2] = -[X_2, X_1]. \qquad 9.3$$

9.2 and 9.3 imply the following relation:

$$[\underset{\sim}{G}, \underset{\sim}{G}] \otimes (f_1 f_2 - f_2 f_1) = 0 \qquad 9.4$$

$$\text{for all } f_1, f_2 \in F.$$

The fact that $\underset{\sim}{G}$ is non-abelian now forces:

$$f_1 f_2 = f_2 f_1$$

which implies that the algebra structure on F is commutative.

<u>Exercise</u>. Show, in a similar way, that the Jacobi identity for the bracket $[\ ,\ ]'$ implies that the algebra structure on F is associative.

<u>Exercise</u>. Show, conversely, that if F is a commutative associative algebra, then 9.1 defines $\underset{\sim}{G}_F$ as a Lie algebra over the real numbers, called the <u>gauge Lie algebra</u> defined by $\underset{\sim}{G}$ and F.

These exercises complete the proof of Theorem 9.1. Suppose that F is a commutative associative algebra. Notice further that $\underset{\sim}{G}_F$ is an F-module; where the module structure is defined by the following formula.

$$f(X \otimes f_1) = X \otimes (f f_1) \qquad 9.5$$

From now on, let us denote the Lie bracket operation on $\underset{\sim}{G}_F$ given by 9.1 with the symbol [,] also, i.e. we leave off the prime. Notice then that, given $X \in \underset{\sim}{G}_F$, the operator $\text{Ad}\, X : \underset{\sim}{G}_F \to \underset{\sim}{G}_F$ is a zero order differential operator. In my paper [13], I call "current algebras" these F-modules which have real Lie algebra structures such that Ad $\mathfrak{X}$ is a differential operator of arbitrary order.

Now, let us suppose that F is the algebra of C^{∞}, real-valued functions on a manifold M. An important general problem for current algebra theory can be stated as follows:

Given an F-module Γ which has a Lie algebra structure which makes it a current algebra over F, can one construct a group whose Lie algebra is Γ ?

In the case where $\Gamma = \underset{\sim}{G}_F$ this can be readily solved via the "gauge group" construction. Let G be a connected Lie group whose Lie algebra is $\underset{\sim}{G}$. Let G_F denote the space of C^{∞} maps, $\phi: M \to G$. Make G_F into a group with the following operations:

$$(\phi_1\phi_2)(p) = \phi_1(p)\phi_2(p)$$
$$(\phi_1)^{-1}(p) = \phi_1(p)^{-1} \qquad 9.6$$

for $p \in M$, $\phi_1, \phi_2 \in G_F$.

To show that $\underset{\sim}{G}_F$ is the Lie algebra of G_F, we must show that there is a one-one correspondence between those parameter subgroups of G_F and the elements of $\underset{\sim}{G}_F$, with the usual relations holding between the products and commutaters of one parameter subgroup and the bracket operation 9.1 in $\underset{\sim}{G}_F$. Let us set up this correspondence in the following way:

Given $\gamma \in \underset{\sim}{G}_F$, write γ in the following form:

$$\gamma = \mathfrak{X}_1 \otimes f_1 + \dots + \mathfrak{X}_n \otimes f_n \ , \qquad 9.7$$

with $\mathfrak{X}_1, \dots, \mathfrak{X}_n \in \underset{\sim}{G}$; $f_1, \dots, f_n \in F$. Associate with γ the following one parameter subgroup $t \to \phi_t$ of G_F:

$$\phi_t(p) = \exp(tf_1(p)X_1+\ldots+tf_n(p)X_n), \qquad 9.8$$

$$\text{for } -\infty<t<\infty,\ p \in M.$$

<u>Exercise</u>. If γ,γ' are elements of $\underset{\sim}{G}_F$, with $t \to \phi_t$, ϕ_t'th parameter subgroups of G_F defined by 9.8 show that:

$$\phi''_t = \lim_{n\to\infty}(\phi_{t/n}\phi'_{t/n})^n, \qquad 9.9$$

$$\phi'''_t = \lim_{n\to\infty}[\phi_{\sqrt{t/n}}\ \phi'_{\sqrt{t/n}}\ \phi_{-\sqrt{tn}}\ \phi'_{-\sqrt{t/n}}]^n, \qquad 9.10$$

where ϕ''_t, ϕ'''_t are the one parameter subgroups of G_F associated via formula 9.8 with the elements

$$\gamma'' \equiv \gamma + \gamma';\gamma''' \equiv [\gamma,\gamma']$$

of $\underset{\sim}{G}_F$.

<u>Exercise</u>. Show that every one parameter subgroup of G_F is of the form indicated in formula 9.8.

These exercises then will complete the proof that $\underset{\sim}{G}_F$ can be identified with the Lie algebra of G_F.

There is an alternate way of defining $\underset{\sim}{G}_F$ that is often useful.

<u>Exercise</u>. Show that $\underset{\sim}{G}_F$ may be identified with the space of C^∞ map $\underset{\sim}{\phi}:M \to \underset{\sim}{G}$, with the bracket defined by the following formula:

$$[\underset{\sim}{\phi},\underset{\sim}{\phi}'] = [\underset{\sim}{\phi}(p),\underset{\sim}{\phi}'(p)] \ . \qquad 9.11$$

Show that with this identification the one parameter subgroup $t \to \phi_t$ defined by an element $\underset{\sim}{\phi} \in \underset{\sim}{G}_F$ is given by the following formula:

$$\underset{\sim}{\phi}_t(p) = \exp(t\underset{\sim}{\phi}(p)) \qquad 9.12$$

$$\text{for } -\infty<t<\infty,\ p \in M.$$

This is a more "geometric" way of defining $\underset{\sim}{G}_F$. It may be generalized in the following direction:

Let $\pi:E\to M$ be a vector bundle over M. This bundle will be called a Lie algebra bundle if each fiber $\pi^{-1}(p)$ has the structure of a Lie algebra. The space Γ of cross-sections may then be made into a Lie algebra using the following formula:

$$[\gamma_1,\gamma_2](p) = [\gamma_1(p),\gamma_2(p)] \quad \text{for } \gamma_1,\gamma_2 \in \Gamma,\ p \in M. \qquad 9.13$$

Remark: The definition - via 9.11 - of $\underset{\sim}{G}_F$ as the space of maps $\phi:M\to\underset{\sim}{G}$ corresponds to the choice of $E=M\times\underset{\sim}{G}$, the product bundle. Thus, in this case, the Lie algebra structure of the fiber $\pi^{-1}(p)$ does not vary from point to point. More general cases are possible using the theory of "deformations" of Lie algebras. (See my papers "Analytic continuation of group representations").

We now turn to a study of a general method of realizing current algebras as the Lie algebras of infinite dimensional Lie groups.

10. REPRESENTATIONS OF CURRENT ALGEBRA AS LIE ALGEBRAS OF VECTOR FIELDS

Let M be a manifold, and let F be the algebra of C^∞, real-valued functions on M. Let Γ be an F-module, which is also a real Lie algebra.

As for any Lie algebra, one can discuss "representation" of Γ, as Lie algebra homomorphisms into various other sorts of Lie algebras. In this section we shall discuss one sort of such representation that seems to be of physical and geometric interest.

Let E be a manifold, with $\pi:E\to M$ a fiber space map of E onto M. Let $V(E)$ denote the C^∞ vector fields on E. We shall consider $V(E)$ an F-module in the following way:

$$f\mathfrak{X} = \pi^*(f)\mathfrak{X} \qquad 10.1$$

$$\text{for } \mathfrak{X} \in V(E),\ f \in F(\equiv F(M)).$$

<u>Definition. By a representation of Γ as vector fields</u> is meant an R-linear mapping $\gamma \to \mathfrak{X}_\gamma$ of $\Gamma \to V(E)$ which is a differential operator in the F-module sense described in Chapter I of Volume I.

The simplest example leads us back to the geometric meaning of the "gauge group". Let N be a manifold, G a Lie group which acts as a transformation group on N. Then, the Lie algebra $\underset{\sim}{G}$ acts as a Lie algebra of vector fields on N. Let $\mathfrak{X}_1,\dots,\mathfrak{X}_n$ be a basis for $\underset{\sim}{G}$.

Let E be the product M×N. Let $\Gamma = \underset{\sim}{G}_F = \underset{\sim}{G} \otimes F$, the gauge Lie algebra. Consider the set of vector fields $\mathfrak{X}$ on E which can be written in the following form:

$$\mathfrak{X} = f_1\mathfrak{X}_1 + \dots + f_n\mathfrak{X}_n\ , \qquad 10.2$$

where $f_1,\dots,f_n$ are arbitrary C^∞ functions on M. Since the $\mathfrak{X}_1,\dots,\mathfrak{X}_n$ are vector fields on E that are tangent to N, we have:

$$\mathfrak{X}_i(f_j) = 0 \quad \text{for} \quad 1 \le j,\ j \le n\ .$$

In particular, we see that the set of vector fields of the form 10.2 on E is isomorphic to $\underset{\sim}{G}_F$, i.e. to $\gamma = \mathfrak{X}_1 \otimes f_1 + \dots + X_n \otimes f_n$ we associate the vector field $\mathfrak{X}_f \equiv$ right hand side of 10.2. We see further that the map $\gamma \to \mathfrak{X}_\gamma$ is a zeroth order differential operator.

<u>Problem</u>. Classify all Lie algebra homomorphisms: $\underset{\sim}{G}_F \to V(E)$ that are zero - the order differential operators as F-modules.

This realization of $\underset{\sim}{G}_F$ as a Lie algebra of vector fields on E=M×N is also useful as a way of constructing a Lie group whose Lie algebra is $\underset{\sim}{G}_F$. For the set of all vector fields of form 10.2 forms a Lie subalgebra of $V(E)$. One can then define G_F as the group of diffeomorphisms of E generated by this Lie algebra.

<u>Exercise</u>. Identifying G_F as in Section 9 as the group of maps $\phi: M \to G$, make G_F act as a transformation group on $E=M\times N$ so that the Lie algebra of vector fields so generated is precisely those of the form 10.2.

The next step in the program should be to study representations: $\underset{\sim}{G}_F \to V(E)$ which are higher order differential operators. We shall now study a typical case of this type.

Let us suppose that $M=R^3$. (This is the important case for physics). Denote a point of M by x, with components (x_i), $1\le i,y\le 3$. Let $\partial_i = \frac{\partial}{\partial x_i}$. Suppose again that $E=M\times N$. Let $\underset{\sim}{G}$ be a finite dimensional Lie algebra with a basis (X_a), $1\le a\le n$, and with Lie algebra structure relations of the following form:

$$[X_a, X_b] = C_{abc}\, X_c \qquad 10.3$$

(Here, the C_{abc} are the structure constants of the Lie algebra $\underset{\sim}{G}$. The summation convention is in force on the indices a and i).

Suppose now that Y_{ai} are vector fields on N. Define a map

$$\alpha: \underset{\sim}{G}_F \to V(E)$$

by the following formula:

$$\alpha(X_a \otimes f) = f\, Y_a + \partial_i(f) Y_{ai} . \qquad 10.4$$

Then,

$$\alpha([X_a \otimes f_1, X_b \otimes f_2])$$

$$= C_{abc}(f_1 f_2\, Y_c + \partial_i(f_1 f_2) Y_{ci}) \qquad 10.5$$

$$
\begin{aligned}
&[\alpha(\underset{\sim}{X}_a \otimes f_1),\ \alpha(\underset{\sim}{X}_b \otimes f_2)] \\
&=[f_1 Y_a+\partial_i(f_1)Y_{ai}, f_2 Y_b+\partial_j(f_2)Y_{bj}] \\
&=f_1 f_2 [Y_a, Y_b] \\
&+\partial_i(f_1) f_2 [Y_{ai}, Y_b] \\
&+f_1 \partial_j(f_2)[Y_a, Y_{bj}] \\
&+\partial_i(f_1)\partial_j(f_2)[Y_{ai}, Y_{bj}].
\end{aligned} \tag{10.6}
$$

Thus,

$$
\begin{aligned}
&\alpha([\underset{\sim}{X}_a \otimes f_1,\ \underset{\sim}{X}_b \otimes f_2]) \\
&-[\alpha(\underset{\sim}{X}_a \otimes f_1),\ \alpha(\underset{\sim}{X}_b \otimes f_2)] \\
&=f_1 f_2([Y_a, Y_b]-C_{abc}\, Y_c) \\
&+\partial_i(f_1) f_2(C_{abc}\, Y_{ci}-[Y_{ai}, Y_b]) \\
&+f_1 \partial_i(f_2)(C_{abc}\, Y_{ci}-[Y_a, Y_{bi}]) \\
&+\partial_i(f_1)\partial_j(f_2)[Y_{ai}, Y_{bj}].
\end{aligned} \tag{10.7}
$$

We can make several deductions from this formula:

<u>Theorem 10.1</u>. α defined by formula 10.3 is a Lie algebra homomorphism of the gauge Lie algebra $\underset{\sim}{G}_F$ into the Lie algebra $V(E)$ of vector fields on $E=R^3\times N$ if and only if the following relations are satisfied:

$$[Y_a, Y_b] = C_{abc}\, Y_c \tag{10.8}$$

$$[Y_{ai}, Y_{bj}] = 0 \tag{10.9}$$

$$[Y_a, Y_{bi}] = C_{abc}\, Y_{ci} \tag{10.10}$$

Relations 10.8-10.10 define a new Lie algebra, which we shall study in Section 11. (Notice that this Lie algebra already appeared in the work presented in LAQM on representations of the gauge Lie algebra).

Let us now investigate more general possibilities for the right hand side of 10.6 than relations 10.7-10.9. To do this, set:

$$Y_{abi} = C_{abc} Y_{ci} - [Y_{ai}, Y_b] \qquad 10.11$$

Then,

$$\begin{aligned} &C_{abc} Y_{ci} - [Y_a, Y_{bi}] \\ &= -C_{bac} Y_{ci} + [Y_{bi}, Y_a] = -Y_{bai} \end{aligned} \qquad 10.12$$

Let us impose the following conditions:

$$[Y_{ai}, Y_{bi}] = 0 \qquad 10.13$$

$$[Y_c, Y_{abi}] = 0 = [Y_{cj}, Y_{abi}] \qquad 10.14$$

With these assumptions, we see that $\alpha(\underset{\sim}{G}_F)$ generates a Lie algebra that we shall call $\underset{\sim}{L}$, with the elements of the form:

$$f\ Y_{abi} \qquad 10.15$$

in the center of $\underset{\sim}{L}$. Let us denote by $\underset{\sim}{K}$ the elements of $\underset{\sim}{L}$ of form 10.14.

Let $\underset{\sim}{K}'$ denote the subspace of elements of $\underset{\sim}{K}$ of the form 10.14, with:

$$\int_{R^3} f(x)\,dx = 0 \qquad 10.16$$

Thus, $\underset{\sim}{K}'$ forms an ideal of $\underset{\sim}{L}$. Let $\underset{\sim}{L}'$ denote the quotient Lie algebra $\underset{\sim}{L}/\underset{\sim}{K}'$. Denote by $f\ Y'_a$, $f\ Y'_{ai}$, $f\ Y'_{abi}$ the image of the elements $f\ Y_a$, $f\ Y_{ai}$, $f\ Y_{abi}$ in $\underset{\sim}{L}'$. Let $\alpha':\underset{\sim}{G}_F \to \underset{\sim}{L}'$ denote the map

$\alpha: \underset{\sim}{G}_F \to \underset{\sim}{L}$, followed by the quotient projection map: $\underset{\sim}{L} \to \underset{\sim}{L}'$. Thus, we have the following relations:

$$\alpha'([X_a \otimes f_1, X_b \otimes f_2]) - [\alpha'(X_a \otimes f_1), \alpha'(X_b \otimes f_2)]$$
$$= f_1 f_2([Y_a, Y_b]' - C_{abc} Y'_c)$$
$$+ \int_{R^3} \partial_i(f_1) f_2 \, dx \, Y'_{abi} - \int_R f_1 \partial_i(f_2) dx \, Y'_{bai}$$

Hence, after integrating by parts,

$$\alpha'([X_a \otimes f_1, X_b \otimes f_2]) - [\alpha'(X_a \otimes f_1), \alpha'(X_b \otimes f_2)]$$
$$= f_1 f_2([Y_a, Y_b]' - C_{abc} Y'_c)$$
$$+ \int_{R^3} \partial_i(f_1) f_2(x) dx (Y'_{abi} + Y'_{bai}) \qquad 10.17$$

Notice that the second term on the right hand side of 10.16 is a typical example of what the physcists call a "Schwinger term". Thus, we see that we have devised a way of realizing the "current algebras" that appear in quantum field theory and elementary particle physics geometrically as Lie algebras of vector fields. The way is then open to constructing the groups whose Lie algebras are the current algebras by means of geometry, i.e. finding the groups of diffeomorphisms of E generated by the Lie algebra of vector fields used to represent the current algebra.

Problem and Example. Consider a typical current algebra from the geometric point of view. For example, the following one is typical:

$$[V_o^\alpha(x), V_o^\beta(y)] = C_{\alpha\beta\gamma} V_o^\gamma \delta(x-y) \qquad 10.18$$

$$[V_o^\alpha(x), V_i^\beta(y)] = C_{\alpha\beta\gamma} V^\gamma \delta(x-y) \qquad 10.19$$
$$+ \partial_j^x (V_{ij}^{\alpha\beta}(x,y))$$

$$[V_i^\alpha(x), V_j^\alpha(y)] = 0 \qquad 10.20$$

In 10.17-10.19, the current algebra relations are presented in the intuitive form preferred by the physicists. Here, "x","y" denote elements of R^3. α,β denote "internal symmetry" indices. Let F denote the compactly supported, C^∞, real-valued functions on R^3. Set:

$$V_o^\alpha(f) = \int_{R^3} V_o^\alpha(x) f(x) dx \qquad 10.21$$

$$V_j^\alpha(f) = \int_{R^3} V_i^\alpha(x) f(x) dx \qquad 10.22$$

$$V_{ij}^{\alpha\beta}(f_1, f_2) = \int_{R^3 \times R^3} V_{ij}^{\alpha\beta}(x,y) f_1(x) f_2(y) dx\, dy \qquad 10.23$$

Then, relations 10.17-10.19 take the following integrated form:

$$[V_o^\alpha(f_1), V_o^\beta(f_2)] = C_{\alpha\beta\gamma} V_o^\gamma(f_1 f_2) \qquad 10.24$$

$$[V_o^\alpha(f_1), V_i^\beta(f_2)] = C_{\alpha\beta\gamma} V_i^\gamma(f_1 f_2) + V_{ij}^{\alpha\beta}(\partial_j(f_1) f_2) \qquad 10.25$$

$$[V_i^\alpha(f_1), V_j^\beta(f_2)] = 0 \qquad 10.26$$

The second term on the right hand side of 10.24 is the "Schwinger term". Without it, the relations 10.23-10.25 define a gauge Lie algebra G_F. (Notice that G is <u>not</u> the "internal symmetry" Lie algebra with structure constants $C_{\alpha\beta\gamma}$, but a larger Lie algebra with the internal symmetry as a subalgebra.) Now, as the "problem" discuss ways of realizing these relations as Lie algebras of vector fields. An important special case is the "Sugawara model," which is obtained by the following special form of the Schwerger term:

$$V_{ij}^{\alpha\beta}(x,y) = \delta_{ij}\ \delta_{\alpha\beta}\ \delta(x-y). \qquad 10.27$$

Then, 10.24 takes the following form:

$$[V_o^{\alpha}(f_1), V_i^{\beta}(f_2)] = C_{\alpha\beta\gamma} V_i^{\alpha}(f_1 f_2) + \int_{R^3} \partial_i(f_1) f_2(x)dx \qquad 10.28$$

Discuss the conditions this places on the Lie algebra $\underset{\sim}{L}'$. Can one find a global group whose Lie algebra is defined by relations 10.23, 10.25, 10.27?

Remark: Realizations of the Sugawara model by means of inhomogeneous differential operators have been discussed already in my paper, "Current algebra, Sugawara model, and differential geometry". What might be done is to translate the more concrete way given there of realizing the commutation relations into the framework sketched above.

11. THE GROUP STRUCTURE ON THE JET SPACES OF A LIE GROUP

In my paper titled "Current algebra, Sugawara model, and differential geometry", it was found that the appropriate "configuration space" for the realization of the "Sugawara model" current algebra commutation relations was a Lie group manifold. Then, the "phase space" is an appropriate jet space of mappings of a manifold (in field-theoretic application, typically R^3) into the group. In this section we shall discuss in an intrinsic geometric way some of the mathematical ideas underlying this construction, without further explicit mention of the possible current algebra applications.

