Maciej Błaszak

Quantum versus Classical Mechanics and Integrability Problems

towards a unification of approaches and tools

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To Magdalen and Lucas

Preface

It is well known for physicists that in order to describe dynamical systems of finite number of degrees of freedom in the macro- and micro-scale, classical and quantum mechanics, respectively, were developed. They are among the best recognized physical theories whose correctness has been confirmed experimentally with high accuracy. The classical Hamiltonian mechanics has been developed since the first half of the nineteenth century, while quantum Hamiltonian mechanics has been developed since the first half of the twentieth century. Reviewing textbooks presenting both theories, the reader finds two fundamental inconsistencies concerning quantum theory versus the classical one. Both inconsistencies are related to mathematical language used in the quantum model. In classical mechanics, every measurable quantity (observable) is represented by a smooth real-valued function on a phase space, while in quantum mechanics every observable is represented by a self-adjoint operator in a Hilbert space over configuration space. It means that mathematical languages of both theories are drastically different. On the other hand, as both theories describe similar systems in the macro- and micro-scale, we expect that related formalisms should transform one into the other when we change the scale in both directions. So how to achieve it when the languages are not compatible? The second inconsistency is even worse. On the one hand, the modern classical Hamiltonian mechanics is formulated in a coordinate independent way, in the language of appropriate tensor fields. On the other hand, even contemporary textbooks of quantum mechanics formulate the theory mostly in Cartesian coordinates, like in the case of classical mechanics and classical electrodynamics in the nineteenth century. It is obvious that quantum mechanics requires a modern coordinate free formulation. One could say that the so-called geometric quantization fulfills that requirement at least to some extent. Although it is true, due to its mathematical complexity it was never adopted to the level of standard textbooks of quantum mechanics. We believe that the presented theory is an interesting alternative to the geometric quantization approach, strongly unifying formalisms from classical and quantum level.

Thus, the first aim of this book is a common, coordinate free formulation of classical and quantum Hamiltonian mechanics, in the frame of common mathematical language. In the presented formulation, quantum mechanics appears as an appropriate deformation of classical mechanics. It means that quantum formalism reduces to classical formalism when the deformation parameter tends to zero. What is more, the model presented in the book solves two inconsistencies mentioned above. To be more precise, presented formalism covers only the bosonic sector of classical and quantum Hamiltonian mechanics. The fermionic sector, involving Grassmann variables, although worthy of separate presentation, is beyond the scope of the book.

Obviously, the idea of deformation quantization is not new and was developed in many papers during the last few decades, but mainly by mathematicians for their own purposes. It was less appreciated by physicists. In this book, we formulate a coordinate free model of quantum bosonic Hamiltonian systems in Riemannian spaces, based on the mathematical idea of deformation quantization, as a complete physical theory with an appropriate mathematical accuracy.

The second aim of the book is related to the particular class of dynamical systems considered on both the classical and quantum level. It is well known that the number of classical and quantum problems which can be solved analytically, i.e., by quadratures, is very limited. So, it is very important to develop the theory which will allow for a deeper understanding of classical and quantum integrability. Thus, the second aim of the book is the presentation of the modern separability theory on both the classical and quantum level.

On the classical level, the Hamilton-Jacobi theory is one of the most powerful methods of integration by quadratures a wide class of systems described by nonlinear ordinary differential equations. The theory in question is closely related to the Liouville integrable Hamiltonian systems. The main difficulty in that approach is that it demands distinguished coordinates, so-called separation coordinates, to work effectively. In this book, we present a modern geometric separability theory, based on bi-Poissonian and bi-presymplectic representations of finite dimensional Liouville integrable systems. This approach leads to the construction of separation coordinates in a systematic way. For the sake of the physical interest, we mainly concentrate on the class of Hamiltonians quadratic in momenta.

We also develop the modern quantum separability theory. Actually, we present the formalism which allows us to find a separable quantization of quadratic in momenta classical separable systems. After such quantization, quantum stationary Schrödinger equations also separate and respective quantized constants of motion commute.

In order to make the text consistent and self-contained, we start from the compact overview of mathematical tools necessary for understanding the remaining part of the book. Moreover, because the book is dedicated mainly to physicists, despite its mathematical nature, we resigned from highlighting definitions, theorems, or lemmas. Nevertheless, all the claims presented are either proved or the reader is referred to the literature where the proof is available. There are two highlighted pieces of text. The first are examples, which illustrate the presented theory. The

second are observations, which contain the most important messages for the reader, resulting from the presented theory. We also boxed the most important formulas from each chapter.

Poznań, Poland Maciej Błaszak

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Chapter 1
Introduction

Classical and quantum Hamiltonian mechanics belong to the most important physical theories which are able to model with an incredible precision various physical processes which take place in the real world, from astronomical macro scale to atomic and molecular micro scale. Historically, classical Hamiltonian mechanics grew out from Newtonian (later on Lagrangian) mechanics, describing particle dynamics under influence of potential forces, in the form of second order ordinary differential equations (ODE's) in base Euclidian (Riemannian) space. Simple, n second order ODE's on a base space was replaced by $2n$ first order ODE's on a phase space, parametrized by n position coordinates and n momentum coordinates. In such formulation, the flow, governed by conserved total energy (classical Hamiltonian) of the system, represented particle dynamics on the phase space. Such Hamiltonians consisted of the kinetic part, quadratic in momenta, and the potential part, position dependent. Since then, the Hamiltonian mechanics has developed into an independent general theory allowing to describe a much wider class of dynamical systems than only particle dynamics on some configuration space (base space). Actually, it is a theory of Hamiltonian flows on Poisson manifolds M, governed by arbitrary smooth real valued functions (Hamiltonians) on M. In consequence, considered dynamical systems are subject to Poisson geometry. Obviously, for particular Poisson manifolds and particular Hamiltonians, one can adopt the Riemannian geometry to Hamiltonian formalism, regarding a Poisson manifold as a cotangent bundle to some Riemannian space and momentum part of Hamiltonian as defined by a respective metric tensor. Nevertheless, on the general level of the Hamiltonian formalism, there is no related Riemannian geometry and hence there is no configuration space where the dynamics could be transferred.

As a consequence, the most fundamental quantization procedure of classical Hamiltonian mechanics should take place on a Poisson manifold in the form of smooth deformation of a classical formalism. As a result, one should obtain a theory of quantum Hamiltonian flows on quantum Poisson manifolds, governed by quantum Hamiltonian functions. All these objects should be deformations of their

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classical counterparts. To be more precise, one should deform a Poisson algebra of classical observables with a Poisson bracket as a Lie bracket to a new Lie algebra with a Lie bracket being deformation of the Poisson bracket. As a consequence we present the reader the theory of quantum flows on a phase space being an appropriate deformation of the classical theory of flows on the same space.

On the other hand, in order to unify the languages of both theories, it is reasonable to begin on the classical level from a bit more extended formalism than the classical Hamiltonian mechanics, i.e. from the so called statistical Hamiltonian mechanics. There are a few advantages of such an extension. First of all, it is a more realistic theory as in any physical experiment the physical measurement is made with finite accuracy. Secondly, such an extension allows for introducing the concept of classical states in the form of probability distributions on the phase space. Obviously, after quantization we get quantum states in the form of pseudo-probability distributions on the phase space as well, being appropriate deformations of classical states. As a consequence, on the classical level one can introduce notions familiar from the quantum level, i.e. pure, mixed and coherent classical states, classical uncertainty relations, classical Schrödinger picture for time evolution of classical states and classical Heisenberg picture for time evolution of classical observables. As the deformation formalism presented in the book is coordinate independent, the obtained quantum Hamiltonian theory is also formulated in a coordinate free form.

For the distinguished class of systems, when we can adopt a Riemannian geometry on the classical level, we construct a Riemannian representation (generalized position representation) of general quantum formalism, where quantum observables are represented by self-adjoint operators acting in a Hilbert space over Riemannian space, with the measure induced by an appropriate metric tensor. In the particular case of Euclidian space and Cartesian coordinates, our representation is reduced to the standard formulation of quantum mechanics from textbooks. Obviously, as in Riemannian representation we also have a coordinate free formulation of quantization procedure, so we demonstrate how to properly quantize a given Hamiltonian system in arbitrary curvilinear coordinates.

The quantization procedures performed in the general setting have more advantages than the ones presented till now. Actually, it allows to control the types of admissible quantizations. Any admissible deformation of classical Poisson algebra is related to the particular quantization. Hence, from the mathematical point of view, we have infinitely many admissible quantizations of classical Hamiltonian systems. Each deformation has its own representation in a Hilbert space over phase space, where quantum observables are represented by an appropriate self-adjoint operators. Various quantizations differ from each other by different orderings of position and momentum operators. So, each deformation is in one to one correspondence with an appropriate ordering of operators of position and momenta. This is why the ordering formalism is treated with a particular attention. The same ordering structure is induced to Riemannian representation, if such exists. Now, which quantization properly describes quantum behavior of real physical systems can only be decided by physical experiments. On the other hand, our experimental experience

is mainly related to the flat configuration space, which is not enough to chose a unique quantization in the general case. In the book, for a given metric tensor, we construct a two-parameter family of admissible quantizations, containing in particular a majority of quantizations considered in literature and derived with the help of various approaches. All these quantizations are reduced to a standard Weyl quantization in the case of the flat configuration space and 'natural' Hamiltonian function being the sum of kinetic and potential parts.

A large part of the book is dedicated to the problem of integrability and in particular separability in Hamiltonian mechanics, which allows for integration of particular nonlinear ODE's on a classical level and particular linear PDE's on a quantum level. On the classical level we first define a class of so called Liouville integrable systems and then develop a theory of Stäckel transforms, which allows for systematic construction of new Liouville integrable systems from the old ones. What is interesting, the flows of Stäckel related systems are transformed into each other by an appropriate transformation between evolution parameters, depending on points from a phase space.

Besides, we present a modern version of Hamilton-Jacobi (HJ) separability theory of Liouville integrable systems, based on the so called separation relations. Such formulation allows to classify separable systems of Stäckel type. In order to apply the HJ method, one has to find a complete integral of the so called HJ equation for generating function of canonical transformation, necessary for further linearization of the considered system. This equation is in general nonlinear PDE, and the only effective method of solving this equation turns out to be the method of separation of variables, carried out in a distinguished coordinate system. In these particular coordinates HJ equation admits a complete integral in the form of a sum of functions depending on one variable only, determined by a set of first order ODE's, called separation equations. In the presented approach we show that for general separability, in order to find separation equations, one HJ equation is not sufficient and it is necessary to consider simultaneously all HJ equations generated by functionally independent constants of motion which are in involution. Only such a set of nonlinear PDE's, written in separation coordinates, can be transformed through purely algebraic operations into a set of separation equations, i.e. a set of nonlinear first order ODE's, each one of one variable. The class of systems for which, in order to find the separation equations a single HJ equation is sufficient, is very restrictive. Nevertheless, in standard textbooks on classical mechanics that particular case (which is a historical one) is the only case considered.

The main problem of the classical HJ theory is the construction of transformation to separation coordinates in which the method works effectively. For many decades there was no general theory allowing to identify the separation coordinates for a given Liouville integrable system. Only recently, at the turn of the twentieth and twenty-first century, a few constructive theories of separation coordinates have appeared. In this book we present two of them, related to particular geometric properties of Liouville integrable systems.

The first one is the so called bi-Hamiltonian separability theory. It enables to construct a transformation from original coordinates to separation coordinates and to derive appropriate separation relations, directly linked to the searched separation equations. It was clear from the very beginning that in order to find a transformation to separation coordinates for Liouville integrable systems, some extra information is required. In this case it is a bi-Hamiltonian representation of the considered system. It means that a given vector field has two different Hamiltonian representations in the same coordinate system. Actually, there exist two different Hamiltonians (constants of motion) and two different Poisson bi-vectors which define the same vector field. In general, both Poisson bi-vectors are degenerate. When the considered vector field is defined on symplectic manifold it means that there exists its extension to Poisson manifold where bi-Hamiltonian formulation is available. The construction of separation coordinates is related with the projection of the second Poisson structure onto symplectic leaves of the first Poisson structure. Thus, the reduction theory for Poisson tensor fields is required and so is presented in the book with particular care. Once we have reduced both Poisson tensors into a symplectic leaf of the first one, we construct the so called recursion operator being the product of symplectic form related to the first reduced Poisson bi-vector with the second reduced bi-vector. It is a second order tensor field of $(1, 1)$ type. Its eigenvalues define the first half of separation coordinates, while conjugate momenta represent the second half of separation coordinates.

The second separability theory presented in the book is a bi-presymplectic (bi-inverse-Hamiltonian in particular) theory. It is an alternative geometric approach, based on the fact that the majority (possible all) of Liouville integrable systems on symplectic manifold admit the extension to higher dimensional presymplectic manifold, where there exists an additional presymplectic two-form such that the differential of Hamiltonian has two different inverse-Hamiltonian representations in the same coordinate frame. Now, the construction of separation coordinates relies on the restriction of both presymplectic two-forms to the original manifold where we have now two symplectic two-forms and a related recursion operator. The further procedure leading to separation coordinates is analogous as in the bi-Hamiltonian model.

The universality of both, mentioned above, separability theories relies on the fact that any separable system (Stäckel system), defined by a separation curve, has bi-Hamiltonian and bi-inverse-Hamiltonian extensions.

On the quantum level, by quantum separability we understand the integration by quadratures of a stationary Schröedinger equation (Helmholtz equation in the case of arbitrary metric tensor). Actually, in separation coordinates each eigenfunction of quantum Hamiltonian multiplicatively separates into a product of one-dimensional functions and in consequence, a multi-dimensional eigenvalue problem splits into an appropriate number of one-dimensional problems. In this book we present a modern quantum separability theory being a generalization of Roberson and Eisenhart approach. As the result we prove that for any classical Stäckel system with all constants of motion quadratic in momenta and for which Stäckel matrix consists of monomials in position coordinates, there exist infinitely many quantizations,

parametrized by n arbitrary functions, each of one variable, that turn this system into a quantum separable Stäckel system. Moreover, separable quantizations are constructed in explicit form.

The book is composed of eight chapters and each chapter is divided into sections and then subsections.

Chapter 2 presents a brief survey of differential calculus, necessary for further considerations. We review the concept of tensor fields over finite dimensional manifolds. Exterior algebras of forms and multi-vectors, important in Hamiltonian theory are presented with a special care. Then the transformation properties of tensor fields via push-forward and pull-back given by an arbitrary local diffeomorphisms are derived. Next, the theory of Lie transport and Lie derivatives of various tensor fields is presented. A subsequent section is devoted to a linear connection and covariant derivatives of tensor fields. Some important formulas, necessary for the theory developed, are collected there. Finally, in the last section, the concept of symplectic manifolds and symplectic connections is briey sketched.

In Chap. 3 a particle dynamics on Riemannian space, its variational construction and the standard Hamiltonian representation is briefly reminded to the reader. The particle representation of higher order variational problems is also presented. In the next section, a coordinate free formulation of Hamiltonian mechanics on Poisson manifold and inverse-Hamiltonian mechanics on presymplectic manifold is systematically derived and then related to each other. The notion of Hamiltonian flows on symplectic manifold is discussed. Then the extension of classical Hamiltonian mechanics to statistical Hamiltonian mechanics is presented, allowing to introduce on a classical level the notions of states, Schröedinger and Heisenberg representations of classical dynamics, ucertainity relations and other notions familiar to the reader from quantum level. Finally, in the last section, a geometric reduction theory of Poisson bi-vectors onto submanifolds is discussed with a great care.

Chapter 4 is dedicated to Liouville integrable systems with particular attention to the ones that are separable. First, the notion of Liouville integrability (superintegrability in particular) is introduced and recently formulated theory of multi-parameter Stäckel transforms is discussed, allowing for the construction of new integrable (superintegrable) systems from the old ones in any coordinate frame. Then the modern formulation of Hamilton-Jacobi theory is presented, built on the notion of separation relations as the fundamental objects. A particular attention is paid to separable systems (Stäckel systems) with all constants of motion being quadratic in momenta. Such systems are classified, adopted to Riemannian (pseudo-Riemannian in general) geometry and then integrated by quadratures. Finally, the explicit form of Stäckel transforms relating systems from different classes as well as systems from the same class of presented classification is constructed.

In Chap. 5 are described two geometric separability theories allowing for explicit construction of separable coordinates for considered Liouville integrable system. First, bi-Hamiltonian theory is presented. Bi-Hamiltonian chains of vector fields are defined on bi-Poisson manifold and the formalism of their reduction to quasi-bi-Hamiltonian chains on bi-symplectic submanifold is discussed. Then the construction of separation coordinates and separation relations is presented in

details. Second, a dual to bi-Hamiltonian, i.e. bi-presymplectic separability theory is described. Bi-inverse-Hamiltonian chains of closed one-forms are defined on bi-presymplectic manifold and the formalism of their reduction, again to quasi-bi-Hamiltonian chains on bi-symplectic submanifold, is demonstrated. A separate section is dedicated to bi-Hamiltonian and bi-presymplectic systems on $\mathbb{R}^3$. In the last section is considered a subclass of Stäckel systems whose geodesic parts are defined by Killing tensors of some flat metrices. In that case, particularly important from the physical point of view, a transformation from separation coordinates to flat coordinates is derived in explicit form for all admissible flat metrices.

In Chap. 6 formalism of quantum deformations of classical Poisson algebra of smooth complex-valued functions on Poisson manifold is discussed. The main attention is paid to symplectic case when the phase space is a cotangent bundle of some pseudo-Riemannian manifold. A class of isomorphic star-algebras is constructed in the form of appropriate deformations of commutative algebra of functions on a phase space. For every star-algebra a Lie bracket in the form of star-commutator represents a quantum Poisson bracket being an appropriate deformation of classical Poisson bracket. Moreover, it is explained why equivalent quantum Poisson algebras lead to non-equivalent quantizations of classical Hamiltonian systems. Finally, in the last section, operator representations in a Hilbert space over phase space is constructed for every quantum Poisson algebra considered in the previous section. As constructed representations are related to different ordering rules, so the general theory of orderings is also presented.

Chapter 7 contains the formulation of quantum Hamiltonian mechanics on a phase space. Quantum states over phase space are defined. Their time evolution is presented in the frame of Schröedinger picture and the time evolution of quantum observables is presented in the frame of Heisenberg picture. Quantum Hamiltonian equations of motion are represented by nonlinear (in general) ODE's from the space of star-functions. The set of solutions of quantum Hamiltonian equations defines a quantum flow on the phase space. Like in the classical case, each quantum trajectory from the quantum flow represents a one-parameter group of quantum symplectomorphisms (quantum canonical transformations), but contrary to classical case, the group multiplication differs from a simple composition of maps. Various cases of quantum trajectories are consider with particular care.

Finally, in Chap. 8, the situation when a particular Riemannian geometry is adopted to Hamiltonian dynamics is analyzed. In such a case, the class of quantizations on a phase space has a Riemannian (position) representation in a form of self-adjoint operators acting in a Hilbert space over respective configuration space (Riemannian space). A two-parameter family of admissible quantizations in curved configuration space is constructed in a coordinate free way. It allows to quantize any classical system in arbitrary coordinate frame according to various quantization procedures. A simple but instructive example of quantization of hydrogen atom directly in spherical coordinates is presented. The last section of that chapter is dedicated to modern quantum separability theory. First, quantum integrability and quantum stationary separability are defined. Then, it is proved that for all considered classically separable systems with constants of motion quadratic in momenta, there

always exists a distinguished family of metric tensors and related quantizations which preserve separability on a quantum stationary level, i.e. eigenfunctions of respective Hamiltonian operator separate multiplicatively onto functions of one variable and multi-dimensional eigenvalue problem splits into one-dimensional problems. The problem of quantum superintegrability and quantum R-separability is also discussed.

Chapter 2
Basic Mathematical Tools

In this chapter we briefly discuss some elements of differential calculus which are important for understanding the content of this book. The reader who is familiar with the theory of tensor fields, Riemannian geometry and symplectic (Poisson) geometry can skip that part, keeping in mind that all important formulas of these formalisms are collected in this chapter. The reader who is less familiar with these mathematical tools will find here necessary knowledge presented in a compact form. For a more comprehensive treatment of the subject we refer the reader to the literature [1, 71, 116, 178, 255, 258].

2.1 Linear Tensor Algebra

Let V has an algebraic structure of a *linear space* of dimension $\dim V = n$, with *vectors* as elements. Denote by V^* a set of linear forms (linear maps) $\alpha : V \longrightarrow \mathbb{R}$, where $\mathbb{R}$ are real numbers, which form a linear space in its own right, called *dual space*, also of dimension $\dim V^* = n$. Its elements are called *covectors* (1-*forms*):

$$\alpha : \quad v \longrightarrow \alpha(v) \in \mathbb{R}, \quad v \in V, \ \alpha \in V^*.$$

There is no distinguished canonical isomorphism between V and V^*, but there is such an isomorphism between V and $(V^*)^*$:

$$f : V \longrightarrow (V^*)^* \text{ such that } f(v)(\alpha) := \alpha(v).$$

So, both vectors and 1-forms can be treated equally as linear maps:

$$V \ni v \longrightarrow v(\alpha) = \langle \alpha, v \rangle \in \mathbb{R} : V^* \longrightarrow \mathbb{R},$$

$$V^* \ni \alpha \longrightarrow \alpha(v) = \langle \alpha, v \rangle \in \mathbb{R} : V \longrightarrow \mathbb{R},$$

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where the bilinear map

$$\langle .,. \rangle : V^* \times V \longrightarrow \mathbb{R} \tag{2.1.1}$$

is known as a *duality map*, pairing V and V^*.

A tensor of type (r, s) on V is called a multilinear map

$$\mathcal{T} : \underbrace{V^* \times \ldots \times V^*}_{r} \times \underbrace{V \times \ldots \times V}_{s} \longrightarrow \mathbb{R},$$

$$(v_1, \ldots, v_r, \alpha_1, \ldots, \alpha_s) \longrightarrow T(v_1, \ldots, v_r, \alpha_1, \ldots, \alpha_s) \in \mathbb{R},$$

$$T(\ldots, v + \lambda w, \ldots) = T(\ldots, v, \ldots) + \lambda T(\ldots, w, \ldots).$$

The set of (r, s) tensors will be denoted by $\mathcal{T}^{(r,s)}(V)$ and its elements are called *tensors* which are r-times covariant and s-times contravariant, wherein $\mathcal{T}^{(0,0)}(V) := \mathbb{R}$. The set $\mathcal{T}^{(r,s)}(V)$ is also a linear space in its own right. How to calculate values of various tensors using the duality map? In order to do it we have to introduce the notion of tensor product.

The *tensor product* $\otimes$ of linear spaces $\mathcal{T}^{(r,s)}(V)$ and $\mathcal{T}^{(p,q)}(V)$ we call a map

$$\boxed{\otimes : \mathcal{T}^{(r,s)}(V) \times \mathcal{T}^{(p,q)}(V) \longrightarrow \mathcal{T}^{(r+p,s+q)}(V),}$$

such that

$$\begin{aligned}(A \otimes B)(v_1, \ldots, v_s, w_1, \ldots, w_q, \alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_q) \\ = A(v_1, \ldots, v_s, \alpha_1, \ldots, \alpha_r) B(w_1, \ldots, w_q, \beta_1, \ldots, \beta_q),\end{aligned}$$

where $A \in \mathcal{T}^{(r,s)}(V)$ and $B \in \mathcal{T}^{(p,q)}(V)$. As a consequence, the linear spaces $\mathcal{T}^{(r,s)}(V)$ can be represented by an appropriate tensor product

$$\boxed{\mathcal{T}^{(r,s)}(V) = \underbrace{V^* \otimes \ldots \otimes V^*}_{r} \otimes \underbrace{V \otimes \ldots \otimes V}_{s}} \tag{2.1.2}$$

and hence any tensor $T^{(r,s)}(V)$ is of the respective form

$$T^{(r,s)} = \gamma_1 \otimes \ldots \otimes \gamma_r \otimes z_1 \otimes \ldots \otimes z_s, \quad \gamma_i \in V^*, \ z_i \in V,$$

so

$$T^{(r,s)}(v_1, \ldots, v_r, \alpha_1, \ldots, \alpha_s) = \langle \gamma_1, v_1 \rangle \ldots \langle \gamma_r, v_r \rangle \langle \alpha_1, z_1 \rangle \ldots \langle \alpha_s, z_s \rangle .$$

Let $\{e_j\}$ be a basis in V and $\{e^j\}$ so called dual basis in V^* defined by

$$\left\langle e^k, e_j \right\rangle = \delta^k_j, \quad k, j = 1, \ldots, n.$$

For arbitrary vector $v \in V$ and covector $\alpha \in V^*$

$$v = v^k e_k, \quad \alpha = \alpha_k e^k,$$

where Einstein summation convention is used. From (2.1) follows that

$$\langle \alpha, v \rangle = v^i \alpha_i, \quad v^k = \left\langle e^k, v \right\rangle, \quad \alpha_i = \langle \alpha, e_i \rangle .$$

So, components of the tensor $T^{(r,s)} \in \mathcal{T}^{(r,s)}(V)$, in a given basis, are of the form

$$\begin{aligned} T^{(r,s)}(e_{k_1}, \ldots, e_{k_r}, e^{j_1}, \ldots, e^{j_s}) &= (\gamma_1 \otimes \ldots \otimes \gamma_r \otimes v_1 \otimes \ldots \otimes v_s)(e_{k_1}, \ldots, e^{j_s}) \\ &= \langle \gamma_1, e_{k_1} \rangle \ldots \left\langle e^{j_s}, v_s \right\rangle = T^{j_1 \ldots j_s}_{k_1 \ldots k_r}, \end{aligned}$$

while the tensor itself is as follows

$$\boxed{T^{(r,s)} = T^{j_1 \ldots j_s}_{k_1 \ldots k_r} e^{k_1} \otimes \ldots \otimes e^{k_r} \otimes e_{j_1} \otimes \ldots \otimes e_{j_s}.}$$

A direct sum of all linear spaces $\mathcal{T}^{(r,s)}(V)$

$$\mathcal{T}(V) := \bigoplus_{r,s=0}^{\infty} \mathcal{T}^{(r,s)}(V)$$

constitutes noncommutative, associative algebra with respect to tensor multiplication.

Example 2.1 Let us consider second order tensors

$$T^{(0,2)} = z_1 \otimes z_2, \quad T^{(2,0)} = \gamma_1 \otimes \gamma_2, \quad T^{(1,1)} = \gamma \otimes z, \qquad z_1, z_2, z \in V, \quad \gamma_1, \gamma_2, \gamma \in V^*,$$

then

$$T^{(0,2)}(\alpha_1, \alpha_2) = \langle \alpha_1, z_1 \rangle \langle \alpha_2, z_2 \rangle , \quad T^{(2,0)}(v_1, v_2) = \left\langle \gamma_1, v_1 \right\rangle \left\langle \gamma_2, v_2 \right\rangle ,$$

$$T^{(1,1)}(\alpha, v) = \langle \alpha, z \rangle \langle \gamma, v \rangle ,$$

where $v_1, v_2, v \in V$, $\alpha_1, \alpha_2, \alpha \in V^*$. In our basis we have

$$T^{(0,2)} = z_1 \otimes z_2 = z_1^i z_2^j e_i \otimes e_j = T^{ij} e_i \otimes e_j,$$
$$T^{(2,0)} = \gamma_1 \otimes \gamma_2 = \gamma_{1i}\gamma_{2j} e^i \otimes e^j = T_{ij} e^i \otimes e^j,$$
$$T^{(1,1)} = \gamma \otimes z = \gamma_j z^i e^j \otimes e_i = T^i_j e^j \otimes e_i.$$

The operation producing tensors from tensors is said to be a tensor operation. So far we have at our disposal a linear combination of tensors and a tensor product of tensors. Another important tensor operation is *contraction* defined as follows

$$C : T^{(r,s)}(V) \longrightarrow T^{(r-1,s-1)}(V), \quad T \longrightarrow C(T) := T(\ldots, e_k, \ldots, e^k, \ldots),$$

where the pairing (2.1.1) is applied to the ith V^* factor and the jth V factor in (2.1.2).

Some tensors will play an important role in our further considerations. One of them is so called *metric tensor*. It is a second order covariant tensor $g \in \mathcal{T}^{(2,0)}(V)$, which is symmetric and non-degenerate, i.e.

$$g(v, w) = g(w, v) \qquad \text{symmetric},$$

$$\bigwedge_{w \in V} g(v, w) = 0 \Longrightarrow v = 0, \quad \text{non-degenerate: } \det(g_{ij}) \neq 0.$$

Sometimes one demands stronger requirement, namely to be positive definite

$$\bigwedge_{v \in V} g(v, v) \geq 0.$$

Metric tensors which are not positive definite are said to be *pseudo-metric tensors*. As it is well known from the linear algebra, any nondegenerate symmetric matrix takes in a suitable basis a canonical form

$$g = diag(\underbrace{1, \ldots, 1}_{n_+}, \underbrace{-1, \ldots, -1}_{n_-}). \tag{2.1.3}$$

Then, one says that the metric tensor g has signature (n_+, n_-).

Metric tensor defines a *scalar product* in V

$$v \cdot w := g(v, w) = g_{ij} v^i w^j. \tag{2.1.4}$$

A linear space over complex numbers with scalar product (2.1.4) is called a *unitary space*. A basis in which the metric tensor g takes the canonical form (2.1.3) is called an orthonormal basis.

In a given basis denote the matrix (G^{ij}) as the inverse matrix to (g_{ij}), i.e.

$$g_{ik}G^{kj} = \delta_i^j.$$

G^{ij} constitute the components of contravariant metric tensor $G \in \mathcal{T}^{(0,2)}(V)$.

Consider the maps

$$\begin{array}{ll} V^* \to V: & \alpha \to G\alpha := G(\alpha, \cdot) \\ V \to V^*: & v \to gv := g(v, \cdot) \end{array} \tag{2.1.5}$$

They are inverses to each other and define isomorphism between linear spaces V and V^*, which in arbitrary basis takes the form

$$\alpha_i = g_{ij}v^j, \quad v^i = G^{ij}\alpha_j, \quad v \in V,\ \alpha \in V^*.$$

2.2 Tensor Fields

Let Q be a smooth manifold. In many applications we will consider a very simple case when $Q = \mathbb{R}^n$. Then, let $\mathcal{F}(Q) := \{f : Q \longrightarrow \mathbb{R}\}$ be a set of smooth functions on Q. $\mathcal{F}(Q)$ constitutes a commutative and associative algebra with respect to point (local) multiplication. For an open set $U \subset Q$, homeomorphism

$$\varphi : U \longrightarrow \mathbb{R}^n(x^1, \ldots, x^n)$$

is called a chart of local coordinates. In local coordinates

$$f_\varphi \equiv f \circ \varphi^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}$$

will be identified with f.

Let T_xQ denote the tangent space to Q in a point $x \in Q$. T_xQ has a structure of a linear space. Moreover, let T_x^*Q denote the dual space. Define the sets TQ and T^*Q over Q as

$$TQ := \bigcup_{x\in M} T_xQ, \quad \bigwedge_{x\in M} T_xQ = V,$$

$$T^*Q := \bigcup_{x\in M} T_x^*Q, \quad \bigwedge_{x\in M} T_x^*Q = V^*,$$

i.e. all vectors at all points $x \in Q$ are regarded as points of a new set TQ, while all covectors at all points $x \in Q$ are regarded as points of a new set T^*Q, with natural

surjective maps

$$\pi : TQ \to Q, \quad T_xQ \ni v \to x,$$
$$\tau : T^*Q \to Q, \quad T_x^*Q \ni p \to x.$$

TQ and T^*Q have a natural structure of smooth manifolds induced by differential structure (maximal atlas) on Q. Besides, if $\varphi : Q \supset U \longrightarrow \mathbb{R}^n(x^1, \ldots, x^n)$ is a local chart on U, where x^i are local coordinates, then on the domain $\tilde{U} := \pi^{-1}(U) \subset TQ$ one can introduce *canonical coordinates* as follows. If $v \in \tilde{U} \Longrightarrow v(x) \in T_xQ$ for $x \in U$ then

$$v = v^i \frac{\partial}{\partial x^i} \mid_x = v^i(x) \frac{\partial}{\partial x^i}, \tag{2.2.1}$$

so

$$\tilde{\varphi} : TQ \supset \tilde{U} \longrightarrow \mathbb{R}^{2n}(x^1, \ldots, x^n, v^1(x), \ldots, v^n(x))$$

is a local chart with canonical coordinates $(x^1, \ldots, x^n, v^1(x), \ldots, v^n(x))$. In a similar way, for the domain $\tilde{U} := \tau^{-1}(U) \subset T^*Q$

$$\tilde{\varphi} : T^*Q \supset \tilde{U} \longrightarrow \mathbb{R}^{2n}(x^1, \ldots, x^n, p_1(x), \ldots, p_n(x))$$

is also a local chart with canonical coordinates $(x^1, \ldots, x^n, p_1(x), \ldots, p_n(x))$, where $p(x) \in T_x^*Q$. Its decomposition with respect to the coordinate basis reads

$$p = p_i dx^i \mid_x = p_i(x) dx^i. \tag{2.2.2}$$

For special cases, when $Q = \mathbb{R}^n$, or Q is a Lie group or Q is a set contractible to a point, then TQ (T^*Q) is diffeomorphic to $Q \times \mathbb{R}^n$ and represents trivial tangent (cotangent) *bundle,* while T_xQ and T_x^*Q are respective *fibers* over a point x. Locally, we always deal with trivial bundles.

The simplest tensor fields are:

1. scalar field: $f : Q \longrightarrow \mathbb{R}; \quad f(x) \in \mathbb{R},$
2. vector field: $v : Q \longrightarrow TQ; \quad v(x) \in T_xQ,$
3. covector field: $\alpha : Q \longrightarrow T^*Q; \quad \alpha(x) \in T_x^*Q.$

The dual map on a smooth manifold is defined by:

$$\boxed{\langle \alpha, v \rangle : T^*Q \times TQ \longrightarrow \mathcal{F}(Q); \quad \langle \alpha, v \rangle (x) \in \mathbb{R}.} \tag{2.2.3}$$

Notice, that $\langle \alpha, v \rangle$ is $\mathcal{F}(Q)$-bilinear mapping:

$$\bigwedge_{f \in \mathcal{F}(M)} \langle \alpha, v + fw \rangle = \langle \alpha, v \rangle + f \langle \alpha, w \rangle ,$$

$$\bigwedge_{f \in \mathcal{F}(M)} \langle \alpha + f\beta, v \rangle = \langle \alpha, v \rangle + f \langle \beta, v \rangle .$$

Thus, vectors and covectors constitute a linear space over $\mathbb{R}$, while vector fields and covector fields constitute a module over algebra $\mathcal{F}(Q)$.

Tensor fields of (r, s)-type can be treated as maps

$$\boxed{\mathcal{T}^{(r,s)}(Q) : \underbrace{T^*Q \times \ldots \times T^*Q}_{r} \times \underbrace{TQ \times \ldots \times TQ}_{s} \longrightarrow \mathcal{F}(Q),} \tag{2.2.4}$$

which are $\mathcal{F}(Q)$-linear in each argument. A direct sum

$$\mathcal{T}(Q) := \bigoplus_{r,s=0}^{\infty} \mathcal{T}^{(r,s)}(Q)$$

with tensor product $\otimes$ acting point-wise, constitutes the algebra of tensor fields on Q. The algebra is associative and noncommutative.

A vector field v over $U \subset Q$ ($v \in \tilde{U} \subset TQ$) in local coordinate chart $\varphi : U \longrightarrow \mathbb{R}^n(x^1, \ldots, x^n)$, according to (2.2.1), is of the form

$$v(x) = v^i(x)\partial_i \equiv v^i(x)\frac{\partial}{\partial x^i}, \qquad \left\{ \frac{\partial}{\partial x^i} \right\}_{i=1,\ldots,n} \text{the base (frame) in } \mathcal{T}^{(1,0)}(U),$$

where functions $v^i(x)$ represent vector field components. Alternatively, vector fields can be considered to be mappings in the algebra $\mathcal{F}(Q)$

$$v : \mathcal{F}(Q) \longrightarrow \mathcal{F}(Q),$$

$$\mathcal{F}(Q) \ni f \longrightarrow v(f) = v^i(x)\frac{\partial f}{\partial x^i} \in \mathcal{F}(Q)$$

of derivative type, i.e. such that are linear and fulfill the *Leibniz rule*

$$v(f + \lambda h) = v(f) + \lambda v(h), \qquad f, h \in \mathcal{F}(Q), \quad \lambda \in \mathbb{R}$$

$$v(fh) = v(f)h + fv(h).$$

A particular case of covector field over $U \subset Q$ is a differential df of function $f : Q \to \mathbb{R}$, which acts on a vector field v in the following way

$$\langle df, v\rangle = v(f).$$

Coordinate differentials $dx^i \in T^{(0,1)}(U)$ constitute a dual basis (coframe) of covector fields

$$\left\langle dx^i, \partial_j \right\rangle = \partial_j(x^i) = \delta^i_j,$$

so

$$\alpha(x) = \alpha_i(x) dx^i, \quad \alpha_i(x) = \langle \alpha, \partial_i \rangle ,$$

is consistent with (2.2.2), and in particular

$$\alpha = df = \frac{\partial f}{\partial x^i} dx^i.$$

The dual map in a chosen basis is of the form

$$\langle \alpha, v\rangle (x) = \alpha_i(x) v^i(x).$$

If $T(Q) \in \mathcal{T}^{(r,s)}(Q)$, then in a local basis on $U \in Q$

$$\boxed{T^{(r,s)}(U) = T^{j_1 \ldots j_s}_{k_1 \ldots k_r} dx^{k_1} \otimes \ldots \otimes dx^{k_r} \otimes \partial_{j_1} \otimes \ldots \otimes \partial_{j_s}.} \tag{2.2.5}$$

Notice, that second order tensor fields $\mathcal{T}^{(2,0)}(Q)$, $\mathcal{T}^{(0,2)}(Q)$, $\mathcal{T}^{(1,1)}(Q)$ can be treated as appropriate maps:

1. $\mathcal{T}^{(0,2)}(Q) : T^*Q \longrightarrow TQ$,
2. $\mathcal{T}^{(2,0)}(Q) : TQ \longrightarrow T^*Q$,
3. $\mathcal{T}^{(1,1)}(Q) : TQ \longrightarrow TQ$, $\mathcal{T}^{(1,1)}(Q) : T^*Q \longrightarrow T^*Q$.

Let

$$A^{(0,2)} = A^{ij} \partial_i \otimes \partial_j, \; A^{(2,0)} = A_{ij} dx^i \otimes dx^j, \; A^{(1,1)} = A^j_i dx^i \otimes \partial_j,$$

then, we will use the following notation

$$\begin{aligned} A^{(0,2)}\alpha = C(A^{(0,2)} \otimes \alpha) = v, \quad A^{(0,2)}(\beta, \alpha) = \left\langle \beta, A^{(0,2)}\alpha \right\rangle = \langle \beta, v\rangle , \\ A^{(2,0)}v = C(A^{(2,0)} \otimes v) = \alpha, \quad A^{(2,0)}(w, v) = \left\langle A^{(2,0)}v, w \right\rangle = \langle \alpha, w\rangle , \end{aligned} \tag{2.2.6}$$

$$A^{(1,1)}v = C(A^{(1,1)} \otimes v) = w, \quad A^{(1,1)}(v,\alpha) = \left\langle \alpha, A^{(1,1)}v \right\rangle = \langle \alpha, w \rangle ,$$

$$A^{(1,1)}\alpha = C(A^{(1,1)} \otimes \alpha) = \beta, \quad A^{(1,1)}(v,\alpha) = \left\langle A^{(1,1)}\alpha, v \right\rangle = \langle \beta, v \rangle ,$$

where for the tensor $A^{(1,1)}$ the first contraction is made over the contravariant index (the upper one) while the second contraction is made over covariant index (the lower one). Thus, later in the book, the symbols of second order tensors, like $A^{(0,2)}$ one, will denote the bilinear map in the notation $A^{(0,2)}(\alpha(x), \beta(x)) \in \mathbb{R}$ and simultaneously will denote the map between one-forms and vectors in the notation $A^{(0,2)}\alpha(x) = v(x)$, where $\alpha(x), \beta(x) \in T_x^*Q,\ v(x) \in T_xQ$.

Tensor fields on smooth manifolds are important objects, as they are our tools in many further considerations. What particular manifold we choose and what tensor fields it is endowed with depends on the physical context in which the tools will be used. In our case we mainly concentrate on pseudo-Riemannian manifolds, Poisson manifolds and symplectic manifolds. Although tensor fields themselves are coordinate free objects, in order to make any particular calculations we have to fix a coordinate chart. We also have to know the transformation rule for tensor field components when we pass from one coordinate frame to another one. Here we only recall the basic formulas. Consider a local coordinate transformation on $U \subset Q$

$$x^i \longrightarrow \bar{x}^i(x), \quad J^i_j(x) = \frac{\partial \bar{x}^i}{\partial x^j} \quad \text{Jacobian of the transformation.}$$

Then, one can show that

$$d\bar{x}^i = J^i_j(x)dx^j, \quad \bar{\alpha}_i(\bar{x}) = (J^{-1})^j_i(x)\alpha_j(x),$$

$$\bar{\partial}_i = (J^{-1})^j_i\partial_j, \quad \bar{v}^i(\bar{x}) = J^i_j(x)v^j(x),$$

$$\bar{T}^{i_1\ldots i_s}_{j_1\ldots j_r}(\bar{x}) = J^{i_1}_{k_1}(x)\ldots J^{i_s}_{k_s}(x)(J^{-1})^{l_1}_{j_1}(x)\ldots(J^{-1})^{l_r}_{j_r}(x)T^{k_1\ldots k_s}_{l_1\ldots l_r}(x).$$

In particular, in matrix notation

$$d\bar{x} = Jdx, \quad \partial = J^T\bar{\partial},$$

$$\bar{v} = Jv, \quad \bar{\alpha} = (J^{-1})^T\alpha,$$

$$\bar{A}^{ij} = J^i_r J^j_s A^{rs} \Longleftrightarrow \bar{A} = JAJ^T,$$

$$\bar{A}_{ij} = (J^{-1})^r_i(J^{-1})^s_j A_{rs} \Longleftrightarrow \bar{A} = (J^{-1})^T AJ^{-1},$$

$$\bar{A}^i_j = J^i_r(J^{-1})^s_j A^r_s \Longleftrightarrow \bar{A} = JAJ^{-1}.$$

Tensor field $g \in \mathcal{T}^{(2,0)}(Q)$ such that in arbitrary point P the tensor g is the metric tensor in T_PQ, is called the metric tensor field, and the pair (Q, g) is

called Riemann or *pseudo-Riemann space*. In the simplest case, when $Q = \mathbb{R}^n$, in orthonormal basis we have

$$g_{ij}(x) = \eta_{ij} = diag(\underbrace{1, \ldots, 1}_{n_+}, \underbrace{-1, \ldots, -1}_{n_-}).$$

In particular

$$(\mathbb{R}^n, \eta_{ij}) \equiv \mathbb{E}^{n_+, n_-} - \textit{pseudo-Euclidean space},$$
$$(\mathbb{R}^4, \eta_{ij}) \equiv \mathbb{E}^{1,3} - \textit{Minkowski space},$$
$$(\mathbb{R}^n, \delta_{ij}) \equiv \mathbb{E}^n - \textit{Euclidean space}.$$

A metric tensor field identify elements of tangent bundle with elements of cotangent bundle

$$g(v^i \partial_i) = v_i dx^i, \quad v_i = g_{ij} v^j,$$
$$G(\alpha_i dx^i) = \alpha^i \partial_i, \quad \alpha^i = G^{ij} \alpha_j, \quad G = g^{-1}.$$

2.3 Differential Forms and Multi-Vectors

There are two subclasses of tensor fields particularly important from the point of view of various physical theories (models). One is the class of totally antisymmetric and fully covariant tensor fields and the other is the class of totally antisymmetric and fully contravariant tensor fields. They are main tools in the construction of mathematical models of important physical processes. One of such models is Hamiltonian mechanics, the subject of a few chapters of our book. Here we only briefly review some basic facts about these objects, important for our further applications, referring the reader to standard textbooks for more details.

Differential k-form in a point $x \in Q$ is called k-linear antisymmetric map

$$\omega(x) : T_x Q \times \ldots \times T_x Q \longrightarrow \mathbb{R},$$

$$\omega(x)(\ldots v \ldots w \ldots) = -\omega(x)(\ldots w \ldots v \ldots).$$

It means, that k-forms can be identified with totally antisymmetric covariant tensor fields $T^{(k,0)}(Q)$. Let us denote the space of k-forms by $\Omega^k(Q)$. From definition 0-forms are scalar fields on Q, i.e. $\Omega^0(Q) \equiv \mathcal{F}(Q)$ and 1-forms are covectors, that is $\Omega^1(Q) = \mathcal{T}^{(1,0)}(Q)$. As from the antisymmetry of a given form follows that

$$\omega(x)(\ldots v \ldots v \ldots) = 0,$$

so, on n dimensional manifold Q, nontrivial k-forms are these for which $k = 0, \ldots, n$. So, a direct sum

$$\Omega(Q) := \bigoplus_{k=0}^{n} \Omega^k(Q) \tag{2.3.1}$$

constitutes a space of differential forms.

Any covariant tensor field $T^{(k,0)} \in \mathcal{T}^{(k,0)}(Q)$ can be antisymmetrized by means of antisymmetrization (alternating) operator $\mathcal{A}$: $\mathcal{A}T^{(k,0)}(Q) \in \Omega^k(Q)$

$$(\mathcal{A}T)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma} (sgn\, \sigma) T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}), \tag{2.3.2}$$

where σ are permutations (elements of symmetric group S_k) of indices $(1, \ldots, k)$ and $sgn\, \sigma$ means a number of transpositions building a given permutation σ. In local coordinates, components of the tensor $\mathcal{A}T$ are expressible by components of the output tensor T in the following way

$$\boxed{(\mathcal{A}T)_{j_1 \ldots j_k} \equiv T_{[j_1 \ldots j_k]} = \frac{1}{k!} \sum_{\sigma} (sgn\, \sigma) T_{\sigma(j_1) \ldots \sigma(j_k)}.} \tag{2.3.3}$$

For example, for $k = 2, 3$ we have

$$T_{[ab]} = \frac{1}{2!}(T_{ab} - T_{ba}),$$
$$T_{[abc]} = \frac{1}{3!}(T_{abc} + T_{cab} + T_{bca} - T_{bac} - T_{cba} - T_{acb}).$$

Notice that analogously to the above construction, any tensor $T^{(k,0)} \in \mathcal{T}^{(k,0)}(Q)$ can be symmetrized by means of symmetrization operator $\mathcal{S}$

$$(\mathcal{S}T)(v_1, \ldots, v_k) := \frac{1}{k!} \sum_{\sigma} T(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

In local coordinates, components of the tensor $\mathcal{S}T$ are expressible by components of the output tensor T in the following way

$$\boxed{(\mathcal{S}T)_{j_1 \ldots j_k} \equiv T_{(j_1 \ldots j_k)} = \frac{1}{k!} \sum_{\sigma} T_{\sigma(j_1) \ldots \sigma(j_k)}.} \tag{2.3.4}$$

For example, for $k = 2, 3$ we have

$$T_{(ab)} = \frac{1}{2!}(T_{ab} + T_{ba}),$$
$$T_{(abc)} = \frac{1}{3!}(T_{abc} + T_{cab} + T_{bca} + T_{bac} + T_{cba} + T_{acb}).$$

The operation which lowers the degree of a differential form is the *interior product* (or the Cartan product) w.r. to a vector field v. The interior product $i_v\omega$ of a k-form ω with a vector field v is a $(k-1)$-form defined by the relation

$$\boxed{i_v\omega(x)(v_1, \ldots, v_{k-1}) = \omega(x)(v, v_1, \ldots, v_{k-1}),} \tag{2.3.5}$$

with components

$$(i_v\omega)_{a\ldots b} = v^c\omega_{ca\ldots b}.$$

In particular, by definition, interior products of 0-form f, 1-form α and two-form ω are

$$i_v f = 0, \quad i_v\alpha = \alpha(v) = \langle\alpha, v\rangle\,, \quad \alpha \in \Omega^1(Q),$$

$$\begin{aligned}(i_v\omega)(v_1) &= \omega(v, v_1) = -\omega(v_1, v)\\ &\overset{(2.2.6)}{=} \langle -\omega v, v_1\rangle = \langle\alpha, v_1\rangle\,, \quad \omega \in \Omega^2(Q),\end{aligned}$$

and thus

$$i_v\omega = -\omega v = v^T\omega = \alpha.$$

The following properties of the interior product hold:

1. $i_v i_w = -i_w i_v \Longrightarrow (i_v)^2 = 0$,
2. $i_{v+\lambda w} = i_v + \lambda i_w$.

The operation which increases the degree of a differential form is an exterior differentiation. The *exterior derivative* of a k-form ω is a $(k+1)$-form $d\omega$ such that

$$\boxed{d\omega(x)(v_1, \ldots, v_{k+1}) = \sum_{i=1}^{k+1}(-1)^{i+1}\omega'(x)[v_i](v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k),} \tag{2.3.6}$$

where $\omega'(x)[v_i]$ denotes the derivative of a k-form ω into the direction of vector field v_i, given by

$$(\omega'(x)[v_i])_{j_1\dots j_k} = \frac{\partial \omega_{j_1\dots j_k}(x)}{\partial x^r}(v_i)^r$$

(see (2.5.7) and (2.5.8) for the case of arbitrary tensor fields). Thus d provides a map

$$d : \Omega^k(Q) \longrightarrow \Omega^{k+1}(Q).$$

Exterior derivative of 0-form $f(x)$, one-form $\alpha(x)$ and two-form $\omega(x)$ are given by

$$df(x)(v) = f'(x)[v] = \langle df, v\rangle(x),$$

$$\begin{aligned} d\alpha(x)(v_1, v_2) &= \alpha'(x)[v_1](v_2) - \alpha'(x)[v_2](v_1) \\ &= \langle \alpha'[v_1], v_2\rangle(x) - \langle \alpha'[v_2], v_1\rangle(x), \end{aligned}$$

$$\begin{aligned} d\omega(x)(v_1, v_2, v_3) &= \omega'(x)[v_1](v_2, v_3) - \omega'(x)[v_2](v_1, v_3) + \omega'(x)[v_3](v_1, v_2) \\ &= \omega'(x)[v_1](v_2, v_3) + \omega'(x)[v_2](v_3, v_1) + \omega'(x)[v_3](v_1, v_2) \\ &= \langle \omega'[v_1]v_2, v_3\rangle(x) + \langle \omega'[v_2]v_3, v_1\rangle(x) + \langle \omega'[v_3]v_1, v_2\rangle(x) \end{aligned}$$

and respectively in the coordinate basis

$$(df)_i = \partial_i f, \tag{2.3.7a}$$

$$(d\alpha)_{ij} = \partial_i \alpha_j - \partial_j \alpha_i, \tag{2.3.7b}$$

$$(d\omega)_{ijk} = \partial_i \omega_{jk} + \partial_k \omega_{ij} + \partial_j \omega_{ki}, \qquad \omega_{ij} = -\omega_{ji}. \tag{2.3.7c}$$

Generally, the component rule with respect to the coordinate basis is as follows

$$\boxed{(d\omega)_{i\dots jr} = (-1)^k (k+1)\omega_{[i\dots j,r]}, \qquad \omega \in \Omega^k(Q),}$$

where we used the notation $\omega_{i\dots j,r} \equiv \partial_r \omega_{i\dots j}$, and which agrees with formulas (2.3.7). The exterior derivative d is a linear map which is nilpotent

$$dd = 0 \Longrightarrow d(d\omega) = 0$$

and so, k-form ω with $d\omega = 0$ is called a *closed form* and k-form ω is called an *exact form* if there exists $(k-1)$-form η such that $\omega = d\eta$.

The space of differential forms $\Omega(Q)$ on a manifold endowed with the exterior derivative constitutes a complex

$$0 \to \Omega^0(Q) \stackrel{d^0}{\to} \Omega^1(Q) \stackrel{d^1}{\to} \ldots \stackrel{d^{n-1}}{\to} \Omega^n(Q) \stackrel{d^n}{\to} 0.$$

This complex is called *de Rham complex* of a manifold Q. As the composition of two adjacent maps dd gives zero, hence

$$\operatorname{Im} d^{k-1} \subset \ker d^k,$$

and moreover elements of $\ker d^k$ are closed k-forms while elements of $\operatorname{Im} d^{k-1}$ are exact k-forms. The quotient

$$H^k(Q) := \ker d^k / \operatorname{Im} d^{k-1}$$

is a linear space in its own right, called k-th *de Rhama cohomology group*. In a special case, when $\ker d^k = \operatorname{Im} d^{k-1}$, all closed forms are exact. Actually, according to Poincare Lemma, if Q is of the star shape ($\bigwedge_{x \in Q} \{\lambda x : 0 \leq \lambda \leq 1\} \subset Q$), or more generally Q is contractible to the point, then each closed k-form is exact. For example, it is the case for any manifold diffeomorphic to $\mathbb{R}^n$, or in particular for an open neighborhood of an arbitrary point x on an arbitrary, possibly non-contractible, manifold.

For any $r \geq 0$ and $s \geq 0$ such that $s + r \leq n$ there exists an exterior product of forms

$$\boxed{\wedge : \Omega^s(Q) \times \Omega^r(Q) \longrightarrow \Omega^{s+r}(Q), \quad (\omega, \eta) \longrightarrow \omega \wedge \eta,} \tag{2.3.8}$$

such that

$$\boxed{(\omega \wedge \eta)(x)(v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+s}) = \frac{(r+s)!}{r!s!} \mathcal{A}(\omega \otimes \eta)(x)(v_1, \ldots, v_r, v_{r+1}, \ldots, v_{r+s}),}$$

where $\mathcal{A}$ is an alternating operator (2.3.2), of the following properties

1. $\omega \wedge \eta = (-1)^{rs} \eta \wedge \omega$,
2. $(\omega \wedge \eta) \wedge \zeta = \omega \wedge (\eta \wedge \zeta)$,
3. $\omega \wedge (\eta + \zeta) = \omega \wedge \eta + \omega \wedge \zeta$,
4. $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^s \omega \wedge d\eta$,
5. $i_v(\omega \wedge \eta) = i_v \omega \wedge \eta + (-1)^s \omega \wedge i_v \eta$.

The space of differential forms $\Omega(Q)$ (2.3.1) together with the exterior product constitutes the so called *exterior algebra of forms*. We know from (2.2.5) that the tensors $dx^{s_1} \otimes \ldots \otimes dx^{s_k}$ constitute the basis of $\mathcal{T}^{(k,0)}(Q)$ which is of dimension n^k. Since the dimension of the subspace of k-forms is $\binom{n}{k}$, the induced basis

$dx^{s_1} \wedge \ldots \wedge dx^{s_k} := k!\mathcal{A}(dx^{s_1} \otimes \ldots \otimes dx^{s_k})$ is an admissible basis in $\Omega^k(Q)$. For example, basic 2-forms and 3-forms are as follows:

$$dx^i \wedge dx^j = dx^i \otimes dx^j - dx^j \otimes dx^i,$$

$$\begin{aligned} dx^i \wedge dx^j \wedge dx^k = {} & dx^i \otimes dx^j \otimes dx^k + dx^k \otimes dx^i \otimes dx^j + dx^j \otimes dx^k \otimes dx^i \\ & - dx^j \otimes dx^i \otimes dx^k - dx^k \otimes dx^j \otimes dx^i - dx^i \otimes dx^k \otimes dx^j. \end{aligned}$$

Hence, in such a defined basis, arbitrary two-form is represented by

$$\omega = \sum_{i>j} \omega_{ij} dx^i \wedge dx^j = \frac{1}{2!} \omega_{ij} dx^i \wedge dx^j,$$

and arbitrary 3-form by

$$\omega = \sum_{i>j>k} \omega_{ijk} dx^i \wedge dx^j \wedge dx^k = \frac{1}{3!} \omega_{ijk} dx^i \wedge dx^j \wedge dx^k.$$

Finally, the exterior derivative of k-form

$$\omega = \omega_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

is the following $(k+1)$-form

$$d\omega = d\omega_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}.$$

Example 2.2 Let $Q = \mathbb{R}^3$ with basis $\{\partial_x, \partial_y, \partial_z\}$ and its dual $\{dx, dy, dz\}$. Exterior derivative of 0-form $f(x)$ (differential of f):

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Exterior derivative of 1-form $\alpha = f dx + g dy + h dz$:

$$d\alpha = (g_x - f_y) dx \wedge dy + (h_x - f_z) dx \wedge dz + (h_y - g_z) dy \wedge dz.$$

Exterior derivative of 2-form $\omega = f dx \wedge dy + g dy \wedge dz + h dx \wedge dz$:

$$d\omega = (g_x - h_y + f_z) dx \wedge dy \wedge dz.$$

For two 1-forms $\alpha = \alpha_1 dx + \alpha_2 dy + \alpha_3 dz$ and $\beta = \beta_1 dx + \beta_2 dy + \beta_3 dz$, the external product is

$$\alpha \wedge \beta = (\alpha_1 \beta_2 - \alpha_2 \beta_1) dx \wedge dy + (\alpha_1 \beta_3 - \alpha_3 \beta_1) dx \wedge dz + (\alpha_2 \beta_3 - \alpha_3 \beta_2) dy \wedge dz.$$

For 2-form $\omega = f dx\wedge dy+g dy\wedge dz+h dx\wedge dz$ and vector field $V = u\partial_x+v\partial_y+w\partial_z$ the interior product is

$$i_V\omega = -(fv + hw)dx + (fu - gw)dy + (hu + gv)dz.$$

In matrix representation the last formula takes the form

$$\omega = \begin{pmatrix} 0 & f & h \\ -f & 0 & g \\ -h & -g & 0 \end{pmatrix}, \quad V = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

$$i_V\omega = -\omega V = \begin{pmatrix} -fv - hw \\ fu - gw \\ hu + gv \end{pmatrix}.$$

Objects dual to differential forms on manifold Q are *multi-vectors*. By definition, multi-vectors are totally antisymmetric and fully contravariant tensor fields:

$$\pi(x) : T_x^*Q \times \ldots \times T_x^*Q \longrightarrow \mathbb{R},$$

$$\pi(x)(\ldots\alpha\ldots\beta\ldots) = -\pi(x)(\ldots\beta\ldots\alpha\ldots).$$

Let us denote the space of k-vectors by $\Lambda^k(Q)$. In particular 1-vectors are ordinary vector fields, that is $\Lambda^1(Q) = \mathcal{T}^{(0,1)}(Q)$. As from the antisymmetry of a given multi-vector follows that

$$\pi(x)(\ldots\alpha\ldots\alpha\ldots) = 0,$$

so, on n dimensional manifold Q, nontrivial k-vectors are these for which $k = 0, \ldots, n$. Thus, a direct sum

$$\Lambda(Q) := \bigoplus_{k=0}^{n} \Lambda^k(Q)$$

constitutes a space of multi-vectors.

In a complete analogy to the covariant exterior algebra of forms one can define contravariant exterior algebra of multi-vectors with the respective basis

$$\partial_{i_1} \wedge \ldots \wedge \partial_{i_k}, \qquad 1 \le i_1 \le \ldots \le i_k \le n.$$

For example 2-vector (*bi-vector*) in the above basis is of the form

$$\pi(x) = \sum_{i>j} \pi^{ij}(x)\partial_i \wedge \partial_j.$$

Like any k-form can be represented by the exterior product of 1-forms (covectors)

$$\omega(x) = \alpha_1(x) \wedge \ldots \wedge \alpha_k(x),$$

so every k-vector can be represented by the exterior product of 1-vectors (vectors)

$$\pi(x) = v_1(x) \wedge \ldots \wedge v_k(x).$$

It is a well known product (bracket)

$$[.,.] : \Lambda^1(Q) \times \Lambda^1(Q) \to \Lambda^1(Q) \tag{2.3.9}$$

between vector fields, called commutator (Lie bracket) (2.5.12), (2.5.13). There exists a natural generalization of (2.3.9) onto multi-vectors

$$\boxed{[.,.]_S : \Lambda^k(Q) \times \Lambda^l(Q) \to \Lambda^{k+l-1}(Q)} \tag{2.3.10}$$

called *Schouten-Nijenhuis bracket* [230, 271]. It is a bilinear skew-symmetric map identical to the ordinary Lie bracket in the case of vector fields. Let $v_1, \ldots, v_k, w_1, \ldots, w_l$ be vector fields over Q, $\pi_k = v_1(x) \wedge \ldots \wedge v_k(x)$ and $\pi_l = w_1(x) \wedge \ldots \wedge w_l(x)$. Then the bracket is defined by

$$\boxed{\begin{aligned}[\pi_k, \pi_l]_S &= [v_1(x) \wedge \ldots \wedge v_k(x), w_1(x) \wedge \ldots \wedge w_l(x)]_S \\ &= \sum_{i,j} (-1)^{i+j} v_1 \wedge \ldots \hat{v}_i \ldots \wedge v_k \wedge [v_i, w_j] \wedge w_1 \wedge \ldots \hat{w}_j \ldots \wedge w_l\end{aligned}} \tag{2.3.11}$$

where $\hat{v}_i, \hat{w}_j$ denote the absence of v_i and w_j. From definition (2.3.11) it follows that:

1. $[\pi_k, \pi_l]_S = (-1)^{kl}[\pi_l, \pi_k]_S$,
2. $[\pi_k, \pi_r \wedge \pi_l]_S = [\pi_k, \pi_r]_S \wedge \pi_l + (-1)^{(k-1)r} \pi_r \wedge [\pi_k, \pi_l]_S$,
3. $(-1)^{k(l-1)}[\pi_k, [\pi_r, \pi_l]_S]_S + (-1)^{l(r-1)}[\pi_l, [\pi_k, \pi_r]_S]_S + (-1)^{r(k-1)}[\pi_r, [\pi_l, \pi_k]_S]_S = 0$.

For example, using the above properties of the Schouten-Nijenhuis bracket one can easily show that if v, w are some vector fields and π is a bi-vector, then

$$[v \wedge w, \pi]_S = w \wedge [v, \pi]_S - v \wedge [w, \pi]_S. \tag{2.3.12}$$

Let $(x^1, \ldots, x^n)$ be local coordinates on Q, $K \in \Lambda^k(Q)$ and $R \in \Lambda^r(Q)$ be of the form

$$K = \frac{1}{k!} K^{i_1 \ldots i_k} \partial_{i_1} \wedge \ldots \wedge \partial_{i_k}, \quad R = \frac{1}{r!} R^{i_1 \ldots i_r} \partial_{i_1} \wedge \ldots \wedge \partial_{i_r}$$

and

$$[K, R]_S = \frac{1}{(k+r-1)!}[K, R]_S^{l_1 \ldots l_{k+r-1}} \partial_{l_1} \wedge \ldots \wedge \partial_{l_{k+r-1}}$$

then

$$\boxed{[K, R]_S^{l_1 \ldots l_{k+r-1}} = \frac{1}{k!r!}\left(k K^{n[l_1 \ldots} \partial_n R^{l_k \ldots l_{k+r-1}]} + (-1)^k r R^{n[l_1 \ldots} \partial_n K^{l_r \ldots l_{k+r-1}]}\right)} \tag{2.3.13}$$

where $[\ldots]$ is the antisymmetrization operator over the indices.

For more details on Shouten-Nijenhuis bracket we refer the reader to the literature [230, 251, 271].

2.4 Mappings of Tensor Fields Between Manifolds

Let f be a smooth map

$$M \xrightarrow{f} N \tag{2.4.1}$$

from manifold M to manifold N for which the inverse map f^{-1} may not exist. The *push-forward* (direct image) under f of a tensor field will be denoted by f_*, while the *pull-back* (inverse image) under f of a tensor field will be denoted by f^*. For the function $\psi : N \longrightarrow \mathbb{R}$

$$M \xrightarrow{f} N \xrightarrow{\psi} \mathbb{R}$$

the admissible composition of maps $\psi \circ f : M \longrightarrow \mathbb{R}$, i.e.

$$f^*\psi = \psi \circ f$$

represents a pull-back of function ψ on N to function $\psi \circ f$ on M. Let $\{x^i\}$ be a local coordinate on M and $\{y^a\}$ be a local coordinate on N. In local coordinates $f : x \longrightarrow y$ and

$$f^*\psi(x) = \psi(y(x)).$$

For the composition of maps $M \xrightarrow{f} N \xrightarrow{h} S$ there holds $(h \circ f)^* = f^* \circ h^*$. Moreover,

$$f^* : \mathcal{F}(N) \longrightarrow \mathcal{F}(M)$$

represents a morphism between algebras of functions, as

$$f^*(\psi_1 + \lambda\psi_2) = f^*\psi_1 + \lambda f^*\psi_2, \quad f^*(\psi_1\psi_2) = (f^*\psi_1)(f^*\psi_2).$$

Vector fields are transported from M to N. For vectors the natural push-forward is defined by

$$f_* : T_x M \longrightarrow T_{f(x)} N,$$

such that for arbitrary function $\psi : N \longrightarrow \mathbb{R}$

$$(f_* v)(\psi) := v(f^*\psi).$$

In local coordinates we have

$$v(f^*\psi) = v^i \partial_i(\psi(y(x)) = v^i \frac{\partial \psi}{\partial y^a} \frac{\partial y^a}{\partial x^i} = J_i^a v^i \partial_a \psi$$

$$\Longrightarrow f_* v = (J_i^a v^i)\partial_a,$$

where $f' \equiv J_i^a := \frac{\partial y^a(x)}{\partial x^i}$ means the derivative of the map (Jacobian of the map). For $\dim M \leq \dim N$ the mapped basis vectors $f_*\partial_i = J_i^a \partial_a$ are linearly independent if $J_i^a(x)$ has maximum rank, i.e. f_* is injective.

In contrast, for 1-forms (covectors) natural is the pull-back defined as

$$f^* : T^*_{f(x)} N \longrightarrow T^*_x M,$$

such that

$$\left\langle f^*\alpha, v \right\rangle := \langle \alpha, f_* v \rangle, \quad v \in TM, \ \ \alpha \in T^*N.$$

In coordinates we have

$$f^* dy^a = J_i^a dx^i, \qquad f^*\alpha = f^*(\alpha_a dy^a) = (\alpha_a J_i^a) dx^i.$$

The pull-back of arbitrary covariant tensor field

$$f^* : \mathcal{T}^{(r,0)}(N) \longrightarrow \mathcal{T}^{(r,0)}(M)$$

is defined as follows

$$(f^* T^{(r,0)})(v, \ldots, w) := T^{(r,0)}(f_* v_1, \ldots, f_* v_r), \qquad v_1(x), \ldots, v_r(x) \in T_x M$$

and in coordinates takes the form

$$\boxed{T^{(r,0)} = T_{a_1\ldots a_r}(y)dy^{a_1}\otimes\ldots\otimes dy^{a_r}}$$

$$\Downarrow$$

$$\boxed{f^*T^{(r,0)} = T_{a_1\ldots a_r}(y(x))J^{a_1}_{i_1}(x)\ldots J^{a_r}_{i_r}(x)dx^{i_1}\otimes\ldots\otimes dx^{i_r}.}$$

In a similar way, the push-forward of arbitrary contravariant tensor field

$$f_* : \mathcal{T}^{(0,s)}(M) \longrightarrow \mathcal{T}^{(0,s)}(N) \tag{2.4.2}$$

is defined as follows

$$(f_*T^{(0,s)})(\alpha_1,\ldots,\alpha_s) := T(f^*\alpha_1,\ldots,f^*\alpha_s), \qquad \alpha_1(y),\alpha_s(y)\in T_yN$$

and in coordinates takes the form

$$\boxed{T^{(0,s)} = T^{i_1\ldots i_s}(x)\partial_{i_1}\otimes\ldots\otimes\partial_{i_s}}$$

$$\Downarrow$$

$$\boxed{f_*T^{(0,s)} = T^{i_1\ldots i_s}(x)J^{a_1}_{i_1}(x)\ldots J^{a_s}_{i_s}(x)\partial_{a_1}\otimes\ldots\otimes\partial_{a_s}.} \tag{2.4.3}$$

In order to push-forward any covariant tensor field or pull-back a contravariant tensor field, f has to be a diffeomorphism, i.e. we assume that f is a differentiable bijection, whose inverse is also differentiable. In general, we rarely have to do with global diffeomorphism of manifolds. For our further considerations the existence of local diffeomorphism will be a sufficient assumption.

If f is a diffeomorphism, the push-forward with respect to f is the same as pull-back with respect to f^{-1}, so

$$\alpha\in T^*M:\ f_*\alpha = (f^{-1})^*\alpha = ((J^{-1})^i_a\alpha_i)dy^a,$$

and

$$v\in TN:\ f^*v = (f^{-1})_*v = ((J^{-1})^i_a v^a)\partial_i.$$

Then, the pull-back of a tensor field of arbitrary type (r,s)

$$f^* : \mathcal{T}^{(r,s)}(N) \longrightarrow \mathcal{T}^{(r,s)}(M)$$

is defined by the relation

$$(f^*T^{(r,s)})(v_1,\ldots,v_r,\alpha_1,\ldots,\alpha_s) := T^{(r,s)}(f_*v_1,\ldots,f_*v_r,(f^{-1})^*\alpha_1,\ldots,(f^{-1})^*\alpha_s),$$

where $v_1,\ldots,v_r \in TM$, $\alpha_1,\ldots,\alpha_s \in T^*M$. In a local basis it takes the form

$$\boxed{T^{(r,s)} = T^{a_1\ldots a_s}_{b_1\ldots b_r}(y)dy^{b_1}\otimes\ldots\otimes\partial_{a_s}}$$

$$\Downarrow$$

$$\boxed{f^*T^{(r,s)} = T^{a_1\ldots a_s}_{b_1\ldots b_r}(y(x))J^{b_1}_{j_1}(x)\ldots(J^{-1})^{i_s}_{a_s}(x)dx^{j_1}\otimes\ldots\otimes\partial_{i_s}.}$$

In a similar fashion, the push-forward of a tensor field of arbitrary type (r,s)

$$f_* : \mathcal{T}^{(r,s)}(M) \longrightarrow \mathcal{T}^{(r,s)}(N)$$

is defined by the relation

$$(f_*T^{(r,s)})(v_1,\ldots,v_r,\alpha_1,\ldots,\alpha_s) := T^{(r,s)}((f^{-1})_*v_1,\ldots,(f^{-1})_*v_r,f^*\alpha_1,\ldots,f^*\alpha_s),$$

where $v_1,\ldots,v_r \in TN$, $\alpha_1,\ldots,\alpha_s \in T^*N$ and hence, in local coordinates,

$$\boxed{T(r,s) = T^{i_1\ldots i_s}_{j_1\ldots j_r}(x)dx^{j_1}\otimes\ldots\otimes\partial_{i_s}}$$

$$\downarrow$$

$$\boxed{f_*T(r,s) = T^{i_1\ldots i_s}_{j_1\ldots j_r}(x)J^{j_1}_{b_1}(x)\ldots(J^{-1})^{a_s}_{i_s}(x)dy^{b_1}\otimes\ldots\otimes\partial_{a_s}.}$$

Example 2.3 Induced metric tensor on a sphere $M = (S^2, g_{ij})$ from its embedding into $N = (E^3, \delta_{ij})$:

$$f : x = R\cos\varphi\sin\theta,\ y = R\sin\varphi\sin\theta,\ z = R\cos\theta,\quad R = const.$$

The differential of the map is

$$J = \begin{pmatrix} -R\sin\varphi\sin\theta & R\cos\varphi\cos\theta \\ R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \\ 0 & -R\sin\theta \end{pmatrix},$$

hence

$$g = f^*\delta = J^T\delta J = \begin{pmatrix} R^2\sin^2\theta & 0 \\ 0 & R^2 \end{pmatrix} \Longleftrightarrow g = R^2(\sin^2\theta\, d\varphi\otimes d\varphi + d\theta\otimes d\theta).$$

2.5 Lie Derivative of Tensor Fields

A *curve* on a manifold Q is a smooth map

$$\gamma : \mathbb{R}[t] \ni I \longrightarrow Q.$$

If $\varphi : \Theta \longrightarrow \mathbb{R}^n[x^1, \ldots, x^n]$ is a chart, i.e. $(x^1, \ldots, x^n)$ are local coordinates on $\Theta \subset Q$, one obtains a local representation of a curve γ

$$\gamma_\varphi \equiv \varphi \circ \gamma : \mathbb{R}[t] \longrightarrow \mathbb{R}^n[x^1, \ldots, x^n],$$

i.e. a curve on $\mathbb{R}^n$

$$t \longrightarrow (x^1(t), \ldots, x^n(t)),$$

which will be identified with γ. Vector $v(t)$ tangent to the curve $\gamma(t)$

$$\dot{\gamma} := \frac{d\gamma}{dt} = v(t), \qquad \dot{\gamma}(P) = v_P,$$

can be interpreted as a *velocity vector* of a dynamical system, whose *trajectory* is represented by a curve $\gamma(t)$. Vector v_P is a velocity vector at the point P on a curve. The metric tensor is the essential element for the concept of the *length of a curve* on (Q, g) between the points $\gamma(t_1)$ and $\gamma(t_2)$

$$L[\gamma] = \int_{t_1}^{t_2} dt \sqrt{g(\dot{\gamma}, \dot{\gamma})}.$$

Let us consider a one-parameter set of maps

$$I \times Q \longrightarrow Q,$$

which in a local basis $(x^1, \ldots, x^n)$ takes the form

$$(t, x) \longrightarrow \Phi(t, x), \quad \Phi(0, x) = x \tag{2.5.1}$$

and defines a *transport* on the manifold Q. For fixed x_0, $\Phi(t, x_0)$ represents a curve on Q which passes through x_0. In general, $\Phi(t, x)$ can be represented by particular solutions of a set of partial differential equations of evolutionary type

$$\frac{\partial \Phi^i}{\partial t} = K^i(\Phi, \Phi_x, \Phi_{xx}, \ldots), \quad i = 1, \ldots, n$$

with initial condition $\Phi^i(0, x) = x^i$.

Now the important question appears: when, using the notion of transport, can one define a vector field on Q, or at least locally on $U \subset Q$? It is possible when with arbitrary point $x \in U$ it can be related one and only one vector. Geometrically it means that curves which fill U do not intersect. Then we say that the transport is represented by a *flow*. So the next question is: when is the transport (2.5.1) represented by a flow? The answer is as follows: the one-parameter set of maps (2.5.1) represents a flow if it is *one-parameter Lie group*

$$\boxed{\bigwedge_{x\in Q}\bigwedge_{t_1,t_2\in I}\Phi(0,x)=x,\quad \Phi(t_2,x)\cdot\Phi(t_1,x)=\Phi(t_1+t_2,x),} \tag{2.5.2}$$

where dot means a group multiplication. In particular, when the group multiplication is point-wise (local), i.e. its value depends on x only, a vector field generated by a flow is represented by a contravariant tensor field of type $(0,1)$: $v = T^{(0,1)}$ and will be discussed in detail in Sect. 3.2.4. In particular, such flows represent classical Hamiltonian dynamics in a phase space. When the group multiplication is nonlocal, the flow cannot be described by local tensor algebra presented in this section. In particular, such flows are related with quantum Hamiltonian dynamics in a phase space and will be described in Sect. 7.2.3.

Let us consider a flow on Q

$$\mathbb{R}\times Q\longrightarrow Q:(t,x)\longrightarrow \psi(t,x)$$

such that

$$\boxed{\bigwedge_{x\in Q}\bigwedge_{t_1,t_2\in\mathbb{R}}\psi(0,x)=x,\quad \psi(t_1,x)\cdot\psi(t_2,x)=\psi(t_1,\psi(t_2,x))=\psi(t_1+t_2,x),} \tag{2.5.3}$$

i.e. the group multiplication is point-wise in the form of the composition of maps. It assigns diffeomorphism ϕ_t to every $t \in \mathbb{R}$

$$\phi_t : Q\longrightarrow Q, \tag{2.5.4}$$

$$\bar{x}=\phi_t\cdot x=\phi_t(x)=\psi(t,x),\quad \phi_{t_2}\cdot\phi_{t_1}=\phi_{t_1+t_2}. \tag{2.5.5}$$

It is a special case of the map (2.4.1) when $N = Q$, $f = \phi_t$ and $\phi_t^{-1} = \phi_{-t}$.

The flow is completely determined by its *infinitesimal generator*

$$v(x)=\frac{d}{dt}\psi(t,x)_{|t=0}=\frac{d}{dt}\phi_t(x)_{|t=0},$$

i.e. a vector field on Q. At arbitrary t

$$\frac{d}{dt}\phi_t = v \circ \phi_t, \tag{2.5.6}$$

which means that for any initial condition $x(0)$

$$x(t) = \phi_t \cdot x(0) = \psi(t, x(0))$$

is an integral curve of the *dynamical system*

$$\dot{x} \equiv x_t = v(x), \quad t - \textit{evolution parameter}.$$

Example 2.4 Consider the one-parameter group of rotations in the plane:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \phi_t \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos t - y\sin t \\ x\sin t - y\cos t \end{pmatrix} = \begin{pmatrix} \psi_1(t, x, y) \\ \psi_2(t, x, y) \end{pmatrix},$$

where the infinitesimal generator of this flow has two components

$$v^1(x) = \frac{d}{dt}\psi_1(t, x)_{|t=0} = -y, \quad v^2(x) = \frac{d}{dt}\psi_2(t, x)_{|t=0} = x.$$

It is not difficult to verify that indeed $\psi(t, x, y)$ is a flow (a set of integral curves) of the dynamical system

$$x_t = -y, \ y_t = x.$$

Let $T \in \mathcal{T}^{(r,s)}(Q)$ and $v \in \mathcal{T}^{(1,0)}(Q)$ be a vector field, then a *directional derivative* of the tensor field T in the direction of v is defined as follows

$$\boxed{T'(x)[v] := \frac{d}{dt}T(\phi_t x)_{|t=0} = \lim_{t\to 0}\frac{T(\phi_t x) - T(x)}{t} = \lim_{t\to 0}\frac{T(\bar{x}) - T(x)}{t},} \tag{2.5.7}$$

where ϕ_t is the flow generated by v.

For $|t| << 1$ from definition

$$T(\phi_t x) = T(x) + T'(x)[v]\cdot t + O(t^2).$$

On the other hand

$$\bar{x} = \phi_t(x) = x + \frac{d}{dt}\phi_t(x)_{|t=0}\cdot t + O(t^2) = x + v(x)\cdot t + O(t^2),$$

so

$$T(\bar{x}) = T(x + v(x) \cdot t + O(t^2)) = T(x) + \left(\frac{\partial T}{\partial x^i} v^i\right) t + O(t^2)$$

and in consequence

$$\boxed{(T'(x)[v])^{i_1 \ldots i_s}_{j_1 \ldots j_r} = \frac{\partial T^{i_1 \ldots i_s}_{j_1 \ldots j_r}(x)}{\partial x^i} v^i(x).} \tag{2.5.8}$$

The Lie transport is a process of displacing a given geometric object along a flow. More precisely, for a given flow (2.5.4) and an arbitrary tensor field $T^{(r,s)}$, the *Lie transport* of $T^{(r,s)}$ along ϕ_t is a push-forward

$$\phi_{t*} : T^{(r,s)}(Q) \longrightarrow T^{(r,s)}(Q)$$

and a pull-back

$$\phi_t^* : T^{(r,s)}(Q) \longrightarrow T^{(r,s)}(Q),$$

where from (2.5.5) follows that

$$\phi_t^* = \left(\phi_t^{-1}\right)_* = \phi_{-t*}. \tag{2.5.9}$$

For a given flow ϕ_t it may happen that for particular tensor fields T

$$\phi_t^* T = T \Longrightarrow \phi_{t*} T = T.$$

Then, we say that such tensor fields are tensor invariants of the flow ϕ_t. As the next step, the natural question appears: how to measure the rate of change of arbitrary tensor field T along a flow ϕ_t? The concept of the Lie derivative gives the solution to that problem.

Let ϕ_t be a flow, $v(x)$ its infinitesimal generator, and $T^{(r,s)}(x)$ an arbitrary tensor field. The *Lie derivative* of a tensor field $T(x)$ along the flow ϕ_t is defined as follows

$$\boxed{L_v T(x) := \frac{d}{dt}(\phi_t^* T)(x)_{|t=0} = \frac{d}{dt}(\phi_{-t*} T)(x)_{|t=0}.} \tag{2.5.10}$$

Hence,

$$\begin{aligned} L_v T(x) &= \frac{d}{dt}(\phi_{-t*} T)(x)_{|t=0} = -\frac{d}{dt}(\phi_{t*} T)(x)_{|t=0} \\ &= \lim_{t \to 0} \frac{T(x) - (\phi_{t*} T)(x)}{t} = \lim_{t \to 0} \frac{T(x) - \bar{T}(x)}{t}, \end{aligned} \tag{2.5.11}$$

and the recipe how to calculate it is as follows: push forward a tensor to the point $\bar{x}$ according to the transformation low

$$T(x) \xrightarrow{\phi_{t*}} \bar{T}(\bar{x}),$$

pull back its argument from $\bar{x}$ to x

$$\bar{x} \xrightarrow{\phi_{-t}} x$$

and compare T with $\bar{T}$ in the point x. Hence, indeed the Lie derivative measures the change of tensor field along a given curve.

From the definition it follows, that for $|t| << 1$

$$(\phi_t^* T)(x) = T(x) + t \cdot L_v T(x) + O(t^2),$$

$$(\phi_{t*} T)(x) = T(x) - t \cdot L_v T(x) + O(t^2).$$

The Lie derivative has the following properties:

1. $L_v : \mathcal{T}^{(r,s)}(Q) \longrightarrow \mathcal{T}^{(r,s)}(Q)$,
2. $L_v(A + \lambda B) = L_v A + \lambda L_v B, \quad \lambda = const.$, linearity,
3. $L_v(A \otimes B) = (L_v A) \otimes B + A \otimes L_v B$, Leibniz rule.

So L_v considered to be the map

$$L_v : \mathcal{T}(Q) \longrightarrow \mathcal{T}(Q)$$

is of a derivative type on the algebra $\mathcal{T}(Q)$ and maps (r, s) type tensors into (r, s) type tensors.

How to derive the explicit form of the Lie derivative for various tensor fields $T^{(r,s)}(x)$? We will use the definition and the following known relations

$$\phi_t(x) = x + t \cdot v(x) + O(t^2), \quad \phi_t'(x) = 1 + t \cdot v'(x) + O(t^2),$$

(where the prime means the derivation) and

$$\bar{T}(\bar{x}) = \bar{T}(x) + T'(x)[v] + O(t^2),$$

$$\lim_{t \to 0} \bar{T}(\bar{x}) = \lim_{t \to 0} \bar{T}(x) = T(x).$$

For a scalar field $T(x) = T^{(0,0)}(x) = f(x)$ push-forward is of the form

$$\bar{f} = \phi_{t*} f = f \circ \phi_{-t} \Longrightarrow \bar{f}(\bar{x}) = f(x)$$

and moreover

$$f(x) = \bar{f}(\bar{x}) = \bar{f}(x + tv + O(t^2)) = \bar{f}(x) + t \cdot \bar{f}'(x)[v] + O(t^2).$$

As the result

$$f(x) - \bar{f}(x) = t \cdot \bar{f}'(x)[v] + O(t^2)$$

$$\Downarrow$$

$$L_v f = f'(x)[v] = v(f).$$

For a vector field $T(x) = T^{(1,0)}(x) = w(x)$ push-forward takes the form

$$\bar{w}(\bar{x}) = \phi_{t*} w(x) = \phi_t'[w(x)] = w(x) + t \cdot v'(x)[w(x)] + O(t^2),$$

so

$$\bar{w}(\bar{x}) = \bar{w}(x) + t \cdot w'(x)[v(x)] + O(t^2),$$

$$\Downarrow$$

$$L_v w = w'[v] - v'[w] := [v, w]. \tag{2.5.12}$$

Thus, if v and w are vector fields, then $u = [v, w]$ is also a vector field called the *commutator* of v and w, such that

$$[v, w](f) = v(w(f)) - w(v(f)) = u(f), \quad u^i = (\partial_k w^i) v^k - (\partial_k v^i) w^k. \tag{2.5.13}$$

For 1-forms: $T(x) = T^{(0,1)}(x) = \alpha(x)$ we have

$$\alpha(x) = \phi_t^* \bar{\alpha}(\bar{x}) = \phi_t'^T \bar{\alpha}(\bar{x}) = \bar{\alpha}(\bar{x}) + t \cdot v'^T(x)[\bar{\alpha}(\bar{x})] + O(t^2)$$

and in consequence

$$\bar{\alpha}(\bar{x}) = \bar{\alpha}(x) + t \cdot \bar{\alpha}'(x)[v(x)] + O(t^2),$$

$$\Downarrow$$

$$L_v \alpha = \alpha'[v] + v'^T[\alpha].$$

In local coordinates, considered Lie derivatives take the form

$$L_v f(x) = \frac{\partial f}{\partial x^k} v^k(x), \tag{2.5.14}$$

$$(L_v w(x))^i = \frac{\partial w^i}{\partial x^j} v^j - \frac{\partial v^i}{\partial x^j} w^j, \quad L_v \partial_i = -\frac{\partial v^j}{\partial x^i} \partial_j, \tag{2.5.15}$$

$$(L_v \alpha(x))_i = \frac{\partial w_i}{\partial x^j} v^j + \frac{\partial v^j}{\partial x^i} \alpha_j, \quad L_v dx^i = \frac{\partial v^i}{\partial x^j} dx^j. \tag{2.5.16}$$

The general formula for Lie derivative of an arbitrary tensor field follows from the Leibniz rule and the formulas (2.5.14)–(2.5.16)

$$\begin{aligned} L_v \left(T^{i_1 \dots i_s}_{j_1 \dots j_r} \partial_{i_1} \otimes \dots \otimes dx^{j_r} \right) =& L_v \left(T^{i_1 \dots i_s}_{j_1 \dots j_r} \right) \partial_{i_1} \otimes \dots \otimes dx^{j_r} + T^{i_1 \dots i_s}_{j_1 \dots j_r} L_v(\partial_{i_1}) \otimes \dots \otimes dx^{j_r} \\ &+ \dots + T^{i_1 \dots i_s}_{j_1 \dots j_r} \partial_{i_1} \otimes \dots \otimes L_v(dx^{j_r}), \end{aligned}$$

and hence

$$\begin{aligned} \left(L_v T^{(r,s)}(x) \right)^{i_1 \dots i_s}_{j_1 \dots j_r} =& \frac{\partial T^{i_1 \dots i_s}_{j_1 \dots j_r}}{\partial x^m} v^m + \frac{\partial v^m}{\partial x^{j_1}} T^{i_1 \dots i_s}_{m \dots j_r} + \dots + \frac{\partial v^m}{\partial x^{j_r}} T^{i_1 \dots i_s}_{j_1 \dots m} \\ &- \frac{\partial v^{i_1}}{\partial x^m} T^{m \dots i_s}_{j_1 \dots j_r} - \dots - \frac{\partial v^{i_s}}{\partial x^m} T^{i_1 \dots m}_{j_1 \dots j_r}. \end{aligned} \tag{2.5.17}$$

In particular, for second order tensors, we have

$$\begin{aligned} L_v T^i_j &= \frac{\partial T^i_j}{\partial x^m} v^m + \frac{\partial v^m}{\partial x^j} T^i_m - \frac{\partial v^i}{\partial x^m} T^m_j, \\ L_v T^{ij} &= \frac{\partial T^{ij}}{\partial x^m} v^m - \frac{\partial v^i}{\partial x^m} T^{mj} - \frac{\partial v^j}{\partial x^m} T^{im}, \\ L_v T_{ij} &= \frac{\partial T_{ij}}{\partial x^m} v^m + \frac{\partial v^m}{\partial x^i} T_{mj} + \frac{\partial v^m}{\partial x^j} T_{im}. \end{aligned}$$

Notice that the Lie derivative is not an $\mathcal{F}$-linear map, as for any $f \in \mathcal{F}(Q)$

$$L_{fv} \neq f L_v, \tag{2.5.18}$$

which follows immediately from the formula (2.5.17). For example, if $f \in \mathcal{F}(Q)$, $v, w \in \mathcal{T}^{(0,1)}(Q)$ and $\pi \in \Lambda^2(Q)$, then

$$L_{fv} w = f L_v w - w(f) v, \quad L_{fv} \pi = f L_v \pi - \pi df \wedge v. \tag{2.5.19}$$

For the Lie derivative L_v, the exterior derivative d, the interior product i_v and for arbitrary k-form ω, the following relations hold

1.

$$L_v\omega = (i_v\, d + d\, i_v)\omega, \tag{2.5.20}$$

2.

$$d\, L_v\omega = L_v d\omega, \tag{2.5.21}$$

3.

$$i_v L_v\omega = L_v i_v\omega. \tag{2.5.22}$$

To show the first relation we proceed by induction on k. For $k = 0$ we have $\omega = f$, $i_v f = 0$ and $L_v f = \langle df, v\rangle = i_v f$. Now assume that (2.5.20) holds for k. Then a $(k+1)$-form may be written as $\alpha \wedge \omega$, where ω is a k-form and α is a 1-form. Hence

$$\begin{aligned}(i_v\, d + d\, i_v)(\alpha \wedge \omega) &= i_v(d\alpha \wedge \omega) - i_v(\alpha \wedge d\omega) + d(i_v\alpha \wedge \omega) - d(\alpha \wedge i_v\omega)\\ &= i_v d\alpha \wedge \omega + d\alpha \wedge i_v\omega - i_v\alpha \wedge d\omega - \alpha \wedge i_v d\omega\\ &\quad + di_v\alpha \wedge \omega + i_v\alpha \wedge d\omega - d\alpha \wedge i_v\omega - \alpha \wedge di_v\omega\\ &= \alpha \wedge L_v\omega + L_v\alpha \wedge \omega = L_v(\alpha \wedge \omega)\end{aligned}$$

by our inductive assumption and the properties of the external product. To show the second relation we apply the formula (2.5.20) and the fact that $d^2 = 0$

$$L_v d = d\, i_v d + i_v d^2 = d\, i_v d = d(L_v - di_v) = dL_v - d^2 i_v = dL_v.$$

Finally, to show the third relation we apply the formula (2.5.20) and the fact that $i_v i_v = 0$

$$L_v i_v = (d\, i_v + i_v d)i_v = i_v d\, i_v = i_v(L_v - i_v\, d) = i_v L_v.$$

A linear space V with a bilinear product $[.,.] : V \times V \longrightarrow V$ which is antisymmetric

$$[a, b] = -[b, a], \quad a, b \in V \tag{2.5.23}$$

and satisfies the so called *Jacobi identity*

$$\boxed{[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0,} \tag{2.5.24}$$

is called the *Lie algebra*.

The linear space of vector fields on Q is an example of the Lie algebra with a Lie product being the commutator. Antisymmetry follows from definition (2.5.12) and the Jacobi identity is a consequence of the fact that the second directional derivative is a symmetric bilinear form.

Another example is related to Lie derivatives themselves. First notice that Lie derivatives constitute a linear space. The commutator

$$[L_{v_1}, L_{v_2}] = L_{v_1} L_{v_2} - L_{v_2} L_{v_1}$$

endows this vector space in a natural way with a Lie algebra structure. The map $v \to L_v$ is a Lie algebra isomorphism from the Lie algebra of vector fields onto the Lie algebra of Lie derivatives

$$[L_{v_1}, L_{v_2}] = L_{[v_1, v_2]}.$$

At the end of this section let us mention that the flow ϕ_t itself, as well as its pull-back ϕ_t^* may in turn be often expressed in a useful form of the exponent. Indeed, for any flow ϕ_t and arbitrary t (2.5.6)

$$\boxed{\frac{d}{dt}\phi_t = v \circ \phi_t.} \tag{2.5.25}$$

The formal solution of this equation takes the form

$$\phi_t(x) = e^{tv} \Longrightarrow x(t) = e^{tv} x(0).$$

In a similar way, from relations (2.5.10) and (2.5.11) it follows that

$$\boxed{\frac{d}{dt}\phi_t^* = L_v \circ \phi_t^*,} \tag{2.5.26}$$

with formal solution

$$\phi_t^* = e^{tL_v} \Longrightarrow T(x(t)) = e^{tL_v} T(x(0)).$$

2.6 Linear Connection and Covariant Derivative

A linear combination of two vectors from different spaces does not make sense. However, if $A : W \longrightarrow V$ is an isomorphism of the mentioned linear spaces, then the operation

$$v + \lambda A(w), \quad v \in V, \ w \in W$$

is admissible. A simple example of such problems is given by the *acceleration* of a point mass which measures the change of velocity vector in time. Let us consider an acceleration vector in $\mathbb{R}^2$. In order to measure a change of the velocity $v(t)$ in a moment t at the point $r(t)$ we make a parallel shift of the vector $v(t+\varepsilon)$ in a moment $t+\varepsilon$ from the point $r(t+\varepsilon)$ back to the point $r(t)$ and then compare both velocity vectors, as now they belong to the same linear space.

The procedure which seems "obvious" for $\mathbb{R}^2$ becomes completely unclear if we pass to the sphere S^2 for example. How can one define the parallel shift in such a case? A similar problem appears with the generalization of the notion of a straight-line from $\mathbb{R}^2$ to S^2. The solution to such problems is given by the theory of *linear connection*, which allows us to extend the notion of a *parallel transport* onto smooth manifold Q. In what follows, the manifold Q equipped with a linear connection will be denoted by (Q, ∇). We say that a rule of parallel transport is given on a manifold Q if, for an arbitrary curve γ on Q and two points x, y on the curve, there is a prescription which assigns uniquely to vectors in x vectors in y, i.e.

$$\tau^{\gamma}_{y,x} : T_x Q \longrightarrow T_y Q, \quad v \longrightarrow \tau^{\gamma}_{y,x} v.$$

As it has to be a generalization of the parallel shift from $\mathbb{R}^n$, we demand some natural restrictive conditions on τ like linearity

$$\tau^{\gamma}_{y,x}(v + \lambda w) = \tau^{\gamma}_{y,x}(v) + \lambda \tau^{\gamma}_{y,x}(w),$$

composition property

$$\tau^{\gamma}_{z,y} \circ \tau^{\gamma}_{y,x} = \tau^{\gamma}_{z,x}$$

and in particular

$$\tau^{\gamma}_{x,x} = id., \quad \left(\tau^{\gamma}_{y,x}\right)^{-1} = \tau^{\gamma}_{x,y}.$$

Note that the rule of parallel transport needs as an input not only the edge points x, y but also a path connecting them. In general, the resulting transported vector may be different for different passes between x and y. We will see that the path dependence of the parallel transport is an important and typical phenomenon and it enables one to speak about the *curvature* of the manifold (Q, ∇).

In the case of the Lie transport along the curve γ we have introduced the notion of the Lie derivative along the vector field $v = \dot{\gamma}$. In a similar way, for the parallel transport along the curve γ, we can introduce the so called *absolute derivative* called also a *covariant derivative* along the vector field $v = \dot{\gamma}$:

$$\boxed{\frac{DT(t)}{Dt} := \lim_{t \to 0} \frac{T^{\parallel}_{\varepsilon}(t) - T(t)}{\varepsilon} \equiv \nabla_{\dot{\gamma}} T = \nabla_v T,} \tag{2.6.1}$$

where $T_{\varepsilon}^{\parallel}(t)$ denotes the tensor field transported parallel backwards from the point $\gamma(t+\varepsilon)$ to the point $\gamma(t)$. If $T = w(\gamma(t))$ is a vector field on γ, such that its absolute derivative along γ vanishes

$$\nabla_v w = 0, \tag{2.6.2}$$

we call it *autoparallel*. It means that $w(\gamma(t)) = w_{\varepsilon}^{\parallel}(\gamma(t))$. In practice, the situation is reverse. First, we define the connection, i.e. the covariant derivative, and then the parallel transport is defined by the demand of the condition (2.6.2). The vanishing of the covariant derivative on a curve means that the field $w(t)$ may be regarded in such a way that its values everywhere on γ arose by a parallel transport of its value at a single fixed point into all the points of the curve.

Let us be more specific with the definition of a linear connection on the manifold Q. With each vector field v on Q one may associate an operator ∇_v, the covariant derivative along the field v, having the following properties:

1. it is a linear operator on the tensor algebra, which preserves the tensor degree

$$\nabla_v : T^{(r,s)}(Q) \longrightarrow T^{(r,s)}(Q),$$

$$\nabla_v(A + \lambda B) = \nabla_v A + \lambda \nabla_v B, \qquad A, B \in T^{(r,s)}(Q),$$

2. on a tensor product it fulfills the Leibniz rule

$$\nabla_v(A \otimes B) = (\nabla_v A) \otimes B + A \otimes \nabla_v B, \quad A \in T^{(r,s)}(Q),\ B \in T^{(r',s')}(Q),$$

3. for a scalar field $T^{(0,0)} = f$ we demand the following property

$$\nabla_v f = v(f) \equiv L_v f,$$

4. it commutes with contractions

$$\nabla_v \circ C = C \circ \nabla_v,$$

5. it is $\mathcal{F}$-linear

$$\nabla_{v+fw} = \nabla_v + f \cdot \nabla_w,$$

which is the only property that differs the covariant derivative from the Lie derivative (see (2.5.18)).

The covariant derivative is uniquely specified by the coefficients of linear connection $\Gamma_{ij}^{k}(x)$ with respect to the frame field $\{\partial_i\}$

$$\nabla_j \partial_i =: \Gamma_{ij}^{k} \partial_k, \tag{2.6.3}$$

where $\Gamma^k_{ij}(x)$ are called the *Christoffel symbols of the second kind*. Hence

$$\nabla_w \partial_i = \nabla_{(w^k \partial_k)} \partial_i = w^k \nabla_k \partial_i = (\Gamma^j_{ik} w^k) \partial_j .$$

Then, one can show that

$$\nabla_j dx^i = -\Gamma^i_{kj} dx^k, \quad \nabla_w dx^i = -(\Gamma^i_{jk} w^k) dx^j . \tag{2.6.4}$$

Indeed, since

$$\begin{aligned} 0 = \nabla_w \delta^i_j = \nabla_w \left\langle dx^i, \partial_j \right\rangle &\overset{2,4}{=} \left\langle \nabla_w dx^i, \partial_j \right\rangle + \left\langle dx^i, \nabla_w \partial_j \right\rangle \\ &= \left\langle \nabla_w dx^i, \partial_j \right\rangle + \left\langle dx^i, \Gamma^l_{jk} w^k \partial_l \right\rangle = (\nabla_w dx^i)_j + \Gamma^i_{jk} w^k \end{aligned}$$

we obtain (2.6.4).

The *coefficients of the linear connection* Γ^k_{ij} have one upper index and two lower indices. Nevertheless, they are not components of any (2, 1)-tensor. Indeed, for a given coordinate transformation $x^i \longrightarrow \bar{x}^i(x),\ i = 1, \ldots, n$ we have

$$\begin{aligned} \Gamma'^k_{ij} \partial'_k = \nabla'_i \partial'_j &= (J^{-1})^k_i \nabla_k (J^{-1})^l_j \partial_l \\ &= (J^{-1})^k_i \left[\left(\partial_k (J^{-1})^l_j \right) \partial_l + (J^{-1})^l_j \nabla_k \partial_l \right] \\ &= (J^{-1})^k_i \left[J^s_k \left(\partial'_s (J^{-1})^l_j \right) \partial_l + (J^{-1})^l_j \Gamma^s_{lk} \partial_l \right] \\ &= \left[\left(\partial'_i (J^{-1})^l_j \right) J^k_l + (J^{-1})^k_i (J^{-1})^l_j \Gamma^s_{lk} J^k_s \right] \partial'_k \end{aligned}$$

$$\Downarrow$$

$$\begin{aligned} \bar{\Gamma}^k_{ij} &= \left(\partial'_i (J^{-1})^l_j \right) J^k_l + (J^{-1})^k_i (J^{-1})^l_j \Gamma^s_{lk} J^k_s \\ &= \frac{\partial \bar{x}^k}{\partial x^l} \frac{\partial^2 x^l}{\partial \bar{x}^j \partial \bar{x}^i} + \frac{\partial \bar{x}^k}{\partial x^r} \frac{\partial x^s}{\partial \bar{x}^i} \frac{\partial x^m}{\partial \bar{x}^j} \Gamma^r_{sm}, \end{aligned} \tag{2.6.5}$$

where $J^i_k = \frac{\partial \bar{x}^i}{\partial x^k}$ and the first term in (2.6.5) is "non-tensorial".

Applying properties **1**–**3** we get

$$\begin{aligned} \nabla_w \left(T^{i_1 \ldots i_s}_{j_1 \ldots j_r} \partial_{i_1} \otimes \ldots \otimes dx^{j_r} \right) = w \left(T^{i_1 \ldots i_s}_{j_1 \ldots j_r} \right) \partial_{i_1} \otimes \ldots \otimes dx^{j_r} \\ + T^{i_1 \ldots i_s}_{j_1 \ldots j_r} \left(\nabla_w \partial_{i_1} \right) \otimes \ldots \otimes dx^{j_r} \\ + \ldots + T^{i_1 \ldots i_s}_{j_1 \ldots j_r} \partial_{i_1} \otimes \ldots \otimes (\nabla_w dx^{j_r}), \end{aligned}$$

and thus, applying formulas (2.6.3)–(2.6.4), the component formula for the covariant derivative of a general tensor field T into the direction of a vector field w has the form

$$\boxed{(\nabla_w T)^{i_1\dots i_s}_{j_1\dots j_r} = w^m \frac{\partial T^{i_1\dots i_s}_{j_1\dots j_r}}{\partial x^m} + \Gamma^{i_1}_{nm} w^m T^{n\dots i_s}_{j_1\dots j_r} + \dots - \Gamma^{n}_{j_r m} w^m T^{i_1\dots i_s}_{j_1\dots n}.} \tag{2.6.6}$$

For vector fields and 1-forms we have

$$\nabla_w v = \left(\frac{\partial v^k}{\partial x^i} w^i + w^i v^j \Gamma^k_{ij} \right) \partial_k, \tag{2.6.7}$$

$$\nabla_w \alpha = \left(\frac{\partial \alpha_k}{\partial x^i} w^i - w^i \alpha_j \Gamma^j_{ki} \right) dx^k.$$

The $\mathcal{F}$-linearity of the operator ∇_w with respect to w enables one to introduce the operation of the covariant gradient

$$\nabla : T^{(s,r)}(Q) \longrightarrow T^{(s+1,r)}(Q),$$

$$(\nabla T)(v, \dots, w, \alpha, \dots) = (\nabla_w T)(v, \dots, \alpha, \dots), \tag{2.6.8}$$

$$(\nabla T)^{i_1\dots i_s}_{j_1\dots j_r m} = (\nabla_m T)^{i_1\dots i_s}_{j_1\dots j_r} =: T^{i_1\dots i_s}_{j_1\dots j_r ;m}. \tag{2.6.9}$$

From (2.6.6) it follows that

$$\boxed{T^{i_1\dots i_s}_{j_1\dots j_r ;m} = \frac{\partial T^{i_1\dots i_s}_{j_1\dots j_r}}{\partial x^m} + \Gamma^{i_1}_{nm} T^{m\dots i_s}_{j_1\dots j_r} + \dots + \Gamma^{i_s}_{nm} T^{i_1\dots n}_{j_1\dots j_r} - \Gamma^{n}_{j_1 m} T^{i_1\dots i_s}_{n\dots j_r} - \Gamma^{n}_{j_r m} T^{i_1\dots i_s}_{j_1\dots n}}$$

and in particular

$$\begin{aligned}
\nabla_k f &= \frac{\partial f}{\partial x^k} := f_{,k}, \\
\nabla_k v^a &= \partial_k v^a + \Gamma^a_{bk} v^b := v^a{}_{;k}, \\
\nabla_k \alpha_a &= \partial_k \alpha_a - \Gamma^b_{ak} \alpha_b := \alpha_{a;k}, \\
\nabla_k T^{ab} &= \partial_k T^{ab} + \Gamma^a_{ck} T^{cb} + \Gamma^b_{ck} T^{ac} := T^{ab}{}_{;k}, \\
\nabla_k T_{ab} &= \partial_k T_{ab} - \Gamma^c_{ak} T_{cb} - \Gamma^c_{bk} T_{ac} := T_{ab;k}, \\
\nabla_k T^a_b &= \partial_k T^a_b + \Gamma^a_{ck} T^c_b - \Gamma^c_{bk} T^a_c := T^a_b{}_{;k}.
\end{aligned} \tag{2.6.10}$$

Notice, that for a scalar field f, the *covariant gradient* (2.6.10) coincides with an "ordinary" gradient, i.e. $grad\ f := \nabla f = df$. The covariant derivative along w

may be written in terms of the covariant gradient as

$$(\nabla_m T)^{i_1\ldots i_s}_{j_1\ldots j_r} = w^m T^{i_1\ldots i_s}_{j_1\ldots j_r;m}.$$

Using relations (2.6.8) and (2.6.9) one can calculate higher order covariant derivatives of an arbitrary tensor field. In particular, for the scalar field f we get

$$\begin{aligned}
(\nabla f)_{j_1} &= f_{,j_1} \\
(\nabla\nabla f)_{j_1 j_2} &= (\nabla f)_{j_1,j_2} - \Gamma^k_{j_1 j_2}(\nabla f)_k = f_{,j_1 j_2} - \Gamma^k_{j_1 j_2} f_{,k} \\
(\nabla\nabla\nabla f)_{j_1 j_2 j_3} &= (\nabla\nabla f)_{j_1 j_2,j_3} - \Gamma^k_{j_1 j_3}(\nabla\nabla f)_{k j_2} - \Gamma^k_{j_2 j_3}(\nabla\nabla f)_{j_1 k} \\
&= f_{,j_1 j_2 j_3} - \Gamma^k_{j_1 j_2,j_3} f_{,k} - \Gamma^k_{j_2 j_3} f_{,j_1 k} - \Gamma^k_{j_1 j_3} f_{,j_2 k} - \Gamma^k_{j_1 j_2} f_{,j_3 k} \\
&\quad + \Gamma^k_{j_1 j_3}\Gamma^s_{k j_2} f_{,s} + \Gamma^k_{j_2 j_3}\Gamma^s_{j_1 k} f_{,s}
\end{aligned}$$

Consider the manifold Q endowed with a pair of structures, the metric tensor g, which enables us to measure the length of the vectors and the linear connection ∇, which enables us to transport the vectors along paths. We are interested in a particular class of connections which preserve the length of vectors under parallel transport, like in the case of the parallel shift in $\mathbb{E}^n$. Let γ be an arbitrary curve in (Q, g, ∇) and v an autoparallel field

$$\nabla_{\dot\gamma} v = 0.$$

The requirement of preservation of the length of v by parallel transport may be stated as

$$\nabla_{\dot\gamma}(g(v, v)) = 0 \quad \text{if} \quad \nabla_{\dot\gamma} v = 0.$$

If this is to be true for an arbitrary curve γ and an arbitrary initial vector v, then for any two vector fields w, v one should demand

$$\nabla_w(g(v, v)) = 0 \quad \text{if} \quad \nabla_w v = 0,$$

which is a particular example of the more general assumption that for any three vector fields w, v, u the covariant derivative should obey

$$\nabla_w(g(v, u)) = 0 \quad \text{if} \quad \nabla_w v = 0 = \nabla_w u.$$

$$\Updownarrow$$

$$\nabla g = 0, \qquad g_{ij;k} = 0. \tag{2.6.11}$$

A connection ∇ which satisfies equations (2.6.11) is called the *metric connection*. The requirement (2.6.11) imposes $\frac{1}{2}n^2(n+1)$ constraints on n^3 Christoffel symbols Γ^i_{jk}, following from symmetry of metric tensor g. From (2.6.11) we have

$$\nabla_k g_{ij} \equiv g_{ij;k} = \partial_k g_{ij} - \Gamma^l_{ik} g_{lj} - \Gamma^l_{jk} g_{il} = 0.$$

Let (Q, ∇) be a manifold with a linear connection. The map

$$T : T^{(1,0)}(Q) \times T^{(1,0)}(Q) \longrightarrow T^{(1,0)}(Q),$$

$$T(v, w) := \nabla_w v - \nabla_v w - [w, v]$$

defines the so called *torsion tensor* of the connection $\nabla : T^{(2,1)}(Q)$. The tensor is antisymmetric in the lower indices

$$T(w, v) = -T(v, w), \quad \text{i.e. } T^i_{jk} = -T^i_{kj}$$

and so it has $\frac{1}{2}n^2(n-1)$ independent components

$$\left\langle dx^i, T(\partial_j, \partial_k) \right\rangle \equiv T^i_{jk} = \Gamma^i_{kj} - \Gamma^i_{jk}. \tag{2.6.12}$$

If the torsion of the connection vanishes, i.e.

$$\nabla_w v - \nabla_v w = [w, v]$$

then the Christoffel symbols are symmetric in the lower indices

$$\Gamma^i_{kj} = \Gamma^i_{jk}$$

and the connection is called symmetric or *torsionless*. If the connection is required to be at the same time metric and symmetric, it imposes n^3 constraints on n^3 symbols $\Gamma^i_{jk}(x)$, hence determines the connection uniquely. In order to construct this connection explicitly, let us introduce the Christoffel symbols of the first kind as

$$\Gamma_{ijk} := g_{il}\Gamma^l_{jk}. \tag{2.6.13}$$

Then, the connection which is metric and symmetric satisfies

$$\Gamma_{ijk} + \Gamma_{jik} = \partial_k g_{ij} \equiv g_{ij,k}, \tag{2.6.14}$$

$$\Gamma_{ijk} - \Gamma_{ikj} = 0$$

and the two relations result in

$$g_{ij,k} + g_{ik,j} - g_{jk,i} = 2\Gamma_{ijk}, \tag{2.6.15}$$

and eventually

$$\boxed{\Gamma^i_{jk} = \frac{1}{2}G^{il}(g_{lj,k} + g_{lk,j} - g_{jk,l})} \tag{2.6.16}$$

where $(G^{ij}) := (g_{ij})^{-1}$. It means that the connection coefficients are determined uniquely by the metric tensor. This distinguished linear connection on Riemannian manifolds is called the *Levi-Civita connection*. In particular, if we chose $Q = \mathbb{E}^n$ in the Cartesian coordinates $\Gamma^i_{jk} = 0$ and

$$\nabla_i \to \partial_i,\ \nabla T \to T',\ \nabla_w T \to T'[w].$$

Example 2.5 For $Q = \mathbb{E}^2$ in polar coordinates (r, ϕ), the metric tensor and non-vanishing Levi-Civita connection coefficients are as follows

$$g = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad \Gamma^r_{\phi\phi} = -r,\ \Gamma^\phi_{r\phi} = \frac{1}{r}.$$

For $Q = \mathbb{E}^3$ in spherical coordinates (r, θ, ϕ) we have respectively

$$g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix},$$

$$\Gamma^r_{\theta\theta} = -r,\ \Gamma^r_{\phi\phi} = -r\sin^2\theta,$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r},\ \Gamma^\phi_{\phi\theta} = \cot\theta,$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r},\ \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta.$$

Killing vectors represent these flows along which the metric tensor is invariant, i.e.

$$L_\xi g = 0 \Longrightarrow \xi\text{-Killing vector} \tag{2.6.17}$$

or in component expressions (2.5.17)

$$g_{ij,k}\xi^k + g_{jk}\xi^k_{,i} + g_{ik}\xi^k_{,j} = 0. \tag{2.6.18}$$

Using relations $\nabla_k g_{ij} = 0$, Killing equations (2.6.18) can be written in terms of covariant derivatives

$$\boxed{\nabla_i \xi_j + \nabla_j \xi_i := \nabla_{(i}\xi_{j)} = 0, \quad \xi_i = g_{ij}\xi^j.} \tag{2.6.19}$$

Indeed

$$\begin{aligned}
0 = \nabla_i \xi_j + \nabla_j \xi_i &= (g_{jk}\xi^k)_{;i} + (g_{ik}\xi^k)_{;j} \\
&= g_{jk;i}\xi^k + g_{jk}\xi^k{}_{;i} + g_{ik;j}\xi^k + g_{ik}\xi^k{}_{;j} \\
&= (g_{jk,i} - \Gamma^l_{ji}g_{lk} - \Gamma^l_{ki}g_{jl})\xi^k + g_{jk}(\xi^k_{,i} + \Gamma^k_{li}\xi^l) \\
&\quad +(g_{ik,j} - \Gamma^l_{ij}g_{lk} - \Gamma^l_{kj}g_{il})\xi^k + g_{ik}(\xi^k_{,j} + \Gamma^k_{lj}\xi^l) \\
&= (g_{jk,i} - \Gamma_{kji} - \Gamma_{jki})\xi^k + (g_{ik,j} - \Gamma_{kij} - \Gamma_{ikj})\xi^k \\
&\quad (\Gamma_{ikj} + \Gamma_{jki})\xi^k + g_{jk}\xi^k_{,i} + g_{ik}\xi^k_{,j} \\
&= g_{ij,k}\xi^k + g_{jk}\xi^k_{,i} + g_{ik}\xi^k_{,j},
\end{aligned}$$

where we used formulas (2.6.10), (2.6.13), (2.6.14) and the relation

$$\Gamma_{ikj} + \Gamma_{jki} = g_{ij,k}$$

which follows from (2.6.15).

In a complete analogy to equations (2.6.19), the *Killing tensor* of order m is called a symmetric tensor K, whose components are solutions of the following Killing equations

$$\boxed{\nabla_{(i_1} K_{i_2 \ldots i_{m+1})} = 0.} \tag{2.6.20}$$

Killing tensors of order 2 will be analyzed in Sect. 4.3.

Let us consider one more geometric object, important for the construction of classical and quantum Hamiltonian systems. The parallel transport of a vector, as well as an arbitrary tensor, depends in general on the path along which it is performed. An alternative formulation of the same property is that if the tensor is transported along a closed path (a loop), the resulting tensor may differ from the initial one. Then we say that the considered manifold Q has nonzero curvature with respect to the chosen connection ∇. A useful object "measuring" the curvature (see literature [71, 116, 258] for a deeper insight into the problem) is the *curvature operator* $R(w, v)$ expressed by

$$R(w, v) := \nabla_w \nabla_v - \nabla_v \nabla_w - \nabla_{[w,v]} \equiv [\nabla_w, \nabla_v] - \nabla_{[w,v]}.$$

It is a derivation of the tensor algebra $\mathcal{T}(Q)$, which commutes with the contraction, vanishes on scalar fields: $R(w, v)f = 0,\ f \in \mathcal{F}(Q)$ and depends $\mathcal{F}(Q)$-linearly on both w and v. With the operator $R(w, v)$ one can relate a $R^{(3,1)}(Q)$ tensor, called the *Riemann tensor* (*curvature tensor*)

$$R(u, w, v, \alpha) := \langle \alpha, R(w, v)u \rangle \equiv \left\langle \alpha, \left([\nabla_w, \nabla_v] - \nabla_{[w,v]} \right) u \right\rangle .$$

Its components are expressed by connection coefficients in the following way

$$\boxed{R^i_{jkl} = \left\langle dx^i, (\nabla_k \nabla_l - \nabla_l \nabla_k)\partial_j \right\rangle = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl}\Gamma^i_{mk} - \Gamma^m_{jk}\Gamma^i_{ml}.} \tag{2.6.21}$$

Moreover, in particular we have

$$R(w, v)u = (R^i_{jkl} w^k v^l u^j)\partial_i \tag{2.6.22}$$

and antisymmetry in the last pair of indices

$$R^a_{bcd} = -R^a_{bdc}.$$

For $u = \partial_i$ and $v = \partial_j$ the curvature operator reduces to the commutator of the covariant derivatives

$$R(\partial_i, \partial_j) = \nabla_i \nabla_j - \nabla_j \nabla_i = [\nabla_i, \nabla_j].$$

As the result, one can show that the commutator of the coordinate covariant derivatives acts on the coordinate basis as

$$[\nabla_i, \nabla_j]\partial_k = R^l_{kij}\partial_l, \qquad [\nabla_i, \nabla_j]dx^k = -R^k_{lij}dx^l.$$

The first relation follows from (2.6.22) while the second one is a consequence of the identity

$$0 = R(w, v)\,\langle \alpha, u \rangle = \langle R(w, v)\alpha, u \rangle + \langle \alpha, R(w, v)u \rangle .$$

Thus, for an arbitrary tensor field $T^{(r,s)}$ there holds

$$\boxed{\begin{aligned} &T^{i_1 \ldots i_s}_{j_1 \ldots j_r;kl} - T^{i_1 \ldots i_s}_{j_1 \ldots j_r;lk} = ([\nabla_k, \nabla_l]T)^{i_1 \ldots i_s}_{j_1 \ldots j_r} \\ &= T^{m \ldots i_s}_{j_1 \ldots j_r} R^{i_1}_{mlk} + \ldots + T^{i_1 \ldots m}_{j_1 \ldots j_r} R^{i_s}_{mlk} - T^{i_1 \ldots i_s}_{m \ldots j_r} R^m_{j_1 lk} - \ldots - T^{i_1 \ldots i_s}_{j_1 \ldots m} R^m_{j_r lk}, \end{aligned}} \tag{2.6.23}$$

as

$$[\nabla_k, \nabla_l]T = T^{i_1 \ldots i_s}_{j_1 \ldots j_r}[\nabla_k, \nabla_l]dx^{j_1} \otimes \ldots \otimes [\nabla_k, \nabla_l]\partial_{i_s}.$$

The curvature tensor, as a tensor of $(3, 1)$-type, admits three contractions R^i_{ikl}, R^i_{jil}, R^i_{jki} all the resulting tensors being of $(2, 0)$-type. It follows from the antisymmetry of the last pair of indices that the second contraction differs from the third one only in a sign and it turns out that the first one vanishes for the Levi-Civite connection

$$R^i_{ikl} = 0, \quad R^i_{jil} = -R^i_{jki} = R_{jl} \quad \textit{Ricci curvature tensor.}$$

In the case of a Riemannian manifold a further contraction is possible and one can define a scalar field

$$R := R^a_a \equiv g^{ab}R_{ba} \quad \textit{scalar curvature.}$$

Obviously, in E^{n_+,n_-} Riemann tensor, Ricci tensor and scalar curvature vanishes.

At the end of this section let us briefly remind the notion of *normal coordinates*. The basic idea behind Riemann normal coordinates is to use the geodesics through a given point to define the coordinates for nearby points. Let the given point be P and consider some nearby point P_1. If P_1 is close enough to P then there exists a unique geodesic joining P_1 to P. The construction is as follows. On a manifold with connection (Q, ∇) define the *exponential map* (centered at point $P \in Q$)

$$\exp : T_P Q \to Q \qquad v \mapsto \exp v := \gamma_v(1) \equiv P_1$$

where $\gamma_v(t)$ is the geodesic. So one assigns to a vector v the point P_1 from Q which we arrive at $t = 1$, if at time $t = 0$ we start from the point P with the initial velocity v and all the time the motion is uniform and straight-line (i.e. along a geodesic). The coordinate representation of the exponential map is

$$\exp : v^i \to x^i(v^1, \ldots, v^n) \equiv x^i(P) + v^i - \frac{1}{2}\Gamma^i_{jk}(P)v^j v^k + \ldots$$

The exp maps bijectively (diffeomorphically) some neighborhood of zero in $T_P Q$ to some neighborhood of the point P. Moreover, the uniform straight-line motion in the tangent space is mapped to the uniform straight-line motion on a Riemann space

$$\exp(vt) = \gamma_v(t).$$

The fact that a neighborhood of a point P may be diffeomorphically mapped on a neighborhood of the zero in a linear space $T_P Q$ means in practice that we have local coordinates in the neighborhood of the point P. The most important property of the coordinates constructed in such a way is the vanishing of all Christoffel symbols in

the point P. So, let exp be exponential map centered at $P \in Q$. If in $T_P Q$ a basis e_i is fixed, we may introduce Riemann normal coordinates in the neighborhood of the point P according to the prescription

$$P_1 = (x^1, \ldots, x^n) \Longleftrightarrow P_1 = \exp(v) \equiv \exp(x^i e_i)$$

So a geodesic is constructed starting in P ($t = 0$) and passing through the point P_1 ($t = 1$) which is to be assigned coordinates. The geodesic has the unique initial velocity v with components v^i with respect to e_i. These components are identified by definition as the coordinates x^i. In these coordinates the geodesic $\gamma_v(t)$ reads $x^i(t) = v^i t$. Moreover, for any symmetric connection (the Levi-Civite in particular) $\Gamma^i_{jk}(P) = 0$ and $g_{ij,k}(P) = 0$, so that in the neighborhood of P

$$g_{ij}(x) = g_{ij}(P) + \frac{1}{2} g_{ij,kl}(P) x^k x^l + \ldots$$

i.e. the linear term is missing in the expansion.

2.7 Symplectic Manifolds and Symplectic Connections

An arbitrary closed and non-degenerate two-form ω on the manifold M is called a symplectic form and the pair (M, ω) is a *symplectic manifold*. The detailed consideration of such objects, both in non-degenerate and degenerate cases, is presented in Sect. 3.2. Here we consider a particular family of simplectic manifolds, i.e. cotangent bundles T^*Q of arbitrary manifolds Q. As was presented in Sect. 2.2, the manifold $M = T^*Q$ itself carries local coordinates (x^i, p_i) where the x are coordinates on the base Q and the p are the coordinates in the fibre. The most important canonical object on T^*Q is the canonical 1-form θ. Its pointwise definition is as follows. Let $p \in T^*Q$ and $w \in T_p T^*Q$, then

$$\langle \theta, w \rangle := \langle p, \tau_* w \rangle. \tag{2.7.1}$$

We thus first project the vector w to $x = \tau(p) \in Q$ and then insert it into the 1-form $p \in T^*_x Q \equiv \tau^{-1}(x)$, which corresponds to the point $p \in T^*Q$. The θ form in canonical coordinates (x^i, p_i) on T^*Q is given by

$$\theta = p_i dx^i.$$

Moreover, on T^*Q there exists a natural exact symplectic form ω given by $\omega = d\theta$ or in canonical coordinates

$$\omega = dp_i \wedge dx^i.$$

Thus, T^*Q is a symplectic manifold. In this book, on both classical and quantum level, the manifold Q is taken as a pseudo-Riemannian manifold. The manifold Q represents a configuration space of a given dynamical system. If $\dim Q = n$ then $\dim T^*Q = 2n$ and further on we will use the following notation: by Latin letters $i, j, k, \ldots$ we will denote indices ranging from 1 to n, by letters $\bar{\imath}, \bar{\jmath}, \bar{k}, \ldots$ indices ranging from n to $2n$ and by Greek letters $\alpha, \beta, \gamma, \ldots$ indices ranging from 1 to $2n$. Thus we have $\alpha = (i, \bar{\imath})$ and $\bar{\imath} = n + i$, respectively.

Let (Q, g, ∇) be a Riemannian space and T^*Q its cotangent bundle. Let V be a union of n-dimensional linear spaces tangent to the fibres of T^*Q. It is an integrable distribution on T^*Q which we called the *vertical distribution*. Let us remind that for a manifold Q, a k-dimensional distribution is called a subset $D \subset TQ$ of the tangent bundle such that for every $x \in Q$ a D_x is k-dimensional subspace of T_xQ. A distribution is differentiable if it is spanned by k vector fields $X_1(x), \ldots, X_k(x)$. A distribution is integrable if it can be spanned by commuting vector fields.

A torsionless linear connection ∇ on Q determines uniquely on T^*Q an n-dimensional distribution complementary to V. This distribution is called the *horizontal distribution* associated with ∇ and is denoted by H. Obviously

$$V \oplus H = TT^*Q$$

and the pair (H, V) defines the so called almost product structure on T^*Q [267].

Let $\{\tau^{-1}(U), (x^i, p_i)\}$ be an induced coordinate system on T^*Q. The horizontal distribution H restricted to $\tau^{-1}(U)$ is spanned by the n independent vector fields

$$D_j \equiv \frac{\delta}{\delta x^j} = \frac{\partial}{\partial x^j} + \Gamma_{ji}\frac{\partial}{\partial p_i}, \quad \Gamma_{ji} = p_k\Gamma^k_{ji}.$$

The vertical distribution V restricted to $\tau^{-1}(U)$ is spanned by the n independent vector fields

$$D_{\bar{\jmath}} \equiv D^j = \frac{\partial}{\partial p_j},$$

It follows that $\{D_\alpha\} = \{D_j, D^j\} = \{\frac{\delta}{\delta x^j}, \frac{\partial}{\partial p_j}\}$ constitute a frame on $\tau^{-1}(U)$. As the frame is adopted to the almost product structure (H, V) it is called *adopted frame* on $\tau^{-1}(U)$. The coframe $\{D^\alpha\} = \{dx^j, \delta p_j\}$, dual to the adopted frame, is given by

$$\delta p_j = -\Gamma_{ji}dx^i + dp_j.$$

Another natural frame on $\tau^{-1}(U)$ is the one related to canonical coordinates $(\xi^\alpha) = (x^i, p_i)$, called *Darboux frame* $\{\partial_\alpha\} = \{\partial_j, \partial_{\bar{\jmath}}\} \equiv \{\partial_j, \partial^j\} = \{\frac{\partial}{\partial x^j}, \frac{\partial}{\partial p_j}\}$ with related dual coframe $\{dx^j, dp_j\}$. Both frames and coframes are related in the

following way

$$\begin{pmatrix} \frac{\delta}{\delta x} \\ \frac{\partial}{\partial p} \end{pmatrix} = \begin{pmatrix} I_n & \Gamma \\ 0 & I_n \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial p} \end{pmatrix},$$

$$\begin{pmatrix} dx \\ \delta p \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ -\Gamma & I_n \end{pmatrix} \begin{pmatrix} dx \\ dp \end{pmatrix},$$

where $(\Gamma)_{ij} = \Gamma_{ij}$ and $\Gamma^T = \Gamma$.

Starting from a torsionless linear connection ∇ on Q, such that the components in U of ∇ are Γ^k_{ji}, the symmetric tensor of $(2,0)$-type on $\tau^{-1}(U)$ with component matrix

$$\tilde{g} = \begin{pmatrix} -2\Gamma & I_n \\ I_n & 0 \end{pmatrix} \tag{2.7.2}$$

in Darboux frame is called the Riemann extension of ∇. It follows that the corresponding component matrix in the adopted frame is

$$\tilde{g} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\tilde{\nabla}$ be the Riemannian connection on T^*Q associated with the Riemann extension $\tilde{g}$. It is called a complete lift of ∇ to T^*Q [268]. The non-zero connection coefficients $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ in Darboux frame, calculated according to (2.6.16), where

$$\tilde{G} = \tilde{g}^{-1} = \begin{pmatrix} 0 & I_n \\ I_n & 2\Gamma \end{pmatrix},$$

are

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \quad \tilde{\Gamma}^{\bar{k}}_{i\bar{j}} = -\Gamma^j_{ik}, \tag{2.7.3}$$

$$\tilde{\Gamma}^{\bar{k}}_{ij} = p_l(2\Gamma^r_{ij}\Gamma^l_{rk} + \Gamma^l_{ij,k} - \Gamma^l_{ki,j} - \Gamma^l_{jk,i}) = p_l(\Gamma^r_{ij}\Gamma^l_{rk} + \Gamma^r_{ik}\Gamma^l_{rj} - \Gamma^l_{jk,i} - R^l_{ijk}).$$

The connection coefficients in the adopted frame cannot be calculated according to formula (2.6.16) as D_α operators do not commute. The right formulas are more complicated and we refer the reader to the literature [202, 267]. The non-zero components $\tilde{\Gamma}^{\alpha}_{\beta\gamma}$ of the complete lift $\tilde{\nabla}$ in the adopted frame are

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij}, \quad \tilde{\Gamma}^{\bar{k}}_{i\bar{j}} = -\Gamma^j_{ik}, \quad \tilde{\Gamma}^{\bar{k}}_{ij} = -p_l R^l_{kij}.$$

If, for example, $\tilde{X}$ is a vector field on T^*Q whose frame components are $\tilde{X}^\alpha$, then

$$\tilde{\nabla}_\beta \tilde{X}^\alpha = D_\beta(\tilde{X}^\alpha) + \tilde{\Gamma}^\alpha_{\beta\gamma}\tilde{X}^\gamma$$

are exactly the frame components of the covariant derivative $\tilde{\nabla}\tilde{X}$ of $\tilde{X}$.

A *symplectic connection* on T^*Q is a symmetric linear connection $\nabla^{(S)}$ on T^*Q such that $\nabla^{(S)}\omega = 0$. So, the symplectic connection is defined locally by

$$\nabla^{(S)}_\delta \omega_{\alpha\beta} = \partial_\delta \omega_{\alpha\beta} - \omega_{\kappa\beta}\,\bar{\Gamma}^\kappa_{\alpha\delta} - \omega_{\alpha\kappa}\,\bar{\Gamma}^\kappa_{\beta\delta} = 0, \quad \bar{\Gamma}^\delta_{\alpha\beta} = \bar{\Gamma}^\delta_{\beta\alpha}, \quad \alpha, \beta, \delta = 1, \ldots, 2n,$$

where $\bar{\Gamma}^\delta_{\alpha\beta}$ are the local components of $\nabla^{(S)}$. In Darboux coordinates, as

$$(\omega_{\alpha\beta}) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix},$$

we have

$$\bar{\Gamma}_{\beta\alpha\delta} - \bar{\Gamma}_{\alpha\beta\delta} = 0, \quad \bar{\Gamma}_{\alpha\beta\delta} - \bar{\Gamma}_{\alpha\delta\beta} = 0, \tag{2.7.4}$$

where

$$\bar{\Gamma}_{\alpha\beta\delta} := \omega_{\alpha\kappa}\bar{\Gamma}^\kappa_{\beta\delta}.$$

From (2.7.4) one infers that if (M, ω) is a symplectic manifold and ∇ a linear connection on M, then ∇ is a symplectic connection if and only if for every Darboux coordinate system in M the components $\Gamma_{\alpha\beta\delta}$ of ∇ are totally symmetric with respect to the indices (α, β, δ).

Let $\tilde{\nabla}$ be any symmetric linear connection on M and $\tilde{\Gamma}^\delta_{\alpha\beta} = \tilde{\Gamma}^\delta_{\beta\alpha}$ its local components. Then one can verify that

$$\bar{\Gamma}^\delta_{\alpha\beta} := \tilde{\Gamma}^\delta_{\alpha\beta} - \tfrac{1}{3}\omega^{\delta\kappa}(\tilde{\nabla}_\alpha\omega_{\beta\kappa} + \tilde{\nabla}_\beta\omega_{\alpha\kappa}), \quad \omega^{\alpha\kappa}\omega_{\kappa\beta} = \delta^\alpha_\beta,$$

are the components of a symplectic connection [119]. Then,

$$\begin{aligned}\bar{\Gamma}_{\delta\alpha\beta} &= \tilde{\Gamma}_{\delta\alpha\beta} - \tfrac{1}{3}(\tilde{\nabla}_\alpha\omega_{\beta\delta} + \tilde{\nabla}_\beta\omega_{\alpha\delta}) \\ &= \tfrac{1}{3}(\tilde{\Gamma}_{\delta\alpha\beta} + \tilde{\Gamma}_{\beta\delta\alpha} + \tilde{\Gamma}_{\alpha\beta\delta}) - \tfrac{1}{3}(\partial_\alpha\omega_{\beta\delta} + \partial_\beta\omega_{\alpha\delta}),\end{aligned}$$

where $\Gamma_{\alpha\beta\delta} := \omega_{\alpha\kappa}\Gamma^\kappa_{\beta\delta}$. Hence, in any Darboux coordinates

$$\bar{\Gamma}_{\delta\alpha\beta} = \tfrac{1}{3}(\tilde{\Gamma}_{\delta\alpha\beta} + \tilde{\Gamma}_{\beta\delta\alpha} + \tilde{\Gamma}_{\alpha\beta\delta}) \tag{2.7.5}$$

we call this symplectic connection the symplectic connection on M induced by the symmetric linear connection $\tilde{\nabla}$ on M.

Assume that M is the cotangent bundle T^*Q over Q, where Q is n-dimensional differentiable manifold endowed with a Riemannian metric g. From the previous considerations we know that (T^*Q, ω) with ω defined by (2.7.1) is a symplectic manifold, but also $(T^*Q, \tilde{g})$ with $\tilde{g}$ defined by (2.7.2) is a Riemannian manifold and the Levi-Civita connection of the metric $\tilde{g}$ on T^*Q is given by (2.7.3). This connection induces the symplectic connection on T^*Q according to (2.7.5). Thus, we conclude that in the presented case the symplectic connection on T^*Q is induced by the Levi-Civita connection on Q [54, 55, 219].

From (2.7.3) and (2.7.5) one finds the components $\bar{\Gamma}^{\delta}_{\alpha\beta}$ of the symplectic connection $\nabla^{(S)}$ on T^*Q in Darboux frame in terms of the Christoffel symbols Γ^k_{ij} of the metric g on Q

$$\boxed{\begin{aligned} &\bar{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk}, \quad \bar{\Gamma}^{\bar{i}}_{\bar{j}k} = -\Gamma^{j}_{ik}, \quad \bar{\Gamma}^{\bar{i}}_{j\bar{k}} = -\Gamma^{k}_{ji}, \\ &\bar{\Gamma}^{\bar{i}}_{jk} = p_l(\Gamma^{r}_{jk}\Gamma^{l}_{ri} + \Gamma^{r}_{ik}\Gamma^{l}_{rj} - \Gamma^{l}_{ij,k} - \tfrac{1}{3}R^{l}_{ijk} - \tfrac{1}{3}R^{l}_{jik}), \end{aligned}} \tag{2.7.6}$$

with the remaining components equal zero. In the adopted frame $\{D_i, D^j\}$ the connection $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ takes the form

$$\boxed{\bar{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk}, \quad \bar{\Gamma}^{\bar{i}}_{\bar{j}k} = -\Gamma^{j}_{ik}, \quad \bar{\Gamma}^{\bar{i}}_{jk} = -\frac{1}{3}p_l(R^{l}_{ijk} + R^{l}_{jik}),} \tag{2.7.7}$$

with the remaining components equal zero. Straightforward but tedious calculations lead to the following components $\bar{R}^{\alpha}_{\beta\gamma\delta}$ for the curvature tensor of the symplectic torsionless connection $\nabla^{(S)}$ given by (2.7.6)

$$\begin{aligned} &\bar{R}^{i}_{jkl} = R^{i}_{jkl}, \quad \bar{R}^{\bar{i}}_{jk\bar{l}} = \tfrac{2}{3}R^{l}_{(ij)k}, \\ &\bar{R}^{\bar{i}}_{jkl} = -\tfrac{1}{3}p_r\left(R^{r}_{jkl;i} + R^{r}_{ikl;j} - 6\Gamma^{r}_{s(i}R^{s}_{j)kl} + 4R^{s}_{(ij)[k}\Gamma^{r}_{l]s}\right), \end{aligned} \tag{2.7.8}$$

with all remaining independent components equal zero, where $(\cdot,\cdot)$ and $[\cdot,\cdot]$ stand for the symmetrization and anti-symmetrization, respectively. From (2.7.8) it is possible to calculate the components of the Ricci curvature tensor, $\bar{R}_{\alpha\beta} = \bar{R}^{\gamma}_{\alpha\gamma\beta}$, receiving

$$\bar{R}_{ij} = \tfrac{2}{3}R_{ij}, \quad \bar{R}_{i\bar{j}} = K_{\bar{i}j} = \bar{R}_{\bar{i}\bar{j}} = 0. \tag{2.7.9}$$

In an analogical way as it was done in the previous section, Riemann normal coordinates can be introduced on T^*Q with respect to the symplectic connection $\nabla^{(S)}$.

Chapter 3
Classical Hamiltonian Mechanics

In this chapter we present the basic facts about the underlying structure of classical Hamiltonian mechanics and in particular statistical Hamiltonian mechanics. The theory is formulated in the frame of Poisson geometry and presymplectic geometry. On the level of statistical Hamiltonian mechanics we introduce the language and notions familiar from the quantum level in order to further unify both theories. In particular we consider such issues as Hamiltonian representation of variational problems of arbitrary order as well as the reduction of Poisson bi-vectors on submanifolds, important for further separability theory.

3.1 Lagrange Formalism and Canonical Hamiltonian Formalism

In this section we briefly remind the reader equations of motion of a particle in Riemannian space Q and their relation with extremals of an appropriate functional of first order. We also remind the reader a canonical Hamiltonian representation of considered dynamics in cotangent bundle T^*Q. What is important, the Hamiltonian formalism is not necessarily related directly to some dynamics on configuration space. We illustrate that fact presenting Ostrogradsky Hamiltonian representation in a phase space for extremals of functionals of arbitrary order. We also present how to adopt a flat Riemannian geometry to Ostrogradsky representation in the case of functionals of a single variable.

3.1.1 Equations of Motion in Riemann Space

Having been equipped with the theory of the linear connection, we may return to the concept of acceleration. Let (Q, g) be a Riemannian space and $\gamma(t)$ be a curve

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in Q, parametrized by t. If we realize what is actually performed with the velocity field $v(t)$ on a curve in order to compute the acceleration $a(t)$, we immediately conclude that the acceleration at a given point on the curve is the absolute derivative of the velocity field along the curve, or in other words the covariant derivative of the velocity along the velocity itself

$$a = \nabla_{\dot\gamma}\dot\gamma = \nabla_v v, \qquad v := \dot\gamma\text{-the velocity vector.}$$

A particularly interesting case arises when acceleration vanishes $a = 0$. Such curves represent trajectories of a *uniform motion* (a free particle motion), i.e. are a reasonable generalization of the notion of *straight lines* from $\mathbb{R}^n$. The respective curves, characterized by the equation

$$\nabla_{\dot\gamma}\dot\gamma = 0,$$

are called *affinely parametrized geodesics*. The geodesic equation in local coordinates takes the form

$$\ddot x^i + \Gamma^i_{jk}\dot x^j \dot x^j \equiv x^i_{tt} + \Gamma^i_{jk} x^j_t x^k_t = 0, \quad i = 1, \ldots, n \tag{3.1.1}$$

so we get a system of n ordinary second-order differential equations for the unknown functions $x^i(t)$, which parametrize γ in these coordinates. Equations (3.1.1) follow immediately from (2.6.7). The geodesics in $\mathbb{R}^n$, when expressed in Cartesian coordinates, are "ordinary" straight lines

$$\ddot x^i \equiv x^i_{tt} = 0 \Longrightarrow x^i(t) = x^i_0 + v^i_0 t, \quad x^i_0 = x^i(0),\ v^i_0 = x^i_t(0).$$

In the general case, the first two terms of the expansion in t of the coordinate representation of a geodesic are

$$x^i(t) = x^i(0) + \dot x^i(0)t + \tfrac{1}{2}\ddot x^i(0)t^2 + \ldots = x^i_0 + v^i_0 t - \tfrac{1}{2}\Gamma^i_{jk} v^j_0 v^k_0 t^2 + \ldots$$

Now, let us come back to the issue of the parametrization of geodesics. One may also traverse the geodesic path, which corresponds to the uniform straight line motion, non-uniformly. Although the acceleration does not vanish in this case, it remains tangent to the path

$$\nabla_{\dot\gamma}\dot\gamma \sim \dot\gamma \Longrightarrow \nabla_{\dot\gamma}\dot\gamma = f(t)\dot\gamma.$$

Let γ_a be an affinely parametrized geodesic and let $\gamma := \gamma_a \circ \sigma$ be a reparametrized curve $\gamma(t) = \gamma_a(\sigma(t))$, $\sigma'(t) > 0$, then

$$\nabla_{\dot\gamma}\dot\gamma = \sigma''(t)\dot\gamma.$$

Indeed, as $\dot{\gamma} = \sigma'(t)\dot{\gamma}_a$ so

$$\begin{aligned}\nabla_{\dot{\gamma}}\dot{\gamma} &= \sigma'\nabla_{\dot{\gamma}_a}(\sigma'\dot{\gamma}_a) = \sigma'(\sigma''\dot{\gamma}_a + \sigma'\nabla_{\dot{\gamma}_a}\dot{\gamma}_a) = \sigma''\sigma'\dot{\gamma}_a \\ &= \sigma''(t)\dot{\gamma}.\end{aligned}$$

Affine reparametrization $\sigma(t) = \alpha t + \beta$ does not spoil the affine parametrization of a geodesic. Thus, the affinely parametrized geodesic curve represents the motion of a free particle of unit mass and is the solution of equations of motion

$$a = 0 \;\Leftrightarrow\; a^i = 0,\ i = 1, \ldots n,$$

where a is the acceleration vector.

If $F(x) = F^i(x)\partial_i$ is a force (vector) field acting on a particle, then the parametric trajectory of a particle is a solution of the generalized Newton equation in a Riemannian space (Q, g): $a = F$, which in local coordinates takes the form

$$\boxed{x^i_{tt} + \Gamma^i_{jk}x^j_t x^k_t = F^i(x).}$$

For the *potential force field*, when $F(x)$ is a gradient vector field of a (by definition negative) scalar potential (*potential energy*) $V(x)$

$$F(x) = -G\,dV(x) \Longrightarrow F^i(x) = -G^{ij}\partial_j V(x), \quad i = 1, \ldots, n,$$

equations of motion take the form

$$\boxed{x^i_{tt} + \Gamma^i_{jk}x^j_t x^k_t = -G^{ij}\partial_j V(x), \quad i = 1, \ldots, n.} \tag{3.1.2}$$

In the case of a *flat Riemannian space* (pseudo-Euclidean space) there exist flat coordinate systems for which $\Gamma^i_{jk} = 0$. A flat coordinate system which is orthonormal, i.e. $g(\partial_i, \partial_j) = \pm 1$, is called *pseudo-Euclidean coordinate system*. In the particular case of Euclidean space $\mathbb{E}^n$ and Euclidean (Cartesian) coordinates, equations of motion (3.1.2) turn into the well known Newton equations

$$x^i_{tt} = -\partial_i V(x) = F^i(x), \quad i = 1, \ldots, n.$$

Now we show how to derive equations of motion (3.1.2) from a variational problem. In a Riemann space (Q, g) let us consider a functional (*action integral*) for the unit mass particle motion $\gamma(t)$

$$S[\gamma] := \int_{t_1}^{t_2} \left[\tfrac{1}{2}g(\dot{\gamma}, \dot{\gamma}) - V(x)\right] dt \equiv \int_{t_1}^{t_2} L(x, x_t)dt, \tag{3.1.3}$$

where

$$L(x, x_t) = \tfrac{1}{2} g_{ij} x_t^i x_t^j - V(x) = T(x, v) - V(x). \tag{3.1.4}$$

The functional density L is called *Lagrangian*, $T(x, v = x_t)$ is the *kinetic energy* of the particle and $V(x)$ the respective potential energy. Trajectories which are extremals of the functional (3.1.3) are solutions of the so called Euler-Lagrange equations

$$\boxed{\delta S[\gamma] = 0 \Longleftrightarrow \frac{\delta S}{\delta x^i} = 0 \Longleftrightarrow \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x_t^i} = 0, \quad i = 1, \ldots, n,} \tag{3.1.5}$$

where $\delta S[\gamma]$ is a covector being a variational derivative of the functional (3.1.3) (see next subsection for details of the derivation).

The most important is that Euler-Lagrange equations (3.1.5) are equivalent to equations of motion (3.1.2) as they share the same set of solutions. Indeed, applying the relation (2.6.16) to the Levi-Civita connection, we have

$$\begin{aligned}
0 &= \frac{\partial L}{\partial x^k} - \frac{d}{dt}\frac{\partial L}{\partial x_t^k} \\
&= \tfrac{1}{2} g_{ij,k} x_t^i x_t^j - \tfrac{1}{2}\tfrac{d}{dt}(g_{kj} x_t^j + g_{ik} x_t^i) - \partial_k V \\
&= \tfrac{1}{2}(g_{ij,k} x_t^i x_t^j - g_{kj,r} x_t^r x_t^j - g_{kj} x_{tt}^j - g_{ik,r} x_t^i x_t^r - g_{ik} x_{tt}^i) - \partial_k V \\
&= -g_{kr} x_{tt}^r - \tfrac{1}{2}(g_{ik,j} + g_{kj,i} - g_{ij,k}) x_t^i x_t^j - \partial_k V \\
&= -g_{kr}\left(x_{tt}^r + \Gamma_{ij}^r x_t^i x_t^j\right) - \partial_k V(x) \\
&= -g_{kr}\left(x_{tt}^r + \Gamma_{ij}^r x_t^i x_t^j + G^{rk}\partial_k V(x)\right).
\end{aligned}$$

The equivalence follows from the assumption that $\det g \neq 0$.

3.1.2 Hamiltonian Representation of Variational Problems

Let us remind the reader of the classical problem from analytical mechanics: how to transform a Lagrangian representation of equations of motion (3.1.5), being a system of n ordinary differential equations (ODE's) of the second order on Q (nonlinear in general), into a system of $2n$ ordinary differential equations of the first order on T^*Q? First, let us notice that equations (3.1.5) can be immediately

transformed into a system of $2n$ ordinary differential equations of the first order on TQ

$$x_t^i = v^i, \quad i = 1, \ldots, n, \tag{3.1.6a}$$

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = 0, \quad i = 1, \ldots, n. \tag{3.1.6b}$$

Next, we transform equations (3.1.6) from TQ to the equivalent equations of motion on T^*Q. Notice that the metric tensor g maps any vector $v \in T_xQ$ onto the dual covector $p \in T^*Q$ called generalized momentum

$$p_k = g_{ki}v^i = \frac{\partial L}{\partial v^k}. \tag{3.1.7}$$

Let us define the following function on T^*Q

$$H(x, p) = p_k v^k(x, p) - L, \tag{3.1.8}$$

where

$$v^i = G^{ik}p_k$$

and follows from (3.1.7) and the invertibility of the metric g. For the Lagrangian L of the form (3.1.4), i.e.

$$L(x, v) = \tfrac{1}{2}g_{ij}v^i v^j - V(x),$$

we get immediately

$$H(x, p) = \tfrac{1}{2}G^{ij}(x)p_i p_j + V(x) = T + V. \tag{3.1.9}$$

Hamiltonian (3.1.9), or more precisely *classical Hamiltonian*, will be further called a *natural Hamiltonian*, as it is the sum of kinetic energy T and potential energy V of a particle of unit mass.

On T^*Q equations of motion (3.1.6a), (3.1.6b) take the form

$$\boxed{x_t^i = \frac{\partial H}{\partial p_i}, \quad (p_i)_t = -\frac{\partial H}{\partial x^i}.} \tag{3.1.10}$$

Equations of motion (3.1.6) and (3.1.10) are equivalent, and relations (3.1.7), (3.1.8) are called the *Legendre transformation*, invertible as far as g is non-degenerate. In fact, as

$$\frac{\partial L}{\partial x^i} = -\frac{\partial H}{\partial x^i}, \quad v^i = \frac{\partial H}{\partial p_i}, \quad p_i = \frac{\partial L}{\partial v^i},$$

so

$$0 = x_t^i - v^i = x_t^i - \frac{\partial H}{\partial p_i},$$

and

$$0 = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial v^i} = -\frac{\partial H}{\partial x^i} - (p_i)_t.$$

More information about Legendre transformation as well as their coordinate free formulation the reader finds in the literature [116, 178, 258].

Equations of motion (3.1.10) are called *canonical Hamiltonian equations* of motion on a space $M = T^*Q$, called further the *phase space*. Coordinates (x, p) are called *canonical coordinates* (*Darboux coordinates*). A coordinate p_i, constructed with the help of transformation (3.1.7), and called generalized momentum coordinate (fiber coordinate), canonically conjugates with a position coordinate x^i.

