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THE CONCEPTUAL FOUNDATIONS OF QUANTUM MECHANICS

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> *"There is no end to our searchings: . . . .*
> *No generous mind stops within itself. Its pursuits are without limit; its food is wonder, the chase, ambiguity."*
>
> MONTAIGNE

Preface

There are two complementary methods, broadly speaking, used by authors to communicate physics. In the first of these, the "formal method," the manipulatory skills relating to the subject matter of interest are emphasized. It is tacitly assumed that a mastery of the techniques required to solve the standard problems will lead automatically and rapidly to an understanding of the physical meanings of the techniques and their products. In the second, the "conceptual method," physical meanings are investigated carefully with little attention to technique. Once the basic concepts are clearly understood, or so it is assumed, manipulative skills will take care of themselves. Graduates of the formal method calculate easily (at least on textbook problems) but often they know not what they compute. The conceptual method, on the other hand, produces philosophical wranglers who can tear subtle ideas to shreds but who are unable, perhaps, to draw a fresh conclusion from even the most fruitful stock of ideas.

This book, as its title intimates, makes use of the conceptual method and shares in the weaknesses and, hopefully, also in some of the strengths of that method. The book is not (and this negation deserves emphasis) a text designed to teach the theory of quantum mechanics; at the end of the book we shall be only at the edges of the formal structure and explicit content of the theory. It is not possible to learn from this book how to solve even the simplest of the problems of quantum mechanics.

But considerable care and attention is given in what follows to an analysis of the physical meaning and conceptual consequences of the Heisenberg principle (Chapters 4, 5, and 6) and, in particular, to a close examination of the incompatibility of pairs of observables. In Chapters 7 and 8, the effects of the concept of incompatibility on the meanings of "measurement," "property," "state," "indeterminism," etc. are studied. An important step beyond what may reasonably be inferred from the Heisenberg principle is taken in Chapter 9, where the existence and great significance of "probability amplitudes" are discussed. But we refrain even after this fairly considerable preparation from going on to the implicitly promised land. The structure of quantum mechanics is not formulated here and its explicit content for classical observables is not given. To repeat what has already been emphasized, this book is not intended as an exposition of quantum mechanical theory.

To whom and for what, then, is the book of value? Two distinct groups may profit from it. Students who have been exposed to highly formal expositions and are only too painfully aware that they know not what they compute may find answers to some of their questions in the following pages. And those blocked from the normal paths which lead to the quantum mechanics by lack of mathematical skills may be able to get here at least a partial understanding of some of the significant ideas of a most important and fascinating theory.

This book is a revised and expanded version of the monograph I wrote while a member of the Conference on the New Instructional Materials in Physics. The Conference was held at Seattle in the summer of 1965 under the auspices of The Commission on College Physics and the University of Washington. The materials produced at the Conference were subsequently given limited publication by the University of Washington Press. I am indebted to The Commission on College Physics and the University of Washington Press for permission to make use of portions of the monograph in the preparation of this book.

It is a pleasure to acknowledge my extensive debts to Professor

Walter C. Michels. It was at his suggestion that I undertook to revise and expand my monograph for publication in the Momentum Series. I owe him thanks also for his careful and able editing of my manuscript.

I can best remember again the many who helped me at the Seattle Conference by quoting the last paragraph of the preface to my monograph. "I wish to thank Walter C. Michels and Ernest Henley for their helpful comments and criticisms. I am grateful also to Jack Ludwig and Ralph Caplan for showing me that it is easier to read English than Academese; the numerous changes they suggested greatly improved the style of the monograph. My thanks are due also to the officers of The Commission on College Physics and the University of Washington for their support and assistance during the pleasant, stimulating and productive months of the 'Writing Conference.' "

LEONARD EISENBUD

Contents

1 *The Failure of Classical Theory*

> *"For what, by nature and by training,*
> *We loved, has little strength remaining."*
>
> *W. H. Auden*

Between Newton's pioneering construction of classical mechanics in the seventeenth century and Maxwell's development of the electromagnetic field in the nineteenth century, classical physics – or perhaps better, deterministic physics – had met with no experiences that challenged its fundamental concepts. Naturally there were at the close of the classical period problems aplenty, but their solutions were seen as possible within the boundaries of existing theory. But between 1895 and 1925, penetration into the realm of microphysics – the physics of the "elementary" constituents of matter and radiation – uncovered numerous phenomena that stubbornly resisted interpretation within the classical conceptual framework. The conflict during this period between theory and observation rocked the very foundations on which classical theory had been built. It was found that to build a successful theory of microphysical processes – the quantum mechanics – it was necessary to eliminate or modify several classical assumptions that for centuries had been considered essential to any scientific theory.

The great difficulties that classical theory faced may be appreciated from the following brief review[1]. In classical mechanics some of the quantitative properties with which we normally deal (such as energy and angular momentum) are continuously variable. Experiments such as those of Franck and Hertz and of Stern and Gerlach indicate, however, that dynamical properties that are continuously variable in classical theory are sometimes limited to discrete sets of values, i.e., are "quantized," in microphysical systems. Thus the internal energy or the magnetic moment of a bound atomic system is not continuously variable but may take on only discrete sets of "possible" values. Similarly, the total energy in any sample of radiation of frequency ν is found to be an integral multiple of the "photon energy" $h\nu$. Such quantized dynamical properties find no comfortable place within classical theoretical structures.

Bohr attempted to graft quantization to classical physics by imposing restrictive principles and thereby to limit the possible motions of an atomic system to a subset of the motions permitted by classical theory. The early results achieved by the application of Bohr's ingenious quantization rules were encouraging. But the resulting theory was incomplete and internally inconsistent. Its successes were few and its failures glaringly evident.

If, for example, an atomic system has only a limited set of motions, and these have quite different energies, how does it change its energy from one to another permitted value in processes of emission or absorption of radiation – or in collisions with other material systems – without taking on intermediate values? Does the photon, in an emission process, simply spring into existence with a simultaneous and discontinuous change in the energy of an atom? If so, what determines when this disruptive event occurs? And so on. The marriage of classical mechanics to restrictive principles for the purpose of producing quantization was, after an all too brief honeymoon, forced, unstable, and quarrelsome.

Electromagnetism had its own troubles. Maxwell's theory provided no home for the photon; consequently, it could give no description of the phenomena associated with the photoelectric and Compton effects [1, 2, 3, 4, 5, 6, 7, 8, and Note *]. The predictions of electromagnetic theory were in conflict with Rutherford's model of the atom in which electrons are assumed to move in orbits about a central nucleus; radiation from the accelerated motion of an orbiting electron would drain energy rapidly from the atom and the electron would spiral down into the nucleus in a time of the order of 10^{-12} seconds! Classical theory could provide no explanation whatever of the remarkable stability of atomic properties. How, for example, is it possible for atoms to maintain fixed properties indefinitely, despite the numerous collisions they suffer as a result of thermal agitation?

Classical theory never quite learned to get along with either the quantization of radiation in photons, or the quantization of various properties of material systems, although it managed a somewhat uneasy relationship for a while. Toward the so-called wave-particle dualism, however, classical theory showed an enduring antipathy. Since early in the nineteenth century when the diffraction properties of radiation

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[1] It is assumed that the reader has studied the following topics at a level of treatment maintained in courses in general physics. The topics are listed in the order of their importance for the work of this monograph:

(a) Photoelectric and Compton effects; experiments and photon interpretation.
(b) Classical diffraction and interference phenomena; experiments and interpretation by Huygen's principle.
(c) Rutherford model of atoms and Bohr theory of the hydrogen atom (circular orbits).
(d) Electron beam experiments for the measurement of electron charge and mass.
(e) Franck-Hertz and Stern-Gerlach experiments; interpretation in terms of quantization.
(f) Elementary features of radioactive decay.
(g) Polarization of radiation.
(h) Blackbody radiation.

(Quantization and the constant h first appeared in physics in Planck's treatment of blackbody radiation. Much can be learned from an analysis of different physical interpretations of Planck's theory. Unfortunately, the background required for an understanding of such an analysis is likely not to be available to the students for which this monograph is designed.)

A list of references where treatments of the foregoing topics may be found are listed at the end of the book. Numbers appearing in brackets throughout this volume are intended to guide the reader to this list of references.

* See bottom of footnote 1.

were extensively observed, it was supposed that light (later recognized as electromagnetic radiation) was propagated as a transverse wave. The long argument between particle and wave theories which had gone on since Newton's time (Newton, it will be remembered, advocated a particle theory) seemed to be settled by the phenomena of diffraction and interference for which only the wave theory had a convincing explanation. Maxwell's highly successful electromagnetic theory of radiation served to deepen the belief that radiation must be regarded as a wave phenomenon. But it seemed to be impossible to explain the photoelectric and Compton effects by means of a wave theory. Indeed these effects appeared to be understood far more easily on the assumption that radiation consists of entities (photons) that are more like particles than waves.

When the study of the emission, absorption, and scattering of radiation by atoms and electrons made clear the significance of the photon characteristics of radiation, physicists were faced with a disturbing dilemma. If radiation consists of photons, how can diffraction be explained? If the diffraction phenomena indicate that light is a wave, how can the particlelike properties of radiation (photoelectric and Compton effects) be understood?

Concern with the issues raised by the existence of both the wavelike and particlelike properties of radiation deepened considerably when it was found, quite unexpectedly, that precisely the same issues arise in the analysis of electron behavior. From the time of its discovery at the end of the nineteenth century, the electron had been considered to be a very good approximation to a Newtonian point particle and many observations were, of course, consistent with this assumption. But when it was discovered that in the transmission of a beam of electrons through a crystal, diffraction patterns of the same character as those previously observed with X rays were produced, it became clear that electrons were not to be so sharply differentiated from radiant energy, as had previously been supposed [1, Chapter 5; 5, Chapter 6; 8, Chapter 4; 15, 16]. The need for a consistent theory that could comprehend the so-called dualistic (wave-particle) behavior of electrons and radiation became painfully obvious.

The diffraction of electrons and the particlelike properties of radiation indicated that far more than mere patchwork would be required to construct a consistent theory of microphysical phenomena. Fortunately, at almost the same time that the electron diffraction was observed, Heisenberg, Schrödinger, Dirac and others were groping toward the formulation of a remarkably successful theory – the quantum mechanics – to replace the now badly mauled classical theories of matter and radiation.

In the course of the construction of the new quantum mechanics it became clear that certain of the tacit assumptions of classical physics, though entirely consistent with experience in "macrophysics," are simply wrong in the microphysical realm. To obtain a consistent theory of microphysical systems it was necessary to make revolutionary changes in several of the assumptions on which the structure of classical theory is based. It is not now our object to recount yet again the many failures of classical theory but to describe the conceptual shift, the alteration in basic assumptions, that is required for an understanding of the behavior of the elementary constituents of matter and radiation[2].

[2] An epitome of the discussion to follow is presented in Section 9.1. The reader may find it helpful to use this epitome as a guide to the structure of the monograph.

2 Consequences of a Mistrust of Theory

"Prohibit sharply the rehearsed response."

W. H. Auden

The interpretation of experimental observations on microphysical systems depends heavily on theory. If we do not have an adequate theory – and classical theory is certainly not adequate in the realm of microphysics – such interpretations are necessarily suspect. Properties of microsystems are usually "observed" by interactions with complex macroscopic apparatus that produce scintillations on a screen, clicks in a counter, vapor droplets in a cloud chamber, readings on various kinds of meters, etc. The raw observations are interpreted by chains of theoretical argument; but if the applicability of the theory employed for analysis is uncertain, the interpretations are also uncertain.

Suppose, for example, that a certain sample of radiation, when passed through a diffraction grating, is deflected on passage by an angle θ. The observation of the angle of deflection is generally interpreted as a measurement of the wavelength of the radiation; the wavelength, λ, is

calculated [9, 10, 11] from

$$\lambda = d \sin \theta, \tag{2-1}$$

where d is the spacing between the lines of the grating. Also, since radiation in free space has the speed c (speed of light), the deflection is further interpreted as a measurement of the frequency, ν, by the use of

$$\nu = \frac{c}{\lambda}. \tag{2-2}$$

But the distance between crests of a waveform in space has not been measured and the number of oscillations per second of the wave field at a point has not been counted. Instead, a theory that the incident radiation is a wave phenomenon has been assumed; from the theory together with the measurement of the angle of deflection, θ, and the line separation, d, the wavelength (Eq. 2-1) and frequency (Eq. 2-2) of the radiation are deduced.

However, it is clear from the photoelectric and Compton effects, as well as from a variety of other observations, that photons are not correctly described by a wave theory. Since it is precisely the wave theory that permits us to infer a wavelength from a measurement of θ, its application in this instance is at least doubtful. But whatever it measures, the deflection through θ of the incident radiation tells us something quite definite about the radiation.

From a knowledge of θ and d, we can predict the energy of the electrons released by the photoelectric effect. Thus it is known from experiment that the energy transfer, ΔE, per photon, is

$$\Delta E = hc/d \sin \theta \tag{2-3}$$

(which is just $h\nu$, of course, with ν given by Eqs. 2-1 and 2-2). Also, the characteristics of the diffraction patterns produced by crystals, slits, etc., vary smoothly with the angle θ through which the radiation is deflected by the grating. The deflection θ tells us something very important about the radiation, but unless a correct theory justifies the association, we cannot assert that the deflection measures a wavelength, or a frequency.

Of course, it is possible that the wavelength and frequency as determined by Eqs. 2-1 and 2-2 have significance in relation to a theory which does correctly describe the behavior of photons. But at this prequantum mechanics stage of our discussion, in which we recognize the failure of the wave theory of radiation and have as yet no adequate theory with which to replace it, we are certainly free to question whether the numbers λ, ν, derived from θ by Eqs. 2-1 and 2-2 have anything whatever to do with the physical properties of radiation.

Consider a second example. Early this century, Rutherford recognized that the distribution of the charges within atoms could be explored ("observed") through the study of the scattering of energetic α particles by matter. If it is assumed that the forces that deflect an α particle when passing an atom are electrostatic, the character of the scattering by matter is determined by the arrangement of the charges within the atoms of which the scatterer is constructed.

Geiger and Marsden (in Rutherford's laboratory) investigated the general characteristics of α particle scattering by means of an arrangement such as that shown schematically in Fig. 2-1. The scatterers they used consisted of sheets of heavy metals which were thin enough to permit most of the incident α particles to go through the scatterer without sensible deflection. The observations consisted in counting at various angles relative to the direction of the incident beam the number of scintillations produced on a zinc sulfide screen by the arrival of scattered α particles. To Rutherford's considerable surprise, Geiger and Marsden discovered that a small, but far from negligible, fraction of the α particles were scattered through angles larger than 90°.

Analysis of the experiments indicated that these large angle scattering events could not have arisen from a number of successive small angle deflections; apparently an energetic α particle (with millions of electron volts of kinetic energy) could be deflected through a large angle as the result of a collision with a single atom of the scattering material.

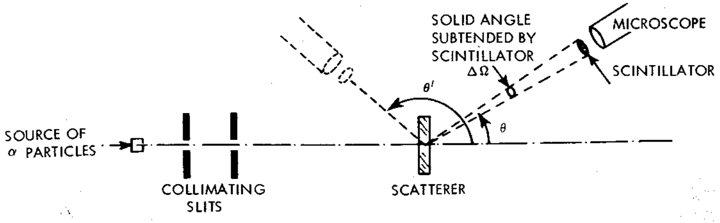

Fig. 2-1. Schematic of Rutherford Experiment.

Rutherford realized that the observations of Geiger and Marsden could not be explained by the atom models then under consideration. In 1911, he proposed a new model [12], the now well-known nuclear or planetary model, which provides a source for the strong forces required to produce the large deflections suffered by α particles in passing through matter. It was assumed that the positive charge required to balance the negative charges of the planetary electrons of the atom is concentrated in a nuclear core of diameter several orders of magnitude smaller than the diameter of the atom and that the nuclear core contains essentially the entire atomic mass. The electrons were assumed to move in orbits about the core under the influence of the attractive forces between the positive nuclear core and the negative electrons.

With this model, Rutherford calculated the form of the distribution of the scattered α particles that would result if real atoms corresponded to his model. Geiger and Marsden [13] undertook to check his predictions by a painstaking quantitative study of the scattering of α particles of various energies by sheets of different heavy metals. The agreement of experiment with theory left nothing to be desired. Effectively it had been "observed" that atoms have a structure like that of the model Rutherford created.

Despite the many difficulties posed by the model (e.g., the electrons must radiate energy in their motions about the nucleus, according to electromagnetic theory, and spiral very rapidly down into the nucleus), it was clearly necessary to take it seriously. Its acceptance was complete when the atomic theory of Bohr, which is based on Rutherford's model, proved so fruitful. (Bohr's first paper and the results of the work of Geiger and Marsden that verified Rutherford's predictions were both published in 1913.) The small, positively charged, massy nuclear core first imagined by Rutherford remains today at the center of theories of atomic structure.

Naturally, Rutherford assumed in his calculations that the motion resulting from a collision of an α particle with an atom could be described by Newtonian mechanics. His model specified the form of the force law to be used in the classical equations of motion. However, since experience in microphysics indicates that it is doubtful whether classical conceptions are applicable to atomic processes, the validity of Rutherford's "observations" of the nuclear core of the atom is open to serious question. Perhaps a model of the atom very different from that imagined by Rutherford together with a correct theory of microphysical processes would duplicate his predictions. Such coincidences are no doubt rare but they are not unknown. As it happens, the quantum mechanics – a successful theory of microphysics – together with the physical assumptions introduced by Rutherford leads to exactly the same scattering distributions as those obtained from classical theory[1]. We are being led away from the point, however, by the intrinsic interest of the story. For our present purposes it is essential to note only the strong interplay between theory and experiment in Rutherford's "observation" of the nuclear core of the atom.

The point needs no reiteration. Other examples abound. What we conclude is that in the realm of microphysics, where measurement is necessarily indirect, our concepts derive from a close interweaving of theory with observation. In this combination, the role of theory grows as investigation goes on. Ultimately, theory becomes so familiar that we hardly realize its importance in the interpretation of observation. The deflection of radiation by an angle, θ, on passing through a grating is almost automatically interpreted as a measurement of wavelength, and the rôle of theory in the measurement is almost forgotten. When theory fails, however, the familiar connections between its constructs and what is observed are broken. We must then return to naked observations and their observed interrelations, and try to build from them new and successful theoretical structures.

The task is enormously difficult. Thinking without preconceptions is probably impossible. Our very language has been conditioned by our experiences with macroscopic phenomena and, consequently, may be ill adapted to the needs of microphysics. The sheer task of describing raw observation is awkward and toilsome. Thus we say we are "observing" the diffraction of electrons when actually we are reading meters connected to certain pieces of physical apparatus and measuring the intensity pattern on a photographic plate. But the very use of the term "diffraction" commits us to thinking in terms of a wave theory and conjures up images not at all connected with what is observed.