Let G be a Lie group, and let M be a manifold. Let Φ be the space of C^∞ mappings: $M \to G$. Recall the notion of "r-th order of contact" of two such mappings. For example, if $r=1$, then $\phi_1, \phi_2 \in \Gamma$ are said to have <u>first order contact at a point $p \in M$</u> if:

$$\begin{aligned} \phi_1(p) &= \phi_2(p) \\ \phi_{1*}(p) &= \phi_{2*}(p) \qquad 11.1 \\ &\text{for all } v \in M_p. \end{aligned}$$

(In fact, in this section we shall deal only with the case of first order jets, for the sake of simplicity).

Define $\Theta^r(M,G)$ as the quotient of $M\times\Phi$ by the following equivalence relation:

(p_1,ϕ_1) is equivalent to
(p_2,ϕ_2) if and only if
$p_1=p_2$ 11.2
ϕ_1 and ϕ_2 agree to the r-th
order at $p_1=p_2$.

Let π denote the map: $M\times\Phi \to M$ which is the Cartesian product projection. 11.2 shows that π is constant on the equivalence classes, and passes to the quotient to define a map, which we also denote by π, of $\Theta^r(M,G) \to M$. This will be called the <u>projection map</u>.

For $p \in M$, let $\pi^{-1}(p)$ be the fiber of π above p. We will denote this space by $\Theta^r(M,G)(p)$. The main result is now:

<u>Theorem 11.1</u>. The Lie group structure on G induces a Lie group structure on the fiber $\Theta^r(M,G)(p)$, for each point $p \in M$.

<u>Proof</u>. We shall give the proof only for $r=1$. Notice that Φ, the space of all mappings:$M\to G$, is a group, with the group structure given by point-wise multiplication. (This is, of course, just the "gauge

group" defined earlier). Let p_o be a fixed point of M, and let 1 be the map: $M \to G$ such that:

$$1(p) = e \quad \text{for all} \quad p \in M.$$

(e is the identity element of the group G).

Let Φ_o denote the set of all $\phi_o \in \Phi$ such that ϕ_o has first order contact with 1 at p_o, i.e. such that:

$$\phi_o(p_o) = e; \phi_{o*}(M_{p_o}) = 0 \qquad 11.3$$

<u>Lemma 11.2</u>. Φ_o defined by conditions 11.3 is an invariant subgroup of Φ.

<u>Proof</u>. Let $\phi \in \Phi$, $\phi_o \in \Phi_o$. We must show that $\phi\, \phi_o\, \phi^{-1}$ belongs to Φ_o, i.e. that it satisfies condition 11.3.

Now,

$$(\phi\, \phi_o\, \phi^{-1})(p_o) = \phi(p_o)\phi_o(p_o)\phi(p_o)^{-1} = e$$

Suppose that $t \to \sigma(t)$ is a curve in M, such that $\sigma(0) = p_o$, with $v \in M_{p_o}$ its tangent vector. Then, the tangent vector to the curve

$$t \to (\phi\, \phi_o\, \phi^{-1})(\sigma(t)) \qquad 11.4$$

at $t=0$ is the element $(\phi\, \phi_o\, \phi^{-1})_*(v)$ of M_{p_o}. The curve 11.4 is now, by the meaning of multiplication in Φ, just the curve:

$$t \to \phi(\sigma(t))\phi_o(\sigma(t))\phi(\sigma(t))^{-1} \qquad 11.5$$

Now, the right hand side of 11.5 can be written as follows:

$$t \to (\phi(\sigma(t))\phi(p_o)^{-1})(\phi(p_o)\phi_o(\sigma(t)\phi(p_o)^{-1})(\phi(p_o)\phi(\sigma(t))^{-1}$$

Thus, we have:

$$(\phi\, \phi_o\, \phi^{-1})(\sigma(t)) = g(t)g_o(t)g(t)^{-1}, \qquad 11.6$$

where:

$$g(t) = \phi(\sigma(t))\phi(p_o)^{-1} \tag{11.7}$$

$$g_o(t) = \phi(p_o)\phi_o(\sigma(t))\phi(p_o)^{-1} \tag{11.8}$$

The curves $t \to g(t)$, $g_o(t)$ are curves in G, equal to the identity element at $t=0$. Their tangent vectors at $t=0$ are then elements $X, Y \in \underset{\sim}{G}$. (Identify $\underset{\sim}{G}$, the Lie algebra of G, with the tangent space of G at the identity element).

Exercise. Suppose that $t \to g_1(t)$, $g_2(t)$ are curves in G, equal to e at $t=0$, with tangent vectors $X, Y \in \underset{\sim}{G}$ at $t=0$. Show that the tangent vector $t=0$ of the curve $t \to g_1(t)g_2(t)$ (i.e. the product of the curves) is the element $X + Y$ of $\underset{\sim}{G}$.

As a result of this exercise, 11.3, 11.6 and 11.8, we see that the tangent vector at $t=0$ of the curve given by 11.5 is zero. This proves that:

$$(\phi\ \phi_o\ \phi^{-1})_*(M_{p_o}) = 0,$$

hence completes the proof of Lemma 1.2.

The completion of the proof of Theorem 11.1 is now routine.

Exercise. Show that $\Theta^1(M,G)(p_o)$ is equal to the space of cosets of the group Φ by the subgroup Φ_o, i.e. that two maps $\phi_1,\phi_2{:}M \to G$ meet to the first order at p_o if and only if: $\phi_1\phi_2^{-1} \in \Phi_o$.

Having thus identified $\Theta^1(M,G)(p_o)$ with the coset space Φ/Φ_o, we see that it inherits a group structure from the fact that Φ_o is an invariant subgroup.

Exercise. Suppose that $M=R$, $p_o=0$. Identify $\Theta^1(M,G)(p_o)$ with the tangent bundle $T(G)$ to G. Let $\alpha{:}G\times G \to G$ be the map that defines the group low on G. Identify

$$T(G \times G) \text{ with } T(G)\otimes T(G).$$

Then, α_* maps $T(G) \otimes T(G)$ linearly into $T(G)$. Show that this map defines a group structure on $T(G)$, and that this group structure is the one which is defined on $\Theta^1(M,G)(p_o)$ by Theorem 11.1.

Exercise. Extend Theorem 11.1 to define a group structure on $\Theta^r(M,G)(p_o)$, for all integers r.

Problem. Suppose that M is a product manifold $M_1 \times M_2$. Determine how the group structure on $\Theta^r(M,G)$ is determined by the group structures on $\Theta^r(M,G)$ and $\Theta^r(M_2,G)$.

Now, let β be the map: $\Theta^1(M,G)(p_o) \to G$ defined by the condition:

$$\beta(\phi) = \phi(p_o) \qquad 11.9$$

for all $\phi \in \Phi$. Notice that β is a group homomorphism of the group $\Theta^1(M,G)(p_o)$ into the group G.

Theorem 11.3. The homomorphism β defines $\Theta^1(M,G)(p_o)$ as an abelian extension of G, i.e. the kernel of β is an abelian invariant subgroup of $\Theta^1(M,G)(p_o)$.

Proof. Let ϕ_1,ϕ_2 be map:$M \to G$, such that:

$$\phi_1(p_o) = e = \phi_2(p_o). \qquad 11.10$$

Then, the images of ϕ_1,ϕ_2 in $\Theta^1(M,G)(p_o)$ lies in the kernel of β. Set:

$$\phi = \phi_1\phi_2\phi_1^{-1}\phi_2^{-1} \qquad 11.11$$

To show that the kernel is abelian, we must show that the image of ϕ in $\Theta^1(M,G)(p_o)$ is the identity element, i.e. that

$$\phi_*(M_{p_o}) = 0 \qquad 11.12$$

Exercise. Show that:

$$(\phi_1)_*^{-1}(v) = -\phi_{1*}(v) \qquad 11.13$$

$$(\phi_1\phi_2)_*(v) = \phi_{1*}(v) + \phi_{2*}(v) \qquad 11.14$$

$$\text{for} \quad v \in M_{p_o}.$$

Repeated application of 11.13-11.14 shows that:

$$\phi_*(v) = \phi_{1*}(v) + \phi_{2*}(v) - \phi_{1*}(v) - \phi_{2*}(v) = 0,$$

$$\text{for all} \quad v \in M_{p_o},$$

which proves 11.12, hence completes the proof of the theorem.

<u>Problem</u>. Suppose $\underset{\sim}{G}$ has a basis (Y_a), such that:

$$[Y_a, Y_b] = C_{abc}\, Y_c\ , \quad 1 \le a,b,c \le n = \dim \underset{\sim}{G}.$$

Let $M=R^3$, with $p=(0)$. Let $\underset{\sim}{G}^1$ denote the Lie group $\Theta^1(M,G)(p_o)$. Show that G^1 has a basis labelled (Y'_a, Y'_{ai}), $1 \le i \le 3$, such that:

$$[Y'_a, Y'_b] = C_{abc}\, Y'_c$$

$$[Y'_{ai}, Y'_{bj}] = 0 \qquad 11.15$$

$$[Y'_a, Y'_{bi}] = C_{abc}\, Y'_{ci}$$

$$\beta_*(Y'_a) = Y_a$$

$$\beta_*(Y'_{ai}) = 0 \qquad 11.16$$

Notice then that the Lie algebra $\underset{\sim}{G}^1$ is the one defined by conditions 10.7-10.9, determining a representation of the gauge Lie algebra $\underset{\sim}{G}_F$. In turn, notice that the ideas presented in my paper "Current algebras, Sugawara model, and differential geometry" involve closely the jet space $\Theta^1(R^3,G)$, since G is the configuration space of the dynamical system which gives rise to the Sugawara model current commutation relations.

Problem. The ideas of this section may be generalized in various directions. (Notice that they are not "deep", but invoke what mathematicians call "generalized nonsense".) One significant and elegant generalization might be described as follows:

Let $\pi: E \to M$ be a fiber space mapping, with a Lie group structure given on each fiber $\pi^{-1}(p)$. Let Φ be the space of C^∞ cross-section maps $\phi: M \to E$. Use the group structure on each fiber to define the product of two elements of Φ. Say that this structure defines a bundle of Lie groups if the product $\phi_1\phi_2$ is also C^∞, for $\phi_1, \phi_2 \in \Phi$. Define the jet spaces $\theta^r(E)$ as usual. (See VB and Vol. I of this treatie). Let $\pi: \theta^r(E) \to M$, also denote the jet space projection map. Show that π defines $\theta^r(E)$ as a bundle of Lie groups, in such a way as to reduce to the set up described in detail whose for the case where E is the product $M \times E$.

Problem. Suppose that $\pi: E \to M$ is a bundle of groups, as defined in the previous section. Let $\pi': E' \to M$ denote another fiber space whose hose space is M. Let us say that the bundle of group E acts on the fiber space E' if, for each $p \in M$, the group $E(p)$ is given as a transformation group on the fiber $E'(p)$. Formulate this precisely, paying particular attention to the proper "smoothness" notions as p varies over M. Show that the jet bundle of groups $\theta^r(E)$ defined in the previous problem acts in a natural way on the jet bundles $\theta^r(E')$. Develop as many of the formal properties of this action as you can, or have patience for.

Chapter V

TOPICS IN LINEAR QUANTUM FIELD THEORY

1. INTRODUCTION

Quantum field theory is a difficult, complicated and frustrating subject. In my previous books I have attempted to clarify various mathematical topics that play a role in it and in related areas of elementary particle physics. In this chapter I will attempt a more systematic exposition of the material to be found in the physicist's treatises concerning linear quantum fields. (For example, those by Bjorken and Drell, Bogoluibov and Shirkov, Gasiorowicz, Lurie, Schweber, and Speer).

Now, most of the material in these treatises is concerned with the theory of linear quantum fields, and their associated symmetries. The theory of non-linear, i.e. "interacting", fields is then treated by perturbation theory.
In this Part III I will only deal with the linear case. Non-linear situations will be treated in a later volume.

It is useful to keep in mind the general geometric framework for understanding "quantization" which is described in my previous books, particularly LAQM, VB, LMP Vol. II, and PALG. The most general viewpoint emphasizes two spaces, the "states" and "observables", a duality between them, an algebraic structure on the observables, and representations of the observables by means of operators in a Hilbert space. Quantum field theory is probably characterized, in this general setting, by a certain "local" (relative to space-time) structure for the states and observables. However, we shall not attempt to keep at this level of abstraction.

Instead, consider a "classical" field. It is convenient to use the fiber space notions developed in GPS and in VB, Vol. I to describe it. Suppose then that M is a manifold, and that $\pi:E \to M$ is a fiber space, with M as base space. (In the typical physical applications, M will be the "flat" space-time R^4, although in such applications as cosmology more general possibilities are important.) Let Γ denote the space of cross-section maps $\gamma:M \to E$, and let $J^r(E)$, r an integer, denote the space of 1-jets of cross-sections. Recall that an <u>r-th order system of differential equations for cross-sections</u> is a subset D of $J^r(E)$ defined by setting a finite number of functions equal to zero. Let Γ_D denote the space of cross-section maps $\gamma:M \to E$ which are solutions of D, i.e. such that:

$$J^r(\gamma)(M) \subset D \qquad 1.1$$

(Recall that $J^r(\gamma)$, the <u>r-jet of γ</u>, is a cross-section map: $M \to J^r(E)$).

The element of Γ_D are the <u>states</u> of the classical fields. The <u>observables</u> are defined as certain types of real-valued functions on Γ_D. A general method for defining observables and a "Poisson bracket" algebraic structure on them, in terms of differential forms, has been described in LAQM and VB, Vol. I. In fact, for the systems considered most often by physicists, which are typically linear or "weakly" non-linear, such a general framework is awkward, since it does not directly take into account the simplifying feature of linearity. Accordingly, we shall sketch how to replace this framework by a more satisfactory one in the case E is a vector bundle with the real numbers as scalars, and D is a system of linear differential equations, i.e. Γ_D is an R-linear subspace of Γ.

Thus, suppose that each fiber $E(x)=\pi^{-1}(x)$, for $x \in M$, is a real vector space. Let us suppose that it is a finite dimensional vector space, for simplicity, although there would be no particular conceptual difficulty in handling the infinite dimensional case. (There would be technical complications, of course). Let E^d be the dual vector bundle over M, i.e. for $x \in M$, the fiber $E^d(x)$ is the dual vector space of the vector space $E(x)$. Let Γ^d denote the cross-sections of E^d which have compact support. For each $\gamma^d \in \Gamma^d$, define a real-valued function: $\Gamma_D \to R$ as follows:

$$\gamma \to \int_M \gamma^d(x)(\gamma(x))dx \ . \qquad 1.2$$

(Assume that "dx" is a fixed volume element differential form on M.)
Let Γ_D^d denote the space of all $\gamma^d \in \Gamma^d$ such that the function on Γ_D defined by 1.2 is identically zero.

Thus, an "observable" can be identified with an element of the quotient vector space Γ^d/Γ_D^d. In fact, what the physicists usually do is to deal directly with the "observables" and forget the "classical" states. Algebraic structures are imposed directly on the observables (via "commutation relations"), and the "quantum" states are defined as elements of the Hilbert spaces on which one represents the algebraic structure defined by the observables.

This is rather abstract. The physicists usually think of a quantum field more concretely as a collection of operator-valued "functions" of space time points. It is easy, however, to pass back and forth between the two ways of looking at quantum fields. Suppose, for example, that E is the product $R^4 \times R^n$. An element of Γ is a C^∞ mapping $x \to (\gamma_a(x))$, $1 \le a \le n$ of $R^4 \to R^n$. Γ_D denotes the subset of those satisfying a given system of linear partial differential equations.

Γ^d can be identified with the space of compact support, C^∞ mappings $x \to (f_a(x))$ of $R^4 \to R^n$. The function $f : \Gamma_D \to R$ defined by 1.2 then takes the following explicit form:

$$f(\gamma) = \int_{R^4} (f_1(x)\gamma_1(x) + \cdots + f_n(x)\gamma_n(x))dx \; . \qquad 1.3$$

Suppose now that the f's are provided with a suitable algebraic structure, and then represented by operators A_f in a Hilbert space H. One can define, formally, operator-valued "functions" $x \to \phi_a(x)$ of the point x by the rule:

$$A_f = \int_{R^4} (f_1(x)\phi_1(x) + \cdots + f_n(x)\phi_n(x))dx \; . \qquad 1.4$$

The operators $\phi_1(x), \ldots, \phi_n(x)$ will then satisfy, as functions of x, the same system of linear partial differential equations as do the "classical" fields $(\gamma_a(x))$. (This is not quite obvious; for general systems it is not even clear that there is a proof available, in terms of the available mathematical tools. However, it will be proved for the Klein-Gordon field in the next section. No doubt the proof for the other equations of importance in physics would follow similar lines).

2. THE NEUTRAL KLEIN-GORDON FIELD

This is the simplest example of a free relativistic quantum field. Let x denote a point of R^4, the real space-time manifold. Choose μ,ν as indices running from 0 to 3. Choose i,j as "space" indices, running from 1 to 3. Let x_μ denote the components of x; x_0 is the "time" component, x_i are the "space" components. Set $\partial_\mu = \frac{\partial}{\partial x_\mu}$;

$$\Box = \partial_0^2 - \sum_{i=1}^{3} \partial_i^2 \, . \qquad 2.1$$

Let Γ denote the space of real valued, C^∞ functions $\gamma: R^4 \to R$. Let Γ_D denote those elements of Γ which satisfy the following equation (called the Klein-Gordon equation):

$$\Box\, \gamma + m^2 \gamma = 0 \qquad 2.2$$

(m is a real positive number; physically, it is the mass of the field).

Let Γ^d denote the space of real-valued, compactly supported, C^∞ functions $f: R^4 \to R$. Let f define a real-valued function on Γ_D as follows:

$$f(\gamma) = \int_{R^4} f(x)\gamma(x)\,dx \, . \qquad 2.3$$

Theorem 2.1. The linear function f defines on Γ_D, via formula 2.3, is identically zero if and only if

$$f \in (\Box + m^2)(\Gamma^d). \qquad 2.4$$

Proof. Let p denote another real four-vector, with components (p_μ). (Physically, p denotes a relativistic "energy-momentum" vector. Define $x \cdot p$ as the Lorentz invariant inner product:

$$x \cdot p = x_0 p_0 - x_1 p_1 - x_2 p_2 - x_3 p_3 \, . \tag{2.5}$$

Let $p \to \hat{f}(p)$ denote the Fourier transform if $x \to f(x)$, i.e.

$$\hat{f}(p) = \frac{1}{(2\pi)^2} \int_{R^4} e^{-x \cdot p} f(x)\, dx \tag{2.6}$$

$f(p)$ exists because $x \to f(x)$ is rapidly decreasing.

Now, if 2.4 holds, it is obvious that the right hand side of 2.3 is zero for every function γ which is a solution of $(\Box + m^2)(\gamma)=0$. Conversely, let us suppose that 2.3 is zero for every such solution. Let us construct a family of such solutions.

Let N denote the submanifold of R^4 consisting of the points $p \in R^4$ such that:

$$p^2 = m^2. \tag{2.7}$$

Physically, N is the <u>mass-shell hyperboloid</u>. Let $d_N p$ denote the Lorentz-invariant volume element of N given by the following formula:

$$d_N p = \frac{dp_1 \wedge dp_2 \wedge dp_3}{|p_0|} \, . \tag{2.8}$$

For each C^∞, compact-support function $p \quad \psi(p)$ on N, set:

$$\gamma(x) = \frac{1}{(2\pi)^2} \int_N e^{ip \cdot x} \psi(p)\, d_N p \, . \tag{2.9}$$

<u>Exercise</u>. Show that γ is a real C^∞ solution of the Klein-Gordon equation if:

$$\psi(p)^* = \psi(-p). \tag{2.10}$$

Show that:

$$\int f(x)\gamma(x)dx = \int_N \hat{f}(p)^*\psi(p)d_Np \ . \qquad 2.11$$

Thus, if $f(\Gamma_D)=0$, then 2.10 implies that:

$$\hat{f}(p) = 0 \text{ for all } p \in N. \qquad 2.12$$

Problem. If 2.11 holds, show that there is a C^∞ function $\hat{f}_1(p)$ on R^4 whose Fourier transform is of compact support such that:

$$\hat{f}(p) = (-p^2+m^2)\hat{f}_1(p). \qquad 2.13$$

(Hint: For this one has to know the "Paley-Wiener theorem" which characterizes the class of functions of p which are Fourier transforms of functions of compact support. See Yosida [1]).