Now, we briefly demonstrate that the Hamiltonian formalism is not necessarily directly related with particles dynamics, so in fact it is much more universal. Let us consider the class of functionals

$$S = \int_{t_1}^{t_2} L(x, x_t, x_{tt}, \ldots)dt \tag{3.1.11}$$

where functional densities $L[x] = L(x, x_t, x_{tt}, \ldots)$ are differential functions of arbitrary order. Consider the following problem: for which $x(t)$ the functional (3.1.11) attains a local extremum, minimum in particular. A necessary condition of the extremum takes the form

$$\frac{d}{d\epsilon} S(x + \epsilon\eta)_{|\epsilon=0} = 0,$$

where $\eta(t)$ is an arbitrary function that has at least as many t-derivatives as $x(t)$ does and vanishes at the endpoints t_1 and t_2. Then we have

$$\begin{aligned}
\frac{d}{d\epsilon} S(x + \epsilon\eta)_{|\epsilon=0} &= \int_{t_1}^{t_2} \frac{d}{d\epsilon} L(x + \epsilon\eta)_{|\epsilon=0}\, dt \\
&= \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i}\eta^i + \frac{\partial L}{\partial x_t^i}\frac{d\eta^i}{dt} + \frac{\partial L}{\partial x_{tt}^i}\frac{d^2\eta^i}{dt^2} + \ldots \right) dt \\
&\overset{\text{by parts}}{=} \int_{t_1}^{t_2} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x_t^i} + \frac{d^2}{dt^2}\frac{\partial L}{\partial x_{tt}^i} - \ldots \right) \eta^i dt \\
&= \int_{t_1}^{t_2} \sum_{i=1}^{n} \frac{\delta S}{\delta x^i}\eta^i dt = \int_{t_1}^{t_2} \langle \delta S, \eta \rangle\, dt
\end{aligned}$$

where $\delta = \left(\frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n}\right)$ and

$$\frac{\delta S}{\delta x^i} = \sum_{k\geq 0}\left(-\frac{d}{dt}\right)^k \frac{\partial L}{\partial x^i_{kt}}, \qquad x^i_{kt} = \frac{d^k}{dt^k}x^i \tag{3.1.12}$$

is the variational derivative of functional (3.1.11) with respect to $x^i(t)$ [212]. From arbitrariness of η it follows that functional (3.1.11) attains a local extremum for $x^i(t)$ being solution of the following system of ODE's

$$\boxed{\frac{\delta S}{\delta x^i} = \sum_{k\geq 0}\left(-\frac{d}{dt}\right)^k \frac{\partial L}{\partial x^i_{kt}} = 0, \quad i = 1, \ldots, n} \tag{3.1.13}$$

known as Euler-Lagrange equations. In particular, for $L = L(x, x_t)$, equations (3.1.13) reduce to the form (3.1.5) considered in the previous subsection. Notice that

$$\ker \delta = \operatorname{Im} \frac{d}{dt}. \tag{3.1.14}$$

Example 3.1 Consider a two dimensional case with the notation $x^1 = x$, $x^2 = y$. For density

$$L[x(t), y(t)] = -\tfrac{1}{2}x_t^2 + 2y_{tt}^2 + \tfrac{1}{3}y^2 x_t + \tfrac{1}{6}x^3$$

we have

$$\frac{\delta L}{\delta x} = \tfrac{1}{2}x^2 - \frac{d}{dt}\left(\tfrac{1}{3}y^2 - x_t\right) = \tfrac{1}{2}x^2 - \tfrac{2}{3}yy_t + x_{tt},$$
$$\frac{\delta L}{\delta y} = \tfrac{2}{3}yx_t + \frac{d^2}{dt^2}(4y_{tt}) = \tfrac{2}{3}yx_t + 4y_{4t}.$$

If on the other hand

$$L[x(t), y(t)] = \frac{d}{dt}(xy + x_t y_t) = x_t y + xy_t + x_{tt}y_t + x_t y_{tt}$$

then

$$\frac{\delta L}{\delta x} = y_t - \frac{d}{dt}(y + y_{tt}) + \frac{d^2}{dt^2}y_t = 0,$$
$$\frac{\delta L}{\delta y} = x_t - \frac{d}{dt}(x + x_{tt}) + \frac{d^2}{dt^2}x_t = 0$$

what illustrates the relation (3.1.14).

To be more specific, let us consider an action functional S, whose density is represented by a Lagrangian of m-th order

$$L = L\left(x, x_t, \ldots, x_{mt}\right)$$

and which fulfills the non degeneracy condition

$$\det\left(\frac{\partial^2 L}{\partial x^i_{mt}\partial x^j_{mt}}\right) \neq 0.$$

The condition $\delta S = 0$ is equivalent with n ODE's

$$\frac{\delta L}{\delta x^i} = 0, \quad i = 1, \ldots, n \tag{3.1.15}$$

of order $2m$ (nonlinear in general), which are not related directly to any particle dynamics. Nevertheless, there exists a generalized Legendre transformation to $2mn$ dimensional phase space, where equations (3.1.15) are represented by a canonical Hamiltonian equations

$$(q^{j,k})_t = \frac{\partial H}{\partial p_{j,k}}, \quad \left(p_{j,k}\right)_t = -\frac{\partial H}{\partial q^{j,k}} \tag{3.1.16}$$

in terms of the so called canonical *Ostrogradsky variables* (q, p) [262]

$$q^{j,k} := x^j_{(k-1)t}, \quad k = 1, \ldots, m, \quad j = 1, \ldots, n, \tag{3.1.17}$$

$$p_{j,k} = \frac{\delta L}{\delta x^j_{kt}} := \sum_{i=0}^{m-k}\left(-\frac{d}{dt}\right)^i \frac{\partial L}{\partial x^j_{(k+i)t}} \tag{3.1.18}$$

with the Hamiltonian function in the form

$$H(q, p) = \sum_{j=1}^{n}\left[p_{j,m}q_t^{j,m} + \sum_{k=1}^{m-1} p_{j,k}q^{j,k+1}\right] - L. \tag{3.1.19}$$

Example 3.2 Consider the action

$$S[x(t)] = \int\left(\tfrac{1}{2}x_{tt}^2 - 5xx_t^2 + \tfrac{5}{2}x^4\right)dt,$$

generated by the second order Lagrangian on the one-dimensional manifold and the related 4-th order ODE

$$\frac{\delta L}{\delta x} = x_{4t} + 10xx_{2t} + 5x_t^2 + 10x^3 = 0.$$

Then, Ostrogradsky coordinates are of the form

$$q^1 = x,\ q^2 = x_t,\quad p_1 = \frac{\delta L}{\delta x_t} = -10xx_t - x_{3t},\quad p_2 = \frac{\delta L}{\delta x_{tt}} = x_{tt}$$

and the respective Hamiltonian

$$H(q, p) = p_2 q_t^2 + p_1 q^2 - L = \tfrac{1}{2}p_2^2 + q^2 p_1 + 5q^1(q^2)^2 - \tfrac{5}{2}(q^1)^4.$$

Thus, four Hamiltonian equations (3.1.16) of the first order

$$\begin{aligned}
(q^1)_t &= \frac{\partial H}{\partial p_1} = q^2,\\
(q^2)_t &= \frac{\partial H}{\partial p_2} = p_2,\\
(p_1)_t &= -\frac{\partial H}{\partial q_1} = 10(q^1)^3 - 5(q^2)^2,\\
(p_2)_t &= -\frac{\partial H}{\partial q_2} = -p_1 - 10q^1 q^2
\end{aligned}$$

are equivalent to one equation of the fourth order.

Example 3.3 Consider the action

$$S[x(t)] = \int \left(\tfrac{1}{2}x_{3t}^2 - 7xx_{tt}^2 + 35x^2x_t^2 - 7x^5\right) dt,$$

generated by the third order Lagrangian on the one-dimensional manifold and the related 6-th order ODE

$$x_{6t} + 14xx_{4t} + 28x_t x_{3t} + 21x_{tt}^2 + 70x^2x_{tt} + 70xx_t^2 + 35x^4 = 0.$$

Then, Ostrogradsky coordinates are of the form

$$q^1 = x,\ q^2 = x_t,\ q^3 = x_{tt},\quad p_1 = \frac{\delta L}{\delta x_t} = x_{5t} + 14xx_{3t} + 14x_t x_{tt} + 70x^2x_t,$$

$$p_2 = \frac{\delta L}{\delta x_{tt}} = -x_{4t} - 14xx_{tt},\quad p_3 = \frac{\delta L}{\delta x_{3t}} = x_{3t}$$

and the Hamiltonian (3.1.19)

$$H(q, p) = p_3 q_t^3 + p_1 q^2 + p_2 q^3 - L = \tfrac{1}{2}p_3^2 + q^2 p_1 + q^3 p_2 + 7q^1(q^3)^2 - 35(q^1)^2(q^2)^2 + 7(q^1)^5.$$

Again, six Hamiltonian equations of the first order

$$
\begin{aligned}
(q^1)_t &= \frac{\partial H}{\partial p_1} = q^2,\\
(q^2)_t &= \frac{\partial H}{\partial p_2} = q^3,\\
(q^3)_t &= \frac{\partial H}{\partial p_3} = p_3,\\
(p_1)_t &= -\frac{\partial H}{\partial q_1} = -7(q^3)^2 + 70q^1(q^2)^2 - 35(q^1)^4,\\
(p_2)_t &= -\frac{\partial H}{\partial q_2} = -p_1 + 70(q^1)^2 q^2,\\
(p_3)_t &= -\frac{\partial H}{\partial q_3} = -p_3 - 14q^1 q^3,
\end{aligned}
$$

are equivalent to one equation of the 6-th order.

The presented examples demonstrate that the Ostrogradsky parametrization (3.1.17), (3.1.18) of the phase space $M = T^*Q$, where (q^i) are local coordinates in some Q, is "non physical" in the sense that it does not describe any particle dynamics (3.1.2) on Q, as Hamiltonians (3.1.19) are not natural Hamiltonians (3.1.9).

3.1.3 Newton Representation of Variational Problems

In this subsection we present a particular parametrization for higher order ODE's of a single variable, which turns them into a set of Newton equations in pseudo-Euclidean space and then, as a consequence, into a Hamiltonian representation with natural Hamiltonian function (3.1.9) [41]. In other words, we show the equivalence between a single variable variational problem of arbitrary order with particle dynamics in pseudo-Euclidean space. Later on we will call such constructions the process of adaptation of Euclidean (Riemannian in general) geometry to Poisson geometry (see Sect. 4.3) and will be important for the construction of position representation (Riemannian representation) of quantum mechanics.

Let us consider some $2n$-th order ordinary differential equations of a single variable x in the form

$$x_{2nt} + \gamma_{n+1}[x] = 0, \tag{3.1.20}$$

where $\gamma_{n+1}[x] = \gamma_{n+1}(x, x_t, \ldots, x_{2(n-1)t})$ is a differential polynomial. We assume that (3.1.20) is homogeneous with respect to the scaling transformation

$$x \to \varepsilon x, \quad t \to \varepsilon^{-1/2} t. \tag{3.1.21}$$

For these equations we introduce a new parametrization which turns them into a set of Newton equations of the form

$$\begin{aligned} (r^k)_{tt} &= r^{k+1} + f_k(r^1, \ldots, r^k), \qquad k = 1, \ldots, n-1, \\ (r^n)_{tt} &= f_n(r^1, \ldots, r^n), \end{aligned} \tag{3.1.22}$$

where each variable r^k scales as $r^k \to \varepsilon^k r^k$ and functions f_k are homogeneous polynomials of order $k+1$ with respect to this scaling transformation. If equations (3.1.22) are Lagrangian then its Lagrangian function has an indefinite kinetic energy term and scales as $L \to \varepsilon^{n+2} L$.

Let us start from the 4-th order equations. A general 4-th order equation which is homogeneous of order ε^3 with respect to the scaling (3.1.21), has the form

$$x_{4t} + a_1 x x_{tt} + a_2 x_t^2 + a_3 x^3 = 0, \quad a_1, a_2, a_3 = const. \tag{3.1.23}$$

We are looking for a Newton representation

$$(r^1)_{tt} = r^2 + b_1 (r^1)^2, \quad (r^2)_{tt} = b_2 r^1 r^2 + b_3 (r^1)^3, \quad b_1, b_2, b_3 = const. \tag{3.1.24}$$

If $r^1 = x$ then $r^2 = x_{tt} - b_1 x^2$ and equations (3.1.24) are equivalent to (3.1.23) if

$$a_1 = -2b_1 - b_2, \quad a_2 = -2b_1, \quad a_3 = b_1 b_2 - b_3.$$

Thus every Eq. (3.1.23) admits the Newton representation (3.1.24), where

$$b_1 = -\tfrac{1}{2} a_2, \quad b_2 = a_2 - a_1, \quad b_3 = \tfrac{1}{2} a_1 a_2 - \tfrac{1}{2} a_2^2 - a_3.$$

If we assume that Eq. (3.1.23) is Lagrangian, with

$$L[x] = \tfrac{1}{2} x_{tt}^2 + a x x_t^2 + b x^4, \quad a, b = const., \tag{3.1.25}$$

(homogeneous of order ε^4) then its Euler-Lagrange equation

$$\frac{\delta L}{\delta x} = x_{4t} - 2a x x_{tt} - a x_t^2 + 4b x^3 = 0 \tag{3.1.26}$$

is equivalent to

$$(r^1)_{tt} = r^2 + \tfrac{1}{2}a(r^1)^2, \quad (r^2)_{tt} = ar^1r^2 + (\tfrac{1}{2}a^2 - 4b)(r^1)^3, \tag{3.1.27}$$

which also follows from the natural Lagrangian

$$L[r] = r_t^T g r_t - V(r) = r_t^1 r_t^2 + \tfrac{1}{2}a(r^1)^2 r^2 + (\tfrac{1}{8}a^2 - b)(r^1)^4 + \tfrac{1}{2}(r^2)^2, \tag{3.1.28}$$

where

$$g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

so (r^1, r^2) are flat, non-orthogonal coordinates. Notice that $L[r(x)] = -L[x]$ modulo total derivative. System (3.1.27) is also Hamiltonian with the Hamiltonian function

$$H = y_1 y_2 - \tfrac{1}{2}a(r^1)^2 r^2 + (b - \tfrac{1}{8}a^2)(r^1)^4 - \tfrac{1}{2}(r^2)^2, \tag{3.1.29}$$

where $y_1 = \left(r^2\right)_t$, $y_2 = \left(r^1\right)_t$ are conjugate momentum. So, the natural Hamiltonian representation of the variational problem (3.1.25), (3.1.26) is as follows

$$\begin{aligned} (r^1)_t &= y_2, \\ (r^2)_t &= y_1, \\ (y_1)_t &= r^1 r^2 + (\tfrac{1}{2}a^2 - 4b)(r^1)^3, \\ (y_2)_t &= \tfrac{1}{2}a(r^1)^2 + r^2, \end{aligned}$$

as

$$\begin{aligned} &r^1 = x, \quad r^2 = x_{tt} - \tfrac{1}{2}ax^2, \\ &y_1 = (r^2)_t = x_{3t} - axx_t, \quad y_2 = (r^1)_t = x_t. \end{aligned}$$

On the other hand, the Hamiltonian representation of Lagrangian dynamics (3.1.25) in Ostrogradsky representation (3.1.17), (3.1.18) takes the form

$$q^1 = x, \quad q^2 = x_t, \quad p_2 = \frac{\delta L}{\delta x_{tt}} = x_{tt},$$

$$p_1 = \frac{\delta L}{\delta x_t} = 2axx_t - x_{3t},$$

$$H = p_2 q_t^2 + p_1 q^2 - L = \tfrac{1}{2}p_2^2 + q^2 p_1 - aq^1(q^2)^2 - b(q^1)^4$$

and hence

$$\begin{aligned}(q^1)_t &= q^2,\\ (q^2)_t &= p_2,\\ (p_1)_t &= a(q^2)^2 + 4b(q^1)^3,\\ (p_2)_t &= -p_1 + 2aq^1q^2.\end{aligned}$$

Both representations are related by the following canonical transformation (see Sect. 4.1.4 for details) of the phase space coordinates

$$r^1 = q^1,\ \ r^2 = p_2 - \tfrac{1}{2}a(q^1)^2,$$

$$y_1 = -p_1 + aq^1q^2,\quad y_2 = q^2.$$

Example 3.4 The Newton representation of the variational problem from Example 3.2 is as follows. As $a = -5$ and $b = \frac{5}{2}$ then

$$\begin{aligned}r^1 &= x,\ \ r^2 = x_{tt} + \tfrac{5}{2}x^2,\\ y_1 &= r^2_t = x_{3t} + 5xx_t,\ \ y_2 = r^1_t = x_t,\\ H &= y_1y_2 + \tfrac{5}{2}(r^1)^2r^2 - \tfrac{5}{8}(r^1)^4 - \tfrac{1}{2}(r^2)^2,\end{aligned}$$

and hence

$$\begin{aligned}(r^1)_t &= y_2,\\ (r^2)_t &= y_1,\\ (y_1)_t &= r^1r^2 + \tfrac{5}{2}(r^1)^3,\\ (y_2)_t &= -\tfrac{5}{2}(r^1)^2 + r^2.\end{aligned}$$

Both Ostrogradsky and Newton representations are related by the following transformation

$$r^1 = q^1,\ \ r^2 = p_2 + \tfrac{5}{2}(q^1)^2,$$

$$y_1 = -p_1 - 5q^1q^2,\quad y_2 = q^2.$$

The 4-th order case suggests that there is a unique Newton representation (3.1.22) for the ODE in the form (3.1.20). In fact this is an exceptional case. For higher

order ODE's we have a lot of freedom in the construction of admissible Newton representations. To see this let us consider the 6-th order case in detail. The most general equation (3.1.20) of 6-th order is

$$x_{6t} + a_1 x x_{4t} + a_2 x_t x_{3t} + a_3 x_{tt}^2 + a_4 x x_t^2 + a_5 x^2 x_{tt} + a_6 x^4 = 0. \tag{3.1.30}$$

The admissible Newton representation reads

$$\begin{aligned}
(r^1)_{tt} &= r^2 + b_1 (r^1)^2, \\
(r^2)_{tt} &= r^3 + b_2 r^1 r^2 + b_3 (r^1)^3, \\
(r^3)_{tt} &= b_4 r^1 r^3 + b_5 (r^1)^2 r^2 + b_6 (r^1)^4 + b_7 (r^2)^2,
\end{aligned} \tag{3.1.31}$$

where

$$r^1 = x, \quad r^2 = x_{tt} - b_2 x^2, \quad r^3 = x_{4t} - 2b_1 x_t^2 - (2b_1 + b_2) x x_{tt} + (b_1 b_2 - b_3) x^3.$$

Equivalence of (3.1.31) and (3.1.30) requires that

$$\begin{aligned}
&a_1 = -(2b_1 + b_2 + b_4), \quad a_2 = -(8b_1 + 2b_2), \quad a_3 = -(6b_1 + b_2 + b_7), \\
&a_4 = 6b_1 b_2 + 2b_1 b_4 - 6b_3, \quad a_5 = 3b_1 b_2 + 2b_1 b_4 + b_2 b_4 + 2b_1 b_7 - 3b_3 - b_5, \\
&a_6 = b_3 b_4 + b_1 b_5 - b_1 b_2 b_4 - b_1^2 b_7 - b_6.
\end{aligned} \tag{3.1.32}$$

This is an undetermined system of equations, so we have some freedom in the choice of the Newton representation (3.1.31). We can fix one b_i coefficient arbitrarily and then solve (3.1.32) with respect to the other b_j coefficients. For example, if we choose a solution with fixed b_1, then we obtain the Newton equations (3.1.31) parametrized by b_1

$$\begin{aligned}
b_2 &= -\tfrac{1}{2} a_2 - 4b_1, \\
b_3 &= -\tfrac{1}{6} a_4 - \tfrac{1}{3}(a_1 + a_2) b_1 - \tfrac{10}{3} b_1^2, \\
b_4 &= -a_1 + \tfrac{1}{2} a_2 + 2b_1, \\
b_5 &= \tfrac{1}{2} a_4 - a_5 - \tfrac{1}{4} a_2^2 + \tfrac{1}{2} a_1 a_2 + (3a_1 - \tfrac{3}{2} a_2 - 2a_3) b_1 - 10 b_1^2, \\
b_6 &= -a_6 + \tfrac{1}{6} a_1 a_4 - \tfrac{1}{12} a_2 a_4 + (\tfrac{1}{6} a_4 - a_5 + \tfrac{1}{3} a_1^2 - \tfrac{1}{6} a_2^2 + \tfrac{1}{6} a_1 a_2) b_1 \\
&\quad + (\tfrac{5}{3} a_1 - \tfrac{4}{3} a_2 - a_3) b_1^2 - \tfrac{20}{3} b_1^3, \\
b_7 &= \tfrac{1}{2} a_2 - a_3 - 2b_1.
\end{aligned}$$

Let us consider the subclass of Lagrangian systems. The most general Lagrangian for (3.1.30) (up to a total t-derivative) reads

$$L[x] = \tfrac{1}{2}x_{3t}^2 + axx_{tt}^2 + bx^2x_t^2 + dx^5, \tag{3.1.33}$$

where a, b, d are some suitable constants, and its Euler-Lagrange equation yields

$$\frac{\delta L}{\delta x} = x_{6t} - 2axx_{4t} - 4ax_tx_{3t} - 3ax_{tt}^2 + 2bx^2x_{tt} + 2bxx_t^2 - 5dx^4 = 0. \tag{3.1.34}$$

Then, in terms of Newton variables, we obtain the following (b_1-parametrized) equations

$$\begin{aligned}
(r^1)_{tt} &= r^2 + b_1(r^1)^2,\\
(r^2)_{tt} &= r^3 + (2a - 4b_1)r^1r^2 + (-\tfrac{1}{3}b + 2ab_1 - \tfrac{10}{3}b_1^2)(r^1)^3,\\
(r^3)_{tt} &= 2b_1r^1r^3 + (-b + 6ab_1 - 10b_1^2)(r^1)^2r^2 + (5d - \tfrac{5}{3}bb_1 + 5a^2b_1^2 - \tfrac{20}{3}b_1^3)(r^1)^4\\
&\quad + (a - 2b_1)(r^2)^2,
\end{aligned} \tag{3.1.35}$$

with the natural Lagrangian

$$\begin{aligned}
L[r] &= r_t^T g r_t - V(r)\\
&= \tfrac{1}{2}(r_t^2)^2 + r_t^1r_t^3 + r^2r^3 + b_1(r^1)^2r^3 + (-\frac{1}{3}b + 2ab_1 - \tfrac{10}{3}b_1^2)(r^1)^3r^2\\
&\quad + (a - 2b_1)r^1(r^2)^2 + (d - \tfrac{1}{3}bb_1 + a^2b_1^2 - \tfrac{4}{3}b_1^3)(r^1)^5
\end{aligned} \tag{3.1.36}$$

where

$$g = \begin{pmatrix} 0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0 \end{pmatrix},$$

so (r^1, r^2, r^3) are again flat, non-orthogonal coordinates. Notice that $L[r] = L[x]$ modulo total derivative. System (3.1.35) is also Hamiltonian with the Hamiltonian function

$$\begin{aligned}
H &= \tfrac{1}{2}y_2^2 + y_1y_3 - r^2r^3 - b_1(r^1)^2r^3 + (\frac{1}{3}b - 2ab_1 + \tfrac{10}{3}b_1^2)(r^1)^3r^2\\
&\quad - (a - 2b_1)r^1(r^2)^2 + (-d + \tfrac{1}{3}bb_1 - a^2b_1^2 + \frac{4}{3}b_1^3)(r^1)^5,
\end{aligned} \tag{3.1.37}$$

where $y_1 = \left(r^3\right)_t$, $y_2 = \left(r^2\right)_t$ and $y_3 = \left(r^1\right)_t$ are conjugate momenta. So, the natural Hamiltonian representation of the variational problem (3.1.25), (3.1.26) is as follows

$$
\begin{aligned}
(r^1)_t &= y_3,\\
(r^2)_t &= y_2,\\
(r^3)_t &= y_1,\\
(y_1)_t &= 2b_1 r^1 r^3 - (b - 6ab_1 + 10b_1^2)(r^1)^2 r^2 + (a - 2b_1)(r^2)^2\\
&\quad + 5(d - \tfrac{1}{3}bb_1 + a^2 b_1^2 - \tfrac{4}{3}b_1^3)(r^1)^4,\\
(y_2)_t &= r^3 - (\tfrac{1}{3}b - 2ab_1 + \tfrac{10}{3}b_1^2)(r^1)^3 + 2(a - 2b_1) r^1 r^2,\\
(y_3)_t &= r^2 + b_1 (r^1)^2,
\end{aligned}
$$

as

$$
r^1 = x, \quad r^2 = x_{tt} - b_1 x^2,
$$

$$
r^3 = x_{4t} - 2b_1 x_t^2 + 2(b_1 - a) x x_{tt} + \tfrac{1}{3}(b - 2b_1^2) x^3,
$$

$$
y_1 = x_{5t} - 2(a + b_1) x_t x_{tt} + 2(b_1 - a) x x_{3t} + (b - 2b_1^2) x^2 x_t,
$$

$$
y_2 = x_{3t} - 2b_1 x x_t, \quad y_3 = x_t.
$$

On the other hand, the Hamiltonian representation of Lagrangian dynamics (3.1.33) in Ostrogradsky representation (3.1.17), (3.1.18) takes the form

$$
q^1 = x, \quad q^2 = x_t, \quad q^3 = x_{tt},
$$

$$
p_1 = \frac{\delta L}{\delta x_t} = x_{5t} - 2a x x_{3t} - 2a x_t x_{tt} + 2b x^2 x_t,
$$

$$
p_2 = \frac{\delta L}{\delta x_{tt}} = -x_{4t} + 2a x x_{tt}, \quad p_3 = \frac{\delta L}{\delta x_{3t}} = x_{3t},
$$

$$
\begin{aligned}
H &= p_3 q_t^3 + p_2 q^3 + p_1 q^2 - L\\
&= \tfrac{1}{2} p_3^2 + q^3 p_2 + q^2 p_1 - a q^1 (q^3)^2 - b (q^1)^2 (q^2)^2 - d (q^1)^5
\end{aligned}
$$

and hence

$$
\begin{aligned}
(q^1)_t &= q^2,\\
(q^2)_t &= q^3,\\
(q^3)_t &= p_3,\\
(p_1)_t &= a(q^3)^2 + 2bq^1(q^2)^2 + 5d(q^1)^4,\\
(p_2)_t &= -p_1 + 2b(q^1)^2q^2,\\
(p_3)_t &= -p_2 + 2aq^1q^3.
\end{aligned}
$$

Both representations are related by the following transformation of the phase space coordinates

$$
\begin{aligned}
r^1 &= q^1,\\
r^2 &= q^3 - b_1(q^1)^2,\\
r^3 &= -p_2 - 2b_1(q^2)^2 + 2b_1q^1q^3 + \tfrac{1}{3}(b - 2b_1^2)(q^1)^3,\\
y_1 &= p_1 + 2b_1q^1p_3 - 2b_1q^2q^3 - (b + 2b_1^2)(q^1)^2q^2,\\
y_2 &= p_3 - 2b_1q^1q^2,\\
y_3 &= q^2,
\end{aligned}
$$

which again is the canonical transformation.

Example 3.5 The Newton representation of the variational problem from Example 3.3 is as follows. As $a = d = -7$ and $b = 35$ then, for $b_1 = \frac{1}{2}a$, we find

$$r^1 = x, \quad r^2 = x_{tt} + \tfrac{7}{2}x^2,$$

$$r^3 = x_{4t} + 7x_t^2 + 7xx_{tt} + \tfrac{7}{2}x^3,$$

$$y_1 = x_{5t} + 21x_tx_{tt} + 7xx_{3t} + \frac{21}{2}x^2x_t,$$

$$y_2 = x_{3t} + 7xx_t, \quad y_3 = x_t.$$

$$H = \tfrac{1}{2}y_2^2 + y_1y_3 - r^2r^3 + \tfrac{7}{2}(r^1)^2r^3 + \tfrac{7}{2}(r^1)^3r^2 - \tfrac{21}{4}(r^1)^5,$$

and hence

$$
\begin{aligned}
(r^1)_t &= y_3,\\
(r^2)_t &= y_2,\\
(r^3)_t &= y_1,\\
(y_1)_t &= -7r^1r^3 - \tfrac{21}{2}(r^1)^2r^2 + \tfrac{105}{4}(r^1)^4,\\
(y_2)_t &= r^3 - \tfrac{7}{2}(r^1)^3,\\
(y_3)_t &= r^2 - \tfrac{7}{2}(r^1)^2.
\end{aligned}
$$

Both Ostrogradsky and Newton representations are related by the following transformation

$$
\begin{aligned}
r^1 &= q^1,\\
r^2 &= q^3 + \tfrac{7}{2}(q^1)^2,\\
r^3 &= -p_2 + 7(q^2)^2 - 7q^1q^3 + \tfrac{7}{2}(q^1)^3,\\
y_1 &= p_1 - 7q^1p_3 + 7q^2q^3 - \tfrac{119}{2}(q^1)^2q^2,\\
y_2 &= p_3 + 7q^1q^2,\\
y_3 &= q^2.
\end{aligned}
$$

3.2 Coordinate Free Formulation of Hamiltonian Mechanics

In the previous section we reminded the reader of the standard canonical formulation of classical Hamiltonian mechanics via an appropriate Legendre transformation from Lagrangian representation. Nevertheless, the modern Hamiltonian mechanics is formulated independently of the Lagrangian formalism (variational problems) and what important, in a coordinate free way. In consequence, Hamiltonian formalism not always has to be related with some dynamics in a Riemanian space. As will be presented in the following section, Hamiltonian mechanics will be formulated in the language of Poisson geometry (presymplectic geometry) not Riemaniann geometry. It means that dynamical systems it describes, are from Poisson manifolds. Nevertheless, in some particular class of examples, when the Poisson manifolds will be chosen as the cotangent bundle of some Riemaniann spaces, the Riemannian geometry can be adopted to Poisson geometry with great freedom in choosing an appropriate metric tensors. That observation will be crucial for further quantization procedure developed in the second part of the book.

3.2.1 Poisson and Presymplectic Manifolds

Consider a manifold M of $\dim M = m$, the algebra $\mathcal{F}(M)$ of smooth, real-valued functions on M and a duality map $\langle ., . \rangle : T^*M \times TM \longrightarrow \mathcal{F}(M)$. A *Poisson tensor* Π of co-rank r on M is a bi-vector $\Pi \in \Lambda^2(M)$ with vanishing Schouten-Nijenhuis bracket [180]:

$$[\Pi, \Pi]_S = 0, \tag{3.2.1}$$

whose kernel is spanned by exact one-forms

$$\ker \Pi = Sp\{dc_i\}_{i=1,\dots,r},$$

i.e. $\Pi dc_i = 0$ according to the notation in (2.2.6). In a local coordinate system $(\zeta^1, \dots, \zeta^m)$ on M we have

$$\boxed{\Pi = \sum_{i<j}^{m} \Pi^{ij} \frac{\partial}{\partial \zeta^i} \wedge \frac{\partial}{\partial \zeta^j},} \tag{3.2.2}$$

while the Poisson property (3.2.1) takes the form of the so called Jacobi equation

$$\boxed{\Pi^{jl}\partial_l \Pi^{ik} + \Pi^{il}\partial_l \Pi^{kj} + \Pi^{kl}\partial_l \Pi^{ji} = 0, \quad \partial_i := \frac{\partial}{\partial \zeta^i}} \tag{3.2.3}$$

and is derived as a particular case of the formula (2.3.13) for $r = k = 2$ and $R = K = \Pi$. Any smooth function $c(\zeta) \in \mathcal{F}(M)$ is called a *Casimir function* of the Poisson tensor Π if $\Pi dc = 0$.

Having a Poisson tensor we can define a Hamiltonian vector fields on M. A vector field X_H related to a function $H \in \mathcal{F}(M)$ by the relation

$$\boxed{X_H = \Pi\, dH,} \tag{3.2.4}$$

(consistent with the notation defined by (2.2.6)), is called the *Hamiltonian vector field* with respect to the Poisson tensor Π. The function H is called a Hamiltonian function or simply a Hamiltonian.

Poisson tensor Π induces the Lie algebra structure into the associative algebra of functions $\mathcal{F}(M)$ through a particular Lie bracket

$$\{., .\} : \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M),$$

$$\boxed{\{F, G\}_\Pi := \Pi(dF, dG) = \langle dF, \Pi\, dG \rangle\,,} \tag{3.2.5}$$

that additionally satisfies Leibniz rule, i.e. the bracket is also a derivation for multiplication in the algebra of functions. Such bracket is called a *Poisson bracket*. Indeed

1. $\{F, G\}_\Pi = -\{G, F\}_\Pi$, antisymmetry,
2. $\{F, GH\}_\Pi = \{F, G\}_\Pi H + G\{F, H\}_\Pi$, Leibniz rule,
3. $\{F, \{H, G\}_\Pi\}_\Pi + \{H, \{G, F\}_\Pi\}_\Pi + \{G, \{F, H\}_\Pi\}_\Pi = 0$, Jacobi identity $\Longleftrightarrow$ $[\Pi, \Pi]_S = 0$.

Antisymmetry is obvious. The Leibniz rule follows from the fact that $d(FG) = (dF)G + F(dG)$. To show the Jacobi identity notice that

$$\begin{aligned}\{F, \{H, G\}_\Pi\}_\Pi &= \{F, dH_i \Pi^{ij} dG_j\}_\Pi = \{F, \Pi^{ij} H_{,i} G_{,j}\}_\Pi = \Pi^{rs} F_{,r} (\Pi^{ij} H_{,i} G_{,j})_{,s} \\ &= \Pi^{rs} \Pi^{ij} F_{,r} H_{,is} G_{,j} + \Pi^{rs} \Pi^{ij} F_{,r} G_{,js} H_{,i} + \Pi^{rs} \Pi^{ij}_{,s} F_{,r} H_{,i} G_{,j}\end{aligned}$$

and moreover $\Pi^{ij} = -\Pi^{ji}$, $F_{,ij} = F_{,ji}$, where $\Pi^{ij}_{,s} \equiv \partial_s \Pi^{ij}$ and $F_{,ij} = \partial_i \partial_j F$. Thus

$$\{F, \{H, G\}_\Pi\}_\Pi + c.p. = (\Pi^{rs} \Pi^{ij}_{,s} + \Pi^{is} \Pi^{jr}_{,s} + \Pi^{js} \Pi^{ri}_{,s}) F_{,r} H_{,i} G_{,j} = 0 \qquad (3.2.6)$$

according to (3.2.3). Such an algebra is called a *Poisson algebra* and a pair (M, Π) is called a *Poisson manifold*.

The distinguished representation (canonical representation) of arbitrary Poisson tensor is described by the *Darboux theorem*. It claims that if a Poisson bi-vector Π on m-dimensional manifold M ($\dim M = m = 2n + r$) has constant rank $2n$ on some open domain, then there are local coordinates $(\zeta^1, \ldots, \zeta^m) = (\xi^1, \ldots, \xi^{2n}, c_1, \ldots, c_r)$, where c_i are Casimir coordinates, such that

$$\Pi = \sum_{i=1}^{n} \partial_{\xi^i} \wedge \partial_{\xi^{n+i}}, \qquad \left(\Pi^{ij}\right) = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (3.2.7)$$

where I_n is n-dimensional unit matrix.

The Casimir functions $c_i(\zeta)$ of Π define a *foliation* of M denoted further by S. This foliation consists of the leaves $S_\nu = \{\zeta \in M : c_i(\zeta) = \nu_i, i = 1, \ldots, r\}$, $\nu = (\nu_r, \ldots, \nu_r)$. From the Darboux theorem it is obvious that the restriction of Π to any leaf S_ν, i.e. $\pi_\nu := \Pi_{|S_\nu}$ is a non degenerate Poisson bi-vector. Such a foliation is called the *symplectic foliation* of a Poisson manifold (M, Π) and S_ν is called a *symplectic leave*.

Further, a *presymplectic form* Ω on M is defined by a two-form that is closed, i.e. $d\Omega = 0$, degenerate in general. Thus, in a local coordinate system $(\zeta^1, \ldots, \zeta^m)$ on

M the two-form Ω is represented by

$$\boxed{\Omega = \sum_{i<j}^{m} \Omega_{ij} d\zeta^i \wedge d\zeta^j,} \tag{3.2.8}$$

while the closeness condition takes the form

$$\boxed{\partial_i \Omega_{jk} + \partial_k \Omega_{ij} + \partial_j \Omega_{ki} = 0.} \tag{3.2.9}$$

Moreover, the kernel of any presymplectic form is an integrable distribution

$$\boxed{\ker \Omega = Sp\{Z_i\}_{i=1,\ldots,r}, \quad [Z_i, Z_j] = 0.} \tag{3.2.10}$$

A pair (M, Ω) is called a *presymplectic manifold*.

Like for Poisson bi-vectors, there exists the Darboux theorem for presymplectic forms. It claims that if a presymplectic form Ω on m-dimensional manifold M ($\dim M = m = 2n + r$) has a constant rank $2n$ on some open domain, then there are local coordinates $(\zeta^1, \ldots, \zeta^m) = (\xi^1, \ldots, \xi^{2n}, c_1, \ldots, c_r)$, such that

$$\Omega = \sum_{i=1}^{n} d\xi^{n+i} \wedge d\xi^i, \quad (\Omega_{ij}) = \begin{pmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2.11}$$

Coordinates, in which Π and Ω take the canonical forms (3.2.7), (3.2.11) are canonical coordinates (Darboux coordinates). Canonical coordinates can be equivalently defined in the following way. Coordinates $(\zeta^1, \ldots, \zeta^m)$ are canonical if and only if

$$\{\zeta^i, \zeta^j\}_\Pi = \Pi^{ij} \tag{3.2.12}$$

where Π^{ij} are given by (3.2.7).

A vector field X^F related to a function $F \in \mathcal{F}(M)$ by the relation

$$\boxed{\Omega X^F = dF} \tag{3.2.13}$$

is called the *inverse Hamiltonian vector field* with respect to the presymplectic form Ω.

Any non-degenerate ($r = 0$) closed two form ω on M is called a *symplectic* form. The inverse of a symplectic form is a non-degenerate Poisson bi-vector π, called an *implectic* operator which satisfies

$$(\omega\pi)\alpha = \alpha, \qquad \alpha \in T^\star M$$
$$(\pi\omega)v = v, \qquad v \in TM.$$

Thus, if some ω satisfies relations (3.2.9) then its inverse satisfies respective relations (3.2.3). In such a case

$$\omega^{-1} = \pi \Longrightarrow X_H = X^H. \tag{3.2.14}$$

Notice that in what follows we will use small letters to denote non-degenerate Poisson and symplectic tensors and the capital letters otherwise. Hence, in the non-degenerate case, the theory of Hamiltonian systems coincides with the theory of inverse Hamiltonian systems. Moreover, a symplectic operator ω defines the same Poisson bracket as the related implectic operator π

$$\boxed{\begin{aligned}\{F, G\}^{\omega} &:= \omega(X_F, X_G) = \langle \omega X_F, X_G\rangle = \left\langle \omega X^F, X_G \right\rangle = \langle dF, X_G\rangle \\ &= \langle dF, \pi dG\rangle = \pi(dF, dG) = \{F, G\}_{\pi}.\end{aligned}} \tag{3.2.15}$$

Darboux theorem suggests an admissible construction of symplectic two-forms and implectic bi-vectors with the help of appropriate 1-forms and vector fields. Such a construction will be particulary important for further quantization process. Let $(X_1, \ldots, X_{2n})$ be a collection of $2n$ linearly independent vector fields on M, which commute $[X_\beta, X_\gamma] = 0$. Then

$$\pi = \sum_{i=1}^{n} X_i \wedge X_{n+i}$$

is a Poisson bivector, as from the commutativity of vector fields follows immediately the vanishing of related Schouten-Nijenhuis bracket $[\pi, \pi]_S = 0$.

Let $(\alpha_1, \ldots, \alpha_{2n})$ be a collection of 1-forms, linearly independent on M, which are exact: $\alpha_\beta = df_\beta$. Then

$$\omega = \sum_{i=1}^{n} \alpha_{n+i} \wedge \alpha_i$$

is a closed two-form. Besides, if α_β is a 1-form dual to vector field X_β, i.e. when $\langle \alpha_\beta, X_\gamma \rangle = X_\gamma(\alpha_\beta) = \delta_{\beta\gamma}$, then

$$\omega\pi = \pi\omega = I_{2n}.$$

Classical Hamiltonian dynamical system on a Poisson manifold is defined by a system of the first order ODE's in a coordinate free form (tensorial form)

$$\xi_t = X_H(\xi) = \Pi\, dH \implies \xi^i_t = \Pi^{ij}(dH)_j, \tag{3.2.16}$$

Hence, the solutions of the system (3.2.16) are represented by a classical flow, generated by the Hamiltonian vector field X_H. Alternatively, equations of motion (3.2.16) can be represented by an appropriate Poisson bracket

$$\boxed{\xi^i_t = \{\xi^i, H\}_\Pi} \tag{3.2.17}$$

as

$$\{\xi^i, H\}_\Pi = \xi^i_{,k}\Pi^{kj}H_{,j} = \delta^i_k\Pi^{kj}H_{,j} = \Pi^{ij}H_{,j} = \Pi^{ij}(dH)_j.$$

In Sect. 3.1.2 we considered the particular (but very important from physical point of view) example of a Poisson manifold being a phase space $M = T^*Q$, $\dim M = 2n$, i.e. a cotangent bundle to some Riemannian space (Q, g) (configuration space) of $\dim Q = n$, with canonical parametrization $(\xi) = (x^1, \dots, x^n, p_1, \dots, p_n)$ which means that

$$\{x^i, x^j\} = \{p_i, p_j\} = 0, \quad \{x^i, p_j\} = \delta^i_j.$$

Then, equations of motion (3.1.2)

$$x^i_{tt} + \Gamma^i_{jk}x^j_t x^k_t = -G^{ij}\partial_i V(x), \quad i = 1, \dots, n$$

of a particle, moving in a base space Q under the influence of a force of potential $V(x)$, have the canonical Hamiltonian representation in the phase space T^*Q in a form

$$\boxed{\begin{pmatrix} x \\ p \end{pmatrix}_t = \begin{pmatrix} \frac{\partial H}{\partial p} \\ -\frac{\partial H}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \pi dH} \tag{3.2.18}$$

with natural Hamiltonian (3.1.9). Notice that in a canonical basis, Hamiltonian vector fields are given by the formula

$$X_H = \left(\partial_{p_i}H\right)\partial_{x^i} - \left(\partial_{x^i}H\right)\partial_{p_i}$$

and Poisson bracket of a pair of functions $A(x, p)$, $B(x, p)$ by the formula

$$\boxed{\{A, B\} = \frac{\partial A}{\partial x^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial x^i}.} \tag{3.2.19}$$

Example 3.6 Let us consider the Hénon-Heiles system [146] in the phase space $\mathbb{R}^4 = T^*\mathbb{E}^2$ and the canonical representation $(x, y, p_x p_y)$, with the Hamiltonian

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + x^3 + \tfrac{1}{2}xy^2.$$

Equations of motion are of the form

$$\begin{pmatrix} x \\ y \\ p_x \\ p_y \end{pmatrix}_t = \begin{pmatrix} p_x \\ p_y \\ -3x^2 - \frac{1}{2}y^2 \\ -xy^2 \end{pmatrix} = \pi\, dH,$$

where

$$\pi = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \tag{3.2.20}$$

The next example presents a more general case of a Poisson manifold, not related to any particular configuration space.

Example 3.7 Let us consider the Euler equations of motion of a rigid body, so called Euler top, as an example of Hamiltonian dynamics on a three dimensional Poisson manifold (in this case being the Lie algebra $so(3)$)

$$\begin{aligned} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}_t &= \begin{pmatrix} \frac{I_2 - I_3}{I_2 I_3}\omega_2\omega_3 \\ \frac{I_3 - I_1}{I_1 I_3}\omega_1\omega_3 \\ \frac{I_1 - I_2}{I_1 I_2}\omega_1\omega_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} d\left(\frac{1}{2}\frac{\omega_1^2}{I_1} + \frac{1}{2}\frac{\omega_2^2}{I_2} + \frac{1}{2}\frac{\omega_3^2}{I_3}\right) \\ &= \Pi\, dH, \end{aligned}$$

where ω_i are components of body angular momentum and I_i are the moments of inertia about the coordinate axis. As the manifold is of odd dimension, the Poisson tensor is degenerate and the Casimir function is of the form $c = \omega_1^2 + \omega_2^2 + \omega_3^2$.

Finally, let us recall two identities important for further considerations. Let Π be a Poisson bi-vector and Ω be a closed two-form, then

$$\boxed{L_{\Pi\alpha}\Pi + \Pi\, d\alpha\, \Pi = 0, \qquad L_v\Omega = d(\Omega v),} \tag{3.2.21}$$

where $v \in TM$ and $\alpha \in T^*M$. The second identity follows immediately from relations (2.5.20)–(2.5.22)

$$L_v\Omega = (d\, i_v + i_v d)\Omega = d\, i_v\Omega = d(\Omega v),$$

while the proof of the first identity is a bit more involved and can be found in [24].

3.2.2 Tensor Invariants of Hamiltonian Systems

In this subsection we turn our attention to various tensor invariants of a given Hamiltonian flow. Assume that (M, Π) is a given Poisson manifold and $X_H = \Pi\, dH$ is a respective Hamiltonian vector field, which in a local basis $(\zeta^1, \ldots, \zeta^m)$ takes the form

$$X_H = \Pi^{ij}\left(\partial_j H\right)\partial_i = X_H^i \partial_i, \quad i, j = 1, \ldots, m.$$

As was demonstrated in Sect. 2.5, a change of considered tensor field T along a Hamiltonian flow ϕ_t^H, generated by a vector field X_H, is measured by the Lie derivative L_{X_H}

$$\boxed{\frac{d}{dt}T = \frac{\partial}{\partial t}T + L_{X_H}T.} \tag{3.2.22}$$

So, a tensor field T, which does not depend explicitly on t, is invariant along the flow ϕ_t^H if

$$L_{X_H}T = 0.$$

The invariant scalar field F is called a *constant of motion*. Thus $F \in \mathcal{F}(M)$ is the constant of motion if

$$0 = L_{X_H}F = X_H(F) = \Pi^{ij}\left(\partial_j H\right)(\partial_i F) = \{F, H\}_\Pi.$$

We say that F is in involution with H. It means that if we fix a trajectory of the flow, F has a constant value for all points from that trajectory.

Observation 1 *This is a proper moment to observe why bi-vectors with a vanishing Schouten-Nijenhuis bracket (3.2.1) lead to a natural, coordinate free, generalization (3.2.4) of canonical Hamiltonian equations of motion (3.2.18). In the canonical formulation (3.2.18), with the Poisson tensor in the form of a constant bi-vector, the related Poisson bracket of two constants of motion is either equal to zero or to another constant of motion. To keep that property for a non-constant bi-vector, the Jacobi property (3.2.6) is required. On the other hand, Jacobi identity follows from Jacobi equation (3.2.3) which is a consequence of vanishing of the Schouten-Nijenhuis bracket (3.2.1).*

Invariant vector field Y which fulfils the condition

$$0 = L_{X_H}Y = [X_H, Y] \tag{3.2.23}$$

is called a *symmetry* (or more precisely a *symmetry generator*) of a Hamiltonian flow generated by X_H. If by ϕ_t we denote the flow generated by X_H and by ϕ_τ the flow

generated by Y, then ϕ_τ is the one-parameter group of symmetries of Hamiltonian flow ϕ_t if

$$\phi_t \circ \phi_\tau - \phi_\tau \circ \phi_t = 0.$$

On the level of respective infinitesimal generators it means their commutativity (3.2.23).

Poisson tensor Π itself is tensor invariant of type $(0, 2)$, i.e.

$$L_{X_H} \Pi = 0.$$

It follows from the fact that $[X_H, X_H] = 0$ and $\Pi(dH, dH) = 0$:

$$\begin{aligned} 0 = L_{X_H} X_H = L_{X_H} \Pi\, dH = (L_{X_H} \Pi) dH + \Pi(L_{X_H} dH) \\ = (L_{X_H} \Pi) dH + \Pi\, d(L_{X_H} H) = (L_{X_H} \Pi) dH \Longrightarrow L_{X_H} \Pi = 0. \end{aligned}$$

Assume now that F is a scalar field. Then we have

$$\boxed{\Pi\, d\{H, F\}_\Pi \overset{(3.2.2)}{=} \Pi\, dL_{X_F} H \overset{(3.2.2)}{=} L_{X_F} \Pi\, dH = L_{X_F} X_H = -[X_H, X_F],} \tag{3.2.24}$$

where $X_F = \Pi\, dF$. It means that the map $\Pi\, d : \mathcal{F}(M) \to T^{(0,1)}(M)$ is an anti-homomorphism of the Poisson algebra of scalar fields into the Lie algebra of vector fields. It also means that a respective Hamiltonian vector field X_F such that $[X_H, X_F] = 0$, is related with any constant of motion F i.e. X_F is a symmetry generator of the flow. It is nothing but the expression of the Noether theorem in the Hamiltonian formulation. It says that for the arbitrary Hamiltonian system, a Hamiltonian symmetry is related with the constant of motion and vice versa, a constant of motion is related with the Hamiltonian symmetry. Observe that the number of all symmetries of a given flow is larger in general than the number of constants of motion, as beside Hamiltonian symmetries there may exist non-Hamiltonian symmetries as well.

When the Poisson bi-vector is impectic, then in an obvious way the tensor field $T^{(2,0)} = \omega = \pi^{-1}$ is also the invariant of considered Hamiltonian flow

$$L_{X_H} \omega = 0.$$

3.2.3 *Dual Poisson-Presymplectic Pairs*

As was shown in the previous subsection, if the Poisson structure is nondegenerate, i.e. if the rank of the Poisson tensor is equal to the dimension of a manifold, then

the manifold becomes a symplectic manifold with a symplectic structure being just the inverse of the Poisson structure. In such a case there exists an alternative (dual) description of Hamiltonian vector fields in the language of symplectic geometry as inverse Hamiltonian systems coincide with Hamiltonian systems (3.2.14). So, a natural question arises, whether one can relate Hamiltonian and inverse Hamiltonian pictures in the degenerate case, when there is no natural inverse of the Poisson tensor [101]. For such a case we introduce the notion of dual Poisson-presymplectic structures [38].

A pair (π, ω) is called a dual implectic-symplectic pair on M if π is a a non-degenerate Poisson tensor, ω is a non-degenerate closed two-form and the following partition of unity holds on TM, respectively on T^*M: $I = \pi\omega$ and $I = \omega\pi$. So, in the non-degenerate case, dual implectic-symplectic pair is a pair of mutually inverse operators on M.

Let us extend these considerations onto a degenerate case. In order to do it let us introduce the concept of a *dual pair* [38]. Consider a manifold M of an arbitrary dimension m. A pair of tensor fields (Π, Ω) on M of co-rank r, where Π is a Poisson bi-vector and Ω is a closed two-form, is called a dual pair (Poisson-presymplectic pair) if there exists r exact one-forms $\alpha_i = dc_i$ and r linearly independent commuting vector fields Z_i, such that the following conditions are satisfied:

1. $\alpha_i(Z_j) = Z_j(c_i) = \delta_{ij}, i, j = 1, 2 \ldots r.$
2. $\ker \Pi = Sp\{dc_i : i = 1, \ldots r\}.$
3. $\ker \Omega = Sp\{Z_i : i = 1, \ldots r\}.$
4. The following partition of unity holds on TM, on T^*M respectively

$$\boxed{I = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes dc_i, \qquad I = \Omega\Pi + \sum_{i=1}^{r} dc_i \otimes Z_i.} \tag{3.2.25}$$

Condition 1 of the above definition implies that the distribution $\mathcal{Z}$ spanned by the vector fields Z_i is transversal to the symplectic foliation S. Thus, for any $x \in M$ we have

$$T_xM = T_xS_\nu \oplus \mathcal{Z}_x, \quad T_x^*M = T_x^*S_\nu \oplus \mathcal{Z}_x^*$$

where S_ν is a leaf from the foliation S that passes through x, the symbol $\oplus$ denotes the direct sum of the vector spaces, $\mathcal{Z}_x$ is the subspace of T_xM spanned by the vectors Z_i at this point, $T_x^*S_\nu$ is the annihilator of $\mathcal{Z}_x$ and $\mathcal{Z}_x^*$ is the annihilator of T_xS_ν. Condition 2 of the above definition implies that $\mathrm{Im}(\Pi) = TS$, Condition 3 means that $\mathrm{Im}(\Omega) = T^*S$ and Condition 4 describes the degree of degeneracy of our pair.

A presymplectic form Ω plays the role of an 'inverse' of Poisson bivector Π in the sense that on any symplectic leaf of the foliation defined by $\ker \Pi$, the restrictions of Ω and Π are inverses of each other. Contrary to the non-degenerate case, for a given

Poisson tensor Π the choice of its dual is not unique. Also for a given presymplectic form Ω the choice of a dual Poisson tensor is not unique either. We will come back to that problem at the end of this subsection.

For the degenerate case the Hamiltonian and the inverse Hamiltonian vector fields are defined in the same way as for the non-degenerate case, but for degenerate structures the notion of Hamiltonian and inverse Hamiltonian vector fields do not coincide any more. For any degenerate dual pair one can find a Hamiltonian vector field that is not inverse Hamiltonian and an inverse Hamiltonian vector field that is not a Hamiltonian one. Actually, assume that (Π, Ω) is a dual pair, $X_F = \Pi\, dF$ is a Hamiltonian vector field and $dF = \Omega X^F$ is an inverse Hamiltonian one-form, where X^F is an inverse Hamiltonian vector field. Having applied Ω to both sides of the Hamiltonian vector field, Π to both sides of the inverse Hamiltonian one-form and using the decomposition (3.2.25) we find that

$$\boxed{dF = \Omega(X_F) + \sum_{i=1}^{r} Z_i(F) dc_i, \qquad X_F = X^F - \sum_{i=1}^{r} X^F(c_i) Z_i.} \tag{3.2.26}$$

It means that an inverse Hamiltonian vector field X^F is simultaneously a Hamiltonian vector field X_F, i.e. $X^F = X_F$, if dF is annihilated by $\ker(\Omega)$ and X^F is annihilated by $\ker(\Pi)$. Besides, for any dual pair (Π, Ω), the following useful relations hold

$$\boxed{L_{X_F}\Pi = 0, \quad L_{X^F}\Omega = 0, \quad L_{Z_i}\Pi = 0, \quad L_{Z_i}\Omega = 0, \quad i - 1, \ldots, r.} \tag{3.2.27}$$

The first relation was proved in (3.2.2). The second one follows immediately from (3.2.21)

$$L_{X^F}\Omega = d(\Omega X^F) = d(dF) = 0.$$

For the third one we have

$$\begin{aligned} 0 &= L_{Z_i}(\Pi dc_j) = L_{Z_i}(\Pi) dc_j + \Pi L_{Z_i}(dc_j) \\ &= L_{Z_i}(\Pi) dc_j + \Pi d(L_{Z_i} c_j) = L_{Z_i}(\Pi) dc_j + \Pi\, d(\delta_{ij}) \\ &= L_{Z_i}(\Pi) dc_j \Longrightarrow L_{Z_i}\Pi = 0. \end{aligned}$$

Finally, the forth relation also follows from (3.2.21)

$$L_{Z_i}\Omega = d(\Omega Z_j) = 0.$$

In the previous subsection we proved that in nondegenerate case the Poisson bracket (3.2.5) can be alternatively expressed by a related symplectic form (3.2.15).

The same formula is valid in a degenerate case, when the inverse of implectic bi-vector is substituted by any dual presymplectic form

$$\boxed{\begin{aligned}\{F, G\}^{\Omega} &:= \Omega(X_F, X_G) = \left\langle \Omega X^F, X_G \right\rangle = \langle dF, X_G \rangle = \langle dF, \Pi\, dG \rangle \\ &= \Pi(dF, dG) = \{F, G\}_{\Pi}\end{aligned}} \tag{3.2.28}$$

as, although $X_F \neq X^F$, but $\langle \Omega X_F, \Pi\, dG \rangle = \left\langle \Omega X^F, \Pi\, dG \right\rangle$. Notice also that $\Omega(X_F, X_G) = \Omega(X^F, X^G)$.

Now, let us consider the problem of a 'gauge freedom' for a duality property. In other words: given a dual pair (Π, Ω) how can we deform Ω to a new presymplectic form Ω' so that (Π, Ω') is again a dual pair, or how can we deform Π to a new Poisson tensor Π' so that (Π', Ω) is also a dual pair?

Let Π be a fixed Poisson tensor and Ω be a dual presympectic form. Assume that $\alpha_i = dc_i \in \ker \Pi$, $Z_i \in \ker \Omega$ and $Z_i(c_j) = \delta_{ij}$, $i, j = 1, \ldots, r$. Define a new closed 2-form

$$\boxed{\Omega' = \Omega + \sum_{i=1}^{r} dc_i \wedge df_i,} \tag{3.2.29}$$

where $f_i \in C(M)$. Then (Π, Ω') is a dual pair, with $\ker(\Omega') = Sp\left\{Z_i' = Z_i + \Pi\, df_i\right\}$, provided that

$$\boxed{\Pi(df_i, df_j) + Z_j(f_i) - Z_i(f_j) = 0 \ \text{ for all } \ i, j.} \tag{3.2.30}$$

Now define a new bi-vector

$$\boxed{\Pi' = \Pi + \sum_{i=1}^{r} Z_i \wedge X_i, \quad \Omega X_i = dF_i,} \tag{3.2.31}$$

then Π' is Poisson and (Π', Ω) is again a dual pair, with $\ker \Pi' = Sp\{dc' = dc_i + dF_i\}$, provided that

$$\boxed{\Omega(X_i, X_j) + X_j(c_i) - X_i(c_j) = 0} \tag{3.2.32}$$

for any pair of indices i, j and

$$Z_k(X_j(c_i)) = 0$$

for any triple of indices i, j, k. In addition, notice that from (3.2.31) follows that

$$Z_i(F_j) = 0 \text{ and } X_i = \Pi\, dF_i + \sum_{k=1}^{r} X_i(c_k) Z_k.$$

Indeed, to prove duality of (Π, Ω'), as we have

$$\Omega'\Pi = \Omega\Pi + \sum_{i=1}^{r} dc_i \otimes \Pi\, df_i = I + \sum_{i=1}^{r} dc_i \otimes (Z_i + \Pi\, df_i)$$

and $Z_i'(c_j) = \delta_{ij}$, it remains to show that $Z_i' \in \ker \Omega'$. In fact, as

$$\begin{aligned}
\Omega' Z_i' &= (\Omega + \sum_{k=1}^{r} dc_k \wedge df_k)(Z_i + \Pi\, df_i) \\
&= \Omega\Pi df_i + \sum_{k=1}^{r} Z_i(f_k) dc_k - df_i + \sum_{k=1}^{r} \Pi(df_k, df_i) dc_k \\
&= \sum_{k=1}^{r} [\Pi(df_k, df_i) + Z_i(f_k) - Z_k(f_i)]\, dc_k
\end{aligned}$$

it vanishes under condition (3.2.30).

On the other hand, to prove duality of (Π', Ω), notice that for (3.2.31) we have

$$\Pi'\Omega = \Pi\Omega + \sum_{i=1}^{r} Z_i \otimes dF_i = I + \sum_{i=1}^{r} Z_i \otimes (dc_i + dF_i)$$

and $Z_i(c_j') = \delta_{ij}$. Also $c_i' \in \ker \Pi'$ as

$$\begin{aligned}
\Pi' dc_k' &= (\Pi + \sum_{i=1}^{r} Z_i \wedge X_i)(dc_k + dF_k) \\
&= \Pi dF_k + \sum_{i=1}^{r} X_i(c_k) Z_i - X_k + \sum_{i=1}^{r} X_i(F_k) Z_i \\
&= \sum_{i=1}^{r} X_i(c_k) Z_i - \sum_{i=1}^{r} X_k(c_i) Z_i + \sum_{i=1}^{r} X_i(F_k) Z_i \\
&= \sum_{i=1}^{r} [\Omega(X_k, X_i) + X_i(c_k) - X_k(c_i)]\, Z_i
\end{aligned}$$

and vanishes under condition (3.2.32). Finally, it remains to prove that (3.2.31) is Poisson. Using formula (2.3.11) and in particular (2.3.12) we have

$$\begin{aligned}[\Pi', \Pi']_S &= 2\sum_{i=1}^{r}[Z_i \wedge X_i, \Pi]_S + \sum_{i,k=1}^{r}[Z_i \wedge X_i, Z_k \wedge X_k]_S \\ &= 2\sum_{i=1}^{r}[Z_i \wedge X_i, \Pi]_S + \sum_{i,k=1}^{r} Z_i \wedge [X_i, X_k] \wedge Z_k - 2\sum_{i,k=1}^{r} X_i \wedge [Z_i, X_k] \wedge Z_k \\ &= 2\sum_{i=1}^{r}[Z_i \wedge X_i, \Pi]_S + \sum_{i,k=1}^{r} Z_i \wedge [X_i, X_k] \wedge Z_k\end{aligned}$$

where $[Z_i, X_k] = 0$ follows from condition (3.2.3) and relations (3.2.3). Now, using relation

$$[X, \Pi]_S = L_X \Pi, \quad X \in \Lambda^1(M), \ \Pi \in \Lambda^2(M) \tag{3.2.33}$$

relation (3.2.28) and the form (3.2.3) we get

$$\begin{aligned}2\sum_{i=1}^{r}[Z_i \wedge X_i, \Pi]_S &= 2\sum_{i,k=1}^{r} Z_i \wedge \Pi d(X_i(c_k)) \wedge Z_k \\ &= \sum_{i,k=1}^{r} Z_i \wedge \Pi d(X_i(c_k) - X_k(c_i)) \wedge Z_k\end{aligned}$$

and

$$[X_i, X_k] = \Pi d\Omega(X_k, X_i)$$

so finally

$$[\Pi', \Pi']_S = \sum_{i,k=1}^{r} Z_i \wedge \Pi d(\Omega(X_k, X_i) + X_i(c_k) - X_k(c_i)) \wedge Z_k$$

and vanishes under condition (3.2.32).

3.2.4 *Classical Hamiltonian Flows on Symplectic Manifold*

In order to compare classical and quantum dynamical systems, let us analyze classical Hamiltonian flows from the point of view of transport equations [93]. Consider a transport (2.5.1) on the phase space in local Darboux coordinates (x, p)

carried out by some functions

$$Q(x, p, t), \quad Q(x, p, 0) = x, \tag{3.2.34a}$$

$$P(x, p, t), \quad P(x, p, 0) = p, \tag{3.2.34b}$$

being a solution of the following system of partial differential equations

$$\boxed{(Q^i)_t(x, p, t) = \{Q^i(x, p, t), H(x, p)\}_{\pi(x,p)}, \quad Q(x, p, 0) = x,} \tag{3.2.35a}$$

$$\boxed{(P_i)_t(x, p, t) = \{P_i(x, p, t), H(x, p)\}_{\pi(x,p)}, \quad P(x, p, 0) = p,} \tag{3.2.35b}$$

where π is given by (3.2.20). These equations are equations of motion of a *classical Hamiltonian transport* on a phase space.

What is important, the system of PDE's (3.2.35) is equivalent to the system of ordinary differential equations

$$\boxed{(Q^i)_t = \{Q^i, H(Q, P)\}_{\pi(Q,P)}, \quad Q^i(0) = x^i,} \tag{3.2.36a}$$

$$\boxed{(P_i)_t = \{P_i, H(Q, P)\}_{\pi(Q,P)}, \quad P_i(0) = p_i,} \tag{3.2.36b}$$

i.e. classical Hamiltonian equations of motion. Indeed, from a previous section it follows that both the function H and the bi-vector π are invariants of the Hamiltonian flow, thus $H(x, p) = H(Q, P)$ and $\pi(x, p) = \pi(Q, P)$. Hence, PDE's (3.2.35) can be written in the form of ODE's (3.2.36).

As the implectic bi-vector π is an invariant, so the flow

$$\phi_t^H : \phi_t^H(x, p) = (Q(x, p, t), P(x, p, t))$$

is a one-parameter group of canonical transformations *(symplectomorphisms)*. It means that from relations

$$\{x^i, p_j\}_{\pi(x,p)} = \delta^i_j$$

follows that

$$\boxed{\{Q^i(x, p, t), P_j(x, p, t)\}_{\pi(x,p)} = \delta^i_j.} \tag{3.2.37}$$

In other words, (3.2.34) represents a *classical trajectory* passing at $t = 0$ through the point (x, p).

The time evolution of any function $A(x, p) \in \mathcal{F}(M)$ is given by the equation

$$\frac{dA}{dt} \equiv A_t = L_{X_H} A = \{A, H\}_\pi. \tag{3.2.38}$$

On the other hand, we know that the pull back of a function

$$\phi_t^* A = A \circ \phi_t,$$

which is a consequence of the fact that the product of two diffeomorphisms $\phi_{t_1} \cdot \phi_{t_2}$ is a simple composition of maps (2.5.3). As the result, the time evolution of any function $A(x, p)$ along the Hamiltonian flow ϕ_t is given by

$$A(t) = A(Q(x, p, t), P(x, p, t)). \tag{3.2.39}$$

It means that once we have solutions of Hamiltonian equations of motion (3.2.36), the time evolution of arbitrary function $A(x, p) \in \mathcal{F}(M)$ is given by (3.2.39) which obviously solves the Eq. (3.2.38) as

$$\begin{aligned} \frac{d}{dt} A &= \frac{\partial A}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial A}{\partial P}\frac{\partial P}{\partial t} = \frac{\partial A}{\partial Q}\{Q, H\}_\pi + \frac{\partial A}{\partial P}\{P, H\}_\pi \\ &= \{A, H\}_\pi. \end{aligned}$$

Notice that if function A depends explicitly on t, then $A(Q(x, p, t), P(x, p, t), t)$ solves the equation

$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial t} + L_{X_H} A = \frac{\partial A}{\partial t} + \{A, H\}_\pi,} \tag{3.2.40}$$

i.e. a particular case of the Eq. (3.2.22).

3.3 Statistical Hamiltonian Mechanics

In order to relate classical and quantum Hamiltonian mechanics to each other, we have to unify the language of both theories. For this purpose we have to extend the standard classical Hamiltonian mechanics to its more realistic (more 'physical') version, i.e. statistical Hamiltonian mechanics. Then, on the classical level, one can define such notions as observables, states, uncertainty relations, coherence et cetera, familiar to the reader from the quantum level.

3.3.1 Classical Poisson Algebra and Classical States

Let us consider a Poisson manifold (M, π) (symplectic manifold $(M, \omega = \pi^{-1})$ in particular), where M is a differential manifold of dimension $\dim M = 2n$ and π is real, nondegenerate bi-vector, of the vanishing Schouten-Nijenhuis bracket $[\pi, \pi]_S = 0$. Then, let us consider the space of complex valued functions $C^\infty(M, \mathbb{C})$ on M. The space $C^\infty(M, \mathbb{C})$ has a structure of commutative algebra with involution, where the product is a simple point-wise multiplication

$$(f \cdot g)(\xi) = f(\xi)g(\xi), \quad \xi \in M,$$

and the involution is a complex conjugation $f \to \bar{f}$.

Recall that the involution in algebra A is represented by anti-automorphism, whose square is an identity,

$$* : A \longrightarrow A$$

such that for $a, b \in A$

1. $(a + b)^* = a^* + b^*$,
2. $(ab)^* = b^* a^*$,
3. $1^* = 1$,
4. $(a^*)^* = a$.

The Poisson tensor π defines in the algebra $C^\infty(M, \mathbb{C})$ the Poisson bracket (3.2.1), (3.2.5), which is a particular realization of a Lie bracket. Such a double algebra with involution $(C^\infty(M), \cdot, \{.,.\}_\pi, ^-)$ is called a *classical Poisson algebra* and will be denoted by $\mathcal{A}_C$. In particular, two Poisson manifolds are diffeomorphic if and only if the related Poisson algebras are isomorphic. The elements of $\mathcal{A}_C$, which are self-adjoint with respect to the involution, i.e. real functions, are called *classical observables*. They constitute the subalgebra of $\mathcal{A}_C$.

The points in a phase space (M, π) represent states of the classical system. Each point in M can be interpreted as generalized positions and momenta of particles composing a classical system. Values of generalized positions and momenta of the particles can be extracted from a point in M by writing this point in canonical coordinates (x, p). Then, x^i are values of generalized positions and p_i are values of generalized momenta.

When the exact state of the system is not known, but only a probability that the state is in a given region of the phase space, then there is a natural need to extend the concept of a state to take into account such a situation. In fact it is a very physical situation as any measurement is performed with finite accuracy. The natural way of doing this is to define states as probabilistic measures defined on a σ-algebra $\mathcal{B}(M)$ of Borel subsets of M. In such a chosen setting, points ξ of the phase space are

identified with Dirac measures δ_ξ

$$\delta_\xi(E) = \begin{cases} 1 & \text{for } \xi \in E \\ 0 & \text{for } \xi \notin E \end{cases}, \quad E \in \mathcal{B}(M).$$

In what follows Dirac measures will be called pure states while other probabilistic measures mixed states, respectively. Further on we will be interested in such probabilistic measures μ that can be written in a form $d\mu = \rho\, d\Omega_\omega$, where ρ is an integrable function on M satisfying

$$\int_M \rho\, d\Omega_\omega = 1 \qquad \text{(normalization)},$$

$$\rho \geq 0 \qquad \text{(positive-definiteness)}$$

and thus can be identified with functions ρ. Here $d\Omega_\omega$ is a measure induced by a volume form

$$\Omega_\omega := \frac{1}{n!}\underbrace{\omega \wedge \ldots . \wedge \omega}_{n}.$$

In local coordinates $(\xi^1, \ldots, \xi^{2n})$ on M

$$d\Omega_\omega = \pm\sqrt{|\det \omega|}d\xi^1 \wedge \ldots . \wedge d\xi^{2n} \tag{3.3.1}$$

and in canonical coordinates (x, p), the measure (3.3.1) turns into Lebesgue measure

$$d\Omega_\omega = dx^1 \ldots dx^n dp_1 \ldots dp_n.$$

In particular, for Dirac measures we will use a notation $d\delta_\xi(\xi') = \delta(\xi - \xi')d\Omega_\omega(\xi')$, where $\delta(\xi - \xi')$ is Dirac delta distribution.

Observe that states can be alternatively defined as those "functions" ρ, where $\rho \in L^1(M)$ or $\rho = \delta_\xi$, which satisfy:

1. $\rho = \bar{\rho}$ (self-conjugation),
2. $\int_M \rho\, d\Omega_\omega = 1$ (normalization),
3. $\int_M \bar{f} \cdot f \cdot \rho\, d\Omega_\omega \geq 0, \quad f \in C_0^\infty(M) \Longleftrightarrow \rho \geq 0$ (positive-definiteness),

where $C_0^\infty(M)$ denotes a space of all smooth functions defined on M with compact support.

Classical states form a convex set. *Pure classical states* are defined as extreme points of the set of states, i.e. as those states which cannot be written as convex linear combinations of some other states. It can be proved that such a characterization of pure states is equivalent with the definition of pure states as Dirac measures. In

such a case, as will be proved later, statistical Hamiltonian mechanics reduces to the ordinary Hamiltonian mechanics. States which are not pure are called *mixed classical states*.

For $M = T^*E^{n_+,n_-}$ classical pure and mixed states can be alternatively defined as follows. Let us introduce a multiplication between states $\rho_1, \rho_2 \in L^1(M)$ in the form of convolution

$$\boxed{\rho_1 * \rho_2 = \int_M \rho_1(\xi')\rho_2(\xi - \xi')d\xi'.} \tag{3.3.2}$$

The space $L^1(M)$ is closed with respect to that product and so form the algebra which is commutative, associative and distributive. No algebra of functions possesses an identity for the convolution. The lack of identity is typically not a major inconvenience, since most collections of functions on which the convolution is performed can be convolved with a Dirac delta distribution δ or, at the very least (as is the case of L^1) admit approximations to the identity. Specifically,

$$\rho * \delta = \rho.$$

Pure states are defined as these which are idempotent

$$\boxed{\rho * \rho = \rho.} \tag{3.3.3}$$

In fact, states which fulfill (3.3.3) are just Dirac delta distributions. Any mixed state is a convex combination of pure states

$$\boxed{\rho = \sum_n p_n \rho^{(n)}_{\text{pure}}, \quad p_n \geq 0, \quad \sum_n p_n = 1.} \tag{3.3.4}$$

The summation in (3.3.4) can be substituted by integration over M such that $p \geq 0$, $\int_M p\, d\Omega_\omega = 1$

$$\rho(\xi) = \int_M p(\xi')\rho_{\text{pure}}(\xi')\, d\Omega_\omega = \int_M p(\xi')\delta(\xi - \xi')d\xi' = p(\xi),$$

where p fulfills the properties **1**-**3** of the classical state.

For a given observable $A \in C^\infty(M)$ and state μ ($d\mu = \rho\, d\Omega_\omega$) the expectation value of the observable A in the state μ is defined by

$$\boxed{\langle A \rangle_\mu := \int_M A\, d\mu = \int_M (A \cdot \rho)\, d\Omega_\omega.} \tag{3.3.5}$$

Note, that the expectation value of the observable A in a pure state δ_ξ is equal $A(\xi)$. Indeed,

$$\langle A\rangle_{\delta_\xi} = \int_M A(\xi')\delta(\xi - \xi')\, d\Omega_\omega(\xi') = A(\xi).$$

The expectation values of observables play an important role in the theory as they are identified with measurements in a physical experiment.

Thus, it is clear that in classical statistical mechanics appear integrals over a phase space which cannot be considered in arbitrary local coordinates, since doing this would change the values of integrals. For example, if $\psi\colon M \supset \mathcal{O} \to \mathbb{R}^{2n}$, $\psi\colon \xi \mapsto (\xi^1, \ldots, \xi^{2n})$ is a coordinate chart, then in general

$$\int_M f\, d\Omega_\omega \neq \int_{\psi(\mathcal{O})} f(\psi^{-1}(\xi))\, d\xi$$

where f is a function defined on M and $d\Omega_\omega$ is a measure induced by the Liouville form Ω_ω. These integrals will be equal only when $M \setminus \mathcal{O}$ is of measure zero. For this reason we introduce the following distinguished class of coordinates. A coordinate system $\psi\colon M \supset \mathcal{O} \to \mathbb{R}^{2n}$ on a symplectic manifold (M, ω) is called almost global if $M \setminus \mathcal{O}$ is of measure zero with respect to the measure $d\Omega_\omega$. Similarly, if (Q, g) is a Riemannian manifold representing a configuration space, then by an almost global coordinate system on Q we mean a coordinate system defined on an open subset $U \subset Q$ such that $Q \setminus U$ is of measure zero with respect to the measure induced by the metric volume form ω_g. It can be proved that an almost global coordinate system on Q induces a canonical coordinate system on T^*Q with the same property.

Observation 2 *Classical Hamiltonian statistical mechanics can be well formulated on Poisson manifolds (simplectic manifolds in particular) which admit almost global coordinate systems.*

More general manifolds, where a full integration measure exists, are beyond the scope of the book.

Example 3.8 Let $Q = E^3$ and consider Cartesian coordinates (x, y, z). Consider also spherical coordinates (r, θ, ϕ) related to the Cartesian coordinates by a transformation $\phi\colon (0, \infty) \times (0, \pi) \times (0, 2\pi) \to \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^3 \setminus \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 0, y = 0\}$, $\phi\colon (r, \theta, \phi) \mapsto (x, y, z)$,

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta. \end{aligned}$$

In the Cartesian coordinates (x, y, z) the metric volume form ω_g on E^3 is equal $dx \wedge dy \wedge dz$, and the corresponding measure $d\omega_g$ takes the form of the Lebesgue

measure $dxdydz$. It can be seen that a set $\mathbb{R}^3 \setminus \mathcal{O}$ is of Lebesgue-measure zero, hence the spherical coordinates (r, θ, ϕ) are almost global on E^3.

Let (x, y, z, p_x, p_y, p_z) be canonical coordinates on T^*E^3 induced by the Cartesian coordinates (x, y, z) on E^3. Canonical coordinates $(r, \theta, \phi, p_r, p_\theta, p_\phi)$ on T^*E^3 induced by the spherical coordinates (r, θ, ϕ) are related to the Cartesian coordinates (x, y, z, p_x, p_y, p_z) by a transformation $T: (0, \infty) \times (0, \pi) \times (0, 2\pi) \times \mathbb{R}^3 \to \hat{\mathcal{O}} = \mathcal{O} \times \mathbb{R}^3$, $T: (r, \theta, \phi, p_r, p_\theta, p_\phi) \mapsto (x, y, z, p_x, p_y, p_z)$,

$$\begin{aligned}
x &= r \sin\theta \cos\phi, \\
y &= r \sin\theta \sin\phi, \\
z &= r \cos\theta, \\
p_x &= \frac{r p_r \sin^2\theta \cos\phi + p_\theta \sin\theta \cos\theta \cos\phi - p_\phi \sin\phi}{r \sin\theta}, \\
p_y &= \frac{r p_r \sin^2\theta \sin\phi + p_\theta \sin\theta \cos\theta \sin\phi + p_\phi \cos\phi}{r \sin\theta}, \\
p_z &= \frac{r p_r \cos\theta - p_\theta \sin\theta}{r}.
\end{aligned} \tag{3.3.6}$$

In the Cartesian coordinates (x, y, z, p_x, p_y, p_z) the Liouville form Ω_ω on T^*E^3 is equal $dx \wedge dy \wedge dz \wedge dp_x \wedge dp_y \wedge dp_z$, and the corresponding measure $d\Omega_\omega$ takes the form of the Lebesgue measure $dxdydzdp_xdp_ydp_z$. It can be seen that a set $\mathbb{R}^6 \setminus \hat{\mathcal{O}}$ is of Lebesgue-measure zero, hence the canonical coordinates $(r, \theta, \phi, p_r, p_\theta, p_\phi)$ are almost global on T^*E^3.

3.3.2 Time Evolution of Classical Systems

For a given classical Hamiltonian system (M, π, H) there exists a dual description of its time evolution. The first one is known as the *classical Heisenberg picture*, The second one, known as *classical Schrödinger picture*, in which observables remain still whereas mixed states undergo a time development.

As was mentioned in Sect. 3.2.4, the Hamiltonian H governs the time evolution of the dynamical system. Actually, H generates a Hamiltonian field X_H. Then integral curves $\xi(t)$ of the vector field X_H, i.e. curves on M which satisfy Hamiltonian equation of motion

$$\boxed{\xi_t = X_H(\xi(t)) = \pi\, dH(\xi(t)),} \tag{3.3.7}$$

represent positions of points $\xi \in M$ for every instance of time t. All integral curves of a Hamiltonian field X_H generate a map $\phi_t^H : M \to M$, called a phase flow or a

Hamiltonian flow, by a prescription that for each point $\xi \in M$ a curve

$$\xi(t) = \phi_t^H(\xi) \tag{3.3.8}$$

is an integral curve of (3.3.7) passing through that point at time $t = 0$. Equation (3.3.7) is a Hamiltonian equation and integral curves of the Hamiltonian field are classical trajectories. For Darboux coordinates $\xi = (x, p)$, the flow (3.3.8) takes the form (3.2.4) while the Hamiltonian equation of motion (3.3.7) the form (3.2.36), respectively. As we will see later on, the Hamiltonian equation of motion (3.3.7) has a twofold interpretation (see Observation 3).

An equation of motion of mixed states can be derived from the probability conservation law, i.e. from the assumption that every probabilistic measure μ (mixed state) has to be constant along any trajectory in the phase space

$$\mu(t)(E) = \mu(t + \beta t)(\Phi_{\beta t}^H(E)), \quad E \in \mathcal{B}(M).$$

That property, written in terms of the pull-back of a measure, takes the form

$$\mu(t) = (\Phi_{\beta t}^H)^* \mu(t + \beta t). \tag{3.3.9}$$

From (3.3.9) it follows that

$$\begin{aligned} 0 &= \lim_{\beta t \to 0} \frac{(\Phi_{\beta t}^H)^* \mu(t + \beta t) - \mu(t)}{\beta t} = \frac{d}{ds}(\Phi_s^H)^* \mu(t + s) \mid_{s=0} \\ &= \frac{d}{ds}(\Phi_0^H)^* \mu(t + s) \mid_{s=0} + \frac{d}{ds}(\Phi_s^H)^* \mu(t) \mid_{s=0}, \end{aligned}$$

which further implies that

$$\boxed{\frac{\partial \mu}{\partial t} + L_{X_H} \mu = 0,} \tag{3.3.10}$$

where $L_{X_H} \mu$ denotes a Lie derivative of the measure μ in the direction of the vector field X_H. Equation (3.3.10) is called a *Liouville equation* and it describes the time evolution of the state μ.

When a mixed state μ takes a form $d\mu = \rho \, d\Omega_\omega$, for a smooth function ρ, then the Liouville equation (3.3.10) can be written in an alternative form. Actually, from (3.3.10) we have

$$0 = \frac{\partial}{\partial t}(\rho(t)\Omega_\omega) + L_{X_H}(\rho(t)\Omega_\omega) = \left(\frac{\partial \rho}{\partial t}(t) + L_{X_H}\rho(t)\right)\Omega_\omega,$$

where the fact that $L_{X_H}\Omega_\omega = 0$, following from (3.2.2) and (3.3.1), was used. It further implies that

$$0 = \frac{\partial \rho}{\partial t} + L_{X_H}\rho = \frac{\partial \rho}{\partial t} + X_H \rho = \frac{\partial \rho}{\partial t} + \{\rho, H\}_\pi .$$

Hence, the following time evolution equation for the function ρ corresponding to the state μ was derived

$$\boxed{\frac{\partial \rho}{\partial t} - \{H, \rho\}_\pi = 0.} \tag{3.3.11}$$

Notice that Liouville equation (3.3.11), for time evolution of a classical mixed state, is a single linear PDE and represents the Schrödinger picture of classical Hamiltonian evolution.

For a pure state $\delta_{x(t)}$ the Liouville equation (3.3.10) is equivalent to the Hamiltonian equation (3.3.7). Indeed, from (3.3.10) it follows that

$$0 = \frac{\partial \delta_{\xi(t)}}{\partial t} + \frac{d}{ds}(\Phi_s^H)^* \delta_{\xi(t)} \mid_{s=0} = \frac{\partial \delta_{\xi(t)}}{\partial t} + \frac{d}{ds}\delta_{\Phi^H_{-s}(\xi(t))} \mid_{s=0}$$

and from the above equation we get

$$0 = \dot{\xi}(t) - \frac{d}{ds}\Phi_s^H(\xi(t)) \mid_{s=0} = \dot{\xi}(t) - X_H(\xi(t)),$$

which is just the Eq. (3.3.7).

Until now the states undergo the time development whereas the observables do not. As we have just mentioned, this is a counterpart of a well known Schrödinger picture in quantum mechanics. There is also a dual representation in which states remain still whereas the observables undergo the time development. This approach corresponds in quantum mechanics to the Heisenberg picture. A pull-back of the Hamiltonian flow $U_t^H = (\Phi_t^H)^* = e^{tL_{X_H}}$ is, for every t, an automorphism of the algebra of observables $\mathcal{A}_C(M)$ (it preserves the linear structure as well as the point-wise product and the Poisson bracket). Its action on an arbitrary observable $A \in \mathcal{A}_C(M)$ is interpreted as the time evolution of A

$$A(t) = U_t^H A(0) = e^{tL_{X_H}} A(0) = e^{tX_H} A(0) = e^{-t\{H, \cdot\}_\pi} A(0). \tag{3.3.12}$$

Differentiating Eq. (3.3.12) with respect to t we obtain the known time evolution equation (3.2.38) for an observable A

$$\frac{dA}{dt}(t) - \{A(t), H\}_\pi = 0. \tag{3.3.13}$$

Contrary to Liouville equation (3.3.11), Eq. (3.3.13) is a single nonlinear ODE and represents the Heisenberg picture of classical Hamiltonian evolution of an observable. In particular, for coordinate observables $A = \xi$, we get

$$\frac{d\xi}{dt}(t) - \{\xi(t), H\}_\pi = 0 \overset{(3.2.1)}{\Longleftrightarrow} \xi_t = \pi\, dH(\xi)$$

i.e. again the Hamiltonian equation (3.3.7).

For any observable A, both presented approaches to the time development yield the same predictions concerning the results of measurements, since

$$\begin{aligned}\langle A(0)\rangle_{\mu(t)} &= \int_M A(0)\, d\mu(t) = \int_M A(0)\, d\Big((\Phi^H_{-t})^*\mu(0)\Big) = \int_M (\Phi^H_t)^* A(0)\, d\mu(0)\\ &= \int_M A(t)\, d\mu(0) = \langle A(t)\rangle_{\mu(0)}.\end{aligned}$$

From (3.3.13) follows that the time evolution of the expectation value of observable A in a mixed state $\rho(t)$, i.e. $\langle A\rangle_{\rho(t)}$, fulfills the following equation

$$\frac{d}{dt}\langle A\rangle_{\rho(t)} - \langle\{A, H\}_\pi\rangle_{\rho(t)} = 0. \tag{3.3.14}$$

Indeed

$$\int_M A(\xi)\frac{\partial\rho}{\partial t}(\xi, t)d\Omega_\omega = \frac{d}{dt}\int_M A(\xi)\rho(\xi, t)d\Omega_\omega = \frac{d}{dt}\langle A\rangle_{\rho(t)},$$

$$\begin{aligned}\int_M A(\xi)\{H, \rho\}(\xi, t)d\Omega_\omega &= \int_M \{H, A\rho\}(\xi, t)d\Omega_\omega - \int_M \{H, A\}(\xi)\rho(\xi, t)d\Omega_\omega\\ &= -\int_M \{H, A\}(\xi)\rho(\xi, t)d\Omega_\omega = \frac{d}{dt}\langle A\rangle_{\rho(t)}.\end{aligned}$$

In second equality we used the fact that on M

$$\begin{aligned}\int_M \{H, A\rho\}(\xi, t)d\Omega_\omega &= \int_M \left(\frac{d}{dt} - \frac{\partial}{\partial t}\right)(A\rho)(\xi, t)d\Omega_\omega\\ &= \left(\frac{d}{dt} - \frac{d}{dt}\right)\langle A\rangle_{\rho(t)} = 0.\end{aligned}$$

Observation 3 *From our considerations follows that in classical statistical Hamiltonian mechanics time development can be realized by two equivalent schemes. Actually, we can either evolve in time a mixed state according to linear PDE (3.3.11) or evolve in time an observable according to nonlinear ODE (3.3.13). Exceptional*

are classical Hamiltonian equations of motion which represent simultaneously time evolution of classical pure stat in the frame of classical Schrödinger picture and time evolution of position and momentum observables in the frame of classical Heisenberg picture.

In Hamiltonian statistical mechanics the classical uncertainty relations for canonical observables of position x^i and momentum p_j in a state ρ are of the form

$$\boxed{\triangle x^i \triangle p_j \geq 0, \quad i, j = 1, \ldots, n,} \tag{3.3.15}$$

where

$$\triangle A = \sqrt{\langle A^2 \rangle_\rho - \langle A \rangle_\rho^2}$$

is a standard deviation. The above uncertainty relations state that it is possible to measure simultaneously the position and momentum of a particle with an arbitrary precision. In particular, the equality in (3.3.15) takes place for pure states. On the other hand, all states which minimize (3.3.15) will be called *classical coherent states*. So, on a classical level, arbitrary pure state

$$\boxed{\rho_C(x', p') = \delta(x' - x)\delta(p' - p),} \tag{3.3.16}$$

is simultaneously a coherent state. That property will not be preserved on a quantum level.

Observation 4 *According to the presented model, "physics" is represented by measurable objects, i.e. expectation values of observables in any classical state. On the other hand, for pure coherent classical state (3.3.16) Eqs. (3.3.13) and (3.3.14) coincide. In particular* $Q^i(t) = \langle Q^i(t) \rangle_{\rho_C}$ *and* $P_i(t) = \langle P_i(t) \rangle_{\rho_C}$, *so classical trajectory (3.2.34) represents simultaneously the time evolution of expectation values of position and momentum in a classical pure coherent state (3.3.16).*

3.4 Reduction of Poisson Structures and Hamiltonian Systems Onto Submanifolds

The reduction theory of dynamical systems consists of two branches: the first branch deals with constrained Lagrangian systems, the second one with constrained Hamiltonian systems. In the Lagrangian approach one considers separately the case of holonomic constraints, i.e. the constraints which may depend on velocities, but only in such a way that the equations of constraints can be integrated to eliminate velocities, and the non-holonomic case [195]. In many cases authors first consider the Lagrangian formulation and then pass to the corresponding Hamiltonian formulation (see the example [252]).

The reduction theory in the Hamiltonian context has been initiated by P.A.M. Dirac, who in his famous paper [94] described a method of reducing a given Poisson bracket onto a submanifold given by some constraints φ provided they were of "second class". In this approach the classical notion of holonomic constraints is usually not introduced as in this context on arbitrary Poisson manifold (M, Π) there is no obvious division of variables between "position" and "momentum". The ideas of Dirac were developed in many papers, among others in [122, 135, 179, 180, 197, 198, 237] (see also the literature quoted there).

A very general geometric interpretation of the Poisson reduction procedure has been investigated in [197] by Marsden and Ratiu. Nevertheless, it is presented in an abstract, formal way which is made hard to handle with particular "physical" reduction problems. In this section, following [190], we present a constructive, computable method of reducing (locally) a given Poisson tensor Π to any regular submanifold (foliation) S. The idea of the method is to chose a distribution $\mathcal{Z}$ (not necessarily integrable) that is transversal to the foliation S, i.e. at any $x \in M$ it completes $T_x S$ to $T_x M$. Moreover, the choice of $\mathcal{Z}$ is such that makes the operator Π $\mathcal{Z}$-invariant (see definitions below) and allows to deform the Poisson tensor Π to a new Poisson tensor Π_D such that its image will be tangent to the submanifold S. This new operator Π_D will be always Poisson (and so its natural restriction to S will be Poisson). In consequence, we obtain a method of reducing a Hamiltonian system on M to a Hamiltonian system on every leaf S_ν of the foliation S. This reduced system strongly depends on the choice of the distribution $\mathcal{Z}$. As a special case we obtain the classical Dirac reduction of the Hamiltonian system. All our considerations will be local in the sense that our manifold M is perhaps only an open submanifold of a larger manifold. The presented construction is equivalent to the reduction method proposed by Marsden and Ratiu. However, it has advantages: it can be performed simultaneously on any leaf S_ν of the foliation S, it is constructive and it is formulated in the language of Poisson bi-vectors rather than Poisson brackets. The reader can also find some elements of the proposed scheme in [85, 112, 113] in the context of Poisson pencils.

The detailed analysis of the reduction procedures for Poisson bi-vectors will be crucial for a bi-Hamiltonian separability theory of integrable systems, developed in Chap. 5.

3.4.1 Geometric Reduction of Poisson Bi-Vectors

Let us consider a smooth manifold M of the finite dimension m and a foliation S of M consisting of the leaves S_ν parametrized by $\nu \in \mathbb{R}^r$ (so that $r \in \mathbb{N}$ is the codimension of every leaf S_ν). Moreover, consider a regular distribution $\mathcal{Z}$ on M (i.e. a smooth collection of the spaces $\mathcal{Z}_x \subset T_x M$ where ν is such that $x \in S_\nu$) such that it completes every $T_x S$ to $T_x M$ in the sense that

$$T_x M = T_x S_\nu \oplus \mathcal{Z}_x \tag{3.4.1}$$

for every x in M. In consequence, every vector field X on the manifold M has a unique decomposition $X = X_\parallel + X_\perp$ such that for every x in M the vector $\left(X_\parallel\right)_x \in T_x S$ ($X_\parallel$ is tangent to the leaves of the foliation S) while $(X_\perp)_x \in \mathcal{Z}_x$ ($X_\perp$ is contained in the distribution $\mathcal{Z}$). The splitting (3.4.1) induces the respective splitting of the dual space $T_x^* M$:

$$T_x^* M = T_x^* S_v \oplus \mathcal{Z}_x^* \tag{3.4.2}$$

where $T_x^* S_v$ is the annihilator of $\mathcal{Z}_x$ while the space $\mathcal{Z}_x^*$ is the annihilator of $T_x S_v$. Thus, any one-form α on M has a unique decomposition $\alpha = \alpha_\parallel + \alpha_\perp$ such that $\left(\alpha_\parallel\right)_x \in T_x^* S$ ($\alpha_\parallel$ annihilates the vectors from $\mathcal{Z}$) while $(\alpha_\perp)_x \in \mathcal{Z}_x^*$ ($\alpha_\perp$ annihilates the vectors tangent to the foliation S). We will call $X_\parallel$ and $\alpha_\parallel$ projections of X and α (respectively) on the foliation S. Abusing notation a bit we will write that $X \subset TS$ if $X = X_\parallel$, $X \subset \mathcal{Z}$ if $X = X_\perp$ and similarly for one forms: $\alpha \subset T^* S$ if $\alpha = \alpha_\parallel$, $\alpha \subset \mathcal{Z}^*$ if $\alpha = \alpha_\perp$.

Assume now that our manifold M is equipped with a Poisson bi-vector Π. A smooth real-valued function F on M is called $\mathcal{Z}$-invariant if the Lie derivative $L_Z F = 0$ for any vector field $Z \subset \mathcal{Z}$. We will now adopt the following definition. The operator Π is said to be $\mathcal{Z}$-invariant if $L_Z \{F, G\}_\Pi = 0$ for any pair of $\mathcal{Z}$-invariant functions F and G and every vector field $Z \subset \mathcal{Z}$.

Observe that our definition does not necessarily mean that $L_Z \Pi = 0$ for all vector fields $Z \subset \mathcal{Z}$, as for any pair F, G of $\mathcal{Z}$-invariant functions the condition $L_Z \{F, G\}_\Pi = 0$ means only that the function $\langle dF, (L_Z \Pi)\, dG\rangle$ vanishes. Thus, Π does not have to be an invariant of the distribution $\mathcal{Z}$ to be $\mathcal{Z}$-invariant in our meaning. Notice also, that the above definition is equivalent to the statement that for any pair $\alpha, \beta \subset T^* S$ we have $\langle \alpha, (L_Z \Pi)\, \beta\rangle = 0$ (since if F is $\mathcal{Z}$-invariant then $dF \subset T^* S$).

Let us assume that the distribution $\mathcal{Z}$ is spanned by r vector fields Z_i and that for operator Π there exist vector fields W_{ij}, $j = 1, \ldots r$ such that

$$\boxed{L_{Z_i} \Pi = \sum_{j=1}^{r} W_{ij} \wedge Z_j.} \tag{3.4.3}$$

It is obvious that this definition does not depend on the choice of the basis in $\mathcal{Z}$ (although the vector fields W_{ij} do). If the operator Π fulfills the condition (3.4.3), then it is $\mathcal{Z}$-invariant, as then for any two one-forms $\alpha, \beta \subset T^* S$

$$\langle \alpha, (L_Z \Pi)\, \beta\rangle = \sum_{j=1}^{r} \left\langle \alpha, \left(W_{ij} \wedge Z_j\right) \beta \right\rangle = 0,$$

since α and β annihilate all the vector fields Z_i. The converse statement is not true in general.

For a Poisson tensor Π on M let us define the following bi-vector:

$$\Pi_D(\alpha, \beta) = \Pi(\alpha_{||}, \beta_{||}) \text{ for any pair } \alpha, \beta \text{ of one-forms.} \tag{3.4.4}$$

We will call the bi-vector Π_D a deformation of Π. Observe that it always exists and that it is uniquely defined once the foliation S and the distribution $\mathcal{Z}$ are given. Its image lies always in TS. $\Pi_D(\alpha) \subset TS$ for any one-form α on M, i.e. the image of Π_D is tangent to the foliation S. To prove it we have to show that $\langle \beta, \Pi_D \alpha \rangle = 0$ for any $\beta \subset \mathcal{Z}^*$, but

$$\langle \beta, \Pi_D \alpha \rangle = \Pi_D(\beta, \alpha) = \Pi\left(\beta_{\|}, \alpha_{\|}\right) = 0$$

since $\beta_{\|} = 0$ for every $\beta \subset \mathcal{Z}^*$.

Thus, the deformed bi-vector Π_D has its image in TS and if we regard it as mapping from one-forms to vector fields on M then it can be naturally restricted to a bi-vector π_{S_ν} on every leaf S_ν of S by simply restricting its domain to S_ν :

$$\pi_{S_\nu} = \Pi_{D|S_\nu}.$$

Besides, Π_D induces a new bracket for functions on M

$$\boxed{\{F, G\}_{\Pi_D} = \Pi_D(dF, dG) = \Pi((dF)_{||}, (dG)_{||}).} \tag{3.4.5}$$

Obviously, the bi-vector Π_D (and thus even π_{S_ν}) does not have to be Poisson. However, it turns out that if Π is $\mathcal{Z}$-invariant then Π_D (and thus every π_{S_ν}) is Poisson. Obviously, this bracket is antisymmetric and satisfies the Leibniz property. It remains to show, that it also satisfies the Jacobi identity for any functions F, G, H. Using the definition of Π_D, this condition can be written as

$$\left\langle \left(d\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle\right)_{\|}, \Pi(dH)_{\|}\right\rangle + \text{cycl.} = 0.$$

However, for any vector field $Z \subset \mathcal{Z}$ we have

$$\left\langle d\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle, Z\right\rangle = Z\left(\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle\right) = L_Z\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle = 0$$

due to the assumed $\mathcal{Z}$-invariance of Π. This means that $d\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle \subset T^*S$, so that

$$\left(d\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle\right)_{\|} = d\left\langle (dF)_{\|}, \Pi(dG)_{\|}\right\rangle,$$

and thus condition (3.4.1) turns out to be the Jacobi identity for Π, which is satisfied since Π is Poisson.

Thus, given a foliation S on M and a transversal distribution $\mathcal{Z}$ on M, such that (3.4.1) is satisfied, we can reduce any Poisson bi-vector Π that is $\mathcal{Z}$-invariant to a Poisson bi-vector π_{S_ν} on the leaf S_ν of S by deforming Π to Π_D and then by restricting Π_D to S_ν. This construction yields the same operator π_{S_ν} as in the approach of Marsden and Ratiu [197] and besides we will show how this construction can be easily realized in practice.

Notice, that in a trivial case when the foliation S coincides with the symplectic foliation of Π then $\Pi_D = \Pi$, since in this case $\Pi((dF)_\perp) = 0$ for any function F and so

$$\{F, G\}_{\Pi_D} = \Pi((dF)_{||}, (dG)_{||}) = \Pi(dF, dG) = \{F, G\}_\Pi .$$

In this case π_{S_ν} is the standard reduction of Π on its symplectic leaf S_ν.

Let us consider a few special cases of our general construction. We firstly observe that the annihilator $\mathcal{Z}^*$ of TS is defined as soon as the foliation S is determined, so we do not need to specify a particular $\mathcal{Z}$ in order to define $\mathcal{Z}^*$. The distribution $\mathcal{D} = \Pi(\mathcal{Z}^*)$ (so that $\mathcal{D}_x = \Pi\left(\mathcal{Z}_x^*\right)$) is called further on a *Dirac distribution* associated with the foliation S. Thus, the distribution $\mathcal{D}$ is determined by S and by Π. A priori, two limit cases are most interesting. If $TM = \mathcal{D} \oplus TS$ we say that we are in the Dirac case, while if $\mathcal{D} \subset TS$ we say that we are in the tangent case. In the Dirac case we have a canonical choice of $\mathcal{Z}$, i.e. we can choose $\mathcal{Z} = \mathcal{D}$. In this case Π is automatically $\mathcal{Z}$-invariant, since $\mathcal{Z}$ is spanned by the vector fields Hamiltonian with respect to Π. Nevertheless, we can also choose some other distribution $\mathcal{Z} \neq \mathcal{D}$ (non-canonical choices). Contrary to the previous case, in the tangent case we have no canonical choice of $\mathcal{Z}$ and we are free to find a distribution $\mathcal{Z}$ that makes Π $\mathcal{Z}$-invariant. Anyway, in both cases we have many non-equivalent deformations Π_D and hence many non-equivalent projections π_{S_ν}. Generically, the distribution $\mathcal{D}$ will not be tangent to S, but it will not suffice to span TM together with TS.

Let us now suppose that the foliation S of M is parametrized by the set of r functionally independent real valued functions $\varphi_i(x)$ so that its leaves have the form $S_\nu = \{x \in M : \varphi_i(x) = \nu_i, \nu_i \in \mathbb{R}, i = 1, \ldots, r\}$ where r is the codimension of the foliation. The one-forms $d\varphi_i$ constitute a basis in $\mathcal{Z}^*$. Then, the Dirac distribution $\mathcal{D}$ is spanned by k (possibly dependent) Hamiltonian vector fields $X_i = \Pi d\varphi_i$. Let us denote a basis of $\mathcal{Z}$ dual to the basis $\{d\varphi_i\}$ in $\mathcal{Z}^*$ by Z_i, i.e. $Z_i(\varphi_j) = \delta_{ij}$. Our projections $X_\parallel$ and $\alpha_\parallel$ are then given by

$$X_{||} = X - \sum_{i=1}^{r} X(\varphi_i) Z_i,$$

(obviously $X_{||}(\varphi_i) = 0$ for all i so that indeed this vector field is tangent to the leaves of S) and by

$$\alpha_{||} = \alpha - \sum_{i=1}^{r} \alpha(Z_i) d\varphi_i$$

(and obviously $\alpha_{||}(Z_i) = 0$ for all i). Thus

$$\begin{aligned}\Pi(\alpha_{||}, \beta_{||}) &= \Pi\left(\alpha - \sum_{i=1}^{r} \alpha(Z_i) d\varphi_i, \beta - \sum_{j=1}^{r} \beta(Z_j) d\varphi_j\right) \\ &= \Pi(\alpha, \beta) - \sum_{j=1}^{r} \beta(Z_j)\Pi(\alpha, d\varphi_j) - \sum_{i=1}^{r} \alpha(Z_i)\Pi(d\varphi_i, \beta) + \\ &+ \sum_{i,j=1}^{r} \alpha(Z_i)\beta(Z_j)\Pi(d\varphi_i, d\varphi_j),\end{aligned}$$

and so the deformation Π_D can be expressed by

$$\boxed{\Pi_D = \Pi - \sum_{i=1}^{r} X_i \wedge Z_i + \tfrac{1}{2} \sum_{i,j=1}^{r} \varphi_{ij} Z_i \wedge Z_j,} \tag{3.4.6}$$

where the functions φ_{ij} are defined as

$$\varphi_{ij} = \{\varphi_i, \varphi_j\}_\Pi = X_j(\varphi_i) = \Pi(d\varphi_i, d\varphi_j).$$

In the canonical Dirac case ($\mathcal{Z} = \mathcal{D}$), all the vector fields X_i are transversal to the foliation S and are moreover linearly independent. It happens precisely when $\det(\varphi_{ij}) \neq 0$ and the functions φ_i are then 'second class constraints' in the terminology of Dirac. The vector fields Z_i (the dual basis to $\{d\varphi_i\}$) can be expressed through the vector fields X_i by

$$Z_i = \sum_{j=1}^{r} (\varphi^{-1})_{ji} X_j, \quad i = 1, \ldots, r$$

as indeed

$$Z_j(\varphi_i) = \sum_{s=1}^{r} (\varphi^{-1})_{sj} X_s(\varphi_i) = \sum_{s=1}^{r} (\varphi^{-1})_{sj} \varphi_{is} = \delta_{ij}.$$

Moreover, in this case the deformation (3.4.6) attains the following form

$$\boxed{\Pi_D = \Pi - \tfrac{1}{2} \sum_{i=1}^{r} X_i \wedge Z_i} \tag{3.4.7}$$

and Π_D is Poisson as Π is $\mathcal{Z}$-invariant (3.4.3). It follows from (2.5.19), (3.2.2) and (3.2.24) as we have

$$\begin{aligned} L_{Z_m}\Pi &= \sum_{j=1}^{r}(\varphi^{-1})_{jm}L_{X_j}\Pi - \sum_{j=1}^{r}\Pi d(\varphi^{-1})_{jm}\wedge X_j \\ &= \sum_{j=1}^{r}[X_j, X_m]\wedge X_j = \sum_{j=1}^{r} W_{mj}\wedge Z_j, \quad m = 1,\dots,r. \end{aligned}$$

Π_D defines the following bracket on $C^\infty(\mathcal{M})$

$$\boxed{\{F,G\}_{\Pi_D} = \{F,G\}_\Pi - \sum_{i,j=1}^{r}\{F,\varphi_i\}_\Pi(\varphi^{-1})_{ij}\{\varphi_j,G\}_\Pi,} \tag{3.4.8}$$

which is just the well known *Dirac bracket* [94] related to the bracket $\{.,.\}_\Pi$. Notice also that for the Dirac-Poisson tensor (3.4.7) constraints φ_i are its Casimir functions, as

$$\begin{aligned} \Pi_D d\varphi_m &= \left(\Pi - \tfrac{1}{2}\sum_{i=1}^{r} X_i\wedge Z_i\right) d\varphi_m \\ &= X_m - \tfrac{1}{2}\sum_{i=1}^{r} Z_i(\varphi_m)X_i + \tfrac{1}{2}\sum_{i=1}^{r} X_i(\varphi_m)Z_i \\ &= X_m - \tfrac{1}{2}\sum_{i=1}^{r}\delta_{im}X_i + \tfrac{1}{2}\sum_{i=1}^{r}\varphi_{mi}\sum_{j=1}^{r}(\varphi^{-1})_{ji}X_j \\ &= X_m - \tfrac{1}{2}X_m - \tfrac{1}{2}\sum_{i,j=1}^{r}\varphi_{mi}(\varphi^{-1})_{ij}X_j \\ &= X_m - \tfrac{1}{2}X_m - \tfrac{1}{2}\sum_{j=1}^{r}\delta_{mj}X_j = 0. \end{aligned} \tag{3.4.9}$$

In the tangent case all the vector fields X_i are tangent to the foliation S and the deformation (3.4.6) attains the form

$$\Pi_D = \Pi - \sum_{i=1}^{r} X_i\wedge Z_i \tag{3.4.10}$$

where transversal distribution $\mathcal{Z}$ has to be chosen in a form which makes Π $\mathcal{Z}$-invariant (3.4.3). Thus,

$$\begin{aligned} L_{Z_m}\Pi =& L_{Z_m}(\Pi_D + \sum_{i=1}^{r} X_i \wedge Z_i) \\ =& L_{Z_m}\Pi_D + \sum_{i=1}^{r}[Z_m, X_i] \wedge Z_i + \sum_{i=1}^{r} X_i \wedge [Z_m, Z_i] \end{aligned}$$

and the condition of being $\mathcal{Z}$-invariant is as follows

$$L_{Z_m}\Pi_D + \sum_{i=1}^{r} X_i \wedge [Z_m, Z_i] = 0.$$

Its strong solution, when $\mathcal{Z}$ is an integrable distribution,

$$L_{Z_m}\Pi_D = 0, \quad [Z_m, Z_i] = 0, \quad m, i = 1, \ldots, r, \tag{3.4.11}$$

will be discussed in detail in the next chapter in the frame of the bi-Hamiltonian separability theory. Observe, that (3.4.11) is equivalent to the condition

$$L_{Z_m}\Pi = \sum_{i=1}^{r}[Z_m, X_i] \wedge Z_i,$$

where $[Z_m, X_i] := W_{mi}$ (3.4.3). Also here the constraints φ_i are Casimir functions of the Π_D Poisson tensor (3.4.10) as

$$\begin{aligned} \Pi_D d\varphi_m &= \left(\Pi - \sum_{i=1}^{r} X_i \wedge Z_i\right) d\varphi_m \\ &= X_m - \sum_{i=1}^{r} Z_i(\varphi_m) X_i + \sum_{i=1}^{r} X_i(\varphi_m) Z_i \\ &= X_m - \sum_{i=1}^{r} \delta_{im} X_i = 0. \end{aligned} \tag{3.4.12}$$

3.4.2 Reduction of Hamiltonian Dynamics

Let us begin this subsection by stating a well-known theorem about the relation between the Dirac deformation Π_D of Π and the dynamic imposed by the con-

straints. Suppose, thus, that our manifold M is a cotangent bundle of a Riemannian manifold (Q, g) with a covariant metric tensor g. Denote the corresponding contravariant metric tensor by G. Let us consider a particle moving in our Riemannian manifold Q according to Lagrangian equations of motion

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x_t^i} = 0, \quad i = 1, \ldots, n, \tag{3.4.13}$$

with a Lagrangian function $L(x, x_t) = \frac{1}{2} g_{ij} x_t^i x_t^j - V(x)$. As we know, this leads to Hamiltonian equations of motion on $M = T^*Q$

$$(x^i)_t = \{x^i, H\}_\Pi\,, \ (p_i)_t = \{p_i, H\}_\Pi \tag{3.4.14}$$

with the Hamiltonian $H = \frac{1}{2} p^T G p + V(x)$ and with the canonical Poisson tensor Π. Now, suppose that this particle is subordinated to some holonomic constraints on Q

$$\varphi_k(x) = 0, \quad k = 1, \ldots, s \tag{3.4.15}$$

defining a submanifold Q_0 of Q. One often makes a physical assumption here that the surface Q_0 acts on our system with a reaction force $R(x, x_t)$ that is orthogonal to Q_0 and such that the trajectories of the constrained system

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\frac{\partial L}{\partial x_t^i} = R_i(x, x_t) \tag{3.4.16}$$

that start on Q_0 remain on Q_0. The velocity $x_t = v = v^i \frac{\partial}{\partial x^i}$ of this particle must then remain tangent to this submanifold so that

$$0 = \langle d\varphi_k, v \rangle = \frac{\partial \varphi_k}{\partial x^i} v^i$$

and since $v^i = G^{ij} p_j$ the motion of the particle is constrained not only by the s relations (3.4.15) but also by the s relations

$$\varphi_{s+k}(x, p) \equiv G^{ij}(q) \frac{\partial \varphi_k(x)}{\partial x^i} p_j = \left(d\varphi_k\right)^T G p = 0, \qquad k = 1, \ldots, s, \tag{3.4.17}$$

that are the lift of (3.4.15) to M. We call the constraints (3.4.17) a g-consequence of the constraints (3.4.15), as they are natural differential consequences of (3.4.15) at a given metric tensor g. The constraints (3.4.15)–(3.4.17) define a submanifold S

of M of dimension $n-2s$ and so modify the (3.4.14) to

$$(x^i)_t = \{x^i, H\}_\Pi\ ,\ (p_i)_t = \{p_i, H\}_\Pi + R_i(x,p). \tag{3.4.18}$$

The equations (3.4.18) are Hamiltonian ones and can be written in the form

$$(x^i)_t = \{x_i, H\}_{\Pi_D}\ ,\ \ (p_i)_t = \{p_i, H\}_{\Pi_D} \tag{3.4.19}$$

where $\Pi_D = \Pi - \frac{1}{2}\sum_{i=1}^{r} X_i \wedge Z_i$ is the Dirac deformation (3.4.7) of Π given by the constraints $\varphi_i,\ i = 1,\ldots,2s = r$. Thus, the response of the Lagrangian system (3.4.13) subordinated to the reaction forces R is accounted for by the related Dirac deformation of the Poisson tensor Π. Below we will only sketch the proof in the case of a pair of constraints.

The reaction force R can be calculated by differentiating the assumed identity $\varphi(x(t)) \equiv 0$ twice with respect to t and eliminating the second derivatives $\left(x^i\right)_{tt}$ applying equations (3.4.16) and by using the demand that the force is orthogonal to Q_0. After some calculations we obtain the result

$$R(q,p) = \frac{1}{(d\varphi)^T\, G\, (d\varphi)}\left((d\varphi)^T\, G\, dV - (p^T G) H_\varphi(Gp) + A\right) d\varphi, \tag{3.4.20}$$

where $\left(H_\varphi\right)_{ij} = \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ is the Hessian of φ and $A = A(x,p)$ is given by

$$A = \frac{\partial \varphi}{\partial x_s}\Gamma^s_{ij} G^{ir} G^{jm} p_r p_m.$$

Notice that it vanishes in flat coordinates when all Christoffel's symbols Γ^i_{jk} are equal to zero. On the other hand, calculating the explicit form of (3.4.19) on the submanifold Q_0 given by the constraints φ_1, φ_2, leads to the equations (3.4.18) with R given by (3.4.20).

Consider now a Hamiltonian vector field $X = \Pi dH$ on M. We constantly assume that we have a smooth, regular foliation S on M and a regular distribution $\mathcal{Z}$ on M such that (3.4.1) is satisfied. Thus, the corresponding Π_D defined by (3.4.4) is Poisson and has its image tangent to the foliation S, so that it can be properly restricted on every leaf S_ν of S. Then, we call the vector field $X_D = \Pi_D dH$ the Hamiltonian projection of the Hamiltonian vector field $X = \Pi dH$.

The vector field X_D lives on every leaf of the foliation, i.e. its restriction on the leaf S_ν is tangent to S_ν. Moreover, on the leaf S_ν it coincides with the Hamiltonian vector field $\pi_{S_\nu} dh$:

$$\Pi_D dH_{|S_\nu} = \pi_{S_\nu} dh,$$

where $h = H_{|S_\nu}$ is the restriction of the Hamiltonian H to the leaf S_ν. To see this it is enough to choose a parametrization $\{\varphi_i\}$ of S and to pass on M to any system of coordinates of the form (x, φ). In these coordinates the bi-vector Π_D has a matrix form with a non-zero upper-left block coinciding with the matrix form of π_{S_ν} and with the remaining terms equal to zero.

There is a connection between $X = \Pi dH$ and its Hamiltonian projection $X_D = \Pi_D dH$. If $X = \Pi dH$, $X_D = \Pi_D dH$ and $X_i = \Pi d\varphi_i$ then

$$X_D = X_\parallel - \sum_{i=1}^{r} Z_i(H) X_{i\parallel}. \tag{3.4.21}$$

Indeed, a direct calculation shows

$$\begin{aligned} X_D = \Pi_D dH &= X - \sum_{i=1}^{r} (Z_i(H)X_i - X_i(H)Z_i) + \sum_{i,j=1}^{r} \varphi_{ij} Z_j(H) Z_i \\ &= X_\parallel - \sum_{i=1}^{r} Z_i(H) \left(X_i - \sum_{j=1}^{r} \varphi_{ji} Z_j \right), \end{aligned}$$

where the last equality is due to

$$X_i(H) = \langle dH, \Pi d\varphi_i \rangle = -\langle d\varphi_i, \Pi dH \rangle = -X(\varphi_i).$$

Since $\varphi_{ji} = X_i(\varphi_j)$ it yields (3.4.21).

Notice that the difference between X_D and $X_\parallel$ is the term $\sum_{i=1}^{r} Z_i(H) X_{i\parallel}$ that is tangent to the foliation S, as it should be. Since for the Dirac case $X_{i\parallel} = 0$, so $X_D = X_\parallel$ and the Hamiltonian projection is just the natural projection (in the sense of direct sum) along the distribution $\mathcal{Z}$.

The term $X_\parallel$ in X_D has a natural physical interpretation: it describes the evolution of the system $X = \Pi dH$ imposed with the constraints given by φ_i. In a non-Dirac reduction case, the physical meaning of the second term in X_D is not clear, although it should represent an additional force (friction) acting on the system and following from a chosen reduction.

Let us now consider the degeneracy of the Dirac deformation Π_D given by (3.4.7). Suppose that the real valued functions c_i, $i = 1, \ldots, s$ on M span the kernel of the operator Π and are such that the functions $\{c_i, \varphi_j\}$ are functionally independent. Then,

1. the constraints φ_i and the 'old' Casimirs c_i are all Casimirs of Π_D,
2. any Casimir of Π_D must be of the form $C(c_1, \ldots, c_s, \varphi_1, \ldots \varphi_r)$.

The proof of **1** is just a calculation partially done in (3.4.9). For c_i functions we have

$$\begin{aligned}\Pi_D dc_r &= \left(\Pi - \tfrac{1}{2}\sum_{i=1}^{r} X_i \wedge Z_i\right) dc_r \\ &= \Pi dc_r - \tfrac{1}{2}\sum_{i=1}^{r} Z_i(c_r)X_i + \tfrac{1}{2}\sum_{i=1}^{r} X_i(c_r)Z_i \\ &= -\tfrac{1}{2}\sum_{i,j=1}^{r} (\varphi^{-1})_{ji} X_j(c_r)X_i = 0.\end{aligned}$$

To prove **2** let us complete the functionally independent functions $\{c_i, \varphi_j\}$ to a coordinate system $\{c_i, \varphi_j, x_k\}$ on M. Suppose that a function $C = C(c, \varphi, x)$ is a Casimir of $\Pi_D : \Pi_D dC = 0$. Then, according to (3.4.21), $(\Pi dC)_{\|} = 0$, i.e. $\Pi dC \subset \mathcal{Z}$. In the Dirac case the distribution $\mathcal{Z}$ is spanned by the vector fields X_i so that there exist functions α_i such that

$$\Pi dC = \sum_{i=1}^{r} \alpha_i X_i = \sum_{i=1}^{r} \alpha_i \Pi d\varphi_i = \Pi\left(\sum_{i=1}^{r} \alpha_i d\varphi_i\right).$$

Thus, $\Pi\left(dC - \sum_{i=1}^{r} \alpha_i d\varphi_i\right) = 0$ or $dC = \sum_{i=1}^{r} \alpha_i d\varphi_i + \sum_{i=1}^{s} \beta_i dc_i$ which proves **2**.

Concluding, we state that the Dirac deformation (3.4.7) preserves all Casimirs of Π tensor and introduces new Casimirs φ_i and that no other Casimirs arise in this process. The situation in the case of general deformation (3.4.4) (or (3.4.6)) is more complicated, since the Casimirs of Π does not have to survive and moreover new Casimirs, different from φ_i ones, can arise. We can only state that in the general case the function C is a Casimir of Π_D if and only if the vector field $Y = \Pi dC$ satisfies the relation

$$Y_{\|} = \sum_{i=1}^{r} Z_i(C) X_{i\|}$$

which for the Dirac case degenerates to the already discussed condition $Y_{\|} = 0$.

Here we illustrate the presented formalism in the Dirac case, both canonical and non-canonical. The tangent case will be considered in the next chapter. Consider the following system of the second order Newton equations

$$\begin{aligned} x_{tt} &= -10x^2 + 4y \\ y_{tt} &= -16xy + 10x^3 + 4z \\ z_{tt} &= -20xz - 8y^2 + 30x^2y - 15x^4. \end{aligned} \tag{3.4.22}$$

What is interesting this system is the so called first Newton representation of the seventh-order stationary flow of the Korteweg-de Vries hierarchy [225], with the corresponding Lagrangian

$$L = x_t z_t + \tfrac{1}{2} y_t^2 + 4yz - 10x^2 z - 8xy^2 + 10x^3 y - 3x^5.$$

The Legendre transformation (3.1.7), (3.1.2) with

$$\begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \\ z_t \end{pmatrix}$$

leads to a Hamiltonian representation:

$$\frac{d}{dt}\left(x, y, z, p_x, p_y, p_z\right)^T = X = \pi dH,$$

where π is the canonical Poisson tensor on a 6-dimensional phase space $T^*\mathbb{E}^{2,1}$ with the Hamiltonian

$$H = p_x p_z + \tfrac{1}{2} p_y^2 + 10x^2 z - 4yz + 8xy^2 - 10x^3 y + 3x^5.$$

Notice that

$$G = \begin{pmatrix} 0\,0\,1 \\ 0\,1\,0 \\ 1\,0\,0 \end{pmatrix},$$

so (x, y, z) are flat but nonorthogonal coordinates on $\mathbb{E}^{2,1}$. Consider also a foliation $\mathcal{S}$ given by a pair of constraints

$$\varphi_1 = z + xy, \quad \varphi_2 = (d\varphi_1)^T Gp = p_x + xp_y + yp_z.$$

where φ_2 is the lift of φ_1 from the configuration space Q to T^*Q, with respect to the antidiagonal metric tensor G. The vector fields $X_i = \Pi d\varphi_i$ have the form

$$X_1 = -y\frac{\partial}{\partial p_x} - x\frac{\partial}{\partial p_y} - \frac{\partial}{\partial p_z}, \; X_2 = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} - p_y\frac{\partial}{\partial p_x} - p_z\frac{\partial}{\partial p_y}$$

and they are transversal to S, so that we have the Dirac case.

Example 3.9 Here we consider a Dirac case with canonical reduction of the system (3.4.22). In this case the distribution $\mathcal{Z} = \mathcal{D} = Sp\{X_i\}$ makes Π $\mathcal{Z}$-invariant. The

basis in $\mathcal{Z}$ that is dual to $\{d\varphi_i\}$ is

$$Z_1 = \frac{1}{\varphi_{12}} X_2,\ Z_2 = -\frac{1}{\varphi_{12}} X_1,$$

where

$$\varphi_{12} = \{\varphi_1, \varphi_2\}_\Pi = 2y + x^2.$$

In the adapted coordinate system the Dirac deformation Π_D given by (3.4.7) attains $(x, y, p_y, p_z, \varphi_1, \varphi_2)$ the form

$$\Pi_D = \frac{1}{2y + x^2} \begin{pmatrix} 0 & 0 & -x & -1 & 0 & 0 \\ 0 & 0 & 2y & -x & 0 & 0 \\ x & -2y & 0 & p_z & 0 & 0 \\ 1 & x & -p_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It has, as it should, two Casimirs φ_1, φ_2 and we can easily restrict Π_D to the operator π_{S_ν} on S_ν. If we parametrize S_ν with the coordinates (x, y, p_y, p_z) (the constraints φ_1, φ_2 are constant on every S_ν) then

$$\pi_{S_\nu} = \frac{1}{2y + x^2} \begin{pmatrix} 0 & 0 & -x & -1 \\ 0 & 0 & 2y & -x \\ x & -2y & 0 & p_z \\ 1 & x & -p_z & 0 \end{pmatrix},$$

which, in accordance with the theory, is non-degenerate. Observe that this expression actually does not depend on the choice of the leaf S_ν in the foliation S. On every leaf S_ν the Hamiltonian projection $X_D = \Pi_D dH$ attains the form

$$\pi_{S_\nu} dh_{S_\nu} = \frac{1}{2y + x^2} \begin{pmatrix} (2y + x^2) p_z \\ (2y + x^2) p_y \\ -2x p_y p_z - 20x^3 y - 36x y^2 + 15x^5 \\ -2 p_y p_z - 36x^2 y + 12 y^2 - 5x^4 \end{pmatrix}$$

where

$$h_{S_\nu} = \tfrac{1}{2} p_y^2 - x p_y p_z - y p_y^2 - 20x^3 y + 12 x y^2 + 3x^5. \tag{3.4.23}$$

Example 3.10 Now we consider a Dirac case with non-canonical reduction of the system (3.4.22). Let us choose a non-canonical distribution $\mathcal{Z}$, for which Π is also

$\mathcal{Z}$-invariant. Since the operator Π has a very simple form, so any pair of constant fields will span such a distribution, as then the condition (3.4.3) is trivially satisfied. Thus, let us take

$$\mathcal{Z} = Sp\left\{\frac{\partial}{\partial z} + \frac{\partial}{\partial p_z}, \frac{\partial}{\partial z} + \frac{\partial}{\partial p_x}\right\}$$

(observe that this distribution is integrable). We have now to change the basis in $\mathcal{Z}$ to a new basis $\{Z_1, Z_2\}$ such that the condition $Z_i(\varphi_j) = \delta_{ij}$ is satisfied. A simple calculation yields

$$Z_1 = \frac{\partial}{\partial z} + \frac{1}{1-y}\left(-y\frac{\partial}{\partial p_x} + \frac{\partial}{\partial p_z}\right),\ Z_2 = \frac{1}{1-y}\left(\frac{\partial}{\partial p_x} - \frac{\partial}{\partial p_z}\right).$$

Now, the general deformation (3.4.6) defined by the above distribution attains the following form in the adopted coordinates $\left(x, y, p_y, p_z, \varphi_1, \varphi_2\right)$

$$\Pi_D = \frac{1}{y-1}\begin{pmatrix} 0 & 0 & 0 & -1 & 0\,0 \\ 0 & 0 & y-1 & -x & 0\,0 \\ 0 & 1-y & 0 & p_z - x & 0\,0 \\ 1 & x & x-p_z & 0 & 0\,0 \\ 0 & 0 & 0 & 0 & 0\,0 \\ 0 & 0 & 0 & 0 & 0\,0 \end{pmatrix}$$

and thus the restricted operator π_{S_ν} on the leaf S_ν parametrized with the coordinates $\left(x, y, p_y, p_z\right)$ is

$$\pi_{S_\nu} = \frac{1}{y-1}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & y-1 & -x \\ 0 & 1-y & 0 & p_z - x \\ 1 & x & x-p_z & 0 \end{pmatrix}$$

and is non-degenerate. Again, this expression does not depend on the choice of the leaf S_ν in the foliation S. The Hamiltonian projection $X_D = \Pi_D dH$ attains the following form on every leaf S_ν

$$\pi_{S_\nu} dh_{S_\nu} = \frac{1}{y-1}\begin{pmatrix} xp_y + 2yp_z \\ (y + x^2 - 1)p_y + (xy + x)p_z \\ -(y+1)p_y^2 - xp_y p_z + x^2 p_y + 2xyp_z + 4x(6y + 5x^2 y - 6y^2 - 5x^2) \\ -xp_y^2 + xp_z^2 - 2p_y p_x + xp_y - x^2 p_z - 36x^2 y + 12y^2 - 5x^4 \end{pmatrix}$$

where h_{S_ν} is given by (3.4.23).

Example 3.11 Here we consider a second Dirac case with non-canonical Poisson reduction of the system (3.4.22). Consider yet another distribution that makes Π $\mathcal{Z}$-invariant, namely

$$\mathcal{Z} = Sp\left\{\frac{\partial}{\partial z} + \frac{\partial}{\partial p_z}, \frac{\partial}{\partial x}\right\}.$$

The appropriate basis of $\mathcal{Z}$ is given by

$$Z_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial p_x} + \frac{\partial}{\partial p_z},\ Z_2 = \frac{\partial}{\partial p_x}.$$

so that we have $Z_i(\varphi_j) = \delta_{ij}$. The general deformation (3.4.6) defined by the above distribution yields in the adopted coordinates $(x, y, p_y, p_z, \varphi_1, \varphi_2)$

$$\Pi_D = \frac{1}{y-1}\begin{pmatrix} 0 & 0 & 0 & 0\,0\,0 \\ 0 & 0 & 1 & 0\,0\,0 \\ 0 & -1 & 0 & x\,0\,0 \\ 0 & 0 & -x & 0\,0\,0 \\ 0 & 0 & 0 & 0\,0\,0 \\ 0 & 0 & 0 & 0\,0\,0 \end{pmatrix}$$

so that the restricted operator π_{S_v} on the leaf S_v (again parametrized with the coordinates (x, y, p_y, p_z)) is

$$\pi_{S_v} = \frac{1}{y-1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & x \\ 0 & 0 & -x & 0 \end{pmatrix},$$

and is degenerate this time, with Casimir function $c = x$. The Hamiltonian projection $X_D = \Pi_D dH$ attains on every leaf S_v the form

$$\pi_{S_v} dh_{S_v} = \frac{1}{y-1}\begin{pmatrix} 0 \\ p_y - xp_z \\ p_y^2 - x^2 p_y - 2xyp_z + 20x^3 - 24xy \\ -xp_y + x^2 p_z \end{pmatrix}$$

where h_{S_v} is given by (3.4.23).

Thus, given a foliation S, by choosing different distributions $\mathcal{Z}$ we can obtain several different Hamiltonian projections of our original Hamiltonian system, not only just the well known Dirac canonical reduction. The tangent case of the reduction procedure will be considered in details in Chap. 5.

Chapter 4
Classical Integrable and Separable Hamiltonian Systems

In this chapter we introduce the concept of classical integrability of Hamiltonian systems and then develop the separability theory of such systems based on the notion of separation relations introduced by Sklyanin [235]. Separation relations are the most fundamental objects of modern separability theory as well as allow for classification of all separable systems. We concentrate our attention on the subclass of separable systems for which all constants of motion are quadratic in momenta. This class of systems is most interesting from the physical point of view on both classical and quantum level.

4.1 Integrable Hamiltonian Systems

Once we have a Hamiltonian system, modeling some physical phenomena, we would like to solve it in order to get admissible trajectories of time evolution. We could do it systematically if the system under consideration is linear. Unfortunately, it happens only in simplest models or in linear approximation of the considered realistic models. In a generic situation Hamiltonian systems are nonlinear. As a consequence, even simple nonlinear dynamical systems can exhibit a completely unpredictable behavior, which might seem to be random, despite the fact that they are fundamentally deterministic. This seemingly unpredictable behavior has been called chaos. For such systems the distance between two, arbitrary close to each other, different initial conditions may diverge exponentially fast. As a consequence, trajectories of chaotic systems are not expressible by elementary functions or even special functions. Thus in order to find particular trajectories numerical methods have to be implemented. On the other hand, among all nonlinear Hamiltonian systems there exists a subclass of systems which despite their nonlinearity are integrable by quadratures. We call them nonlinear integrable systems and they are the subject of our further consideration, on both classical and quantum level.

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4.1.1 Liouville Integrability

Let M be a symplectic manifold of dimension $\dim M = 2n$ with the defined Hamiltonian system (3.2.1). Assume that the considered system posseses n global (defined on the whole M, possible modulo some set of measure zero) constants of motion $H_1, \ldots, H_n$ (first integrals) all in involution (Poisson commute) $\{H_i, H_j\}_\pi = 0$, and functionally independent in the sense that dH_i are linearly independent. Such a system is called *Liouville integrable*.

Let (Q, g, ∇) be a Riemannian space with Levi-Civita connection generated by a covariant metric tensor g. As was mentioned in Sect. 2.6, a symmetric covariant tensor $K \in T^{(m,0)}(Q)$ is called a Killing tensor of the metric g when it fulfills the condition (2.6.20)

$$\nabla_{(i_1} K_{i_2 \ldots i_{m+1})} = 0. \tag{4.1.1}$$

If the function

$$T = \tfrac{1}{2} G^{ij} p_i p_j,$$

where G is a contravariant metric tensor, represents Hamiltonian (kinetic energy) of a free particle in the space (Q, g) and (x, p) is a canonical parametrization of the phase space $M = T^*Q$, then for arbitrary contravariant Killing tensor $K \in T^{(0,m)}(Q)$

$$K^{i_1 \ldots i_m} = G^{i_1 j_1} \ldots G^{i_m j_m} K_{j_1 \ldots j_m},$$

the function

$$F(x, p) = K^{i_1 \ldots i_m} p_{i_1} \ldots p_{i_m}$$

is a constant of motion: $\{F, T\}_\pi = 0$ [97, 239].

To see that important relation, first let us notice that the Schouten-Nijenhuis bracket, introduced in Sect. 2.3 for multi-vectors, can be defined for symmetric contravariant tensor fields in a completely analogical way. It is sufficient to replace the wedge-product by a respective symmetric product. Let $S^k(Q)$ be the space of contravariant symmetric tensor fields of order k on Q. Let (x^i) be a local coordinate system on Q, $K \in S^k(Q)$ and $R \in S^r(Q)$, then their Schouten-Nijenhuis bracket $[K, R]_S$ takes the form

$$\boxed{[K, R]_S^{l_1 \ldots l_{k+r-1}} = \frac{1}{k!r!} \left[k K^{n(l_1 \ldots} \partial_n R^{l_k \ldots l_{k+r-1})} - r R^{n(l_1 \ldots} \partial_n K^{l_r \ldots l_{k+r-1})} \right]} \tag{4.1.2}$$

where $(\ldots)$ is the symmetrization operator over the indices (compare with (2.3.13)). If we now define the following functions

$$F_K = \frac{1}{k!} K^{i_1 \ldots i_k} p_{i_1} \ldots p_{i_k}, \quad F_R = \frac{1}{r!} R^{i_1 \ldots i_r} p_{i_1} \ldots p_{i_r}$$

on T^*Q, where p_i are momenta conjugate to position x^i then, by a simple inspection, we find that

$$\boxed{\{F_K, F_R\} = -[K, R]_S^{i_1 \ldots i_{k+r-1}} p_{i_1} \ldots p_{i_{k+r-1}},} \tag{4.1.3}$$

where $\{.,.\}$ is a canonical Poisson bracket (3.2.19) and hence

$$\{F_K, F_R\} = 0 \Longleftrightarrow [K, R]_S = 0.$$

The formula (4.1.2) has the remarkable property that the right-hand side is invariant under the substitution $\partial_i \to \nabla_i$ where ∇ denotes the covariant derivative with respect to any torsionless linear connection. It is in fact this property that the right-hand side of this formula indeed gives the components of a rank $k + r - 1$ tensor field. Thus for each choice of a symmetric linear connection, one can write down the above formula in an invariant way without reference to any coordinate system [211]. So, lowering indices in covariant form of (4.1.2), substituting $R = G$ and using the fact that $\nabla_i g_{jk} = 0$, we immediately get that $[K, G] = 0$ if and only if the condition (4.1.1) is fulfilled, i.e. when K is a Killing tensor of metric g.

Example 4.1 Consider the Hénon-Heiles system from the Example 3.6. For Liouville integrability two constants of motion are desired

$$H_1 = T_1 + V_1 = \tfrac{1}{2} p_x^2 + \tfrac{1}{2} p_y^2 + \tfrac{1}{2} x y^2 + x^3,$$

$$H_2 = T_2 + V_2 = \tfrac{1}{2} y p_x p_y - \tfrac{1}{2} x p_y^2 + \frac{1}{4} x^2 y^2 + \tfrac{1}{16} y^4.$$

It is easy to check that they are in involution with respect to the canonical Poisson bi-vector. Obviously, geodesic Hamiltonians

$$T_1 = \tfrac{1}{2} p_x^2 + \tfrac{1}{2} p_y^2, \quad T_2 = \tfrac{1}{2} y p_x p_y - \tfrac{1}{2} x p_y^2$$

Poisson commute as well $\{T_1, T_2\} = 0$, so in Cartesian coordinates (x, y) both metric tensor (g_{ij}) and its inverse (G^{ij}) are represented by a 2×2 unite matrix, while both the second order covariant (K_{ij}) and the second order contravariant $(K^{ij}) = (K_{sr} G^{si} G^{rj})$ Killing tensors are represented by

$$\begin{pmatrix} 0 & \frac{1}{4} y \\ \frac{1}{4} y & -\frac{1}{2} x \end{pmatrix}.$$

Our next example illustrates how rare such systems are, i.e. what is the ratio between chaotic and integrable nonlinear Hamiltonian systems.

Example 4.2 Consider the two parameter family of Hénon-Heiles systems, generated by

$$H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + axy^2 - \tfrac{1}{3}bx^3.$$

The simplicity of its potential together with the richness of dynamical behavior for different values of a and b turned it into a standard testing ground for many methods of studying two-dimensional Hamiltonian systems. Usually the system is nonintegrable. There are only three integrable cases, where there exists the second global constant of motion [42]

$$\begin{aligned} b = -6a: &\quad F = \tfrac{1}{2}yp_xp_y - \tfrac{1}{2}xp_y^2 + \tfrac{1}{2}ax^2y^2 + \tfrac{1}{8}ay^4, \\ b = -a: &\quad F = p_xp_y + ax^2y + \tfrac{1}{3}ay^3, \\ b = -16a: &\quad F = 3p_y^4 + 12axy^2p_y^2 - 4ay^3p_xp_y - 4a^2x^2y^2 - \tfrac{2}{3}a^2y^6. \end{aligned}$$

The first case with $a = \frac{1}{2}$ was considered in the previous example.

Consider a Liouville integrable systems on a $2n$-dimensional symplectic manifold (M, π, H), with n constants of motion $(F_1, \ldots, F_n)$ which Poisson commute. Let us fix $a \in \mathbb{R}^n$ and consider the map

$$f : M \longrightarrow \mathbb{R}^n, \quad f = (F_1, \ldots, F_n).$$

From the assumptions on dF_i it is a regular map, so $M_a = f^{-1}(a)$ is a smooth, n-dimensional submanifold of M for any $a \in \mathbb{R}^n$. The Arnold-Liouville theorem says that if M_a is compact and connected, then it is diffeomorphic to n-dimensional torus $\mathbf{T}^n$. Moreover, in the neighborhood of the torus, in M there exist particular coordinates $(\varphi^1, \ldots, \varphi^n, I_1, \ldots, I_n)$ $(0 \le \varphi^i < 2\pi)$, called *action-angle coordinates*, in which equations of motion are linear with respect to the evolution parameter t

$$(I_k)_t = 0, \quad (\varphi^k)_t = \frac{\partial H(I_1, \ldots, I_n)}{\partial I_k} = \omega_k = const.$$

The motion on the torus is quasi-periodic. For the periodicity, an additional condition is necessary

$$\sum_{i=1}^{n} m_i\omega_i = 0, \qquad m_i \in \mathbb{Z}.$$

Assume that the considered Liouville integrable system is given in a canonical coordinate chart (x, p) and that the system of equations $F_j(x, p) = a_j$ can be

solved almost globally (globally in particular) for the momenta p_i

$$p_i = p_i(x, a) \tag{4.1.4}$$

and the relations $F_i(x,\, p(x, a)) = a_i$ hold identically. Now, let us differentiate these identities with respect to x_j

$$\frac{\partial F_i}{\partial x^j} + \sum_{r=1}^{n} \frac{\partial F_i}{\partial p_r} \frac{\partial p_r}{\partial x^j} = 0,$$

then multiply the resulting equations by $\partial F_s / \partial p_j$ and sum over j

$$\sum_{j=1}^{n} \frac{\partial F_s}{\partial p_j} \frac{\partial F_i}{\partial x^j} + \sum_{r,j=1}^{n} \frac{\partial F_i}{\partial p_r} \frac{\partial F_s}{\partial p_j} \frac{\partial p_r}{\partial x^j} = 0,$$

swap the indices and subtract $(si) - (is)$ obtaining finally

$$\{F_i, F_s\} + \sum_{r,j=1}^{n} \left(\frac{\partial F_s}{\partial p_j} \frac{\partial F_i}{\partial p_r} \frac{\partial p_r}{\partial x^j} - \frac{\partial F_i}{\partial p_j} \frac{\partial F_s}{\partial p_r} \frac{\partial p_r}{\partial x^j} \right) = 0.$$

The first term vanishes by the assumption and rearranging the indices in the second term we get

$$\sum_{r,j=1}^{n} \frac{\partial F_s}{\partial p_j} \frac{\partial F_i}{\partial p_r} \left(\frac{\partial p_r}{\partial x^j} - \frac{\partial p_j}{\partial x^r} \right) = 0.$$

From the invertibility of matrices $\partial F_s / \partial p_j$ we finally get

$$\frac{\partial p_r}{\partial x^j} - \frac{\partial p_j}{\partial x^r} = 0. \tag{4.1.5}$$

This condition implies, due to the Stokes theorem, that

$$\oint p_j \, dx^j = 0$$

for any closed contractible curve on the torus $\mathbf{T}^n$. There are n closed curves Γ_k which cannot be contracted down to a point, i.e. the corresponding integrals do not vanish and in consequence we can define the action coordinates

$$I_k := \frac{1}{2\pi} \oint_{\Gamma_k} p_j \, dx^j, \tag{4.1.6}$$

where the closed curve Γ_k is the k-th basic cycle of the torus $\mathbf{T}^n$

$$\Gamma_k = \{(\phi_1, \ldots, \phi_n) \in \mathbf{T}^n; 0 \le \phi_k < 2\pi,\ \phi_i = \text{const.}\ i \ne k\},$$

and ϕ are some coordinates on $\mathbf{T}^n$. The actions I_k are also first integrals as the r.h.s. of (4.1.6) due to (4.1.4) only depend on $a_i = F_i$ which are first integrals as well.

Now we can construct the angle coordinates φ_i canonically conjugated to the actions using a generating function (see Sect. 4.1.4)

$$W(x, I) = \int_{x_0}^{x} p_j dx^j$$

where x_0 is a fixed point on the torus. This definition does not depend on a path joining x_0 and x as a consequence of (4.1.5) and Stoke's theorem. Choosing different x_0 just adds a constant to W thus leaving the angles

$$\varphi^k := \frac{\partial W}{\partial I_k}$$

invariant. The angles are periodic coordinates with a period 2π. Indeed if we add a k-th cycle C_k to the path C between x_0 and x then

$$W(x, I) = \int_{C \cup C_k} p_j dx^j = \int_C p_j dx^j + \int_{C_k} p_j dx^j = W(x, I) + 2\pi I_k$$

so

$$\varphi^k = \frac{\partial W}{\partial I_k} = \varphi^k + 2\pi.$$

The transformations are canonical and invertible from construction

$$x = x(\varphi, I),\ p = p(\varphi, I) \quad \text{and} \quad \varphi = \varphi(x, p),\ I = I(x, p).$$

The dynamics is given by

$$(I_k)_t = \{I_k, H\} = 0 \Longrightarrow H = H(I_1, \ldots, I_n)$$

$$(\varphi^k)_t = \{\varphi^k, H\} = \frac{\partial H(I_1, \ldots, I_n)}{\partial I_k} = \omega_k = const$$

and thus we have reconstructed the equations (4.1.1).

So, "in principle" the system is integrable in quadratures. In practice, we can do it explicitly only in particular cases as the explicit construction of generating function (4.1.1) is far from being trivial. For example, we can do it in the case when

a Hamiltonian H separates into a sum of n one-dimensional Hamiltonians h_i

$$H(x, p) = \sum_{i=1}^{n} h_i(x^i, p_i).$$

In order to integrate by quadratures the arbitrary Liouville integrable system, we need an extra property, i.e the existence of the so called separation coordinates. Then, such Liouville integrable systems are called *separable systems*. The theory of separable systems is developed in the next chapter.

Example 4.3 Let us consider n-dimensional harmonic oscillator described by the following n Newton equations of motion in Cartesian coordinates

$$m x^i_{tt} + \beta_i x^i = 0, \quad i = 1, \ldots, n,$$

where m is a particle mass and $\beta_i \in \mathbb{R}_+$ are elastic constants, with well known solutions

$$x^i(t) = A_i \sin(\omega_i t + \varphi_i^0), \quad \omega_i = \sqrt{\frac{\beta_i}{m}},$$

parametrized by $2n$ constants $(A_1, \ldots, A_n, \varphi_0^1, \ldots, \varphi_0^n)$, i.e. amplitudes A_i and phases φ_0^i. The considered system is equivalent to Liouville integrable systems generated by n functions in involution

$$h_i = \frac{1}{2}\left(\frac{1}{m} p_i^2 + \beta_i (x^i)^2\right), \quad i = 1, \ldots, n,$$

with the Hamiltonian function

$$H(x, p) = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} \frac{1}{2}\left(\frac{1}{m} p_i^2 + \beta_i (x^i)^2\right).$$

Equations of motion are in the form

$$(x^i)_t = \{x^i, H\} = \frac{\partial H}{\partial p_i} = \frac{1}{m} p_i,$$

$$(p_i)_t = \{p_i, H\} = -\frac{\partial H}{\partial x^i} = -\beta_i x^i, \qquad i = 1, \ldots, n.$$

and split into n two-dimensional problems, each from $\mathbb{R}^2$. Let us consider the k-th problem. Different choices of the energy $h_k = a_k$ give a foliation of $\mathbb{R}^2$ by ellipses l_{a_k}

$$\frac{1}{2}\left(\frac{1}{m} p_k^2 + \beta_k (x^k)^2\right) = a_k.$$

For a fixed value of a_k we can take $\Gamma_k = l_{a_k}$. Then

$$I_k = \frac{1}{2\pi}\oint_{\Gamma_k} p_k\, dx^k = \frac{1}{2\pi}\iint_{S_k} dx^k dp_k = \frac{S_k}{2\pi} = \sqrt{\frac{m}{\beta_k}}a_k = \frac{a_k}{\omega_k}$$

where we used the Stoke's theorem. S_k is the area of ellipse l_{a_k} and there is no summation over k under the integral. The Hamiltonian expressed in new variables is

$$H = \sum_{i=1}^{n} h_i = \sum_{i=1}^{n} \omega_i I_i$$

and

$$(\varphi^i)_t = \frac{\partial H}{\partial I_i} = \omega_i, \quad (I_i)_t = -\frac{\partial H}{\partial \varphi^i} = 0$$

$$\Downarrow$$

$$\varphi^i(t) = \omega_i t + \varphi_0^i, \quad I_i = \frac{a_i}{\omega_i} = const.$$

To complete the picture we need to relate (x, p) and (φ, I) coordinates. We already know that

$$I_k = \frac{1}{2}\left(\frac{p_k^2}{m\omega_k} + m\omega_k (x^k)^2\right).$$

Thus the generating function is (with a chosen sign)

$$W(x^k, I_k) = \int p_k dx^k = \int \sqrt{2mI_k\omega_k - m^2\omega_k^2 (x^k)^2}\, dx^k$$

and

$$\varphi^k = \frac{\partial W}{\partial I_k} = \int \frac{m\omega_k}{\sqrt{2mI_k\omega_k - m^2\omega_k^2 (x^k)^2}} dx^k = \arcsin\left(\sqrt{\frac{m\omega_k}{2I_k}}x^k\right) - \varphi_0^k.$$

This gives

$$x^k = \sqrt{\frac{2I_k}{m\omega_k}}\sin(\varphi^k + \varphi_0^k)$$

and finally we recover the solution

$$x^i(t) = A_i \sin(\omega_i t + \varphi_0^i) = \frac{1}{\omega_i}\sqrt{\frac{2a_i}{m}} \sin(\omega_i t + \varphi_0^i),$$

$$p_i(t) = m\frac{dx^i(t)}{dt} = m\omega_i A_i \cos(\omega_i t + \varphi_0^i) = \sqrt{2ma_i} \cos(\omega_i t + \varphi_0^i),$$

with a new parametrization $(a_1, \ldots, a_n, \varphi_0^1, \ldots, \varphi_0^n)$, i.e. fixed values of constants of motion $F_i \equiv \omega_i I_i = a_i = const$ and phases φ_0^i. The respective action-angle coordinates (φ, I) are related to canonical coordinates (x, p) in the following way

$$x^i = \sqrt{\frac{2I_i}{\omega_i}} \sin\varphi^i, \quad p_i = \sqrt{2I_i\omega_i} \sin\varphi^i, \quad i = 1, \ldots, n.$$

The n-dimensional submanifold, such that $F_i = a_i = const.$ for $i = 1, \ldots, n$, is the n-dimensional tori with the coordinate system $(\varphi^1, \ldots, \varphi^n)$. Fixing a point on the tori $(\varphi_0^1, \ldots, \varphi_0^n)$ we chose a particular trajectory. Trajectories are closed when the condition (4.1.1) is fulfilled.