This leads us into a bind: If we stick to the language of pure observation, discussion becomes ponderous and inordinately wearisome; if we proceed rapidly through the introduction of theoretical constructs, the risk of misconception and error is large. In spite of its dangers, though, the latter course must be chosen. We shall use, for example, derived quantities, such as the λ and ν in Eqs. 2-1 and 2-2, but try at the same time to keep in mind the experimental and mathematical procedures by which these quantities are obtained. One must try to avoid thinking of λ and ν as the wavelength and frequency of some physical wave. These symbols refer to quantities which are obtained from raw observations by easy computations; they are used because they help to express, in a simple way, observed relations among phenomena.

In our effort to shake loose from the grip of classical conceptions, we have probably overemphasized the extent of the failure of classical theory. Electromagnetic theory certainly correlates a vast set of observations on the properties of radiation. Many characteristics of atomic systems – e.g., the normal Zeeman effect – may be interpreted successfully by means of Newtonian mechanics. The Bohr theory of the hydrogen atom, a classical theory to which a restrictive rule is added, enjoyed many successes. Though many fundamental classical conceptions are inadequate for the needs of microphysics, it is beyond question that a correct theory of entities on the atomic scale (electrons, atoms, photons, etc.) must overlap classical descriptions in some way.

---

[1] This most happy coincidence between the results of quantum mechanics and classical mechanics for the scattering distribution occurs only for the inverse square law of force. For other force laws, the predictions of the two theories may differ considerably. It is amusing to speculate on how long the development of atomic physics might have been held up if the two theories had led to markedly different scattering distributions for the inverse square law of force.

3 Properties of Electrons, Photons; The De Broglie Relations

> *"Does he wear a turban, a fez, or a hat;*
> *Does he sleep on a mattress, a bed, or a mat?"*
>
> Edward Lear

It is not our purpose to review all the observations on microphysical entities in order to expose those interpretations that are tainted by unwarranted use of classical concepts; most of such a review would teach us little. Many conclusions about the microphysical realm are consistent with so great a variety of experiments, or are so largely independent of theoretical interpretation, that they are hardly subject to doubt.

It is obvious that the entities named electrons play an important role in atomic constitution. We know that the charge, e, and rest mass, m, of the electron [5,8] are

$$e \cong -4.8 \cdot 10^{-10}\,\text{esu} \cong -1.6 \cdot 10^{-19}\,\text{coulombs}$$

$$m \cong 9.1 \cdot 10^{-28}\,\text{gm}.$$

The energy, E, for free electrons is related to the momentum, p, by

$$E = p^2/2m \quad (E \ll mc^2\text{; nonrelativistic approximation}), \tag{3-1a}$$

$$E = c[p^2 + (mc)^2]^{1/2} \quad \text{(relativistic)}. \tag{3-1b}$$

The velocity, v, of free electrons is found to be related to the momentum as in classical mechanics:

$$v = p/m \quad (v \ll c) \tag{3-2}$$

$$v/(1 - v^2/c^2)^{1/2} = p/m \quad \text{(relativistic)}$$

Electron beams are deflected under the influence of laboratory-produced electric and magnetic fields[1]. The deflections may be computed using Newton's second law $\mathbf{F} = d\mathbf{p}/dt$ with

$$\mathbf{F} = e\left[\mathbf{E} + \frac{\mathbf{v} \times \mathbf{H}}{c}\right] \tag{3-3}$$

where **E**, **H** are the electric and magnetic fields at the location of the electron.

Photons may be interpreted as the elementary constituents of radiant energy; they are uncharged and are transmitted in free space with the velocity of light. On interaction with material systems (electrons, atoms, etc.), they transfer specific quantities of momentum and energy. The relation between these quantities is found to be

$$E = cp. \tag{3-4}$$

This relation follows from Eq. 3-1b if we set $m = 0$; consequently the photon is said to be an entity with zero rest mass.

The possible energy values for photons range over an infinite continuum. It is found that photons to which one may assign a "frequency," ν, have a *uniquely correlated energy* given by

$$E = h\nu \tag{3-5a}$$

---

[1] The word "laboratory" here means that the experiments are conducted on a macroscopic scale with macroscopic controls. The Newtonian concept of force loses its meaning for electron "motions" on a microscopic scale (see Chapter 6).

(where h, of course, is Planck's constant). By the use of Eq. 3-4 and the relation $\lambda\nu = c$, we have

$$p = E/c = h\nu/c = h/\lambda.^{2}$$

Photons exhibit properties associated with the term "polarization."[3] The angular momentum carried by circularly polarized photons has been measured and found to be $h/2\pi$.

It is important to recognize that a photon of "frequency" ν is as indecomposable as an electron. Just as we never find parts of an electron, we never find photons of a given frequency (i.e., photons which are deflected by the same angle θ in a given grating) with a fraction of the energy $h\nu$. All efforts to cut a photon into parts each with the same "frequency" but with fractions of the energy $h\nu$ have failed. Like electrons, photons may suffer energy changes in collision processes (e.g., the Compton effect), but in such changes the "frequency" is also altered.

Most of the properties we have discussed for electrons and photons enter into *conservation laws*. A multitude of observations are consistent with the assumption that the general conservation laws – for energy, momentum, angular momentum, charge – are valid in the microphysical realm. These conservation laws make it possible to give clear meanings to the measurements of conserved quantities, meanings which are independent of the nature of more detailed theory.

3.1 The de Broglie Relations. Both electrons and photons (indeed all "particles") produce "diffraction effects" under suitably arranged conditions. The diffraction effects are similar in some respects to those produced by classical waves and may be described, in part, by a classical wave theory. An examination of the diffraction effects indicates that the "wavelength" which must be employed by the classical wave theory to describe the observations is related to the observed momentum of the incident beam of electrons or photons by

$$\lambda = h/p \qquad (3\text{-}6a)$$

We shall have occasion to associate a "frequency," ν, as well as a wavelength, λ, with electrons of momentum p; this "frequency" (which will not be interpreted as the number of oscillations per second of a physical wave) may be related to the electron energy by

$$\nu = E/h \qquad (3\text{-}6b)$$

Thus the relations 3-6a, 3-6b between E, p, and ν, λ hold for photons and for electrons.

The equations

$$\lambda = h/p \qquad \nu = E/h, \qquad (3\text{-}6c)$$

which connect the "particle properties" p, E,[4] with the "wave properties" λ, ν, are generally known as the "de Broglie relations."

Some years before the observation of electron diffraction, de Broglie, in his doctoral thesis, suggested that the properties of electrons might be understood better on the assumption that the electron constituted a wave phenomenon of some kind [14]. He assumed that the wavelength and frequency of his "electron waves" are related to the energy and momentum of electrons in exactly the same way as the wavelength and frequency of photons are related to their momentum and energy. De Broglie published his theory in 1924. His conceptions were considered rather fanciful until, about a year later, Schrödinger took them up and extended them into his system of "Wave Mechanics." It was not until 1927 that the phenomenon of electron "diffraction" was observed and the relation $\lambda = h/p$, first suggested for electrons by de Broglie, was confirmed [15, 16].

That the de Broglie relations hold for such dissimilar entities as electrons and photons is most remarkable. These relations express compactly the connection between the wavelike properties associated with λ, ν, and the particlelike properties associated with E, p. Note in particular that the connection involves the new constant h. The magnitude of h determines the conditions under which the wavelike properties of electrons will be important (see Chapters 5 and 6).

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[2] Properties of energy and momentum are attributed to classical electromagnetic fields. However, the energy and momentum are interpreted as spread continuously over the field. One speaks, in the theory, of energy, momentum *densities*, and of the *flow* of energy, momentum across surfaces. In the interaction with matter of a wave of frequency ν (and this quantity has an unambiguous meaning in the theory), energy and momentum are *continuously* transferred rather than in the lumps $h\nu$, h/λ as in the photon picture.

[3] Electrons also have properties analogous to the polarization properties of photons; these properties are referred to by the name "spin." We do not wish now to enter into a discussion of the spin properties of electrons.

[4] See footnote 2, this chapter.

The measured value of h is

$$h = 6.62 \cdot 10^{-27} \text{ erg} \cdot \text{sec}$$
$$= 4.14 \cdot 10^{-15} \text{ eV} \cdot \text{sec.}$$

We find from Eq. 3-6c, on setting $p = \sqrt{2mE}$, (for electrons with nonrelativistic energies) that

$$\lambda \cong 12.3\, E^{-\frac{1}{2}} \quad [\text{Å}(\text{eV})^{\frac{1}{2}}] \tag{3-7}$$

(Å = angstrom; eV = electron volts). Thus, for scattering centers with separations of the order of an angstrom – such spacings occur in crystals – diffraction effects are large for the full range of electron energies below 10^5 eV. For electron energies of the order of 10^6 eV, however, the wavelength is approximately 0.01Å; hence diffraction effects about objects of atomic dimensions (of the order of an angstrom) will be small and the electrons will appear to behave like particles. For photons the relation $E = h\nu = hc/\lambda$ leads to

$$E \cong 1.2 \cdot 10^4 \lambda^{-1} (\text{eV Å}). \tag{3-8}$$

For visible radiation with $\lambda \cong 5000$ angstroms, the photon energy is roughly 2 eV. An energy transfer of this magnitude has significant effects for atoms; consequently, the particlelike nature of visible radiation is important in atomic processes. For smaller wavelengths – X rays, γ rays – the particlelike nature of photons becomes more pronounced. If we consider, however, radiation in the microwave region with $\lambda \sim 10^8$ angstroms (= 1 cm), the photon energy is only 10^{-4} eV. In the normal processes of transmission and reception of such radiation the photon character of the radiation would be difficult to observe[5].

---

[5] It is worth recalling that it is not always an easy matter to determine whether a given process which transmits energy is more suitably described as a stream of particles or as a wave process. The debate that began in Newton's time over the character of light went on for more than a century. Newton, on failing to detect diffraction effects, favored a particle theory of light and his hunch was backed by most physicists of his time. The issue appeared to be settled against Newton's view early in the nineteenth century by the work of Young and Fresnel on diffraction. However, after the photon characteristics of radiation were discovered, it was recognized that the nature of light was not yet fully understood.

4 An Analysis of Electron Diffraction

"But whether that be true or no
The Devil any of you know."

Samuel Butler

Perhaps the most surprising of all the strange phenomena that have been observed in the microphysical realm are those associated with the so-called wave-particle dualism. It is in this area that classical concepts fail most clearly. A detailed analysis of certain diffraction and interference phenomena will clarify and isolate this failure.

A typical experiment on the diffraction of electrons [8, 15, 16, 17, 18] is arranged as shown in Fig. 4-1. An electron source – say a hot filament similar to that used in radio tubes – is put behind a pair of plates in which there are circular holes with diameter of the order of millimeters (a macroscopic dimension). The plates are separated by about 10 cm; between the filament and the first plate a potential difference of the order of several hundred volts is maintained. Some of the electrons that are "boiled" out of the heated filament find their way

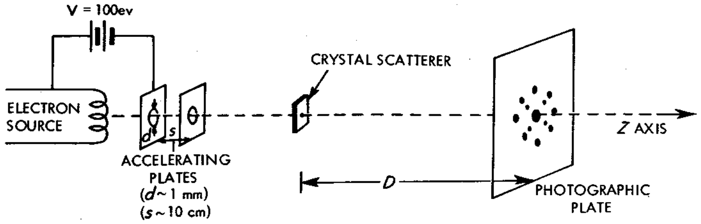

Fig. 4-1. Schematic of electron diffraction experiment.

through the hole in the first plate; in transmission from the filament to the plate the electrons are subject to a fairly uniform accelerating field and gain an energy of say 100 electron volts. The electrons that get through the hole in the second plate constitute a well-collimated beam. The number of electrons per second entering this beam may be controlled by a variety of devices.

The energy distribution and the geometric characteristics of the beam of electrons that emerge from the accelerating system may be directly tested. Suppose that the experimental arrangement determines the energy to an accuracy of 1%. Since $p = \sqrt{2mE}$, the momentum magnitude is accurate to about 0.5%. The size of a cross section of the beam may be examined at various distances from the exit hole by examining the effects of the beam on a photographic plate. This cross section is found to vary with distance (to within the accuracy of the observation) as one would expect on the assumption that electrons move on straight line paths (Fig. 4-2). It is easily seen that the angle, δ (in radians), of the conical region in which the electrons are found is given approximately by the ratio, d/s, of the hole diameter, d, to the plate separation, s. For the demonsions, $d \sim 1$ mm, $s \sim 10$ cm, this angle is 10^{-3} radian. Thus the transverse component (i.e., the component in the plane perpendicular to the direction of the electron beam) of the momentum of an electron is less than about $10^{-3}\, p_0$, where $p_0 \cong \sqrt{2mE}$ is the average value of the component of the momentum along the direction of the beam (Fig. 4-3).

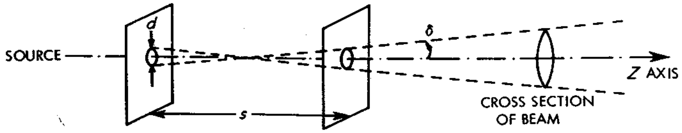

Fig. 4-2. Angular spread of electron beam.

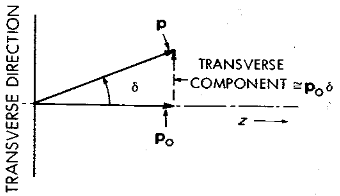

Fig. 4-3. Transverse component of electron momentum.

On the axis of the electron beam produced as described above, a thin section of a crystal is placed. A photographic plate is put at a convenient distance, D, from the crystal along the direction of the electron beam; D may be of the order of 10 cm. When the filament is hot, electrons emerge from the accelerating system, strike the crystal and suffer deflections. The photographic plate serves as a detector for the location of the electrons in a plane transverse to the beam. A photograph of the pattern that appears on the plate in an experiment of this sort (if a powder target rather than a single crystal is used) is shown in Fig. 4-4.

The character of the pattern produced on the photographic plate is independent of the intensity of the incident electron beam; if the number of electrons per second is cut by a fraction f and the time of the exposure of the photographic plate is increased by the factor f^{-1} (so that the total number of electrons recorded on the plate is kept constant), the picture produced is unchanged. The same general pattern is found on the photographic plate if a beam of X rays (photons) of suitable wavelength is employed in place of the electron beam (Fig. 4-5). The form of the observed pattern depends on the structure of the crystal scatterer employed in the experiment.

If the arrangement and spacings of the atoms within the crystal are known, the pattern observed on the photographic plate can be calculated by assuming that the incident beam may be replaced by a

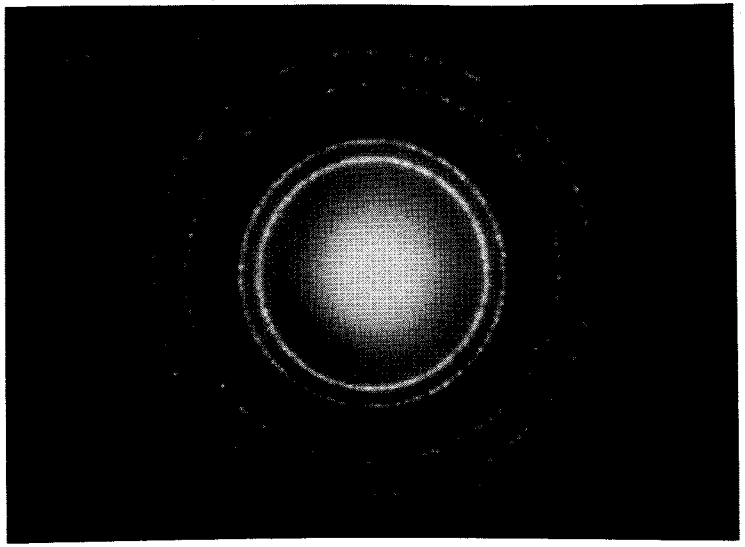

Fig. 4-4. Photograph of the diffraction pattern produced by passing a beam of electrons through a sample of aluminum powder. The electron wavelength employed was 0.15 Å. (*Courtesy Film Studio, Education Development Center, Inc.*)

plane wave propagating in the direction of the beam with a wavelength given by the de Broglie relation, $h/p_0 \cong h/(2mE)^{1/2}$ where E is the incident energy of the electrons. In the calculation, one simply assumes that each atomic site is the source of a secondary wave that emerges radially from the atom (Fig. 4-6). It is found that constructive interference of the waves from the numerous atomic sites occurs only at the positions at which the darkening of the plate is observed.

We have described the electron diffraction experiment essentially as it was first performed by Davisson and Germer and by Thomson. For the purposes of the analysis to follow we shall consider the experiment in an idealized form. The incident beam in this experiment is assumed to have well-defined energy and momentum and a cross section such as shown in Fig. 4-7. We replace the crystal by a plate on which narrow

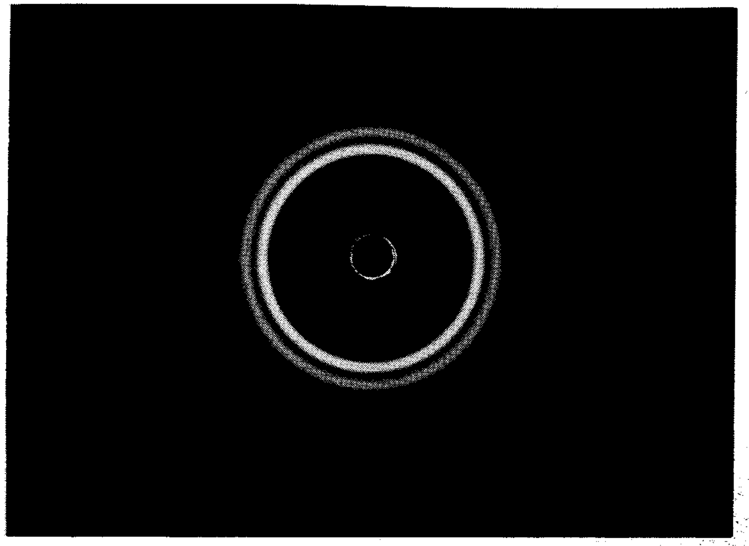

Fig. 4-5. Photograph of the diffraction pattern produced by passing a beam of X-rays ($\lambda = 0.71$ Å) through a sample of aluminum powder. (*Courtesy Film Studio, Education Development Center, Inc.*)

slits are cut out. The geometry of these slits and the arrangement of the detector (a photographic plate and a sheet of scintillating material or an electron counter) is shown in Fig. 4-7. An experiment with an arrangement of this kind, in which laboratory manufactured slits were used, was performed relatively recently by Jönsson [19], who used slits with a width $\tau \cong 0.2$ micron and spacing $\Delta \cong 1.5$ microns. His photographs of the diffraction patterns produced by a single slit and by a double slit are shown in Figs. 4-8 and 4-9. The observed patterns are similar to the patterns obtained by the diffraction of light [9, 10, 11].