Now, applying inverse Fourier transforms to 2.12 shows that f satisfies 2.4.

Thus, we may say that a suitable space of observables for the free Klein-Gordon particle of mass m is the following vector space

$$F' = F/(\Box + m^2)(F) \ , \qquad 2.14$$

where F denotes the compact support, C^∞ functions on R^4. (The quotient vector space 2.14 is called the cokernel of the linear transformation $(\Box + m^2)$.)

Now, we want to define a "Poisson bracket" structure for F'. Since F' consists of the linear observables this will be defined by a skew-symmetric, R-bilinear map $\omega: F'\times F' \to R$, or, equivalently, by a skew-symmetric, bilinear map $\omega: F\times F \to R$ such that:

$$\omega((\Box + m^2)F,F) = 0. \qquad 2.15$$

Such a map could be defined by means of the symplectic structure on the state space of the classical problem. This approach has been already described in PALG.

However, for purposes of generalization, it is more interesting to forget about the states, and to try to define the form ω directly on the observables. We shall discuss this, beginning with the physicist's point of view.

The physicists think of a free Klein-Gordon field of mass m as a Hermitian-operator valued "function" $x \to \phi(x)$ such that:

$$(\Box + m^2)(\phi(x)) = 0 \qquad 2.16$$

$$[\phi(x),\phi(y)] = -i\Delta(x-y), \qquad 2.17$$

$$\Delta(x) = \frac{1}{2i(2\pi)^2} \int_N e^{ip\cdot x} \epsilon_0(p) d_N p, \qquad 2.18$$

where ϵ_0 is the function on N such that:

$$\epsilon_0(p) = \begin{cases} 1 \text{ if } p_0>0 \\ -1 \text{ if } p_0<0 \end{cases} .$$

Using 2.7, and writing:

$$x = (t,\vec{x}) \ , \qquad 2.19$$

$$p = \left(\pm\sqrt{\vec{p}^2+m^2}, \vec{p}\right) \text{ for } p \in N, \qquad 2.20$$

we have, using 2.7,

$$\Delta(t,\vec{x}) = \frac{1}{(2\pi)^2} \int_{R^3} \frac{\sin\sqrt{p^2+m^2}t}{\sqrt{\vec{p}^2+m^2}} \ e^{-i\vec{p}\cdot x} \ d^3\vec{p} \qquad 2.21$$

Δ is one of the family of "invariant functions" or "propagators" which play a key role in the physicists approach to quantum field theory. It is really a "distribution" or "generalized function", of course.

Exeridse. Show that Δ can be defined as a distribution (in the Schwartz sense) on rapidly decreasing functions on R^4.

Now, define the map $\omega: F\times F \to R$

$$\omega(f_1,f_2) = \int_{R^4\times R^4} \Delta(x-y) f_1(x) f_2(y) dx\, dy \qquad 2.22$$

Exercise. Show that ω can be defined also by the following momentun space integral:

$$\omega(f_1,f_2) = \frac{1}{2i}\int_N e_0(p)\hat{f}_1(-p)\hat{f}_2(p) d_N p \ , \qquad 2.23$$

where $\hat{f}_1, \hat{f}_2$ denote the Fourier transform of $f_1, f_2 \in F$.

Problem. Discuss how formula 2.20 can be used to extend ω to larger classes of functions. For example, the extension to the rapidly decreasing functions is immediate.

Remark: In the physics literature, 2.15 and 2.17 are most frequently written in the following "covariant" form:

$$\Delta(x) = \frac{1}{2i(2\pi)^2}\int_{R^4} e^{ip\cdot x} e_0(p)\delta(m^2-p^2) d^4p \qquad 2.24$$

$$\omega(f_1,f_2) = \frac{1}{2i(2\pi)^2}\int_{R^4} e_0(p)\delta(m^2-p^2)\hat{f}_1(-p)\hat{f}_2(p) d^4p \qquad 2.25$$

Now, we can start the main result concerning this type of free quantum field. Its proof is discussed in detail in PALG.

Theorem 2.1. There is a Hilbert space H, and a linear mapping $f \to \phi(f)$ of F into skew-Hermitian operators on H, such that:

$$\phi((\Box + m^2)F) = 0 \qquad 2.26$$

$$[\phi(f_1),\phi(f_2)] = -i\omega(f_1,f_2) \qquad 2.27$$

for f_1, $f_2 \in F$.

Further, this object fulfills all the conditions needed to define a "quantum field" in the precise sense of Wightman. (See Streater-Wightman [1] and Jost [1]). If one defines the objects $\phi(x)$ "symbolically" as operators on H as follows:

$$\phi(f) = \int \phi(x)f(x)dx \ , \qquad 2.28$$

then they have all the properties taken for granted in the physicists approach.

In PALG, this result was proved by Fourier transforming f to a function $p \to \hat{f}(p)$ on R^4, then restricting f to be a function on the upper sheet N_+ of the hyperboloid $p^2=m^2$. H' is chosen to be the rapidly decreasing, C^∞, complex-valued functions on N_+, with the Hilbert space structure that defined by the volume element $d_N p$, H is defined to be the Boson Fock space associated with H', i.e. the Hilbert space of symmetric tensors on H'. Finally, $\phi(f)$ is defined as the difference of the "annihilation" and "creation" vectors defined by f. As explained in PALG, this amounts more abstractly to constructing the Heisenberg Lie algebra $\underset{\sim}{G}=F \oplus R$, with the Lie bracket defined by the form ω; constructing its complexification $\underset{\sim}{G}_c=\underset{\sim}{G} + i\underset{\sim}{G}$; splitting up $\underset{\sim}{G}_c$ as a direct sum $\underset{\sim}{G}^+ \oplus \underset{\sim}{G}^-$, and then requiring that the representation ϕ be such that G^- annihilate a vector of H, which is called the <u>vacuum state</u>. This algebraic structure for $\underset{\sim}{G}_c$ can be exhibited more directly in terms of x-space, without going to momentum space. This point of view will be useful for other quantization problems as well as for studying "current algebras" and "Feynman rules".

From the physicist's point of view, it can be described most readily as follows: Formally write:

$$\phi(x) = \phi^+(x) - \phi^-(x), \tag{2.29}$$

with:

$$\phi^-(x)^* = \phi^+(x) \tag{2.30}$$

$$[\phi^+(x),\phi^+(y)] = 0 = [\phi^-(x),\phi^-(y)]$$

$$[\phi^+(x),\phi^-(y)] = \Delta_+(x-y), \tag{2.31}$$

where $\Delta_+(x)$ is defined as follows:

$$\Delta_+(x) = \frac{1}{2(2\pi)^2} \int_{R^4} e^{ip\cdot x}\theta_0(p)\delta(p^2-m^2)dp, \tag{2.32}$$

with:

$$\theta_0(p) = \begin{cases} 1 \text{ if } p_0>0 \\ 0 \text{ otherwise} . \end{cases} \tag{2.33}$$

Using 2.21, and the relation

$$e_0(p) = \theta_0(p)-\theta_0(-p) \tag{2.34}$$

note that:

$$i\Delta(x) = \Delta_+(x) - \Delta_+(-x) \tag{2.35}$$

$$\Delta_+(x)^* = \Delta(-x) \ . \tag{2.36}$$

Let us now compute; using 2.26 as the definition of $\phi(x)$:

$$[\phi(x),\phi(g)] = [\phi^+(x)-\phi^-(x),\phi^+(y)-\phi^-(y)]$$

$$= -\Delta_+(x-y) + \Delta_+(y-x)$$

$$= \text{, using 2.32, } -i\Delta(x-y)\text{, which is}$$

relation 2.14. Thus, we have <u>derived</u> the commutation relations 2.14 assuming the commutation relations 2.28.

We can convert the commutation relations 2.28 into a more legitimate form by using F again. For $f \in F$, set:

$$\phi_+(f) = \int \phi_+(x)f(x)dx$$

$$\phi_-(f) = \int \phi_-(x)f(x)dx \ .$$

Then, we have:

$$[\phi_+(f_1),\phi_+(f_2)] = 0 = [\phi_-(f_1),\phi_-(f_2)] \tag{2.37}$$

$$[\phi_+(f_1),\phi_-(f_2)] = \omega_+(f_1,f_2),$$

where:

$$\omega_+(f_1',f_2) = \int \Delta_+(x-y)f_1(x)f_2(y)dx\ dy \ . \tag{2.38}$$

Note that:

$$\omega_+(f_1,f_2)^* = \omega_+(f_2,f_1) \ . \tag{2.39}$$

Thus, ω_+ is an R-bilinear map: $F\times F \to C$ which is Hermitian symmetric, i.e. satisfies 2.36. Further, if $\hat{f}_1$, $\hat{f}_2$ denote the

Fourier transforms,

$$\omega_+(f_1,f_2) = \tfrac{1}{2}\int\theta_0(p)\delta(p^2-m^2)\hat{f}_1(p)\hat{f}_2(p)^*dp. \qquad 2.40$$

<u>Exercise</u>: Prove 2.37.

In particular, note from 2.37 that:

$$\omega_+(f,f) > 0 \text{ if } f\neq 0 . \qquad 2.41$$

$$\omega(f_1,f_2) = \frac{1}{i}(\omega_+(f_1,f_2)-\omega_+(f_1,f_2)^*). \qquad 2.42$$

Putting the commutation relations for the Klein-Gordon field in this algebraic form enables us to abstract a general algebraic framework, within which one can probably encompass the most general linear quantum fields. (Of course, the formalism must also be modified to accomodate Fermions, as well as the Boson case treated above. We shall tend to this later on).

3. A GENERAL ALGEBRAIC FRAMEWORK FOR BOSON LINEAR QUANTUM FIELDS

Let Γ be a vector space over the real numbers as a field of scalars. Let ω_+ be an R-bilinear, complex-valued, map: $\Gamma \times \Gamma \to C$, which is Hermitian symmetric, i.e. which satisfies the following relation:

$$\omega_+(\gamma_1,\gamma_2)^* = \omega_+(\gamma_2,\gamma_1) \quad 3.1$$

for $\gamma_1,\gamma_2 \in \Gamma$.

Construct a real Lie algebra $\underset{\sim}{G}$ as follows:

As a vector space $\underset{\sim}{G}$ is a direct sum

$$\Gamma \oplus \Gamma \oplus C \ . \quad 3.2$$

For notational convenience, $\gamma \in \Gamma$, we denote by γ^+, γ^- the elements of G_Γ: defined in the following way:

$$\gamma^+ = \gamma \oplus 0 \oplus 0 \quad 3.3$$

$$\gamma^- = 0 \oplus \gamma \oplus 0 \ . \quad 3.4$$

Denote by Γ^+, Γ^- the linear subspace of $\underset{\sim}{G}$ as follows:

$$\Gamma^+ = \{\gamma^+ : \gamma \in \Gamma\}$$
$$\Gamma^- = \{\gamma^- : \gamma \in \Gamma\}. \quad 3.5$$

Thus, $\underset{\sim}{G}$ is, as a vector space, the direct sum of Γ^+, Γ^- and C.

Define the Lie bracket structure on $\underset{\sim}{G}$ as follows:

$$[\Gamma^+,\Gamma^+] = 0 = [\Gamma^-,\Gamma^-] \quad 3.6$$

$$[\gamma_1^+,\gamma_2^-] = i\omega_+(\gamma_1,\gamma_2) \quad 3.7$$

for γ_1, $\gamma_2 \in \Gamma$.

$$[\underset{\sim}{G}, C] = 0. \quad 3.8$$

Then, $\underset{\sim}{G}$ is a Heisenberg Lie algebra, i.e., the derived Lie algebra $[\underset{\sim}{G},\underset{\sim}{G}]$ is contained in the center of $\underset{\sim}{G}$. (See PALG for a discussion of the elementary properties of Heisenberg Lie algebras, and the associated groups).

Now, let Γ^0 be the linear subspace of $\underset{\sim}{G}$ defined as follows:

$$\Gamma^0 = \{\gamma^+ - \gamma^- : \gamma \in \Gamma\}. \qquad 3.9$$

Let us calculate the Lie bracket for the subspace Γ^0; using 3.6-3.7:

$$\begin{aligned} &[\gamma_1^+ - \gamma_1^-,\ \gamma_2^+ - \gamma_2^-] \\ &= i\omega_+(\gamma_1,\gamma_2) + i\omega_+(\gamma_2,\gamma_1) \\ &= -i(\omega_+(\gamma_1,\gamma_2) - \omega_+(\gamma_1,\gamma_2)^*) \qquad 3.10 \end{aligned}$$

$$\text{for } \gamma_1, \gamma_2 \in \Gamma.$$

Notice that the right hand side of 3.10 is a real number. To make this relation more explicit, define $\omega:\Gamma\times\Gamma \to R$ as follows:

$$\omega(\gamma_1,\gamma_2) = -i(\omega_+(\gamma_1,\gamma_2) - \omega_+(\gamma_1,\gamma_2)^*) \qquad 3.11$$

$$\text{for } \gamma_1, \gamma_2.$$

Then, ω is a skew-symmetric, bilinear form. For $\gamma \in \Gamma$, set:

$$\gamma^0 = \gamma^+ - \gamma^- \qquad 3.12$$

$$\underset{\sim}{G}^0 = \Gamma^0 + R. \qquad 3.13$$

Then, 3.10 translates into the following relation:

$$[\gamma_1^0,\gamma_2^0] = \omega(\gamma_1,\gamma_2) \qquad 3.14$$

$$\text{for } \gamma_1, \gamma_2 \in \Gamma^0.$$

In particular, notice that $\underset{\sim}{G}^0$ is a Lie subalgebra of $\underset{\sim}{G}$. The pair $(\underset{\sim}{G}^0,\underset{\sim}{G})$ of Heisenberg Lie algebras is said to be a (real) Heisenberg Lie algebra structure equipped with an annihilation-creation operator structure. As an abbreviation, we will call this structure a Heisenberg-Fock Lie algebra pair.

Definition. A vacuum state representation of such a Heisenberg-Fock pair $(\underset{\sim}{G}^0,\underset{\sim}{G})$ is a representation ϕ of $\underset{\sim}{G}$ by complex-linear operators on a complex vector space H such that:

$$\phi(c) = -icx(\text{identity operator}) \qquad 3.15$$

for $c \in C$.

H has a vector ψ_0 such that:

$$\phi(\Gamma^-)(\psi_0) = 0. \qquad 3.16$$

The vector ψ_0 is called a vacuum state of the representation. (We do not necessarily require that it be unique).

Exercise. Show that such vacuum state representations may be constructed purely algebraically by considering the universal enveloping algebra $U(\underset{\sim}{G})$ of $\underset{\sim}{G}$ (with complex numbers as scalars), and dividing out by left ideals which contain the abelian subalgebra Γ^- of $\underset{\sim}{G}$.

Now, the representations considered by physicists - at least for free fields - have a stronger property, which we define as follows.

Definition. Let $(\underset{\sim}{G},\phi,H,\psi_0)$ be a vacuum state representation of the Heisenberg-Fock pair $(\underset{\sim}{G},\underset{\sim}{G}^0)$. We will call this representation a Fock representation if the following conditions are satisfied:

H has a real-bilinear form: $H \times H \to C$

$$(\psi_1, \psi_2) \to \langle\psi_1|\psi_2\rangle \qquad 3.17$$

such that:

$$\langle\psi_1|c\psi_2\rangle = c\langle\psi_1|\psi_2\rangle = \langle c^*\psi_1|\psi_2\rangle \qquad 3.18$$

for $c \in C; \psi_1, \psi_2 \in H$

$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_1\rangle^*. \qquad 3.19$$

The operator $\phi(\underset{\sim}{G}^0)$ are skew-Hermitian with respect to the form $\langle\,|\,\rangle$, i.e.

$$\langle\psi_1|\phi(X)\psi_2\rangle = -\langle\phi(X)\psi_1|\psi_2\rangle \qquad 3.20$$

for $\psi_1, \psi_2 \in H$, $X \in \underset{\sim}{G}^0$.

The vacuum state ψ_0 is unique (up to a scalar multiple,) and satisfies:

$$\langle\psi_0|\psi_0\rangle \neq 0 \;. \qquad 3.21$$

For $\gamma \in \Gamma$,

$$\phi(\gamma^+)^* = \phi(\gamma^-), \qquad 3.22$$

where * denotes the Hermitian adjoint for operators on H with respect to the Hermitian form $\langle\,|\,\rangle$ on H.

Remark: Notice that we do not require that the inner product $\langle\,|\,\rangle$ be positive, i.e. that it define a "Hilbert space". Without this condition, it may be called a Hilbert space of indefinite sign.

Research Problem. Let G be the Heisenberg group whose Lie algebra is $\underset{\sim}{G}$. (See PALG for the definition of the group-law on G. As a set, $\underset{\sim}{G}$ is equal to G.) Investigate using "analytic vector" techniques, the integrability of the representation of G to give a "global" representation of G.

There is a relation between the positivity of the form $\gamma \to \omega_+(\gamma,\gamma)$ and the positivity of H that is very important for the physics. Let us suppose that the form $<|>$ on H is at least positive on ψ_0. We can then normalize ψ_0 so that:

$$<\psi_0|\psi_0> = 1.$$

Thus, for $\gamma \in \Gamma$,

$$<\phi(\gamma^+)\psi_0|\phi(\gamma^+)\psi_0>$$

$$= <\phi(\gamma^0)\psi_0|\phi(\gamma^+)\psi_0>$$

$$= <\psi_0|\phi(\gamma^-)\phi(\gamma^+)\psi_0>$$

(using 3.22)

$$= <\psi_0|[\phi(\gamma^-),\phi(\gamma^+)]\psi_0>$$

(using 3.16)

$$= <\psi_0|\omega_+(\gamma,\gamma)\psi_0> \text{ (using 3.7 and 3.15)}$$

$$= \omega_+(\gamma,\gamma). \tag{3.23}$$

The subspace $\phi(\Gamma^+)(\psi_0)$ H is called the single particle subspace of H. Thus, we see that the form $<|>$ is positive on the single particle subspace of H if and only if the form $\gamma \to \omega_+(\gamma,\gamma)$ is positive.

<u>Exercise</u>. Define the n-particle subspaces of H as the subspaces obtained by applying polynomials in n creation operators. (The operators $\phi(\gamma^+)$ are called <u>creation operators</u>, for obvious physical reasons; they are interpreted physically as the operators that generate "particles" out of the "vacuum" state ψ_0. Show that $<|>$ is positive on the n-particle subspaces, $n=2,3\ldots$, if the form $\gamma \to \omega_+(\gamma,\gamma)$ is positive.

<u>Problem</u>. Suppose that Γ splits up into a direct sum of subspaces $\Gamma_1 \oplus \Gamma_2$, such that:

$$\begin{aligned} &\omega_+(\Gamma_1,\Gamma_2) = 0 \\ &\omega_+(\gamma_1,\gamma_1) > 0 \\ &\omega_+(\gamma_2,\gamma_2) < 0 \end{aligned} \qquad 3.23$$

for $\gamma_1 \in \Gamma_1, \gamma_2 \in \Gamma_2$.

Analyze how this impasses a split up of H into subspaces on which the form $<|>$ is positive and negative.

<u>Research problem</u>. In the interesting infinite dimensional cases, the form ω_+ may not split up into the "direct sum" form 3.23, at least in a natural way. Investigate - possibly through examples - how such more subtle phenomena are reflected in the sign properties of $<|>$ on H.

The question now arises of actually constructing Fock representations of a Heisenberg-Fock pair $(\underset{\sim}{G},\underset{\sim}{G}^0)$ of Lie algebras. In practice, i.e. in the simple free-field and finite dimensional problems, this may be done in the following way: Exhibit a complex Hilbert space H' - possibly of indefinite sign - and a real linear map:

$$\alpha:\Gamma \to H'$$

such that:

$$\langle\alpha(\gamma_1)|\alpha(\gamma_2)\rangle' = \omega_+(\gamma_1,\gamma_2) \qquad 3.24$$

$$\text{for } \gamma_1,\ \gamma_2 \in \Gamma,$$

where $\langle\ |\ \rangle'$ is the Hilbert form on H'. Thus, one can construct H as the Boson Fock space associated with H', i.e. as the space of symmetric tensors $S(H')$ on H'. (See VB, Vol. I and Guichardet [1]).