4.1.2 Superintegrability

In the previous subsection we introduced the notion of Liouville integrability with n constants of motion in involution on the $2n$-dimensional phase space. So, the natural question appears whether there are systems with a bigger number of global, single-valued integrals of motion. The answer is positive. A real-valued function H_1 on a $2n$-dimensional manifold (phase space) $M = T^*Q$ is called a *classical superintegrable* Hamiltonian if it belongs to a set of n Poisson-commuting functions $H_1, \ldots, H_n$ (constants of motion, so that $\{H_i, H_j\} = 0$ for all $i, j = 1, \ldots, n$) and for which there exist $0 < k < n$ additional functions $H_{n+1}, \ldots, H_{n+k}$, globally defined on M, that Poisson-commute with the Hamiltonian H_1 and such that all the functions $H_1, \ldots, H_{n+k}$ constitute a functionally independent set of functions. Obviously, in general, $\{H_i, H_j\} \neq 0$ for $1 < i \leq n$ and $j > n$. In particular, when $k = n - 1$, such systems are called *maximally superintegrable*.

The reader can meet in literature the weaker notion of maximal superintegrability which requires that there exists $2n - 1$ functions, one of which (the Hamiltonian) commutes with all of the others. In this sense, maximal superintegrability coincides with a particular case of the so called non-commutative integrability [52, 120, 201], which requires $2n - r$ integrals, r of which commute with all of the integrals. Such a class of systems is beyond the scope of the book.

In our further considerations we will limit ourselves to the so called polynomially superintegrable systems, with functionally independent and globally defined con-

stants of motion that are polynomial functions of the momenta. Let us analyze such a case on examples familiar to everybody of two dimensional harmonic oscillator and Kepler problem, respectively.

Example 4.4 The harmonic oscillator in $\mathbb{R}^2$, with equal component frequencies, can be described as the Hamiltonian system in the four dimensional phase space with Cartesian coordinates (x^1, x^2) and conjugate momenta (p_1, p_2), where

$$H_1 = \tfrac{1}{2}(p_1^2 + p_2^2) + \tfrac{1}{2}\left(\omega^2 (x^1)^2 + \omega^2 (x^2)^2\right).$$

We show how superintegrability alone determines the orbits. Here $n = 2$ and $2n - 1 = 3$ so three constants of the motion are required for superintegrability. Consider the following functions

$$H_2 = x^1 p_2 - x^2 p_1, \quad H_3 = \tfrac{1}{2}(-p_1^2 + p_2^2) + \tfrac{1}{2}\left(-\omega^2 (x^1)^2 + \omega^2 (x^2)^2\right), \quad H_4 = \omega^2 x^1 x^2 + p_1 p_2$$

which are in involution with Hamiltonian with respect to the canonical Poisson bracket: $\{H_1, H_i\} = 0$, $i = 2, 3, 4$, where H_2 represents the angular momentum (angular momentum is conserved in any Hamiltonian system with potential that depends only on the radial distance) and H_3 and H_4 represent extra quadratic in momenta constants of motion. The remaining nonzero Poisson brackets are

$$\{H_2, H_3\} = -2H_4, \quad \{H_2, H_4\} = 2H_3, \quad \{H_3, H_4\} = -2\omega^2 H_2.$$

We have found 4 constants of the motion and only 3 can be functionally independent. Indeed, one finds that

$$H_1^2 - H_3^2 = H_4^2 + \omega^2 H_2^2.$$

Let us fix the value of total energy $H_1 \equiv E_1$, $H_3 \equiv E_3 < E_1$ and choose $H_4 \equiv 0$. From the last choice we get $\omega^4 (x^1)^2 (x^2)^2 = p_1^2 p_2^2$ while from the first two choices we get $p_1^2 = a_1^2 - \omega^2 (x^1)^2$, $p_2^2 = a_2^2 - \omega^2 (x^2)^2$, where $a_1^2 = E_1 - E_3$, $a_2^2 = E_1 + E_3$. Eliminating p coordinates, we get the elliptical orbit in (x^1, x^2)-plane

$$\frac{\omega^2}{a_1^2}(x^1)^2 + \frac{\omega^2}{a_2^2}(x^2)^2 = 1.$$

Example 4.5 The Kepler problem is a specific case of the two body problem for which one of the bodies is stationary relative to the other and the bodies interact according to an inverse square law. The motion of two isolated bodies satisfies this condition to good approximation if one is significantly more massive than the other. Since all orbits lie in a plane (like in the previous example) we can write the Hamiltonian system again in the four dimensional phase space with Cartesian

coordinates (x_1, x_2) and conjugate momenta (p_1, p_2)

$$H_1 = \tfrac{1}{2}(p_1^2 + p_2^2) - \frac{a}{\sqrt{(x^1)^2 + (x^2)^2}}, \quad a > 0.$$

We show how superintegrability alone implies Kepler's laws and determines the orbits. Here $n = 2$ and $2n - 1 = 3$ so again three constants of the motion are required for superintegrability. Consider the following functions

$$H_2 = x^1 p_2 - x^2 p_1, \quad H_3 = H_2 p_2 - \frac{a x^1}{\sqrt{(x^1)^2 + (x^2)^2}}, \quad H_4 = -H_2 p_1 - \frac{a x^2}{\sqrt{(x^1)^2 + (x^2)^2}}$$

which are in involution with the Hamiltonian with respect to the canonical Poisson bracket: $\{H_1, H_i\} = 0$, $i = 2, 3, 4$, where H_2 represents the angular momentum (as the Hamiltonian again depends only on the radial distance) and H_3 and H_4 represent two components of the so called Laplace-Runge-Lenz vector $\mathbf{e} = (H_3, H_4)$ in (x^1, x^2)-plane for the Kepler problem. For an elliptical orbit or a hyperbolic trajectory the Laplace-Runge-Lenz vector is directed along the axis formed by the origin $(x^1, x^2) = (0, 0)$ and the perihelion (point of closest approach) of the trajectory to the origin. The perihelion is time-invariant in the Kepler problem, so the direction in which $\mathbf{e}$ points must be a constant of the motion. The remaining nonzero Poisson brackets are

$$\{H_2, H_3\} = -H_4, \quad \{H_2, H_4\} = H_3, \quad \{H_3, H_4\} = 2H_1 H_2.$$

We have found 4 constants of the motion and again only 3 can be functionally independent. Indeed, the length squared of the Laplace vector is determined by the energy and angular momentum

$$H_3^2 + H_4^2 = 2H_1 H_2 + a^2.$$

Let us fix the value of of total energy $H_1 \equiv E$, the angular momentum $H_2 \equiv L$ and choose $H_4 \equiv 0$. For such a choice the Laplace-Runge-Lenz vector is pointed in the direction of the positive x_1-axis and $E_3^2 = 2EL^2 + a^2$. Then, we have

$$p_1 = -\frac{a x^2}{L\sqrt{(x^1)^2 + (x^2)^2}}, \quad p_2 = \frac{E_3}{L} + \frac{a x^1}{L\sqrt{(x^1)^2 + (x^2)^2}}.$$

The expression for H_2 allows us to write

$$L = \frac{E_3 x^1}{L} + \frac{a(x^1)^2}{L\sqrt{(x^1)^2 + (x^2)^2}} + \frac{a(x^2)^2}{L\sqrt{(x^1)^2 + (x^2)^2}} = \frac{E_3 x^1}{L} + \frac{a\sqrt{(x^1)^2 + (x^2)^2}}{L},$$

which after rearrangement and squaring becomes

$$\left(1-\frac{E_3^2}{a^2}\right)(x^1)^2+\frac{2LE_3}{a^2}x^1+(x^2)^2=\frac{L^4}{a^2}. \tag{4.1.7}$$

These are conic sections of the plane (x^1, x^2), so our trajectories are ellipses, parabolas, and hyperbolas, depending on the constants of the motion E, L.

The first Kepler's law says that planetary orbits are planar ellipses with the sun positioned at a focus. Indeed we found that only closed trajectories, or orbits, are elliptical.

The second Kepler's law says that a planetary orbit sweeps out equal areas in equal time, so it is a statement of conservation of the angular momentum. Indeed, let us introduce polar coordinates such that $x^1 = r\cos\varphi, x^2 = r\sin\varphi$ then, along the trajectory,

$$H_2 = L = x^1(t)p_2(t) - x^2(t)p_1(t) = x^1\frac{dx^2}{dt} - x^2\frac{dx^1}{dt} = r^2\frac{d\varphi}{dt}.$$

The area traced out from time 0 to time t is

$$A(t) = \tfrac{1}{2}\int_{\varphi(0)}^{\varphi(t)} r^2 d\varphi.$$

Differentiating with respect to time

$$\frac{d}{dt}A(t) = \tfrac{1}{2}r^2\frac{d\varphi}{dt} = \tfrac{1}{2}L,$$

so the rate is constant.

The third Kepler's law says that the square of the period of an orbit is proportional to the cube of the length of the semi-major axis of the ellipse. Again, it is only valid for closed trajectories: ellipses. We may write the period T of such an orbit in terms of the constants of the motion. Explicit evaluation for $\varphi(0) = 0, \varphi(T) = 2\pi$, yields $A(T) = \frac{1}{2}LT$ as the area of the ellipse. Kepler's third law follows from the equations (4.1.7) and the simple calculus expression for the area of an ellipse.

The reader can find other examples of superintegrable systems with two degrees of freedom and various methods of their construction for instance in [79, 163, 185, 196, 249], while superintegrable systems with three and more degrees of freedom can be found in [8, 9, 64–66, 110, 134, 164, 166, 168] and in literature quoted there.

Suppose that we have an integrable system, i.e. n functionally independent Hamiltonians on a $2n$-dimensional phase space M that pairwise commute: $\{H_i, H_j\} = 0$ for all $i, j = 1, \ldots, n$. If there exists an additional function P commuting to a constant with one of the Hamiltonians, say with H_1 (so that

$\{H_1, P\} = c$) and if the $(n-1)$ functions

$$H_{n+i} = \{H_{i+1}, P\}, \ i = 1, \ldots, n-1$$

together with all H_i are functionally independent, then the system becomes maximally superintegrable since then by the Jacobi identity

$$\{H_{n+i}, H_1\} = -\{\{P, H_1\}, H_{i+1}\} - \{\{H_1, H_{i+1}\}, P\} = 0, \quad i = 1, \ldots, n-1.$$

If moreover the first n integrals of motion H_i are quadratic in momenta and if P is linear in momenta, like in our two examples, then the resulting $(n-1)$ extra integrals of motion H_{n+i} are also quadratic in momenta.

Suppose that $(x, p) = (x^1, \ldots, x^n, p_1 \ldots, p_n)$ are Darboux (canonical) coordinates on a $2n$-dimensional phase space $M = T^*Q$. Consider two functions on M:

$$H = \tfrac{1}{2} p_i A^{ij}(x) p_j + U(x) \text{ with } A = A^T \text{ and } P = y^i(x) p_i.$$

Then

$$\boxed{\{H, P\} = \tfrac{1}{2} p_i \left(L_Y A\right)^{ij} p_j + Y(U),} \tag{4.1.8}$$

where Y is the vector field on Q given by $Y = y^i(x)\frac{\partial}{\partial x^i}$ and where L_Y is the Lie derivative (on Q) along Y. Thus, one can say that H and P commute if the corresponding vector field Y is the Killing vector (2.6.17) for the metric defined by the $(2,0)$-tensor A (i.e. if $L_Y A = 0$) and if moreover $Y(U) = 0$. Thus, the simplest way of the construction of additional constants of motion is via an appropriate Killing vector. We will come back to such a construction of superintegrable systems later on. More complex methods of construction of superintegrable systems are beyond the scope of that book and we send the reader to the literature mentioned above.

4.1.3 Stäckel Transform

Stäckel transform is a functional transform that transforms a given Liouville integrable system into a new Liouville integrable system on the same Poisson manifold. Actuall, this is essentially a transformation that sends an n-tuple of functions in involution on a $2n$-dimensional symplectic manifold into another n-tuple of functions on the same manifold, and these n new functions are again in involution. It also generates a corresponding reciprocal transform between solutions of these Stäckel-related systems, confined to proper submanifolds of the phase space. The construction in the restricted form (called coupling constant methamorphosis) was

first described in [58, 150] and then generalized to the final form in [39, 231]. Applied to a Stäckel separable system, this transformation yields a new Stäckel separable system, which explains its name. We will develop the theory of Stäckel transform of separable systems in the next chapter.

Consider a Poisson manifold (M, π) and suppose we have r functions (Hamiltonians) $h_i : M \to \mathbb{R}$ on M, each depending on $k \leq r$ parameters $\alpha_1, \ldots, \alpha_k$ so that

$$h_i = h_i(\xi, \alpha_1, \ldots, \alpha_k), \quad i = 1, \ldots, r, \tag{4.1.9}$$

where $\xi \in M$. From r functions in (4.1.9) let us now choose k functions h_{s_i}, $i = 1, \ldots, k$, where $S = \{s_1, \ldots, s_k\} \subset \{1, \ldots, r\}$. Assume that

$$\det\left(\partial h_{s_i}/\partial \alpha_j\right) \neq 0 \tag{4.1.10}$$

and that the system of equations

$$h_{s_i}(\xi, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \quad i = 1, \ldots, k, \tag{4.1.11}$$

where $\tilde{\alpha}_i$ is another set of k free parameters, or values of Hamiltonians h_{s_i}, can be solved almost globally (globally in particular) for the parameters α_i yielding

$$\boxed{\alpha_i = \tilde{h}_{s_i}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \quad i = 1, \ldots, k.} \tag{4.1.12}$$

The right hand sides of these solutions define k new functions $\tilde{h}_{s_i}$ on M, each depending on k parameters $\tilde{\alpha}_i$. Finally, let us define $(r - k)$ functions $\tilde{h}_i$ with $i = 1, \ldots, r$ and such that $i \notin \{s_1, \ldots, s_k\}$ by, substituting $\tilde{h}_{s_i}$ instead of α_i in h_i for $i \notin \{s_1, \ldots, s_k\}$

$$\boxed{\tilde{h}_i = h_i|_{\alpha_1 \to \tilde{h}_{s_1}, \ldots, \alpha_k \to \tilde{h}_{s_k}}, \quad i = 1, \ldots, r, \ \ i \notin \{s_1, \ldots, s_k\}.} \tag{4.1.13}$$

The functions $\tilde{h}_i = \tilde{h}_i(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$, $i = 1, \ldots, r$, defined through (4.1.12) and (4.1.13) are called the generalized Stäckel transform of the functions (4.1.9) with respect to the indices $\{s_1, \ldots, s_k\}$ (or with respect to the functions $h_{s_1}, \ldots h_{s_k}$). Note that unless we extend the manifold M this operation cannot be obtained by any change of coordinates. It is also easy to see that if we perform again the Stäckel transform on the functions $\tilde{h}_i$ with respect to $\tilde{h}_{s_i}$ we will receive back the functions h_i in (4.1.9). Note also that neither k nor r are related to the dimension of the manifold M.

Example 4.6 Here we consider the simplest situation when $k = r = 1$. Let us take, after [150], the Fokas-Lagerström potential on the four-dimensional phase space M

with coordinates (x, y, p_x, p_y) :

$$h = \tfrac{1}{2}(p_x^2 + p_y^2) - \tfrac{2}{3}\alpha(xy)^{-2/3}.$$

Solving the equation $h = \tilde{\alpha}$ with respect to the only parameter α one obtains

$$\alpha = \tfrac{3}{4}(xy)^{2/3}(p_x^2 + p_y^2) - \tfrac{3}{2}\tilde{\alpha}(xy)^{2/3} \equiv \tilde{h}$$

which can be shown to be equivalent to the axially symmetric potential [150].

The importance of Stäckel transform relies on two properties that make it so useful for the study of integrable systems, i.e. it preserves functional independence and involutivity with respect to the Poisson tensor π.

In the special but nonetheless important case when functions (4.1.9) depend linearly on parameters α_i it is possible to write down the Stäckel transform in an explicit form. Suppose therefore that the functions in (4.1.9) are of the form

$$h_i = H_i + \sum_{j=1}^{k} \alpha_j H_i^{(j)}, \quad i = 1, \ldots, r. \tag{4.1.14}$$

The equations (4.1.11) defining the first part of the Stäckel transform take then the form of a system of k linear equations in unknowns $\alpha_1, \ldots, \alpha_k$

$$H_{s_i} + \sum_{j=1}^{k} \alpha_j H_{s_i}^{(j)} = \tilde{\alpha}_i, \quad i = 1, \ldots, k,$$

with the determinant solution for $\alpha_i = \tilde{h}_{s_i}$ of the form:

$$\tilde{h}_{s_i} = \det W_i / \det W = \tilde{H}_{s_i} + \sum_{j=1}^{k} \tilde{\alpha}_j \tilde{H}_{s_i}^{(j)}, \tag{4.1.15}$$

where

$$W = \begin{pmatrix} H_{s_1}^{(1)} & \cdots & H_{s_1}^{(k)} \\ \vdots & \ddots & \vdots \\ H_{s_k}^{(1)} & \cdots & H_{s_k}^{(k)} \end{pmatrix}$$

is the $k \times k$ matrix $\left(\partial h_{s_i}/\partial \alpha_j\right)$ given in (4.1.10) (so that $\det W \neq 0$) and where W_i are obtained from W by replacing $H_{s_j}^{(i)}$ in the i-th column by $\tilde{\alpha}_j - H_{s_j}$ for all $j = 1, \ldots, k$. The second part of the transformation, i.e. formulas (4.1.13), is as

follows

$$\tilde{h}_i = H_i + \sum_{j=1}^{k} \tilde{h}_{s_j} H_i^{(j)} = \tilde{H}_i + \sum_{j=1}^{k} \tilde{\alpha}_j \tilde{H}_i^{(j)}, \quad i = 1, \ldots, r, \quad i \notin \{s_1, \ldots, s_k\}$$

where $\tilde{h}_{s_i}$ are given by (4.1.15).

For the Stäckel transform in this case, after setting all the α and $\tilde{\alpha}$ equal to zero we obtain the following formulas relating Hamiltonians H and $\tilde{H}$

$$\begin{aligned} &H_{s_i} + \tilde{H}_{s_1} H_{s_i}^{(1)} + \ldots + \tilde{H}_{s_k} H_{s_i}^{(k)} = 0, \qquad s_i \in S, \\ &H_j + \tilde{H}_{s_1} H_j^{(1)} + \ldots + \tilde{H}_{s_k} H_j^{(k)} = \tilde{H}_j, \qquad j \notin S. \end{aligned} \tag{4.1.16}$$

Relations (4.1.16) can be written in a matrix form

$$H = A\tilde{H} \tag{4.1.17}$$

where $H = (H_1, \ldots, H_r)^T$ and $\tilde{H} = (\tilde{H}_1, \ldots, \tilde{H}_r)^T$ and where the $r \times r$ matrix A is given by

$$A_{ij} = \delta_{ij} \text{ for } j \notin \{s_1, \ldots, s_k\}, \; A_{is_j} = -\frac{\partial h_i}{\partial \alpha_j} = -H_i^{(j)} \text{ for } j = 1, \ldots, k \tag{4.1.18}$$

Notice that relation (4.1.17) is valid on the whole M.

Let us now discuss the Stäckel transform between Hamiltonians and the corresponding reciprocal transform between respective solutions of two Liouville integrable systems. Suppose that $\dim M = 2n$ and that we have exactly $r = n$ functionally independent Hamiltonians

$$h_i = h_i(\xi, \alpha_1, \ldots, \alpha_k), \quad i = 1, \ldots, n$$

that depend on $k \leq n$ parameters α_i and that are for all values of α_i in involution with respect to a nondegenerate Poisson bi-vector π: $\{h_i, h_j\}_\pi = 0$ for all i, j. These functions yield n Hamiltonian systems on M:

$$\frac{d\xi}{dt_i} = \pi dh_i \equiv X_i, \quad i = 1, \ldots, n \tag{4.1.19}$$

so that X_i are n commuting Hamiltonian vector fields on M. Consider now a new set of n Hamiltonians $\tilde{h}_i$ obtained from h_i by a Stäckel transform performed with respect to $h_{s_1}, \ldots, h_{s_k}$. These functions define a set of Hamiltonian flows on M, the vector fields of which are given by

$$\frac{d\xi}{d\tilde{t}_i} = \pi d\tilde{h}_i \equiv \tilde{X}_i, \quad i = 1, \ldots, n. \tag{4.1.20}$$

Let us now analyze the relation between the Hamiltonian systems (4.1.19) and (4.1.20). In order to study this relation it is important to realize that both systems (4.1.19) and (4.1.20) are multiparameter and the relation between them can thus only be found if one fixes the values of both all α_i and all $\tilde{\alpha}_i$ which means that the sought relation can only exists on the $(2n-k)$-dimensional submanifolds $M_{\alpha,\tilde{\alpha}}$ given by (4.1.11):

$$M_{\alpha,\tilde{\alpha}} = \left\{\xi \in M : h_{s_i}(\xi, \alpha_1, \ldots, \alpha_k) = \tilde{\alpha}_i, \;\; i = 1, \ldots k\right\} \tag{4.1.21}$$

Note that the surfaces $M_{\alpha,\tilde{\alpha}}$ depend on the simultaneous choice of $2k$ parameters α_i and $\tilde{\alpha}_i$ and that its codimension is k (so that $\dim M_{\alpha,\tilde{\alpha}} = 2n - k \geq n$). Note also that due to the equivalence between (4.1.11) and (4.1.12) the surfaces $M_{\alpha,\tilde{\alpha}}$ can equivalently be defined through

$$M_{\alpha,\tilde{\alpha}} = \left\{\xi \in M : \tilde{h}_{s_i}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) = \alpha_i, \;\; i = 1, \ldots k\right\} \tag{4.1.22}$$

Observation 5 *Through each point ξ in M there passes infinitely many submanifolds $M_{\alpha,\tilde{\alpha}}$. If we fix the values of all the parameters α_i we can for any ξ always find some values of the parameters $\tilde{\alpha}_i$ so that $\xi \in M_{\alpha,\tilde{\alpha}}$; and vice versa, if we fix $\tilde{\alpha}_i$, for any given ξ we can find α_i so that $\xi \in M_{\alpha,\tilde{\alpha}}$.*

As it follows from (4.1.11) and (4.1.12) the following identity is valid on M and for all values of parameters $\tilde{\alpha}_i$:

$$h_{s_i}(\xi, \tilde{h}_{s_1}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \ldots, \tilde{h}_{s_k}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)) \equiv \tilde{\alpha}_i, \;\; i = 1, \ldots, k. \tag{4.1.23}$$

Moreover, the second part of the transformation, given by (4.1.13), can be written as the following identity on M, valid again for all values of $\tilde{\alpha}_i$,

$$\tilde{h}_i(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \equiv h_i(\xi, \tilde{h}_{s_1}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \ldots, \tilde{h}_{s_k}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_S)), \tag{4.1.24}$$

where $i = 1, \ldots, n, \;\; i \notin \{s_1, \ldots, s_k\}$. Differentiating (4.1.23) with respect to ξ we get that on each $M_{\alpha,\tilde{\alpha}}$

$$\boxed{dh_{s_i} = -\sum_{j=1}^{k} \frac{\partial h_{s_i}}{\partial \alpha_j} d\tilde{h}_{s_j}, \;\; i = 1, \ldots, k} \tag{4.1.25}$$

while differentiation of (4.1.24) on $M_{\alpha,\tilde{\alpha}}$ gives

$$\boxed{dh_i = d\tilde{h}_i - \sum_{j=1}^{k} \frac{\partial h_i}{\partial \alpha_j} d\tilde{h}_{s_j}, \;\; i = 1, \ldots, n, \;\; i \notin \{s_1, \ldots, s_k\}\,.} \tag{4.1.26}$$

Obviously, in equations (4.1.25) and (4.1.26) we make the substitution (4.1.12). The transformation (4.1.25)–(4.1.26) on $M_{\alpha,\tilde{\alpha}}$ can be written in a matrix form as

$$dh = Ad\tilde{h} \tag{4.1.27}$$

where $dh = (dh_1, \ldots, dh_n)^T$, $d\tilde{h} = (d\tilde{h}_1, \ldots, d\tilde{h}_n)^T$ and where the $n \times n$ matrix A is given by

$$A_{ij} = \delta_{ij} \text{ for } j \notin \{s_1, \ldots, s_k\}, A_{is_j} = -\frac{\partial h_i}{\partial \alpha_j} \text{ for } j = 1, \ldots, k \tag{4.1.28}$$

(compare with (4.1.18).

From the structure of the matrix A it follows that

$$\det A = \pm \det\left(\frac{\partial h_{s_i}}{\partial \alpha_j}\right)$$

so that $\det A \neq 0$ due to the assumption (4.1.10). In consequence, the relation (4.1.27) can be inverted yielding $d\tilde{h} = A^{-1}dh$. This leads to important two properties. Actually, if the functions h_i are functionally independent then so are $\tilde{h}_i$ and if the functions h_i are in involution with respect to the Poisson tensor π (for all values of α_i), then the functions $\tilde{h}_i$ are also in involution with respect to π for all values of $\tilde{\alpha}_i$.

For the first property, assume that h_i are functionally independent for all values of α_i. Consider the differentials $d\tilde{h}_i$ at a given point $\xi \in M$ and for some arbitrary values of $\tilde{\alpha}_i$. Due to Observation 5 one can always find values of α_i such that $\xi \in M_{\alpha,\tilde{\alpha}}$. By (4.1.27) and by the fact that $\det A \neq 0$ the differentials $d\tilde{h}_i$ linearly independent at ξ (since dh_i are) and since ξ ia arbitrary, $\tilde{h}_i$ are functionally independent on the whole M.

For the second property, assume $\{h_i, h_j\}_\pi = 0$ for all $i, j = 1, \ldots, n$ and for all values of α_i. Then, as in the proof of the first statement, at any $\xi \in M$ we can choose an appropriate $M_{\alpha,\tilde{\alpha}}$ so that (4.1.27) is valid and so

$$\begin{aligned}\left\{\tilde{h}_i, \tilde{h}_j\right\}_\pi &= \left\langle d\tilde{h}_i, \pi d\tilde{h}_j\right\rangle = \left\langle \sum_{l_1=1}^{n} \left(A^{-1}\right)_{il_1} dh_{l_1}, \pi \sum_{l_2=1}^{n} \left(A^{-1}\right)_{jl_2} dh_{l_2}\right\rangle \\ &= \sum_{l_1,l_2=1}^{n} \left(A^{-1}\right)_{il_1} \left(A^{-1}\right)_{jl_2} \left\langle dh_{l_1}, \pi dh_{l_2}\right\rangle \\ &= \sum_{l_1,l_2=1}^{n} \left(A^{-1}\right)_{il_1} \left(A^{-1}\right)_{jl_2} \left\{h_{l_1}, h_{l_2}\right\}_\pi = 0.\end{aligned}$$

Thus the system (4.1.20) is again Liouville integrable. It proves that Stäckel transform maps a Liouville integrable system into a Liouville integrable system.

From (4.1.25)–(4.1.26) it follows that vector fields $X_i = \pi dh_i$ and $\tilde{X}_i = \pi d\tilde{h}_i$ are on the appropriate $M_{\alpha,\tilde{\alpha}}$ related by the following transformation

$$\boxed{X_{s_i} = -\sum_{j=1}^{k} \frac{\partial h_{s_i}}{\partial \alpha_j} \tilde{X}_{s_j}, \quad i = 1, \ldots, k} \tag{4.1.29a}$$

$$\boxed{X_i = \tilde{X}_i - \sum_{j=1}^{k} \frac{\partial h_i}{\partial \alpha_j} \tilde{X}_{s_j}, \quad i = 1, \ldots, n, \quad i \notin \{s_1, \ldots, s_k\}.} \tag{4.1.29b}$$

This means that the Hamiltonian vector fields X_i and $\tilde{X}_i$ span the same n-dimensional distribution on each $M_{\alpha,\tilde{\alpha}}$ and also that the vector fields X_{s_i} and $\tilde{X}_{s_i}$ span the same k-dimensional sub-distribution of the above distribution. The transformation (4.1.29) can be written on $M_{\alpha,\tilde{\alpha}}$ in the matrix form

$$X = A\tilde{X}$$

where $X = (X_1, \ldots, X_n)^T$, $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)^T$, the $n \times n$ matrix A is given above and substitution (4.1.12) is performed.

All the vector fields X_i and $\tilde{X}_i$ are naturally tangent to the corresponding $M_{\alpha,\tilde{\alpha}}$ so that if $\xi_0 \in M_{\alpha,\tilde{\alpha}}$ then the multiparameter simultaneous solution

$$\xi = \xi(t_1, \ldots, t_n, \xi_0) \tag{4.1.30}$$

of all equations in (4.1.19) starting at ξ_0 for $t = 0$, will always remain in $M_{\alpha,\tilde{\alpha}}$ and the same is also true for multiparameter solutions of (4.1.20).

The relations (4.1.29) can be reformulated in the dual language, that of reciprocal multi-time transformations. The reciprocal transformation $\tilde{t}_i = \tilde{t}_i(t_1, \ldots, t_n, \xi)$, $i = 1, \ldots, n$ given on $M_{\alpha,\tilde{\alpha}}$ by

$$d\tilde{t} = A^T dt, \tag{4.1.31}$$

where $dt = (dt_1, \ldots, dt_n)^T$ and $d\tilde{t} = (d\tilde{t}_1, \ldots, d\tilde{t}_n)^T$, transforms the n-parameter solutions (4.1.30) of the system (4.1.19) to the n-parameter solutions $\tilde{\xi} = \tilde{\xi}(\tilde{t}_1, \ldots, \tilde{t}_n, \xi_0)$ of the system (4.1.20), with the same initial condition $\xi(0) = \xi_0$, in the sense that for any $\xi_0 \in M_{\alpha,\tilde{\alpha}}$ we have

$$\tilde{\xi}(\tilde{t}_1(t_1, \ldots, t_n, \xi_0), \ldots, \tilde{t}_n(t_1, \ldots, t_n, \xi_0), \xi_0) = \xi(t_1, \ldots, t_n, \xi_0)$$

for all values of t_i sufficiently close to zero.

The transformation (4.1.31) is well defined since the right hand side of (4.1.31) is an exact differential, as it follows from the above construction. Thus it is possible, at least locally, to integrate (4.1.31) and obtain an explicit transformation $\tilde{t}_i = \tilde{t}_i(t_1, \ldots, t_n, \xi)$ that takes multi-time solutions of all Hamiltonian systems (4.1.19) to multi-time solutions of all the systems in (4.1.20).

In a specific case which is important for further considerations, when $k = n$ and so that the Stäckel transform consists only of the first part (4.1.12), the matrix A simplifies to the form

$$A_{ij} = -\frac{\partial h_i}{\partial \alpha_j}, \ i, j = 1, \ldots, n \tag{4.1.32}$$

so that the formulas (4.1.29) simplify to the single formula

$$X_i = -\sum_{j=1}^{n} \frac{\partial h_i}{\partial \alpha_j} \tilde{X}_j, \quad i = 1, \ldots, n,$$

and (4.1.31) can be explicitly written as

$$d\tilde{t}_i = -\sum_{j=1}^{n} \frac{\partial h_j}{\partial \alpha_i} dt_j, \quad i = 1, \ldots, n. \tag{4.1.33}$$

In this case our manifolds $M_{\alpha,\tilde{\alpha}}$ become level surfaces for all the Hamiltonians $h_i(\xi, \alpha)$ and also level surfaces for all the Hamiltonians $\tilde{h}_i(\xi, \tilde{\alpha})$.

As a simple illustration of the above results, consider the Hamiltonian systems on a four-dimensional phase space $M = \mathbb{R}^4$ with the coordinates $\xi = (x, y, p_x, p_y)$ and canonical Poisson structure. For our first example let $k = 1$, $r = 2$, $s_1 = 2$, $\alpha_1 \equiv \alpha$ and $\tilde{\alpha}_1 \equiv \tilde{\alpha}$ with nonlinear α-dependence.

Example 4.7 Consider the Hamiltonian

$$h_1 = \tfrac{1}{2} p_x^2 + \tfrac{1}{2} p_y^2 + \frac{\alpha(x^2 - y^2)}{y} p_y - 2\alpha^2 x^2,$$

which is Liouville integrable because it Poisson commutes with

$$h_2 = \frac{x p_y - y p_x - 2\alpha x y}{p_y}.$$

Relation (4.1.11) for $h_2 = \tilde{\alpha}$ takes the form

$$\frac{x p_y - y p_x - 2\tilde{h}_2 x y}{y} = \tilde{\alpha},$$

whence

$$\tilde{h}_2 = \frac{xp_y - yp_x - \tilde{\alpha}p_y}{2xy},$$

and therefore by virtue of (4.1.13) we have

$$\tilde{h}_1 = \frac{x^2 + y^2 - 2\tilde{\alpha}x}{2xy} p_x p_y + \frac{\tilde{\alpha}(x^2 - \tilde{\alpha}x + y^2)}{2xy^2} p_y^2.$$

The relation $\{h_1, h_2\} = 0$ implies $\{\tilde{h}_1, \tilde{h}_2\} = 0$, so $\tilde{h}_1$ is a Liouville integrable just like h_1. The latter equality can be readily verified by straightforward computation. Interestingly enough, in this example the generalized Stäckel transform sends the Hamiltonian h_1 into a natural geodesic Hamiltonian $\tilde{h}_1$, but the metric associated with $\tilde{h}_1$ is not flat and, moreover, has nonconstant scalar curvature unlike the metric associated with h_1. Moreover, the reciprocal transformation (4.1.31), (4.1.28)

$$\begin{aligned} \tilde{t}_1 &= t_1 \\ d\tilde{t}_2 &= \left(\frac{y^2 - x^2}{y} p_y + 4\alpha x^2 \right) dt_1 + \frac{2xy}{p_y} dt_2 \\ &= \left(-2xp_x + \frac{(x^2 - 2\tilde{\alpha}x + y^2)p_y}{y} \right) dt_1 + \frac{2xy}{p_y} dt_2 \end{aligned}$$

takes the equations of motion for h_1 and h_2, with the respective evolution parameters t_1 and t_2, restricted onto the common level surface $M_{\alpha\tilde{\alpha}} = \{\xi \in \mathbb{R}^4 | h_2(\xi, \alpha) = \tilde{\alpha}\}$ into the equations of motion for $\tilde{h}_1$ and $\tilde{h}_2$, with the respective evolution parameters $\tilde{t}_1$ and $\tilde{t}_2$, restricted onto the same common level surface $M_{\alpha\tilde{\alpha}} = \{\xi \in \mathbb{R}^4 | \tilde{h}_2(\xi, \tilde{\alpha}) = \alpha\}$.

For the second example we set $k = r = 2$.

Example 4.8 Consider the extended Hénon–Heiles system with the Hamiltonian

$$h_1 = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + \alpha_1 \left(x^3 + \tfrac{1}{2}xy^2 \right) + \alpha_2 x,$$

which Poisson commutes with

$$h_2 = \tfrac{1}{2}y^2 p_x p_y - \tfrac{1}{2}xp_y^2 + \alpha_1 \left(\tfrac{1}{4}x^2y^2 + \tfrac{1}{16}x^4 \right) + \tfrac{1}{4}\alpha_2 y^2.$$

Let $s_1 = 1$, $s_2 = 2$, $k = r = 2$. Then (4.1.15) yield the following commuting deformation of h_1 and h_2:

$$\tilde{h}_1 = \tfrac{2}{xy^2}p_x^2 - \tfrac{8}{y^3}p_x p_y - \tfrac{2(y^2+4x^2)}{xy^4}p_y^2 + \tfrac{4}{xy^2}\tilde{\alpha}_1 - \tfrac{16}{y^4}\tilde{\alpha}_2,$$

$$\tilde{h}_2 = -\tfrac{y^2+4x^2}{2xy^2}p_x^2 - \tfrac{4(y^2+2x^2)}{y^3}p_x p_y + \tfrac{16x^4+12x^2y^2+y^4}{xy^4}p_x^2$$
$$-\tfrac{y^2+4x^2}{q_1 q_2^2}\tilde{\alpha}_1 + \tfrac{8(y^2+2x^2)}{y^4}\tilde{\alpha}_2.$$

Using (4.1.32), (4.1.33) and proceeding in analogy with the previous example we readily find that the reciprocal transformation takes the form

$$d\tilde{t}_1 = \left(x^3 + \tfrac{1}{2}xy^2\right)dt_1 + \left(\tfrac{1}{16}x^4 + \tfrac{1}{4}x^2y^2\right)dt_2, \quad d\tilde{t}_2 = x dt_1 + \tfrac{1}{4}y^2 dt_2.$$

But what about superintegrability? Adding to Hamiltonians (4.1.3) extra constants of motion $h_{n+i} = h_{n+i}(\xi, \alpha_1, \ldots, \alpha_k)$, $i = 1, \ldots, n-1$ that commute with h_1 and differentiating extra identities

$$\tilde{h}_{n+i}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k) \equiv h_{n+i}(\xi, \tilde{h}_{s_1}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k), \ldots, \tilde{h}_{s_k}(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)),$$

with respect to ξ we get

$$dh_{n+i} = d\tilde{h}_{n+i} - \sum_{j=1}^{k} \frac{\partial h_{n+i}}{\partial \alpha_j} d\tilde{h}_{s_j}, \quad i = 1, \ldots, n-1.$$

The transformation matrix A (4.1.27), now $(2n-1)\times(2n-1)$, is given by the same formula (4.1.28) and again on $M_{\alpha\tilde{\alpha}}$ we find

$$\{\tilde{h}_i, \tilde{h}_j\}_\pi = \sum_{l_1, l_2 = 1}^{2n-1} \left(A^{-1}\right)_{il_1} \left(A^{-1}\right)_{jl_2} \{h_{l_1}, h_{l_2}\}_\pi .$$

In order to get extra conditions $\{\tilde{h}_1, \tilde{h}_{n+j}\}_\pi = 0$ a strong restriction has to be imposed on the matrix A. Actually, the first row of A must be zero except the element A_{11}. It happens only in the case when $h_1 = h_1(\xi, \alpha_1)$ and $s_1 = 1$. Thus, if all Hamiltonians depend on the same number of parameters, the superintegrability is preserved when that number is equal one and the Stäckel transform is performed with respect to h_1 (see Sect. 4.4.4 for particular examples).

4.1.4 Canonical Transformations

As we presented in the previous chapter, Hamiltonian mechanics is formulated in a coordinate free way. Nevertheless, in order to perform particular calculations, we need a local coordinate system. On the other hand, it is well known that complexity of calculations strongly depends on chosen coordinates. It will become apparent in the next chapter where the separability theory is presented. Thus, the coordinate transformations are a standard element of the further developed theory.

An important class of coordinate transformations on a symplectic manifold is the class of *canonical transformations*, i.e. these which do not change the canonical form of implectic bi-vector (symplectic 2-form). A particular importance of such transformations will be demonstrated in the next chapter. Thus, on $2n$-dimensional phase space M consider some transformation of coordinates

$$(x,\, p) \longrightarrow (x',\, p')$$

such that if $(x,\, p)$ are Darboux coordinates, i.e.

$$\omega = dp_i \wedge dq^i, \quad \pi = \omega^{-1} = \partial_{q^i} \wedge \partial_{p_i},$$

then

$$\omega' = dp'_i \wedge dx'^i, \quad \pi' = \omega'^{-1} = \partial_{x'^i} \wedge \partial_{p'_i}$$

and hence $(x',\, p')$ are also Darboux coordinates. In order to construct such a transformation, consider any smooth function $F(x,\, p')$, called further the *generating function* of canonical transformation, that $\left|\frac{\partial^2 F}{\partial x \partial p'}\right| \neq 0$. Then, define the following transformation

$$p_i = \frac{\partial F}{\partial x^i} \equiv F_{x^i}, \quad x'^i = \frac{\partial F}{\partial p'_i} \equiv F_{p'_i}, \quad i = 1, \ldots, n \tag{4.1.34}$$

which in a matrix form reads

$$dp = F_{xx}dx + F_{p'x}dp', \quad dx' = F_{xp'}dx + F_{p'p'}dp', \tag{4.1.35}$$

where $dp = (dp_1, \ldots, dp_n)^T$, $(F_{xx})_{ij} = F_{x^i x^j} = \partial^2 F/\partial x^j \partial x^i$ and so on. Then, from (4.1.35) and the fact that $F_{xx} = F_{xx}^T$, $F_{p'p'} = F_{p'p'}^T$, $F_{xp'} = F_{p'x}^T$ it follows that

$$dx = F_{xp'}^{-1}dx' - F_{xp'}^{-1}F_{p'p'}dp', \quad dp = F_{xx}F_{xp'}^{-1}dx' + (F_{p'x} - F_{xx}F_{xp'}^{-1}F_{p'p'})dp'$$

$$\Downarrow$$

$$
\begin{aligned}
dp_i \wedge dx^i &= (dp)^T \wedge dx \\
&= (dx')^T F_{p'x}^{-1} F_{xx} \wedge F_{xp'}^{-1} dx' - (dx')^T F_{p'x}^{-1} F_{xx} \wedge F_{xp'}^{-1} F_{p'p'} dp' \\
&\quad + (dp')^T F_{xp'} \wedge F_{xp'}^{-1} dx' - (dp')^T F_{xp'} \wedge F_{xp'}^{-1} F_{p'p'} dp' \\
&\quad + (dp')^T F_{p'p'} F_{p'x}^{-1} F_{xx} \wedge F_{xp'}^{-1} dx' - (dp')^T F_{p'p'} F_{p'x}^{-1} F_{xx} \wedge F_{xp'}^{-1} F_{p'p'} dp' \\
&= \left(dp'\right)^T \wedge dx' = dp'_i \wedge dx'^i .
\end{aligned}
$$

For the last by one equality we also used the fact that $\alpha^T \wedge A\alpha = \alpha^T A \wedge \alpha = 0$ for $A = A^T$ and $\alpha^T A \wedge A^{-1}\beta = \alpha^T \wedge \beta$. Thus in fact the transformation (4.1.34) is canonical. In a similar way one can introduce three other generating functions of canonical transformations

$$
F(x, x') \Longrightarrow p_i = F_{x^i},\ p'_i = -F_{x'^i}, \quad i = 1, \ldots, n, \tag{4.1.36a}
$$

$$
F(p, p') \Longrightarrow x^i = -F_{p_i}, \quad x'^i = -F_{p'_i}, \quad i = 1, \ldots, n, \tag{4.1.36b}
$$

$$
F(p, x') \Longrightarrow x_i = F_{p^i},\ p'_i = F_{x'^i} \quad i = 1, \ldots, n. \tag{4.1.36c}
$$

The important subclass of canonical transformations on a phase space consists of these canonical transformations, which are generated by transformations on the underlying configuration space Q. Let (x, p) be some Darboux coordinates on $M = T^*Q$. Besides, consider any coordinate transformation on Q

$$
x \longrightarrow x' : x' = \phi(x) \Longrightarrow x'^i = \phi^i(x), \quad i = 1, \ldots, n.
$$

The generating function of respective canonical transformation on $M : (x, p) \to (x', p')$ is of the form

$$
F(x, p') = \phi^i(x) p'_i,
$$

and hence

$$
\begin{aligned}
x'^i &= \phi^i(x), \\
p'_i &= \left[J^{-1}(x)\right]_i^j p_j, \quad i = 1, \ldots, n,
\end{aligned} \tag{4.1.37}
$$

where $J(x)$ is the Jacobian of the map $\phi(x)$. Canonical transformations (4.1.37) are called *point transformations*.

4.2 The Modern Formulation of Hamilton-Jacobi Theory

The theory of finite dimensional conservative integrable systems has a long history, starting from the works of Lagrange, Hamilton and Jacobi in the first half of the nineteenth century. In fact the Hamilton-Jacobi (HJ) theory is one of the most powerful methods of integration by quadratures a wide class of systems described by nonlinear ordinary differential equations, with a long history as part of analytical mechanics. The theory in question is closely related to the Liouville integrable Hamiltonian systems. The main difficulty of the HJ approach is that it demands distinguished coordinates, so called *separation coordinates*, in order to work effectively.

There are some milestones of that theory. First, in 1891 Stäckel initiated a program of classification of separable systems presenting conditions for separability of the HJ equations in orthogonal coordinates [240–242]. Then, in 1904 Levi-Civita found a test for the separability of a Hamiltonian dynamics in a given system of canonical coordinates [177]. The next was Eisenhart [104–106], who in 1934 inserted a separability theory in the context of the Riemannian geometry, making it coordinate free and introducing the crucial objects of the theory, i.e. Killing tensors. This approach was then developed by Woodhouse [265], Klanins [158, 162], and others. Finally, in 1992, Benenti [14–16] constructed a particular but very important subclass of separable systems, based on the so called special conformal Killing tensors.

4.2.1 Linearization of Hamiltonian Dynamics

Let us consider a Liouville integrable system on $2n$ dimensional symplectic manifold M, defined by n Poisson commuting Hamiltonians $\{H_i, H_j\}_\pi = 0,\ i, j = 1, \ldots, n$, with related Hamiltonian equations of motion

$$\xi_{t_i} = X_{H_i} = \pi dH_i, \quad i = 1, \ldots, n, \quad \xi = (x, p)^T, \tag{4.2.1}$$

where t_i is the evolution parameter of the i-th equation. Assume that (x, p) are local Darboux coordinates. The time independent HJ method of integration by quadratures of the system (4.2.1) relies on its linearization through an appropriate canonical transformation

$$(x, p) \longrightarrow (b, a), \quad a_i = H_i, \quad i = 1, \ldots, n. \tag{4.2.2}$$

In order to find coordinates b^i, canonically conjugated with a_i, it is necessary to find respective generating function $W(x, a)$ of the transformation (4.2.2). Then,

according to (4.1.34)

$$b^i = \frac{\partial W}{\partial a_i}, \quad p_i = \frac{\partial W}{\partial x^i},$$

where function $W(x, a)$ is the solution of the system of n Hamilton-Jacobi equations, generated by constants of motion H_i,

$$\boxed{H_i(x^1, \ldots, x^n, \frac{\partial W}{\partial x^1}, \ldots, \frac{\partial W}{\partial x^n}) = a_i, \quad i = 1, \ldots, n.} \tag{4.2.3}$$

Observation 6 *Historically, as well as in standard textbooks from classical mechanics (including Wikipedia) the reader finds a single HJ equation, for a distinguished Hamiltonian H. A single HJ equation is sufficient to find a generating function W only in simplest cases. Further on, we also analyze these simplest cases explicitly. Nevertheless, for the majority of separable systems, all n HJ equations (4.2.3) are necessary in order to find the generating function W and in consequence to integrate by quadratures equations (4.2.1).*

In (b, a) representation, n evolution equations (4.2.1) become trivial (linear)

$$b^j_{t_i} = \frac{\partial H_i}{\partial a_j} = \delta_{ij}, \quad (a_j)_{t_i} = -\frac{\partial H_i}{\partial b_j} = 0, \quad H_i = a_i, \quad ij = 1, \ldots, n.$$

where

$$b^j(x, a) = \frac{\partial W}{\partial x^j} = t_j + c_j, \quad c_j \in \mathbb{R}. \tag{4.2.4}$$

Equations (4.2.4) define implicit solutions of equations (4.2.1) in original coordinates (x, p). Solving the system of equations (4.2.4) with respect to x^i (so called *inverse Jacobi problem*), we reconstruct classical trajectories in the explicit form

$$x^i = x^i(t_1, \ldots, t_n, a_1, \ldots, a_n, c_1, \ldots, c_n), \quad p_i = p_i(x, x_t, ..). \tag{4.2.5}$$

Unfortunately, for the majority of separable systems, with the exception of a few elementary systems, (4.2.5) cannot be expresed by elementary functions and the theory of special functions like Riemannian theta functions is necessary [139] (see also [6]). Thus, in general, searching for a solution of the inverse Jacobi problem is a complex mathematical problem from the algebraic geometry and goes beyond the scope of the book.

Observation 7 *It is very important for the reader to keep in mind that once we are considering separable systems on a phase space, the x^i and p_i are functions which depend on n different evolution parameters t_j (4.2.5) and solve simultaneously all evolution equations (4.2.1). Obviously, for a k-th evolution system only t_k*

plays the role of a variable while the remaining $t_{j\neq k}$ are just parameters. This observation was crucial for understanding the concept of a reciprocal transform, linking solutions of Stäckel related Liouville integrable systems, where all evolution parameters were simultaneously involved in the transformation (4.1.31).

So, where is the problem if the separation procedure is so clear? The main difficulty in applying the presented method to a given Liouville integrable system in some canonical coordinates (x, p) is in solving the system (4.2.3) for W. In general this is a hopeless task, as (4.2.3) is a very complicated system of nonlinear coupled partial differential equations. In essence, the only hitherto known way of overcoming this difficulty is to find distinguished canonical coordinates, denoted here by (λ, μ) and defined almost globally on M, for which there exist n relations

$$\boxed{\varphi_i(\lambda^i, \mu_i; a_1, \ldots, a_n) = 0, \qquad i = 1, \ldots, n, \ a_i \in \mathbb{R}, \quad \det\left[\frac{\partial \varphi_i}{\partial a_j}\right] \neq 0,} \tag{4.2.6}$$

such that each of these relations involves only a single pair of canonical coordinates [235] and with additional assumption that we can solve these equations for a_i not only locally but almost globally on M. Thus, in the domain of (λ, μ), a_i are expressed in the form

$$a_i = H_i(\lambda, \mu), \qquad i = 1, \ldots, n.$$

If the functions $W_i(\lambda^i, a)$ are solutions of a system of n decoupled ODE's, called *separation equations*, obtained from (4.2.6) by substituting $\mu_i = \frac{dW_i(\lambda^i, a)}{d\lambda^i}$

$$\boxed{\varphi_i\left(\lambda^i, \mu_i = \frac{dW_i(\lambda^i, a)}{d\lambda^i}, a_1, \ldots, a_n\right) = 0, \quad i = 1, \ldots, n,} \tag{4.2.7}$$

then the function

$$W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda^i, a)$$

is an additively separable solution of all the equations (4.2.7). It is also a solution of all Hamilton-Jacobi equations (4.2.3) because solving (4.2.6) to the form $a_i = H_i(\lambda, \mu)$ is a purely algebraic operation.

The Hamiltonian functions $H_i(\lambda, \mu)$ Poisson commute as a consequence of separation relations (4.2.6). Indeed, differentiating equations (4.2.6) with respect to (λ, μ) coordinates we get

$$\frac{\partial \varphi_k}{\partial \lambda^i} + \sum_{r=1}^{n} \frac{\partial \varphi_k}{\partial a_r} \frac{\partial H_r}{\partial \lambda^i} = 0, \qquad \frac{\partial \varphi_k}{\partial \mu_i} + \sum_{r=1}^{n} \frac{\partial \varphi_k}{\partial a_r} \frac{\partial H_r}{\partial \mu_i} = 0$$

so

$$\frac{\partial H_r}{\partial \lambda^i} = -\sum_{s=1}^{n} A_s^r \frac{\partial \varphi_k}{\partial \lambda^i}, \quad \frac{\partial H_r}{\partial \mu_i} = -\sum_{s=1}^{n} A_s^r \frac{\partial \varphi_k}{\partial \mu_i}$$

where (A_s^r) is a matrix being the inverse of the matrix $(\partial \varphi_s / \partial a_r)$. In consequence

$$\begin{aligned}
\{H_r, H_s\} &= \sum_{k=1}^{n} \left(\frac{\partial H_r}{\partial \lambda^k} \frac{\partial H_s}{\partial \mu_k} - \frac{\partial H_r}{\partial \mu_k} \frac{\partial H_s}{\partial \lambda^k} \right) \\
&= \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} A_i^r \frac{\partial \varphi_i}{\partial \lambda^k} A_j^s \frac{\partial \varphi_j}{\partial \mu_k} - \sum_{i,j=1}^{n} A_i^r \frac{\partial \varphi_i}{\partial \mu_k} A_j^s \frac{\partial \varphi_j}{\partial \lambda^k} \right) \\
&= \sum_{i,j=1}^{n} A_i^r A_j^s \sum_{k=1}^{n} \left(\frac{\partial \varphi_i}{\partial \lambda^k} \frac{\partial \varphi_j}{\partial \mu_k} - \frac{\partial \varphi_i}{\partial \mu_k} \frac{\partial \varphi_j}{\partial \lambda^k} \right) \\
&= \sum_{i,j=1}^{n} A_i^r A_j^s \{\varphi_i, \varphi_j\} = 0.
\end{aligned}$$

The distinguished coordinates (λ, μ) for which the original Hamilton-Jacobi equations (4.2.3) are equivalent to a set of separation equations (4.2.7) are called the *separation coordinates*.

Of course, the original Jacobi formulation of the method was a bit different from the one presented above, and was adopted to a particular class of Hamiltonians, nevertheless it contained all important ingredients of the method. Jacobi himself doubted whether there exists a systematic method for the construction of separation coordinates. Indeed, for many decades of development of the separability theory, the method did not exist. Only recently, at the end of the twentieth century and at the beginning of the twenty-first century, after more than 100 years of efforts, a few constructive methods have appeared. Some of them are the subject of the next chapter.

4.2.2 Stäckel Systems

In what follows we restrict ourselves to considering a special case of (4.2.6) when all separation relations are affine in constants H_i:

$$\boxed{\sum_{k=1}^{n} S_{ik}(\lambda^i, \mu_i) H_k = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \ldots, n,} \tag{4.2.8}$$

where S_{ik} and ψ_i are arbitrary smooth functions of their arguments. The relations (4.2.8) are called the generalized *Stäckel separation relations* and the related dynamical systems are called the *Stäckel separable* ones. The matrix $S = (S_{ik})$ is called a *generalized Stäckel matrix*. The reason behind this name is the fact that the conditions (4.2.8) with S_{ik} being μ-independent and ψ_i being quadratic in momenta μ are equivalent to the original Stäckel conditions for separability of Hamiltonians H_i. To recover the explicit Stäckel form of the Hamiltonians it is sufficient to solve the linear system (4.2.8) with respect to H_i.

Although the restriction of linearity appears to be very strong, nevertheless for all separable systems known from the literature (at least to the knowledge of the author), the general separation relations can be reduced to the form (4.2.8) upon suitable choice of constents of motion H_i.

If in separation relations (4.2.8) we further assume that $S_{ik}(\lambda^i, \mu_i) = S_k(\lambda^i, \mu_i)$ and $\psi_i(\lambda^i, \mu_i) = \psi(\lambda^i, \mu_i)$ then the separation relations can be represented by n copies of a single curve

$$\sum_{k=1}^{n} S_k(\lambda, \mu) H_k = \psi(\lambda, \mu) \tag{4.2.9}$$

in (λ, μ) plane, called a *separation curve*. The copies in question are obtained by setting $\lambda = \lambda^i$ and $\mu = \mu_i$ for $i = 1, \ldots, n$.

Finally, let us point out that in fact, with a given Stäckel system one can relate the whole set of admissible separation coordinates (ξ, ζ), related to (λ, μ) by canonical transformations of the form

$$\lambda^i = \lambda^i(\xi^i, \zeta_i), \quad \mu_i = \mu_i(\xi^i, \zeta_i). \tag{4.2.10}$$

It is a reason why in the literature the reader can meet the notion of separable web instead of separation coordinates.

As the separation relations play the fundamental role in the Hamilton-Jacobi theory, it is natural to employ them for classification of Stäckel systems. The form of separation relations (4.2.8) allows us to classify the associated Stäckel systems. Actually, any given class of Stäckel separable systems can be represented by a fixed Stäckel matrix S and the particular form of ψ.

In our further considerations we mainly restrict to a particular subclass of separation relations (4.2.8), for which

$$S_{ik}(\lambda^i, \mu_i) = S_{ik}(\lambda^i), \quad \psi_i(\lambda^i, \mu_i) = \tfrac{1}{2} f_i(\lambda^i)\mu_i^2 + \sigma_i(\lambda^i), \tag{4.2.11}$$

i.e. the Stäckel matrix is μ independent and ψ functions are quadratic in momenta μ. Nevertheless, this subclass of Stäckel separation relations contains a majority of known from analytical mechanics separable Hamiltonian systems. Also these Stäckel systems will be most interesting for the further quantization procedure.

In order to illustrate, let us consider three important cases of separation relations (4.2.9), (4.2.11).

The first case is the simplest one, when the Hamiltonian itself separates into the sum of Hamiltonians of one degree of freedom each: $H = \sum_{i=1}^{n} H_i(\lambda^i, \mu_i)$ where

$$H_i = \tfrac{1}{2}\mu_i^2 + \sigma_i(\lambda^i), \tag{4.2.12}$$

so

$$H = \sum_{i=1}^{n} H_i = \tfrac{1}{2}\sum_{i=1}^{n} \mu_i^2 + \sum_{i=1}^{n} \sigma_i(\lambda^i) = T + V.$$

Equations (4.2.12) are simultaneously particularly simple separation relations generated by a Stäckel matrix equal to the unit matrix.

In the second case of separable relations we assume that $H_{n+1-i}(\lambda, \mu) = H_{n+1-i}(\lambda^1, \ldots, \lambda^i, \mu_1, \ldots, \mu_i)$ and chose $H = H_1(\lambda, \mu)$. Let us consider the following separation relations

$$\boxed{H_i - \zeta_i(\lambda^i)H_{i+1} = \tfrac{1}{2}\mu_i^2 + \sigma_i(\lambda^i), \quad i = 1, \ldots, n} \tag{4.2.13}$$

$$\Downarrow$$

$$\begin{aligned} H_1 - \zeta_1(\lambda^1)H_2 &= \tfrac{1}{2}\mu_1^2 + \sigma_1(\lambda^1), \\ H_2 - \zeta_2(\lambda^2)H_3 &= \tfrac{1}{2}\mu_2^2 + \sigma_2(\lambda^2), \\ &\vdots \\ H_n &= \tfrac{1}{2}\mu_n^2 + \sigma_n(\lambda^n) \end{aligned} \tag{4.2.14}$$

generated by a Stäckel matrix of the form

$$S = \begin{pmatrix} 1 & -\zeta_1(\lambda^1) & 0 & 0 & 0 \\ 0 & 1 & -\zeta_2(\lambda^2) & 0 & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & -\zeta_n(\lambda^n) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote $h_i = \frac{1}{2}\mu_i^2 + \sigma_i(\lambda^i)$, then

$$H_k = T_k + V_k = h_k + \zeta_k(\lambda^k)H_{k+1}, \quad H_n = h_n$$

and hence

$$H_k = h_k + \zeta_k h_{k+1} + \zeta_k \zeta_{k+1} h_{k+2} + \ldots + \zeta_k \ldots \zeta_{n-1} h_n,$$

which gives the common formula for kinetic T_k and potential V_k parts of the Hamiltonian H_k

$$\begin{aligned} T_k =& \tfrac{1}{2}(\mu_k^2 + \zeta_k \mu_{k+1}^2 + \zeta_k \zeta_{k+1} \mu_{k+2}^2 + \ldots + \zeta_k \ldots \zeta_{n-1} \mu_n^2), \\ V_k =& \sigma_k + \zeta_k \sigma_{k+1} + \zeta_k \zeta_{k+1} \sigma_{k+2} + \ldots + \zeta_k \ldots \zeta_{n-1} \sigma_n. \end{aligned} \tag{4.2.15}$$

Notice that separation relations (4.2.13) can be transformed with the help of point transformation (4.2.10) to new separation relations of the form

$$H_i \lambda^i + H_{i+1} = \tfrac{1}{2} f_i(\lambda^i) \mu_i^2 + \gamma_i(\lambda^i), \quad i = 1, \ldots, n.$$

Finally, the third case is a generic case, when $H_i = H_i(\lambda^1, \ldots, \mu_n)$ and $H = H_1$. Consider separation relations (4.2.9), with the condition (4.2.11), of the irreducible form

$$\boxed{\sum_{k=1}^{n} H_k (\lambda^i)^{\gamma_k} = \tfrac{1}{2} f_i(\lambda^i) \mu_i^2 + \sigma_i(\lambda^i), \quad i = 1, \ldots, n,} \tag{4.2.16}$$

where $\gamma_1 > \gamma_2 > \cdots > \gamma_n = 0$, $\gamma_i \in \mathbb{Z}_+$ and f_i, σ_i are rational functions. Irreducibility means that the set $\{\gamma_1, \ldots, \gamma_{n-1}\}$ of integers does not have a common divisor α. Otherwise, separation curve (4.2.16) can be reduced to the one with $\gamma_i \to \frac{\gamma_i}{\alpha} \in \mathbb{Z}_+$ by a transformation $\lambda \mapsto \lambda^{\frac{1}{\alpha}}$.

Hamiltonian functions H_i are solutions of the system (4.2.16)

$$\mathbf{H} = S_\gamma^{-1} \mathbf{U},$$

where $\mathbf{H} = (H_1, \ldots, H_n)^T$, $\mathbf{U} = (\frac{1}{2}\varphi_1(\lambda^1)\mu_1^2 + \sigma_1(\lambda^1), \ldots, \frac{1}{2}\varphi_n(\lambda^n)\mu_n^2 + \sigma_n(\lambda^n))^T$ is the Stäckel vector and

$$S_\gamma = \begin{pmatrix} (\lambda^1)^{\gamma_1} & (\lambda^1)^{\gamma_2} & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ (\lambda^n)^{\gamma_1} & (\lambda^n)^{\gamma_2} & \cdots & 1 \end{pmatrix}$$

is the Stäckel matrix.

At the end of that subsection let us briefly mention systems from classes, where separation curves are different from quadratic in momenta. In the first class of

systems momenta enter exponentially

$$\boxed{\sum_{j=1}^{n} H_j \lambda^{n-j} = \exp(a\mu) + \exp(-b\mu) + \sigma(\lambda), \quad a, b \in \mathbb{R}_+, \quad i = 1, \ldots, n,} \tag{4.2.17}$$

where σ defines a separable potential. This class includes such systems as the periodic Toda lattice [115], the KdV dressing chain [28], the Ruijsenaars-Schneider system [27] and others. In the second class momenta enter cubic

$$\boxed{\mu \sum_{j=1}^{n_1} H_j^{(1)} \lambda^{n_1-j} + \sum_{j=1}^{n_2} H_j^{(2)} \lambda^{n_2-j} = \mu^3 + \mu\sigma_1(\lambda) + \sigma_2(\lambda), \qquad i = 1, \ldots, n,}$$

where $\mu\sigma_1$ and σ_2 give rise to the separable potentials. We also know some particular examples from the classes. For instance, stationary flows of nonlinear PDE's, known as the Boussinesq hierarchy, belong to the class with $n_1 = 1$ and $n_2 = n - 1$ [27, 114], while dynamical system on loop algebra $\hat{\mathfrak{sl}}(3)$ belongs to the class with $n_1 = 2$ and $n_2 = 4$ [113].

4.2.3 *Inverse Jacobi Problem*

Let us integrate by quadratures separable equations whose separation relations were considered in the previous subsection. In the first case

$$H = \sum_{i=1}^{n} \left(\frac{1}{2m_i} p_i^2 + \sigma_i(x^i) \right). \tag{4.2.18}$$

In this case separation coordinates are just Cartesian coordinates x^i and conjugated momenta p_i. Separation equations (4.2.3) for the generating function W are of the form

$$\frac{1}{2m_i} \left(\frac{dW_i}{dx^i} \right)^2 + \sigma_i(x^i) = a_i, \quad i = 1, \ldots, n$$

with the solution

$$W_i = \int \sqrt{2m_i(a_i - \sigma_i(x^i))}\, dx^i, \quad i = 1, \ldots, n,$$

thus

$$b^i = \frac{dW_i}{da_i} = \int \frac{m_i dx^i}{\sqrt{2m_i(a_i - \sigma_i(x^i))}} = t + c_i.$$

Example 4.9 Consider n dimensional harmonic oscillator, analyzed in Example 4.3, now from the point of view of separability theory. It has Hamiltonian of the form (4.2.18) with potential

$$\sigma_i(x^i) = \tfrac{1}{2}\beta_i (x^i)^2.$$

Then, the implicit solution (4.2.3) is expressible in elementary functions

$$t + c_i = \int \frac{m_i dx^i}{\sqrt{2m_i(a_i - \frac{1}{2}\beta_i (x^i)^2}} = \sqrt{\frac{m_i}{\beta_i}} \arcsin \sqrt{\frac{\beta_i}{2a_i}} x^i,$$

and can be inverted to the form

$$x^i = A_i \sin(\omega_i t + \varphi_i), \quad A_i = \sqrt{\frac{2a_i}{\beta_i}}, \quad \omega_i = \sqrt{\frac{\beta_i}{m_i}}, \quad \varphi_i = \omega_i c_i$$

known from the Example 4.3.

As the second case consider separation relations (4.2.13), for which the generating function is of the form

$$W(\lambda, a) = \sum_{i=1}^{n} W_i(\lambda^i, a_{i+1}, a_i), \quad H_i = a_i. \tag{4.2.19}$$

From relations (4.2.7) and (4.2.13) follows that

$$\frac{dW_k}{d\lambda^k} = \sqrt{a_k - \zeta_k(\lambda^k) a_{k+1} - \sigma_k(\lambda^k)}$$

and so

$$\begin{aligned} b^k = \frac{\partial W}{\partial a_k} &= \frac{1}{2} \int \frac{d\lambda^k}{\sqrt{a_k - \zeta_k(\lambda^k) a_{k+1} - \sigma_k(\lambda^k)}} \\ &\quad + \frac{1}{2} \int \frac{\zeta_{k-1}(\lambda^{k-1}) d\lambda^{k-1}}{\sqrt{a_{k-1} - \zeta_{k-1}(\lambda^{k-1}) a_k - \sigma_{k-1}(\lambda^{k-1})}} \\ &= t_k + c_k, \quad k = 1, \ldots, n. \end{aligned} \tag{4.2.20}$$

where $\alpha_{n+1} = 0$. Hamiltonian dynamical systems from $\mathbb{R}^3$, which separate in spherical and cylindrical coordinates belong to this class.

Example 4.10 Consider one particle natural Hamiltonian in $T^*\mathbb{R}^3$ in the Cartesian representation

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_z^2 + V(x, y, z).$$

The point transformation to spherical coordinates

$$(r, \phi, \theta, p_r, p_\theta, p_\phi) \longrightarrow (x, y, z, p_x, p_y, p_z)$$

is given by (3.3.6) and the Hamiltonian takes the form

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r, \phi, \theta).$$

Assume that the potential $V(x, y, z)$ in spherical coordinates has the following structure

$$V(r, \phi, \theta) = \sigma_r(r) + \frac{\sigma_\theta(\theta)}{r^2} + \frac{\sigma_\phi(\phi)}{r^2\sin^2\theta},$$

then, the case (4.2.14) of separation relations is realized where $n = 3$, $(\lambda^1, \lambda^2, \lambda^3) = (r, \theta, \phi)$, $\zeta_r(r) = \frac{1}{r^2}$, $\zeta_\theta(\theta) = \frac{1}{\sin^2\theta}$. Hence, the separation relations (4.2.14) are of the form

$$\begin{aligned} H_1 - \frac{1}{r^2}H_2 &= \frac{1}{2m}p_r^2 + \sigma_r(r), \\ H_2 - \frac{1}{\sin^2\theta}H_3 &= \frac{1}{2m}p_\theta^2 + \sigma_\theta(\theta), \\ H_3 &= \frac{1}{2m}p_\phi^2 + \sigma_\phi(\phi), \end{aligned}$$

and

$$\begin{aligned} H_1 = H = h_r + \zeta_r H_2 &= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + \sigma_r(r) + \frac{\sigma_\theta(\theta)}{r^2} + \frac{\sigma_\phi(\phi)}{r^2\sin^2\theta}, \\ H_2 = h_\theta + \zeta_\theta H_3 &= \frac{1}{2m}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + \sigma_\theta(\theta) + \frac{\sigma_\phi(\phi)}{\sin^2\theta}, \\ H_3 = h_\phi &= \frac{1}{2m}p_\phi^2 + \sigma_\phi(\phi). \end{aligned} \tag{4.2.21}$$

Generating function W additively separates

$$W = W_\phi(\phi, a_3) + W_\theta(\theta, a_2, a_3) + W_r(r, a_1, a_2),$$

and implicit solutions (4.2.20) are of the form

$$b^1 = \sqrt{\frac{m}{2}} \int \frac{rdr}{\sqrt{r^2(a_1 - \sigma_r(r)) - a_2}} = t_1 + c_1,$$

$$b^2 = \sqrt{\frac{m}{2}} \int \frac{\sin\theta d\theta}{\sqrt{\sin^2\theta(a_2 - \sigma_\theta(\theta)) - a_3}} - \sqrt{\frac{m}{2}} \int \frac{dr}{r\sqrt{r^2(a_1 - \sigma_r(r)) - a_2}} = t_2 + c_2,$$

$$b^3 = \sqrt{\frac{m}{2}} \int \frac{d\phi}{\sqrt{a_3 - \sigma_\phi(\phi)}} - \sqrt{\frac{m}{2}} \int \frac{d\theta}{\sin\theta\sqrt{\sin^2\theta(a_2 - \sigma_\theta(\theta)) - a_3}} = t_3 + c_3.$$

Let us mention that this example is a key example of the HJ theory, presented in the majority of textbooks from analytical mechanics, here presented from the point of view of separation relations.

Finally, as the third case, let us consider the generic case of separation relations (4.2.16), for which $W(\lambda, a) = \sum_i W_i(\lambda^i, a)$, so separation equations (4.2.7) take the form

$$\tfrac{1}{2} f_i(\lambda^i) \left(\frac{dW_i}{d\lambda^i}\right)^2 = -\sigma_i(\lambda^i) + \sum_{k=1}^{n} a_k \left(\lambda^i\right)^{\gamma_k} \equiv P(\lambda^i, a).$$

The solution of (4.2.3) is as follows

$$W_i = \int \sqrt{\frac{2P(\lambda^i, a)}{f_i(\lambda^i)}} d\lambda^i$$

and thus

$$b^k = \frac{\partial W}{\partial a_k} = \sum_{j=1}^{n} \int \frac{(\lambda^j)^{\gamma_k} d\lambda^j}{\sqrt{R_j(\lambda^j, a)}} = t_k + c_k, \quad k = 1, \ldots, n,$$

where $R_j(\lambda^j, a) = 2 f_j(\lambda^j) P(\lambda^j, a)$.

Summarizing, one can say that on the level surface $M_a = \{\xi \in M : H_i = a_i \in \mathbf{R}\}$ the multi-time solutions $\lambda_i = \lambda_i(t_1, \ldots, t_n, \xi_0)$ of all Hamiltonian systems defined by the separation relations (4.2.16), or equivalently by all Hamiltonians (4.2.2), attain the following Abel-Jacobi differential form

$$\boxed{dt = S_\gamma^T \frac{d\lambda}{\sqrt{R(\lambda, a)}},} \tag{4.2.22}$$

where dt is a column vector with components dt_i and $d\lambda/\sqrt{R(\lambda, a)}$ means a column vector with components $d\lambda_j/\sqrt{R_j(\lambda^j, a)}$. Note that solutions (4.2.22) define in a standard (canonical) way the corresponding multi-time solutions for the momenta $\mu_i = \mu_i(t_1, \ldots, t_n, \xi_0)$.

Example 4.11 Consider once more the Hénon-Heiles system from Example 4.1

$$H_1 = H = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + x^3 + \tfrac{1}{2}xy^2,$$
$$H_2 = \tfrac{1}{2}yp_x p_y - \tfrac{1}{2}xp_y^2 + \tfrac{1}{4}x^2y^2 + \tfrac{1}{16}x^4.$$

The transformation between Cartesian and separation coordinates is of the form

$$x = \lambda^1 + \lambda^2, \quad y = 2\sqrt{-\lambda^1\lambda^2},$$

$$p_x = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \quad p_y = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right).$$

Hamiltonians H_1 and H_2 in canonical coordinates (λ, μ) are of the form

$$H_1 = \frac{1}{2}\frac{\lambda^1}{\lambda^1 - \lambda^2}\mu_1^2 + \frac{1}{2}\frac{\lambda^2}{\lambda^2 - \lambda^1}\mu_2^2 + \left(\lambda^1\right)^3 + \left(\lambda^1\right)^2\lambda^2 + \lambda^1\left(\lambda^2\right)^2 + \left(\lambda^2\right)^3,$$
$$H_1 = \frac{1}{2}\frac{\lambda^1\lambda^2}{\lambda^1 - \lambda^2}\mu_1^2 + \frac{1}{2}\frac{\lambda^1\lambda^2}{\lambda^2 - \lambda^1}\mu_2^2 - \lambda^1\lambda^2[\left(\lambda^1\right)^2 + \lambda^1\lambda^2 + \left(\lambda^2\right)^2].$$

and fulfill the following separation relations

$$H_1\lambda^1 + H_2 = \tfrac{1}{2}\lambda^1\mu_1^2 + (\lambda^1)^4,$$
$$H_1\lambda^2 + H_2 = \tfrac{1}{2}\lambda^2\mu_2^2 + (\lambda^2)^4$$

so, the implicit solution expressed by separation coordinates is as follows

$$t_1 + c_1 = \int \frac{\lambda^1 d\lambda^1}{\sqrt{R(\lambda^1, a)}} + \int \frac{\lambda^2 d\lambda^2}{\sqrt{R(\lambda^2, a)}},$$
$$t_2 + c_2 = \int \frac{d\lambda^1}{\sqrt{R(\lambda^1, a)}} + \int \frac{d\lambda^2}{\sqrt{R(\lambda^2, a)}},$$

where $R(\lambda, a) = 2\lambda(a_2 + a_1\lambda - \lambda^4)$. Obtaining the explicit solution in Cartesian coordinates is a far from being a trivial task and we skip it here.

In our last example we miraculously found the transformation to separation coordinates. In fact, the systematic construction of a transformation relating some

natural coordinates (pseudo-Euclidean for example) with separation coordinates, was the most challenging problem of the separability theory for over 100 years and will be discussed in the next chapter.

4.3 Stäckel Systems in Riemannian Geometry

The class of Liouville integrable and separable systems considered in the previous section consists of functions quadratic in momenta. So, we can adopt a Riemannian geometry to the Poisson geometry of such systems. It is not a unique procedure and that fact will be crucial for further separable quantization of Stäckel systems, considered in Sect. 8.2. Here we mainly concentrate on a natural choice, which allows us to identify a distinguished metric tensor and related Killing tensors from kinetic parts of Poisson commuting Hamiltonians. We also derive the algorithmic construction of separable potentials with the help of an appropriate recursion matrix. We analyze in details a particular important class of Stäckel systems, so called Benenti class. The significance of that class becomes clear in the next section.

4.3.1 Killing Tensors and Separable Potentials

Consider the separable system defined by the following separation relations

$$\boxed{\sum_{k=1}^{n} H_k S_{ik}(\lambda^i) = \tfrac{1}{2} f_i(\lambda^i)\mu_i^2 + \sigma_i(\lambda^i), \quad i = 1, \ldots, n,} \tag{4.3.1}$$

linear in H_i, with the following solution

$$H_r = \tfrac{1}{2} \sum_{i=1}^{n} A_r^{ii} \mu_i^2 + V_r, \quad r = 1, \ldots, n. \tag{4.3.2}$$

It is useful for further applications to relate a symplectic geometry (Poisson in general) with a Riemannian geometry in the context of the separability theory. Let us consider the Hamiltonians (4.3.2) as functions from the phase space T^*Q, where Q is a Riemannian space (Q, g), written in local coordinates (λ, μ). Then, Hamiltonians adopted to such an interpretation, can be written in the form

$$\boxed{H_r = \tfrac{1}{2} \sum_{i=1}^{n} A_r^{ii} \mu_i^2 + V_r = \tfrac{1}{2} \sum_{i=1}^{n} (B_r G)^{ii} \mu_i^2 + V_r, \quad r = 1, \ldots, n.} \tag{4.3.3}$$

Such freedom in the interpretation of Hamiltonians H_i will be crucial for the quantum separability theory developed in Sect. 8.2. Here, on the classical level, we make a useful "natural" choice, identifying A_1 with a contravariant metric tensor G, i.e.

$$\boxed{H_r = T_r + V_r = \frac{1}{2}\sum_{i=1}^{n}(K_r G)^{ii}\mu_i^2 + V_r, \quad r = 1, \ldots, n} \tag{4.3.4}$$

where, from the construction, $B_1 = K_1 = I$, $A_1 \equiv G$ and $B_r = K_r$ represent second order Killing tensors of $(1,1)$ type for the metric $g = G^{-1}$. T_r represent geodesic Hamiltonians and V_r separable potentials. Notice that in separation coordinates metric tensor G and Killing tensors K_r are represented by diagonal matrices. Moreover, $V_r(\lambda)$ represent separable potentials. In fact we have n such choices, where A_r is identified with a metric tensor G_r and $A_{i\neq r}$ are respective Killing tensors. As all constants of motion are functions quadratic in momenta μ_i, so all Hamiltonian systems generated by H_r have their own Legendre transformation (3.1.7), (3.1.8), generated by the metric tensor $G_r \equiv A_r$, and the related second order Newton equations of motion can be represented by the following equations

$$\boxed{\lambda^i_{t_r t_r} + (\Gamma_r)^i_{jk}\lambda^j_{t_r}\lambda^k_{t_r} = -G_r^{ij}\,\partial_j V_r, \quad r = 1, \ldots, n.} \tag{4.3.5}$$

The form of separation relations (4.3.1) allows for a classification of such a type of systems. Each class is defined by a particular choice of Stäckel matrix S. Systems inside a class are parametrized by functions f and σ. Actually, functions f parametrize *separable metrics* (called sometimes Stäckel metrics) while functions σ parametrize *separable potentials*. Indeed, the Stäckel matrix for separation relations (4.3.1) is of the form

$$S = \begin{pmatrix} S_{11}\left(\lambda^1\right) & S_{12}\left(\lambda^1\right) & \cdots & S_{1n}\left(\lambda^1\right) \\ \vdots & \vdots & \ldots & \vdots \\ S_{n1}\left(\lambda^n\right) & S_{n2}\left(\lambda^n\right) & \cdots & S_{nn}\left(\lambda^n\right) \end{pmatrix}, \tag{4.3.6}$$

hence tensors A_r and potentials $V_r(\lambda)$ can be expressed as

$$\boxed{A_r = diag\left((S^{-1})_{r1}f_1(\lambda^1), \ldots, (S^{-1})_{rn}f_n(\lambda^n)\right), \quad V_r(\lambda) = (S^{-1})_{ri}\sigma_i(\lambda^i).} \tag{4.3.7}$$

Besides, for a "natural" choice $G = A_1$, we have

$$\boxed{K_r = diag\left(\frac{(S^{-1})_{r1}}{(S^{-1})_{11}}, \ldots, \frac{(S^{-1})_{rn}}{(S^{-1})_{1n}}\right)}$$

$$\Downarrow \tag{4.3.8}$$

$$(K_r)^i_i = \frac{(S^{-1})_{ri}}{(S^{-1})_{1i}} = \frac{D_{ir}}{D_{i1}},$$

where D_{kj} are respective cofactors of matrix S. Notice, that for Stäckel matrix S in the form (4.3.6) we have

$$\boxed{\frac{\partial}{\partial \lambda_i}(K_r)^i_i = 0, \quad i = 1, \ldots, n.} \tag{4.3.9}$$

Moreover, one can check that metric $G = A_1$ is non-flat in general. We will come back to that problem later on.

Let us consider in greater detail two types of separation relations from the previous subsection. For separation relations (4.2.13) $f_i(\lambda^i) = 1$ and non-zero elements of Stäckel matrix are of the form

$$S_{i,i} = 1, \quad S_{i,i+1} = -\zeta_i(\lambda^i)\,,$$

while non-zero elements of the inverse Stäckel matrix are

$$(S^{-1})_{ii} = 1, \quad (S^{-1})_{ij} = \prod_{k=i}^{j-1} \zeta_k(\lambda^k), \quad i < j.$$

For example, for $n = 4$ we have

$$S = \begin{pmatrix} 1 & -\zeta_1 & 0 & 0 \\ 0 & 1 & -\zeta_2 & 0 \\ 0 & 0 & 1 & -\zeta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1 & \zeta_1 & \zeta_1\zeta_2 & \zeta_1\zeta_2\zeta_3 \\ 0 & 1 & \zeta_2 & \zeta_2\zeta_3 \\ 0 & 0 & 1 & \zeta_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The first row of the matrix S^{-1} represents elements of diagonal metric G

$$G = diag(1, \zeta_1, \zeta_1\zeta_2, \ldots, \zeta_1 \ldots \zeta_{n-1}),$$

with respective Killing tensors of (1, 1) type

$$K_r = diag\left(\underbrace{0,\ldots,0}_{r-1},\frac{1}{\xi_r},\ldots,\frac{1}{\xi_r}\right),\quad \xi_r=\prod_{k=1}^{r-1}\zeta_k(\lambda^k).$$

Separable potentials are given by relation (4.2.15).

Example 4.12 Let us supplement the Example 4.10 of separable particle dynamics in $\mathbb{R}^3$ by some extra information. In spherical coordinates (r,ϕ,θ) (3.3.6) the metric tensor has the form

$$G=\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{r^2} & 0\\ 0 & 0 & \frac{1}{r^2\sin^2\theta}\end{pmatrix},$$

the related Killing tensors of (1, 1) type are

$$K_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\quad K_2=\begin{pmatrix}0&0&0\\0&r^2&0\\0&0&r^2\end{pmatrix},\quad K_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&r^2\sin^2\theta\end{pmatrix},$$

while admissible separable potentials are of the form

$$\begin{aligned}V_1&=\sigma_r(r)+\frac{\sigma_\theta(\theta)}{r^2}+\frac{\sigma_\phi(\phi)}{r^2\sin^2\theta},\\ V_2&=\sigma_\theta(\theta)+\frac{\sigma_\phi(\phi)}{\sin^2\theta},\\ V_3&=\sigma_\phi(\phi),\end{aligned}$$

where $\sigma_r(r),\sigma_\theta(\theta),\sigma_\phi(\phi)$ are smooth functions of its arguments.

As a second particular case of separation relations (4.3.1) consider systems whose separable relations are n copies of irreducible separation curve

$$\sum_{k=1}^{n}H_k\lambda^{\gamma_k}=f(\lambda)\left[\tfrac{1}{2}\mu^2+\kappa(\lambda)\right]=\tfrac{1}{2}f(\lambda)\mu^2+\sigma(\lambda),\tag{4.3.10}$$

where $\gamma_1>\gamma_2>\ldots>\gamma_n$, $\gamma_i\in\mathbb{Z}_+$, with normalization $\gamma_n=0$, being a particular case of (4.2.16). We also assume in our further considerations the meromorphic form of functions $f(\lambda)$ and $\sigma(\lambda)$. The explicit form of metric tensor G and respective Killing tensors K_r will be presented in the next subsection. Here we demonstrate the construction of separable potentials with the help of the so called *recursion matrix*. Notice first that from the form of separation relations (4.3.1)

follows that they split into a geodesic part and a potential part, respectively. The potential part of the separation curve takes the form

$$V_1^{(\sigma)}\lambda^{\gamma_1} + V_2^{(\sigma)}\lambda^{\gamma_2} + \ldots + V_n^{(\sigma)}\lambda^{\gamma_n} = \sigma(\lambda). \tag{4.3.11}$$

Let us begin from the so called *basic potentials*, i.e. $\sigma(\lambda) = \lambda^k$, $k \in \mathbb{Z}$. Then, the n copies of the separation curve (4.3.11) with $(\lambda, \mu) = (\lambda^i, \mu^i)_{i=1,\ldots,n}$ can be written in a matrix form

$$S_\gamma V^{(k)} = \Lambda^k (1, \ldots, 1)^T,$$

where $V^{(k)} = (V_1^{(k)}, \ldots, V_n^{(k)})^T$, $\Lambda = diag(\lambda^1, \ldots, \lambda^n)$ and Stäckel matrix S_γ, according to the normalization in (4.3.10), is of the form

$$S_\gamma = \begin{pmatrix} (\lambda^1)^{\gamma_1} & (\lambda^1)^{\gamma_2} & \cdots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ (\lambda^n)^{\gamma_1} & (\lambda^n)^{\gamma_2} & \cdots & 1 \end{pmatrix}.$$

As a consequence

$$V^{(0)} = S_\gamma^{-1}(1, \ldots, 1)^T = (0, \ldots 0, 1)^T,$$

which follows from the form of Stäckel matrix (4.3.6), and hence

$$S_\gamma V^{(1)} = \Lambda(1, \ldots, 1)^T = \Lambda S_\gamma V^{(0)},$$

so

$$V^{(1)} = S_\gamma^{-1} \Lambda S_\gamma V^{(0)} = F_\gamma V^{(0)},$$

where

$$F_\gamma = S_\gamma^{-1} \Lambda S_\gamma$$

is called the recursion matrix for separation relations (4.3.10). Indeed

$$V^{(k)} = F_\gamma^k V^{(0)}, \quad k \in \mathbb{Z} \tag{4.3.12}$$

as from the assumption about the invertibility of Stäckel matrix S_γ follows the invertibility of recursion matrix F_γ. Besides

$$V^{(\sigma)} = \sigma(F_\gamma) V^{(0)}, \tag{4.3.13}$$

for any meromorphic function $\sigma(\lambda)$ in (4.3.11). Notice that relation (4.3.13) is of simple matrix form so it is valid in any coordinate frame on Q. Moreover, since

$V^{(0)} = (0, 0, \ldots, 0, 1)^T$, we find that the potentials (4.3.12) are trivial for $k = \gamma_1, \ldots, \gamma_n$ as

$$V_r^{(\gamma_j)} = \delta_{\gamma_r, \gamma_j}.$$

4.3.2 *Benenti Class of Separable Systems*

A particular important role, among all considered Stäckel systems (4.3.10), plays the so called Benenti class, defined by

$$(\gamma_1, \ldots, \gamma_n) = (n-1, n-2, \ldots, 0),$$

hence by separation relations of the form

$$\boxed{\sum_{r=1}^{n} H_r (\lambda^i)^{n-r} = \tfrac{1}{2} f_i(\lambda^i)\mu_i^2 + \sigma(\lambda^i), \quad i = 1, \ldots, n,} \tag{4.3.14}$$

with solution

$$\boxed{H_r = \tfrac{1}{2}(K_r G)^{ii}\mu_i^2 + V_r^{(\sigma)}, \quad r = 1, \ldots, n.} \tag{4.3.15}$$

For this class of systems the Stäckel matrix

$$S = \begin{pmatrix} (\lambda^1)^{n-1} & (\lambda^1)^{n-2} & \cdots & 1 \\ \vdots & \vdots & \ldots & \vdots \\ (\lambda^n)^{n-1} & (\lambda^n)^{n-2} & \cdots & 1 \end{pmatrix}$$

is the Vandermonde matrix and metric tensors are of the form

$$G^{ii} = \boxed{\frac{f_i(\lambda^i)}{\Delta_i}, \quad \Delta_i = \prod_{k \neq i} (\lambda^i - \lambda^k), \quad i = 1, \ldots, n,} \tag{4.3.16}$$

i.e. are parametrized by n functions of one variable $f_i(\lambda^i)$.

All metric tensors (4.3.16) have a common set of Killing tensors of (1, 1) type

$$\boxed{(K_r)_i^i = -\frac{\partial \rho_r}{\partial \lambda^i}, \quad r = 1, \ldots, n,} \tag{4.3.17}$$

where $\rho_r(\lambda)$ are signed elementary symmetric polynomials in λ (later in the book called Viète polynomials, for short):

$$\rho_i(\lambda) = (-1)^i \sum_{1 \le n_1 < n_2 < \ldots < n_i \le n} \lambda_{n_1} \ldots \lambda_{n_i}, \quad i = 1, \ldots, n \tag{4.3.18}$$

so in particular

$$\rho_1 = -(\lambda^1 + \ldots + \lambda^n), \ldots, \rho_n = (-1)^n \lambda^1 \lambda^2 \ldots \lambda^n.$$

It follows from (4.3.17) and (4.3.18) that Killing tensors and Viète polynomials fulfill the following matrix equations

$$\boxed{\rho_r I_n = K_{r+1} - \Lambda K_r, \quad r = 1, \ldots, n-1, \quad \rho_n I_n = -\Lambda K_n.} \tag{4.3.19}$$

In particular, when separation relations (4.3.14) are represented by n copies of the separation curve

$$\sum_{r=1}^{n} H_r \lambda^{n-r} = \tfrac{1}{2} f(\lambda)\mu^2 + \sigma(\lambda), \quad i = 1, \ldots, n,$$

metric tensors (4.3.16) are represented by

$$G = f(\Lambda) G_0, \quad (G_0)^{ii} = \frac{1}{\Delta_i}.$$

Moreover, the potential part of separation relations (4.3.14) is generated by

$$V_1^{(\sigma)} \lambda^{n-1} + V_2^{(\sigma)} \lambda^{n-2} + \ldots + V_n^{(\sigma)} = \sigma(\lambda).$$

Hence, the recursion matrix $F = S^{-1} \Lambda S$ attains the simple form

$$F = \begin{pmatrix} -\rho_1 & 1 & & \\ -\rho_2 & & \ddots & \\ \vdots & & & 1 \\ -\rho_n & 0 & \cdots & 0 \end{pmatrix} \tag{4.3.20}$$

with the inverse

$$F^{-1} = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{\rho_n} \\ 1 & & & -\frac{\rho_1}{\rho_n} \\ & \ddots & & \vdots \\ & & 1 & -\frac{\rho_{n-1}}{\rho_n} \end{pmatrix} \tag{4.3.21}$$

and for the hierarchy of basic potentials $\sigma(\lambda) = \lambda^k$ we have

$$V^{(k)} = F^k V^{(0)}, \quad k \in \mathbb{Z}. \tag{4.3.22}$$

Since $V^{(0)} = (0, 0, \ldots, 0, 1)^T$, we easily obtain that the potentials (4.3.22) are trivial for $k = 1, \ldots, n-1$ as

$$V_r^{(k)} = \delta_{r,n-k}.$$

The first nontrivial positive potentials are

$$\begin{aligned} V^{(n)} &= F^n V^{(0)} = (-\rho_1, \ldots, -\rho_n)^T \Longrightarrow V_r^{(n)} = -\rho_r, \\ V^{(n+1)} &= F^{n+1} V^{(0)} = (\rho_1^2 - \rho_2, \rho_1\rho_2 - \rho_3, \ldots, \rho_1\rho_n)^T \\ &\Longrightarrow V_r^{(n+1)} = \rho_1\rho_r - \rho_{r+1}, \end{aligned}$$

while the first nontrivial negative potentials take the form

$$V^{(-1)} = F^{-1}V^{(0)} = (-1/\rho_n, -\rho_1/\rho_n, \ldots, -\rho_{n-1}/\rho_n)^T \Longrightarrow V_r^{(-1)} = -\frac{\rho_{r-1}}{\rho_n},$$

$$V^{(-2)} = F^{-2}V^{(0)} \Longrightarrow V_r^{(-2)} = \frac{\rho_{r-1}\rho_{n-1} - \rho_{r-2}\rho_n}{\rho_n^2},$$

where $\rho_0 := 1$ and $\rho_r = 0$ for $r < 0$. Notice that the recursion matrix (4.3.20) and its inverse (4.3.21) are constructed from respective vectors of potentials

$$F = (V^{(n)}, V^{(n-1)}, \ldots, V^{(1)}), \quad F^{-1} = (V^{(n-2)}, \ldots, V^{(0)}, V^{(-1)}).$$

From (4.3.22) follows immediately a simple recursive formula for components of positive potentials $V^{(k)}$, $k \in \mathbb{N}$

$$\boxed{V_r^{(k+1)} = V_{r+1}^{(k)} - \rho_r V_1^{(k)}, \quad r = 1, \ldots, n-1, \quad V_n^{(k+1)} = -\rho_n V_1^{(k)},} \tag{4.3.23}$$

while from (4.3.21) follows a simple recursive formula for components of negative potentials $V^{(-k)}$, $k \in \mathbb{N}$

$$\boxed{V_r^{(-k-1)} = V_{r-1}^{(-k)} - \frac{\rho_{r-1}}{\rho_n} V_n^{(-k)}, \quad r = 2, \ldots, n, \quad V_1^{(-k-1)} = -\frac{1}{\rho_n} V_n^{(-k)},} \tag{4.3.24}$$

The Benenti class is a distinguished class among separable systems of type (4.3.10), which contains a subclass of flat metrics, hence a lot of separable

systems, known from analytical mechanics, belong to that class. One can show that Stäckel matrices (4.3.16) are flat when

$$f_i(\lambda^i) = f(\lambda^i) = \prod_{k=1}^{m}(\lambda^i - \beta_k), \quad m = 0, 1, \ldots, n$$

is a real polynomial. If the polynomial $f(\lambda^i)$ is of order $n+1$, the metric G is of constant curvature: $R = const$.

Observation 8 *Note, that any Stäckel system (4.3.1) of two degrees of freedom can be transformed to the Benenti class (4.3.14) through the point transformation* $\lambda^i \to (\lambda^i)^{\frac{1}{\gamma_1}}$ *(4.1.37). In a consequence any Stäckel system of two degrees of freedom with both integrals of motion quadratic in momenta belongs to the Benenti class.*

A typical representative of this class is the Henon Heiles system from Example 4.11. More examples will appear in the following sections.

4.3.3 Superintegrability in Benenti Class

Let us investigate the problem of superintegrability inside the Benenti class of separable systems. In order to simplify the problem we restrict the considerations to systems generated by separation curves of the form

$$\sum_{k\in I}\alpha_k\lambda^k + \sum_{r=1}^{n} H_r\lambda^{n-r} = \tfrac{1}{2}\lambda^m\mu^2, \quad m = 0, \ldots, n+1. \tag{4.3.25}$$

where $I \subset \mathbf{Z}$ is some finite index set, numerating nontrivial basic potentials. The coordinates (λ, μ), convenient for the integrability procedure, are inconvenient for any other purpose as the components of metric tensors, Killing tensors and separable potentials are rational functions making computations very complicated. We will therefore perform the search of other Darboux coordinates. The simplest choice are Viète coordinates

$$\boxed{q^i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^{n}\frac{(\lambda^k)^{n-i}\,\mu_k}{\Delta_k}.} \tag{4.3.26}$$

Since the transformation from (λ, μ) to (q, p) is a point transformation the coordinates (q, p) are also Darboux coordinates for our Poisson tensor. It can be shown [44] that in the (q, p)-coordinates

$$(\Lambda)^i_j = -\delta^1_j q^i + \delta^{i+1}_j, \qquad (G_0)^{ij} = q^{i+j-n-1}$$

so components of metric tensors $G_m = \Lambda^m G_0$ take the form

$$(G_m)^{ij} = \begin{cases} q^{i+j+m-n-1}, & i,j = 1,\dots,n-m \\ -q^{i+j+m-n-1}, & i,j = n-m+1,\dots,n \\ 0 \quad \text{otherwise} \end{cases} \qquad m = 1,\dots,n$$

$$(G_{n+1})^{ij} = q^i q^j - q^{i+j}, \quad i,j = 1,\dots,n, \tag{4.3.27}$$

where we set $q^0 = 1$ and $q^r = 0$ for $r > n$ or $r < 0$. An advantage of these new coordinates is that the geodesic parts of H_i are polynomial in q. Moreover, separable potentials are given by the same formula (4.3.22) under substitution $q_i = \rho_i(\lambda)$.

Example 4.13 For $n = 3$ in Viète coordinates (4.3.26) we have

$$\Lambda = \begin{pmatrix} -q^1 & 1 & 0 \\ -q^2 & 0 & 1 \\ -q^3 & 0 & 0 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q^1 \\ 1 & q^1 & q^2 \end{pmatrix}$$

and hence the metric tensors G_j have the form

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q^1 & 0 \\ 0 & 0 & -q^3 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -q^2 & -q^3 \\ 0 & -q^3 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} -q^1 & -q^2 & -q^3 \\ -q^2 & -q^3 & 0 \\ -q^3 & 0 & 0 \end{pmatrix}, \quad G_4 = \begin{pmatrix} (q^1)^2 - q^2 & q^1q^2 - q^3 & q^1q^3 \\ q^1q^2 - q^3 & (q^2)^2 & q^2q^3 \\ q^1q^3 & q^2q^3 & (q^3)^2 \end{pmatrix}$$

In accordance with (4.3.2), the metric tensors $G_0,\dots,G_3$ are flat, while the metric G_4 is of constant curvature.

Let us come back to superintegrability. According to (4.1.2) and (4.1.8) we are looking for additional constants of motion of H_1, linear in momenta, generated by Killing vectors of the respective metric tensor. Nontrivial Killing vectors are expected for flat and constant curvature metrices and Viète coordinates are very useful for such a search. In Viète coordinates, for the geodesic part of H_1, i.e. $T_1 = \frac{1}{2}G_m^{ij}p_i p_j$, the simplest constants of motion P_r linear in momenta, generated by Killing vectors of the metric G_m (4.3.27) with components linear in positions,

are of the form [193]

$$\boxed{\begin{aligned} P_r &= \sum_{k=1}^{r} k q^{r-k} p_{n-m-k+1}, \quad m = 0, \dots, n-1, \quad r = 1, \dots, n-m, \\ P_r &= \sum_{k=1}^{n-r+1} k q^{r+k-1} p_{n-m+k+1}, \quad m = 2, \dots, n+1, \quad r = n-m+2, \dots, n \end{aligned}} \tag{4.3.28}$$

Formulas (4.3.28) can be verified by direct computation. Other constants P are generated by Killing vectors with components being higher order polynomials of position coordinates.

Example 4.14 For the metric tensors G_m from Example 4.13 we have the following sets of Killing vectors Y and related functions P (4.3.28) that commute with geodesic Hamiltonian T:

$$G_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & q^1 \\ 1 & q^1 & q^2 \end{pmatrix}: \quad Y_1 = (0,0,1),\ Y_2 = (0,2,q^1),\ Y_3 = (3,2q^1,q^2)$$

$$\begin{aligned} T =& p_1 p_3 + \tfrac{1}{2} p_2^2 + q^1 p_2 p_3 + \tfrac{1}{2} q^2 p_3^2, \\ P_1 =& p_3, \quad P_2 = 2p_2 + q^1 p_3, \quad P_3 = 3p_1 + 2q^1 p_2 + q^2 p_3, \end{aligned}$$

$$G_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & q^1 & 0 \\ 0 & 0 & -q^3 \end{pmatrix}: \quad Y_1 = (0,1,0),\ Y_2 = (2,q^1,0),$$

$$T = p_1 p_2 + \tfrac{1}{2} q^1 p_2^2 - \tfrac{1}{2} q^3 p_3^2, \quad P_1 = p_2, \quad P_2 = 2p_1 + q^1 p_2,$$

$$G_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -q^2 & -q^3 \\ 0 & -q^3 & 0 \end{pmatrix}: \quad Y_1 = (1,0,0),\ Y_3 = (0,0,q^3),$$

$$T = \tfrac{1}{2} p_1^2 - \tfrac{1}{2} q^2 p_2^2 - q^3 p_2 p_3, \quad P_1 = p_1, \quad P_3 = q^3 p_3,$$

$$G_3 = \begin{pmatrix} -q^1 & -q^2 & -q^3 \\ -q^2 & -q^3 & 0 \\ -q^3 & 0 & 0 \end{pmatrix}: \quad Y_1 = (0,q^3,0),\ Y_3 = (0,q^2,2q^3),$$

$$T = -\tfrac{1}{2} q^1 p_1^2 - q^2 p_1 p_2 - q^3 p_1 p_3 - \tfrac{1}{2} q^3 p_2^2, \quad P_1 = q^3 p_2, \quad P_3 = q^2 p_2 + 2q^3 p_3,$$

$$G_4 = \begin{pmatrix} (q^1)^2 - q^2 & q^1q^2 - q^3 & q^1q^3 \\ q^1q^2 - q^3 & (q^2)^2 & q^2q^3 \\ q^1q^3 & q^2q^3 & (q^3)^2 \end{pmatrix}:$$

$$Y_1 = (0, q^3, 0),\ Y_2 = (0, q^2, 2q^3),\ Y_3 = (q^1, 2q^2, 3q^3),$$

$$T = (q^1q^2 - q^3)p_1p_2 + \tfrac{1}{2}((q^1)^2 - q^2)p_1^2 + \tfrac{1}{2}(q^2)^2p_2^2 + q^1q^3p_1p_3 + \tfrac{1}{2}(q^3)^2p_3^2 + q^2q^3p_2p_3,$$

$$P_1 = q^3p_1,\quad P_2 = q^2p_1 + 2q^3p_2,\quad P_3 = q^1p_1 + 2q^2p_2 + 3q^3p_3.$$

In order to find superintegrable potentials we have to use the condition $Y(U) = c \Leftrightarrow \{H_1, P\}$ (4.1.2)–(4.1.8). The highest number of such potentials is detected by $P = P_1 = p_{n-m}$ for $m = 0, \ldots, n-1$ and by $P = P_n = q^n p_{n-m+2}$ for $m = n, n+1$. In consequence, the Stäckel system (4.3.25) is maximally superintegrable in the following cases:

(i) case $m \in \{0, \ldots, n-1\}$: if $I \subset \{2n - m - 1, \ldots, n\} \cup \{-1, \ldots, -m\}$,
(ii) case $m = n$: if $I \subset \{n\} \cup \{-1, \ldots, -n+2\}$,
(iii) case $m = n+1$ (the case of constant curvature): if $I \subset \{-1, \ldots, -n+1\}$.

The additional integrals h_{n+r} commuting with h_1 have the structure

$$\boxed{h_{n+r} = \tfrac{1}{2}p_i\,(L_Y A_{r+1})^{ij}\,p_j + Y(U_{r+1}),\quad r = 1, \ldots, n-1,} \tag{4.3.29}$$

where Y is a vector field on Q given by

(i) for $m \in \{0, \ldots, n-1\}$: $Y = \frac{\partial}{\partial q^{n-m}}$,
(ii) for $m = n$: $Y = q^n \frac{\partial}{\partial q^2}$,
(iii) for $m = n+1$: $Y = q^n \frac{\partial}{\partial q^1}$.

The above result provides us with a sufficient condition for maximal superintegrability of Stäckel systems of constant curvature (flat in particular) in case when $f(\lambda)$ is a monomial of maximal order $n+1$. In consequence, the case (i) yields an n-parameter family of maximally superintegrable systems, parametrized by $\{\alpha_{-m}, \ldots, \alpha_{-1}, \alpha_n, \ldots, \alpha_{2n-m-1}\}$, $m = 0, \ldots, n-1$, where α_j parametrize families of nontrivial superintegrable potentials U in (4.3.25). Similarly, in the cases (ii) and(iii) we get an appropriate $(n-1)$-parameter families of superintegrable systems [43]. The reader can find a more general case of that classification, i.e. the polynomial case of $f(\lambda)$ in (4.3.25), in [40].

The geodesic parts

$$T_{n+r} = \tfrac{1}{2}A^{ij}_{n+r}(\lambda)\mu_i\mu_j,\ r = 1, \ldots, n-1$$

of additional integrals of motion $h_{n+r} = \{h_{r+1}, P\}$ with $r = 1, \ldots, n-1$, written in the separation coordinates (λ, μ), are of the form

(i) for $0 \leq m \leq n-1$:

$$A^{ij}_{n+r} = -\frac{\partial^2 \rho_r}{\partial \lambda^i \partial \lambda^j} \frac{(\lambda^i)^m (\lambda^j)^m}{\Delta_i \Delta_j}, \ i \neq j,$$

$$A^{ii}_{n+r} = \frac{(\lambda^i)^m}{\Delta_i} \sum_{j=1}^{n} \frac{\partial^2 \rho_r}{\partial \lambda^i \partial \lambda^j} \frac{(\lambda^j)^m}{\Delta_j},$$

where $\rho_r(\lambda)$ are given by (4.3.18) while Δ_i by (4.3.16)

(ii) for $m = n, n+1$

$$A^{ij}_{n+r} = -\frac{\partial^2 \rho_r}{\partial \lambda^i \partial \lambda^j} \frac{\partial \rho_n}{\partial \lambda^i} \frac{1}{\lambda^j} \frac{(\lambda^i)^m (\lambda^j)^m}{\Delta_i \Delta_j}, \ i \neq j,$$

$$A^{ii}_{n+r} = \frac{(\lambda^i)^m}{\Delta_i} \sum_{j=1}^{n} \frac{\partial^2 \rho_r}{\partial \lambda^i \partial \lambda^j} \frac{(\lambda^j)^m}{\Delta_j} \frac{\partial \rho_n}{\partial \lambda^j} \frac{1}{\lambda^j}$$

Let us illustrate the above considerations by some examples.

Example 4.15 Consider the flat case $n = 3, m = 1$, (4.3.25) with $I = \{-1, 3, 4\}$, so commuting Hamiltonians h_i are given by a separation curve

$$\alpha_4 \lambda^4 + \alpha_3 \lambda^3 + h_1 \lambda^2 + h_2 \lambda + h_3 + \alpha_{-1} \lambda^{-1} = \tfrac{1}{2} \lambda \mu^2. \tag{4.3.30}$$

Then, the corresponding Stäckel Hamiltonians attain in Viète coordinates the form

$$h_1 = p_1 p_2 + \tfrac{1}{2} q^1 p_2^2 - \tfrac{1}{2} q^3 p_3^2 + \alpha_{-1} V_1^{(-1)}(q) + \alpha_3 V_1^{(3)}(q) + \alpha_4 V_1^{(4)}(q),$$

$$h_2 = \tfrac{1}{2} p_1^2 + \tfrac{1}{2}((q^1)^2 - q^2) p_2^2 - \tfrac{1}{2} q^1 q^3 p_3^2 + q^1 p_1 p_2 - q^3 p_2 p_3 + \alpha_{-1} V_2^{(-1)}(q) + \alpha_3 V_2^{(3)}(q) + \alpha_4 V_2^{(4)}(q),$$

$$h_3 = -\tfrac{1}{2} q^3 p_2^2 - \tfrac{1}{2} q^2 q^3 p_3^2 - q^3 p_1 p_3 - q^1 q^3 p_2 p_3 + \alpha_{-1} V_3^{(-1)}(q) + \alpha_3 V_3^{(3)}(q) + \alpha_4 V_3^{(4)}(q),$$

where

$$V_1^{(-1)}(q) = \frac{1}{q^3}, \quad V_2^{(-1)}(q) = \frac{q^1}{q^3}, \quad V_3^{(-1)}(q) = \frac{q^2}{q^3},$$

$$V_1^{(3)}(q) = q^1, \quad V_2^{(3)}(q) = q^2, \quad V_3^{(3)}(q) = q^3,$$

$$V_1^{(4)}(q) = -(q^1)^2 + q^2, \quad V_2^{(4)}(q) = -q^1 q^2 + q^3, \quad V_3^{(4)}(q) = -q^1 q^3.$$

As $Y = \frac{\partial}{\partial q_2}$ so that $P = p_2$ and thus

$$\{h_1, P\} = \begin{cases} 0 \text{ for } k = -1 \\ 0 \text{ for } k = 3 \\ \alpha_4 \text{ for } k = 4 \end{cases}$$

Hence, the system is maximally superintegrable with additional constants of motion for h_1 given by:

$$h_4 = \{h_2, P\} = -\tfrac{1}{2}p_2^2 + \alpha_3 - \alpha_4 q^1,$$
$$h_5 = \{h_3, P\} = -\tfrac{1}{2}q^3 p_3^2 + \frac{\alpha_{-1}}{q^3}.$$

Consider now the point transformation from (q, p)-coordinates to new coordinates (x, y, z, p_x, p_y, p_z) such that

$$q^1 = x, \quad q^2 = y + \tfrac{1}{4}x^2, \quad q^3 = -\tfrac{1}{4}z^2 \tag{4.3.31}$$

while (p_x, p_y, p_z) are new conjugated momenta. Then (x, y, z) are flat but non-orthogonal coordinates for the metric $G_1 = A_1$ (the reader can find more about flat coordinates in Sect. 5.5). In new coordinates we get

$$G = G_1 = \begin{pmatrix} 0\,1\,0 \\ 1\,0\,0 \\ 0\,0\,1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -\frac{1}{2}x & 1 & 0 \\ -y & -\frac{1}{2}x & -\frac{1}{2}z \\ -\frac{1}{2}z & 0 & 0 \end{pmatrix} \tag{4.3.32}$$

while the first three commuting Hamiltonians become

$$\begin{aligned} h_1 &= p_x p_y + \tfrac{1}{2}p_z^2 + \alpha_{-1}V_1^{(-1)} + \alpha_3 V_1^{(3)} + \alpha_4 V_1^{(4)} \\ h_2 &= \tfrac{1}{2}p_x^2 - \tfrac{1}{2}y p_y^2 + \tfrac{1}{2}x p_z^2 + \tfrac{1}{2}x p_x p_y - \tfrac{1}{2}z p_y p_z + \alpha_{-1}V_2^{(-1)} + \alpha_3 V_2^{(3)} + \alpha_4 V_2^{(4)} \\ h_3 &= \tfrac{1}{8}z^2 p_y^2 + \tfrac{1}{2}(y + \tfrac{1}{4}x^2)p_z^2 - \tfrac{1}{2}z p_x p_z - \tfrac{1}{4}x z p_y p_z + \alpha_{-1}V_3^{(-1)} + \alpha_3 V_3^{(3)} + \alpha_4 V_3^{(4)} \end{aligned} \tag{4.3.33}$$

with $V_k^{(s)} = V_k^{(s)}(x, y, z)$ of the form

$$V_1^{(-1)} = \frac{4}{z^2}, \quad V_2^{(-1)} = \frac{4x}{z^2}, \quad V_3^{(-1)} = \frac{x^2 + 4y}{z^2}$$

$$V_1^{(3)} = x, \quad V_2^{(3)} = \left(y + \tfrac{1}{4}x^2\right), \quad V_3^{(3)} = -\tfrac{1}{4}z^2 \tag{4.3.34}$$

$$V_1^{(4)} = y - \tfrac{3}{4}x^2, \quad V_2^{(4)} = -\left(xy + \tfrac{1}{4}x^3 + \tfrac{1}{4}z^2\right), \quad V_3^{(4)} = \tfrac{1}{4}xz^2$$

After the transformation to flat coordinates we have $P = p_y$, and $Y = \frac{\partial}{\partial y}$ so the additional constants of motion h_{n+i} of h_1 are:

$$h_4 = \{h_2, P\} = -\tfrac{1}{2}p_y^2 + \alpha_3 - \alpha_4 x, \quad h_5 = \{h_3, P\} = \tfrac{1}{2}p_z^2 + \frac{4\alpha_{-1}}{z^2} \tag{4.3.35}$$

4.4 Stäckel Transform for Separable Systems

In Sect. 4.1 we presented the theory of the Stäckel transform allowing for the construction of new Liouville integrable system from a given Liouville integrable system. Here we apply that theory to Stäckel systems considered in the previous section. We show that all Stäckel systems form any γ-class (4.2.16) are Stäckel equivalent with appropriate systems from Benenti class. Even more, systems from Benenti class are also Stäckel related. Finally, using constructed Stäckel transforms we analyze the geometric structure of Hamiltonians from the arbitrary γ-class. That structure will be important for the process of separable quantization of the considered systems.

4.4.1 Reciprocal Equivalence with Benenti Class

We will now turn to the fundamental property of all Stäckel systems generated by separation relations of the form (4.2.16). As we will show in this subsection, any class of Stäckel systems with the particular choice of $\gamma = (\gamma_1, \ldots, \gamma_{n-1}, 0)$ is related to Benenti class with $\gamma = (n-1, \ldots, 1, 0)$ by a single Stäckel transform and in such a way those solutions of respective systems from both classes are related by a reciprocal transform. As we mentioned above, Hamiltonians (4.3.4) do not depend on any additional parameters α_i so, in order to perform a Stäckel transform, we have to embed it into a parameter-dependent system. Of course, there are infinitely many ways of embedding of our Stäckel system into an n-parameter system but the choice below is natural in the sense that the corresponding Stäckel transform maps a Stäckel system into a new Stäckel system.

Consider n Hamiltonians $h_i = h_i(\lambda, \mu, \alpha)$ from the Benenti class, defined by the separation curve

$$\boxed{R^{-1}(\lambda)\sum_{j=1}^{n}\alpha_j\lambda^{\gamma_j} + \sum_{j=1}^{n}h_j\lambda^{n-j} = f(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right]} \tag{4.4.1}$$

where we do admit the possibility that some or all of γ_k coincide with some $n-k$ and where $R(\lambda)$ is an arbitrary meromorphic function of one variable so that

$$R(\lambda)=\prod_{j=1}^{k_1}(\lambda-\beta_j)\prod_{j=1}^{k_2}(\lambda-\beta'_j)^{-1}$$

for some (complex in general) constants $\beta_1,\ldots,\beta_{k_1}$ and $\beta'_1,\ldots,\beta'_{k_2}$. This function can be generalized to a matrix function i.e. we define, for any $n\times n$ matrix A (λ-dependent or not)

$$R(A)=\prod_{r=1}^{k_1}(A-\beta_r I_n)\prod_{r=1}^{k_2}(A-\beta'_r I_n)^{-1} \tag{4.4.2}$$

(as all the terms in (4.4.2) commute so that there is no ordering problem here). The relations (4.4.1) written in a matrix form are as follows

$$R^{-1}(\Lambda)S_\gamma\alpha+Sh=U$$

where S_γ and S are two Stäckel matrices given by (4.3.2) and (4.2.2) respectively (so that $\left(S_\gamma\right)_{ij}=(\lambda^i)^{\gamma_j}$ and $(S)_{ij}=(\lambda^i)^{n-j}$), $h=(h_1,\ldots,h_n)^T$ is the column vector consisting of Hamiltonians h_i, $\alpha=(\alpha_1,\ldots,\alpha_n)^T$, U is the column vector given by $U_i=f(\lambda^i)\left[\mu_i^2+\kappa(\lambda^i)\right]$ and where $\Lambda=\mathrm{diag}(\lambda^1,\ldots,\lambda^n)$.

Solving (4.4.1) with respect to h we obtain

$$h=S^{-1}U-S^{-1}R^{-1}(\Lambda)S_\gamma\alpha=S^{-1}U-S^{-1}R^{-1}(\Lambda)SS^{-1}S_\gamma\alpha. \tag{4.4.3}$$

In the notation as above

$$S^{-1}R(\Lambda)S=R(F),$$

where F is given by (4.3.20). We show (4.4.1) for $R(\lambda)=\lambda-\beta$ as

$$S^{-1}\left(\Lambda-\beta I\right)S=S^{-1}\Lambda S-\beta I=F-\beta I=R(F).$$

The general statement follows by developing the above derivation. Thus, introducing the shorthand notation

$$W_\gamma^{-1}:=S^{-1}S_\gamma$$

formula (4.4.3) can be written as

$$\boxed{h=H-R^{-1}(F)W_\gamma^{-1}\alpha} \tag{4.4.4}$$

where $H=S^{-1}U$ is the part of h that is independent of parameters α_i (cf. (4.3.4)).

Let us shortly analyze the structure of the matrix $W_\gamma^{-1} = S^{-1}S_\gamma$. Assume that $\sigma(\lambda)$ in (4.3.2) is a polynomial of the form $\sigma(\lambda) = \sum_{i=1}^n \alpha_i \lambda^{\gamma_i}$. Then, as it follows from (4.3.2) and from the definition of potentials $V^{(k)}$ (4.3.22), we have

$$V^{(\sigma)} = \sum_{i=1}^n \alpha_i V^{(\gamma_i)}$$

where $V^{(\gamma_i)} = (V_1^{(\gamma_i)}, \ldots, V_n^{(\gamma_i)})^T$ are basic potentials from Benenti class. On the other hand, the formula (4.3.2) can be written as

$$SV^{(\sigma)} = S_\gamma \alpha$$

so that $V^{(\sigma)} = S^{-1}S_\gamma \alpha = W_\gamma^{-1}\alpha$ which implies that

$$\boxed{\left(W_\gamma^{-1}\right)_{ij} = V_i^{(\gamma_j)},} \tag{4.4.5}$$

where $V^{(\gamma_j)} = F^{\gamma_j} V^{(0)}$ are the basic potential from Benenti class. Therefore, the formula (4.4.4) can be written in the form

$$h_i = H_i - \sum_{j,k=1}^n \left(R^{-1}(F)\right)_{ij} V_j^{(\gamma_k)} \alpha_k, \quad i = 1, \ldots, n.$$

Let us now perform an n-parameter Stäckel transform of the system given by the separation curve (4.4.1). Since the number of parameters α_i and the number of Hamiltonians h_i is equal to n, the Stäckel transform consists only of part (4.1.12) and is generated by the relation $h = \tilde{\alpha}$ (which implies $\tilde{h} = \alpha$) in the vector notation. The n-parameter Stäckel transform generated by $h = \tilde{\alpha}$ transforms the set of n Hamiltonians h defined by (4.4.1) into the following set of Hamiltonians

$$\boxed{\tilde{h} = W_\gamma R(F) H - W_\gamma R(F)\tilde{\alpha}} \tag{4.4.6}$$

(where $\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n)^T$ and similarly $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)^T$) which constitute a new Stäckel system with the separation curve of the form

$$\boxed{R(\lambda)\sum_{j=1}^n \tilde{\alpha}_j \lambda^{n-j} + \sum_{j=1}^n \tilde{h}_j \lambda^{\gamma_j} = \tilde{f}(\lambda)\left[\mu^2 + \kappa(\lambda)\right],} \tag{4.4.7}$$

where $\tilde{f}(\lambda) := R(\lambda) f(\lambda)$. Moreover, the reciprocal transformation

$$\boxed{d\tilde{t} = \left[R(F^T) W_\gamma^T\right]^{-1} dt} \tag{4.4.8}$$

transforms n-time solutions $\xi = \xi(t_1, \ldots, t_n, \xi_0)$ of the system (4.4.1) into n-time solutions $\tilde{\xi} = \tilde{\xi}(\tilde{t}_1, \ldots, \tilde{t}_n, \xi_0)$ of the system (4.4.7). Note that in spite of the fact that we introduced both systems in the (λ, μ)-variables the matrix formulas (4.4.6) and (4.4.8) are not of tensor type and that they are coordinate-free. They can be therefore freely applied in any coordinate system on M, which will be used in the examples further on.

In order to verify the formulas (4.4.6)–(4.4.8) let us multiply the curve (4.4.1) by $R(\lambda)$ obtaining

$$\sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j} + R(\lambda) \sum_{j=1}^{n} h_j \lambda^{n-j} = R(\lambda) f(\lambda) \left[\mu^2 + \kappa(\lambda)\right]$$

which after the Stäckel transform $h = \tilde{\alpha}$ (so that $\tilde{h} = \alpha$) obviously attains the form (4.4.7). Let us therefore show the matrix form are

$$R(\Lambda) S \tilde{\alpha} + S_\gamma \tilde{h} = R(\Lambda) U.$$

Solving this with respect to $\tilde{h}$ we get

$$\begin{aligned}
\tilde{h} &= S_\gamma^{-1} R(\Lambda) U - S_\gamma^{-1} R(\Lambda) S \tilde{\alpha} = S_\gamma^{-1} R(\Lambda) S H - S_\gamma^{-1} R(\Lambda) S \tilde{\alpha} \\
&= \left(S_\gamma^{-1} R(\Lambda) S_\gamma\right) \left(S^{-1} S_\gamma\right)^{-1} H - \left(S_\gamma^{-1} R(\Lambda) S_\gamma\right) \left(S^{-1} S_\gamma\right)^{-1} \tilde{\alpha} \\
&= R(F_\gamma) W_\gamma H - R(F_\gamma) W_\gamma \tilde{\alpha}
\end{aligned}$$

so the only remaining thing is to show that $R(F_\gamma) W_\gamma = W_\gamma R(F)$. We show it for $R(\lambda) = \lambda - \beta$ as a general statement can be proved in a similar way

$$\begin{aligned}
(F_\gamma - \beta) W_\gamma &= S_\gamma^{-1} (\Lambda - \beta I) S_\gamma S_\gamma^{-1} S = S_\gamma^{-1} (\Lambda - \beta I) S \\
&= S_\gamma^{-1} S S^{-1} (\Lambda - \beta I) S = W_\gamma (F - \beta).
\end{aligned}$$

Finally, the formula (4.4.8) is obtained by inserting (4.4.4) into (4.1.33)

$$d\tilde{t} = -\left(\frac{\partial h}{\partial \alpha}\right)^T dt = \left[R(F^T) W_\gamma^T\right]^{-1} dt,$$

where we use the equality $R(A)^T = R(A^T)$.

Let us also present an alternative way of proving the formula (4.4.8), directly involving solutions of (4.4.1) and (4.4.7). It follows from (4.2.22) and from the above considerations that the multi-time solutions of the systems (4.4.1) and (4.4.7) on any common level surface

$$M_{\alpha,\tilde{\alpha}} = \left\{\xi \in M : h_i(\xi, \alpha_1, \ldots, \alpha_n) = \tilde{\alpha}_i, \tilde{h}_i(\xi, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) = \alpha_i \; i = 1, \ldots n\right\}$$

take the form

$$dt = S^T \frac{d\lambda}{\sqrt{2f(\lambda)P(\lambda, \tilde{\alpha}, \alpha)}}, \qquad d\tilde{t} = S_\gamma^T \frac{d\lambda}{\sqrt{2R(\lambda)f(\lambda)\tilde{P}(\lambda, \alpha, \tilde{\alpha})}}, \tag{4.4.9}$$

where

$$P(\lambda, \tilde{\alpha}, \alpha) = -f(\lambda)\kappa(\lambda) + \sum_{j=1}^{n} \tilde{\alpha}_j \lambda^{n-j} + R^{-1}(\lambda) \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j},$$

$$\tilde{P}(\lambda, \alpha, \tilde{\alpha}) = -\tilde{f}(\lambda)\kappa(\lambda) + R(\lambda) \sum_{j=1}^{n} \tilde{\alpha}_j \lambda^{n-j} + \sum_{j=1}^{n} \alpha_j \lambda^{\gamma_j}.$$

Now, we can see that $\tilde{P}(\lambda, \alpha, \tilde{\alpha}) = R(\lambda)P(\lambda, \tilde{\alpha}, \alpha)$ as $R(\lambda)f(\lambda) = \tilde{f}(\lambda)$, so that, by (4.4.9) and by the fact that $R(\Lambda)$ is symmetric,

$$d\tilde{t} = S_\gamma^T \frac{d\lambda}{\sqrt{2R(\lambda)f(\lambda)\tilde{P}(\lambda, \alpha, \tilde{\alpha})}} = S_\gamma^T \frac{d\lambda}{\sqrt{2R^2(\lambda)f(\lambda)P(\lambda, \tilde{\alpha}, \alpha)}}$$

$$= S_\gamma^T R^{-1}(\Lambda) \frac{d\lambda}{\sqrt{2f(\lambda)P(\lambda, \tilde{\alpha}, \alpha)}} = S_\gamma^T R^{-1}(\Lambda) \left(S^T\right)^{-1} dt = \left(S^{-1} R^{-1}(\Lambda) S_\gamma\right)^T dt$$

$$= \left(S^{-1} S_\gamma S_\gamma^{-1} R^{-1}(\Lambda) S_\gamma\right)^T dt = (W_\gamma^{-1} R^{-1}(F_\gamma))^T dt$$

and hence

$$d\tilde{t} = R^{-1}(F_\gamma^T) \left(W_\gamma^T\right)^{-1} dt = \left[W_\gamma^T R(F_\gamma^T)\right]^{-1} dt = \left[R(F^T) W_\gamma^T\right]^{-1} dt$$

$$= \left[W_\gamma^T R(F^T)\right]^{-1} dt,$$

which is what we wanted to prove.

Our formulas contain two special cases: when $\gamma = (n-1, \ldots, 1, 0)$ and when $R = 1$. In the first case we relate systems belonging to the same Benenti class and differ by f and σ. The matrix $W_\gamma = I_n$ so that the formulas (4.4.4) and (4.4.6)

become

$$\boxed{\begin{aligned} h &= H - R^{-1}(F)\alpha, \\ \tilde{h} &= R(F)H - R(F)\tilde{\alpha}, \end{aligned}} \tag{4.4.10}$$

while (4.4.8) attains the form

$$\boxed{d\tilde{t} = R^{-1}(F^T)dt.} \tag{4.4.11}$$

In the second case ($R = 1$) we relate systems from different classes, i.e. $(\gamma_1, \ldots, \gamma_n)$ and Benenti class $(n-1, \ldots, 1, 0)$ respectively, which share the same f and σ. The formulas (4.4.4) and (4.4.6) become

$$\boxed{\begin{aligned} h &= H - W_\gamma^{-1}\alpha, \\ \tilde{h} &= W_\gamma H - W_\gamma\tilde{\alpha}, \end{aligned}} \tag{4.4.12}$$

while the formula (4.4.8) attains the form

$$\boxed{d\tilde{t} = \left[W_\gamma^T\right]^{-1} dt.} \tag{4.4.13}$$

Thus, the general transformation between the systems (4.4.1) and (4.4.7) can be regarded as a composition of two transformations: a map between two Stäckel systems from the Beneti class with different f (i.e. metrics) and the transformation between two Stäckel systems from different classes, sharing the same f.

4.4.2 Elementary Stäckel Transforms in Benenti Class

In the previous subsection we presented the construction of equivalent separable systems, in the sense that their sets of Hamiltonians were related by the Stäckel transform (4.4.6) and respective solutions are related by an appropriate reciprocal transformation (4.4.8) (coordinate dependent reparametrization of evolution parameters). If we skip the demand of reciprocal relations between solutions and restrict only to Stäckel related systems, then on the level of Stäckel transforms we can formulate the following statement. Any separable system generated by a separation curve of the form

$$\sum_{j=1}^{n} \bar{H}_j \lambda^{\gamma_j} = \bar{f}(\lambda)\mu^2 + \bar{\sigma}(\lambda) \tag{4.4.14}$$

is Stäckel-related to a Benenti system generated by a separation curve

$$\sum_{j=1}^{n} H_j \lambda^{n-j} = f(\lambda)\mu^2 + \sigma(\lambda) \tag{4.4.15}$$

under the condition

$$\bar{f}(\lambda)\sigma(\lambda) = \bar{\sigma}(\lambda) f(\lambda) \Longrightarrow \sigma(\lambda) = f(\lambda)\kappa(\lambda), \quad \bar{\sigma}(\lambda) = \bar{f}(\lambda)\kappa(\lambda), \tag{4.4.16}$$

where the single Stäckel transform is of the form

$$\boxed{\bar{H} = W_\gamma R(F) H} \tag{4.4.17}$$

with $R(\lambda) = \frac{\bar{f}(\lambda)}{f(\lambda)}$. In particular, any two geodesic Stäckel systems (i.e. with $\sigma = \bar{\sigma} = 0$) are connected by the Stäckel transform (4.4.17). The statement follows directly from (4.4.4) and (4.4.6) for the particular case $\alpha = \bar{\alpha} = 0$ (compare with (4.1.17)). In this sense our Stäckel transform transforms the parameter-free Liouville-integrable system (4.4.15) into another parameter-free Liouville integrable system (4.4.14), although their solutions are not related by any reciprocal transformation.

The condition (4.4.16) splits all Stäckel systems generated by (4.3.10) into equivalence classes since it is an equivalence relation. Indeed, if $\frac{f_2(\lambda)}{f_1(\lambda)} = \frac{\sigma_2(\lambda)}{\sigma_1(\lambda)} = R_1(\lambda)$ and $\frac{f_3(\lambda)}{f_2(\lambda)} = \frac{\sigma_3(\lambda)}{\sigma_2(\lambda)} = R_2(\lambda)$ then $\frac{f_3(\lambda)}{f_1(\lambda)} = \frac{\sigma_3(\lambda)}{\sigma_1(\lambda)} = R_2(\lambda)R_1(\lambda)$ so the relation is transitive. Further, if $\frac{f_2(\lambda)}{f_1(\lambda)} = \frac{\sigma_2(\lambda)}{\sigma_1(\lambda)} = R(\lambda)$ then $\frac{f_1(\lambda)}{f_2(\lambda)} = \frac{\sigma_1(\lambda)}{\sigma_2(\lambda)} = \frac{1}{R(\lambda)}$ so the relation is reflexive. Finally, $\frac{f_1(\lambda)}{f_1(\lambda)} = \frac{\sigma_1(\lambda)}{\sigma_1(\lambda)} = 1$ so the relation is symmetric.

The general Stäckel transform (4.4.17) is a composition of two elementary Stäckel transforms. The first one is the Stäckel transform inside Benenti class, relating two systems with different $f(\lambda)$

$$\begin{gathered} H_1\lambda^{n-1} + H_2\lambda^{n-2} + \cdots + H_n = f(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right] \\ \Big\downarrow R(F),\ W_\gamma = I_n \\ \bar{H}_1\lambda^{n-1} + \bar{H}_2\lambda^{n-2} + \cdots + \bar{H}_n = \bar{f}(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right], \end{gathered} \tag{4.4.18}$$

where

$$\bar{H} = R(F)H, \quad R(F) = \bar{f}(F) f^{-1}(F). \tag{4.4.19}$$

The second one ($R = 1$) relates systems from Benenti class with systems from $(\gamma_1, \ldots, \gamma_n)$ class, which share the same $f(\lambda)$

$$\begin{gathered} H_1\lambda^{n-1} + H_2\lambda^{n-2} + \cdots + H_n = f(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right] \\ \Big\downarrow W_\gamma,\ R(F) = I_n \\ \bar{H}_1\lambda^{\gamma_1} + \bar{H}_2\lambda^{\gamma_2} + \cdots + \bar{H}_n = f(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right], \end{gathered} \tag{4.4.20}$$

where

$$\bar{H} = W_\gamma H. \tag{4.4.21}$$

Let us analyze the structure of the matrix $W_\gamma = S_\gamma^{-1} S$ in a similar fashion as we analyzed the structure of the matrix $W_\gamma^{-1} = S^{-1} S_\gamma$ in the previous subsection. Assume that $\sigma(\lambda)$ in (4.3.11) is a polynomial of the form $\sigma(\lambda) = \sum_{i=1}^{n} \bar{\alpha}_i \lambda^{n-i}$. Then, as it follows from (4.3.11) and from the definition of potentials $\bar{V}^{(k)}$, we have

$$\bar{V}^{(\sigma)} = \sum_{i=1}^{n} \alpha_i \bar{V}^{(n-i)},$$

where $\bar{V}^{(n-i)} = (\bar{V}_1^{(n-i)}, \ldots, \bar{V}_n^{(n-i)})^T$ are basic potentials from γ-class (4.3.10). On the other hand, the formula (4.3.11) can now be written as

$$S_\gamma^{(\sigma)} \bar{V} = S\bar{\alpha},$$

so that $\bar{V}^{(\sigma)} = S_\gamma^{-1} S\bar{\alpha} = W_\gamma \bar{\alpha}$, which implies that

$$\left(W_\gamma\right)_{ij} = \bar{V}_i^{(n-j)}, \tag{4.4.22}$$

where $\bar{V}^{(n-j)} = F_\gamma^{n-j} V^{(0)}$ in accordance with (4.3.13).

4.4.3 The Structure of Stäckel Hamiltonians from γ-Classes

The separation curve for γ-class of Stäckel systems for $\gamma = (\gamma_1, \ldots, \gamma_{n-1}, 0)$ is of the form

$$\bar{H}_1\lambda^{\gamma_1} + \bar{H}_2\lambda^{\gamma_2} + \cdots + \bar{H}_n = f(\lambda)\left[\tfrac{1}{2}\mu^2 + \kappa(\lambda)\right] = \tfrac{1}{2} f(\lambda)\mu^2 + \sigma(\lambda), \tag{4.4.23}$$

with Hamiltonians

$$\bar{H}_r = \tfrac{1}{2}\sum\nolimits_i \bar{A}_r^{ii}\mu_i^2 + \bar{V}_r^{(\sigma)}, \quad r = 1, \ldots, n. \tag{4.4.24}$$

According to our previous considerations, tensors $\bar{A}_r$ and separable potentials $\bar{V}^{(\sigma)}$ can be calculated either directly from the separation curve (4.4.23) with the help of (4.3.7) and (4.3.13) or by the Stäckel transform (4.4.20), (4.4.21) from Beneti class.

For the sake of the further quantum separability theory, it is important to reveal in a greater detail the structure of tensors $\bar{A}_r$. In order to do it in this subsection we adopt the alternative derivation of Stäckel transform (4.4.20), developed in [30]. Let us start with the separation curve (4.4.23) with the following notation

$$\boxed{\tilde{H}_1\lambda^{(n+k)-1} + \tilde{H}_2\lambda^{(n+k)-2} + \ldots + \tilde{H}_{n+k} = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda),} \tag{4.4.25}$$

with missing k monomials $\tilde{H}_{m_1}\lambda^{(n+k)-m_1}$, $\tilde{H}_{m_2}\lambda^{(n+k)-m_2}, \ldots, \tilde{H}_{m_k}\lambda^{(n+k)-m_k}$, $1 < m_1 < \ldots < m_k < n+k-1$, $k \in \mathbb{N}$, i.e. $\tilde{H}_{m_1} = \tilde{H}_{m_2} = \ldots = \tilde{H}_{m_k} = 0$, and the separation curve for Benenti systems with the same right hand side

$$H_1\lambda^{n-1} + H_2\lambda^{n-2} + \ldots + H_n = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda). \tag{4.4.26}$$

As for the basic potentials of Benenti class

$$V_1^{(n+k)}\lambda^{n-1} + \ldots + V_n^{(n+k)} = \lambda^{n+k},$$

substituting this relation to (4.4.25) for $\lambda^{(n+k)-1}, \ldots, \lambda^n$ and comparing the obtained relation with (4.4.26) we get

$$H_r = \tilde{H}_{r+k} + V_r^{(n+k-1)}\tilde{H}_1 + V_r^{(n+k-2)}\tilde{H}_2 + \ldots + V_r^{(n)}\tilde{H}_k, \quad r = 1, \ldots, n, \tag{4.4.27}$$

where $\tilde{H}_{m_1} = \ldots = \tilde{H}_{m_k} = 0$ and $V_r^{(m)}$ are appropriate basic potentials. Notice that fixing the numbers $m_1, \ldots, m_k$ we chose in a unique way the numbers $\gamma_1, \ldots, \gamma_n$, i.e. a γ-class.

The inverse formula to the (4.4.27) one, is given by a following determinant form

$$\boxed{\tilde{H}_r = \frac{\begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{m_1-k} & \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{m_k-k} & \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}}{\begin{vmatrix} \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}},} \tag{4.4.28}$$

where ρ_i are Viète polynomials (4.3.18), where $\rho_0 = 1$, $\rho_r = 0$ for $r > n$ and $r < 0$. The constants m_i are those for which the corresponding monomials λ^{n+k-m_i} are missing in the left hand side of (4.4.25).

In order to verify the formula (4.4.28), first we select from (4.4.27) k equations containing $\tilde{H}_{m_1}, \ldots, \tilde{H}_{m_k}$

$$\begin{aligned} H_{m_1-k} &= V_{m_1-k}^{(n+k-1)}\tilde{H}_1 + \ldots + V_{m_1-k}^{(n)}\tilde{H}_k, \\ &\vdots \\ H_{m_k-k} &= V_{m_k-k}^{(n+k-1)}\tilde{H}_1 + \ldots + V_{m_k-k}^{(n)}\tilde{H}_k. \end{aligned}$$

The solution with respect to $\tilde{H}_i$, $i = 1, \ldots, k$ is given by a determinant form

$$\tilde{H}_i = \frac{D_i}{D}, \quad i = 1, \ldots, n,$$

where

$$D = \begin{vmatrix} V_{m_1-k}^{(n+k-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots \\ V_{m_k-k}^{(n+k-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}$$

and

$$D_i = (-1)^{i+1} \begin{vmatrix} H_{m_1-k} & V_{m_1-k}^{(n+k-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ H_{m_k-k} & V_{m_k-k}^{(n+k-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}$$

with the column $(V_{m_1-k}^{(n+k-i)}, \ldots, V_{m_k-k}^{(n+k-i)})^T$ missing. Substituting this result to (4.4.27) we get

$$\tilde{H}_r = \frac{H_{r-k}D - V_{r-k}^{(n+k-1)}D_1 - \ldots - V_{r-k}^{(n)}D_k}{D}$$

$$= \frac{\begin{vmatrix} H_{r-k} & V_{r-k}^{(n+k-1)} & \cdots & V_{r-k}^{(n)} \\ H_{m_1-k} & V_{m_1-k}^{(n+k-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ H_{m_k-k} & V_{m_k-k}^{(n+k-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}}{\begin{vmatrix} V_{m_1-k}^{(n+k-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots \\ V_{m_k-k}^{(n+k-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}}$$

$$= \frac{\begin{vmatrix} H_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ H_{m_1-k} & \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ H_{m_k-k} & \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}}{\begin{vmatrix} \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}}.$$

The last step is valid due to the fact that $V_i^{(n)} = -\rho_i$, the form of the recursion formula for basic potentials (4.3.23) and the properties of determinants. It allows us to replace the arbitrary potential $V_s^{(n+k-i)}$ in determinants by the $V_{s+k-i}^{(n)} = -\rho_{s+k-i}$ one. For each recursive step we have

$$\begin{vmatrix} \cdots & V_{m_1-k}^{(n+k-i)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V_{m_k-k}^{(n+k-i)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} \cdots & V_{m_1-k+1}^{(n+k-i-1)} - \rho_{m_1-k} V_1^{(n+k-i-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V_{m_k-k+1}^{(n+k-i-1)} - \rho_{m_k-k} V_1^{(n+k-i-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} \cdots & V_{m_1-k+1}^{(n+k-i-1)} & \cdots & V_{m_1-k}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & V_{m_k-k+1}^{(n+k-i-1)} & \cdots & V_{m_k-k}^{(n)} \end{vmatrix}.$$

The formula (4.4.28) applies separately to the geodesic and the potential parts. Moreover it is valid in any local coordinates. So first, let us look at n geodesic Hamiltonians $\tilde{T}_r,\ r = 1, \ldots, n+k,\ r \neq m_1, \ldots, m_k$. Introducing the abbreviation

$$\varphi(m_1, \ldots, m_k) = \begin{vmatrix} \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots \\ \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}, \tag{4.4.29}$$

one finds

$$\tilde{T}_r = \frac{1}{\varphi} \begin{vmatrix} T_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ T_{m_1-k} & \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ T_{m_k-k} & \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}.$$

Using the known relations for Killing tensors (4.3.19), the convention $K_r = 0$ for $r > n$ and $r < 1$ and the property of determinants we get (in a coordinate free form)

$$\tilde{T}_r = \frac{1}{2} p^T \begin{vmatrix} K_{r-k} & \rho_{r-1} I & \cdots & \rho_{r-k} I \\ K_{m_1-k} & \rho_{m_1-1} I & \cdots & \rho_{m_1-k} I \\ \cdots & \cdots & \cdots & \cdots \\ K_{m_k-k} & \rho_{m_k-1} I & \cdots & \rho_{m_k-k} I \end{vmatrix} \frac{1}{\varphi} G p$$

$$= \frac{1}{2} p^T \begin{vmatrix} K_{r-k} & K_r & \cdots & K_{r-k+1} \\ K_{m_1-k} & K_{m_1} & \cdots & K_{m_1-k+1} \\ \cdots & \cdots & \cdots & \cdots \\ K_{m_k-k} & K_{m_k} & \cdots & K_{m_k-k+1} \end{vmatrix} \frac{1}{\varphi} G p$$

$$= (-1)^k \frac{1}{2} p^T \begin{vmatrix} K_r & \cdots & K_{r-k+1} & K_{r-k} \\ K_{m_1} & \cdots & K_{m_1-k+1} & K_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ K_{m_k} & \cdots & K_{m_k-k+1} & K_{m_k-k} \end{vmatrix} \frac{1}{\varphi} G p \tag{4.4.30}$$

$$= \frac{1}{2} p^T \frac{1}{\varphi} M_r G p,$$

where

$$M_r = (-1)^k \begin{vmatrix} K_r & \cdots & K_{r-k+1} & K_{r-k} \\ K_{m_1} & \cdots & K_{m_1-k+1} & K_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ K_{m_k} & \cdots & K_{m_k-k+1} & K_{m_k-k} \end{vmatrix}, \quad r = 1, \ldots, n+k, \tag{4.4.31}$$

$I \equiv I_n$ and $K_m, \rho_m I$ in determinant calculations are treated as symbols not matrices. Again the formula (4.4.30) is valid in any local coordinate frame. Thus, the structure of tensors $\tilde{A}_r$ in Hamiltonians (4.4.25) is as follows

$$\tilde{A}_r = \frac{1}{\varphi(m_1, \ldots, m_k)} M_r G, \tag{4.4.32}$$

where G is a metric tensor (4.3.16) from Benenti class, M_r tensors are polynomial functions (4.4.31) of Killing tensors (4.3.17) and function φ is determined by (4.4.29). From the construction of M_r tensors, it follows that in the separation coordinate

$$\frac{\partial}{\partial \lambda_i} (M_r)^i_i = 0, \quad i = 1, \ldots, n. \tag{4.4.33}$$

The structure of $\tilde{A}_r$ tensors, revealed above, will be crucial for the quantum separability theory developed in Sect. 8.2. Additionally, the basic deformed potentials can be calculated form the general formula

$$\tilde{V}_r = \frac{1}{\varphi(m_1,\ldots,m_k)} \begin{vmatrix} V_{r-k} & \rho_{r-1} & \cdots & \rho_{r-k} \\ V_{m_1-k} & \rho_{m_1-1} & \cdots & \rho_{m_1-k} \\ \cdots & \cdots & \cdots & \cdots \\ V_{m_k-k} & \rho_{m_k-1} & \cdots & \rho_{m_k-k} \end{vmatrix}.$$

Let us consider in detail the simplest nontrivial case $k = 1$, $1 < m_1 < n$, where Hamiltonians $\bar{H} = \bar{T} + \bar{V}$ and $\tilde{H} = \tilde{T} + \tilde{V}$ from (4.4.23) and (4.4.25) are related by

$$\begin{aligned} \bar{H}_r &= \tilde{H}_r, \quad r = 1,\ldots,m_1-1, \\ \bar{H}_r &= \tilde{H}_{r+1}, \quad r = m_1,\ldots,n \end{aligned} \tag{4.4.34}$$

and will be expressed by respective elements of Benenti Hamiltonians H_r (4.4.26), (4.3.15). The formula (4.4.28) applies separately to the geodesic and the potential parts

$$\tilde{T}_r = T_{r-1} - \frac{\rho_{r-1}}{\rho_{m_1-1}} T_{m_1-1}, \tag{4.4.35a}$$

$$\tilde{V}_r = V_{r-1} - \frac{\rho_{r-1}}{\rho_{m_1-1}} V_{m_1-1}. \tag{4.4.35b}$$

First let us consider the geodesic parts, which according to (4.4.30) take the form

$$\tilde{T}_r = \tfrac{1}{2} p^T \frac{1}{\rho_{m_1-1}} M_r G p, \quad M_r = K_{m_1} K_{r-1} - K_{m_1-1} K_r,$$

where $\varphi(m_1) = \rho_{m_1-1}$ and $\bar{T}_r$ are related with $\tilde{T}_r$ by relation (4.4.34). Now, let us analyze the basic potentials $\tilde{V}_r^{(m)}$ related with $\sigma(\lambda) = \lambda^m$, $m \in \mathbb{Z}$ in (4.4.25). From (4.3.2)–(4.3.23) and (4.4.35a) we have

$$\tilde{V}_r^{(m)} = \delta_{r-1,n-m}, \quad m < n+1, \quad m \neq n+1-m_1$$

and first nontrivial potentials

$$\tilde{V}_r^{(n+1-m_1)} = \delta_{r,m_1} - \frac{\rho_{r-1}}{\rho_{m_1-1}}, \qquad \tilde{V}_r^{(n+1)} = -\rho_r + \frac{\rho_{r-1}\rho_{m_1}}{\rho_{m_1-1}}.$$

Again $\bar{V}_r$ are related with $\tilde{V}_r$ by relations (4.4.34), so for $m < n+1, \quad m \neq n+1-m_1$

$$\bar{V}_r^{(m)} = \delta_{r-1,n-m}, \quad r = 1, \ldots, m_1 - 1,$$

$$\bar{V}_r^{(m)} = \delta_{r,n-m}, \quad r = m_1, \ldots, n,$$

then

$$\bar{V}_r^{(n+1-m_1)} = -\frac{\rho_{r-1}}{\rho_{m_1-1}}, \quad r = 1, \ldots, m_1 - 1,$$

$$\bar{V}_r^{(n+1-m_1)} = -\frac{\rho_r}{\rho_{m_1-1}}, \quad r = m_1, \ldots, n,$$

and

$$\bar{V}_r^{(n+1)} = -\rho_r + \frac{\rho_{r-1}\rho_{m_1}}{\rho_{m_1-1}}, \quad r = 1, \ldots, m_1 - 1,$$

$$\bar{V}_r^{(n+1)} = -\rho_{r+1} + \frac{\rho_r \rho_{m_1}}{\rho_{m_1-1}}, \quad r = m_1, \ldots, n,$$

respectively. The recursion matrix (4.3.1), expressed by vectors of potentials $\bar{V}^{(m)}$ takes the form

$$F = S^{-1}\Lambda S = (\bar{V}^{(n+1)}, \ldots, \bar{V}^{(n+3-m_1)}, \bar{V}^{(n+1-m_1)}, \ldots, \bar{V}^{(1)}).$$

We illustrate the general case $k > 1$ by one simple example.