The observations are deducible from a classical wave theory in which the wavelength employed is $\lambda = h/p$, where p is the momentum of the incident electrons. If a single slit rather than a double slit is used, the first minimum occurs at a position P (see Fig. 4-10) such that the distances from the two edges of the slit to P differ by a wavelength. It is

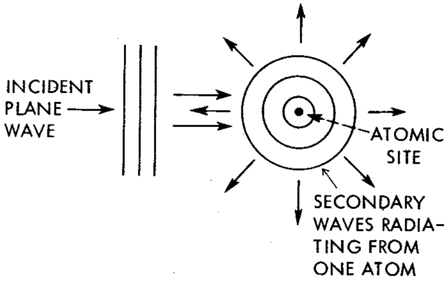

Fig. 4-6. Scattering of an incident wave from an atomic site.

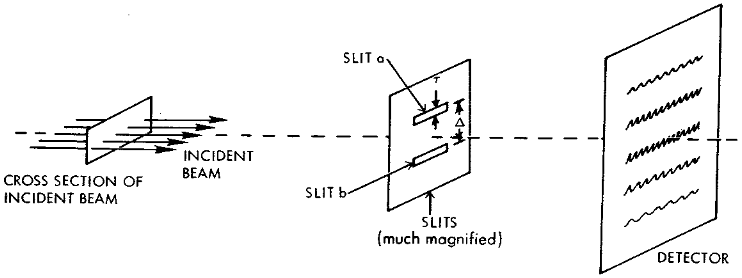

Fig. 4-7. Idealized diffraction experiment.

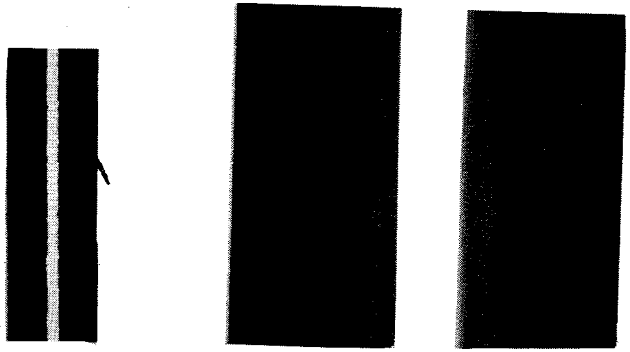

Fig. 4-8. Photographs of double slit and resulting diffraction pattern in Jönsson's experiment. From Jönsson, *Zeitschrift für Physik,* 161, 454, 1961.

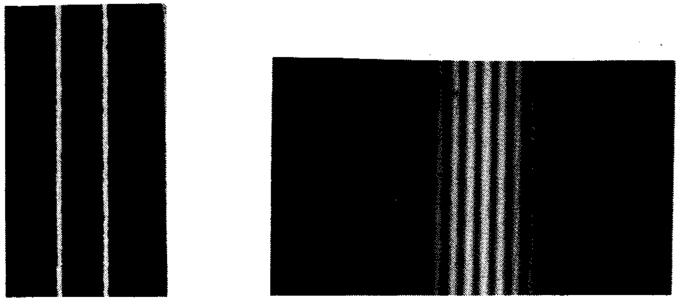

Fig. 4-9. Photographs of double slit and resulting diffraction pattern in Jönsson's experiment. Also from Jönsson, *Zeitschrift für Physik,* 161, 454, 1961.

easily seen from the figure that $\theta \approx \lambda/\tau$ (for small θ) so that the distance OP $\approx D(\lambda/\tau)$. (The general structure of the intensity pattern predicted by the wave theory is shown in Fig. 4-10.) When two slits are used in the experiment, a succession of maxima and minima are produced as a result of the interference of the waves emerging from the separate slits. If the slit widths are small compared with the separation,

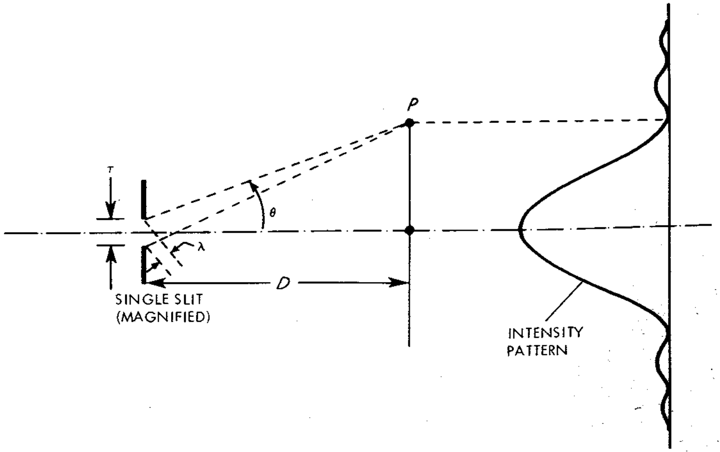

Fig. 4-10. Single-slit diffraction.

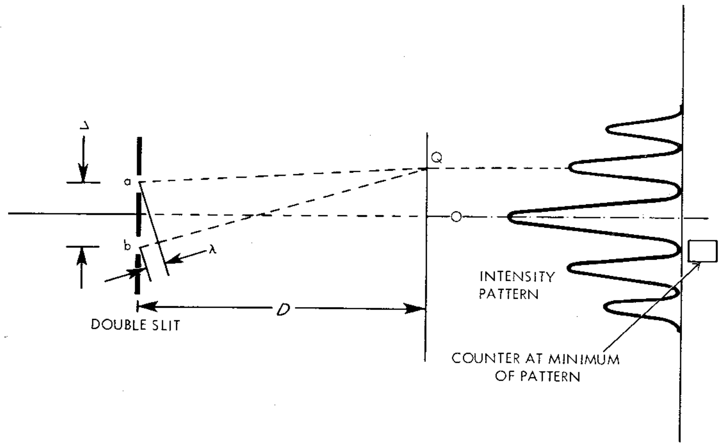

Fig. 4-11. Double-slit interference.

Δ, the slits may be treated as line sources. The position of the first maximum of the intensity (off the central maximum) is at a point Q (Fig. 4-11); the distances from the two slits to Q differ by a wavelength. Thus OQ $\approx D(\lambda/\Delta)$. If Δ is of the order of 5τ, several maxima and minima are obtained in the two-slit pattern covering the region of the central maximum obtained with a single slit.

Remember that the structure of the patterns produced in these experiments is independent of the intensity of the incident beam. Therefore, one may use a beam so weak that the probability of an electron's being within a centimeter of the slit at any instant is 10^{-3} or less. In other words, the diffraction experiments may be performed under conditions such that there is a negligibly small probability that more than one electron at a time will be near the slits.

So far there seems to be no argument against considering the electron as a manifestation of some sort of wave motion. However, the wave model breaks down completely as soon as we consider what happens at the detector. If a scintillating screen is used instead of a photographic plate, each electron produces at the screen a well-localized scintillation[1] while successive electrons appear to fall more or less at random on the screen. Thus an electron does not produce a diffraction pattern; the patterns are formed from the distribution on the detector of a large number of electrons. The intensity at some position on the photographic plate is proportional to the number of electrons that are deflected into this position. The diffraction pattern, in short, is not the property of one electron but of an ensemble of similarly prepared electrons. Classical wave theory does not predict this feature of our observations.

When we take into account the indivisibility of an electron, the structure of the two-slit pattern presents a paradox. Since the electron cannot be divided, it presumably must pass through one or the other of the two slits. Surely the open character of the slit through which the electron does not pass cannot affect the path the electron takes in getting to the detector[2]. We conclude, therefore, that the two-slit pattern should be exactly the same as that which would be produced if we exposed a photographic plate for a time *T* with only the "a" slit open and then, for the same period, with only the "b" slit open.

The pattern observed with both slits open, however, is very different from that obtained by superposing two single-slit patterns (see Fig. 4-12). The difference between the result predicted (on the assumption that each electron goes through one or the other of the slits) and the pattern observed may be shown rather dramatically by putting a counter at the first minimum of the two-slit pattern (Fig. 4-11). With both slits open, few counts per second are recorded; if one of the slits is closed, however, the counting rate increases! On the basis of the foregoing argument, the closing of a slit could never increase the counting rate for any position of the counter. Apparently something very much like an interference effect occurs; to obtain interference we must have influences simultaneously from both slits; i.e., the electron must somehow get through both slits. But presumably the electron is an indivisible entity and cannot go through two separated slits. The paradox appears to be unshakable.

It could be considered ridiculous simply to guess at what is going on as the electron passes a double slit. Under the conditions of the

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[1] Recall that in an observation of an electron we always find a whole electron or none at all; a fraction of the electron charge or mass is never observed.

[2] The diffraction pattern is wholly independent of the nature of the "opaque" materials which form the slit, or of the distribution of the materials in the opaque regions.

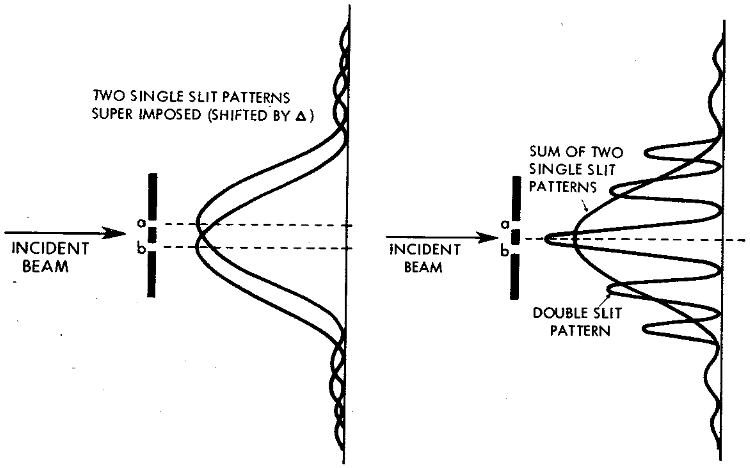

Fig. 4-12. Comparison of sum of two single-slit patterns with double-slit pattern.

diffraction experiment, different electrons reach the detecting screen at very different points. We have as yet no idea of what determines where any particular electron that gets through the slits will be found at the detector. Why don't we arrange to observe the passage of the electron? By suitable observations we could determine definitely whether an electron can or cannot somehow get through both slits and how the electrons which enter the slits along different paths are deflected.

Sadly, it is impossible to observe the precise path of an electron[3] (as Chapter 6 will show). Suppose, however, we try for the moment something a little less ambitious. In the experimental arrangement of Fig. 4-7, an incident beam with a width, t, of the order of a millimeter is used whereas the slit width, τ, and the slit separation, d, are orders of magnitude smaller; the incident beam simply blankets the slits. Why not use a beam narrow by comparison with the width τ and arrange to move it across the slits? We could then have electrons pass an edge or

the middle of one of the slits and see where such well-aimed electrons reach the detector.

No doubt the reader has already noticed the errors on which the foregoing suggestion is based. If we try to aim the electrons accurately by the use of narrow slits, new diffraction effects are introduced. A pair of very narrow aligned slits does not determine a correspondingly narrow beam for the electrons. If the openings in the accelerating plates are narrow and a detecting screen is placed at the position of the diffraction slits, we do not find a geometric image of the slit on the second accelerator plate just as we do not find geometric images of the slits a, b in our diffraction experiment. The diffraction phenomenon severely limits our ability to aim electrons.

Indeed we employ openings in the accelerator plates with relatively large dimensions ($\approx$ 1 mm) to avoid introducing significant diffraction effects in the incident beam. For electrons with energies of the order of volts and a slit width of about a millimeter, the angle through which the emerging electrons are spread by diffraction effects is about 10^{-6} radian ($\cong \lambda/\tau$) (Fig. 4-7), a small angle compared with the spread, δ, (see Fig. 4-2) arising from the geometry of the slits.

Let us consider a different experiment. We noted that the interference phenomenon could not be understood at all if it was assumed that the electron passed either slit a or slit b, but never both. The issue raised by this consideration may be investigated. Immediately behind slits a and b, place detectors A, B capable of "observing" the passage of an electron. The detectors must be such that if slit a is closed detector A never responds, while if a is open and b closed each electron that passes is detected. Obviously the same must be true if we interchange a, A with b, B in the foregoing. (An example of a pair of detectors with these properties is considered in Section 5.1. See, in particular, Fig. 5-5.)

We assume again that the incident beam is so weak that two electrons are never simultaneously in the immediate neighborhood of the slits. If in the course of the passage of the electron beam, the A, B detectors never (or rarely) respond "simultaneously" – i.e., within an interval short by comparison with the average interval between the arrival of successive electrons in the beam – it must be concluded that each electron either gets through a or through b. Simultaneous passage

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[3] Approximate electron paths are observed in cloud and bubble chamber pictures. The limitations on the observability of electron path is explained below and in Chapter 6.

of an electron through both slits would be indicated by a response of both A and B to the passage of a single electron.

This experiment – the diffraction experiment with the A, B detectors – is possible in principle, but forbiddingly difficult in practice. It has never been performed. However, a wide range of experience indicates that an electron is never detected simultaneously at two separate positions. Suppose then that in this "Gedanken experiment" (or pencil-and-paper experiment) the detectors A, B never respond simultaneously; i.e., the electrons are indeed found either behind a or behind b. Under these circumstances, surely the pattern observed on the photographic plate must consist of the simple superposition of two single-slit patterns.

Fortunately for our peace of mind, there is good reason to believe that in the experiment using detectors, the pattern that would be observed (if the experiment could be done) is a superposition of two single slit patterns. The process of observing position – as the next chapter will show – cannot be performed without influencing the electron. The experiment with the detectors is physically different from the experiment without them. The analysis in Section 5.1 will show that if detectors are arranged to change the electron's properties as little as is possible (consistent with the requirement that a response of A cannot occur if slit a is closed), the pattern produced at the photographic plate becomes roughly a simple superposition of two single-slit patterns. In the experiment with detectors, then, the results are consistent with expectation.

Notice that the observation of the electron passage does not help us to understand the normal two-slit pattern. The presence of the detectors radically alters the distribution produced on the photographic plate. To understand the ordinary two-slit diffraction we are forced to the peculiar assumption that, in the normal experiment, the electrons somehow are influenced by both slits (i.e., get through both slits) despite the fact that a determination of location always discovers the electron at one or the other of the two slits. Reasons for believing that this idea constitutes more than a simple confession of complete defeat will be advanced in Chapters 6 and 7.

Effects similar to those described above arise frequently in the microphysical realm. With a certain experimental arrangement an interesting phenomenon is observed. If, however, we attempt to

examine the processes that give rise to the phenomenon of interest, we discover that the examination alters the previous observations radically[4]. Nature seems determined to prevent us from discovering some of her secrets.

We do not appear to have come near our destination – an understanding of the two-slit interference experiment – but perhaps the scenery along the way has been interesting. Note in particular the following two very remarkable features of the diffraction phenomena:

1. The diffraction pattern is not a property of a single electron but rather the property of a large collection (ensemble) of electrons.
2. The laws of nature conspire to prevent us from examining the details of the processes that occur in the diffraction experiment. We cannot arrange, for example, both to produce the two-slit pattern and to know with certainty how each electron gets through the slits.

More can be learned from an examination of the one-slit diffraction pattern (Fig. 4-10). The component of momentum along the incident direction of the electron is not altered in its passage through the rigid slit. By using the arrangement pictured in Fig. 4-13, we could prove that the electrons which reach the detector are deflected on passing the diffracting slit by some angle θ (of course, the openings in the collimating plates must be large – say about a millimeter wide – so as not to produce further diffraction). The y component of the momentum after deflection by θ (Fig. 4-14) is $p_0 \tan\theta \gtrsim p_0 \sin\theta$, where p_0 is the incident electron momentum. From previous work (see Fig. 4-10) we know that the bulk of the deflections experienced by an ensemble of electrons in passing the slit lie between the limits, $\pm\theta'$, where θ' is given by

$$\sin\theta' \cong \lambda/\tau$$

($\lambda = h/p_0$ and τ is the slit width). Hence the y components of the momenta of the electrons after passage range roughly between $\pm p_0 \tan \theta'$; also $p_0 \tan\theta' \gtrsim p_0 \sin\theta' = p_0 (\lambda/\tau)$.

Now we do not know and cannot follow in detail (for the same reasons that prevented our closer examination of the formation of the two-slit pattern) how any one electron will be deflected in passing the slit. The foregoing analysis shows that after passage of an electron we

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[4] Students should not be surprised by such effects. They often complain (perhaps justly) that frequent tests of their knowledge tend to destroy the knowledge they may have acquired.

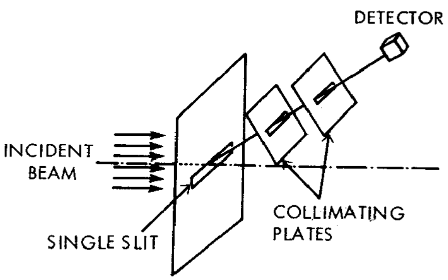

Fig. 4-13. Change of electron momentum on passing slit.

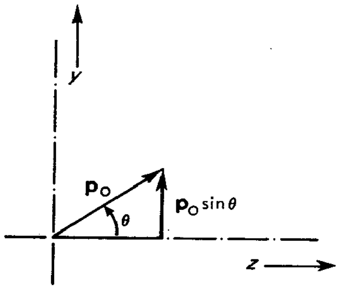

Fig. 4-14. The y-component of momentum after passing slit.

can only assert that the y component of momentum is found somewhere within the range $\pm p_0 \lambda/\tau$ which, since $\lambda = h/p_0$, is $\pm h/\tau$. Thus we cannot predict with precision what the value of the measured y component of an electron which gets through the slit will be. We shall indicate the uncertainty[5] or dispersion of the y component of the

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[5] The "uncertainty" is defined as the root mean square of the deviation from the mean value. If the distribution of the y component of the momentum for an ensemble of electrons diffracted by the slit is $w(p_y)$, i.e., if $w(p_y)dp_y$ is the probability of finding the component p_y in the interval dp_y, then the uncertainty is

$$\Delta p_y = [\int w(p_y)(p_y - \bar{p}_y)^2 dp_y]^{1/2}$$

where $\bar{p}_y$ is the mean value of p_y in the distribution. In the case we are considering, $\bar{p}_y = 0$.

momentum by Δp_y; its value is roughly $\gtrsim h/\tau$. Note that τ also measures an uncertainty. We do not know where a single electron will come through the slit but obviously it must be found on observation at the plane of the slit somewhere within the width τ; τ then roughly determines an uncertainty in the y component of position of an electron in passage through the slit. Setting $\tau \cong \Delta y$, the uncertainty in the y component of position, we have $\Delta p_y \gtrsim h/\Delta y$, or

$$\Delta y \Delta p_y \gtrsim h. \tag{4-1}$$

This result may be summarized as follows: If an electron of known momentum passes a slit of width Δy, we can only predict the y component of the momentum after passage to within an uncertainty Δp_y, where Δy and Δp_y are related by (4-1). For an electron with an energy of the order of 100 eV ($2\lambda \approx 1$ Å), the uncertainty Δp_y introduced in passage through a slit of macroscopic size (say $\Delta y \approx$ 1 mm) is negligible by comparison with the momentum $p_0 = \sqrt{2mE}$; we have $\Delta p_y/p_0 \cong h/p_0\Delta y = \lambda/\Delta y \approx 10^{-8}/10^{-1} = 10^{-7}$. But if Δy is of the order of angstroms, the uncertainty Δp_y is of the same order as p_0 itself!