Exercise. Show explicitly how $\underset{\sim}{G}$ is represented as operators in $H=S(H')$, in terms of the standard annihilation creation operators on $S(H')$.

Exercise. In the case where $(\underset{\sim}{G},\underset{\sim}{G}^0)$ is the Heisenberg-Fock pair constructed in Section 2 for the linear observables of the free (neutral) Klein-Gordon field, construct α explicitly, defining H' as the space of complex-valued functions on the upper part of the mass-shell hyperboloid in momentum space.

So far in this chapter we have been considering Γ to be an abstract vector space. To specialize to linear quantum fields, we may take Γ and Γ' to be the space of compact-support C^∞ cross sections of vector bundles E, E' on a manifold M. Let $D:\Gamma' \to \Gamma$ be a linear differential operator. We would then require that the form $\omega_+:\Gamma\times\Gamma \to C$ needed to make the definitions also satisfy the following conditions:

$$\omega_+(D(\Gamma'),\Gamma) = 0. \qquad 3.25$$

Further, if G is a group of vector bundle automorphisms which leaves D invariant, and if we wanted the corresponding "quantum field" to be invariant under G, we would require that the form ω_+ be invariant under G. In sufficiently simple situations, this completely determines ω_+, or at least strongly restricts its possibilities.

Problem. Let G be the connected Poincaré group, $E=R^4\times R$, D the Klein-Gordon differential operator. Investigate how G-invariance determines ω_+. Also, investigate possible invariance under the full Poincaré group, i.e. the group including parity and time-reversal.

Problem. Investigate "causality" properties of the free quantum field associated with the neutral Klein-Gordon equation, i.e. investigate the regions x and y vary over such that:

$$[\phi(x),\phi(y)] = 0.$$

Problem. Carry out a parallel development to that in Section 2, where one assumes anti-commutation relations:

$$[\phi(x),\phi(y)]_+ = \Delta'(x-y)$$

for the Klein-Gordon equation. Show that "causality" would be violated if such commutation relations were adopted.

4. THE CHARGE OPERATOR AND CHARGE CONJUGATION FOR THE KLEIN-GORDON FIELD

The algebraic formalism developed in Section 3 was only designed to cover the case of "neutral" quantum fields. I have found it rather confusing to understand in the physics literature precisely what is meant by a "charged" particle. Accordingly, we shall begin with the simplest example, the so-called "charged" or "complex" Klein-Gordon field. (Notice that one of the confusing features of the physics literature is that physicists are so casual in distinguishing between "real" and "complex" vector spaces, despite the fact that the most interesting experimental effects, such as "time invariance", "charge conjugation" and "antiparticles" involve relatively subtle algebraic distinctions between real and complex numbers).

Let us begin with two neutral non-interacting Klein-Gordon fields $x \to \phi_1(x)$, $\phi_2(x)$ with the same mass m, as explained in Section 2. We shall treat them first from the physicist's point of view. They can essentially be characterized by their decomposition into creation and annihilation operators, and the commutation relations:

$$\phi_1(x) = \phi_1^+(x) - \phi_1^-(x)$$

$$\phi_2(x) = \phi_2^+(x) - \phi_2^-(x)$$

$$[\phi_a^+(x), \phi_a^+(y)] = 0 = [\phi_a^-(x), \phi_a^-(y)] \qquad 4.1$$

$$[\phi_a^+(x), \phi_a^-(y)] = \Delta^+(x-y)$$

$$[\phi_a(x), \phi_a(y)] = i\Delta(x-y)$$

$$\text{for } a = 1,2.$$

$$[\phi_1(x), \phi_2(y)] = 0.$$

Suppose that the $\phi_a^+(x), \phi_a^-(x), \phi_a(x)$ are operators in a Hilbert space H, satisfying these commutation relations, with $\phi_a(x)$ Hermitian.

<u>Definition</u>. A skew-Hermitian operator $Q: H \to H$ is a <u>charge operator</u> for the field (ϕ_1, ϕ_2) if the following conditions are satisfied:

$$[Q, \phi_1(x)] = \phi_2(x)$$
$$[Q, \phi_2(x)] = -\phi_1(x). \tag{4.2}$$

An element $\psi \in H$ has <u>charge q</u> if:

$$Q(\psi) = -i\, q\, \psi. \tag{4.3}$$

Now, let:

$$\phi(x) = \phi_1(x) + i\, \phi_2(x).$$

$x \to \phi(x)$ is no longer skew-Hermitian, of course. It satisfies the following sort of commutation relations:

$$[\phi(x), \phi(y)] = [\phi_1(x) + i\, \phi_2(x), \phi_1(y) + i\, \phi_2(y)]$$
$$= i\, \Delta(x-y) - i\, \Delta(x-y) = 0 \tag{4.4}$$

$$[\phi(x), \phi(y)^*] = 2i\, \Delta(x-y). \tag{4.5}$$

Now, consider the decomposition of $\phi(x)$ into creation and annihilation operators: Set:

$$\phi^+(x) = \phi_1^+(x) + i\, \phi_2^+(x)$$
$$\phi^-(x) = \phi_1^-(x) + i\, \phi_2^-(x). \tag{4.6}$$

4.2 implies the following commutation relations:

$$[Q, \phi(x)] = \phi_2(x) - i\, \phi_1(x)$$
$$[Q, \phi(x)^*] = i\, \phi(x)^*. \tag{4.7}$$

We shall now show that $x \to \phi(x)$ may be regarded physically as a "field" of "charge" +1.

Let us suppose that these operators act on a Hilbert H, with a "vacuum" state ψ_0 such that:

$$\phi_1^-(x)(\psi_0) = 0 = \phi_2^-(x)(\psi_0) \qquad 4.8$$

$$Q(\psi_0) = 0.$$

Then,

$$\phi(x)(\psi_0) = \phi^+(x)(\psi_0)$$

$$Q\ \phi(x)(\psi_0) = -i\ \phi(x)(\psi_0). \qquad 4.9$$

According to 4.3, this means that the state $\phi(x)(\psi_0)$ has charge +1. Similarly, $\phi(x)^*(\psi_0)$ has charge -1. In other words, $\phi(x)$ <u>creates out of the vacuum a particle of charge +1</u>, while $\phi(x)^*$ creates one of charge -1.

<u>Definition</u>. A unitary operator $C:H \to H$ is a <u>charge conjugation operator</u> if:

$$C(\psi_0) = \psi_0 \qquad 4.10$$

$$C\ \phi(x)C^{-1} = \phi(x)^*. \qquad 4.11$$

Suppose such a charge conjugation operator exists. Then,

$$C(\phi(x)\psi_0) = C\ \phi(x)C^{-1}\psi_0$$
$$= \phi(x)^*\psi_0.$$

C thus converts particles of charge q into particles of charge $-q$. If ψ is a state of H representing a "particle" of charge q, then $C(\psi)$ is called the <u>anti-particle</u>.

<u>Exercise</u>. If the fields act in the standard Boson Fock space manner, show that such a charge conjugation operator may be con-

structed.

The next step in the program might be to abstract from this elementary material the notions of "charge operator" and "charge-conjugation operator" to more general quantum fields. Before this is done, however, it would be better to understand the Dirac equation, where the theory of "charge" is more subtle. We turn to this task in the next section.

Problem. Translate these ideas into the language of "Heisenberg-Fock Lie algebras" described in Section 3.

5. JORDAN ALGEBRAS AND ANTI-COMMUTATION RELATIONS

As we have seen, to describe certain types of quantum fields it suffices to construct certain types of Lie algebras, and then to construct certain types of representations of these Lie algebras. Now, certain types of fields - Dirac field is the simplest - satisfy anticommutation relations. The corresponding abstract structure is a "Jordan algebra". We shall now present a brief treatment of elementary algebraic facts about Jordan algebras that we shall need.

Let H be a vector space over a field of scalars. If X_1,X_2 are linear operators on H, define the anticommutator of X_1,X_2 as follows:

$$\{X_1,X_2\} = \tfrac{1}{2}(X_1,X_2+X_2X_1). \qquad 5.1$$

(Note that in using $\{\ ,\ \}$ to denote this anticommutator we are running the risk of confusion with the "Poisson bracket" of two classical-mechanics observables. However, since the notation $\{\ ,\ \}$ for the anticommutator is standard in the physical literature, it is hoped that the reader can sort out the possible ambiguity for himself).

Definition. A space $\underset{\sim}{J}$ of operators on H is a linear Jordan algebra if, for $X_1,X_2 \in \underset{\sim}{J}$

$$X_1+X_2 \text{ and } \{X_1,X_2\} \in \underset{\sim}{J}.$$

(Of course, we also require that a scalar times an $X \in \underset{\sim}{J}$ belong to $\underset{\sim}{J}$. We shall compress notation by not specifying the field of scalars chosen at each stage).

We will attempt to abstract out of this definition the concept of "Jordan algebra", independently of its realization as an algebra of operators, just as the "Lie algebra" concept was abstracted out of Lie algebras of operators (with the commutator playing the basic role, of course). To do this, we will look for a replacement for the "Jacobi identity" for Lie algebras.

Suppose then that J is a linear Jordan algebra. Let $X,Y,Z \in J$. Since they are linear operators on H, the commutator $[X,Y]=XY-YX$ makes sense as an operator on H, although of course does not necessarily belong to J. However, we shall prove the following identity:

$$[X,\{Y,Z\}] = \{[X,Y],Z\} + \{Y,[X,Z]\}. \qquad 5.2$$

<u>Proof</u>. The left hand side of 5.2 is:

$$X(YZ+ZY)-(YZ+ZY)X.$$

The right hand side is:

$$\begin{aligned}&(XY-YX)Z+Z(XY-YX)\\&+ Y(XZ-ZX)+(XZ-ZX)Y\\&= XYZ-ZYX-YZX+XZY .\end{aligned}$$

One sees now that this is equal to the left hand side.

<u>Remark</u>. Rule 5.2 can be summed up by saying that $\text{Ad } X: Y \to [X,Y]$ is a <u>derivation</u> of the Jordan algebra.

Now, we have the following identity:

$$\{\{X^2,Y\},X\} = \{X^2,\{Y,X\}\}. \qquad 5.3$$

Proof. The left hand side of 5.3 is:

$$X(X^2Y+YX^2)+(X^2Y+YX^2)X.$$

The right hand side is:

$$X^2(YX+XY)+(YX+XY)X^2.$$

We see explicitly that equality holds.

We also obviously have:

$$\{X,Y\} = \{Y,X\} \quad 5.4$$

$$X^2 = \tfrac{1}{2}\{X,X\}. \quad 5.5$$

Definition. Identities 5.3-5.5 provide the necessary data to formulate the idea of an abstract Jordan algebra. It may now be defined as a vector space, $\underset{\sim}{J}$, with a bilinear map

$$\{\ ,\ \}:\underset{\sim}{J}\times\underset{\sim}{J} \to \underset{\sim}{J},$$

satisfying 5.3-5.5.

Having arrived at the Jordan algebra notion via an associative algebra, we can now reverse the process and associate an associative algebra with any Jordan algebras so that the Jordan product appears as the anticommutator.

Abstract Jordan algebras and their universal enveloping algebras.

Having defined an abstract Jordan algebra $\underset{\sim}{J}$ via properties 5.3-5.5, let us define $U(\underset{\sim}{J})$, its universal enveloping algebra, following the pattern used to define the universal enveloping algebra of a Lie algebra.

Let $T(\underset{\sim}{J})$ be the tensor algebra associated with the vector space $\underset{\sim}{J}$. Recall that it is the linear combination of monomials of arbitrary degrees of the form $X_1 \otimes \cdots \otimes X_r$, with the associative-algebra product defined as follows:

$$(X_1 \otimes \cdots \otimes X_r)(Y_1 \otimes \cdots \otimes Y_s)$$
$$= X_1 \otimes \cdots \otimes X_r \otimes Y_1 \otimes \cdots \otimes Y_s \ .$$

Let $\underset{\sim}{I}$ be the two-sided ideal of $T(\underset{\sim}{J})$ generated by all elements of the following form:

$$2\{X,Y\}-XY-YX \ , \qquad 5.6$$

where X,Y range over $\underset{\sim}{J}$.

Let $U(\underset{\sim}{J})$ be the quotient of $T(\underset{\sim}{J})$ by the ideal $\underset{\sim}{I}$. It inherits an associative-algebra structure, (the quotient map $T(\underset{\sim}{J}) \rightarrow U(\underset{\sim}{J})$ is a homomorphism in the sense of associative algebras), that we denote by: XY. In view of the fact that I contains all elements of the form 5.6, within $U(\underset{\sim}{J})$ we have:

$$2\{X,Y\} = XY-YX. \qquad 5.7$$

Thus, we have a linear mapping $\underset{\sim}{J} \rightarrow U(\underset{\sim}{G})$ of the Jordan algebra into the associative algebra $U(\underset{\sim}{J})$ so that the abstractly-given Jordan algebra on $\underset{\sim}{J}$ goes over to the anticommutator in the associative algebra structure of $\underset{\sim}{J}$. (We shall call such a mapping an <u>associative algebra</u> representation of $\underset{\sim}{J}$.)

<u>Exercise</u>. Discuss the "universality" properties of this representation with respect to the "category" of all associative algebra representations.

<u>Remark</u>: The definitive treatise for this area is that by

Jacobson [1]. The relation between a liner Jordan algebra and its realization in terms of its enveloping algebra is more complicated than for the analogous case of a Lie algebra.

One can construct "gauge" Jordan algebras analogously to "gauge" Lie lagebras.(See LAQM for the "gauge Lie algebra" concept).

Exercise. Suppose that $\underset{\sim}{J}$ is a Jordan algebra, and that F is a commutative, associative algebra. Set:

$$\underset{\sim}{J}_F = \underset{\sim}{J} \otimes F .$$

Define an algebra-type product for J_F as follows:

$$\{X \otimes f_1, Y \otimes f_2\} = \{X,Y\} \otimes f_1 f_2 \qquad 5.8$$

for X, Y $\in \underset{\sim}{J}$; $f_1, f_2 \in F$.

Show that 5.8 defines $\underset{\sim}{J}_F$ as a Jordan algebra. We now turn to another standard example of a Jordan algebra.

The Clifford algebra of a quadratic form.

Let V be a vector space over a field (in the sense of algebra) of scalars that we denote by K. (In the applications K will be either R or C. In any case, we will assume that K has characteristic zero). Let $\beta: V\times V \to K$ be a symmetric, bilinear form on V , with values in K. (Such an object is also called a "quadratic form", since it is determined by its values $v \to \beta(v,v)$ on the diagonal subspace of $V\times V$).

Now, set:

$$\underset{\sim}{J} = V \oplus K. \qquad 5.9$$

Define a Jordan algebra structure on $\underset{\sim}{J}$ as follows:

$$\{v_1,v_2\} = \beta(v_1,v_2) \tag{5.10}$$

for $v_1,v_2 \in V$.

$$\{v,k\} = kv \tag{5.11}$$

$$\{k_1,k_2\} = k_1k_2 \tag{5.12}$$

for $v,v_1,v_2 \in V$; $k,k_1,k_2 \in K$.

The unit element "1" of K is then a unit of $\underset{\sim}{J}$, i.e.

$$\{1,X\} = X$$

for all $X \in \underset{\sim}{J}$.

Now, let $U(\underset{\sim}{J})$ denote the universal enveloping algebra of $\underset{\sim}{J}$. Let $C(\beta)$ denote the quotient algebra defined by dividing by the 2-sided ideal generated by all elements of the form

$$X-X\,1\ .$$

Definition. The associative algebra $C(\beta)$ defined in this way is called the Clifford algebra of the quadratic form β.

A more direct way of defining $C(\underset{\sim}{J})$ would go as follows: Let $T(V)$ be the associative tensor algebra defined by the vector space V, i.e. an element of $T(V)$ is of the form:

$$k \oplus v_1 \oplus (v_1' \otimes v_2) \oplus \cdots$$

The multiplication in $T(V)$ is defined as follows:

$$k_1(k \oplus v_1 \oplus (v_1' \otimes v_2) \oplus \cdots)$$
$$= k_1k \oplus k_1v_1 \oplus (k_1v_1' \otimes v_2) \oplus \ .$$

$$v(k \oplus v_1 \oplus (v_1' \otimes v_2) \oplus \cdots)$$
$$= kv \oplus (v \otimes v_1) \oplus (v \otimes v_1' \otimes v_2) + \ldots$$

$C(\beta)$ is now the quotient of $T(V)$ by the two-sided ideal generated by all elements of the form:

$$\tfrac{1}{2}(v_1 \otimes v_2 + v_2 \otimes v_1) - \beta(v_1,v_2).$$

Exercise. If V is n dimensional, show that the dimension of $C(\beta)$, as a vector space, is 2^n. If β is identically zero, show that $C(\beta)$ is isomorphic (as an algebra) with the Grassman algebra of V, i.e. with the algebra (under wedge product $\wedge$) of skew-symmetric tensors on N.

Exercise. If $K=R$, and if V is even dimensional, show that faithful representations of $C(\beta)$ by linear transformations may be constructed by making V into a complex Hilbert space (possibly of indefinite sign), $(v_1,v_2) \to \langle v_1|v_2\rangle$, such that:

$$\beta(v_1,v_2) = \tfrac{1}{2}(\langle v_1|v_2\rangle + \langle v_2|v_1\rangle),$$

constructing the Fermion Fock space, and representing $C(\beta)$ via annihilation and creation operators.

Problem. Can the Fermion Fock space representations defined in the previous exercise be generalized to more general ground fields K?

Derivations and automorphisms of Jordan algebras.

Definition. Let $\underset{\sim}{J}$ be a Jordan algebra, with the Jordan bracket denoted by $\{X,Y\}$. A vector space automorphism $A:\underset{\sim}{J} \to \underset{\sim}{J}$ is an automorphism if:

$$A\{X,Y\} = \{AX,AY\} \qquad 5.13$$

for $X,Y \in \underset{\sim}{J}$.

A linear map $B:\underset{\sim}{J} \to \underset{\sim}{J}$ is a <u>derivation</u> if:

$$B\{X,Y\} = \{BX,Y\} + \{X,BY\} \qquad 5.14$$

for $X,Y \in \underset{\sim}{J}$.

<u>Exercise</u>. If B_1, B_2 are derivations of $\underset{\sim}{J}$, show that the commutator $[B_1,B_2]=B_1B_2-B_2B_1$ is also a derivation.

Thus, the space of derivation of $\underset{\sim}{J}$ forms a Lie algebra, called the <u>derivation Lie algebra</u> of $\underset{\sim}{J}$, denoted by $\underset{\sim}{L}(\underset{\sim}{J})$.

The derivation Lie algebra of $\underset{\sim}{J}$ should be regarded as the Lie algebra of the group of automorphisms of $\underset{\sim}{J}$. To see this, suppose that $t \to A(t)$, $-\infty<t<\infty$, is a parameter family of automorphisms, i.e. that:

$$A(t)\{X,Y\} = \{A(t)X,A(t)Y\}. \qquad 5.15$$

Differentiate both sides of 5.15 with respect to t

$$\frac{dA}{dt}\{X,Y\} = \{\frac{dA}{dt}X,AY\} + \{AX,\frac{dA}{dt}Y\}. \qquad 5.16$$

Apply $A(t)^{-1}$ to both sides of 5.16, using 5.15, and set:

$$B(t) = A(t)^{-1}\frac{dA}{dt}. \qquad 5.17$$

The result is:

$$B(t)\{X,Y\} = \{B(t)X,Y\} + \{X,B(t)Y\} \qquad 5.18$$

for all $X, Y \in \underset{\sim}{J}$, all t.

5.18 then shows that:

$$B(t) \in \underset{\sim}{L}(\underset{\sim}{J}) \text{ for all } t. \qquad 5.19$$

The curve $t \to B(t)$ in $\underset{\sim}{L}(\underset{\sim}{J})$ associated with the curve $t \to A(t)$ in the group of automorphisms of $\underset{\sim}{J}$ is called its infinitesimal generator.

Derivations of quadratic forms.