Example 4.16 Consider the case of $\gamma = (4, 1, 0)$, i.e. $n = 3$, $k = 2$, $m_1 = 2$, $n_2 = 3$ and thus the separation curve in the form

$$\bar{H}_1\lambda^4 + \bar{H}_2\lambda + \bar{H}_3 = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda).$$

The related Hamiltonians from Beneti class are given by the separation curve

$$H_1\lambda^2 + H_2\lambda + H_3 = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda).$$

According to (4.4.28) and (4.4.29)

$$\varphi = \begin{vmatrix} \rho_1 & 1 \\ \rho_2 & \rho_1 \end{vmatrix} = \rho_1^2 - \rho_2,$$

$$\bar{H}_1 = \tilde{H}_1 = \frac{1}{\varphi}\begin{vmatrix} 0 & 1 & 0 \\ 0 & \rho_1 & 1 \\ H_1 & \rho_2 & \rho_1 \end{vmatrix} = \frac{1}{\rho_1^2 - \rho_2}H_1,$$

$$\bar{H}_2 = \tilde{H}_4 = \frac{1}{\varphi}\begin{vmatrix} H_2 & \rho_3 & \rho_2 \\ 0 & \rho_1 & 1 \\ H_1 & \rho_2 & \rho_1 \end{vmatrix} = H_2 + \frac{\rho_3 - \rho_1\rho_2}{\rho_1^2 - \rho_2} H_1,$$

$$\bar{H}_3 = \tilde{H}_5 = \frac{1}{\varphi}\begin{vmatrix} H_3 & 0 & \rho_3 \\ 0 & \rho_1 & 1 \\ H_1 & \rho_2 & \rho_1 \end{vmatrix} = H_3 - \frac{\rho_1\rho_3}{\rho_1^2 - \rho_2} H_1.$$

Observe that Hamiltonians $\bar{H}_i$ can be alternatively constructed according to (4.4.21) and (4.4.5)

$$\bar{H} = W_\gamma^{-1} H$$

where

$$W_\gamma = \begin{pmatrix} V_1^{(4)} & V_1^{(1)} & V_1^{(0)} \\ V_2^{(4)} & V_2^{(1)} & V_2^{(0)} \\ V_3^{(4)} & V_3^{(1)} & V_3^{(0)} \end{pmatrix} = \begin{pmatrix} \rho_1^2 - \rho_2 & 0 & 0 \\ \rho_1\rho_2 - \rho_3 & 1 & 0 \\ \rho_1\rho_3 & 0 & 1 \end{pmatrix}.$$

According to (4.4.30), geodesic parts of Hamiltonians $\bar{H}_i$ are $\bar{T}_i = \frac{1}{2}\frac{1}{\varphi} p^T \bar{M}_r G p$, where

$$\bar{M}_1 = M_1 = \begin{vmatrix} I & 0 & 0 \\ K_2 & I & 0 \\ K_3 & K_2 & I \end{vmatrix} = I,$$

$$\bar{M}_2 = M_4 = \begin{vmatrix} 0 & K_3 & K_2 \\ K_2 & I & 0 \\ K_3 & K_2 & I \end{vmatrix} = K_2^3 - 2K_2K_3,$$

$$\bar{M}_3 = M_5 = \begin{vmatrix} 0 & 0 & K_3 \\ K_2 & I & 0 \\ K_3 & K_2 & I \end{vmatrix} = K_2^2 K_3 - K_3^2.$$

Metric tensor G and Killing tensors K_i are from the Benenti class. Separable potentials are constructed according to formulas

$$\bar{V}_1 = \tilde{V}_1 = \frac{1}{\rho_1^2 - \rho_2} V_1,$$

$$\bar{V}_2 = \tilde{V}_4 = V_2 + \frac{\rho_3 - \rho_1\rho_2}{\rho_1^2 - \rho_2} V_1,$$

$$\bar{V}_3 = \tilde{V}_5 = V_3 - \frac{\rho_1\rho_3}{\rho_1^2 - \rho_2} V_1.$$

The simplest potentials are as follows

$$\bar{V}^{(1)} = (0, 1, 0)^T,$$

$$\bar{V}^{(2)} = \left(\frac{1}{\rho_1^2 - \rho_2}, \frac{\rho_3 - \rho_1\rho_2}{\rho_1^2 - \rho_2}, -\frac{\rho_1\rho_3}{\rho_1^2 - \rho_2} \right)^T,$$

$$\bar{V}^{(3)} = \left(-\frac{\rho_1}{\rho_1^2 - \rho_2}, \frac{\rho_2^2 - \rho_1\rho_3}{\rho_1^2 - \rho_2}, \frac{\rho_2\rho_3}{\rho_1^2 - \rho_2} \right)^T,$$

$$\bar{V}^{(4)} = (1, 0, 0)^T,$$

$$\bar{V}^{(5)} = \left(\frac{2\rho_1\rho_2 - \rho_1^3 - \rho_3}{\rho_1^2 - \rho_2}, \frac{2\rho_1\rho_2\rho_3 - \rho_2^3 - \rho_3^2}{\rho_1^2 - \rho_2}, \frac{\rho_1\rho_3^2 - \rho_2^2\rho_3}{\rho_1^2 - \rho_2} \right)^T$$

and the recursion matrix takes the form

$$F = S^{-1}\Lambda S = (\bar{V}^{(5)}, \bar{V}^{(2)}, \bar{V}^{(1)}) = \begin{pmatrix} \frac{2\rho_1\rho_2 - \rho_1^3 - \rho_3}{\rho_1^2 - \rho_2} & \frac{1}{\rho_1^2 - \rho_2} & 0 \\ \frac{2\rho_1\rho_2\rho_3 - \rho_2^3 - \rho_3^2}{\rho_1^2 - \rho_2} & \frac{\rho_3 - \rho_1\rho_2}{\rho_1^2 - \rho_2} & 1 \\ \frac{\rho_1\rho_3^2 - \rho_2^2\rho_3}{\rho_1^2 - \rho_2} & -\frac{\rho_1\rho_3}{\rho_1^2 - \rho_2} & 0 \end{pmatrix}.$$

4.4.4 *Stäckel Transform of Superintegrable Systems*

In this subsection we perform Stäckel transforms of systems considered in Sect. 4.3.3 that preserve maximal superintegrability. According to the results of Sect. 4.1.3, the Hamiltonian h_1 of the considered system can only depend on one parameter $h_1 = h_1(x, \alpha)$. It is then natural to choose one of the α_k in (4.3.25) as this parameter.

Consider thus a maximally superintegrable system $(h_1, \ldots, h_{2n-1})$ with the first n commuting Hamiltonians $h_1, \ldots, h_n$ defined by a separation curve

$$\sum_{s \in I} \alpha_s \lambda^s + h_1 \lambda^{n-1} + h_2 \lambda^{n-2} + \ldots + h_n = \tfrac{1}{2}\lambda^m \mu^2, \quad m = 0, \ldots, n+1,$$

where the index set I was found in Sect. 4.3.3 and where the higher integrals h_{n+r} are constructed as usual through $h_{n+r} = \{h_{r+1}, P\}$. Let us now choose one of the parameters α_s, with $s \in I$, say α_k, (we will suppose that $k \geq n$ or $k < 0$ otherwise the corresponding potential is trivial, as explained earlier) and define the functions H_r, $r = 1, \ldots, 2n-1$, through

$$h_r = H_r + \alpha_k V_r^{(k)}, \ r = 1, \ldots, 2n-1.$$

We now perform the Stäckel transform on this system $(h_1, \ldots, h_{2n-1})$ with respect to the chosen parameter α_k. Thus, we first solve the relation $h_1 = \tilde{\alpha}$ i.e. $H_1 + \alpha_k V_1^{(k)} = \tilde{\alpha}$ with respect to α_k which yields

$$\tilde{h}_1 = \alpha_k = -\frac{1}{V_1^{(k)}} H_1 + \tilde{\alpha} \frac{1}{V_1^{(k)}} \tag{4.4.36}$$

and then replace α_k with $\tilde{h}_1$ in all the remaining Hamiltonians:

$$\tilde{h}_r = H_r - \frac{V_r^{(k)}}{V_1^{(k)}} H_1 + \tilde{\alpha} \frac{V_r^{(k)}}{V_1^{(k)}}, \quad r = 2, \ldots, 2n-1 \tag{4.4.37}$$

We obtain in this way a new superintegrable system $(\tilde{h}_1, \ldots, \tilde{h}_{2n-1})$ where the first n commuting Hamiltonians $\tilde{h}_r$ are defined by the following separation curve

$$\tilde{h}_1 \lambda^k + \sum_{s \in I,\, s \neq k} \alpha_s \lambda^s + \tilde{\alpha} \lambda^{n-1} + \tilde{h}_2 \lambda^{n-2} \ldots + \tilde{h}_n = \tfrac{1}{2} \lambda^m \mu^2, \quad m = 0, \ldots, n+1 \tag{4.4.38}$$

since on the level of the separation relations our Stäckel transform replaces α_k with $\tilde{h}_1$ and h_1 with $\tilde{\alpha}$. For $k \geq n$ the system (4.4.38) is no longer in the Benenti class (4.3.25). On the other hand, for $k < 0$, the separation curve (4.4.38) attains after the consecutive point transformation given by

$$\lambda \to 1/\lambda, \mu \to -\lambda^2 \mu,$$

the form

$$\tilde{\alpha} \lambda^{-1} + \sum_{s \in I,\, s \neq k} \alpha_s \lambda^{n-2-s} + \tilde{h}_1 \lambda^{n-k-2} + \tilde{h}_n \lambda^{n-2} + \ldots + \tilde{h}_2 = \tfrac{1}{2} \lambda^{n-m+2} \mu^2.$$

Thus, for $k = -1$, it again attains the form (4.3.25), i.e. belongs to the Benenti class, while for $k < -1$ we deal with particular γ-class. Notice that the transformation (4.4.4) does not change the separation web of the system.

Denoting

$$\tilde{h}_r = \widetilde{H}_r + \tilde{\alpha} \tilde{V}_r, \; r = 1, \ldots, 2n-1$$

(where $\tilde{h}_r$ for $r = 1, \ldots, n$ are defined by (4.4.37) while $\tilde{h}_r$ for $r = n+1, \ldots, 2n-1$ are obtained through $\tilde{h}_{n+r} = \{\tilde{h}_{r+1}, P\}$) we find from (4.4.37) that

$$\tilde{V}_r = V_r - \frac{V_r^{(k)}}{V_1^{(k)}} V_1, \quad r = 2, \ldots, 2n-1$$

and from (4.4.36) it also follows that the geodesic part $\tilde{T}_1$ of $\tilde{h}_1$ has the form

$$\tilde{T}_1 = \tilde{G}^{ij} p_i p_j, \quad \tilde{G} = -\frac{1}{V_1^{(k)}} G$$

It means that the metric $\tilde{G}$ is a *conformal deformation* of either a flat or a constant curvature metric G.

It is interesting to find the cases of superintegrable systems when the metric $\tilde{G}$ is actually flat or of constant curvature as well. The following results hold:

(i) For $0 \leq m \leq n-1$ and $k \in \{-m, \ldots, -1, n, \ldots, 2n-m-1\}$ the metric $\tilde{G}$ in (4.4.4) is flat for $k \in \{-[m/2], \ldots, -1, n, \ldots, n-1+[(n-m)/2]\}$, where $[\cdot]$ denotes the integer part. Moreover, for $m = 1$ and $k = -1$ $\tilde{G}$ is of constant curvature. Otherwise $\tilde{G}$ is conformally flat.

(ii) For $m = n$ and $k \in \{-(n-2), \ldots, -1, n\}$ the metric $\tilde{G}$ in (4.4.4) is flat for $k \in \{-[n/2], \ldots, -1\}$. Otherwise $\tilde{G}$ is conformally flat.

(iii) For $m = n+1$ and $k \in \{-(n-1), \ldots, -1\}$ the metric $\tilde{G}$ in (4.4.4) is flat for $k \in \{-[(n+1)/2], \ldots, -1\}$. Otherwise $\tilde{G}$ is conformally flat.

For the proof we send the reader to [40].

If $Y(V_1^{(k)}) = 0$ then $Y(1/V_1^{(k)}) = 0$ and due to (4.4.4) also $L_Y \tilde{G} = 0$ so that $\{\tilde{h}_1, P\} = 0$ as well and the same P as in the "non-tilde"-case (i.e. before the Stäckel transform) can be used as an alternative definition of extra Hamiltonians through $\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\}$, $r = 1, \ldots, n-1$. This is, however, no longer true if $Y(V_1^{(k)}) = c \neq 0$ and it happens only in the case when $m < n$ and $k = 2n-m-1$. In consequence, if $Y(V_1^{(k)}) = 0$ then both sets of extra integrals of motion:

$$\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\}, \quad r = 1, \ldots, n-1$$

and

$$\tilde{h}_{n+r} = h_{n+r}|_{\alpha = \tilde{h}_1(\tilde{\alpha})}, r = 1, \ldots, n-1$$

coincide. Indeed, on one hand, according to (4.4.37) and due to the fact that $\{\tilde{h}_1, P\} = 0$ we have

$$\begin{aligned}
\bar{h}_{n+r} = \{\tilde{h}_{r+1}, P\} &= \left\{ H_{r+1} - \frac{V_{r+1}^{(k)}}{V_1^{(k)}} H_1 + \tilde{\alpha} \frac{V_{r+1}^{(k)}}{V_1^{(k)}}, P \right\} \\
&= \{H_{r+1}, P\} - \frac{H_1}{V_1^{(k)}} \left\{ V_{r+1}^{(k)}, P \right\} + \frac{\tilde{\alpha}}{V_1^{(k)}} \left\{ V_{r+1}^{(k)}, P \right\} \\
&= \{H_{r+1}, P\} + \tilde{h}_1 \left\{ V_{r+1}^{(k)}, P \right\}
\end{aligned}$$

On the other hand, due to

$$\tilde{h}_{n+r} = h_{n+r}|_{\alpha=\widetilde{h}_1(\widetilde{\alpha})} = \{h_{r+1}, P\}|_{\alpha=\widetilde{h}_1(\widetilde{\alpha})} = \{H_{r+1}, P\} + \alpha \left\{V_{r+1}^{(k)}, P\right\}\Big|_{\alpha=\widetilde{h}_1(\widetilde{\alpha})}$$

which yields the same result.

Thus, if $Y(V_1^{(k)}) = 0$, the diagram below commutes

$$\begin{array}{ccc} (h_1, \ldots, h_n) & \xrightarrow{P} & (h_1, \ldots, h_{2n-1}) \text{ with } h_{n+r} = \{h_{r+1}, P\} \\ \Big| & & \Big| \\ \text{Stäckel transform} & & \text{Stäckel transform} \\ \downarrow & & \downarrow \\ \left(\tilde{h}_1, \ldots, \tilde{h}_n\right) & \xrightarrow{P} & \left(\tilde{h}_1, \ldots, \tilde{h}_{2n-1}\right) \text{ with } \tilde{h}_{n+r} = \left\{\tilde{h}_{r+1}, P\right\} \end{array}$$

Example 4.17 Let us apply the relations (4.4.36)–(4.4.37) to perform the Stäckel transform on the system from Example 4.15. To keep the formulas simple, we assume that all the α_s in (4.3.30) are zero except the transformation parameter α_k. Thus, we consider again the system given by the separation curve

$$\alpha_k \lambda^k + h_1 \lambda^2 + h_2 \lambda + h_3 = \tfrac{1}{2}\lambda\mu^2,$$

with $k = -1, 3$ or 4, respectively. Applying the Stäckel transform to the resulting Hamiltonians from Example 4.15 we obtain a maximally superintegrable system with the separation curve of the form:

$$\tilde{h}_1 \lambda^k + \tilde{\alpha}\lambda^2 + \tilde{h}_2 \lambda + \tilde{h}_3 = \tfrac{1}{2}\lambda\mu^2, \tag{4.4.39}$$

Again we perform our calculations in non-orthogonal flat coordinates (x, y, z, p_x, p_y, p_z). Explicitly, we obtain for $k = -1$

$$\begin{aligned} \tilde{h}_1 &= \tfrac{1}{8}z^2 p_z^2 + \tfrac{1}{4}z^2 p_x p_y - \tfrac{1}{4}\tilde{\alpha}z^2 \\ \tilde{h}_2 &= \tfrac{1}{2}p_x^2 - \tfrac{1}{2}y p_y^2 - \tfrac{1}{2}x p_x p_y - \tfrac{1}{2}z p_y p_z + \tilde{\alpha}x \\ \tilde{h}_3 &= \tfrac{1}{8}z^2 p_y^2 - \left(\tfrac{1}{4}x^2 + y\right) p_x p_y - \tfrac{1}{2}z p_x p_z - \tfrac{1}{4}x z p_y p_z + \tfrac{1}{4}\tilde{\alpha}\left(x^2 + 4y\right) \\ \tilde{h}_4 &= -\tfrac{1}{2}p_y^2 \\ \tilde{h}_5 &= -p_x p_y + \tilde{\alpha} \end{aligned} \tag{4.4.40}$$

for $k = 3$

$$\begin{aligned} \tilde{h}_1 &= -\tfrac{1}{x}p_x p_y - \tfrac{1}{2}\tfrac{1}{x}p_z^2 + \tilde{\alpha}\tfrac{1}{x} \\ \tilde{h}_2 &= \tfrac{1}{2}p_x^2 + \tfrac{1}{4}\tfrac{x^2-4y}{x}p_x p_y - \tfrac{1}{2}y p_y^2 - \tfrac{1}{2}z p_y p_z + \tfrac{1}{8}\tfrac{3x^2-4y}{x}p_z^2 + \tfrac{1}{4}\tilde{\alpha}\tfrac{x^2+4y}{x} \end{aligned}$$

$$\tilde{h}_3 = \tfrac{1}{4}\tfrac{z^2}{x} p_x p_y - \tfrac{1}{2} z p_x p_z + \tfrac{1}{8} z^2 p_y^2 - \tfrac{1}{4} x z p_y p_z + \tfrac{1}{8}\tfrac{x^3+4xy+z^2}{x} p_z^2 - \tfrac{1}{4}\tilde{\alpha}\tfrac{z^2}{x}$$
$$\tilde{h}_4 = -\tfrac{1}{x} p_x p_y - \tfrac{1}{2} p_y^2 - \tfrac{1}{2}\tfrac{1}{x} p_z^2 + \tilde{\alpha}\tfrac{1}{x}$$
$$\tilde{h}_5 = \tfrac{1}{2} p_z^2$$

and for $k = 4$

$$\tilde{h}_1 = -\tfrac{1}{y-\frac{3}{4}x^2} p_x p_y - \tfrac{1}{2}\tfrac{1}{y-\frac{3}{4}x^2} p_z^2 + \tilde{\alpha}\tfrac{1}{y-\frac{3}{4}x^2}$$
$$\tilde{h}_2 = \tfrac{1}{2} p_x^2 - \tfrac{1}{2} y p_y^2 - \tfrac{1}{8}\tfrac{2x^3-8xy-z^2}{y-\frac{3}{4}x^2} p_z^2 - \tfrac{1}{8}\tfrac{x^3-12xy-2z^2}{y-\frac{3}{4}x^2} p_x p_y - \tfrac{1}{2} z p_y p_z$$
$$- \tilde{\alpha}\tfrac{xy+\frac{1}{4}x^3+\frac{1}{4}z^2}{y-\frac{3}{4}x^2}$$
$$\tilde{h}_3 = \tfrac{1}{8} z^2 p_y^2 - \tfrac{1}{32}\tfrac{3x^4+8x^2y+4xz^2+16y^2}{y-\frac{3}{4}x^2} p_z^2 + \tfrac{1}{4}\tfrac{xz^2}{y-\frac{3}{4}x^2} p_x p_y - \tfrac{1}{2} z p_x p_z - \tfrac{1}{4} x z p_y p_z$$
$$+ \tfrac{1}{4}\tilde{\alpha}\tfrac{xz^2}{y-\frac{3}{4}x^2} \tag{4.4.41}$$
$$\tilde{h}_4 = -\tfrac{1}{2} p_y^2 + \tfrac{1}{2}\tfrac{x}{y-\frac{3}{4}x^2} p_z^2 + \tfrac{x}{y-\frac{3}{4}x^2} p_x p_y - \tilde{\alpha}\tfrac{x}{y-\frac{3}{4}x^2}$$
$$\tilde{h}_5 = \tfrac{1}{2} p_z^2$$

According to (i) of that subsection the metrics of $\tilde{h}_1$ are of constant curvature, flat and conformally flat, respectively.

Chapter 5
Classical Separability Theory

As was analysed in the previous chapter, once we find separation coordinates for a Liouville integrable system, we can integrate the system by quadratures through an appropriate separation relations. The fundamental problem in the Hamilton–Jacobi method is the systematic construction of transformation from some "natural" coordinates to separation coordinates. As was demonstrated in the previous chapter, such coordinates like Cartesian, spherical or cylindrical are separation coordinates only in very special cases. In general, separation coordinates are much less obvious and completely unpredictable. So the question about the existence of a systematic method for the construction of separation coordinates is very important. Indeed, for many decades of development of the separability theory, the method did not exist. Only recently, at the end of the twentieth century, after more than 100 years of efforts, a few different constructive methods were suggested. Obviously, the knowledge of all constants of motion for a given Liouville integrable system is not enough. Some extra information is required.

The first constructive theory of separated coordinates for Hamiltonian systems was made by Sklyanin [235]. He adopted the method of the Soliton theory, i.e. the Lax representation and the so called r-matrix approach, to systematic derivation of separation coordinates. In that theory involutive functions appear as coefficients of the characteristic equation (*spectral curve*) of Lax matrix. The method was successfully applied to the separation of variables for many integrable systems [107, 108, 175, 233–235, 272, 273]. The detailed description of r-matrix approach to separability theory and related mathematical tools the reader can find in the book [6], so we skip it here.

Then, a modern geometric theory of separability on bi-Poisson manifolds [23, 26–28, 112–115, 189], and bi-presymplectic manifolds [29, 32, 46] was constructed. In that approach we require from symmetries to constitute appropriate bi-Hamiltonian chains or from conserved 1-forms to constitute appropriate bi-presymplectic chains, both on an extended phase space. The geometric approach

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for the construction of separation coordinates is at least as powerful as the r-matrix approach and as the less known to readers, will be presented with all details in the following sections.

5.1 Bi-Hamiltonian Separability Theory

Presented here the modern geometric separability theory relies on the fact that any Liouville integrable and separable system, with separation relations generated by a separation curve, possesses two different Hamiltonian representations in any local coordinates, or can be extended to such representations. Thus, for a given Liouville integrable system its bi-Hamiltonian representation is sufficient for the construction of a transformation from original coordinates to separation coordinates. During that process we will reduce the second Poisson bi-vector onto the symplectic foliation of the first one. As in the considered case, the Dirac distribution is tangent to the foliation (see the results of Sect. 3.4), we have to find an appropriate transversal distribution $\mathcal{Z}$ along which the reduction will be performed. This is the only non algorithmic element of the construction.

5.1.1 Bi-Hamiltonian Liouville Integrable Systems

The idea of bi-Hamiltonian systems was introduced by Magri [186] in the context of the so called Soliton systems, i.e. integrable nonlinear PDE's (see [23] for review of bi-Hamiltonian field systems). Then it was transferred to the theory of finite dimensional systems [4] (see also [23] and the literature quoted there).

The first step in a geometric separability theory is to find the bi-Hamiltonian representation of a given Liouville integrable system. This is closely related to the notion of Poisson pencils of a particular type and their Casimirs. Let us consider a manifold M and two Poisson tensors Π_0 and Π_1. A linear combination $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$ ($\lambda \in \mathbb{R}$) is called a *Poisson pencil* if the operator Π_λ is Poissonian for any value of the parameter λ. In this case we say that Π_0 and Π_1 are *compatible*. A pair of Poisson bi-vectors Π_0 and Π_1 is compatible if and only if one of the following equivalent conditions is satisfied

1.

$$[\Pi_0, \Pi_1]_S = 0, \tag{5.1.1}$$

2.

$$\big(\{\{f, g\}_{\Pi_0}, h\}_{\Pi_1} + \{\{f, g\}_{\Pi_0}, h\}_{\Pi_1}\big) + c.p. = 0, \tag{5.1.2}$$

3.

$$[X_f, Y_g] + [Y_f, X_g] = X_{\{f,g\}_{\Pi_1}} + Y_{\{f,g\}_{\Pi_0}}, \quad X_f = \Pi_0 df, \quad Y_f = \Pi_1 df. \tag{5.1.3}$$

In fact, each of these conditions follows from the application of the corresponding conditions (3.2.1), (3.2.6), (3.2.24) for a single Poisson structure to a Poisson pencil. For example we have

$$0 = [\Pi_\lambda, \Pi_\lambda]_S = [\Pi_1, \Pi_1]_S - 2\lambda[\Pi_0, \Pi_1]_S + \lambda^2[\Pi_0, \Pi_0]_S \Longrightarrow [\Pi_0, \Pi_1]_S = 0.$$

Given a Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$ we can often construct a sequence of vector fields X_i on M that have a twofold Hamiltonian form (the so-called *bi-Hamiltonian chain*)

$$\boxed{X_i = \Pi_1 dh_i = \Pi_0 dh_{i+1}} \tag{5.1.4}$$

where $h_i : M \to \mathbb{R}$ are called the Hamiltonians of the chain (5.1.4) and where i is a discrete index. This sequence of vector fields may or may not truncate (depending on the existence of Casimir functions).

Let us consider a bi-Poisson manifold (M, Π_0, Π_1) of $\dim M = m = 2n + r$ where Π_0, Π_1 is a pair of compatible Poisson tensors of rank $2n$. Moreover we assume that the Poisson pencil Π_λ admits r, polynomial with respect to the pencil parameter λ, Casimir functions of the form

$$h^{(j)}(\lambda) = \sum_{i=0}^{n_j} h_i^{(j)} \lambda^{n_j - i}, \qquad j = 1, \ldots, r, \tag{5.1.5}$$

such that $n_1 + \ldots + n_r = n$ and $h_i^{(j)}$ are functionally independent. The collection of n bi-Hamiltonian vector fields

$$\boxed{\Pi_\lambda dh^{(j)}(\lambda) = 0 \Longleftrightarrow \begin{array}{r} \Pi_0 dh_0^{(j)} = 0 \\ \Pi_0 dh_1^{(j)} = X_1^{(j)} = \Pi_1 dh_0^{(j)} \\ \vdots \\ \Pi_0 dh_{n_j}^{(j)} = X_{n_j}^{(j)} = \Pi_1 dh_{n_j-1}^{(j)} \\ 0 = \Pi_1 dh_{n_j}^{(j)} \end{array}} \tag{5.1.6}$$

where $j = 1, \ldots, r$, is called the Gel'fand–Zakharevich (GZ) system of pure Kronecker type (see [123, 124, 216, 217] and the references quoted therein). Notice that each chain starts from a Casimir of Π_0 and terminates with a Casimir of Π_1. It is the Liouville integrable system as all $h_i^{(j)}$ are functionally independent from the

definition and moreover pairwise commute with respect to both Poisson structures

$$X_i^{(j)}(h_l^{(k)}) = \Pi_0(dh_l^{(k)}, dh_i^{(j)}) = \Pi_1(dh_l^{(k)}, dh_{i-1}^{(j)}) = 0. \tag{5.1.7}$$

In fact, as

$$\begin{aligned}\Pi_0(dh_l^{(k)}, dh_i^{(j)}) &= \Pi_1(dh_l^{(k)}, dh_{i-1}^{(j)}) = -\Pi_1(dh_{i-1}^{(j)}, dh_l^{(k)}) \\ &= -\Pi_0(dh_{i-1}^{(j)}, dh_{l-1}^{(k)}) = \Pi_0(dh_{l-1}^{(k)}, dh_{i-1}^{(j)}),\end{aligned}$$

so after an appropriate number of iterations we get either $\Pi_0(dh_0^{(k)}, dh_s^{(j)}) = 0$ or $\Pi_0(dh_s^{(k)}, dh_0^{(j)}) = 0$.

Below we illustrate our considerations with a few instructive examples. First three examples are bi-Hamiltonian chains written in some canonical coordinates.

Example 5.1 The bi-Hamiltonian extension of the Henon–Heiles system.

Let us consider the integrable case of the Henon–Heiles system considered in previous examples with an extra parameter c

$$x_{tt} = -3x^2 - \tfrac{1}{2}y^2 + c, \quad y_{tt} = -xy. \tag{5.1.8}$$

In the Hamiltonian representation the system (5.1.8) belongs to one-Casimir bi-Hamiltonian chain on a 5-dimensional extended phase space parametrized by (x, y, p_x, p_y, c)

$$\begin{aligned}\Pi_0 dh_0 &= 0 \\ \Pi_0 dh_1 &= X_1 = \Pi_1 dh_0 \\ \Pi_0 dh_2 &= X_2 = \Pi_1 dh_1 \\ 0 &= \Pi_1 dh_2,\end{aligned} \tag{5.1.9}$$

where

$$\begin{aligned}h_0 &= c, \\ h_1 &= \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + \tfrac{1}{2}xy^2 + x^3 - cx, \\ h_2 &= \tfrac{1}{2}yp_xp_y - \tfrac{1}{2}xp_y^2 + \tfrac{1}{16}y^4 + \tfrac{1}{4}x^2y^2 - \tfrac{1}{4}cy^2,\end{aligned}$$

and both Poisson structures are

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & 0 & x & \frac{1}{2}y & p_x \\ 0 & 0 & \frac{1}{2}y & 0 & p_y \\ -x & -\frac{1}{2}y & 0 & \frac{1}{2}p_y & -h_{1,x} \\ -\frac{1}{2}y & 0 & -\frac{1}{2}p_y & 0 & -h_{1,y} \\ -p_x & -p_y & h_{1,x} & h_{1,y} & 0 \end{pmatrix}.$$

One can check that indeed $[\Pi_0, \Pi_1]_S = 0$. Notice that the last column of Π_1 is built of Hamiltonian vector field X_1, which represents Hamiltonian dynamics (5.1.8). The chain (5.1.9) represents the Casimir $h(\lambda) = c\lambda^2 + h_1\lambda + h_2$ of the Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$: $\Pi_\lambda dh(\lambda) = 0$.

Example 5.2 The bi-Hamiltonian extension of the Kepler problem in the plane. Let us consider the classical problem of a particle in the plane under the influence of the Kepler potential (Example 4.5) and an additional homogeneous field force. The Hamiltonian function reads

$$h_1 = H_1 - cy = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 - \frac{\alpha}{\sqrt{x^2+y^2}} - cy, \quad \alpha = const.$$

There is a second independent integral of the motion

$$h_2 = -H_4 - \tfrac{1}{4}cx^2 = -\tfrac{1}{2}yp_x^2 + \tfrac{1}{2}xp_xp_y + \tfrac{1}{2}\frac{\alpha y}{\sqrt{x^2+y^2}} - \tfrac{1}{4}cx^2,$$

which together with $h_0 = c$ allows us to construct a bi-Hamiltonian chain (5.1.9) with the second Poisson structure in the form

$$\Pi_1 = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2}x & p_x \\ 0 & 0 & \frac{1}{2}x & y & p_y \\ 0 & -\frac{1}{2}x & 0 & -\frac{1}{2}p_x & -h_{1,x} \\ -\frac{1}{2}x & -y & \frac{1}{2}p_x & 0 & -h_{1,y} \\ -p_x & -p_y & h_{1,x} & h_{1,y} & 0 \end{pmatrix}.$$

Example 5.3 Two-Casimir bi-Hamiltonian chains. Consider the two-parameter Lagrangian system

$$L = \tfrac{1}{2}xx_t^2 + \tfrac{1}{2}xy_t^2 - x^2 - \tfrac{1}{2}y^2 + c_1(x + \tfrac{1}{4}x^{-1}y^2) + c_2x^{-1}. \tag{5.1.10}$$

In the Hamiltonian representation, the system generated by (5.1.10) belongs to the two-Casimir bi-Hamiltonian chain on a 6-dimensional extended phase space parametrized by $(x, y, p_x, p_y, c_1, c_2)$

$$\begin{aligned} \Pi_0 dh_0^{(1)} &= 0 & \Pi_0 dh_0^{(2)} &= 0 \\ \Pi_0 dh_1^{(1)} &= X_1^{(1)} = \Pi_1 dh_0^{(1)} & \Pi_0 dh_1^{(2)} &= X_1^{(1)} = \Pi_1 dh_0^{(2)} \\ 0 &= \Pi_1 dh_1^{(1)} & 0 &= \Pi_1 dh_1^{(2)} \end{aligned} \tag{5.1.11}$$

where

$$h_0^{(1)} = c_1,$$
$$h_1^{(1)} = \tfrac{1}{2}x^{-1}p_x^2 + \tfrac{1}{2}x^{-1}p_y^2 + x^2 + \tfrac{1}{2}y^2 - c_1(x + \tfrac{1}{4}x^{-1}y^2) - c_2x^{-1},$$

$$h_0^{(2)} = c_2,$$
$$h_1^{(2)} = -\tfrac{1}{8}x^{-1}y^2p_x^2 + \tfrac{1}{2}yp_xp_y - \frac{1}{8}x^{-1}y^2p_y^2 - \tfrac{1}{16}y^4 + \tfrac{1}{16}c_1x^{-1}y^4 + \tfrac{1}{4}c_2x^{-1}y^2,$$

and both Poisson structures are

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & x & \frac{1}{2}y & \partial_{p_x}h_1^{(1)} & \partial_{p_x}h_1^{(2)} \\ 0 & 0 & \frac{1}{2}y & 0 & \partial_{p_y}h_1^{(1)} & \partial_{p_y}h_1^{(2)} \\ -x & -\frac{1}{2}y & 0 & \frac{1}{2}p_y & -\partial_x h_1^{(1)} & -\partial_x h_1^{(2)} \\ -\frac{1}{2}y & 0 & -\frac{1}{2}p_y & 0 & -\partial_y h_1^{(1)} & -\partial_y h_1^{(2)} \\ -\partial_{p_x}h_1^{(1)} & -\partial_{p_y}h_1^{(1)} & \partial_x h_1^{(1)} & \partial_y h_1^{(1)} & 0 & 0 \\ -\partial_{p_x}h_1^{(2)} & -\partial_{p_y}h_1^{(2)} & \partial_x h_1^{(2)} & \partial_y h_1^{(2)} & 0 & 0 \end{pmatrix}.$$

Notice that two last columns of Π_1 are built of Hamiltonian vector fields $X_1^{(1)}$ and $X_1^{(2)}$. The chains (5.1.11) represent two Casimirs $h^{(1)}(\lambda) = c_1\lambda + h_1^{(1)}$ and $h^{(2)}(\lambda) = c_2\lambda + h_1^{(2)}$ of the Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$: $\Pi_\lambda dh^{(i)}(\lambda) = 0,\ i = 1, 2.$

The last example is a bi-Hamiltonian chain written in a non-canonical representation.

Example 5.4 The Euler top, considered in Example 3.7, has the following bi-Hamiltonian representation

$$h_0 = \tfrac{1}{2}\omega_1^2 + \tfrac{1}{2}\omega_2^2 + \tfrac{1}{2}\omega_3^2,$$
$$h_1 = \tfrac{1}{2}a_1\omega_1^2 + \tfrac{1}{2}a_2\omega_2^2 + \tfrac{1}{2}a_3\omega_3^2,$$

$$\Pi_0 = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & a_3\omega_3 & -a_2\omega_2 \\ -a_3\omega_3 & 0 & a_1\omega_1 \\ a_2\omega_2 & -a_1\omega_1 & 0 \end{pmatrix},$$

$$\begin{aligned} \Pi_0 dh_0 &= 0 \\ \Pi_0 dh_1 &= X = \Pi_1 dh_0 \\ 0 &= \Pi_1 dh_1 \end{aligned}$$

where

$$X = \begin{pmatrix} (a_3 - a_2)\omega_2\omega_3 \\ (a_1 - a_3)\omega_1\omega_3 \\ (a_2 - a_1)\omega_1\omega_2 \end{pmatrix}, \qquad a_i = I_i^{-1}.$$

Although the presented examples show the existence of bi-Hamiltonian representation of some Liouville integrable systems, nevertheless in order to make the bi-Hamiltonian separability theory relevant, such a representation has to be a common feature inside integrable systems. Fortunately, it is the case, i.e. any Stäckel system with separation relations of the general form

$$\sum_{j=1}^{n} H_j(\lambda^i)^{\gamma_j}(\mu_i)^{\delta_j} = \psi_i(\lambda^i, \mu_i), \quad \gamma_j, \delta_j \in \mathbb{N}, \qquad i = 1, \ldots, n, \tag{5.1.12}$$

has an appropriate extension to the bi-Hamiltonian chains of Gel'fand–Zakharevich type (5.1.6) [31]. In order to construct such an extension, let us rearrange the l.h.s. of (5.1.12) collecting terms and renumerating Hamiltonians as follows:

$$\boxed{\sum_{k=1}^{r} \beta^k(\lambda^i, \mu_i) H^{(k)}(\lambda^i) = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \ldots, n,} \tag{5.1.13}$$

where

$$\boxed{H^{(k)}(\lambda) = \sum_{i=1}^{n_k} H_i^{(k)} \lambda^{n_k - i}, \qquad n_1 + \cdots + n_r = n}$$

$\beta^k(\lambda^i, \mu_i)$ are respective monomials of it arguments and let impose the normalization $\beta^r(\lambda^i, \mu_i) = 1$. The matrix S is uniquely defined by r functions $\beta^k = \beta^k(\lambda, \mu)$, $k = 1, \ldots, r$, and the partition $(n_1, \ldots, n_r)$ of n. Note that in our normalization we have $\beta^r = 1$.

All Hamiltonian systems $H_i^{(k)}$ are defined on $2n$-dimensional phase space parametrized by their separation coordinates (λ, μ). Now, let us extend the phase space by r extra coordinates $(c_1, \ldots, c_r)$ to new, $(2n + r)$-dimensional space. Moreover, let us extend Hamiltonians $H_i^{(k)}(\lambda, \mu)$ to a new Hamiltonians by adding terms linear in c_i:

$$\boxed{h_i^{(k)}(\lambda, \mu, c) = H_i^{(k)}(\lambda, \mu) + \sum_{m=1}^{r} F_i^{(k,m)}(\lambda, \mu) c_m,} \tag{5.1.14}$$

which fulfills the following, new separation relations

$$\sum_{k=1}^{r} \beta^k(\lambda^i, \mu_i) h^{(k)}(\lambda^i) = \psi_i(\lambda^i, \mu_i), \qquad i = 1, \ldots, n, \tag{5.1.15}$$

$$h^{(k)}(\lambda) = \sum_{i=0}^{n_k} h_i^{(k)} \lambda^{n_k - i}, \qquad n_1 + \cdots + n_r = n, \tag{5.1.16}$$

where $h_0^{(k)} = c_k$. It means that $F_i^{(k,r)}$ can be treated as additional potentials generated by extra terms $\beta^k(\lambda^i, \mu_i)(\lambda^i)^{n_k} c_k$ in separation relations (5.1.15). So, the potentials $F_i^{(k,m)}$ are solutions of the set of linear algebraic equations

$$\beta^m(\lambda^i, \mu_i)(\lambda^i)^{n_m} + \sum_{k=1}^{r} \beta^k(\lambda^i, \mu_i) F^{(k,m)}(\lambda^i) = 0, \qquad m=1, \ldots, r, \quad i=1, \ldots, n, \tag{5.1.17}$$

where

$$F^{(k,m)}(\lambda) = \sum_{j=1}^{n_k} F_j^{(k,m)} \lambda^{n_k - j}, \qquad n_1 + \cdots + n_r = n.$$

Such extended Hamiltonian functions form bi-Hamiltonian chains (5.1.6) on a $(2n+r)$-dimensional extended phase space, where

$$\Pi_0 = \left(\begin{array}{cc|c} 0 & I & 0 \cdots 0 \\ -I & 0 & \\ \hline 0 & & \\ \vdots & & 0 \\ 0 & & \end{array}\right), \tag{5.1.18a}$$

$$\Pi_1 = \left(\begin{array}{cc|c} 0 & \Lambda & X_1^{(1)} \cdots X_1^{(r)} \\ -\Lambda & 0 & \\ \hline -\left(X_1^{(1)}\right)^T & & \\ \vdots & & 0 \\ -\left(X_1^{(r)}\right)^T & & \end{array}\right). \tag{5.1.18b}$$

The proof that Poisson tensors Π_0 and Π_1 form a Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0$ with $h^{(k)}(\lambda)$ (5.1.16) as Casimirs

$$\boxed{\Pi_\lambda dh^{(k)}(\lambda) = 0, \qquad k = 1, \ldots, r,} \tag{5.1.18c}$$

is laborious and technical so we send the interested reader to the original paper [31]. In particular, a lift to bi-Hamiltonian chains can be done for the class (4.3.10) of separation relations considered in Sect. 4.3.

To illustrate the method, let us consider such a lift for particular classes of Stäckel systems. The simplest one is the Benenti class where $r = 1 \Longrightarrow k = m = 1$, $\beta^1(\lambda^i, \mu_i) = 1$, hence Eqs. (5.1.17) for $F_i^{(1,1)} \equiv F_i$ reduce to

$$(\lambda^i)^n + F_1(\lambda^i)^{n-1} + \ldots + F_n = 0, \quad i = 1, \ldots, n$$

$$\Downarrow$$

$$F_i = -V_i^{(n)},$$

where $V^{(n)}$ is given by (4.3.23). Thus,

$$h_i(\lambda, \mu, c) = H_i(\lambda, \mu) - V_i^{(n)}(\lambda)c = H_i(\lambda, \mu) + \rho_i(\lambda)c$$

and we have one bi-Hamiltonian chain (5.1.6) in a $(2n + 1)$-dimensional extended phase space. It explains, for example, the appropriate extension of the Henon–Heiles system from Example 5.1, where the transformation to Cartesian coordinates is given in Example 4.11. The bi-Hamiltonian representation of systems from Benenti class was first constructed in [156].

Another class of separation relations for which we construct the extension (5.1.14) is given by a separation curve (4.4.23) of the form

$$\bar{H}_1\lambda^n + \ldots + \bar{H}_{n_1}\lambda^{n+1-n_1} + \bar{H}_{n_1+1}\lambda^{n-1-n_1} + \cdots + \bar{H}_n = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda), \tag{5.1.19}$$

or equivalently by (4.4.25) with $k = 1, 1 < m_1 = n_1 + 1 < n$, discussed in Sect. 4.3. Adopting the notation (5.1.13), the separation curve (5.1.19) takes the form

$$\lambda^{n+1-n_1}(H_1^{(1)}\lambda^{n_1-1} + \ldots + H_{n_1}^{(1)}) + (H_1^{(2)}\lambda^{n_2-1} + \ldots + H_{n_2}^{(1)}) = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda), \quad n_1 + n_2 = n,$$

where

$$H_j^{(1)} = \bar{H}_j, \quad j = 1, \ldots, n_1,$$
$$H_j^{(2)} = \bar{H}_{n_1+j}, \quad j = 1, \ldots, n_2,$$

thus $\beta^1(\lambda) = \lambda^{n+1-n_1}$ and $\beta^2(\lambda) = 1$. So, the potentials $F_i^{(k,1)}$ are solutions of the set of linear algebraic equations being n copies of

$$\lambda^{n+1-n_1}(\lambda^{n_1} + F_1^{(1,1)}\lambda^{n_1-1} + \ldots + F_{n_1}^{(1,1)}) + (F_1^{(2,1)}\lambda^{n_2-1} + \ldots + F_{n_2}^{(2,1)}) = 0,$$

while the potentials $F_i^{(k,2)}$ are solutions of the set of linear algebraic equations being n copies of

$$\lambda^{n+1-n_1}(F_1^{(1,2)}\lambda^{n_1-1} + \ldots + F_{n_1}^{(1,2)}) + (\lambda^{n_2} + F_1^{(2,2)}\lambda^{n_2-1} + \ldots + F_{n_2}^{(2,2)}) = 0.$$

Thus

$$\begin{pmatrix} F^{(1,1)} \\ F^{(2,1)} \end{pmatrix} = -\bar{V}^{(n+1)}, \qquad \begin{pmatrix} F^{(1,2)} \\ F^{(2,2)} \end{pmatrix} = -\bar{V}^{(n+1-m_1)} = -\bar{V}^{(n_2)}$$

and according to (4.4.3) and (4.4.3)

$$\begin{aligned} F_r^{(1,1)} &= \rho_r - \frac{\rho_{r-1}\rho_{n_1+1}}{\rho_{n_1}}, \quad r = 1, \ldots, n_1, \\ F_r^{(2,1)} &= \rho_{r+1} - \frac{\rho_{n_1+r}\rho_{n_1+1}}{\rho_{n_1}}, \quad r = 1, \ldots, n_2, \\ F_r^{(1,2)} &= \frac{\rho_{r-1}}{\rho_{n_1}}, \quad r = 1, \ldots, n_1, \\ F_r^{(2,2)} &= \frac{\rho_{n_1+r}}{\rho_{n_1}}, \quad r = 1, \ldots, n_2. \end{aligned} \tag{5.1.20}$$

Extended Hamiltonians

$$\begin{aligned} h_i^{(1)}(\lambda, \mu, c) &= H_i^{(1)}(\lambda, \mu) + F_i^{(1,1)}(\lambda, \mu)c_1 + F_i^{(1,2)}(\lambda, \mu)c_2, \quad i = 1, \ldots, n_1, \\ h_i^{(2)}(\lambda, \mu, c) &= H_i^{(2)}(\lambda, \mu) + F_i^{(2,1)}(\lambda, \mu)c_1 + F_i^{(2,2)}(\lambda, \mu)c_2, \quad i = 1, \ldots, n_2, \end{aligned}$$

belong to two bi-Hamiltonian chains (5.1.6) in a $(2n + 2)$-dimensional extended phase space.

5.1.2 *Reduction of Poisson Pencils onto Symplectic Leaves*

The second step of our geometric separability theory is as follows. The necessary condition for separability of Hamiltonian systems generated by functions $h_i^{(j)}$ (5.1.5) is a Poisson projection of the pencil Π_λ onto a symplectic foliation S of Π_0, that is a $2n$-dimensional submanifold defined by fixed values of Casimir

functions of Π_0 : $h_0^{(1)} = c_1, \ldots, h_0^{(r)} = c_r$. Thus, every leave S_c is a submanifold of a codimension r in M. The projection is done through the particular realization of the reduction schema considered in Sect. 3.4.1. As our constraints $\varphi_i = h_0^{(k)} = c_k,\ k = 1, \ldots, r$ are Casimirs of Π_0 its reduction is trivial. From the property that all Casimirs c_i of Π_0 are in involution with respect to Π_1 : $\{dh_0^{(k)}, dh_0^{(j)}\}_{\Pi_1} = \Pi_1(dh_0^{(k)}, dh_0^{(j)}) = 0$ we have to reduce Π_1 according to the tangent case (Dirac reduction is useless in that case). Actually, we are looking for an appropriate distribution $\mathcal{Z}$, transversal to the foliation S that the deformation of Π_1 in the form

$$\boxed{\Pi_D = \Pi_1 - \sum_{k=1}^{r} X_1^{(k)} \wedge Z_k, \qquad X_1^{(k)} = \Pi_0 dh_1^{(k)} = \Pi_1 dh_0^{(k)},} \tag{5.1.21}$$

fulfills the following conditions:

1. the image of Π_D is tangent to the foliation S,
2. Π_D is Poisson,
3. Π_D is compatible with Π_0,
4. all constants of motion are in involution with respect to Π_D :

$$\{dh_l^{(k)}, dh_i^{(j)}\}_{\Pi_D} = \Pi_D(dh_l^{(k)}, dh_i^{(j)}) = 0. \tag{5.1.22}$$

The first condition means that $\mathrm{Im}(\Pi_D) \subset TS$, i.e. that for every 1-form α and $k = 1, \ldots, r$,

$$\left\langle dh_0^{(k)}, \Pi_D \alpha \right\rangle = 0$$

and it follows directly from the fact that $h_0^{(k)}$ are Casimirs of Π_D (3.4.12). The second condition is fulfilled if $\mathcal{Z}$ is an integrable distribution $\big([Z_i, Z_j] = 0\big)$ and $L_{Z_i}\Pi_D = 0$ or equivalently fulfills relation (3.4.1) for $i = 1, \ldots, r$. For the compatibility of Π_D and Π_0 we have the condition

$$\begin{aligned} 0 = [\Pi_D, \Pi_0]_S &= [\Pi_1, \Pi_0]_S - \sum_{k=1}^{r} [X_1^{(k)} \wedge Z_k, \Pi_0] \\ &= \sum_{k=1}^{r} \left(-L_{X_1^{(k)}} \Pi_0 \wedge Z_k + X_1^{(k)} \wedge L_{Z_k} \Pi_0 \right), \end{aligned} \tag{5.1.23}$$

where the last equality follows from the properties of Schouten–Nijenhuis bracket (3.2.33). As $L_{X_1^{(k)}} \Pi_0 = 0$ (3.2.2), the strong solution of (5.1.23) is

$$L_{Z_k} \Pi_0 = 0, \quad k = 1, \ldots, r. \tag{5.1.24}$$

Finally, the involutivity (5.1.22) follows from the form of Π_D (5.1.21)

$$\begin{aligned}\Pi_D(dh_l^{(k)}, dh_i^{(j)}) &= \Pi(dh_l^{(k)}, dh_i^{(j)}) + \sum_{k=1}^{r} \left\langle dh_l^{(r)}, (X_1^{(k)} \wedge Z_k) dh_i^{(j)} \right\rangle \\ &= \sum_{k=1}^{r} \left(Z_k(h_i^{(j)}) X_1^{(k)}(h_l^{(r)}) - X_1^{(k)}(h_i^{(j)}) Z_k(h_l^{(r)}) \right) \\ &= 0\end{aligned}$$

according to (5.1.7). Notice, that if the compatibility condition (5.1.24) is fulfilled, then the condition (3.4.1) takes the form

$$L_{Z_k}\Pi_1 = \sum_{i=1}^{r} [Z_k, X_1^{(i)}] \wedge Z_i = \sum_{i=1}^{r} Y_{ki} \wedge Z_i, \quad Y_{ki} = \Pi_0 d(Z_k(h_1^{(i)})). \tag{5.1.25}$$

Assume that conditions 1–4 are fulfilled. Then, bi-Hamiltonian chains on M take the form

$$\boxed{\begin{aligned}\Pi_D dh_i^{(j)} &= \Pi_0 dh_{i+1}^{(j)} - \sum_{k=1}^{r} Z_k(dh_i^{(j)}) \Pi_0 dh_1^{(k)} \\ &= \Pi_0 dh_{i+1}^{(j)} - \sum_{k=1}^{r} F_i^{(j,k)} \Pi_0 dh_1^{(k)},\end{aligned}} \tag{5.1.26}$$

where $F_i^{(j,k)} = Z_k(dh_i^{(j)})$ and can be restricted to any leave S_c of Π_0

$$\boxed{\pi_1 dh_i^{(j)} = \pi_0 dh_{i+1}^{(j)} - \sum_{k=1}^{r} F_i^{(j,k)} \pi_0 dh_1^{(k)},} \tag{5.1.27}$$

where

$$\pi_1 = \Pi_{D|S_c}, \quad \pi_0 = \Pi_{0|S_c}$$

are Poisson restrictions of Π_D and Π_0 to S_c and $h_i^{(j)}$ are functions on S_c with constant values of c_k coordinates. Obviously, from the construction, π_0 and π_1 are compatible Poisson tensors on S_c and π_0 is nondegenerate. Hence, $\omega_0 = \pi_0^{-1}$ is symplectic two-form on S_c. Moreover, on S_c we have

$$\pi_0(dh_i^{(j)}, dh_l^{(k)}) = 0, \quad \pi_1(dh_i^{(j)}, dh_l^{(k)}) = 0.$$

The involutivity with respect to π_0 is obvious, while the involutivity with respect to π_1 follows from (5.1.22). Relations (5.1.27) are called the *quasi-bi-Hamiltonian* representation of the Liouville integrable system [60, 203].

Application of the presented reduction procedure to bi-Hamiltonian chains for Hamiltonians (5.1.14), which are written directly in separation coordinates (λ, μ, c) is a trivial task. With a natural choice of transversal distribution $\mathcal{Z}$ by $Z_i = \frac{\partial}{\partial c_i}$, $i = 1, \ldots, r$, on each symplectic leaf of Π_0 (5.1.18a), i.e. $c_i = const, i = 1, \ldots, r$, Hamiltonians (5.1.14) form a quasi-bi-Hamiltonian chains (5.1.27) with $F_i^{(j,k)}$ being solutions of (5.1.17) and with two nondegenerate Poisson structures

$$\pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix},$$

being reductions of (5.1.18). Notice that in particular, for $c_i = 0, i = 1, \ldots, r$, Hamiltonians (5.1.13) form the same quasi-bi-Hamiltonian chains on a bi-Poisson manifold (M, π_0, π_1).

5.1.3 Bi-Simplectic Manifolds

Let us consider a bi-Poisson manifold (S, π_0, π_1), where π_0 is invertible and compatible with π_1. Then consider a following $(1, 1)$-tensor field

$$N = \pi_1 \pi_0^{-1} = \pi_1 \omega_0, \quad \omega_0 = \pi_0^{-1}, \tag{5.1.28}$$

called a *recursion operator* with the dual

$$N^* = N^T = \omega_0 \pi_1. \tag{5.1.29}$$

Notice that

$$N\pi_0 = \pi_1, \qquad N^* \omega_0 = \omega_0 \pi_1 \omega_0 := \omega_1. \tag{5.1.30}$$

First, we show that from the compatibility of π_0 and π_1 follows that the Nijenhuis torsion $T(N)$ of tensor N vanishes on the image of π_0. We recall that the Nijenhuis torsion of a $(1, 1)$-tensor N on a manifold S is the $(1, 2)$-tensor $T(N)$, such that for all vector fields $v \in TS$

$$\boxed{T(N)v = L_{Nv} N - N L_v N.} \tag{5.1.31}$$

The alternative definition is given by the formula

$$\boxed{T(N)(v, w) = [Nv, Nw] - N[Nv, w] - N[v, Nw] + N^2[v, w].} \tag{5.1.32}$$

Indeed

$$\begin{aligned} T(N)(v,w) &= L_{Nv}(Nw) - NL_{Nv}w - NL_v(Nw) + N^2L_vw \\ &= (L_{Nv}N)w - N(L_vN)w = (L_{Nv}N - NL_vN)w. \end{aligned}$$

In order to show the vanishing of $T(N)$ it is enough to show that $T(N)$ vanishes on any pair of vectors $(\pi_0 df, \pi_0 dh)$ where $f, h \in \mathcal{F}(S)$. In fact, using the notation from (5.1.3)

$$\begin{aligned} T(N)(\pi_0 df, \pi_0 dh) =& [N\pi_0 df, N\pi_0 dh] - N[N\pi_0 df, \pi_0 dh] - N[\pi_0 df, N\pi_0 dh] \\ &+ N^2[\pi_0 df, \pi_0 dh] \\ =& [Y_f, Y_h] - N[Y_f, X_h] - N[X_f, Y_h] + N^2[X_f, X_h] \\ =& Y_{\{f,h\}_{\pi_1}} - N(X_{\{f,h\}_{\pi_1}} + Y_{\{f,h\}_{\pi_0}}) + N^2 X_{\{f,h\}_{\pi_0}}, \end{aligned}$$

which vanishes since $NX_g = Y_g$, $g \in \mathcal{F}(S)$. Bi-Poisson manifolds (S, π_0, π_1) with the tensor π_0 invertible and the tensor $N = \pi_1\pi_0^{-1}$ of vanishing torsion, are also called ωN manifolds and were studied for example in [173] and [187].

Second, we show that from the compatibility of π_0 and π_1 follows that the second two-form ω_1 (5.1.30) is closed on S. First observe that as π_0 and $\omega_0 = \pi_0^{-1}$ are nondegenerate on S, so any vector field $v \in TS$ can be represented by $v = \pi_0\gamma$ for some $\gamma \in T^*S$ and any γ can be represented by $\gamma = \omega_0 v$ for some $v \in TS$. Moreover, from relations (3.2.21) follows that for closed ω_0 we have $L_v\omega_0 = d(\omega_0 v) = d\gamma$ and for Poisson pencil π_λ

$$0 = L_{\pi_\lambda\gamma}\pi_\lambda + \pi_\lambda(d\gamma)\pi_\lambda$$

$$\Updownarrow$$

$$\begin{aligned} 0 &= L_{\pi_1\gamma}\pi_0 + L_{\pi_0\gamma}\pi_1 + \pi_1(d\gamma)\pi_0 + \pi_0(d\gamma)\pi_1 \\ &= L_{\pi_0\omega_0\pi_1\gamma}\pi_0 + L_v\pi_1 + \pi_1(d\gamma)\pi_0 + \pi_0(d\gamma)\pi_1 \\ &= -\pi_0 d(\omega_0\pi_1\gamma)\pi_0 + L_v\pi_1 + \pi_1(d\gamma)\pi_0 + \pi_0(d\gamma)\pi_1. \end{aligned}$$

Multiplying from left and right by ω_0 we get

$$\begin{aligned} 0 &= -d(\omega_0\pi_1\gamma) + \omega_0(L_v\pi_1)\omega_0 + \omega_0\pi_1(d\gamma) + (d\gamma)\pi_1 \\ &= -d(\omega_0\pi_1\omega_0 v) + \omega_0(L_v\pi_1)\omega_0 + \omega_0\pi_1 L_v\omega_0 + (L_v\omega_0)\pi_1\omega_0 \\ &= -d(\omega_0\pi_1\omega_0 v) + L_v(\omega_0\pi_1\omega_0) \end{aligned}$$

and hence, according to (3.2.21), $\omega_1 = \omega_0\pi_1\omega_0$ is closed.

In our further considerations we will assume that the recursion operator N has, at every point of S, n distinct and different from zero double eigenvalues $\lambda_1, \ldots, \lambda_n$, functionally independent as a functions of oryginal variables. Such an operator will be called a *regular recursion operator*. In such a generic case the second closed two-form ω_1 is symplectic.

Concluding, if S is a bi-Poisson manifold (S, π_0, π_1) where π_0 is invertible and compatible with π_1, and where $N = \pi_1 \pi_0^{-1}$ is a regular recursion operator, then S is simultaneously a bi-symplectic manifold (S, ω_0, ω_1), where both symplectic forms are given by

$$\omega_0 = \pi_0^{-1}, \quad \omega_1 = \omega_0 \pi_1 \omega_0 = \pi_0^{-1} \pi_1 \pi_0^{-1}.$$

Now we are prepared to introduce the notion of compatibility for closed forms, at least in the case when one of them is symplectic. The compatibility of two presymplectic forms will be defined in the next subsection. Let S be a manifold of even dimension with a symplectic-implectic pair $(\omega_0, \pi_0 = \omega_0^{-1})$, i.e a nondegenerate case of a dual pair (3.2.25). We say that a closed two-form ω_1 is *d-compatible* with a symplectic form ω_0 if $\pi_0 \omega_1 \pi_0$ is a Poisson tensor. We say that a Poisson tensor π_1 is d-compatible with an implectic tensor π_0 if $\omega_0 \pi_1 \omega_0$ is closed.

What is important, in the case considered, i.e. when π_0 is nondegenerate, the notions of d-compatibility and compatibility of Poisson tensors are equivalent. Actually, we proved that if π_0 and π_1 are compatible then are d-compatible. As the presented proof works in both directions, so in this case, from d-compatibility follows also compatibility of π_0 and π_1.

Let (S, ω) be a symplectic manifold, with $\dim S = 2n$. A Lagrangian distribution on S is a n-dimensional distribution D such that $\omega(\mathbf{x}_1, \mathbf{x}_2) = 0$ for all vector fields $\mathbf{x}_1, \mathbf{x}_2 \in D$. So, for the quasi-bi-Hamiltonian system (5.1.27) and closed two-forms ω_0, ω_1 (5.1.3), using the relation (5.1.2) and notation $\mathbf{x}_i^{(j)} = \pi_0 dh_i^{(j)}$, we have

$$\begin{aligned} \omega_0(\mathbf{x}_i^{(j)}, \mathbf{x}_l^{(k)}) &= \left\langle \omega_0 \mathbf{x}_i^{(j)}, \mathbf{x}_l^{(k)} \right\rangle = \left\langle \omega_0 \pi_0 dh_i^{(j)}, \pi_0 dh_l^{(k)} \right\rangle \\ &= \left\langle dh_i^{(j)}, \pi_0 dh_l^{(k)} \right\rangle = \pi_0(dh_i^{(j)}, dh_l^{(k)}) = 0, \end{aligned}$$

and

$$\begin{aligned} \omega_1(\mathbf{x}_i^{(j)}, \mathbf{x}_l^{(k)}) &= \left\langle \omega_1 \mathbf{x}_i^{(j)}, \mathbf{x}_l^{(k)} \right\rangle = \left\langle \omega_0 \pi_1 \omega_0 \mathbf{x}_i^{(j)}, \mathbf{x}_l^{(k)} \right\rangle \\ &= \left\langle \omega_0 \pi_1 dh_i^{(j)}, \mathbf{x}_l^{(k)} \right\rangle = -\left\langle \omega_0 \mathbf{x}_l^{(k)}, \pi_1 dh_i^{(j)} \right\rangle \\ &= -\left\langle dh_l^{(k)}, \pi_1 dh_i^{(j)} \right\rangle = \pi_1(dh_i^{(j)}, dh_l^{(k)}) = 0. \end{aligned}$$

Hence, in the generic case, the n-dimensional integrable distribution

$$D = Sp\{\mathbf{x}_i^{(j)} = \pi_0 dh_i^{(j)}\} \subset TS$$

is bi-Lagrangian as it is Lagrangian with respect to two different symplectic forms.

Now, using relations (5.1.28)–(5.1.30), quasi-bi-Hamiltonian chains (5.1.27) are equivalent to quasi-bi-symplectic chains

$$\omega_1 \mathbf{x}_i^{(j)} = \omega_0 \mathbf{x}_{i+1}^{(j)} - \sum_{k=1}^{r} F_i^{(j,k)} \omega_0 \mathbf{x}_1^{(k)}. \tag{5.1.33}$$

There are also two other equivalent representations

$$N\mathbf{x}_i^{(j)} = \mathbf{x}_{i+1}^{(j)} - \sum_{k=1}^{m} F_i^{(j,k)} \mathbf{x}_1^{(k)} \tag{5.1.34}$$

and

$$\boxed{N^* dh_i^{(j)} = dh_{i+1}^{(j)} - \sum_{k=1}^{r} F_i^{(j,k)} dh_1^{(k)}.} \tag{5.1.35}$$

Renumbering Hamiltonian functions $(h_1^{(1)}, \ldots, h_{n_r}^{(r)}) = (h_1, \ldots, h_n)$ and respective vector fields $(\mathbf{x}_1^{(1)}, \ldots, \mathbf{x}_{n_r}^{(r)}) = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ relations (5.1.34) and (5.1.35) can be written in compact forms

$$N\mathbf{x}_i = \sum_{j=1}^{n} F_{ij} \mathbf{x}_j, \quad i = 1, \ldots, n, \tag{5.1.36}$$

and

$$\boxed{N^* dh_i = \sum_{j=1}^{n} F_{ij} dh_j, \quad i = 1, \ldots, n,} \tag{5.1.37}$$

where matrix F is called a *control matrix* [113]. Relation (5.1.36) says that bi-Lagrangian distribution (5.1.3) is invariant with respect to N and relation (5.1.37) says that the subspace spanned by $(dh_1, \ldots, dh_n)$ is invariant with respect to N^*. In particular, for $c_i = 0$, $i = 1, \ldots, r$, the subspace spanned by $(dH_1, \ldots, dH_n)$ is also invariant with respect to N^*

$$N^* dH_i = \sum_{j=1}^{n} F_{ij} dH_j, \quad i = 1, \ldots, n.$$

5.1.4 Stäckel Separability on Bi-Symplectic Manifolds

A set of local coordinates (ξ, ζ) on the bi-symplectic manifold (ωN manifold) is called a set of *Darboux-Nijenhuis (DN) coordinates* if they are canonical with respect to a symplectic form ω_0

$$\omega_0 = \sum_{i=1}^{n} d\zeta_i \wedge d\xi^i$$

and diagonalize the recursion operator

$$N = \sum_{i=1}^{n} \lambda^i(\xi^i, \zeta_i) \left(\partial_{\xi^i} \otimes d\xi^i + \partial_{\zeta_i} \otimes d\zeta_i \right),$$

thus

$$\pi_1 = \sum_{i=1}^{n} \lambda^i \partial_{\xi^i} \wedge \partial_{\zeta_i}.$$

The existence of DN coordinates is a consequence of vanishing of the Nijenhuis torsion $T(N)$ of operator N [123, 247, 250]. The double eigenvalues λ^i follow from the implectic-symplectic factorization of N (5.1.28). The coordinates $(\lambda^1, \ldots, \lambda^n, \mu_1, \ldots \mu_n)$, where μ_i are momenta canonically conjugate to positions λ^i, are called special DN coordinates and will be identified with separation coordinates. It means that in the (λ, μ) coordinates

$$\pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \Lambda_n & 0 \\ 0 & \Lambda_n \end{pmatrix},$$

where $\Lambda_n = diag(\lambda_1, \ldots, \lambda_n)$, and their differentials span the $T^*\mathcal{N}$ which is an eigenspace of N^* (the adjoint of N), as

$$N^* d\lambda_i = \lambda_i d\lambda_i, \quad N^* d\mu_i = \lambda_i d\mu_i, \quad i = 1, \ldots, n.$$

A function f on ωN manifold is said to be a Stäckel function if its differential is an eigenfunction of N^*

$$N^* df = \lambda^i df.$$

The immediate consequence of such a definition is that $f = f(\lambda^i, \mu_i)$. As the elements of Stäckel matrix are Stäckel functions, hence

$$N^* dS = \Lambda dS \Longleftrightarrow N^* dS_{ij} = \lambda^i dS_{ij}.$$

In previous subsections, using the reduction procedure, we proved that the subspace $(dh_1, \ldots, dh_n)$, spanned by differentials of Stäckel Hamiltonians (5.1.14) or in particular (5.1.13) ones, is invariant with respect to N^*. Here we show the simple alternative proof, revealing simultaneously the structure of the control matrix F (5.1.37). Let us start with general Stäckel separation relations (4.2.9)

$$\varphi_i(\lambda^i, \mu_i, h_1, \ldots, h_n) = \sum_{k=1}^{n} S_i^k(\lambda^i, \mu_i)h_k - \psi(\lambda^i, \mu_i) = 0, \quad i = 1, \ldots, n. \tag{5.1.38}$$

Differentiate the relations (5.1.38)

$$\frac{\partial \varphi_i}{\partial \lambda^i} d\lambda^i + \frac{\partial \varphi_i}{\partial \mu_i} d\mu_i + \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} dh_j = 0,$$

and then apply N^* to obtain

$$\frac{\partial \varphi_i}{\partial \lambda^i} \lambda^i d\lambda^i + \frac{\partial \varphi_i}{\partial \mu_i} \lambda^i d\mu_i + \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} N^* dh_j = 0.$$

It follows that

$$\sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} N^* dh_j = -\lambda^i \left(\frac{\partial \varphi_i}{\partial \lambda^i} d\lambda^i + \frac{\partial \varphi_i}{\partial \mu_i} d\mu_i \right) = \lambda^i \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial h_j} dh_j,$$

or in the matrix form

$$SN^* dh = \Lambda S dh \Longrightarrow F = S^{-1} \Lambda S, \tag{5.1.39}$$

where $S_{ij} = \frac{\partial \varphi_i}{\partial h_j}$ are elements of the Stäckel matrix, $dh = (dh_1, \ldots, dh_n)^T$, $N^* dh = (N^* dh_1, \ldots, N^* dh_n)^T$ and so, Eq. (5.1.39) coincides with Eqs. (5.1.37). Thus, the control matrix F is nothing but the recursion matrix (4.3.1), considered in the previous chapter for particular separation relations (4.3.10). From (5.1.39) follows that F matrix has n distinct eigenvalues λ^i and the i-th row $S_i := (S_{i1}, \ldots, S_{in})$ of a Stäckel matrix S is a related left eigenvector

$$S_i F = \lambda^i S_i.$$

Moreover, for the Stäckel separability, the control matrix F satisfies

$$N^* dF = F dF \Longleftrightarrow N^* dF_{ij} = \sum_{k=1}^{n} F_{ik} dF_{kj}, \quad i, j = 1, \ldots, n. \tag{5.1.40}$$

Indeed

$$\begin{aligned} N^*dF &= N^*d(S^{-1}\Lambda S) = N^*(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS) \\ &= -S^{-1}\Lambda dSS^{-1}\Lambda S + S^{-1}\Lambda d\Lambda S + S^{-1}\Lambda^2 dS \\ &= S^{-1}\Lambda S(-S^{-1}dSS^{-1}\Lambda S + S^{-1}d\Lambda S + S^{-1}\Lambda dS) = FdF. \end{aligned}$$

Condition (5.1.40) is also sufficient for the Stäckel separability. Actually, let $(h_1, \ldots, h_n)$ be independent functions, defining a bi-Lagrangian foliation on a regular ωN manifold (5.1.37). If the control matrix F fulfills additionally relation (5.1.40), then the left eigenvectors of F, if suitably normalized, form a Stäckel matrix and the functions $(h_1, \ldots, h_n)$ are a Stäckel separable in DN coordinates. The reader can find the proof of the sufficient condition in [113].

So, up to now, we have demonstrated that a n-tuple $(h_1^{(1)}, \ldots, h_{n_r}^{(r)})$ of functions which fulfill separation relations (5.1.15) form Gel'fand–Zakharevich bi-Hamiltonian chains (5.1.6). On the other hand, we have demonstrated that an n-tuple $(h_1^{(1)}, \ldots, h_{n_r}^{(r)})$ of functions from Gel'fand–Zakharevich bi-Hamiltonian chains (5.1.6) are a Stäckel separable if a Poisson pencil Π_λ is reducible onto a symplectic foliation S_c of Π_0 (5.1.34)–(5.1.37), the recursion operator N on each leave is regular and the control matrix F fulfills the condition (5.1.40).

Now, for a quasi-bi-Hamiltonian chains (5.1.27), given in some local coordinates (x, p) on a $2n$-dimensional phase space, we construct a transformation to separation coordinates (λ, μ), i.e. special DN coordinates and find the explicit form of separation relations (5.1.15) following [113]. First, we find λ^i coordinates which, by definition, are roots of the minimal polynomial of N or, equivalently, roots of a characteristic polynomial of the control matrix F :

$$\begin{aligned} 0 &= \sqrt{\det(\lambda I - N(x, p))} \\ &= \det(\lambda I - F(x, p)) = \lambda^n + \rho_1\lambda^{n-1} + \ldots + \rho_n \equiv B(\lambda), \end{aligned} \tag{5.1.41}$$

where $\rho_i(\lambda)$ are Viète polynomials (4.3.18). Before we go further on, we will show the alternative construction of λ^i coordinates. Let us define the $r \times r$ matrix $F(\lambda)$

$$F(\lambda) = \begin{pmatrix} Z_1(h^{(1)}(\lambda)) & \cdots & Z_1(h^{(r)}(\lambda)) \\ \vdots & & \vdots \\ Z_r(h^{(1)}(\lambda)) & \cdots & Z_r(h^{(r)}(\lambda)) \end{pmatrix}.$$

The eigenvalues of N (F) can be easily obtained from the matrix $F(\lambda)$. Actually, the determinant of $F(\lambda)$ is the characteristic polynomial of F

$$\det F(\lambda) = \det(\lambda I - F) = B(\lambda). \tag{5.1.42}$$

In order to show it, let us differentiate the separation relations (5.1.15) with respect to Z_k

$$0 = \sum_{k=1}^{r} \beta^k(\lambda^i)\partial_{Z_m} h^{(k)}(\lambda^i) = (\beta^1(\lambda^i), \ldots, \beta^r(\lambda^i))F(\lambda^i), \quad m = 1, \ldots, r, \quad i = 1, \ldots, n. \tag{5.1.43}$$

This shows that $\det F(\lambda^i) = 0$. Since $\det F(\lambda)$ is a monic degree n polynomial and λ^i are distinct, we conclude that (5.1.42) holds. Moreover, from Eqs. (5.1.43) we can calculate Casimir multipliers β^k from separation relations (5.1.15). Obviously, Eqs. (5.1.43) have infinitely many solutions with respect to β^k unless we chose one by fixing the normalization, $\beta^r = 1, i = 1, \ldots, n$ in our case.

Finally, let us describe a procedure for constructing conjugate momenta μ_i. First observe that

$$Y = \pi_0 d\rho_1 = \sum_{i=1}^{n} \frac{\partial}{\partial \mu_i}, \quad \rho_1 = -\tfrac{1}{2}\mathrm{tr}N = -\mathrm{tr}F = -(\lambda^1 + \ldots + \lambda^n).$$

Then, notice that $(\beta^1(\lambda^i), \ldots, \beta^r(\lambda^i))F(\lambda^i) = \psi_i(\lambda^i, \mu_i)$ is a Stäckel function. Applying successively Y vector to $(\beta^1(\lambda^i), \ldots, \beta^r(\lambda^i))F(\lambda^i)$ find such $f_i(x, p, \lambda^i)$ that

$$Y(f_i(x, p, \lambda^i)) = 1, \quad i = 1, \ldots, n$$

and define

$$\mu_i = f_i(x, p, \lambda^i), \quad i = 1, \ldots, n \tag{5.1.44}$$

which is an admissible canonical momentum conjugated with λ^i.

In order to demonstrate how efficient the presented procedure of separability is, let us start with a few instructive examples. More systematic application of bi-Hamiltonian separability theory is presented in the next section. First of all, notice that if the Poisson structure Π_0 is canonical and

$$\frac{\partial}{\partial c_k}\Pi_1^{ij} = 0, \quad i, j = 1, \ldots, 2n, \quad k = 1, \ldots, r, \tag{5.1.45}$$

then a desired transversal distribution is given simple by $\{Z_k = \partial_{c_k}\}_{k=1,\ldots,r}$. In fact, if we define Π_D

$$\Pi_D = \Pi_1 - \sum_{k=1}^{r} X_1^{(k)} \wedge \frac{\partial}{\partial c_k},$$

so that

$$\Pi_D^{ij} = \begin{cases} \Pi_1^{ij}, & i,j = 1,\ldots,2n, \\ \\ 0, & 2n < i,j \leq 2n+r, \end{cases}$$

then, according to considerations from Sect. 3.4.1, Π_D is Poisson and compatible with Π_0 if $L_{Z_k}\Pi_D = 0,\ k = 1,\ldots,r$, which reduces to (5.1.45). It is the case of our first following three examples. The first example is a one-Casimir case with a separation curve.

Example 5.5 Let us separate the Henon–Heiles system from Example 5.1. It is a one-Casimir case with one-dimensional transversal distribution generated by $Z = \frac{\partial}{\partial c}$. Indeed, one can check that

$$L_Z\Pi_0 = 0, \quad L_Z\Pi_1 = Y \wedge Z,$$

where

$$Y = [Z, X_1] = \Pi_0 d(Z(h_1)) = \frac{\partial}{\partial p_x}.$$

The Casimir multiplier is trivial and hence, $F(\lambda)$ matrix is one-dimensional

$$F(\lambda) = \frac{\partial}{\partial c}h(\lambda) = \lambda^2 - x\lambda - \tfrac{1}{4}y^2 = \lambda^2 + \rho_1\lambda + \rho_2$$

$$\Downarrow$$

$$\lambda^1 + \lambda^2 = x, \quad \lambda^1\lambda^2 = -\tfrac{1}{4}y^2.$$

Alternatively, one obtains the same result from (5.1.41), where

$$N = \pi_1\pi_0^{-1} = \begin{pmatrix} 0 & 0 & x & \frac{1}{2}y \\ 0 & 0 & \frac{1}{2}y & 0 \\ -x & -\frac{1}{2}y & 0 & \frac{1}{2}p_y \\ -\frac{1}{2}y & 0 & -\frac{1}{2}p_y & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} x & \frac{1}{2}y & 0 & 0 \\ \frac{1}{2}y & 0 & 0 & 0 \\ 0 & \frac{1}{2}p_y & x & \frac{1}{2}y \\ -\frac{1}{2}p_y & 0 & \frac{1}{2}y & 0 \end{pmatrix}$$

and

$$F = \begin{pmatrix} x & 1 \\ \frac{1}{4}y^2 & 0 \end{pmatrix}.$$

As (x, p) are canonical coordinates and the transformation to separation coordinates is a point transformation, we could calculate the missing part of a canonical transformation using formula (4.1.37). Nevertheless, we demonstrate how works the general construction (5.1.44). First notice that

$$Y = \pi_0 d\rho_1 = \frac{\partial}{\partial p_x}$$

and

$$Y(h(\lambda)) = p_x\lambda + \tfrac{1}{2}yp_y, \quad Y^2(h(\lambda)) = \lambda,$$

hence

$$Y\left(\frac{Y(h(\lambda))}{Y^2(h(\lambda))}\right) = 1.$$

As a result

$$\mu_1 = \frac{Y(h(\lambda^1))}{Y^2(h(\lambda^1))} = p_x + \tfrac{1}{2}\frac{yp_y}{\lambda^1} = p_x - \frac{(-\lambda^1\lambda^2)^{1/2}}{\lambda^1}p_y,$$

$$\mu_2 = \frac{Y(h(\lambda^2))}{Y^2(h(\lambda^2))} = p_x + \tfrac{1}{2}\frac{yp_y}{\lambda^2} = p_x - \frac{(-\lambda^1\lambda^2)^{1/2}}{\lambda^2}p_y,$$

so that

$$p_x = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}, \quad p_y = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right)$$

and we reconstructed the transformation to separation coordinates from Example 4.11, given there ad hoc. Evaluating $h(\lambda^i)$ we recover separation relations in the form

$$c(\lambda^1)^2 + h_1\lambda^1 + h_2 = \tfrac{1}{2}\lambda^1\mu_1^2 + (\lambda^1)^4,$$
$$c(\lambda^2)^2 + h_1\lambda^2 + h_2 = \tfrac{1}{2}\lambda^2\mu_2^2 + (\lambda^2)^4$$

which are two copies of the separation curve

$$c\lambda^2 + h_1\lambda + h_2 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4.$$

The second example is also a one-Casimir case without a separation curve.

Example 5.6 Let us separate the Kepler problem on the plane considered in Example 5.2. Like in previous example, it is a one-Casimir case with one-dimensional transversal distribution generated by $Z = \frac{\partial}{\partial c}$. As

$$F = \lambda^2 - y\lambda - \tfrac{1}{4}x^2,$$

so

$$\lambda^1 + \lambda^2 = y, \quad \lambda^1\lambda^2 = -\tfrac{1}{4}x^2.$$

Moreover

$$Y = \pi_0 d\rho_1 = \frac{\partial}{\partial p_y},$$

hence

$$\mu_1 = \frac{Y(h(\lambda^1))}{Y^2(h(\lambda^1))} = p_y + \frac{1}{2}\frac{xp_x}{\lambda^1} = p_y - \frac{(-\lambda^1\lambda^2)^{1/2}}{\lambda^1}p_x,$$

$$\mu_2 = \frac{Y(h(\lambda^2))}{Y^2(h(\lambda^2))} = p_y + \frac{1}{2}\frac{xp_x}{\lambda^2} = p_y - \frac{(-\lambda^1\lambda^2)^{1/2}}{\lambda^2}p_x,$$

so that

$$p_x = \sqrt{-\lambda^1\lambda^2}\left(\frac{\mu_1}{\lambda^1 - \lambda^2} + \frac{\mu_2}{\lambda^2 - \lambda^1}\right), \quad p_y = \frac{\lambda^1\mu_1}{\lambda^1 - \lambda^2} + \frac{\lambda^2\mu_2}{\lambda^2 - \lambda^1}.$$

Again, evaluating $h(\lambda^i)$ we recover separation relations in the form

$$c(\lambda^1)^2 + h_1\lambda^1 + h_2 = \tfrac{1}{2}\lambda^1\mu_1^2 - \tfrac{1}{2}a,$$
$$c(\lambda^2)^2 + h_1\lambda^2 + h_2 = \tfrac{1}{2}\lambda^2\mu_2^2 + \tfrac{1}{2}a$$

for $\lambda^1 > \lambda^2$ and $a \to -a$ for $\lambda^2 > \lambda^1$. These separation relations are not related to any separation curve.

The third example is a two-Casimir case again with separation curve.

Example 5.7 Let us separate the system from Example 5.3. It is two-Casimir case with two-dimensional transversal distribution generated by $Z_1 = \frac{\partial}{\partial c_1}$, $Z_2 = \frac{\partial}{\partial c_2}$, hence

$$F(\lambda) = \begin{pmatrix} \lambda - (x + \frac{1}{4}x^{-1}y^2) & \frac{1}{16}x^{-1}y^4 \\ -x^{-1} & \lambda + \frac{1}{4}x^{-1}y^2 \end{pmatrix}$$

and

$$\det F(\lambda) = \lambda^2 - x\lambda - \tfrac{1}{4}y^2 \implies \lambda^1 + \lambda^2 = x, \quad \lambda^1\lambda^2 = -\tfrac{1}{4}y^2, \tag{5.1.46}$$

so the transformation to separation coordinates is the same as in the Henon–Heiles case from Example 5.5. The Casimir multiplier is calculated from

$$\begin{pmatrix} \beta_i^1 & 1 \end{pmatrix} F(\lambda) = \begin{pmatrix} \beta_i^1[\lambda - (x + \frac{1}{4}x^{-1}y^2)] - x^{-1} \\ \frac{1}{16}\beta_i^1 x^{-1}y^4 + \lambda + \frac{1}{4}x^{-1}y^2 \end{pmatrix} = 0.$$

Substituting (5.1.46), both equations give the same result $\beta_i^1 = (\lambda^i)^2$. Evaluating $\beta_i^1 h^{(1)}(\lambda^i) + h^{(2)}(\lambda)$ we recover separation relations in the form

$$\begin{aligned} c_1(\lambda^1)^3 + h_1^{(1)}(\lambda^1)^2 + c_2\lambda^1 + h_1^{(2)} &= \tfrac{1}{2}\lambda^1\mu_1^2 + (\lambda^1)^4, \\ c_1(\lambda^2)^3 + h_1^{(1)}(\lambda^2)^2 + c_2\lambda^2 + h_1^{(2)} &= \tfrac{1}{2}\lambda^2\mu_2^2 + (\lambda^2)^4, \end{aligned}$$

which are two copies of the separation curve

$$c_1\lambda^3 + h_1^{(1)}\lambda^2 + c_2\lambda + h_1^{(2)} = \frac{1}{2}\lambda\mu^2 + \lambda^4.$$

Finally, we consider bi-Hamiltonian separability of the Euler top from Example 5.4

Example 5.8 Although the bi-Hamiltonian representation of Euler top (see Example 5.4) seems relatively simple, nevertheless, it is not such a simple task to find a one-dimensional distribution Z of the desired properties. An appropriate vector field is of the form

$$Z = \begin{pmatrix} \frac{\omega_1}{\omega_1^2+\omega_2^2} \\ \frac{\omega_2}{\omega_1^2+\omega_2^2} \\ \\ 0 \end{pmatrix}.$$

Indeed, as according to (2.5.8)

$$\Pi_0'[Z] = \begin{pmatrix} 0 & 0 & \frac{\omega_2}{\omega_1^2+\omega_2^2} \\ 0 & 0 & -\frac{\omega_1}{\omega_1^2+\omega_2^2} \\ -\frac{\omega 2}{\omega_1^2+\omega_2^2} & \frac{\omega_1}{\omega_1^2+\omega_2^2} & 0 \end{pmatrix},$$

$$\Pi_1'[Z] = \begin{pmatrix} 0 & 0 & -\frac{a_2\omega_2}{\omega_1^2+\omega_2^2} \\ 0 & 0 & \frac{a_1\omega_1}{\omega_1^2+\omega_2^2} \\ \frac{a_2\omega 2}{\omega_1^2+\omega_2^2} & -\frac{a_1\omega_1}{\omega_1^2+\omega_2^2} & 0 \end{pmatrix},$$

$$Z' = \begin{pmatrix} -\frac{\omega_1^2-\omega_2^2}{(\omega_1^2+\omega_2^2)^2} & -\frac{2\omega_1\omega_2}{(\omega_1^2+\omega_2^2)^2} & 0 \\ -\frac{2\omega_1\omega_2}{(\omega_1^2+\omega_2^2)^2} & \frac{\omega_1^2-\omega_2^2}{(\omega_1^2+\omega_2^2)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then, one can check that

$$L_Z\Pi_0 = 0, \quad L_Z\Pi_1 = Y \wedge Z,$$

where

$$Y = [Z, X] = \Pi_0 d(Z(h_1)) = \begin{pmatrix} \frac{2(a_1-a_2)\omega_1^2\omega_2\omega_3}{(\omega_1^2+\omega_2^2)^2} \\ \frac{2(a_1-a_2)\omega_1\omega_2^2\omega_3}{(\omega_1^2+\omega_2^2)^2} \\ -\frac{2(a_1-a_2)\omega_1\omega_2}{\omega_1^2+\omega_2^2} \end{pmatrix}$$

hence, according to (5.1.24) and (5.1.25), the Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_1$ can be reduced along the distribution Z. Now,

$$F(\lambda) = Z(\lambda h_0 + h_1) = \lambda + \frac{a_1\omega_1^2 + a_2\omega_2^2}{\omega_1^2 + \omega_2^2}$$

and

$$\det F(\lambda) = 0 \Longrightarrow \lambda^1 = -\frac{a_1\omega_1^2 + a_2\omega_2^2}{\omega_1^2 + \omega_2^2}.$$

Then,

$$\Pi_0 d(-\lambda_1) = \Pi_0 d(Z(h_1)) = Y$$

and one can check that

$$Y\left(\frac{Y(h(\lambda))}{Y^2(h(\lambda))}\right) = 1 \Longrightarrow \mu_1 = \frac{Y(h(\lambda))}{Y^2(h(\lambda))} = \tfrac{1}{2}\frac{\omega_3(\omega_1^2+\omega_2^2)}{(a_2-a_1)\omega_1\omega_2},$$

giving the following separation relation

$$c\lambda^1 + h_1 = -2(\lambda^1 + a_1)(\lambda^1 + a_2)(\lambda^1 + a_3)\mu_1^2.$$

The transformation between original coordinates $(\omega_1, \omega_2, \omega_3)$ and separation coordinates (λ^1, μ_1, c) is as follows

$$\lambda^1 = -\frac{a_1\omega_1^2 + a_2\omega_2^2}{\omega_1^2+\omega_2^2}, \quad \mu_1 = \tfrac{1}{2}\frac{\omega_3(\omega_1^2+\omega_2^2)}{(a_2-a_1)\omega_1\omega_2}, \quad c = \tfrac{1}{2}\omega_1^2 + \tfrac{1}{2}\omega_2^2 + \tfrac{1}{2}\omega_3^2.$$

There is a famous generalization of Euler top from $so(3)$ to $so(m)$, constructed by Manakov [188], which has also a bi-Hamiltonian formulation of the GZ type [117]. Nevertheless, it has not been separated yet. We even do not have any proof that it can be separated. Although we know [31] that separable systems have a bi-Hamiltonian extension to GZ chains, the inverse statement is not proved. As will be presented in the next section, the necessary and sufficient condition for separability of GZ chains is the so called d-compatibility of two Poisson structures, which is a stronger demand than ordinary compatibility.

5.2 Application of Bi-Hamiltonian Separability Theory

A reach source of finite dimensional bi-Hamiltonian systems are hierarchies of bi-Hamiltonian field systems, i.e. nonlinear PDE's known as Soliton equations. A systematic methods of their construction the reader can find in [23] and in literature quoted there. In particular, the construction of bi-Hamiltonian representation for stationary flows and constraint flows of Soliton field equations is presented. Thus we skip that class of examples. Here we will illustrate the bi-Hamiltonian separability theory, described in previous sections, on another classes of systems with arbitrary number of degrees of freedom.

5.2.1 Elliptic Separable Potentials

Consider a Liouville integrable geodesic Hamiltonians in $M = \mathbb{R}^{2n}$, given in Euclidean coordinates x^i and conjugate momenta p_i by

$$T_r = \frac{1}{2}\sum_{i=1}^{n}\left[\frac{\partial \rho_r(\beta)}{\partial \beta_i} - \frac{1}{4}\sum_{k=1,k\neq i}^{n}\frac{\partial^2 \rho_r(\beta)}{\partial \beta_i \partial \beta_k}(x^k)^2\right]p_i^2 + \frac{1}{8}\sum_{i,j=1,i\neq j}^{n}\frac{\partial^2 \rho_r(\beta)}{\partial \beta_i \partial \beta_j}x^i x^j p_i p_j, \tag{5.2.1}$$

for $r = 1, \ldots, n$, where $\rho_r(\beta)$ are Viète polynomials of n parameters β_i (signed elementary symmetric polynomials of β_i (5.5.3)). The extended systems in $M = \mathbb{R}^{2n+1}$

$$h_r = T_r + c\left[\rho_r(\beta) - \frac{1}{4}\sum_{i=1}^{n}\frac{\partial \rho_r(\beta)}{\partial \beta_i}(x^i)^2\right] \tag{5.2.2}$$

form a bi-Hamiltonian chain [24, 224]

$$\begin{aligned}
\Pi_0 dh_0 &= 0\\
\Pi_0 dh_1 &= X_1 = \Pi_1 dh_0\\
&\vdots\\
\Pi_0 dh_n &= X_n = \Pi_1 dh_{n-1}\\
0 &= \Pi_1 dh_n
\end{aligned} \tag{5.2.3}$$

where

$$\Pi_0 = \begin{pmatrix} 0 & I_n & 0\\ -I_n & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & B - \frac{1}{4}x\otimes x & p\\ -B + \frac{1}{4}x\otimes x & \frac{1}{4}p\otimes x - \frac{1}{4}x\otimes p & -\frac{1}{2}cx\\ -p & \frac{1}{2}cx & 0\end{pmatrix}, \tag{5.2.4}$$

$x = (x^1, \ldots, x^n)^T$, $p = (p_1, \ldots, p_n)^T$, $B = \mathrm{diag}(\beta_1, \ldots, \beta_n)$ and $h_0 = c$. Notice that

$$h_1 = \frac{1}{2}\sum_{i=1}^{n}p_i^2 - \frac{1}{4}c\sum_{i=1}^{n}\left[\beta_i + (x^i)^2\right].$$

According to the projection procedure

$$Z = \frac{\partial}{\partial c}, \quad Y = \frac{1}{2}\sum_{i=1}^{n}x^i\frac{\partial}{\partial p_i}$$

and

$$Z(h_0) = 1, \quad Y(h_0) = 0, \quad L_Z \Pi_0 = 0, \quad L_Z \Pi_1 = Y \wedge Z.$$

The first part of the transformation to separation coordinates is a point transformation

$$\rho_r(\lambda) = \rho_r(\beta) - \frac{1}{4} \sum_{i=1}^{n} \frac{\partial \rho_r(\beta)}{\partial \beta_i} (x^i)^2, \qquad r = 1, \ldots, n. \tag{5.2.5}$$

As is proved in Sect. 5.5.1 (see formulas (5.5.11), (5.5.20) and (5.5.13) for $\varepsilon = -1$), the relations (5.2.5) are equivalent to the following one

$$1 + \frac{1}{4} \sum_{j=1}^{n} \frac{(x^j)^2}{z - \beta_j} \equiv \frac{\prod_{j=1}^{n} (z - \lambda^j)}{\prod_{j=1}^{n} (z - \beta_j)}$$

which defines the generalized elliptic coordinates $(\lambda^1, \ldots, \lambda^n)$. Moreover,

$$(x^j)^2 = 4 \frac{\prod_{k=1}^{n} (\beta_j - \lambda^k)}{\prod_{\substack{k=1 \\ k \neq j}}^{n} (\beta_j - \beta_k)}, \qquad \mu_j = \frac{\partial x^i}{\partial \lambda^j} p_i, \qquad j = 1, \ldots n \tag{5.2.6}$$

and we get the following form of Hamiltonians (5.2.2) in separation coordinates

$$h_r = (-1)^r \frac{1}{2} \sum_{i=1}^{n} \frac{\partial \rho_r(\lambda)}{\partial \lambda^i} \frac{\prod_{k=1}^{n} (\lambda^i - \beta_k) \mu_i^2 + c \lambda_i^n}{\prod_{k=1, k \neq i}^{n} (\lambda^i - \lambda^k)}.$$

In consequence, the separation curve takes the form

$$c\lambda^n + h_1 \lambda^{n-1} + \ldots + h_n = \frac{1}{2} \prod_{k=1}^{n} (\lambda - \beta_k) \mu^2$$

and belongs to the Benenti class.

In the next step let us add some potentials to Hamiltonian functions (5.2.2)

$$h_r = T_r + V_r(x) + c\left[\rho_r(\beta) - \frac{1}{4}\sum_{i=1}^{n}\frac{\partial \rho_r(\beta)}{\partial \beta_i}(x^i)^2\right]. \tag{5.2.7}$$

It change the tensor Π_1 to the form

$$\Pi_1 = \begin{pmatrix} 0 & B - \frac{1}{4}x \otimes x & p \\ -B + \frac{1}{4}x \otimes x & \frac{1}{4}p \otimes x - \frac{1}{4}x \otimes p & -\frac{\partial}{\partial x}V_1(x) - \frac{1}{2}cx \\ -p & \frac{\partial}{\partial x}V_1(x) + \frac{1}{2}cx & 0 \end{pmatrix}. \tag{5.2.8}$$

As we demand from new Hamiltonians to constitute the bi-Hamiltonian chain (5.2.3), so Π_1 has to be a Poisson tensor and Jacobi equations (3.2.3) impose restrictions on $V_1(x)$ in the form

$$(\beta_i - \beta_j)\frac{\partial^2 V_1}{\partial x^i \partial x^j} + \frac{3}{4}\left(x^j\frac{\partial V_1}{\partial x^i} - x^i\frac{\partial V_1}{\partial x^j}\right) + \frac{1}{4}\sum_{k=1}^{n}\left(x^j x^k\frac{\partial^2 V_1}{\partial x^i \partial x^k} - x^i x^k\frac{\partial^2 V_1}{\partial x^j \partial x^k}\right) = 0, \tag{5.2.9}$$

where $i, j = 1, \ldots, n$. Notice that remaining potentials V_r are uniquely determined by V_1 and the chain (5.2.3). Such potentials are known as *elliptic separable potentials*, as sepatate in generalized elliptic coordinates, and are constructing systematically in Sect. 5.5.1.

Example 5.9 Garnier system with n degrees of freedom. Its Hamiltonian representation in canonical coordinates (x, p) is of the form [24, 68]

$$\begin{aligned} (x^i)_t &= p_i, \quad i = 1, \ldots, n, \\ (p_i)_t &= -\frac{1}{4}x^i\sum_{k=1}^{n}(x^k)^2 + \frac{1}{2}(\beta_i - c)x^i, \quad i = 1, \ldots, n \end{aligned}$$

generated by the Hamiltonian function

$$h_1 = \frac{1}{2}\sum_{i=1}^{n}p_i^2 + \frac{1}{16}\left[\sum_{i=1}^{n}(x^i)^2\right]^2 - \frac{1}{4}\sum_{i=1}^{n}\beta_i(x^i)^2 - c\sum_{i=1}^{n}\left[\beta_i - \frac{1}{4}(x^i)^2\right].$$

The potential

$$V_1 = \frac{1}{16}\left[\sum_{i=1}^{n}(x^i)^2\right]^2 - \frac{1}{4}\sum_{i=1}^{n}\beta_i(x^i)^2$$

fulfills condition (5.2.9) so h_1 belongs to bi-Hamiltonian hierarchy for elliptic potentials. For simplicity we illustrate the separability procedure for the case of $n = 2$. The two dimensional Garnier system

$$\begin{aligned}
(x^1)_t &= p_1\\
(x^2)_t &= p_2\\
(p_1)_t &= -\frac{1}{4}x^1\left[(x^1)^2+(x^2)^2\right]+\frac{1}{2}(\beta_1-c)x^1\\
(p_2)_t &= -\frac{1}{4}x^2\left[(x^1)^2+(x^2)^2\right]+\frac{1}{2}(\beta_2-c)x^2\\
c_t &= 0
\end{aligned}$$

is a Liouville integrable system with two constants of motion

$$h_1=\frac{1}{2}p_1^2+\frac{1}{2}p_2^2+\frac{1}{16}\left[(x^1)^2+(x^2)^2\right]^2-\frac{1}{4}\beta_1(x^1)^2-\frac{1}{4}\beta_2(x^2)^2+c\left[-\beta_1-\beta_2+\frac{1}{4}(x^1)^2+\frac{1}{4}(x^2)^2\right],$$

$$\begin{aligned}
h_2 = &-\frac{1}{2}\beta_2p_1^2-\frac{1}{2}\beta_1p_2^2+\frac{1}{8}(x^2p_1-x^1p_2)^2-\frac{1}{16}\left[(x^1)^2+(x^2)^2\right]\left[\beta_2(x^1)^2+\beta_1(x^2)^2\right]\\
&+\frac{1}{4}\beta_1\beta_2\left[(x^1)^2+(x^2)^2\right]-c\left[-\beta_1\beta_2+\frac{1}{4}\beta_2(x^1)^2+\frac{1}{4}\beta_1(x^2)^2\right]
\end{aligned}$$

which together with $h_0 = c$ belong to the bi-Hamiltonian chain (5.2.3), where

$$\Pi_0 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ -1 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & \beta_1-\frac{1}{4}(x^1)^2 & -\frac{1}{4}x^1x^2 & p_1\\ 0 & 0 & -\frac{1}{4}x^1x^2 & \beta_2-\frac{1}{4}(x^2)^2 & p_2\\ -\beta_1+\frac{1}{4}(x^1)^2 & \frac{1}{4}x^1x^2 & 0 & \frac{1}{4}(x^2p_1-x^1p_2) & -\frac{\partial h_1}{\partial x^1}\\ \frac{1}{4}(x^1)^2 & -\beta_2+\frac{1}{4}(x^2)^2 & -\frac{1}{4}(x^2p_1-x^1p_2) & 0 & -\frac{\partial h_1}{\partial x^2}\\ -p_1 & -p_2 & \frac{\partial h_1}{\partial x^1} & \frac{\partial h_1}{\partial x^2} & 0 \end{pmatrix}$$

From (5.2.6) we find the transformation to separation coordinates

$$x^1 = 2\sqrt{\frac{(\beta_1-\lambda^1)(\beta_1-\lambda^2)}{\beta_1-\beta_2}}, \qquad x^2 = 2\sqrt{\frac{(\beta_2-\lambda^1)(\beta_2-\lambda^2)}{\beta_2-\beta_1}},$$

$$p_1 = \sqrt{\frac{(\beta_1 - \lambda^1)(\beta_1 - \lambda^2)}{\beta_1 - \beta_2}} \left[\frac{(\beta_1 - \lambda^1)\mu_1}{\lambda^1 - \lambda^2} + \frac{(\beta_1 - \lambda^2)\mu_2}{\lambda^2 - \lambda^1} \right],$$

$$p_2 = \sqrt{\frac{(\beta_2 - \lambda^1)(\beta_2 - \lambda^2)}{\beta_2 - \beta_1}} \left[\frac{(\beta_2 - \lambda^1)\mu_1}{\lambda^1 - \lambda^2} + \frac{(\beta_2 - \lambda^2)\mu_2}{\lambda^2 - \lambda^1} \right]$$

and the separation curve of the form

$$c\lambda^2 + h_1\lambda + h_2 = (\lambda - \beta_1)(\lambda - \beta_2)(\frac{1}{2}\mu^2 + \lambda).$$

5.2.2 Bi-Cofactor Systems

Let us consider the system of differential equations in Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, g = I_n)$ of Newton form

$$x^i_{tt} = F^i(x), \quad i = 1, \ldots, n. \tag{5.2.10}$$

Notice, that in $\mathbb{E}^n$ and Euclidean coordinates, rising and lowering indices of any second order tensor with the help of metric tensor does not change its matrix representation. Thus, in the further notation, for a second order tensor A : $A^{ij}(x) = A^i_j(x) = A_{ij}(x)$. The function

$$H = \frac{1}{2}A^{ij}p_i p_j + V,$$

where $p_i = x^i_t$, $A = A(x)$ is a nondegenerate symmetric matrix and $V = V(x)$ is some function, is an integral of motion for the system (5.2.10) if and only if the following two conditions hold

$$\partial_i A^{jk} + \partial_k A^{ij} + \partial_j A^{ki} = 0, \quad i, j, k = 1, \ldots, n \tag{5.2.11}$$

and

$$A_{ij}F^j + (dV)_i = 0, \quad i = 1, \ldots, n. \tag{5.2.12}$$

Equations (5.2.11) implies that the matrix A is a Killing tensor of the Euclidean metric. We will restrict ourselves to a class of solutions of (5.2.11) that have the form

$$A = \mathrm{cof}(J)$$

with

$$J = \alpha\, x \otimes x + \beta \otimes x + x \otimes \beta + \gamma,$$

where cof means the cofactor matrix (so that $\mathrm{cof}(J)J = \det(J)$), $x = (x^1, \ldots, x^n)^T$, α is a real constant, $\beta = (\beta_1, \ldots, \beta_n)^T$ is a column vector of constants and γ is a symmetric $n \times n$ constant matrix. Equations (5.2.12) imply that the force F can be written in the *quasi-potential* form $F = -A^{-1}dV$.

In the case when the system (5.2.10) has the second, functionally independent of H, integral of motion of the form

$$\bar{H} = \frac{1}{2}\bar{A}^{ij} p_i p_j + \bar{V},$$

with an invertible matrix $\bar{A}(x)$, then it can be written in a quasi-potential form in two distinct ways. Actually, a system of equations

$$x_{tt} = F = -A^{-1}dV = -\bar{A}^{-1}d\bar{V} \tag{5.2.13}$$

where A and $\bar{A}$ are two linearly independent matrices of the cofactor form

$$A = \mathrm{cof}(J), \quad J = \alpha\, x \otimes x + \beta \otimes x + x \otimes \beta + \gamma$$

$$\bar{A} = \mathrm{cof}(\bar{J}), \quad \bar{J} = \bar{\alpha}\, x \otimes x + \bar{\beta} \otimes x + x \otimes \bar{\beta} + \bar{\gamma}$$

where $V = V(x)$ and $\bar{V} = \bar{V}(x)$ are two scalar functions, is called a *bi-cofactor system* [184, 194, 225].

If the Newton system (5.2.10) has a bi-cofactor form (5.2.13) then it has n integrals of motion of the form

$$H_k = \frac{1}{2}(A_k)^{ij} p_i p_j + V_k, \quad k = 1, \ldots, n$$

where the matrices A_k are defined as coefficients in the polynomial expansion of $\mathrm{cof}(J + \lambda\bar{J})$ with respect to the parameter λ [184]

$$\mathrm{cof}(J + \lambda\bar{J}) = \sum_{k=1}^{n} A_k \lambda^{n-k}$$

with $A_1 = \mathrm{cof}(\bar{J})$, $A_n = \mathrm{cof}(J)$ and where $V_1 = \bar{V}$ and $V_n = V$. Consequently, such a system can be written in a quasi-potential form in n distinct ways

$$x_{tt} = F = -A_k^{-1}dV_k, \quad k = 1, \ldots, n. \tag{5.2.14}$$

The admissible potentials V_k can be constructed recursively [184] through the formula

$$V^{(r+1)}(\lambda) = \frac{\det(J + \lambda \bar{J})}{\det(\bar{J})} V_1^{(r)} - \lambda V^{(r)}(\lambda)$$

and its inverse

$$V^{(r)}(\lambda) = \frac{1}{\lambda} \left(\frac{\det(J + \lambda \bar{J})}{\det(J)} V_n^{(r+1)} - V^{(r+1)}(\lambda) \right),$$

where $V^{(r)}(\lambda) = \sum_{k=1}^{n} V_k^{(r)} \lambda^{n-k}$. Starting from a trivial potential $V_i^{(0)} = \delta_{in}$ it is possible to construct the infinite hierarchy of positive and negative potentials $V_k^{(r)}$, $r \in \mathbb{Z}$.

The bi-cofactor system (5.2.13) can be embedded in a bi-Hamiltonian system on $M = T^*\mathbb{E}^n \times \mathbb{R}$. Actually, consider a following pencil $\Pi_\lambda = \Pi_1 - \lambda \Pi_0$

$$\Pi_0 = \begin{pmatrix} 0 & -\bar{J} & 0 \\ \bar{J} & -\bar{R} & -2c\bar{N} \\ 0 & c\bar{N} & 0 \end{pmatrix}, \quad \Pi_1 = \begin{pmatrix} 0 & J & p \\ -J & R & F + 2cN \\ -p & -F - cN & 0 \end{pmatrix}. \tag{5.2.15}$$

The $n \times n$ symmetric matrices J and $\bar{J}$ are exactly the matrices that defined our system (5.2.13), the $n \times 1$ matric N and $\bar{N}$ are given by

$$N = \alpha\, x + \beta, \qquad \bar{N} = \bar{\alpha}\, x + \bar{\beta}$$

and the $n \times n$ matrices R and $\bar{R}$ are defined by

$$R = N \otimes p - p \otimes N, \qquad \bar{R} = \bar{N} \otimes p - p \otimes \bar{N}.$$

It is a Poisson pencil due to the fact that the term F can be represented as (5.2.14). The bi-Hamiltonian chain (5.2.3) is generated by the Casimir function h_λ of Π_λ of the form

$$h_\lambda = \frac{1}{2} p^T \operatorname{cof}(J + \lambda \bar{J}) + V(\lambda) - c \det(J + \lambda \bar{J}),$$

where $V(\lambda) = V^{(r)}(\lambda)$, $r \in \mathbb{Z}$. Thus, $h_\lambda = \sum_{k=0}^{n} h_k \lambda^{n-k}$ with functions $h_k(x, p, c)$ given by

$$h_k(x, p, c) = H_k(x, p) - c D_k, \quad k = 1, \ldots, n, \quad h_0(x, c) = -c D_0$$

and with D_i defined as

$$\sum_{k=0}^{n} D_k \lambda^{n-k} = \det(J + \lambda \bar{J}),$$

so that $D_0 = \det(\bar{J})$ and $D_n = \det(J)$. Notice that coordinates (x, p, c) are non-canonical coordinates for both Poisson structures (5.2.15).

In order to separate the bi-Hamiltonian system we pass to new coordinates $(x, p, c) \longrightarrow (x, p, c')$, where $c' = h_0 = -cD_0$. In new coordinates the bi-Hamiltonian chain (5.2.3) is generated by

$$\Pi_0 = \begin{pmatrix} 0 & -\bar{J} & 0 \\ \bar{J} & -\bar{R} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & J & -\det(\bar{J})p \\ -J & R & -\det(\bar{J})F + c'(N - J\bar{J}^{-1}\bar{N}) \\ \det(\bar{J})p & \det(\bar{J})F - c'(N - J\bar{J}^{-1}\bar{N}) & 0 \end{pmatrix}$$

and

$$h_k(x, p, c') = H_k(x, p) + c' \frac{D_k}{D_0}, \quad k = 1, \ldots, n, \quad h_0(x, c) = c'. \tag{5.2.16}$$

For this new representation the projection procedure is as follows [189]

$$Z = \frac{\partial}{\partial c'}, \quad Y = \Pi_0 d(Z(h_1)) = \Pi_0 d\left(\frac{D_1}{D_0}\right),$$

$$L_Z \Pi_0 = 0, \quad L_Z \Pi_1 = Y \wedge Z,$$

and

$$Y^3\left(h_{\lambda_i}\right) = 0, \quad i = 1, \ldots, n. \tag{5.2.17}$$

From (5.2.16) follows that

$$Z(h_\lambda) = \frac{\det(J + \lambda \bar{J})}{\det(\bar{J})}$$

so $\lambda_i(x)$ are roots of $\det(J + \lambda \bar{J}) = 0$. From (5.2.17) follows that

$$\mu_i = \frac{Y\left(h_{\lambda^i}\right)}{Y^2\left(h_{\lambda^i}\right)}, \quad i = 1, \ldots, n. \tag{5.2.18}$$

Example 5.10 We consider 2-dimensional bi-cofactor system

$$x^1_{tt} = \frac{2x^1(x^2+1)}{[(x^1)^2 - 2x^2]},$$
$$x^2_{tt} = \frac{(x^2)^2}{[(x^1)^2 - 2x^2]},$$

generated by a pair of matrices

$$\bar{J} = \begin{pmatrix} 1 & x^1 \\ x^1 & 2x^2 \end{pmatrix}, \quad J = \begin{pmatrix} (x^1)^2+1 & x^1x^2 \\ x^1x^2 & (x^2)^2 \end{pmatrix}$$

and a pair of potential functions

$$V = \frac{x^2(x^2+2)}{2x^2 - (x^1)^2}, \quad \bar{V} = \frac{(x^2)^2}{2x^2 - (x^1)^2}.$$

Two constants of motion are

$$H_1 = x^2p_1^2 - x^1p_1p_2 + \frac{1}{2}p_2^2 + \frac{x^2(x^2+2)}{2x^2 - (x^1)^2},$$
$$H_2 = \frac{1}{2}(x^1)^2p_1^2 - x^1x^2p_1p_2 + \frac{1}{2}[(x^1)^2+1]p_2^2 + \frac{(x^2)^2}{2x^2 - (x^1)^2},$$

where $p_1 = x^1_t$, $p_2 = x^2_t$. As

$$\bar{\alpha} = 0, \quad \alpha = 1, \quad \bar{\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} \quad \bar{\gamma} = \gamma = \begin{pmatrix} 1\,0 \\ 0\,0 \end{pmatrix},$$

the extended Hamiltonians

$$h_1 = H_1 - c\,x^2(x^2+2), \quad h_2 = H_2 - c\,(x^2)^2, \quad \text{and} \quad h_0 = c[(x^1)^2 - 2x^2]$$

form the bi-Hamiltonian chain (5.2.3) with respect to two Poisson tensors

$$\Pi_0 = \begin{pmatrix} 0 & 0 & -1 & -x^1 & 0 \\ 0 & 0 & -x^1 & -2x^2 & 0 \\ 1 & x^1 & 0 & p_1 & 0 \\ x^1 & 2x^2 & -p_1 & 0 & -2c \\ 0 & 0 & 0 & 2c & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & 0 & (x^1)^2+1 & x^1x^2 & p_1 \\ 0 & 0 & x^1x^2 & (x^2)^2 & p_2 \\ -(x^1)^2-1 & -x^1x^2 & 0 & x^1p_2 - x^2p_1 & \frac{2x^1(x^1+1)}{[(x^1)^2-2x^2]^2} + 2cx^1 \\ -x^1x^2 & -(x^2)^2 & -x^1p_2 + x^2p_1 & 0 & \frac{2(x^2)^2}{[(x^1)^2-2x^2]^2} + 2cx^2 \\ & & * & & 0 \end{pmatrix}.$$

According to presented procedure, the first part of transformation to separation coordinates is given by

$$\frac{\det(J+\lambda\bar{J})}{\det(\bar{J})} = 0 \Longrightarrow \lambda^1+\lambda^2 = \frac{x^2(x^2+2)}{(x^1)^2-2x^2}, \quad \lambda^1\lambda^2 = -\frac{(x^2)^2}{(x^1)^2-2x^2}.$$

As

$$Y = \frac{2x^1(x^1+1)}{(x^1)^2-2x^2}\frac{\partial}{\partial p_1} + \frac{2(x^2)^2}{(x^1)^2-2x^2}\frac{\partial}{\partial p_2},$$

the second part of transformation to separation coordinates, according to (5.2.18), takes the form

$$\mu_i = \frac{1}{2}\frac{[(x^1)^2-2x^2]\,x^1x^2(x^2+\lambda^i x^2+2\lambda^i)p_1}{x^2\left[(x^2)^3+(x^1)^2x^2+2\lambda^i(x^1)^2+\lambda^i(x^2)^3+2\lambda^i(x^1)^2x^2\right]} + \frac{1}{2}\frac{[(x^1)^2-2x^2][(x^2)^2-(x^1)^2x^2+\lambda^i(x^2)^2-\lambda^i(x^1)^2-\lambda^i(x^1)^2x^2]p_2}{x^2\left[(x^2)^3+(x^1)^2x^2+2\lambda^i(x^1)^2+\lambda^i(x^2)^3+2\lambda^i(x^1)^2x^2\right]}, \quad i=1,2.$$

The solution is as follows

$$x^1 = -2\frac{\sqrt{-\lambda^1\lambda^2(\lambda^1+1)(\lambda^2+1)}}{\lambda^1+\lambda^2+\lambda^1\lambda^2}, \quad x^2 = -2\frac{\lambda^1\lambda^2}{\lambda^1+\lambda^2+\lambda^1\lambda^2},$$

$$p_1 = \sqrt{-\lambda^1\lambda^2(\lambda^1+1)(\lambda^2+1)}\left[\frac{(\lambda^1\lambda^2-\lambda^1+\lambda^2)\mu_1}{\lambda^2(\lambda^1-\lambda^2)} + \frac{(\lambda^1\lambda^2+\lambda^1-\lambda^2)\mu_2}{\lambda^1(\lambda^2-\lambda^1)}\right],$$

$$p_2 = -2\lambda^1\lambda^2\left[\frac{(\lambda^1+1)\mu_1}{\lambda^1-\lambda^2} + \frac{(\lambda^2+1)\mu_2}{\lambda^2-\lambda^1}\right]$$

with separation curve

$$h_0\lambda^2 + h_1\lambda + h_2 = 2\lambda^2(\lambda+1)\mu^2 + \lambda^2.$$

The theory of bi-cofactor systems extended from Euclidean space $\mathbb{E}^n$ to Riemannian space the reader can find in [70, 72, 73, 191].

5.2.3 Non-Periodic Toda Lattice

Let us consider a dynamical system on $M = \mathbb{R}^{2n+1}$

$$\begin{aligned} (a_i)_t &= a_i(b_i - b_{i+1}) = X_1^i, \qquad i = 1, \ldots, n \\ (b_i)_t &= 2(a_i^2 - a_{i-1}^2) = X_1^{n+i}, \qquad i = 1, \ldots, n+1 \end{aligned} \tag{5.2.19}$$

where $(a_1, \ldots, a_n, b_1, \ldots, b_{n+1})$ are some coordinates on M. The system (5.2.19) is known as non-periodic Toda lattice in Flaschka coordinates [121]. For arbitrary n it is bi-Hamiltonian Liouville integrable system [77], with first Poisson tensor given by the following non-zero elements

$$\Pi_0: \quad \{a_i, b_i\}_{\Pi_0} = -a_i, \quad \{a_i, b_{i+1}\}_{\Pi_0} = a_i$$

and the second Poisson tensor given by

$$\Pi_1: \quad \{a_i, a_{i+1}\}_{\Pi_1} = \tfrac{1}{2} a_i a_{i+1}, \quad \{a_i, b_i\}_{\Pi_1} = -a_i b_i,$$

$$\{a_i, b_{i+1}\}_{\Pi_0} = a_i b_{i+1}, \quad \{b_i, b_{i+1}\}_{\Pi_0} = 2a_i^2.$$

The bi-Hamiltonian chain is given by the Casimir of Poisson pencil $\Pi_\lambda = \Pi_1 - \lambda\Pi_0 : \Pi_\lambda dh_\lambda^{(n)} = 0$, where

$$dh_\lambda^{(n)} = \det(\lambda I - L_n) = \lambda^{n+1} + h_0^{(n)}\lambda^n + h_1^{(n)}\lambda^{n-1} + \ldots + h_n^{(n)},$$

$$L_n = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ 0 & a_2 & \ddots & \ddots & 0 \\ 0 & 0 & \ddots & b_n & a_n \\ 0 & 0 & 0 & a_n & b_{n+1} \end{pmatrix},$$

$h_0^{(n)}$ is the Casimir of Π_0 while $h_n^{(n)}$ is the Casimir of Π_1. The system (5.2.19) is generated by Hamiltonian vector field $X_1 = \Pi_0 dh_1^{(n)}$. Moreover, Hamiltonian functions $h_i^{(n+1)}$ can be constructed recursively by

$$h_i^{(n)} = h_i^{(n-1)} - b_{n+1}h_{i-1}^{(n-1)} - a_n^2 h_{i-2}^{(n-2)} \tag{5.2.20}$$

with $h_{-1}^{(m)} \equiv 1$.

The projection procedure for the Poisson pencil of non-periodic Toda lattice is given by

$$Z = -\frac{\partial}{\partial b_{n+1}}, \quad Y = a_n \frac{\partial}{\partial a_n}$$

as

$$Z(h_0^{(n)}) = 1, \quad L_Z \Pi_1 = Y \wedge Z$$

and thus

$$\Pi_{1D} = \Pi_1 - X_1 \wedge Z.$$

Poisson structures projected on symplectic leaves $h_0^{(n)} = const$ parametrized by $(a_1, \ldots, a_n, b_1, \ldots, b_n)$, are nondegenerate tensors π_0, π_1 and minimal polynomial of $N = \pi_1 \pi_0^{-1}$ (5.1.41) is equal $h_\lambda^{(n-1)}$ as according to (5.2.20) $Z(h_\lambda^{(n)}) = h_\lambda^{(n-1)}$. Also from (5.2.20) it follows that

$$Y(h_\lambda^{(n)}) = -2a_n^2 (h_\lambda^{(n-2)}) \Longrightarrow Y\left[\tfrac{1}{2}\ln\left(-2a_n^2 h_\lambda^{(n-2)}\right)\right] = 1.$$

In consequence, transformation to separation coordinates is given by the following relations

$$\rho_i(\lambda) = h_i^{(n-1)}, \quad \mu_i = \tfrac{1}{2}\ln\left(-2a_n^2 h_{\lambda_i}^{(n-2)}\right), \quad i = 1, \ldots, n \tag{5.2.21}$$

and separation curve takes the form

$$\lambda^{n+1} + h_0^{(n)}\lambda^n + h_1^{(n)}\lambda^{n-1} + \ldots + h_n^{(n)} = \tfrac{1}{2}\exp(2\mu),$$

see Eq. (4.2.2).

Example 5.11 Non-periodic Toda system with $n = 2$. The dynamical system

$$\begin{aligned}
(a_1)_t &= a_1(b_1 - b_2)\\
(a_2)_t &= a_2(b_2 - b_3)\\
(b_1)_t &= -2a_1^2\\
(b_2)_t &= 2(a_1^2 - a_2^2)\\
(b_3)_t &= 2a_2^2
\end{aligned}$$

is bi-Hamiltonian and Liouville integrable with constants of motion of the form

$$\begin{aligned} h_0 =& -b_1 - b_2 - b_3 \\ h_1 =& b_1b_2 + b_1b_3 + b_2b_3 - a_1^2 - a_2^2 \\ h_2 =& -b_1b_2b_3 + a_1^2b_3 + a_2^2b_1 \end{aligned}$$

and two Poisson tensors

$$\Pi_0 = \begin{pmatrix} 0 & 0 & -a_1 & a_1 & 0 \\ 0 & 0 & 0 & -a_2 & a_2 \\ a_1 & 0 & 0 & 0 & 0 \\ -a_1 & a_2 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 & 0 \end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & \frac{1}{2}a_1a_2 & -a_1b_1 & a_1b_2 & 0 \\ -\frac{1}{2}a_1a_2 & 0 & 0 & -a_2b_2 & a_2b_3 \\ a_1b_1 & 0 & 0 & 2a_1^2 & 0 \\ -a_1b_2 & a_2b_2 & -2a_1^2 & 0 & 2a_2^2 \\ 0 & -a_2b_3 & 0 & -2a_2^2 & 0 \end{pmatrix}.$$

The transformation to separation coordinates, calculated from (5.2.21) of the form

$$\lambda^1 + \lambda^2 = b_1 + b_2, \quad \lambda^1\lambda^2 = b_1b_2 - a_2^2, \quad c = -b_1 - b_2 - b_3,$$

$$\mu_1 = \frac{1}{2}\ln(-2a_2^2(\lambda^1 - b_1)), \quad \mu_2 = \frac{1}{2}\ln(-2a_2^2(\lambda^2 - b_1)),$$

is given by

$$b_1 = \frac{\lambda^2\exp(2\mu_1) - \lambda^1\exp(2\mu_2)}{\exp(2\mu_1) - \exp(2\mu_2)}, \quad b_2 = \frac{\lambda^1\exp(2\mu_1) - \lambda^2\exp(2\mu_2)}{\exp(2\mu_1) - \exp(2\mu_2)}, \quad b_3 = -\lambda^1 - \lambda^2 - c,$$

$$a_1^2 = \frac{\exp(2\mu_1 + 2\mu_2)(\lambda^1 - \lambda^2)^2}{(\exp(2\mu_1) - \exp(2\mu_2))^2}, \quad a_2^2 = -\frac{1}{2}\frac{\exp(2\mu_1) - \exp(2\mu_2)}{\lambda^1 - \lambda^2}.$$

As $h_{\lambda^i} = h_0(\lambda^i)^2 + h_1\lambda^i + h_2 = \frac{1}{2}\exp(2\mu_i) - (\lambda^i)^3$, $i = 1, 2$, the separation curve takes the form

$$\lambda^3 + h_0\lambda^2 + h_1\lambda + h_2 = \tfrac{1}{2}\exp(2\mu).$$

5.2.4 *Dressing Chain*

Consider the so called *dressing chain*

$$(v_k + v_{k+1})_x = v_k^2 - v_{k+1}^2 + \alpha_k, \qquad v_k = v_k(x), \qquad \alpha_k = const \tag{5.2.22}$$

for the Schrödinger equation

$$\Psi_{xx} = (u - \lambda)\Psi. \tag{5.2.23}$$

If u_k is a sequence of solutions of (5.2.23), generated by a chain of Darboux transformations and $u_k = (v_k)_x + v_k^2 + \beta_k$ is a Miura map to the modified fields v_k, then the new fields v_k are related among themselves through the chain of Eqs. (5.2.22), where $\alpha_k = \beta_k - \beta_{k+1}$ (see [232] for details of the construction). Closing the chain by $v_k \equiv v_{k+N}$, $\alpha_k \equiv \alpha_{k+N}$ and by assumption that $\sum_{k=1}^{N} \alpha_k = 0$, we obtain a finite dimensional dynamical system

$$(v_k + v_{k+1})_x = v_k^2 - v_{k+1}^2 + \beta_k - \beta_{k+1}, \qquad k = 1, \ldots, N,$$

where x plays a role of evolution parameter. As was shown in [253], for $N = 2n+1$ and variables $g_k = v_k + v_{k+1}$, it is a bi-Hamiltonian system. The nonzero matrix elements of both Poisson tensors are the following

$$\begin{aligned} \{g_k, g_{k-1}\}_{\Pi_0} &= 1, \\ \{g_k, g_j\}_{\Pi_1} &= (-1)^{j-k} g_k g_j, \quad j \neq k \pm 1, \\ \{g_k, g_{k-1}\}_{\Pi_1} &= g_k g_{k-1} + \beta_i, \end{aligned}$$

and the Casimir of the pencil $\Pi_\lambda = \Pi_1 - \lambda \Pi_0$ is given by

$$h_\lambda = h_0\lambda^n + h_1\lambda^{n-1} + \ldots + h_n = (-1)^N \left[\prod_{j=1}^{N} \left(1 + \zeta_{j+1} \frac{\partial^2}{\partial g_j \partial g_{j+1}} \right) \right] \prod_{k=1}^{N} g_k, \quad \zeta_i = \beta_i - \lambda.$$

The projection procedure for the Poisson pencil of the dressing chain is given by [28]

$$Z = -\frac{\partial}{\partial g_N}, \qquad Y = \sum_{i=1}^{N-1} (-1)^{i+1} g_i \frac{\partial}{\partial g_i} + \left(\sum_{i=1}^{N-1} (-1)^i g_i \right) \frac{\partial}{\partial g_N}$$

as then

$$Z(h_0) = 1, \quad Y(h_0) = 0, \quad L_Z \Pi_0 = 0, \quad L_Z \Pi_1 = Y \wedge Z.$$

The first part of the transformation to the separation coordinates is

$$c = h_0, \quad \rho_k(\lambda) = \frac{\partial h_k}{\partial g_N}, \quad k = 1, \ldots, n. \tag{5.2.24}$$

The following property of the Casimir h_λ

$$Y^3(h_\lambda) = Y(h_\lambda) \Longrightarrow Y \ln(Y(h_\lambda) + Y^2(h_\lambda)) = 1$$

gives the second part of the transformation

$$\mu_k = \ln(Y(h_{\lambda_k}) + Y^2(h_{\lambda_k})), \quad k = 1, \ldots, n. \tag{5.2.25}$$

Moreover, low dimensional examples suggest the separation curve of the form

$$h_0\lambda^n + h_1\lambda^{n-1} + \ldots + h_n = 2\prod_{i=1}^{N}(\lambda - \beta_i)\exp(-\mu) + \tfrac{1}{2}\exp(\mu),$$

see Eq. (4.2.2).

Example 5.12 The dressing chain for $N = 5$. The dynamical system

$$\begin{aligned}
(g_1)_x &= g_1(-g_2 + g_3 - g_4 + g_5) + \beta_1 - \beta_2\\
(g_2)_x &= g_2(g_1 - g_3 + g_4 - g_5) + \beta_2 - \beta_3\\
(g_3)_x &= g_3(-g_1 + g_2 - g_4 + g_5) + \beta_3 - \beta_4\\
(g_4)_x &= g_4(g_1 - g_2 + g_3 - g_5) + \beta_4 - \beta_5\\
(g_5)_x &= g_5(-g_1 + g_2 - g_3 + g_4) + \beta_5 - \beta_1
\end{aligned}$$

is bi-Hamiltonian and Liouville integrable with constants of motion of the form

$$\begin{aligned}
h_0 &= g_1 + g_2 + g_3 + g_4 + g_5,\\
h_1 &= -g_1g_2g_3 - g_2g_3g_4 - g_3g_4g_5 - g_4g_5g_1 - g_5g_1g_2\\
&\quad -g_1(\beta_3 + \beta_5) - g_2(\beta_4 + \beta_1) - g_3(\beta_5 + \beta_2) - g_4(\beta_1 + \beta_3) - g_5(\beta_2 + \beta_4),\\
h_2 &= g_1g_2g_3g_4g_5 + \beta_1g_2g_3g_4 + \beta_2g_3g_4g_5 + \beta_3g_4g_5g_1 + \beta_4g_5g_1g_2 + \beta_5g_1g_2g_3\\
&\quad +\beta_3\beta_5g_1 + \beta_1\beta_4g_2 + \beta_2\beta_5g_3 + \beta_1\beta_3g_4 + \beta_2\beta_4g_5
\end{aligned}$$

and two Poisson tensors

$$\Pi_0 = \begin{pmatrix}
0 & -1 & 0 & 0 & 1\\
1 & 0 & -1 & 0 & 0\\
0 & 1 & 0 & -1 & 0\\
0 & 0 & 1 & 0 & -1\\
-1 & 0 & 0 & 1 & 0
\end{pmatrix},$$

$$\Pi_1 = \begin{pmatrix} 0 & -g_1g_2-\beta_2 & g_1g_3 & -g_1g_4 & g_1g_5+\beta_1 \\ g_1g_2+\beta_2 & 0 & -g_2g_3-\beta_3 & g_2g_4 & -g_2g_5 \\ -g_1g_3 & g_2g_3+\beta_3 & 0 & -g_3g_4-\beta_4 & g_3g_5 \\ g_1g_4 & -g_2g_4 & g_3g_4+\beta_4 & 0 & -g_4g_5-\beta_5 \\ -g_1g_5-\beta_1 & g_2g_5 & -g_3g_5 & g_4g_5+\beta_5 & 0 \end{pmatrix}.$$

The transformation to separation coordinates, calculated from (5.2.24) and (5.2.25), is of the form

$$g_1 = \frac{1}{2}\frac{(\lambda_2-\lambda_1)\exp(\mu_1+\mu_2)}{(\lambda_2-\beta_3)(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_3)(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)},$$

$$g_2 = 2\frac{(\lambda_2-\beta_2)(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_2)(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)}{(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)}$$
$$\times\frac{(\lambda_2-\beta_3)(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_3)(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)}{(\lambda_2-\lambda_1)\exp(\mu_1+\mu_2)},$$

$$g_3 = -\frac{1}{2}\frac{(\lambda_2-\beta_3)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_3)(\lambda_1-\beta_5)\exp(\mu_2)}{(\lambda_2-\beta_3)(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_3)(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)}$$
$$\times\frac{(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)}{(\lambda_2-\lambda_1)(\lambda_1-\beta_5)(\lambda_2-\beta_5)},$$

$$g_4 = -2\frac{(\lambda_2-\lambda_1)(\lambda_1-\beta_4)(\lambda_1-\beta_5)(\lambda_2-\beta_4)(\lambda_2-\beta_5)}{(\lambda_2-\beta_4)(\lambda_2-\beta_5)\exp(\mu_1)-(\lambda_1-\beta_4)(\lambda_1-\beta_5)\exp(\mu_2)},$$

$$g_5 = c-g_1-g_2-g_3-g_4$$

and the separation curve is

$$c\lambda^2+h_1\lambda+h_2 = 2(\lambda-\beta_1)(\lambda-\beta_2)(\lambda-\beta_3)(\lambda-\beta_4)(\lambda-\beta_5)\exp(-\mu)+\tfrac{1}{2}\exp(\mu).$$

5.3 Bi-Presymplectic Separability Theory

The bi-Poisson formulation of finite dimensional integrable Hamiltonian systems has been systematically developed in the previous two sections. It has been found that most of the known Liouville integrable systems have more then one Hamiltonian representation. Moreover, in the majority of known cases, both Poisson structures of a given flow are degenerate. For such systems, related bi-Poisson (bi-Hamiltonian) commuting vector fields belong to one or more bi-Hamiltonian chains starting and terminating with Casimirs of respective Poisson structures. The most important aspect of such a construction is its relation to the geometric separability theory. Having a bi-Hamiltonian representation of a given system, the sufficient condition for the existence of separation coordinates is the reducibility of one of the Poisson structures onto a symplectic leaf of the other one. Unfortunately, this

procedure is non-algorithmic and has to be considered independently from case to case and is related with finding an appropriate integrable distribution $\mathcal{Z}$, transversal to symplectic foliation. Moreover, we do not have any proof that it is always possible for any GZ system. Anyway, once the reduction is done, the remaining procedure of the construction of separation coordinates is algorithmic. The relevance of bi-Hamiltonian formalism in separability theory confirms the fact that the arbitrary Stäckel system, defined by an appropriate separation relations (4.2.9), has a bi-Hamiltonian extension [31].

On the other hand, it is well known from the classical mechanics that if the Poisson structure is nondegenerate, i.e. if the rank of the Poisson tensor is equal to the dimension of a phase space, then the phase space becomes a symplectic manifold with a symplectic structure being just the inverse of the Poisson structure (see the previous chapter). In such a case there exists an alternative (dual) description of Hamiltonian vector fields in the language of symplectic geometry. So, a natural question arises, whether one can construct such a dual picture in the degenerate case, when there is no natural inverse of the Poisson tensor. For such tensors the notion of dual presymplectic structures was developed in Sect. 3.2.3.

The presymplectic picture is especially interesting in the case of Liouville integrable systems. As was mentioned above, there is a well developed bi-Hamiltonian theory of such systems, based on Poisson pencils of the GZ type, whose Casimir functions are polynomials with respect to pencil parameters and the related separability theory. The important question is whether it is possible to formulate a dual, bi-presymplectic (bi-inverse-Hamiltonian in particular) theory of Liouville integrable systems with the related separability theory and how both theories are related to each other.

The following section presents the general bi-presymplectic theory of Liouville integrable systems when the co-rank of presymplectic forms is equal and fixed. The whole formalism is based on the notion of *d-compatibility* of presymplectic forms and *d-compatibility* of Poisson bi-vectors. What is important, in the new formalism the construction of separation coordinates is simply algorithmic, once the bi-presymplectic chain is given. Finally it is shown that any Stäckel system, defined by an appropriate separation relations, has a bi-inverse-Hamiltonian representation, which confirms the relevance of the presented formalism.

5.3.1 D-Compatibility of Closed Two-Forms and Poisson Bi-Vectors

In the following section we develop a concept of d-compatibility which is crucial for our further considerations. The notion of d-compatibility for a non-degenerate case was introduced in Sect. 5.1.3. Here we extend the notion of d-compatibility onto the degenerate case.

A closed two-form Ω_0 is d-compatible with a closed two-form Ω_1 if there exists a Poisson tensor Π_0, dual to Ω_0, such that $\Pi_0\Omega_1\Pi_0$ is Poisson. Then we say that the pair (Ω_0, Ω_1) is d-compatible with respect to Π_0. Analogically, a Poisson tensor Π_0 is d-compatible with a Poisson tensor Π_1 if there exists a presymplectic form Ω_0, dual to Π_0, such that $\Omega_0\Pi_1\Omega_0$ is closed. Then we say that the pair (Π_0, Π_1) is d-compatible with respect to Ω_0.

Comparing the notions of compatibility and d-compatibility for Poisson pair (Π_0, Π_1) we have shown in Sect. 5.1.3 that when Π_0 is non-degenerate both notions are equivalent, but for a degenerate case the notion of d-compatibility is the stronger one. Actually, let us consider the following identity, proved in [29],

$$\boxed{\begin{aligned} &L_{(\Pi_\lambda)\gamma}\Pi_\lambda + (\Pi_\lambda)\, d\gamma\, (\Pi_\lambda) \\ &= \lambda\{L_\tau(\Omega_0\Pi_1\Omega_0) - d(\Omega_0\Pi_1\Omega_0\tau) - \textstyle\sum_i [\Omega_0(L_{Z_i}\Pi_1)\Omega_0]\tau \wedge dc_i \\ &\quad - \textstyle\sum_i \tau(c_i)[\Omega_0(L_{Z_i}\Pi_1)\Omega_0]\}, \end{aligned}} \tag{5.3.1}$$

where $\Pi_\lambda = \Pi_1 + \lambda\Pi_0$ is a Poisson pencil, Π_0, Π_1 are Poisson tensors, (Π_0, Ω_0) is a dual pair, where $dc_i \in \ker\Pi_0$, $Z_i \in \ker\Omega_0$, $\tau \in TM$ and $\gamma = \Omega_0\tau \in T^*M$. Assume first that Π_0 and Π_1 are d-compatible with respect to Ω_0. Then $\Omega_0\Pi_1\Omega_0$ is closed and hence

$$\boxed{L_\tau(\Omega_0\Pi_1\Omega_0) - d(\Omega_0\Pi_1\Omega_0\tau) = 0, \qquad \tau \in TM.} \tag{5.3.2}$$

In particular, for $\tau = Z_i$, relation (5.3.2) gives

$$\boxed{\Omega_0\left(L_{Z_i}\Pi_1\right)\Omega_0 = 0, \qquad i = 1, \ldots, r,} \tag{5.3.3}$$

so as a result

$$\boxed{L_{(\Pi_\lambda)\gamma}\Pi_\lambda + (\Pi_\lambda)\, d\gamma\, (\Pi_\lambda) = 0} \tag{5.3.4}$$

and $\Pi_1 + \lambda\Pi_0$ is Poisson according to (3.2.21). On the other hand, from the compatibility relation (5.3.4) the d-compatibility (5.3.2) follows under additional conditions (5.3.3).

Observation 9 *From the above construction follows that d-compatibility of a Poisson pair* (Π_0, Π_1) *with respect to* Ω_0*, dual to* Π_0*, guarantees not only ordinary compatibility of the pair* (Π_0, Π_1)*, but also a Poisson reduction of* Π_1 *onto the symplectic foliation of* Π_0 *along the distribution* $\mathcal{Z} = \ker\Omega_0$*. So, in a generic case, bi-Hamiltonian chains of Liouville integrable systems, with a d-compatible pair of Poisson structures are separable.*

Before we pass to bi-presymplectic chains and the related separability theory, we need some relations between d-compatible Poisson bi-vectors and d-compatible

presymplectic two-forms. Thus, let a Poisson tensor Π_0 and a closed two-form Ω_0 form a dual pair. Moreover, let $Z_k \in \ker \Omega_0$, $dc_k \in \ker \Pi_0$ and $Z_k(c_i) = \delta_{ki}$, $k, i = 1, \ldots, r$. Then, we have the following statements.

1. If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0 and

$$\Pi_1(dc_i, dc_j) = 0,$$

then forms Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible with respect to Π_0.
2. If Ω_1 is a closed two-form d-compatible with Ω_0 with respect to Π_0 and

$$\Omega_1(Z_i, Z_j) = 0 \ \text{ and } \ \Pi_0\Omega_1 Z_i = \Pi_0 dh^{(i)} \tag{5.3.5}$$

for some functions $h^{(i)} \in h(M)$, $i = 1, \ldots, r$, then Poisson tensors Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0$ are d-compatible with respect to Ω_0, provided that

$$Z_i(h^{(k)}) = Z_k(h^{(i)}), \qquad k, i = 1, \ldots, r. \tag{5.3.6}$$

In the first case, from the definition, Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0$ are d-compatible if

$$\Pi = \Pi_0\Omega_1\Pi_0 = \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0 = (I - \sum_{i=1}^{r} Z_i \otimes dc_i)\Pi_1(I - \sum_{j=1}^{r} dc_j \otimes Z_j)$$

$$= \Pi_1 - \sum_{i=1}^{r} X_i \wedge Z_i + \tfrac{1}{2}\sum_{i,j=1}^{r} \Pi_1(dc_i, dc_j)Z_i \wedge Z_j$$

$$= \Pi_1 - \sum_{i=1}^{r} X_i \wedge Z_i,$$

where $X_i = \Pi_1 dc_i$, is a Poisson. From (3.2.27) and (5.3.3) it follows that $L_{Z_i}\Pi = 0$ and hence

$$L_{Z_k}\Pi_1 = \sum_{i=1}^{r} [Z_k, X_i] \wedge Z_i.$$

Using relations (3.2.33) we get

$$[\Pi, \Pi]_S = [\Pi_1, \Pi_1]_S - 2[\sum_{i=1}^{r} X_i \wedge Z_i, \Pi_1]_S + [\sum_{i=1}^{r} X_i \wedge Z_i, \sum_j X_j \wedge Z_j]_S$$

$$= -2\sum_{i=1}^{r} (Z_i \wedge [X_i, \Pi_1]_S - X_i \wedge [Z_i, \Pi_1]_S)$$

$$+\sum_{i,j=1}^{r}(Z_i\wedge[X_i,X_j\wedge Z_j]_S-X_i\wedge[Z_i,X_j\wedge Z_j]_S)$$
$$=2\sum_{i=1}^{r}X_i\wedge L_{Z_i}\Pi_D+\sum_{i,j=1}^{r}(Z_i\wedge X_j\wedge[X_i,Z_j]-X_i\wedge[Z_i,X_j]\wedge Z_j)$$
$$=\sum_{i,j=1}^{r}(2X_i\wedge[Z_i,X_j]\wedge Z_j+Z_i\wedge X_j\wedge[X_i,Z_j]-X_i\wedge[Z_i,X_j]\wedge Z_j)$$
$$=0. \tag{5.3.7}$$

In the second case, from the d-compatibility of Ω_0 and Ω_1 it follows that Π_1 is a Poisson. Then,

$$\Omega_0\Pi_1\Omega_0=\Omega_0\Pi_0\Omega_1\Pi_0\Omega_0=(I-\sum_{k=1}^{r}dc_k\otimes Z_k)\Omega_1(I-\sum_{i=1}^{r}Z_i\otimes dc_i)$$
$$=\Omega_1+\sum_{k=1}^{r}dc_k\wedge\Omega_1(Z_k)+\frac{1}{2}\sum_{i,k=1}^{r}\Omega_1(Z_i,Z_k)dc_k\wedge dc_i.$$

From the assumption (5.3.5) and decompositions (3.2.25) it follows that

$$\Omega_1Z_k=dh^{(k)}+\sum_{i=1}^{r}\Big[\Omega_1(Z_k,Z_i)-Z_i(h^{(k)})\Big]dc_i=dh^{(k)}-\sum_{i=1}^{r}Z_i(h^{(k)})dc_i,$$

hence,

$$\Omega_0\Pi_1\Omega_0=\Omega_1+\sum_{k=1}^{r}dc_k\wedge dh^{(k)}\ -\sum_{i,k=1}^{r}\Big[Z_i(h^{(k)})\Big]dc_k\wedge dc_i$$
$$=\Omega_1+\sum_{k=1}^{r}dc_k\wedge dh^{(k)}\ +\tfrac{1}{2}\sum_{i,k=1}^{r}\Big[Z_k(h^{(i)})-Z_i(h^{(k)})\Big]dc_k\wedge dc_i$$

and under condition (5.3.6) $\Omega_0\Pi_1\Omega_0$ is closed.

As a consequence of previous relations, two other statements can be proved. Let again a Poisson tensor Π_0 and a closed two-form Ω_0 form a dual pair, where $Z_k\equiv Y_0^{(k)}\in\ker\Omega_0$, $dc_k\equiv h_0^{(k)}\in\ker\Pi_0$ and $Y_0^{(k)}(h_0^{(i)})=\delta_{ki}$, $k,i=1,\dots,r$. Then, we have what follows.

1. If Π_1 is a Poisson tensor d-compatible with Π_0 with respect to Ω_0,

$$X_1^{(k)}=\Pi_1dh_0^{(k)}=\Pi_0dh_1^{(k)},\qquad k=1,\dots,r$$

are bi-Hamiltonian vector fields for some functions $h_1^{(k)}$ and

$$\Pi_1(dh_0^{(i)}, dh_0^{(j)}) = 0, \qquad i, j = 1, \ldots, r$$

then Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0 + \sum_{k=1}^{r} dh_1^{(k)} \wedge dh_0^{(k)}$ is a d-compatible pair of presymplectic forms with respect to Π_0.

2. If Ω_1 is a presymplectic form d-compatible with Ω_0 with respect to Π_0,

$$\beta_1^{(k)} = \Omega_1 Y_0^{(k)} = \Omega_0 Y_1^{(k)}, \qquad k = 1, \ldots, r \tag{5.3.8}$$

are bi-presymplectic one-forms for some vector fields $Y_1^{(k)}$ such that

$$X_1^{(k)} = \Pi_0\beta_1^{(k)} = \Pi_0 dh_1^{(k)}, \quad h_1^{(k)} \in F(M), \quad k = 1, \ldots, r \tag{5.3.9}$$

and

$$\Omega_1(Y_0^{(i)}, Y_0^{(j)}) = 0, \qquad i, j = 1, \ldots, r, \tag{5.3.10}$$

then Π_0 and $\Pi_1 = \Pi_0\Omega_1\Pi_0 + \sum_{k=1}^{r} X_1^{(k)} \wedge Y_0^{(k)}$, are d-compatible Poisson tensors with respect to Ω_0, provided that

$$Y_0^{(s)}(h_1^{(k)}) = Y_0^{(k)}(h_1^{(s)}), \tag{5.3.11a}$$

$$Y_1^{(s)}(h_0^{(k)}) = Y_0^{(k)}(h_1^{(s)}), \qquad k, s = 1, \ldots, r, \tag{5.3.11b}$$

$$\Pi_0\beta_2^{(k)} \equiv \Pi_0\Omega_1 Y_1^{(k)} = \Pi_0 dh_2^{(k)}, \quad h_2^{(k)} \in F(M). \tag{5.3.11c}$$

In the first case obviously $d\Omega_1 = 0$ and from the definition, Ω_0 and $\Omega_1 = \Omega_0\Pi_1\Omega_0 + \sum_{k=1}^{r} dh_1^{(k)} \wedge dh_0^{(k)}$ are d-compatible as

$$\begin{aligned}\Pi_0\Omega_1\Pi_0 &= \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0 + \Pi_0\left(\sum_{k=1}^{r} dh_1^{(k)} \wedge dh_0^{(k)}\right)\Pi_0 \\ &= \Pi_0\Omega_0\Pi_1\Omega_0\Pi_0 = \Pi\end{aligned}$$

is a Poisson according to (5.3.7).

In the second case, the proof is technical and laborious so we skip it and refer the interested reader to the original paper [45].

5.3.2 Bi-Presymplectic Chains

Now we are ready to investigate main properties of bi-presymplectic chains. Assume we have a pair of presymplectic forms (Ω_0, Ω_1), d-compatible with respect to some Π_0 dual to Ω_0, both of rank $2n$ and co-rank r. Assume further that they form bi-presymplectic chains of one-forms

$$\boxed{\beta_i^{(k)} = \Omega_0 Y_i^{(k)} = \Omega_1 Y_{i-1}^{(k)}, \quad i = 1, 2, \ldots, n_k} \tag{5.3.12}$$

where $k = 1, \ldots, r$, $n_1 + \ldots + n_r = n$ and each chain starts with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminates with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1. Then:

1.

$$\boxed{\Omega_0(Y_i^{(k)}, Y_j^{(m)}) = \Omega_1(Y_i^{(k)}, Y_j^{(m)}) = 0,} \tag{5.3.13}$$

for $k, m = 1, \ldots, r$, $i = 1, 2, \ldots, n_k$, $j = 1, 2, \ldots, n_m$.

2. Moreover, let us assume that

$$X_i^{(k)} = \Pi_0 \beta_i^{(k)} = \Pi_0 dh_i^{(k)}, \tag{5.3.14}$$

for $k = 1, \ldots, r$, $i = 1, 2, \ldots, n_k$, which implies

$$\beta_i^{(k)} = dh_i^{(k)} - \sum_{m=1}^{r} Y_0^{(m)}(h_i^{(k)}) dh_0^{(m)}, \tag{5.3.15}$$

$$Y_i^{(k)} = X_i^{(k)} + \sum_{m=1}^{r} Y_i^{(k)}(h_0^{(m)}) Y_0^{(m)}, \quad Y_i^{(k)}(h_0^{(m)}) \neq 0, \tag{5.3.16}$$

where $\Pi_0 dh_0 = 0$. Then,

$$\Pi_0(dh_i^{(k)}, dh_j^{(m)}) = 0, \quad [X_i^{(k)}, X_j^{(m)}] = 0 \tag{5.3.17}$$

and bi-presymplectic one-forms (5.3.12) define a Liouville integrable system

$$\xi_{t_{k,i}} = X_i^{(k)}(\xi) = \Pi_0 \beta_i^{(k)}(\xi) = \Pi_0 dh_i^{(k)}(\xi)$$

for $k = 1, \ldots, r$, $i = 1, \ldots, n_k$.

3. Additionally, if

$$Y_0^{(m)}(h_1^{(k)}) = Y_0^{(k)}(h_1^{(m)}), \tag{5.3.18a}$$

$$Y_i^{(m)}(h_0^{(k)}) = Y_0^{(k)}(h_i^{(m)}), \tag{5.3.18b}$$

where $k, m = 1, \ldots, r$, $i = 1, 2, \ldots, n_m$, then vector fields $X_i^{(k)}$ (5.3.14) form bi-Hamiltonian chains

$$X_i^{(k)} = \Pi_0 dh_i^{(k)} = \Pi_1 dh_{i-1}^{(k)}, \tag{5.3.19}$$

where

$$\Pi_1 = \Pi_0 \Omega_1 \Pi_0 + \sum_{m=1}^{r} X_1^{(m)} \wedge Y_0^{(m)}, \tag{5.3.20}$$

$k, m = 1, \ldots, r$, $i = 1, 2, \ldots, n_k$ and $n_1 + \ldots + n_r = n$. The chain starts with $h_0^{(k)}$, a Casimir of Π_0, and terminates with $h_{n_k}^{(k)}$, a Casimir of Π_1. Moreover, the Poisson pair (Π_0, Π_1) is d-compatible with respect to Ω_0.

4. Contrary, if

$$Y_0^{(k)}(h_i^{(m)}) = 0,$$

for all admissible values of k, m and i, chains (5.3.12) are bi-inverse-Hamiltonian as $\beta_i^{(k)} = dh_i^{(k)}$ and obviously $X_i^{(k)}$ are not bi-Hamiltonian according to violation of condition (5.3.18b). Such Hamiltonians will be denoted by $H_i^{(k)}$ according to the notation from previous sections.

From (5.3.12) we have

$$\Omega_0(Y_i^{(k)}, Y_j^{(r)}) = \Omega_0(Y_{i-1}^{(k)}, Y_{j+1}^{(r)}),$$
$$\Omega_1(Y_i^{(k)}, Y_j^{(r)}) = \Omega_1(Y_{i+1}^{(k)}, Y_{j-1}^{(r)})$$
$$\Omega_0(Y_i^{(k)}, Y_j^{(r)}) = \Omega_1(Y_{i-1}^{(k)}, Y_j^{(r)}).$$

Then, the first property (5.3.13) follows from

$$\Omega_0(Y_0^{(k)}, Y_i^{(r)}) = 0, \quad \Omega_1(Y_{n_k}^{(k)}, Y_i^{(r)}) = 0.$$

From properties of dual pair (Π_0, Ω_0), if $X_i^{(k)} = \Pi_0 dh_i^{(k)}$ then

$$\Pi_0(dh_i^{(k)}, dh_j^{(r)}) = \Omega_0(X_i^{(k)}, X_j^{(r)}).$$

On the other hand as $X_i^{(k)} = Y_i^{(k)} + \sum_{m=1}^{r} \alpha_m^{(k)} Y_0^{(m)}$, where $\alpha_m^{(k)}$ are appropriate functions. So the second property follows from the fact that

$$\Omega_0(X_i^{(k)}, X_j^{(m)}) = \Omega_0(Y_i^{(k)}, Y_j^{(m)}).$$

The property (5.3.19) is proved as follows

$$\begin{aligned} X_i^{(k)} &= \Pi_0 dh_i^{(k)} = \Pi_0 \Omega_1 Y_{i-1}^{(k)} \\ &= \Pi_0 \Omega_1 (X_{i-1}^{(k)} + \sum_{m=1}^{r} Y_{i-1}^{(k)}(h_0^{(m)}) Y_0^{(m)}) \\ &= \Pi_0 \Omega_1 \Pi_0 dh_{i-1}^{(k)} + \sum_{m=1}^{r} Y_{i-1}^{(k)}(h_0^{(m)}) X_1^{(m)} \\ &\overset{(5.3.18b)}{=} (\Pi_0 \Omega_1 \Pi_0 + \sum_{m=1}^{r} X_1^{(m)} \wedge Y_0^{(m)}) dh_{i-1}^{(k)} \\ &= \Pi_1 dh_{i-1}^{(k)}. \end{aligned}$$

Moreover, Π_0 and Π_1 are d-compatible Poisson tensors provided that (5.3.18) are fulfilled (see (5.3.8)–(5.3.11c)). We also have

$$\begin{aligned} \Pi_1 dh_{n_k}^{(k)} &= (\Pi_0 \Omega_1 \Pi_0 + \sum_{m=1}^{r} X_1^{(m)} \wedge Y_0^{(m)}) dh_{n_k}^{(k)} = \Pi_0 \Omega_1 X_{n_k}^{(k)} + \sum_{m=1}^{r} Y_0^{(m)}(h_{n_k}^{(k)}) X_1^{(m)} \\ &\overset{(5.3.16)}{=} \Pi_0 \Omega_1 (Y_{n_k}^{(k)} - \sum_{m=1}^{r} Y_{n_k}^{(k)}(h_0^{(m)}) Y_0^{(m)}) + \sum_{m=1}^{r} Y_0^{(m)}(h_{n_k}^{(k)}) X_1^{(m)} \\ &= -\sum_{m=1}^{r} Y_{n_k}^{(k)}(h_0^{(m)}) X_1^{(m)} + \sum_{m=1}^{r} Y_0^{(m)}(h_{n_k}^{(k)}) X_1^{(m)} \overset{(5.3.18b)}{=} 0. \end{aligned}$$

As will be demonstrated in the next subsection, for Stäckel systems conditions (5.3.18) are violated, so the only case when Stäckel system has simultaneously a bi-Hamiltonian and a bi-presymplectic representation is the case of co-rank $r = 1$, when conditions (5.3.18) reduce to a trivial one (see examples from next section).