Eq. 4-1 suggests that if we arrange to determine a component of the position of an electron, we can do so only at the expense of our prior knowledge of the corresponding component of the momentum. If this implication of our work is generally true it is of enormous significance. To make a prediction in classical mechanics we must be given "initial conditions"; for a single particle these conditions are the position and momentum. If we cannot know both the position and momentum of the particle our mechanics loses its power to predict. Now Eq. 4-1 follows essentially from the de Broglie relation, $\lambda = h/p$, and the meaning of λ in relation to diffraction effects. Do the de Broglie relations imply basic limitations on our capacity to determine the position and momentum of an electron simultaneously? This question is examined in the next chapter.

5 Heisenberg's Principle of Indeterminacy

"O, swear not by the moon, the inconstant moon."

William Shakespeare

We want to measure both the position and momentum of an electron. Let's try to make the problem as simple as we can. Suppose that the momentum has already been measured and that after the measurement the electron is "free" so that the electron momentum does not change with time. The assumption of momentum conservation is surely as safe as any that can be made in physics. If the measured component along the x axis has the value p_x, then a subsequent measurement of the x component of momentum is certain to yield the value p_x. All that now remains is to measure the position by procedures that do not change the measured momentum or, if the position measurement does alter the momentum, to measure the position in such a way that both the position and the momentum are known after the measurement.

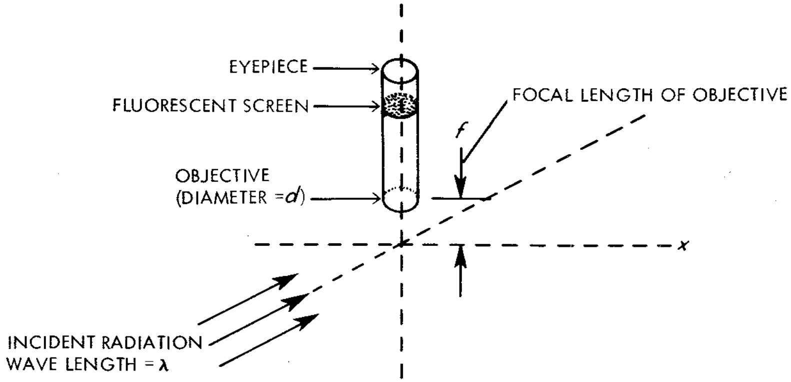

Fig. 5-1. Position measurement with a microscope.

To make the position measurement we shall use a microscope. The stage of the microscope is illuminated with radiation of some definite wavelength, λ, directed along an axis perpendicular to the axis of the microscope (Fig. 5-1). A fluorescent screen that scintillates when a photon of wavelength λ falls upon it is placed in the microscope so that an image of an object illuminated by the radiation is formed on the screen. We observe the fluorescent screen through the eyepiece.

If we are lucky we discover, after a period of watching, a scintillation at some point P on the screen. This effect results from the scattering of a photon in the incident radiation by the electron into the microscope. From the observed location of the scintillation we must determine the location of the electron at the time the radiation was scattered. Of course, the scattering event will change the previously measured electron momentum so that we must also determine, if possible, the new value of the momentum of the electron after the measurement.

As is known, a point source of radiation does not produce a point image in a microscope [10, Chapter 4]. Diffraction effects lead to an image that is spread over a small circular region. The radius of this circular image depends on the diameter, d, of the objective lens (see Fig. 5-2) and the wavelength of the radiation from the point source. The first minimum of the diffraction pattern formed on the image plane for a point source of radiation occurs at a distance r from the center of the pattern. Roughly speaking, this minimum is formed where the paths of the radiation that pass through opposite ends of a diameter of the objective (see the dotted paths in Fig. 5-2) differ by a wavelength.

It is not difficult to show [10, Chapter 4] that the angle subtended by the image area of radius r at the objective lens is given by

$$\delta \cong \lambda/d, \tag{5-1}$$

where d is the diameter of the objective lens. It must be realized that the extended image of a point source is produced by a large number of photons striking the image plane at well-defined points (or, more exactly, regions small compared with the area of the image). In the observation of a single scintillation, a single photon is detected at, say, P. However, photons scattered from a range of different locations may also give rise to scintillations at P. Thus (see Fig. 5-3), the scintillation at P might have arisen from a photon emitted from sources at S_1 or S_2, or from any point between S_1 and S_2. The observation of a scintillation at P implies that the photon was scattered from some point along the axis within the range Δx, (Fig. 5-3). This uncertainty in the location of the point at which the electron scattered the photon is approximately $f\delta$, where f is the focal length of the objective and δ is given by Eq. 5-1. Consequently,

$$\Delta x \approx \lambda(f/d). \tag{5-2}$$

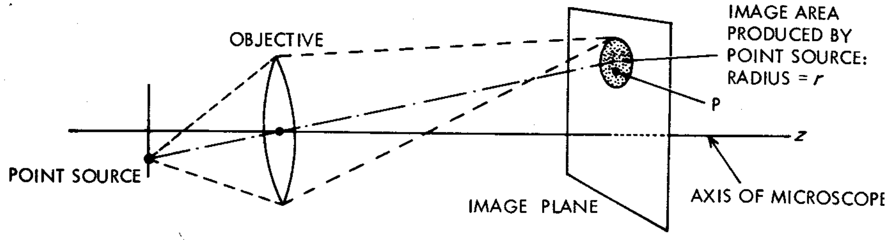

Fig. 5-2. Image of a point source in a microscope.

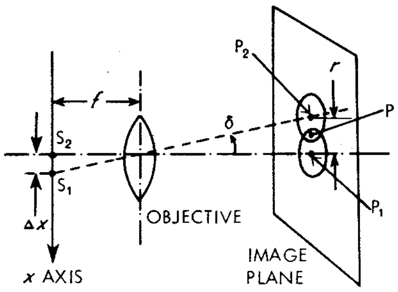

Fig. 5-3. Determination of position uncertainty in microscope measurement.

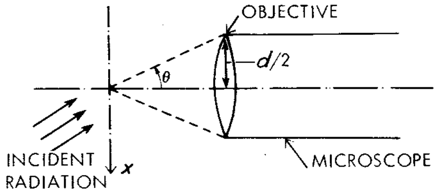

Fig. 5-4. Momentum uncertainty after position measurement.

Consider now the change produced in the electron momentum when the photon that reaches P is scattered by the electron. This process (Compton effect) may be described by assuming that the incident photon is a particle with momentum h/λ and energy $E = h\nu$ ($\nu = c/\lambda$), and that momentum and energy are conserved in the collision. To get into the microscope the photon must be scattered into a cone of angle θ (Fig. 5-4). Thus the components of the momentum of the scattered photon along the x axis lie between $\pm$ $(h/\lambda)\sin\theta$. (We neglect the change of λ in the collision.) Also, $\sin\theta \approx \tan\theta = d/2f$. Since the total momentum of electron and photon is conserved we may conclude that the momentum after the observation lies between $p_0 + h/\lambda\,(d/2f)$ and $p_0 - h/\lambda\,(d/2f)$, where p_0 is the measured momentum before the scattering. The uncertainty in the x component of the momentum of the electron after collision is, therefore,

$$\Delta p_x \approx \frac{h}{\lambda}\frac{\mathrm{d}}{\mathrm{f}}. \tag{5-3}$$

For the product of the uncertainties in position and momentum we find

$$\Delta x \, \Delta p_x \approx (\lambda f/d)(h/\lambda)(d/f) = h, \tag{5-4}$$

which is just what was found from the analysis of the single-slit diffraction (Eq. 4-1). The uncertainties in the position and associated momentum component of the electron as measured by the microscope are inversely related. If we arrange to make Δx small (by making λ small or θ large), Δp_x gets large. Similarly, if Δp_x is made small (by making λ large θ small), Δx becomes correspondingly large. If the position of an electron with an energy of the order of tens or hundreds of electron volts is to be determined within an uncertainty of macroscopic size (say $\Delta x \sim 1$ mm $= 10^7$ Å), the uncertainty in p specified by (4-1) is negligible. Using Eqs. 4-1, 2-6, and 2-7,

$$\Delta p_x/p_x = h/(p_x \Delta x) = \lambda/\Delta x \cong 12.3/(E^{1/2}\, \Delta x) \sim 12.3/(10 \cdot 10^7) \sim 10^{-7}.$$

This implies, of course, an enormously precise determination of p. If, however, we wish to determine the position of an electron within an uncertainty as large as an atom ($\Delta x \sim 1$ Å), the uncertainty in p_x is of the order of p_x.

The relation 4-1, first derived by Heisenberg [20, 21] through an analysis of the observation of position with a microscope and other modes of position, momentum measurement, is known as the Heisenberg uncertainty relation. Heisenberg's discussion of the uncertainty relation contributed greatly to an understanding of the physical meaning of the formal structure of quantum mechanics. The mathematical formalism of quantum mechanics was discovered in 1925 by Heisenberg, Schrödinger, and Dirac but it took about two years of intensive study after this discovery to appreciate the full physical significance of the new formalism[1].

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[1] It is an interesting feature of the historical development of quantum mechanics that the mathematical structure of the theory was invented before its physical interpretation was completely understood. One would expect a new theory to grow out of new physical insights into the nature of phenomena, with the mathematical structure built afterwards to give precise quantitative expression to these insights. Some of the reasons for the reversed order in the case of the quantum mechanics are considered in Chapters 6, 7, and 8.

Of course, the failure of the microscope method does not imply by itself that a precise simultaneous measurement of position and momentum is impossible. Perhaps the desired measurements can be made by other methods. However, the analyses of all proposed devices for making the measurements – a number of most ingenious methods have been investigated – lead uniformly to Heisenberg's relation (4-1). These investigations suggest strongly that the limitation specified by the uncertainty relation expresses a general law applicable not only to electrons and photons but to all microphysical entities.

In the argument leading to Eq. 5-4, a wave theory was used to obtain the position uncertainty[2], Δx, while to obtain the momentum uncertainty, Δp_x, a particle theory of the collision of a photon with the electron was employed. This admixture of wave and particle considerations occurs in the analysis of every method designed to measure position and momentum.

Naturally, a paper-and-pencil analysis of a measurement presupposes some theoretical foundation. Exactly what are the theoretical assumptions employed? They are simply the de Broglie relations with the following interpretation: In processes of energy and momentum transfer, electrons and photons (as well as other microphysical entities) are to be treated as particles, while in processes of transmission between spatially separated interactions they are to be treated as waves. All relevant observations are consistent with this interpretation of the de Broglie relations.

5.1 Supplement to Chapter 4. In our investigation of the two-slit diffraction experiment we postponed consideration of the effects attendant on observations designed to determine whether the electron goes through slit a, slit b, or possibly in some manner, through both (see Fig. 4-7). Suppose we arrange devices capable of detecting a passing electron in front of the slits (see Fig. 5-5). To be specific, suppose that we have a source S emitting radiation of wavelength, λ, (between the diffracting slits and the detectors in the arrangement of Fig. 4-7) which

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[2] The diffraction in the microscope was analyzed by a wave theory. Remember, however, that this theory does not describe all the features of the diffraction of photons.

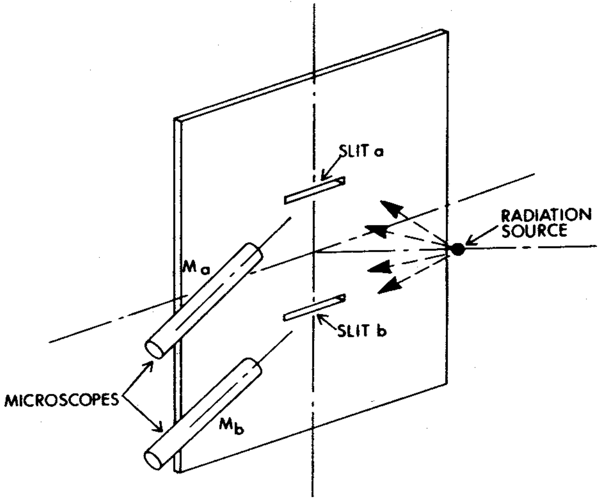

Fig. 5-5. Observation to determine through which slit an electron passes.

illuminates both slits. We have also a pair of microscopes, M_a and M_b, which are focused on the regions in front of the slits. The microscopes are to be arranged so that the detection of an electron by M_a (or M_b) implies that the electron was in the neighborhood of slit a (or slit b). Of course, we arrange that if the electron beam is off, no photons can be scattered into the microscopes.

To be able to distinguish between the passage of an electron through slit a or slit b, we must be sure that the detection of a photon in microscope M_a cannot be interpreted as rising from the scattering of a photon by an electron in the neighborhood of slit b. Consequently the microscopes must be capable of locating electrons within an uncertainty Δy smaller than the separation, d, between the slits. Such measurements give rise to an uncertain change in the momentum of the electron; the uncertainty in the y component of the momentum introduced by the observation is

$$\Delta p_y \cong h/\Delta y \gtrsim h/d.$$

This momentum uncertainty along the y axis implies an uncertainty, $\Delta\theta$, in the direction taken by the electron after passage through the slit.

This angular uncertainty, $\Delta\theta$, is roughly $\Delta p_y/p$, where p may be taken as the magnitude of the incident momentum. Thus, making use of the de Broglie relation, $\lambda = h/p$,

$$\Delta\theta \sim \Delta p_y/p \sim h/(p\,\Delta y) \gtrsim h/(pd) = \lambda/d.$$

But the angular separation at the slits of successive maxima in the interference pattern when the detectors are absent (see Fig. 4-11) is also λ/d. We have, therefore, $\Delta\theta \gtrsim \theta$, where θ is the angular separation (measured from the slits) of successive maxima in the normal two-slit pattern. In determining locations by the microscope, we introduce a new directional uncertainty (superimposed on the directions taken by the electrons after getting through the slits) of such size as to smear out the characteristic structure of the two-slit pattern. Thus, when the interference experiment is performed with instruments capable of determining which of the two slits the electron passes in its path from source to detector, the characteristic two-slit interference pattern is blurred and the pattern obtained is like the superposition of two single-slit diffraction distributions.

5.2 Continuation: Heisenberg's Principle. Assume now that the Heisenberg relation (4-1) expresses a fact of nature – i.e., that the laws of physics make it impossible to know both the momentum and position of an electron with uncertainties smaller than those permitted by Eq. 4-1. What consequences follow? Can the uncertainty relation help us see how a rational theory of microphysical processes might be formulated?

One consequence is immediate, of obvious importance, and independent of further theoretical assumptions. In principle, either the position or the momentum of an electron can be measured with arbitrary accuracy. The accuracy within which both properties can be known simultaneously, however, is limited by Eq. 4-1. Suppose that the momentum of an electron has been measured precisely; according to the Heisenberg relation, the position at which the electron will be found on measurement cannot be predicted. However, in an exact observation of position the electron is found at some definite location (after which the result of a momentum measurement cannot be predicted). The point of all this is that the Heisenberg relation, or

Heisenberg principle, implies that it is not possible to arrange initial conditions so as to be able to predict the results of all possible observations that might be made on an electron. If position (or momentum) is known, the result of an exact momentum (or position) measurement is unpredictable. If position and momentum are known within the uncertainties Δx and Δp_x, the result of either an exact position measurement or an exact momentum measurement cannot be predicted, although a precise measurement of position will yield a result somewhere in the range Δx or, if momentum is measured precisely, its value will be found in the range Δp_x.

This unpredictability of the properties of an electron may remind us of the unpredictable features noted in the study of diffraction phenomena. Equipment could not be arranged to ensure that an electron that gets through the slits will arrive at some particular point on the detecting screen. The Heisenberg principle suggests that the indeterminate behavior of electrons encountered in the diffraction study is by no means peculiar to the phenomenon of diffraction. Indeterminate behavior, it would seem, must be expected throughout the realm of microphysics. But how can one possibly have a science for indeterminate – i.e., unpredictable – behavior?

The diffraction studies of Chapter 4 suggest an answer. The positions at which individual electrons appear on the detecting screen in the diffraction process cannot be predicted. Or, put in another way, the diffraction apparatus cannot be arranged so that every electron in the incident beam will arrive at the same point on the detecting screen. However, and this is the crucial consideration, the distribution of position at the detector, i.e., the diffraction pattern, can be predicted. If 20 physicists perform the diffraction experiment at different places and different times, they all find the same pattern. The nature of the electron source, the manner of detection, and the methods used for producing the incident beam have no influence on the pattern. Whenever and however an incident beam with a given well-defined momentum is incident on a given crystal scatterer, the same distribution in position at the detector is found; *the incident momentum uniquely determines the final distribution*. It is clearly possible to predict not the locations of single electrons but the distribution in position of a large collection or ensemble of electrons.

In effect, a theory of microphysical processes must have a statistical character. In general, only statistical properties (i.e., ensemble properties) rather than the properties of single entities are determinate in the realm of microphysics. Often a somewhat loose use of language obscures this fact. We have already explained what is meant when we speak of the wave property of electrons as exhibited in the diffraction experiments. It is not a single electron with definite momentum that is similar to a wave but, rather, a large ensemble of electrons all with definite momentum.

An additional example of the determinate properties of ensembles of microphysical entities will be helpful. A free neutron undergoes spontaneous transformation into a proton with the emission of an electron and a neutrino. It is said that the "half-life" of a neutron is 12 minutes. This is a statement not about individual neutrons but about ensembles of neutrons. If, initially, we have a set of neutrons (at rest), some transform before one minute has passed while others have stubbornly refused to change into protons even after a lapse of an hour. We cannot predict precisely when any individual neutron will change to a proton, but regardless of the prior history of the neutrons collected in an ensemble, half the ensemble will be transformed into protons at the end of 12 minutes. The term "half-life," obviously, is an ensemble property.

Let us summarize briefly. The Heisenberg principle implies that there is an irreducible indeterminateness in the behavior of microphysical systems. Experimental conditions cannot be arranged to ensure the outcome of all observations. However, the behavior of suitably prepared ensembles is found to be regular and lawful.

6 Interpretations of the Heisenberg Principle

"All things counter, original, spare, strange,
Whatever is fickle, freckled (who knows how?)
With swift, slow; sweet, sour, adazzle, dim;"

Gerard Manley Hopkins

The physical meaning of the Heisenberg principle is not obvious from its content. It may be interpreted several different ways. Two classical attitudes toward the principle are sketched in Subsections 6.1 and 6.2. An interpretation that breaks sharply with classical conceptions – the interpretation that leads to quantum mechanics – is introduced in Section 6.3.