As an example of derivations, suppose that $\underset{\sim}{J}$ is constructed as a direct sum $V \oplus R$, where V is a real vector space, with a symmetric, bilinear form $\beta: V \times V \to R$ such that the Jordan bracket takes the following form:

$$\{v_1, v_2\} = \beta(v_1, v_2) \quad \text{for } v_1, v_2 \in V \qquad 5.20$$
$$\{v, c\} = v \text{ for } v \in V,\ c \in R.$$

Let $V \wedge V$ denote the exterior product of two copies of V. Let $\underset{\sim}{L}$ denote the Lie algebra of derivations of $\underset{\sim}{J}$. Define a linear mapping $\alpha: V \wedge V \to \underset{\sim}{L}$ as follows:

$$\alpha(v_1, v_2)(v) = \beta(v, v_2)v_1 - \beta(v, v_1)v_2 \qquad 5.21$$
$$\alpha(v_1, v_2)(c) = 0$$
$$\text{for } v_1, v_2,\ v \in V,\ c \in R.$$

We leave it to the reader to show that this map is well defined by formula 5.21.

Exercise. Show that $\alpha(V \wedge V)$ is a Lie subalgebra of $\underset{\sim}{L}$. In case V is finite dimensional, describe the Lie algebra structure of $\underset{\sim}{L}$.

Exercise. Compute a closed formula for the Lie algebra structure on $V \wedge V$ such that α becomes a Lie algebra homomorphism.

These constructions can most readily be understood in terms of the Clifford algebra $C(\beta)$. If $v_1, v_2 \in V \subset C(\beta)$, then consider

v_1v_2 as the product of the elements relative to the Clifford algebra structure. $Ad(v_1v_2): v \to v_1v_2v - vv_1v_2$ then maps $V \oplus R$ into itself, and this mapping is essentially the mapping defined by formula 5.21.

Exercise. Consider the elements $C^2(\beta)$ consisting of polynomials in V of degree at most two. Show that $C^2(\beta)$ forms a Lie algebra under commutator. Describe the relation between this Lie algebra and L.

The Jordan algebras are important in quantum mechanics because they are the abstract mathematical structures whose representations define the "anticommutation relations" satisfied by certain types of quantum fields, i.e. those which describe particles having the "Fermion"property. We now turn to study a simple situation of this sort.

6. LINEAR QUANTUM FIELDS SATISFYING ANTI-COMMUTATION RELATIONS

Let us now use notations similar to those of quantum field theory books. Let μ,ν denote "space-time" indices, ranging from 0 to 3. Let i,j denote "space" indices, ranging from 1 to 3. Let $x=(x_\mu)$ denote space-time points, with components (x_μ), i.e. $x_0=t$ is the "time" component, $(x_j)=\vec{x} \in R^3$ is the "space" component. Thus, we can write:

$$x = (t,\vec{x}).$$

Set:

$$\partial_\mu = \frac{\partial}{\partial x_\mu} .$$

Let a,b be further indices running from 1 to n. Adopt the summation convention on all the indices introduced up to now. Consider operator-valued "functions" $\phi_a(x)$ of space-time points x and the discrete indices a, acting in a Hilbert space H. (Typically, a will be composed of two sorts of indices; one, of a "vector, tensor or spinor" nature linked to Lorentz group covariance and the property the physicists call "spin", the other set decoupled from Lorentz transformations, and called "internal symmetry" indices. For example, these indices may range over representations of SU(2) ("isospin") or SU(3). However, we will attempt in this chapter to develop general methods for dealing with broad classes of quantum fields).

Let us suppose that the $\phi_a(x)$ satisfy a first order, linear, constant coefficient differential equation of the following sort:

$$A^{\mu}_{ab}\partial_{\mu}\phi_b(x) + A_{ab}\phi_b(x) = 0. \qquad 6.1$$

Let us also suppose equal-time anticommutation relations of the following type:

$$\{\phi_a(t,\vec{x}),\phi_b(t,\vec{y})\} = \beta_{ab}\delta(\vec{x}-\vec{y}), \qquad 6.2$$

$$\text{for } \vec{x},\ \vec{y} \in R^3.$$

In 6.1, 6.2, the A^{μ}_{ab}, A_{ab}, β_{ab} are constant matrices with complex coefficients while 6.2 has the usual interpretation in terms of "generalized functions". We shall investigate the conditions for compatibility between 6.1 and 6.2.

First, differentiate both sides of 6.2 with respect to t:

$$\{\partial_0\phi_a(t,\vec{x}),\phi_b(t,\vec{y})\} + \{\phi_a(t,\vec{x}),\partial_0\phi_b(t,\vec{y})\} = 0 . \qquad 6.3$$

From 6.3, we have:

$$A^0_{bb'}\{A^0_{aa'}\ \partial_0\phi_{a'}(t,\vec{x}),\phi_{b'}(t,\vec{y})\} + A^0_{aa'}\{\phi_{a'}(t,\vec{x}),A^0_{bb'}\partial_0\phi_{b'}(t,\vec{y})\} = 0.$$

Using the field equations 6.1, this gives

$$A^0_{bb'}\{A^i_{aa'}\partial_i\phi_{a'}(t,\vec{x})+A_{aa'}\phi_{a'}(t,\vec{x}),\phi_{b'}(t,\vec{y})\} + A^0_{aa'}\{\phi_{a'}(t,\vec{x}),A^i_{bb'}\partial_i\phi_{b'}(t,\vec{y})+A_{bb'}\phi_{b'}(t,\vec{y})\} = 0.$$

Using the postulated equal time commutation relations 6.2, we have:

$$\begin{aligned} & A^0_{bb'}A^i_{aa'}\beta_{a'b'}\partial^x_i\delta(\vec{x}-\vec{y}) \\ & + A^0_{bb'}A_{aa'}\beta_{a'b'}\delta(\vec{x}-\vec{y}) \\ & + A^0_{aa'}A^i_{bb'}\beta_{a'b'}\partial^y_i\delta(\vec{x}-\vec{y}) \\ & + A^0_{aa'}A_{bb'}\beta_{a'b'}\delta(\vec{x}-\vec{y}) = 0. \end{aligned} \qquad 6.4$$

Now, we can use the following relation, whose proof is left as an exercise:

$$\begin{aligned} \partial^x_i\delta(\vec{x}-\vec{y}) &\equiv \frac{\partial}{\partial x_i}\,\delta(\vec{x}-\vec{y}) \\ &= -\partial^y_i\delta(\vec{x}-\vec{y}) \equiv -\frac{\partial}{\partial y_i}\,\delta(\vec{x}-\vec{y}). \end{aligned} \qquad 6.5$$

(Of course, in 6.5 one should follow the conventional rules for differentiation of "generalized functions").

Using 6.5, relations 6.4 are equivalent to the following purely algebraic relations:

$$(A^0_{bb'}A_{aa'} + A^0_{aa'}A_{bb'})\beta_{a'b'} = 0 \qquad 6.6$$

$$(A^0_{bb'}A^i_{aa'} - A^0_{aa'}A^i_{bb'})\beta_{a'b'} = 0 \; . \qquad 6.7$$

Let us now put relations 6.6-6.7 into coordinate-free form. Let V be a complex vector space of dimension n. Let (v_a) be a basis for V. Let A^μ, A be linear transformations: $V \to V$ such that:

$$A^\mu(v_a) = A^\mu_{ab}v_b \qquad 6.8$$

$$A(v_a) = A_{ab}v_b \qquad 6.9$$

Let β be the bilinear form: $V\times V \to C$ defined:

$$\beta(v_a, v_b) = \beta_{ab} \ . \qquad 6.10$$

Then, 6.6 takes the following form:

$$\begin{aligned} 0 &= (A^0_{bb'}A_{aa'} + A^0_{aa'}A_{bb'})\beta(v_{a'}, v_{b'}) \\ &= \beta(A(v_a), A^0(v_b)) + \beta(A^0(v_a), A(v_b)). \end{aligned} \qquad 6.11$$

6.7 takes the following form:

$$\begin{aligned} 0 &= (A^0_{bb'}A^i_{aa'} - A^0_{aa'}A^i_{bb'})\beta(v_{a'}, v_{b'}) \\ &= \beta(A^i(v_a), A^0(v_b)) - \beta(A^0(v_a), A^i(v_b)). \end{aligned} \qquad 6.12$$

Notice that the steps are reversible, i.e. if 6.11-6.12 are satisfied, then the equal time anticommutation relations 6.2 may be set up, and they are compatible with the field equations 6.1. Let us sum up as follows:

Theorem 6.1. Suppose that V is a complex vector space, with A^μ, A a collection of linear transformations: $V \to V$. Then, a system of equal-time anticommutation relations for the linear quantum field determined by the field equation:

$$A^\mu \partial_\mu + A$$

is determined by a symmetric, bilinear form $\beta: V\times V \to C$ such that:

$$\beta(A(v_1), A^0(v_2)) + \beta(A^0(v_1), A(v_2)) = 0 \qquad 6.13$$

$$\beta(A^i(v_1), A^0(v_2)) = \beta(A^0(v_1), A^i(v_2)) \qquad 6.14$$

for all $v_1, v_2 \in V$.

<u>Remark</u>: The same argument could be used to characterize equal time <u>commutation</u> relations of the same form as 6.2. Then, the matrix (β_{ab}) and the form β would be skew-symmetric, and 6.13-6.14 would be satisfied.

Let us now suppose 6.13-6.14 are satisfied, and see how the field equations 6.1 and the equal-time anticommutation relations can be characterized more algebraically in terms of infinite dimensional Jordan algebras. Let Γ denote the vector space of compact-support, C^∞ maps: $R^3 \to V$. Thus, each $\gamma \in \Gamma$ is of the form:

$$\gamma(\vec{x}) = \gamma_1(\vec{x})v_1+\cdots+\gamma_n(\vec{x})v_n, \qquad 6.15$$

where $(\gamma_a(\vec{x}))$, the <u>components</u> of γ, C^∞, complex-valued, compact by supported functions of $\vec{x} \in R^3$. For each $\gamma \in \Gamma$, construct the curve $t \to \phi_t(\gamma)$ in the space of operators in H, defined as follows:

$$\phi_t(\gamma) = \int_{R^3} \gamma_a(\vec{x})\phi_a(t,\vec{x})d\vec{x} \ . \qquad 6.16$$

Let us now compute the anticommutation relations 6.2 in terms of the objects defined by 6.16. Suppose $\gamma'=\gamma'_a v_a$ is another element of Γ. Then,

$$\{\phi_t(\gamma),\phi_t(\gamma')\}$$

$$= \iint_{R^3\times R^3} \gamma_a(\vec{x})\gamma'_b(\vec{y})\{\phi_a(t,\vec{x}),\phi_b(t,\vec{y})\}d\vec{x}\ d\vec{y}$$

=, using 6.2,

$$\int_{R^3} \gamma_a(\vec{x})\gamma'_b(\vec{y})\beta_{ab}\ d\vec{x}$$

$$= \int_{R^3} \beta(\gamma(\vec{x}),\gamma'(\vec{x})d\vec{x} \ . \qquad 6.17$$

Thus, 6.17 is the basis-independent form of the anticommutation relations 6.2.

We can also write the field equations 6.1 in basis free form. Using 6.8, 6.9 and 6.16, we have:

$$\frac{\partial}{\partial t} \phi_t(A^0(\gamma)) = \int_{R^3} \gamma_a(\vec{x}) A^0_{ab} \partial_0 \phi_b(t,\vec{x}) d\vec{x}$$

$$= -\int_{R^3} \gamma_a(\vec{x}) (A_{ab}\phi_b(t,\vec{x}) + A^i_{ab} \partial_i \phi_b(t,\vec{x})) d\vec{x}$$

$$= -\phi_t(A(\gamma)) - \phi_t(A^i \partial_i(\gamma)) \ . \qquad 6.18$$

We can interpret these formulas in the following way: Define a symmetric, bilinear form $\underset{\sim}{\beta}: \Gamma\times\Gamma \to C$ as follows:

$$\underset{\sim}{\beta}(\gamma,\gamma') = \int \beta(\gamma(\vec{x}),\gamma'(\vec{x})) d\vec{x} \ . \qquad 6.19$$

Set:

$$\underset{\sim}{J} = \Gamma \oplus C.$$

Make $\underset{\sim}{J}$ into a Jordan algebra as follows:

$$\{\gamma,\gamma'\} = \underset{\sim}{\beta}(\gamma,\gamma') \qquad 6.20$$

$$\{\gamma,c\} = \gamma$$

$$\text{for } \gamma,\gamma' \in \Gamma,\ c \in C \ .$$

Then, giving the "linear quantum field" in the traditional physicist's form, of equations 6.1-6.2, is equivalent to giving a one-parameter homomorphism $t \to \phi_t$ of J into the Jordan algebra of operators on the Hilbert space H, which satisfies the differential equations 6.18. We shall call this method of interpretation of what is meant by a linear quantum field the <u>Heisenberg picture interpretation</u>. The equations 6.18 can be satisfied by postulating the following form:

$$\phi_t(\gamma) = \phi(\gamma_t), \tag{6.21}$$

where $t \to \gamma_t$ is a curve in Γ such that:

$$A^0 \frac{\partial}{\partial t}(\gamma_t) + A^i \partial_i(\gamma_t) + A\gamma_t = 0, \tag{6.22}$$

and where ϕ is a fixed homomorphism of $\underset{\sim}{J}$ into the operators on H. To satisfy the commutation relations 6.17, we must have the following conditions:

$$\frac{\partial}{\partial t} \underset{\sim}{\beta}(\gamma_t, \gamma_t') = 0 . \tag{6.23}$$

Relations 6.13-6.14 give a set of necessary conditions for 6.23 to be satisfied. Showing that they are sufficient would involve a detailed study of the "Cauchy problem" for the partial differential equations 6.22, a task we shall not go into at the moment. Notice that what we have essentially done is to solve the "quantum" mechanical equations of motion, 6.1, in terms of the equations of motion, 6.22, of the underlying "classical" field. This is feasible because the classical equations are linear. (A simple example of this phenomenon involving systems with a finite number of degrees of freedom is the harmonic oscillator: Once the classical equations of motion are solved, the quantum ones are solved also by passing to the "Heisenberg" picture.)

In this section we have merely touched on what should be a major subject of study. However, we normally emphasize the approach to quantum field theory which treats space and time on a more symmetric footing.

7. FOCK STRUCTURES FOR JORDAN ALGEBRAS DEFINED BY QUADRATIC FORMS

The Jordan algebras encountered in quantum mechanics are most often encountered via the "Fock space" construction. In this section, we shall present a slightly more abstract version of this construction, for use in the next section when we return to the study of linear quantum fields.

Let V be a real vector space, with $\beta: V\times V \to R$ a symmetirc, real-valued bilinear form. Let $\underset{\sim}{J}$ denote, as a vector space, the direct sum $V \oplus R$. Make $\underset{\sim}{J}$ into a Jordan algebra as follows:

$$\{v_1,v_2\} = \beta(v_1,v_2)$$

$$\{v,c\} = v \qquad 7.1$$

$$\text{for } v,\ v_1,\ v_2 \in V;\ c \in R.$$

<u>Definition</u>. A <u>Fock structure</u> for $\underset{\sim}{J}$ is defined by an R-bilinear map

$$\alpha: V\times V \to C\ , \qquad 7.2$$

such that:

$$\alpha(v_1,v_2)^* = \alpha(v_1,v_2) \qquad 7.3$$

$$\beta(v_1,v_2) = \alpha(v_1,v_2)+\alpha(v_2,v_1) \qquad 7.4$$

Given such a Fock structure, we shall define a new real Jordan algebra $\underset{\sim}{J}'$ which contains $\underset{\sim}{J}$ as a subalgebra. $\underset{\sim}{J}'$ will be called the <u>Fock</u> or <u>annihilation-creation Jordan algebra</u> associated with $\underset{\sim}{J}$ and α.

As a vector space, $\underset{\sim}{J}'$ is the direct sum of two copies of V and C. Denote one such copy of V by V^+, the other by V^-.

Thus,

$$\underset{\sim}{J}' = V^+ \oplus V^- \oplus C. \qquad 7.4$$

Define the Jordan bracket in $\underset{\sim}{J}'$ as follows:

$$\{V^+,V^+\} = 0 = \{V^-,V^-\} \qquad 7.5$$

$$\{v_1^+,v_2^-\} = \alpha(v_1,v_2) \qquad 7.6$$

for v_1, v_2 e V.

$$\{c,v^+\} = v^+; \{c,v^-\} = v^- \qquad 7.7$$

for v e V.

Define $\underset{\sim}{J}$, the Jordan algebra with structure relations 7.1, as a subalgebra of $\underset{\sim}{J}'$, as follows:

Assign:

$$v = v^+ + v^-. \qquad 7.8$$

Then, using 7.4-7.6, we have:

$$\begin{aligned}\{v_1,v_2\} &= \{v_1^+ + v_1^-, v_2^+ + v_2^-\} \\ &= \{v_1^-,v_2^+\} + \{v_1^+,v_2^-\} \\ &= \{v_2^+,v_1^-\} + \{v_1^+,v_2^-\} \\ &= \alpha(v_2,v_1) + \alpha(v_1,v_2) \\ &= \beta(v_1,v_2),\end{aligned}$$

which shows that the mapping

$$v \to v^+ + v^-$$

$$R \to R \subset C$$

indeed defines $\underset{\sim}{J} = V \oplus R$ as a Jordan subalgebra of $\underset{\sim}{J}'$.

Once such a Fock structure is superimposed on $\underset{\sim}{J}$, it can be used to construct linear operator representations of the Jordan algebra $\underset{\sim}{J}$. Let $\underset{\sim}{J}'$ be a Fock algebra containing $\underset{\sim}{J}$. Let us say that a linear representation $\rho:\underset{\sim}{J}' \to$ (operators on a Hilbert space H) is a <u>Fock representation</u> if the following conditions are satisfied:

$$\rho(v^+)^* = \rho(v^-) \qquad 7.9$$

for $v \in V$

$$\rho(c) = c\times(\text{identity operator}) \qquad 7.10$$

for $c \in C$.

There is a vector $\psi_0 \in H$ such that:

$$\rho(V^-)(\psi_0) = 0 \qquad 7.11$$

$$\langle\psi_0|\psi_0\rangle = 1 \qquad 7.12$$

$$\langle\rho(v_1^+)\psi_0|\rho(v_2^+)\psi_0\rangle$$

$$= \alpha(v_2,v_1) \qquad 7.13$$

for $v_1,v_2 \in V$.

For $v \in V$, $\rho(v)$ is a bounded operator. 7.14

<u>Remark</u>. The more explicit Fermion Fock representation has all of these properties. Let us recall how it is constructed: (See VB, Vol. 1, Chapter IX). Suppose that V itself has a Hilbert space structure $\langle\ |\ \rangle$, such that

$$\langle v_1|v_2\rangle = 2\alpha(v_1,v_2) \qquad 7.15$$

for $v_1,v_2 \in V$.

Let $H=A(V)$, the space of skew-symmetric tensors on V, $\rho(v^+)$, $\rho(v^-)$ the operators of "creation" and "annihilation" by $v \in V$, defined in VB, Vol. I, Chapter IX. (The fact that elements of H, the "multiparticle" states of V, are chosen as skew-symmetric tensors is the reason for associating the name "Fermion" with the process. Physically, "Fermions" are particles which satisfy the "Fermi exclusion principle", i.e. no two particles can be in exactly the same state. In terms of the mathematics of quantum mechanics, this is usually interpreted by saying that the "wave function" of a system of identical particles which are Fermions changes sign when the particles are permuted. When suitably translated into Hilbert space language, this means that the "multiparticle" states are chosen as the skew-symmetric tensors on the single particle states).

Exercise. Using properties 7.9-7.13, compute the inner products between various elements of H built up by applying "creation" operators $\rho(V^+)$ to the "vacuum state" ψ_0. Construct an orthonormal basis for H using these formulas.

Exercise. Prove that 7.14 is a consequence of the properties 7.9-7.13, if one assumes that the operators in $\rho(\underset{\sim}{J})$ applied to ψ_0 span all of H.

Exercise. Compute the matrix elements functions

$$t \to \langle\psi_0|\exp(t\rho(v)|\psi_0\rangle \qquad 7.16$$

Linear quantum fields in terms of modules.

Let F denote the algebra (over the real numbers) of C^∞, compact-support, real valued functions: $R^4 \to R$. As in Section 6, we shall denote points of R^4 by "x", so that an element $f \in F$ is defined by a function $x \to f(x)$.

Let Γ, Γ' be F-modules, and let $D:\Gamma' \to \Gamma$ be a differential operator. We shall call the tuple (Γ,Γ',D) a linear field. We shall attempt to define the basic concepts of quantum field theory in terms of this mathematical structure.