Finally we show that arbitrary Liouville integrable system (5.3.14), (5.3.17), which has a bi-presymplectic representation on $(2n+r)$-dimensional extended phase space, has also quasi-bi-Hamiltonian representation on any symplectic leaf of its Poisson structure Π_0. Actually, from (5.3.12), (5.3.16) follows that

$$\begin{aligned} \Pi_0 dh_{i+1}^{(k)} &= \Pi_0 \beta_{i+1}^{(k)} = \Pi_0 \Omega_1 Y_i^{(k)} \\ &= \Pi_0 \Omega_1 (X_i^{(k)} + \sum_{m=1}^{r} Y_i^{(k)}(h_0^{(m)}) Y_0^{(m)}) \end{aligned}$$

$$= \Pi_0\Omega_1\Pi_0 dh_i^{(k)} + \sum_{m=1}^{r} Y_i^{(k)}(h_0^{(m)})X_1^{(m)}$$

$$= \Pi_0\Omega_1\Pi_0 dh_i^{(k)} + \sum_{m=1}^{r} Y_i^{(k)}(h_0^{(m)})\Pi_0 dh_1^{(m)},$$

hence on a $(2n + r)$-dimensional extended phase space we have a quasi-bi-Hamiltonian representation

$$\Pi_D dh_i^{(k)} = \Pi_0 dh_{i+1}^{(k)} - \sum_{m=1}^{r} F_i^{(k,m)}\Pi_0 dh_1^{(m)}, \tag{5.3.21}$$

where $\Pi_D = \Pi_0\Omega_1\Pi_0$, $F_i^{(k,m)} = Y_i^{(k)}(h_0^{(m)})$ (compare with (5.1.26)).

Notice that both Poisson structures Π_0 and $\Pi_D = \Pi_0\Omega_1\Pi_0$ share the same Casimirs $h_0^{(k)}$, so the quasi-bi-Hamiltonian dynamics can be restricted immediately to any common leaf S_c of dimension $2n$

$$\pi_1 dh_{c,i}^{(k)} = \pi_0 dh_{c,i+1}^{(k)} - \sum_{m=1}^{r} F_i^{(k,m)}\pi_0 dh_{c,1}^{(m)}, \qquad i = 1, \ldots, n,$$

where ω_1, $\pi_1 = \pi_0\omega_1\pi_0$, π_0 are restrictions of Ω_1, Π_D, Π_0 and $h_{i|S_c}^{(k)} = h_{c,i}^{(k)}$, respectively. As ω_1 is closed, hence π_0 and π_1 are compatible and we deal with a Stäckel system whose separation coordinates are eigenvalues of the recursion operator $N = \pi_1\pi_0^{-1}$, provided that N has n distinct and functionally independent eigenvalues at any point of S_v, i.e. we are in a generic case. We will come back to separable systems in next subsections.

The advantage of a bi-presymplectic representation of the Liouville integrable system, when compared to bi-Hamiltonian ones, is that the existence of the first guarantees that the system is separable and the construction of separation coordinates is purely algorithmic (in a generic case), while the bi-Hamiltonian representation does not guarantee the existence of the quasi-bi-Hamiltonian representation and hence separability of the system in question. Moreover, the projection of the second Poisson structure onto the symplectic foliation of the first one, in order to construct a quasi-bi-Hamiltonian representation, necessary for separability, is far from being trivial non-algorithmic procedure, as was demonstrated in the previous section.

5.3.3 *Bi-Inverse-Hamiltonian Representation of Stäckel Systems*

As was shown in the previous section, the Stäckel Hamiltonians defined by separation relations (5.1.13) admit on M the following quasi bi-Hamiltonian chains

in (λ, μ) representation

$$\pi_1 dH_j^{(k)} = \pi_0\, dH_{j+1}^{(k)} - \sum_{l=1}^{r} F_j^{(k,l)}\, \pi_0\, dH_1^{(l)}, \quad H_{n_k+1}^{(k)} = 0, \quad k = 1, \ldots, r, \quad j = 1, \ldots, n_k, \tag{5.3.22}$$

with compatible Poisson tensors π_0 and π_1

$$\pi_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad \pi_1 = \begin{pmatrix} 0 & \Lambda_n \\ -\Lambda_n & 0 \end{pmatrix}, \quad \Lambda_n = diag(\lambda^1, \ldots, \lambda^n)$$

and the expansion coefficients $F_j^{(k,l)}$ (appropriate basic potentials) being solutions of the set of linear algebraic Eqs. (5.1.17).

Now we show how to lift (5.3.22) to a bi-inverse-Hamiltonian representation on the extended phase space. Let us consider the following symplectic forms on M

$$\omega_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \qquad \omega_1 = \begin{pmatrix} 0 & -\Lambda_n \\ \Lambda_n & 0 \end{pmatrix}.$$

Observe that (π_0, ω_0) constitutes a non degenerate dual implectic-symplectic pair as $\omega_0 = \pi_0^{-1}$, π_0 and $\pi_1 = \pi_0\omega_1\pi_0$ are d-compatible with respect to ω_0 and ω_0 and $\omega_1 = \omega_0\pi_1\omega_0$ are d-compatible with respect to π_0. Besides, quasi bi-Hamiltonian chains (5.3.22) have equivalent quasi bi-inverse-Hamiltonian representations (5.1.33)

$$\omega_1 \mathbf{x}_i^{(k)} = \omega_0\, \mathbf{x}_{i+1}^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)}\, \omega_0\, \mathbf{x}_1^{(l)}, \quad \mathbf{x}_{n_k+1}^{(k)} = 0, \quad k{=}1, \ldots, r, \quad i{=}1, \ldots, n_k, \tag{5.3.23}$$

where

$$\mathbf{x}_i^{(k)} = \pi_0 dH_i^{(k)}, \qquad dH_i^{(k)} = \omega_0 \mathbf{x}_i^{(k)}.$$

Let us lift the whole construction to the extended phase space

$$M \to \mathcal{M} : (\lambda, \mu) \to (\lambda, \mu, c), \tag{5.3.24}$$

where $c_1, \ldots, c_r$ are extra coordinates and $\dim \mathcal{M} = 2n + r$. Then, on $\mathcal{M}$

$$\omega_0 \to \Omega_0 = \left(\begin{array}{c|c} \omega_0 & 0 \\ \hline 0 & 0 \end{array}\right), \qquad \pi_0 \to \Pi_0 = \left(\begin{array}{c|c} \pi_0 & 0 \\ \hline 0 & 0 \end{array}\right),$$

where

$$\ker \Omega_0 = Sp\{Y_0^{(k)}\},\ \ k = 1, \ldots, r, \quad Y_0^{(k)} = \frac{\partial}{\partial c_k}, \quad \Omega_0 Y_0^{(k)} = 0$$

and

$$\ker \Pi_0 = Sp\{dc_k\},\ \ k = 1, \ldots, r, \qquad \Pi_0 dc_k = 0, \quad Y_0^{(k)}(c_j) = \delta_{kj}.$$

Obviously, (Π_0, Ω_0) is a dual Poisson-presymplectic pair on $\mathcal{M}$. In the same fashion we lift

$$\omega_1 \to \Omega_{1D}, \quad \pi_1 \to \Pi_{1D}, \quad \mathbf{x}_i^{(k)} \to X_i^{(k)} = (\mathbf{x}_i^{(k)}, 0)^T,$$

where $\ker \Omega_{1D} = \ker \Omega_0$ and $\ker \Pi_{1D} = \ker \Pi_0$. On $\mathcal{M}$ quasi bi-inverse-Hamiltonian chains (5.3.23) take the form

$$\Omega_{1D} X_i^{(k)} = \Omega_0\, X_{i+1}^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)}\, \Omega_0\, X_1^{(l)}, \quad X_{n_k+1}^{(k)} = 0, \quad k = 1, \ldots, r, \quad i = 1, \ldots, n_k.$$

Let us define the following presymplectic two-forms

$$\boxed{\Omega_1 = \Omega_{1D} + \sum_{k=1}^{r} dH_1^{(k)} \wedge dc_k} \tag{5.3.25}$$

and the set of vector fields

$$\boxed{Y_i^{(k)} = X_i^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)} Y_0^{(l)}.} \tag{5.3.26}$$

Then, we have

$$\begin{aligned}
\Omega_0 Y_{i+1}^{(k)} &= dH_{i+1}^{(k)} \\
&= \Omega_0 X_{i+1}^{(k)} = \Omega_{1D} X_i^{(k)} + \sum_{l=1}^{r} F_i^{(k,l)}\, \Omega_0\, X_1^{(l)} \\
&= (\Omega_1 - \sum_{l=1}^{r} dH_1^{(l)} \wedge dc_l)(Y_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)} Y_0^{(l)}) + \sum_{l=1}^{r} F_i^{(k,l)} dH_1^{(l)} \\
&= \Omega_1 Y_i^{(k)} - \sum_{l=1}^{r} F_i^{(k,l)} \Omega_1 Y_0^{(l)} - \sum_{l=1}^{r} Y_i^{(k)}(c_l) dH_1^{(l)} + \sum_{l=1}^{r} Y_i^{(k)}(H_1^{(l)}) dc_l
\end{aligned}$$

$$+\sum_{l=1}^{r} F_i^{(k,l)} dH_1^{(l)} - \sum_{l,m=1}^{r} F_i^{(k,m)} Y_0^{(m)}(H_1^{(l)}) dc_l + \sum_{l=1}^{r} F_i^{(k,l)} dH_1^{(l)}$$
$$= \Omega_1 Y_i^{(k)},$$

as

$$\Omega_1 Y_0^{(l)} = \sum_{k=1}^{r} (dH_1^{(k)} \wedge dc_k) Y_0^{(l)} = dH_1^{(l)},$$

$$Y_i^{(k)}(H_1^{(l)}) = 0, \quad Y_i^{(k)}(c_l) = F_i^{(k,l)}, \quad Y_0^{(r)}(c_k) = \delta_{rk}.$$

Hence, on $\mathcal{M}$, differentials $dH_i^{(k)}$ form a bi-inverse-Hamiltonian hierarchies

$$\boxed{\begin{aligned} &\Omega_0 Y_0^{(k)} = 0 \\ &\Omega_0 Y_1^{(k)} = dH_1^{(k)} = \Omega_1 Y_0^{(k)} \\ &\qquad\qquad \vdots \qquad\qquad\qquad k = 1, \ldots, r, \\ &\Omega_0 Y_{n_k}^{(k)} = dH_{n_k}^{(k)} = \Omega_1 Y_{n_k-1}^{(k)} \\ &\qquad\qquad 0 = \Omega_1 Y_{n_k}^{(k)} \end{aligned}} \tag{5.3.27}$$

which start with a kernel vector field $Y_0^{(k)}$ of Ω_0 and terminate with a kernel vector field $Y_{n_k}^{(k)}$ of Ω_1. Indeed

$$\Omega_1 Y_{n_k}^{(k)} = (\Omega_{1D} + \sum_{m=1}^{r} dH_1^{(m)} \wedge dc_m)(X_{n_k}^{(k)} + \sum_{m=1}^{r} F_{n_k}^{(k,m)} Y_0^{(m)})$$
$$= -\sum_{m=1}^{r} F_{n_k}^{(k,m)} dH_1^{(m)} + \sum_{m=1}^{r} F_{n_k}^{(k,m)} dH_1^{(m)} = 0,$$

what follows from (5.3.17), (5.3.25), (5.3.26) and the fact that $\Omega_{1D} Y_0^{(k)} = 0$. Moreover, Ω_0 and Ω_1 are d-compatible with respect to Π_0, as

$$\Pi_0 \Omega_1 \Pi_0 = \Pi_0 \Omega_{1D} \Pi_0 = \Pi_{1D}$$

which is a Poisson. According to conditions (5.3.18) vector fields $X_i^{(k)}$ are not bi-Hamiltonian as $Y_j^{(k)}(c_l) = F_j^{(k,l)} \neq 0$ and $Y_0^{(k)}(H_i^{(k)}) = 0$, which violates condition (5.3.18b).

In order to lift (5.3.22) to a respective bi-Hamiltonian representation on the extended phase space $\mathcal{M}$ (5.3.24), one has to extend the original Hamiltonians $H_j^{(k)}$

$$H_j^{(k)} \to h_j^{(k)} = H_j^{(k)} + \sum_{l=1}^{r} F_j^{(k,l)} c_l, \quad i = 1, \ldots, n,$$

where functions $F_j^{(k,l)}$ are appropriate potentials calculated from Eqs. (5.1.17). Then, on $\mathcal{M}$, vector fields $X_j^{(k)} = \Pi_0 dh_j^{(k)}$ form a bi-Hamiltonian chains

$$\Pi_0 dh_{j+1}^{(k)} = X_{j+1}^{(k)} = \Pi_1 dh_j^{(k)}, \quad j = 1, 2, \ldots, n_k, \quad k = 1, \ldots, r,$$

where

$$\Pi_1 = \Pi_{1D} + \sum_{m=1}^{r} X_1^{(m)} \wedge Y_0^{(m)}$$

is a Poisson tensor compatible with Π_0 one. Each chain starts with the Casimir of Π_0, i.e. $H_0^{(k)} = c_k$, and terminates with the Casimir of Π_1, i.e. $H_{n_k}^{(k)}$. The reader finds the details of the construction in [31]. Poisson tensors Π_0 and Π_1 are d-compatible with respect to Ω_0 as

$$\Omega_0 \Pi_1 \Omega_0 = \Omega_0 \Pi_{1D} \Omega_0 = \Omega_{1D}$$

is closed. As was proved in [29], bi-Hamiltonian chains (5.3.3) have no bi-presymplectic counterparts as the conditions (5.3.18) are not satisfied. Indeed

$$Y_0^{(k)}(H_1^{(m)}) = F_1^{(m,k)} \neq F_1^{(k,m)} = Y_0^{(m)}(H_1^{(k)})$$

as coefficients $F_1^{(k,l)}$ are solutions of the set of linear algebraic Eqs. (5.1.17) and all are different in general. The only exception is the case of co-rank one ($r = 1$), as then (5.3.18) is trivially fulfilled.

Example 5.13 The bi-inverse-Hamiltonian of the Henon–Heiles system.

Let us consider the integrable case of the Henon–Heiles system considered in Examples 3.6 and 4.1 with the following constants of motion

$$H_1 = H = \tfrac{1}{2} p_x^2 + \tfrac{1}{2} p_y^2 + \tfrac{1}{2} x y^2 + x^3,$$

$$H_2 = \tfrac{1}{2} y p_x p_y - \tfrac{1}{2} x p_y^2 + \tfrac{1}{4} x^2 y^2 + \tfrac{1}{16} y^4.$$

On 5-dimensional extended phase space parametrized by (x, y, p_x, p_y, c) differentials dH_1 and dH_2 have a bi-inverse-Hamiltonian representation of the form

$$\begin{aligned}\Omega_0 Y_0 &= 0\\ \Omega_0 Y_1 &= dH_1 = \Omega_1 Y_0\\ \Omega_0 Y_2 &= dH_2 = \Omega_1 Y_1\\ 0 &= \Omega_1 Y_2\end{aligned}$$

where vector fields Y_i are

$$\begin{aligned}Y_0 &= (0,0,0,0,1)^T\\ Y_1 &= X_1 + F_1 Y_0 = (p_x, p_y, -3x^2 - \tfrac{1}{2}y^2, -xy, -x)^T\\ Y_2 &= X_2 + F_2 Y_0 = (\tfrac{1}{2}yp_y, \tfrac{1}{2}yp_x - xp_x, \tfrac{1}{2}p_y^2 - \tfrac{1}{2}xy^2,\\ &\quad -\tfrac{1}{2}p_x p_y - \tfrac{1}{4}y^3 - \tfrac{1}{2}x^2 y, -\tfrac{1}{4}y^2)^T,\end{aligned}$$

presymplectic forms

$$\Omega_0 = \begin{pmatrix} 0&0&-1&0&0\\ 0&0&0&-1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&0 \end{pmatrix},$$

$$\Omega_1 = \begin{pmatrix} 0 & -\frac{1}{2}p_y & -x & -\frac{1}{2}y & 3x^2 + \frac{1}{2}y^2\\ \frac{1}{2}p_y & 0 & -\frac{1}{2}y & 0 & xy\\ x & \frac{1}{2}y & 0 & 0 & p_x\\ \frac{1}{2}q_y & 0 & 0 & 0 & p_y\\ -3x^2 - \frac{1}{2}y^2 & -xy & -p_x & -p_y & 0 \end{pmatrix}.$$

are d-compatible with respect to the canonical Poisson tensor Π_0 dual to Ω_0 one, $X_i = \Pi_0 dH_i$ and expansion coefficients (5.1.17) $F_1 \equiv F_1^{(1,1)}$, $F_2 \equiv F_2^{(1,1)}$ are respective basic potentials $F_1 = -V_1^{(2)} = -x$, $F_2 = -V_2^{(2)} = -\frac{1}{4}y^2$. The chain starts with a kernel vector field Y_0 of Ω_0 and terminates with a kernel vector field Y_2 of Ω_1. On $\mathbb{R}^4$ we have immediately

$$\omega_0 = \Omega_0|_{\mathbb{R}^4} = \begin{pmatrix} 0&0&-1&0\\ 0&0&0&-1\\ 1&0&0&0\\ 0&1&0&0 \end{pmatrix},$$

$$\omega_1 = \Omega_1|_{\mathbb{R}^4} \begin{pmatrix} 0 & -\frac{1}{2}p_y & -x & -\frac{1}{2}y \\ \frac{1}{2}p_y & 0 & -\frac{1}{2}y & 0 \\ x & \frac{1}{2}y & 0 & 0 \\ \frac{1}{2}y & 0 & 0 & 0 \end{pmatrix}$$

and the quasi-bi-Hamiltonian representation takes the form (5.3.22)

$$\pi_1 dH_1 = \pi_0 dH_2 - F_1 \pi_0 dH_1, \quad \pi_1 dH_2 = -F_2 \pi_0 dH_1,$$

where

$$\pi_0 = \Pi_0|_{\mathbb{R}^4} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \omega_0^{-1},$$

$$\pi_1 = \Pi_0 \Omega_1 \Pi_0|_{\mathbb{R}^4} = \begin{pmatrix} 0 & 0 & x & \frac{1}{2}y \\ 0 & 0 & \frac{1}{2}y & 0 \\ -x & -\frac{1}{2}y & 0 & \frac{1}{2}p_y \\ -\frac{1}{2}y & 0 & -\frac{1}{2}p_y & 0 \end{pmatrix} = \pi_0 \omega_1 \pi_0,$$

Example 5.14 Consider the separation relations on a 6-dimensional phase space M given by the following geodesic separation curve

$$T_1 \lambda^2 + T_2 \lambda + T_3 = \tfrac{1}{2}\mu^2.$$

This curve corresponds to the geodesic motion for a classical Stäckel system (of Benenti type). As in this example $r = 1$, we use the notation $T_i^{(1)} \equiv T_i$. The transformation to flat coordinates (x, y, z) of associated metric (see Sect. 5.5) follows from the point transformation

$$\begin{aligned} \rho_1 &= x = -\lambda^1 - \lambda^2 - \lambda^3, \\ \rho_2 &= \tfrac{1}{4}x^2 + y = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3, \\ \rho_3 &= \tfrac{1}{2}xy + z = -\lambda^1\lambda^2\lambda^3. \end{aligned}$$

In the canonical coordinates (x, y, z, p_x, p_y, p_z) Hamiltonians take the form

$$\begin{aligned} T_1 &= p_x p_z + \tfrac{1}{2}p_y^2, \\ T_2 &= p_x p_y + \tfrac{1}{2}x p_y^2 + \tfrac{1}{2}x p_x p_z - \tfrac{1}{2}y p_y p_z - \tfrac{1}{2}z p_z^2, \\ T_3 &= \tfrac{1}{2}p_x^2 + \tfrac{1}{8}x^2 p_y^2 + \tfrac{1}{8}y^2 p_z^2 + \tfrac{1}{2}x p_x p_y + \tfrac{1}{2}y p_x p_z - (\tfrac{1}{4}xy + z) p_y p_z, \end{aligned} \tag{5.3.28}$$

and admit a quasi bi-inverse-Hamiltonian representation (5.3.23)

$$\begin{aligned}\omega_1\mathbf{x}_1 &= \omega_0\mathbf{x}_2 - F_1\omega_0\mathbf{x}_1,\\ \omega_1\mathbf{x}_2 &= \omega_0\mathbf{x}_3 - F_2\omega_0\mathbf{x}_1,\\ \omega_1\mathbf{x}_3 &= -F_3\omega_0\mathbf{x}_1,\end{aligned}$$

with symplectic operators ω_0 and ω_1 of the form

$$\omega_0 = \pi_0^{-1} = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}, \tag{5.3.29}$$

$$\omega_1 = \begin{pmatrix} 0 & \frac{1}{2}p_y & \frac{1}{2}p_z & \frac{1}{2}x & \frac{1}{2}y & z \\ -\frac{1}{2}p_y & 0 & 0 & -1 & 0 & \frac{1}{2}y \\ -\frac{1}{2}p_z & 0 & 0 & 0 & -1 & \frac{1}{2}x \\ -\frac{1}{2}x & 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}y & 0 & 1 & 0 & 0 & 0 \\ -z & -\frac{1}{2}y & -\frac{1}{2}x & 0 & 0 & 0 \end{pmatrix}, \tag{5.3.30}$$

and Hamiltonian vector fields $\mathbf{x}_i = \pi_0 dT_i,\ i = 1, 2, 3$. The expansion coefficients $F_i^{(1,1)} \equiv F_i$ are respective basic potentials $F_i = -V_i^{(3)} = \rho_i$, i.e. the solution of equations (5.1.17)

$$(\lambda^i)^3 + F_1(\lambda^i)^2 + F_2\lambda^i + F_3 = 0, \qquad i = 1, 2, 3$$

which in flat coordinates are

$$F_1 = \rho_1 = x, \quad F_2 = \rho_2 = \tfrac{1}{4}x^2 + y, \quad F_3 = \sigma_3 = \tfrac{1}{2}xy + z.$$

On the extended phase space $\mathcal{M}$ of dimension seven, with an additional coordinate c, the differentials dH_i form a bi-inverse-Hamiltonian chain

$$\begin{aligned}\Omega_0 Y_0 &= 0\\ \Omega_0 Y_1 &= dT_1 = \Omega_1 Y_0\\ \Omega_0 Y_2 &= dT_2 = \Omega_1 Y_1\\ \Omega_0 Y_3 &= dT_3 = \Omega_1 Y_2\\ 0 &= \Omega_1 Y_3,\end{aligned}$$

with presymplectic forms

$$\Omega_0 = \left(\begin{array}{c|c} \omega_0 & 0 \\ \hline 0 & 0 \end{array}\right) \quad , \quad \Omega_1 = \left(\begin{array}{c|c} \omega_1 & dT_1 \\ \hline -(dT_1)^T & 0 \end{array}\right)$$

d-compatible with respect to

$$\Pi_0 = \begin{pmatrix} 0 & I_3 & 0 \\ -I_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and vector fields

$$Y_0 = (0, \ldots, 0, 1)^T, \quad Y_i = X_i + F_i Y_0, \quad i = 1, 2, 3,$$

where $X_i = \Pi_0 dT_i = (\mathbf{x}_i, 0)^T$.

Example 5.15 Consider the separation relations on a 6-dimensional phase space given by the following geodesic separation curve

$$\bar{T}_1\lambda^3 + \bar{T}_2\lambda^2 + \bar{T}_3 \equiv \lambda^2(T_1^{(1)}\lambda + T_2^{(1)}) + T_1^{(2)} = \tfrac{1}{2}\mu^2$$

representing geodesic motion for a classical Stäckel system (this time of the non-Benenti type). Actually it is the case (5.1.19) from the previous section when $n = 3$, $n_1 = 2$ and $\sigma(\lambda) = 0$. Using the coordinates, the geodesic Hamiltonians T_i, and the functions σ_i from the previous example we find, according to (4.4.34) and (4.4.35) for $n = 3,\ m_1 = 3$, that

$$\begin{aligned} T_1^{(1)} &= -\tfrac{1}{\sigma_2} T_2, \\ T_2^{(1)} &= T_1 - \tfrac{\sigma_1}{\sigma_2} T_2, \\ T_1^{(2)} &= T_3 - \tfrac{\sigma_3}{\sigma_2} T_2 \end{aligned}$$

and thus we see that the Hamiltonians $T_i^{(k)}$ are related to T_j (5.3.28) through the Stäckel transform (4.4.28) (see Sect. 4.4.3 for details). They admit a quasi bi-inverse-Hamiltonian representation (5.3.23)

$$\begin{aligned} \omega_1 \mathbf{x}_1^{(1)} &= \omega_0 \mathbf{x}_2^{(1)} - F_1^{(1,1)} \omega_0 \mathbf{x}_1^{(1)} - F_1^{(1,2)} \omega_0 \mathbf{x}_1^{(2)}, \\ \omega_1 \mathbf{x}_2^{(1)} &= -F_2^{(1,1)} \omega_0 \mathbf{x}_1^{(1)} - F_2^{(1,2)} \omega_0 \mathbf{x}_1^{(2)}, \\ \omega_1 \mathbf{x}_1^{(2)} &= -F_1^{(2,1)} \omega_0 \mathbf{x}_1^{(1)} - F_1^{(2,2)} \omega_0 \mathbf{x}_1^{(2)} \end{aligned}$$

with the presymplectic forms (5.3.29), (5.3.30) and Hamiltonian vector fields $\mathbf{x}_i^{(k)} = \pi_0 dT_i^{(k)}$. The expansion coefficients, according to (5.1.17), are solutions of three copies of respective equations

$$\lambda^2(\lambda^2 + F_1^{(1,1)}\lambda + F_2^{(1,1)}) + F_1^{(2,1)} = 0$$

and

$$\lambda^2(F_1^{(1,1)}\lambda + F_2^{(1,1)}) + (\lambda + F_1^{(2,1)}) = 0$$

for $\lambda = \lambda_1, \lambda_2, \lambda_3$, i.e. from (5.1.20) we get

$$F_1^{(1,1)} = \rho_1 - \tfrac{\rho_3}{\rho_2}, \quad F_2^{(1,1)} = \rho_2 - \tfrac{\rho_1\rho_3}{\rho_2}, \quad F_1^{(2,1)} = \tfrac{\rho_3^2}{\rho_2},$$

$$F_1^{(1,2)} = \tfrac{1}{\rho_2}, \quad F_2^{(1,2)} = \tfrac{\rho_1}{\rho_2}, \quad F_1^{(2,2)} = \tfrac{\rho_3}{\rho_2}.$$

On the extended phase space $\mathcal{M}$ of dimension eight, with additional coordinates c_1 and c_2, the differentials $dT_i^{(k)}$ form a bi-inverse-Hamiltonian chains

$$\begin{array}{lll} \Omega_0 Y_0^{(1)} = 0 & & \\ \Omega_0 Y_1^{(1)} = dT_1^{(1)} = \Omega_1 Y_0^{(1)} & & \Omega_0 Y_0^{(2)} = 0 \\ \Omega_0 Y_2^{(1)} = dT_2^{(1)} = \Omega_1 Y_1^{(1)} & & \Omega_0 Y_1^{(2)} = dT_1^{(2)} = \Omega_1 Y_0^{(2)} \\ 0 = \Omega_1 Y_2^{(1)} & & 0 = \Omega_1 Y_1^{(2)}, \end{array}$$

with the presymplectic forms

$$\Omega_0 = \left(\begin{array}{c|cc} \omega_0 & 0 & 0 \\ \hline 0 & & \\ 0 & & 0 \end{array}\right), \quad \Omega_1 = \left(\begin{array}{c|cc} \omega_1 & dT_1^{(1)} & dT_1^{(2)} \\ \hline -(dT_1^{(1)})^T & & \\ -(dT_1^{(2)})^T & & 0 \end{array}\right)$$

d-compatible with respect to Π_0 and vector fields

$$Y_0^{(1)} = (0, \ldots, 0, 1, 0)^T, \quad Y_0^{(2)} = (0, \ldots, 0, 0, 1)^T, \quad Y_1^{(1)} = X_1^{(1)} + F_1^{(1,1)} Y_0^{(1)} + F_1^{(1,2)} Y_0^{(2)},$$

$$Y_2^{(1)} = X_2^{(1)} + F_2^{(1,1)} Y_0^{(1)} + F_2^{(1,2)} Y_0^{(2)}, \quad Y_1^{(2)} = X_1^{(2)} + F_1^{(2,1)} Y_0^{(1)} + F_1^{(2,2)} Y_0^{(2)},$$

where again $X_i^{(j)} = \Pi_0 dT_i^{(j)} = (\mathbf{x}_i^{(j)}, 0)^T$.

Example 5.16 Consider Hamiltonians with elliptic potentials, described in Sect. 5.2.1

$$H_r = \frac{1}{2}\sum_{i=1}^{n}\left[-\frac{\partial\rho_r(\beta)}{\partial\beta_i} - \frac{1}{4}\sum_{k=1,k\neq i}^{n}\frac{\partial^2\rho_r(\beta)}{\partial\beta_i\partial\beta_k}(x^k)^2\right]p_i^2 + \frac{1}{8}\sum_{i,j=1,i\neq j}^{n}\frac{\partial^2\rho_r(\beta)}{\partial\beta_i\partial\beta_j}x^i x^j p_i p_j + V_r(x),$$

where $r = 1, \ldots, n$. They form the bi-inverse-Hamiltonian chain

$$\begin{aligned}
\Omega_0 Y_0 &= 0 \\
\Omega_0 Y_1 &= dH_1 = \Omega_1 Y_0 \\
&\vdots \\
\Omega_0 Y_n &= dH_2 = \Omega_1 Y_{n-1} \\
0 &= \Omega_1 Y_n
\end{aligned}$$

where

$$\Omega_0 = \begin{pmatrix} 0 & -I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \Omega_1 = \begin{pmatrix} \frac{1}{4} p \otimes x - \frac{1}{4} x \otimes p & -B + \frac{1}{4} x \otimes x & \frac{\partial V_1(x)}{\partial x} \\ B - \frac{1}{4} x \otimes x & 0 & p \\ -\frac{\partial V_1(x)}{\partial x} & -p & 0 \end{pmatrix}$$

and

$$Y_r = \begin{pmatrix} (X_r)^1 \\ \vdots \\ (X_r)^{2n} \\ \rho_r(\beta) - \frac{1}{4} \sum_{i=1}^{n} \frac{\partial \rho_r(\beta)}{\partial \beta_i} (x^i)^2 \end{pmatrix}, \qquad X_r = \Pi_0 dH_r, \qquad \Pi_0 = \begin{pmatrix} 0 & I_n & 0 \\ -I_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The presymplectic forms Ω_1 and Ω_0 are d-compatible with respect to Π_0 if and only if the potential function $V_1(x)$ satisfies the Eqs. (5.2.9).

5.4 Bi-Hamiltonian and Bi-Presymplectic Theory in $\mathbb{R}^3$

The simplest realization of the theory presented in previous two sections takes place in $M = \mathbb{R}^3$. We consider Poisson bi-vectors and presymplectic forms which in this case have a particularly convenient description. In consequence we get a simple condition for respective compatibility. We analyze bi-Hamiltonian and bi-presymplectic chains with particular care and find conditions for their equivalence. We illustrate that case by a few instructive examples.

5.4.1 Poisson and Presymplectic Structures in $\mathbb{R}^3$

In $\mathbb{R}^3$, parametrized by coordinates $(\zeta^1, \zeta^2, \zeta^3)$, any Poisson bi-vector Π can be represented by the following form [5]

$$\boxed{\Pi^{ij} = \mu \epsilon^{ijk} \partial_k F \Longrightarrow \Pi = \begin{pmatrix} 0 & \mu\partial_3 F & -\mu\partial_2 F \\ -\mu\partial_3 F & 0 & \mu\partial_1 F \\ \mu\partial_2 F & -\mu\partial_1 F & 0 \end{pmatrix}.} \tag{5.4.1}$$

Here μ and F are some differentiable functions in $\mathbb{R}^3$ and ϵ^{ijk} is a Levi-Civita symbol

$$\epsilon^{ijk} = \begin{cases} 0 & \text{when some indices coincide} \\ 1 & \text{for even permutations of } i, j, k \\ -1 & \text{for odd permutations of } i, j, k. \end{cases}$$

Note that for the above Poisson tensor we have $\Pi dF = 0$ that is the kernel of Π is spanned by the form dF.

In order to prove the representation (5.4.1) put $\Pi^{12} = u$, $\Pi^{31} = v$ and $\Pi^{23} = w$, respectively. Then the Jacobi equation (3.2.3) takes the form

$$u\partial_1 v - v\partial_1 u + w\partial_2 u - u\partial_2 w + v\partial_3 w - w\partial_3 v = 0. \tag{5.4.2}$$

First, assume that $u \neq 0$, let $\kappa = v/u$ and $\xi = w/u$, then Eq. (5.4.2) can be written as

$$\partial_1 \kappa - \partial_2 \xi + \kappa\partial_3 \xi - \xi\partial_3 \kappa = 0,$$

or in a more suitable form

$$(\partial_1 - \xi\partial_3)\kappa - (\partial_2 - \kappa\partial_3)\xi = 0. \tag{5.4.3}$$

Introducing differential operators

$$D_1 = \partial_1 - \xi\partial_3, \quad D_2 = \partial_2 - \kappa\partial_3,$$

one can write Eq. (5.4.3) as

$$D_1\kappa - D_2\xi = 0. \tag{5.4.4}$$

If (5.4.4) is satisfied, it is easy to show that the operators D_1 and D_2 commute and hence there exist new coordinates $(\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3)$, such that

$$D_1 = \partial_{\bar{\zeta}^1}, \quad D_2 = \partial_{\bar{\zeta}^2}. \tag{5.4.5}$$

Let F be a common invariant function of D_1 and D_2

$$D_1 F = D_2 F = 0, \tag{5.4.6}$$

then the coordinates $(\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3)$ are given by

$$\bar{\zeta}^1 = \zeta^1, \quad \bar{\zeta}^2 = \zeta^2, \quad \bar{\zeta}^3 = F.$$

Moreover, from (5.4.6) we get

$$\xi = \frac{\partial_1 F}{\partial_3 F}, \quad \kappa = \frac{\partial_1 F}{\partial_3 F}. \tag{5.4.7}$$

Using (5.4.7), the entries of matrix Π, in the coordinates $(\bar{\zeta}^1, \bar{\zeta}^2, \bar{\zeta}^3)$, can be written as

$$u = \mu \partial_3 F, \quad v = \mu \partial_2 F, \quad w = \mu \partial_1 F. \tag{5.4.8}$$

Thus matrix Π has the form (5.4.1).

So far we assumed that $u \neq 0$. If $u = 0$ then the Jacobi equation (3.2.3) becomes simpler

$$v \partial_3 w - w \partial_3 v = 0,$$

which has the simple solution $w = v\varphi(\zeta^1, \zeta^2)$, where φ is an arbitrary differentiable function of ζ^1 and ζ^2. This class is also covered by the general solution (5.4.1) by letting F be independent of ζ^3.

Let Poisson tensors Π_0 and Π_1 be given by $(\Pi_0)^{ij} = \mu_0 \epsilon^{ijk} \partial_k H_0$ and $(\Pi_1)^{ij} = \mu_1 \epsilon^{ijk} \partial_k H_1$, where μ_0, μ_1 and H_0, H_1 are some differentiable functions. Then Π_0 and Π_1 are compatible if and only if there exists a differentiable function $\Phi(H_0, H_1)$ such that

$$\mu_1 = \mu_0 \frac{\partial_{H_1} \Phi}{\partial_{H_0} \Phi} \tag{5.4.9}$$

provided that $\partial_{H_1} \Phi = \partial \Phi / \partial H_1 \neq 0$ and $\partial_{H_0} \Phi = \partial \Phi / \partial H_0 \neq 0$ [5]. Indeed, compatibility means that the linear combination of Π_0 and Π_1 has the form (5.4.1).

Let us consider the Poisson structure (5.4.1) with $F = \Phi(H_0, H_1)$. Then

$$\Pi = \mu\epsilon^{ijk}\partial_k F = \mu\epsilon^{ijk}\partial_k\left[\frac{\partial\Phi}{\partial H_0}\partial_k H_0 + \frac{\partial\Phi}{\partial H_1}\partial_k H_1\right]$$
$$= \mu\frac{\partial\Phi}{\partial H_0}\epsilon^{ijk}\partial_k H_0 + \mu\frac{\partial\Phi}{\partial H_1}\epsilon^{ijk}\partial_k H_1,$$

hence

$$\mu_0 = \mu\frac{\partial\Phi}{\partial H_0}, \quad \mu_1 = \mu\frac{\partial\Phi}{\partial H_1}$$

and after elimination of μ we get condition (5.4.9).

For example, it follows that a Poisson tensor Π_0, given by μ and a function H_0, and a Poisson tensor Π_1, given by $-\mu$ and a function H_1, are compatible. One should take $\Phi = H_0 - H_1$.

The presymplectic forms in $\mathbb{R}^3$ are described in the following way. Any closed two-form Ω in $\mathbb{R}^3$ has the form

$$\boxed{\Omega_{ij} = \nu\epsilon_{ijk}Y^k \Longrightarrow \Omega = \begin{pmatrix} 0 & \nu Y^3 & -\nu Y^2 \\ -\nu Y^3 & 0 & \nu Y^1 \\ \nu Y^2 & -\nu Y^1 & 0 \end{pmatrix},} \tag{5.4.10}$$

where ν is a differentiable function and $Y = (Y^1, Y^2, Y^3)^T$ is a vector, such that νY is a divergence free vector

$$\boxed{\operatorname{div}\nu Y = \partial_i(\nu Y^i) = 0 \Longleftrightarrow \nu\operatorname{div} Y + Y(\nu) = 0.} \tag{5.4.11}$$

Equation (5.4.11) has also a strong solution

$$\operatorname{div} Y = 0, \;\; Y(\nu) = 0. \tag{5.4.12}$$

Note that for the above presymplectic form we have $\Omega Y = 0$, which means that the kernel of Ω is spanned by the vector Y. A particular case of (5.4.10) for $\nu = 1$ was considered in [46].

Next, let us consider a dual pair. For a Poisson tensor $\Pi^{ij} = \mu\epsilon^{ijk}\partial_k F$ and a presymplectic form $\Omega_{ij} = \nu\epsilon_{ijk}Y^k$ the pair (Π, Ω) is a dual pair if and only if

$$Y(F) = Y^i\partial_i F = 1, \quad \nu = -\mu^{-1}. \tag{5.4.13}$$

Indeed, the form Ω is dual to the Poisson tensor Π if the following partition of the unit operator holds

$$I = \Pi\Omega + Y \otimes dF.$$

The above equality is fulfilled under condition (5.4.13), as

$$\begin{aligned}(\Pi\Omega)^i_s &= \mu\nu\varepsilon^{ijk}\varepsilon_{jsm}F_{,k}Y^m = \mu\nu(\delta^k_s\delta^i_m - \delta^i_s\delta^k_m)F_{,k}Y^m \\ &= \mu\nu F_{,s}Y^i - \mu\nu F_{,k}Y^k\delta^i_s,\end{aligned}$$

where we used the following property of the Levi-Civita symbol

$$\varepsilon^{ikj}\varepsilon_{jsm} = \delta^i_s\delta^k_m - \delta^k_s\delta^i_m. \tag{5.4.14}$$

Consider a dual pair (Π_0, Ω_0), where the Poisson tensor Π_0 is given by $(\Pi_0)^{ij} = \mu_0\epsilon^{ijk}\partial_k H_0$ and the presymplectic form Ω_0 is given by $(\Omega_0)_{ij} = -\mu_0^{-1}\epsilon_{ijk}Y_0^k$. Then, the Poisson tensor Π_1, $(\Pi_1)^{ij} = \mu_1\epsilon^{ijk}\partial_k H_1$, is d-compatible with the Poisson tensor Π_0 if

$$Y_0\left(\frac{\mu_1}{\mu_0}Y_0(H_1)\right) = 0. \tag{5.4.15}$$

The condition (5.4.15) follows from the fact that the two-form

$$\Omega_0\Pi_1\Omega_0 = \frac{\mu_1}{\mu_0}Y_0(H_1)\Omega_0$$

is closed under condition (5.4.15) which follows from (5.4.12).

Consider the same dual pair (Π_0, Ω_0). Then, the presymplectic form Ω_1, $(\Omega_1)_{ij} = \nu_1\epsilon_{ijk}Y_1^k$, is d-compatible with the presymplectic form Ω_0 if

$$Y_1(H_0) \neq 0. \tag{5.4.16}$$

The condition (5.4.16) follows from the fact that the bi-vector

$$\Pi_0\Omega_1\Pi_0 = \mu_0\nu_1 Y_1(H_0)\Pi_0$$

is a Poisson if $Y_1(H_0) \neq 0$.

5.4.2 *Bi-Hamiltonian and Bi-Presymplectic Chains in* $\mathbb{R}^3$

Suppose we have two compatible Poisson structures Π_0 and Π_1 in an open domain of $\mathbb{R}^3$, given by

$$\boxed{(\Pi_0)^{ij} = \mu\,\epsilon^{ijk}\partial_k H_0 \quad \text{and} \quad (\Pi_1)^{ij} = -\mu\,\epsilon^{ijk}\partial_k H_1, \qquad i, j = 1, 2, 3.} \tag{5.4.17}$$

The Casimirs of the Π_0 and Π_1 are dH_0 and dH_1 respectively. Then we can consider a bi-Hamiltonian chain

$$\boxed{\begin{aligned}\Pi_0 dH_0 &= 0\\ \Pi_0 dH_1 &= X = \Pi_1 dH_0\\ 0 &= \Pi_1 dH_1.\end{aligned}} \tag{5.4.18}$$

It follows from the fact that

$$\begin{aligned}(\Pi_0 dH_1)^i &= \mu\,\epsilon^{ijk}(\partial_k H_0)(\partial_j H_1),\\ (\Pi_1 dH_0)^i &= -\mu\,\epsilon^{ijk}(\partial_k H_1)(\partial_j H_0) = \mu\,\epsilon^{ijk}(\partial_k H_0)(\partial_j H_1).\end{aligned}$$

From the construction it follows that in $\mathbb{R}^3$ any Hamiltonian vector field X is simultaneously a bi-Hamiltonian.

Moreover, if the presymplectic form Ω_0, given by $(\Omega_0)_{ij} = -\mu^{-1}\epsilon_{ijk}Y_0^k$, is a dual to Poisson structure Π_0 with additional condition (5.4.15)

$$Y_0(Y_0(H_1)) = 0, \tag{5.4.19}$$

then the pair (Π_0, Π_1) is d-compatible with respect to Ω_0.

Consider closed two-forms Ω_0 and Ω_1 in an open domain of $\mathbb{R}^3$, given in terms of vectors Y_0, Y_1 and a function ν by

$$\boxed{(\Omega_0)_{ij} = \nu\epsilon_{ijk}Y_0^k \quad \text{where} \quad \partial_k(\nu Y_0^k) = 0, \;\; i,j = 1,2,3} \tag{5.4.20}$$

and

$$\boxed{(\Omega_1)_{ij} = -\nu\epsilon_{ijk}Y_1^k \quad \text{where} \quad \partial_k(\nu Y_1^k) = 0, \;\; i,j = 1,2,3.} \tag{5.4.21}$$

It is easy to see that in $\mathbb{R}^3$ any two such presymplectic forms give a bi-presymplectic chain

$$\boxed{\begin{aligned}\Omega_0 Y_0 &= 0\\ \Omega_0 Y_1 &= \beta = \Omega_1 Y_0\\ 0 &= \Omega_1 Y_1.\end{aligned}} \tag{5.4.22}$$

Again it follows from the fact that

$$\begin{aligned}(\Omega_0 Y_1)_i &= \nu\varepsilon_{ijk}Y_0^k Y_1^j,\\ (\Omega_1 Y_0)_i &= -\nu\varepsilon_{ijk}Y_1^k Y_0^j = \nu\varepsilon_{ijk}Y_0^k Y_1^j.\end{aligned}$$

If a Poisson tensor Π_0 given by $\Pi_0^{ij} = -\frac{1}{\nu}\epsilon^{ijk}\partial_k H_0$ is dual to Ω_0 and such that $Y_1(H_0) \neq 0$, then Ω_0 and Ω_1 are d-compatible with respect to Π_0.

For a d-compatible bi-Hamiltonian chain (5.4.18), (5.4.19) we can construct a corresponding bi-presymplectic chain. Take a pair of presymplectic structures in the form $(\Omega_0)_{ij} = -\mu^{-1}\epsilon_{ijk}Y_0^k$, $(\Omega_1)_{ij} = \mu^{-1}\epsilon_{ijk}Y_1^k$ where

$$Y_1 = X + Y_0(H_1)Y_0. \tag{5.4.23}$$

The two-form Ω_0 is presymplectic dual to Π_0 and the 2-form Ω_1 is presymplectic as

$$\begin{aligned}\operatorname{div}(\mu^{-1}Y_1) &= \operatorname{div}(\mu^{-1}X) + \operatorname{div}(Y_0(H_1)\mu^{-1}Y_0)\\ &= \operatorname{div}(\mu^{-1}X) + \operatorname{div}(\mu^{-1}Y_0) + \mu^{-1}Y_0(Y_0(H_1)) = 0.\end{aligned}$$

They form a bi-presymplectic chain (5.4.22) with

$$\beta = \Omega_0 Y_1 = \Omega_0 X = \Omega_0\Pi_0 dH_1 = (I - dH_0 \otimes Y_0)dH_1 = dH_1 - Y_0(H_1)dH_0. \tag{5.4.24}$$

For a d-compatible bi-presymplectic chain (5.4.22) we can construct a corresponding bi-Hamiltonian chain provided that

$$X = \Pi_0\beta = \Pi_0 dH_1, \tag{5.4.25}$$

where Π_0, given by $\Pi_0^{ij} = -\frac{1}{\nu}\epsilon^{ijk}\partial_k H_0$, is dual to Ω_0 and such that $Y_1(H_0) \neq 0$. Following previous considerations, every Hamiltonian system in $\mathbb{R}^3$ has a bi-Hamiltonian representation. Thus the vector field $X = \Pi_0 dH_1$ can be also written as $X = \Pi_1 dH_0$, where $(\Pi_1)^{ij} = \frac{1}{\nu}\epsilon^{ijk}\partial_k H_1$ for $i, j = 1, 2, 3$.

Relations (5.3.18)–(5.3.20) from the previous section also give the bi-Hamiltonian representation of the vector field X. Let us show that these two representations coincide provided that

$$Y_0(H_1) = Y_1(H_0). \tag{5.4.26}$$

According to (5.3.20)

$$\Pi_1 = \Pi_0\Omega_1\Pi_0 + X \wedge Y_0 \tag{5.4.27}$$

that is

$$\Pi_1^{ij} = \tfrac{1}{\nu}Y_1(H_0)\epsilon^{ijk}\partial_k H_0 + (X^i Y_0^j - X^j Y_0^i), \quad i, j = 1, 2, 3.$$

The property (5.4.14) allows us to write

$$X^i Y_0^j - X^j Y_0^i = \epsilon^{ijk} W_k, \quad W_k = \varepsilon_{krs} X^r Y^s$$

and

$$\Pi_1^{ij} = -\tfrac{1}{\nu} Y_1(H_0)\epsilon^{ijk}\partial_k H_0 + \epsilon^{ijk} W_k = \epsilon^{ijk}(-\tfrac{1}{\nu} Y_1(H_0)\partial_k H_0 + W_k).$$

Since $X^i = -\frac{1}{\nu}\epsilon^{ijk}\Pi_0^k \partial_j H_1$, we can put

$$\begin{aligned} W_k &= \varepsilon_{krs} X^r Y^s = -\tfrac{1}{\nu}\epsilon_{krs}\epsilon^{rij}(\partial_j H_0)(\partial_i H_1) Y_0^s = -\tfrac{1}{\nu}(\delta_k^j \delta_s^i - \delta_k^i \delta_s^j)(\partial_j H_0)(\partial_i H_1) Y_0^s \\ &= -\tfrac{1}{\nu} Y_0(H_1)\partial_k H_0 + \tfrac{1}{\nu} Y_0(H_0)\partial_k H_1. \end{aligned}$$

Using the above equality for W_k and condition (5.4.26) we get

$$\Pi_1^{ij} = \tfrac{1}{\nu} Y_1(H_0)\epsilon^{ijk}\partial_k H_0 - \tfrac{1}{\nu} Y_0(H_1)\epsilon^{ijk}\partial_k H_0 + \tfrac{1}{\nu}\epsilon^{ijk}\partial_k H_1 = \tfrac{1}{\nu}\epsilon^{ijk}\partial_k H_1.$$

Finally, we show that there might exist presymplectic chains that do not admit a dual bi-Hamiltonian representation. Consider closed 2-forms Ω_0 and Ω_1 in $\mathbb{R}^3$, given by

$$\Omega_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \Omega_1 = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}.$$

where a, b and c are the functions of x_1, x_2 and x_3. Their kernels are spanned by vectors $Y_0 = (0, 0, 1)^t$ and $Y_1 = (a, b, c)^t$ respectively. Since $\operatorname{div} Y_1 = 0$ then we have

$$\partial_1 a + \partial_2 b + \partial_3 c = 0.$$

Let us take a Poisson tensor Π_0 in the form

$$\Pi_0 = \begin{pmatrix} 0 & \partial_3 H_0 & -\partial_2 H_0 \\ -\partial_3 H_0 & 0 & \partial_1 H_0 \\ \partial_2 H_0 & -\partial_1 H_0 & 0 \end{pmatrix},$$

where H_0 is an arbitrary function of ζ^1, ζ^2 and ζ^3. If $\partial_3 H_0 = 1$, then one can show that Π_0 and Ω_0 are dual and Ω_0 and Ω_1 are d-compatible with respect to Π_0 and

$$H_0 = \zeta^3 + h_0(\zeta^1, \zeta^2). \tag{5.4.28}$$

The forms Ω_0 and Ω_1 create a presymplectic chain

$$\begin{aligned} \Omega_0 Y_0 &= 0 \\ \Omega_0 Y_1 &= \beta = \Omega_1 Y_0 \\ 0 &= \Omega_1 Y_1, \end{aligned} \tag{5.4.29}$$

where $\beta = (b, -a, 0)^t$.

Now, let us consider a vector field X

$$X = \Pi_0 \beta = (a, b, 0)^t.$$

We find that an additional condition

$$X = \Pi_0 d H_1,$$

gives

$$a = \partial_2 H_1 - (\partial_2 H_0)(\partial_3 H_1), \tag{5.4.30a}$$

$$b = -\partial_1 H_1 + (\partial_1 H_0)(\partial_3 H_1), \tag{5.4.30b}$$

$$a\,\partial_1 H_0 + b\,\partial_2 H_0 = (\partial_1 H_0)(\partial_2 H_1) - (\partial_2 H_0)(\partial_1 H_1), \tag{5.4.30c}$$

and from the constraint (5.4.26) we get that

$$\partial_3 H_1 = a\,\partial_1 H_0 + b\,\partial_2 H_0 + c\,. \tag{5.4.31}$$

Then, using a and b from the Eqs. (5.4.30a) and (5.4.30b)respectively we show that (5.4.30c) is satisfied. Moreover, using the identity (5.4.30c) in (5.4.31) we get

$$c = \partial_3 H_1 - (\partial_1 H_0)(\partial_2 H_1) + (\partial_2 H_0)(\partial_1 H_1). \tag{5.4.32}$$

As a summary we are left with the Eqs. (5.4.30a), (5.4.30b), (5.4.32) for a,b, and c. When we use a, b and c in (5.4.29) we obtain that

$$\partial_3^2 H_1 = 0. \tag{5.4.33}$$

This is nothing else but the d-compatibility condition (5.4.19), i.e., $Y_0(Y_0(H_1)) = 0$, of the Poisson tensors Π_0 and Π_1. Equation (5.4.33) means that

$$H_1 = h_1(\zeta^1, \zeta^2)\, H_0 + h_2(\zeta^1, \zeta^2) \tag{5.4.34}$$

where h_1 and h_2 are arbitrary functions of ζ^1 and ζ^2. Using (5.4.34) and (5.4.28) we get

$$a = H_0\partial_2 h_1 + \partial_2 h_2 = \zeta^3\partial_2 h_1 + \partial_2 h_2 + f\,\partial_2 h_1,$$
$$b = -H_0\partial_1 h_1 - \partial_1 h_2 = -\zeta^3\partial_1 h_1 - \partial_1 h_2 - f\,\partial_1 h_1,$$
$$c = h_1 - a\,\partial_1 H_0 - b\,\partial_2 H_0 = h_1 - a\partial_1 f - b\partial_2 f.$$

From the above equations follows immediately that a bi-presymplectic chain (5.4.29) has a dual bi-Hamiltonian chain only in the case when functions a, b and c are linear with respect to ζ^3. On the other hand, arbitrary three functions $h_1(\zeta^1, \zeta^2)$, $h_2(\zeta^1, \zeta^2)$ and $f(\zeta^1, \zeta^2)$ determined all bi-presymplectic chains (5.4.29) which have dual bi-Hamiltonian chains.

Observation 10 *Summarizing the results of this section, first observe that any Hamiltonian system in $\mathbb{R}^3$ is simultaneously bi-Hamiltonian with respect to a pair of compatible Poisson bi-vectors (5.4.17) and belongs to the bi-Hamiltonian chain (5.4.18). If in addition Poisson bi-vectors are d-compatible then there exists a related pair of d-compatible presymplectic forms (5.4.20), (5.4.21) and related bi-presymplectic chain (5.4.22), (5.4.24). In particular, when $Y_0(H_1) = 0$ the chain is bi-inverse-Hamiltonian. The opposite is not always true, i.e. once we have a one-form in a bi-presymplectic representation (5.4.22), there exists a related bi-Hamiltonian chain (5.4.18) provided that extra condition (5.4.25) is fulfilled. Finally, observe that any Darboux coordinates are separation coordinates.*

Example 5.17 Consider the harmonic oscillator in the extended phase space $\mathbb{R}^3$, where it has both bi-Hamiltonian and bi-inverse-Hamiltonian representations. In Darboux coordinates (x, p, c) the first (canonical) Hamiltonian representation takes the form

$$\begin{pmatrix} x \\ p \\ c \end{pmatrix}_t = \begin{pmatrix} p \\ -\omega^2 x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 x^2\right) = \Pi_0 d H_1$$

where Π_0 is generated by $\mu_0 = 1$ and $H_0 = c$. The second Hamiltonian representation

$$\begin{pmatrix} x \\ p \\ c \end{pmatrix}_t = \begin{pmatrix} p \\ -\omega^2 x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & -\omega^2 x \\ -p & \omega^2 x & 0 \end{pmatrix} dc = \Pi_1 d H_0$$

is generated by $\mu_1 = -1$ and $H_1 = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$. Poisson tensors Π_0 and Π_1 are d-compatible with respect to the presymplectic form

$$\Omega_0 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

generated by $\nu_0 = -1$ and $Y_0 = (0, 0, 1)^T$, dual to Π_0. So, according to (5.4.23), the second presymplectic form

$$\Omega_1 = \begin{pmatrix} 0 & 0 & \omega^2 x \\ 0 & 0 & p \\ -\omega^2 x & -p & 0 \end{pmatrix}$$

generated by $\nu_1 = 1$ and $Y_1 = (p, -\omega^2 x, 0)$, is d-compatible with Ω_0 with respect to Π_0. Both presymplectic forms lead to the bi-inverse-Hamiltonian representation of a harmonic oscillator

$$\begin{aligned} dH_1 \equiv \begin{pmatrix} \omega^2 x \\ p \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ -\omega^2 x \\ 0 \end{pmatrix} = \Omega_0 Y_1 \\ &= \begin{pmatrix} 0 & 0 & \omega^2 x \\ 0 & 0 & p \\ -\omega^2 x & -p & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \Omega_1 Y_0. \end{aligned} \tag{5.4.35}$$

However, the constructed representation does not fit to the schema from the previous section as the reduction procedure of Π_1 onto symplectic foliation of Π_0 along the transversal distribution $\mathcal{Z} = \ker \Omega_0 = Y_0$ does not reconstruct separation coordinates $\lambda^1 = x, \mu_1 = p$. Simply $\omega_1 = \Omega_1|_{\mathbb{R}^2} = 0$, $\pi_1 = \pi_0 \omega_1 \pi_0 = 0$ and hence $N = \pi_1 \omega_0 = 0$. In order to fit in the schema presented in Sects. 5.1 and 5.3 one has to extend the harmonic oscillator Hamiltonian H_1 by an extra term

$$H_1 = \tfrac{1}{2}\left(p^2 + \omega^2 x^2\right) - cx.$$

For the extended harmonic oscillator its bi-Hamiltonian representation takes the form

$$\begin{aligned} \begin{pmatrix} x \\ p \\ c \end{pmatrix}_t &= \begin{pmatrix} p \\ -\omega^2 x + c \\ 0 \end{pmatrix} = X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} d\left(\tfrac{1}{2}p^2 + \tfrac{1}{2}\omega^2 x^2 - cx\right) = \Pi_0 dH_1 \\ &= \begin{pmatrix} 0 & x & p \\ -x & 0 & -\omega^2 x + c \\ -p & \omega^2 x - c & 0 \end{pmatrix} dc = \Pi_1 dH_0 \end{aligned} \tag{5.4.36}$$

where Π_0 is generated by $\mu_0 = 1, H_0 = c$ and Π_1 by $\mu_1 = -1, H_1 = \frac{1}{2}\left(p^2 + \omega^2 x^2\right) - cx$. The related bi-presymplectic representation takes the form

$$\beta \equiv \begin{pmatrix} \omega^2 x - c \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ -\omega^2 x + c \\ -x \end{pmatrix} = \Omega_0 Y_1$$

$$= \begin{pmatrix} 0 & -x & \omega^2 x - c \\ x & 0 & p \\ -\omega^2 x + c & -p & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \Omega_1 Y_0$$

where Ω_0 is generated by $\nu = -1, Y_0 = (0, 0, 1)$ and Ω_1 by $\nu = 1, Y_1 = X + Y_0(H_1)Y_0 = (p, -\omega^2 x + c, -x)^T$. Notice that $\beta \neq dH_1$ so it is not a bi-inverse-Hamiltonian representation. We make it bi-inverse-Hamiltonian putting $c = 0$

$$dH_1 \equiv \begin{pmatrix} \omega^2 x \\ p \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p \\ -\omega^2 x \\ -x \end{pmatrix} = \Omega_0 Y_1$$

$$= \begin{pmatrix} 0 & -x & \omega^2 x \\ x & 0 & p \\ -\omega^2 x & -p & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \Omega_1 Y_0$$

as new Ω_1 is still a presymplectic two-form, generated by $\nu_1 = 1, Y_1 = X + Y_0(H_1)Y_0 = X = (p, -\omega^2 x, 0)^T$. Notice that there is no related bi-Hamiltonian representation (5.4.36) of a standard harmonic oscillator, as a bi-vector

$$\begin{pmatrix} 0 & x & p \\ -x & 0 & -\omega^2 x \\ -p & \omega^2 x & 0 \end{pmatrix}$$

is not Poisson any more. The reduction of Ω_0 and Ω_1 onto any symplectic leave of Π_0 gives

$$\omega_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \omega_1 = \begin{pmatrix} 0 & -x \\ x & 0 \end{pmatrix}$$

and hence the recursion operator

$$N = \pi_0 \omega_1 = \omega_0^{-1} \omega_1 = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$$

reconstructs trivial information that $\lambda^1 = x$.

In the presented example bi-Hamiltonian and bi-inverse-Hamiltonian representations of a harmonic oscillator were irrelevant, as from the beginning the system was written in separation coordinates. In the next examples both representations are crucial for the construction of separation coordinates.

Example 5.18 Consider the Lorentz system [5, 141]

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}_t = \begin{pmatrix} \frac{1}{2}y \\ -xz \\ xy \end{pmatrix}$$

It admits a bi-Hamiltonian representation (5.4.18) with $H_0 = \frac{1}{4}z - \frac{1}{4}x^2$, $\mu_0 = 1$ and $H_1 = y^2 + z^2$, $\mu_1 = -1$, so

$$\Pi_0^{ij} = \varepsilon^{ijk}\partial_k H_0 \Longrightarrow \Pi_0 = \begin{pmatrix} 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & -\frac{1}{2}x \\ 0 & \frac{1}{2}x & 0 \end{pmatrix}$$

and

$$\Pi_1^{ij} = -\varepsilon^{ijk}\partial_k H_1 \Longrightarrow \Pi_1 = \begin{pmatrix} 0 & -2z & 2y \\ 2z & 0 & 0 \\ -2y & 0 & 0 \end{pmatrix}.$$

The form Ω_0 dual to Π_0 and compatible with Π_1 is given by

$$\Omega_0 = -\begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ b & -\alpha & 0 \end{pmatrix},$$

where the vector $Y_0 = (\alpha, \beta, \gamma)^T$. The conditions on α, β and γ are

$$\operatorname{div} Y_0 = \partial_x\alpha + \partial_y\beta + \partial_z\gamma = 0, \quad Y_0(H_0) = \tfrac{1}{4}\gamma - \tfrac{1}{2}x\alpha = 1.$$

One can find Ω_1 having determined Y_1 from (5.4.23)

$$Y_1 = (\tfrac{1}{2}y + 2\alpha\eta, -x\,z + 2\beta\eta, x\,y + 2\gamma\eta)$$

where $\eta = \frac{1}{2}\,Y_0(H_1) = \beta y + \gamma z$. We have an additional constraint on α, β and γ coming from $\operatorname{div} Y_1 = 0$ which reads

$$Y_0(\eta) = \alpha\partial_x\eta + \beta\partial_y\eta + \gamma\partial_z\eta = 0.$$

A simple solution for the above presymplectic structures is given as $\alpha = -2/x,\ \beta = -2y/x^2,\ \gamma = 0$. Thus we have a pair of presymplectic forms given by

$$\nu_0 = -1, \quad Y_0 = \begin{pmatrix} -2x^{-1} \\ -2yx^{-2} \\ 0 \end{pmatrix}$$

and

$$\nu_1 = 1, \quad Y_1 = \begin{pmatrix} \frac{1}{2}y + 8y^2x^{-3} \\ z + 8y^3x^{-3} \\ xy \end{pmatrix},$$

so

$$(\Omega_0)_{ij} = -\varepsilon_{ijk}Y_0^k \Longrightarrow \Omega_0 = \begin{pmatrix} 0 & 0 & -2yx^{-2} \\ 0 & 0 & 2x^{-1} \\ 2yx^{-2} & -2x^{-1} & 0 \end{pmatrix}$$

and

$$(\Omega_1)_{ij} = \varepsilon_{ijk}Y_1^k \Longrightarrow$$

$$\Omega_1 = \begin{pmatrix} 0 & xy & -z - 8y^3x^{-3} \\ -xy & 0 & \frac{1}{2}y + 8y^2x^{-3} \\ z + 8y^3x^{-3} & -\frac{1}{2}y - 8y^2x^{-3} & 0 \end{pmatrix}$$

are d-compatible presymplectic forms with respect to Π_0. According to Observation 9, the reduction of the Poisson structure Π_1 onto symplectic foliation of Π_0 is given along the distribution $\mathcal{Z} = \ker \Omega_0 = Y_0$. So, following the procedure described by (5.1.4)–(5.1.44), we find that

$$F(\lambda) = Y_0(H(\lambda)) = Y_0(H_0\lambda + H_1) = -4x^{-2}y^2,$$

$$Y = \Pi_0 d(Y_0(H_1)) = \begin{pmatrix} -2x^{-2}y \\ -2x^{-3}y^2 \\ -4x^{-1}y \end{pmatrix},$$

$$Y\left(\frac{Y(H(\lambda))}{Y^2(H(\lambda))}\right) = 1$$

and the transformation to separation coordinates (λ^1, μ_1, c) is as follows

$$\lambda^1 = 4x^{-2}y^2, \quad \mu_1 = -\tfrac{1}{8}x^{-1}y - \tfrac{1}{4}xy^{-1}z, \quad c = H_0 = \tfrac{1}{4}z - \tfrac{1}{4}x^2.$$

The respective separation relation takes the form

$$H_0\lambda^1 + H_1 = 4\lambda^1\mu_1^2 - \frac{1}{64}(\lambda^1)^2.$$

As another example let us come back to the Euler top from Examples 5.4 and 5.8.

Example 5.19 Both Poisson structures of Euler top from Example 5.4 are generated by the formula $\Pi_l^{ij} = \mu_l \epsilon^{ijk}\partial_k H_l$, $l = 0, 1$, where $\mu_0 = -1$, $H_0 = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_2^2$ and $\mu_1 = 1$, $H_1 = \frac{1}{2}a_1\omega_1^2 + \frac{1}{2}a_2\omega_1^2 + \frac{1}{2}a_3\omega_1^2$. The bi-presymplectic representation of Euler top can be found by an analogical procedure as in the case of the Lorentz system. Thus we have a pair of presymplectic forms given by

$$\nu_0 = 1, \quad Y_0 = \begin{pmatrix} \frac{\omega_1}{\omega_1^2+\omega_2^2} \\ \frac{\omega_2}{\omega_1^2+\omega_2^2} \\ 0 \end{pmatrix}$$

and

$$\nu_1 = -1, \quad Y_1 = \begin{pmatrix} (a_3 - a_2)\omega_2\omega_3 + \frac{(a_1\omega_1^2+a_2\omega_2^2)\omega_1}{(\omega_1^2+\omega_2^2)^2} \\ (a_1 - a_3)\omega_1\omega_3 + \frac{(a_1\omega_1^2+a_2\omega_2^2)\omega_2}{(\omega_1^2+\omega_2^2)^2} \\ (a_2 - a_1)\omega_1\omega_2 \end{pmatrix}.$$

The transversal distribution Z, which appeared ad hoc in Example 5.8, is just equal to Y_0 from the above construction.

5.5 Direct Transformation from Separation to Flat Coordinates

In previous sections we constructed in a systematic way a transformation from original coordinates (x, p) (flat in particular) to separation coordinates (λ, μ), provided a bi-Hamiltonian or a bi-inverse-Hamiltonian representation of the considered Liouville integrable systems is given. The presented construction of separation coordinates is general and covers all Stäckel systems. Nevertheless, there exists an alternative approach to the problem. Actually, we can start from the system written in separation coordinates, constructed from a given separation relations, and then find the transformation to some distinguished coordinates in which all commuting Hamiltonians take a particularly simple form. For the class of systems considered in previous chapters, flat coordinates of particular Stäckel matrices are such distinguished coordinates.

The search for flat coordinates for systems that we a priori know are flat is not an easy task. In this section we construct separable flat systems of Stäckel type directly from scratch i.e. from an appropriate separation curve or an appropriate set of separation relations and then find the transformation to flat coordinates [44, 192]. We also establish the signature of metric tensors of these systems as in the majority of cases we deal with pseudo-Euclidian spaces. Moreover, we present in these new coordinates the explicit form of many important geometric objects connected to these flat Stäckel systems like metric tensors, Killing tensors or separable potentials. Notice, that once we construct the set of Stäckel systems in some flat coordinates, we can construct all other Stäckel systems in the same coordinates, using the Stäckel transform described in the previous chapter.

Our construction contains in particular two known cases of pure Euclidean metrics, when separation coordinates are either Jacobi elliptic coordinates [157] or Jacobi parabolic coordinates. All other cases are related with pseudo-Euclidean metrics.

In the following section we consider separation curves (4.3.14) in the form

$$\sum_{j=1}^{n} H_j \lambda^{n-j} = \tfrac{1}{2} f(\lambda)\mu^2 + \sigma(\lambda) = B_m(\lambda)\left(\tfrac{1}{2}\mu^2 + \lambda^k\right), \quad m \in \mathbf{N}, k \in \mathbf{Z} \tag{5.5.1}$$

where

$$B_m(\lambda) = \sum_{j=0}^{m} \lambda^{m-j} \rho_j^{(m)}(\beta) \equiv \prod_{j=1}^{m} (\lambda - \beta_j) \tag{5.5.2}$$

is a real polynomial of order m in λ with possibly complex roots β_j that are all assumed to be different. A particular case of degeneracy will be considered in a separate subsection. The real coefficients $\rho_j^{(m)}(\beta)$ are thus Viète polynomials of the possibly complex constants $\beta_1, \ldots, \beta_m$

$$\rho_j^{(m)}(\beta) = (-1)^j \sum_{1 \le s_1 < s_2 < \ldots < s_j \le m} \beta_{s_1} \ldots \beta_{s_j}, \quad j = 1, \ldots, m. \tag{5.5.3}$$

The Hamiltonians H_i generated by the separation curve (5.5.1) were considered in Sect. 4.3.2. Let us remind that Hamiltonians H_i have the form

$$H_i = \tfrac{1}{2}\mu^T K_i G \mu + U_i(\lambda) \;\; i = 1, \ldots, n, \tag{5.5.4}$$

with the metric tensor G and the Killing tensors K_i given explicitly through

$$G = \operatorname{diag}\left(\frac{f(\lambda^1)}{\Delta_1}, \ldots, \frac{f(\lambda^n)}{\Delta_n}\right) = \operatorname{diag}\left(\frac{B_m(\lambda^1)}{\Delta_1}, \ldots, \frac{B_m(\lambda^n)}{\Delta_n}\right), \quad \Delta_i = \prod_{j \neq i} (\lambda^i - \lambda^j)$$
$$K_i = -\operatorname{diag}\left(\frac{\partial q^i}{\partial \lambda^1}, \cdots, \frac{\partial q^i}{\partial \lambda^n}\right) \quad i = 1, \ldots, n \tag{5.5.5}$$

Here and below $q^i = q^i(\lambda)$ are Viète polynomials (4.3.18) in the variables $\lambda^1, \ldots, \lambda^n$:

$$q^i(\lambda) = (-1)^i \sum_{1 \leq s_1 < s_2 < \ldots < s_i \leq n} \lambda^{s_1} \ldots \lambda^{s_i}, \quad i = 1, \ldots, n \tag{5.5.6}$$

(cf. (5.5.3)) that can also be considered as new coordinates on the Riemannian manifold Q and in the previous chapter we referred to them as Viète coordinates.

As was mentioned in Sect. 4.3.2, the metric (5.5.5) is flat only for $m \leq n$ and is of constant curvature for $m = n + 1$. For higher m it has a non-constant curvature. Thus, it is meaningful to seek for flat coordinates for Benenti systems only in case when $m = 0, \ldots, n$.

Let us now turn our attention to the separable potentials $U_i(\lambda)$ in (5.5.4) in the Benenti case. The potentials $U_i(\lambda)$ depend on the constants m and k (as well as on the dimension n) so will be denoted by $U_i^{(m,k)}$, while the column vector with components $U_i^{(m,k)}$ will be denoted by $U^{(m,k)}$

$$U^{(m,k)} = \left(U_1^{(m,k)}, \ldots, U_n^{(m,k)}\right)^T.$$

By solving (5.5.1) with respect to H_i one obtains that

$$U^{(m,k)} = \sum_{j=0}^{m} \rho_j^{(m)}(\beta) V^{(m-j+k)} \tag{5.5.7}$$

where the column vector $V^{(k)}$ represents the so called basic separable potentials related to $\sigma(\lambda) = \lambda^k$ which was constructed recursively (4.3.22) by

$$V^{(k)} = F^k V^{(0)} \tag{5.5.8}$$

with the recursion matrix F (4.3.20) of the form

$$F = \begin{pmatrix} -q^1 & 1 & & \\ -q^2 & & \ddots & \\ \vdots & & & 1 \\ -q^n & 0 & \cdots & 0 \end{pmatrix} \tag{5.5.9}$$

and with $V^{(0)} = (0, 0, \ldots, 0, 1)^T$. Note that the formulas (5.5.7)–(5.5.9) are non tensorial so are the same in an arbitrary coordinate system. Note also that for $m = 0$ we have $U_r^{(0,k)} = V_r^{(k)}$ so that for $m = 0$ both families of potentials coincide. The potentials U are naturally linear combinations of the basic separable potentials V determined by our specific choice of the function $\sigma(\lambda)$ in (5.5.1). This choice is motivated by the fact that the potentials U in flat coordinates generalize the well known potentials as it will be demonstrated below.

5.5.1 *Flat Coordinates for Real Non Degenerate Case*

In this subsection we construct flat coordinates of G in the case where all the roots $\beta_1, \ldots, \beta_m$ of $B_m(\lambda)$ are real and distinct.

Consider thus the following generating function [192]

$$\boxed{\sum_{j=0}^{n-m} z^{n-m-j} a^j - \frac{1}{4}\varepsilon \sum_{j=1}^{m} \frac{(x^j)^2}{z-\beta_j} \equiv \frac{\prod_{j=1}^{n}(z-\lambda^j)}{\prod_{j=1}^{m}(z-\beta_j)}} \tag{5.5.10}$$

where $\varepsilon = +1$ or $\varepsilon = -1$ and where the identity is taken with respect to the variable z. As will be shown below, this function defines locally an invertible map between variables $(\lambda^1, \ldots, \lambda^n)$ and new variables $(x^1, \ldots, x^m, a^1, \ldots, a^{n-m})$ on the flat Riemannian manifold Q whereas the choice of the sign of ε is governed by the actual sign of the variables in a given region of Q. A simple way to see this is to multiply both sides of (5.5.10) by $B_m(z) \equiv \prod_{j=1}^{m}(z-\beta_j)$ and compare the coefficients of polynomials on both sides of the equation. We find that $a_0 = 1$, so in the particular case $m = n$ the generating function (5.5.10) attains the form

$$1 - \frac{1}{4}\varepsilon \sum_{j=1}^{n} \frac{(x^j)^2}{z-\beta_j} \equiv \frac{\prod_{j=1}^{n}(z-\lambda^j)}{\prod_{j=1}^{n}(z-\beta_j)} \tag{5.5.11}$$

which in the regions of the manifold Q when $\varepsilon < 0$ is nothing else than the well known transformation (see [157]) between the coordinates $(x^1, \ldots, x^n)$ and the Jacobi elliptic coordinates $(\lambda^1, \ldots, \lambda^n)$.

In the case $m = n - 1$ the function (5.5.10) becomes

$$z + a^1 - \frac{1}{4}\varepsilon \sum_{j=1}^{n-1} \frac{(x^j)^2}{z - \beta_j} \equiv \frac{\prod_{j=1}^{n}(z - \lambda^j)}{\prod_{j=1}^{m}(z - \beta_j)}$$

which is commonly known as the generating function for transformation between the coordinates $\left(x^1, \ldots, x^{n-1}, a^1\right)$ and the Jacobi parabolic coordinates $\left(\lambda^1, \ldots, \lambda^n\right)$. In the case $m = 0$ we consider instead of (5.5.10) the generating function of the form

$$\sum_{j=0}^{n} z^{n-j} a^j \equiv \prod_{j=1}^{n}(z - \lambda^j) \tag{5.5.12}$$

so that $a^i(\lambda) = q^i(\lambda)$ i.e. the variables $\left(a^1, \ldots, a^n\right)$ coincide then with the Viète coordinates (5.5.6) while the variables x_i are not present at all. One can say that this function is a variant of (5.5.10) with both ε and all β_i not present.

Let us now investigate the map between coordinates $\left(\lambda^1, \ldots, \lambda^n\right)$ and $(x^1, \ldots, x^m, a^1, \ldots, a^{n-m})$ in a general case. Such a map is given by

$$\boxed{(x^j)^2 = -4\varepsilon \frac{\prod_{k=1}^{n}(\beta_j - \lambda^k)}{\prod_{\substack{k=1 \\ k \neq j}}^{m}(\beta_j - \beta_k)}, \quad j = 1, \ldots m} \tag{5.5.13}$$

$$\boxed{\begin{pmatrix} a^1 \\ \vdots \\ a^{n-m} \end{pmatrix} = M \begin{pmatrix} q^1(\lambda) - \rho_1(\beta) \\ \vdots \\ q^{n-m}(\lambda) - \rho_{n-m}(\beta) \end{pmatrix}} \tag{5.5.14}$$

where M is a square matrix with the following entries

$$M_{ij} = \begin{cases} V_1^{(m,m-1+i-j)} & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases} \quad \text{with } i, j = 1, \ldots, n - m$$

where $V_1^{(m,m-1+i-j)}$ are basic separable potentials given by (5.5.8) with the dimension n replaced by m.

To show (5.5.13) let us first multiply both sides of (5.5.10) by $B_m(z) = \prod_{k=1}^{m}(z-\beta_k)$ receiving

$$B_m(z)\sum_{k=0}^{n-m} z^{n-m-k}a^k - \frac{1}{4}\varepsilon B_m(z)\sum_{k=1}^{m}\frac{(x^k)^2}{z-\beta_k} \equiv \prod_{k=1}^{n}(z-\lambda^k). \tag{5.5.15}$$

Then, let us insert $z = \beta_j$ in (5.5.15) and since $B_m(\beta_j) = 0$ we obtain

$$-\frac{1}{4}\varepsilon (x^j)^2 \prod_{\substack{k=1\\k\neq j}}^{m}(\beta_j - \beta_k) = \prod_{k=1}^{n}(\beta_j - \lambda^k)$$

from which (and since $1/\varepsilon = \varepsilon$) we obtain (5.5.13). The formula (5.5.14) can be obtained by comparison of coefficients of polynomials in (5.5.15).

By a direct comparison of the coefficients in (5.5.15) one can also find that

$$\boxed{q^i = \sum_{j=0}^{n-m}\rho_{i-j}a^j + \frac{1}{4}\varepsilon\sum_{j=1}^{m}\frac{\partial\rho_{i-(n-m)}}{\partial\beta_j}(x^j)^2,\ i=1,\ldots,n} \tag{5.5.16}$$

which represents the map from the variables $\left(x^1,\ldots,x^m,a^1,\ldots,a^{n-m}\right)$ to the Viète variables (5.5.6) and where we use the notation $\rho_i = 0$ for $i < 0$ or for $i > m$ and $\rho_0 = 1$.

Let us now move on to the problem of finding flat coordinates for the metric G generated by (5.5.1). In order to do it consider the polynomial map [44]

$$\boxed{a^i = r^i + \tfrac{1}{4}\sum_{j=1}^{i-1} r^j r^{i-j},\quad i=1,\ldots,n-m} \tag{5.5.17}$$

from the variables $\left(r^1,\ldots,r^{n-m}\right)$ to $\left(a^1,\ldots,a^{n-m}\right)$. This map is invertible due to its triangular structure and its inverse is also a polynomial map. Then, combining the maps (5.5.13)–(5.5.14) and (5.5.17) we find the map between variables $(\lambda^1,\ldots,\lambda^n)$ and $(x^1,\ldots,x^m,r^1,\ldots r^{n-m})$.

We are now in position to formulate the following statement. The metric G defined by (5.5.5) in coordinates $\left(x^1,\ldots,x^m,r^1,\ldots r^{n-m}\right)$ attains the form

$$G = \begin{pmatrix} \varepsilon I_{m\times m} & 0_{m\times(n-m)} \\ 0_{(n-m)\times m} & J_{(n-m)\times(n-m)} \end{pmatrix} \tag{5.5.18}$$

where $I_{k\times k}$ denotes the $k\times k$ identity matrix and $J_{k\times k}$ denotes the $k\times k$ matrix given by $(J_{k\times k})_{ij} = \delta_{i,k-j+1}$ i.e. with entries equal to 0 everywhere except on the antidiagonal where all the entries are equal to 1.

The form (5.5.18) can be proved by direct but tedious calculation. Thus, the variables $\left(x^1, \ldots, x^m, r^1, \ldots r^{n-m}\right)$ as a whole are flat but not orthogonal coordinates for the metric G. Actually, they consist of the orthogonal part $\left(x^1, \ldots, x^m\right)$ and the not orthogonal part $\left(r^1, \ldots r^{n-m}\right)$. It is now elementary to find the transformation from coordinates $\left(x^1, \ldots, x^m, r^1, \ldots r^{n-m}\right)$ to the pseudo-Euclidean coordinates for G. However, we skip that transformation as formulas for Killing tensors and potentials become much less transparent in orthogonal flat coordinates.

The signature (n_+, n_-) (2.1.3) (where n_+ and n_- is the number of positive respective negative eigenvalues of G) of the metric G in the considered case is given by

$$(n_+, n_-) = \left(n - \left[\frac{n-m}{2}\right], \left[\frac{n-m}{2}\right]\right) \text{ in the region where } \varepsilon = +1$$

$$(n_+, n_-) = \left(n - m - \left[\frac{n-m}{2}\right], \left[\frac{n-m}{2}\right] + m\right) \text{ in the region where } \varepsilon = -1$$

where $[\alpha]$ denotes the integer part of the number α. This means that the metric G is Euclidean (in the appropriate regions, where $\varepsilon = +1$) only in the elliptic case and in the parabolic case, i.e. for $m = n$ and $m = n - 1$, otherwise it is pseudo-Euclidean. Note also that in the case $m = 0$ both expressions coincide.

We will now investigate the structure of the Killing tensors $A_i = K_i G$ and separable potentials V in flat coordinates $(x^1, \ldots, x^m, r^1, \ldots r^{n-m})$ in the elliptic case $(x^1, \ldots, x^n)$ and in the parabolic case $(x^1, \ldots, x^{n-1}, r)$. In the case of arbitrary m the formulas become very complicated and non-transparent, so we only present some results concerning the simplest separable potentials.

Let us start with the elliptic case $m = n$. The form of the $(0, 2)$-type tensors A_r in flat coordinates can be calculated by the usual transformation rules for tensors and in flat orthogonal coordinates $\left(x^1, \ldots, x^n\right)$ attains the form

$$A_s^{ij} = \frac{1}{4}\frac{\partial^2 \rho_s}{\partial \beta_i \partial \beta_j} x^i x^j, \quad i \neq j$$

$$A_s^{ii} = \varepsilon \frac{\partial \rho_s}{\partial \beta_i} - \frac{1}{4}\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\partial^2 \rho_s}{\partial \beta_i \partial \beta_k}(x^k)^2 \tag{5.5.19}$$

(no summation over repeated indices is performed here) where $\rho_s = \rho_s^{(n)}(\beta_1, \ldots, \beta_n)$ is given by (5.5.3). Notice that matrix elements of tensors A_r are quadratic functions of Cartesian coordinates.

It is not possible to present the general formula for the potentials $U_r^{(n,k)}$ in flat coordinates but we can at least present a few first potentials with low k. Let $x =$

$(x^1, \dots, x^n)^T$ and by $(\cdot, \cdot)$ we denote the usual scalar product in $\mathbb{R}^n$. Further, denote

$$\Gamma_s = -\operatorname{diag}\left(\frac{\partial \rho_s^{(n)}}{\partial \beta_1}, \cdots, \frac{\partial \rho_s^{(n)}}{\partial \beta_n}\right) \quad s = 1, \dots, n$$

$$B = \operatorname{diag}(\beta_1, \dots, \beta_n)$$

$$W = 1 + \tfrac{1}{4}\varepsilon(x, B^{-1}x)$$

where $\rho_s = 0$ for $s < 0$ and $s > m$. In the above notation, where (5.5.16) reduces to

$$q^i(\lambda) = \rho_i(\beta) + \tfrac{1}{4}\varepsilon \sum_{j=1}^{n} \frac{\partial \rho_i}{\partial \beta_j}(x^j)^2, \ i = 1, \dots, n \tag{5.5.20}$$

after some calculations we obtain

$$\begin{aligned}
U_s^{(n,2)} &= \tfrac{1}{4}\varepsilon\left(\Gamma_s x, B^2 x\right) + \tfrac{1}{16}\left(\Gamma_s x, x\right)(x, Bx) + \tfrac{1}{64}\varepsilon(\Gamma_s x, x)(x, x)^2 \\
&\quad + \tfrac{1}{16}(x, x)(\Gamma_s x, Bx), \\
U_s^{(n,1)} &= \tfrac{1}{4}\varepsilon(\Gamma_s x, Bx) + \tfrac{1}{16}\left(\Gamma_s x, x\right)(x, x), \\
U_s^{(n,0)} &= \tfrac{1}{4}\varepsilon(\Gamma_s x, x), \\
U_s^{(n,-1)} &= \tfrac{1}{4}\varepsilon\frac{(\Gamma_s x, B^{-1}x)}{W}, \\
U_s^{(n,-2)} &= \frac{1}{W^2}\left(\tfrac{1}{4}\varepsilon\left(\Gamma_s x, B^{-2}x\right) + \tfrac{1}{16}\left(\Gamma_{s-1}x, B^{-1}x\right)(x, B^{-1}x)\right. \\
&\quad \left. - \tfrac{1}{16}(\Gamma_{s-1}x, x)(x, B^{-2}x)^2\right).
\end{aligned}$$

For a higher positive or negative k these potentials quickly become very complicated functions of their arguments. Since $\Gamma_1 = I$ and $\Gamma_0 = 0$ (due to (5.5.3)) we have in particular

$$\begin{aligned}
U_1^{(n,2)} &= \tfrac{1}{4}\varepsilon\left(x, B^2 x\right) + \tfrac{1}{8}(x, Bx)(x, x) + \tfrac{1}{64}\varepsilon(x, x)^3, \\
U_1^{(n,1)} &= \tfrac{1}{4}\varepsilon(x, Bx) + \tfrac{1}{16}(x, x)^2, \\
U_1^{(n,0)} &= \tfrac{1}{4}\varepsilon(x, x), \\
U_1^{(n,-1)} &= \tfrac{1}{4}\varepsilon\frac{(x, B^{-1}x)}{W}, \\
U_1^{(n,-2)} &= \tfrac{1}{4}\varepsilon\frac{\left(x, B^{-2}x\right)}{W^2}.
\end{aligned}$$

This family of potentials, known as elliptic separable potentials, has been obtained for the first time in [264] (see also [3]). The potential $U_1^{(n,1)}$ is the well known Garnier potential while $U_1^{(n,0)}$ is just a harmonic oscillator. Note that both in the Killing tensors A_s and in the potentials $U_s^{(n,k)}$ the sign ε is present only at terms with odd powers of (x, x) which is clearly due to (5.5.13).

Let us now turn to the parabolic case $m = n - 1$. In this case the structure of the Killing tensors A_s in flat coordinates $(x^1, \dots, x^{n-1}, r)$ is more complicated and attains the form

$$A_s^{ij} = \frac{1}{4}\frac{\partial^2 \rho_{s-1}}{\partial\beta_i \partial\beta_j} x^i x^j, \quad i \neq j, \ i, j = 1, \dots, n-1,$$

$$A_s^{ii} = -\varepsilon \frac{\partial \rho_s}{\partial \beta_i} - \frac{1}{4}\sum_{\substack{k=1 \\ k\neq i}}^{n-1} \frac{\partial^2 \rho_{s-1}}{\partial\beta_i \partial\beta_k}(x^k)^2 - \varepsilon \frac{\partial \rho_{s-1}}{\partial \beta_i} r, \ i = 1, \dots, n-1,$$

$$A_s^{in} = A_s^{ni} = \frac{1}{2}\frac{\partial \rho_{s-1}}{\partial \beta_i} x^i, \ i = 1, \dots, n-1,$$

$$A_s^{nn} = \rho_{s-1}(\beta),$$

(again, with no summation over repeated indices) where $\rho_s = \rho_s^{(n-1)}(\beta_1, \dots, \beta_{n-1})$ is given by (5.5.3). Also in that case matrix elements of tensors A_s are quadratic functions of Cartesian coordinates.

We will now construct the potentials $U_s^{(n-1,k)}$. Let us slightly change the notation:

$$\Gamma_s = -\operatorname{diag}\left(\frac{\partial \rho_s^{(n-1)}}{\partial \beta_1}, \cdots, \frac{\partial \rho_s^{(n-1)}}{\partial \beta_{n-1}}\right) \quad s = 1, \dots, n$$

$$B = \operatorname{diag}(\beta_1, \dots, \beta_{n-1}),$$

$$W = r + \tfrac{1}{4}\varepsilon \left(x, B^{-1}x\right),$$

while $(\cdot, \cdot)$ stands now for the standard scalar product in $\mathbb{R}^{n-1}$. In the above notation, where (5.5.16) reduces to

$$q^i = \rho_{i-1} r + \tfrac{1}{4}\varepsilon \sum_{j=1}^{n-1} \frac{\partial \rho_{i-1}}{\partial \beta_j}(x^j)^2, \ i = 1, \dots, n$$

we receive, after some calculations

$$
\begin{aligned}
U_s^{(n-1,3)} &= -\rho_{s-1} r^3 + \tfrac{1}{4}\varepsilon\,(\Gamma_{s-1}x, x)\, r^2 - \varepsilon \left(\tfrac{1}{2}\,(\Gamma_s x, x) - \tfrac{1}{4} \sum_{j=1}^{n-1} \rho_{s-j-1}(x, B^j x) \right) r \\
&\quad + \tfrac{1}{4}\varepsilon\,(\Gamma_s x, Bx) + \tfrac{1}{16}\,(\Gamma_{s-1}x, x)\,(x, x), \\
U_s^{(n-1,2)} &= \rho_{s-1} r^2 - \tfrac{1}{4}\varepsilon\,(\Gamma_{s-1}x, x)\, r + \tfrac{1}{4}\varepsilon\,(\Gamma_s x, x)\,, \\
U_s^{(n-1,1)} &= -\rho_{s-1} r + \tfrac{1}{4}\varepsilon\,(\Gamma_{s-1}x, x)\,, \\
U_s^{(n-1,0)} &= \rho_{s-1}, \\
U_s^{(n-1,-1)} &= \frac{1}{W}\left(-\rho_{s-1} + \tfrac{1}{4}\varepsilon\left(\Gamma_{s-1}x, B^{-1}x\right)\right), \\
U_s^{(n-1,-2)} &= \frac{1}{W^2}\left(\rho_{s-1} + \tfrac{1}{4}\varepsilon\left(\Gamma_s x, B^{-2}x\right) - \tfrac{1}{2}\varepsilon\left(\Gamma_{s-1}x, B^{-1}x\right) + \tfrac{1}{4}\varepsilon\left(\Gamma_{s-1}x, B^{-2}x\right) r \right. \\
&\quad \left. + \tfrac{1}{16}\left(\Gamma_{s-2}x, B^{-1}x\right)\left(x, B^{-1}x\right) - \tfrac{1}{16}\,(\Gamma_{s-2}x, x)\left(x, B^{-2}x\right)\right),
\end{aligned}
$$

(with $s = 1, \ldots, n$) and again these formulas become quickly very complicated for a higher positive or negative k. In the particular case $s = 1$, since $\rho_0 = 1$, $\Gamma_1 = I$ and $\Gamma_0 = 0$ we get

$$
\begin{aligned}
U_1^{(n-1,4)} &= r^4 + \tfrac{3}{4}\varepsilon(x, x) r^2 - \tfrac{1}{2}\varepsilon(x, Bx) r + \tfrac{1}{4}\varepsilon(x, B^2 x) + \tfrac{1}{16}(x, x)^2, \\
U_1^{(n-1,3)} &= -r^3 + \tfrac{1}{2}\varepsilon\,(x, x)\, r + \frac{1}{4}\varepsilon\,(x, Bx)\,, \\
U_1^{(n-1,2)} &= r^2 + \tfrac{1}{4}\varepsilon\,(x, x)\,, \\
U_1^{(n-1,1)} &= -r, \\
U_1^{(n-1,0)} &= 1, \\
U_1^{(n-1,-1)} &= -\frac{1}{W}, \\
U_1^{(n-1,-2)} &= \frac{1 + \tfrac{1}{4}\varepsilon\left(x, B^{-2}x\right)}{W^2}.
\end{aligned}
$$

Again, in the above formulas the sign ε is present only at terms with odd powers of (x, x). All these potentials are known as parabolic separable potentials.

For arbitrary $0 \le m \le n-2$ the form of the Killing tensors A_s is not so transparent and we will omit it here. Let us, however, present some results on separable potentials $U^{(m,k)}$. In the case $m = 0$ the variables are $(r^1, \ldots, r^n)$ and $U_r^{(0,k)} = V_r^{(k)}$ where $V_r^{(k)}$ are functions of q^i given by (5.5.8)–(5.5.9). Thus,

remember that for $m = 0$ we have $q^i = a^i$ (5.5.12),

$$U^{(0,k)}(r) = V^{(k)}(r) = \begin{pmatrix} -a^1 & 1 & & \\ -a^2 & & \ddots & \\ \vdots & & & 1 \\ -a^n & 0 & \cdots & 0 \end{pmatrix}^k \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ k \in \mathbf{Z}$$

with

$$a_i = r_i + \tfrac{1}{4} \sum_{j=1}^{i-1} r_j r_{i-j}, \quad i = 1, \ldots, n$$

and so the first nontrivial potential is $V^{(0,n)} = (-a_1(r), \ldots, -a_n(r))^T$.

The situation is much more complex for arbitrary m such that $0 < m < n - 1$. Before we present some results in this generic case, let us introduce a notation similar to that from previous cases. We denote

$$\Gamma_s = -\operatorname{diag}\left(\frac{\partial \rho_s^{(m)}}{\partial \beta_1}, \cdots, \frac{\partial \rho_s^{(m)}}{\partial \beta_m} \right) \quad s = 1, \ldots, m$$

$$B = \operatorname{diag}(\beta_1, \ldots, \beta_m)$$

and to shorten the notation we denote $\rho_s^{(m)}(\beta_1, \ldots, \beta_m)$ by ρ_s. The variables are now $(x_1, \ldots, x_m, r_1 \ldots, r_{n-m})$ and $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^{n-m}$. Introduce now the column vector of potentials $V^{(l)} = V^{(l)}(r_1, \ldots . r_{n-m})$ given by

$$V^{(l)}(r) = \begin{pmatrix} -a_1(r) & 1 & & \\ -a_2(r) & & \ddots & \\ \vdots & & & 1 \\ -a_{n-m}(r) & 0 & \cdots & 0 \end{pmatrix}^l \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ l \in \mathbf{Z}$$

with

$$a^i = r^i + \tfrac{1}{4} \sum_{j=1}^{i-1} r^j r^{i-j}, \quad i = 1, \ldots, n - m$$

so that the last trivial potential is $V^{(n-m-1)} = (1, \ldots, 0)^T$. After some calculations we obtain

$$\vdots$$

$$U_k^{(m,n-m+2)} = \sum_{j=1}^{n-m} \rho_{k-j} V_j^{(n-m+2)}(r) + \tfrac{1}{4}\varepsilon \sum_{j=0}^{2} \left(\Gamma_{k-(n-m-j)}x, x\right) V_1^{(n-m+1-j)}(r)$$

$$U_k^{(m,n-m+1)} = \sum_{j=1}^{n-m} \rho_{k-j} V_j^{(n-m+1)}(r) + \tfrac{1}{4}\varepsilon \sum_{j=0}^{1} \left(\Gamma_{k-(n-m-j)}x, x\right) V_1^{(n-m-j)}(r)$$

$$U_k^{(m,n-m)} = \sum_{j=1}^{n-m} \rho_{k-j} V_j^{(n-m)}(r) + \tfrac{1}{4}\varepsilon \left(\Gamma_{k-(n-m)}x, x\right) V_1^{(n-m-1)}(r)$$

$$U_k^{(m,n-m-1)} = \rho_{k-1} V_1^{(n-m-1)}(r) = \text{const.}$$

$$\vdots$$

$$U_k^{(m,-1)} = \frac{-\sum_{j=1}^{n-m} \rho_{k-j} a_{j-1}(r) + \frac{1}{4}\varepsilon(\Gamma_{k-(n-m)}x, B^{-1}x)}{a_{n-m}(r) + \frac{1}{4}\varepsilon(x, B^{-1}x)}$$

$$\vdots$$

Potentials higher than $U_k^{(m,n-m+2)}$ as well as lower than $U_k^{(m,-1)}$ contain terms at least quadratic in (x, x) and are too complicated to present them in the explicit form.

The reader can find the case of complex roots β_j (5.5.2) in [192].

5.5.2 *Flat Coordinates for Particular Degeneration*

Our second case under consideration is represented by the separation curve in the form (5.5.1) with a maximal degeneracy of (5.5.2), given by $\beta_1 = \ldots = \beta_m = 0$, i.e.

$$H_1\lambda^{n-1} + H_2\lambda^{n-2} + \cdots + H_n = \tfrac{1}{2}\lambda^m\mu^2 + \lambda^l, \quad l \in \mathbb{Z}, \quad n \in \mathbb{N}$$

with $B_m(\lambda) = \lambda^m$. Solving these equations for fixed m and l yields n functions denoted here by $H_r^{(m,l)}(\lambda, \mu)$. In this case, the passage to flat coordinates is as follows [44]. The first step of the construction is to perform the canonical transformation from the (λ, μ) to the Viète coordinates (q, p) (4.3.26)

$$q^i = \rho_i(\lambda), \quad p_i = -\sum_{k=1}^{n} (\lambda^k)^{n-i} \mu_k/\Delta_k, \quad i = 1, \ldots, n.$$

In the (q, p) coordinates flat metrics are (4.3.27)

$$(G_m)^{ij} = \begin{cases} q^{i+j+m-n-1}, & i, j = 1, \dots, n-m \\ -q^{i+j+m-n-1}, & i, j = n-m+1, \dots, n \\ 0 \quad \text{otherwise} \end{cases}$$

where we set $q^0 = 1$, $q^k = 0$ for $k < 0$ or $k > 0$ and $m = 0, \dots, n$.

At the second step, we fix the value of m and perform a canonical transformation from the (q, p) to the (r, s) coordinates:

$$\boxed{\begin{aligned} q^i &= r^i + \tfrac{1}{4} \sum_{j=1}^{i-1} r^j r^{i-j}, \quad i = 1, \dots, n-m, \\ q^i &= -\tfrac{1}{4} \sum_{j=i}^{n} r^j r^{n-j+i}, \quad i = n-m+1, \dots, n, \\ s_k &= \sum_{i=1}^{n} \frac{\partial q^i}{\partial r^k} p_i, \quad k = 1, \dots, n. \end{aligned}} \tag{5.5.21}$$

It is straightforward to verify that for any given m, $0 \le m \le n$, the metrics G_m in the coordinates r^i defined by (5.5.21) takes the form

$$\boxed{(G_m)^{kl} = \left(\delta^{k+l}_{n-m+1} + \delta^{k+l}_{2n-m+1} \right).} \tag{5.5.22}$$

Notice that although the canonical coordinates (r, s) are still nonorthogonal, the metric tensor G_m is constant in these coordinates. In order to bring G_m into a canonical form, with $+1$ and -1 at the diagonal and zeros off the diagonal, we must perform one more canonical transformation from flat not orthogonal coordinates (r, s) to pseudo-Euclidian coordinates (x, p_x), defined as follows (here $d \equiv [(n-m)/2]$). For any given m, $0 \le m \le n$, the transformation defined by the formulas

$$p_{x^k} = \frac{\partial r^i}{\partial x^k} s_i, \quad k = 1, \dots, n,$$

$$\begin{aligned} &r^i = (x^i + x^{n-i+1})/\sqrt{2}, \quad i = 1, \dots, d, \\ &r^i = (x^{n-m-i+1} - x^{m+i})/\sqrt{2}, \quad i = n-m-d+1, \dots, n-m, \\ &r^i = (x^{i-n+m+d} + x^{2n-m-d-i+1})/\sqrt{2}, \quad i = n-m+1, \dots, n-m+[m/2], \\ &r^i = (x^{n+1+d-i} - x^{i-d})/\sqrt{2}, \quad i = n+1-[m/2], \dots, n, \\ &\text{if } n-m \text{ is odd and } m \text{ is even then } r^{d+1} = x^{d+[m/2]+1}, \\ &\text{if } n-m \text{ is even and } m \text{ is odd then } r^{n-m+[m/2]+1} = x^{d+[m/2]+1}, \\ &\text{if both } n-m \text{ and } m \text{ are odd then } r^{d+1} = x^{d+[m/2]+1},\ r^{n-m+[m/2]+1} = x^{d+[m/2]+2}, \end{aligned}$$

brings the metrics G_m into the form

$$G_m^{ij} = \begin{cases} +1, \text{if } i = j \text{ and } i = 1, \ldots, n - [(n-m)/2] - [m/2], \\ -1, \text{if } i = j \text{ and } i = n - [(n-m)/2] - [m/2] + 1, \ldots, n, \\ 0 \quad \text{otherwise.} \end{cases}$$

and we have

$$H_1^{(m,l)} = \tfrac{1}{2} \left(\sum_{j=1}^{n-d-[m/2]} p_{x_j}^2 - \sum_{j=n-d-[m/2]+1}^{n} p_{x_j}^2 \right) + V_1^{(l)}, \quad m = 0, \ldots, n.$$

As a final remark notice that the metrics G_m, $m = 0, \ldots, n$, are in general pseudo-Euclidean with the signature

$$(n_+, n_-) = \left(n - \left[\frac{n-m}{2} \right] - \left[\frac{m}{2} \right], \left[\frac{n-m}{2} \right] + \left[\frac{m}{2} \right] \right).$$

Example 5.20 Let us illustrate our results for $n = 4$ [44]. In the (q, p) coordinates we have

$$G_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q^1 \\ 0 & 1 & q^1 & q^2 \\ 1 & q^1 & q^2 & q^3 \end{pmatrix}, \; G_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & q^1 & 0 \\ 1 & q^1 & q^2 & 0 \\ 0 & 0 & 0 & -q^4 \end{pmatrix}, \; G_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & q^1 & 0 & 0 \\ 0 & 0 & -q^3 & -q^4 \\ 0 & 0 & -q^4 & 0 \end{pmatrix},$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -q^2 & -q^3 & -q^4 \\ 0 & -q^3 & -q^4 & 0 \\ 0 & -q^4 & 0 & 0 \end{pmatrix}, \; G_4 = \begin{pmatrix} -q^1 & -q^2 & -q^3 & -q^4 \\ -q^2 & -q^3 & -q^4 & 0 \\ -q^3 & -q^4 & 0 & 0 \\ -q^4 & 0 & 0 & 0 \end{pmatrix},$$

with the simplest nontrivial potentials being

$$V_1^{(-3)} = (q^2 q^4 - (q^3)^2)/(q^4)^3, \quad V_1^{(-2)} = q^3/(q^4)^2, \quad V_1^{(-1)} = 1/q^4, \quad V_1^{(4)} = -q^1,$$

$$V_1^{(5)} = -q^2 + (q^1)^2, \; V_1^{(6)} = -q^3 + 2q^1 q^2 - (q^1)^3, \; V_1^{(7)} = -q^4 + 2q^1 q^3 + (q^2)^2 - 3(q^1)^2 q^2 + (q^1)^4.$$

For $m = 0$ in the (r, y) coordinates we have

$$q^1 = r^1, \quad q^2 = \tfrac{1}{4}(r^1)^2 + r^2, \quad q^3 = \tfrac{1}{2} r^1 r^2 + r^3, \quad q^4 = \tfrac{1}{2} r^1 r^3 + \tfrac{1}{4}(r^2)^2 + r^4,$$

$$G_0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

while in the $(x,\, p_x)$ coordinates

$$q^1 = \tfrac{1}{\sqrt{2}}\left(x^1 + x^4\right), \quad q^2 = \tfrac{1}{8}\left(x^1 + x^4\right)^2 + \tfrac{1}{\sqrt{2}}\left(x^2 + x^3\right),$$
$$q^3 = \tfrac{1}{4}\left(x^1x^2 + x^1x^3 + x^4x^2 + x^4x^3\right) + \tfrac{1}{\sqrt{2}}\left(x^2 - x^3\right),$$
$$q^4 = \tfrac{1}{4}\left(x^1x^2 - x^1x^3 + x^4x^2 - x^4x^3\right) + \tfrac{1}{8}\left(x^2 + x^3\right)^2 + \tfrac{1}{\sqrt{2}}\left(x^1 - x^4\right),$$

$$G_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and, for instance, the Hamiltonian $H_1^{(0,6)}$ reads

$$H_1^{(0,6)} = \tfrac{1}{2}\left(p_{x^1}^2 + p_{x^2}^2 - p_{x^3}^2 - p_{x^4}^2\right) + \tfrac{3}{4}\left(x^1x^2 + x^1x^3 + x^4x^2 + x^4x^3\right)$$
$$+ \tfrac{1}{\sqrt{2}}\left(x^3 - x^2\right) - \tfrac{1}{4\sqrt{2}}\left(x^1 + x^4\right)^3.$$

For another choice $m = n - 1 = 3$ in the (r, s) coordinates we obtain

$$q^1 = r^1,\ q^2 = -\tfrac{1}{4}(r^3)^2 - \tfrac{1}{2}r^2r^4,\ \ q^3 = -\tfrac{1}{2}r^3r^4,\ \ q^4 = -\tfrac{1}{4}(r^4)^2,$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and in the $(x,\, p_x)$ coordinates we have

$$q^1 = x^2,\ \ q^2 = -\tfrac{1}{4}\left((x^1)^2 + (z)^2 - (x^4)^2\right),\ q^3 = -\tfrac{1}{2\sqrt{2}}\,x^3\left(x^1 - x^4\right),$$

$$q^4 = -\tfrac{1}{8}\left(x^1 - x^4\right)^2,$$

$$G_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and, for example, the Hamiltonian $H_1^{(3,6)}$ takes the form

$$H_1^{(3,6)} = \tfrac{1}{2}\left(p_{x^1}^2 + p_{x^2}^2 + p_{x^3}^2 - p_{x^4}^2\right) - \tfrac{1}{2}\left((x^1)^2 + 2(x^2)^2 + (x^3)^2 - (x^4)^2\right)x^2 + \tfrac{1}{2\sqrt{2}}\left(x^1 - x^4\right)x^3.$$

5.5.3 Stäckel Transform in Flat Coordinates

In a few previous subsections the explicit transformation to flat coordinates was presented for separable systems from Benenti class (5.5.1) with flat metrics. In fact, using Stäckel transform (4.4.17), we can write down any Stäckel system (4.3.2) in these coordinates. We will illustrate the whole procedure for a few particular cases.

First, let us analyze the construction of systems from Benenti class when the metric $G \equiv A_1$ is non-flat. In fact we have to construct the geodesic parts as potentials do not depend on the metric tensor and are constructed according to (5.5.8), (5.5.9).

Example 5.21 As the first example let us consider a pseudo-Euclidean space $E^{2,1}$ with signature $(++-)$ and flat non-orthogonal coordinates (r^1, r^2, r^3) such that

$$\bar{g} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Then, consider the following Stäckel geodesic system on $T^*E^{2,1}$

$$\begin{aligned}
\bar{T}_1 &= \bar{G}^{ij}s_i s_j = s_1 s_3 + \tfrac{1}{2}s_2^2,\\
\bar{T}_2 &= (\bar{K}_2\bar{G})^{ij}s_i s_j = \tfrac{1}{8}(r^1)^2 s_1^2 - \tfrac{1}{4}r^1 r^3 s_2^2 + \tfrac{1}{8}(r^3)^2 s_3^2 + \left(\tfrac{1}{4}r^1 r^2 + 1\right)s_1 s_2\\
&\quad - \tfrac{1}{4}\left(r^1 r^3 + (r^2)^2\right)s_1 s_3 - \tfrac{1}{4}r^2 r^3 s_2 s_3,\\
\bar{T}_3 &= (\bar{K}_3\bar{G})^{ij}s_i s_j = \left(\tfrac{1}{4}r^1 r^2 + \tfrac{1}{2}\right)s_1^2 - \tfrac{1}{4}r^1 r^3 s_1 s_2 - \tfrac{1}{4}r^2 r^3 s_1 s_3 + \tfrac{1}{4}(r^3)^2 s_2 s_3.
\end{aligned}$$

One can check that $\{\bar{T}_i, \bar{T}_j\} = 0$. The transformation to separation coordinates (λ, μ) is generated by (5.5.21)

$$\begin{aligned}
\lambda^1 + \lambda^2 + \lambda^3 &= \tfrac{1}{2}r^1 r^3 + \tfrac{1}{4}(r^2)^2,\\
\lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3 &= -\tfrac{1}{2}r^2 r^3,\\
\lambda^1\lambda^2\lambda^3 &= \tfrac{1}{4}(r^3)^2
\end{aligned} \tag{5.5.23}$$

and the related separation curve is

$$\bar{T}_1\lambda^2 + \bar{T}_2\lambda + \bar{T}_3 = \tfrac{1}{2}\lambda^3\mu^2.$$

The recursion matrix F (5.5.9) in r-coordinates is

$$F = \begin{pmatrix} \frac{1}{2}r^1r^3 + \frac{1}{4}(r^2)^2 & 1 & 0 \\ \frac{1}{2}r^2r^3 & 0 & 1 \\ \frac{1}{4}(r^3)^2 & 0 & 0 \end{pmatrix},$$

so separable potentials $\bar{V}_r^{(k)}$ are given by (4.3.22). For example, the first nontrivial potential is

$$\bar{V}^{(3)} = F^3\bar{V}^{(0)} = \begin{pmatrix} \frac{1}{2}r^1r^3 + \frac{1}{4}(r^2)^2 \\ \frac{1}{2}r^2r^3 \\ \frac{1}{4}(r^3)^2 \end{pmatrix}$$

and the separation curve for Hamiltonians $\bar{H}_i = \bar{T}_i + \bar{V}_i^{(k)}$, $i = 1, 2, 3$, takes the form

$$\bar{H}_1\lambda^2 + \bar{H}_2\lambda + \bar{H}_3 = \tfrac{1}{2}\lambda^3\mu^2 + \lambda^k.$$

Now, let us consider the following Stäckel transform

$$\begin{array}{c} \bar{H}_1\lambda^2 + \bar{H}_2\lambda + \bar{H}_3 = \frac{1}{2}\lambda^3\mu^2 + \lambda^{r-s+3} \\ \Big\downarrow \; R(F) = F^{s-3} \\ H_1\lambda^2 + H_2\lambda + H_3 = \frac{1}{2}\lambda^s\mu^2 + \lambda^r \end{array}$$

so, $H = F^{s-3}\bar{H}$ and in particular, for $s = 4$ and $r = 4$, we have for $H_i = T_i + V_i^{(4)}$

$$T_1 = \tfrac{1}{8}(r^1)^2s_1^2 + \tfrac{1}{8}(r^2)^2s_2^2 + \tfrac{1}{8}(r^3)^2s_3^2 + \left(\tfrac{1}{4}r^1r^2 + 1\right)s_1s_2 + \tfrac{1}{4}r^1r^3s_1s_3 + \tfrac{1}{4}r^2r^3s_2s_3,$$

$$T_2 = \left(\tfrac{1}{4}r^1r^2 + \tfrac{1}{2}\right)s_1^2 + \tfrac{1}{4}r^2r^3s_2^2 - \tfrac{1}{4}r^1r^3s_1s_2 + \tfrac{1}{4}r^2r^3s_1s_3 + \tfrac{1}{4}(r^3)^2s_2s_3,$$

$$T_3 = \tfrac{1}{4}(r^3)^2s_1s_3 + \tfrac{1}{8}(r^3)^2s_2^2$$

and

$$V_1^{(4)} = \tfrac{1}{4}(r^1)^2(r^3)^2 + \tfrac{1}{4}r^1(r^2)^2r^3 + \tfrac{1}{16}(r^2)^4 + \tfrac{1}{2}r^2r^3,$$

$$V_2^{(4)} = \tfrac{1}{4}r^1r^2(r^3)^2 + \tfrac{1}{8}(r^2)^3r^3 + \tfrac{1}{4}(r^3)^2,$$

$$V_3^{(4)} = \tfrac{1}{16}(r^3)^2\left(2r^1r^3 + (r^2)^2\right).$$

Of course, again canonical transformation generated by (5.5.23) is a transformation to separation coordinates.

Now we present how to construct in Euclidean coordinates and conjugate momenta systems from the subclass given by the separation curve of the form

$$\sum_{j=1}^{n} \bar{H}_j \lambda^{n-j} = \tfrac{1}{2} f(\lambda)\mu^2 + \sigma(\lambda) = B_{n+1}(\lambda)\left(\tfrac{1}{2}\mu^2 + \lambda^k\right), \quad m \in \mathbf{N}, k \in \mathbf{Z} \tag{5.5.24}$$

where

$$B_{n+1}(\lambda) = \prod_{j=1}^{n+1} (\lambda - \beta_j)$$

is a real polynomial of order $n+1$ in λ with real roots β_j. As was mentioned in the previous subsection, in this case the metric tensor (5.5.5) is of constant curvature while Euclidean coordinates are flat orthogonal coordinates for systems described by a separation curve (5.5.1) with $m = n$. Hence, according to (4.4.19)

$$\bar{H} = R(F)H = (F - \beta_{n+1} I)H$$

and in particular

$$\bar{H}_k = -\rho_k H_1 - \beta_{n+1} H_k + H_{k+1}, \quad k = 1, \ldots, n.$$

Using formulas (5.5.19), (5.5.20) and the following relation between Viète polynomials (5.5.3)

$$-\beta_{n+1}\rho_k^{(n)} + \rho_{k+1}^{(n)} = \rho_{k+1}^{(n+1)}$$

tensors A_k of the geodesic part take in Euclidean coordinates the form

$$A_s^{ij} = \frac{1}{4}\frac{\partial^2 \rho_{s+1}^{(n+1)}}{\partial\beta_i \partial\beta_j} x^i x^j, \quad i \neq j$$

$$A_s^{ii} = -\varepsilon \frac{\partial \rho_{s+1}^{(n+1)}}{\partial \beta_i} - \frac{1}{4}\sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{\partial}{\partial \beta_k}\left(\rho_s^{(n)} + \frac{\partial \rho_{s+1}^{(n+1)}}{\partial \beta_i}\right)(x^k)^2.$$

Example 5.22 Neumann system for $n = 3$. Relations (5.5.24) for $\sigma(\lambda) = c\lambda^n$ are separation relations for the Neumann system [25, 204, 218, 223] well known from the analytical mechanics. Hence, according to our considerations, three commuting Hamiltonians of the three dimensional Neumann system written in Euclidean

coordinates (x, y, z)

$$H_r = \tfrac{1}{2} A_r^{ij} p_i p_j + V_r, \quad r = 1, 2, 3$$

are given by

$$A_1 = \frac{1}{4}\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix} - \varepsilon \begin{pmatrix} \beta_2+\beta_3+\beta_4 & 0 & 0 \\ 0 & \beta_1+\beta_3+\beta_4 & 0 \\ 0 & 0 & \beta_1+\beta_2+\beta_4 \end{pmatrix}$$

$$A_2 = -\frac{1}{4}\begin{pmatrix} (\beta_2+\beta_3)x^2 \\ +(\beta_1-\beta_4)(y^2+z^2) & (\beta_3+\beta_4)xy & (\beta_2+\beta_4)xz \\ (\beta_3+\beta_4)xy & (\beta_1+\beta_3)y^2 \\ & +(\beta_2-\beta_4)(x^2+z^2) & (\beta_1+\beta_4)yz \\ (\beta_2+\beta_4)xz & (\beta_1+\beta_4)yz & (\beta_1+\beta_3)z^2 \\ & & +(\beta_3-\beta_4)(x^2+y^2) \end{pmatrix}$$
$$+ \varepsilon \begin{pmatrix} \beta_2\beta_3+\beta_2\beta_4+\beta_3\beta_4 & 0 & 0 \\ 0 & \beta_1\beta_3+\beta_1\beta_4+\beta_3\beta_4 & 0 \\ 0 & 0 & \beta_1\beta_2+\beta_1\beta_4+\beta_2\beta_4 \end{pmatrix}$$

$$A_3 = \frac{1}{4}\begin{pmatrix} (\beta_2\beta_3)x^2 \\ +(\beta_1\beta_3-\beta_3\beta_4)y^2 & \beta_3\beta_4 xy & \beta_2\beta_4 yz \\ +(\beta_1\beta_2-\beta_2\beta_4)z^2 \\ & (\beta_1\beta_3)y^2 \\ \beta_3\beta_4 xy & +(\beta_2\beta_3-\beta_3\beta_4)x^2 & \beta_1\beta_4 yz \\ & +(\beta_1\beta_2-\beta_1\beta_4)z^2 \\ & & (\beta_1\beta_2)z^2 \\ \beta_2\beta_4 xz & \beta_1\beta_4 yz & +(\beta_2\beta_3-\beta_2\beta_4)x^2 \\ & & +(\beta_1\beta_3-\beta_1\beta_4)y^2 \end{pmatrix}$$
$$- \varepsilon \begin{pmatrix} \beta_2\beta_3\beta_4 & 0 & 0 \\ 0 & \beta_1\beta_3\beta_4 & 0 \\ 0 & 0 & \beta_1\beta_2\beta_4 \end{pmatrix}$$

and

$$V_1 = -\tfrac{1}{4}\varepsilon c\left[x^2+y^2+z^2\right],$$
$$V_2 = \tfrac{1}{4}\varepsilon c\left[(\beta_2+\beta_3)x^2+(\beta_1+\beta_3)y^2+(\beta_1+\beta_2)z^2\right],$$
$$V_3 = -\tfrac{1}{4}\varepsilon c\left[\beta_2\beta_3 x^2+\beta_1\beta_3 y^2+\beta_1\beta_2 z^2\right].$$

Now let us consider the Stäckel transform (4.4.20) in flat coordinates. Again we illustrate the method using the example of a system with three degrees of freedom.

Example 5.23 Consider the Stäckel system for $n = 3$ given by the separation curve of the form

$$H_1\lambda^2 + H_2\lambda + H_3 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4 \tag{5.5.25}$$

and flat coordinates (r^1, r^2, r^3) defined by (cf. 5.5.21)

$$\rho_1 = -\left(\lambda^1 + \lambda^2 + \lambda^3\right) = r^1$$
$$\rho_2 = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3 = r^2 + \tfrac{1}{4}(r^1)^2$$
$$\rho_3 = -\lambda^1\lambda^2\lambda^3 = -\tfrac{1}{4}(r^3)^2.$$

Solving the relations (5.5.25) with respect to the Hamiltonians H_i and passing to the variables r^i we receive $H_r = A_r^{ij} s_i s_j + V_r(r)$, where y_i are momenta conjugate to r^i, where the tensors A_r have the form

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{1}{2}r^1 & 0 \\ \frac{1}{2}r^1 & -r^2 & -\frac{1}{2}r^3 \\ 0 & -\frac{1}{2}r^3 & r^1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & -\frac{1}{2}r^3 \\ 0 & \frac{1}{4}(r^3)^2 & -\frac{1}{4}r^1r^3 \\ -\frac{1}{2}r^3 & -\frac{1}{4}r^1r^3 & \frac{1}{4}(r^1)^2 + r^2 \end{pmatrix}$$

and respective potentials are

$$V_1 = -\tfrac{3}{4}(r^1)^2 + r^2$$
$$V_2 = -\tfrac{1}{4}(r^1)^3 - r^1r^2 - \tfrac{1}{4}(r^3)^2$$
$$V_3 = \tfrac{1}{4}r^1(r^3)^2.$$

According to (4.4.28), the Stäckel transform to Hamiltonians $\bar{H}_i$ from separation curve

$$\bar{H}_1\lambda_i^3 + \bar{H}_2\lambda_i + \bar{H}_3 = \tfrac{1}{2}\lambda_i\mu_i^2 + \lambda_i^4, \quad i = 1, 2, 3$$

is given by

$$\bar{H}_1 = -\frac{1}{\rho_1}H_1, \quad \bar{H}_2 = H_2 - \frac{\rho_2}{\rho_1}H_1, \quad \bar{H}_3 = H_3 - \frac{\rho_3}{\rho_1}H_1.$$

Hence, we obtain

$$A_1 = \begin{pmatrix} 0 & -\frac{1}{r^1} & 0 \\ -\frac{1}{r^1} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{1}{4}r^1 - \frac{r^2}{r^1} & 0 \\ \frac{1}{4}r^1 - \frac{r^2}{r^1} & -r^2 & -\frac{1}{2}r^3 \\ 0 & -\frac{1}{2}r^3 & \frac{3}{4}r^1 - \frac{r^2}{r^1} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \frac{1}{4}\frac{(r^3)^2}{r^1} & -\frac{1}{2}r^3 \\ \frac{1}{4}\frac{(r^3)^2}{r^1} & \frac{1}{4}(r^3)^2 & -\frac{1}{4}r^1r^3 \\ -\frac{1}{2}r^3 & -\frac{1}{4}r^1r^3 & \frac{1}{4}(r^1)^2 + r^2 + \frac{1}{4}\frac{(r^3)^2}{r^1} \end{pmatrix}$$

with the corresponding potentials

$$V_1(x) = -\tfrac{3}{4}r^1 + \frac{r^2}{r^1}$$

$$V_2(x) = \tfrac{1}{16}(r^1)^3 + \tfrac{1}{2}r^1r^2 + \tfrac{1}{4}(r^3)^2 + \frac{(r^2)^2}{r^1}$$

$$V_3(x) = -\tfrac{1}{16}r^1(r^3)^2 - \tfrac{1}{4}\frac{r^2(r^3)^2}{r^1}.$$

In Sect. 8.2 we will demonstrate how to quantize all systems constructed in this section in a way preserving quantum separability.

Chapter 6
Deformation Theory of Classical Poisson Algebras

Classical physics explains matter and energy on a scale familiar to human experience, including the behavior of astronomical bodies. It remains the key to measurement for much of modern science and technology. However, toward the end of the nineteenth century, scientists discovered phenomena in both the large (macro) and the small (micro) worlds that classical physics could not explain. It means for example, that for a proper description of conservative dynamics in micro-scale the classical Hamiltonian mechanics has to be modified (deformed) to a new theory whose predictions are in agrement with experiments in micro-scale of atoms and molecules. Summarizing experimental results from that level we observe that on the microscopic level the classical uncertainty relations (3.3.15) are violated and have to be modified to a new *quantum uncertainty relations*

$$\boxed{\triangle x^i \triangle p_j \geq \tfrac{1}{2}\hbar\delta^i_j, \quad i,j=1,\ldots,n} \tag{6.0.1}$$

where $\hbar$ is a fundamental constant (the Planck constant) determined in a physical experiment. This is a fundamental change compared to classical mechanics, where the only source of different from zero uncertainty is our limited knowledge of the system. In a quantum system, different from zero uncertainty follows directly from the physical nature of the system itself.

From our previous consideration it is clear that the modern classical Hamiltonian mechanics is formulated on the Poisson manifold in a coordinate free way. So, it is natural to formulate its quantum modification on a quantum analog of the Poisson manifold (symplectic manifold in particular) and obviously in a coordinate free way as well. Such a theory cannot be formulated from the very beginning in a Riemannian space (the Euclidean space in particular) as a priori such a structure does not exist in a classical Poisson geometry. On the other hand, as we know from previous chapters, one can adopt various Riemannian geometries to a given Poisson geometry, identifying the phase space with cotangent bundle to a chosen Riemannian basic space (configuration space). In consequence, on the quantum

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level, we also expect to construct respective Riemannian representations of the quantum Hamiltonian mechanics, which will reduce in the case of the Euclidean space and Cartesian coordinates to a standard formulation of quantum mechanics from many textbooks.

6.1 Star-Algebras

In this section we construct in a systematic way star-algebras and related quantum Poisson algebras which are admissible from the point of view of our quantization assumption (6.0.1). On an appropriate symplectic manifolds we derive star-products in vector representation, covariant representation and integral representation, respectively. We present a detailed discussion on equivalence of the constructed star-algebras and the explicit form of the related morphisms.

6.1.1 Preliminaries

One of the admissible realizations of the quantization procedure is deformation quantization developed in [11–13]. Their work was based on earlier works of Weyl [260, 261], Wigner [263], Groenewold [140], Moyal [205] and Berezin [19–21] on the physical side and of Gerstenhaber [125–129] and Gerstenhaber and Schack [130] deformation theory of associatives algebras on the mathematical side (see also later papers of Bordemann et. al [53–57]). Since then many efforts have been made in order to develop the phase space quantum mechanics [54, 59, 74, 75, 80, 81, 83, 90–92, 111, 152, 153, 207, 213, 236, 256, 269, 270], see also [96, 142, 143] for recent reviews. In this approach quantum mechanics is formulated as a deformation of classical mechanics. Such a procedure results in a quantum theory described in a geometric language similar to that of its classical counterpart. This allows the introduction of many concepts from the classical theory like trajectories, observables or states into quantum mechanics. Moreover, the formalism of deformation quantization gives a smooth passage from the classical to quantum theory, which makes it easy to investigate the classical limit of quantum mechanics.

The main element of deformation quantization is a formal deformation of a Poisson algebra $C^\infty(M)$ of smooth complex-valued functions defined on a phase space M (symplectic manifold). The formal deformation procedure is based on the Gerstenhaber's theory of deformation of rings and algebras [126]. Let $\mathbb{C}[\![\nu]\!]$ and $C^\infty(M)[\![\nu]\!]$ denote the ring of formal power series in a parameter ν with coefficients in $\mathbb{C}$ and $C^\infty(M)$, respectively. The space $C^\infty(M)[\![\nu]\!]$ is a $\mathbb{C}[\![\nu]\!]$-module. A *star-product* on a symplectic manifold $(M, \omega = \pi^{-1})$ is defined as a bilinear map

$$C^\infty(M) \times C^\infty(M) \to C^\infty(M)[\![\nu]\!], \quad (f, g) \mapsto f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \tag{6.1.1}$$

which extends $\mathbb{C}[\![\nu]\!]$-linearly to $C^\infty(M)[\![\nu]\!] \times C^\infty(M)[\![\nu]\!]$, with the following properties

1. C_k are bi-differential operators,
2. $(f \star g) \star h = f \star (g \star h)$ (associativity),
3. $C_0(f, g) = fg$, $C_1(f, g) - C_1(g, f) = \{f, g\}_\pi$,
4. $1 \star f = f \star 1 = f$.

Then, a deformed Poisson bracket is defined by the formula

$$\boxed{[\![f, g]\!]_\star := \frac{1}{\nu}[f, g]_\star = \frac{1}{\nu}(f \star g - g \star f),} \tag{6.1.2}$$

and a formal involution as an anti-linear map

$$\boxed{C^\infty(M) \to C^\infty(M)[\![\nu]\!], \quad f \mapsto f^* = \sum_{k=0}^{\infty} \nu^k B_k(f),} \tag{6.1.3}$$

extending $\mathbb{C}[\![\nu]\!]$-anti-linearly to $C^\infty(M)[\![\nu]\!]$, where

1. B_k are anti-linear operators,
2. $(f \star g)^* = g^* \star f^*$,
3. $(f^*)^* = f$,
4. $B_0(f) = \bar{f}$.

From the above definitions it follows that the $\star$-product, the quantum Poisson bracket $[\![\,\cdot\,,\,\cdot\,]\!]_\star$, and involution $*$ are deformations of the point-wise product of functions $\cdot$, Poisson bracket $\{\,\cdot\,,\,\cdot\,\}_\pi$, and complex-conjugation:

$$\boxed{\begin{aligned} f \star g &= fg + o(\nu), \\ [\![f, g]\!]_\star &= \{f, g\}_\pi + o(\nu), \\ f^* &= \bar{f} + o(\nu). \end{aligned}} \tag{6.1.4}$$

On the other hand, the associativity of the $\star$-product implies the following relations that the bi-differential operators C_k have to satisfy

$$\sum_{n=0}^{k} \big(C_n(C_{k-n}(f, g), h) - C_n(f, C_{k-n}(g, h))\big) = 0, \quad k = 1, 2, \ldots .$$

Moreover, in what follows the deformation parameter ν will be chosen in the form $i\hbar$ in order to make the presented theory compatible with quantum uncertainty relations (6.0.1).

For our further considerations star-products of the following form are particularly interesting

$$\boxed{f \star g = \sum_{k=0}^{\infty} \left(\frac{i\hbar}{2}\right)^k C_k(f,g),} \tag{6.1.5}$$

which beside the conditions (1–4) from definition (6.1.1) fulfill additionally the following ones

1. $\overline{C_k(f,g)} = (-1)^k C_k(g,f)$,
2. $\overline{C_k(f,g)} = C_k(\bar{f},\bar{g})$,
3. $\int_M C_k(f,g)\, d\Omega = 0$ for $f, g \in C_0^\infty(M)$ and $k = 1, 2, \ldots$.

$C_0^\infty(M)$ denotes the space of smooth compactly supported functions on M, and $d\Omega$ is the Liouville measure induced by the Liouville form Ω_ω. Conditions 1 and 2 imply that like in the classical case, the complex-conjugation is a proper involution, and from condition 3 follows that the $\star$-product under the integral sign reduces to the ordinary point-wise product

$$\boxed{\int_M f \star g\, d\Omega = \int_M fg\, d\Omega, \quad f, g \in C_0^\infty(M).} \tag{6.1.6}$$

However, we will not limit ourselves only to star-products of the form (6.1.5) and we will also consider other star-products, in particular, those for which an involution differs from the complex-conjugation.

The symplectic manifold M is completely described by the Poisson algebra $C^\infty(M)$, hence the deformation of the Poisson algebra can be though of as a deformation of a geometrical structure of the symplectic manifold M. Thus by the deformation of $C^\infty(M)$ to some non-commutative algebra we can think of it as describing a non-commutative symplectic manifold.

From a historical point of view, the existence of a star-product on the arbitrary symplectic manifold was first proved in 1983 by De Wilde and Lecomte [84]. Later Fedosov [118] and Omori [213], independently, gave a proof of the existence of a star-product on a symplectic manifold using the framework of Weyl bundles. Finally, in 1997, Kontsevich [172] proved the existence of a star-product on the arbitrary Poisson manifold.

Let $\star$ and $\star'$ be two star-products on a symplectic manifold (M, ω). Both star-products are said to be equivalent if there exists a series

$$S = \sum_{k=0}^{\infty} \nu^k S_k, \quad S_0 = \mathrm{id}, \tag{6.1.7}$$

where S_k are differential operators on $C^\infty(M)$, such that

$$S(f \star g) = Sf \star' Sg. \tag{6.1.8}$$

On the other hand, having a star-product on (M, ω) and a morphism (6.1.7) one can define a new star-product on (M, ω) by the formula (6.1.8). One can check that the new star-product indeed satisfies conditions 1–4 from the definition of a star-product.

From the mathematical point of view the study of equivalences of star-products was described in the language of Hochschild cohomologies [126]. The relation of equivalence of star-products is an equivalence relation, so the set of all star-products on any symplectic manifold is divided into disjoint equivalence classes. Due to results of Nest and Tsygan [210], Bertelson et al. [22] and Deligne [86], the equivalence classes of star-products on a symplectic manifold M are parametrized by formal series of elements from the second de Rham cohomology group of M, i.e. $H^2(M; \mathbb{C})[\![\nu]\!]$. In particular, on a symplectic manifold M for which the second de Rham cohomology group $H^2(M; \mathbb{C})$ vanishes all star-products are equivalent.

If we choose a coordinate system on a domain $\mathcal{O} \subset M$ of the Poisson manifold M, then a given star-product can be written locally in this coordinate system. The simplest case is when a coordinate representation of the star-product is in the form of a Moyal star-product (6.1.33) [205]. To each star-product corresponds a particular class of coordinate systems, namely quantum canonical coordinate systems. Coordinates which are canonical with respect to a one star-product do not have to be canonical with respect to the other star-product. If $\mathcal{O} \subset M$ is a domain of a coordinate system $\varphi\colon \mathcal{O} \to \mathbb{R}^{2N}$ then equivalence classes of star-products written in these coordinates are parametrized by elements of $H^2(\mathcal{O}; \mathbb{C})[\![\hbar]\!]$. The Moyal star-product is in one of these classes. Let us denote this class by $\mathcal{S}(\mathcal{O}, \varphi)$. So every star-product on M whose coordinate representation with respect to the coordinate chart $(\mathcal{O}, \varphi)$ is in the class $\mathcal{S}(\mathcal{O}, \varphi)$ is locally equivalent with the Moyal star-product. For part of these star-products, the coordinates $(\mathcal{O}, \varphi)$ are quantum canonical (see (6.1.15)), like for the Moyal product. We will denote the class of such star-products by $\mathcal{S}_{qc}(\mathcal{O}, \varphi)$.

The star-products in $\mathcal{S}_{qc}(\mathcal{O}, \varphi)$ will be used to perform nonequivalent quantizations of the classical Hamiltonian system. For this reason the knowledge of morphisms relating the star-products in $\mathcal{S}_{qc}(\mathcal{O}, \varphi)$ with the Moyal product can help in establishing the relations between the received nonequivalent quantizations. Moreover, the fact that these star-products are equivalent with the Moyal product is useful when constructing particular realizations of quantizations. For this reason we will need an explicit form of S which will be constructed order by order. Fortunately, for classical Hamiltonian functions polynomial in momenta, the $\hbar$-expansion of S has only the finite number of terms which will give the non-zero contribution when acting on such functions. For instance, we will show that to calculate the action of S on functions at most cubic in momenta we only need S to the second order in $\hbar$.

Let (M, ω, H) be a classical Hamiltonian system. Such a system can be quantized in the framework of deformation quantization. Actually, the classical

Poisson algebra $\mathcal{A}_C = (C^\infty(M), \cdot, \{\cdot, \cdot\}, \bar{})$ is deformed to a *quantum Poisson algebra* $\mathcal{A}_Q = (C^\infty(M)[\![\nu]\!], \star, [\![\cdot, \cdot]\!], *)$, with the deformation parameter $\nu = i\hbar$. In analogy to classical case, elements of $C^\infty(M)[\![\nu]\!]$, self-adjoint with respect to the involution $*$ from $\mathcal{A}_Q$ form the set of admissible *quantum observables*. So, like on the classical level, an observable corresponds to every measurable quantity. The correspondence between measurable quantities and self-adjoint elements of $C^\infty(M)[\![\nu]\!]$ is fixed by the particular choice of quantization, so can vary depending on the chosen quantization. What is important, quantum observables do not have to coincide with functions in the classical case and in general they will be an $\hbar$-deformations of classical observables. They do not even have to be real valued if for the considered quantization the involution from $\mathcal{A}_Q$ is not the complex-conjugation. Note that to each classical observable corresponds the whole family of quantum observables, related to various admissible quantizations, which will reduce to the same classical observable in the classical limit. It means that if A_C is a classical observable then the related quantum observables corresponding to it are of the form

$$A_Q = A_C + \sum_{k=1}^{\infty} \hbar^k A_k$$

for some functions $A_k \in C^\infty(M)$.

On a mathematical level there is no way of telling which assignment of measurable quantities to elements of $C^\infty(M)[\![\hbar]\!]$ is appropriate for a given star-product. This can be only verified through experiment, i.e. on a physical level. On the other hand, there is a very restrictive number of known physical quantum systems, being counterparts of some classical systems. They are mainly described by the so called natural Hamiltonians

$$H(x, p) = \tfrac{1}{2} G^{ij}(x) p_i p_j + V(x),$$

with flat metrics G on a configuration space. The knowledge of quantization of such systems is not enough to fix uniquely the quantization in arbitrary Riemann space and is the source of ambiguities. In consequence, in the literature one meets various versions of quantizations which coincide for the class of natural flat Hamiltonians.

Observation 11 *A choice of quantization of the classical Hamiltonian system is fixed by a choice of both, the $\star$-product and the particular assignment $A \to A(\hbar)$ to classical observables A their quantum counterparts $A(\hbar)$. Moreover, two quantizations $(\star, A(\hbar))$ and $(\star', A'(\hbar))$ are equivalent if there exists an isomorphism S of their quantum Poisson algebras, i.e. when both star-products are related by (6.1.7), (6.1.8), and respective quantum observables are related by $A'(\hbar) = S(A(\hbar))$.*

In what follows let $d\Omega_\hbar(\xi) = \frac{d\Omega(\xi)}{(2\pi\hbar)^n}$ be the normalized Liouville measure and $L^2(M, d\Omega_\hbar)$ a Hilbert space of functions defined on the phase space M and square

integrable with respect to the measure $d\Omega_\hbar$, where the scalar product is given by

$$\boxed{\langle f, g\rangle = \int_M \overline{f(\xi)} g(\xi)\, d\Omega_\hbar.} \tag{6.1.9}$$

So far we considered a quantum Poisson algebra as a formal algebra and so we did not worry about the convergence of formal series appearing during the process of formal quantization. However, such an approach is not entirely physical as observables should be functions on a phase space not formal power series and hence, complete quantum theory requires to investigate the problem of convergence. This is an important mathematical problem which is beyond the scope of the book, nevertheless let us give some remarks about the convergence of formal power series appearing in the definition of star-products.

Let $\star$ be a star-product on (M, ω). In general it is not possible to find a topology on $C^\infty(M)$ such that the $\star$-product will be convergent for every pair of smooth functions. What can be done is to search for a subspace $\mathcal{A} \subset C^\infty(M)$ with appropriately chosen topology such that the $\star$-product will be convergent. In general, functions from $\mathcal{A}$ can depend implicitly on $\hbar$. Additionally, we will require that there exists a subalgebra $\mathcal{G} \subset \mathcal{A}$ such that $\mathcal{G}$ is a dense subset of $L^2(M, d\Omega_\hbar)$, and for $f, g \in \mathcal{G}$ there holds

$$\|f \star g\| \leq \|f\| \|g\|. \tag{6.1.10}$$

From (6.1.10) results that the $\star$-product is continuous on the subspace $\mathcal{G} \times \mathcal{G}$ with respect to the L^2-topology and in consequence can be uniquely extended to the continuous star-product on the whole space $L^2(M, d\Omega_\hbar)$ satisfying (6.1.10) for every $f, g \in L^2(M, d\Omega_\hbar)$, which follows from the fact that $\mathcal{G}$ is dense in $L^2(M, d\Omega_\hbar)$.

The convergence of a $\star$-product in $L^2(M, d\Omega_\hbar)$ is closely related with the existence of integral representation of such a product (see the next subsection). The reader can find more on the convergence of deformation quantization in the literature. For example, in [208, 209, 226] the authors study the convergence in the framework of C^*-algebras (this is usually referred to as strict deformation quantization). A non-formal deformation quantization developed in the framework of Fréchet-Poisson algebras is studied in [214, 215]. Worth noting are also papers [136, 238] where the convergence of a Moyal product on suitable spaces of functions is investigated.

The Hilbert space $L^2(M, d\Omega_\hbar)$ with some convergent $\star$-product (6.1.5) has a structure of an algebra, denoted hereafter by $\mathcal{L}$. From properties (1−3) of (6.1.5) it is clear that for the algebra $\mathcal{L} = (L^2(M, d\Omega_\hbar), \star)$ the complex-conjugation is an involution, that under the integral sign the star-product of two functions from $L^2(M, d\Omega_\hbar)$ reduces to the point-wise product and that the following relation holds

$$\langle g, f \star h\rangle = \langle \bar{f} \star g, h\rangle, \quad f, g, h \in L^2(M, d\Omega_\hbar).$$

For any $f \in \mathcal{A}$ such that $D(f)$ is a subspace of $\mathcal{A}$, dense in $L^2(M, d\Omega_\hbar)$, and such that for every $\rho \in D(f)$, $f \star \rho \in L^2(M, d\Omega_\hbar)$, we can associate a densely defined operator $f\star$ on the Hilbert space $L^2(M, d\Omega_\hbar)$ with the domain equal $D(f)$ and which satisfies

$$(f\star)^\dagger = \bar{f}\star.$$

Defining a trace functional by the formula

$$\boxed{\mathrm{Tr}(f) = \int_M f(\xi)\, d\Omega_\hbar(\xi), \qquad f \in L^1(M, d\Omega_\hbar),} \tag{6.1.11}$$

the $\star$-product in the algebra $\mathcal{L}$ obey the following property: the ideal $\mathcal{L}^1 = \mathcal{L} \star \mathcal{L}$ is a subset of $L^1(M, d\Omega_\hbar)$ and

$$\boxed{\mathrm{Tr}(\bar{f} \star g) = \langle f, g\rangle} \tag{6.1.12}$$

for any $f, g \in L^2(M, d\Omega_\hbar)$.

So far the star-products have been represented by formal series of bi-differential operators. Then, using a respective topology on an appropriate space of smooth functions, these series could be made convergent. In such a way we can introduce a star-product on a subspace of $C^\infty(M)$ and then transfer it to the Hilbert space $L^2(M, d\Omega_\hbar)$. There is however alternative way of introducing star-products, presented in [170]. One can first define a star-product on some subspace $\mathcal{G} \subset C^\infty(M)$ of smooth functions, which is at the same time required to be a dense subspace in $L^2(M, d\Omega_\hbar)$. Then $\mathcal{G}$ should be endowed with a topology. Moreover, the star-product should be continuous in $\mathcal{G}$ and it is usually defined by an integral formula. From there it can be extended to a continuous star-product on the whole space $L^2(M, d\Omega_\hbar)$.

The geometrical language which was used to deform classical Poisson algebra allowed for the deformation presented in a coordinate free way. However, in a full analogy with the classical case, it is possible to develop the deformed theory in any particular coordinate system. Let $M \supset U \to V \subset \mathbb{R}^{2n}$, $\xi \mapsto (\xi^1(\xi), \ldots, \xi^{2n}(\xi))$ be a local coordinate system on the symplectic manifold M. In analogy with the classical case this coordinate system is called quantum canonical if there holds

$$[\![\xi^\alpha, \xi^\beta]\!] = \pi^{\alpha\beta}, \tag{6.1.13}$$

where

$$(\pi^{\alpha\beta}) = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}. \tag{6.1.14}$$

In the case when $M = T^*Q$ we will denote a *quantum canonical coordinate system*, like in the classical case, by

$$(x^1, \ldots, x^n, p_1, \ldots, p_n) \equiv (x^i, p_j).$$

Then the *quantum canonicity condition* (6.1.13) takes the form

$$\boxed{[\![x^i, x^j]\!] = [\![p_i, p_j]\!] = 0, \quad [\![x^i, p_j]\!] = \delta^i_j.} \tag{6.1.15}$$

Note that in the limit $\hbar \to 0$ a quantum canonical coordinate system reduces to a classical canonical one. If (x^i, p_j) and (x'^i, p'_j) are two quantum canonical coordinate systems then the transformation $(x^i, p_j) \mapsto (x'^i, p'_j)$ is called a *quantum canonical transformation* and will be considered with a special care in our further considerations.

Let us derive the condition on a coordinate system $(\xi^1, \ldots, \xi^{2n})$ which has to be satisfied to make it simultaneously classical and quantum canonical coordinate system. A coordinate system $(\xi^1, \ldots, \xi^{2n})$ is classical and quantum canonical for star-algebra (6.1.1), (6.1.2) if and only if

$$\{\xi^\alpha, \xi^\beta\} = C_1(\xi^\alpha, \xi^\beta) - C_1(\xi^\beta, \xi^\alpha) = \pi^{\alpha\beta}, \tag{6.1.16a}$$

$$C_k(\xi^\alpha, \xi^\beta) = C_k(\xi^\beta, \xi^\alpha), \quad k = 2, 3, \ldots, \tag{6.1.16b}$$

for every $\alpha, \beta = 1, \ldots, 2n$, where C_k are bi-differential operators from the expansion (6.1.1) of the $\star$-product. Indeed, from (3.2.12) and condition 3 from the definition (6.1.1) of a $\star$-product we get (6.1.16a). In accordance with (6.1.13) a coordinate system $(\xi^1, \ldots, \xi^{2n})$ is a quantum canonical coordinate system if and only if

$$[\xi^\alpha, \xi^\beta] = \xi^\alpha \star \xi^\beta - \xi^\beta \star \xi^\alpha = i\hbar\pi^{\alpha\beta}.$$

The above condition can be written in the form

$$\sum_{k=0}^{\infty} \nu^k \left(C_k(\xi^\alpha, \xi^\beta) - C_k(\xi^\beta, \xi^\alpha)\right) = i\hbar\pi^{\alpha\beta}$$

and is equivalent with the system (6.1.16a), (6.1.16b). In particular

$$C_k(\xi^\alpha, \xi^\beta) = C_k(\xi^\beta, \xi^\alpha) = 0, \quad k = 2, 3, \ldots .$$

For the distinguished class of $\star$-products in the form (6.1.5) the symmetry

$$C_k(f, g) = (-1)^k C_k(g, f)$$

reduces conditions (6.1.16a), (6.1.16b) to the simpler form

$$C_1(\xi^\alpha, \xi^\beta) = \pi^{\alpha\beta},$$
$$C_k(\xi^\alpha, \xi^\beta) = 0, \quad k = 3, 5, \ldots .$$

If $(\xi^1, \ldots, \xi^{2n})$ is a coordinate system on some domain $V \subset M$ then we can write elements of $C^\infty(M)[[\hbar]]$ in these coordinates and get a formal power series in $C^\infty(V)[[\hbar]]$. In particular, if $f = \sum_{k=0}^{\infty} \hbar^k f_k \in C^\infty(M)[[\hbar]]$, then by writing each $f_k \in C^\infty(M)$ in the coordinates $(\xi^1, \ldots, \xi^{2n})$ we get a formal power series in $C^\infty(V)[[\hbar]]$. Analogically, we can write a $\star$-product on M in the coordinates $(\xi^1, \ldots, \xi^{2n})$ receiving a star-product on a subset V. Further on we will denote such star-product by $\star^{(\xi)}$.

Let us make some remarks about admissible domains of coordinate systems. If one is interested only in the investigation of a geometry of the classical Hamiltonian system (M, ω, H), then one can consider coordinate systems defined on arbitrary open subsets U of a phase space M. However, for quantum systems, even on such a general level of the geometry of the system, this does not remain true since star-products, considered in a non-formal setting by an integral representation are not local. The same situation occurs on both classical and quantum levels, when one investigates integrals over the phase space in order to calculate expectation values of observables. Then it cannot be done in an arbitrary coordinate system. The reason for this is that, in general the values of integrals will change if the integration will be performed over a subset $U \subset M$ where coordinates are defined. That problem was already discussed on a classical level. The only coordinate systems in which it is meaningful to consider integration are almost global coordinate systems (cf. Observation 6). In most general situation, when integration is defined in several charts, a full integration measure on M may exists as well. Nevertheless, that cases are beyond the scope of the book and we confine to manifolds with almost global coordinate system, where both classical Hamiltonian statistical mechanics and quantum Hamiltonian mechanics can be always well formulated.