6.1 Classical Statistics. If we accept the principles of classical theory, it is natural to argue that even if the simultaneous position and momentum cannot be measured to better than the accuracy indicated by the Heisenberg relation (4-1), nevertheless the electron must have, at each instant, both a well-defined position and a well-defined momentum. In other words, the uncertainty principle does not in itself prevent

us from considering the electron as a Newtonian particle. The wave-particle dualism and its consequence, the Heisenberg principle, merely imply, in this classical view, that the position and the momentum of an electron cannot be measured simultaneously.

But since position and momentum are the initial conditions required to predict the path of an electron, accurate prediction is rendered impossible and the motion of an electron is necessarily uncertain. We may, however, try to make statistical predictions by the methods of statistical mechanics. When we deal with a sample of a gas, we do not know, for practical reasons, the positions and momenta of the molecules that constitute the gas. Nevertheless, relations between such statistical properties as the pressure and the temperature of the gas, for example, can be derived.

When an object subject to some set of conditions is studied by the methods of statistical mechanics – whether it is a sample of a gas or a single electron – a large ensemble of objects of the same kind (all subject to the same conditions) rather than a single object is considered. The ensemble may be a theoretical entity (as in the case of a gas sample, since we do not collect for study 10^9 or so samples of the gas) or a natural entity, as is often the case in microphysics where observations on a single system (atom, electron) are impractical, while observations on large collections of these systems are relatively simple. In any case we attempt to calculate, or observe (if a physical ensemble is available), the distribution of the properties of interest over the whole ensemble rather than the properties of a single object.

We review the statistical method for treating a single electron (now considered as a Newtonian particle) moving in one dimension. It will be useful for this purpose to introduce a "phase space" diagram (Fig. 6-1). A point in this diagram (a "state point") represents a single electron, with position as given on the abscissa and momentum as given on the ordinate. Suppose our electron to be subject to an external force of some kind. The motion of the electron is completely determined once its position and momentum at some time, say $t = 0$, are given. Both position and momentum change, in general, under the influence of the external force and hence the state point in the phase space diagram traces out a definite path.

When we have imperfect knowledge about the initial state of the

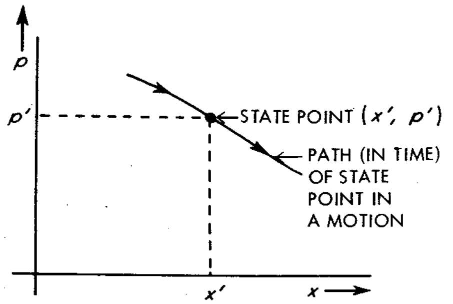

Fig. 6-1. Phase space diagram for motion of a point particle in one dimension.

electron we introduce an ensemble of electrons, with each electron having some definite position and momentum, so as to represent the range of possible initial conditions that are consistent with the available information; i.e., each electron in the ensemble chosen has a chance of being the electron we wish to study. An ensemble is represented in the phase-space diagram by a large number of points, each of which is a state point for a single electron.

Figure 6-2 shows a phase-space representation of an ensemble that could be used to study the motion of an electron (in the statistical sense) if all that is known is that the uncertainty in x is Δx about the neighborhood of x_0, the uncertainty in p is Δp about the momentum p_0, and the x and p distributions are uncorrelated. (Absence of correlation means that the distribution of the points along the p axis is the same for every chosen value of x within the range Δx.)

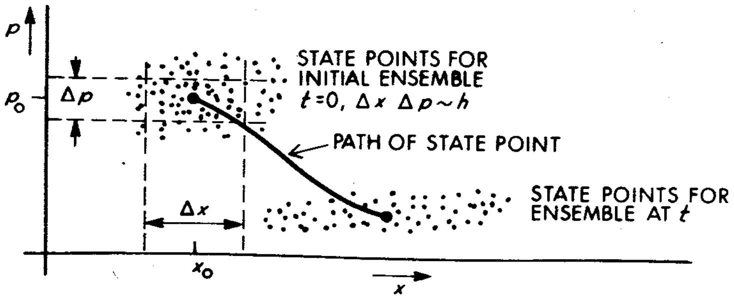

Fig. 6-2. Representation of an ensemble of electrons in phase space.

From the assumed initial ensemble and the laws of motion the position of all state points at any later time and the distributions in x, p, or any other dynamical variable of interest can be calculated. Thus the initial ($t = 0$) characteristics of the ensemble determine the average value of the position in the ensemble at time t, the uncertainty in position at t, etc.

The Heisenberg principle asserts that when measurements are made as accurately as possible, there remain position and momentum uncertainties Δx, Δp such that $\Delta x \Delta p \sim h$. Therefore, to make a statistical analysis of the motions of an electron, it must be replaced by an ensemble of electrons with a phase space representation like that of Fig. 6-2. (We use an uncorrelated distribution since observations can tell us nothing about possible correlations.) Note that the Heisenberg principle says that no matter how the position and momentum of an electron are measured, the ensemble of phase-space points required for statistical treatment must cover an area in phase space roughly of magnitude $h(\cong \Delta x \Delta p)$.

It seems highly improbable that the foregoing classical statistical theory could solve the conceptual problems raised by microphysical observations. The phenomena of quantization would still require the addition of restrictive rules of some kind (as in the theory of Bohr). The introduction of such rules would lead to the same objections as were generated by the Bohr theory (see Chapter 1). The difficulties are more apparent if we consider the wavelike properties of electrons. How, for instance, could a statistical theory handle the two-slit diffraction experiment? The theory considers electrons to be classical particles; each electron would be pictured as going through one or the other of the two slits. Thus the theory could only lead to the false prediction that the two-slit pattern is a simple superposition of two one-slit patterns.

6.2 Hidden Variables. The Heisenberg principle implies that the behavior of electrons is indeterminate. What is the cause of this indeterminacy? Are we to understand that on the microphysical level there is an essential play of chance or, to paraphrase Einstein, that "God plays dice with elementary phenomena"? "Chance" is merely a word we use when we try to hide our ignorance. Surely (says the classical physicist) the position at which each electron hits the screen in a diffraction experiment is uniquely determined by some specific set of conditions, even if we have not yet discovered exactly what these conditions are.

Macrophysical systems may seem to have an indeterminate behavior if we do not take into consideration all the details of their structure. Suppose, for example, that we have a set of boxes all with the same dimensions and all of the same mass containing differently oriented gyroscopes with different angular momenta. Under external torques of the same magnitude and direction (relative to the box geometry), different boxes will exhibit different responses. An observer who is able to examine only the outside features of this collection of boxes might claim that they are identical but that their behavior is indeterminate. A physicist who made such an interpretation without even thinking that the boxes might be systems with different internal properties would be a simpleton. We who know about the internal gyros have no difficulty in explaining the variation in the behavior of the different boxes. Obviously if we do not know and control all the variables on which the behavior of a set of systems depends, the same external influences may lead to different consequences simply because the different systems have different internal properties.

Might this not be the origin of the indeterminacy implied by Heisenberg's principle? We don't really know in any direct sense what an electron is. When a photon registers in the microscope we speak of a "position" measurement. From the "position" measurement alone it is not possible to predict the result of a subsequent "momentum" observation. But might there not be as yet unknown measurable properties – i.e., hidden variables (corresponding to the gyros in the example above) – which, if measured together with "position," would enable us to predict the result of a subsequent "momentum" measurement with precision and certainty?

It is impossible to prove or disprove the existence of the hidden variables needed to make the behavior of microphysical entities determinate[1]. As yet, however, no one has turned up any trace of such hidden properties. It is only the brute fact of the appearance of indeterminacy in the behavior of microphysical systems that suggests the existence of hidden variables. But the proof of a pudding, it is said, is in the eating. More than 40 years ago (1925), a highly successful nondeterministic theory (quantum mechanics) of microphysical systems was formulated. In all the work which has gone on since 1925, no evidence for the existence of hidden variables that might serve to reestablish determinism has been found.

6.3 A Nonclassical Interpretation of the Heisenberg Principle. The Heisenberg principle states that the exact simultaneous values of the position and momentum of an electron cannot be observed. Recall also that the motion of electrons in diffraction processes is not observable; it could not even be determined through which of two slits each electron passes without completely changing the resulting diffraction pattern. Now the concepts of simultaneous position and momentum, and of a path through one slit or another derive from the conception of an electron as a classical (Newtonian) particle. Are there other features of the mental images we form of microphysical processes that are unobservable? From the Heisenberg principle it follows that a large number of the concepts of classical physics have no observational counterpart in the realm of microphysics.

Suppose we wished to observe the path or orbit of the electron in a hydrogen atom. To measure the path we must determine the positions of the electron at a number of successive instants. Since the atom has a diameter of about 1 angstrom, any meaningful position measurement must be made with an uncertainty, Δx, of no more than about 0.1 angstrom. By the Heisenberg relation such a position measurement

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[1] It was shown by von Neumann [22], with the aid of certain rather general and apparently harmless assumptions, that hidden variables are inconsistent with the results of quantum mechanics. For many years it was believed that his "proof" showed that hidden variables could not exist. But any theorem is no more than an expression of the hidden content of the assumptions on which it is based. With a slight change in von Neumann's assumptions, J. S. Bell [*Rev. Mod. Phys.* 38, 447 (1966)] showed that it is possible for hidden variables to be consistent with the statements of quantum mechanics. Unfortunately we cannot pursue these questions further in this brief introduction. Interested readers will find a discussion of present attitudes toward the problem of hidden variables (as well as of other issues relating to the conceptual foundations of quantum mechanics) in a recent review article: Ballentine, R. E., *Rev. Mod. Phys.* **42**, 358 (1970).

introduces an uncertain change in the electron's momentum of about $h/\Delta x$, where $\Delta x \sim 10^{-9}$ cm. A momentum change of this magnitude corresponds to a change in energy which is about 10 times the ionization energy of hydrogen. Thus the first measurement of the sequence required to determine the electron's orbit is almost certain to ionize the atom. In principle we may determine the position of the electron (with arbitrary accuracy) in the atom at some time, but we cannot observe the electron path.

A further deduction may be made from this example. In the analysis of the impossibility of observing the orbit we noted that the position measurement resulted in the possibility of a large and uncertain transfer of energy to the atom (corresponding to an uncertain alteration in momentum). This suggests that the exact energy of the atom and the position of its electron cannot be known simultaneously. Suppose the energy has been measured. After the measurement (we assume energy conservation) the energy is constant. Hence we should know both energy and position if the position can be measured without change, or with a determinate change, of the energy. The Heisenberg principle implies that these requirements cannot be met. The measurement of the position will alter the energy of the system in a manner that we cannot completely control. By analogous considerations it could be shown that the energy and momentum of an electron in an atom cannot be known simultaneously. It appears that our inability to measure pairs of variables simultaneously is by no means limited to the position, momentum pair. Indeed, our findings suggest that simultaneous measurability of quantities is the exception rather than the rule in microphysics.

In classical mechanics the concepts of velocity and acceleration play important roles. Force is related directly to acceleration, and the force on an object is measured by observing, in one aspect or another, the acceleration the force induces. But at the level of atomic dimensions, velocities, accelerations and, consequently, forces are unobservable[2].

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[2] If the position components at t_1, t_2 are x_1, x_2, then the average velocity in the interval $t_2 - t_1$ is $(x_2 - x_1)/(t_2 - t_1)$. For this average velocity to be a reasonable approximation of the electron velocity in the interval $t_2 - t_1$, the distance $x_2 - x_1$ must be small compared with the distance over which the velocity changes markedly; thus $x_2 - x_1$ must be considerably smaller than an

We have considered how the Heisenberg principle (deduced from the de Broglie relations) affects the observability or measurability of features of classical mechanical systems. Classical electromagnetic concepts may be investigated in the same spirit. We shall not attempt an analysis here but simply quote results [21]. The analysis shows that electric and magnetic fields cannot be measured simultaneously with accuracy. To predict (by classical theory) the development in time of an electromagnetic field, it is necessary to know both fields at the same time. Our inability to measure the fields simultaneously thus undercuts the possibility of prediction in electromagnetism in the same way that our inability to measure position and momentum simultaneously prevents prediction for mechanical systems.

Consider a somewhat different consequence of the Heisenberg principle. When we think of an "electron" we are likely to generate a mental picture of a tiny object which has a definite location in space and is either at rest or moving in some direction with a definite speed relative to a coordinate frame. Is the physical electron an entity that corresponds to this picture? Maybe it is. But we cannot establish the

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atomic diameter, since in moving across an atomic diameter the velocity must change sign. Take $x_2 - x_1 \sim 0.1$ angstrom. To obtain a measure of this difference within about 10% accuracy, the positions x_1, x_2 must be known with uncertainties no larger than 0.01 angstrom. Such accurate position measurements lead to large and uncertain momentum and energy changes. For a position accuracy of 0.1 angstrom, we calculated that the associated energy uncertainty is about 10 times the ionization energy; for an accuracy in position of 0.01 angstrom, the momentum uncertainty is increased by a factor of 10 and, since the energy depends on the square of the momentum, the energy alteration can be as large as $(100)(10) = 1000$ times the energy required for ionization. Clearly the atom will be ionized in the first position measurement and our effort to obtain the velocity of the electron in its orbit fails completely. Since acceleration is the rate of change of velocity and velocity is not measurable, neither is acceleration. Moreover, force, which is determined by the acceleration it produces, cannot be measured.

The preceding remarks refer to electrons within atoms. If, as in many electron beam experiments (see the next to the last paragraph of this section), it is sufficient to measure position to an accuracy of about a millimeter (rather than 10^{-8} mm as is required to obtain the path within an atom), the concepts of path, velocity, acceleration, and force can be given approximate meanings. Anyone who has seen tracks of electrons in cloud chambers or photographic emulsions has had visible proof that the concept of electron path is sometimes meaningful.

truth of the picture by observation. Our mental model implies that the electron is an entity capable of having both a precise position and a precise momentum. But since position and momentum cannot be measured simultaneously and exactly, we cannot prove that the electron can have both a precise position and a precise momentum simultaneously[3].

It is possible to go considerably further in the direction taken by the foregoing paragraphs. However, a sufficient number of instances have been cited to indicate that only a remarkably small subset of classically meaningful properties are observable in microphysical phenomena.

What are we to make of all these examples of our incapacity to observe and measure? It is depressing to think of so many failures. But before we go off to cry in our beer about our many deficiencies, it is worth reflecting that the failure to observe an expected phenomenon does not necessarily imply an incapacity of some kind; it may be that the phenomenon just wasn't there to be observed.

Let us return then to the usual optimism of physicists and deny failure. Instead of looking upon the Heisenberg principle as an indicator of incapacity we shall boldly assume that an electron is an entity that simply does not have an exact simultaneous position and momentum. (To avoid its frequent repetition, the awkward phrase "exact simultaneous position and momentum" will be designated by "*X*".) On our new assumption, the words "an electron with the property *X*" is as meaningless as the phrase "a square of radius *r*." This assumption seems harmless enough – after all, the analysis of diffraction phenomena in Chapter 4 indicated clearly that an electron is not a simple Newtonian particle – but it contains packages of dynamite.

Consider an electron that has been prepared with a precise momentum; our assumption forces the conclusion that this electron does not have a position. For if the electron with definite momentum is located somewhere, then it has both position and momentum simultaneously (even though the position is unknown); i.e., the electron has the property *X*, and this, by the assumption introduced above, is impossible. An argument of the same kind would show that an electron with a definite position does not have a momentum property. Clearly, if *X* is not an electron property, electrons do not have positions or momentum under all circumstances.

It is not difficult to invent situations with features analogous to those of the preceding paragraph. For example, a plane string figure cannot have simultaneously both a precise "radius" and a precise "side length." If the string figure is a circle it has a radius but it does not have a property of side length; if, however, the figure is a square then it has an exact "side length" but it does not have a radius. For a second example, consider certain motions of an infinitely long, stretched string. If the motion is characterized by an exact wavelength, no meaning is associated with the idea of a sharp location of the wave; when the wavelength is exact the wave is infinite in extent (Fig. 6-3a). When, however, a sharply defined pulse (Fig. 6-3b) travels down the wire, the position of the disturbance (at some instant) has a fairly well-defined meaning, but the string has no precisely assignable wavelength.

However, the problem posed by the assumption that *X* does not exist for electrons has features the examples above fail to encompass. There is no reason to doubt that, whatever the circumstances, a position (or momentum) measurement can always be made successfully on an electron. We are consequently led to what seems an absurd

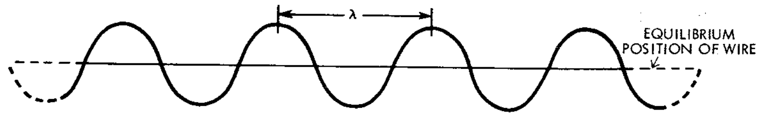

Fig. 6-3a. The deflection of a wire at some instant during a motion. At this instant, the motion has a wavelength but not a location property.

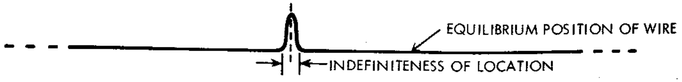

Fig. 6-3b. The deflection of the same wire as that of Fig. 6-3a at some instant in a different motion. The disturbance has a fairly well-defined location but has no wavelength property.

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[3] When the restrictions of the Heisenberg principle are not significant, position and momentum can be measured with enough accuracy as to make a particle model of the electron and a good approximation. Such a model, for example, is used in the interpretation of experiments designed to measure the charge-to-mass ratio of the electron.

conclusion: Although an electron with precise momentum cannot be said to have a position, nevertheless a suitable measurement will find the electron at some definite position (the words "position" and "momentum" may be interchanged). For classical physics such a conclusion is nonsensical; it simply is not possible to measure a property that an object does not have. For example, no one would dream of measuring the wavelength of a classical particle; the measurements that can be made on such a particle could never lead to a wavelength. Similarly it is not possible to find on observation a wave with a precise wavelength at a sharply defined location[4].

The ideas developed above, if disturbing, are nevertheless not without attractive features. In the two-slit diffraction experiment (Chapter 4), both slits appear to have an influence on the behavior of the electrons that get through; the two-slit pattern is quite different from the sum of two single-slit patterns. If the electron always has a sharply defined (even if unknown) location, it seems to be impossible to understand how an electron can "know" whether the slit through which it does not pass is open or closed.

Recall that the incident electrons in the diffraction experiments must have well-defined momentum. By the assumption considered in the previous paragraphs, the electrons in the incident beam do not have a sharp location. It is therefore at least conceivable that both slits play a role in the transmission process. Of course these thoughts do not constitute a theory of the diffraction experiments; they simply indicate that certain of the paradoxes considered in Chapter 4 can be avoided if position is not always a meaningful property of electrons.

Is it possible, however, to conceive that an electron does not always have a definite location even though a measurement of position will always discover it at some place? The training we have all had from birth in macrophysics makes this idea very difficult to accept; it is in fundamental conflict with deeply held presuppositions about the meanings of the concepts of "measurement" and "property." Unless these presuppositions can be shown to be false, the assumption that electrons cannot have a precise simultaneous position and momentum must be abandoned.