First, the space of observables is the quotient vector space

$$\Gamma/D(\Gamma'). \qquad 7.17$$

A Bose structure for the observables is defined by a skew-symmetric, R-bilinear, real-valued form on the observables. Alternately (and more conveniently) it can be defined as a skew-symmetric, R-bilinear form, typically denoted by $\omega:\Gamma\times\Gamma \to R$, such that:

$$\omega(D(\Gamma'),\Gamma) = 0. \qquad 7.18$$

Similarly a Fermi structure for the observables is defined by a symmetric, R-bilinear form $\beta:\Gamma\times\Gamma \to R$ such that:

$$\beta(D(\Gamma'),\Gamma) = 0. \qquad 7.19$$

A Fock structure for the observables is defined by an R-bilinear map

$$\alpha:\Gamma\times\Gamma \to C$$

which satisfies the following conditions:

$$\alpha(D(\Gamma'),\Gamma) = 0 \qquad 7.20$$

$$\alpha(\gamma_1,\gamma_2) = \alpha(\gamma_2,\gamma_1)^* \qquad 7.21$$

for $\gamma_1,\gamma_2 \in \Gamma$.

Given such a Fock structure, one can define a Bose and Fermi structure by separators α into its real and imaginary parts:

$$\alpha = \beta + i\,\omega \qquad 7.22$$

We can now define the "classical states". Let Γ^d, Γ'^d denote the dual F-modules of Γ and Γ'. Thus, an element $\gamma^d \in \Gamma^d$ is an F-linear map: $\Gamma \to F$. Define a bilinear map: $\Gamma\times\Gamma^d \to R$ as follows:

$$(\gamma,\gamma^d)\to\int_{R^4} \gamma^d(\gamma)(x)dx^4 \qquad 7.23$$

Denote the right hand side of 7.6 by the following notation:

$$<\gamma,\gamma^d>.$$

Similarly, define a pairing: $\Gamma'\times\Gamma'^d \to R$. Let us suppose that the differential operator $D:\Gamma' \to \Gamma$ has a <u>dual differential operator</u>

$$D^T:\Gamma^d \to \Gamma'^d$$

with the following property:

$$<D(\gamma'),\gamma^d> = <\gamma',D^T(\gamma^d)>. \qquad 7.24$$

<u>Definition</u>. A classical state of the system (Γ,Γ',D) is an element $\gamma^d \in \Gamma^d$ such that:

$$D^T(\gamma^d) = 0. \qquad 7.25$$

We can now show that each classical state associated with the system (Γ,Γ',D) determines a real-valued function on the observables 7.1. In fact, notice that the function

$$\gamma \to \langle\gamma,\gamma^d\rangle \qquad 7.26$$

satisfies

$$\langle D\gamma',\gamma^d\rangle = \langle\gamma',D^T(\gamma^d)\rangle = 0,$$

hence the function 7.8 passes to the quotient to define a linear function

$$\Gamma(D(\Gamma) \to R.$$

Let us now turn to the study of linear quantum fields in the constant-coefficient case, where one can use the Fourier transform as a tool.

8. FOURIER TRANSFORM FOCK STRUCTURES FOR QUANTUM FIELDS DEFINED BY CONSTANT COEFFICIENT DIFFERENTIAL OPERATORS

Let us now specialize the general framework discussed in Section 7. Let M denote R^4, the space-time manifold, with $x=(x_\mu)$, $0\le\mu$, $\nu\le 3$, denoting a point of M. Let V be a real vector space of dimension n, and let Γ denote the C^∞, rapidly decreasing mappings $\gamma:M\to V$, considered as a module over F, the C^∞ compact support real-valued functions on M. Let D be a first order, constant coefficient differential operator: $\Gamma\to\Gamma$, of the following form:

$$D(\gamma) = A^\mu \partial_\mu(\gamma) + B\gamma \ , \qquad 8.1$$

where A^μ, B are linear transformations: $V\to V$, and where $\partial_\mu = \frac{\partial}{\partial x_\mu}$.

We are interested in the vector space $\Gamma/D(\Gamma)$, which we identify with the "observables" of the system. To study it, introduce the Fourier transform in the following way.

Let $\hat{M}$ denote another copy of R^4, with a point of $\hat{M}$ denoted by $p=(p_\mu)$. (Physically, $\hat{M}$ is the space of relativistic energy-momentum vectors). Set:

$$x\cdot p = x_0p_0 - x_1\theta_1 - x_2p_2 - x_3p_3 .$$

Let $V_c = V\otimes C = V\oplus iV$ denote the "complexification" of V. If $v=v_1+iv_2$ is an element of V_i, then define its "complex conjugate" $\bar{v}$ as v_1-iv_2. Let $\hat{\Gamma}$ denote the space of C^∞, rapidly decreasing functions $\hat{f}:\hat{M}\to V_c$, such that:

$$\overline{\hat{f}(p)} = \hat{f}(-p) \qquad 8.2$$

for $p\in\hat{M}$.

If $\gamma \in \Gamma$, define its Fourier transform $\hat{\gamma} \in \hat{\Gamma}$ as follows:

$$\hat{\gamma}(p) = \frac{1}{(2\pi)^2}\int_{R^4} e^{-ip\cdot x}\gamma(x)dx \ . \qquad 8.3$$

Then, this transform sets up an isomorphism between Γ and $\hat{\Gamma}$. D goes over by Fourier transform to the following zero-th order differential operator $\hat{D}:\hat{\Gamma} \to \hat{\Gamma}$:

$$\hat{D}(\hat{\gamma})(p) = A(p)(\hat{\gamma}(p)) + B\hat{\gamma}(p), \qquad 8.4$$

where:

$$A(p) = i(p_0A^0 - p_1A^1 - p_2A^2 - p_3A^3). \qquad 8.5$$

Notice that, for each $p \in R^4$, $A(p)$ is a linear transformation: $V_c \to V_c$. Further

$$A(p)^* = A(-p). \qquad 8.6$$

(Notice that, conversely, 8.6 is the condition guaranteeing that the operator $\hat{D}$, defined via formula 8.4, actually maps $\hat{\Gamma}$ into $\hat{\Gamma}$).

Now, let N be a subset of points $p \in \hat{M}$ satisfying the following condition:

$$\text{determinant}(A(p)+B) = 0. \qquad 8.7$$

Let us assume that N is an oriented submanifold of, with a volume element differential form denoted by

$$d_Np.$$

Condition 8.7 then is equivalent (since V is finite dimensional) to requiring that:

$$(A(p)+B)(V_c) \neq V_c. \qquad 8.8$$

We shall now define a Fock structure on Γ with the aid of a certain type of algebraic structure on V_c. Suppose that, for each $p \in N$, α_p is a Hermitian-symmetric form: $V_c \times V_c \to C$ such that:

$$\alpha_p(A(p)+B)(V_c),V_c) = 0. \qquad 8.9$$

Define a map

$$\alpha:\Gamma\times\Gamma \to C$$

as follows:

$$\alpha(\gamma_1,\gamma_2) = \int_N \alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))d_N p. \qquad 8.10$$

We see that formula 8.10 satisfies all the conditions needed to assure that α defines a Fock structure for (Γ,D), in the sense defined in Section 7.

Of course, there are many possible choices of the forms $p \to \alpha_p$ determining the Fock structure. In practice, the choice is fixed much more rigidly by conditions of Lorentz invariance, and the other qualitative properties that are desirable on physical grounds (such as causality). We shall now examine the conditions imposed by Lorentz invariance.

Let L be the simply connected covering group of the Lorentz group; as a matrix group L can be identified with $SL(2,C)$. (See LMP, Vol. II). L is a 2-fold covering group of $SO^+(1,3)$, the connected component of the Lorentz group. As such, L acts as a transformation group on M and $\hat{M}$ (both identified with R^4 of course) so that:

$$\ell x\cdot\ell x = \ell p\cdot\ell x = \ell p\cdot\ell p$$

$$\text{for } \ell \in L,\ x \in M,\ p \in \hat{M}.$$

Suppose also given a real-linear action ρ of L on V. It can be extended to a complex linear action on $V_c = V + iV$.

<u>Definition</u>. The differential operator D is said to be invariant under $\rho(L)$ if:

$$\rho(\ell)A(p)\rho(\ell^{-1}) = A(\ell p) \qquad 8.11$$

for $\ell \in L$, $p \in M$.

$$\rho(\ell)B\rho(\ell)^{-1} = B \qquad 8.12$$

for $\ell \in L$.

<u>Remark</u>: Conditions 8.11 mean, in the physicist's jargon, that the operators A^{μ} on V transform "like a four vector" under $\rho(L)$, while B transforms "like a scalar". One can find in the books by Gelfand, Minlos and Shapiro [1] and Naimark [1] the "classical" analysis of the conditions this imposes on the representation $\rho(L)$ and the operators A^{μ}, B. In VB, Vol. I, I have given a presentation of the theory which emphasizes certain general features, such as the split up of V^c into the direct sum of two subspaces (which physicists call "the spaces of "dotted" and "undotted" spinors), and the Lie algebra of linear transformations on V generated by the operators A^{μ}, B and $\rho(L)$.

Suppose now that the representation $\rho(L)$ satisfies 8.11-8.12, and that the family of $p \to \alpha_p$ of Hermitian forms on V_c defines a Fock structure for Γ. We shall say that the Fock structure is <u>invariant under $\rho(L)$</u> if the following conditions are satisfied:

$$\ell p \in N \text{ for } \ell \in L,\ p \in N \qquad 8.13$$

$$\alpha_{\ell p}(\rho(\ell)v_1, \rho(\ell)v_2) = \alpha_p(v_1, v_2) \qquad 8.14$$

for $p \in M$; $v_1, v_2 \in V_c$; $\ell \in L$.

Let us suppose that L acts transitively on N. We shall show how the system $p \to \alpha_p$ of Hermitian forms which satisfies 8.14 is determined by this value at one point. Let p_0 be a fixed point of N. Let K be the isotropy subgroup of L at p_0. (Hence, N is identified with the coset space $L(K)$.

Set:

$$\alpha_0 = \alpha_{p_0} \ .$$

Then, the invariance condition 8.14, when cut down to K, implies the following condition:

$$\alpha_0(\rho(k)v_1, \rho(k)v_2) = \rho_0(v_1, v_2) \qquad 8.15$$

for v_1, $v_2 \in V_c$.

Set:

$$V^0 = (A(p_0)+B)(V_c). \qquad 8.16$$

Then, conditions 8.11-8.12 imply that:

$$\rho(K)(V^0) \subset V^0 \ . \qquad 8.17$$

Hence, $\rho(K)$ passes to the quotient to define a representation $\rho'(K)$ of K in

$$V' = V_c/V^0$$

α_0 also passes to the quotient; condition 8.16 implies that it is a Hermitian symmetric form on V' which is invariant under $\rho(K)$.

Now, we can write the form α on Γ, defined by 8.10, in an elegant form that exhibits more clearly the role that L plays. Let $d\ell$ be the bi-invariant volume element on L. Let us also suppose that K is compact. Recall that L is assumed to act transitively on N.

Exercise. Show that the volume element $d_N p$ on N and L can be chosen so that:

$$\int_N f(p)d_N p = \int_L f(\ell p_0)d\ell \qquad 8.18$$

for each rapidly decreasing function f on N.

With formula 8.16, we can now write formula 8.10 as follows:

$$\alpha(\gamma_1,\gamma_2) = \int_L \alpha_{\ell p_0}(\hat{\gamma}_1(\ell p_0),\hat{\gamma}_2(\ell p_0))d\ell$$

=, using 8.14,

$$\int_L \alpha_0(\rho(\ell^{-1})\hat{\gamma}_1(\ell p_0),\rho(\ell)^{-1}\hat{\gamma}_2(\ell p_0))d\ell. \qquad 8.19$$

Notice that this formula frees us from the use of the differential operator D. We could now start off with an arbitrary Hermitian-symmetric form α_0 on V_c, and define α via formula 8.17.

Exercise. Define a representation σ of L on Γ as follows:

$$\sigma(\ell)(\gamma)(x) = \rho(\ell)^{-1}\gamma(\ell^{-1}x). \qquad 8.20$$

With α defined by 8.17, show that:

$$\alpha(\sigma(\ell)\gamma_1,\sigma(\ell)\gamma_2) = \alpha(\gamma_1,\gamma_2), \qquad 8.21$$

i.e. that the form $\alpha:\Gamma\times\Gamma\to C$ is invariant under the action of L.

We can also write 8.17 in terms of the γ_1, γ_2 instead of their Fourier transforms. To do this, use 8.3:

$$\hat{\gamma}_1(\ell p_0) = \frac{1}{(2\pi)^2}\int_{R^4} e^{-i\ell p_0\cdot x}\gamma_1(x)dx$$

$$= \frac{1}{(2\pi)^2}\int_{R^4} e^{-ip_0\cdot \ell^{-1}x}\gamma_1(x)dx$$

$$= \frac{1}{(2\pi)^2}\int_{R^4} e^{-ip_0\cdot x}\gamma_1(\ell x)dx. \qquad 8.22$$

Now, for $\ell \in L$, set:

$$\sigma(\ell)(\hat{\gamma})(p) = \rho(\ell)\hat{\gamma}(\ell^{-1}p). \qquad 8.23$$

Thus, we can rewrite 8.17 as follows:

$$\alpha(\gamma_1,\gamma_2) = \int_L \alpha_0(\sigma(\ell)(\hat{\gamma}_1)(p_0), \sigma(\ell)(\hat{\gamma}_2)(p_0))d\ell. \qquad 8.24$$

<u>Remark</u>: The reason for formula 8.21 can best be understood in terms of vector bundle theory. (See LGP and VB). Let E be the product vector bundle $\hat{M}\times V_c$. Let L act on E as follows:

$$\ell(\hat{p},v) = (\ell\hat{p},\rho(\ell)v) \qquad 8.25$$

$$\text{for } \ell \in L,\ \hat{p} \in \hat{M},\ v \in V_c.$$

Then, 8.23 defines a linear action of L on E. 8.21 defines

$\sigma(L)$ as the natural action of L on the cross-sections of E.

Note that:

$$\sigma(\ell)\hat{\gamma} = \widehat{\sigma(\ell)\gamma} \ , \qquad 8.26$$

i.e. the Fourier transform map $\gamma \to \hat{\gamma}$ intertwinces the action of L. Using 8.17 and 8.3, we have:

$$\alpha(\gamma_1,\gamma_2) = \frac{1}{(2\pi)^4} \int_L \iint_{R^4 \times R^4} e^{i\ell p_0\cdot(x-y)} \alpha_0(\rho(\ell^{-1})\gamma_1(x),\rho(\ell^{-1})\gamma_2(y))dx\, dy \ . \qquad 8.27$$

Now, for each $x \in R^4$, let us define a bilinear mapping $\Delta_+(x):V_c \times V_c \to C$ as follows:

$$\Delta_+(x)(v_1,v_2) = \frac{1}{(2\pi)^4} \int_L e^{i\ell p_0\cdot x} \alpha_0(\rho(\ell)^{-1}v_1,\rho(\ell)^{-1}v_2)d\ell. \qquad 8.28$$

Then, combining 8.25 and 8.26, we have:

$$\alpha(\gamma_1,\gamma_2) = \int_{R^4}\int_{R^4} \Delta_+(x-y)(\gamma_1(x),\gamma_2(y))dx\, dy. \qquad 8.29$$

For each x, $\Delta_+(x)$ is a Hermitian symmetric bilinear form on V_c. Of course, it will have singularities as a function of x. (For example, notice that, for $x=0$, the defining integral on the right hand side of 8.6 diverges, since the volume of L, which is a non-compact group, is infinite). Further,

$$\Delta_+(x)^* = \Delta_+(-x) \ . \qquad 8.30$$

<u>Remark</u>: Notice that, if $V=R$, $\rho(L)$=identity, that $\Delta_+(x)$ reduces to the "positive frequency" commutator function used in the quantum field theory textbooks to quantize the neutral spin

zero field. (See Schweber [1] for the most complete description of this topic). Explicitly, we have:

$$\Delta^{+}(x) = \frac{1}{(2\pi)^4}\int_{R^4} \theta_0(p)e^{ip\cdot x}dp. \qquad 8.31$$

Thus, formulas 8.26 and 8.27 constitute a far reaching group-theoretic generalization of this. Once we have derived these formulas, we are not limited to the situation with which we started, but may start with V_c an arbitrary complex vector space, carrying a representation $\rho(L)$ of the Lorentz group L (or any extension of L), with α_0 an arbitrary Hermitian symmetric form. This is one way of understanding the fundamental group-theoretic nature of the process of quantization of linear fields.

Another important generalization would be to replace M by a general coset space G/L. Then the integral 8.26 defining the commutator function $\Delta_{+}(x)$ by what I have called (in LGP, Chapter 13) the "group-theoretic version of the Fourier transform". In intend to go into these extensive generalizations and ramifications at a later point in this treatise.

Finally, let us remark that the "derivation" of formula 8.27 is not complete, because we have not justified the interchange of limits in 8.25. Of course, one might argue that this is a traditional bit of sloppiness in mathematical physics, that can be readily justified by being more careful and systematically considering Δ_{+} as a "generalized function", i.e. essentially define Δ_+ by the formula 8.27. One tip-off that this is necessary for a rigorous mathematical interpretation is that the defining integral 8.26 for Δ_+ diverges for certain values of x. Hoever, in the spin-zero case, one can, in fact, go beyond this general remark and compute precisely the form of $\Delta_+(x)$ and its singularities. (See Schweber [1], p.182. Notice that he gives explicit formulas for $\Delta(x)$ and $\Delta^{(1)}(x)$ in terms of Bessel functions, and that $\Delta_+(x)$ can be written in terms of $\Delta(x)$ and $\Delta^{(1)}(x)$.) It is obviously an important problem to do this for the generalized "positive frequency commutator function" $\Delta_+(x)$ defined by formula 8.26. In turn, this involves a major expedition into Lie group harmonic analysis theory. (For example, notice that formula 8.26 also makes sense if V_c is infinite dimensional, and the analysis of the singularties of $\Delta_+(x)$ poses even more complicated harmonic analysis questions in this case.) Again, I plan to go into this in more detail in a later volume; at this point I believe it is more urgent to continue to elaborate the fascinating general principles underlying quantum field theory.

9. A GENERAL PRINCIPLE FOR THE CONSTRUCTION OF FOCK STRUCTURES ON OBSERVABLES DEFINED BY PARTIAL DIFFERENTIAL EQUATIONS

In practice, i.e. in the quantum field theory textbooks, the commutator (or anticommutator) functions for the fields of spin $\frac{1}{2}$ or higher are constructed in terms of the corresponding functions for the spin 0 field. Now, I don't know of any general theorem why this should be so; partly it seems to be the most convenient choice and partly it follows from the historical path, i.e. the wave equations defining the higher spin fields were chosen so that their components satisfied the Klein-Gordon equation. (Notice, however, that this fact depends on the geometric fact that Minkowski space has zero curvature. Presumably if one systematically attempted to define quantum field theory on curved Riemannian manifolds, this simplification would not be available. Lichnerowicz [1] has developed a formalism to extend the theory in this direction.) In this section we present a few remarks that explain the general background for this phenomenon.

Let F be a commutative, associative algebra over the real numbers, and let Γ be an F-module. Let $D:\Gamma \to \Gamma$ be a differential operator. (See GPS , Chapter I for the definition of a differential operator in this general context.) As before, the real vector space $\Gamma/D(\Gamma)$ will be defined as the <u>space of observables</u> of the classical field defined by D.

A <u>commutator form</u> for the observables will be defined as an R-bilinear map $\alpha:\Gamma\times\Gamma \to C$ such that:

$$\alpha(D(\Gamma),\Gamma) = 0. \qquad 9.1$$

(For the moment, we do not assume any particular symmetry or Hermitian symmetry properties for α, as we have done in previous sections).

Now, suppose that D' is a differential operator: $\Gamma \to \Gamma$. Set:

$$D'' = D'D. \tag{9.2}$$

Suppose that α'' is an R-bilinear map: $\Gamma\times\Gamma \to C$ such that:

$$\alpha''(D''(\Gamma),\Gamma) = 0, \tag{9.3}$$

i.e. α'' is a commutator function for the differential operator D''.