6.1.2 Fourier Transforms

Fourier transforms play an important role in our further considerations. Thus, it is a proper moment for a choice of a notation and a convention for various Fourier transforms used in the following sections on the quantum level. In what follows we will define the Fourier transform of a function $f \in L^1(\mathbb{R}^{2n})$ by a formula

$$\boxed{(\mathcal{F}f)(\zeta) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} f(\xi) e^{-\frac{i}{\hbar}\zeta_\mu \xi^\mu} d\xi = \int_{\mathbb{R}^{2n}} f(\xi) e^{-\frac{i}{\hbar}\zeta_\mu \xi^\mu} d\Omega_\hbar(\xi)} \tag{6.1.17}$$

and the inverse Fourier transform by

$$(\mathcal{F}^{-1}f)(\xi) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} f(\xi)e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta = \int_{\mathbb{R}^{2n}} f(\zeta)e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}d\Omega_\hbar(\zeta) \tag{6.1.18}$$

The Fourier transform has the following properties

$$\mathcal{F}(\partial_{\xi^\mu}f)(\zeta) = \frac{i}{\hbar}\zeta_\mu\mathcal{F}f(\zeta), \tag{6.1.19}$$

$$\mathcal{F}(f\cdot g) = \mathcal{F}f * \mathcal{F}g, \tag{6.1.20}$$

where $*$ is a convolution of functions defined by

$$\boxed{(f*g)(\xi) = \int_{\mathbb{R}^{2n}} f(\xi')g(\xi-\xi')\,d\Omega_\hbar(\xi') = \int_{\mathbb{R}^{2n}} f(\xi-\xi')g(\xi')\,d\Omega_\hbar(\xi').} \tag{6.1.21}$$

For further considerations we also need a notion of a symplectic Fourier transform. Thus, for a function $f \in L^1(\mathbb{R}^{2n})$ we define a symplectic Fourier transform of f by a formula

$$\boxed{\mathcal{F}_\omega f(\xi') = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} f(\xi)e^{-\frac{i}{\hbar}\omega(\xi',\xi)}\,d\xi = \int_{\mathbb{R}^{2n}} f(\xi)e^{-\frac{i}{\hbar}\omega(\xi',\xi)}d\Omega_\hbar(\xi),} \tag{6.1.22}$$

where ω is a canonical symplectic form on $\mathbb{R}^{2N}$ given by $\omega(\xi',\xi) = \omega_{\alpha\beta}\xi'^\alpha\xi^\beta$ where

$$(\omega_{\alpha\beta}) = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

Note that $\mathcal{F}_\omega f(\xi') = \mathcal{F}f(\omega^T\xi')$.

In $(x,\,p)$ representation we have

$$\mathcal{F}_\omega f(p',x') = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x,\,p)\exp\left[-\frac{i}{\hbar}\left(p'_kx^k - x'^kp_k\right)\right]dxdp,$$

$$\mathcal{F}_\omega^{-1} f(x,\,p) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(p',x')\exp\frac{i}{\hbar}\left(p'_kx^k - x'^kp_k\right)dx'dp'.$$

Besides, in this convention

$$\mathcal{F}_\omega(\partial_x^n \partial_p^m f)(p', x') = \left(\frac{i}{\hbar} p'\right)^n \left(-\frac{i}{\hbar} x'\right)^m \mathcal{F}_\omega f(p', x'),$$
$$\mathcal{F}_\omega(x^n p^m f)(p', x') = \left(i\hbar\partial_{p'}\right)^n (-i\hbar\partial_{x'})^m \mathcal{F}_\omega f(p', x'),$$

$$\mathcal{F}_\omega^{-1}(\partial_{p'}^n \partial_{x'}^m f)(x, p) = \left(-\frac{i}{\hbar} x\right)^n \left(\frac{i}{\hbar} p\right)^m \mathcal{F}_\omega f(x, p),$$
$$\mathcal{F}_\omega^{-1}(p'^n x'^m f)(x, p) = (-i\hbar\partial_x)^n \left(i\hbar\partial_p\right)^m \mathcal{F}_\omega f(x, p),$$

and

$$\delta(x - x_0) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \exp\left(\frac{i}{\hbar}(x^k - x_0^k) p_k\right) dp,$$
$$\delta(p - p_0) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \exp\left(-\frac{i}{\hbar}(p_k - p_{0k}) x^k\right) dx.$$

Finally, let

$$\tilde{f}(x, x') = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} f(x, p) e^{\frac{i}{\hbar} p_k x'^k} dp \tag{6.1.23}$$

and

$$\tilde{f}(p', p) = \int_{\mathbb{R}^n} f(x, p) e^{-\frac{i}{\hbar} p'_k x^k} dx, \tag{6.1.24}$$

where $\tilde{f}$ denotes the Fourier transform of a function $f \in L^1(\mathbb{R}^{2n})$ with respect to n momentum coordinates p or n position coordinates x, respectively. The inverse transforms are

$$f(x, p) = \int_{\mathbb{R}^n} \tilde{f}(x, x') e^{-\frac{i}{\hbar} p_k x'^k} dx' = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n} \tilde{f}(p', p) e^{\frac{i}{\hbar} p'_k x^k} dp'. \tag{6.1.25}$$

6.1.3 Star Products on $\mathbb{R}^{2n}$

Let us consider a $2n$-dimensional simplectic manifold $M = \mathbb{R}^{2n}$ with Darboux coordinates $(\xi^1, \ldots, \xi^{2n})$. In particular we can adapt the Euclidean geometry to the construction in the sense that the first n Darboux coordinates ξ^i are a Cartesian coordinate system $(x^1, \ldots, x^n)$ on E^n while the remaining Darboux coordinates are fiber coordinates, i.e. momenta p_i conjugated to x^i. That is $M = T^*E^n$ and

$(\xi^1, \ldots, \xi^{2n}) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$. In these coordinates the symplectic form ω takes the canonical form $dp_i \wedge dx^i$. Also the Poisson tensor (implectic bi-vector) $\pi = \omega^{-1}$ related to the symplectic form ω can be written in the form

$$\pi = \pi^{\mu\nu} \partial_{\xi^\mu} \otimes \partial_{\xi^\nu} = \partial_{x^i} \wedge \partial_{p_i}. \tag{6.1.26}$$

In Sect. 3.2.1 was shown that the Poisson tensor π can be decomposed into a wedge product of pair-wise commuting on M vector fields (3.2.1)

$$\boxed{\pi = \pi^{\mu\nu} D_\mu \otimes D_\nu = \sum_{i=1}^{n} X_i \wedge Y_i,} \tag{6.1.27}$$

where $X_i = D_i$ and $Y_i = D_{n+i}$ for $i = 1, \ldots, n$, as the Poisson property $[\pi, \pi]_S = 0$ follows directly from commutativity of vector fields $D_\mu : [D_\mu, D_\nu] = 0$. The canonical Poisson bi-vector (6.1.26) can be represented by these D_α for which

$$X_i \wedge Y_i = D_i \wedge D_{n+i} = \partial_{x^i} \wedge \partial_{p_i}.$$

The related Poisson bracket $\{f, g\}_\pi$ can be written as

$$\begin{aligned} \{f, g\}_\pi &= \pi(df, dg) = \sum_{i=1}^{n} [X_i(f) Y_i(g) - Y_i(f) X_i(g)] \\ &= f \sum_{i=1}^{n} (\overleftarrow{X}_i \overrightarrow{Y}_i - \overleftarrow{Y}_i \overrightarrow{X}_i) g = f \left(\sum_{i=1}^{n} \overleftarrow{X}_i \wedge \overrightarrow{Y}_i \right) g, \end{aligned}$$

In what follows we will define an important family of star-products on the symplectic manifold T^*E^n. Let (D_μ) be a sequence of pair-wise commuting global vector fields from the decomposition (6.1.27) of the Poisson tensor π. Define a star-product by the formula

$$\boxed{\begin{aligned} f \star g &= f \exp\left(\tfrac{1}{2} i\hbar \pi^{\mu\nu} \overleftarrow{D}_\mu \overrightarrow{D}_\nu \right) g = f \exp\left(\tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{X}_k \wedge \overrightarrow{Y}_k \right) g \\ &= f \exp\left(\tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{X}_k \overrightarrow{Y}_k - \tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{Y}_k \overrightarrow{X}_k \right) g = f \cdot g + O(\hbar), \end{aligned}} \tag{6.1.28}$$

and aquantum Poisson bracket by a respective formula

$$[\![f, g]\!]_\star = \frac{1}{i\hbar} [f, g]_\star = \frac{1}{i\hbar} (f \star g - g \star f) = \{f, g\}_\pi + O(\hbar). \tag{6.1.29}$$

From the commutativity of vector fields X_i, Y_i follows the associativity of the star-product and hence the quantum bracket (6.1.29) is a Lie bracket as the Jacobi identity (2.5.24) is fulfilled by any commutator built up on an associative multiplication. Besides, the quantum involution in the star-algebra (6.1.28) is the complex conjugation as from (6.1.28) we get immediately that

$$\overline{f \star g} = \bar{g} \star \bar{f}. \tag{6.1.30}$$

In the simplest case $n = 1$, for example, the product (6.1.28) can be written in the form

$$\boxed{\begin{aligned} f \star g &= f \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{X}\,\overrightarrow{Y} - \tfrac{1}{2} i\hbar \overleftarrow{Y}\,\overrightarrow{X}\right) g \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{m=0}^{k} \binom{k}{m} (-1)^m (X^{k-m} Y^m f)(X^m Y^{k-m} g). \end{aligned}} \tag{6.1.31}$$

However, the decomposition (6.1.27) of the Poisson tensor π is not unique. There are different sequences of commuting vector fields X_i', Y_i', $i = 1, \ldots, n$ on $M = T^* E^n$ such that

$$\pi = \sum_{i=1}^{n} X_i \wedge Y_i = \sum_{i=1}^{n} X_i' \wedge Y_i'. \tag{6.1.32}$$

So, with the arbitrary canonical Poisson tensor π (6.1.26) one can relate the whole family of star-products (6.1.28), parametrized by appropriate sets of commuting vector fields from the decomposition (6.1.32).

The constructed family of star-products consists of equivalent star-products, which is a direct consequence of the fact that the second de Rham cohomology group $H^2(\mathbb{R}^{2n}; \mathbb{C})$ vanishes. In particular, for any pair of $\star$-algebras generated by respective sets of commuting vector fields (X_i, Y_i) and (X_i', Y_i') there exists an isomorphism S (6.1.7), (6.1.8) of both algebras.

Example 6.1 Let us consider the Poisson manifold $T^*\mathbb{R} \cong \mathbb{R}^2$ with the canonical Poisson tensor π in a Darboux coordinate system (x, p). Consider the following vector fields

$$\begin{aligned} X &= \partial_x, \quad Y = \partial_p, \\ X' &= x^2 \partial_x - 2xp\partial_p, \quad Y' = x^{-2} \partial_p. \end{aligned}$$

It can be checked immediately that $[X, Y] = 0$, $[X', Y'] = 0$ and that

$$\pi = X \wedge Y = X' \wedge Y' = \partial_x \wedge \partial_p.$$

Star-products induced by vector fields X, Y and X', Y' are equivalent and the morphism S up to $\hbar^2$ is given by

$$S = \mathrm{id} + \frac{\hbar^2}{4}\left(2x^{-2}\partial_p^2 + x^{-2}p\partial_p^3 - x^{-1}\partial_x\partial_p^2\right) + O(\hbar^4).$$

Note that vector fields X, Y and X', Y' are related by a canonical transformation $T\colon (x, p) \mapsto T(x, p) = (-x^{-1}, x^2 p)$:

$$(Xf)\circ T = X'(f\circ T), \quad (Yf)\circ T = Y'(f\circ T),$$

for $f \in C^\infty(\mathbb{R}^2)$. We will investigate systematically the construction of star-products via canonical transformations in the next subsection.

For a given sequence of vector fields (D_μ) from the decomposition (6.1.27) of the Poisson tensor π there exists a coordinate system $(\xi^1, \dots, \xi^{2n})$ in which D_μ are coordinate vector fields, i.e. $D_\mu = \partial_{\xi^\mu}$. Such a coordinate system is of course a Darboux coordinate system associated with the Poisson tensor π. In these coordinates the star-product (6.1.28) takes the form

$$\begin{aligned} f \star_M g &= f \exp\left(\frac{i\hbar}{2}\pi^{\mu\nu}\overleftarrow{\partial}_{\xi^\mu}\overrightarrow{\partial}_{\xi^\nu}\right) g \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1}\cdots\pi^{\mu_k\nu_k}(\partial_{\xi^{\mu_1}}\cdots\partial_{\xi^{\mu_k}} f)(\partial_{\xi^{\nu_1}}\cdots\partial_{\xi^{\nu_k}} g), \end{aligned} \tag{6.1.33}$$

where $\pi^{\mu\nu}$ is the Poisson tensor in Darboux coordinates (6.1.14). The star-product (6.1.33) was first considered by [140, 205], and [19] and is usually called a *Moyal product*. The coordinate system $(\xi^1, \dots, \xi^{2n})$ will be called a natural coordinate system of the $\star$-product.

The family of $\star$-products (6.1.28) belongs to a particular class of products defined by (6.1.5). The property 1 follows immediately from the form of (6.1.28) and the expansion of the exponents (see (6.1.31) for example). The property 2 was mentioned in (6.1.30). Finally, we prove the property 3 by writing the product (6.1.28) in its natural coordinate system (6.1.33). Let

$$C_k(f, g) = \frac{1}{k!}\pi^{\mu_1\nu_1}\cdots\pi^{\mu_k\nu_k}(\partial_{\xi^{\mu_1}}\cdots\partial_{\xi^{\mu_k}} f)(\partial_{\xi^{\nu_1}}\cdots\partial_{\xi^{\nu_k}} g).$$

Then, using integration by parts we get for $k = 1, 2, \dots$

$$\begin{aligned} \int_{\mathbb{R}^{2N}} C_k(f, g)\, d\xi = -\int_{\mathbb{R}^{2N}} \frac{1}{k!}\pi^{\nu_1\mu_1}\pi^{\mu_2\nu_2}\cdots\pi^{\mu_k\nu_k}(\partial_{\xi^{\nu_1}}\partial_{\xi^{\mu_2}}\cdots\partial_{\xi^{\mu_k}} f) \\ \times (\partial_{\xi^{\mu_1}}\partial_{\xi^{\nu_2}}\cdots\partial_{\xi^{\nu_k}} g)\, d\xi = -\int_{\mathbb{R}^{2N}} C_k(f, g)\, d\xi \end{aligned}$$

and so

$$\int_{\mathbb{R}^{2N}} C_k(f,g)\,d\xi = 0,$$

which proves 3.

In the simplest case $n = 1$, for example, the Moyal product (6.1.33) in (x, p) representation takes the form

$$\begin{aligned} f \star_M g &= f \exp\left[\tfrac{1}{2} i\hbar \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)\right] g \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{m=0}^{k} \binom{k}{m} (-1)^m (\partial_x^{k-m}\, \partial_p^m f)(\partial_x^m\, \partial_p^{k-m} g) \end{aligned} \tag{6.1.34}$$

and hence

$$\begin{aligned} [\![f,g]\!]_M &= \frac{2}{\hbar}\, f \sin\left[\tfrac{1}{2}\hbar \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)\right] g \\ &= \{f,g\}_\pi + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\hbar}{2}\right)^{2n} f \left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)^{2n+1} g. \end{aligned} \tag{6.1.35}$$

The structure of the symplectic manifold T^*E^n distinguishes one product from the presented family of star-products, namely the one for which the natural coordinate system is the Cartesian coordinate system. Such a star-product is indeed uniquely defined since coordinate vector fields of Cartesian coordinate systems are related to each other by linear symplectic transformations and such transformations do not change the star-product (6.1.33), as will be shown later. This distinguished star-product will be called a *canonical star-product* on T^*E^n. As will be proved in the next chapter, such a particular deformation of the classical Poisson algebra is equivalent with standard Weyl quantization of classical mechanics.

As was mentioned above, in a given classical canonical coordinate system $(\xi^1, \ldots, \xi^{2n})$, one of the products (6.1.28) takes the form of Moyal representation (6.1.33). In this case evidently classical canonical coordinates are simultaneously quantum canonical coordinates. Besides, all other star-products (6.1.28) are generated from the Moyal one (6.1.33) by classical canonical transformations (see Example 6.1). Obviously, not every classical canonical transformation generates a new star-product (6.1.28) for which coordinates $(\xi^1, \ldots, \xi^{2n})$ are quantum canonical. That problem will be considered in details in the next subsection.

According to Observation 11, the choice of quantization consists of the choice of a $\star$-product and simultaneously the choice of a quantum observable $A_Q(\hbar)$. For a particular class of $\star$-products (6.1.28), or more general (6.1.5), when the involution is a complex conjugation, the simplest admissible choice of quantum observables is

given by

$$A_Q(\hbar) = A_C.$$

Now let us present a family of star-products on T^*E^n, which are not in the form (6.1.28) and for which the complex-conjugation is not the involution. Let $X_1, \ldots, X_n, Y_1, \ldots, Y_n$ be a set of pair-wise commuting global vector fields from the decomposition (6.1.27) of the Poisson tensor π. Define a star-product (in fact a three-parameter family of star-products) by the formula

$$\boxed{\begin{aligned} f \star_{\sigma,\alpha,\beta} g = f \exp\Bigg(& i\hbar\left(\tfrac{1}{2} - \sigma\right) \sum_{j=1}^{n} \overleftarrow{X}_j \overrightarrow{Y}_j - i\hbar\left(\tfrac{1}{2} + \sigma\right) \sum_{j=1}^{n} \overleftarrow{Y}_j \overrightarrow{X}_j \\ & + \hbar\alpha \sum_{j=1}^{n} \overleftarrow{X}_j \overrightarrow{X}_j + \hbar\beta \sum_{j=1}^{n} \overleftarrow{Y}_j \overrightarrow{Y}_j \Bigg) g, \end{aligned}} \tag{6.1.36}$$

where $\sigma, \alpha, \beta \in \mathbb{R}$. The star-product (6.1.36) is equivalent with the star-product (6.1.28) corresponding to the same sequence (X_i, Y_j) of vector fields. A morphism S (6.1.7), (6.1.8) giving this equivalence is of the form

$$\boxed{S_{\sigma,\alpha,\beta} = \exp\left(-i\hbar\sigma \sum_{j=1}^{n} X_j Y_j + \tfrac{1}{2}\hbar\alpha \sum_{j=1}^{n} X_j X_j + \tfrac{1}{2}\hbar\beta \sum_{j=1}^{n} Y_j Y_j \right).} \tag{6.1.37}$$

For the simplicity of formulas we will prove it for a particular case of $\alpha = \beta = 0$, when

$$f \star_\sigma g = f \exp\left(i\hbar\left(\tfrac{1}{2} - \sigma\right) \sum_{j=1}^{n} \overleftarrow{X}_j \overrightarrow{Y}_j - i\hbar\left(\tfrac{1}{2} + \sigma\right) \sum_{j=1}^{n} \overleftarrow{Y}_j \overrightarrow{X}_j \right) g$$

and

$$S_\sigma = \exp\left(-i\hbar\sigma \sum_{j=1}^{n} X_j Y_j \right).$$

Using the notation

$$f \exp\left(\overleftarrow{X}\, \overrightarrow{Y} \right) g := \exp\left(X^f Y^g \right)(fg)$$

and observing that for commuting vector fields X, Y

$$\exp(XY)(fg) = \exp\left[\left(X^f + X^g\right)\left(Y^f + Y^g\right)\right](fg)$$

we get

$$\begin{aligned} f \star_\sigma g &= S_\sigma(S_\sigma^{-1} f \star S_\sigma^{-1} g) \\ &= \exp\left[-i\hbar\sigma \sum_{j=1}^{n}\left(X_j^f + X_j^g\right)\left(Y_j^f + Y_j^g\right)\right] \exp\left[i\hbar\sigma \sum_{j=1}^{n} X_j^f Y_j^f\right] \\ &\exp\left[\tfrac{1}{2} i\hbar \sum_{j=1}^{n}\left(X_j^f Y_j^g - X_j^g Y_j^f\right)\right] \exp\left[i\hbar\sigma \sum_{j=1}^{n} X_j^g Y_j^g\right](fg) \\ &= \exp\left[i\hbar\left(\tfrac{1}{2} - \sigma\right)\sum_{j=1}^{n} X_j^f Y_j^g - i\hbar\left(\tfrac{1}{2} + \sigma\right)\sum_{j=1}^{n} X_j^g Y_j^f\right](fg) \\ &= f \exp\left[i\hbar\left(\tfrac{1}{2} - \sigma\right)\sum_{j=1}^{n} \overleftarrow{X}_j \overrightarrow{Y}_j - i\hbar\left(\tfrac{1}{2} + \sigma\right)\sum_{j=1}^{n} \overleftarrow{Y}_j \overrightarrow{X}_j\right] g. \end{aligned}$$

The involution for the $\star_{\sigma,\alpha,\beta}$-product as well as for other star-products of the general form (6.1.1) is constructed from the following considerations. Assume that in a $\star$-algebra the involution is the complex-conjugation. Let a $\star'$-algebra be isomorphic to the previous one by S. Then, the involution in $\star'$-algebra, induced by involution in $\star$-algebra, takes the form

$$\boxed{f^* = S\overline{S^{-1} f}.} \tag{6.1.38}$$

Indeed

$$\begin{aligned} \left(f \star' g\right)^* &= S\overline{S^{-1}\left(f \star' g\right)} = S\overline{(S^{-1} f \star S^{-1} g)} \\ &= S(\overline{S^{-1} g} \star \overline{S^{-1} f}) = S\overline{S^{-1} g} \star' S\overline{S^{-1} f}) \\ &= g^* \star' f^*. \end{aligned}$$

Now, the same question, concerning a choice of a quantum observable according to Observation 11, appears for $\star$-products with involution given by (6.1.38). The simplest admissible deformation of a classical observable A_C, which is self-adjoint with respect to (6.1.38) is given by

$$\boxed{A_Q(\hbar) = W(\hbar) A_C, \quad S\overline{S^{-1}\overline{W}} = W.} \tag{6.1.39}$$

Indeed, for such $W(\hbar)$ we have $A_Q^* = A_Q$. In particular, for

$$\boxed{S\overline{S^{-1}} = \exp iB \Longrightarrow W = \exp \tfrac{1}{2} iB,}$$

where B is a real differential operator.

The involution of algebra (6.1.36), induced by the involution of algebra (6.1.28), according to (6.1.38) takes the form

$$f^* = \exp\left(-2i\hbar\sigma \sum_{j=1}^{n} X_j Y_j\right) \bar{f} \tag{6.1.40}$$

and

$$W(\hbar) = \exp\left(-i\hbar\sigma \sum_{j=1}^{n} X_j Y_j\right). \tag{6.1.41}$$

From (6.1.40) it is evident that for $\sigma \neq 0$ the involution for the $\star_{\sigma,\alpha,\beta}$-product is different from the complex-conjugation and functions self-adjoint with respect to it can be complex in general .

Example 6.2 In particular let us consider a quantization given by the $\star_{\sigma,\alpha,\beta}$-product for a one-dimensional case ($n = 1$) and in a natural coordinate system when $X = \partial_x$ and $Y = \partial_p$. Then

$$f^* = \exp\left(-2i\hbar\sigma\,\partial_x\partial_p\right)\bar{f}, \qquad W(\hbar) = \exp(-i\hbar\sigma\,\partial_x\partial_p)$$

and choosing for instance $A_C = xp^2$ we get

$$A_Q(\hbar) = W(\hbar)A_C = xp^2 - 2i\hbar\sigma\, p,$$

which evidently is self-adjoint: $A_Q^*(\hbar) = A_Q(\hbar)$.

The last question, related to the definition of involution different from the complex conjugation, is about the existence of a canonical choice of the involution. Such a problem is related directly with the existence of a canonical star-product on the considered manifold. From previous considerations we know that a canonical star-product exists on T^*E^n, i.e. the one which in Cartesian coordinates takes the form of the Moyal product. Thus, on T^*E^n, we also have a canonical choice of S in (6.1.39) as this related the considered star-product with the canonical star-product.

Let us consider a larger family of star-products related to the decomposition (6.1.27) of the classical Poisson tensor π by the formula

$$\begin{aligned} f \star g = & f \exp\Bigg(\tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{X}_k \overrightarrow{Y}_k - \tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{Y}_k \overrightarrow{X}_k \\ & + P(\overleftarrow{X}_1 + \overrightarrow{X}_1, \ldots, \overleftarrow{Y}_n + \overrightarrow{Y}_n; \hbar) \\ & - P(\overleftarrow{X}_1, \ldots, \overleftarrow{Y}_n; \hbar) - P(\overrightarrow{X}_1, \ldots, \overrightarrow{Y}_n; \hbar) \Bigg) g, \end{aligned} \tag{6.1.42}$$

where P is a polynomial of $2n$ arguments with coefficients dependent on $\hbar$. An isomorphism S intertwining the $\star$-product (6.1.42) with the $\star$-product (6.1.28) reads

$$S = \exp\left(P(X_1, \ldots, Y_n; \hbar)\right).$$

Obviously the family (6.1.36) is the particular case of (6.1.42) with S given by (6.1.37). If additionally the condition

$$\overline{P(X_1, \ldots, Y_n)} = P(Y_1, \ldots, X_n)$$

is fulfilled, then the complex-conjugation is the involution for this product as well. In particular let us take $P(X_1, \ldots, Y_n; \hbar) = -\frac{1}{8} b\hbar^2 \sum_{k,j=1}^{n} X_k X_j Y_k Y_j$, $b \in \mathbb{R}$. Then the $\star$-product (6.1.42) takes the form

$$\begin{aligned} f \star g = & f \exp\Bigg(\tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{X}_k \overrightarrow{Y}_k - \tfrac{1}{2} i\hbar \sum_{k=1}^{n} \overleftarrow{Y}_k \overrightarrow{X}_k \\ & + \tfrac{1}{8} b\hbar^2 \sum_{k,j=1}^{n} (\overleftarrow{X}_k \overleftarrow{Y}_k \overleftarrow{X}_j \overleftarrow{Y}_j + \overrightarrow{X}_k \overrightarrow{Y}_k \overrightarrow{X}_j \overrightarrow{Y}_j) \\ & - \tfrac{1}{8} b\hbar^2 \sum_{k,j=1}^{n} (\overleftarrow{X}_k + \overrightarrow{X}_k)(\overleftarrow{Y}_k + \overrightarrow{Y}_k)(\overleftarrow{X}_j + \overrightarrow{X}_j)(\overleftarrow{Y}_j + \overrightarrow{Y}_j) \Bigg) g \end{aligned} \tag{6.1.43}$$

and will play an important role in our further considerations.

For a given sequence of vector fields (X_i, Y_i), from the decomposition (6.1.27) of the Poisson tensor π, there exists a coordinate system $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ in which $X_i = \partial_{x^i}$, $Y_i = \partial_{p_i}$ are coordinate vector fields. Such a coordinate system, as was mentioned earlier, is a Darboux coordinate system associated with the Poisson

tensor π. In these coordinates the star-products (6.1.42) take the form

$$f\star g = f\exp\left(\tfrac{1}{2}i\hbar\left(\overleftarrow{\partial}_{x^k}\overrightarrow{\partial}_{p_k}-\overleftarrow{\partial}_{p_k}\overrightarrow{\partial}_{x^k}\right)+P(\overleftarrow{\partial}_{x^1}+\overrightarrow{\partial}_{x^1},\dots,\overleftarrow{\partial}_{p_n}+\overrightarrow{\partial}_{p_n};\hbar)\right.$$
$$\left.-P(\overleftarrow{\partial}_{x^1},\dots,\overleftarrow{\partial}_{p_n};\hbar)-P(\overrightarrow{\partial}_{x^1},\dots,\overrightarrow{\partial}_{p_n};\hbar)\right)g$$

while the star-product (6.1.43) the respective form

$$f\star g = f\exp\left(\tfrac{1}{2}i\hbar\left(\overleftarrow{\partial}_{x^k}\overrightarrow{\partial}_{p_k}-\overleftarrow{\partial}_{p_k}\overrightarrow{\partial}_{x^k}\right)\right.$$
$$+\tfrac{1}{8}b\hbar^2(\overleftarrow{\partial}_{x^k}\overleftarrow{\partial}_{p_k}\overleftarrow{\partial}_{x^j}\overleftarrow{\partial}_{p_j}+\overrightarrow{\partial}_{x^k}\overrightarrow{\partial}_{p_k}\overrightarrow{\partial}_{x^j}\overrightarrow{\partial}_{p_j})$$
$$\left.-\tfrac{1}{8}b\hbar^2(\overleftarrow{\partial}_{x^k}+\overrightarrow{\partial}_{x^k})(\overleftarrow{\partial}_{p_k}+\overrightarrow{\partial}_{p_k})(\overleftarrow{\partial}_{x^j}+\overrightarrow{\partial}_{x^j})(\overleftarrow{\partial}_{p_j}+\overrightarrow{\partial}_{p_j})\right)g.$$

Again, all other star-products (6.1.42) are constructed by classical canonical transformations of coordinates.

An integral representation of a star-product is very important and particularly useful for further considerations. In order to do it let us chose the Fourier transform and the convolution defined by formulas (6.1.17)–(6.1.21). Next we need to specify a space $\mathcal{G}$ where the series (6.1.1) ((6.1.7) in particular) is convergent. Let $\mathcal{G} = \mathcal{F}(C_0^\infty(\mathbb{R}^{2n}))$ be the Fourier image of the space of smooth functions on $\mathbb{R}^{2n}$ with the compact support, where the Moyal product (6.1.33) is convergent [136]. We prove the following integral form of that product

$$\boxed{(f\star_M g)(\xi)=\frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} f(\xi+\xi')g(\xi+\xi'')e^{-\frac{2i}{\hbar}\omega_{\mu\nu}\xi'^{\mu}\xi''^{\nu}}\,d\xi'\,d\xi''.} \tag{6.1.44}$$

Indeed, using the properties (6.1.19) and (6.1.20) of the Fourier transform, the Moyal product (6.1.33) can be written in the following representation

$$(f\star_M g)(\xi)=\mathcal{F}^{-1}\mathcal{F}(f\star_M g)(\xi)=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}}\mathcal{F}(f\star_M g)(\zeta)e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta$$
$$=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k\pi^{\mu_1\nu_1}\dots\pi^{\mu_k\nu_k}\int_{\mathbb{R}^{2N}}\mathcal{F}(\partial_{\xi^{\mu_1}}\cdots\partial_{\xi^{\mu_k}}f)(\zeta')$$
$$\times\,\mathcal{F}(\partial_{\xi^{\nu_1}}\cdots\partial_{\xi^{\nu_k}}g)(\zeta-\zeta')e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta'\,d\zeta$$
$$=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k\pi^{\mu_1\nu_1}\dots\pi^{\mu_k\nu_k}\int_{\mathbb{R}^{2N}}\frac{i}{\hbar}\zeta'_{\mu_1}\cdots\frac{i}{\hbar}\zeta'_{\mu_k}\mathcal{F}f(\zeta')$$

$$
\begin{aligned}
&\times \frac{i}{\hbar}(\zeta_{\nu_1}-\zeta'_{\nu_1})\cdots\frac{i}{\hbar}(\zeta_{\nu_k}-\zeta'_{\nu_k})\mathcal{F}g(\zeta-\zeta')e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta'\,d\zeta\\
&=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{-i}{2\hbar}\right)^k\left(\pi^{\mu\nu}\zeta'_\mu(\zeta_\nu-\zeta'_\nu)\right)^k\\
&\times\mathcal{F}f(\zeta')\mathcal{F}g(\zeta-\zeta')e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta'\,d\zeta\\
&=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\mathcal{F}f(\zeta')\mathcal{F}g(\zeta-\zeta')e^{-\frac{i}{2\hbar}\pi^{\mu\nu}\zeta'_\mu(\zeta_\nu-\zeta'_\nu)}e^{\frac{i}{\hbar}\zeta_\mu\xi^\mu}\,d\zeta'\,d\zeta.
\end{aligned}
$$

After the change of variables

$$
\begin{aligned}
\zeta'_\mu &\to \zeta'_\mu,\\
\zeta_\mu &\to \zeta_\mu+\zeta'_\mu
\end{aligned}
$$

we get

$$
\begin{aligned}
(f\star_M g)(\xi)&=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}\mathcal{F}f(\zeta')\mathcal{F}g(\zeta)e^{\frac{i}{\hbar}(\xi^\mu+\frac{1}{2}\pi^{\mu\nu}\zeta'_\nu)\zeta_\mu}e^{\frac{i}{\hbar}\zeta'_\mu\xi^\mu}\,d\zeta'\,d\zeta\\
&=\frac{1}{(2\pi\hbar)^{n}}\int_{\mathbb{R}^{2n}}\mathcal{F}f(\zeta')g(\xi+\tfrac{1}{2}\pi\zeta')e^{\frac{i}{\hbar}\zeta'_\mu\xi^\mu}\,d\zeta'\\
&=\frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}f(\xi')g(\xi+\tfrac{1}{2}\pi\zeta')e^{\frac{i}{\hbar}\zeta'_\mu\xi^\mu}e^{-\frac{i}{\hbar}\zeta'_\mu\xi'^\mu}\,d\xi'\,d\zeta'.
\end{aligned}
$$

Performing another change of variables

$$
\begin{aligned}
\xi'^\mu &\to \xi^\mu+\xi'^\mu,\\
\zeta'_\mu &\to 2\omega_{\mu\nu}\xi''^\nu
\end{aligned}
$$

and observing that $\pi\omega=I_n$ we receive the result. Notice that in the representation $M=T^*\mathbb{R}^n$ with canonical basis $(x^1,\dots,x^n,p_1,\dots,p_n)$, the star product (6.1.44) takes the form [7, 152, 269]

$$
\begin{aligned}
&(f\star_M g)(x,p) \qquad\qquad (6.1.45)\\
&=\frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}f(x+x',p+p')g(x+x'',p+p'')e^{-\frac{2i}{\hbar}(x''^kp'_k-x'^kp''_k)}\,dx'dx''dp'dp''\\
&=\frac{1}{(\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}}f(x',p')g(x'',p'')e^{-\frac{2i}{\hbar}[(p'_k-p_k)(x''^k-x^k)-(x'^k-x^k)(p''_k-p_k)]}\,dx'dx''dp'dp''.
\end{aligned}
$$

The integral form of the Moyal product is also valid for the space of Schwartz functions $\mathcal{S}$. Moreover, it can be shown that $\star_M$ is continuous on $\mathcal{S}$ and that for

$f, g \in \mathcal{S}$ we have $f \star_M g \in \mathcal{S}$ and

$$\|f \star_M g\|_{L^2} \leq \|f\|_{L^2}\|g\|_{L^2},$$

see e.g. [33, 136]. The extension of the Moyal product from $\mathcal{G}$ to a continuous star-product on $\mathcal{S}$ is unique since $\mathcal{G}$ is dense in $\mathcal{S}$. Hence the Schwartz space $\mathcal{S}$ is an algebra with respect to the Moyal product. From (6.1.3) follows that the Moyal product is continuous with respect to the L^2-topology and thus can be uniquely extended to a continuous star-product on $L^2(\mathbb{R}^{2N})$ making it an algebra $\mathcal{L} = L^2(\mathbb{R}^{2N}, \star_M)$.

Under substitution $x' \to \frac{1}{2}x'$, $x'' \to -\frac{1}{2}x''$, the integral form (6.1.45) of the Moyal product can be written in the following way

$$\boxed{(f \star_M g)(x, p) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{f}(x + \tfrac{1}{2}x', x'')\tilde{g}(x - \tfrac{1}{2}x'', x')e^{-\frac{i}{\hbar}(x'^k + x''^k)p_k}\, dx'\, dx''},$$

(6.1.46)

where $\tilde{f}$ denotes the Fourier transform of f in the momentum variable (6.1.23). Note, that the Moyal product on $\mathcal{G}$ is not local, which can be seen from its integral form (6.1.44). For a fixed $\xi \in \mathbb{R}^{2n}$ the value of the integral in (6.1.44) depends on the values of functions f and g far away from ξ.

The Moyal product (6.1.33) is also a valid star-product on the symplectic manifold $M = T^*U = U \times \mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$. This is a direct consequence of the fact that the Moyal product is a series of bi-differential operators which are local operators. For $f, g \in C_0^\infty(M)$ the integral form (6.1.44) of the product still makes sense, since f and g can be uniquely extended to smooth functions on the whole space $\mathbb{R}^{2n}$ with the same supports as f and g respectively (simply by putting the functions f and g equal 0 outside $U \times \mathbb{R}^n$). In such a case the expression (6.1.44) still can be formally expanded to the series (6.1.33). For $f, g \in \mathcal{G}$ formula (6.1.46) makes sense and defines the Moyal product of functions f and g. Moreover, $f \star_M g$ is smooth and hence $f \star_M g \in \mathcal{G}$, i.e. $\mathcal{G}$ is an algebra with respect to $\star_M$.

Finally, at the end of that subsection, let us prove that any function $f \in \mathcal{A}_Q$ can be expanded into an appropriate $\star$-power series. For simplicity we do it for the case $n = 1$, but the generalization onto arbitrary n is straightforward. We show that for any $f(x, p) \in \mathcal{A}_Q$

$$f = \sum_{n,m=0}^{\infty} a_{nm} \underbrace{x \star \cdots \star x}_{n} \star \underbrace{p \star \cdots \star p}_{m}, \tag{6.1.47}$$

where $a_{nm} \in \mathbb{C}$. First we prove it for the Moyal algebra. From relations

$$x \star_M x = x \cdot x, \quad p \star_M p = p \cdot p, \quad x \star_M p = x \cdot p + \tfrac{1}{2}i\hbar$$

we get the recurrence relations

$$x\cdot(\underbrace{x\star_M\cdots\star_M x}_{n}\star_M\underbrace{p\star_M\cdots\star_M p}_{m})=\underbrace{x\star_M\cdots\star_M x}_{n+1}\star_M\underbrace{p\star_M\cdots\star_M p}_{m}$$
$$-\tfrac{1}{2}i\hbar m\,\underbrace{x\star_M\cdots\star_M x}_{n}\star_M\underbrace{p\star_M\cdots\star_M p}_{m-1},$$
$$p\cdot(\underbrace{x\star_M\cdots\star_M x}_{n}\star_M\underbrace{p\star_M\cdots\star_M p}_{m})=\underbrace{x\star_M\cdots\star_M x}_{n}\star_M\underbrace{p\star_M\cdots\star_M p}_{m+1}$$
$$-\tfrac{1}{2}i\hbar n\,\underbrace{x\star_M\cdots\star_M x}_{n-1}\star_M\underbrace{p\star_M\cdots\star_M p}_{m}$$

and hence, it follows that monomials $x^n p^m$ can be written as $\star_M$-polynomials. Thus after expanding f into the power series it is seen that f can be written in the form (6.1.47). Then, for any other $\star$-product, related to the Moyal one by (6.1.8), we have

$$Sf=\sum_{n,m=0}^{\infty}a_{nm}\,\underbrace{x\star\cdots\star x}_{n}\star\underbrace{p\star\cdots\star p}_{m},$$

and hence

$$f=\sum_{n,m=0}^{\infty}a'_{nm}\,\underbrace{x\star\cdots\star x}_{n}\star\underbrace{p\star\cdots\star p}_{m},$$

where

$$f'=S^{-1}f=\sum_{n,m=0}^{\infty}a'_{nm}\,\underbrace{q\star_M\cdots\star_M q}_{n}\star_M\underbrace{p\star_M\cdots\star_M p}_{m}.$$

Particular examples of morphism S, either explicitly or up to some order of $\hbar$, are constructed in next subsections.

6.1.4 Canonical Transformations of Star-Products

As previously, let us consider a symplectic manifold $M=T^*U=U\times\mathbb{R}^n$, where U is an open subset of $\mathbb{R}^n$. A transformation of phase space coordinates is defined as in classical mechanics, i.e., as a smooth bijective map $T:M\supset U\ni(x,p)\to(x',p')\in W\subset M$. The transformed star-product, denoted hereafter by $\star_T$, should

fulfill the following natural condition

$$\boxed{(f \star g) \circ T = (f \circ T) \star_T (g \circ T), \quad f, g \in \mathcal{A}_Q.}$$

Assume that (x', p') are classical canonical coordinates. When the star-product in (x', p') coordinates is the Moyal product

$$f \star'_M g = f \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{\partial}_{x'^j} \overrightarrow{\partial}_{p'_j} - \frac{1}{2} i\hbar \overleftarrow{\partial}_{p'_j} \overrightarrow{\partial}_{x'^j}\right) g \tag{6.1.48}$$

hence (x', p') are also quantum canonical coordinates. Then, the $\star_T$-product is given by the following formula (6.1.33)

$$\boxed{f \star_T g = f \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{D}_{x^j} \overrightarrow{D}_{p_j} - \tfrac{1}{2} i\hbar \overleftarrow{D}_{p_j} \overrightarrow{D}_{x^j}\right) g,} \tag{6.1.49}$$

where vector fields D_{x^j}, D_{p_j} are derivations $\partial_{x'^j}$, $\partial_{p'_j}$ transformed by the transformation T according to the rule

$$\boxed{\begin{aligned} (\partial_{x'^j} f) \circ T &= D_{x^j}(f \circ T), \quad f \in C^\infty(M), \\ (\partial_{p'_j} f) \circ T &= D_{p_j}(f \circ T), \quad f \in C^\infty(M). \end{aligned}}$$

Notice that the Moyal product (6.1.48) in (x', p') coordinates transforms into a non-Moyal product (6.1.49) in new coordinates (x, p), more complicated in general.

To this moment we considered general transformations of coordinates. In what follows we will focus on an important class of transformations, namely these which are classical and/or quantum canonical transformations. In classical mechanics a canonical transformation is such a transformation T of phase space coordinates which transforms the system from one canonical coordinate system to the other. In other words, T is a canonical transformation if it preserves the form of the Poisson bracket (see Sect. 4.1.4), i.e.,

$$\{x^i, p_j\} = \delta^i_j,$$

where $\{\cdot, \cdot\}$ denotes a Poisson bracket transformed by T to the new coordinate system:

$$\boxed{\{f, g\} = \{f \circ T^{-1}, g \circ T^{-1}\}' \circ T, \quad f, g \in C^\infty(M).}$$

Canonical transformations for the quantum Poisson bracket are defined in a similar manner [35, 75, 87, 91, 147]. Namely, a quantum canonical transformation is such a transformation T of coordinates which preserves the form of the quantum

Poisson bracket, i.e.,

$$[\![x^i, p_j]\!]_\star = \delta^i_j,$$

where $[\![\cdot, \cdot]\!]_\star$ denotes a deformed Poisson bracket transformed by T to the new coordinate system:

$$\boxed{[\![f, g]\!]_\star = [\![f \circ T^{-1}, g \circ T^{-1}]\!]_{\star'} \circ T, \quad f, g \in C^\infty(M).}$$

Assume now that (x, p) are canonical coordinates for a nondegenerate Poisson tensor π and $\star_M$ is a Moyal product in these coordinates. Let us denote

$$T(x, p) = (Q^1(x, p), \ldots, Q^n(x, p), P_1(x, p), \ldots, P_n(x, p)),$$

then T defines a classical canonical transformation if

$$\boxed{\begin{aligned} &\left\{Q^i(x, p), P_j(x, p)\right\}_\pi = \delta^i_j, \\ &\left\{Q^i(x, p), Q^j(x, p)\right\}_\pi = \left\{P_i(x, p), P_j(x, p)\right\}_\pi = 0 \end{aligned}} \tag{6.1.50}$$

and T defines a quantum canonical transformation if

$$\boxed{\begin{aligned} &[\![Q^i(x, p), P_j(x, p)]\!]_{\star_M} = \delta^i_j, \\ &\left[\!\left[Q^i(x, p), Q^j(x, p)\right]\!\right]_{\star_M} = [\![P_i(x, p), P_j(x, p)]\!]_{\star_M} = 0. \end{aligned}} \tag{6.1.51}$$

There are transformations which are either classically canonical or quantum canonical. The following two examples illustrate such cases.

Example 6.3 A transformation which is only classically canonical [93]. Let us define such a transformation by

$$Q(x, p) = x \exp(2\alpha xp), \quad P(x, p) = p \exp(-2\alpha xp),$$

where $\alpha \neq 0$ is a real parameter. It is obviously classically canonical transformation as

$$\{Q(x, p), P(x, p)\} = 1.$$

However, it is not a quantum canonical with respect to the Moyal bracket. To check this let us first compute $Q(x, p) \star_M P(x, p)$ using the kernel representation (6.1.45).

Substituting $t = i\tau$, we get after some integrations and integration by parts

$$Q(x,p) \star_M P(x,p) = \frac{xp + \frac{1}{2}i\hbar}{[1+(\alpha\hbar)^2]^2}, \quad P(x,p) \star_M Q(x,p) = \frac{xp - \frac{1}{2}i\hbar}{[1+(\alpha\hbar)^2]^2}$$

and hence

$$[\![Q(x,p), P(x,p)]\!]_{\star_M} = [1+(\alpha\hbar)^2]^{-2} \neq 1.$$

Example 6.4 A transformation which is only quantum canonical [93]. Let us define such a transformation by

$$Q(x,p) = \beta \exp\left(\frac{x}{\beta}\right), \quad P(x,p) = \exp\left(-\frac{x}{\beta}\right)\left[p + \gamma \sinh\left(\frac{2\beta\pi p}{\hbar}\right)\right],$$

where β and γ are positive real constants. Obviously this is not a classically canonical transformation as

$$\{Q(x,p), P(x,p)\} = 1 + \frac{2\beta\gamma\pi}{\hbar}\cosh\left(\frac{2\beta\pi p}{\hbar}\right) > 1.$$

On the other hand we have

$$Q(x,p)\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)^{2n+1} P(x,p) = \beta\gamma\left(\frac{2\pi}{\hbar}\right)^{2n+1}\cosh\left(\frac{2\beta\pi p}{\hbar}\right).$$

Consequently, from (6.1.35),

$$\begin{aligned}
&[\![Q(x,p), P(x,p)]\!]_{\star_M} \\
&= \{Q(x,p), P(x,p)\} \\
&\quad + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\hbar}{2}\right)^{2n} Q(x,p)\left(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x\right)^{2n+1} P(x,p) \\
&= 1 + \frac{2\beta\gamma}{\hbar}\cosh\left(\frac{2\beta\pi p}{\hbar}\right)\sin\pi = 1,
\end{aligned}$$

and so this is a quantum canonical transformation.

In what follows we will present four important classes of nonlinear canonical transformations (both classical and quantum) [35]. The linear case is considered separately. A sufficient condition for such a class of transformations is a linearity in one set of arguments of an appropriate generating function (see Sect. 4.1.4).

Let us begin with the case of two-dimensional phase space and generating function of the form

$$F_1(x', x) = x'\phi_1(x) + \phi_2(x).$$

The related transformation is expressed by equations

$$p' = \frac{\partial F_1}{\partial x'}(x', x), \quad p = -\frac{\partial F_1}{\partial x}(x', x),$$

where ϕ_1 is a smooth bijective function and ϕ_2 is a smooth function. The above equations lead to a class of transformations in the form

$$T_1(x, p) = (Q_1(x, p), P_1(x, p)) = \Big(-\big(\phi_1'(x)\big)^{-1} p - \big(\phi_1'(x)\big)^{-1}\phi_2'(x), \phi_1(x)\Big), \tag{6.1.52}$$

where $\phi'(x) = \frac{d}{dx}\phi(x)$. Note, that generating function

$$\tilde{F}_1(x', x) = -x\phi_1(x') - \phi_2(x')$$

induces a transformation $\tilde{T}_1$ being an inverse transformation to T_1.

The second class of transformations

$$x' = -\frac{\partial F_2}{\partial p'}(p', p), \quad x = \frac{\partial F_2}{\partial p}(p', p),$$

is generated by functions

$$F_2(p', p) = -p'\phi_1(p) - \phi_2(p)$$

which give

$$T_2(x, p) = (Q_2(x, p), P_2(x, p)) = \Big(\phi_1(p), -\big(\phi_1'(p)\big)^{-1} x - \big(\phi_1'(p)\big)^{-1}\phi_2'(p)\Big).$$

In this case functions

$$\tilde{F}_2(p', p) = p\phi_1(p') + \phi_2(p')$$

generate transformation $\tilde{T}_2$ being an inverse transformation to T_2.

The third and fourth classes of transformations are generated by

$$F_3(x', p) = x'\phi_1(p) + \phi_2(p),$$
$$F_4(x, p') = -p'\phi_1(x) - \phi_2(x),$$

with related transformations expressed by the equations

$$p' = \frac{\partial F_3}{\partial x'}(x', p), \quad x = \frac{\partial F_3}{\partial p}(x', p),$$

and

$$x' = -\frac{\partial F_4}{\partial p'}(x, p'), \quad p = -\frac{\partial F_4}{\partial x}(x, p').$$

The above equations give transformations in the form

$$T_3(x, p) = (Q_3(x, p), P_3(x, p)) = \Big(\big(\phi_1'(p)\big)^{-1} x - \big(\phi_1'(p)\big)^{-1} \phi_2'(p), \phi_1(p)\Big), \tag{6.1.53}$$

$$T_4(x, p) = (Q_4(x, p), P_4(x, p)) = \Big(\phi_1(x), \big(\phi_1'(x)\big)^{-1} p - \big(\phi_1'(x)\big)^{-1} \phi_2'(x)\Big). \tag{6.1.54}$$

Observe that functions

$$\tilde{F}_3(x', p) = -p\phi_1(x') - \phi_2(x'),$$
$$\tilde{F}_4(x, p') = x\phi_1(p') + \phi_2(p')$$

generate transformations $\tilde{T}_3 = T_4^{-1}$ and $\tilde{T}_4 = T_3^{-1}$.

Note, that the transformations T_2, T_3 and T_4 can be constructed from T_1 with the help of an interchange of variables transformation $I(x, p) = (-p, x)$ being a special case of the transformation T_1 generated by a function $F(x', x) = xx'$:

$$T_2 = I \circ T_1 \circ I^{-1},$$
$$T_3 = T_1 \circ I^{-1},$$
$$T_4 = I^{-1} \circ T_1.$$

Thus instead of considering the transformation theory for transformations T_1, T_2, T_3 and T_4 it is enough to consider only the transformation T_1 and its appropriate compositions with I.

The four presented classes of transformations are obviously classically canonical. It can be shown that they are also quantum canonical for Moyal product as

$$[\![Q_i(x, p), P_i(x, p)]\!]_{\star_M} = 1.$$

They belong to the intersection of the set of classical canonical transformations of a given Poisson tensor π and the set of quantum canonical transformations of its related quantum deformation.

The presented considerations can be easily extended to a $2n$-dimensional case. As an example let us present the transformation T_1 (6.1.52) in $2n$-dimensions. Consider a generating function

$$F(x', x) = x'^i(\phi_1)_i(x) + \phi_2(x),$$

where $\phi_1 = ((\phi_1)_1, \ldots, (\phi_1)_n)\colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth bijective function and $\phi_2\colon \mathbb{R}^n \to \mathbb{R}$ a smooth function, and $x = (x^1, \ldots, x^n)$. The function F generates the transformation $T(x, p) = (x', p')$ of the form

$$\begin{aligned} x'^i &= -p_j[(\phi_1'(x))^{-1}]^{ji} - [\phi_2'(x)]_j[(\phi_1'(x))^{-1}]^{ji}, \\ p_i' &= (\phi_1)_i(x), \end{aligned}$$

where $[(\phi_1'(x))^{-1}]^{ji}$ is the inverse of the Jacobian matrix $[\phi_1'(x)]_{ij} = \frac{\partial(\phi_1)_i}{\partial x^j}(x)$ of ϕ_1 and $[\phi_2'(x)]_j = \frac{\partial\phi_2}{\partial x^j}(x)$ is the Jacobian matrix of ϕ_2. A calculation shows that this transformation is also quantum canonical for the Moyal product. All other cases can be extended in a similar fashion.

In what follows the well known linear transformations of coordinates of a quantum phase space will be reconsidered in the frame of formalism just developed. For $n = 1$ the linear transformation is a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by the equation

$$T(x, p) = (dx - bp, -cx + ap),$$

where $a, b, c, d \in \mathbb{R}$. Moreover, it is assumed that $ad - bc = 1$, which makes this transformation canonical both on a classical and quantum level, i.e. it preserves both the Poisson bracket and the star-commutator. The inverse transformation is given by the following equation

$$T^{-1}(x', p') = (ax' + bp', cx' + dp').$$

The linear transformation T is generated by a function $F(x', x) = \frac{1}{b}xx' - \frac{a}{2b}x'^2 - \frac{d}{2b}x^2$, i.e.

$$p' = \frac{\partial F}{\partial x'}(x', x), \quad p = -\frac{\partial F}{\partial x}(x', x),$$

where $(x', p') = T(x, p)$.

For a given function $f \in C^\infty(\mathbb{R}^2)$ the derivatives of the function f transform as follows

$$\frac{\partial f}{\partial x'} \circ T = a\frac{\partial}{\partial x}(f \circ T) + c\frac{\partial}{\partial p}(f \circ T),$$
$$\frac{\partial f}{\partial p'} \circ T = b\frac{\partial}{\partial x}(f \circ T) + d\frac{\partial}{\partial p}(f \circ T).$$

Using the above formulae one finds that the linear transformation T preserves the $\star$-product, i.e. the $\star$-product does not change after the transformation of coordinates

$$(f \star g) \circ T = (f \circ T) \star (g \circ T), \quad f, g \in C^\infty(\mathbb{R}^2).$$

Of course, in this case the isomorphism $S_T = 1$. The extension onto arbitrary n is straightforward.

Let us return to the transformed Moyal product (6.1.49). If new coordinates (x, p) are quantum canonical with respect to the transformed product (6.1.49) then, as will be proved in the next subsection, there exists a morphism S_T of the form

$$S_T = \mathrm{id} + \sum_{k=1}^{\infty} \hbar^k S_k,$$

where S_k are differential operators on $C^\infty(\mathbb{R}^{2n})[\![\hbar]\!]$, such that

$$\boxed{S_T(f \star_M^{(x,p)} g) = S_T f \star_T^{(x,p)} S_T g,} \tag{6.1.55a}$$

$$\boxed{S_T x^i = x^i, \quad S_T p_i = p_i, \quad i = 1, \ldots, n,} \tag{6.1.55b}$$

where $\star_M^{(x,p)}$ is a star-product which in the coordinates $(\xi) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ is of the form of the Moyal product (6.1.33). This fact is crucial as it means that in new coordinates we again can use the Moyal product with deformed properly quantum observables.

For deriving the form of the automorphism S_T for particular transformations T, other forms of the conditions (6.1.55) will be more useful. The conditions (6.1.55) are fulfilled if and only if the conditions

$$\boxed{(\hat{q}_T)^j = S_T(\hat{q}_M)^j S_T^{-1},} \tag{6.1.56a}$$

$$\boxed{(\hat{p}_T)_j = S_T(\hat{p}_M)_j S_T^{-1},} \tag{6.1.56b}$$

are fulfilled, where $(\hat{q}_T)^j = x^j \star_T$, $(\hat{p}_T)_j = p_j \star_T$, $(\hat{q}_M)^j = x^j \star_M$ and $(\hat{p}_M)_j = p_j \star_M$ are operators of position and momentum in $\star_M$ and $\star_T$

quantizations, respectively. Notice, that according to (6.1.34)

$$\boxed{(\hat{q}_M)^j = x^j + \tfrac{1}{2} i\hbar \partial_{p_j}, \quad (\hat{p}_M)_j = p_j - \tfrac{1}{2} i\hbar \partial_{x^j},} \tag{6.1.57}$$

and are known as "Bopp shifts" (see Chapter 18 in [82], for example).

For simplicity we will present the proof for a two-dimensional case ($n = 1$). If the conditions (6.1.55) are fulfilled then the conditions (6.1.56a), (6.1.56b) are fulfilled as well, as for any $f \in \mathcal{A}_Q$

$$\begin{aligned} \hat{q}_T f &= x \star_T f = S_T S_T^{-1}(x \star_T f) = S_T(S_T^{-1} x \star_M S_T^{-1} f) \\ &= S_T(x \star_M S_T^{-1} f) = [S_T(x \star_M) S_T^{-1}] f = (S_T \hat{q} S_T^{-1}) f \end{aligned}$$

with similar calculations for (6.1.56b). Assume now, that the conditions (6.1.56a), (6.1.56b) are fulfilled. Then it follows that (6.1.55a) will be satisfied for every f in the form of a $\star$-monomial $x \star_M \cdots \star_M x \star_M p \star_M \cdots \star_M p$ as

$$\begin{aligned} & S_T((\underbrace{x \star_M \cdots \star_M x}_{n} \star_M \underbrace{p \star_M \cdots \star_M p}_{m}) \star_M g) \\ &= S_T x \star_T S_T(\underbrace{x \star_M \cdots \star_M x}_{n-1} \star_M \underbrace{p \star_M \cdots \star_M p}_{m} \star_M g) \\ &= \underbrace{S_T x \star_T \cdots \star_T S_T x}_{n} \star_T \underbrace{S_T p \star_T \cdots \star_T S_T p}_{m} \star_T S_T g \\ &= \underbrace{S_T x \star_T \cdots \star_T S_T x}_{n} \star_T \underbrace{S_T p \star_T \cdots \star_T S_T p}_{m-2} \star_T S_T(p \star_M p) \star_T S_T g \\ &= S_T(\underbrace{x \star_M \cdots \star_M x}_{n} \star_M \underbrace{p \star_M \cdots \star_M p}_{m}) \star_T S_T g. \end{aligned}$$

From the linearity of S_T and the fact that the general $f \in \mathcal{A}_Q$ can be written in the form of the series (6.1.47) follows the condition (6.1.55a). The condition (6.1.55b) can be received by calculating left and right sides of (6.1.56a), (6.1.56b) on function identically equal 1.

If $\bar{T} = T$, then for both star-algebras (6.1.48) and (6.1.49) the complex conjugation is an involution. As a consequence $\bar{S_T} = S_T$. More generally, such transformations do not change the involution (6.1.38) for the class of a star-product (6.1.42). Indeed

$$f^{*_T} = S S_T \overline{(S S_T)^{-1}} f = S S_T \overline{S_T^{-1} S^{-1}} \bar{f} = S \overline{S^{-1}} f = f^*.$$

Let us analyze the following instructive case [35] of a particular class of nonlinear transformations of coordinates

$$\boxed{T(x,p) = (-ap - a\phi'(x), a^{-1}x),}$$

i.e. the case of T_1 (6.1.52) generated by the function $F(x',x) = a^{-1}xx' + \phi(x)$ ($a \in \mathbb{R}$, $a \neq 0$, ϕ being an arbitrary smooth function). This transformation is a classical canonical transformation. For a given function $f \in C^\infty(\mathbb{R}^2)$ the derivatives of the function f transform as follows

$$\frac{\partial f}{\partial x'} \circ T = -\frac{1}{a}\frac{\partial}{\partial p}(f \circ T),$$
$$\frac{\partial f}{\partial p'} \circ T = a\frac{\partial}{\partial x}(f \circ T) - a\phi''(x)\frac{\partial}{\partial p}(f \circ T).$$

Hence the Moyal $\star'$-product transforms to the following one

$$\boxed{f \star_T g = f \exp\left(\tfrac{1}{2}i\hbar\overleftarrow{D_x}\overrightarrow{D_p} - \tfrac{1}{2}i\hbar\overleftarrow{D_p}\overrightarrow{D_x}\right) g,}$$

where

$$\boxed{\begin{aligned} D_x &= a^{-1}\partial_p, \\ D_p &= a\partial_x - a\phi''(x)\partial_p. \end{aligned}}$$

In consequence, one calculates that

$$\boxed{\begin{aligned} \hat{q}_T &= x \star_T = x + \tfrac{1}{2}i\hbar\partial_p, \\ \hat{p}_T &= p \star_T = p - \tfrac{1}{2}i\hbar\partial_x - \sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \phi^{(n+1)}(x)\partial_p^n, \end{aligned}}$$

and moreover

$$[\hat{q}_T, \hat{p}_T] = i\hbar.$$

The above equation shows that the transformation T is a quantum canonical transformation for the Moyal product.

The isomorphism S_T intertwining the Moyal $\star_M$-product with the $\star_T$-product in (x, p) coordinates is given by [35]

$$\boxed{S_T = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}(-1)^n \left(\frac{\hbar}{2}\right)^{2n} \phi^{(2n+1)}(x)\partial_p^{2n+1}\right).} \tag{6.1.58}$$

Indeed, from (6.1.56a), (6.1.56b) it is enough to prove that

$$\hat{q}_T = S_T \hat{q}_M S_T^{-1},$$
$$\hat{p}_T = S_T \hat{p}_M S_T^{-1}.$$

Since $S_T = e^{\hat{A}}$ where

$$\hat{A} = -\sum_{n=1}^{\infty} \frac{1}{(2n+1)!}(-1)^n \left(\frac{\hbar}{2}\right)^{2n} \phi^{(2n+1)}(x)\partial_p^{2n+1}$$

the above equations, from the Hadamard's lemma, take the form

$$\hat{q}_T = e^{[\hat{A}, \cdot]}\hat{q}_M, \tag{6.1.59}$$
$$\hat{p}_T = e^{[\hat{A}, \cdot]}\hat{p}_M. \tag{6.1.60}$$

One can calculate that $[\hat{A}, \hat{q}_M] = 0$ and thus

$$e^{[\hat{A}, \cdot]}\hat{q}_M = \hat{q}_M = \hat{q}_T,$$

which proves (6.1.59). On the other hand, one finds that

$$\begin{aligned}[\hat{A}, \hat{p}_M] &= -\sum_{n=1}^{\infty} \frac{1}{(2n)!}(-1)^n \left(\frac{\hbar}{2}\right)^{2n} \phi^{(2n+1)}(x)\partial_p^{2n} \\ &\quad - \tfrac{1}{2} i\hbar \sum_{n=1}^{\infty} \frac{1}{(2n+1)!}(-1)^n \left(\frac{\hbar}{2}\right)^{2n} \phi^{(2n+2)}(x)\partial_p^{2n+1} \\ &= -\sum_{n=2}^{\infty} \frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \phi^{(n+1)}(x)\partial_p^n\end{aligned}$$

and

$$[\hat{A}, [\hat{A}, \hat{p}_M]] = 0.$$

Thus

$$e^{[\hat{A},\cdot]}\hat{p}_M = p - \tfrac{1}{2}i\hbar\partial_x - \sum_{n=2}^{\infty}\frac{1}{n!}\left(\frac{i\hbar}{2}\right)^n \phi^{(n+1)}(x)\partial_p^n = \hat{p}_T,$$

which proves (6.1.60).

Let us derive the more compact formula for S_T, transforming (6.1.58) as follows

$$\begin{aligned}
S_T &= \exp\left(-\sum_{n=1}^{\infty}\frac{1}{(n+1)!}\left(\frac{i\hbar}{2}\right)^n \frac{1+(-1)^n}{2}\phi^{(n+1)}(x)\partial_p^{n+1}\right)\\
&= \exp\left[\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{1}{n!}\left(1-(-1)^n\right)\phi^{(n)}(x)\left(\tfrac{1}{2}i\hbar\partial_p\right)^n\right]\\
&= \exp\left[\frac{i}{\hbar}\left(\phi\left(x+\tfrac{1}{2}i\hbar\partial_p\right)-\phi(x)-\phi'(x)\tfrac{1}{2}i\hbar\partial_p-\phi\left(x-\tfrac{1}{2}i\hbar\partial_p\right)\right.\right.\\
&\qquad \left.\left.+\phi(x)-\phi'(x)\tfrac{1}{2}i\hbar\partial_p\right)\right]\\
&= \exp\left[\frac{i}{\hbar}\left(\phi\left(x+\tfrac{1}{2}i\hbar\partial_p\right)-\phi\left(x-\tfrac{1}{2}i\hbar\partial_p\right)-i\hbar\phi'(x)\partial_p\right)\right]\\
&= \exp\left(\frac{i}{\hbar}\left[\phi(\hat{q}_M)-\phi'(x)\hat{q}_M-\phi(\hat{q}_M^*)+\phi'(x)\hat{q}_M^*\right]\right).
\end{aligned}$$

Observe that the map S_T preserves the involution, i.e.,

$$S_T(\bar{f}) = (\overline{S_T f}), \quad f \in \mathcal{A}_Q,$$

which can be immediately seen from the fact that $S_T = \bar{S}_T$.

The class of point transformations on the phase space $M = T^*E^n$ is particularly important for our further considerations. Let us start from the simplest case of $n = 1$. Then, the transformation takes the form

$$\boxed{T(x,p) = (\phi(x), (\phi'(x))^{-1}p)}$$

generated by functions $F(x,p') = -p'\phi(x)$ (ϕ being an arbitrary smooth bijective function). These transformations from construction are classical canonical transformations of class T_4 (6.1.54) where $\phi_1 \equiv \phi$ and $\phi_2 = 0$. For a given function $f \in C^\infty(\mathbb{R}^2)$ the derivatives of the function f transform as follows

$$\begin{aligned}
\frac{\partial f}{\partial x'}\circ T &= (\phi'(x))^{-2}\phi''(x)p\frac{\partial}{\partial p}(f\circ T) + (\phi'(x))^{-1}\frac{\partial}{\partial x}(f\circ T),\\
\frac{\partial f}{\partial p'}\circ T &= \phi'(x)\frac{\partial}{\partial p}(f\circ T).
\end{aligned}$$

In consequence, the Moyal $\star'$-product transforms to the following product

$$\boxed{f \star_T g = f \exp\left(\tfrac{1}{2}i\hbar \overleftarrow{D}_x \overrightarrow{D}_p - \tfrac{1}{2}i\hbar \overleftarrow{D}_p \overrightarrow{D}_x\right) g,} \tag{6.1.61}$$

where

$$\boxed{\begin{aligned} D_x &= (\phi'(x))^{-1}\partial_x + (\phi'(x))^{-2}\phi''(x)p\partial_p, \\ D_p &= \phi'(x)\partial_p. \end{aligned}}$$

To the third order in $\hbar$ the operators $\hat{q}_T$ and $\hat{p}_T$ are of the form

$$\boxed{\begin{aligned} \hat{q}_T &= x + \tfrac{1}{2}i\hbar\partial_p + \tfrac{1}{8}\hbar^2\Gamma(x)\partial_p^2 - \frac{1}{48}i\hbar^3\left(2\Gamma^2(x) - \Gamma'(x)\right)\partial_p^3 + O(\hbar^4), \\ \hat{p}_T &= p - \tfrac{1}{2}i\hbar\partial_x + \tfrac{1}{8}\hbar^2\left[2\Gamma^2(x)\partial_p + 2\Gamma(x)\partial_x\partial_p + \left(2\Gamma^2(x) - \Gamma'(x)\right)p\partial_p^2\right] \\ &\quad + \frac{1}{48}i\hbar^3\left[6\Gamma(x)\Gamma'(x)\partial_p^2 + 3\Gamma'(x)\partial_x\partial_p^2 - (\Gamma''(x) + 4\Gamma(x)\Gamma'(x))p\partial_p^3\right] + O(\hbar^4), \end{aligned}}$$

where $\Gamma(x) = (\phi'(x))^{-1}\phi''(x)$. Moreover, it can be verified [35] that

$$[\hat{q}_T, \hat{p}_T] = i\hbar,$$

which proves that T is a quantum canonical transformation.

Up to the second order in $\hbar$ the isomorphism S_T reads

$$\boxed{S_T = 1 + \frac{\hbar^2}{4!}\left[3\Gamma^2(x)\partial_p^2 + 3\Gamma(x)\partial_x\partial_p^2 + (2\Gamma^2(x) - \Gamma'(x))p\partial_p^3\right] + O(\hbar^4).}$$

Example 6.5 Consider a classical point transformation of coordinates [35] $T : (\mathbb{R} \setminus \{0\}) \times \mathbb{R} \to (\mathbb{R} \setminus \{0\}) \times \mathbb{R}$, $T(x, p) = (x', p')$ where

$$x' = \begin{cases} \sqrt{|2x|}, & x > 0 \\ -\sqrt{|2x|}, & x < 0 \end{cases}, \quad p' = p\sqrt{|2x|},$$

with the inverse T^{-1} in the form

$$x = \begin{cases} \frac{1}{2}x'^2, & x' > 0 \\ -\frac{1}{2}x'^2, & x' < 0 \end{cases}, \quad p = |x'|^{-1}p',$$

The operators $\hat{q}_T$ and $\hat{p}_T$ take then the form

$$\hat{q}_T = x + \tfrac{1}{2}i\hbar\partial_p - \tfrac{1}{8}\hbar^2 \operatorname{sgn}(x)|2x|^{-1}\partial_p^2,$$

$$\hat{p}_T = p - \tfrac{1}{2}i\hbar\partial_x + \sum_{n=1}^{\infty}\left(-\frac{i\hbar}{2}\right)^{n+1}\Big((\operatorname{sgn}(x))^n|2x|^{-n}\partial_x\partial_p^n - n(\operatorname{sgn}(x))^{n+1}|2x|^{-n-1}\partial_p^n\Big),$$

and the isomorphism S_T is expressed by the formula

$$S_T = \exp\left(\sum_{n=1}^{\infty}(-1)^n\left(\frac{\hbar}{2}\right)^{2n}\left(A_n \operatorname{sgn}(x)|2x|^{-2n+1}\partial_x\partial_p^{2n} - B_n|2x|^{-2n}\partial_p^{2n}\right)\right),$$

where A_n and B_n are rational constants given recursively by

$$A_n = \frac{1}{2n}\left(1 - \sum_{k=2}^{n}\frac{1}{k!}A_{2n-1}^{(k)}\right),$$

$$B_n = \frac{1}{2n}\left(2n - 1 - \sum_{k=2}^{n}\frac{1}{k!}B_{2n-1}^{(k)}\right),$$

where

$$A_{2n-1}^{(k)} = \sum_{m=1}^{n-1} 4(n-2m)A_{n-m}A_{2m-1}^{(k-1)},$$

$$B_{2n-1}^{(k)} = \sum_{m=1}^{n-1}\Big(4(n-m)B_{n-m}A_{2m-1}^{(k-1)} - 4mA_{n-m}B_{2m-1}^{(k-1)}\Big)$$

for $k = 2, 3, \ldots, n$ and $n = 2, 3, \ldots$, and

$$A_1^{(k)} = B_1^{(k)} = 0, \quad k = 2, 3, \ldots,$$

$$A_{2n-1}^{(1)} = 2nA_n, \quad B_{2n-1}^{(1)} = 2nB_n, \quad n = 1, 2, \ldots .$$

The values of a few first constants A_n and B_n are

$$A_1 = \tfrac{1}{2}, \qquad A_2 = \tfrac{1}{4}, \qquad A_3 = \tfrac{1}{4}, \qquad A_4 = \frac{7}{24},$$

$$B_1 = \tfrac{1}{2}, \qquad B_2 = \tfrac{3}{4}, \qquad B_3 = \frac{5}{4}, \qquad B_4 = \frac{49}{24}.$$

Now, let us pass to a multi-dimensional case of point transformation on $M = T^*E^n$

$$\boxed{\begin{aligned} T:\ x'^i &= \phi^i(x), \\ p'_i &= \left[\phi'(x)^{-1}\right]_i^j p_j, \end{aligned}} \tag{6.1.62}$$

where $\phi'(x)^{-1}$ is an inverse of Jacobian $\left[\phi'(x)\right]_j^i = \frac{\partial \phi^i}{\partial x^j}$, x' are Cartesian coordinates and x are curvilinear coordinates on E^n. The Moyal product

$$\star_M^{(x',p')} = \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{\partial}_{x'^i} \overrightarrow{\partial}_{p'_i} - \tfrac{1}{2} i\hbar \overleftarrow{\partial}_{p'_i} \overrightarrow{\partial}_{x'^i}\right) \tag{6.1.63}$$

transforms to a new star-product

$$\boxed{\star^{(x,p)} = \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{D}_{x^i} \overrightarrow{D}_{p_i} - \tfrac{1}{2} i\hbar \overleftarrow{D}_{p_i} \overrightarrow{D}_{x^i}\right) = \exp \tfrac{1}{2} i\hbar \pi^{\alpha\beta} \overleftarrow{D}_\alpha \overrightarrow{D}_\beta,} \tag{6.1.64}$$

such that

$$(f \star_M^{(x',p')} g) \circ T = (f \circ T) \star^{(x,p)} (g \circ T),$$

where

$$\boxed{\begin{aligned} D_i = D_{x^i} &= \left[\phi'(x)^{-1}\right]_i^j \partial_{x^j} + \left[\phi'(x)^{-1}\right]_i^j \left[\phi'(x)^{-1}\right]_k^r \left[\phi''(x)\right]_{jl}^k p_r \partial_{p_l}, \\ D_{n+i} = D_{p_i} &= \left[\phi'(x)\right]_j^i \partial_{p_j}, \end{aligned}} \tag{6.1.65}$$

and $\left[\phi''(x)\right]_{jl}^k = \frac{\partial^2 \phi^k}{\partial x^j \partial x^l}$ is the Hessian of ϕ. Up to quadratic terms in $\hbar$, the isomorphism S_T relating $\star_M^{(x,p)}$-algebra (6.1.33) and $\star^{(x,p)}$-algebra (6.1.64) respectively, is of the form

$$\boxed{\begin{aligned} S_T = id + \frac{\hbar^2}{4!}[&3\Gamma^i_{lj}(x)\Gamma^l_{ik}(x)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}(x)\partial_{x^i}\partial_{p_j}\partial_{p_k} \\ &+ \left(2\Gamma^i_{nl}(x)\Gamma^n_{jk}(x) - \partial_{x^l}\Gamma^i_{jk}(x)\right) p_i \partial_{p_j}\partial_{p_k}\partial_{p_l}] + O(\hbar^4) \end{aligned}} \tag{6.1.66}$$

and will be crucial for our further considerations. To the second order in $\hbar$ the operators $\hat{q}_T^j$ and $\hat{p}_{jT}$ are

$$(\hat{q}_T)^j = x^j + \tfrac{1}{2}i\hbar\partial_{p_j} + \tfrac{1}{8}\hbar^2\Gamma^j_{kn}(x)\partial_{p_k}\partial_{p_n} + O(\hbar^3), \tag{6.1.67a}$$

$$\begin{aligned}(\hat{p}_T)_j =& p_j - \tfrac{1}{2}i\hbar\partial_{x^j} + \tfrac{1}{8}\hbar^2\Big\{2\Gamma^m_{nj}(x)\Gamma^n_{mk}(x)\partial_{p_k} + 2\Gamma^k_{jn}(x)\partial_{x^k}\partial_{p_n} \\ &+ [\Gamma^k_{nl}(x)\Gamma^n_{mj}(x) + \Gamma^k_{nj}(x)\Gamma^n_{ml}(x) - \Gamma^k_{mj,l}(x)]p_k\partial_{p_m}\partial_{p_l}\Big\} + O(\hbar^3),\end{aligned} \tag{6.1.67b}$$

where

$$\Gamma^i_{jk}(x) = [(\phi'(x)^{-1}]^i_r[\phi''(x)]^r_{jk}$$

are coefficients of the Levi-Civita connection on E^n in curvilinear coordinates (see (2.6.5)).

In what follows let us transform the vector representation (6.1.64) of the $\star^{(x,p)}$-product to its covariant representation. Taking into account the explicit form of vector fields (6.1.65), the $\star^{(x,p)}$-product (6.1.64) can be written in the form

$$\boxed{\begin{aligned}f\star^{(x,p)}g &= \sum_{n,m=0}^{\infty}\frac{1}{n!m!}(-1)^m\left(\frac{i\hbar}{2}\right)^{n+m}(D^{j_1\dots j_m}_{i_1\dots i_n}f)(D^{i_1\dots i_n}_{j_1\dots j_m}g) \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k\sum_{n=0}^{k}\binom{k}{n}(-1)^{k-n}(D^{j_1\dots j_{k-n}}_{i_1\dots i_n}f)(D^{i_1\dots i_n}_{j_1\dots j_{k-n}}g),\end{aligned}} \tag{6.1.68}$$

where operators $D^{j_1\dots j_m}_{i_1\dots i_n}$ are given by recursion formulas [37]

$$\begin{aligned}D^{j_1\dots j_m}_{i_1\dots i_{n+1}}f &= D_{i_{n+1}}(D^{j_1\dots j_m}_{i_1\dots i_n}f) - \Gamma^k_{i_1i_{n+1}}D^{j_1\dots j_m}_{k\dots i_n}f - \dots - \Gamma^k_{i_ni_{n+1}}D^{j_1\dots j_m}_{i_1\dots k}f \\ &\quad + \Gamma^{j_1}_{ki_{n+1}}D^{k\dots j_m}_{i_1\dots i_n}f + \dots + \Gamma^{j_m}_{ki_{n+1}}D^{j_1\dots k}_{i_1\dots i_n}f,\end{aligned} \tag{6.1.69a}$$

$$D^{j_1\dots j_{m+1}}_{i_1\dots i_n}f = D^{j_{m+1}}(D^{j_1\dots j_m}_{i_1\dots i_n}f), \tag{6.1.69b}$$

$$D_if = \partial_{x^i}f + \Gamma^k_{ij}p_k\partial_{p_j}f, \tag{6.1.69c}$$

$$D^jf = \partial_{p_j}f, \tag{6.1.69d}$$

and where $\{D_i, D^j\}$ is a so called adopted frame on $\mathbb{R}^{2n} = T^*E^n$ (see Sect. 2.7). Note that the upper indices in the operator $D^{j_1\dots j_m}_{i_1\dots i_n}$ commute with the lower indices, i.e. it does not matter if, when calculating $D^{j_1\dots j_m}_{i_1\dots i_n}f$, we first use formula (6.1.69a) and then (6.1.69b) or vice verse.

Equation (6.1.68) takes the form

$$f \star^{(x,p)} g \tag{6.1.70}$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{n=0}^{k} \binom{k}{n} (-1)^{k-n} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{i_1 \ldots i_n \bar{j}_1 \ldots \bar{j}_{k-n}} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\bar{\imath}_1 \ldots \bar{\imath}_n j_1 \ldots j_{k-n}},$$

where $\bar{\imath} = n + i$ and $\bar{\nabla}$ is a linear connection on the symplectic manifold T^*E^N, whose components in the frame $\{D_i, D^j\}$ are

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk}, \quad \bar{\Gamma}^{\bar{\imath}}_{\bar{j}k} = -\Gamma^j_{ik} \tag{6.1.71}$$

with the remaining components equal zero. Thus, Eq. (6.1.70) can be written in the form

$$f \star^{(x,p)} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \sum_{n=0}^{k} \binom{k}{n} A^{\mu_1\nu_1} \cdots A^{\mu_n\nu_n} B^{\mu_{n+1}\nu_{n+1}} \cdots B^{\mu_k\nu_k}$$
$$(\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k}, \tag{6.1.72}$$

where

$$A = \begin{pmatrix} 0_n & I_n \\ 0_n & 0_n \end{pmatrix}, \quad B = \begin{pmatrix} 0_n & 0_n \\ -I_n & 0_n \end{pmatrix}.$$

Then, Eq. (6.1.72) can be transformed to the form

$$f \star^{(x,p)} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k (A+B)^{\mu_1\nu_1} \cdots (A+B)^{\mu_k\nu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k}$$

and introducing

$$\pi = A + B = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

we finally receive

$$\boxed{f \star^{(x,p)} g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k}.} \tag{6.1.73}$$

Since $D_i \wedge D^j = \partial_{x^i} \wedge \partial_{p_j}, \pi^{\mu\nu}$ are components of the Poisson tensor in the Darboux frame $\{\partial_{x^i}, \partial_{p_j}\}$ as well as in the adopted frame $\{D_i, D^j\}$.

The Christoffel symbols of the linear connection $\bar{\nabla}$ in the Darboux coordinate frame take the form

$$\begin{aligned} &\bar{\Gamma}^{i}_{jk} = \Gamma^{i}_{jk}, \quad \bar{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}} = -\Gamma^{j}_{ik}, \quad \bar{\Gamma}^{\bar{i}}_{j\bar{k}} = -\Gamma^{k}_{ji}, \\ &\bar{\Gamma}^{\bar{i}}_{jk} = p_l(\Gamma^{r}_{jk}\Gamma^{l}_{ri} + \Gamma^{r}_{ik}\Gamma^{l}_{rj} - \Gamma^{l}_{ij,k}), \end{aligned} \tag{6.1.74}$$

with the remaining components equal zero. Comparing (6.1.74) and (6.1.71) with (2.7.7) and (2.7.6) we immediately recognize that $\bar{\nabla} = \nabla^{(S)}$ is a flat symplectic connection on T^*E^n induced by the Levi-Civita connection on E^n. Thus we wrote the canonical star-product on T^*E^n in a covariant form (6.1.73) with respect to an appropriate symplectic connection. Notice that the isomorphism S_T (6.1.66) between $\star_M^{(x,p)}$-algebra (6.1.33) and $\star^{(x,p)}$-algebra (6.1.73), expressed by a symplectic connection in the Darboux frame, is of the form

$$\boxed{S = id + \frac{\hbar^2}{4!}\left(-\bar{\Gamma}_{\alpha\beta\gamma}\partial^\alpha\partial^b\partial^\gamma + \frac{3}{2}\bar{\Gamma}^{\mu}_{\nu\alpha}\bar{\Gamma}^{\nu}_{\mu\beta}\partial^\alpha\partial^\beta\right) + O(\hbar^4),} \tag{6.1.75}$$

where $\partial^\alpha = \pi^{\alpha\beta}\partial_{\xi^\beta}$, $(\xi^1, \ldots, \xi^{2n}) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ and $\bar{\Gamma}_{\alpha\beta\gamma} = \omega_{\alpha\delta}\bar{\Gamma}^{\delta}_{\beta\gamma}$, while (6.1.67a) reads

$$\boxed{\xi^\alpha \star = \xi^\alpha + \frac{1}{2}i\hbar\partial^\alpha + \left(\frac{i\hbar}{2}\right)^2\left(-\tfrac{1}{2}\bar{\Gamma}^{\alpha}_{\mu_1\mu_2}\partial^{\mu_1}\partial^{\mu_2} - \tfrac{1}{2}\bar{\Gamma}^{\nu_1}_{\mu_1\mu_2}\bar{\Gamma}^{\mu_2}_{\nu_1\nu_2}\partial^{\nu_2}\right) + O(\hbar^3).} \tag{6.1.76}$$

As we will see in the next subsection, on a symplectic manifold endowed with a symplectic torsionless connection it is possible to distinguish a whole family of star-products. In the majority of physically interesting cases a symplectic manifold is taken in the form of a cotangent bundle to a Riemannian configuration space. In such a case there exists a distinguished connection, induced by the Levi-Civita connection from the related Riemannian space, and thus a family of star-products which can be used in the process of quantization.

The star-product (6.1.64) is also an admissible star-product on more general symplectic manifolds. Let us consider a symplectic manifold M whose Poisson tensor can be written in the form (6.1.27). Additionally, let us assume that the first de Rham cohomology class $H^1(M)$ vanishes, which guarantees the existence of global natural coordinate systems associated to the star-products (6.1.64). On such a symplectic manifold M the product (6.1.64) is a proper star-product, which can also be written in a covariant form (6.1.73) with an appropriate linear connection $\bar{\nabla}$. However, in this case there is no distinguished star-product from the family

of products (6.1.64). To distinguish a star-product we have to select a set of commuting vector fields (D_μ) from the decomposition (6.1.27) of the Poisson tensor, or equivalently, by choosing a flat torsionless symplectic linear connection $\bar{\nabla}$ on M.

6.1.5 Star Products on Symplectic Manifolds

Let now $(M, \omega = \pi^{-1})$ be a general symplectic manifold with $\star$-product (6.1.1). The following statement will be crucial for our further considerations. If on M exists a coordinate system $M \supset U \to V \subset \mathbb{R}^{2n}$, $\xi \mapsto (\xi^1, \ldots, \xi^{2n})$ such that it is at the same time classical with respect to the classical Poisson bracket and quantum canonical with respect to the quantum bracket (6.1.2), then there exists a unique series S of the form

$$S = \mathrm{id} + \sum_{k=1}^{\infty} \hbar^k S_k,$$

where S_k are differential operators on $C^\infty(U)[\![\hbar]\!]$, such that

$$S(f \star_M^{(\xi)} g) = Sf \star^{(\xi)} Sg, \tag{6.1.77a}$$

$$S\xi^\alpha = \xi^\alpha, \tag{6.1.77b}$$

where $\star_M^{(\xi)}$ is a star-product which in the coordinates $(\xi^1, \ldots, \xi^{2n})$ is of the form of the Moyal product (6.1.33). The operators S_k will satisfy the following recurrence relations for $k \geq 1$

$$[S_k, \xi^\alpha](f) = \tfrac{1}{2} \sum_{l=1}^{k} \big(C_l(\xi^\alpha, S_{k-l}(f)) + C_l(S_{k-l}(f), \xi^\alpha)\big), \quad f \in C^\infty(M), \tag{6.1.78}$$

where operators C_l are defined in (6.1.1). The reader can find the proof of that statement in [99].

If the $\star$-product satisfies also the parity condition (6.1.5)

$$C_k(f, g) = (-1)^k C_k(g, f), \quad f, g \in C^\infty(M), \tag{6.1.79}$$

then relations (6.1.78) take the form

$$[S_{2k+1}, \xi^\alpha](f) = 0, \tag{6.1.80a}$$

$$[S_{2k}, \xi^\alpha](f) = \sum_{l=1}^{k} C_{2l}(\xi^\alpha, S_{2(k-l)}(f)), \tag{6.1.80b}$$

for $k \geq 1$. Thus, in this special case only terms of the even order in the expansion of S are non-zero and they are given by (6.1.80b).

A direct calculation shows that the solution of (6.1.78) is of the form

$$S_k = \sum_{n=1}^{\infty} \frac{1}{n!} [\xi^{\alpha_1}, \ldots, [\xi^{\alpha_{n-1}}, F_k^{\alpha_n}]] \partial_{\xi^{\alpha_1}} \cdots \partial_{\xi^{\alpha_n}}, \tag{6.1.81}$$

where

$$F_k^{\alpha}(f) = \tfrac{1}{2} \sum_{l=1}^{k} \big(C_l(\xi^{\alpha}, S_{k-l}(f)) + C_l(S_{k-l}(f), \xi^{\alpha}) \big). \tag{6.1.82}$$

Indeed,

$$\begin{aligned}
[S_k, \xi^{\alpha}] &= -\sum_{n=1}^{\infty} \frac{1}{n!} [\xi^{\alpha}, [\xi^{\beta_1}, \ldots, [\xi^{\beta_{n-1}}, F_k^{\beta_n}]]] \partial_{\beta_1} \cdots \partial_{\beta_n} \\
&\quad + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} [\xi^{\beta_1}, \ldots, [\xi^{\beta_{n-1}}, F_k^{\alpha}]] \partial_{\beta_1} \cdots \partial_{\beta_{n-1}} \\
&= -\sum_{n=1}^{\infty} \frac{1}{n!} [\xi^{\beta_1}, \ldots, [\xi^{\beta_n}, F_k^{\alpha}]] \partial_{\beta_1} \cdots \partial_{\beta_n} \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{n!} [\xi^{\beta_1}, \ldots, [\xi^{\beta_n}, F_k^{\alpha}]] \partial_{\beta_1} \cdots \partial_{\beta_n} = F_k^{\alpha}.
\end{aligned}$$

Observe that when C_k are finite order bi-differential operators then the sum in (6.1.81) will be finite.

If the $\star$-product satisfies additionally the parity condition (6.1.79), then (6.1.82) reduces to the form

$$F_{2k}^{\alpha}(f) = \sum_{l=1}^{k} \big(C_{2l}(\xi^{\alpha}, S_{2(k-l)}(f)) \big). \tag{6.1.83}$$

Further on we will consider such $\star$-products for which, for every almost global classical and quantum canonical coordinate system $M \supset U \to V \subset \mathbb{R}^{2n}$, $\xi \mapsto (\xi^1, \ldots, \xi^{2n})$, the associated series S such that $S = \bar{S}$ giving the equivalence with a Moyal product has the property that for every $f \in C_0^{\infty}(V)$ the series $S(f)$ is convergent to an element of $L^2(V, d\Omega_{\hbar})$ and

$$\int_V Sf \, d\Omega_{\hbar} = \int_V f \, d\Omega_{\hbar}, \quad f \in C_0^{\infty}(V). \tag{6.1.84}$$

From (6.1.84) it follows that

$$\langle Sf, Sg\rangle = \langle f, g\rangle, \quad f, g \in C_0^\infty(V).$$

Indeed, we have

$$\begin{aligned}\langle Sf, Sg\rangle &= \int_V \overline{Sf} Sg \, d\Omega_\hbar = \int_V S\bar{f} \star^{(x)} Sg \, d\Omega_\hbar \\ &= \int_V S(\bar{f} \star_M^{(\xi)} g) \, d\Omega_\hbar = \int_V \bar{f} \star_M^{(\xi)} g \, d\Omega_\hbar = \langle f, g\rangle.\end{aligned}$$

The above property imposed on the series S guaranties that S can be uniquely extended to a unitary operator defined on the whole Hilbert space $L^2(V, d\Omega_\hbar)$ and satisfying

$$S(f \star_M^{(\xi)} g) = Sf \star^{(\xi)} Sg, \quad f, g \in L^2(V, d\Omega_\hbar).$$

In the case when $(\xi^1, \ldots, \xi^{2n})$ is a purely quantum canonical coordinate system, i.e. it is not at the same time classical canonical, then it must depend on $\hbar$ and will be a deformation of a classical canonical coordinate system. The components $\pi^{\mu\nu}$ of the Poisson tensor π for such purely quantum canonical coordinate system will be $\hbar$ dependent and can be expanded in the series

$$\pi^{\mu\nu}(\hbar) = \pi_0^{\mu\nu} + \hbar\pi_1^{\mu\nu} + \hbar^2\pi_2^{\mu\nu} + O(\hbar^3), \tag{6.1.85}$$

where $\pi_0^{\mu\nu}$ components are of the form (6.1.14). In consequence, the bi-differential operators C_k from the expansion (6.1.1) of the $\star$-product written in the coordinates $(\xi^1, \ldots, \xi^{2n})$ will depend on $\hbar$. Expanding C_k in the power series of $\hbar$ allows to write the $\star^{(\xi)}$-product in the form

$$f \star^{(\xi)} g = \sum_{k=0}^{\infty} \hbar^k C_k'(f, g),$$

where C_k' are new bi-differential operators which do not depend on $\hbar$, and satisfy conditions **1–4** of (6.1.1). Moreover, in condition **3** the Poisson bracket, in accordance with (6.1.85), is associated with the Poisson tensor π_0. In consequence, the $\star^{(\xi)}$-product can be regarded as a coordinate representation, with respect to the coordinate system $(\xi^1, \ldots, \xi^{2n})$, of a star-product on a Poisson manifold (V, π_0). The coordinates $(\xi^1, \ldots, \xi^{2n})$ are then classical and quantum canonical. Thus, our statement is also valid for a purely quantum canonical coordinate system $(\xi^1, \ldots, \xi^{2n})$, but the Moyal product $\star_M^{(\xi)}$ will no longer be associated with the Poisson tensor $\pi(\hbar)$, but with another Poisson tensor.

The current considerations can be summarized as follows.

Observation 12 *Quantum Hamiltonian mechanics is formulated on simplectic manifolds which admit almost global coordinate systems. Choosing such a coordinate system* $(\xi^1, \ldots, \xi^{2n})$*, a family of admissible quantizations is represented by a family of* $\star^{(\xi)}$*-products equivalent through S with the Moyal product* $\star_M^{(\xi)}$ *(6.1.33) and such that (6.1.77b) and (6.1.84) are fulfilled. Moreover, a canonical choice of quantum observables for* $\star^{(\xi)}$*-product with involution in the form of complex-conjugation is given by* $A_Q(\hbar) = A_C$*, and for* $\star^{(\xi)}$*-product with other involution by* $A_Q(\hbar) = W(\hbar)A_C$ *(6.1.39) respectively. In consequence, all these nonequivalent quantizations can be represented by a single Moyal product* $\star_M^{(\xi)}$ *and a family of assignments of quantum observables* $A_S(\hbar) = SA_Q(\hbar)$ *(see Observation 11).*

The star-product (6.1.28) can be defined on more general symplectic manifolds M, different from $M = \mathbb{R}^{2n}$. Moreover, we adopt the Riemannian geometry to our construction. Let (Q, g) be an n-dimensional flat pseudo-Riemannian manifold with a property that every two points of Q can be connected by exactly one geodesic. On such a manifold, called the *simple connected pseud-Riemannian manifold*, there exists a global Riemann normal coordinate system $(x^1, \ldots, x^n)$, which are flat coordinates of the metric tensor g. Every such a coordinate system is parametrized by a point $x \in Q$ and a basis $e_1, \ldots, e_n$ in Q. Using the flatness of the manifold Q one can show that Riemann normal coordinate systems transform according to the formula

$$x'^i = A^i_j x^j + x^i_0, \tag{6.1.86}$$

where x^i_0 are the coordinates of the origin of the second coordinate system from the perspective of the first coordinate system, and A^i_j is a matrix which transforms the basis $e_1, \ldots, e_n$ of the first coordinate system to a parallel transported basis $e'_1, \ldots, e'_n$ of the second coordinate system, respectively.

The Riemann normal coordinate system $(x^i) = (x^1, \ldots, x^n)$ induces a global canonical coordinate system $(\xi^\alpha) = (x^i, p_i) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ on a symplectic manifold $M = T^*Q$. We will call this coordinate system a Riemann normal coordinate system on T^*Q. The canonical Poisson tensor π on T^*Q in Riemann normal coordinates can be globally written in the form (6.1.26).

Using the coordinate vector fields of the Riemann normal coordinate system on T^*Q we can introduce a star-product on the symplectic manifold T^*Q by the formula (6.1.33) and hence the Riemann normal coordinate system is a natural coordinate system for this star-product. Such a star-product is independent of the choice of the Riemann normal coordinate system as according to (6.1.86) coordinate vector fields of Riemann normal coordinate systems are related to each other by linear symplectic transformations which do not change the star-product. Thus, on the symplectic manifold T^*Q, for any flat connection ∇ on Q there is a distinguished star-product from the family of star-products (6.1.28) having Moyal representation (6.1.33) in any Riemann normal coordinate system on T^*Q induced

by Riemann normal coordinates of ∇. We will call this product a canonical star-product on T^*Q related to the flat connection ∇.

For Riemann normal coordinates the Christoffel symbols Γ^i_{jk} of the Levi-Civita connection ∇ on Q vanish, so g is a symmetric constant matrix. In consequence the Christoffel symbols $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ of the lift (6.1.74) of the connection ∇ to a symplectic connection $\bar{\nabla}$ on T^*Q vanish as well. This shows that the canonical star-product on T^*Q can be written in a covariant form

$$\boxed{\begin{aligned} f \star g &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (\partial_{\xi^{\mu_1}} \cdots \partial_{\xi^{\mu_k}} f)(\partial_{\xi^{\nu_1}} \cdots \partial_{\xi^{\nu_k}} g), \end{aligned}} \tag{6.1.87}$$

as for Riemann normal coordinates both products coincide. From flatness property of the linear connection $\bar{\nabla}$ follows the associativity of the star-product (6.1.87).

Now, let us consider some automorphism ϕ of (Q, g). It induces a new flat metric

$$\boxed{g^{\phi} = \left(\phi'^{-1}\right)^T g \phi'^{-1}} \tag{6.1.88}$$

of the same signature as g and new connection ∇^{ϕ} with different from zero Christoffel symbols in Riemann normal coordinates of the old connection ∇. The new connection ∇_{ϕ} induces a new symplectic connection $\bar{\nabla}_{\phi}$ on T^*Q and a new star-product

$$\boxed{\begin{aligned} f \star g &= f \exp\left(\tfrac{1}{2} i\hbar \pi^{\alpha\beta} \overleftarrow{\bar{\nabla}^{\phi}}_{\alpha} \overrightarrow{\bar{\nabla}^{\phi}}_{\beta}\right) g \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (\underbrace{\bar{\nabla}^{\phi} \cdots \bar{\nabla}^{\phi}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla}^{\phi} \cdots \bar{\nabla}^{\phi}}_{k} g)_{\nu_1 \ldots \nu_k} \end{aligned}} \tag{6.1.89}$$

being a covariant representation of the product (6.1.28)

$$\begin{aligned} f \star g &= f \exp\left(\tfrac{1}{2} i\hbar \pi^{\alpha\beta} \overleftarrow{D}_{\alpha} \overrightarrow{D}_{\beta}\right) g \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (D_{\xi^{\mu_1}} \cdots D_{\xi^{\mu_k}} f)(D_{\xi^{\nu_1}} \cdots D_{\xi^{\nu_k}} g) \end{aligned} \tag{6.1.90}$$

where $D_\alpha \equiv D_{\xi^\alpha}$ are given by (6.1.65). Thus, we have constructed a family of star-products (6.1.89), (6.1.90) induced by a family of flat connections on the base space Q and hence by a family of flat symplectic connections on T^*Q.

It is also possible to distinguish covariant star-products on more general symplectic manifolds. Let Q be an n-dimensional flat Riemannian manifold, and let us take as a symplectic manifold M the cotangent bundle to Q, $M = T^*Q$. According to (6.1.74), we can lift a flat Levi-Civita connection ∇ on Q to a flat torsionless symplectic connection $\bar{\nabla}$ on M. In analogy to (6.1.73) we can define a local canonical star-product on M by the formula

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k}. \tag{6.1.91}$$

On can prove that the star-product (6.1.91) is associative (see [12]), so it is a proper star-product on M. What is interesting, in a case of a non-flat connection $\bar{\nabla}$ the star-product (6.1.91) in general fails to be associative and thus has to be modified (deformed).

The star-product (6.1.91) can be written in a different, coordinate free form. Let $\overline{\exp}\colon TM \to M$ be an exponential map of the connection $\bar{\nabla}$. For every $\xi \in M$ there exists a neighborhood $U \subset M$ of ξ on which $\overline{\exp}_\xi$ is a diffeomorphism of an open subset V of the tangent space $T_\xi M$ onto U. Diffeomorphism $\overline{\exp}_\xi$ can be used to locally represent each function $f \in C^\infty(M)$ as a smooth function defined on the vector space $T_\xi M$. On each vector space there exists a canonical star-product, namely the Moyal product $\star_M$, thus it is natural to define on M a star-product by the following formula

$$(f \star g)(\xi) = (\overline{\exp}_\xi^* f \star \overline{\exp}_\xi^* g)(0), \tag{6.1.92}$$

where the pull-back $\overline{\exp}_\xi^* f = f \circ \overline{\exp}_\xi$. Using the formula

$$\frac{\partial^k}{\partial y^{\mu_1} \cdots \partial y^{\mu_k}} f(\overline{\exp}_\xi(y))\Big|_{y=0} = (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k}(\xi) \tag{6.1.93}$$

one can see that the star-product (6.1.92) is equal to (6.1.91).

In what follows we are mainly interested in certain manifolds Q for which the star-product (6.1.91) can be written in an integral form. For this purpose a Riemannian manifold (Q, g) will be called *almost geodesically simply connected* if for every $x \in Q$ there exists a neighborhood $U \subset Q$ of x such that $Q \setminus U$ is of measure zero with respect to the measure induced by the metric volume form ω_g, and every point in U can be connected with x by a unique geodesic. By analogy we define the notion of an almost geodesically simply connected symplectic manifold (M, ω) equipped with a torsionless symplectic connection. In that case in

the definition we replace the metric volume form ω_g by Liouville volume form Ω_ω. If Q is almost geodesically simply connected then T^*Q has the same property. The simplest nontrivial example of almost geodesically simply connected Riemannian manifold is the sphere.

If $M = T^*Q$ is almost geodesically simply connected then for every $\xi \in M$ exists a neighborhood $U \subset M$ such that $M \setminus U$ is of measure zero and $\overline{\exp}_\xi$ is a diffeomorphism of an open subset $V \subset T_\xi M$ onto U. If $f \in C_0^\infty(M)$ is a smooth function with the compact support then $\overline{\exp}_\xi^* f \in C_0^\infty(V)$ has the same property. The function $\overline{\exp}_\xi^* f$ can be uniquely extended to a smooth function on the whole tangent space $T_\xi M$ with the same support as $\overline{\exp}_\xi^* f$, just by putting the function $\overline{\exp}_\xi^* f$ equal 0 outside V. Thus according to (6.1.92) and the known integral form of the Moyal product (6.1.44) it follows that for $f, g \in C_0^\infty(M)$ the $\star$-product can be written in the following integral form

$$\boxed{(f \star g)(\xi) = \frac{1}{(\pi\hbar)^{2N}} \int_{T_\xi M} \int_{T_\xi M} f(\overline{\exp}_\xi(u)) g(\overline{\exp}_\xi(v)) e^{-\frac{2i}{\hbar}\omega_\xi(u,v)}\, du\, dv.} \tag{6.1.94}$$

Note that the crucial assumption that $M \setminus U$ is of measure zero guaranties that the above integral form of the $\star$-product indeed expands to the series (6.1.91).

An important property of the star-product (6.1.91) is that for a given classical and quantum canonical coordinate system (x, p) it is equivalent with the Moyal product associated with the same coordinates (x, p). The morphism S is constructed according to general formulas (6.1.81) and (6.1.83). For a star-product (6.1.91) the calculation shows that the operators $C_k(\xi^\alpha, \cdot)$ take the form

$$C_k(x^j, \cdot) = \frac{1}{k!}\left(\frac{i}{2}\right)^k (\underbrace{\nabla \cdots \nabla}_{k} x^j)_{j_1 \ldots j_k} \partial_{p_{j_1}} \cdots \partial_{p_{j_k}}, \tag{6.1.95a}$$

$$\begin{aligned} C_{k+1}(p_j, \cdot) = {} & \frac{1}{(k+1)!}\left(\frac{i}{2}\right)^{k+1} \left(\gamma^r_{jj_1\ldots j_{k+1}} - (k+1)\gamma^l_{j(j_1\ldots j_k}\Gamma^r_{j_{k+1})l}\right) p_r \partial_{p_{j_1}} \cdots \partial_{p_{j_{k+1}}} \\ & - \frac{1}{k!}\left(\frac{i}{2}\right)^{k+1} \gamma^l_{jj_1\ldots j_k} \partial_{q^l} \partial_{p_{j_1}} \cdots \partial_{p_{j_k}} \\ & - \frac{1}{(k-1)!}\left(\frac{i}{2}\right)^{k+1} \gamma^r_{jl(j_1\ldots j_{k-1}}\Gamma^l_{j_k)r} \partial_{p_{j_1}} \cdots \partial_{p_{j_k}}, \end{aligned} \tag{6.1.95b}$$

where functions $\gamma^l_{jj_1\ldots j_k}$ are given recursively by

$$\begin{aligned} \gamma^l_{jj_1\ldots j_{k+1}} &= \gamma^l_{j(j_1\ldots j_k, j_{k+1})} + \gamma^r_{j(j_1\ldots j_k}\Gamma^l_{j_{k+1})r} - k\gamma^l_{jr(j_1\ldots j_{k-1}}\Gamma^r_{j_k j_{k+1})}, \\ \gamma^l_{jj_1} &= \Gamma^l_{jj_1}. \end{aligned} \tag{6.1.96}$$

The round bracket () means as usually the symmetrization with respect to a group of indices. Indeed, using (6.1.74) one receives that

$$(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k}x^j)_{j_1\ldots j_k} = (\underbrace{\nabla\cdots\nabla}_{k}x^j)_{j_1\ldots j_k}, \tag{6.1.97a}$$

$$(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k+1}p_j)_{j_1\ldots j_{k+1}} = \left(\gamma^r_{jj_1\ldots j_{k+1}} - (k+1)\gamma^l_{j(j_1\ldots j_k}\Gamma^r_{j_{k+1})l}\right)p_r, \tag{6.1.97b}$$

$$(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k+1}p_j)_{\bar{l}j_1\ldots j_k} = \gamma^l_{jj_1\ldots j_k}, \tag{6.1.97c}$$

where remaining terms are equal zero, and

$$(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k}f)_{\bar{j}_1\ldots\bar{j}_k} = \partial_{p_{j_1}}\cdots\partial_{p_{j_k}}f, \tag{6.1.98a}$$

$$(\underbrace{\tilde{\nabla}\cdots\tilde{\nabla}}_{k+1}f)_{l\bar{j}_1\ldots\bar{j}_k} = \partial_{x^l}\partial_{p_{j_1}}\cdots\partial_{p_{j_k}}g + \Gamma^{j_1}_{lj}\partial_{p_j}\cdots\partial_{p_{j_k}}f + \cdots + \Gamma^{j_k}_{lj}\partial_{p_{j_1}}\cdots\partial_{p_j}f. \tag{6.1.98b}$$

From (6.1.97) and (6.1.98) one receives (6.1.95).

Let us calculate the second order terms in the $\hbar$ expansion of the morphism S. As $F_2^\alpha = C_2(\xi^\alpha,\cdot)$ and

$$\begin{aligned}
C_2(x^i,\cdot) &= \tfrac{1}{8}\Gamma^i_{kj}(x)\partial_{p_k}\partial_{p_j},\\
C_2(p_i,\cdot) &= \tfrac{1}{4}\Gamma^m_{li}(x)\Gamma^l_{mk}(x)\partial_{p_k} + \tfrac{1}{4}\Gamma^k_{il}(x)\partial_{x^k}\partial_{p_l}\\
&\quad + \tfrac{1}{8}[\Gamma^k_{rl}(x)\Gamma^r_{mi}(x) + \Gamma^k_{ri}(x)\Gamma^r_{ml}(x) - \Gamma^k_{mi,l}(x)]p_k\partial_{p_m}\partial_{p_l}
\end{aligned}$$

hence, from (6.1.81) and (6.1.83) we find that

$$\begin{aligned}
S_2 &= C_2(\xi^\gamma,\cdot)\partial_{\xi^\gamma} + \tfrac{1}{2}\left[\xi^\alpha, C_2(\xi^\gamma,\cdot)\right]\partial_{\xi^\alpha}\partial_{\xi^\gamma} + \tfrac{1}{6}\left[\xi^\alpha,\left[\xi^\beta, C_2(\xi^\gamma,\cdot)\right]\right]\partial_{\xi^\alpha}\partial_{\xi^\beta}\partial_{\xi^\gamma}\\
&= \tfrac{1}{8}\Gamma^i_{jk}\partial_{x^i}\partial_{p_j}\partial_{p_k} + \tfrac{1}{8}\Gamma^i_{lj}\Gamma^l_{ik}\partial_{p_j}\partial_{p_k} + \frac{1}{24}\left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right)p_i\partial_{p_j}\partial_{p_k}\partial_{p_l}
\end{aligned}$$

(confront with (6.1.66)). The complexity of terms S_k grow rapidly for $k > 2$. The reader can find the complicated explicit form of S_4, calculated using a computer algebra program, in [99].

Now, let us describe a procedure of constructing star-products on a symplectic manifold $M = T^*Q$ over a non-flat almost geodesically simply connected pseudo-Riemannian manifold (Q, g). In such a case, in analogy to the flat case, we will use a symplectic connection $\bar{\nabla}$ on T^*Q induced from a Levi-Civita connection ∇ on Q, in order to derive a star-product. However, for a curved linear connection $\bar{\nabla}$ a star-product in the form (6.1.91) is not a proper star-product as it is not associative

and (6.1.93) is not valid any more. In consequence we have to deform the star-product (6.1.91) in such a way that for a curved linear connection $\bar{\nabla}$ it would remain associative. In addition, we would like the new star-product to be equivalent with the Moyal product for every classical and quantum canonical coordinate system.

In what follows, any admissible star-product will be constructed from an appropriate morphism S acting on the Moyal product (see Observation 13). We will present the construction up to the third order in $\hbar$. Let us take the admissible one parameter family of morphisms S_a which in the flat case coincide with the known formula (6.1.75)

$$\boxed{S_a = \mathrm{id} + \hbar^2 \left(-\frac{1}{24} \bar{\Gamma}_{\alpha\beta\gamma} \partial^\alpha \partial^\beta \partial^\gamma + \tfrac{1}{16} (\bar{\Gamma}^\mu_{\nu\alpha} \bar{\Gamma}^\nu_{\mu\beta} + a \bar{R}_{\alpha\beta}) \partial^\alpha \partial^\beta \right) + O(\hbar^4),} \tag{6.1.99}$$

where a is a real parameter, $\bar{R}_{\alpha\beta}$ is the Ricci curvature tensor (2.7.9) and the components of the non-flat symplectic connection in the Darboux coordinate frame are given by (2.7.6). Then, after some cumbersome calculations, we will receive the one-parameter family of star-products in the form

$$\boxed{f \star_a g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2} \right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} \Big((\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} f)_{\mu_1 \ldots \mu_k} (\underbrace{\bar{\nabla} \cdots \bar{\nabla}}_{k} g)_{\nu_1 \ldots \nu_k} + B_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k}(f, g) \Big),} \tag{6.1.100}$$

equivalent with the Moyal product, up to third order in $\hbar$, where $B_{\mu_1 \ldots \mu_k \nu_1 \ldots \nu_k}$ are bilinear operators given by [37]

$$\begin{aligned}
B_0(f, g) &= 0, \\
B_{\mu_1\nu_1}(f, g) &= 0, \\
B_{\mu_1\mu_2\nu_1\nu_2}(f, g) &= -3a \bar{R}_{\mu_1\mu_2} (\bar{\nabla}_{\nu_1} f)(\bar{\nabla}_{\nu_2} g), \\
B_{\mu_1\mu_2\mu_3\nu_1\nu_2\nu_3}(f, g) &= -\bar{R}_{\nu_1\nu_2\nu_3\alpha} \pi^{\alpha\beta} (\bar{\nabla}\bar{\nabla}\bar{\nabla} f)_{\mu_1\mu_2\mu_3} (\bar{\nabla}_\beta g) \\
&\quad - \bar{R}_{\mu_1\mu_2\mu_3\alpha} \pi^{\alpha\beta} (\bar{\nabla}_\beta f)(\bar{\nabla}\bar{\nabla}\bar{\nabla} g)_{\nu_1\nu_2\nu_3} \\
&\quad - \tfrac{9}{2} a \bar{R}_{\mu_1\mu_2;\mu_3} (\bar{\nabla}_{\nu_3} f)(\bar{\nabla}\bar{\nabla} g)_{\nu_1\nu_2} \\
&\quad + \tfrac{9}{2} a \bar{R}_{\mu_1\mu_2;\mu_3} (\bar{\nabla}\bar{\nabla} f)_{\nu_1\nu_2} (\bar{\nabla}_{\nu_3} g) \\
&\quad + 9a \bar{R}_{\mu_2\nu_3} (\bar{\nabla}\bar{\nabla} f)_{\mu_1\mu_3} (\bar{\nabla}\bar{\nabla} g)_{\nu_1\nu_2} \\
&\quad + \bar{R}_{\mu_1\mu_2\mu_3\alpha} \bar{R}_{\nu_1\nu_2\nu_3\gamma} \pi^{\alpha\beta} \pi^{\gamma\delta} (\bar{\nabla}_\beta f)(\bar{\nabla}_\delta g),
\end{aligned}$$

where $\bar{R}_{\alpha\beta\gamma\delta} = \omega_{\alpha\lambda}\bar{R}^{\lambda}_{\beta\gamma\delta}$ is the curvature tensor. Clearly for the flat linear connection $\bar{\nabla}$ the product (6.1.100) reduces to (6.1.91) ones.

In a special case $a = 0$ the star-product (6.1.100) reduces to

$$f \star g = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{i\hbar}{2}\right)^k \pi^{\mu_1\nu_1} \cdots \pi^{\mu_k\nu_k} (D_{\mu_1\ldots\mu_k} f)(D_{\nu_1\ldots\nu_k} g), \tag{6.1.101}$$

where $D_{\mu_1\ldots\mu_k}$ are linear operators mapping functions to k-times covariant tensor fields according to the formulas

$$D_0 f = f, \tag{6.1.102a}$$

$$D_{\mu_1} f = \bar{\nabla}_{\mu_1} f, \tag{6.1.102b}$$

$$D_{\mu_1\mu_2} f = (\bar{\nabla}\bar{\nabla} f)_{\mu_1\mu_2}, \tag{6.1.102c}$$

$$D_{\mu_1\mu_2\mu_3} f = (\bar{\nabla}\bar{\nabla}\bar{\nabla} f)_{\mu_1\mu_2\mu_3} - \bar{R}_{\mu_1\mu_2\mu_3\alpha}\pi^{\alpha\beta}\bar{\nabla}_\beta f. \tag{6.1.102d}$$

A direct calculation, with the help of the Ricci identity

$$\bar{R}_{\alpha\beta\gamma\delta} + \bar{R}_{\alpha\gamma\delta\beta} + \bar{R}_{\alpha\delta\beta\gamma} = 0,$$

shows that operators (6.1.102) are symmetric with respect to indices $\mu_1, \mu_2, \ldots$. The reader can verify that the star-product (6.1.101), up to at least third order in $\hbar$, is a Fedosov star-product associated with the Weyl curvature form $\Omega = \omega$ [118]. It should be also noted that for $a \neq 0$ the star-product (6.1.100) is not a Fedosov star-product.

From the presented construction it is clear that when the configuration space Q is curved there is no single natural star-product on T^*Q but rather the whole family of natural star-products. In the considered case (see formula (6.1.99)) the natural star-products are parametrized by a real number a. Notice that also the Fedosov construction of star-products has freedom in taking different Weyl curvature forms Ω [118].

Let us extend the introduced family of star-products on $M = T^*Q$ in the way which will be important for the formalism developed in the next chapter. Using (2.7.6) and (2.7.9) the formula (6.1.99) can be rewritten in the Darboux chart

$$\begin{aligned} S = \mathrm{id} + \frac{\hbar^2}{4!}\Big[& 3\left(\Gamma^i_{lj}\Gamma^l_{ik} + aR_{jk}\right)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{x^i}\partial_{p_j}\partial_{p_k} \\ & + \left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right) p_i \partial_{p_j}\partial_{p_k}\partial_{p_l}\Big] + O(\hbar^4). \end{aligned} \tag{6.1.103}$$

Now let us generalize the formula (6.1.103) in the following way

$$\begin{aligned} S = \mathrm{id} + \frac{\hbar^2}{4!}\Big[& 3\left(\Gamma^i_{lj}\Gamma^l_{ik} + aR_{jk}\right)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{x^i}\partial_{p_j}\partial_{p_k} \\ & + \left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right) p_i\partial_{p_j}\partial_{p_k}\partial_{p_l} \\ & - 3b\partial_{p_j}(\partial_{x^j} + \Gamma^i_{jl}p_i\partial_{p_l})\partial_{p_k}(\partial_{x^k} + \Gamma^r_{kn}p_r\partial_{p_n})\Big] + O(\hbar^4), \end{aligned} \tag{6.1.104}$$

where b is a real parameter. This two-parameter family of morphisms will be crucial for quantizations considered in the next chapter. For a symplectic manifold T^*E^n and Cartesian coordinates (x^i, p_j) all Christoffel symbols $\Gamma^i_{jk} = 0$ and the morphism S (6.1.104) take the form

$$S = \mathrm{id} - \frac{\hbar^2}{8}b\partial_{x^j}\partial_{p_j}\partial_{x^k}\partial_{p_k} + O(\hbar^4),$$

and can be regarded as the expansion of the following morphism

$$S = \exp\left(-\frac{\hbar^2}{8}b\partial_{x^j}\partial_{p_j}\partial_{x^k}\partial_{p_k}\right). \tag{6.1.105}$$

The morphism S (6.1.105) induces a star-product which takes the form

$$\begin{aligned} f\star g = f\exp\Big(& \tfrac{1}{2}i\hbar\left(\overleftarrow{\partial}_{x^k}\overrightarrow{\partial}_{p_k} - \overleftarrow{\partial}_{p_k}\overrightarrow{\partial}_{x^k}\right) \\ & + \tfrac{1}{8}b\hbar^2(\overleftarrow{\partial}_{x^k}\overleftarrow{\partial}_{p_k}\overleftarrow{\partial}_{x^j}\overleftarrow{\partial}_{p_j} + \overrightarrow{\partial}_{x^k}\overrightarrow{\partial}_{p_k}\overrightarrow{\partial}_{x^j}\overrightarrow{\partial}_{p_j}) \\ & - \tfrac{1}{8}b\hbar^2(\overleftarrow{\partial}_{x^k} + \overrightarrow{\partial}_{x^k})(\overleftarrow{\partial}_{p_k} + \overrightarrow{\partial}_{p_k})(\overleftarrow{\partial}_{x^j} + \overrightarrow{\partial}_{x^j})(\overleftarrow{\partial}_{p_j} + \overrightarrow{\partial}_{p_j})\Big)g \end{aligned}$$

considered in Sect. 6.1.3. In general, the star-product induced by the morphism S (6.1.104) for $a = 1$ and $b = 1$ leads to what will be called in Sect. 8.1.6 a "minimal" quantization. Moreover, the same quantization will be used in Sect. 8.2 in order to investigate the quantum integrability and quantum separability of classical Stäckel systems.

Observation 13 *As long as a symplectic manifold T^*Q has the flat base manifold Q, a lot of star-products have a compact explicit representation. However, it is lost in the case of a non-flat base. Fortunately, such explicit representations are not necessary in many physically interesting cases. Observe that for the quantum theory, once we chose a local coordinate system, all admissible quantizations (star-products) should be equivalent with the one which has the Moyal representation in*

these coordinates. Thus, they are uniquely specified by an appropriate morphism S. Obviously, for quantization of classical Hamiltonian, represented by an arbitrary smooth function on T^*Q, a complete S in explicit form is necessary. But, in the case of "physical" Hamiltonians, represented by functions polynomial in momenta, a few first terms in $\hbar$ expansion of S are sufficient. For example, for a two-parameter family of quantizations defined by (6.1.104), the knowledge of S_2 is sufficient for quantizations of Hamiltonians which are linear, quadratic and cubic in momenta.

6.2 Operator Representation of Star-Algebras and Related Orderings

In the previous section we constructed a broad class of quantum Poisson algebras $\mathcal{A}_Q = (C^\infty(M)[[\hbar]], \star)$. Observables were represented by particular elements of A_Q, self-adjoint with respect to involution. In the following section we construct an operator representation of the algebra $\mathcal{A}_Q$ in the Hilbert space $L^2(M, d\Omega_\hbar)$. Actually, with any element of $C^\infty(M)[[\hbar]]$ we associate an operator

$$A \mapsto \hat{A} \equiv A\star \tag{6.2.1}$$

defined on the Hilbert space $L^2(M, d\Omega_\hbar)$. We construct both, integral and differential representations of any operator (6.2.1) for the class of quantum algebras considered in the previous section. We prove that for each star-algebra $\mathcal{A}_Q$ any operator $\hat{A}$ is represented by appropriately ordered operators of position and momenta

$$\boxed{\hat{A} = A\star = A(x\star, p\star) = A(\hat{q}, \hat{p}).}$$

Thus, with each quantization we relate some ordering.

6.2.1 Weyl Ordering for Moyal Product

Let us take as a phase space M the symplectic vector space $(\mathbb{R}^{2n}, \omega)$, where ω is a symplectic form. Moreover, let us consider on M a star-product which in canonical coordinates $(\xi^1, \ldots, \xi^{2n})$ takes the form of Moyal product (6.1.33). For any element of $C^\infty(\mathbb{R}^{2n})[[\hbar]]$ we can associate the operator defined on the Hilbert space $L^2(\mathbb{R}^{2n}, d\Omega_\hbar)$ by the prescription

$$A \mapsto A \star_M . \tag{6.2.2}$$

Formula (6.2.2) gives us a representation of the algebra $\mathcal{A}_Q = (C^\infty(\mathbb{R}^{2n})[\![\hbar]\!], \star_M)$ in the Hilbert space $L^2(\mathbb{R}^{2n}, d\Omega_\hbar)$. In what follows we will derive the form of the operators $A \star_M$.

Let A be an element of the space $\mathcal{S}(\mathbb{R}^{2n})$ of Schwartz functions. The operator $A \star_M$ can be written in the following integral form

$$\boxed{A \star_M = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(p', x') e^{\frac{i}{\hbar}\left[p'_j (\hat{q}_M)^j - x'^j (\hat{p}_M)_j\right]} dx'\, dp',} \tag{6.2.3}$$

where

$$\boxed{(\hat{q}_M)^j = x^i \star_M = x^i + \tfrac{1}{2} i\hbar \partial_{p_i}, \quad (\hat{p}_M)_j = p_i \star_M = p_i - \tfrac{1}{2} i\hbar \partial_{x^i}}$$

are operators of position and momenta, respectively and $\mathcal{F}_\omega$ is the symplectic Fourier transform (6.1.23). The formula (6.2.3) can be derived as follows. For $\rho \in L^2(\mathbb{R}^{2n})$, using the identity

$$e^{y^i \partial_{z^i}} \rho(z) = \rho(z + y), \quad (y^1, \ldots, y^{2n}) \in \mathbb{R}^{2n}$$

and the Baker-Campbell-Hausdorff formula

$$e^{\hat{a}+\hat{b}} = e^{\hat{a}} e^{\hat{b}} e^{-\frac{1}{2}[\hat{a},\hat{b}]}, \quad e^{\hat{a}} e^{\hat{b}} = e^{\hat{b}} e^{\hat{a}} e^{[\hat{a},\hat{b}]}, \quad [\hat{a}, \hat{b}] = const.,$$

we receive that

$$\begin{aligned}
e^{\frac{i}{\hbar}\left[p'_j (\hat{q}_M)^j - x'^j (\hat{p}_M)_j\right]} \rho(x, p) &= e^{-\frac{i}{2\hbar} x'^i p'_i} e^{\frac{i}{\hbar} p'_j (\hat{q}_M)^j} e^{-\frac{i}{\hbar} x'^j (\hat{p}_M)_j} \rho(x, p) \\
&= e^{-\frac{i}{2\hbar} x'^j p'_j} e^{\frac{i}{\hbar} p'_j (x^j + \frac{1}{2} i\hbar \partial_{p_j})} e^{-\frac{i}{\hbar} x'^j (p_j - \frac{1}{2} i\hbar \partial_{x^j})} \rho(x, p) \\
&= e^{-\frac{i}{2\hbar} x'^j p'_j} e^{\frac{i}{\hbar} p'_j x^j} e^{-\frac{1}{2} p'_j \partial_{p_j}} e^{-\frac{i}{\hbar} x'^j p_j} e^{-\frac{1}{2} q'^j \partial_{x^j}} \rho(x, p) \\
&= e^{\frac{i}{\hbar}(p'_j x^j - x'^j p_j)} e^{-\frac{1}{2} p'_j \partial_{p_j}} e^{-\frac{1}{2} x'^j \partial_{x^j}} \rho(x, p) \\
&= e^{\frac{i}{\hbar}(p'_j x^j - x'^j p_j)} \rho(x - \tfrac{1}{2} x', p - \tfrac{1}{2} p').
\end{aligned}$$

From the above result we find that

$$\begin{aligned}
&\left[\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(p', x') e^{\frac{i}{\hbar}\left[p'_j (\hat{q}_M)^j - x'^j (\hat{p}_M)_j\right]} dx'\, dp'\right] \rho(x, p) = \\
&= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(p', x') \rho(x - \tfrac{1}{2} x', p - \tfrac{1}{2} p') e^{\frac{i}{\hbar}(p'_j x^j - x'^j p_j)} dx'\, dp'
\end{aligned}$$

$$= \frac{1}{(2\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} A(x'', p'')\rho(x - \tfrac{1}{2}x', p - \tfrac{1}{2}p') \times e^{\frac{i}{\hbar}(p'_j(x^j - x''^j) - x'^j(p_j - p''_j))}\, dx'\, dp'\, dx''\, dp''.$$

After changing variables

$$x'j \to -2x'j, \quad x''j \to x^j + x''j,$$
$$p'_j \to -2p'_j, \quad p''_j \to p_j + p''_j,$$

the above equation can be written in a form

$$\left[\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(x', p') e^{\frac{i}{\hbar}\left[p'_j(\hat{q}_M)^j - x'^j(\hat{p}_M)_j\right]} dx'\, dp'\right] \rho(x, p)$$
$$= \frac{1}{(\pi\hbar)^{2n}} \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^{2n}} A(x + x'', p + p'')\rho(x + x', p + p') e^{-\frac{2i}{\hbar}(x'^j p''_j - p'_j x''^j)}\, dx'\, dp'\, dx''\, dp'',$$

which is the known integral form (6.1.45) of the Moyal star-product $A \star_M \rho$.

The formula (6.2.3) represents the Weyl ordering (symmetric ordering) of A as a function of quantum canonical operators $\hat{q}_M$, $\hat{p}_M$, as will be shown in a moment, so we will use the following notation

$$\boxed{A(q, p)\star_M := A_W(\hat{q}_M, \hat{p}_M).} \tag{6.2.4}$$

It was first proposed by Weyl [260] for the symmetric ordering and formally it works by substituting variables q^i, p_j with operators $\hat{q}^i$, $\hat{p}_j$ and appropriately ordering them. The symmetric-ordered function A of operators $\hat{q}^i$, $\hat{p}_j$ (6.2.3) will be shortly denoted by a formula

$$\boxed{A_W(\hat{q}, \hat{p}) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(p', x')\hat{T}(p', x')\, dx'\, dp',} \tag{6.2.5}$$

where

$$\boxed{\hat{T}(p', x') = e^{\frac{i}{\hbar}(p'_i\hat{q}^i - x'^i\hat{p}_i)} = e^{\frac{i}{\hbar}p'_i\hat{q}^i} e^{-\frac{i}{\hbar}x'^i\hat{p}_i} e^{-\frac{1}{2}\frac{i}{\hbar}x'^i p'_i}} \tag{6.2.6}$$

is a unitary operator for every $(p', x') \in \mathbb{R}^{2n}$.

These functions A which are polynomial in momenta are particulary interesting from the point of view of further applications. For $A(x,\,p) = K^{j_1\cdots j_m}(x)p_{j_1}\cdots p_{j_m}$, where $K^{j_1\cdots j_m}$ is a symmetric tensor field on $\mathbb{R}^n$, we get

$$\boxed{A_W(\hat{q},\,\hat{p}) = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\hat{p}_{j_1}\cdots\hat{p}_{j_k}K^{j_1\cdots j_m}(\hat{q})\hat{p}_{j_{k+1}}\cdots\hat{p}_{j_m}.} \tag{6.2.7}$$

Indeed, from (6.2.5) for $(p',\,x') = (\eta,\,\xi)$ we get

$$\begin{aligned}
A_W(\hat{q},\,\hat{p}) &= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} K^{j_1\cdots j_m}(x)p_{j_1}\cdots p_{j_n}e^{-\frac{i}{\hbar}(\eta_j x^j-\xi^j p_j)}\hat{T}(\eta,\,\xi)\,dx\,dp\,d\xi\,d\eta\\
&= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} K^{j_1\cdots j_m}(x)\left((-i\hbar)^m\partial_{\xi^{j_1}}\cdots\partial_{\xi^{j_m}}e^{-\frac{i}{\hbar}(\eta_j x^j-\xi^j p_j)}\right)\\
&\qquad\qquad\times\hat{T}(\eta,\,\xi)\,dx\,dp\,d\xi\,d\eta\\
&= \frac{1}{(2\pi\hbar)^{2n}}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} K^{j_1\cdots j_m}(x)e^{-\frac{i}{\hbar}(\eta_j x^j-\xi^j p_j)}\left((i\hbar)^m\partial_{\xi^{j_1}}\cdots\partial_{\xi^{j_m}}\hat{T}(\eta,\,\xi)\right)\\
&\qquad\qquad\times\,dx\,dp\,d\xi\,d\eta.
\end{aligned} \tag{6.2.8}$$

Using the Baker-Campbell-Hausdorff formula the operator $\hat{T}(\eta,\,\xi)$ can be written in a form

$$\hat{T}(\eta,\,\xi) = e^{-\frac{1}{2}\frac{i}{\hbar}\xi^j\hat{p}_j}e^{\frac{i}{\hbar}\eta_j\hat{q}^j}e^{-\frac{1}{2}\frac{i}{\hbar}\xi^j\hat{p}_j}. \tag{6.2.9}$$

From (6.2.6) and the Leibniz's formula we get

$$\begin{aligned}
&(i\hbar)^m\partial_{\xi^{j_1}}\cdots\partial_{\xi^{j_m}}\hat{T}(\eta,\,\xi) = (i\hbar)^m\partial_{\xi^{j_1}}\cdots\partial_{\xi^{j_m}}e^{-\frac{1}{2}\frac{i}{\hbar}\xi^j\hat{p}_j}e^{\frac{i}{\hbar}\eta_j\hat{q}^j}e^{-\frac{1}{2}\frac{i}{\hbar}\xi^j\hat{p}_j}\\
&= \sum_{k=0}^{m}\binom{m}{k}\left(\frac{1}{2}\right)^k\hat{p}_{j_1}\cdots\hat{p}_{j_k}e^{-\frac{i}{\hbar}\frac{1}{2}\xi^j\hat{p}_j}e^{\frac{i}{\hbar}\eta_j\hat{q}^j}\left(\frac{1}{2}\right)^{m-k}\hat{p}_{j_{k+1}}\cdots\hat{p}_{j_m}e^{-\frac{i}{\hbar}\frac{1}{2}\xi^j\hat{p}_j}.
\end{aligned} \tag{6.2.10}$$

Substituting (6.2.10) into (6.2.8) and performing integration with the help of (6.1.23) and (6.1.23) we get the result. In particular, for $n = 1$, we get

$$\boxed{\left(f(\hat{q})\hat{p}^m\right)_W = \frac{1}{2^m}\sum_{k=0}^{m}\binom{m}{k}\hat{p}^k f(\hat{q})\hat{p}^{m-k},} \tag{6.2.11}$$

thus

$$
\begin{aligned}
(f\hat{p})_W &= \tfrac{1}{2}(f\hat{p}+\hat{p}f),\\
(f\hat{p}^2)_W &= \tfrac{1}{4}(f\hat{p}^2+\hat{p}^2f)+\tfrac{1}{2}\hat{p}f\hat{p},\\
(f\hat{p}^3)_W &= \tfrac{1}{8}(f\hat{p}^3+\hat{p}^3f)+\frac{3}{8}(\hat{p}f\hat{p}^2+\hat{p}^2f\hat{p}),\\
(f\hat{p}^4)_W &= \tfrac{1}{16}(f\hat{p}^4+\hat{p}^4f)+\tfrac{1}{4}(\hat{p}f\hat{p}^3+\hat{p}^3f\hat{p})+\frac{3}{8}\hat{p}^2f\hat{p}^2,\\
&\vdots
\end{aligned}
$$

A general differential representation of (6.2.5) for any canonical operators $\hat{q}$, $\hat{p}$ takes the form

$$
\boxed{A_W(\hat{q},\hat{p}) = A(-i\hbar\partial_\eta, i\hbar\partial_\xi)\hat{T}(\eta,\xi)_{|\xi=\eta=0}} \tag{6.2.12}
$$

from which we get immediately (6.2.7) once we choose $\hat{T}(\eta,\xi)$ in representation (6.2.9). The proof of (6.2.12) is based on the following equality

$$
f(i\hbar\partial_x, -i\hbar\partial_p)g(x,p)_{|x=0} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2}\mathcal{F}_\omega f(x,p)g(x,p)dx\,dp
$$

for a pair of coordinates(x,p) and admissible functions f and g.

The adjoint $\dagger$ in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^{2n}, d\Omega_\hbar)$ is defined, according to (6.1.1), by

$$
\boxed{\langle\rho_1, A\star_M\rho_2\rangle := \left\langle (A\star_M)^\dagger\rho_1, \rho_2\right\rangle,}
$$

where

$$
\langle\rho_1, A\star_M\rho_2\rangle = \int_M \bar{\rho}_1\,(A\star_M\rho_2)\,d\Omega_\hbar,
$$

thus we get

$$
(A\star_M)^\dagger = \bar{A}\star_M = \bar{A}_W(\hat{q},\hat{p}).
$$

6.2.2 General Theory of Orderings

Let S be an isomorphism of quantum algebras $(\mathcal{A}_Q, \star_M)$ and $(\mathcal{A}_Q, \star)$ on $(\mathbb{R}^{2N}, \omega)$, given by (6.1.77), which does not change the value of integral

$$\int_{\mathbb{R}^{2n}} SA d\Omega_\hbar = \int_{\mathbb{R}^{2n}} A d\Omega_\hbar \tag{6.2.13}$$

for any integrable function $A \in \mathcal{A}_Q$. Such types of isomorphisms are generated, for example, by coordinate transformations which are simultaneously classical and quantum canonical (see the previous subsection). Isomorphism S defines a new $\star_S$-product

$$f \star_S g = S(S^{-1} f \star_M S^{-1} g), \quad f, g \in \mathcal{A}_Q.$$

Involution in $\star_S$-algebra is defined by (6.1.38). Besides, let $f, g \in \mathcal{A}_Q$ be such that $f \star_S g$ and $g \star_S f$ are integrable functions. Then, from (6.1.6) for Moyal product and (6.2.13) follows that

$$\int_{\mathbb{R}^{2n}} (f \star_S g)\, d\Omega_\hbar = \int_{\mathbb{R}^{2n}} (g \star_S f)\, d\Omega_\hbar.$$

We demonstrate now how to construct an operator representation of $\star_S$-algebra [33]. Let us define an S-order of operator function through the relation

$$\boxed{A(q, p) \star_S = A_S(\hat{q}, \hat{p}) := (S^{-1}A)_W(\hat{q}, \hat{p}),} \tag{6.2.14}$$

where $[\hat{q}^k, \hat{p}_j] = i\hbar\delta^k_j$. The conjugation of such an ordering is of the form

$$\begin{aligned} \left[A_S(\hat{q}, \hat{p})\right]^\dagger &= \left[(S^{-1}A)_W(\hat{q}, \hat{p})\right]^\dagger \\ &= (\overline{S^{-1}A})_W(\hat{q}, \hat{p}) \overset{(6.1.38)}{=} (S^{-1}A^*)_W(\hat{q}, \hat{p}) \\ &= \left(A^*\right)_S(\hat{q}, \hat{p}), \end{aligned}$$

where $*$ is a new involution (6.1.38) of $\star_S$-algebra.

Let us introduce new operators

$$(\hat{q}_S)^j := x^j \star_S, \quad (\hat{p}_S)_j := p_j \star_S,$$

then, according to (6.1.56a) and (6.1.56b), we get

$$\hat{q}_S = S\hat{q}_M S^{-1}, \quad \hat{p}_S = S\hat{p}_M S^{-1}. \tag{6.2.15}$$

Obviously, from (6.2.15) follows that

$$[(\hat{q}_S)^k, (\hat{p}_S)_j] = i\hbar\delta^k_j.$$

Now, for any $A \in \mathcal{A}_Q$ we have

$$\boxed{\begin{aligned} A\star_S &= A_S(\hat{q}_S, \hat{p}_S) = (S^{-1}A)_W(\hat{q}_S, \hat{p}_S) \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega(S^{-1}A)(\eta,\xi)\exp\frac{i}{\hbar}[\eta_j(\hat{q}_S)^j - \xi^j(\hat{p}_S)_j]d\xi\, d\eta \end{aligned}} \tag{6.2.16}$$

as

$$\begin{aligned} A\star_S &= SS^{-1}(A\star_S) = S(S^{-1}A)_W(\hat{q},\hat{p})S^{-1} \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega(S^{-1}A)(\eta,\xi)Se^{\frac{i}{\hbar}[\eta_j(\hat{q}_M)^j - \xi^j(\hat{p}_M)_j]}\, S^{-1}d\xi\, d\eta \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega(S^{-1}A)(\eta,\xi)e^{\frac{i}{\hbar}[\eta_j S(\hat{q}_M)^j S^{-1} - \xi^j S(\hat{p}_M)_j S^{-1}]}d\xi\, d\eta \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega(S^{-1}A)(\eta,\xi)\exp\frac{i}{\hbar}[\eta_j(\hat{q}_S)^j - \xi^j(\hat{p}_S)_j]d\xi\, d\eta \\ &= A_S(\hat{q}_S,\hat{p}_S). \end{aligned}$$

Moreover, the related differential representation of (6.2.16), for any canonical operators $\hat{q}$, $\hat{p}$, takes the form

$$\boxed{A_S(\hat{q},\hat{p}) = \left(S^{-1}A\right)(-i\hbar\partial_\eta, i\hbar\partial_\xi)\hat{T}(\eta,\xi)_{|\xi=\eta=0}} \tag{6.2.17}$$

where $\hat{T}(\xi,\eta)$ is given by (6.2.6) or equivalently by (6.2.9).

The S-ordering rule (6.2.16) is very general and contains as special cases all ordering rules found in the literature. In particular, for a class of S for which

$$S^{-1} = F(i\hbar\partial_p, -i\hbar\partial_x), \tag{6.2.18}$$

where $F: \mathbb{R}^{2n} \to \mathbb{C}$ is a general analytic function such that $F(0) = 1$, the S-ordered function of operators $\hat{q}^i$, $\hat{p}_j$ can be presented alternatively by the formula

$$\begin{aligned} A_S(\hat{q},\hat{p}) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega\left(S^{-1}A\right)(\eta,\xi)e^{\frac{i}{\hbar}(\eta_j\hat{q}^j - \xi^j\hat{p}_j)}\, d\xi\, d\eta \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega\left(FA\right)(\eta,\xi)e^{\frac{i}{\hbar}(\eta_j\hat{q}^j - \xi^j\hat{p}_j)}\, d\xi\, d\eta \end{aligned}$$

$$= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(\eta,\xi) F(\eta,\xi) e^{\frac{i}{\hbar}(\eta_j \hat{q}^j - \xi^j \hat{p}_j)}\, d\xi\, d\eta \tag{6.2.19}$$

$$= \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(\eta,\xi) \hat{T}_F(\eta,\xi)\, d\xi\, d\eta. \tag{6.2.20}$$

The above formula was first considered by Cohen [69]. Thus, it is clear that the very broad family of orderings considered in [69] is a special case of the introduced family of orderings (6.2.16). In this case, alternative to (6.2.17) differential representation of (6.2.20) for any canonical operators $\hat{q}$, $\hat{p}$ takes the form

$$A_S(\hat{q},\hat{p}) = A(-i\hbar\partial_\eta, i\hbar\partial_\xi)\hat{T}_F(\eta,\xi)_{|\xi=\eta=0}, \quad \hat{T}_F(\eta,\xi) = \hat{T}(\eta,\xi)F(\eta,\xi)\,. \tag{6.2.21}$$

Now, let us illustrate various admissible orderings by a few particular cases. As the first case let us consider $\star_{\sigma,\alpha,\beta}$-algebra (6.1.3) on the symplectic manifold $M = (\mathbb{R}^{2n}, \omega)$ in natural coordinates (x, p)

$$\boxed{\begin{aligned} f \star_{\sigma,\alpha,\beta} g = & f \exp\Big(i\hbar\left(\tfrac{1}{2}-\sigma\right) \overleftarrow{\partial}_{x^j} \overrightarrow{\partial}_{p_j} - i\hbar\left(\tfrac{1}{2}+\sigma\right) \overleftarrow{\partial}_{p^j} \overrightarrow{\partial}_{x_j} \\ & + \hbar\alpha \overleftarrow{\partial}_{x^j} \overrightarrow{\partial}_{x^j} + \hbar\beta \overleftarrow{\partial}_{p_j} \overrightarrow{\partial}_{p_j} \Big) g, \end{aligned}} \tag{6.2.22}$$

related with $\star_M$-product by isomorphism $S_{\sigma,\alpha,\beta}$ (6.1.37)

$$\boxed{S_{\sigma,\alpha,\beta} = \exp \sum_{j=1}^{n} (-i\hbar\sigma\partial_{x^j}\partial_{p_j} + \tfrac{1}{2}\hbar\alpha\partial_{x^j}^2 + \tfrac{1}{2}\hbar\beta\partial_{p_j}^2).} \tag{6.2.23}$$

Observables $A(\hbar)$ in $\star_M$-algebra are real functions on M $A(\hbar) = A_C$, i.e. functions self-adjoint with respect to complex conjugation, while in $\star_{\sigma,\alpha,\beta}$-algebra observables were chosen as the set of functions $A(\hbar) = W(\hbar)A_C$ (6.1.41) which are complex functions, self-adjoint with respect to involution (6.1.40)

$$\boxed{A^* = \exp \sum_{j=1}^{n} \left(-2i\hbar\sigma\partial_{x^j}\partial_{p_j}\right) \bar{A}.}$$

For such a choice the Moyal quantization and $\star_{\sigma,\alpha,\beta}$-quantization are nonequivalent quantizations. The choice of quantum observables in the form $A(\hbar) = S_{\sigma,\alpha,\beta}(\hbar)A_C$ makes them isomorphic.

Besides

$$\left(\hat{q}_S\right)^j = x^j \star_{\sigma,\alpha,\beta} = S_{\sigma,\alpha,\beta}\left(\hat{q}_M\right)^j S^{-1}_{\sigma,\alpha,\beta} = x^j + i\hbar(\tfrac{1}{2}-\sigma)\partial_{p_j} + \hbar\alpha\partial_{x^j}, \tag{6.2.24}$$

$$\left(\hat{p}_S\right)_j = p_j \star_{\sigma,\alpha,\beta} = S_{\sigma,\alpha,\beta}\left(\hat{p}_M\right)_j S^{-1}_{\sigma,\alpha,\beta} = p_j - i\hbar(\tfrac{1}{2}+\sigma)\partial_{x^j} + \hbar\beta\partial_{p_j} \tag{6.2.25}$$

and

$$[\left(\hat{q}_S\right)^j, \left(\hat{p}_S\right)_k] = i\hbar\delta^j_k.$$

$S_{\sigma,\alpha,\beta}$-ordering is expressible by the Cohen formula

$$\boxed{\begin{aligned} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) =& \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} \mathcal{F}_\omega A(\eta,\xi)\exp\frac{i}{\hbar}\left(\eta_j\hat{q}^j - \xi^j\hat{p}_j\right) \\ &\times\exp\left(\frac{i}{\hbar}\sigma\xi^j\eta_j + \frac{1}{2}\frac{\alpha}{\hbar}\sum_{j=1}^n\eta_j^2 + \frac{1}{2}\frac{\beta}{\hbar}\sum_{j=1}^n\xi^{j2}\right)d\xi\,d\eta. \end{aligned}} \tag{6.2.26}$$

The integral formula (6.2.16) gives us immediately the integral representation of a respective $\star_{\sigma,\alpha,\beta}$-product in the form

$$\boxed{\begin{aligned} &(f\star_{\sigma,\alpha,\beta}g)(x,p) = \\ &\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}}\mathcal{F}_\omega f(\eta,\xi)g(x-(\tfrac{1}{2}+\sigma)\xi + i\alpha\eta, p-(\tfrac{1}{2}-\sigma)\eta - i\beta\xi)e^{\frac{i}{\hbar}(\eta_j x^j - \xi^j p_j)}d\xi\,d\eta. \end{aligned}}$$

or equivalently, for $\sigma^2-\alpha\beta \neq \frac{1}{4}$, in the form

$$\boxed{\begin{aligned} &(f\star_{\sigma,\alpha,\beta}g)(x,p) = \\ &\frac{1}{(2\pi\hbar)^{2n}\left|\frac{1}{4}+\alpha\beta-\sigma^2\right|^n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^{2n}} f(x',p')g(x'',p'')K(x,p;x',p',x'',p'')dx'dp'dx''dp'', \end{aligned}}$$

with the kernel

$$\begin{aligned} K(x,p;x',p',x'',p'') = \exp\Bigg[&-\frac{i}{\hbar}\frac{1}{\left(\frac{1}{4}+\alpha\beta-\sigma^2\right)}\sum_{j=1}^n\Big[\Big((\tfrac{1}{2}+\sigma)(p''_j-p_j) - i\beta(x''^j - x^j)\Big) \\ &\times(x'^j - x^j) - \Big((\tfrac{1}{2}-\sigma)(x''^j-x^j) + i\alpha(p''_j-p_j)\Big)(p'_j-p_j)\Big]\Bigg]. \end{aligned}$$

The representation (6.2.2) follows from (6.2.2) after substitution

$$\mathcal{F}_\omega f(\eta,\xi)=\frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^{2n}} f(x',p')\exp\left[-\frac{i}{\hbar}\left(\eta_j x'^j-\xi^j p'_j\right)\right]dx'dp'$$

and change of coordinates from ξ^j,η_j to

$$x''^j=x^j-(\tfrac{1}{2}+\sigma)\xi^j+i\alpha\eta_j,\quad p''_j=p_j-(\tfrac{1}{2}-\sigma)\eta_j-i\beta\xi^j.$$

We have two admissible differential representations of $S_{\sigma,\alpha,\beta}$-ordering of operators (6.2.24) and (6.2.25), given by formulas (6.2.17) and (6.2.21), respectively. Let us consider more carefully the case $\alpha=\beta=0$ and in particular the differential formula (6.2.21)

$$A_\sigma(\hat q,\hat p)=A(-i\hbar\partial_\eta,i\hbar\partial_\xi)\hat T_\sigma(\eta,\xi)_{|\xi=\eta=0},\quad \hat T_\sigma(\eta,\xi)=\hat T(\eta,\xi)\exp\frac{i}{\hbar}\sigma\xi^j\eta_j\,.$$

Using the Baker-Campbell-Hausdorff formula the operator $\hat T_\sigma(\eta,\xi)$ can be written in a form

$$\hat T_\sigma(\eta,\xi)=e^{-\frac{i}{\hbar}(\frac{1}{2}-\sigma)\xi^j\hat p_j}e^{\frac{i}{\hbar}\eta_j\hat q^j}e^{-\frac{i}{\hbar}(\frac{1}{2}+\sigma)\xi^j\hat p_j}$$

and then, for $A(x,p)=K^{j_1\cdots j_m}(x)p_{j_1}\cdots p_{j_m}$, we get

$$\boxed{A_\sigma(\hat q,\hat p)=\sum_{k=0}^{m}\binom{m}{k}\left(\tfrac{1}{2}-\sigma\right)^k\left(\tfrac{1}{2}+\sigma\right)^{m-k}\hat p_{j_1}\cdots\hat p_{j_k}K^{j_1\cdots j_m}(\hat q)\hat p_{j_{k+1}}\cdots\hat p_{j_m}.}\tag{6.2.27}$$

The best known from the literature particular orderings (6.2.27) are symmetric (Weyl) ordering $W:\sigma=0$ (6.2.11), standard ordering $St:\sigma=\frac{1}{2}$ and anti-standard ordering $ASt:\sigma=-\frac{1}{2}$. For $n=1$ and $A=f(x)p^m$

$$A_{St}(\hat q,\hat p)=f(\hat q)\hat p^m,\qquad A_{ASt}(\hat q,\hat p)=\hat p^m f(\hat q).$$

Example 6.6 For the simplest monomial $A=qp$ and σ-ordering (6.2.27) we have

$$A_\sigma(\hat q,\hat p)=\left(\tfrac{1}{2}+\sigma\right)\hat q\hat p+\left(\tfrac{1}{2}-\sigma\right)\hat p\hat q.$$

Observe, that we can construct an equivalent formula for Weyl ordering (6.2.11) using isomorphism S of quantum algebras $(\mathcal{A}_Q,\star_{St})$ and $(\mathcal{A}_Q,\star_M)$ on $(\mathbb{R}^2,\omega)$,

$$S=\exp i\hbar\tfrac{1}{2}\partial_x\partial_p$$

and differential formula (6.2.14). Actually, for $A = f(x)p^m$

$$\boxed{\begin{aligned} A_W(\hat{q},\hat{p}) &= (S^{-1}A)_{St}(\hat{q},\hat{p}) = \sum_{k=0}^{m}\binom{m}{k}\frac{1}{2^k}(-i\hbar)^k\left(f^{(k)}(x)p^{m-k}\right)_{St} \\ &= \sum_{k=0}^{m}\binom{m}{k}\frac{1}{2^k}(-i\hbar)^k f^{(k)}(\hat{q})\hat{p}^{m-k}, \end{aligned}} \tag{6.2.28}$$

and one can prove the equivalence with (6.2.11) observing that $[\hat{p}, f(\hat{q})] = -i\hbar f'(\hat{q})$.