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[4] On the plane string figure of radius r it is senseless to measure "side length."

The classical assumptions relating to the concepts of measurement and property are criticized in the next chapter, where it is shown that the ideas developed in the preceding paragraphs are neither internally inconsistent nor controvertible by observation. For our present introductory purposes the following brief but suggestive remarks must suffice. In an accurate position measurement, strong interactions occur between the electron and the apparatus employed to effect the measurement. In the microscope method (Chapter 5), for example, an energetic photon must be scattered off the electron; the more accurate the measurement, the more energetic the photon must be. Just how the rough treatment suffered by an electron in the course of measurement modifies its characteristics cannot be known. All that we have available are observable manifestations – scintillations, counts – from which properties are inferred. Immediately after a position measurement on an electron with momentum p, which locates the electron at a definite position, a second position measurement is certain to find the electron at the same position; after the first measurement, therefore, the electron is said to "have" a position property (see Chapter 7). (After the measurement the electron no longer has a momentum property.) But by the very construction of the statement, it is impossible to establish by observation that just prior to the position measurement the electron had the location at which it was subsequently discovered[5].

It is useful to recall that the central object of theory is to provide a set of general rules from which the observable relations among phenomena may be deduced. No theory legitimately can be asked to do more. Certainly theories need not be required to make statements about imagined but unobservable features of phenomena. Anyone who would assert that electrons are red but that the property of redness is unobservable, and then would go on to wonder how the redness of

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[5] An extension of the example of the string figure provides a rough analogy. Obviously the figure cannot simultaneously have an exact "radius" and an exact "side length." If it has a radius then it does not have a side length. Suppose, however, that when a measurement of side length is made, an interaction between the observing apparatus and the string occurs which forces the figure into a square. Then the side length is measurable on a circular figure even though prior to the measurement the figure does not have the measured property. In the course of measurement the figure gains a "side length" and loses a "radius."

electrons influences other unobservable properties of microphysical phenomena would be considered slightly mad.

There may, however, be good reasons for the introduction of unobservable properties into theoretical considerations. If, without affecting the predictive power of a theory, we can make it simpler, or applicable to a wider range of observable phenomena, or more beautiful, by introducing unobservable features, why not do so? No one is bothered, for example, by the unobservability of the "inside" of a completely closed box or the widely used but unobservable concept of continuity. Equally, when theory fails, as classical theory fails in the realm of microphysics, its imagined but unobservable aspects may be cut away if there is reason to believe that the excision will permit the construction of a more successful or more beautiful theory. We need not remain enslaved to concepts of classical theory, no matter how successful they are for macrophysics, if they are unobservable in the vastly different realm of microphysics[6].

The Heisenberg principle shows us how to escape from the tight chains of the classical system of thought without danger of conflict with experience. If an aspect of classical theory is unobservable we are free to try to modify it or even to junk it altogether. Of course a declaration of freedom from unobservable classical concepts does not constitute a theory; it remains to be seen whether the new freedom conferred by the Heisenberg principle will help us to find a useful theory of microphysical processes.

The thoughts of the past several paragraphs have been qualitative. The Heisenberg principle contains, however, a quantitative aspect that has not yet been taken into account. The principle says much more than that "precise position and momentum cannot be observed simultaneously." It tells us that the product of the uncertainties in

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[6] Nowadays we are accustomed to think so familiarly of atoms and electrons that we tend not to appreciate the enormous change in orders of magnitude that occur when we pass from the macrophysical to the microphysical realms. Normal macrophysical magnitudes are: length ~1 cm; mass ~1 g; time interval ~1 sec; charge ~1 microcoulomb. The factors which convert these macromagnitudes to proper microphysical magnitudes (for atoms) are 10^{-8} for length, 10^{-27} for mass, 10^{-16} for interval, 10^{-13} for charge. To get some impression of the difference a factor such as 10^{27} can make, note that 10^{27} meters is equal to 100 billion light years, a distance about 10 times the size of the universe.

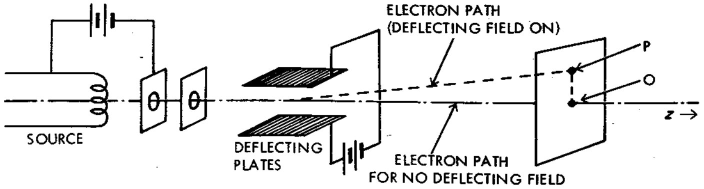

Fig. 6-4. Deflection of an electron beam by an electric field. In such experiments, classical concepts provide quite accurate predictions.

position and momentum cannot be made smaller than $h \sim 6 \cdot 10^{-27}$ g cm²/sec[7]. For electrons with energy of the order of hundreds of electron volts (or smaller) and position uncertainties of the order of atomic dimensions, the restrictions imposed by the Heisenberg principle are very important; the classical notion of simultaneous position and momentum is unobservable even in the sense of a rough approximation.

But consider an experiment on the deflection of electrons in an electric field by means of the apparatus shown schematically in Fig. 6-4. We may suppose the source and accelerating plates to be those used in the diffraction experiment (Fig. 4-1). A pair of plates between which an electric field may be generated is arranged so that when no field is present the electron beam passes between the plates and is detected at O. When the field is turned on, the electrons arrive at P. The magnitude of the electric field and the geometry of the apparatus are chosen so that the deflection, OP, is several centimeters. In the discussion of the diffraction experiment it was shown that the momentum of the electrons in the beam is determined to within an uncertainty of about

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[7] This refers to one component of position and momentum. For each of the three orthogonal components the relation is the same. However, the x component of position and any orthogonal component of momentum are measurable simultaneously. Thus, if we locate an electron in a volume $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$, the uncertainties in the components of momentum $\Delta p_x, \Delta p_y, \Delta p_z$ are such that $\Delta x \Delta p_x \Delta y \Delta p_y \Delta z \Delta p_z \sim h^3$; i.e., $\Delta V(\Delta p_x \Delta p_y \Delta p_z) \sim h^3$.

The "phase space" (see Subsection 6.1) for an electron in three dimensions is a six-dimensional coordinate frame. In this "space" a point describes the three components of position and the three components of momentum of an electron. The Heisenberg principle states that location within a (six-dimensional) "volume" in phase space is unobservable if the "volume" is smaller than h^3.

0.5% of the value of the momentum; the diffraction from 1 mm slits is negligible so that position in the x, y plane is determined to within a millimeter. For all the purposes of the observations to be made in the experiment, the position and momentum are both quite well defined (i.e., percentage errors are small). Under these conditions the whole set of classical conceptions that are unobservable in the microscopic realm can be given quite accurate meanings. The path of the electron beam may be observed (naturally location measurements will be made with an accuracy $\Delta x \sim 1$ mm), the velocity and acceleration of the electrons may be measured with accuracy sufficient for our objectives, and Newtonian mechanics may be used to describe (approximately) the course of the motion[8].

The foregoing complex of considerations may be brought together in a new interpretation of the Heisenberg principle. When the restrictions implied by $\Delta x \Delta p \sim h$ are effectively negligible (as in the example above), classical dynamical concepts have approximate significance. But in the realm of microphysics – i.e., where the restrictions of the uncertainty relations are important, as they clearly are for electrons in atoms – many classical ideas lose even approximate meaning. In particular, microphysical entities such as electrons do not have the property of exact simultaneous position and momentum.

In the construction of a new theory of microphysical phenomena the Heisenberg principle may be used to free us from the constrictions of imagined but unobservable classical concepts. However, the new theory must be capable of showing that classical concepts have an approximate validity wherever the restrictions imposed by $\Delta x \Delta p \sim h$ are unimportant. The role played by Planck's constant, h, in this interpretation is interesting. It provides a quantitative measure by which we judge whether a particular experiment or phenomenon may be interpreted classically (in some approximate sense), or whether we are free to try to develop a theoretical description of a nonclassical character.

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[8] Many of the electron properties given in Chapter 3 are obtained by experiments in which classical conceptions are applicable.

7 Dynamical Properties of Microsystems

"Imagination is often at war with reason and with fact."

Benjamin Jowett

We must be careful about trying to make even a small change in a tightly organized structure; the attempt often leads to almost complete reconstruction. The alteration of a sentence in the middle of a paragraph, for example, may infuriatingly necessitate extensive reworking of several pages of manuscript. Similarly the changes in classical modes of thinking required by the nonclassical interpretation of the Heisenberg principle effectively force us to re-examine and reconstruct a number of the concepts on which classical physics is based to make them useful for the description of microphysical processes.

There are two coupled pairs of ideas that we wish to examine: "determinism" and "state" and "measurement" and "property." We have noted that from the Heisenberg principle it follows that microphysical processes are indeterminate. Does not indeterminacy mean the complete absence of lawfulness? What is the meaning of a theory for systems with indeterminate behavior? What sort of structure can such a theory have? These questions and others related to them will be examined in Chapter 8.

From the nonclassical interpretation of the Heisenberg principle proposed in the last section, we drew a seemingly paradoxical conclusion: The electron need not always have a position even though a precise measurement of the electron's location always discovers it in some definite place. In short, it is possible to measure a property (position) of the electron that the electron does not have. In certain circumstances, then, it is incorrect and misleading to assert that the electron under consideration is located anywhere at all; a search, nevertheless, will find it at some definite place.

These ideas are wholly at odds with classical thought in which measurement simply reveals the properties the system objectively possesses. We have seen that certain pairs of properties (e.g., position and momentum of an electron, electron position and energy of a hydrogen atom, etc.) cannot be measured simultaneously. Some qualitative understanding of this fact follows from the realization that in microphysical measurements there are generally strong interactions between system and apparatus of measurement which modify the measured system in an indeterminate manner. But if we follow the ideas of Subsection 6.3 we shall not interpret the failure to find simultaneous properties as a failure to discover the existing properties of a system, but rather as an indication that the system simply does not have the property pairs in question. This feature of microphysics is also in conflict with classical conceptions.

Clearly the ideas of "property" and "measurement" do not have the same meanings in microphysics that they enjoy in classical physics. A re-examination of these ideas and the formulation of concepts appropriate for the consideration of microphysical systems is essential if the confusion that results from the unconscious application of classical ideas is to be avoided. A number of questions require answers. What is meant by the phrase "the electrons in this beam have a definite momentum," or, in more general terms, by "the system S has the property q'"? How are the properties of microphysical systems discovered? Can a system have unobservable properties such as simultaneous position and momentum for an electron? What, if anything, is the meaning of an unobservable property?

7.1 Objective Properties. In microphysics, the fundamental classical assumption of objective physical properties of systems fails. The basis for this assumption and the reasons for its failure are sketched below.

As I write I cannot see Mars but, though I don't know where it is, I believe that Mars is "out there" somewhere. On what observational grounds is this belief justifiable? How can I know that Mars has a position if I don't know where to find it? Similar questions are often raised in beginning courses in philosophy. "How," the old chestnut goes, "do I know that the tree in my garden is still there when I'm not looking at it, or feeling its bark, or otherwise observing it? How do I know that the properties I see when I look at the tree are still 'out there' when I no longer give it attention?" The very structure of these questions makes it impossible to establish by observation either of the statements: "All the properties of the tree remain unchanged even when I'm off on a trip," or "all properties of the tree disappear or change in some unknown way when I'm not around to look at it." Despite my inability to establish the facts, however, I happily continue to believe that the tree placidly remains in my garden when I'm not observing it[1].

My attitude may be partially rationalized, perhaps, as follows. I have a theory that is consistent with all my observations of the tree; it is the simplest consistent theory I know. For example: Whether or not I look at the tree I know that on any sunny day I can find shade in its neighborhood; since the tree has some dead branches which may fall in a strong wind, I take care not to be in its neighborhood on windy days. In short, there is no aspect of my experience in conflict with the assumption that the tree is "out there" whether or not it is observed – i.e., that the tree has what philosophers call an "objective existence." In essence, the same remarks may be made in reference to our original question about the location of Mars. The assumption of the existence of "objective properties" may be justified, for macrophysical objects, in the same way that the objective existence of the tree is justified.

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[1] In fact, unless one is a professional philosopher, any other belief carries with it the danger of getting oneself committed.

Our belief in the objective properties of stones, trees, dogs, etc. rests in part on the fact that they may be observed without altering their properties in the course of observation. Could I be so sure of the objective existence of my tree if after each momentary glance at it the tree sprouted a new branch, or withered, or changed its species? Probably not. Clearly our general belief in the objective properties of the entities of our experience is based on the assumption that these entities may be observed without modifying their properties.

But the arguments that justify the assumption of objectively existent properties are simply invalid in the realm of microphysics. An old joke says that no Irishman is so poor that there isn't another Irishman who can live well on his leavings. Similarly, classical physics assumes (but not as a joke) that there is no physical system, however small, but what there isn't another one with which the first may be observed without changing its properties. The preceding sections suggest that this assumption is false in the realm of microphysics. For example, an observation of position within an accuracy of 0.1 Å on a free electron with a kinetic energy of 10 eV necessarily has cataclysmic consequences for the electron; the measurement may alter the energy by hundreds of electron volts. If the process of observing a tree produced changes such as a cyclone striking the tree might make, the processes of tree observation and accurate electron position measurement would have analogous features.

A more subtle example of the modification of properties by the process of observation may be drawn from the diffraction experiments described in Chapter 4. Observation of whether each electron passes slit a or slit b (Fig. 4-7) cannot be effected without completely changing the two-slit pattern obtained when no observation of electron passage is made. Clearly, not all the measurable properties of microphysical entities can be observed "gently."

The argument for objective properties based on the consistency of theory also fails. No adequate theory of the behavior of electrons is consistent with the assumption that the electron always has a location. In fact, that assumption gets us into trouble by leading to the false expectation of a two-slit interference pattern consisting of a simple sum of two single-slit patterns. Since the assumption of an objective location property leads to difficulties and since, by the nature of the question, it cannot be known that an electron always has a meaningful location property, a continued insistence that the electron must always be at some well-defined place can only be interpreted as a form of masochism. Remember that with the unobservable creations of our imagination we are free to do whatever is useful or comforting.

Thus the classical arguments for the existence of unknown objective properties fails for microphysical systems. The assumption that an electron, for example, must always have some position or some momentum and hence simultaneous position and momentum must be avoided despite the fact that a position (or momentum) measurement will always reveal some definite result.

7.2 Measurement and Property. To "know" a property of a system implies the knowledge of the outcome of a specific experiment on the system, namely that experiment which constitutes a measurement of the property. When the system is brought into interaction with an instrument capable of distinguishing the presence or absence of the property, the instrumental response will indicate that the system has the considered property. Note that each property is associated with a measuring instrument or mode of observation by which the presence of the property may be tested. To know a property then is to know the response to the associated apparatus for the measurement of the property if system and apparatus are brought into interaction.

With one important modification this conception of property can be carried over into microphysics. Instead of single systems our considerations will be applied to large ensembles of systems; for the most part we shall speak of the properties of ensembles. The reasons for this shift will appear in the work of this and the following section. Obviously it is related to the indeterminacy in the behavior of single microphysical systems and the regular behavior of suitable ensembles.

Naturally the large ensembles that are to be the objects of our considerations will not be composed of mixtures of such different systems as free electrons, uranium nuclei, cats, and dogs; instead they will be ensembles of systems of the "same kind" – hydrogen atoms, or methane molecules, etc. Each ensemble of interest will be characterized by a set of fixed or static (time independent) properties (e.g., system composition) that are the same for all the systems included in the ensemble. Naturally this does not imply that all systems in an ensemble with given fixed properties (systems of the same kind) are necessarily identical in all measurable properties. Systems of the same kind will differ in their so-called dynamical or variable properties. Our main concern is with these dynamical properties in ensembles of systems of a given kind.

We wish then to examine with care and detail the meaning for microphysics of the assertion: The systems (all of the same kind) in a given ensemble have a certain dynamical property q' (say). As we have seen, to know a property means first of all to know the outcome of a suitable measurement or test for the property. The test will consist, in general, of some arrangement of apparatus, directions for arranging an interaction between the apparatus and the systems to be observed, and a specification of two classes of observable responses that result from the interaction of the apparatus and the observed system. The two response classes may be labeled "yes" and "no" (or "y" and "n"). We mean by a "test instrument for property q'" – the test will be indicated by the symbol $\mathscr{T}(q')$ – the apparatus, the method for arranging interaction, and the specification of the "yes" and "no" responses. For example, the test for the property, "the x component of position of an electron is x' with uncertainty Δx," might consist of a microscope, a source of illumination, directions for location and orientation of the microscope and the sources, and the specification of the "yes" and "no" responses as the appearance of a scintillation within (yes) or outside (no) a specific area on a screen. The apparatus $\mathscr{T}(q')$ forms a part of the definition of the property q'.

The systems in an ensemble will be said to have property q' if, and only if, it is certain that the application of the test $\mathscr{T}(q')$ to every system in the ensemble will result in a "yes" response. This definition is empty, however, unless it is possible to produce ensembles of systems with property q', i.e., to prepare collections of systems such that the interaction of each system in the collection with $\mathscr{T}(q')$ results in a "yes" response. The process of producing an ensemble with the property q' will be called "preparation" or "preparatory measurement" of property q'. In general, a preparation also requires physical apparatus, methods for its disposition, and specification of classes of "yes" and "no" responses. For example, an ensemble of photons may be prepared with the property, "the energy of each photon is 2 eV within an uncertainty of 0.1 eV" by directing a beam of radiation at a grating (Fig. 7-1) and selecting, by means of a suitable arrangement of slits, only the photons that are deflected by a particular angle θ' (within an uncertainty $\Delta\theta$). Here the "yes" response is successful passage of the slits, the "no" response is failure to pass the slits[2]. An apparatus that prepares an ensemble with property q' will be designated by the symbol $\mathscr{P}(q')$. Both a preparation, $\mathscr{P}(q')$, and a test, $\mathscr{T}(q')$, are essential to the determination of the meaning of the property q'. The property is undefined if no preparation, $\mathscr{P}(q')$, exists for a presumed test $\mathscr{T}(q')$, or if no test exists for a presumed preparation, $\mathscr{P}(q')$. Note that a test process looks backward; there need be no interest in the properties of the tested systems after the test is over. (The photoelectric effect, for example, could be used in a test for the energy of photons. In the test the photons are absorbed, and no photon properties remain.) In the process of preparation, however, we look to the future; the property of the systems after preparation is of concern. For this reason the $\mathscr{P}(q')$ process is also called a "predictive measurement."

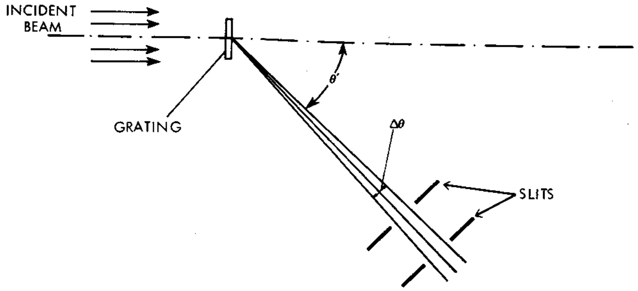

Fig. 7-1. A preparatory measurement for photon energy

Often the same instrument may serve both as a test and as a preparation of a property q'; an instrument of such versatility will be

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[2] Quite frequently preparations in microphysics have characteristics similar to those of this example. An apparatus that prepares q' transmits only systems that test successfully for q'; transmission is the "yes" response.