Now, define α as an R-bilinear map: $\Gamma\times\Gamma \to C$ as follows:

$$\alpha(\gamma_1,\gamma_2) = \alpha''(D'(\gamma_1),\gamma_2) \tag{9.4}$$

$$\text{for } \gamma_1,\gamma_2 \in \Gamma.$$

Notice that the following identity follows immediately from 9.2 and 9.3:

$$\alpha(D\Gamma,\Gamma) = 0 . \tag{9.5}$$

Thus formula 9.4 gives a method of defining a commutator form for the differential operator D in terms of the commutator form α'' for the differential operator D''.

Let us descendfrom these generalities to consider the traditional case, where D'' is the Klein-Gordon differential operator. Let V be a real vector space, and let F be the space of rapidly decreasing, real-valued functions $f:R^4 \to R$. Let $x=(x_\mu)$, $0\leq\mu$, $\nu\leq 3$, denote a point of R^4; $\partial_\mu=\frac{\partial}{\partial x_\mu}$. Let Γ denote the space of rapidly decreasing maps $\gamma : x \to \gamma(x)$ of $R^4 \to V$. Set:

$$D(\gamma) = A^{\mu}\partial_{\mu}(\gamma) + A\gamma, \qquad 9.6$$

where A^{μ}, A are constant-coefficient linear maps: $V \to V$. (Adopt the summation convention on the indices μ,ν). Suppose that D' is of similar form:

$$D' = B^{\mu}\partial_{\mu} + B. \qquad 9.7$$

However, let us impose the condition that D'' is the Klein-Gordon operator:

$$D'' = g_{\mu\nu}\partial_{\mu}\partial_{\nu} + m^2, \qquad 9.8$$

where

$$g_{\mu\nu} = \begin{cases} 0 \text{ if } \mu \neq \nu \\ 1 \text{ if } \mu=\nu=0 \\ -1 \text{ if } 1 \leq \mu=\nu \leq 3 \end{cases} .$$

Let us work out the conditions that $D''=D'D$:

$$\begin{aligned} D'D &= (B^{\mu}\partial_{\mu}+B)(A^{\mu}\partial_{\nu}+A) \\ &= B^{\mu\ \nu}\partial_{\mu}\partial_{\nu}+(BA^{\nu}+B^{\nu}A)\partial_{\nu} \qquad 9.9 \\ &\quad +BA . \end{aligned}$$

Thus, we have proved the following result:

Theorem 9.1. Let D be a first order, constant coefficient differential operator of form 9.7. Let us say that a differential operator D' is a Klein-Gordon divisor of D if $D'D$=right hand side of 9.9. (The terminology is that used by Takahashi [1]). Such a "divisor" is of the first order, constant coefficient form 9.7 if and only if the following conditions are satisfied:

$$B^\mu A^\nu + B^\nu A^\mu = 2g_{\mu\nu}$$

$$BA^\mu + B^\mu A = 0 \qquad 9.10$$

$$BA = m^2.$$

Remark: The most familiar special case of these conditions is the one first considered by Dirac. Recall that Dirac's motivation for the introduction of what is now called the Dirac equation for the electron field was that he wanted to introduce a differential equation $D=0$ which was of first order in all space-time derivatives (in order to satisfy "special relativity" and to support a "probability" interpretation similar to that of the Schrodinger equation) such that the components of the solutions γ of $D(\gamma)=0$, also satisfied the Klein-Gordon equation. The simplest assumption that would reproduce this behavior involves the following specializations:

$$A = im \qquad 9.11$$

(m is a real scalar, of course).

$$B = -im \qquad 9.12$$

$$B^\mu = A^\mu. \qquad 9.13$$

With these assumptions the last two conditions of 9.11 are automatically satisfied, and everything reduces to the following conditions on the A^μ:

$$A^\mu A^\nu + A^\nu A^\mu \equiv 2\{A^\mu, A^\nu\} = g_{\mu\nu}. \qquad 9.14$$

Condition 9.15 in turn means that the A^μ determine a representation by operators on V of the Clifford algebra associated with the symmetric bilinear form on R^4 defined by the metric tensor $(g_{\mu\nu})$. Thus, if one requires that V be finite dimensional and that the operator D be "irreducible" in the sense defined in Chapter VIII of VB, Vol. I (i.e. that the A^μ generate a Lie algebra of operators on V which acts irreducibly) there is but one choice for D, the Dirac equation described in all the quantum field theory textbooks.

Now, let us return to the more general case where conditions 9.11 are satisfied. Since D" is the Klein-Gordon operator, we may choose a commutator form

$$\alpha'': \Gamma \times \Gamma \to C$$

in the following way:

$$\alpha''(\gamma_1,\gamma_2) = \int_{R^4 \times R^4} \Delta_+(x-y)\beta_0(\gamma_1(x),\gamma_2(y))dx\,dy\,. \qquad 9.15$$

In 9.16, $\Delta_+(x)$ is the "positive frequency" commutator function for the scalar Klein-Gordon equation (i.e. the generalized function determining the Fock structure for the scalar Klein-Gordon field), and β_0 is an arbitrary symmetric, R-bilinear form: $V \times V \to R$. (Of course, if one imposes invariance under a group of transformations on V, β would be determined more explicitly). If α is now defined by formula 9.4, we have:

$$\alpha(\gamma_1,\gamma_2) \equiv \alpha''(D'(\gamma_1),\gamma_2)$$

$$= \text{, using 9.16}$$

$$\iint \Delta_+(x-y)\beta_0(B^\mu \partial_\mu(\gamma_1)(x)+B\gamma_1(x),\gamma_2(y))dx\ dy$$

$$= -\iint \frac{\partial}{\partial x_\mu}(\Delta_+(x-y))\beta_0((B^\mu\gamma_1+B\gamma_1)(x),\gamma_2(y))dx\ dy. \qquad 9.16$$

This is a general basis-free formula for the commutator form; special cases are described in considerable detail in quantum field theory textbooks. (Schweber [1] and Takahashi [1] have the most complete treatment of this topic).

Exercise. Work out the Fourier-transformed version of 9.17, and the condition needed to be satisfied in order that 9.17 define α as a Fock structure form for (Γ,D).

Research Problem. Separate α into its real and imaginary parts $\beta+i\omega$. In terms of formula 9.17 (or its Fourier transformed version) work out the "causality" properties of β and ω.

10. GENERAL REMARKS ABOUT SYMMETRIES AND CHARGES ASSOCIATED WITH LINEAR QUANTUM FIELDS

Let F be a commutative, associative algebra over the real numbers, and let Γ,Γ' be F-modules. Let $D:\Gamma' \to \Gamma$ be a linear differential operator.

Definition. A quantization of the observables associated with the differential operator D is defined by a Hilbert space H and an R-linear map $\gamma \to \phi(\gamma)$ of Γ into the space of linear operators on H such that:

$$\phi(D\Gamma') = 0. \qquad 10.1$$

Let us suppose that such a quantization is given. An automorphism of the observable vector space associated with the differential operator D is defined as a vector space automorphism $A:\Gamma \to \Gamma$ such that:

$$A(D(\Gamma')) = D(\Gamma'). \qquad 10.2$$

Such an automorphism is (unitarily) realizable if there is a unitary operator $U:H \to H$ such that:

$$\phi(A(\gamma)) = U\phi(\gamma)U^{-1} \qquad 10.3$$

$$\text{for } \gamma \in \Gamma.$$

An endomorphism of the observables is defined as an R-linear transformation $B:\Gamma \to \Gamma$ such that:

$$B(D(\Gamma')) \subset D(\Gamma'). \qquad 10.4$$

Such an endomorphism is said to be realizable (via a skew-Hermitian operator) if there is a skew-Hermitian operator $W:H \to H$

such that:

$$\phi(B\gamma) = [W, \phi(\gamma)] \qquad 10.5$$

for all $\gamma \in \Gamma$.

The unitary and skew-Hermitian operators obtained in this way are very important in classifying elementary particles and their interactions. (Typical examples of such U's would be the P,C operations, i.e. the unitary operators corresponding to parity and charge conjugation. Typical examples of such W's would be the charge, hypercharge, strangeness, isotopic spin operators, i.e. all those quantum observables which are "additive" with respect to interactions).

There is another possibility that is important for the applications to elementary particle physics. Recall then an R-linear automorphism $U: H \to H$ is said to be anti-unitary if it satisfies the following conditions:

$$U(c\psi) = c^* U(\psi) \qquad 10.6$$

for $\psi \in H$, $c \in C$.

$$\langle U(\psi_1) | U(\psi_2) \rangle = \langle \psi_2 | \psi_1 \rangle \qquad 10.7$$

for $\psi_1, \psi_2 \in H$.

One may then say that an observable automorphism A is anti-unitarily realizable if 10.3 is satisfied, with U anti-unitary. For example, the "time reversal operator", usually denoted by T, is anti-unitary. (This is necessary in order that the "energy operator" associated with the Lie algebra of the Poincaré group have a positive spectrum).

Later on, we shall study in more detail the algebraic questions associated with these concepts, and the related problems concerning the classification and interaction of elementary particles. In the rest of this section, we shall restrict attention to an important special case, namely that associated with the concept for a "charged" particle. (In Section 2 we have already studied this in the simplest case, the charged, scalar Klein-Gordon particle).

Let us suppose that Γ has a complex vector space structure. For each real number λ, let $A(\lambda)$ be the following automorphism of Γ:

$$A(\lambda)(\gamma) = e^{i\lambda}\gamma. \qquad 10.8$$

Suppose that it satisfied 10.2, i.e. is a D-observable automorphism. Then, $\lambda \to A(\lambda)$ is a one parameter group of observable automorphisms. A one parameter group $\lambda \to U(\lambda)$ of unitary operators on H such that:

$$U(\lambda)\phi(\gamma)U(-\lambda) = \phi(e^{i\lambda}\gamma) \quad \text{for each } \lambda \in R. \qquad 10.9$$

is called the <u>charge unitary group</u>.

Its infinitesimal generator would be a skew-Hermitian operator $Q:H \to H$ such that:

$$[Q,\phi(\gamma)] = \phi(i\gamma) \qquad 10.10$$

$$\text{for } \gamma \in \Gamma.$$

Such a Q is called a <u>charge operator</u>. The eigenstates of Q are called the <u>charge states</u>. If $\psi \in H$ is such a state, with:

$$Q(\psi) = -iq\psi \ , \qquad 10.11$$

then q is called the <u>charge</u> of the state (or "particle" which is assigned to that state).

If Γ' is another F-module with a complex vector space structure, and if D is an R-linear differential operator: $\Gamma' \to \Gamma$ which is also complex-linear, we see that $\lambda \to A(\lambda)$ defined by 10.8 will be an observable symmetry, i.e. will satisfy 10.2.

Let us now turn to the construction of the charge operator. In Section 2, we have presented the traditional way of looking at this question for the complex Klein-Gordon field. We shall now abstract from this example a general method for constructing a charge operator. (It is not clear to me whether the method presented here covers all cases of physical interest. I have found the whole question of the "charge" operators and "charge conjugation" to be very unclear in the physics literature).

Let Γ_c be a complex vector space. (The module structure of the observables will play no role here, so we will forget it.) Let $D_0:\Gamma_c \to \Gamma_c$ be a complex-linear operator. (We will for simplicity, ignore the general case where D_0 is a map between different spaces.)

<u>Definition</u>. A subspace $\Gamma_0 \subset \Gamma_c$ is called a <u>neutral subspace</u> for the complex differential operator D_0 if the following conditions are satisfied:

$$\Gamma_0 \text{ is a real linear subspace of } \Gamma_c, \quad \text{i.e. } c\Gamma_0 \subset \Gamma_0 \text{ for } c \in R. \qquad 10.12$$

$$\Gamma = \Gamma_0 + i\Gamma_0 ; \Gamma_0 \subset i\Gamma_0 = 0, \qquad 10.13$$

i.e. Γ is the "complexification" of Γ_0.

$$D_0(\Gamma_0) \subset \Gamma_0 . \qquad 10.14$$

Now, let

$$\Gamma = \Gamma_0 \oplus \Gamma_0 . \qquad 10.15$$

In 10.13, the direct sum is that in the sense of real vector spaces. Hence, Γ is considered to be a real vector space. (If Γ_c is considered as a real vector space, it is isomorphic to Γ, of course, but to be as clear as possible we shall keep the ideas separate). Let D be the R-linear map:$\Gamma \to \Gamma$ defined as follows:

$$D(\gamma_1 \oplus \gamma_2) = D_0(\gamma_1) \oplus D_0(\gamma_2) \qquad 10.16$$

$$\text{for } \gamma_1, \gamma_2 \in \Gamma_0 .$$

Suppose that H_0 is a Hilbert space, and that $\gamma \to \phi_0(\gamma)$ is a quantization of D_0 restricted to Γ_0 by skew-Hermitian operators on H, i.e. for $\gamma \in \Gamma_0$, $\phi_0(\gamma)$ is a skew-Hermitian operator on H, and:

$$\phi_0(D_0(\Gamma_0)) = 0. \qquad 10.17$$

Let us construct a quantization of D defined by 10.14. This can be readily done using the tensor product of Hilbert spaces notion, (which we shall discuss in more detail in Chapter III. Set:

$$H = H_0 \otimes H_0 . \qquad 10.18$$

For $\gamma=\gamma_1 \oplus \gamma_2 \in \Gamma$, set:

$$\phi(\gamma) = \phi_0(\gamma_1) \otimes 1+1 \otimes \phi_0(\gamma_2). \qquad 10.19$$

Explicitly, 10.17 means that $\phi(\gamma)$ acts on H as follows:

$$\phi(\gamma)(\psi_1 \otimes \psi_2) = \phi_0(\gamma_1)(\psi_1) \otimes \psi_2+\psi_1 \otimes \phi_0(\gamma_2)(\psi_2) \qquad 10.20$$

$$\text{for } \psi_1, \psi_2 \in H_0.$$

Let B be the linear transformation:$\Gamma \to \Gamma$ defined as follows:

$$B(\gamma_1 \oplus \gamma_2) = -\gamma_2 \oplus (-\gamma_1). \qquad 10.21$$

Then, combining 10.14 with 10.19, we have:

$$B(D(\gamma_1 \oplus \gamma_2)) = B(D_0(\gamma_1) \oplus D_0(\gamma_2))$$

$$= -D_0(\gamma_2) \oplus D_0(\gamma_1) = DB(\gamma_1 \oplus \gamma_2).$$

In particular, B is an endomorphism of the observables associated with the operator D.

<u>Exercise</u>. Suppose Q is a linear operator:$H \to H$ such that:

$$\begin{aligned} [Q,A \otimes 1] &= 1 \otimes A \\ [Q,1 \otimes A] &= -A \otimes 1 \end{aligned} \qquad 10.22$$

for each linear operator $A:H_0 \to H_0$ which is a linear combination of the operators $\phi_0(\Gamma_0)$.

Then, with $\phi(\Gamma)$ defined by 10.17,

$$[Q,\phi(\gamma)] = \phi(B\gamma), \qquad 10.23$$

i.e. Q is a realization by operators on H of the endomorphism B.

<u>Remark</u>. To the best of my knowledge, there is no general way of

constructing Q to satisfy conditions 10.20. In the cases of importance in physics, this can be done readily using a "Fock space" structure for H_0. We shall then have to assume an operator Q exists satisfying 10.21.

Suppose now that $\gamma_c=\gamma_1+i\gamma_2$, with $\gamma_1,\gamma_2 \in \Gamma_0$, is an element of Γ_c. We ask the following question: Can we assign an operator $\phi(\gamma_c)$ to γ_c such that:

$$[Q,\phi(\gamma_c)] = \phi(i\gamma_c)? \qquad 10.24$$

The answer to this is the following formula:

$$\phi(\gamma_c) = \phi(\gamma)+i\phi(B\gamma) \ , \qquad 10.25$$

where $\gamma=\gamma_1 \oplus i\gamma_2$.

Proof of 10.22.

If $\gamma_c=\gamma_1+i\gamma_2$, $\gamma=\gamma_1 \oplus \gamma_2$, then

$i\gamma_c=i\gamma_1-\gamma_2$; $\gamma'=-\gamma_2 \oplus \gamma_1$,

$B(\gamma)=\gamma'$, $B^2=-(\text{identity})$.

Using 10.23, with $i\gamma_c$ replacing γ_c,

$$\phi(i\gamma_c) = \phi(\gamma')+i\phi(B\gamma')$$

$$[Q,\phi(\gamma_c)] = \phi(B\gamma)+i\phi(B^2\gamma)$$

$$\phi(\gamma')+i\phi(B\gamma').$$

This proves 10.22.

Let us sum up as follows:

Theorem 10.1. Let Γ_c be a complex vector space, $D:\Gamma_c \to \Gamma_c$ a complex-linear operator. Under the assumption there is a neutral subspace $\Gamma_0 \subset \Gamma_c$ for the operator D, and the other conditions described above are satisfied, there is a quantization $\gamma_c \to \phi(\gamma_c)$ of D by non-Hermanian operators on a Hilbert space H, and a skew-Hermitian "charge" operator $Q:H \to H$ such that:

$$[Q,\phi(\gamma_c)] = \phi(i\gamma_c) \quad \text{for all } \gamma_c \in \Gamma_c. \tag{10.26}$$

<u>Further Remarks</u>. 10.23 shows that the Hermitian part of $\phi(\gamma_c)$,

$$\tfrac{1}{2}(\phi(\gamma_c)+\phi(\gamma_c)^*) = \phi(\gamma) , \tag{10.27}$$

defines the quantization of the corresponding "real" fields, $\{\phi(\gamma)\}$:

From 10.23 we can also derive the commutation relations of the "complex" fields $\phi(\gamma_c)$ in terms of those of the "real" fields $\phi(\gamma)$. Suppose that ω is an R-bilinear mapping: $\Gamma\times\Gamma \to R$ such that:

$$[\phi(\gamma),\phi(\gamma')] = i\omega(\gamma,\gamma') \quad \text{for } \gamma,\ \gamma' \in \Gamma. \tag{10.28}$$

Then,

$$[\phi(\gamma_c),\phi(\gamma_c')] = [\phi(\gamma)+i\phi(B\gamma),\phi(\gamma')+i\phi(B\gamma')]$$
$$= i(\omega(\gamma,\gamma')-\omega(B\gamma,B\gamma')+i\omega(B\gamma,\gamma') + i\omega(\gamma,B\ \gamma') \tag{10.29}$$

$$[\phi(\gamma_c)^*,\phi(\gamma_c')] = [\phi(\gamma)-i\phi(B\gamma),\phi(\gamma')+i\phi(B\gamma')]$$
$$= i(\omega(\gamma,\gamma')+\omega(B\gamma,B\gamma')+i\omega(\gamma,B\gamma')-i\omega(B\gamma,\gamma')). \tag{10.30}$$

The commutation relations simplify dramatically if the following conditions are satisfied:

$$\omega(B\gamma,\gamma')+\omega(\gamma,B\gamma') = 0 \qquad 10.31$$

for γ, $\gamma' \in \Gamma$.

Exercise. Show that 10.27 is a necessary condition for the existence of a skew-Hermitian charge operator Q satisfying 10.24 (with 10.23 satisfied, of course).

With 10.27, we have:

$$[\phi(\gamma_C),\phi(\gamma_C')] = 0 \qquad 10.32$$

for γ_C, $\gamma_0' \in \Gamma_C$

$$[\phi(\gamma_C)^*,\phi(\gamma_C')] = 2i(\omega(\gamma,\gamma')+i\omega(\gamma,B\gamma')). \qquad 10.33$$

Set:

$$\tfrac{1}{2}\,\alpha(\gamma_C,\gamma_C') = i\omega(\gamma,\gamma')-i\omega(\gamma,B\gamma').$$

Exercise. Show that α is a Hermitian-symmetric, R-bilinear map: $\Gamma_C \times \Gamma_C \to C$.

Notice that the commutation relations 10.28-10.29 can be put into the following elegant form:

$$[\phi(\gamma_C),\phi(\gamma_C')] = 0$$
$$[\phi(\gamma_C)^*,\phi(\gamma_C')] = \alpha(\gamma_C,\gamma_C') \qquad 10.34$$

for $\gamma_C,\gamma_C' \in \Gamma_C$.

It is curious (and confusing) that Hermitian symmetric forms occur in two distinct places in the theory of linear quantum fields: In the description of the "Fock" structure of skew-Hermitian fields (as we have seen earlier in this chapter) and, as in 10.30, in the commutation relations of "complex", i.e. non skew-Hermitian fields.