In the general case, when $\alpha \neq 0$ and $\beta \neq 0$, formula (6.2.17) is more suitable for the construction of the appropriate ordering.

Example 6.7 For $n = 1$ and $A = qp + \frac{1}{2}q^2 + \frac{1}{2}p^2$ we get for (σ,α,β)-ordering (6.2.17)

$$\begin{aligned} A_{\sigma,\alpha,\beta}(\hat{q},\hat{p}) &= (S^{-1}A)_W(\hat{q},\hat{p}) = \left(\tfrac{1}{2}\hat{q}^2 + \tfrac{1}{2}\hat{p}^2 + \hat{q}\hat{p} + i\hbar\sigma - \tfrac{1}{2}\hbar\alpha - \tfrac{1}{2}\hbar\beta\right)_W \\ &= \tfrac{1}{2}\hat{q}^2 + \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \sigma[\hat{q},\hat{p}] + \tfrac{1}{2}i\alpha[\hat{q},\hat{p}] + \tfrac{1}{2}i\beta[\hat{q},\hat{p}] \\ &= \tfrac{1}{2}\hat{q}^2 + \tfrac{1}{2}\hat{p}^2 + \left(\tfrac{1}{2} + \sigma + \tfrac{1}{2}i\alpha + \tfrac{1}{2}i\beta\right)\hat{q}\hat{p} + \left(\frac{1}{2} - \sigma - \tfrac{1}{2}i\alpha - \tfrac{1}{2}i\beta\right)\hat{p}\hat{q}. \end{aligned}$$

For $n = 1$, in holomorphic coordinates [136]

$$\boxed{a(q,p) = \frac{\omega q + ip}{\sqrt{2\omega}}, \quad \bar{a}(q,p) = \frac{\omega q - ip}{\sqrt{2\omega}}, \quad \omega > 0}$$

the $\star_{\sigma,\alpha,\beta}$-product takes the form

$$\begin{aligned} f \star_{\sigma,\alpha,\beta} g = f \exp\Bigg[&\tfrac{1}{2}\hbar\left(1 + \alpha\omega + \frac{\beta}{\omega}\right)\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}} - \tfrac{1}{2}\hbar\left(1 - \alpha\omega - \frac{\beta}{\omega}\right)\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a \\ &+\tfrac{1}{2}\hbar\left(2\sigma + \alpha\omega - \frac{\beta}{\omega}\right)\overleftarrow{\partial}_a\overrightarrow{\partial}_a - \tfrac{1}{2}\hbar\left(2\sigma - \alpha\omega + \frac{\beta}{\omega}\right)\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a\Bigg] g. \end{aligned}$$

Let us consider a particular case of the star-product (6.2.2), when $\sigma = 0$, $\alpha = -\frac{\lambda}{\omega}$ and $\beta = -\lambda\omega$

$$\boxed{f \star_\lambda g = f \exp\left[\hbar\left(\tfrac{1}{2} - \lambda\right)\overleftarrow{\partial}_a\overrightarrow{\partial}_{\bar{a}} - \hbar\left(\tfrac{1}{2} + \lambda\right)\overleftarrow{\partial}_{\bar{a}}\overrightarrow{\partial}_a\right] g.} \tag{6.2.29}$$

Moreover, the operators $f \star_\lambda$ can be written in a form

$$f \star_\lambda = f_\lambda(\hat{a}, \hat{a}^\dagger) = \frac{1}{\pi\hbar} \int_{\mathbb{R}^2} \mathcal{F}f(w, \bar{w}) e^{\hbar^{-1}(w\hat{a}^\dagger - \bar{w}\hat{a} + \lambda|w|^2)}\, d^2w,$$

where $\hat{a} = a \star_\lambda$, $\hat{a}^\dagger = \bar{a} \star_\lambda$ are operators of annihilation and creation, and

$$\mathcal{F}f(w, \bar{w}) = \frac{1}{\pi\hbar} \int_{\mathbb{R}^2} f(z, \bar{z}) e^{\hbar^{-1}(z\bar{w} - \bar{z}w)}\, d^2z,$$

where $d^2z = d(\mathrm{Re}\, z)\, d(\mathrm{Im}\, z)$, is the symplectic Fourier transform in holomorphic coordinates. The star-product (6.2.29) and the operator function $f_\lambda(\hat{a}, \hat{a}^\dagger)$ are widely used in quantum optics. In particular, for $\lambda = \frac{1}{2}$ ($\sigma = 0, \alpha = -\frac{1}{2\omega}, \beta = -\frac{\omega}{2}$) we have the so called normal ordering N while for $\lambda = -\frac{1}{2}$ ($\sigma = 0, \alpha = \frac{1}{2\omega}, \beta = \frac{\omega}{2}$) we have the so called anti-normal ordering AN. For monomial $f = a^n \bar{a}^m$

$$f_N(\hat{a}, \hat{a}^\dagger) = (\hat{a}^\dagger)^m \hat{a}^n, \qquad f_{AN}(\hat{a}, \hat{a}^\dagger) = \hat{a}^n (\hat{a}^\dagger)^m.$$

The choice $\sigma = 0, \alpha = \frac{1}{2\kappa}, \beta = \frac{\kappa}{2}$ turns the product (6.2.2) into

$$\boxed{\begin{aligned} f \star_\kappa g = & f \exp\left[\tfrac{1}{2}\hbar\left(1 + \tfrac{\omega^2+\kappa^2}{2\kappa\omega}\right) \overleftarrow{\partial}_a \overrightarrow{\partial}_{\bar{a}} - \tfrac{1}{2}\hbar\left(1 - \tfrac{\omega^2+\kappa^2}{2\kappa\omega}\right) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a \right. \\ & \left. + \tfrac{1}{2}\hbar\left(\tfrac{\omega^2-\kappa^2}{2\kappa\omega}\right) \overleftarrow{\partial}_a \overrightarrow{\partial}_a + \tfrac{1}{2}\hbar\left(\tfrac{\omega^2-\kappa^2}{2\kappa\omega}\right) \overleftarrow{\partial}_{\bar{a}} \overrightarrow{\partial}_a \right] g, \end{aligned}}$$

and related ordering is called generalized anti-normal (Husimi) ordering which collapses onto anti-normal ordering for $\kappa = \omega$.

In general, morphisms S are not of the form (6.2.18). As an example in two-dimensional case ($n = 1$) the following two parameter family of morphisms may serve

$$S = \exp\left(-i\hbar a \partial_q \partial_p + i\hbar b q \partial_p^2\right), \tag{6.2.30}$$

where $a, b \in \mathbb{R}$.

Example 6.8 To illustrate the S-ordering rule (6.2.30) let us consider a function $A(q, p) = \frac{1}{2}p^2 + qp$. Then, one finds that

$$(S^{-1}A)(q, p) = \tfrac{1}{2}p^2 + qp + i\hbar a - i\hbar b q$$

and for canonical $\hat{q}$, $\hat{p}$ operators

$$A_S(\hat{q}, \hat{p}) = \tfrac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) + \frac{1}{2}\hat{p}^2 + a[\hat{q}, \hat{p}] - b[\hat{q}, \hat{p}]\hat{q}$$
$$= \left(\tfrac{1}{2} + a\right)\hat{q}\hat{p} + \left(\tfrac{1}{2} - a\right)\hat{p}\hat{q} + \tfrac{1}{2}\hat{p}^2 - b\hat{q}\hat{p}\hat{q} + b\hat{p}\hat{q}^2.$$

Let us now consider a general phase space in the form of a cotangent bundle T^*Q to an almost geodesically simple connected pseudo-Riemannian manifold Q, and a general $\star$-product (6.1.1) defined on it such that for any canonical coordinates (x^i, p_j) on T^*Q the $\star^{(x,p)}$-product is equivalent with the Moyal product. Using the related morphism S and performing analogical considerations as for (6.2.14), we get for a quantum observable $A(x, p, \hbar)$ polynomial in momenta an appropriate operator

$$A(x, p, \hbar) \star^{(x,p)} = A_S(\hat{q}, \hat{p}),$$

acting in the Hilbert space $L^2(T^*Q, d\Omega_\hbar)$. Observe that for star-products considered in the previous subsection the action $S^{-1}A$ of the morphism S on a function A polynomial in momenta was again a function polynomial in momenta. Thus, to a general star-product on T^*Q written in canonical coordinates corresponds an S-ordering of operators of position and momenta.

Chapter 7
Quantum Hamiltonian Mechanics on Symplectic Manifolds

In the previous chapter we presented the general theory of quantum deformations of classical Poisson algebras. In the following chapter we develop a deformation procedure applied to classical statistical Hamiltonian mechanics (described in Sect. 3.3) in order to construct its quantum analogue on the phase space. First, we define quantum states as appropriate deformations of classical states and their time development through the respective deformation of the classical Liouville equation. Then we introduce quantum Hamiltonian equations of motion being a deformation of classical Hamiltonian equations and time development of quantum observables. With particular care we present the theory of quantum flow and quantum trajectories on a phase space together with a wide range of examples which illustrate the presented formalism. Such constructed quantum theories (each related with an appropriate quantum algebra) reduce to a common classical counterpart as deformation parameter $\hbar$ tends to zero: $\hbar \to 0$.

7.1 General Theory of Quantization

7.1.1 Quantum States

By definition, by an analogy with the classical case (cf. Sect. 3.3.1), *quantum states* related to quantum Poisson algebra $\mathcal{A}_Q = (C^\infty(M)[[\hbar]], \star, [[\cdot, \cdot]], *)$ are those functions $\rho \in L^2(M, d\Omega_\hbar)$ which satisfy the following conditions

1. $\rho = \rho^*$ (self-conjugation),
2. $\int_M \rho \, d\Omega_\hbar = \mathrm{Tr}(\rho) = 1$ (normalization),
3. $\int_M f^* \star f \star \rho \, d\Omega_\hbar = \mathrm{Tr}(f^* \star f \star \rho) \geq 0$ for $f \in C_0^\infty(M)$ (positive-definiteness).

Quantum states form a convex subset of the Hilbert space $L^2(M, d\Omega_\hbar)$. *Pure states* are defined as extreme points of the set of states, i.e. as those states which cannot

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be written as convex linear combinations of some other states. Thus ρ_{pure} is a pure state if and only if there do not exist two different states ρ_1 and ρ_2 such that $\rho_{\text{pure}} = p\rho_1 + (1-p)\rho_2$ for some $p \in (0,1)$. A state which is not pure is further called a *mixed state*.

For certain symplectic manifolds M pure states can be alternatively characterized as functions $\rho_{\text{pure}} \in L^2(M, d\Omega_\hbar)$ which are self-conjugated, normalized, and idempotent (cf. Sect. 3.3.1):

$$\boxed{\rho_{\text{pure}} \star \rho_{\text{pure}} = \rho_{\text{pure}}.} \tag{7.1.1}$$

Mixed states $\rho_{\text{mix}} \in L^2(M, d\Omega_\hbar)$ can be characterized as convex linear combinations, possibly infinite, of pure states $\rho^{(\lambda)}_{\text{pure}}$

$$\boxed{\rho_{\text{mix}} = \sum_\lambda p_\lambda \rho^{(\lambda)}_{\text{pure}},} \tag{7.1.2}$$

where $p_\lambda \geq 0$ and $\sum_\lambda p_\lambda = 1$.

The interpretation of pure and mixed states is similar as in classical mechanics. When we have the full knowledge of the state of the system, then the system is described by a pure state. If we only know that the system is in some pure state with some probability, then the system must be described by a mixed state. The quantum states ρ are the analogue of the classical distribution functions representing states of the classical Hamiltonian system and considered in Sect. 3.3.1. The difference between classical and quantum distribution functions is that the latter do not have to be non-negative everywhere. Thus, $\rho(x,p)$ cannot be interpreted as a probability density of finding a particle in a point (x,p) of the phase space. This is a consequence of the fact that x and p coordinates do not commute with respect to the $\star$-multiplication, which yield, from the Heisenberg uncertainty principle, that it is impossible to measure simultaneously the position and momentum of a particle like in classical mechanics. Hence, the point position of a particle in the phase space does not make sense anymore. On the other hand it is possible to introduce the so called *marginal distributions*

$$P(x) = \int (S^{-1}\rho)(x,p) d\mu(p), \quad P(p) = \int (S^{-1}\rho)(x,p) d\mu(x), \tag{7.1.3}$$

which are probabilistic distribution functions and can be interpreted as probability densities that a particle in the phase space has position x or momentum p (see Sect. 8.1.2). In (7.1.3) S-operator links a given $\star$-product with the Moyal one. The result is not surprising as each marginal distribution depends on commuting coordinates only. Note, however, that only in the case of the $\star_M$-product the marginal distributions are received by simple integration of a distribution function with respect to x or p variable. In general, the distribution function first has to be transformed with the isomorphism S.

Finally, for a given observable $A \in C^\infty(M)[\![\hbar]\!]$ and state ρ the expectation value of the observable A in the state ρ is defined by

$$\boxed{\langle A\rangle_\rho = \int_M (A \star \rho)\, d\Omega_\hbar = \mathrm{Tr}(A \star \rho),} \tag{7.1.4}$$

being the analogue of respective classical notion (3.3.5).

7.1.2 Time Evolution of Quantum Systems

The quantum time evolution of a system is governed by a quantum Hamilton function $H \in C^\infty(M)[\![\hbar]\!]$ which is, like in classical mechanics, a distinguished observable, being a deformation of a classical Hamilton function H_C and self-conjugated with respect to involution $*$ of respective $\star$-algebra. Like in the classical theory, there are two equivalent points of view on the time evolution: quantum Schrödinger picture and quantum Heisenberg picture. In the Schrödinger picture states undergo time development while observables do not. An equation of motion for states which is the analogue of classical Liouville equation (3.3.11) takes the form

$$\boxed{\frac{\partial \rho}{\partial t}(t) - [\![H, \rho(t)]\!] = 0 \Longleftrightarrow i\hbar \frac{\partial \rho}{\partial t}(t) - [H, \rho(t)] = 0} \tag{7.1.5}$$

The formal solution of (7.1.5) is of the form

$$\rho(t) = U(t) \star \rho(0) \star U(t)^*, \tag{7.1.6}$$

where

$$U(t) = \exp_\star\left(-\frac{i}{\hbar} t H\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(-\frac{i}{\hbar} t\right)^k \underbrace{H \star \cdots \star H}_{k} \tag{7.1.7}$$

is a *unitary function* as H is self-conjugated

$$H = H^* \to U(t)^* = \overline{U(t)}$$

and hence

$$U(t) \star \overline{U(t)} = \overline{U(t)} \star U(t) = 1. \tag{7.1.8}$$

Here $\phi(t) \equiv \exp_\star tB$ is the noncommutative exponential solution of

$$\frac{\partial \phi}{\partial t} = B \star \phi = \phi \star B, \quad \phi(0) = 1.$$

In consequence, the time evolution of states can be expressed in terms of the one-parameter group of unitary functions $U(t)$. Notice that *quantum Liouville equation* (*von Neumann equation*) on the symplectic manifold is represented like its classical analogue by a linear PDE.

States ρ which do not depend explicitly on time: $\frac{\partial \rho}{\partial t} = 0$, are called stationary states and hence fulfill the relation

$$[H, \rho] = 0. \tag{7.1.9}$$

As will be shown in Sect. 8.1.2, if ρ is a pure state, then (7.1.9) is equivalent to a pair of $\star$-genvalue problems

$$\boxed{H \star \rho = E\rho, \quad \rho \star H = E\rho, \quad E \in \mathbb{R}.} \tag{7.1.10}$$

Notice that E in (7.1.10) is equal to the expectation value of Hamiltonian H in a pure stationary state ρ, i.e. is equal to energy of a system in that state

$$\langle H \rangle_\rho = \int_M (H \star \rho)\, d\Omega_\hbar = E \int_M \rho d\Omega_\hbar = E.$$

From (7.1.5) follows that time evolution of expectation value of observable $A \in \mathcal{A}_Q$ in a state $\rho(t)$, i.a. $\langle A \rangle_{\rho(t)}$, fulfills the following equation of motion

$$\boxed{\frac{d}{dt} \langle A \rangle_{\rho(t)} - \langle [\![A, H]\!] \rangle_{\rho(t)} = 0.} \tag{7.1.11}$$

Indeed

$$\begin{aligned} 0 &= \int_M A \star \left(\frac{\partial \rho}{\partial t} - [\![A, H]\!] \right) d\Omega_\hbar \\ &= \int_M A \star \frac{\partial \rho}{\partial t}(t) d\Omega_\hbar - \int_M A \star \frac{1}{i\hbar} [H \star \rho(t) - \rho(t) \star H] d\Omega_\hbar \\ &= \frac{d}{dt} \int_M A \star \rho(t) d\Omega_\hbar - \int_M \frac{1}{i\hbar} [A \star H - H \star A] \star \rho(t) d\Omega_\hbar \\ &= \frac{d}{dt} \langle A \rangle_{\rho(t)} - \langle [\![A, H]\!] \rangle_{\rho(t)}. \end{aligned}$$

In the Heisenberg picture states remain still whereas observables undergo the time evolution. The time evolution of an observable $A \in C^\infty(M)[\![\hbar]\!]$ is given by

the action of the unitary function $U(t)$ from (7.1.7) on A:

$$A(t) = \overline{U(t)} \star A(0) \star U(t) = e^{-t[\![H, \cdot]\!]} A(0), \tag{7.1.12}$$

where

$$e^{-t[\![H, \cdot]\!]} := \sum_{k=0}^{\infty} \frac{1}{k!} (-t)^k \underbrace{[\![H, [\![H, \ldots [\![H, \cdot]\!] \ldots]\!]}_{k}$$

and $U(t)$ is given by (7.1.7).

Differentiating (7.1.12) with respect to t results in the evolution equation for A:

$$\boxed{\frac{dA}{dt}(t) - [\![A(t), H]\!] = 0 \Longleftrightarrow i\hbar \frac{dA}{dt} - [A(t), H] = 0.} \tag{7.1.13}$$

Equation (7.1.13) is the quantum analogue of the classical equation (3.3.13).

In particular, *quantum Hamiltonian equations* of motion are of the form

$$\boxed{\left(Q^i\right)_t (x, p, t) = [\![Q^i(x, p, t), H(x, p)]\!]_{(x,p)}, \quad Q^i(x, p, 0) = x^i,} \tag{7.1.14a}$$

$$\boxed{(P_i)_t (x, p, t) = [\![P_i(x, p, t), H(x, p)]\!]_{(x,p)}, \quad P_i(x, p, 0) = p_i} \tag{7.1.14b}$$

and, as in the classical case, are nonlinear PDE's and represent a *quantum Hamiltonian transport*. What is important, the system of PDE's (7.1.14) is equivalent to the system of ordinary differential equations but in the space of $\star$-functions. Indeed, any function $A \in \mathcal{A}_Q$ can be expanded in a $\star$-power series (6.1.47). In particular, any monomial $x^n p^m$ can be expressed as an $\star$-polynomial (6.2.16)

$$x^n p^m = \left(x^n p^m\right) \star 1 = \left(x^n p^m\right)_S (x\star, p\star)1. \tag{7.1.15}$$

So, equations (7.1.14) can be written as the system of ODE's in the space of $\star$-functions in the form

$$\boxed{\begin{aligned} Q^i_t &= [\![Q^i(x, p, t), H(x, p)]\!]_{(x,p)} = U(-t) \star \frac{\partial H}{\partial p_i}(x, p) \star U(t) \\ &= U(-t) \star \left[\frac{\partial H}{\partial p_i}(x, p)\right]_S (x\star, p\star) U(t) = \left[\frac{\partial H}{\partial P_i}(Q, P)\right]_S (Q\star, P\star)1, \end{aligned}} \tag{7.1.16a}$$

$$\boxed{P_{it} = [\![P_i(x, p, t), H(x, p)]\!]_{(x,p)} = \left[-\frac{\partial H}{\partial Q_i}(Q, P)\right]_S (Q\star, P\star)1.} \tag{7.1.16b}$$

Notice, that according to Observation 12, equations (7.1.16) can be always transformed to the Moyal case in (x, p) coordinates and Weyl ordering, with transformed Hamiltonian $H(\hbar) = S(\hbar)H$.

Example 7.1 Consider the Hamiltonian system on $M = \mathbb{R}^2$ with the classical Hamiltonian function in canonical coordinates (x, p)

$$H(x, p) = \kappa x^2 p^2, \quad \kappa > 0.$$

Classical equations of motion are

$$Q_t = \{Q(x, p, t), H(x, p)\}_{(x,p)} = \{Q, H(Q, P)\}_{(Q,P)} = \frac{\partial H}{\partial P} = 2\kappa Q^2 P, \quad Q(0) = x,$$

$$P_t = \{P(x, p, t), H(x, p)\}_{(x,p)} = \{P, H(Q, P)\}_{(Q,P)} = -\frac{\partial H}{\partial Q} = -2\kappa Q P^2, \quad P(0) = p.$$

If the quantization is given by $\star$-product which in (x, p) chart takes the Moyal form, then quantum equations of motion, according to (7.1.16), take the form

$$Q_t = [\![Q(x, p, t), H(x, p)]\!]_{(x,p)} = \left(\frac{\partial H}{\partial P}\right)_W (Q\star, P\star)\, 1 = \kappa Q \star Q \star P + \kappa P \star Q \star Q,$$

$$P_t = [\![P(x, p, t), H(x, p)]\!]_{(x,p)} = \left(-\frac{\partial H}{\partial Q}\right)_W (Q\star, P\star)\, 1 = -\kappa Q \star P \star P - \kappa P \star P \star Q,$$

where $Q(0) = x$, $P(0) = p$ and $\star \equiv \star_M^{(x,p)}$. The solution of classical and quantum dynamics will be considered in the next section.

Like in the classical case, both presented approaches to the time evolution yield equal predictions concerning the results of measurements, since from the property of trace (6.1.11)

$$\langle A(0)\rangle_{\rho(t)} = \int_M A(0) \star \rho(t)\, d\Omega_\hbar = \int_M A(t) \star \rho(0)\, d\Omega_\hbar = \langle A(t)\rangle_{\rho(0)}.$$

Observation 14 *Comparing the results of Sects. 3.3.2 and 7.1.2 we observe that the linear aspect of classical and quantum Hamiltonian mechanics is represented by time evolution of states, described on both levels by linear PDE (the so called Schrödinger picture). On the other hand, the nonlinear aspect of both theories is represented by time evolution of observables, described on both levels by nonlinear ODE (the so called Heisenberg picture) defined on an appropriate space of ordinary-functions and star-functions, respectively. Contrary to a classical case, on a quantum level Hamiltonian equations of motion belong only to the Heisenberg picture as pure coherent classical states (3.3.16) are not admissible as quantum states.*

7.2 Quantum Trajectories in Phase Space

The time evolution of a classical Hamiltonian system is fully determined by trajectories (a flow) in a phase space (see Sect. 3.3.2). Once we calculate a classical flow Φ_t for the given system a time evolution of states and observables can be received by simply composing them with Φ_t. A classical flow is defined as a map $\Phi_t : M \to M$ on the phase space M, which at every point $\xi_0 \in M$ gives a trajectory (curve) $\gamma(t) = \Phi_t(\xi_0)$ on M passing through the point ξ_0 and being a solution of the Hamilton's equations (3.2.35). Moreover, any trajectory $\Phi_t(\xi_0)$ has the property of being a classical canonical transformation for every t, and the set $\{\Phi_t\}_{t\in\mathbb{R}}$ have a structure of a group with multiplication being a composition of maps.

From the very beginning of quantum physics, a lot of efforts have been taken to formulate some kind of an analogue of phase space trajectories in quantum mechanics [95]. The most common approaches to quantum dynamics are the de Broglie-Bohm approach [50, 51, 154], the average value approach [181, 266], and the Moyal trajectories approach (see [93, 174] and references therein).

In the following section we develop the theory of Moyal trajectories resulting from quantum Hamiltonian equations (7.1.14). In consequence, the time evolution of observables cannot be given as a simple composition of observables with a quantum flow. For this reason in papers [93] and [174] observables were considered to be $\star$-functions. Then the action of a flow on observables was given as a $\star$-composition.

In the approach presented in this section we treat observables as ordinary functions on a classical phase space. We also present in an explicit form a quantum action of a flow on observables, which is a deformation of the respective classical action. The resulting time dependence of observables gives an appropriate solution of a quantum time evolution equation for observables (7.1.13). Then, we show that a set of quantum symplectomorphisms (quantum flow) has a structure of a group with multiplication (quantum composition) being a deformation of the ordinary composition regarded as a multiplication in a group of classical symplectomorphisms (classical flow) [34]. Such an approach to quantum trajectories has a benefit in that it is not needed to calculate the form of observables as star-functions, but only a quantum action of a given trajectory needs to be found.

7.2.1 Quantum Flow

Let us consider the Moyal quantization of a classical Hamiltonian system (M, π, H), where $M = \mathbb{R}^{2N}$, $\pi = \partial_{x^i} \wedge \partial_{p_i}$, and $H \in C^\infty(M)$ is an arbitrary real function. Then the solution of quantum Hamiltonian equations

$$\left(Q^j\right)_t = [\![Q^j(t), H]\!], \quad \left(P_j\right)_t = [\![P_j(t), H]\!], \tag{7.2.1}$$

where $Q^i(x, p, 0) = x^i$ and $P_j(x, p, 0) = p_j$, i.e., the Heisenberg representation for observables of position and momentum, generates a quantum flow Φ_t in a phase space according to an equation

$$\Phi_t(x, p; \hbar) = (Q(x, p, t; \hbar), P(x, p, t; \hbar)). \tag{7.2.2}$$

For every instance of time t the map Φ_t is a quantum canonical transformation (quantum symplectomorphism) from coordinates (x, p) to new coordinates $x' = Q(x, p, t; \hbar)$, $p' = P(x, p, t; \hbar)$. It means that Φ_t preserves the quantum Poisson bracket $[\![Q^i(t), P_j(t)]\!] = \delta^i_j$, which can be seen from (7.2.3) and the fact that $[\![Q^i(0), P_j(0)]\!] = [\![x^i, p_j]\!] = \delta^i_j$.

The flow Φ_t, treated as a quantum canonical transformation, can act on observables and states as simple composition of maps. Such a classical action can also be used to transform the algebraic structure of the quantum Poisson algebra so that the action will be an isomorphism of the initial algebra and its transformation. So, a star-product $\star_t$ being the Moyal product transformed by Φ_t^{-1} is defined by the formula

$$\boxed{(f \star g) \circ \Phi_t^{-1} = (f \circ \Phi_t^{-1}) \star_t (g \circ \Phi_t^{-1}), \quad f, g \in C^\infty(\mathbb{R}^{2N}).}$$

As we know from our previous considerations, the $\star_t$-product takes the form

$$\boxed{f \star_t g = f \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{D}_{x^i} \overrightarrow{D}_{p_i} - \tfrac{1}{2} i\hbar \overleftarrow{D}_{p_i} \overrightarrow{D}_{x^i} \right) g,}$$

where vector fields D_{x^i}, D_{p_i} are transformations of coordinate vector fields $\partial_{x^i}, \partial_{p_i}$:

$$(\partial_{x^i} f) \circ \Phi_t^{-1} = D_{x^i}(f \circ \Phi_t^{-1}), \quad (\partial_{p_i} f) \circ \Phi_t^{-1} = D_{p_i}(f \circ \Phi_t^{-1}).$$

The most important for our further construction is the observation that the $\star_t$-product is gauge equivalent to the Moyal product. In other words, to a quantum flow Φ_t there corresponds a unique isomorphism S_t satisfying

$$\boxed{\begin{aligned} S_t(f \star g) &= S_t f \star_t S_t g, \\ S_t x^i &= x^i, \quad S_t p_j = p_j, \\ S_t(\bar{f}) &= (\overline{S_t f}). \end{aligned}}$$

Observe, that for the $\star_t$-algebra the involution is also the complex-conjugation.

A formal solution of the time evolution equation (7.1.13) for an observable $A \in \mathcal{A}_Q$ can be expressed by the formula

$$A(t) = e^{-t[\![H, \cdot]\!]} A(0) = e_\star^{\frac{i}{\hbar} tH} \star A(0) \star e_\star^{-\frac{i}{\hbar} tH},$$

(confront with (7.1.12), (7.1.7) and (7.1.2)). In particular, the solution of (7.2.1) is of the form

$$Q^i(t) = e^{-t[\![H,\cdot]\!]}Q^i(0) = e_\star^{\frac{i}{\hbar}tH} \star Q^i(0) \star e_\star^{-\frac{i}{\hbar}tH}, \tag{7.2.3a}$$

$$P_j(t) = e^{-t[\![H,\cdot]\!]}P_j(0) = e_\star^{\frac{i}{\hbar}tH} \star P_j(0) \star e_\star^{-\frac{i}{\hbar}tH}, \tag{7.2.3b}$$

which for the fixed initial condition $Q^i(x,p,0) = x_0^i$ and $P_j(x,p,0) = p_{0j}$ represents a particular quantum trajectory.

A time evolution of an observable $A \in \mathcal{A}_Q$ should be alternatively expressed by action of the quantum flow Φ_t on A. The composition of Φ_t with observables, i.e. the classical action of Φ_t on observables, does not result in a proper time evolution of observables and thus it is necessary to deform this classical action. It will be proved that a proper action of the quantum flow Φ_t on functions from $\mathcal{A}_Q$ (a pull-back of Φ_t) is given by the new formula

$$\boxed{\Phi_t^* A = (S_t A) \circ \Phi_t,} \tag{7.2.4}$$

where S_t is an isomorphism associated to the quantum canonical transformation Φ_t^{-1}.

The formula (7.2.4) can be proved first by noting that

$$\Phi_t^* Q^i(0) = (S_t Q^i(0)) \circ \Phi_t = Q^i(0) \circ \Phi_t = Q^i(t) = e^{-t[\![H,\cdot]\!]}Q^i(0)$$

and similarly

$$\Phi_t^* P_j(0) = e^{-t[\![H,\cdot]\!]}P_j(0),$$

where the fact that $S_t x^i = x^i$ and $S_t p_j = p_j$ was used, being on the other hand a consequence of the quantum canonicity of Φ_t. Secondly, Φ_t^* given by (7.2.4) is an automorphism of $\mathcal{A}_Q$ as

$$\begin{aligned}\Phi_t^*(A \star B) &= (S_t(A \star B)) \circ \Phi_t = (S_t A \star_t S_t B) \circ \Phi_t \\ &= ((S_t A) \circ \Phi_t) \star ((S_t B) \circ \Phi_t) = \Phi_t^* A \star \Phi_t^* B,\end{aligned}$$

where $\star_t$ denotes a star-product transformed by Φ_t^{-1}. Thus

$$\Phi_t^* = e^{-t[\![H,\cdot]\!]} \tag{7.2.5}$$

holds true since, as was proved earlier, every function from $\mathcal{A}_Q$ can be presented as a $\star$-power series.

In a complete analogy with classical theory one can define a quantum Hamiltonian vector field by $\zeta_H = [\![\cdot, H]\!]$. Then (7.2.5) states that Φ_t is a flow of

the quantum Hamiltonian vector field ζ_H. Moreover, in an analogy with classical mechanics, $\{\Phi_t\}$ is a one-parameter group of quantum canonical transformations with respect to a new multiplication defined by

$$\boxed{\Phi_{t_1} \cdot \Phi_{t_2} = (S_{t_2}\Phi_{t_1}) \circ \Phi_{t_2},} \tag{7.2.6}$$

where $S_{t_2}\Phi_{t_1}$ denotes a map $\mathbb{R}^{2N} \to \mathbb{R}^{2N}$ given by the formula

$$S_{t_2}\Phi_{t_1} = (S_{t_2}Q^1(t_1), \ldots, S_{t_2}P_N(t_1)),$$

where $\Phi_{t_1} = (Q^1(t_1), \ldots, Q^N(t_1), P_1(t_1), \ldots, P_N(t_1))$. Multiplication defined in such a way satisfies properties similar to their classical counterparts (composition):

$$\Phi_0 = \mathrm{id}, \quad \Phi_{t_1} \cdot \Phi_{t_2} = \Phi_{t_1+t_2},$$

proving that $\{\Phi_t\}$ is a group. Further on we will call it a quantum composition. The quantum composition rule given by (7.2.6) is properly defined since it respects the quantum pull-back of flows:

$$(\Phi_{t_1} \cdot \Phi_{t_2})^* = \Phi_{t_2}^* \circ \Phi_{t_1}^*. \tag{7.2.7}$$

Indeed, it is enough to prove (7.2.7) for an arbitrary $\star$-monomial. For simplicity we will present the proof for a two-dimensional case and for a $\star$-monomial $x \star p$. Using the fact that $S_t x = x$ and $S_t p = p$ for every t one calculates that

$$\begin{aligned}(\Phi_{t_2}^* \circ \Phi_{t_1}^*)(x \star p) &= \Phi_{t_2}^*\big((S_{t_1}(x \star p)) \circ \Phi_{t_1}\big) = \Phi_{t_2}^*\big((x \star_{t_1} p) \circ \Phi_{t_1}\big) \\ &= \Phi_{t_2}^*\big(Q(t_1) \star P(t_1)\big) = \big(S_{t_2}(Q(t_1) \star P(t_1))\big) \circ \Phi_{t_2} \\ &= \big(S_{t_2}Q(t_1) \star_{t_2} S_{t_2}P(t_1)\big) \circ \Phi_{t_2} = (x \star_{t_2,t_1} p) \circ S_{t_2}\Phi_{t_1} \circ \Phi_{t_2},\end{aligned}$$

where $\star_{t_1}$, $\star_{t_2}$, denote Moyal products transformed, respectively, by transformations $\Phi_{t_1}^{-1}$, $\Phi_{t_2}^{-1}$, and $\star_{t_2,t_1}$ denotes the $\star_{t_2}$-product transformed by $(S_{t_2}\Phi_{t_1})^{-1}$. From the relation $S_{T_1 \circ T_2} = S_{T_1,T_2}S_{T_1}$ valid for any transformations T_1, T_2 defined on the whole phase space ($S_{T_1 \circ T_2}$ is an isomorphism intertwining star-products $\star$ and $\star_{T_1 \circ T_2}$, S_{T_1,T_2} intertwines $\star_{T_1}$ with $\star_{T_1 \circ T_2}$, and S_{T_1} intertwines $\star$ with $\star_{T_1}$, where $\star_{T_1}$ and $\star_{T_1 \circ T_2}$ are the Moyal products transformed, respectively, by transformations T_1 and $T_1 \circ T_2$), one finds that

$$S_{(\Phi_{t_1}\Phi_{t_2})^{-1}}(x \star p) = S_{\Phi_{t_2}^{-1},(S_{t_2}\Phi_{t_1})^{-1}}S_{t_2}(x \star p) = S_{\Phi_{t_2}^{-1},(S_{t_2}\Phi_{t_1})^{-1}}(x \star_{t_2} p) = x \star_{t_2,t_1} p$$

and hence

$$(\Phi_{t_2}^* \circ \Phi_{t_1}^*)(x \star p) = S_{(\Phi_{t_1}\Phi_{t_2})^{-1}}(x \star p) \circ S_{t_2}\Phi_{t_1} \circ \Phi_{t_2} = (\Phi_{t_1} \cdot \Phi_{t_2})^*(x \star p).$$

As a direct consequence of these considerations and the fact that for the Moyal product

$$\begin{aligned}[\![H,\cdot]\!] =& \{H,\cdot\} - \frac{1}{24}\hbar^2\left[\left(\partial_{x^i}\partial_{x^j}\partial_{x^k}H\right)\partial_{p_i}\partial_{p_j}\partial_{p_k}\right.\\ &- (\partial_{p_i}\partial_{p_j}\partial_{p_k}H)\partial_{x^i}\partial_{x^j}\partial_{x^k} - 3\left(\partial_{x^i}\partial_{x^j}\partial_{p_k}H\right)\partial_{p_i}\partial_{p_j}\partial_{x^k}\\ &\left.+\ 3(\partial_{p_i}\partial_{p_j}\partial_{x^k}H)\partial_{x^i}\partial_{x^j}\partial_{p_k}\right] + O(\hbar^4)\end{aligned} \tag{7.2.8}$$

yields the following observation.

Observation 15 *Quantum trajectories of the linear Hamiltonian systems coincide with classical trajectories. It follows from the fact that for the Hamiltonian functions being quadratic polynomials of phase space coordinates:* $[|H,\cdot|] = \{H,\cdot\}$ *(7.2.8). Besides, because solutions* $Q(t)$ *and* $P(t)$ *are linear in* x^i, p_i, *so* $S_t = 1$. *It means that the quantum group multiplication* (7.2.6) *(quantum composition) coincides with the classical composition* (7.2.10) *and in consequence, the quantum time evolution of any observable* A *is the same as the classical time evolution of* A. *For such systems, the only difference between the classical and quantum dynamics relies on different admissible states in which the evolution takes place. On the other hand, even in such simplest cases, classical and quantum systems differ fundamentally on the level of stationary states.*

In the limit $\hbar \to 0$, (7.2.3) reduces to classical phase space trajectories

$$\begin{aligned} Q^j(t) &= e^{-t\{H,\cdot\}}Q^j(0), \quad & P_j(t) &= e^{-t\{H,\cdot\}}P_j(0),\\ Q^j(x,p,0) &= x^j, \quad & P_j(x,p,0) &= p_j, \end{aligned}$$

which are formal solutions of classical Hamiltonian equations

$$\left(Q^j\right)_t = \{Q^j(t),H\}, \quad \left(P_j\right)_t = \{P_j(t),H\}.$$

In a more explicit form classical trajectories are represented by a flow (classical symplectomorphism)

$$\Phi_t(x,p) = (Q(x,p,t),P(x,p,t)), \tag{7.2.9}$$

which is an $\hbar \to 0$ limit of the quantum flow (7.2.2) (quantum symplectomorphism). An action of the classical flow Φ_t on functions from $\mathcal{A}_C$ (a pull-back of Φ_t) is just a simple composition of functions with Φ_t, being an $\hbar \to 0$ limit of (7.2.4)

$$\Phi_t^* A = A \circ \Phi_t.$$

$\{\Phi_t\}$ forms a one-parameter group of canonical transformations, preserving a classical Poisson bracket: $\{Q^i(t), P_j(t)\} = \delta^i_j$, with multiplication being an ordinary composition of maps

$$\Phi_{t_1} \cdot \Phi_{t_2} = \Phi_{t_1} \circ \Phi_{t_2}, \tag{7.2.10}$$

which is the $\hbar \to 0$ limit of quantum composition (7.2.6).

7.2.2 *Quantum Dynamics with Classical Trajectories*

In the following subsection we analyze fairly accurately a few simple examples of quantum systems, with quantization defined by the Moyal product, for which classical and quantum trajectories coincide. Let us start from a free particle in one dimension. The free particle is a system, whose time evolution is governed by a Hamiltonian

$$H(x, p) = \tfrac{1}{2}p^2,$$

where the mass of the particle $m = 1$. This Hamiltonian describes only the kinetic energy of the particle. It does not contain any terms describing the potential energy, i.e. there are no forces acting on the particle (the particle is free).

From relation (7.2.8) it follows that $\Phi^*_t(\hbar) = \Phi^*_t(0)$ as

$$\boxed{\exp\left(-t\,[\![H, \cdot]\!]\right) = \exp\left(-t\{H, \cdot\}\right).}$$

Thus, a common quantum and classical flow of a free particle is of the form $\Phi_t(x, p; \hbar) = \Phi_t(x, p) = (Q(t), P(t))$, where

$$Q(t) = x + pt, \quad P(t) = p. \tag{7.2.11}$$

Besides, because $S_t = 1$ for any linear transformation, so the time evolution of any classical and quantum observable $A(x, p)$ is given by

$$A(t) = A(Q(t), P(t)). \tag{7.2.12}$$

So, what is a difference between the classical and quantum free dynamics? Let me remind that both, on the classical and the quantum level, "physics" are represented by expectation values of observables $\langle A \rangle_\rho$ in a chosen admissible state ρ. For the classical system, Eqs. (7.2.11) and (7.2.12) represent simultaneously the dynamics of expectation values of observables (position and momentum in particular) in the pure coherent classical state

$$\rho_C(x', p') = \delta(x' - x)\delta(p' - p), \tag{7.2.13}$$

as

$$\langle Q(t)\rangle_{\rho_C} = \int_{\mathbb{R}^2} \rho_C(x', p') Q(x', p', t) dx' dp' = Q(x, p, t),$$
$$\langle P(t)\rangle_{\rho_C} = \int_{\mathbb{R}^2} \rho_C(x', p') P(x', p', t) dx' dp' = P(x, p, t)$$

and

$$(\Delta Q(t))^2 = \left\langle Q^2(t)\right\rangle_{\rho_C} - \langle Q(t)\rangle^2_{\rho_C} = 0, \quad (\Delta P(t))^2 = \left\langle P^2(t)\right\rangle_{\rho_C} - \langle P(t)\rangle^2_{\rho_C} = 0. \tag{7.2.14}$$

On the other hand, the state (7.2.13) is not an admissible quantum state.

Let us consider a one-parameter family of pure quantum states ρ of the form

$$\rho_Q(x', p', \gamma) = 2\exp\left(-\frac{\gamma}{\hbar}(x' - x)^2\right)\exp\left(-\frac{1}{\gamma\hbar}(p' - p)^2\right), \qquad \gamma \in \mathbb{R}_+. \tag{7.2.15}$$

They all are pure states as one can show by direct calculations that

$$\int_{\mathbb{R}^2} \rho_Q(x', p', \gamma) d\Omega'_\hbar = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \rho_Q(x', p', \gamma) dx' dp' = 1, \tag{7.2.16}$$

and

$$\rho_Q(x', p', \gamma) \star \rho_Q(x', p', \gamma) = \rho_Q(x', p', \gamma). \tag{7.2.17}$$

States (7.2.15) are simultaneously coherent states as they minimize quantum uncertainty relation: $\Delta x \Delta p = \frac{1}{2}\hbar$. It follows directly from the property of the Gauss distribution

$$f(z; \sigma, \mu) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(z - \mu)^2}{2\sigma^2}\right),$$

for which

$$\langle z\rangle_f = \int_{\mathbb{R}} z f dz = \mu, \quad \left\langle z^2\right\rangle_f = \int_{\mathbb{R}} z^2 f dz = \mu^2 + \sigma^2,$$

and then

$$(\Delta x)^2 = \left\langle x^2\right\rangle_{\rho_Q} - \langle x\rangle^2_{\rho_Q} = \frac{\hbar}{2\gamma}, \quad (\Delta p)^2 = \left\langle p^2\right\rangle_{\rho_Q} - \langle p\rangle^2_{\rho_Q} = \frac{\gamma\hbar}{2}.$$

So, in the case of the initial quantum coherent state (7.2.15), the time evolution of the expectation value of position and momentum takes the form

$$\langle Q(t)\rangle_{\rho_Q} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \rho_Q(x', p')Q(x', p', t)dx'dp' = Q(x, p, t), \quad (7.2.18a)$$

$$\langle P(t)\rangle_{\rho_Q} = \frac{1}{2\pi\hbar}\int_{\mathbb{R}^2} \rho_Q(x', p')P(x', p', t)dx'dp' = P(x, p, t), \quad (7.2.18b)$$

and coincides with the classical time evolution of the expectation value of position and momentum in a classical pure coherent state $\rho_C = \delta(x' - x)\delta(p' - p)$. Nevertheless, contrary to the classical case, for the quantum system we get

$$\begin{aligned}
\langle Q(t)\rangle_{\rho_Q} &= \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} x \star_M \rho_Q dx'dp' = x + pt,\\
\langle P(t)\rangle_{\rho_Q} &= \frac{1}{2\pi\hbar}\iint_{\mathbb{R}^2} p \star_M \rho_Q dx'dp' = p,\\
\Delta Q(t) &= \sqrt{\langle Q^2\rangle_{\rho_Q} - \langle Q\rangle^2_{\rho_Q}} = \sqrt{(\Delta x)^2 + (\Delta p)^2 t^2},\\
\Delta P(t) &= \sqrt{\langle P^2\rangle_{\rho_Q} - \langle P\rangle^2_{\rho_Q}} = \Delta p
\end{aligned}$$

where

$$(\Delta x)^2 = \frac{1}{2}\frac{\hbar}{\gamma}, \quad (\Delta p)^2 = \frac{1}{2}\gamma\hbar$$

and in consequence

$$\Delta Q(t)\Delta P(t) = \tfrac{1}{2}\hbar\sqrt{1 + \gamma^2 t^2} \geq \tfrac{1}{2}\hbar.$$

Note that during the time evolution the uncertainty of the momentum $\Delta P(t)$ of the free particle described by the state (7.2.15) does not change in time and is equal to its initial value Δp, whereas the uncertainty of the position $\Delta Q(t)$ initially equal Δx increases in time. Note also that the uncertainties of the position and momentum satisfy the Heisenberg uncertainty principle, i.e. $\Delta Q(t)\Delta P(t) \geq \frac{\hbar}{2}$. Moreover, initially the free particle is in a state which minimizes the Heisenberg uncertainty principle since $\Delta Q(0)\Delta P(0) = \Delta x \Delta p = \frac{\hbar}{2}$. It is also worth noting that the expectation value of the momentum $\langle P(t)\rangle_{\rho_Q}$ is constant and equal p, whereas the expectation value of the position $\langle Q(t)\rangle_{\rho_Q}$ is equal $x + pt$. Hence, the time evolution of the free particle described by the state (7.2.15) can be interpreted as the movement of the particle along a straight line with the constant momentum equal p, similarly as in the classical case. The difference between the classical and quantum case is that in the quantum case there is some uncertainty of the position and momentum, in contrast to the classical case where the position and momentum

are known precisely. Observe also that for any admissible value of γ the coherence is not preserved during time evolution.

It is interesting to calculate to which classical state the state (7.2.15) converges in the limit $\hbar \to 0^+$. The limit has to be calculated in the distributional sense, i.e. one has to calculate the limit $\lim_{\hbar\to 0^+}\langle \rho_Q, \phi\rangle = \lim_{\hbar\to 0^+} Tr(\rho_Q(\hbar)\phi)$ for every test function ϕ. One easily calculates that

$$\lim_{\hbar\to 0^+}\langle \rho_Q, \phi\rangle = \phi(x, p).$$

Hence

$$\lim_{\hbar\to 0^+} \rho_Q = \delta(x' - x)\delta(p' - p),$$

where the scalar product $\langle\cdot,\cdot\rangle$ is given by (6.1.9). The above equation implies that the state ρ_Q (7.2.15), describing a quantum free particle, converges in the limit $\hbar \to 0^+$ to the classical pure state describing a classical free particle moving along a straight line with the constant momentum equal p.

Our second example is the harmonic oscillator described by Hamiltonian

$$H(x, p) = \tfrac{1}{2}\left(p^2 + \omega^2 x^2\right), \quad \omega \in \mathbb{R}_+. \tag{7.2.19}$$

Again from relation (7.2.8) it follows that $\Phi_t^*(\hbar) = \Phi_t^*$ and thus a common quantum and classical flow of harmonic oscillator is of the form $\Phi_t(x, p; \hbar) = \Phi_t(x, p) = (Q(t), P(t))$, where

$$Q(t) = x\cos\omega t + \omega^{-1} p \sin\omega t, \quad P(t) = p\cos\omega t - \omega x \sin\omega t. \tag{7.2.20}$$

Moreover, as $S_t = 1$, so the time evolution of any classical and quantum observable $A(x, p)$ is given by (7.2.12). For the classical system, like in the previous example, equations (7.2.20) represent simultaneously the dynamics of expectation values of position and momentum in the pure coherent classical state (7.2.13), for which the minimal classical uncertainty relation (7.2.14) is fulfilled. As the state (7.2.13) is not an admissible quantum state we again consider a one-parameter family of pure and coherent quantum states ρ of the form (7.2.15).

Like in the previous case, the time evolution of the expectation value of position and momentum takes the form (7.2.18) and so coincides with the classical time evolution of expectation value of position and momentum in the classical pure coherent state $\rho_C = \delta(x' - x)\delta(p' - p)$. Nevertheless, contrary to the classical case, for the quantum system we get

$$(\Delta Q(t))^2 = \left\langle Q^2(t)\right\rangle_{\rho_Q} - \langle Q(t)\rangle^2_{\rho_Q} = \frac{\hbar}{2\gamma}\cos^2\omega t + \frac{\gamma\hbar}{2\omega^2}\sin^2\omega t,$$

$$(\Delta P(t))^2 = \left\langle P^2(t)\right\rangle_{\rho_Q} - \langle P(t)\rangle^2_{\rho_Q} = \frac{\gamma\hbar}{2}\cos^2\omega t + \frac{\omega^2\hbar}{2\gamma}\sin^2\omega t$$

and hence

$$\Delta Q(t)\Delta P(t) = \frac{\hbar}{2}\left[\sin^4\omega t + \cos^4\omega t + \left(\frac{\omega^2}{\gamma^2} + \frac{\gamma^2}{\omega^2}\right)\sin^2\omega t\cos^2\omega t\right]^{\frac{1}{2}}.$$

Notice that in this particular case there exists a distinguished coherent state

$$\rho_Q(x', p') = 2\exp\left(-\frac{\omega}{\hbar}(x' - x)^2\right)\exp\left(-\frac{1}{\omega\hbar}(p' - p)^2\right)$$

when $\gamma = \omega$, which remains coherent for arbitrary value of t. Indeed, as

$$(\Delta Q(t))^2 = (\Delta x)^2 = \frac{\hbar}{2\omega}, \quad (\Delta P(t))^2 = (\Delta p)^2 = \frac{\omega\hbar}{2},$$

then

$$\Delta Q(t)\Delta P(t) = \frac{\hbar}{2}.$$

As was proved earlier, in the limit $\hbar \to 0^+$, pure coherent quantum states (7.2.15) converge to the pure coherent classical state (7.2.13).

Let us try to find stationary pure states of the harmonic oscillator. From Sect. 7.1.2 it is known that the stationary pure states are precisely the solutions of the following pair of $\star$-genvalue equations

$$H \star \rho = E\rho, \qquad \rho \star H = E\rho,$$

for $E \in \mathbb{R}$. To solve the above equations it is convenient to introduce new coordinates called *holomorphic coordinates* (6.2.2)

$$a(x, p) = \frac{\omega x + ip}{\sqrt{2\hbar\omega}}, \qquad \bar{a}(x, p) = \frac{\omega x - ip}{\sqrt{2\hbar\omega}}.$$

The functions a and $\bar{a}$ are called the *annihilation* and *creation* functions since they decrease and increase the number of excitations of the vibrational mode with frequency ω (annihilate and create the quanta of vibrations). Note, that $a\star = (\bar{a}\star)^\dagger$, $\bar{a}\star = (a\star)^\dagger$ and

$$[a, \bar{a}] = a \star \bar{a} - \bar{a} \star a = 1.$$

In these new coordinates the function H takes the form

$$H(a, \bar{a}) = \hbar\omega a\bar{a} = \hbar\omega\left(\bar{a} \star a + \tfrac{1}{2}\right) = \hbar\omega\left(a \star \bar{a} - \tfrac{1}{2}\right).$$

Let us consider a more general problem of finding a solution to the following pair of $\star$-genvalue equations

$$H \star \rho_{mn} = E_m \rho_{mn}, \tag{7.2.21a}$$

$$\rho_{mn} \star H = E_n \rho_{mn}, \tag{7.2.21b}$$

where m, n are numbering the $\star$-genvalues of H. It can be shown that m, n are non-negative integer numbers. The energy levels E_n of the harmonic oscillator are equal

$$E_n = (n + \tfrac{1}{2})\hbar\omega.$$

Since $H = \hbar\omega(\bar{a} \star a + \frac{1}{2})$, $\star$-genvalues of the function $N := \bar{a} \star a$ are the natural numbers $n = 0, 1, 2, \ldots$ and $\star$-genfunctions are the $\star$-genfunctions ρ_{mn} of H, i.e.

$$N \star \rho_{mn} = m\rho_{mn}, \quad \rho_{mn} \star N = n\rho_{mn}.$$

Hence, the function $N = \bar{a} \star a$ can be interpreted as an observable of the number of excitations of the vibrational mode with frequency ω.

Moreover, the normalized solutions of Eqs. (7.2.21) can be calculated from the ground state ρ_{00}

$$a \star \rho_{00} = 0$$

according to the equation

$$\rho_{mn} = \frac{1}{\sqrt{m!n!}} \underbrace{\bar{a} \star \ldots \star \bar{a}}_{m} \star \rho_{00} \star \underbrace{a \star \ldots \star a}_{n}$$

and the ground state ρ_{00} takes the form

$$\rho_{00}(a, \bar{a}) = 2\exp(-2a\bar{a}), \qquad \rho_{00}(x, p) = 2\exp\left(-\frac{p^2 + \omega^2 x^2}{\hbar\omega}\right),$$

with normalization given by (7.2.16). The $\star$-genfunctions ρ_{mn} can be now calculated giving

$$\rho_{mn}(a, \bar{a}) = \frac{1}{\sqrt{m!n!}} \sum_{k=0}^{n} (-1)^k k! \binom{m}{k} \binom{n}{k} \frac{1}{2^{2k-n-m}} \bar{a}^{m-k} a^{n-k} \rho_{00}(a, \bar{a}).$$

The above equation can be written alternatively when passing to the polar coordinates (r, θ)

$$\omega x + ip = re^{i\theta}.$$

Then we have

$$a(r,\theta)=\frac{1}{\sqrt{2\hbar\omega}}re^{i\theta},\quad \bar{a}(r,\theta)=\frac{1}{\sqrt{2\hbar\omega}}re^{-i\theta},\quad r^2=p^2+\omega^2x^2,$$

and Eq. (7.2.2) takes the form [10, 111, 140]

$$\rho_{mn}(r,\theta)=2(-1)^n\sqrt{\frac{n!}{m!}}\frac{1}{2^{n-m}}\left(\frac{r}{\sqrt{2\hbar\omega}}\right)^{m-n}\times L_n^{m-n}\left(\frac{2r^2}{\hbar\omega}\right)e^{-i(m-n)\theta}\exp\left(-\frac{r^2}{\hbar\omega}\right),$$

where

$$L_n^s(x)=\frac{x^{-s}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+s}\right)=\sum_{k=0}^{n}(-1)^k\frac{(n+s)!}{(n-k)!(s+k)!k!}x^k$$

are the generalized Laguerre polynomials. The stationary pure states of the harmonic oscillator are of the form

$$\rho_{nn}(r,\theta)=2(-1)^nL_n\left(\frac{2r^2}{\hbar\omega}\right)\exp\left(-\frac{r^2}{\hbar\omega}\right),\tag{7.2.22}$$

where $L_n(x)=L_n^0(x)$ are the Laguerre polynomials. Equation (7.2.22) can be also written in the following form

$$\rho_n(x,p)\equiv\rho_{nn}=2(-1)^nL_n\left(\frac{4H}{\hbar\omega}\right)\exp\left(-\frac{2H}{\hbar\omega}\right).\tag{7.2.23}$$

It is interesting to check to which classical states quantum states ρ_n in the limit $\hbar\to 0^+$ converge. Again it has to be calculated in a distributional sense, hence the limit $\lim_{\hbar\to0^+}\langle\rho_n,\phi\rangle$ has to be calculated for every test function ϕ. One finds that for fixed n

$$\lim_{\hbar\to0^+}\langle\rho_n,\phi\rangle=\phi(0,0)=\langle\delta(x)\delta(p),\phi\rangle$$

and hence

$$\lim_{\hbar\to0^+}\rho_n=\delta(x)\delta(p),$$

i.e. all quantum stationary pure states ρ_n of the harmonic oscillator converge, in the limit $\hbar \to 0^+$, to the single classical state $(x = 0, p = 0)$ describing a particle with the position and momentum equal 0. This result is not surprising as the state $(x = 0, p = 0)$ is the only classical stationary pure state of the harmonic oscillator.

The reader can find other interesting examples of $\star$-genvalue problems and their solutions in [75] and [76]. Moreover, the reader can find the general solution of $\star$-genvalue problem for Hamiltonians quadratic in phase space coordinates in [92].

Previous results of this subsection give us an ambiguous answer to the question of time development of the initial coherent state. We investigated it indirectly, calculating time development of uncertainty relation $\Delta Q(t)\Delta P(t)$, with minimal initial value $\Delta Q(0)\Delta P(0) = \frac{\hbar}{2}$. For a free particle we found that the coherence is not preserved during time evolution (7.2.2) while for harmonic oscillator it is preserved for a distinguished initial coherent state (7.2.15) for $\gamma = \omega$. Thus, let us investigate more systematically that problem for arbitrary linear Hamiltonian system in $\mathbb{R}^2$.

Let us consider a harmonic oscillator described by a Hamiltonian (7.2.19). It is convenient to introduce normalized variables

$$x \to \frac{1}{\sqrt{\omega}}x, \quad p \to \sqrt{\omega}p.$$

In these new variables the Hamiltonian of the harmonic oscillator takes the form

$$H = \tfrac{1}{2}\omega(p^2 + x^2). \tag{7.2.24}$$

Adding to (7.2.24) the interaction term

$$H_I(q, p) = \alpha xp + \tfrac{1}{2}\beta p^2 - \tfrac{1}{2}\beta x^2,$$

where $\alpha, \beta \in \mathbb{R}$ are some constants, we will consider the following Hamiltonian [100]

$$H(q, p) = \tfrac{1}{2}(\omega + \beta)p^2 + \tfrac{1}{2}(\omega - \beta)x^2 + \alpha xp. \tag{7.2.25}$$

Note that any Hamiltonian quadratic in x and p variables is of the above form for some values of constants ω, α and β. It should be noted that this type of Hamiltonian is very often found in quantum optics where admissible coherent and squeezed states of the light are investigated [132, 133, 144, 243, 257].

The classical and quantum Hamilton equations for time evolution of observables of position $Q(t)$ and momentum $P(t)$ take the common form

$$\begin{aligned} Q_t &= \alpha Q + (\beta + \omega)P, \\ P_t &= (\beta - \omega)Q - \alpha P. \end{aligned} \tag{7.2.26}$$

The Hamilton flow in a case $\omega^2 > \alpha^2 + \beta^2$ reads

$$Q(t) = \frac{\alpha x + (\omega + \beta)p}{R}\sin(Rt) + x\cos(Rt),$$
$$P(t) = -\frac{(\omega - \beta)x + \alpha p}{R}\sin(Rt) + p\cos(Rt),$$

where $R = \sqrt{|\omega^2 - \alpha^2 - \beta^2|}$. When $\omega^2 < \alpha^2 + \beta^2$ we get

$$Q(t) = \frac{\alpha x + (\omega + \beta)p}{R}\sinh(Rt) + x\cosh(Rt),$$
$$P(t) = -\frac{(\omega - \beta)x + \alpha p}{R}\sinh(Rt) + p\cosh(Rt),$$

and when $\omega^2 = \alpha^2 + \beta^2$

$$Q(t) = x + (\alpha x + (\omega + \beta)p)t,$$
$$P(t) = p - ((\omega - \beta)x + \alpha p)t.$$

First, we will focus on the case $\omega^2 > \alpha^2 + \beta^2$. In an initial coherent state

$$\rho(q, p) = 2\exp\left(-\frac{\gamma(x' - x)^2}{\hbar}\right)\exp\left(-\frac{(p' - p)^2}{\hbar\gamma}\right)$$

we receive the following formulas for the uncertainties $(\Delta Q)^2$ and $(\Delta P)^2$

$$\begin{aligned}(\Delta Q)^2 &= \frac{\hbar}{2}\gamma^{-1}\left(\frac{\alpha^2 + \gamma^2(\omega + \beta)^2}{R^2}\sin^2(Rt) + \cos^2(Rt) + \frac{2\alpha}{R}\sin(Rt)\cos(Rt)\right)\\ &= \frac{\hbar}{2}\gamma^{-1}\left(1 + \frac{2(\alpha^2 + \beta^2) + 2\gamma^2\omega\beta + (\gamma^2 - 1)(\omega^2 + \beta^2)}{R^2}\sin^2(Rt)\right.\\ &\qquad\left. + \frac{2\alpha}{R}\sin(Rt)\cos(Rt)\right),\\ (\Delta P)^2 &= \frac{\hbar}{2}\gamma\left(\frac{\alpha^2 + \gamma^{-2}(\omega - \beta)^2}{R^2}\sin^2(Rt) + \cos^2(Rt) - \frac{2\alpha}{R}\sin(Rt)\cos(Rt)\right)\\ &= \frac{\hbar}{2}\gamma\left(1 + \frac{2(\alpha^2 + \beta^2) - 2\gamma^{-2}\omega\beta + (\gamma^{-2} - 1)(\omega^2 + \beta^2)}{R^2}\sin^2(Rt)\right.\\ &\qquad\left. - \frac{2\alpha}{R}\sin(Rt)\cos(Rt)\right).\end{aligned}$$

Introducing a parameter $\xi = \alpha + i\beta = re^{i\theta}$ and writing it in polar variables (r, θ) the uncertainties $(\Delta Q)^2$ and $(\Delta P)^2$ take the form

$$(\Delta Q)^2 = \frac{\hbar}{2}\gamma^{-1}\left(1 + \frac{\sin^2(Rt)}{R}\left(\frac{2r^2 + 2\gamma^2\omega r\sin\theta + (\gamma^2 - 1)(\omega^2 + r^2\sin^2\theta)}{R} + 2r\cos\theta\cot(Rt)\right)\right),$$
$$(\Delta P)^2 = \frac{\hbar}{2}\gamma\left(1 + \frac{\sin^2(Rt)}{R}\left(\frac{2r^2 - 2\gamma^{-2}\omega r\sin\theta + (\gamma^{-2} - 1)(\omega^2 + r^2\sin^2\theta)}{R} - 2r\cos\theta\cot(Rt)\right)\right). \tag{7.2.27}$$

From (7.2.27) we find that coherence is not preserved during time evolution when parameters ω, α, β are arbitrary.

In the special case $\gamma = 1$ ($\gamma = \omega$ for old x and p)

$$(\Delta Q)^2 = \frac{\hbar}{2}\left(1 + \frac{2r}{R}\sin^2(Rt)\left(\frac{r + \omega\sin\theta}{R} + \cos\theta\cot(Rt)\right)\right),$$
$$(\Delta P)^2 = \frac{\hbar}{2}\left(1 + \frac{2r}{R}\sin^2(Rt)\left(\frac{r - \omega\sin\theta}{R} - \cos\theta\cot(Rt)\right)\right).$$

and

$$(\Delta q)^2(\Delta p)^2 = \frac{\hbar^2}{4}\left(1 + \frac{4r^2}{R^2}\sin^2(Rt)\left(\frac{\omega}{R}\cos\theta\sin(Rt) - \sin\theta\cos(Rt)\right)^2\right). \tag{7.2.28}$$

From (7.2.28) it follows that the minimization of the Heisenberg uncertainty relation occurs only for $Rt = k\pi$ and $Rt = \arctan(\frac{R}{\omega}\tan\theta) + k\pi$, $k \in \mathbb{Z}$. Thus, we will consider a further reduction. First, let us take $\beta = 0$ and $\alpha > 0$. Then $r = \alpha$ and $\theta = 0$. In this case we receive

$$(\Delta Q)^2 = \frac{\hbar}{2}\left(1 + \frac{2\alpha}{R}\sin^2(Rt)\left(\frac{\alpha}{R} + \cot(Rt)\right)\right),$$
$$(\Delta P)^2 = \frac{\hbar}{2}\left(1 + \frac{2\alpha}{R}\sin^2(Rt)\left(\frac{\alpha}{R} - \cot(Rt)\right)\right),$$

and

$$(\Delta Q)^2(\Delta P)^2 = \frac{\hbar^2}{4}\left(1 + \frac{4\omega^2\alpha^2}{R^4}\sin^4(Rt)\right). \tag{7.2.29}$$

From (7.2.29) it follows that the coherence is preserved during time evolution if additionally $\alpha = 0$, but this is exactly the case of the harmonic oscillator (7.2.24).

Now, let us consider the case $\omega^2 < \alpha^2 + \beta^2$ by taking $\omega = 0$. Then $R = r$ and

$$\begin{aligned} Q(q, p, t) &= (q\cos\theta + p\sin\theta)\sinh(rt) + q\cosh(rt), \\ P(q, p, t) &= (q\sin\theta - p\cos\theta)\sinh(rt) + p\cosh(rt). \end{aligned} \tag{7.2.30}$$

Moreover,

$$\begin{aligned} (\Delta Q)^2 &= \frac{\hbar}{2}\gamma^{-1}\big(\cosh(2rt) + \cos\theta\sinh(2rt) + (\gamma^2 - 1)\sin^2\theta\sinh^2(rt)\big), \\ (\Delta P)^2 &= \frac{\hbar}{2}\gamma\big(\cosh(2rt) - \cos\theta\sinh(2rt) + (\gamma^{-2} - 1)\sin^2\theta\sinh^2(rt)\big). \end{aligned} \tag{7.2.31}$$

If additionally $\gamma = 1$ we get

$$\begin{aligned} (\Delta Q)^2 &= \frac{\hbar}{2}\big(\cosh(2rt) + \cos\theta\sinh(2rt)\big) = \frac{\hbar}{2}\left(e^{2rt}\cos^2\frac{\theta}{2} + e^{-2rt}\sin^2\frac{\theta}{2}\right), \\ (\Delta P)^2 &= \frac{\hbar}{2}\big(\cosh(2rt) - \cos\theta\sinh(2rt)\big) = \frac{\hbar}{2}\left(e^{-2rt}\cos^2\frac{\theta}{2} + e^{2rt}\sin^2\frac{\theta}{2}\right), \end{aligned}$$

and

$$(\Delta Q)^2(\Delta P)^2 = \frac{\hbar^2}{4}\left(1 + \sin^2\theta\sinh^2(2rt)\right). \tag{7.2.32}$$

Again, as it is evident from (7.2.32), the Heisenberg uncertainty relation is not minimized during the whole time evolution. In order to get a minimal uncertainty for any t we have to take $\beta = 0$. Then $\theta = 0$, $r = \alpha$, equations (7.2.30) reduce to

$$\begin{aligned} Q(q, p, t) &= qe^{\alpha t}, \\ P(q, p, t) &= pe^{-\alpha t}, \end{aligned}$$

variances (7.2.31) are

$$\begin{aligned} (\Delta Q)^2 &= \frac{\hbar}{2}\gamma^{-1}e^{2\alpha t}, \\ (\Delta P)^2 &= \frac{\hbar}{2}\gamma e^{-2\alpha t}, \end{aligned}$$

and hence we get a conservation of minimal uncertainty

$$(\Delta Q)^2(\Delta P)^2 = \frac{\hbar^2}{4}.$$

Finally, let us consider a case $\omega^2 = \alpha^2 + \beta^2$. In this case

$$\begin{aligned}(\Delta Q)^2 &= \frac{\hbar}{2}\left(\gamma^{-1}(1+\alpha t)^2 + \gamma(\beta+\omega)^2 t^2\right),\\ (\Delta P)^2 &= \frac{\hbar}{2}\left(\gamma^{-1}(\beta-\omega)^2 t^2 + \gamma(1-\alpha t)^2\right),\end{aligned} \tag{7.2.33}$$

and

$$\begin{aligned}&(\Delta Q)^2(\Delta P)^2 \\ &= \frac{\hbar^2}{4}\left(1 - 2\alpha^2 t^2 + 2\alpha^4 t^4 + \gamma^{-2}(1+\alpha t)^2(\beta-\omega)^2 t^2 + \gamma^2(1-\alpha t)^2(\beta+\omega)^2 t^2\right).\end{aligned} \tag{7.2.34}$$

It can be seen that during time development the variance of position $(\Delta Q)^2$ and momentum $(\Delta P)^2$ increase quadratically with time, so the minimal uncertainty is not preserved for any t except $t = 0$. In the particular case, when $\beta = \omega = \frac{1}{2}$ and $\alpha = 0$, formulas (7.2.33) and (7.2.34) reduce to these for a free particle, considered at the beginning of this subsection.

Note that in a case $\omega^2 > \alpha^2 + \beta^2$ the following one-parameter family of linear canonical transformations of coordinates

$$\begin{aligned}q' &= \frac{Ra + \alpha A}{\omega + \beta} q + Ap,\\ p' &= \frac{\alpha a - RA}{\omega + \beta} q + ap,\end{aligned} \qquad A = \pm\sqrt{\frac{\omega+\beta}{R} - a^2}, \quad a \in \mathbb{R}$$

transforms the Hamiltonian (7.2.25) into the following Hamiltonian of the harmonic oscillator

$$H(q', p') = \tfrac{1}{2} R(p'^2 + q'^2).$$

On the other hand, in a case $\omega^2 < \alpha^2 + \beta^2$ another one-parameter family of linear canonical transformations of coordinates

$$\begin{aligned}q' &= -\frac{R+\alpha}{\omega+\beta} aq - ap,\\ p' &= \frac{R-\alpha}{2Ra} q - \frac{\omega+\beta}{2Ra} p,\end{aligned} \qquad a \in \mathbb{R}$$

transforms the Hamiltonian (7.2.25) into the following Hamiltonian

$$H(q', p') = Rq'p'.$$

For both Hamiltonians $H(q', p')$ the initial coherence is preserved during time evolution.

Observation 16 *From the above considerations it follows that the conservation of coherence property during time development of quantum state is rather rare phenomenon. Even for a three-parameter family of linear quantum Hamiltonian equations (7.2.26) in $\mathbb{R}^2$, initial coherence is preserved during time evolution only for two cases: $\beta = \alpha = 0$ and $\beta = \omega = 0$. So why should we expect such a property for nonlinear quantum Hamiltonian equations? On the other hand, when $\omega^2 > \alpha^2 + \beta^2$ we can always reduce the dynamics to the harmonic oscillator ($\beta = \alpha = 0$), and when $\omega^2 < \alpha^2 + \beta^2$ we can reduce the dynamics to the case $\omega = \beta = 0$, provided that we will be working with new variables q', p'. In the frame of original variables q, p it means that for the considered class of systems there always exist canonically conjugated observables $q' = q'(q, p)$, $p' = p'(q, p)$ for which the minimal uncertainty is preserved during time evolution.*

At the end of this subsection let us consider a system of two degrees of freedom described by the Hamiltonian cubic in phase space coordinates

$$H(x, p) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + kx^1 p_2^2, \tag{7.2.35}$$

where m_1, m_2 are masses of particles and k is a coupling constant. Quantum equations of motion (7.1.16) for observables of position and momentum are of the form

$$\begin{aligned}
(Q^1)_t &= \frac{1}{m_1} P_1, \\
(Q^2)_t &= \frac{1}{m_2} P_2 + 2k Q^1 \star P_2, \\
(P_1)_t &= -k P_2 \star P_2, \\
(P_2)_t &= 0.
\end{aligned} \tag{7.2.36}$$

Hamiltonian (7.2.35) is specific because x^2 is a cyclic coordinate, so P_2 is a constant of motion equal to its initial value $P_2 = p_2$ and in consequence, equations (7.2.36) reduce to their classical counterparts

$$\begin{aligned}
(Q^1)_t &= \frac{1}{m_1} P_1, \\
(Q^2)_t &= \frac{1}{m_2} P_2 + 2k Q^1 P_2, \\
(P_1)_t &= -k (P_2)^2, \\
(P_2)_t &= 0.
\end{aligned}$$

with the solution

$$\begin{aligned} Q^1(t) &= x^1 + \frac{1}{m_1} p_1 t - \frac{k}{2m_1} p_2^2 t^2, \\ P_1(t) &= p_1 - k p_2^2 t, \\ Q^2(t) &= x^2 + \left(\frac{1}{m_2} p_2 + 2k x^1 p_2 \right) t + \frac{k}{m_1} p_1 p_2 t^2 - \frac{k^2}{3m_1} p_2^3 t^3, \\ P_2(t) &= p_2, \end{aligned} \tag{7.2.37}$$

which again represents a common quantum and classical trajectory of the considered two particle system. Hence, the flow (7.2.37) represents a one-parameter family (group) of classical and quantum canonical transformations $T(x^1, x^2, p_1, p_2) = (Q^1, Q^2, P_1, P_2)$ in a four-dimensional phase space $\mathbb{R}^4$ with the following generating function

$$\begin{aligned} F(x^1, x^2, P_1, P_2) =& x^1 P_1 + x^2 P_2 + k t x^1 (P_2)^2 + \frac{1}{2m_1} t (P_1)^2 + \frac{1}{2m_2} t (P_2)^2 \\ &+ \frac{k}{2m_1} t^2 P_1 (P_2)^2 + \frac{k^2}{6m_1} t^3 (P_2)^4. \end{aligned}$$

Note, that this transformation is a four-dimensional example of the transformation generated by F_4 from Sect. 6.1.4. In accordance with (7.2.1) the received quantum flow Φ_t transforms the Moyal product to the following product

$$f \star_t g = f \exp\left(\tfrac{1}{2} i\hbar \overleftarrow{D}_{x^1} \overrightarrow{D}_{p_1} + \tfrac{1}{2} i\hbar \overleftarrow{D}_{x^2} \overrightarrow{D}_{p_2} - \tfrac{1}{2} i\hbar \overleftarrow{D}_{p_1} \overrightarrow{D}_{x^1} - \tfrac{1}{2} i\hbar \overleftarrow{D}_{p_2} \overrightarrow{D}_{x^2} \right) g,$$

where

$$\begin{aligned} D_{x^1} &= \partial_{x^1} + 2kt p_2 \partial_{x^2}, \\ D_{x^2} &= \partial_{x^2}, \\ D_{p_1} &= \partial_{p_1} + \frac{1}{m_1} t \partial_{x^1} + \frac{k}{m_1} t^2 p_2 \partial_{x^2}, \\ D_{p_2} &= \partial_{p_2} - 2kt p_2 \partial_{p_1} - \frac{k}{m_1} t^2 p_2 \partial_{x^1} + \left(\frac{1}{m_2} t + 2kt x^1 - \frac{k}{m_1} t^2 p_1 - \frac{k^2}{m_1} t^3 p_2^2 \right) \partial_{x^2}. \end{aligned}$$

Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product at $t = 0$ with the $\star_t$-product takes the form

$$S_t = \exp\left[\tfrac{1}{4} k\hbar^2 \left(\frac{1}{2} \frac{1}{m_1} t^2 \partial_{x^1} \partial_{x^2}^2 + t \partial_{p_1} \partial_{x^2}^2 + \frac{1}{3} \frac{k p_2}{m_1} t^3 \partial_{x^2}^3 \right) \right].$$

.

It can be also proved that S_t is an isomorphism (unitary operator) of the Hilbert space $L^2(\mathbb{R}^4)$ onto itself.

As in this case $S_{t_2}\Phi_{t_1} = \Phi_{t_1}$, the group multiplication for $\{\Phi_t\}$ is just a composition of maps, as one could expect since Φ_t is simultaneously the classical and quantum trajectory. However, the action of Φ_t on observables and states does not reduce in general to a composition of maps (7.2.1) like in the classical case as now $S_t \neq 1$, which is a direct consequence of the fact that the Hamiltonian is a cubic function of phase space coordinates. As the result, the time evolution of quantum observables is governed by (7.2.4). This shows that for the considered case the time evolution of quantum observables differs in general from the time evolution of classical observables.

One can check by direct calculations that the action of the quantum flow Φ_t on an observable A, given by (7.2.4), indeed describes the quantum time evolution of A. As the illustration of that fact let us take $A(x, p) = x_1 x_2^2$. Then

$$(S_t A)(x, p) = x_1 x_2^2 + \tfrac{1}{4}\hbar^2 \frac{k}{m_1} t^2$$

and it can be checked by direct computation that

$$A(t) = (S_t A) \circ \Phi_t = Q^1(t)(Q^2(t))^2 + \tfrac{1}{4}\hbar^2 \frac{k}{m_1} t^2$$

satisfies the time evolution equation (7.1.13).

7.2.3 *Pure Quantum Trajectories*

We discuss the concept of quantum trajectories on a simple, but far from being trivial, example of a system described by a Hamiltonian

$$H(x, p) = \kappa x^2 p^2.$$

The Moyal dynamics (7.1.16) takes the form (see Example 7.1)

$$\begin{aligned} Q_t &= \kappa Q \star Q \star P + \kappa P \star Q \star Q, \qquad Q(0) = x, \\ P_t &= -\kappa Q \star P \star P - \kappa P \star P \star Q, \qquad P(0) = p. \end{aligned} \tag{7.2.38}$$

We briefly sketch how to find the solution of the considered quantum dynamics [93]. Since $[\![Q, P]\!] = 1$ at any time t, we have

$$\begin{aligned} \frac{d}{dt}(Q \star P) =& Q_t \star P + Q \star P_t \\ =& \kappa P \star Q \star Q \star P - \kappa Q \star P \star P \star Q \end{aligned}$$

$$
\begin{aligned}
&=\kappa P \star Q \star Q \star P - \kappa P \star Q \star Q \star P + i\hbar\kappa P \star Q - i\hbar\kappa P \star Q \\
&=0
\end{aligned}
$$

and so, as in the classical case, $Q \star P$ is a constant of motion

$$
Q(x, p, t) \star^{(x,p)} P(x, p, t) = x \star p = xp + \tfrac{1}{2} i\hbar. \tag{7.2.39}
$$

Substituting (7.2.39) in the Eq. (7.2.38) we get

$$
\begin{aligned}
Q_t &= \kappa Q \star (xp) + \kappa (xp) \star Q, \qquad Q(0) = x, \\
P_t &= -\kappa (xp) \star P - \kappa P \star (xp), \qquad P(0) = p.
\end{aligned}
$$

One can immediately check that the solution of this equations is

$$
\begin{aligned}
Q(x, p, t) &= \exp_\star(\kappa t xp) \star x \star \exp_\star(\kappa t xp) \\
P(x, p, t) &= \exp_\star(-\kappa t xp) \star p \star \exp_\star(-\kappa t xp).
\end{aligned} \tag{7.2.40}
$$

By construction this is a unitary transformation.

In order to go further we need the explicit form of $\star$-exponential from (7.2.40). Following a technique developed in [12], in [92] was derived the $\star$-exponential for any polynomial of second degree in phase space coordinates

$$
H = A_{\alpha\beta}\xi^\alpha \xi^\beta + B_\alpha \xi^\alpha,
$$

where A is a symmetric, nonsingular, $2n \times 2n$ matrix. In a particular case, for factorization

$$
A = S_A^T S_A
$$

such that

$$
S_A \omega S_A^T = a\omega,
$$

where ω is a symplectic matrix (6.1.2) and $a \in \mathbb{C}$, the noncommutative exponential is given by

$$
\boxed{\exp_\star(\gamma H) = [\cos(i\hbar a\gamma)]^{-n} \exp\left\{ \frac{H}{i\hbar a} \tan(i\hbar a\gamma) + \frac{1}{4i\hbar a} B^T A^{-1} B \left[\tan(i\hbar a\gamma) - i\hbar a\gamma\right] \right\}.}
$$

In our case (7.2.40) $H = xp$ and

$$
B = 0, \quad A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}, \quad S_A = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad a = \tfrac{1}{2} i
$$

hence

$$\exp_\star(\pm\kappa t x p) = \sec\left(\tfrac{1}{2}\kappa\hbar t\right)\exp\left[\pm\frac{2}{\hbar}xp\tan\left(\tfrac{1}{2}\kappa\hbar t\right)\right]$$

and so

$$\begin{aligned} Q(x,p,t) &= \exp_\star(\kappa t x p)\star x\star\exp_\star(\kappa t x p)\\ &= \left[1+i\tan\left(\tfrac{1}{2}\kappa\hbar t\right)\right]\exp_\star(\kappa t x p)\star\left[x\exp_\star(\kappa t x p)\right]\\ P(x,p,t) &= \exp_\star(-\kappa t x p)\star p\star\exp_\star(-\kappa t x p)\\ &= \left[1+i\tan\left(\tfrac{1}{2}\kappa\hbar t\right)\right]\exp_\star(-\kappa t x p)\star\left[p\exp_\star(-\kappa t x p)\right]. \end{aligned}$$

Applying the integral representation (6.1.45) of the $\star$-product, after some calculation we get the final solution of quantum Hamiltonian equations (7.2.38) in the form [93]

$$Q(x,p,t;\hbar) = \sec^2(\kappa\hbar t)x\exp\left(\frac{2}{\hbar}\tan(\kappa\hbar t)xp\right), \tag{7.2.41a}$$

$$P(x,p,t;\hbar) = \sec^2(\kappa\hbar t)p\exp\left(-\frac{2}{\hbar}\tan(\kappa\hbar t)xp\right), \tag{7.2.41b}$$

for $t\neq\frac{2k+1}{2}\frac{\pi}{\kappa\hbar}$, $k\in\mathbb{Z}$. This solution (7.2.41) is a deformation of a classical one given by the limit $\hbar\to 0$

$$Q_C(x,p,t) = xe^{2t\kappa xp},\qquad P_C(x,p,t) = pe^{-2t\kappa xp}.$$

The induced quantum flow Φ_t is an example of a flow for which Φ_t, for every $t\neq\frac{k\pi}{\kappa\hbar}$, is not a classical symplectomorphism, since

$$\{Q(t),P(t)\} = \sec^4(\kappa\hbar t)\neq 1.$$

In accordance with (7.2.1) the quantum flow Φ_t transforms the Moyal product to the following product

$$f\star_t g = f\exp\left(\tfrac{1}{2}i\hbar\overleftarrow{D}_x\overrightarrow{D}_p-\tfrac{1}{2}i\hbar\overleftarrow{D}_p\overrightarrow{D}_x\right)g,$$

where

$$\begin{aligned} D_x &= \sec^2(\kappa\hbar t)\big(1+2t\chi(\kappa\hbar t)xp\big)\exp\big(2t\chi(\kappa\hbar t)xp\big)\partial_x\\ &\quad -2t\chi(\kappa\hbar t)\sec^2(\kappa\hbar t)p^2\exp\big(2t\chi(\kappa\hbar t)xp\big)\partial_p, \end{aligned}$$

$$D_p = 2t\chi(\kappa\hbar t)\sec^2(\kappa\hbar t)x^2\exp\big(-2t\chi(\kappa\hbar t)xp\big)\partial_x + \sec^2(\kappa\hbar t)\big(1-2t\chi(\kappa\hbar t)xp\big)\exp\big(-2t\chi(\kappa\hbar t)xp\big)\partial_p,$$

and $\chi(\kappa\hbar t) = \frac{\tan(\kappa\hbar t)}{\kappa\hbar t\sec^4(\kappa\hbar t)}$. Moreover, the isomorphism S_t associated with Φ_t and intertwining the Moyal product with the $\star_t$-product, up to the second order in $\hbar$, takes the form

$$\begin{aligned} S_t = 1 + \hbar^2\kappa^2\Big(&\tfrac{1}{6}(3t^2x^3 + 4t^3x^4p)\partial_x^3 + \tfrac{1}{6}(3t^2p^3 - 4t^3xp^4)\partial_p^3 \\ &+ \tfrac{1}{2}(-tp - t^2xp^2 + 4t^3x^2p^3)\partial_x\partial_p^2 + \tfrac{1}{2}(tx - t^2x^2p - 4t^3x^3p^2)\partial_x^2\partial_p \\ &+ (2t^2x^2 + 2t^3x^3p)\partial_x^2 + (2t^2p^2 - 2t^3xp^3)\partial_p^2 + (-2t^2xp)\partial_x\partial_p\Big) + O(\hbar^4). \end{aligned} \tag{7.2.42}$$

In fact, expanding relations (6.1.77) with respect to $\hbar$ one can show that S_t in the above form satisfies these relations up to $O(\hbar^2)$.

From the fact that Φ_t is a purely quantum trajectory, we deal with the quantum group multiplication (7.2.6) for $\{\Phi_t\}$ as well as the quantum action (7.2.4) of Φ_t on observables and states. Indeed, expanding (7.2.41) with respect to $\hbar$:

$$\begin{aligned} Q(x,p,t;\hbar) &= Q_C\left(1 + \hbar^2\kappa^2\left(t^2 + \tfrac{2}{3}t^3xp\right)\right) + O(\hbar^4), \\ P(x,p,t;\hbar) &= P_C\left(1 + \hbar^2\kappa^2\left(t^2 - \tfrac{2}{3}t^3xp\right)\right) + O(\hbar^4) \end{aligned}$$

and applying isomorphism S_t (7.2.42), the quantum composition law

$$\begin{aligned} Q(t_1+t_2) &= S_{t_2}Q(t_1)\circ\Phi_{t_2} = S_{t_1}Q(t_2)\circ\Phi_{t_1}, \\ P(t_1+t_2) &= S_{t_2}P(t_1)\circ\Phi_{t_2} = S_{t_1}P(t_2)\circ\Phi_{t_1} \end{aligned}$$

holds up to $O(\hbar^2)$. Note also that the flow Φ_t is not defined for all $t \in \mathbb{R}$ as it is singular for $t = \frac{2k+1}{2}\frac{\pi}{\kappa\hbar}$, contrary to classical flows which are globally defined. This is an interesting result showing that in general the quantum time evolution do not have to be defined for all instances of time t.

Observation 17 *Singularities of classical trajectories are not admissible as each classical trajectory represents measurable quantities, actually expectation values of position and momentum of a system in a pure coherent classical state (7.2.13) for all $t \in \mathbb{R}$. On the contrary, pure quantum trajectories themselves are not "physical" objects as states (7.2.13) are not admissible so, singularities of pure quantum trajectories are acceptable.*

Let us come back to the singular quantum trajectory (7.2.41). Through direct integration we can calculate the expectation values of Q and P from (7.2.41) in the coherent state (7.2.15). The result after introducing

$$a(t) = \frac{\cos(\kappa\hbar t)}{\sqrt{\cos(2\kappa\hbar t)}}x + \frac{\gamma^{-1}\sin(\kappa\hbar t)}{\sqrt{\cos(2\kappa\hbar t)}}p,$$
$$b(t) = -\frac{\gamma\sin(\kappa\hbar t)}{\sqrt{\cos(2\kappa\hbar t)}}x + \frac{\cos(\kappa\hbar t)}{\sqrt{\cos(2\kappa\hbar t)}}p,$$

reads

$$\langle Q\rangle_\rho = \frac{a(t)}{\cos(2\kappa\hbar t)}\exp\left(\frac{\gamma}{\hbar}\left(a^2(t) - x^2\right)\right),$$
$$\langle P\rangle_\rho = \frac{b(t)}{\cos(2\kappa\hbar t)}\exp\left(\frac{1}{\hbar\gamma}\left(b^2(t) - p^2\right)\right).$$

Note, that $\langle Q\rangle_\rho$ and $\langle P\rangle_\rho$ are well defined only on intervals $(-\frac{1}{4}+n)\frac{\pi}{\kappa\hbar} < t < (\frac{1}{4}+n)\frac{\pi}{\kappa\hbar}$, $n \in \mathbb{Z}$. This once again shows that time evolution of the considered system is not defined for all values of the evolution parameter t and even time development of expectation values of position and momentum is only well defined on certain intervals of t.

Observation 18 *We have found that for the considered quantum trajectories, expectation values of observables of position and momentum in the coherent state* (7.2.15) *were well defined only on certain intervals of t, which raises problems and questions of interpretation of such a kind of time evolution. If we assume, like in the classical case, that the expectation values of position and momentum have to be smooth functions for any $t \in \mathbb{R}$, then we have two options. Either, for a chosen quantization, there exist quantum states for which our assumption is fulfilled (the state* (7.2.15) *does not belong to that class) or, if there are no such states, our quantization is not 'physical' and we have to chose another quantization which fulfills the imposed assumption.*

The above observation was made from the mathematical point of view. Let's have a look on the problem from the physical side. In other words, let us asses the length of the interval on which time evolution of the system is well defined. Notice that dimension of κ in joule-seconds is $J^{-1}s^{-2}$ so we put $\kappa = |\kappa|\, J^{-1}s^{-2}$ and moreover $\hbar \simeq 10^{-34}Js$. So, the length of the time interval $\frac{\pi}{4\hbar\kappa}$ will be approximately equal $|\kappa|^{-1}\, 0.785 \times 10^{34}s$. Notice that the age of Universe is $0.437 \times 10^{18}s$. So, for a large range of κ singularities appearing in time evolution are nonphysical.

After reading this chapter the reader might be disappointed with a small number of examples of stationary problems of known quantum systems, presented in deformation quantization formalism. The exception was made for the case of quantum harmonic oscillator. The reason is that such calculations directly in a

Hilbert space over the phase space are very cumbersome and complex. Fortunately, at least for the canonical quantization, we can simplify that problem passing to so called position representation of quantum systems. This is the subject of the next chapter in which we present many known and new examples of separable eigenvalue quantum problems defined in an appropriate Hilbert spaces over Riemannian configuration spaces.

Chapter 8
Position Representation of Quantum Mechanics over Riemannian Configuration Space

The last chapter of the book is devoted to two very important issues of the developed quantum theory. The first one is related with systematic construction of the so called position representation of quantum mechanics over an appropriate class of Riemaniann spaces in any admissible local curvilinear coordinates. In particular, for a flat space and Cartesian coordinates we reconstruct the standard quantization procedure from textbooks of quantum mechanics. The second issue of that chapter is related with quantum integrability (quantum superintegrability) and quantum separability. Actually, we present the reader a class of quantizations of classical Stäckel systems considered in previous chapters, which preserve quantum integrability, quantum superintegrability and quantum stationary separability of related quantum Hamiltonian operators.

8.1 Operator Representation over Riemannian Space

In this section we will present a coordinate free construction of a natural operator representation of quantum mechanics [33, 36, 37, 98], which reproduces the usual Hilbert space approach to quantum mechanics. We will be dealing with quantum systems defined on a phase space M in the form of a cotangent bundle T^*Q to the almost geodesically simply connected Riemannian manifold Q. The manifold Q plays the role of a configuration space of the system. The representation will be constructed in a Hilbert space $L^2(Q, d\omega_g)$ of functions on Q, square integrable with respect to a measure $d\mu(x) = |g|^{1/2}(x)\,dx$ induced by the metric volume form ω_g on Q, where $|g|(x) = |\det[g_{ij}(x)]|$, and with standard inner product

$$(\varphi, \psi) = \int_Q \bar{\varphi}(x)\psi(x)d\mu(x). \tag{8.1.1}$$

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The elements of $L^2(Q, d\omega_g)$ are interpreted as wave functions describing the states of the quantum system in position representation.

8.1.1 Moyal Quantization

First, let us consider the phase space $M = T^*U$, where U is an open subset of $\mathbb{R}^n$, such that $\mathbb{R}^n \backslash U$ is of measure zero, endowed with a metric tensor g. Then, we will consider a classical system defined on M and its quantization with respect to the Moyal product on M. The construction of the position representation for such a quantum system we begin with an observation that the Hilbert space $L^2(T^*U, d\Omega_\hbar)$ can be written in the form of a tensor product of the Hilbert space $L^2(U, d\mu)$ and the space dual to it. Thus we start with the explicit construction of this tensor product.

In accordance with the Riesz representation theorem, the Hilbert space $(L^2(U, d\mu))^*$ dual to $L^2(U, d\mu)$ is isomorphic to the Hilbert space $L^2(U, d\mu)$ and can be naturally identified with $L^2(U, d\mu)$ itself. Actually, such linear isomorphism $J\colon L^2(U, d\mu) \to (L^2(U, d\mu))^*$ takes the form $J(\psi) = (\bar{\psi}, \cdot)$ (8.1.1). Let us denote by $L^2(TU)$ the Hilbert space of functions from the tangent bundle $TU = U \times \mathbb{R}^n$, square integrable with respect to the Lebesgue measure on $U \times \mathbb{R}^n$. Then, let us define a bilinear map of Hilbert spaces $\tilde{W}\colon (L^2(U, d\mu))^* \times L^2(U, d\mu) \to L^2(TU)$, which on vectors $\varphi, \psi \in C_0^\infty(U)$ takes the form

$$\boxed{\tilde{W}(\bar{\varphi}, \psi)(x, y) = \overline{\varphi}(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y)\zeta(x, y),} \tag{8.1.2}$$

where

$$\zeta(x, y) = |g|^{1/4}(x - \tfrac{1}{2}y)\,|g|^{1/4}(x + \tfrac{1}{2}y).$$

For $\varphi_1, \psi_1, \varphi_2, \psi_2 \in C_0^\infty(U)$ it holds

$$\boxed{\left\langle \tilde{W}(\bar{\varphi_1}, \psi_1), \tilde{W}(\bar{\varphi_2}, \psi_2) \right\rangle = (\bar{\varphi_1}, \bar{\varphi_2})(\psi_1, \psi_2).} \tag{8.1.3}$$

Indeed,

$$\begin{aligned} \left\langle \tilde{W}(\bar{\varphi_1}, \psi_1), \tilde{W}(\bar{\varphi_2}, \psi_2) \right\rangle &= \int_{U\times\mathbb{R}^n} \overline{\tilde{W}(\bar{\varphi_1}, \psi_1)}(x, y)\tilde{W}(\bar{\varphi_2}, \psi_2)(x, y)\,dx\,dy \\ &= \int_{U\times\mathbb{R}^n} \varphi_1(x - \tfrac{1}{2}y)\overline{\psi}_1(x + \tfrac{1}{2}y)\overline{\varphi}_2(x - \tfrac{1}{2}y)\psi_2(x + \tfrac{1}{2}y) \\ &\quad \times |g|^{1/2}(x - \tfrac{1}{2}y)\,|g|^{1/2}(x + \tfrac{1}{2}y)\,dx\,dy. \end{aligned} \tag{8.1.4}$$

Note that since $\varphi_1, \psi_1, \varphi_2, \psi_2$ have a compact support, the integration in (8.1.4) can be extended to the whole space $\mathbb{R}^n \times \mathbb{R}^n$. Under the following change of variables

$$x' = x - \tfrac{1}{2}y,$$
$$x'' = x + \tfrac{1}{2}y.$$

Equation (8.1.4) transforms to

$$\begin{aligned}\left\langle \tilde{W}(\bar{\varphi_1}, \psi_1), \tilde{W}(\bar{\varphi_2}, \psi_2) \right\rangle &= \int_{\mathbb{R}^n} \overline{\varphi}_2(x')\varphi_1(x')\,|g|^{1/2}(x')\,dx' \\ &\quad \times \int_{\mathbb{R}^n} \overline{\psi}_1(x'')\psi_2(x'')\,|g|^{1/2}(x'')\,dx'' \\ &= (\varphi_1^*, \varphi_2^*)(\psi_1, \psi_2).\end{aligned}$$

From property (8.1.3) follows that $\tilde{W}$ is continuous on $C_0^\infty(U) \times C_0^\infty(U)$. Thus, as $C_0^\infty(U)$ is dense in $L^2(U, d\mu)$, it can be uniquely extended to a bilinear map defined on the whole space $(L^2(U, d\mu))^* \times L^2(U, d\mu)$ and satisfy (8.1.3). One can prove that finite linear combinations of vectors $\tilde{W}(\bar{\varphi}, \psi)$ for $\varphi, \psi \in L^2(U, d\mu)$ create a dense subset of $L^2(TU)$. In conclusion $\tilde{W}$ is a tensor product of Hilbert spaces $(L^2(U, d\mu))^*$ and $L^2(U, d\mu)$.

Next, let us take the Fourier transform (6.1.23) of $\tilde{W}(\bar{\varphi}, \psi)$ in momentum variable p conjugated to position variable y. In such a way we receive a bilinear map of Hilbert spaces $W\colon (L^2(U, d\mu))^* \times L^2(U, d\mu) \to L^2(T^*U, d\Omega_\hbar)$, which on arbitrary vectors $\varphi, \psi \in C_0^\infty(U)$ takes the form

$$\boxed{\begin{aligned}W(\bar{\varphi}, \psi)(x, p) &= \int_{\mathbb{R}^n} \tilde{W}(\bar{\varphi}, \psi)(x, y)e^{-\frac{i}{\hbar}y^k p_k}\,dy \\ &= \int_{\mathbb{R}^n} \overline{\varphi}(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y)\zeta(x, y)e^{-\frac{i}{\hbar}y^k p_k}\,dy.\end{aligned}} \tag{8.1.5}$$

Because the Fourier transform in momentum variable is an isomorphism of the Hilbert space $L^2(T^*U)$ onto the Hilbert space $L^2(TU)$, so W is also a tensor product of Hilbert spaces $(L^2(U, d\mu))^*$ and $L^2(U, d\mu)$. Let us denote this tensor product by $\otimes_W$

$$\boxed{\bar{\varphi} \otimes_W \psi \equiv W(\bar{\varphi}, \psi).}$$

In a case when $U = \mathbb{R}^n$ with a standard metric tensor g in flat coordinates, (8.1.5) is a well known from literature *Wigner transform* [2, 263].

Now let us prove a couple properties of the tensor product $\otimes_W$. For $\varphi, \psi \in L^2(U, d\mu)$ there holds

$$\boxed{\overline{\bar{\varphi} \otimes_W \bar{\psi}} = \bar{\psi} \otimes_W \varphi,} \tag{8.1.6}$$

$$\boxed{\int_{T^*U} (\bar{\varphi} \otimes_W \psi)\, d\Omega_\hbar = (\varphi, \psi).} \tag{8.1.7}$$

Formula (8.1.6) follows immediately from the definition (8.1.5). To prove (8.1.7) it is enough to consider $\varphi, \psi \in C_0^\infty(U)$ as the general case follows from the continuity of the tensor product $\otimes_W$ and the integral, and from the fact that $C_0^\infty(U)$ is dense in $L^2(U, d\mu)$. From (8.1.5) we have that

$$\begin{aligned}
&\frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} (\bar{\varphi} \otimes_W \psi)(x, p)\, dx\, dp = \frac{1}{(2\pi\hbar)^n} \int_{U \times \mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\varphi}(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y) \\
&\quad \times e^{-\frac{i}{\hbar} y^i p_i} \, |g|^{1/4}(x - \tfrac{1}{2}y)\, |g|^{1/4}(x + \tfrac{1}{2}y)\, dy\, dx\, dp \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\varphi}(x - \tfrac{1}{2}y)\psi(x + \tfrac{1}{2}y)\, |g|^{1/4}(x - \tfrac{1}{2}y)\, |g|^{1/4}(x + \tfrac{1}{2}y)\delta(y)\, dy\, dx \\
&= \int_{\mathbb{R}^n} \overline{\varphi}(x)\psi(x)\, |g|^{1/2}(x)\, dx = (\varphi, \psi).
\end{aligned}$$

Let $\rho_1 = \bar{\varphi}_1 \otimes_W \psi_1$ and $\rho_2 = \bar{\varphi}_2 \otimes_W \psi_2$ for $\varphi_1, \psi_1, \varphi_2, \psi_2 \in L^2(U, d\mu)$, then

$$\boxed{\rho_1 \star_M \rho_2 = (\varphi_1, \psi_2)(\bar{\varphi}_2 \otimes_W \psi_1).} \tag{8.1.8}$$

To prove formula (8.1.8) it is sufficient to consider $\varphi_1, \psi_1, \varphi_2, \psi_2 \in C_0^\infty(U)$ since the general case again follows from the continuity of tensor product $\otimes_W$, the Moyal product $\star_M$ and a scalar product, and from the fact that $C_0^\infty(U)$ is dense in $L^2(U, d\mu)$. From (6.1.46) we have that

$$\begin{aligned}
(\rho_1 \star_M \rho_2)(x, p) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{\rho}_1(x + \tfrac{1}{2}y, z)\tilde{\rho}_2(x - \tfrac{1}{2}z, y) e^{-\frac{i}{\hbar}(y^k + z^k)p_k}\, dy\, dz \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \overline{\varphi}_1(x + \tfrac{1}{2}y - \tfrac{1}{2}z)\psi_1(x + \tfrac{1}{2}y + \tfrac{1}{2}z)\, |g|^{1/4}(x + \tfrac{1}{2}y - \tfrac{1}{2}z) \\
&\quad \times |g|^{1/4}(x + \tfrac{1}{2}y + \tfrac{1}{2}z)\overline{\varphi}_2(x - \tfrac{1}{2}y - \tfrac{1}{2}z)\psi_2(x + \tfrac{1}{2}y - \tfrac{1}{2}z) \\
&\quad \times |g|^{1/4}(x - \tfrac{1}{2}y - \tfrac{1}{2}z)\, |g|^{1/4}(x + \tfrac{1}{2}y - \tfrac{1}{2}z) e^{-\frac{i}{\hbar}(y^k + z^k)p_k}\, dy\, dz.
\end{aligned}$$

and after the change of variables

$$\begin{aligned}
x' &= y + z, \\
x'' &= x + \tfrac{1}{2}y - \tfrac{1}{2}z,
\end{aligned}$$

we get

$$\begin{aligned}(\rho_1 \star_M \rho_2)(x,p) &= \int_{\mathbb{R}^N} \overline{\varphi}_1(x'')\psi_2(x'')\,|g|^{1/2}(x'')\,dx'' \\ &\quad\times \int_{\mathbb{R}^n} \overline{\varphi}_2(x-\tfrac{1}{2}x')\psi_1(x+\tfrac{1}{2}x')e^{-\frac{i}{\hbar}x'^k p_k} \\ &\quad\times |g|^{1/4}(x-\tfrac{1}{2}x')\,|g|^{1/4}(x+\tfrac{1}{2}x')\,dx' \\ &= (\varphi_1,\psi_2)(\bar{\varphi}_2 \otimes_W \psi_1)(x,p).\end{aligned}$$

Let $\{\varphi_i\}$ be an orthonormal basis in $L^2(U,d\mu)$, then $\{\rho_{ij}\} = \{\bar{\varphi}_i \otimes_W \varphi_j\}$ is an orthonormal basis in $L^2(T^*U, d\Omega_\hbar)$. From relations (8.1.6), (8.1.7), and (8.1.8) we find that the basis functions ρ_{ij} have the following properties:

$$\boxed{\bar{\rho}_{ij} = \rho_{ji},} \tag{8.1.9a}$$

$$\boxed{\int_{T^*U} \rho_{ij}\,d\Omega_\hbar = \delta_{ij},} \tag{8.1.9b}$$

$$\boxed{\rho_{ij} \star_M \rho_{kl} = \delta_{il}\rho_{kj}.} \tag{8.1.9c}$$

Using the basis $\{\rho_{ij}\}$ the following characterization of quantum states can be proved. Function $\rho \in L^2(T^*U, d\Omega_\hbar)$ is a quantum state, i.e. it satisfies conditions

1. $\rho = \bar{\rho}$,
2. $\int_{T^*U} \rho\,d\Omega_\hbar = 1$,
3. $\int_{T^*U} \bar{f} \star f \star \rho\,d\Omega_\hbar \geq 0$ for $f \in C_0^\infty(M)$

if and only if ρ is in the form

$$\rho = \sum_\lambda p_\lambda(\bar{\varphi}_\lambda \otimes_W \varphi_\lambda),$$

where $\varphi_\lambda \in L^2(U,d\mu)$, $\|\varphi_\lambda\| = 1$, $p_\lambda \geq 0$, and $\sum_\lambda p_\lambda = 1$.

Indeed, function ρ can be written in a form

$$\rho = \sum_{i,j} c_{ij}\rho_{ij},$$

where $c_{ij} \in \mathbb{C}$ and $\{\rho_{ij}\} = \{\bar{\varphi}_i \otimes_W \varphi_j\}$ is an induced basis in $L^2(T^*U, d\Omega_\hbar)$ by the basis $\{\varphi_i\}$ from $L^2(U,d\mu)$. Properties **1**–**3** mean that the coefficient matrix $\check{c} = (c_{ij})$ is self-adjoint ($\check{c} = \check{c}^\dagger$), normalized ($\operatorname{tr}\check{c} = 1$), and positively define ($c_{ii} \geq 0$). Indeed, self-adjointness and normalization follow from (8.1.9a) and (8.1.9b). In

order to prove the positive definite note that **3** is valid for every $f \in L^2(T^*U, d\Omega_\hbar)$ since $C_0^\infty(T^*U)$ is dense in $L^2(T^*U, d\Omega_\hbar)$. So, in particular for basis functions ρ_{kk}, with the help of (8.1.9), we have

$$0 \leq \int_{T^*U} \rho_{kk} \star_M \rho_{kk} \star_M \rho \, d\Omega_\hbar = \int_{T^*U} \rho_{kk} \star_M \rho \, d\Omega_\hbar = \sum_{i,j} c_{ij} \int_{T^*U} \rho_{kk} \star_M \rho_{ij} \, d\Omega_\hbar$$

$$= \sum_{i,j} c_{ij} \int_{T^*U} \delta_{kj} \rho_{ik} \, d\Omega_\hbar = \sum_i c_{ik} \int_{T^*U} \rho_{ik} \, d\Omega_\hbar = \sum_i c_{ik} \delta_{ik} = c_{kk}$$

for any k.

Since the matrix $\check{c}$ is self-adjoint it can be diagonalized, i.e. there exists a unitary matrix $\check{T}$ such that $c_{ij} = \sum_{k,l} T^\dagger_{ik}(p_k \delta_{kl}) T_{lj} = \sum_k \bar{T}_{ki} p_k T_{kj}$ for some $p_k \in \mathbb{R}$. Hence, ρ can be presented in the form

$$\rho = \sum_{i,j,k} \bar{T}_{ki} p_k T_{kj} (\bar{\varphi}_i \otimes_W \varphi_j) = \sum_k p_k \left(\left(\sum_i \bar{T}_{ki} \bar{\varphi}_i \right) \otimes_W \left(\sum_j T_{kj} \varphi_j \right) \right)$$

$$= \sum_k p_k (\bar{\psi}_k \otimes_W \psi_k),$$

where $\psi_k = \sum_i T_{ki} \varphi_i$. The conditions that $c_{ii} \geq 0$ and $\operatorname{tr} \check{c} = 1$ give that $0 \leq p_k \leq 1$ and $\sum_k p_k = 1$.

From the above considerations follows that every pure state has the form

$$\boxed{\rho_{\text{pure}} = \bar{\varphi} \otimes_W \varphi,} \tag{8.1.10}$$

for some normalized $\varphi \in L^2(U, d\mu)$. Conversely, every function ρ of the form (8.1.10) is a pure state. Besides, from relation (8.1.8) follows that every pure state is idempotent

$$\boxed{\rho_{\text{pure}} \star_M \rho_{\text{pure}} = \rho_{\text{pure}}.}$$

The inverse is also true, i.e. that every function $\rho \in L^2(T^*U, d\Omega_\hbar)$ which satisfies

1. $\rho = \bar{\rho}$,
2. $\int_{T^*U} \rho \, d\Omega_\hbar = 1$,
3. $\rho \star_M \rho = \rho$,

is a pure state. Indeed, function ρ can be written in a form

$$\rho = \sum_{i,j} c_{ij} \rho_{ij},$$

where $c_{ij} \in \mathbb{C}$ and $\{\rho_{ij}\} = \{\bar{\varphi}_i \otimes_W \varphi_j\}$ is an induced basis in $L^2(T^*U, d\Omega_\hbar)$. Again, properties **1**–**3** are equivalent to the statement that the coefficient matrix $\check{c} = (c_{ij})$ is self-adjoint ($\check{c} = \check{c}^\dagger$), normalized ($\operatorname{tr}\check{c} = 1$), and idempotent ($\check{c}^2 = \check{c}$). So, since the matrix $\check{c}$ is self-adjoint it can be diagonalized, i.e. there exists a unitary matrix $\check{T}$ such that $c_{ij} = \sum_{k,l} T^\dagger_{ik}(a_k\delta_{kl})T_{lj} = \sum_k \bar{T}_{ki}a_k T_{kj}$ for some $a_k \in \mathbb{R}$. In consequence, ρ takes the form

$$\rho = \sum_{i,j,k} \bar{T}_{ki}a_k T_{kj}(\bar{\varphi}_i \otimes_W \varphi_j) = \sum_k a_k \left(\left(\sum_i \bar{T}_{ki}\bar{\varphi}_i \right) \otimes_W \left(\sum_j T_{kj}\varphi_j \right) \right)$$
$$= \sum_k a_k(\bar{\psi}_k \otimes_W \psi_k),$$

where $\psi_k = \sum_i T_{ki}\varphi_i$. The conditions that $\check{c}^2 = \check{c}$ and $\operatorname{tr}\check{c} = 1$ give that $a_k^2 = a_k$ and $\sum_k a_k = 1$. Hence $a_k = \delta_{k_0 k}$ for some k_0, from which we get $\rho = \bar{\psi}_{k_0} \otimes_W \psi_{k_0}$. Thus indeed ρ is a pure state.

As was just noted, pure states $\rho \in L^2(T^*U, d\Omega_\hbar)$ are of the form $\rho = \bar{\varphi}\otimes_W \varphi$ for normalized $\varphi \in L^2(U, d\mu)$ and in consequence there is a one to one correspondence between pure states and normalized vectors in $L^2(U, d\mu)$. Moreover, we will show that there is also a one to one correspondence between states $\rho \in L^2(T^*U, d\Omega_\hbar)$ and *density operators* $\hat{\rho}$ on $L^2(U, d\mu)$ [33, 98].

From the Sect. 6.2.1 we know that vectors $f \in L^2(T^*U, d\Omega_\hbar)$ can be regarded as operators $f \star_M$ on $L^2(T^*U, d\Omega_\hbar)$ given by the formula

$$(f \star_M)\rho = f \star_M \rho, \quad \rho \in L^2(T^*U, d\Omega_\hbar).$$

Besides, from (6.1.3) it follows that operators $f \star_M$ are bounded with the norm $\|f \star_M\| \leq \|f\|$. Now, we will prove that operators $f \star_M$ can be naturally identified with Hilbert-Schmidt operators on $L^2(U, d\mu)$.

For a Hilbert space $\mathcal{H}$ a bounded operator $\hat{A} \in B(\mathcal{H})$ is called a *Hilbert-Schmidt operator* if $\operatorname{tr}(\hat{A}^\dagger\hat{A}) < \infty$. The space of all Hilbert-Schmidt operators will be denoted by $B_{HS}(\mathcal{H})$ and it forms a Hilbert space with a scalar product given by

$$\boxed{(\hat{A}, \hat{B})_{HS} = \operatorname{tr}(\hat{A}^\dagger\hat{B}), \quad \hat{A}, \hat{B} \in B_{HS}(\mathcal{H}).} \tag{8.1.11}$$

From the relation between the Hilbert-Schmidt norm and the usual operator norm

$$\|\hat{A}\| \leq \|\hat{A}\|_{HS}, \quad \hat{A} \in B_{HS}(\mathcal{H})$$

it follows that the inclusion $B_{HS}(\mathcal{H}) \subset B(\mathcal{H})$ is continuous.

In what follows we show that for every $\rho \in L^2(T^*U, d\Omega_\hbar)$

$$\boxed{\rho \star_M = \hat{1} \otimes_W \hat{\rho},} \tag{8.1.12}$$

where $\hat{\rho} \in B_{HS}(L^2(U, d\mu))$ is a Hilbert-Schmidt operator defined on the Hilbert space $L^2(U, d\mu)$. Conversely, for every $\hat{\rho} \in B_{HS}(L^2(U, d\mu))$ the operator $\hat{1} \otimes_W \hat{\rho}$ is of the form $\rho \star_M$ for some $\rho \in L^2(T^*U, d\Omega_\hbar)$. Moreover, the following properties are fulfilled:

(i) for $\rho = \bar{\varphi} \otimes_W \psi$, $\hat{\rho} = (\varphi, \cdot)\psi$,
(ii) $\bar{\rho} \star_M = \hat{1} \otimes_W \hat{\rho}^\dagger$,
(iii) $\mathrm{Tr}(\rho) \equiv \int_{T^*U} \rho \, d\Omega_\hbar = \mathrm{tr}(\hat{\rho})$,
(iv) for $\rho_1, \rho_2 \in L^2(T^*U, d\Omega_\hbar)$ and $\hat{\rho}_1, \hat{\rho}_2 \in B_{HS}(L^2(U, d\mu))$ such that $\rho_1 \star_M = \hat{1} \otimes_W \hat{\rho}_1$ and $\rho_2 \star_M = \hat{1} \otimes_W \hat{\rho}_2$

$$\langle \rho_1, \rho_2 \rangle = (\hat{\rho}_1, \hat{\rho}_2)_{HS},$$

(v) $\int_{T^*U} \bar{f} \star_M f \star_M \rho \, d\Omega_\hbar \geq 0$ for $f \in C_0^\infty(T^*U)$ if and only if $(\varphi, \hat{\rho}\varphi) \geq 0$ for $\varphi \in L^2(U, d\mu)$.

First, let us consider property (i). From (8.1.8) for basis functions $\rho_{ij} = \bar{\varphi}_i \otimes_W \varphi_j$ we get

$$\begin{aligned} \rho \star_M \rho_{ij} &= (\bar{\varphi} \otimes_W \psi) \star_M (\bar{\varphi}_i \otimes_W \varphi_j) = (\varphi, \varphi_j)(\bar{\varphi}_i \otimes_W \psi) = \bar{\varphi}_i \otimes_W (\hat{\rho}\varphi_j) \\ &= (\hat{1} \otimes_W \hat{\rho})\rho_{ij}, \end{aligned}$$

which proves (i).

Now, note that for a basis $\{\varphi_i\}$ in $L^2(U, d\mu)$ the operators $\hat{\rho}_{ij} = (\varphi_i, \cdot)\varphi_j$ form a basis in the Hilbert space $B_{HS}(L^2(U, d\mu))$ of Hilbert-Schmidt operators. From (i) for basis functions $\rho_{ij} = \bar{\varphi}_i \otimes_W \varphi_j$ we have

$$\boxed{\rho_{ij} \star_M = \hat{1} \otimes_W \hat{\rho}_{ij}.} \tag{8.1.13}$$

Any $\rho \in L^2(T^*U, d\Omega_\hbar)$ can be written in the form $\rho = \sum_{i,j} c_{ij}\rho_{ij}$ for some $c_{ij} \in \mathbb{C}$, so according to (8.1.13) the corresponding Hilbert-Schmidt operator $\hat{\rho}$ is of the form $\hat{\rho} = \sum_{i,j} c_{ij}\hat{\rho}_{ij}$. This proves the relation (8.1.12).

In order to show properties (ii)–(iv) it is enough to prove them for basis functions ρ_{ij}. Property (ii) results from (8.1.9a) and the fact that $\hat{\rho}_{ij}^\dagger = \hat{\rho}_{ji}$. Property (iii) is a consequence of (8.1.9b) and the identity $\mathrm{tr}(\hat{\rho}_{ij}) = \delta_{ij}$. Property (iv) follows from the equality

$$\boxed{\begin{aligned} \langle \rho_{ij}, \rho_{kl} \rangle &= \int_{T^*U} \bar{\rho}_{ij}\rho_{kl} \, d\Omega_\hbar = \int_{T^*U} \rho_{ji} \star_M \rho_{kl} \, d\Omega_\hbar = \int_{T^*U} \delta_{jl}\rho_{ki} \, d\Omega_\hbar = \delta_{jl}\delta_{ik} \\ &= \mathrm{tr}(\hat{\rho}_{ij}^\dagger \hat{\rho}_{kl}) = (\hat{\rho}_{ij}, \hat{\rho}_{kl})_{HS}. \end{aligned}}$$

In order to prove (v) we expand ρ and $\hat{\rho}$ in the corresponding basis: $\rho = \sum_{i,j} c_{ij}\rho_{ij}$ and $\hat{\rho} = \sum_{i,j} c_{ij}\hat{\rho}_{ij}$. The property follows from the fact that the positive-definiteness of ρ and $\hat{\rho}$ is equivalent with the inequality $c_{kk} \geq 0$ for every k.

From the foregoing considerations it follows that the Hilbert spaces $L^2(T^*U, d\Omega_\hbar)$ and $B_{HS}(L^2(U, d\mu))$ are naturally isomorphic, where the isomorphism $\rho \mapsto \hat{\rho}$ is given by $\rho \star_M = \hat{1} \otimes_W \hat{\rho}$. The isomorphism $\rho \mapsto \hat{\rho}$ is in fact a representation of the algebra $\mathcal{L} = (L^2(T^*U, d\Omega_\hbar), \star_M)$ in the Hilbert space $L^2(U, d\mu)$ as it satisfies

$$\boxed{\widehat{\rho_1 \star_M \rho_2} = \hat{\rho}_1\hat{\rho}_2, \quad \hat{\bar{\rho}} = \hat{\rho}^\dagger, \quad \mathrm{Tr}(\rho) = \mathrm{tr}(\hat{\rho}),}$$

where the last property is restricted to the subspace $\mathcal{L}^1 = \mathcal{L} \star_M \mathcal{L}$. Moreover, it follows as well that there is a one to one correspondence between quantum states $\rho \in L^2(T^*U, d\Omega_\hbar)$ and density operators on $L^2(U, d\mu)$, i.e. trace class operators $\hat{\rho}$ satisfying

1. $\hat{\rho}^\dagger = \hat{\rho}$,
2. $\mathrm{tr}(\hat{\rho}) = 1$,
3. $(\varphi, \hat{\rho}\varphi) \geq 0$ for every $\varphi \in L^2(U, d\mu)$.

The density operators represent quantum states in the operator representation of quantum mechanics.

Now let us show that observables $A \in C^\infty(T^*U)[\![\hbar]\!]$ can be naturally identified with appropriate operators defined on the Hilbert space $L^2(U, d\mu)$ [33, 98]. What is important, the presented identification will be in agreement with the Weyl correspondence rule. Let $A \in C^\infty(T^*U)[\![\hbar]\!]$ and $\rho = \bar{\varphi} \otimes_W \psi$ for $\varphi, \psi \in C_0^\infty(U)$. Then

$$\boxed{A \star_M \rho = \bar{\varphi} \otimes_W A_W(\hat{q}, \hat{p})\psi,} \tag{8.1.14a}$$

$$\boxed{\rho \star_M A = \overline{(A_W(\hat{q}, \hat{p})^\dagger \varphi)} \otimes_W \psi,} \tag{8.1.14b}$$

where $A_W(\hat{q}, \hat{p})$ is a symmetrically ordered function of canonical operators of position and momentum

$$\boxed{\hat{q}^j = x^j, \quad \hat{p}_j = -i\hbar(\partial_{x^j} + \tfrac{1}{2}\Gamma^k_{jk}),} \tag{8.1.15}$$

where Γ^i_{jk} are Christoffel symbols of the Levi-Civita connection on U, acting in the Hilbert space $L^2(U, d\mu)$.

Indeed, from (6.1.46) and (8.1.2) we get

$$
\begin{aligned}
(A \star_M \rho)(x,p) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{A}(x+\tfrac{1}{2}x', x'')\tilde{\rho}(x-\tfrac{1}{2}x'', x')e^{-\frac{i}{\hbar}(x'^k+x''^k)p_k}\,dx'\,dx'' \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{A}(x+\tfrac{1}{2}x', x'')\overline{\varphi}(x-\tfrac{1}{2}x''-\tfrac{1}{2}x')\psi(x-\tfrac{1}{2}x''+\tfrac{1}{2}x') \\
&\quad \times |g|^{1/4}(x-\tfrac{1}{2}x''-\tfrac{1}{2}x')\,|g|^{1/4}(x-\tfrac{1}{2}x''+\tfrac{1}{2}x')e^{-\frac{i}{\hbar}(x'^k+x''^k)p_k}\,dx'\,dx''.
\end{aligned}
$$

After the change of variables

$$
\begin{aligned}
x' &\to x''-y, \\
x'' &\to y
\end{aligned}
$$

we receive

$$
\begin{aligned}
&(A \star_M \rho)(x,p) \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{A}(x+\tfrac{1}{2}x''-\tfrac{1}{2}y, y)\overline{\varphi}(x-\tfrac{1}{2}x'')\psi(x+\tfrac{1}{2}x''-y)\,|g|^{1/4}(x-\tfrac{1}{2}x'') \\
&\quad \times |g|^{1/4}(x+\tfrac{1}{2}x''-y)\,|g|^{1/4}(x+\tfrac{1}{2}x'')\,|g|^{-1/4}(x+\tfrac{1}{2}x'')e^{-\frac{i}{\hbar}x''^k p_k}\,dy\,dx'' \\
&= \int_{\mathbb{R}^n} \overline{\varphi}(x-\tfrac{1}{2}x'')(A(\hat{q},\hat{p})\psi)(x+\tfrac{1}{2}x'')\,|g|^{1/4}(x-\tfrac{1}{2}x'')\,|g|^{1/4}(x+\tfrac{1}{2}x'')e^{-\frac{i}{\hbar}x''^k p_k}\,dx'' \\
&= (\bar{\varphi}\otimes_W A(\hat{q},\hat{p})\psi)(x,p),
\end{aligned}
$$

where

$$
\begin{aligned}
(A(\hat{q},\hat{p})\psi)(x+\tfrac{1}{2}x'') &= \int_{\mathbb{R}^n} \tilde{A}(x+\tfrac{1}{2}x''-\tfrac{1}{2}y, y)\psi(x+\tfrac{1}{2}x''-y) \\
&\quad \times |g|^{1/4}(x+\tfrac{1}{2}x''-y)\,|g|^{-1/4}(x+\tfrac{1}{2}x'')dy.
\end{aligned}
$$

Changing $x+\frac{1}{2}x'' \to x$, taking explicit form of the Fourier transform in momentum variable (6.1.23) and changing $y \to -y$, we get

$$
\begin{aligned}
(A(\hat{q},\hat{p})\psi)(x) = \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} A(x+\tfrac{1}{2}y, p)e^{-\frac{i}{\hbar}y^k p_k}\psi(x+y) \\
\times |g|^{1/4}(x+y)\,|g|^{-1/4}(x)dy\,dp. \qquad (8.1.16)
\end{aligned}
$$

What remains to prove is that formula (8.1.16) represents Weyl ordering (6.2.3) of operators $\hat{q}$ and $\hat{p}$ given by (8.1.15). Actually, for $A(x,p) = K^{i_1\ldots i_n}(x)p_{i_1}\cdots p_{i_n}$,

where $K^{i_1\cdots i_n}$ is a symmetric complex tensor field on U, and any $\psi(x) \in \mathcal{H}$, we get

$$\begin{aligned} A_W(\hat{q},\hat{p})\psi(x) &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K^{i_1\cdots i_n}(x+\tfrac{1}{2}y)p_{i_1}\cdots p_{i_n}e^{-\frac{i}{\hbar}y^k p_k}\psi(x+y) \\ &\quad\times |g|^{1/4}(x+y)\,|g|^{-1/4}(x)\,dy\,dp \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K^{i_1\cdots i_n}(x+\tfrac{1}{2}y)\left[(i\hbar)^n\partial_{y^{i_1}}\cdots\partial_{y^{i_n}}e^{-\frac{i}{\hbar}y^k p_k}\right]\psi(x+y) \\ &\quad\times |g|^{1/4}(x+y)\,|g|^{-1/4}(x)\,dy\,dp \\ &= \frac{1}{(2\pi\hbar)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}(-i\hbar)^n\partial_{y^{i_1}}\cdots\partial_{y^{i_n}}\left[K^{i_1\cdots i_n}(x+\tfrac{1}{2}y)\psi(x+y)\xi(x,y)\right] \\ &\quad\times e^{-\frac{i}{\hbar}y^k p_k}\,dy\,dp \\ &= (-i\hbar)^n\partial_{y^{i_1}}\cdots\partial_{y^{i_n}}\left[K^{i_1\cdots i_n}(x+\tfrac{1}{2}y)\psi(x+y)\xi(x,y)\right]\Big|_{y=0}. \end{aligned} \tag{8.1.17}$$

By virtue of the following identity

$$\partial_y\big(g(x+\tfrac{1}{2}y)h(x+y)\big) = (\partial_{x'}+\partial_{x''})\big[g(x+\tfrac{1}{2}x')h(x+\frac{1}{2}x'+\tfrac{1}{2}x'')\big]\Big|_{x'=x''=y}$$

valid for any functions g and h, (8.1.17) can be written in a form

$$\begin{aligned} A_W(\hat{q},\hat{p})\psi(x) = (-i\hbar)^n\sum_{k=0}^{n}\binom{n}{k}\partial_{y^{i_1}}\cdots\partial_{y^{i_k}}\partial_{x'^{i_{k+1}}}\cdots\partial_{x'^{i_n}}\big[K^{i_1\cdots i_n}(x+\tfrac{1}{2}y) \\ \times\psi(x+\tfrac{1}{2}y+\tfrac{1}{2}x')\,|g|^{1/4}(x+\tfrac{1}{2}y+\tfrac{1}{2}x')\,|g|^{-1/4}(x)\big]\Big|_{y=x'=0}. \end{aligned} \tag{8.1.18}$$

Applying the known relation to the metric tensor

$$\frac{\partial\,|g|}{\partial x^j} = 2\,|g|\,\Gamma^k_{jk} \tag{8.1.19}$$

we find that

$$-i\hbar\partial_{x^j}(\psi\,|g|^{1/4}) = -i\hbar\left(\frac{\partial\psi}{\partial x^j}\,|g|^{1/4}+\tfrac{1}{2}\psi\Gamma^k_{jk}\,|g|^{1/4}\right) = (\hat{p}_j\psi)\,|g|^{1/4},$$

where $\hat{p}_j$ is given just by the formula (8.1.15). From this and (8.1.18) we receive

$$\begin{aligned} A_W(\hat{q},\hat{p})\psi(x) = \sum_{k=0}^{n}\binom{n}{k}(-i\hbar)^k\left(\tfrac{1}{2}\right)^{n-k}\partial_{y^{i_1}}\cdots\partial_{y^{i_k}}\big[K^{i_1\ldots i_n}(x+\tfrac{1}{2}y) \\ \times(\hat{p}_{i_{k+1}}\cdots\hat{p}_{i_n}\psi)(x+\tfrac{1}{2}y+\tfrac{1}{2}x')\,|g|^{1/4}(x+\tfrac{1}{2}y+\frac{1}{2}x') \\ \times|g|^{-1/4}(x)\big]\Big|_{y=v=0} \\ = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\hat{p}_{i_1}\cdots\hat{p}_{i_k}K^{i_1\ldots i_n}(\hat{q})\,\hat{p}_{i_{k+1}}\cdots\hat{p}_{i_n}\psi(x) \end{aligned}$$

that recovers formula (6.2.7) and (8.1.14a) is proved.

In order to prove (8.1.14b) we can use (8.1.6) and (8.1.14a) receiving

$$\rho\star_M A = \overline{\bar{A}\star_M\bar{\rho}} = \overline{\bar{\psi}\otimes_W A_W^{\dagger}(\hat{q},\hat{p})\varphi} = \overline{(A_W(\hat{q},\hat{p})^{\dagger}\varphi)}\otimes_W\psi.$$

Observe, that products $A\star_M\rho$ and $\rho\star_M A$ are properly defined by (6.1.46), even though A has no compact support and is not defined on the whole space $\mathbb{R}^n\times\mathbb{R}^n$, since ρ has the form $\bar{\varphi}\otimes_W\psi$ for φ and ψ with a compact support.

From (8.1.14) follows that operators $A\star_M$ can be written as the following tensor product

$$\boxed{A\star_M = \hat{1}\otimes_W A_W(\hat{q},\hat{p}).} \tag{8.1.20}$$

Equation (8.1.20) is an analog of (8.1.12) for functions $A\in C^{\infty}(T^*U)[\![\hbar]\!]$ and it allows to identify functions A with operators $A(\hat{q},\hat{p})$. That way the Weyl correspondence rule in a natural way appears in the operator representation of quantum mechanics.

The map $A\mapsto\hat{A}=A_W(\hat{q},\hat{p})$ has the following properties

$$\boxed{\widehat{A_1\star_M A_2} = \hat{A}_1\hat{A}_2,\quad \hat{\bar{A}} = \hat{A}^{\dagger}}$$

for functions $A_1, A_2\in C^{\infty}(T^*U)[\![\hbar]\!]$, thus we are dealing with representation of the algebra $\mathcal{A}_Q=(C^{\infty}(T^*U)[\![\hbar]\!],\star_M)$ in the Hilbert space $L^2(U,d\mu)$.

Let $A\in C^{\infty}(T^*U)[\![\hbar]\!]$ and $\rho\in L^2(T^*U,d\Omega_\hbar)$, then for $A\star_M\rho\in L^1(T^*U,d\Omega_\hbar)$ we have

$$\boxed{\int_{T^*U}A\star_M\rho\,d\Omega_\hbar = \mathrm{tr}(A_W(\hat{q},\hat{p})\hat{\rho}).} \tag{8.1.21}$$

In particular, if $\rho = \bar{\varphi} \otimes_W \psi$ for $\varphi, \psi \in C_0^\infty(U)$ then

$$\boxed{\int_{T^*U} A \star_M \rho \, d\Omega_\hbar = (\varphi, A_W(\hat{q}, \hat{p})\psi).} \tag{8.1.22}$$

We prove (8.1.21) assuming that $\{\varphi_i\}$ is a basis in $L^2(U, d\mu)$ such that φ_i have a compact support. From (8.1.14) and (8.1.7) then follows that

$$\int_{T^*U} A\rho_{ij} \, d\Omega_\hbar = \int_{T^*U} A \star_M \rho_{ij} \, d\Omega_\hbar = (\varphi_i, A_W(\hat{q}, \hat{p})\varphi_j)$$

for $\rho_{ij} = \bar{\varphi}_i \otimes_W \varphi_j$. The function ρ can be expanded in the basis ρ_{ij}, $\rho = \sum_{i,j} c_{ij}\rho_{ij}$. Using this expansion we find that

$$\int_{T^*U} A \star_M \rho \, d\Omega_\hbar = \int_{T^*U} A\rho \, d\Omega_\hbar = \sum_{i,j} c_{ij} \int_{T^*U} A\rho_{ij} \, d\Omega_\hbar = \sum_{i,j} c_{ij}(\varphi_i, A_W(\hat{q}, \hat{p})\varphi_j)$$

$$= \sum_{i,j} c_{ij} \operatorname{tr}(A_W(\hat{q}, \hat{p})\hat{\rho}_{ij}) = \operatorname{tr}(A_W(\hat{q}, \hat{p})\hat{\rho}).$$

The results of this subsection for a very special case of a flat metric tensor and Cartesian (i.e. flat and orthogonal) coordinates, are well known from the literature and were investigated in many papers, starting from [7, 111, 246]. Here we generalized these results to the case of the arbitrary metric tensor and the arbitrary coordinate system.

Example 8.1 Let us consider the case of a harmonic oscillator and verify relations (8.1.14) for its spectral problem [151]. In the phase space representation, the spectral problem was considered in Sect. 7.2.2. In the position representation, the spectral problem of a harmonic oscillator is presented in any textbook of quantum mechanics. For classical Hamiltonian $H = \frac{1}{2}\left(p^2 + \omega^2 x^2\right)$ and canonical operators (8.1.15)

$$\hat{q} = x, \quad \hat{p} = -i\hbar\partial_x,$$

the eigenvalue problem

$$\hat{H}\varphi_n(x) = \tfrac{1}{2}\left(-\hbar^2\partial_x^2 + \omega^2 x^2\right)\varphi_n(x) = E_n\varphi_n(x), \qquad n \in \mathbb{N}$$

has the following solution

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \qquad \varphi_n(x) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \left(\frac{1}{2^n n!}\right) e^{-\frac{1}{2}a^2x^2} H_n(ax),$$

where H_n is the nth Hermite polynomial, $a = \left(\frac{\omega}{\hbar}\right)^{\frac{1}{2}}$ and $\varphi_n(x)$ are orthonormal functions from the Hilbert space $L^2(\mathbb{R})$. In order to verify the relations (8.1.14) we have to prove that phase space $\star$-genfunctions $\rho_n(x, p)$ (7.2.23) and eigenfunctions $\varphi_n(x)$ are related by (8.1.10), (8.1.5)

$$\rho_n(x, p) = \overline{\varphi}_n(x) \otimes_W \varphi_n(x) = W(\varphi_n^*, \varphi_n) = \int_{\mathbb{R}} \overline{\varphi}_n(x - \tfrac{1}{2}y)\varphi_n(x + \tfrac{1}{2}y)e^{-\frac{i}{\hbar}py}\, dy.$$

Rescaling $y \to -2y$, we find that

$$W(\varphi_n^*, \varphi_n) = \frac{2a}{\sqrt{\pi}} \frac{1}{2^n n!} e^{-\frac{1}{2}a^2x^2} \int_{\mathbb{R}} e^{\frac{2i}{\hbar}py} e^{-\frac{1}{2}a^2y^2} H_n(a(x + y)) H_n(a(x - y)) dy.$$

We note that

$$a^2y^2 - 2ipy/\hbar = a^2(y - ip/a^2\hbar)^2 + p^2/a^2\hbar^2$$

and define a new variable

$$z = a(y - ip/a^2\hbar).$$

Then

$$W(\varphi_n^*, \varphi_n) = \frac{2}{\sqrt{\pi}} \frac{1}{2^n n!} e^{-\frac{1}{2}a^2x^2} e^{b^2} \int_{\mathbb{R}} e^{-z^2} H_n(ax + z + b) H_n(ax - z - b) dz,$$

where $b = ip/a\hbar$. As $H_n(-x) = (-1)^n H_n(x)$, we get

$$W(\varphi_n^*, \varphi_n) = \frac{2}{\sqrt{\pi}} \frac{(-1)^n}{2^n n!} e^{-\frac{1}{2}a^2x^2} e^{b^2} \int_{\mathbb{R}} e^{-z^2} H_n(z + b + ax) H_n(z + b - ax) dz.$$

The integral can be done [137]

$$\int_{\mathbb{R}} e^{-z^2} H_n(z + b + ax) H_n(z + b - ax) dz = 2^n \sqrt{\pi} n! L_n(2(a^2x^2 - b^2))$$

where L_n is the nth Laguerre polynomial. Re-expressing the argument of L_n by x and p we get

$$a^2x^2 - b^2 = \frac{1}{\hbar\omega}(p^2 + \omega^2x^2) = \frac{2H}{\hbar\omega}$$

and finally

$$W(\varphi_n^*, \varphi_n) = 2(-1)^n \exp\left(-\frac{2H}{\hbar\omega}\right) L_n\left(\frac{4H}{\hbar\omega}\right) = \rho_n(x, p).$$

8.1.2 Other Admissible Quantizations

Let us consider n-dimensional almost geodesically simply connected Riemannian manifold forming the configuration space Q and a related phase space $M = T^*Q$. Then, let us consider a classical system defined on M and its quantization by means of a $\star$-product on M. Let $Q \supset U \to V \subset \mathbb{R}^n$, $x \mapsto (x^1, \ldots, x^n)$ be an almost global coordinate system on Q. From the assumption that Q is almost geodesically simply connected such a coordinate system always exists. The coordinate system on Q induces on M an almost global classical canonical coordinate system $T^*U \to T^*V = V \times \mathbb{R}^n$, $(\xi^\alpha) \mapsto (x^1, \ldots, x^n, p_1, \ldots, p_n)$. This coordinate system will be at the same time quantum canonical. Observe that although $\star$-product is not local it still can be written in the coordinates (x, p) since this coordinate system is almost globally defined on M.

The idea behind introducing the operator representation over configuration space Q for the considered quantum system lies in the observation that the quantum system in coordinates (x, p) is equivalent with a system quantized by the Moyal product. If S is an appropriate morphism of this equivalence then S is an unitary operator on the Hilbert space $L^2(T^*V, d\Omega_\hbar)$. Let us introduce a new tensor product $\otimes_S \colon (L^2(V, d\mu))^* \times L^2(V, d\mu) \to L^2(T^*V, d\Omega_\hbar)$ defined by the formula

$$\boxed{\bar{\varphi} \otimes_S \psi = S(\bar{\varphi} \otimes_W \psi), \quad \varphi, \psi \in L^2(V, d\mu)} \tag{8.1.23}$$

and a function A of S-ordered operators $\hat{q}^j$, $\hat{p}_j$

$$\boxed{A_S(\hat{q}, \hat{p}) = (S^{-1}A)_W(\hat{q}, \hat{p}).} \tag{8.1.24}$$

Using properties (6.1.77) and assumption (6.1.84) it can be proved that all previous formulas for the case of a Moyal quantization, presented in the previous subsection, also hold true for a general quantum system in (x, p) coordinates, provided that the tensor product $\otimes_W$ will be replaced by $\otimes_S$ and operators $A_W(\hat{q}, \hat{p})$ by $A_S(\hat{q}, \hat{p})$, respectively. In particular, let $A \in C^\infty(T^*V)[\![\hbar]\!]$ and $\rho = \bar{\varphi} \otimes_S \psi$ for $\varphi, \psi \in C_0^\infty(V)$. Then [33, 98]

$$\boxed{A \star^{(x,p)} \rho = \bar{\varphi} \otimes_S A_S(\hat{q}, \hat{p})\psi,} \tag{8.1.25a}$$

$$\boxed{\rho \star^{(x,p)} A = (\overline{A_S(\hat{q}, \hat{p})^\dagger \varphi}) \otimes_S \psi,} \tag{8.1.25b}$$

where $A_S(\hat{q}, \hat{p})$ is an S-ordered function of canonical operators of position and momentum

$$\hat{q}^j = x^j, \quad \hat{p}_j = -i\hbar(\partial_{x^j} + \tfrac{1}{2}\Gamma^k_{jk}), \tag{8.1.26}$$

acting in the Hilbert space $L^2(V, d\mu)$.

From (6.1.77) and (8.1.14) we find that

$$\boxed{A \star^{(x,p)} \rho = SS^{-1}(A \star^{(x,p)} \rho) = S(S^{-1}A \star_M^{(x,p)} S^{-1}\rho) = S\left(\bar{\varphi} \otimes_W (S^{-1}A)_W(\hat{q}, \hat{p})\psi\right) = \bar{\varphi} \otimes_S A_S(\hat{q}, \hat{p})\psi}$$

which proves (8.1.25a). Equation (8.1.25b) can be proved analogically.

From (8.1.25) follows that operators $A \star^{(x,p)}$ can be written as

$$\boxed{A \star^{(x,p)} = \hat{1} \otimes_S A_S(\hat{q}, \hat{p}).} \tag{8.1.27}$$

Equation (8.1.27) allows us to identify functions $A \in C^\infty(T^*V)[\![\hbar]\!]$ with operators $A_S(\hat{q}, \hat{p})$. Moreover, the map $A \mapsto \hat{A} = A_S(\hat{q}, \hat{p})$ is a representation of the algebra $\mathcal{A}_Q = (C^\infty(T^*V)[\![\hbar]\!], \star^{(x,p)})$ in the Hilbert space $L^2(V, d\mu)$. Moreover, the analog of (8.1.12) holds true, which gives us a representation $\rho \mapsto \hat{\rho}$ of the algebra $\mathcal{L} = (L^2(T^*V, d\Omega_\hbar), \star^{(x,p)})$ in the Hilbert space $L^2(V, d\mu)$ given by

$$\boxed{\rho \star^{(x,p)} = \hat{1} \otimes_S \hat{\rho}.}$$

Furthermore, the following analog of (8.1.21) and (8.1.22) can be proved. Let $A \in C^\infty(T^*V)[\![\hbar]\!]$ and $\rho \in L^2(T^*V, d\Omega_\hbar)$. If $A \star^{(x,p)} \rho \in L^1(T^*V, d\Omega_\hbar)$ then

$$\boxed{\int_{T^*V} A \star^{(x,p)} \rho \, d\Omega_\hbar = \operatorname{tr}(A_S(\hat{q}, \hat{p})\hat{\rho}).} \tag{8.1.28}$$

In particular, if $\rho = \bar{\varphi} \otimes_S \psi$ for $\varphi, \psi \in C_0^\infty(V)$ then

$$\boxed{\int_{T^*V} A \star^{(x,p)} \rho \, d\Omega_\hbar = (\varphi, A_S(\hat{q}, \hat{p})\psi).} \tag{8.1.29}$$

To show it, let $\{\varphi_i\}$ be a basis in $L^2(V, d\mu)$ and $\{\rho_{ij}\} = \{\bar{\varphi}_i \otimes_S \varphi_j\}$ an induced basis in $L^2(T^*V, d\Omega_\hbar)$. Function ρ can be expanded in the basis $\{\rho_{ij}\}$: $\rho = \sum_{i,j} c_{ij}\rho_{ij}$. From (6.1.77), (8.1.21), (8.1.22) and assumption (6.1.84) we find

$$\begin{aligned}\int_{T^*V} A \star^{(x,p)} \rho \, d\Omega_\hbar &= \int_{T^*V} SS^{-1}(A \star^{(x,p)} \rho) \, d\Omega_\hbar = \int_{T^*V} S^{-1}A \star_M^{(x,p)} S^{-1}\rho \, d\Omega_\hbar \\ &= \sum_{i,j} c_{ij} \int_{T^*V} S^{-1}A \star_M^{(x,p)} S^{-1}\rho_{ij} \, d\Omega_\hbar = \sum_{i,j} c_{ij}(\varphi_i, A_S(\hat{q}, \hat{p})\varphi_j) \\ &= \sum_{i,j} c_{ij} \operatorname{tr}(A_S(\hat{q}, \hat{p})\hat{\rho}_{ij}) = \operatorname{tr}(A_S(\hat{q}, \hat{p})\hat{\rho}).\end{aligned}$$

Using the results of this subsection it is possible to prove a pure state stationary equations (7.1.10) from Sect. 7.1.2. Namely for any $A \in \mathcal{A}_Q$, a pure state function $\rho = \bar{\varphi} \otimes_S \varphi \in \mathcal{H}$ satisfies the equation

$$[A, \rho] = 0 \tag{8.1.30}$$

if and only if it satisfies the following pair of $\star_S$-genvalue equations

$$A \star_S \rho = a\rho, \quad \rho \star_S A = a\rho, \tag{8.1.31}$$

for some $a \in \mathbb{C}$.

Indeed, it is obvious that if ρ satisfies (8.1.31) then it also satisfies (8.1.30). Let us assume that ρ satisfies (8.1.30). Hence, it also satisfies

$$A \star_S \rho \star_S \rho = \rho \star_S A \star_S \rho.$$

From the idempotent property of pure states the above equation implies

$$A \star_S \rho = \rho \star_S A \star_S \rho. \tag{8.1.32}$$

From (8.1.25) it follows that

$$A \star_S \rho = \bar{\varphi} \otimes_S A_S(\hat{q}, \hat{p})\varphi. \tag{8.1.33}$$

Now, Eqs. (8.1.32) and (8.1.33), with the help of (8.1.8) which valid for any $\star_S$, give

$$A \star_S \rho = \rho \star_S (A \star_S \rho) = (\varphi, A_S(\hat{q}, \hat{p})\varphi)\rho = a\rho,$$

where $a = (\varphi, A_S(\hat{q}, \hat{p})\varphi)$. The second $\star_S$-genvalue equation can be derived analogically.

Observation 19 *Note that for a general $\star_S$-quantization the operator representation corresponding to a coordinate system gives us the correspondence rule $A \mapsto A_S(\hat{q}, \hat{p})$ which in general differs from the Weyl correspondence rule. The Weyl correspondence rule is associated only with the Moyal quantization. To create an operator representation of arbitrary $\star_S$-quantum system in any coordinates on Q in a consistent way, one should use S-ordering (8.1.24) of the same operators of position and momentum (8.1.26).*

Let us construct the $\otimes_S$-tensor product for particular star-algebras. First, notice that (8.1.23) can be written in an integral form

$$\boxed{\begin{aligned}(\bar\varphi \otimes_S \psi)(x,p) =& S\int dy\exp(-\frac{i}{\hbar}p_i y^i)\bar\varphi(x-\tfrac{1}{2}y)\psi(x+\tfrac{1}{2}y)\zeta(x,y)\\ =&\int dx'dp'dy\exp(-\frac{i}{\hbar}p'_k y^k)\bar\varphi(x'-\tfrac{1}{2}y)\psi(x'+\tfrac{1}{2}y)\\ &|g|^{1/4}(x-\tfrac{1}{2}y)\,|g|^{1/4}(x+\tfrac{1}{2}y)S(x,p,x',p'),\end{aligned}}$$

where $\varphi,\psi\in L^2(\mathbb{R}^n)$ and $S(x,p,x',p')$ is an integral kernel of isomorphism S.

Example 8.2 As an interesting example let us consider the $\star_{\sigma,\alpha,\beta}$-algebra in the simplest case $n=1$ and Cartesian coordinates. Let us start from a simpler case $\alpha=\beta=0$. Then, as we already know,

$$S_\sigma = \exp(-i\hbar\sigma\,\partial_x\partial p)$$

and hence, using (6.2.1) and (6.2.1) we get

$$\begin{aligned}(\bar\varphi\otimes_\sigma\psi)(x,p) &= \exp(-i\hbar\sigma\partial_x\partial p)\int dy\exp(-\frac{i}{\hbar}py)\bar\varphi(x-\tfrac{1}{2}y)\psi(x+\tfrac{1}{2}y)\\ &=\int dy\exp(-\frac{i}{\hbar}py)\exp(-i\hbar\sigma\partial_x\partial p)\exp(-\sigma y\partial_x)\bar\varphi(x-\tfrac{1}{2}y)\psi(x+\tfrac{1}{2}y)\\ &=\int dy\exp(-\frac{i}{\hbar}py)\bar\varphi(x-(\tfrac{1}{2}+\sigma)y)\psi(x+(\tfrac{1}{2}-\sigma)y).\end{aligned}$$

For $\alpha,\beta>0$ we have

$$S_{\alpha,\beta}=\exp(\tfrac{1}{2}\hbar\alpha\partial_x^2+\tfrac{1}{2}\hbar\beta\partial_p^2)$$

$$\Downarrow$$

$$S_{\alpha,\beta}(x,p,x',p')=\frac{1}{2\pi\hbar\sqrt{\alpha\beta}}\exp\left[-\frac{1}{2}\frac{1}{\hbar\alpha}(x-x')^2\right]\exp\left[-\frac{1}{2}\frac{1}{\hbar\beta}(p-p')^2\right],$$

and finally

$$\begin{aligned}&(\bar\varphi\otimes_{\sigma,\alpha,\beta}\psi)(x,p)\\ &=\frac{1}{2\pi\hbar\sqrt{\alpha\beta}}\int dx'dp'dy\exp(-\frac{i}{\hbar}py)\bar\varphi(x-(\tfrac{1}{2}+\sigma)y)\psi(x+(\tfrac{1}{2}-\sigma)y)\\ &\times\exp\left[-\frac{1}{2}\frac{1}{\hbar\alpha}(x-x')^2\right]\exp\left[-\frac{1}{2}\frac{1}{\hbar\beta}(p-p')^2\right].\end{aligned}$$

The diagonal tensor products $(\bar{\varphi} \otimes_S \varphi)(x, p)$ are pure phase space quantum states related to $\star_S$-quantization, known in the literature as pure *phase-space distribution functions (d.f.)*[176]. In particular, for the family of $\star_{\sigma,\alpha,\beta}$-quantizations, we have:

1. $\sigma = \alpha = \beta = 0$: symmetrically ordered d.f. (Wigner function) [263]
2. $\sigma = \frac{1}{2}, \alpha = \beta = 0$: standard ordered d.f. [199]
3. $\sigma = -\frac{1}{2}, \alpha = \beta = 0$: antistandard ordered d.f. (Kirkwood function) [171, 227]
4. $\sigma = 0, \alpha = -\frac{1}{2\omega}, \beta = -\frac{\omega}{2}$: normal ordered d.f. (Glauber-Sudarshan P function) [132, 133, 243]
5. $\sigma = 0, \alpha = \frac{1}{2\omega}, \beta = \frac{\omega}{2}$: antinormal ordered d.f. (Q function) [133]
6. $\sigma = 0, \alpha = -\frac{1}{2\kappa}, \beta = -\frac{\kappa}{2}, \ \kappa \neq \omega$: generalized antinormal ordered d.f. (Husimi function) [155]

So, if $\{\rho_{ij} = \bar{\varphi}_i \otimes_S \varphi_j\}$ represent the orthonormal basis in $\mathcal{H} = L^2(\mathbb{R}^{2n})$, phase-space distribution functions belong to the subspace $\mathcal{H}^\rho \subset \mathcal{H}$ span by

$$\mathcal{H}^\rho = span\{\rho_{ii} \equiv \rho_i = \bar{\varphi}_i \otimes_S \varphi_i\}, \qquad \varphi_i \in L^2(\mathbb{R}^n)$$

obviously isomorphic with $L^2(\mathbb{R}^n)$.