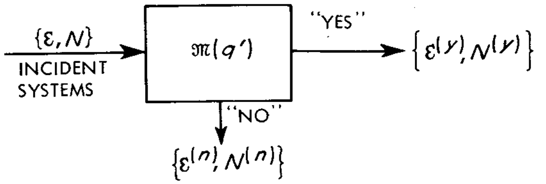

Fig. 7-2. Diagrammatic representation of a measurement instrument $\mathscr{M}(q')$. The incident ensemble, $\mathscr{E}$, is separated into ensembles $\mathscr{E}^{(y)}$, $\mathscr{E}^{(n)}$. ($N, N^{(y)}, N^{(n)}$ are the numbers of systems in the ensembles, $\mathscr{E}$, $\mathscr{E}^{(y)}$, $\mathscr{E}^{(n)}$). The $\mathscr{E}^{(y)}$ ensemble consists of systems that produced "y" responses in $\mathscr{M}(q')$. By definition the $\mathscr{E}^{(y)}$ ensemble has property q'. $\mathscr{M}(q')$ is also a test instrument for property q'; a system that produces a "y" response on interaction with $\mathscr{M}(q')$ is said to exhibit q' on test.

called a "measurement" and designated by the symbol $\mathscr{M}(q')$[3]. We shall assume that an $\mathscr{M}$ instrument exists for every definable system property. A useful diagrammatic representation of an $\mathscr{M}(q')$ instrument is shown in Fig. 7-2.

Consider the test of an ensemble which does not have the property q' with $\mathscr{T}(q')$. If the systems were classical, none of the systems would produce a "yes" response; if the systems, by hypothesis, do not have a q' property they will not test for q'. In microphysics we have a wider set of possibilities. We may have ensembles such that no system produces the "yes" response in a $\mathscr{T}(q')$ test. But there are also ensembles in which sometimes the response to the test for q' is "yes" and sometimes "no." If this behavior were found for classical systems, one would conclude that some of the systems have the property q' and the others do not. For microphysical systems, however, this interpretation is dangerous and is to be avoided in general. In the two-slit interference experiment, we began with an ensemble of electrons prepared with a certain momentum. If we test for position at the slits, we always discover an electron at either one or the other of the two slits. The ensemble then does not have a well-defined position at either slit. But it is improper, as we have argued in previous sections, to conclude that the ensemble consists of electrons with a location property at either one slit or the other.

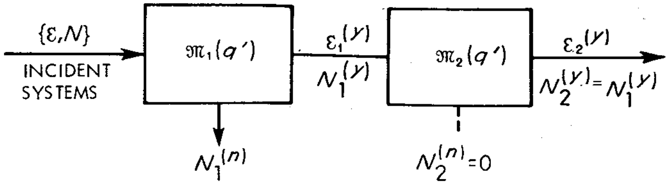

Fig. 7-3. Definition of property q' by means of $\mathscr{M}(q')$ instruments.

If some but not all systems of a given ensemble produce a "yes" response in $\mathscr{T}(q')$ we shall say simply that these systems "exibited the q' response on test." However, we must avoid the conclusion (which may in special cases be true) that the systems that tested successfully for q' in fact had the property q' before the test.

The consideration of the foregoing paragraphs may be summarized in the following definitions:

(a) "A system may have a property q'" or "property q' is measurable" means: There exists an instrument $\mathscr{M}(q')$ such that whenever systems interact successively with two such instruments ($\mathscr{M}_1(q')$, $\mathscr{M}_2(q')$ – see Fig. 7-3) every system that produces response "yes" in $\mathscr{M}_1(q')$ also produces response "yes" in $\mathscr{M}_2(q')$; if N systems are incident on $\mathscr{M}_1(q')$ and $N_1{}^{(y)}$ systems produce response "yes," all these systems produce response "yes" on interaction with $\mathscr{M}_2(q')$[4].

(b) "A system with a given 'history' has the property q'" means: In a large number of previous experiments it has been found that every system (of the same kind) with the given "history" produces a "yes" response on interaction with an $\mathscr{M}(q')$ instrument; i.e., the specific "history" constitutes a preparation of property q'.

Several implications of these definitions are worth noting explicitly:

(1) An unobservable property has no meaning. A property is defined for a system only if instruments that prepare and test the property are given. Often the Heisenberg principle is formulated in terms of the idea

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[3] The definition of a property as formulated in preceding paragraphs in terms of $\mathscr{T}$ and $\mathscr{P}$ instruments is incomplete. The reader may enjoy discovering its ambiguities and supplying the details needed for its completion. The definition in terms of $\mathscr{M}$ instruments, however, is unambiguous. Although $\mathscr{M}$ instruments are convenient for theoretical considerations, and are sometimes useful in the laboratory, most experimental investigations employ preparation ($\mathscr{P}$) and test ($\mathscr{T}$) instruments.

[4] Unless the property q' does not change with time, the interaction with $\mathscr{M}_2$ must follow immediately after the completion of the interaction with $\mathscr{M}_1$.

of unobservable properties; thus it is said that the simultaneous precise position and momentum of an electron is unobservable. It may be useful to formulate the principle again without resort to this objectionable idea.

The Heisenberg Uncertainty Principle. For every real number x' (or p_x') and every positive number Δx (or Δp_x) an electron may have the property, "the x component of position (or momentum) is x' (or p_x') within an uncertainty Δx (or Δp_x) for arbitrary magnitude of the uncertainty." There exists no preparation capable of producing an ensemble with both an x component of position (for any x') with an uncertainty Δx and an x component of momentum (for any p_x') within an uncertainty Δp_x if the product $\Delta x \Delta p_x$ is less than a quantity of the order of Planck's constant, h.

(2) From the observation of a "yes" response in a test (or "measurement") of property q', it does not follow that the system had the property q' just prior to its interaction with the test apparatus; the system may be said to have had property q' before a test only if its history prior to the test constitutes a preparation of q'. Many errors of interpretation stem from a failure to distinguish between the meanings of "a system had property q'" and "a system exhibits property q' on test."

(3) The meaning of sentences such as "the electron does not always have a location property" may be clarified. Suppose that experiments on many ensembles of electrons prepared with property q' ("q' ensembles") reveals that the electron position property, x' (x component), is distributed over a range larger than D in every q' ensemble[5]. In other words, in any q' ensemble, different values of x' are found in tests on different electrons, and the difference between the largest and smallest position values is always greater than D. It follows from preceding definitions that in q' ensembles, electrons do not have the property "location within a range $d<D$"; q' ensembles with the additional property, "location within range d," simply do not exist[6].

The complex abstractions required for the general definition of "property" have surprisingly elementary exemplifications. Let's follow

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[5] See Subsection 7.5 for a definition of the phrase, "distribution of position in an ensemble."

[6] For example, in ensembles with momentum uncertainty Δp_x, electrons do not have a position property with uncertainty Δx unless $\Delta x \, \Delta p_x > h$.

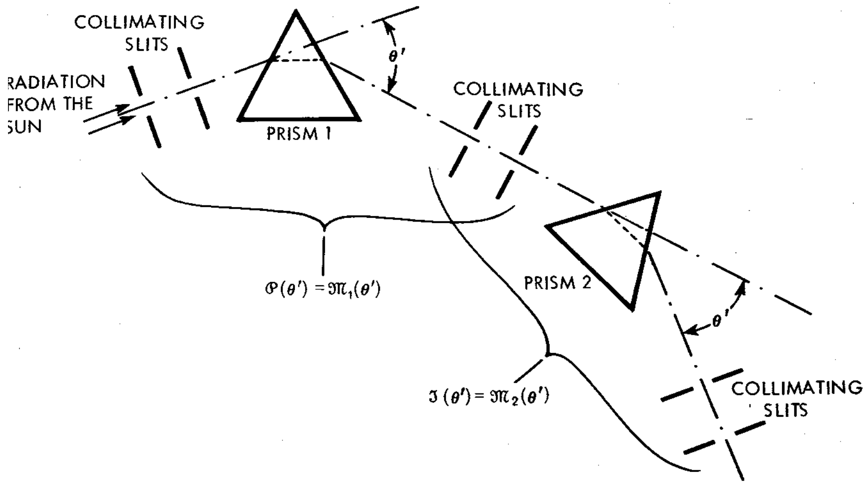

Fig. 7-4. Newton's measurement of color properties.

Newton in his discovery of the property of color. Newton was led, by considerations into which we need not delve (every student of physics must read his fascinating "Opticks"), to pass light from the sun through a prism and to observe its spectrum. No doubt many men had noticed the appearance of color when cut glass was illuminated. But Newton went a step further; he selected a small range of the colors produced by one prism (i.e., the radiation deflected by well-defined angle, θ') by means of slits (Fig. 7-4) and passed this range of color through a second prism. In passing through the second prism the light was not spread out again as the original light from the sun had been spread into a spectrum by the first prism. Instead, the deflection in the second prism was again just θ', the angle through which the selected radiation had been deflected originally. By these observations Newton had discovered a "property" of light.

The first prism (with the slit arrangement defining a deflection θ') constitutes the process of selection the apparatus $\mathscr{P}(\theta')$; the second (with the same slit arrangement) is the test, $\mathscr{T}(\theta')$. Both $\mathscr{P}(\theta')$ and $\mathscr{T}(\theta')$ consist of the same equipment, i.e., the arrangement is also an $\mathscr{M}(\theta')$. The "yes" response in both is passage through the slits defining deflection θ'. Newton discovered that any radiation with a "yes" response in $\mathcal{M}_1$ (θ') leads to a "yes" response in $\mathcal{M}_2$ (θ') (see Fig. 7-4). It is the whole set of these observations which permits us to speak of a property of radiation; the property in question may be formulated as "deflection by θ'," or by the associated "color" as judged by eye, since the deflection angle and color are correlated.

The complexity of the definition of a dynamical property indicates that the discovery of properties is no simple matter. As an example of the difficulty, let's try to find a dynamical property of electrons by imitating Newton. Suppose a beam of electrons is produced by the procedures used in diffraction experiments and directed on a film of material which transmits and scatters the incident electrons. From the scattered electrons, those deflected by an angle θ' are selected for consideration. Do the electrons so prepared have a "property" distinguished by their deflection through θ'? The answer is yes if an $\mathcal{M}$ apparatus exists such that all electrons scattered through θ' produce "yes" responses in $\mathcal{M}$. Since a measurement, $\mathcal{M}$, with the desired property is not known, the preparation does not determine a property of electrons.

It is the business of experiment to search out all observable properties of a system and to discover all the correlations among them. Thus, for example, it is found that photons with the property, "deflection, θ', by a particular prism," also have definite energy. Therefore, the property, "deflection by θ' in the prism," implies the property "a definite energy E'," or "measurement of θ' by the prism" is equivalent to a "measurement of the energy E'."

A very different type of correlation is frequently met. In the diffraction experiment we first produce an ensemble of electrons with a well-defined momentum property (Fig. 4-1). The fluorescent screen (or photographic plate) is a device which tests for a range of position properties on those electrons of the ensemble that get through the slits placed between the source and the screen. Electrons (prepared with the same momentum) are found at different locations on the screen. We do not, therefore, have a unique correlation between the initial momentum property and a tested position property. However, there is a well-defined correlation between the initial momentum and the distribution in the tested positions – i.e., between the initial momentum and the diffraction pattern. Indeed, most of the correlations that are discovered

in the realm of microphysics are of this general character. It is important to recognize that the position distribution does not have meaning for a single electron with definite momentum but is a characteristic of an ensemble of electrons prepared initially with definite momentum.

Theory must provide a set of general rules by which the observable correlations among properties may be deduced. Thus, theory should be able to predict the correlation between deflection, θ', in a prism and photon energy, E', or to predict the correlation between incident momentum and the distribution in position that appears on a photographic plate in a diffraction experiment (given the geometrv of the apparatus). However, theory need not describe imagined but unobservable features of any process; theory need not tell us, for example, how the ensemble of electrons in the incident beam gets through the slits in the process of forming a diffraction pattern.

7.3 Incompatibility. The Heisenberg principle states that it is impossible to prepare an ensemble with position and momentum uncertainties (Δx, Δp_x) such that $\Delta x \Delta p_x$ is less than h (approximately). This principle suggests (Section 5) that many other pairs of properties – e.g., location of the electron in a hydrogen atom, within $\Delta x \sim 0.1$ Å and an exact energy property of the atom – are not simultaneously measurable. We have not yet had occasion to mention microphysical properties that can be measured simultaneously. Examples are easily discovered.

An electron may be located in space and this location may be converted into components along three chosen axes; thus the components x', y', z' of the position of an electron are simultaneously measurable. Similarly, the direction and magnitude of the momentum of an electron may be measured [e.g., by the arrangement employed to produce the incident beam (Fig. 4-1) in the diffraction experiments]. From the direction and magnitude, the three perpendicular components of momentum along the axes of a chosen coordinate frame are easily obtained. Less trivially, the x component of position and y component of momentum may be measured together. A somewhat different example – not quite as trivial as it may seem at first sight – is afforded by the pair of properties "position located in a segment Δx" and

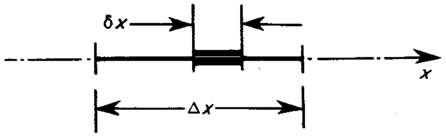

Fig. 7-5. An example of simultaneously measurable properties.

"position located in a segment δx," where δx is inside the Δx segment (Fig. 7-5). A measurement of the property "location in δx" is simultaneously a measure of "location within Δx." For any electron with the "δx" property I am certain to get a "yes" response if I test for the "Δx" property[7] (i.e., if I am certain of finding, on test, a location within δx, it is certain that I shall find, on test, location within Δx). Properties which can be measured simultaneously are said to be compatible. The properties q', r', of a system, are compatible if it is possible to prepare an ensemble of such systems that have simultaneously the properties q', r'; this means that it is certain that if a test for q' is made by $\mathscr{T}(q')$, a "yes" response is obtained and a similar statement is true of tests for r'.

The existence of compatible properties is hardly surprising; classical ideas lead us to expect compatibility. For a Newtonian particle exact position and exact momentum are compatible; in classical theory electric and magnetic fields are compatible. If, on classical assumptions. a system has a property, q', and on observation a property, r', is observed, then properties q' and r' are compatible. For classical theory tacitly assumes that the measurement of r' on a system with property q' can be made with such care as not to change q'; after the r' measurement (preparation), the system has properties q' and r'. (We assume that the r' measurement is made immediately following the measurement of r'.)

When an ensemble of systems prepared in some manner is such that on test for q' no system produces a "yes" response, it is said that the ensemble does not have the property q'. A property q'' is said to be exclusive to (or to exclude) a second property q' if no ensemble with property q'' has the property q'; i.e., whenever an ensemble has property q'' it is certain that q' will not be exhibited on test. It appears to be a physical fact that whenever a property q'' is exclusive to q', then q' is exclusive to q'' – the relation of exclusiveness is symmetric. Thus we may speak of q' and q'' as an exclusive pair of properties. The relation of exclusiveness is well known both in classical physics and in microphysics. Two different electron position properties are exclusive; if an electron is at a position x' it is certain that it is not at x'' if $x'' \neq x'$. Similarly, two different energy values (of a hydrogen atom, say) are exclusive. In general, both in classical physics and microphysics, any two distinct values of a given variable such as position momentum, energy, angular momentum, etc., are exclusive (see Section 7-4).

In classical theory, measurement without disturbance of the properties of the measured system is assumed to be possible. On systems with property r', any second property s' is either never found on measurement or sometimes found. If "never," then r', s' are exclusive. If "sometimes," then there exist systems with both r' and s' properties; i.e., the r', s' properties are compatible. Clearly, then, any two properties, in classical theory, are either compatible or exclusive.

For microphysical systems, however, there are pairs of properties that are neither compatible nor exclusive. Certainly a position x' of an electron and a momentum p' are not compatible; by the Heisenberg principle it is impossible to prepare an electron with both exact position and exact momentum properties. But neither are these properties exclusive. On an ensemble of electrons at a definite position, some electrons with a definite momentum p' may be found on test. Pairs of microphysical properties that are neither compatible nor exclusive will be called incompatible. Any exact position property is incompatible with any exact momentum property. It is easily seen from the symmetry of the relation of exclusion that incompatibility must also be a symmetric relation; if q' is incompatible with r', then r' is incompatible with q' and we may speak of a pair q', r' of incompatible properties.

The fact that pairs of microphysical properties may be incompatible – neither simultaneously measurable nor exclusive – *constitutes the central difference between classical expectation and microphysical experience.*

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[7] This example indicates the nature of the ambiguity referred to in footnote 6 of this chapter. On an ensemble with property, "location in δx," tests for the property, "location within Δx," are certain to produce "yes" response although this is not a test for the property possessed by the ensemble.

In microphysics incompatibility is a far more frequently met relation between properties than is compatibility. Some examples of incompatible properties were indicated in Chapter 6. No exact electron position property is compatible with an exact energy property E' for the hydrogen atom (classically there is a set of position properties compatible with E' and a second set exclusive to E'). Any two components along distinct axes of the angular momentum of a system are incompatible (classically they are compatible). The properties "total energy E'" and "potential energy greater than E'" are incompatible (exclusive classically) in microphysical systems. The pairs x', p' (position, momentum properties) are compatible for any values of x', p' in classical physics, and always incompatible in microphysics. Different polarization properties (of radiation), which are treated in classical physics as exclusive, are generally incompatible; special pairs of polarization properties, however, are exclusive. Two linear polarizations polarized in orthogonal planes are exclusive, as are also left and right circular polarizations.

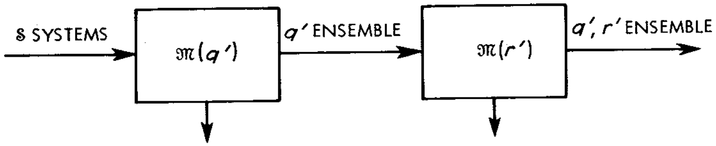

Fig. 7-6. Diagrammatic representations of the meanings of compatible, exclusive, and incompatible properties.

Fig. 7-6a. Compatible Properties. The $\mathcal{M}(q')$ and $\mathcal{M}(r')$ instruments may be interchanged. The ensemble transmitted by the second instrument – systems with "yes" responses in both instruments – has both q' and r' properties.

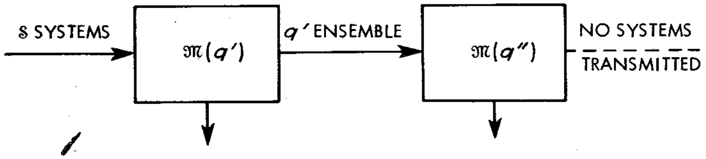

Fig. 7-6b. Exclusive Properties. None of the systems in the q' ensemble produces a "yes" response in $\mathcal{M}(q'')$.