Given such a Hermitian symmetric form $\alpha:\Gamma_c \times \Gamma_c \to C$, one can also postulate anticommutation relations for the fields $\phi(\gamma_c)$, of the following form:

$$\{\phi(\gamma_c),\phi(\gamma_c')\} = 0$$
$$\{\phi(\gamma_c)^*,\phi(\gamma_c')\} = \alpha(\gamma_c,\gamma_c') \qquad 10.35$$
$$\text{for } \gamma_c,\gamma_c' \in \Gamma_c.$$

Thus, a basic problem is to determine all such Hermitian symmetric forms on Γ_c . We shall worry about this later.

In this section we have described some basic properties of the "charge" operator. We now investigate the relations between the "charge" and the "charge conjugation" operator.

11. CHARGE CONJUGATION OPERATOR FOR LINEAR REAL AND COMPLEX QUANTUM FIELDS

We shall now review, and slightly generalize, our basic definitions. Let Γ be a real vector space. Let B be an R-linear map: $\Gamma \to \Gamma$ such that:

$$B^2 = (\text{-identity}) \tag{11.1}$$

Let $\gamma \to \phi(\gamma)$ be an R-linear mapping of Γ into the space of skew-Hermitian operators on a Hilbert space H. Such a mapping is called a <u>quantization</u> of Γ. A skew-Hermitian operator $Q:H \to H$ is said to be a <u>charge operator</u> for the quantization (Γ,ϕ) (relative to the endomorphism B) if the following condition is satisfied:

$$[Q,\phi(\gamma)] = \phi(B\gamma) \tag{11.2}$$

$$\text{for all } \gamma \in \Gamma.$$

We denote the charge operator satisfying 11.2 by: (Q,B).

<u>Definition</u>. Let $C:H \to H$ be a unitary operator, and let $A:\Gamma \to \Gamma$ be an R-linear vector space automorphism. The pair (C,A) is said to be a <u>charge conjugation</u> operation for the charge operator (Q,B) associated with the linear quantum field (Γ,ϕ) if the following conditions are satisfied:

$$CQC^{-1} = -Q. \tag{11.3}$$

$$C\phi(\gamma)C^{-1} = \phi(A\gamma) \tag{11.4}$$

$$\text{for all } \gamma \in \Gamma.$$

$$AB+BA = 0. \tag{11.5}$$

$$C^2 = e^{i\theta}(\text{identity}), \qquad 11.6$$

where θ is a real number.

We can now give a very elegant algebraic interpretation of condition 11.5. Make Γ into a complex vector space as follows:

$$(a+bi)(\gamma) = a\gamma = bB(\gamma) \qquad 11.7$$

for $a, b \in R$.

Exercise. Show that 11.7 does indeed define Γ as a vector space over the complex numbers.

Remark. This algebraic trick -- regarding a complex vector space as a real vector space (of twice the complex dimension) together with a real-linear map whose sequence is twice the dimension-- is very well known in differential geometry. See DGCV, Chapter 32.

Recall (e.g. from LMP, Vol. II, Chapter III) the definition of an "anti-linear" transformation of a complex vector space.

Definition. Let V be a vector space over the complex numbers. A linear transformation $W: V \to V$ is said to be anti-linear (relative to the complex structure of V) if it satisfies the following two conditions:

W is R-linear, i.e.

$$W(av) = aW(v)$$

for $a \in R, v \in V$. 11.8

$$W(iv) = -W(v)$$

for $v \in V$.

<u>Exercise</u>. With the complex structure defined in an Γ via 11.7 by bilinear transformation B, show that condition 11.5 associated with the "charge conjugation" operation (C,A) is equivalent to the condition that A be an anti-linear transformation: $\Gamma \to \Gamma$.

Making Γ into a complex vector space, using formula 11.6, let us define a new "non-skew Hermitian" or "complex" quantum field ϕ_c as follows:

$$\phi_c(\gamma) = \phi(\gamma)+i\phi(B\gamma) \qquad 11.9$$
$$\text{for } \gamma \in \Gamma.$$

Notice that; using 11.6,

$$\begin{aligned} \phi_c(i\gamma) &= \phi_c(-B\gamma) = -\phi(B\gamma)-i\phi(B^2\gamma) \\ &= -\phi(B\gamma)+i\phi(\gamma) \\ &= i\phi_c(\gamma). \end{aligned} \qquad 11.10$$

In particular, notice that 11.10 implies that ϕ_c defines a complex-linear map from the complex vector space Γ to the complex vector space of all linear operators on H. Further if (C,A) is the charge conjugation operator for the quantum field (ϕ,Γ) we have the following relations between (C,A) and the "complex" quantum field (ϕ_c,Γ):

$$\begin{aligned} C\,\phi_c(\gamma)C^{-1} &= \text{,using 11.9 and 11.4,} \\ &\quad \phi(A\gamma)+i\phi(AB\gamma) \\ &= \phi(A\gamma)-i\phi(BA\gamma) \\ &= -\phi_c(A\gamma)^*. \end{aligned} \qquad 11.11$$

We may now abstract from this formula some general definitions, applying to complex quantum fields, independently of their association from real (i.e. skew-Hermitian) ones:

Definition. Let Γ be a complex vector space, H a Hilbert space, $A:\Gamma \to \Gamma$ an anti-linear transformation, Q a skew-Hermitian operator: $H \to H$, and U a unitary operator: $H \to H$. Then, a complex linear quantum field is a complex-linear mapping $\phi_c:\Gamma \to$ (linear operators on H) such that:

$$[Q,\phi_c(\gamma)] = \phi(i\gamma) \qquad 11.12$$

$$CQC^{-1} = -Q.$$

$$C\,\phi_c(\gamma)C^{-1} = \phi_c(A\gamma)^* \qquad 11.13$$

for $\gamma \in \Gamma$.

This definition is independent of the "commutation relations" that the field may satisfy. They may be "axiomatized" in the following way:

Definition. Let Γ, ϕ_0, H, Q, C, A be as in the preceding definition. Then, the field satisfies Bose commutation relations if the following conditions are satisfied:

$$[\phi_c(\gamma_1),\phi_c(\gamma_2)] = 0 \qquad 11.14$$

$$[\phi_c(\gamma_1)^*,\phi_c(\gamma_2)] = \alpha(\gamma_1,\gamma_2), \qquad 11.15$$

where α is a Hermitian-symmetric map: $\Gamma\times\Gamma \to C$.

Similarly, the field satisfies Fermi anti-commutation relations if the following conditions are satisfied:

$$\{\phi_c(\gamma_1),\phi_c(\gamma_2)\} = 0 \qquad 11.16$$

$$\{\phi_c(\gamma_1)^*,\phi_c(\gamma_2)\} = \alpha(\gamma_1,\gamma_2) \qquad 11.17$$

for γ_1, $\gamma_2 \in \Gamma$.

Finally, conditions 11.14 or 11.16 imply compatibility conditions between A and α. Apply $\mathrm{Ad}\, C$ to both sides of 11.14 using 11.12:

$$\alpha(\gamma_1,\gamma_2) = [C\ \phi_c(\gamma_1)^* C^{-1},\ C\ \phi_c(\gamma_2)C^{-1}]$$

= ,using the fact that C is unitary, i.e. $C^*=C^{-1}$

$$[(C\ \phi_c(\gamma_1)C^{-1})^*,\ C\ \phi_c(\gamma_2)C^{-1}]$$

$$= [\phi_c(A\gamma_1),\ \phi_c(A\gamma_2)^*].$$

Hence, we have:

$$\alpha(A\gamma_1,A\gamma_2) = -\alpha(\gamma_2,\gamma_1)\ . \qquad 11.18$$

Recall that condition 11.17 means that A is an <u>anti-unitary operator</u> relative to the form α. (See LMP, Vol. II).

<u>Exercise</u>. Show that relation 11.17 also follows from the Fermi anti-commutation relations 11.15-16.

Let us now see how these abstract ideas apply to vector spaces of observables associated to first order, complex linear differential equations.

12. QUANTIZATION OF COMPLEX LINEAR QUANTUM FIELDS

Let V be a complex vector space. Let M be the space-time manifold, i.e. R^4. Denote points of M by x; their components by (x_μ). Set: $\partial_\mu = \frac{\partial}{\partial x_\mu}$. Let Γ denote the space of rapidly decreasing maps $\gamma : M \to V$. Let $D : \Gamma \to \Gamma$ denote the following first order, constant-coefficient differential operator:

$$D(\gamma) = X^\mu \partial_\mu(\gamma) + X(\gamma), \qquad 12.1$$

where (X^μ, X) are complex-linear transformations of V into itself.

(Note that we are changing the notations slightly; e.g. the Γ here plays the role of Γ_c in Section 11. We shall attempt to work with complex vector spaces and their "complex" quantizations directly, instead of working via skew-Hermitian fields).

We shall then attempt to quantize Γ, via a complex-linear transformation

$$\phi : \Gamma \to \text{(operators on a Hilbert space H)},$$

such that:

$$\phi(D(\Gamma)) = 0. \qquad 12.2$$

(Note that ϕ is what was called ϕ_c in Section 11; we can leave the subscript "c" off, since we shall not deal with Hermitian fields in this section.)

Our first problem is to define an anti-linear map $A : \Gamma \to \Gamma$ such that:

$$A(D(\Gamma))) \quad D(\Gamma). \qquad 12.3$$

We shall not seek the most general solution to this problem, but shall look for A of the following form:

$$A(\gamma)(x) = A(\gamma(x)) \qquad 12.4$$

where A is an anti-linear transformation: $V \to V$ which satisfies 12.8. (For the sake of simplicity of notation, we use "A" both as a linear transformation on Γ and on V). In order to satisfy 12.3, we shall suppose that:

$$AD = \pm DA. \qquad 12.5$$

Using 12.1, we see that the conditions for 12.5 may be written in the following form:

$$AX^{\mu} = \pm X^{\mu}A \qquad 12.6$$

$$AX = \pm XA \,. \qquad 12.7$$

Let us also assume that the differential operator D is irreducible in the sense defined in Chapter VIII of VB, Vol. I, i.e. that the Lie algebra of linear transformations on V generated by real linear combinations of the X^{μ} and X and their commutators acts irreducibly on V. Denote this Lie algebra by $\underset{\sim}{G}$. V is finite dimensional, it follows by a theorem due to E. Cartan (see VB, Vol. II, p.129) that $\underset{\sim}{G}$ is semisimple, or is the direct sum of a semisimple ideal and a one-dimensional center. Thus, A^2 commutes with $\underset{\sim}{G}$.
By Schur's lemma,

$$A^2 = \text{scalar multiple of identity} \;.$$

Then, we can change A by a scalar multiple so that:

$$A^2 = \pm 1. \qquad 12.8$$

In VB, Vol. I, Chapter VIII, Section 7, I have discussed the existence and properties of such an A in case D was the Dirac equation, or had properties similar to it. Let us abstract from that discussion some general conditions.

Suppose that in addition to 12.6-12.7, the X^μ, X satisfy the following conditions:

There exists a complex-linear transformation

$$\delta : V \to V \text{ such that:}$$

$$\delta X^\mu = - X^\mu \delta \qquad 12.9$$

$$\delta X = X\delta . \qquad 12.10$$

<u>Remark</u>: In the case where D is a Lorentz invariant differential operator, such a δ may be constructed - as indicated in VB, Vol. I - by means of the "PT" operator of the complex Lorentz group.

Thus, we have the following alternative:

Either:

$$AX^\mu = X^\mu A;\ AX = XA;\ A^2 = 1 \qquad 12.11$$

or

$$\delta AX^\mu = X^\mu \delta A;\ \delta AX = -X\delta A,\ A^2 = 1 \qquad 12.12$$

or

$$AX^\mu = X^\mu A;\ AX = XA;\ A^2 = -1 \qquad 12.13$$

or

$$\delta AX^\mu = X^\mu \delta A;\ \delta AX = -X\delta A;\ A^2 = -1. \qquad 12.14$$

In case 12.11 is satisfied, the subspace

$$V_0 = \{v \in V : Av=v\}$$

is a real subspace of V, such that:

$$V = V_0 \oplus iV_0;\ X^{\mu}(V_0) \subset V_0;\ X(V_0) \subset V_0. \qquad 12.15$$

These conditions mean that, with Γ_0=set of maps:$R^4 \to V_0$, $D(\Gamma_0) \subset \Gamma_0$ and $\Gamma=\Gamma_0 \oplus i\Gamma_0$, and the quantization of (Γ,D) proceeds, as explained in previous sections, by "complexifying" a skew-Hermitian quantization of Γ_0. The other cases, 1212-14, involve more complicated algebra, which we shall not examine in detail here. (I hope to do this at a later point in this work). For example, if $(BA)^2=1$, notice that BA is an anti-linear involution of V which commutes with the X^{μ} but anti-commutes with X. One can then prove that with respect to a suitable basis of V, the X^{μ} are real matrices, while X is pure imaginary. This is the situation, for example, with the Dirac equation, where it is known that in the "Majorana form" the Dirac matrices can be taken to be real. The rest of the algebraic possibilities must be examined in a similar way.

Let us turn now to the question of the construction of a Hermitian symmetric form $\alpha:\Gamma\times\Gamma \to C$ which is suitable for setting up of "commutation" or "anti-commutation" relations for the fields $\phi(\Gamma)$. For this purpose, it is convenient to introduce the Fourier transform. Let $\hat{M}$ denote the real vector of relativistic momentum four-vectors; $p=(p_{\mu})$ denotes a typical point. For $x \in M$, $p \in \hat{M}$, set:

$$x \cdot p = g_{\mu\nu}x_{\mu}p_{\nu} \equiv x_0p_0-x_1p_1-x_2p_3-x_4p_4.$$

Let $\hat{M}$ denote the space of rapidly decreasing C^{∞} mappings $\hat{\gamma}:\hat{M} \to V$. For $\gamma \in \Gamma$, denote its Fourier transform by $\hat{\gamma}$, defined as follows:

$$\hat{\gamma}(p) = \frac{1}{(2\pi)^2}\int_{R^4} e^{-ip\cdot x}\gamma(x)dx \ . \qquad 12.16$$

Let $\hat{D}:\hat{\Gamma} \to \hat{\Gamma}$ denote the Fourier transform of D:

$$\hat{D}(\hat{\psi})(p) = X(p)\hat{\psi}(p) + X\hat{\psi}(p), \qquad 12.17$$

with:

$$A(p) = ig_{\mu\nu}p_{\mu}X^{\nu} \qquad 12.18$$

($g_{\mu\nu}$ is the Lorentz metric tensor).

Because the Fourier transform sets up an isomorphism between Γ and $\hat{\Gamma}$, $\Gamma/D(\Gamma)$ is isomorphic to $\hat{\Gamma}/D(\hat{\Gamma})$. Let N be the subset of M defined as follows:

$$N = \{p \in M:X(p)+X)(V)\neq V\}. \qquad 12.19$$

Let $\hat{A}$ denote the Fourier transformation of A acting on Γ, where A arises from an anti-linear map:$V \to V$ satisfying 12.6-12.7. From 12.14, we have:

$$\hat{A}(\hat{\gamma})(p) = A(\hat{\gamma}(-p)). \qquad 12.20$$

Let d_Np denote a measure on N. For each $p \in N$, let $\alpha_p:V\times V \to C$ be a Hermitian symmetric form such that:

$$\alpha_p((X(p)+X)(V),V) = 0. \qquad 12.21$$

For $\hat{\gamma}_1, \hat{\gamma}_2 \in \hat{\Gamma}$, set:

$$\alpha(\hat{\gamma}_1,\hat{\gamma}_2) = \int_N \alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))d_Np. \qquad 12.22$$

Then, α defined by 12.21 is a Hermitian symmetric form on $\hat{\Gamma}$, hence, via the isomorphism 12.14, on Γ. Let us impose the key condition 11.17, which links the "charge conjugation" operation to the commutation relations. Combining 12.19 and 12.21, we see that 11.17 implies the following condition:

$$\alpha(\hat{A}\hat{\gamma}_1,\hat{A}\hat{\gamma}_1) = -\alpha(\hat{\gamma}_2,\hat{\gamma}_1),$$

or

$$\int_N \alpha_p(A\hat{\gamma}_1(-p),A\hat{\gamma}_2(-p))d_Np$$

$$= \int_N -\alpha_p(\hat{\gamma}_2(p),\hat{\gamma}_1(p)d_Np.$$

Now, let N_+, N_- be the following subsets of N:

$$N_+ = \{p \in N:p_0>0\} \qquad 12.23$$

$$N_- = \{p \in N:p_0<0\}. \qquad 12.24$$

Let us suppose that $N=N_+\cup N_-$, and that the "total reflection" or "PT" map $p \to -p$ preserves N, maps N_+ onto N_-, and preserves the measure d_Np.
Then, 12.22 takes the following form:

$$\int_N \alpha_{-p}(A\hat{\gamma}_1(p),A\hat{\gamma}_2(p))d_Np$$

$$= \int_N -\alpha_p(\hat{\gamma}_2(p),\hat{\gamma}_1(p))d_Np. \qquad 12.25$$

Let us assume that 12.25 is true because the following relation is true:

$$\alpha_p(v_2,v_1) = -\alpha_{-p}(Av_1,Av_2) \quad . \qquad 12.26$$

for all $p \in N$.

Notice that relation 12.26 determines the form α_p for $p \in N^-$

in terms of α_p for $p \in N^+$. Thus, we have; for γ_1, $\gamma_2 \in \Gamma$,

$$\alpha(\gamma_1,\gamma_2) \equiv \alpha(\hat{\gamma}_1,\hat{\gamma}_2)$$

$$= \int_{N_+}(\alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))+\alpha_{-p}(\hat{\gamma}_1(-p),\hat{\gamma}_2(-p))d_N p$$

$$= \int_{N_+}(\alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))-\alpha_p(A\hat{\gamma}_2(-p),A\hat{\gamma}_1(-p))d_N p$$

$$= \int_{N_+}[\alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))-\alpha_p(\hat{A}(\hat{\gamma}_2)(p),\hat{A}(\hat{\gamma}_1)(p))]d_N p. \qquad 12.27$$

Formula 12.26 is the key to understanding how "anti-particles" are introduced when Γ is quantized. Suppose that we set:

$$\alpha_1(\gamma_1,\gamma_2) = \int_{N_+}\alpha_p(\hat{\gamma}_1(p),\hat{\gamma}_2(p))d_N p \qquad 12.28$$

$$\alpha_2(\gamma_1,\gamma_2) = -\alpha_1(A\gamma_2,A\gamma_1). \qquad 12.29$$

Then, we have:

$$\alpha(\gamma_1,\gamma_2) = \alpha_1(\gamma_1,\gamma_2)+\alpha_2(\gamma_1,\gamma_2). \qquad 12.30$$

Let us suppose that H_1, H_2 are Hilbert spaces, and that ϕ_1, ϕ_2 are complex-linear mappings: $\Gamma \to$ (operators on H_1,H_2) such that:

$$\begin{aligned}
&[\phi_1(\gamma_1),\phi_1(\gamma_2)] = 0\\
&[\phi_1(\gamma_1)^*,\phi_1(\gamma_2)] = \alpha_1(\gamma_1,\gamma_2)\\
&[\phi_2(\gamma_1),\phi_2(\gamma_2)] = 0\\
&[\phi_2(\gamma_1)^*,\phi_2(\gamma_2)] = \alpha_2(\gamma_1,\gamma_2)
\end{aligned} \qquad 12.31$$

for $\gamma_1,\gamma_2 \in \Gamma$.

Now, set:

$$H = H_1 \otimes H_2. \qquad 12.32$$

$$\phi(\gamma) = \phi_1(\gamma) \otimes 1 + 1 \otimes \phi_2(\gamma). \qquad 12.33$$

Exercise. With ϕ defined by 12.32, and with commutation relations 12.30, show that the commutation relations 11.14 are satisfied (with ϕ_c used in 11.14 identified with ϕ).
In particular, 12.31-12.32 shows that the quantum field $\phi(\Gamma)$ is built up as a tensor product of two free non-interacting fields, namely ϕ_1 and ϕ_2. Let us call the states in H_1 "particles", and the states in H_2 their "anti-particles". Notice that the same construction applies in case one deals with the "anti-commutation" relations 11.15-16.
Problem. Show how the charge operator Q and the charge conjugation operator C may now be introduced into H, provided ϕ_1 and ϕ_2 are defined in the standard "Fock" way, with a unique vacuum state.
Exercise. Apply this formalism to the Dirac equation. Verify that, if it is quantized in this way (using anti-commutation relations, of course) the energy operator $\gamma \to i\partial_0(\gamma)$ has a positive spectrum.

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