An arbitrary phase-space distribution function (pure or mixed) $\rho = \sum_\gamma p_\gamma (\bar{\varphi}^{(\gamma)} \otimes_S \varphi^{(\gamma)}) \in \mathcal{H}^\rho$ and normalized $\varphi^{(\gamma)} \in L^2(\mathbb{R}^n)$, fulfills relations (7.1.3). In particular, for Euclidean coordinates,

$$\boxed{\frac{1}{(2\pi\hbar)^n} \int S^{-1}\rho(x, p)dp = \sum_\gamma p_\gamma \left|\varphi^{(\gamma)}(x)\right|^2 = P(x),} \tag{8.1.34a}$$

$$\boxed{\frac{1}{(2\pi\hbar)^n} \int S^{-1}\rho(x, p)dx = \sum_\gamma p_\gamma \left|\mathcal{F}\varphi^{(\gamma)}(p)\right|^2 = P(p),} \tag{8.1.34b}$$

which are marginal distributions. For example, the first relation for a pure state $\rho = (\bar{\varphi} \otimes_S \varphi)$ follows from

$$\begin{aligned}\frac{1}{(2\pi\hbar)^n} \int S^{-1}\rho(x, p)dp &= \frac{1}{(2\pi\hbar)^n} \int\int \exp(-\frac{i}{\hbar} p_k y^k)\bar{\varphi}(x - \frac{1}{2}y)\varphi(x + \tfrac{1}{2}y)dydp \\ &= \int \delta(y)\varphi^*(x - \tfrac{1}{2}y)\varphi(x + \tfrac{1}{2}y)dy = |\varphi(x)|^2 .\end{aligned}$$

As the conclusion we observe that the expectation value of an observable $A \in C^\infty(T^*U)[\![\hbar]\!]$ in a state $\rho \in L^2(T^*U, d\Omega_\hbar)$ in the position representation of quantum mechanics is expressed by the formula

$$\boxed{\langle A\rangle_\rho = \mathrm{tr}(A_S(\hat{q}, \hat{p})\hat{\rho}).} \tag{8.1.35}$$

Moreover, in the position representation, the following equation corresponds to time evolution equation (7.1.5)

$$\boxed{i\hbar\frac{\partial\hat{\rho}}{\partial t}(t) - [H_S(\hat{q},\hat{p}),\hat{\rho}(t)] = 0} \tag{8.1.36}$$

It is known as the *von Naumann equation*. For pure state $\hat{\rho} = (\varphi,\cdot)\,\varphi$ Eq. (8.1.36) takes the form

$$\begin{aligned} &i\hbar(\frac{\partial\varphi}{\partial t}\mid\cdot)\varphi + i\hbar\,(\varphi,\cdot)\,\frac{\partial\varphi}{\partial t} - (\varphi,\cdot)\,H_S(\hat{q},\hat{p})\varphi + \left(H_S(\hat{q},\hat{p})\varphi,\cdot\right)\varphi \\ &= (-i\hbar\frac{\partial\varphi}{\partial t} + H_S(\hat{q},\hat{p})\varphi,\cdot)\varphi + (\varphi,\cdot)\left(i\hbar\frac{\partial\varphi}{\partial t} - H_S(\hat{q},\hat{p})\varphi\right) = 0 \end{aligned}$$

$$\Updownarrow$$

$$\boxed{i\hbar\frac{\partial\varphi}{\partial t} = H_S(\hat{q},\hat{p})\varphi.} \tag{8.1.37}$$

Equation (8.1.37) is nothing but the well known *Schrödinger equation* of time evolution of quantum states in the position representation of quantum mechanics.

The equation for stationary states takes now the form

$$[H_S(\hat{q},\hat{p}),\hat{\rho}] = 0,$$

which for pure states $\hat{\rho} = (\varphi,\cdot)\varphi$ is equivalent to the well known eigenvalue equation

$$\boxed{H_S(\hat{q},\hat{p})\varphi = E\varphi}$$

called the *stationary Schrödinger equation*.

The respective representation of one-parameter group of unitary functions (7.1.7) in the Hilbert space $L^2(\mathbb{R}^n)$ is a one-parameter group of unitary operators

$$U_S(\hat{q},\hat{p},t) = \exp\left(-\frac{i}{\hbar}tH_S(\hat{q},\hat{p})\right). \tag{8.1.38}$$

The time evolution of density operator $\hat{\rho}$ can be alternatively expressed by

$$\hat{\rho}(t) = U_S(\hat{q},\hat{p},t)\hat{\rho}(0)U_S(\hat{q},\hat{p},-t).$$

It is easy to check that such defined $\hat{\rho}(t)$ represents a solution of the von Neumann equation (8.1.36).

With the help of the unitary operator (8.1.38), the time evolution of observable $A_S(\hat{q}, \hat{p})$ can be expressed in analogy to (7.1.12) by

$$A_S(\hat{q}, \hat{p}, t) = U_S(\hat{q}, \hat{p}, -t) A_S(\hat{q}, \hat{p}) U_S(\hat{q}, \hat{p}, t). \tag{8.1.39}$$

The expression (8.1.39) is a formal solution of the equation

$$\boxed{i\hbar \frac{d}{dt} A_S(t) = [A_S(\hat{q}, \hat{p}), H_S(\hat{q}, \hat{p})],} \tag{8.1.40}$$

which is a well known *Heisenberg equation* of time evolution for quantum observables in position representation.

8.1.3 Coordinate Transformation on Configuration Space

Assume that $Q \supset U \to V \subset \mathbb{R}^n$, $x \mapsto (x^1, \ldots, x^n)$ and $Q \supset U' \to V' \subset \mathbb{R}^n$, $x \mapsto (x'^1, \ldots, x'^n)$ are two almost global coordinate systems on Q, and $T^*U \to T^*V = V \times \mathbb{R}^n$, $\xi \mapsto (x^1, \ldots, x^n, p_1, \ldots, p_n)$ and $T^*U' \to T^*V' = V' \times \mathbb{R}^n$, $\xi \mapsto (x'^1, \ldots, x'^n, p'_1, \ldots, p'_n)$ are induced canonical coordinate systems on T^*Q. A map $\phi\colon (x'^1, \ldots, x'^n) \mapsto (x^1, \ldots, x^n)$ is then a transformation of coordinates on the configuration space Q and a map $T\colon (x'^1, \ldots, x'^n, p'_1, \ldots, p'_n) \mapsto (x^1, \ldots, x^n, p_1, \ldots, p_n)$ is a respective canonical transformation of coordinates on the phase space T^*Q. The transformation T is given by the formula (6.1.62).

In what follows we will investigate how the operator representation of quantum mechanics behaves after such changing of coordinates. First, note that a map $\hat{U}_T\colon L^2(V, d\mu) \to L^2(V', d\mu')$ given by

$$\boxed{(\hat{U}_T \psi)(x') = \psi(\phi(x'))} \tag{8.1.41}$$

is an isomorphism of the Hilbert spaces. Moreover, a map $L^2(T^*V, d\Omega_\hbar) \to L^2(T^*V', d\Omega_\hbar)$ given by

$$f \mapsto f \circ T$$

is also an isomorphism of the Hilbert spaces. Let $\otimes_S$ and $\otimes_{S'}$ be tensor products which correspond to star-products $\star^{(x,p)}$ and $\star^{(x',p')}$ respectively. The following statement can be proved. For $\varphi, \psi \in L^2(V, d\mu)$ there holds

$$\boxed{(\bar{\varphi} \otimes_S \psi) \circ T = (\overline{\hat{U}_T \varphi}) \otimes_{S'} \hat{U}_T \psi.} \tag{8.1.42}$$

From relation (8.1.42) follows that operator representations of quantum mechanics corresponding to different coordinate systems are unitarily equivalent. In particular, we get that operators corresponding to the same function $A \in C^\infty(T^*Q)[\![\hbar]\!]$ written in different coordinate systems, are unitarily equivalent. Actually, for $A \in C^\infty(T^*V)[\![\hbar]\!]$ there holds

$$\boxed{A'_{S'}(\hat{q}', \hat{p}') = \hat{U}_T A_S(\hat{q}, \hat{p})\hat{U}_T^{-1},} \tag{8.1.43}$$

where $A' = A \circ T$ and $\hat{q}^j = x^j$, $\hat{p}_j = -i\hbar(\partial_{x^j} + \frac{1}{2}\Gamma^k_{jk})$ and $\hat{q}'^j = x'^j$, $\hat{p}'_j = -i\hbar(\partial_{x'^j} + \frac{1}{2}\Gamma'^k_{jk})$ are respective operators of position and momentum in coordinates (x, p) and (x', p') respectively.

In order to prove (8.1.43), first let $\rho = \bar{\varphi} \otimes_S \psi$. Then, from (8.1.25) and (8.1.42) we get from one side

$$(A \star^{(x,p)} \rho) \circ T = A \star'^{(x',p')} (\rho \circ T) = (\hat{U}_T\varphi)^* \otimes_{S'} A'_{S'}(\hat{q}', \hat{p}')\hat{U}_T\psi$$

and from the other side

$$\begin{aligned}(A \star^{(x,p)} \rho) \circ T &= (\bar{\varphi} \otimes_S A_S(\hat{q}, \hat{p})\psi) \circ T = (\overline{\hat{U}_T\varphi}) \otimes_{S'} \hat{U}_T A_S(\hat{q}, \hat{p})\psi \\ &= (\overline{\hat{U}_T\varphi}) \otimes_{S'} \hat{U}_T A_S(\hat{q}, \hat{p})\hat{U}_T^{-1}\hat{U}_T\psi.\end{aligned}$$

Comparing the above two formulas we get the result.

In particular, let us derive the position and momentum operators (8.1.15) in curvilinear coordinates directly from the canonical transformation. Let

$$\phi : \mathbb{R}^n \supset V \to V' \subset \mathbb{R}^n, \quad x' = \phi(x)$$

be a transformation from curvilinear coordinates $(x^1, \ldots, x^n)$ to pseudo-Euclidian coordinates $(x'^1, \ldots, x'^n)$, such that $\mathbb{R}^n \backslash V'$ is of measure zero. Transformation ϕ induces classical and quantum canonical transformation T (6.1.62). Notice that Hilbert spaces over $\mathbb{R}^n$ and V' are naturally isomorphic.

First, we define position and momentum operators corresponding to the curvilinear coordinate system (x, p) by symmetrically ordered Euclidean operators $\hat{q}'$, $\hat{p}'$ in the following form

$$\boxed{\hat{Q}^j = (Q^j)_W(\hat{q}') = Q^j(\hat{q}'), \quad \hat{P}_j = (P_j)_W(\hat{q}', \hat{p}'), \quad \hat{q}'^j = x'^j, \quad \hat{p}' = -i\hbar\partial_{x'^j}}$$

where $T^{-1}(x', p') = (Q^1(x'), \ldots, P_n(x', p'))$ and

$$\boxed{x^j = Q^j = \left[\phi^{-1}(x')\right]^j, \quad p_j = P_j = p'_i \left[\phi'(\phi^{-1}(x'))\right]^i_j.}$$

These operators are defined on a Hilbert space $L^2(V) \cong L^2(\mathbb{R}^n)$ parametrized by pseudo-Euclidean coordinates x'. Then, we can use operators $\hat{Q}^i$ and $\hat{P}_j$ for the construction of the position representation of a quantum system related to the curvilinear coordinates (x, p).

Let $\hat{U}_T : L^2(\mathbb{R}^n) \to L^2(V, d\mu)$, where $d\mu(x) = \left|\phi'(x)\right| dx = |\det[g(x)]|^{1/2} dx$, be a unitary operator such that

$$(\hat{U}_T\varphi)(x) = \varphi(\phi(x))$$

$$\Downarrow$$

$$(\hat{U}_T\varphi)(x) = (\varphi \circ \phi)(x), \qquad (\hat{U}_T^{-1}\varphi)(x') = \left(\varphi \circ \phi^{-1}\right)(x'),$$

then

$$\boxed{\hat{q}^i := \hat{U}_T \hat{Q}^i \hat{U}_T^{-1} = x^i} \tag{8.1.44}$$

is a new position operator in a Hilbert space $L^2(V, \mu)$ as

$$\begin{aligned}(\hat{q}^i\psi)(x) &= \left(\hat{U}_T \hat{Q}^i \hat{U}_T^{-1}\psi\right)(x) = \left(\hat{Q}^i \hat{U}_T^{-1}\psi \circ \phi\right)(x) \\ &= \left(\hat{Q}^i \hat{U}_T^{-1}\psi\right)(x') = \left[\phi^{-1}(x')\right]^i \left(\psi \circ \phi^{-1}\right)(x') = x^i\psi(x).\end{aligned}$$

New momentum operator $\hat{p}_j$ in $L^2(V, d\mu)$ is of the form

$$\boxed{\hat{p}_j := \hat{U}_T \hat{P}_j \hat{U}_T^{-1} = -i\hbar\left(\partial_{x^j} + \tfrac{1}{2}\Gamma^k_{jk}(x)\right),} \tag{8.1.45}$$

where Γ^i_{jk} are connection coefficients of the metric tensor $g(x)$. Indeed, $P_j = p_j = p'_i\left[\phi'(\phi^{-1}(x'))\right]^i_j$, so

$$\begin{aligned}&(\hat{p}_j\psi)(x) \\ &= \left(\hat{U}_T \hat{P}_j \hat{U}_T^{-1}\psi\right)(x) = \left(\hat{U}_T\left(\frac{1}{2}\hat{p}'_i\frac{\partial\phi^i}{\partial x^j}(\phi^{-1}(\hat{q}')) + \frac{1}{2}\frac{\partial\phi^i}{\partial x^j}(\phi^{-1}(\hat{q}'))\hat{p}'_i\right)\hat{U}_T^{-1}\psi\right)(x) \\ &= -\frac{1}{2}i\hbar\partial_{x'^i}\left[\left(\frac{\partial\phi^i}{\partial x^j}\circ\phi^{-1}\right)(\phi(x))\psi(x)\right] - \frac{1}{2}i\hbar\frac{\partial\phi^i}{\partial x^j}(x)\partial_{x'^i}(\psi\circ\phi^{-1})(\phi(x)) \\ &= -i\hbar\left(\frac{\partial\phi^i}{\partial x^j}(x)\frac{\partial(\phi^{-1})^k}{\partial x'^i}(\phi(x))\frac{\partial\psi}{\partial x^k}(x) + \frac{1}{2}\frac{\partial^2\phi^i}{\partial x^j\partial x^k}(x)\frac{\partial(\phi^{-1})^k}{\partial x'^i}(\phi(x))\psi(x)\right) \\ &= -i\hbar\left(\delta^k_j\frac{\partial}{\partial x^k} + \frac{1}{2}\Gamma^k_{jk}\right)\psi(x),\end{aligned}$$

where we used the relation $(\phi'(x))^{-1} = (\phi^{-1})'(\phi(x))$, from which follows that $\left[(\phi'(x))^{-1}\right]_i^k = \frac{\partial(\phi^{-1})^k}{\partial x'^i}(\phi(x))$, and the formula (2.6.5) for the transformation of connection coefficients between the pseudo-Euclidean and curvilinear basis.

Momentum operator $\hat{p}_j$ (8.1.45) is self-adjoint in the Hilbert space $L^2(V, d\mu)$. Indeed, integrating by parts and using formula (8.1.19) we find

$$\begin{aligned}(\varphi, \hat{p}_j\psi) &= \int \bar{\varphi}(x)\hat{p}_j\psi(x)d\mu(x) = \int \bar{\varphi}(x)(-i\hbar)(\partial_{x^j} + \tfrac{1}{2}\Gamma^k_{jk}(x))\psi(x)\,|g(x)|^{1/2}\,dx\\ &= i\hbar\int \psi(x)\partial_{x^j}(\bar{\varphi}(x))\,|g(x)|^{1/2}\,dx - \tfrac{1}{2}i\hbar\int \Gamma^k_{jk}(x)\varphi^*(x)\psi(x)\,|g(x)|^{1/2}\,dx\\ &= i\hbar\int \psi(x)\partial_{x^j}(\bar{\varphi}(x)\,|g(x)|^{1/2})dx + i\hbar\int \Gamma^k_{jk}(x)\bar{\varphi}(x)\psi(x)\,|g(x)|^{1/2}\,dx\\ &\quad - \tfrac{1}{2}i\hbar\int \Gamma^k_{jk}(x)\bar{\varphi}(x)\psi(x)\,|g(x)|^{1/2}\,dx\\ &= \int(-i\hbar(\partial_{x^j} + \tfrac{1}{2}\Gamma^k_{jk}(x))\bar{\varphi}\psi(x)\,|g(x)|^{1/2}\,dx = (\hat{p}_j\varphi, \psi),\end{aligned}$$

which means that $\hat{p}_j = \hat{p}_j^\dagger$ in $L^2(V, d\mu)$.

8.1.4 Coordinate Free Formulation

In previous subsections we developed the operator representation of quantum mechanics written down in some coordinate systems on the configuration space. In this subsection we will apply our formalism to derive an operator representation in a coordinate free way [37, 98]. Let $\phi\colon Q \supset U \to V \subset \mathbb{R}^n$ and $\Phi\colon T^*U \to T^*V = V \times \mathbb{R}^n$ be an almost global coordinate system on the configuration space Q and the phase space T^*Q, respectively. Since the coordinate system ϕ is almost globally defined in consequence it defines an isomorphism $\hat{U}\colon L^2(Q, d\omega_g) \to L^2(V, d\mu)$ of the Hilbert spaces by

$$\hat{U}\psi = \psi|_U \circ \phi^{-1}.$$

In fact, the restriction $|_U$ is an isomorphism of $L^2(Q, d\omega_g)$ onto $L^2(U, d\mu)$ since for $\psi \in L^2(Q, d\omega_g)$, ψ and $\psi|_U$ are equal almost everywhere and Hilbert spaces of square integrable functions consist of equivalence classes of functions equal almost everywhere. In consequence, the coordinate system Φ defines an isomorphism of the Hilbert space $L^2(T^*Q, d\Omega_\hbar)$ onto the Hilbert space $L^2(T^*V, d\Omega_\hbar)$. Let us now define a tensor product

$$\otimes\colon (L^2(Q, d\omega_g))^* \times L^2(Q, d\omega_g) \to L^2(T^*Q, d\Omega_\hbar)$$

as follows

$$\bar{\varphi} \otimes \psi = ((\overline{\hat{U}\varphi}) \otimes_S \hat{U}\psi) \circ \Phi, \quad \varphi, \psi \in L^2(Q, d\omega_g),$$

where

$$\otimes_S : (L^2(V, d\mu))^* \times L^2(V, d\mu) \to L^2(T^*V, d\Omega_\hbar)$$

is a tensor product defined in the previous subsection, corresponding to the coordinate system Φ. Such a definition of the tensor product $\otimes$ is independent of the choice of a particular coordinate system. Indeed, if $\phi' : Q \supset U' \to V' \subset \mathbb{R}^n$ and $\Phi' : T^*U' \to T^*V' = V' \times \mathbb{R}^n$ is another almost global coordinate system on Q and T^*Q, respectively and $L^2(Q, d\omega_g) \to L^2(V', d\mu')$ a Hilbert space isomorphism induced by ϕ' then $T = \Phi \circ \Phi'^{-1}$ is a canonical transformation of coordinates and $\hat{U}_T = \hat{U}'\hat{U}^{-1}$ is a related unitary operator (8.1.41). Then from (8.1.42) we have

$$\begin{aligned}\bar{\varphi} \otimes \psi &= ((\overline{\hat{U}\varphi}) \otimes_S \hat{U}\psi) \circ T \circ \Phi' = ((\overline{\hat{U}_T\hat{U}\varphi}) \otimes_{S'} \hat{U}_T\hat{U}\psi) \circ \Phi' \\ &= ((\overline{\hat{U}'} \otimes_{S'} \hat{U}'\psi) \circ \Phi'.\end{aligned}$$

The tensor product $\otimes$ inherits all properties of the tensor products $\otimes_S$. In particular, for every $\rho \in L^2(T^*Q, d\Omega_\hbar)$

$$\rho \star = \hat{1} \otimes \hat{\rho},$$

where $\hat{\rho} \in B_{HS}(L^2(Q, d\omega_g))$ is a Hilbert-Schmidt operator defined on the Hilbert space $L^2(Q, d\omega_g)$. Conversely, for every $\hat{\rho} \in B_{HS}(L^2(Q, d\omega_g))$ the operator $\hat{1} \otimes \hat{\rho}$ is of the form $\rho \star$ for some $\rho \in L^2(T^*Q, d\Omega_\hbar)$. The following properties are fulfilled:

(i) for $\rho = \bar{\varphi} \otimes \psi$, $\hat{\rho} = (\varphi, \cdot)\psi$,
(ii) $\bar{\rho} \star = \hat{1} \otimes \hat{\rho}^\dagger$,
(iii) $\mathrm{Tr}(\rho) \equiv \int_{T^*Q} \rho \, d\Omega_\hbar = \mathrm{tr}(\hat{\rho})$,
(iv) for $\rho_1, \rho_2 \in L^2(T^*Q, d\Omega_\hbar)$ and $\hat{\rho}_1, \hat{\rho}_2 \in B_{HS}(L^2(Q, d\omega_g))$ such that $\rho_1 \star = \hat{1} \otimes \hat{\rho}_1$ and $\rho_2 \star = \hat{1} \otimes \hat{\rho}_2$

$$(\rho_1, \rho_2) = (\hat{\rho}_1, \hat{\rho}_2)_{HS},$$

(v) $\int_{T^*Q} \bar{f} \star f \star \rho \, d\Omega_\hbar \geq 0$ for $f \in C_0^\infty(T^*Q)$ if and only if $(\varphi, \hat{\rho}\varphi) \geq 0$ for $\varphi \in L^2(Q, d\omega_g)$.

Let $A \in C^\infty(T^*Q)[\![\hbar]\!]$ and $\rho = \bar{\varphi} \otimes \psi$ for $\varphi, \psi \in C_0^\infty(Q)$. Then

$$\begin{aligned} A \star \rho &= \bar{\varphi} \otimes \hat{A}\psi, \\ \rho \star A &= \overline{(\hat{A}^\dagger \varphi)} \otimes \psi, \end{aligned} \tag{8.1.46}$$

where $\hat{A}$ is an operator acting in the Hilbert space $L^2(Q, d\omega_g)$. Besides, if $(x^1, \ldots, x^n)$ is an almost global coordinate system on Q, $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ a respective canonical coordinate system on T^*Q and $\hat{U}$ a corresponding unitary operator given by (8.1.4), then

$$\hat{U}\hat{A}\hat{U}^{-1} = A_S(\hat{q}, \hat{p}).$$

Let $A \in C^\infty(T^*Q)[\![\hbar]\!]$ and $\rho \in L^2(T^*Q, d\Omega_\hbar)$. If $A \star \rho \in L^1(T^*Q, d\Omega_\hbar)$ then

$$\int_{T^*Q} A \star \rho \, d\Omega_\hbar = \operatorname{tr}(\hat{A}\hat{\rho}).$$

In particular, if $\rho = \bar{\varphi} \otimes \psi$ for $\varphi, \psi \in C_0^\infty(Q)$ then

$$\int_{T^*Q} A \star \rho \, d\Omega_\hbar = (\varphi, \hat{A}\psi).$$

From (8.1.4) follows that the map $\rho \mapsto \hat{\rho}$ is a representation of the algebra $\mathcal{L} = (L^2(T^*Q, d\Omega_\hbar), \star)$ in the Hilbert space $L^2(Q, d\omega_g)$. Furthermore, from relation (8.1.46) follows that functions $A \in C^\infty(T^*Q)[\![\hbar]\!]$ can be identified with operators $\hat{A}$ through the formula

$$A \star = \hat{1} \otimes \hat{A}.$$

Moreover, the map $A \mapsto \hat{A}$ is a representation of the algebra $\mathcal{A}_Q = (C^\infty(T^*Q)[\![\hbar]\!], \star)$ in the Hilbert space $L^2(Q, d\omega_g)$.

We get the following characterization of quantum states. Pure states can be alternatively characterized as these functions $\rho_{\text{pure}} \in L^2(T^*Q, d\Omega_\hbar)$ which are self-conjugated, normalized, and idempotent

$$\begin{aligned} \rho_{\text{pure}} &= \bar{\rho}_{\text{pure}}, \\ \int_{T^*Q} \rho_{\text{pure}} \, d\Omega_\hbar &= 1, \\ \rho_{\text{pure}} \star \rho_{\text{pure}} &= \rho_{\text{pure}}. \end{aligned} \tag{8.1.47}$$

Then mixed states $\rho_{\text{mix}} \in L^2(T^*Q, d\Omega_\hbar)$ are characterized as convex linear combinations, possibly infinite, of some families of pure states $\rho^{(\lambda)}_{\text{pure}}$

$$\rho_{\text{mix}} = \sum_\lambda p_\lambda \rho^{(\lambda)}_{\text{pure}},$$

where $p_\lambda \geq 0$ and $\sum_\lambda p_\lambda = 1$.

If $\phi\colon Q \supset U \to V \subset \mathbb{R}^n$ and $\Phi\colon T^*U \to T^*V = V \times \mathbb{R}^n$ are almost global coordinate systems on the configuration space Q and the phase space T^*Q, respectively, and $\hat{U}\colon L^2(Q, d\omega_g) \to L^2(V, d\mu)$ an isomorphism of the Hilbert spaces given by (8.1.4), then the maps $x^i = \Phi^i$ and $p_j = \Phi^{j+n}$ $(i, j = 1, 2, \ldots, n)$ are observables of position and momentum corresponding to the coordinate system Φ. To the maps x^i, p_j are related operators $\hat{q}^i$, $\hat{p}_j$ defined on the Hilbert space $L^2(Q, d\omega_g)$. The operators $\hat{q}^i = \phi^i$ are of the form of multiplication operators by coordinate functions ϕ^i and they constitute a complete set of commuting observables and thus they can be used to create a position representation corresponding to the coordinate system ϕ. In this representation operators $\hat{q}^i$ take the form of the multiplication operators by a coordinate variable defined on the Hilbert space $L^2(V, d\mu)$. In accordance with (8.1.4) the unitary operator giving this representation is equal $\hat{U}$.

8.1.5 *Quantization of Hydrogen Atom in Curvilinear Coordinates*

In order to illustrate how to quantize the classical system directly in curvilinear coordinates, let us consider a quantum system of the hydrogen atom [36]. A configuration space of such a system is the 3-dimensional Euclidean space E^3 and represents the position in space of an electron of the hydrogen atom. A phase space of the system is T^*E^3 and a Hamiltonian H in Cartesian coordinates takes a form

$$H(x, y, z, p_x, p_y, p_z) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} - \frac{1}{4\pi\epsilon_0}\frac{e^2}{\sqrt{x^2 + y^2 + z^2}}. \tag{8.1.48}$$

A standard Weyl quantization means that as a star-product on T^*E^3 is taken the canonical $\star$-product which in the Cartesian coordinates takes the form of the Moyal product. In the position representation, the Hilbert space of states takes the form of the space $L^2(\mathbb{R}^3)$ of functions on $\mathbb{R}^3$ square integrable with respect to the Lebesgue measure. The canonical operators of position and momentum in Cartesian coordinates take the standard form

$$\begin{aligned} &\hat{q}_x = x, && \hat{q}_y = y, && \hat{q}_z = z, \\ &\hat{p}_x = -i\hbar\partial_x, && \hat{p}_y = -i\hbar\partial_y, && \hat{p}_z = -i\hbar\partial_z, \end{aligned}$$

and the Hamiltonian operator, being a symmetrically ordered function H of the operators of position and momentum, takes the form

$$H(\hat{q}_x, \hat{q}_y, \hat{q}_z, \hat{p}_x, \hat{p}_y, \hat{p}_z) = -\frac{\hbar^2}{2m}(\partial_x^2 + \partial_y^2 + \partial_z^2) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{\sqrt{x^2+y^2+z^2}}, \tag{8.1.49}$$

where $\partial_x^2 + \partial_y^2 + \partial_z^2 = \Delta$ is the Laplace operator in the Cartesian coordinates. This is the quantization procedure presented in any textbook of quantum mechanics.

Now, let us quantize the considered system directly in the spherical coordinates. The Moyal product in Cartesian coordinates, under the point transformation to spherical coordinates (3.3.6), transforms to a star-product of the form (6.1.64), (6.1.65). This star-product is equivalent to the Moyal product, where the equivalence morphism S, by virtue of (6.1.66), takes the form

$$\begin{aligned}
S = \mathrm{id} + \frac{\hbar^2}{4}\Bigg[& \frac{1}{r^2}\partial_{p_r}^2 + \left(\frac{1}{2\tan^2\theta} - 1\right)\partial_{p_\theta}^2 - \partial_{p_\phi}^2 + \frac{1}{r\tan\theta}\partial_{p_r}\partial_{p_\theta} + \frac{1}{r^2}p_\theta\partial_{p_r}^2\partial_{p_\theta} \\
& - \tfrac{1}{2}p_r\partial_{p_r}\partial_{p_\theta}^2 + \frac{2}{r\tan\theta}p_\phi\partial_{p_r}\partial_{p_\theta}\partial_{p_\phi} - \left(\tfrac{1}{2}p_r\sin^2\theta + \frac{1}{r}p_\theta\sin\theta\cos\theta\right)\partial_{p_r}\partial_{p_\phi}^2 - \tfrac{1}{3}p_\theta\partial_{p_\theta}^3 \\
& + \frac{1}{\tan^2\theta}p_\phi\partial_{p_\theta}^2\partial_{p_\phi} - \tfrac{1}{2}p_\theta\partial_{p_\theta}\partial_{p_\phi}^2 - \tfrac{1}{3}p_\phi\partial_{p_\phi}^3 + \frac{1}{r^2}p_\phi\partial_r^2\partial_{p_\phi} - \tfrac{1}{2}r\partial_r\partial_{p_\theta}^2 - \tfrac{1}{2}r\sin^2\theta\partial_r\partial_{p_\phi}^2 \\
& + \frac{1}{r}\partial_\theta\partial_{p_r}\partial_{p_\theta} - \tfrac{1}{2}\sin\theta\cos\theta\partial_\theta\partial_{p_\phi}^2 + \frac{1}{r}\partial_\phi\partial_{p_r}\partial_{p_\phi} + \frac{1}{\tan\theta}\partial_\phi\partial_{p_\theta}\partial_{p_\phi}\Bigg] + O(\hbar^4).
\end{aligned}$$

The classical Hamilton function (8.1.48) in spherical coordinates takes the form

$$H(r, \theta, \phi, p_r, p_\theta, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}, \tag{8.1.50}$$

and the action of the morphism S on the transformed Hamilton function (8.1.50) results in the following new function

$$\begin{aligned}
(S^{-1}H)(r, \theta, \phi, p_r, p_\theta, p_\phi) = & \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r} \\
& - \frac{\hbar^2}{8mr^2}\left(\frac{1}{\sin^2\theta} + 1\right).
\end{aligned} \tag{8.1.51}$$

Note the extra term in (8.1.51) depends on $\hbar^2$. Thus, the quantum system in spherical coordinates can be described by the Hamiltonian (8.1.50) and the star-product in the form (6.1.61), or equivalently by the Hamiltonian (8.1.51) and the Moyal star-product.

In the position representation in spherical coordinates the Hilbert space of states is equal $L^2(V, d\mu)$, where $V = (0, \infty) \times (0, \pi) \times (0, 2\pi)$, $d\mu(r, \theta, \phi) = r^2 \sin\theta \, dr \, d\theta \, d\phi$ and according to (8.1.26) the operators of position and momentum take the form

$$\begin{aligned} &\hat{q}_r = r, && \hat{q}_\theta = \theta, && \hat{q}_\phi = \phi, \\ &\hat{p}_r = -i\hbar\left(\partial_r + \frac{1}{r}\right), && \hat{p}_\theta = -i\hbar\left(\partial_\theta + \frac{1}{2}\cot\theta\right), && \hat{p}_\phi = -i\hbar\partial_\phi. \end{aligned} \tag{8.1.52}$$

The Hamilton operator is calculated as an S-ordered Hamilton function (8.1.50) of operators of position and momentum (8.1.52), or equivalently as a symmetrically ordered function (8.1.51) of these operators:

$$\begin{aligned} H_S(\hat{q}_r, \hat{q}_\theta, \hat{q}_\phi, \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi) &= (S^{-1}H)_M(\hat{q}_r, \hat{q}_\theta, \hat{q}_\phi, \hat{p}_r, \hat{p}_\theta, \hat{p}_\phi) \\ &= -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \frac{1}{\tan\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\right] \\ &\quad - \frac{1}{4\pi\epsilon_0}\frac{e^2}{r}. \end{aligned} \tag{8.1.53}$$

The expression in square brackets is just the well known Laplace operator written in spherical coordinates. A direct computation shows that the operators (8.1.49) and (8.1.53) are unitarily equivalent, where a unitary operator giving this equivalence is equal

$$\hat{U}_T : L^2(\mathbb{R}^3) \to L^2(V, d\mu), \quad (\hat{U}_T\psi)(r, \theta, \phi) = \psi(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta).$$

Note, that the property that the spherical coordinates are almost global is crucial for $\hat{U}_T$ to be a unitary operator. Since the operators (8.1.49) and (8.1.53) are unitarily equivalent they have the same spectra, and thus solving the eigenvalue problem of one of these operators gives the solution to the other one.

8.1.6 Operators Linear, Quadratic and Cubic in Momenta

Suppose now that we want to quantize in any almost global coordinates a Hamiltonian system given on a phase space $M = T^*Q$ that is the cotangent bundle to an appropriate pseudo-Riemannian manifold equipped with a metric tensor g. In what follows we will consider the class of quantizations of a classical system with Hamiltonians polynomial in momenta [36, 37], for which the morphism S giving

the equivalence with the Moyal quantization is in the form (6.1.104)

$$S = \mathrm{id} + \frac{\hbar^2}{4!}\Big(3\left(\Gamma^i_{lj}\Gamma^l_{ik} + aR_{jk}\right)\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{x^i}\partial_{p_j}\partial_{p_k} + \left(2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l}\right)p_i\partial_{p_j}\partial_{p_k}\partial_{p_l}$$
$$- 3b\partial_{p_j}(\partial_{x^j} + \Gamma^i_{jl}p_i\partial_{p_l})\partial_{p_k}(\partial_{x^k} + \Gamma^r_{kn}p_r\partial_{p_n})\Big) + O(\hbar^4),$$

for any classical and quantum canonical coordinate system.

First, let us consider a function H on T^*Q linear in momenta, which in a Darboux coordinate system (x, p) takes the form

$$\boxed{H(x, p) = A^j(x)p_j,} \tag{8.1.54}$$

where A^i are components of a vector field A defined on Q. The action of the morphism S on H leaves function H unchanged:

$$(S^{-1}H)(x, p) = H(x, p).$$

Thus the following self-adjoint operator will correspond from (6.2.7) to function H

$$H_S(\hat{q}, \hat{p}) = \tfrac{1}{2}A^j(\hat{q})\hat{p}_j + \tfrac{1}{2}\hat{p}_j A^j(\hat{q}).$$

By virtue of (8.1.26)

$$\hat{q}^j = x^j, \quad \hat{p}_j = -i\hbar(\partial_j + \tfrac{1}{2}\Gamma^k_{jk}), \quad \partial_j \equiv \partial_{x^j}$$

the above operator can be written in the form

$$H_S(\hat{q}, \hat{p}) = -\tfrac{1}{2}i\hbar\left(2A^j\partial_j + A^i{}_{,j} + \Gamma^k_{jk}A^j\right) = -\tfrac{1}{2}i\hbar\left(2A^j\partial_j + A^j{}_{;j}\right).$$

Finally, one can transform the above operator to the following invariant form

$$\boxed{H_S(\hat{q}, \hat{p}) = -\tfrac{1}{2}i\hbar\left(2A^i\nabla_i + A^i{}_{;i}\right) = -\tfrac{1}{2}i\hbar\left(A^i\nabla_i + \nabla_i A^i\right),} \tag{8.1.55}$$

expressed by covariant derivative operators.

Now, let us consider a function H on T^*Q quadratic in momenta, which in a Darboux coordinate system takes the following form

$$\boxed{H(x, p) = \tfrac{1}{2}A^{ij}(x)p_i p_j + V(x),} \tag{8.1.56}$$

where A^{ij} are components of a symmetric second order contravariant tensor field A defined on Q and V is a smooth function on Q, representing a potential. Now the

action of the morphism S on H results in the following function

$$(S^{-1}H)(x,\,p) = H(x,\,p) - \tfrac{1}{2}\hbar^2\left(\tfrac{1}{4}A^{ij}{}_{,k}\Gamma^k_{ij} + \tfrac{1}{4}A^{ij}\Gamma^k_{li}\Gamma^l_{kj} - \tfrac{1}{4}bA^{ij}{}_{;ij} + \tfrac{1}{4}aA^{ij}R_{ij}\right).$$

The following self-adjoint operator will correspond from the above equation and (6.2.7) to function H (*quantum Hamiltonian*)

$$H_S(\hat{q},\,\hat{p}) = \tfrac{1}{2}\left(\tfrac{1}{4}A^{ij}(\hat{q})\hat{p}_i\hat{p}_j + \tfrac{1}{2}\hat{p}_iA^{ij}(\hat{q})\hat{p}_j + \tfrac{1}{4}\hat{p}_i\hat{p}_jA^{ij}(\hat{q})\right) + V(\hat{q})$$
$$-\tfrac{1}{2}\hbar^2\left(\tfrac{1}{4}A^{ij}{}_{,k}(\hat{q})\Gamma^k_{ij}(\hat{q}) + \tfrac{1}{4}A^{ij}(\hat{q})\Gamma^k_{li}(\hat{q})\Gamma^l_{kj}(\hat{q}) - \tfrac{1}{4}bA^{ij}{}_{;ij}(\hat{q}) + \tfrac{1}{4}aA^{ij}(\hat{q})R_{ij}(\hat{q})\right).$$

Again by virtue of (8.1.26) the above equation can be transformed to the form

$$H_S(\hat{q},\,\hat{p}) = -\tfrac{1}{2}\hbar^2\Big(A^{ij}\partial_i\partial_j + A^{ij}\Gamma^l_{jl}\partial_i + A^{ij}{}_{,i}\partial_j + \tfrac{1}{2}A^{ij}\Gamma^l_{jl,i} + \tfrac{1}{4}A^{ij}\Gamma^k_{ik}\Gamma^l_{jl} + \tfrac{1}{2}A^{ij}{}_{,i}\Gamma^l_{jl}$$
$$+ \tfrac{1}{4}A^{ij}{}_{,ij} + \tfrac{1}{4}A^{ij}{}_{,k}\Gamma^k_{ij} + \tfrac{1}{4}A^{ij}\Gamma^k_{li}\Gamma^l_{kj} - \tfrac{1}{4}bA^{ij}{}_{;ij} + \tfrac{1}{4}aA^{ij}R_{ij}\Big) + V$$

and then, using the equality $A^{ij}{}_{,k} = -A^{rj}\Gamma^i_{rk} - A^{ri}\Gamma^j_{rk} + A^{ij}{}_{;k}$ and relation (2.6.21), it simplifies to

$$H_S(\hat{q},\,\hat{p}) = -\tfrac{1}{2}\hbar^2\Big(A^{ij}\partial_i\partial_j + A^{ij}\Gamma^l_{jl}\partial_i + A^{ij}{}_{,i}\partial_j + \tfrac{1}{4}(1-b)A^{ij}{}_{;ij}$$
$$- \tfrac{1}{4}(1-a)A^{ij}R_{ij}\Big) + V. \tag{8.1.57}$$

Note that (8.1.57) can be written in the following invariant form

$$\boxed{\begin{aligned} H_S(\hat{q},\,\hat{p}) &= -\tfrac{1}{2}\hbar^2\left(\nabla_iA^{ij}\nabla_j + \tfrac{1}{4}(1-b)A^{ij}{}_{;ij} - \tfrac{1}{4}(1-a)A^{ij}R_{ij}\right) + V \\ &= -\tfrac{1}{2}\hbar^2\hat{\Delta}_A + V_{\text{quant}} + V, \end{aligned}} \tag{8.1.58}$$

where $\hat{\Delta}_A = \nabla_iA^{ij}\nabla_j$ is the *pseudo-Laplace operator* and

$$V_{\text{quant}}(x;\,a,\,b) = -\tfrac{1}{8}\hbar^2\left[(1-b)A^{ij}{}_{;ij} - (1-a)A^{ij}R_{ij}\right]$$

represents a quantum correction to classical potential $V(x)$.

Observe that according to (2.6.10) and (8.1.19), the pseudo-Laplace operator $\hat{\Delta}_A$ can be expressed by

$$\boxed{\hat{\Delta}_A = \nabla_i A^{ij} \nabla_j = |g|^{-\frac{1}{2}} \partial_i |g|^{\frac{1}{2}} A^{ij} \partial_j.} \tag{8.1.59}$$

For a special case, when A coincides with the standard metric tensor g on the configuration space, the function H has the form of a natural Hamiltonian (6.1.1), and Eq. (8.1.58) reduces to

$$\boxed{H_S(\hat{q}, \hat{p}) = -\tfrac{1}{2}\hbar^2 \left(G^{ij} \nabla_i \nabla_j - \tfrac{1}{4}(1-a)R \right) + V} \tag{8.1.60}$$

where $\nabla_i G^{ij} \nabla_j = G^{ij} \nabla_i \nabla_j = \hat{\Delta}$ is the *Laplace-Beltrami operator*. Hence, in particular

$$\boxed{\hat{\Delta} = \nabla_i G^{ij} \nabla_j = |g|^{-\frac{1}{2}} \partial_i |g|^{\frac{1}{2}} G^{ij} \partial_j.} \tag{8.1.61}$$

Thus, we get a two-parameter family of quantizations related to a given metric tensor g. Particular examples of Hamiltonian operators (8.1.58) with fixed values of parameters a and b, derived by different methods can be found in the literature [61, 88, 89, 102, 109, 131, 183, 220].

Observation 20 *To the knowledge of the author, for the general non-flat metric tensor g, there is no experimental evidence which quantizations, from the considered family, are realized in nature. On the other hand, for a flat metric tensor g and a natural Hamiltonian* (6.1.1)*, all quantum operators* (8.1.60) *collapse onto a single operator*

$$H(\hat{q}, \hat{p}) = -\tfrac{1}{2}\hbar^2 G^{ij} \nabla_i \nabla_j + V = -\tfrac{1}{2}\hbar^2 |g|^{-\frac{1}{2}} \partial_i |g|^{\frac{1}{2}} G^{ij} \partial_j + V$$

equivalent to a standard Weyl quantization. For a non flat case there exists one distinguished quantization given by $a = b = 1$*, called a minimal quantization, for which quantum correction to the classical potential vanishes:* $V_{quant} = 0$ *and hence*

$$H(\hat{q}, \hat{p}) = -\tfrac{1}{2}\hbar^2 \nabla_i A^{ij} \nabla_j + V = -\tfrac{1}{2}\hbar^2 |g|^{-\frac{1}{2}} \partial_i |g|^{\frac{1}{2}} A^{ij} \partial_j + V. \tag{8.1.62}$$

Finally, let us consider a function H on T^*Q, which in a Darboux coordinate system (x, p) is cubic in momenta (we skip the lower terms in momenta):

$$\boxed{H(x, p) = A^{ijk}(x) p_i p_j p_k,} \tag{8.1.63}$$

where A^{ijk} are components of a symmetric third order contravariant tensor field A defined on Q. In a similar fashion as in the previous case (we skip here a cumbersome calculations) one can derive the form of the corresponding self-adjoint operator:

$$\boxed{\begin{aligned} H_S(\hat{q},\hat{p}) = \tfrac{1}{2}i\hbar^3\Big(&\nabla_i A^{ijk}\nabla_j\nabla_k + \nabla_i\nabla_j A^{ijk}\nabla_k + \tfrac{1}{4}(1-b)\nabla_k A^{ijk}{}_{;ij} \\ &+ \tfrac{1}{4}(1-b)A^{ijk}{}_{;ij}\nabla_k - \tfrac{3}{4}(1-a)\nabla_i A^{ijk}R_{jk} - \tfrac{3}{4}(1-a)A^{ijk}R_{jk}\nabla_i\Big). \end{aligned}} \tag{8.1.64}$$

In particular, for minimal quantization, we get

$$H_S(\hat{q},\hat{p}) = \tfrac{1}{2}i\hbar^3\left(\nabla_i A^{ijk}\nabla_j\nabla_k + \nabla_i\nabla_j A^{ijk}\nabla_k\right),$$

the form considered for the first time in [103], but introduced without any justification from basic principles. Minimal quantization, related to various metric tensors, will be of great importance for the quantum stationary separability theory developed in the next section.

Note that the received operators are defined on the Hilbert space $L^2(V, d\mu)$ and correspond to a given canonical coordinate system (x, p). These operators are written in an invariant form and consequently they can be treated as operators defined on the Hilbert space $L^2(Q, d\omega_g)$. Indeed, using the unitary operator $\hat{U}$ related to the coordinate system (x, p) and given by (8.1.4), in accordance with the formula (8.1.4) we can receive operators defined on the Hilbert space $L^2(Q, d\omega_g)$.

8.2 Quantum Integrability and Stationary Quantum Separability

The notion of classical integrability and classical separability in particular is well recognized and was described in details in the first half of the book. The question about the definition of quantum integrability on the most general level of multiparticle fermionic and bosonic systems is far from being trivial (see [62] and references there). Fortunately, as far as we will concentrate on quantum analogous of classical separable systems, considered in previous chapters, the following restricted definition of quantum integrability will be sufficient for our purpose.

Let us consider a quantum system $(M, \star, H)$ on a $2n$ dimensional phase space M. By the analogy to classical systems, we say that the quantum system is quantum integrable if

1. in the Hilbert space $L^2(M)$, there exist n operators $H_i\star$ (including $H\star$), formally self-adjoint

$$(H_i\star)^\dagger = H_i^*\star = H_i\star, \quad i = 1, \ldots, n$$

which are in quantum involution

$$[\![H_i, H_j]\!] = 0, \quad i, j = 1, \ldots, n,$$

2. which are algebraically independent
3. and in the classical limit

$$H_k\star \overset{\hbar\to 0}{\longrightarrow} H_k, \quad k = 1, \ldots, n$$

functions H_k constitute a Liouville integrable system.

Algebraic independence means that there is no polynomial P built of operators $H_i\star$ and such that

$$P_S(H_1\star, \ldots, H_n\star) = (S^{-1}P)_W(H_1\star, \ldots, H_n\star) = 0,$$

where the morphism S defines the quantization $\star$. In a position representation of quantum mechanics it means that there exist n self-adjoint differential operators $H_i(\hat{q}, \hat{p})$ in the Hilbert space $L^2(Q, d\mu)$, being quantum counterparts of classical, functionally independent Hamiltonians in involution, which are algebraically independent and which commute

$$[H_i(\hat{q}, \hat{p}), H_j(\hat{q}, \hat{p})] = 0. \tag{8.2.1}$$

The restriction to pure differential operators allow us to avoid the consequences of von Neumann theorem [254] in the majority of cases. The theorem says that if a pair of self-adjoint operators A and B commutes, then there is a third self-adjoint operator C such that $A = f(C)$ and $B = g(C)$. However, if A and B are differential operators C usually is not a differential one. But even if it happens, like for the two dimensional quantum oscillator [259], it not contradicts presented definition. It simple means, that for some quantum integrable systems one may alternatively finds a less number of commuting operators, carrying the same information about a quantum system. It is of course a pure quantum phenomenon.

By *quantum dynamical separability* we understand the existence of quantum separation coordinates (λ, μ) which allow efficient linearization of quantum Hamiltonian equations of motion. It means the construction in an explicit form (or at

least implicit form) quantum trajectories, i.e. integration by quadratures quantum equations of motion (7.1.14) or equivalently (7.1.16). It is complex mathematical problem beyond the scope of the book. There is no quantum analog of the theory developed in Chaps. 4 and 5 and so completely different mathematical tools have to be applied. What in literature is known as quantum separability should be rather called *quantum stationary separability*. Actually it means the existence of such quantum Darboux coordinates (λ, μ), in which simultaneously separate n eigenvalue problems

$$H_i(\hat{q}, \hat{p})\Psi = E_i\Psi, \quad i = 1, \ldots, n \tag{8.2.2}$$

of commuting differential operators (8.2.1).

The problem of quantum stationary separability has a long history, starting from the early works of Robertson [228] and Eisenhart [104], and has a reach literature (see for example [17, 18, 47–49, 145, 182, 206, 248] and references therein). In the following section we develop quantum separability theory for quantized Stäckel systems with quadratic in momenta first integrals, considered in Chap. 4, and prove that for any classical Stäckel system from the considered class, there exists an infinite family of quantizations leading to a quantum stationary separable system. The presented theory is a significant extension of already classical results of Robertson [228] and Eisenhart [104]. We want to stress, however, that we do not deal here with spectral theory of the obtained quantum systems, as it requires a separate investigations.

8.2.1 *Separability for Minimal Quantization with Natural Metric*

Let us come back to Stäckel systems considered in Chap. 4 with all constants of motion quadratic in momenta, so fulfilling general separation relations (4.2.9) with condition (4.2.11). First, let us consider a minimal quantization (8.1.62) of classical Stäckel Hamiltonians (4.3.2) in classical separation coordinates

$$H_r = \tfrac{1}{2}\sum_{i=1}^{n} A_r^{ii}\mu_i^2 + V_r = \tfrac{1}{2}\sum_{i=1}^{n} (K_r G)^{ii}\,\mu_i^2 + V_r, \quad r = 1, \ldots, n, \tag{8.2.3}$$

with a natural choice of metric tensor $G \equiv A_1$, where matrices A_r and K_r are given by formulas (4.3.7) and (4.3.8). Notice, that with such choice of metric, tensors K_r are Killing tensors of G. Then, the respective quantum Hamiltonians $\hat{H}_r$ (8.1.62) are given by

$$\hat{H}_r = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} \nabla_i A_r^{ii} \nabla_i + V_r, \quad r = 1, \ldots, n \tag{8.2.4}$$

where ∇_i are operators of Levi-Civita connection associated with the metric $g = G^{-1}$. Hamiltonian operators (8.2.4) are self-adjoint in the Hilbert space $L^2(Q, |\det g|^{1/2} d\lambda)$.

Now let us consider a set of eigenvalue problems (stationary Schrödinger equations) associated with Hamiltonian operators (8.2.4). For the purposes of calculation let us introduce metrically contracted Christoffel symbols of g defined by

$$\boxed{\Gamma_i := G^{jk}\Gamma_{jki} = G^{jk}g_{ir}\Gamma^r_{jk} = \tfrac{1}{2}\partial_i \ln\det|G| - g_{ik}\partial_j G^{jk}.} \tag{8.2.5}$$

As the classical separation coordinates λ are orthogonal coordinates, a lot of formulas simplify. For example, nonzero Christoffel symbols in such coordinates are

$$\Gamma^j_{ij} = \Gamma^j_{ji} = -\tfrac{1}{2}\partial_i \ln G^{jj}, \quad \text{no summation over } j,$$

$$\Gamma^i_{jj} = -\tfrac{1}{2}G^{ii}\partial_i g_{jj}, \quad i \neq j,$$

hence

$$\Gamma_i = \tfrac{1}{2}\partial_i \sum_{k=1}^{n} \ln G^{kk} - \partial_i \ln G^{ii} = \tfrac{1}{2}\partial_i \ln \frac{\prod_{k\neq i} G^{kk}}{G^{ii}} \tag{8.2.6}$$

and

$$\sum_{k=1}^{n} \Gamma^k_{ki} = -\tfrac{1}{2}\partial_i \sum_{k=1}^{n} \ln G^{kk} = -\Gamma_i - \partial_i \ln G^{ii}. \tag{8.2.7}$$

Denoting $(K_r)^i_i \equiv v^i_r$ we get $A^{ii}_r = v^i_r G^{ii}$ and then

$$\begin{aligned}
\nabla_i A^{ii}_r \nabla_i \Psi &= \partial_i (A^{ii}_r \partial_i \Psi) + \Gamma^i_{ij} A^{jj}_r \partial_j \Psi \\
&= A^{ii}_r \partial_i^2 \Psi + (\partial_j A^{jj}_r + \Gamma^i_{ij} A^{jj}_r)\partial_j \Psi \\
&= v^i_r G^{ii} \partial_i^2 \Psi + \left(\partial_j (v^j_r G^{jj}) - v^j_r G^{jj}(\Gamma_j + \partial_j \ln G^{jj})\right)\partial_j \Psi \\
&= v^i_r G^{ii} \partial_i^2 \Psi + G^{jj}(\partial_j v^j_r - v^j_r \Gamma_j)\partial_j \Psi \\
&= v^i_r G^{ii} \left(\partial_i^2 - \Gamma_i \partial_i\right) \Psi \\
&= A^{ii}_r \left(\partial_i^2 - \Gamma_i \partial_i\right) \Psi,
\end{aligned} \tag{8.2.8}$$

where relations (4.3.9) and (8.2.7) were used. Thus, in classical orthogonal separation coordinates λ

$$\boxed{\hat{H}_r\Psi = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} A_r^{ii}\left(\partial_i^2 - \Gamma_i\partial_i\right)\Psi + V_r\Psi, \quad r = 1,\ldots,n.} \tag{8.2.9}$$

Now we demonstrate that the system of n eigenvalue problems (8.2.2) is separable if and only if

$$\boxed{\partial_j\Gamma_i = 0, \;\; i \neq j \Longleftrightarrow \Gamma_i = \Gamma_i(\lambda^i).} \tag{8.2.10}$$

Condition (8.2.10) is known as a *Robertson condition* [228]. What is interesting, it is possible to demonstrate that in orthogonal coordinates, the non-diagonal components R_{ij} of Ricci tensor are expressed by metrically contracted Christoffel symbols Γ_i [17]

$$R_{ij} = \frac{3}{2}\partial_i\Gamma_j, \;\; i \neq j, \tag{8.2.11}$$

so the condition of stationary quantum separability can be reduced to the diagonality condition of the Ricci tensor in classical separation coordinates [104].

Applying the Stäckel matrix (4.3.6) to the system (8.2.2) and using relations (8.2.9) we get

$$S\begin{pmatrix}\hat{H}_1\Psi\\ \vdots\\ \hat{H}_n\Psi\end{pmatrix} = S\begin{pmatrix}E_1\Psi\\ \vdots\\ E_n\Psi\end{pmatrix},$$

and then

$$[E_1S_{1i}(\lambda^i) + E_2S_{2i}(\lambda^i) + \cdots + E_nS_{ni}(\lambda^i)]\Psi = -\tfrac{1}{2}\hbar^2 f_i(\lambda^i)\left[\partial_i^2 - \Gamma_i\partial_i\right]\Psi + \sigma_i(\lambda^i)\Psi,$$

$i = 1,\ldots,n$. From equations (8.2.1) follows that eigenfunction Ψ separates multiplicatively

$$\Psi(\lambda^1,\ldots,\lambda^n,\alpha,\beta,E) = \prod_{i=1}^{n}\psi_i(\lambda^i,\alpha_i,\beta_i,E)$$

if and only if the condition (8.2.10) is fulfilled and system (8.2.1) separates onto n one-dimensional eigenvalue problems (*quantum separation equations*)

$$\begin{aligned}&[E_1 S_{1i}(\lambda^i) + E_2 S_{2i}(\lambda^i) + \cdots + E_n S_{ni}(\lambda^i)]\psi_i(\lambda^i)\\ &= -\frac{1}{2}\hbar^2 f_i(\lambda^i)\left[\frac{d^2}{(d\lambda^i)^2} - \Gamma_i(\lambda^i)\frac{d}{d\lambda^i}\right]\psi_i(\lambda^i) + \sigma_i(\lambda^i)\psi_i(\lambda^i), \quad i = 1, \ldots, n,\end{aligned} \tag{8.2.12}$$

where α_i, β_i are integration constants originating during the process of solving equation i-th in (8.2.12). As a consequence, differential operators $\hat{H}_i$ fulfill *quantum separation relations*

$$\begin{aligned}&S_{1i}(\lambda^i)\hat{H}_1 + S_{2i}(\lambda^i)\hat{H}_2 + \cdots + S_{ni}(\lambda^i)\hat{H}_n\\ &= -\frac{1}{2}\hbar^2 f_i(\lambda^i)\left[\frac{d^2}{(d\lambda^i)^2} - \Gamma_i(\lambda^i)\frac{d}{d\lambda^i}\right] + \sigma_i(\lambda^i), \quad i = 1, \ldots, n.\end{aligned} \tag{8.2.13}$$

Thus, summarizing our considerations, we conclude that n eigenvalue problems (8.2.2) separate in classical orthogonal separation coordinates under an extra condition (8.2.11).

Let us analyze in details the Stäckel systems considered in Chap. 4. We begin with separation relations (4.2.13)

$$H_i - \zeta_i(\lambda^i)H_{i+1} = \tfrac{1}{2}\mu_i^2 + V_i(\lambda^i), \quad i = 1, \ldots, n. \tag{8.2.14}$$

As diagonal elements of the metric tensor G are products of functions of one variable (4.3.1), so from relation (8.2.6) we get

$$\Gamma_i = \tfrac{1}{2}(n-i)\frac{d}{d\lambda^i}\ln\zeta_i(\lambda^i)$$

and Robertson condition is fulfilled.

Example 8.3 For $n = 3$ and spherical coordinates we have the quantum version of systems from Example 4.10. In that case

$$\hat{H}_1 = -\frac{\hbar^2}{2m}\left[\partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left(\partial_\theta^2 + \frac{1}{\tan\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right)\right] + \sigma_r(r) + \frac{\sigma_\theta(\theta)}{r^2} + \frac{\sigma_\phi(\phi)}{r^2\sin^2\theta},$$

$$\hat{H}_2 = -\frac{\hbar^2}{2m}\left(\partial_\theta^2 + \frac{1}{\tan\theta}\partial_\theta + \frac{1}{\sin^2\theta}\partial_\phi^2\right) + \sigma_\theta(\theta) + \frac{\sigma_\phi(\phi)}{\sin^2\theta},$$

$$\hat{H}_3 = -\frac{\hbar^2}{2m}\partial_\phi^2 + \sigma_\phi(\phi)$$

and as

$$\Gamma_r = -\frac{2}{r}, \quad \Gamma_\theta = -\cot\theta, \quad \Gamma_\phi = 0,$$

so, quantum separation relations (8.2.13) are of the form

$$\hat{H}_1 - \frac{1}{r^2}\hat{H}_2 = -\frac{1}{2m}\hbar^2\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right] + \sigma_r(r),$$

$$\hat{H}_2 - \frac{1}{\sin^2\theta}\hat{H}_3 = -\frac{1}{2m}\hbar^2\left[\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta}\right] + \sigma_\theta(\theta),$$

$$\hat{H}_3 = -\frac{1}{2m}\hbar^2\frac{d^2}{d\theta^2} + \sigma_\theta(\theta),$$

while one-dimensional quantum separation equations (8.2.12) take the form

$$(E_1 - \frac{1}{r^2}E_2)\psi(r) = -\frac{1}{2m}\hbar^2\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right]\psi(r) + \sigma_r(r)\psi(r),$$

$$(E_2 - \frac{1}{\sin^2\theta}E_3)\eta(\theta) = -\frac{1}{2m}\hbar^2\left[\frac{d^2}{d\theta^2} + \cot\theta\frac{d}{d\theta}\right]\eta(\theta) + \sigma_\theta(\theta)\eta(\theta),$$

$$E_3\chi(\phi) = -\frac{1}{2m}\hbar^2\frac{d^2}{d\theta^2}\chi(\phi) + \sigma_\theta(\theta)\chi(\phi),$$

and $\Psi = \psi(r)\eta(\theta)\chi(\phi)$.

As another class of the Stäckel systems consider these systems, generated by irreducible separation relations (4.2.16)

$$\sum_{k=1}^{n} H_k(\lambda^i)^{\gamma_k} = \tfrac{1}{2}f_i(\lambda^i)\mu_i^2 + \sigma_i(\lambda^i), \quad i = 1, \ldots, n, \tag{8.2.15}$$

with the Stäckel matrix S_γ given by (4.2.2). From relations (8.2.6) follows that in separation coordinates metrically contracted the Christoffel symbols can be written in the form

$$\Gamma_i = \tfrac{1}{2}\partial_i \ln F_i, \tag{8.2.16}$$

where

$$F_i = \frac{\prod_{k\neq i} G^{kk}}{G^{ii}} = \frac{\prod_{k\neq i}(S_\gamma^{-1})_{1k}}{(S_\gamma^{-1})_{1i}}\frac{\prod_{k\neq i} f_k(\lambda^k)}{f_i(\lambda^i)} = \frac{\delta_i(\lambda)}{D_{i1}(\lambda)}\frac{\prod_{k\neq i} f_k(\lambda^k)}{f_i(\lambda^i)}. \tag{8.2.17}$$

The symbol $D_{i1}(\lambda)$ stands for the signed $(i,1)$-minor of S_γ, which is λ_i-independent, and

$$\delta_i(\lambda) = \frac{\prod_{k\neq i} D_{k1}}{(\det S)^{n-2}}$$

is a rational function of variable λ^i with coefficients being polynomials with respect to the remaining variables λ^k, $k \neq i$.

It can be proved [47] that for general $\gamma = (\gamma_1, \ldots, \gamma_n)$ metrically contracted Christoffel symbols (8.2.16), (8.2.17) do not satisfy the Robertson condition (8.2.10) except the distinguished case of the Benenti class defined by $\gamma = (n-1, \ldots, 1, 0)$

$$\sum_{k=1}^{n} H_k(\lambda^i)^{n-k} = \tfrac{1}{2} f_i(\lambda^i)\mu_i^2 + \sigma_i(\lambda^i), \quad i = 1, \ldots, n, \tag{8.2.18}$$

where the Stäckel matrix is a Vandermonde one (4.3.2) with the metric tensor in the form (4.3.16). Then

$$F_i = \frac{\prod_{k\neq i} G^{kk}}{G^{ii}} = \frac{(-1)^n B_i}{\prod_{k\neq i} B_k} \frac{\prod_{k\neq i} f_k(\lambda^k)}{f_i(\lambda^i)}, \quad B_i = \prod_{k\neq i}(\lambda^i - \lambda^k)$$

and $(-1)^n B_i \prod_{k\neq i} \frac{f_k(\lambda^k)}{B_k}$ term does not depend on λ^i, hence

$$\Gamma_i = \tfrac{1}{2}\partial_i \ln f_i^{-1} = -\frac{1}{2}\frac{f_i'(\lambda^i)}{f_i(\lambda^i)}, \quad f_i'(\lambda^i) = \frac{df_i(\lambda^i)}{d\lambda^i}.$$

Thus, eigenvalue problems for quantum Hamiltonians (8.2.4) from the Benenti class, in a minimal quantization with natural metric $G = A_1$, separate onto n one-dimensional eigenvalue problems

$$\boxed{\begin{aligned}&[E_1(\lambda^i)^{n-1} + E_2(\lambda^i)^{n-2} + \cdots + E_n]\psi_i(\lambda^i)\\ &= -\frac{1}{2}\hbar^2\left[f_i(\lambda^i)\frac{d^2}{(d\lambda^i)^2} + \frac{1}{2}\frac{df_i(\lambda^i)}{d\lambda^i}\frac{d}{d\lambda^i}\right]\psi_i(\lambda^i) + \sigma_i(\lambda^i)\psi_i(\lambda^i).\end{aligned}} \tag{8.2.19}$$

Besides, Hamiltonian operators in separation coordinates fulfill the respective quantum separation relations

$$\boxed{\begin{aligned}&(\lambda^i)^{n-1}\hat{H}_1+(\lambda^i)^{n-2}\hat{H}_2+\cdots+\hat{H}_n\\&=-\frac{1}{2}\hbar^2\left[f_i(\lambda^i)\frac{d^2}{\left(d\lambda^i\right)^2}+\frac{1}{2}\frac{df_i(\lambda^i)}{d\lambda^i}\frac{d}{d\lambda^i}\right]+\sigma_i(\lambda^i),\quad i=1,\ldots,n.\end{aligned}}\tag{8.2.20}$$

Example 8.4 Quantum Hénon-Heiles system. Two constants of motion in the Cartesian and separation coordinates respectively are of the form (see Example 4.11)

$$H_1=T_1+V_1=\tfrac{1}{2}p_x^2+\tfrac{1}{2}p_y^2+x^3+\tfrac{1}{2}xy^2,$$
$$H_2=T_2+V_2=\tfrac{1}{2}yp_xp_y-\tfrac{1}{2}xp_y^2+\tfrac{1}{4}x^2y^2+\tfrac{1}{16}x^4,$$

$$H_1=\frac{1}{2}\frac{\lambda^1}{\lambda^1-\lambda^2}\mu_1^2+\frac{1}{2}\frac{\lambda^2}{\lambda^2-\lambda^1}\mu_2^2+(\lambda^1)^3+(\lambda^1)^2\lambda^2+\lambda^1(\lambda^2)^2+(\lambda^2)^3,$$
$$H_2=\frac{1}{2}\frac{\lambda^1\lambda^2}{\lambda^1-\lambda^2}\mu_1^2+\frac{1}{2}\frac{\lambda^1\lambda^2}{\lambda^2-\lambda^1}\mu_2^2-\lambda^1\lambda^2[(\lambda^1)^2+\lambda^1\lambda^2+(\lambda^2)^2].$$

The respective operators in both maps are

$$\hat{H}_1=-\frac{1}{2}\hbar^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+V_1(x,y),$$
$$\hat{H}_2=-\frac{1}{2}\hbar^2\left(x^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}-x\frac{\partial^2}{\partial y^2}+\frac{1}{2}\frac{\partial}{\partial x}\right)+V_2(x,y),$$

$$\hat{H}_1=-\frac{1}{2}\hbar^2\left[\frac{\lambda^1}{\lambda^1-\lambda^2}\left(\frac{\partial^2}{\left(\partial\lambda^1\right)^2}+\frac{1}{2}\frac{1}{\lambda^1}\frac{\partial}{\partial\lambda^1}\right)+\frac{\lambda^2}{\lambda^2-\lambda^1}\left(\frac{\partial^2}{\left(\partial\lambda^2\right)^2}+\frac{1}{2}\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda^2}\right)\right]$$
$$+V_1(\lambda^1,\lambda^2),$$
$$\hat{H}_2=-\frac{1}{2}\hbar^2\left[\frac{\lambda^1\lambda^2}{\lambda^1-\lambda^2}\left(\frac{\partial^2}{\left(\partial\lambda^1\right)^2}+\frac{1}{2}\frac{1}{\lambda^1}\frac{\partial}{\partial\lambda^1}\right)+\frac{\lambda^1\lambda^2}{\lambda^2-\lambda^1}\left(\frac{\partial^2}{\left(\partial\lambda^2\right)^2}+\frac{1}{2}\frac{1}{\lambda^2}\frac{\partial}{\partial\lambda^2}\right)\right]$$
$$+V_2(\lambda^1,\lambda^2).$$

As for the Hénon-Heiles system separation relations come from the separation curve, so quantum stationary separability reduces to a one-dimensional eigenvalue problem

$$(E_1\lambda+E_2)\psi(\lambda)=-\frac{1}{2}\hbar^2\left[\lambda\frac{d^2}{d\lambda^2}+\frac{1}{2}\frac{d}{d\lambda}\right]\psi(\lambda)+\lambda^4\psi(\lambda).$$

Taking two copies of the solution $\psi(\lambda) : \lambda = \lambda^1, \lambda^2$, $\Psi(\lambda) = \psi(\lambda^1)\psi(\lambda^2)$ is a common eigenfunction of $\hat{H}_1$ and $\hat{H}_2$

$$\hat{H}_1\Psi(\lambda) = E_1\Psi(\lambda), \qquad \hat{H}_2\Psi(\lambda) = E_2\Psi(\lambda).$$

Besides, one can verify that

$$[\hat{H}_1, \hat{H}_2] = 0$$

and in separation coordinates

$$\lambda^1\hat{H}_1 + \hat{H}_2 = -\frac{1}{2}\hbar^2\left[\lambda^1\frac{d^2}{(d\lambda^1)^2} + \frac{1}{2}\frac{d}{d\lambda^1}\right] + (\lambda^1)^4,$$
$$\lambda^2\hat{H}_1 + \hat{H}_2 = -\frac{1}{2}\hbar^2\left[\lambda^2\frac{d^2}{(d\lambda^2)^2} + \frac{1}{2}\frac{d}{d\lambda^2}\right] + (\lambda^2)^4.$$

For the considered quantization the Benenti class is exceptional for the family (8.2.15). One can show [47] that for the remaining Stäckel matrices S_γ (4.2.2), the Robertson condition is not fulfilled as

$$\frac{\partial}{\partial\lambda^i}\delta_i(\lambda) \neq 0.$$

It means that systems from these classes after minimal quantization with respect to natural metric $G = A_1$, are not stationary separable.

Now, let us verify why minimal quantization is optimal for quantum separability. In Sect. 8.1.6 was considered the two-parameter family of quantizations of classical Hamiltonian systems quadratic in momenta

$$H(x, p) = \tfrac{1}{2}K^{ij}p_ip_j + V$$

$$\Downarrow$$

$$\hat{H}(\hat{x}, \hat{p}) = -\tfrac{1}{2}\hbar^2\left(\nabla_iK^{ij}\nabla_j + \tfrac{1}{4}(1-b)K^{ij}{}_{;ij} - \tfrac{1}{4}(1-a)K^{ij}R_{ij}\right) + V.$$

It means that in the process of quantization the quantum correction to classical potential is generated in the form

$$V_{\text{quant}}(x, \hbar) = -\tfrac{1}{2}\hbar^2\left(\tfrac{1}{4}(1-b)K^{ij}{}_{;ij} - \tfrac{1}{4}(1-a)K^{ij}R_{ij}\right). \tag{8.2.21}$$

What is interesting, for systems from the Beneti class with flat metrics $V_{\text{quant}}(x, \hbar) = const$. In consequence, all quantizations (8.2.21) for these systems are equivalent. Differences in quantizations appear in the case of non flat metrics

and hence nontrivial quantum potentials are generated. The problem is that these potentials are not separable. It means that using alternative admissible quantizations instead of the minimal one, we restrict the class of quantum separable Hamiltonians to these with flat metrics.

Let us consider dynamical systems from the Benenti class with metrics (4.3.16) generated by $f_i(\lambda^i) = (\lambda^i)^k$, $k \in \mathbb{Z}$. Then, one can show that

$$K_r^{ij}{}_{;ij} = \tfrac{1}{4}(n+1-r)V_{r-1}^{(k-1)} \tag{8.2.22}$$

where $V_r^{(k)}$ are potentials (4.3.22). In the flat cases, when $0 \leq k \leq n$, $V_{r-1}^{(k-1)} = \delta_{r,n-k}$, $R_{ij} = 0$ and Hamiltonian operators (8.2.21) coincide with these for minimal quantization up to a constant. For non flat cases extra terms R and $K^{ij}R_{ij}$ are complicated functions of position, non expressible by separable potentials and we lose quantum separability. The exceptional non flat case is the one with constant curvature. Then, when $k = n+1$, one can show that

$$K_r^{ij}R_{ij} = -\tfrac{1}{4}(n+1-r)(n-1)V_{r-1}^{(n)}, \quad V_0 \equiv 1.$$

From (8.2.22) we have

$$K_r^{ij}{}_{;ij} = \tfrac{1}{4}(n+1-r)V_{r-1}^{(n)}$$

and the choice

$$\frac{1-b}{1-a} = n-1$$

cancels quantum potential (8.2.21).

Similarly to the classical case, also on the quantum level the notion of quantum integrability is wider than the notion of quantum separability. So, the question arises whether there are quantum Stäckel systems which although are not separable but are integrable, i.e. when

$$[\hat{H}_i, \hat{H}_j] = 0, \quad i,j = 1,\ldots,n. \tag{8.2.23}$$

The problem of commutativity (8.2.23) for Hamiltonians (8.2.4) was considered in detail in [18]. Here we skip the involved calculations, presenting only the final result formulated in separation coordinates. In [18] was proved that Hamiltonians (8.2.4) commute if and only if in classical separation coordinates the so-called *pre-Robertson condition*

$$\boxed{\partial_i R_{ij} - \Gamma_i R_{ij} = 0 \Longleftrightarrow \partial_j\left(\partial_i\Gamma_i - \tfrac{1}{2}\Gamma_i^2\right) = 0, \quad i \neq j} \tag{8.2.24}$$

is fulfilled.

From (8.2.24) follows immediately that the Robertson condition (8.2.10), (8.2.11) implies the pre-Robertson condition (8.2.21), so quantum separability implies quantum integrability, in full analogy to the classical case. But less obvious is the observation that for the class of Stäckel systems related to separation relations (4.2.16), the pre-Robertson condition is satisfied if and only if the Robertson condition holds [47].

8.2.2 *Separability for Minimal Quantization with Adopted Metric*

In the previous subsection, using the natural minimal quantization, we proved the quantum integrability and quantum stationary separability for systems from class (8.2.14) and from the Benenti class (8.2.18). Can we find some quantizations for other classes of systems (8.2.15), preserving quantum integrability and quantum separability? We give the positive answer to that question in the following subsection. To do it, let us first adopt more general Riemannian geometry than the natural one. The specifications below will be motivated by the fact that our quantization procedure will be performed in appropriate pseudo-Riemannian spaces. Thus, from now on we will suppose that the manifold M is a cotangent bundle to a pseudo-Riemannian manifold i.e. $M = T^*Q$ with Q equipped with a metric tensor g. We will also make three additional assumptions:

1. The manifold (Q, g) and the Poisson structure are adapted to each other in the sense that the first n Darboux coordinates λ^i are coordinates on Q while the remaining Darboux coordinates μ_i are fiber coordinates.
2. Coordinates λ^i are orthogonal coordinates for the metric g i.e. g and $G = g^{-1}$ are diagonal (but not necessarily flat) in λ^i.
3. The base manifold Q is almost covered by a single, open and dense in M, chart with coordinates $(\lambda^1, \ldots, \lambda^n)$.

The matrices A_r in (8.2.3) can be interpreted as $(2, 0)$-tensors on Q that can be written as

$$A_r = B_r G, \quad r = 1, \ldots, n$$

where B_r are some $(1, 1)$-tensors on Q. Further, in a very special case, considered in the previous subsection, when $G = A_1$ the tensors B_r are Killing tensors K_r for the metric G. In what follows, we assume that in general $G \neq A_1$.

Suppose we have a Stäckel system written in arbitrary Darboux coordinates (x, p):

$$H_r = \tfrac{1}{2} p^T A_r p + V_r, \quad r = 1, \ldots, n. \tag{8.2.25}$$

Given a metric g we can now perform the minimal quantization of our Stäckel system (8.2.25) as described in the previous subsection. As a result we obtain n quantum Hamiltonians

$$\hat{H}_r = -\tfrac{1}{2}\hbar^2 \nabla_i \, (B_r G)^{ij} \, \nabla_j + V_r, \quad r = 1, \ldots, n \tag{8.2.26}$$

acting in the Hilbert space $L^2(Q, \omega_g)$, $\omega_g = |\det g|^{1/2} \, dx$, where $A_r = B_r G$. Let us rewrite the operators (8.2.26) in separation coordinates (λ, μ) for the classical Stäckel system (8.2.25). We will always assume the conditions **1–3** from the beginning of the subsection. This also means that g and thus G should be diagonal in separation coordinates. Thus, since A_r are diagonal in separation coordinates, so are B_r. Calculating covariant derivatives like in (8.2.8) we obtain

$$\boxed{\begin{aligned} \hat{H}_r &= -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} G^{ii} \left(B_r^{(i)} \partial_i^2 + (\partial_i B_r^{(i)}) \partial_i - B_r^{(i)} \Gamma_i \partial_i \right) + V_r \\ &= -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} A_r^{ii} \left[\partial_i^2 + \left(\tfrac{\partial_i B_r^{(i)}}{B_r^{(i)}} - \Gamma_i \right) \partial_i \right] + V_r \end{aligned}} \tag{8.2.27}$$

where $B_r^{(i)} \equiv (B_r)_i^i$ (no summation). Notice that $\partial_i B_r^{(i)} \neq 0$ for $B_r \neq K_r$.

From (8.2.27) it follows that the necessary and sufficient condition for quantum separability of operators $\hat{H}_r$ takes the form

$$\boxed{\Xi_i = \Xi_i(\lambda^i) \text{ or } \partial_j \Xi_i = 0, \quad j \neq i} \tag{8.2.28}$$

where

$$\Xi_i = \frac{\partial_i B_r^{(i)}}{B_r^{(i)}} - \Gamma_i.$$

The condition (8.2.28) is called the *generalized Robertson condition* [48]. Indeed, due to (4.3.7), the operators (8.2.27) can be written as

$$\hat{H}_r = -\tfrac{1}{2}\hbar^2 \left(S_\gamma^{-1} \right)_r^i f_i(\lambda^i) \left(\partial_i^2 + \Xi_i(\lambda^i) \partial_i \right) + \left(S_\gamma^{-1} \right)_r^i \sigma_i(\lambda^i), \quad r = 1, \ldots, n$$

and then application of the Stäckel matrix S_γ to the system of eigenvalue problems

$$S_\gamma \begin{pmatrix} \hat{H}_1 \Psi \\ \vdots \\ \hat{H}_n \Psi \end{pmatrix} = S_\gamma \begin{pmatrix} E_1 \Psi \\ \vdots \\ E_n \Psi \end{pmatrix}$$

separates it to n one-dimensional quantum separation equations

$$\boxed{\begin{aligned}&\left[E_1(\lambda^i)^{\gamma_1}+E_2(\lambda^i)^{\gamma_2}+\cdots+E_n\right]\psi_i(\lambda^i)\\&=-\frac{1}{2}\hbar^2 f_i(\lambda^i)\left[\frac{d^2\psi_i(\lambda^i)}{(d\lambda^i)^2}+\Xi_i(\lambda^i)\frac{d\psi_i(\lambda^i)}{d\lambda^i}\right]+\sigma_i(\lambda^i)\psi_i(\lambda^i),\end{aligned}} \tag{8.2.29}$$

so that again like in (8.2.1) $\Psi(\lambda^1,\ldots,\lambda^n,E)=\prod_{i=1}^n\psi_i(\lambda^i,E)$ is a common, multiplicatively separable solution of the stationary Schrödinger equations for all $\hat{H}_r$. The related quantum separation relations are of the form

$$\boxed{(\lambda^i)^{\gamma_1}\hat{H}_1+(\lambda^i)^{\gamma_2}\hat{H}_2+\cdots+\hat{H}_n=-\frac{1}{2}\hbar^2 f_i(\lambda^i)\left[\frac{d^2}{(d\lambda^i)^2}+\Xi_i(\lambda^i)\frac{d}{d\lambda^i}\right]+\sigma_i(\lambda^i).}$$

Notice that in both above formulas $\gamma_v=0$ according to our normalization (4.2.16).

In the special case $G=A_1$ (or, in general, G equal to any A_s) $B_r=K_r$ are Killing tensors of g so in λ-coordinates $\partial_i K_r^{(i)}=0$. In consequence the condition (8.2.28) reduces to the Robertson condition for quantum separability (8.2.10).

In the previous subsection we proved that for the case $G=A_1$ the only class of Stäckel systems (8.2.15) for which the Robertson condition (8.2.10) is satisfied is the Benenti class. For all other choices of γ_i in (8.2.15) this condition fails. It means that in order to achieve quantum separability of an arbitrary Stäckel system of the type (8.2.15) we have to consider a broader class of admissible metric tensors g used in the quantization procedure.

Let us now go back to an arbitrary Stäckel system of the form (8.2.15) defined by the choice of the constants $\gamma_1>\gamma_2>\cdots>\gamma_n=0$ and the choice of functions f_i,σ_i. Then, according to results from Sect. 4.4.3, the tensors A_r for this system can be written as (4.4.32)

$$A_r=\frac{1}{\varphi}M_r G_{B,f},\quad r=1,\ldots,n \tag{8.2.30}$$

where $G_{B,f}$ is the corresponding Benenti metric

$$G_{B,f}=diag\left(\frac{f_1(\lambda^1)}{\Delta_1},\ldots,\frac{f_n(\lambda^n)}{\Delta_n}\right), \tag{8.2.31}$$

M_r are some polynomial functions of the Killing tensors K_r and

$$\varphi=\det\begin{pmatrix}\rho_{m_1-1}&\cdots&\rho_{m_1-k}\\ \vdots&\ddots&\vdots\\ \rho_{m_k-1}&\cdots&\rho_{m_k-k}\end{pmatrix}, \tag{8.2.32}$$

where we adapt the notation $\rho_0 = 1$ and $\rho_i = 0$ for $i < 0$ or $i > n$. Let me remind that constants n_i are those for which the corresponding monomials λ^{n+k-n_i} are missing in the left hand side of (8.2.15), i.e. they are "holes" in the sequence $\{\gamma_1 = n+k-1, \gamma_2, \cdots, \gamma_n = 0\}$ numbered from the left. Besides, k is determined from the equation $\gamma_1 = n + k - 1$. Note that if such "holes" are absent (as in the Benenti case) then $\varphi = 1$. For example, if the left hand side of the Stäckel system is $H_1\lambda^4 + H_2\lambda + H_3$, then $n = 3, k = 2, m_1 = 2, m_2 = 3$ and the function φ becomes:

$$\varphi = \det \begin{pmatrix} \rho_1 & \rho_0 \\ \rho_2 & \rho_1 \end{pmatrix} = (\lambda^1)^2 + \left(\lambda^2\right)^2 + (\lambda^3)^2 + \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3.$$

Consider thus a Stäckel system (8.2.15) defined by a fixed choice of $\gamma_1 > \gamma_2 > \cdots > \gamma_n = 0$ and the choice of f_i, σ_i. We will now search for the metric G that satisfies the generalized Robertson condition (8.2.28) for this system. Due to the structure (8.2.30) of A_r we look for G in the form

$$G = u^{-1}(\lambda) G_{B,\theta} \tag{8.2.33}$$

where $G_{B,\theta}$ is the Benenti metric given by (8.2.31) with n arbitrary functions $\theta_i(\lambda^i)$:

$$G_{B,\theta} = diag\left(\frac{\theta_1(\lambda^i)}{\Delta_1}, \ldots, \frac{\theta_n(\lambda^n)}{\Delta_n}\right) \tag{8.2.34}$$

and where u is a function on Q. Albeit this choice is by no means the most general one it will prove to be sufficiently general. The tensors B_r become in this case

$$B_r = \frac{u}{\varphi} M_r G_{B,f}\, g_{B,\theta}$$

where φ is again given by (8.2.32) and where as usual $g_{B,\theta} = G^{-1}_{B,\theta}$. Plugging this into (8.2.28) we get

$$\frac{\partial_i B_r^{(i)}}{B_r^{(i)}} - \Gamma_i = \frac{\kappa_i'(\lambda^i)}{\kappa_i(\lambda^i)}, \quad = 1, \ldots, n \tag{8.2.35}$$

where κ_i are arbitrary functions of one variable (the right hand side is just a convenient for us way of writing an arbitrary function of λ_i). Since for (8.2.33)

$$\Gamma_i = \left(\Gamma_{B,\theta}\right)_i + \left(1 - \tfrac{1}{2}n\right)\frac{\partial_i u}{u}$$

with $\left(\Gamma_{B,\theta}\right)_i$ being the metrically contracted Christoffel symbols for the metric $G_{B,\theta}$, the formula (8.2.35) takes the form

$$\frac{n}{2}\frac{\partial_i u}{u} - \frac{\partial_i \varphi}{\varphi} = \frac{\kappa_i'(\lambda^i)}{\kappa_i(\lambda^i)} + \frac{1}{2}\frac{\theta_i'(\lambda^i)}{\theta_i(\lambda^i)} - \frac{f_i'(\lambda^i)}{f_i(\lambda^i)}, \quad = 1, \ldots, n$$

which has a solution

$$u = \varphi^{\frac{2}{n}} \prod_{i=1}^{n} \left(\frac{|\theta_i|\,\kappa_i^2}{f_i^2} \right)^{\frac{1}{n}} \tag{8.2.36}$$

In order to receive a solution as simple as possible we choose κ_i so that

$$\frac{|\theta_i|\,\kappa_i^2}{f_i^2} = 1$$

(notice that θ_i are still arbitrary) yielding (8.2.36) in the form $u = \varphi^{\frac{2}{n}}$. Thus, applying the procedure of minimal quantization, with

$$\boxed{g = \varphi^{\frac{2}{n}} g_{B,\theta}} \tag{8.2.37}$$

to the Stäckel system (8.2.15) we obtain a quantum system (8.2.26) in separation coordinates

$$\boxed{\hat{H}_r = -\frac{1}{2}\hbar^2 \sum_{i=1}^{n} (S^{-1})_r^i \left[f(\lambda^i)\partial_i^2 + \left(\frac{df(\lambda^i)}{d\lambda^i} - \frac{1}{2}\frac{f(\lambda^i)}{\theta(\lambda^i)}\frac{d\theta(\lambda^i)}{d\lambda^i} \right)\partial_i \right] + \sum_{i=1}^{n} (S^{-1})_r^i \sigma(\lambda^i),}$$

with separation equations of the form

$$\boxed{\begin{aligned} &\left[E_1(\lambda^i)^{\gamma_1} + E_2(\lambda^i)^{\gamma_2} + \cdots + E_n \right] \psi_i(\lambda^i) \\ &= -\frac{1}{2}\hbar^2 f_i(\lambda^i) \left[\frac{d^2\psi_i(\lambda^i)}{(d\lambda^i)^2} + \left(\frac{f_i'(\lambda^i)}{f_i(\lambda^i)} - \frac{1}{2}\frac{\theta_i'(\lambda^i)}{\theta_i(\lambda^i)} \right) \frac{d\psi_i(\lambda^i)}{d\lambda^i} \right] + \sigma_i(\lambda^i)\psi_i(\lambda^i), \end{aligned}}$$

where $i = 1, \ldots, n$. Notice that in particular, we also have an infinite family of separable minimal quantizations of systems from the Beneti class in which case $\varphi = 1$. The most natural choice of the metric (8.2.37) is the one when $\theta_i = f_i$, then

Hamiltonians (8.2.2) take the form

$$\boxed{\hat{H}_r = -\frac{1}{2}\hbar^2 \sum_{i=1}^{n} (S^{-1})^i_r \left[f(\lambda^i)\partial_i^2 + \frac{1}{2}\frac{df(\lambda^i)}{d\lambda^i}\partial_i \right] + \sum_{i=1}^{n} (S^{-1})^i_r \sigma(\lambda^i),} \tag{8.2.38}$$

and separation equations (8.2.2) are

$$\boxed{\begin{aligned} &\left[E_1(\lambda^i)^{\gamma_1} + E_2(\lambda^i)^{\gamma_2} + \cdots + E_n \right] \psi_i(\lambda^i) \\ &= -\frac{1}{2}\hbar^2 f_i(\lambda^i) \left[\frac{d^2\psi_i(\lambda^i)}{(d\lambda^i)^2} + \frac{1}{2}\frac{f_i'(\lambda^i)}{f_i(\lambda^i)}\frac{d\psi_i(\lambda^i)}{d\lambda^i} \right] + \sigma_i(\lambda^i)\psi_i(\lambda^i). \end{aligned}} \tag{8.2.39}$$

Observation 21 *There exists an infinite family of separable quantizations of a Stäckel system (8.2.15) parametrized by n arbitrary functions θ_i of one variable: any Stäckel system (8.2.15) can be separably quantized in the metric (8.2.37) (note that this metric is conformally flat in the case when $g_{B,\theta}$ is flat). Moreover, since for the Benenti class $\varphi = 1$, any Stäckel system from the Benenti class (8.2.18) can be separably quantized in any other metric of the Benenti class (8.2.34), including in particular the subclass of flat metrics.*

Now, the question arises: whether all these separable quantizations are independent from each other? We answer this question in the next subsection.

8.2.3 *Minimal Quantization in Different Hilbert Spaces*

Our goal now is to relate two minimal quantizations induced by different metric tensors. We will need this in order to be able to write systems of commuting operators in various Hilbert spaces with measures induced by different metrics and in particular to relate different separable minimal quantizations.

Consider thus two different metric tensors g and $\bar{g}$. As usual, we will denote their contravariant forms by G and $\bar{G}$, respectively. Each of these metrics induces a minimal quantization (described in Sect. 8.1.6) by morphisms S and $\bar{S}$, respectively, where (cf. (6.1.104) with $a = b = 1$)

$$\begin{aligned} S = 1 + \frac{\hbar^2}{4!}\Big[&3(\Gamma^i_{lj}\Gamma^l_{ik} + R_{jk})\partial_{p_j}\partial_{p_k} + 3\Gamma^i_{jk}\partial_{x_i}\partial_{p_j}\partial_{p_k} + (2\Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{jk,l})p_i\partial_{p_j}\partial_{p_k}\partial_{p_l} \\ &-3\partial_{p_j}(\partial_{x_j} + \Gamma^i_{jl}p_i\partial_{p_l})\partial_{p_k}(\partial_{x_k} + \Gamma^r_{kn}p_r\partial_{p_n}) + O(\hbar^4)\Big], \end{aligned} \tag{8.2.40}$$

and where $\bar{S}$ is given by an analogous expression with Γ^i_{jk} replaced by the Christoffel symbols $\bar{\Gamma}^i_{jk}$ of the Levi-Civita connection induced by $\bar{g}$. For a (classical) observable of the form

$$H(x, p) = \tfrac{1}{2} A^{ij} p_i p_j + V$$

its minimal quantization with respect to g is given by

$$\hat{H} = (S^{-1} H)_W(\hat{x}, \hat{p}) = -\tfrac{1}{2}\hbar^2 \nabla_i A^{ij} \nabla_j + V \tag{8.2.41}$$

and acts in $L^2(Q, \omega_g)$, while its quantization with respect to $\bar{g}$ is given by a similar expression involving $\bar{\nabla}_i$. These are in general two different operators, acting in two different Hilbert spaces: $L^2(Q, \omega_g)$ and $L^2(Q, \omega_{\bar{g}})$, respectively. The Hilbert spaces $L^2(Q, \omega_g)$ and $L^2(Q, \omega_{\bar{g}})$ are however isometric, with the isometry $L^2(Q, \omega_g) \to L^2(Q, \omega_{\bar{g}})$ given by

$$\boxed{\bar{\Psi} = U\Psi = \frac{|\det g|^{1/4}}{|\det \bar{g}|^{1/4}} \Psi} \tag{8.2.42}$$

where $\Psi \in L^2(Q, \omega_g)$ and $\bar{\Psi} \in L^2(Q, \omega_{\bar{g}})$. The isometry (8.2.42) induces a similarity map between operators in both spaces: it maps an operator $\hat{F}$ acting in $L^2(Q, \omega_g)$ to the operator

$$\hat{\bar{F}} = U \hat{F} U^{-1} \tag{8.2.43}$$

acting in $L^2(Q, \omega_{\bar{g}})$.

Suppose that the operator $\hat{H}$ in the Hilbert space $L^2(Q, \omega_g)$ is given by (8.2.41). Then the operator $U\hat{H}U^{-1}$, acting in the Hilbert space $L^2(Q, \omega_{\bar{g}})$, has the form

$$\boxed{U\hat{H}U^{-1} = -\tfrac{1}{2}\hbar^2 \bar{\nabla}_i A^{ij} \bar{\nabla}_j + V(x) + \hbar^2 W} \tag{8.2.44}$$

with function W given by

$$\boxed{W = \tfrac{1}{8} A^{ij} \left(\Gamma^k_{ik} \Gamma^s_{js} - \bar{\Gamma}^k_{ik} \bar{\Gamma}^s_{js} \right) + \tfrac{1}{4} \partial_i \left(A^{ij} \left(\Gamma^k_{jk} - \bar{\Gamma}^k_{jk} \right) \right).} \tag{8.2.45}$$

We will call the term $W(x)$ the *quantum correction term* as it describes what happens to the operator (8.2.41) transformed from $L^2(Q, \omega_g)$ to $L^2(Q, \omega_{\bar{g}})$.

One can prove the relation (8.2.44) by direct calculations of $U\hat{H}U^{-1}$. As

$$U\hat{H}U^{-1} = U \left(-\tfrac{1}{2}\hbar^2 \nabla_i A^{ij} \nabla_j + V \right) U^{-1} = -\tfrac{1}{2}\hbar^2 U \nabla_i A^{ij} \nabla_j U^{-1} + V$$

and using the fact that

$$\frac{\partial U}{\partial x^i} = \tfrac{1}{2} U \left(\Gamma^k_{ik} - \bar{\Gamma}^k_{ik} \right),$$

after some calculations we arrive at (8.2.44)–(8.2.45).

Alternatively, the similarity map (8.2.43) can be calculated using the automorphism $\bar{S} S^{-1}$. From our general theory it follows that quantizing the observable H with respect to g yields an operator that is mapped through (8.2.43) on the operator that we obtain by quantizing the observable $H' = \bar{S} S^{-1} H$ with respect to $\bar{g}$. This yields that the operator (8.2.41) attains in the space in $L^2(Q, \omega_{\bar{g}})$ the form

$$U \hat{H} U^{-1} = (\bar{S}^{-1} H')_W (\hat{\bar{x}}, \hat{\bar{p}}) = (\bar{S}^{-1} \bar{S} S^{-1} H)_W (\hat{\bar{x}}, \hat{\bar{p}}) = (S^{-1} H)_W (\hat{\bar{x}}, \hat{\bar{p}}). \tag{8.2.46}$$

Let us thus explicitly calculate the operator on the right hand side of (8.2.46). Due to (8.2.40) and using the fact that H is second order in momenta (so that the only terms in S^{-1} that act on H are or order up to $\hbar^2$), after some calculations we obtain

$$S^{-1} H = H + \tfrac{1}{2} \hbar^2 \left(\tfrac{1}{4} A^{ij}{}_{,ij} + \tfrac{1}{2} A^{ij}{}_{,i} \Gamma^k_{jk} + \tfrac{1}{2} A^{ij} \Gamma^k_{ik,j} + \tfrac{1}{4} A^{ij} \Gamma^k_{ik} \Gamma^l_{jl} \right) = \bar{S}^{-1} H + \hbar^2 W$$

with

$$W = \tfrac{1}{2} \left[\tfrac{1}{2} A^{ij}{}_{,i} \left(\Gamma^k_{jk} - \bar{\Gamma}^k_{jk} \right) + \tfrac{1}{2} A^{ij} \left(\Gamma^k_{ik,j} - \bar{\Gamma}^k_{ik,j} \right) + \tfrac{1}{4} A^{ij} \left(\Gamma^k_{ik} \Gamma^l_{jl} - \bar{\Gamma}^k_{ik} \bar{\Gamma}^l_{jl} \right) \right]$$

coinciding with $W(x)$ in (8.2.45).

The relation (8.2.45) can be also written in a covariant form as

$$W = \tfrac{1}{8} \left(A^{ij}{}_{;i} G^{ks} g_{ks;j} + A^{ij} G^{ks} g_{ks;ij} + A^{ij} G^{ks}{}_{;i} g_{ks;j} + \tfrac{1}{4} A^{ij} G^{kr} g_{kr;i} G^{sl} g_{sl;j} \right) \tag{8.2.47}$$

where the covariant derivatives are taken with respect to the connection $\bar{\nabla}_i$. The reader can find the proof of formula (8.2.47) in [49].

In separation coordinates formulas simplify drastically. We have

$$\hat{H}_r = -\tfrac{1}{2} \hbar^2 \nabla_i A^{ii}_r \nabla_i + V = -\tfrac{1}{2} \hbar^2 A^{ii}_r [\partial_i^2 + \Xi_i \partial_i] + V_r \tag{8.2.48}$$

and then

$$\bar{\Psi} = U \Psi = U \prod_{i=1}^{n} \psi(\lambda_i) \tag{8.2.49}$$

and

$$\begin{aligned}\hat{\bar{H}}_r &= U\hat{H}_r U^{-1} = -\tfrac{1}{2}\hbar^2 A_r^{ii}\left[\partial_i^2 + \left(\Xi_i - 2\frac{\partial_i U}{U}\right)\partial_i\right] + V_r + \hbar^2 W_r \\ &= -\tfrac{1}{2}\hbar^2 A_r^{ii}\left[\partial_i^2 + \overline{\Xi}_i\partial_i\right] + V_r + \hbar^2 W_r \\ &= -\tfrac{1}{2}\hbar^2 \bar{\nabla}_i A_r^{ii}\bar{\nabla}_i + V_r + \hbar^2 W_r,\end{aligned} \tag{8.2.50}$$

where

$$\boxed{W_r = -\frac{1}{2}\sum_{i=1}^{n} A_r^{ii}\left[2\left(\frac{\partial_i U}{U}\right)^2 - \frac{\partial_i^2 U}{U} + \Xi_i\frac{\partial_i U}{U}\right].} \tag{8.2.51}$$

We will say that operators (8.2.50) are *R-separable*, as their eigenfunctions are of the form (8.2.49), where $U(\lambda)$ is a known function of λ and $\psi(\lambda_i)$ solve quantum separation equations (8.2.2). The reader can find more about R-separability in a historical context in the last subsection.

In order to compare separable quantizations (8.2.2) and (8.2.38) and check whether they are unitary equivalent or not, consider the operators

$$\hat{H}_r = -\frac{1}{2}\hbar^2\sum_{i=1}^{n}(S^{-1})_r^i\left(f(\lambda^i)\partial_i^2 + \frac{1}{2}\frac{df(\lambda^i)}{d\lambda^i}\partial_i\right) + \sum_{i=1}^{n}(S^{-1})_r^i\sigma(\lambda^i), \tag{8.2.52}$$

$$\hat{\bar{H}}_r = -\frac{1}{2}\hbar^2\sum_{i=1}^{n}(S^{-1})_r^i\left[f(\lambda^i)\partial_i^2 + \left(\frac{df(\lambda^i)}{d\lambda^i} - \frac{1}{2}\frac{f(\lambda^i)}{\bar{f}(\lambda^i)}\frac{d\bar{f}(\lambda^i)}{d\lambda^i}\right)\partial_i\right] + \sum_{i=1}^{n}(S^{-1})_r^i\bar{\sigma}(\lambda^i), \tag{8.2.53}$$

(for simplicity we assume that $f_i(\lambda^i) = f(\lambda^i)$ and $\bar{f}_i(\lambda^i) = \bar{f}(\lambda^i)$) and the associated eigenvalue problems

$$\hat{H}_r\Psi = E_r\Psi, \quad \hat{\bar{H}}_r\bar{\Psi} = \bar{E}_r\bar{\Psi}, \quad r = 1,\ldots,n,$$

where $\Psi(\lambda^1,\ldots,\lambda^n) = \prod_{k=1}^n\psi(\lambda^k)$, $\bar{\Psi}(\lambda^1,\ldots,\lambda^n) = \prod_{k=1}^n\bar{\psi}(\lambda^k)$, and $\psi(\lambda^k)$ and $\bar{\psi}(\lambda^k)$ are n copies of one-dimensional eigenvalue problems

$$\left(\sum_{i=1}^{n}E_i\lambda^{\gamma_i}\right)\psi(\lambda) = -\frac{\hbar^2}{2}\left(f(\lambda)\frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{df(\lambda)}{d\lambda}\frac{d\psi(\lambda)}{d\lambda}\right) + \sigma(\lambda)\psi(\lambda), \tag{8.2.54}$$

$$\left(\sum_{i=1}^{n}\bar{E}_i\lambda^{\gamma_i}\right)\bar{\psi}(\lambda)=-\frac{\hbar^2}{2}\left[f(\lambda)\frac{d^2\bar{\psi}(\lambda)}{d\lambda^2}+\left(\frac{df(\lambda)}{d\lambda}-\frac{1}{2}\frac{f(\lambda)}{\bar{f}(\lambda)}\frac{d\bar{f}(\lambda)}{d\lambda}\right)\frac{d\bar{\psi}(\lambda)}{d\lambda}\right]$$
$$+\bar{\sigma}(\lambda)\bar{\psi}(\lambda). \tag{8.2.55}$$

It is easily checked that we also have quantum separation relations

$$\sum_{k=1}^{n}\lambda^{\gamma_k}\hat{H}_k=-\frac{1}{2}\hbar^2\left(f(\lambda)\frac{d^2}{d\lambda^2}+\frac{1}{2}\frac{df(\lambda)}{d\lambda}\frac{d}{d\lambda}\right)+\sigma(\lambda),$$
$$\sum_{k=1}^{n}\lambda^{\gamma_k}\hat{\bar{H}}_k=-\frac{1}{2}\hbar^2\left[f(\lambda)\frac{d^2}{d\lambda^2}+\left(\frac{df(\lambda)}{d\lambda}-\frac{1}{2}\frac{f(\lambda)}{\bar{f}(\lambda)}\frac{d\bar{f}(\lambda)}{d\lambda}\right)\frac{d}{d\lambda}\right]+\bar{\sigma}(\lambda),$$

where, just as in (8.2.54) and (8.2.55), $\lambda=\lambda^i$, $i=1,\ldots,n$.

Moreover, we see from (8.2.52) and (8.2.53) that for a suitable choice of $\bar{\sigma}(\lambda)$ the operators $\hat{H}_r$ and $\hat{\bar{H}}_r$ are related by a transformation (8.2.43)

$$\hat{\bar{H}}_r=U\hat{H}_rU^{-1}. \tag{8.2.56}$$

Indeed, for

$$U=\prod_{i=1}^{n}\left(\frac{\bar{f}(\lambda^i)}{f(\lambda^i)}\right)^{1/4},$$

relation (8.2.56) holds under condition

$$\bar{\sigma}(\lambda^i)=\sigma(\lambda^i)-\frac{\hbar^2}{2}\left(f(\lambda^i)\frac{d^2v(\lambda^i)}{d(\lambda^i)^2}+\frac{1}{2}\frac{df(\lambda^i)}{d\lambda^i}\frac{dv(\lambda^i)}{d\lambda^i}\right),$$

where

$$v(\lambda^i)=\left(\frac{f(\lambda^i)}{\bar{f}(\lambda^i)}\right)^{1/4}.$$

In consequence, the Hamiltonians (8.2.52) and (8.2.53) are the avatars of the same operator, acting on different Hilbert spaces. Notice that if we impose the additional constraint $\bar{\sigma}(\lambda)=\sigma(\lambda)$, the Hamiltonians $\hat{H}_r$ and $\hat{\bar{H}}_r$ become the avatars of the same operator if we choose $\bar{f}(\lambda)$ so that

$$f(\lambda)\frac{d^2v(\lambda)}{d\lambda^2}+\frac{1}{2}\frac{df(\lambda)}{d\lambda}\frac{dv(\lambda)}{d\lambda}=0.$$

It means that

$$v(\lambda) = \int f(\lambda)^{-\frac{1}{2}} d\lambda \rightarrow \bar{f}(\lambda) = a \frac{f(\lambda)}{\left[\int (f(\lambda))^{-1/2} d\lambda\right]^4},$$

where a is an arbitrary constant. Of course, the above results are readily generalized to the case when our separation relations are not copies of the same separation curve.

Now, let us come back to the question from the end of previous subsection. For $\bar{f}(\lambda) = \theta(\lambda)$, $g = \varphi^{\frac{2}{n}} g_{B,f}$, $\bar{g} = \varphi^{\frac{2}{n}} g_{B,\theta}$, observing that Hilbert spaces $L^2(Q, \omega_g)$ and $L^2(Q, \omega_{\bar{g}})$ are isometric, with the isometry given by

$$U = \frac{|\det g|^{1/4}}{|\det \bar{g}|^{1/4}} = \prod_{k=1}^{n} \left(\frac{\theta_k(\lambda^k)}{f_k(\lambda^k)}\right)^{\frac{1}{4}},$$

the avatar of operator (8.2.38) from $L^2(Q, \omega_g)$ is the operator (8.2.2) from $L^2(Q, \omega_{\bar{g}})$ with

$$\theta(\lambda) = a \frac{f(\lambda)}{\left[\int (f(\lambda))^{-1/2} d\lambda\right]^4}.$$

Other separable quantizations (8.2.2) are not equivalent to the (8.2.38) ones.

Observe that there is a distinguished class of separable quantizations related to flat metrics $g_{B,\theta}$. Then, for the Benenti class we are dealing with the class of flat minimal quantizations [48] and for other classes, with conformally flat minimal quantizations. We consider them in the next subsection.

8.2.4 *Quantum Integrability of Stäckel Systems in Various Hilbert Spaces*

We remind the reader that in [18] the authors derived the necessary and sufficient condition for commutativity of quantum Hamiltonians $\hat{H}_r$ of the form (8.2.4) (and with $A_1 = G$) called the pre-Robertson condition (8.2.24), which took the form

$$\partial_i^2 \Gamma_j - \Gamma_i \partial_i \Gamma_j = 0, \quad i \neq j.$$

In our case metric G is not related with any A_r, so analogous calculations to these from [18] lead to the following necessary and sufficient condition for commutativity of $\hat{H}_r$, which we call the generalized pre-Robertson condition [48]

$$\boxed{\partial_i^2 \Xi_j - \Xi_i \partial_i \Xi_j = 0, \quad i \neq j.} \tag{8.2.57}$$

Let us consider a Stäckel system H_r, $r = 1, \ldots, n$ of the form (8.2.15) and let us perform the procedure of minimal quantization of this system in the metric G given by (8.2.37). We obtain then the quantum separable system consisting of n operators $\hat{H}_r$ acting on the Hilbert space $L^2(Q, \omega_g)$, $\omega_g = |\det g|^{1/2} d\lambda$. Since the generalized Robertson condition (8.2.28) implies the generalized pre-Robertson condition (8.2.57) we conclude that this system is also quantum integrable: $[\hat{H}_r, \hat{H}_s] = 0$. Using (8.2.44) we are able to write operators $\hat{H}_r$ in another metric $\bar{G}$ i.e. in the Hilbert space $L^2(Q, \omega_{\bar{g}})$ which yields new quantum operators $\hat{\bar{H}}_r, r = 1, \ldots, n$ that constitute again quantum integrable and quantum R-separable system. Due to the theory developed in the previous subsection we know, that we can equally well take the classical Hamiltonians H_r amended by quantum correction terms, i.e. the functions $H_r + \hbar^2 W_r$ with W_r given by (8.2.45) (or equivalently by (8.2.47)) and minimally quantize them in the metric $\bar{G}$ as this yields the same quantum integrable system $\hat{\bar{H}}_r, r = 1, \ldots, n$.

In Sect. 4.4 we demonstrated that any Stäckel system of the class (8.2.15) can be constructed by an appropriate Stäckel transform of a suitably chosen flat Stäckel system from the Benenti class. Moreover, in Sect. 5.5 we explicitly constructed flat coordinates for any flat Stäckel system. Therefore we are able to write down our original Stäckel system H_r, $r = 1, \ldots, n$ in flat coordinates of the metric $\bar{G}$ of the form (8.2.34) ($\bar{G}$ is flat as soon the (4.3.2) is satisfied). In this specific case, if we apply the standard Weyl quantization (minimal flat quantization) to the Stäckel system $H_r + \hbar^2 W_r$ we will obtain a quantum integrable system. One can also say, alternatively, that if we want to avoid quantum correction terms, we should quantize the original system $H_r, r = 1, \ldots, n$ not by the Weyl quantization but in a suitably chosen minimal quantization with respect to conformally flat metric G. In what follows we will also need a specification of this correction term to the following situation: suppose that $G = \frac{1}{u} G_{B,\theta}$ (where $u = u(x)$) where the metric $G_{B,\theta}$ is flat and suppose that $\bar{G} = G_{B,\theta}$. Then the correction term (8.2.47) attains the form

$$\boxed{W(x) = \frac{n}{8} \partial_i \left[A^{ij} \frac{(\partial_j u)}{u} \right] + \frac{n^2}{32} A^{ij} \frac{(\partial_i u)(\partial_j u)}{u^2}.} \tag{8.2.58}$$

In papers [148, 149] the authors presented some ad hoc calculations generating quantum correction terms that guarantee integrability of quantum systems obtained through the Weyl quantization of some Hamiltonian systems. The presented theory shows how to construct these quantum correction terms in a systematic way (albeit within the class of Stäckel systems not considered in [148, 149]). We will illustrate this on two examples below. It is important to stress that the presented systems cannot be separably quantized in the frame of the classical Robertson-Eisenhart formalism.

Example 8.5 Consider the Stäckel system (8.2.15) for $n = 3$ given by the separation curve of the form:

$$H_1\lambda^3 + H_2\lambda + H_3 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4, \quad i = 1, 2, 3 \tag{8.2.59}$$

so that $\gamma_1 = 3$, $\gamma_2 = 1$, $\gamma_3 = 0$ with $f(\lambda^i) = \lambda^i$ and $\sigma(\lambda^i) = (\lambda^i)^4$. In this case $\varphi = \rho_1(\lambda) = -(\lambda^1 + \lambda^2 + \lambda^3)$. Consider also the corresponding metric $G_{B,f}$ given by (8.2.34). This metric is flat. In the coordinates x, y, z defined through (cf. 4.3.18))

$$\begin{aligned} \rho_1 &\equiv -(\lambda^1 + \lambda^2 + \lambda^3) = x \\ \rho_2 &\equiv \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3 = y + \tfrac{1}{4}x \\ \rho_3 &\equiv -\lambda^1\lambda^2\lambda^3 = -\tfrac{1}{4}z \end{aligned} \tag{8.2.60}$$

the metric $G_{B,f}$ reads

$$G_{B,f} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so $\varphi = x$ and (x, y, z) are flat non-orthogonal coordinates for $G_{B,f}$. Solving the relations (8.2.59) with respect to the Hamiltonians H_i and passing to the variables (x, y, z) we receive $H_r = p^T A_r p + V_r$ where $p = (p_x, p_y, p_z)^T$ and where tensors A_r have the form

$$A_1 = \begin{pmatrix} 0 & -\frac{1}{x} & 0 \\ -\frac{1}{x} & 0 & 0 \\ 0 & 0 & -\frac{1}{x} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \frac{1}{4}x - \frac{y}{x} & 0 \\ \frac{1}{4}x - \frac{y}{x} & -y & -\frac{1}{2}z \\ 0 & -\frac{1}{2}z & \frac{3}{4}x - \frac{y}{x} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & \frac{1}{4}\frac{z}{x} & -\frac{1}{2}z \\ \frac{1}{4}\frac{z}{x} & \frac{1}{4}z & -\frac{1}{4}xz \\ -\frac{1}{2}z & -\frac{1}{4}xz & \frac{1}{4}x + y + \frac{1}{4}\frac{z}{x} \end{pmatrix}$$

with the corresponding rational potentials

$$\begin{aligned} V_1 &= -\tfrac{3}{4}x + \frac{y}{x}, \\ V_2 &= \tfrac{1}{16}x + \tfrac{1}{2}xy + \tfrac{1}{4}z + \frac{y}{x}, \\ V_3 &= -\tfrac{1}{16}xz - \tfrac{1}{4}\frac{yz}{x}. \end{aligned}$$

From the presented theory it follows that we can perform a separable quantization of this system in the conformally flat metric $G = \frac{1}{u} G_{B,f}$ (which means that we choose $\theta = f$) with $u = \varphi^{\frac{2}{n}} = x^{\frac{2}{3}}$. We obtain three commuting operators

$$\widehat{H}_r = -\tfrac{1}{2}\hbar^2 \nabla_i A_r^{ij} \nabla_j + V_r \tag{8.2.61}$$

(where ∇_i is the connection defined by G),acting in the Hilbert space $L^2(Q, \omega_g) = L^2(Q, |x|\, dxdydz)$ ($\omega_g = |\det g|^{1/2} dxdydz = \left|u^{3/2}\right| dxdydz = |x|\, dxdydz$). In the separation coordinates (λ, μ) the separation equations (8.2.39) for $\widehat{H}_r$ attain the form of three copies of

$$(E_1\lambda^3 + E_2\lambda + E_3)\psi(\lambda) = -\frac{1}{2}\hbar^2 \left(\lambda \frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{d\psi(\lambda)}{d\lambda}\right) + \lambda^4 \psi(\lambda). \tag{8.2.62}$$

Let us now rewrite our operators (8.2.61) in the Hilbert space $L^2(Q, \omega_{\bar{g}}) = L^2(Q, dxdydz)$ ($\omega_{\bar{g}} = |\det \bar{g}|^{1/2} dxdydz = dxdydz$) with the flat metric $\bar{G} = G_{B,f}$. From our theory it follows that a suitable way to do it is to quantize our Hamiltonians H_r directly in the metric $\bar{G}$ after amending them by the quantum correction terms W_i given by (8.2.58)

$$W_1 = 0,\ W_2 = -\frac{3}{8}\frac{1}{x^2},\ W_3 = -\frac{1}{8}\frac{1}{x}$$

One can check by direct calculations that the operators

$$\widehat{\bar{H}}_r = -\tfrac{1}{2}\hbar^2 \partial_i A_r^{ij} \partial_j + \hbar^2 W_r + V_r, \quad r = 1, \ldots, n \tag{8.2.63}$$

do indeed commute, thus constituting a quantum integrable system. The operators (8.2.63) are however not quantum separable, contrary to the operators (8.2.61), but are R-separable. It means that in separation coordinates

$$\widehat{\bar{H}}_r \bar{\Psi}(\lambda) = E_r \bar{\Psi}(\lambda), \quad \bar{\Psi}(\lambda) = U\Psi = (\lambda^1 + \lambda^2 + \lambda^3)^{\frac{1}{2}} \psi(\lambda^1)\psi(\lambda^2)\psi(\lambda^3),$$

and $\psi(\lambda^i)$ solves (8.2.62).

Example 8.6 In our second example we consider the Stäckel system generated by the following separation curve

$$H_1\lambda^3 + H_2\lambda^2 + H_3 = \tfrac{1}{2}\lambda\mu^2 + \lambda^4,$$

so that this time $\gamma_1 = 3$, $\gamma_2 = 2$ and $\gamma_3 = 0$ but still with $f(\lambda^i) = \lambda^i$ and $\sigma(\lambda^i) = (\lambda^i)^4$. In this case $\varphi = \rho_2(\lambda) = \lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3$. We consider again the same metric $G_{B,f}$ with the same flat coordinates (x, y, z) given by (8.2.60).

This time the tensors A_r have the form

$$A_1 = \frac{1}{\rho_2}\begin{pmatrix} -1 & -\frac{1}{2}x & 0 \\ -\frac{1}{2}x & y & \frac{1}{2}z \\ 0 & \frac{1}{2}z & -x \end{pmatrix},$$

$$A_2 = \frac{1}{\rho_2}\begin{pmatrix} -x & -\frac{1}{4}x^2 + y & 0 \\ -\frac{1}{4}x^2 + y & yx & \frac{1}{2}xz \\ 0 & \frac{1}{2}xz & -\frac{3}{4}x^2 + y \end{pmatrix},$$

$$A_3 = \frac{1}{4\rho_2}\begin{pmatrix} z^2 & \frac{1}{2}z^2x & -\frac{1}{2}\left(x^2 + 4\,y\right)z \\ \frac{1}{2}z^2x & \frac{1}{4}z^2x^2 & -\frac{1}{4}z\left(x^3 + 4\,yx + 2\,z^2\right) \\ -\frac{1}{2}\left(x^2 + 4\,y\right)z & -\frac{1}{4}z\left(x^3 + 4\,yx + 2\,z^2\right) & \frac{1}{4}x^4 + 2\,x^2y + 4\,y^2 + z^2x \end{pmatrix}$$

where $\rho_2 = y + \frac{1}{4}x$, while the potentials are

$$V_1 = -\frac{1}{4\rho_2(x)}\left(x + 4xy + z\right) = -\frac{x + 4xy + z}{x + 4y},$$

$$V_2 = -\frac{1}{4\rho_2(x)}\left(\frac{1}{4}x + 2xy + xz + 4y\right) = -\frac{\frac{1}{4}x + 2xy + xz + 4y}{x + 4y},$$

$$V_3 = \frac{z}{16\rho_2(x)} = \frac{1}{4}\frac{z}{x + 4y}.$$

Now, let us perform a separable quantization in the conformally flat metric $G = \frac{1}{u}G_{B,f}$ with $u = \varphi^{\frac{2}{n}} = \left(y + \frac{1}{4}x\right)^{\frac{2}{3}}$. We obtain again three commuting operators

$$\widehat{H}_r = -\tfrac{1}{2}\hbar^2\nabla_i A_r^{ij}\nabla_j + V_r$$

(where ∇_i operators are related with the connection defined by G),acting in the Hilbert space $L^2(Q, \left|\rho_2\right| dxdydz)$, while the separation equations (8.2.39) for $\hat{H}_r$ attain the form of three copies of

$$(E_1\lambda^3 + E_2\lambda^2 + E_3)\psi(\lambda) = -\frac{1}{2}\hbar^2\left(\lambda\frac{d^2\psi(\lambda)}{d\lambda^2} + \frac{1}{2}\frac{d\psi(\lambda)}{d\lambda}\right) + \lambda^4\psi(\lambda),$$

with the same right hand side as in the previous example. Rewriting our operators (8.2.61) in the Hilbert space $L^2(Q, dxdydz)$ with quantization in the flat metric $\bar{G} = G_{B,f}$ leads to the following correction terms $W_i(x, y, z)$

$$W_1 = \frac{1}{16}\frac{5x - 4y}{\rho_2^3(x)},$$

$$W_2 = \frac{1}{32}\frac{7x - 20xy}{\rho_2^3(x)},$$
$$W_3 = -\frac{1}{128}\frac{x + 8xy + 13xz + 16xy + 4yz}{\rho_2^3(x)}.$$

Again, the operators

$$\widehat{\bar{H}}_r = -\tfrac{1}{2}\hbar^2 \partial_i A_r^{ij} \partial_j + \hbar^2 W_r + V_r, \quad r = 1, \ldots, n \tag{8.2.64}$$

commute, as it can be checked for example in Maple. Operators (8.2.64) are R-separable and in separation coordinates

$$\widehat{\bar{H}}_r \bar{\Psi}(\lambda) = E_r \bar{\Psi}(\lambda), \quad \bar{\Psi}(\lambda) = (\lambda^1\lambda^2 + \lambda^1\lambda^3 + \lambda^2\lambda^3)^{\frac{1}{2}} \psi(\lambda^1)\psi(\lambda^2)\psi(\lambda^3),$$

where $\psi(\lambda^i)$ solves (8.6).

8.2.5 *Quantization of Superintegrable Stäckel Systems*

In analogy to classical mechanics, a quantum maximally superintegrable Hamiltonian is a self-adjoint differential operator $\hat{h}_1$ acting in an appropriate Hilbert space of functions on the configuration space Q (square integrable with respect to some metric) belonging to a set of n commuting self-adjoint differential operators $\hat{h}_1, \ldots, \hat{h}_n$ acting in the same Hilbert space (so that $[\hat{h}_i, \hat{h}_j] = 0$ for all $i, j = 1, \ldots n$) and such that it also commutes with an additional set of $n - 1$ differential operators $\hat{h}_{n+1}, \ldots, \hat{h}_{2n-1}$ of the finite order. Besides, it is required that all the operators $\hat{h}_1, \ldots, \hat{h}_{2n-1}$ are algebraically independent. There is an extended literature devoted to the construction and investigation of quantum superintegrable systems [67, 138, 165, 167, 169, 221, 229] (see also references of the review paper [200]).

This subsection is devoted to separable quantizations of superintegrable Stäckel systems that were considered in the classical setting in Sects. 4.3.3 and 4.4.4. First, let us formulate the quantum analog of the classical commutator (4.1.8). Suppose that $\hat{h}$ is given by

$$\hat{h} = -\tfrac{1}{2}\hbar^2 \nabla_i A^{ij}(x) \nabla_j + V(x) = -\tfrac{1}{2}\hbar^2 \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} A^{ij}(x) \partial_j + V(x) \tag{8.2.65}$$

and that $Y = y^i(x)\nabla_i$ is a vector field on the Riemannian manifold (Q, g). Then

$$\boxed{\left[\hat{h}, Y\right] = \tfrac{1}{2}\hbar^2 \nabla_i \left(L_Y A\right)^{ij} \nabla_j + \tfrac{1}{2}\hbar^2 A^{ij} \left(\nabla_j \nabla_k y^k\right) \nabla_i - Y(U)}$$

One proves this relation by a direct computation. Thus, a sufficient condition for $[\hat{h}, Y] = c$ is satisfied when Y is a Killing vector for both A and g and if moreover U is constant along Y, that is when

$$L_Y A = 0,\ L_Y g = 0\,,\quad Y(U) = c \tag{8.2.66}$$

(note that $L_Y g = 0$ implies $\nabla_k y^k = 0$).

Suppose we have a quantum integrable system on the configuration space Q, that is a set of n commuting and algebraically independent operators $\hat{h}_1, \ldots, \hat{h}_n$ of the form (8.2.65) acting in the Hilbert space $L^2(Q, |g|^{1/2}\,dx)$ where g is a metric on Q. Suppose also that a vector field Y satisfies (8.2.66) with A_1 and U_1 instead of A and U (so that $[\hat{h}_1, Y] = c$). Then, analogously to the classical case, the operators

$$\boxed{\hat{h}_{n+r} = \left[\hat{h}_{r+1}, Y\right] = \tfrac{1}{2}\hbar^2 \nabla_i \left(L_Y A_{r+1}\right)^{ij} \nabla_j - Y(U_{r+1}),\quad r = 1, \ldots, n-1} \tag{8.2.67}$$

satisfy $[\hat{h}_{n+r}, \hat{h}_1] = 0$ and the system $\hat{h}_1, \ldots, \hat{h}_{2n-1}$ is algebraically independent; that is we obtain a quantum separable and quantum superintegrable system.

We can now apply this formula to construct quantum superintegrable counterparts of classical systems considered in Sects. 4.3.3 and 4.4.4. According to results of previous subsections, for the systems generated by the separation curves (4.3.25) the most natural choice of the metric g is to take $G = A_1$. Then, by construction, $[\hat{h}_i, \hat{h}_j] = 0$ for $i, j = 1, \ldots, n$ while the remaining operators $\hat{h}_{n+r}$ are constructed by the formula (8.2.67) and are - up to a sign - identical with minimal quantization (in the metric G) of the extra integrals h_{n+r} obtained in (4.3.29).

Example 8.7 Consider again separation curve (4.3.30) from Example 4.15. Performing the minimal quantization of the Hamiltonians (4.3.33) in the metric $G = A_1$ i.e. given by (4.3.32), we obtain, in the flat coordinates (4.3.31)

$$\begin{aligned}
\hat{h}_1 &= -\tfrac{1}{2}\hbar^2\left(\partial_x\partial_2 + \tfrac{1}{2}\partial_z^2\right) + \alpha_{-1}V_1^{(-1)} + \alpha_3 V_1^{(3)} + \alpha_4 V_1^{(4)},\\
\hat{h}_2 &= -\tfrac{1}{4}\hbar^2\left(\partial_x^2 - \partial_y y\partial_y + x\partial_z\partial_z + \frac{1}{2}\partial_x x\partial_y + \tfrac{1}{2}x\partial_y\partial_1 - z\tfrac{1}{2}\partial_y\partial_z - \tfrac{1}{2}\partial_z z\partial_y\right)\\
&\quad + \alpha_{-1}V_2^{(-1)} + \alpha_3 V_2^{(3)} + \alpha_4 V_2^{(4)},\\
\hat{h}_3 &= -\tfrac{1}{8}\hbar^2\left(\tfrac{1}{2}z^2\partial_y^2 + \left(2y + \tfrac{1}{2}x^2\right)\partial_3^2 - z\partial_x\partial_z - \partial_z z\partial_x - \frac{1}{2}xz\partial_y\partial_z - \tfrac{1}{2}x\partial_z z\partial_y\right)\\
&\quad + \alpha_{-1}V_3^{(-1)} + \alpha_3 V_3^{(3)} + \alpha_4 V_3^{(4)},
\end{aligned}$$

The respective one dimensional eigenvalue problem, according to (4.3.30) and (8.2.19), is of the form

$$(\alpha_{-1}\lambda^{-1}+\alpha_3\lambda^3+\alpha_4\lambda^4+E_1\lambda^2+E_2\lambda+E_3)\psi(\lambda) = -\frac{1}{2}\hbar^2\left[\lambda\frac{d^2\psi(\lambda)}{d\lambda^2}+\frac{1}{2}\frac{d\psi(\lambda)}{d\lambda}\right]$$

Now $Y = \partial_y$ satisfies the conditions (8.2.66) and the extra operators $\hat{h}_4, \hat{h}_5$ can be obtained either by using the formula (8.2.67) or directly by a minimal quantization of functions h_4, h_5 in (4.3.35). The result is (up to a sign)

$$\hat{h}_4 = \tfrac{1}{4}\hbar^2\partial_y^2 - \alpha_3 + \alpha_4 x, \quad \hat{h}_5 = -\tfrac{1}{4}\hbar^2\partial_z^2 + \frac{4\alpha_{-1}}{z^2}$$

If we want to perform the separable quantization of superintegrable systems obtained by the Stäckel transform, as in Sect. 4.4.4, we have two cases: either the system—after the Stäckel transform—belongs again to the same class (4.3.25) or belongs to the other class, given by the separation relations (4.4.38) that are different from (4.3.25) as soon as $k \neq -1$. In the first case the natural choice of the metric in which we perform the minimal quantization is to take $\tilde{G} = \tilde{A}_1$ i.e. $\tilde{G}$ as given by (4.4.4). In the second case we have to use the metric given by (8.2.37) which in our case is given by $G = \varphi^{1-\frac{2}{n}}\tilde{A}_1$ with $\varphi = -V_1^{(k)}$.

Example 8.8 Let us now minimally quantize the Stäckel Hamiltonians $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ given in (4.4.40), obtained through the Stäckel transform in Example 4.17, generated by the separation curve (4.4.39) with $k = -1$, that is by

$$\tilde{h}_1\lambda^{-1} + \tilde{\alpha}\lambda^2 + \tilde{h}_2\lambda + \tilde{h}_3 = \tfrac{1}{2}\lambda\mu^2,$$

The metric associated with $\tilde{h}_1$

$$\tilde{G} = \tilde{A}_1 = \tfrac{1}{4}z^2\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \tag{8.2.68}$$

is of constant curvature as after applying transformation (4.4.4), in the new separation coordinates the separation curve (4.4.39) turns to

$$\tilde{\alpha}\lambda^{-1} + \tilde{h}_1\lambda^2 + \tilde{h}_3\lambda + \tilde{h}_2 = \tfrac{1}{2}\lambda^4\mu^2,$$

and belongs again to the class (4.3.25). Thus, we have to perform the minimal quantization of this system with respect to the original metric $\tilde{A}_1$ of the system which is just (8.2.68). Observing that $\sqrt{|\tilde{g}|} = 8/z^3$, we obtain the following

quantum superintegrable system (we use the second expression in (8.1.62)):

$$\hat{\tilde{h}}_1 = -\tfrac{1}{4}\hbar^2 z^2\left(\tfrac{1}{2}z\partial_z\tfrac{1}{z}\partial_z + \partial_x\partial_2\right) - \tfrac{1}{4}\tilde{\alpha}z^2,$$

$$\hat{\tilde{h}}_2 = \tfrac{1}{4}\hbar^2\left(-2\partial_x^2 + 2\partial_y y\partial_y + \partial_x x\partial_y + x\partial_y\partial_x + z\partial_y\partial_z + z^3\partial_z\tfrac{1}{z^2}\partial_y\right) + \tilde{\alpha}x,$$

$$\hat{\tilde{h}}_3 = \tfrac{1}{8}\hbar^2\left[-z^2\partial_y^2 + (x^2+4y)\partial_x\partial_y + \partial_x x^2\partial_y + 4\partial_y y\partial_x + 2z\partial_1\partial_z + 2z^3\partial_z\frac{1}{z^2}\partial_x\right.$$
$$\left. xz\partial_y\partial_z + xz^3\partial_z\tfrac{1}{z^2}\partial_x\right] + \tfrac{1}{4}\tilde{\alpha}\left(x^2+4y\right),$$

$$\hat{\tilde{h}}_4 = \tfrac{1}{2}\hbar^2\partial_y^2,$$

$$\hat{\tilde{h}}_5 = \hbar^2\partial_x\partial_y + \tilde{\alpha}$$

Example 8.9 Let us finally minimally quantize the Stäckel Hamiltonians $\tilde{h}_1, \tilde{h}_2, \tilde{h}_3$ given in (4.4.41), obtained through the Stäckel transform in Example 4.17 and generated by separation curve (4.4.39) with $k = 4$.

$$\tilde{h}_1\lambda^4 + \tilde{\alpha}\lambda^2 + \tilde{h}_2\lambda + \tilde{h}_3 = \tfrac{1}{2}\lambda\mu^2,$$

The metric associated with $\tilde{h}_1$

$$\tilde{G} = \tilde{A}_1 = \frac{1}{\frac{3}{4}x^2 - y}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is conformally flat. We have to perform minimal quantization of this system with respect to the metric $G = \varphi^{1-\frac{2}{n}}\tilde{A}_1$ with $\varphi = -V_1^{(4)}$ given by

$$G = \left(-V_1^{(4)}\right)^{1-\frac{2}{3}}\tilde{G} = \left(y - \tfrac{3}{4}x^2\right)^{-\frac{2}{3}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Observing that $\sqrt{|g|} = V_1^{(4)} = y - \frac{3}{4}x^2$, we obtain the following quantum operators (we use again the second expression in (8.1.62)):

$$\hat{\tilde{h}}_1 = \tfrac{1}{2}\hbar^2\left(y - \tfrac{3}{4}x^2\right)^{-1}\left(2\partial_x\partial_y + \partial_z^2\right) + \frac{\tilde{\alpha}}{y - \frac{3}{4}x^2}$$

$$\hat{\tilde{h}}_2 = -\tfrac{1}{2}\hbar^2\left(y - \tfrac{3}{4}x^2\right)^{-1}\sum_{i,j}\partial_i B_2^{ij}\partial_j - \tilde{\alpha}\frac{xy + \frac{1}{4}x^3 + \frac{1}{4}z^2}{y - \frac{3}{4}x^2}$$

$$\hat{\tilde{h}}_3 = -\tfrac{1}{2}\hbar^2\left(y-\tfrac{3}{4}x^2\right)^{-1}\sum_{i,j}\partial_i B_3^{ij}\partial_j + \tfrac{1}{4}\tilde{\alpha}\frac{xz^2}{y-\frac{3}{4}x^2}$$

$$\hat{\tilde{h}}_4 = -\tfrac{1}{2}\hbar^2\left(y-\tfrac{3}{4}x^2\right)^{-1}\left[\partial_x x\partial_y + x\partial_y\partial_x - \partial_y\left(y-\tfrac{3}{4}x^2\right)\partial_y + x\partial_z^2\right] - \tilde{\alpha}\frac{x}{y-\frac{3}{4}x^2}$$

$$\hat{\tilde{h}}_5 = -\tfrac{1}{2}\hbar^2\partial_z^2$$

where

$$B_2 = \begin{pmatrix} y-\frac{3}{4}x^2 & \frac{3}{2}xy-\frac{1}{8}x^3+\frac{1}{4}z^2 & 0 \\ \frac{3}{2}xy-\frac{1}{8}x^3+\frac{1}{4}z^2 & -y\left(y-\frac{3}{4}x^2\right) & -\frac{1}{2}z\left(y-\frac{3}{4}x^2\right) \\ 0 & -\frac{1}{2}z\left(y-\frac{3}{4}x^2\right) & 2xy-\frac{1}{2}x^3+\frac{1}{4}z^2 \end{pmatrix}$$

$$B_3 = \begin{pmatrix} 0 & -\frac{1}{4}xz^2 & -\frac{1}{2}z\left(y-\frac{3}{4}x^2\right) \\ -\frac{1}{4}xz^2 & \frac{1}{4}z^2\left(y-\frac{3}{4}x^2\right) & -\frac{1}{4}xz\left(y-\frac{3}{4}x^2\right) \\ -\frac{1}{2}z\left(y-\frac{3}{4}x^2\right) & -\frac{1}{4}xz\left(y-\frac{3}{4}x^2\right) & -\frac{1}{2}x^2y+y^2-\frac{1}{4}xz^2-\frac{3}{16}x^4 \end{pmatrix}$$

with $B = \sqrt{|g|}A$ in (8.1.62). It can be checked that it is again a quantum superintegrable system.

8.2.6 R-Separability of Stäckel Metrics

In the following subsection we analyze the notion of R-separability, introduced in previous subsections, from a more restrictive point of view, known from the literature. We begin that subsection by a short summary of the most important results from the whole section. Consider separation relations of the form (8.2.15)

$$\sum_{r=1}^{n} H_r(\lambda^i)^{\gamma_r} = \tfrac{1}{2}f_i(\lambda^i)\mu_i^2 + \sigma_i(\lambda^i), \quad i = 1,\ldots,n$$

and related Stäckel Hamiltonians

$$H_r = \tfrac{1}{2}\sum_{i=1}^{n} A_r^{ii}\mu_i^2 + V_r = \tfrac{1}{2}\sum_{i=1}^{n}(S_\gamma^{-1})_r^i f_i(\lambda^i)\mu_i^2 + \sum_{i=1}^{n}(S_\gamma^{-1})_r^i\sigma(\lambda^i), \qquad (8.2.69)$$

where the Stäckel matrix is of the form (4.3.1). In particular, for $\gamma = (n-1, \ldots, 0)$, we deal with the so called Benenti class and then

$$A_r^{ii} = \frac{K_r^{(i)} f_i(\lambda^i)}{\Delta_i}, \quad V_r = \frac{K_r^{(i)} \sigma(\lambda^i)}{\Delta_i},$$

where

$$\Delta_i = \prod_{k \neq i} (\lambda^i - \lambda^k), \quad K_r^{(i)} = -\frac{\partial \rho_r}{\partial \lambda^i}, \quad i = 1, \ldots, n,$$

and $\rho_r(\lambda)$ are the Viète polynomials (4.3.18) so $\partial K_r^{(i)}/\partial \lambda^i = 0$. For any other γ-class we have (4.4.32)

$$A_r^{ii} = \frac{1}{\varphi(\lambda)} \frac{M_r^{(i)} f_i(\lambda^i)}{\Delta_i}, \quad V_r = \frac{1}{\varphi(\lambda)} \frac{M_r^{(i)} \sigma(\lambda^i)}{\Delta_i},$$

where $\varphi(\lambda)$ is the function given by (4.4.29) and $M_r^{(i)}$ are appropriate polynomials of $K_j^{(i)}$, $j = 1, \ldots, n$ (see Sect. 4.4.3), hence again $\partial M_r^{(i)}/\partial \lambda^i = 0$.

Minimal quantization of classical Hamiltonians (8.2.69) leads to the respective quantum Hamiltonians $\hat{H}_r$ (8.1.62) given by

$$\hat{H}_r = -\tfrac{1}{2}\hbar^2 \sum\nolimits_i \nabla_i A_r^{ii} \nabla_i + V_r, \quad r = 1, \ldots, n \tag{8.2.70}$$

where ∇_i are operators of a covariant derivative associated with a metric $g = G^{-1}$ for which λ coordinates are orthogonal coordinates. Hamiltonian operators (8.2.70) are self-adjoint in the Hilbert space $L^2(Q, |\det g|^{1/2}\, d\lambda)$. For the Benenti class, with a natural choice of the metric tensor

$$G^{ii} = A_1^{ii} = \frac{f_i(\lambda^i)}{\Delta_i},$$

we have proved that n multi-dimensional eigenvalue problems

$$\hat{H}_r \Psi = E_r \Psi, \quad r = 1, \ldots, n \tag{8.2.71}$$

for Hamiltonian operators

$$\hat{H}_r = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} \nabla_i A_r^{ii} \nabla_i + V = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} A_r^{ii} \left[\partial_i^2 + \frac{1}{2} \frac{f_i'}{f_i} \partial_i \right] + V_r$$

separate multiplicatively

$$\Psi(\lambda) = \prod_{i=1}^{n} \psi_i(\lambda^i),$$

onto n one-dimensional eigenvalue problems

$$[E_1(\lambda^i)^{n-1} + E_2(\lambda^i)^{n-2} + \cdots + E_n]\psi_i(\lambda^i)$$
$$= -\frac{1}{2}\hbar^2 \left[f_i(\lambda^i) \frac{d^2}{(d\lambda^i)^2} + \frac{1}{2} \frac{df_i(\lambda^i)}{d\lambda^i} \frac{d}{d\lambda^i} \right] \psi_i(\lambda^i) + \sigma_i(\lambda^i)\psi_i(\lambda^i), \quad i = 1, \ldots, n.$$

We also have proved that for any other γ-class, with a choice of the metric tensor in the form

$$\bar{G}^{ii} = \varphi(\lambda)^{-\frac{2}{n}} \frac{f_i(\lambda^i)}{\Delta_i},$$

n multi-dimensional eigenvalue problems (8.2.71) for Hamiltonian operators

$$\hat{\bar{H}}_r = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} \bar{\nabla}_i A_r^{ii} \bar{\nabla}_i + V = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} A_r^{ii} \left[\partial_i^2 + \tfrac{1}{2} \tfrac{f_i'}{f_i} \partial_i \right] + V_r \tag{8.2.72}$$

separate multiplicatively

$$\bar{\Psi}(\lambda) = \prod_{i=1}^{n} \bar{\psi}_i(\lambda^i)$$

onto n one-dimensional eigenvalue problems of the form

$$[E_1(\lambda^i)^{\gamma_1} + E_2(\lambda^i)^{\gamma_2} + \cdots + E_n]\bar{\psi}_i(\lambda^i) \tag{8.2.73}$$
$$= -\frac{1}{2}\hbar^2 \left[f_i(\lambda^i) \frac{d^2}{(d\lambda^i)^2} + \frac{1}{2} \frac{df_i(\lambda^i)}{d\lambda^i} \frac{d}{d\lambda^i} \right] \bar{\psi}_i(\lambda^i) + \sigma_i(\lambda^i)\bar{\psi}_i(\lambda^i), \quad i = 1, \ldots, n.$$

Now, applying the results of Sect. 8.2.3, let us transform the eigenvalue problem to the Hilbert space with a measure generated by a natural metric of γ-class (4.4.32)

$$G^{ii} = A_1^{ii} = \frac{1}{\varphi(\lambda)} \frac{M_1^{(i)} f_i(\lambda^i)}{\Delta_i}. \tag{8.2.74}$$

According to (8.2.42) and (8.2.43) we have

$$\Psi = U^{-1}\bar{\Psi} \equiv R\bar{\Psi} = R \prod_{i=1}^{n} \bar{\psi}_i(\lambda^i), \tag{8.2.75}$$

where

$$U = \varphi^{\frac{1}{2}(\frac{n}{2}-1)} \prod_{k=1}^{n} \left(M_1^{(k)}\right)^{-\frac{1}{4}}. \tag{8.2.76}$$

Thus, like in (8.2.48)–(8.2.51), we get from (8.2.72) and (8.2.76) that

$$\hat{H}_r = U^{-1}\hat{\tilde{H}}_r U = -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} A_r^{ii}\left[\partial_i^2 + \left(2\tfrac{\partial_i U}{U} + \tfrac{1}{2}\tfrac{f_i'}{f_i}\right)\partial_i\right] + V_r + \hbar^2 W_r$$

$$= -\tfrac{1}{2}\hbar^2 \sum_{i=1}^{n} \nabla_i A_r^{ii} \nabla_i + V_r + \hbar^2 W_r,$$

where quantum correction terms are of the form

$$W_r = -\frac{1}{2}\sum_{i=1}^{n} A_r^{ii}\left(\frac{\partial_i^2 U}{U} + \frac{1}{2}\frac{f_i'}{f_i}\frac{\partial_i U}{U}\right) = -\frac{1}{2}\sum_{i=1}^{n}(S_\gamma^{-1})_r^i\left(f_i\frac{\partial_i^2 U}{U} + \frac{1}{2}f_i'\frac{\partial_i U}{U}\right)$$

and operators ∇_i are related to metric tensor (8.2.74).

If $W_1 = 0$, then we say that the stationary Schrödinger equation

$$-\tfrac{1}{2}\hbar^2 \Delta\Psi + (V - E_1)\Psi = 0 \tag{8.2.77}$$

is *R-separable*. It means that eigenfunctions are of the form (8.2.75), where $R(\lambda)$ is a known function and each $\bar{\psi}_i(\lambda^i)$ solves one-dimensional second order ODE ((8.2.73) in our case). Particular cases of Eq. (8.2.77) are the n-dimensional *Laplace equation* ($E_1 = 0$, $V = 0$) and n-dimensional *Helmholtz equation* ($V = 0$).

The notion of R-separability, understood as above, was introduced by Darboux in the second half of nineteenth century and then has been further completed in his monograph [78], where he classified all R-separable cases of the 3-dimensional Euclidean Laplace equations. Some classification results in the 4-dimensional Riemannian space can be found in [159] and [245] and other results on that subject together with particular examples in [63, 160, 161, 222, 244].

Here, according to classical definition, we would like to identify the n-dimensional Stäckel metrics (8.2.74) which lead to R-separation of the Schrödinger equation (8.2.77). First, let us notice that from the form of quantum corrections (8.2.6) and (8.2.76) follows nonexistence of nontrivial case with all $W_r = 0$. Thus, let us concentrate on R-separability of Eq. (8.2.77) and find for which Stäckel metrics (8.2.74) $W_1 = 0$. From the form of trivial potentials for a

given Stäckel matric (4.3.1) it follows that if

$$f_i \frac{\partial_i^2 U}{U} + \frac{1}{2} f_i' \frac{\partial_i U}{U} = \sum_{r=2}^{n} a_r(\lambda)(\lambda^i)^{\gamma_r}, \quad i = 1, \ldots, n,$$

then $W_1 = 0$.

Let us consider particular γ-classes given by

$$\gamma = (n + k - 1, n - 2, n - 3, \ldots, 0), \quad k \in \mathbb{N},$$

and Stäckel Hamiltonians defined by the separation curve of the form

$$H_1 \lambda^{n+k-1} + H_2 \lambda^{n-2} + \ldots + H_n = \tfrac{1}{2} f(\lambda)\mu^2 + \sigma(\lambda).$$

It means that in the notation of Sect. 4.4.3 $m_1 = 2, m_2 = 3, \ldots, m_k = k + 1$ and from (4.4.27) we find that

$$\bar{H}_1 = -\frac{1}{V_1^{(n+k-1)}} H_1$$

$$\Downarrow$$

$$G^{ii} = -\frac{1}{V_1^{(n+k-1)}} \frac{f_i(\lambda^i)}{\Delta_i}$$

$$\Downarrow$$

$$U = \left(-V_1^{(n+k-1)}\right)^{\frac{1}{2}\left(\frac{n}{2}-1\right)}$$

where $V_1^{(n+k-1)}$ is an appropriate potential (4.3.22) such that

$$V_1^{(n+k-1)} = V_1^{(n+k-1)}(\rho_1, \ldots, \rho_k), \quad \text{for } n \geq k$$

and ρ_i are the Viète polynomials (4.3.18). On the other hand, the Viète polynomials fulfil the following relations

$$\frac{\partial \rho_s}{\partial \lambda^i} = -\sum_{r=1}^{s} \rho_{s-r}(\lambda)(\lambda^i)^{r-1}, \quad i = 1, \ldots, n$$

so

$$\frac{\partial_i U}{U} = \sum_{r=0}^{k-1} b_r(\lambda)(\lambda^i)^r$$

and

$$\frac{\partial_i^2 U}{U} = \partial_i\left(\frac{\partial_i U}{U}\right) + \left(\frac{\partial_i U}{U}\right)^2 = \sum_{r=0}^{2(k-1)} c_r(\lambda)(\lambda^i)^r.$$

In consequence, if f_i is a polynomial in λ^i of order α, then

$$f_i\frac{\partial_i^2 U}{U} + \frac{1}{2}f_i'\frac{\partial_i U}{U} = \sum_{r=0}^{2(k-1)+\alpha} a_r(\lambda)(\lambda^i)^r, \quad i = 1,\ldots,n$$

and $W_1(\lambda) = 0$ for $n \geq 2k + \alpha$ where metric (8.2.6) is a conformal deformation of flat metric (8.2.6) from the Benenti class.

In order to illustrate our considerations, let us take the simplest case when $k = 1$ with the separation curve

$$H_1\lambda^n + H_2\lambda^{n-2} + \ldots + H_n = \tfrac{1}{2}f(\lambda)\mu^2 + \sigma(\lambda),$$

where $f(\lambda)$ is a polynomial of order α and σ is a rational function. Then, $V_1^{(n)} = \rho_1$ so the natural Stäckel metric takes the form

$$G^{ii} = A_1^{ii} = -\frac{1}{\rho_1(\lambda)}\frac{f(\lambda^i)}{\Delta_i} = \frac{1}{(\lambda^1 + \ldots + \lambda^n)}\frac{f(\lambda^i)}{\Delta_i}$$

and the stationary Schrödinger equation

$$-\tfrac{1}{2}\hbar^2 G^{ij}\nabla_i\nabla_j\Psi + (V - E_1)\Psi = 0$$

is R-separable for $n \geq 2 + \alpha$, where

$$\Psi = (\lambda^1 + \ldots + \lambda^n)^{\frac{1}{2}(1-\frac{n}{2})}\prod_{i=1}^{n}\psi(\lambda^i)$$

and $\psi(\lambda^i)$ are n copies of one dimensional ODE

$$[E_1\lambda^n + E_2\lambda^{n-2} + \cdots + E_n]\psi(\lambda) = -\frac{1}{2}\hbar^2\left[f(\lambda)\frac{d^2}{(d\lambda)^2} + \frac{1}{2}\frac{df(\lambda)}{d\lambda}\frac{d}{d\lambda}\right]\psi(\lambda) + \sigma(\lambda)\psi(\lambda).$$

In particular, when $\alpha = 0$, the nonzero quantum correction appears only for the last operator

$$W_n = -\frac{1}{8}\left(1 - \frac{n}{2}\right)\left(3 - \frac{n}{2}\right)(\lambda^1 + \ldots + \lambda^n)^{-2}.$$

Such a case for $n = 3$, $\alpha = 0$ and $V = 0$ was considered in [160].

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