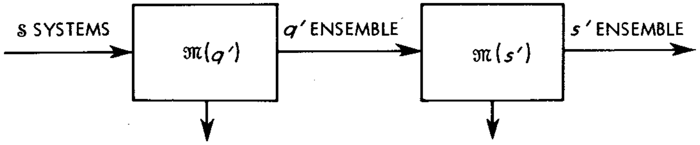

Fig. 7-6c. Incompatible Properties. Some, but not all, of the systems in the q' ensemble produce "yes" responses in $\mathcal{M}(s')$ and form an s' ensemble. However, this ensemble does not have the property q'.

The three types of relations – compatible, exclusive, and incompatible – that have been discussed above are represented diagrammatically in relation to measurement instruments in Fig. 7-6.

7.4 Observables. In the laboratory, tests for specific properties, such as energy E' or position in Δx about x', etc., are less common perhaps than experiments designed to "measure" the "energy," or "position," etc., of an entity. It is necessary to understand what is meant by "energy," "position," in such expressions and to know what is implied by their "measurement."

Usage is not firmly fixed but, if no qualifications are specified, "an energy (or position, etc.) measurement on a system" generally means a simultaneous test for all the exact energy (position, etc.) properties the system may have. The "energy" of electrons in a beam might be "measured," for example, by the apparatus illustrated in Fig. 7-7. An electron incident on the photomultiplier produces a pulse that is displayed on the cathode ray tube of the oscilloscope; from the "height" h' of the observed pulse a unique energy value E' is determined. It is useful to consider this instrument as a test apparatus for any of the energy properties that it "measures." The instrument has the following characteristics: If an ensemble prepared with energy E' is incident upon the apparatus, each electron produces a pulse of height

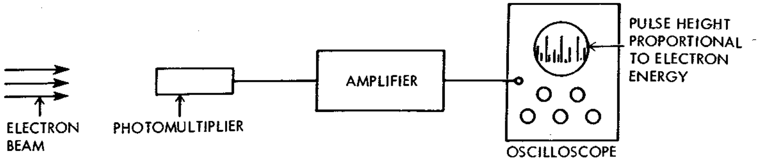

Fig. 7-7. An instrument for testing a range of the energy properties of electrons.

h'; electrons in any ensemble with a property exclusive to E' never produce a pulse of height h'. In the language of Section 7.2, height h' is the "yes" response, and the set of heights other than h' is the class of "no" responses for the property E'. Each of the responses of the instrument – each height – is a "yes" response for one of a set of exclusive properties, the different energy properties of electrons.

The foregoing example illustrates the general characteristics of an instrument that can test simultaneously for a set of exclusive properties – say the set of n properties, $\{q^{(1)}, q^{(2)} \ldots q^{(n)}\}$. The instrument must be capable of $n+1$ responses (or response classes) $R^{(1)}, R^{(2)}, \ldots R^{(n)}, R'$, where $R^{(k)}$ is the "yes" response for the property $q^{(k)}$ and all the other responses comprise the "no" class for this property (k may take any value from 1 to n). The response R' to a system means that none of the properties of the set $\{q^{(1)}, q^{(2)}, \ldots q^{(n)}\}$ is exhibited[8]. As with the tests described in Subsection 7.2, the observation of a response $R^{(i)}$ does not in itself imply that the system tested had (before the test) the property $q^{(i)}$.

It has been noted that the words "energy," "position," etc. (in "energy measurement," "position measurement," etc.) refer to sets of exclusive properties – all exact energy properties, all exact position properties, etc. These sets of properties are called "observables"; the set of all energy properties (of an electron, say), $x', x'', \ldots$ is the position observable, x, of an electron[9]. The property sets belonging to observables have two characteristic features: (1) The different properties in the set are mutually exclusive; (2) if all the properties in the set constituting an observable are tested simultaneously on a system, some one property in the set is certain to be exhibited (e.g., if tests for all possible positions are made simultaneously on an electron, the electron is certain to be "found" somewhere). Any set of properties that satisfies condition (2) is said to be "complete."

The foregoing considerations suggest a general definition: any set of properties – say $q^{(1)}, q^{(2)}, \ldots$ – of a system that are (1) mutually exclusive and (2) complete, is called an observable, q, of the system. An instrument that can test simultaneously for all the properties in observable q will be called a "test of observable q" and denoted by $\mathscr{T}(q)$[10].

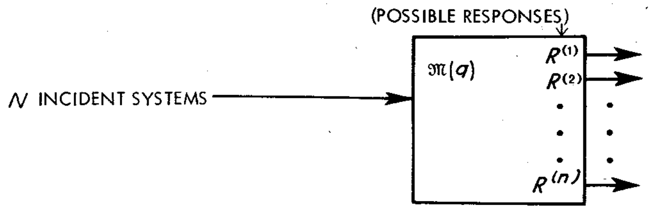

Fig. 7-8. Schematic representation of a measurement instrument for an observable q consisting of a complete set of exclusive properties: $q^{(1)}, q^{(2)}, \ldots q^{(n)}$.

If a test instrument $\mathscr{T}(q)$ for an observable, q, is such that after a test exhibiting any one of its properties – say $q^{(i)}$ – the tested system has the property $q^{(i)}$, the instrument will be said to be a measurement of q and will be denoted by $\mathscr{M}(q)$. Such an instrument not only can test for q, but can be used to prepare systems with any of the properties in the observable. Figure 7-8 provides a diagrammatic representation of an $\mathscr{M}(q)$ instrument[11].

Two observables, q and r, are called "compatible observables" if the pairs of properties $q^{(i)}$; $r^{(j)}$ ($q^{(i)}$ any property of q, $r^{(j)}$ any property of r) are either compatible or exclusive[12]. (For any two observables, r, s, not all the pairs, $r^{(i)}, s^{(j)}$, can be exclusive. (Why not?) Two compatible observables, q, r, uniquely determine a third observable (denoted by "q & r") consisting of the set of all the compatible pairs $q^{(i)}, r^{(j)}$, that may be formed from q and r. Two observables, say q, s, are said to be "incompatible observables" if among the pairs $q^{(i)}, s^{(j)}$, at least one is

[8] A photographic plate, for example, tests simultaneously for the set of positions at which the active grains of the plate are located.

[9] In classical physics it is customary to call such sets of properties "variables" (e.g., "energy variable," "position variable").

[10] Figure 7-7 illustrates a $\mathscr{T}(E)$ instrument for the observable E (energy of electron). Examples of observables other than those already mentioned are easily invented; the pair of properties "position to the left of a given plane" and "position to the right of, or on, the plane" constitutes an observable. Any two exclusive polarization properties, e.g., right and left circular polarizations, form an observable.

[11] A spectroscope is an example of a measurement instrument for the observable, "energy of photons."

[12] Our definitions imply that all classical observables (variables) are compatible.

an incompatible pair. Obviously the position and momentum observables are incompatible; in this case no two position and momentum properties are compatible. Momentum and any polarization observable (for photons) are compatible. Indeed, every momentum property is compatible with every polarization property, i.e., photons may be prepared with any definite momentum and any polarization property. Two polarization observables, however, are incompatible. This fact follows from the observation that it is impossible to prepare photons which have two polarization properties simultaneously.

In classical physics the "dependence of an observable on time" is frequently used to describe the "motion" of a system; e.g., a motion of a free particle in one dimension is described by a function, $x(t)$, which specifies the "position" of the particle at each instant, t. Although the meaning of $x(t)$ is clear, the phrase, "dependence of an observable on time" is ambiguous. An observable is a set of properties (complete and exclusive) for systems of a definite kind, and the definition of an observable makes no reference to time. Therefore, the phrase quoted above is nonsense if interpreted literally. An observable has meaning for all systems of the same kind while functions such as $x(t)$ refer to a particular system in a particular motion. The function $x(t)$ tells us what property (in the set constituting observable x) the particular system under consideration has at any definite time; at t' the system has property x', at t'', the property x'', etc., where, at the different times, we are concerned with the properties in the observable x. We shall call a function, $q_S(t)$, which tells us the property in q that the system S has at time t, for a range of different times, the "motion of q" in S (often we shall use $q(t)$ instead of $q_S(t)$, but the fact that the function refers to a particular system, rather than systems of a certain kind, should be kept in mind.

The system properties that have been considered so far are properties at some time instant. It is possible in classical physics to define properties of a more complex character. Consider, for example (for classical "particles"), the property, "position x' now and position x'', t seconds later." A whole motion, as described by $x(t)$, is also a system property. Such properties may be tested and prepared. The test for a motion $x(t)$, consists of a succession of tests of the position observable at a large number of closely spaced intervals. Also, as is known from classical mechanics, if a motion, $x(t)$, is observed on a particle, other particles of the same kind can be prepared – by selecting the position $x(0)$ and the velocity $v(0)$ that the motion $x(t)$ has at $t = 0$ – so that all the suitably prepared particles have the same motion.

In microphysics, however, motions of observables are not, in general, system properties. Consider an electron. Suppose that the position observable is measured at some instant, $t = t^0$, and again at t'. The position properties exhibited, say, are found to be x^0, x'. By a large extension of the number of measurements over a succession of instants, the function $x(t)$ can, in principle, be tested, but let us restrict ourselves for the moment to the consideration of the pair of properties: x^0 at t^0, and x' at t'. Is this "composite property" a system property? The test of the putative property is clear; it consists of $\mathscr{T}(x^0)$ at t^0 and $\mathscr{T}(x')$ at t'. Beside the test, however, a preparation is required. But electrons cannot be prepared so that a test $\mathscr{T}(x^0)$ at t^0 and $\mathscr{T}(x')$ at t' are both certain to succeed ("yes" responses in both tests). We can prepare property q^0 at t^0. To arrange that q' at t' is also certain, the electrons with property q^0 must be selected for some additional property that is compatible with x^0. In classical physics the additional property is the velocity, or momentum. An additional property for electrons that is compatible with x^0, such that when x^0 and the additional property are prepared the electron is certain to test for x' at t' does not exist. If, now, we consider a motion – a large set of x properties at a large number of instants – the problem is greatly magnified. In general, microphysical systems cannot be prepared so that it is certain that they will execute specific motions. This simply reflects the observation that the behavior of microphysical systems is indeterminate.

Microphysical and classical theories, clearly, must have very significant differences. In classical physics, theory attempts to derive the motions of observables, i.e., functions such as $x(t)$, from simpler system properties (the position and velocity at $t = 0$, say) and general laws. Microphysical theory has no interest in "motions" of "observables" since such motions are not system properties. Functions, such as $x(t)$, which play so large a role in classical physics, are without precise meaning in microphysics.

7.5 Distributions.

Because of the indeterminate characteristics of microsystems, the objects of most interest in microphysics are not single systems but ensembles of systems. In effect, "indeterminacy" means that regular, repeatable behavior in single systems is not found. Ensembles are used to investigate relations among the statistical properties of systems. Examples of interesting statistical regularities have been noted: In diffraction experiments the "distribution in position" of the detected electrons forms a definite pattern; in radioactive decay of neutrons the fraction of the number of neutrons remaining after time t is a definite exponential function of t. Clearly, ensembles and their statistical properties are of importance for microphysical investigations.

The central concept in statistics is that of the distribution of an observable. For the study of microphysics it is necessary to understand not only the meanings of microphysical "property" and "observable" but also the general characteristics of distributions in microphysical ensembles. Because of the differences between classical and microphysical concepts of property, the meaning of a "distribution" for microphysical ensembles is not quite the same as it is for classical ensembles.

Distributions in Classical Ensembles. A distribution is defined relative to a particular ensemble and a particular observable – say ensemble $\mathscr{E}$ and observable q. Examples are: (1) The distribution in "height" (observable) in the "set of all 20-year old males in New York State" (ensemble); (2) the distribution in "heads, tails" (this pair of properties forms the observable) in the "set of a thousand pennies" (ensemble); (3) the distribution in position (observable) of the "molecules in a container at time t'" (ensemble)[13]. To avoid needless complication it is useful to suppose that the observable, q, with which we are concerned is "discrete," i.e., that the observable consists of an enumerable set of properties – for instance, $q^{(1)}$, $q^{(2)}, \ldots$, $q^{(n)} \ldots$[14]. (The set may be finite or infinite.) Let N be the number of systems in $\mathscr{E}$ and $N^{(k)}$ the number of systems in $\mathscr{E}$ with property $q^{(k)}$. The number designated by the symbol $\{q^{(k)} | \mathscr{E}\}$ is defined by:

$$\{q^{(k)} \mid \mathscr{E}\} \equiv N^{(k)}/N = \text{fraction of the systems with property } q^{(k)} \text{ in } \mathscr{E}. \tag{7-1}$$

Note that

$$\sum_k \{q^{(k)} \mid \mathscr{E}\} = \sum_k N^{(k)}/N = \frac{1}{N}\sum_k N^{(k)} = \frac{N}{N} = 1, \tag{7-2}$$

where the sum over k is extended over all the properties in q. The set of fractions $\{q^{(k)} | \mathscr{E}\}$ for $k = 1, 2$, etc., specifies the "distribution of q in $\mathscr{E}$."

The set of numbers $\{ q^{(k)} | \mathscr{E} \}$ for $k = 1, 2, \ldots n \ldots$ may be considered as a function defined on the domain consisting of the properties in the observable q; the ensemble determines, for each $q^{(k)}$, a unique number, $\{ q^k | \mathscr{E}\}$. The function so defined is the distribution function for q in $\mathscr{E}$. This function will be denoted by $\{q | \mathscr{E}\}$; the value of this function at the property $q^{(k)}$ is $\{q^{(k)} | \mathscr{E}\}$. Thus the "distribution of q in $\mathscr{E}$" and the "function $\{q | \mathscr{E}\}$" have the same meaning; both specify the association of the number $\{q^{(k)} | \mathscr{E}\}$ to the property $q^{(k)}$ ($k = 1, 2$, etc.).

The number $\{q^{(k)} | \mathscr{E}\}$, the fraction of the systems in $\mathscr{E}$ with property $q^{(k)}$, is also the probability that a system chosen at random from $\mathscr{E}$ has property $q^{(k)}$. Clearly the probability of choosing any particular system from the ensemble is $1/N$; since $N^{(k)}$ systems in $\mathscr{E}$ have property $q^{(k)}$, the probability that the system has $q^{(k)}$ is $N^{(k)}/N \equiv \{q^{(k)} | \mathscr{E}\}$. Therefore, the function, $\{q | \mathscr{E}\}$, is also called the probability distribution of q in $\mathscr{E}$. (Examples of several $\{q | \mathscr{E}\}$ functions are displayed, in graphical form, in Fig. 7-9.)

Fortunately it is not necessary to measure the observable q on all the entities (systems, objects) in a large ensemble, $\mathscr{E}$, to obtain the distribution $\{q | \mathscr{E}\}$. No one would measure, for example, the heights of all 20-year-old males in the United States to obtain a distribution function. It is sufficient to make measurements on a relatively small, but randomly selected, "sample" of systems from $\mathscr{E}$ (the sample is a "subensemble" of $\mathscr{E}$; we denote it by $\mathscr{E}_s$). The distribution function, $\{q | \mathscr{E}_s\}$, obtained by measurement on the sample, will not, in general, be equal to the "true" distribution $\{q | \mathscr{E}\}$. By taking a sufficiently large

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[13] Note that ensemble properties may change in time as in example 3; time has no significance, however, in relation to example 2.

[14] The restriction to discrete observables is not a very significant one. A continuous observable containing a continuous range of properties is always an idealization and cannot be measured with complete precision. If the x component of position, for example, is measured with an accuracy of about Δ, the observable that is measured is composed (in a rough sense) of a set of nonoverlapping intervals of size Δ covering the whole x axis. This observable is discrete.

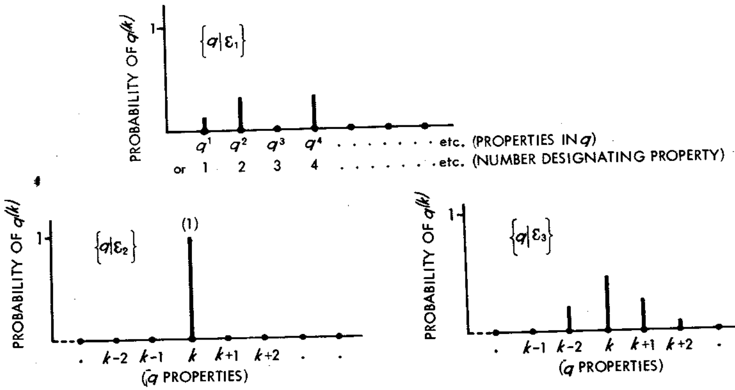

Fig. 7-9(a,b,c). Plots of distributions of q in various ensembles, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$. The distribution of q in $\mathcal{E}_2$ is said to be sharp.

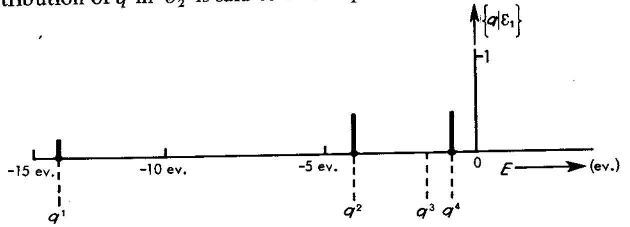

Fig. 7-9d. For quantitative properties the distribution is generally given as a function of the numerical measure of the property. The $\{q|\mathcal{E}_1\}$ of Fig. 9a is plotted on the assumption that $q^{(k)}$ is the property "energy = $-13.6\ \text{eV}/k^2$."

sample, however, we can expect that the sample distribution will resemble the true distribution closely (or, more accurately, the probability that sample distributions differ from ensemble distributions by more than a given small error can be made small). Even when the sample is "sufficiently large," it may still be only a small part of the original ensemble. (We shall not be concerned here with the question of how large a sample must be to achieve a particular level of accuracy or confidence in the distribution.)

Distributions in Microphysical Ensembles. The development in the preceding paragraphs cannot be directly applied to microphysical ensembles. In general, and particularly if nothing is known about the preparation of the ensemble, no clear meaning can be given to the idea that a system in the ensemble has one or another of the set of properties in a microphysical observable, q (see Sections 7-1, 7-2). To be sure, a test for q on a system in $\mathcal{E}$ is certain to "exhibit" one or another of the properties in q, but this does not imply that the system had the property it exhibits (on test) before the test was made. If all the systems in $\mathcal{E}$ are tested, the number $N^{(k)}$ of the systems that exhibit $q^{(k)}$ may be counted for each of the properties in q, and these numbers may be used to define a distribution function as in the classical case.

But what use can be made of the function so obtained? Just after the test we are left with an ensemble quite different, in general, from the one just before the test. Though the number $\{q^k|\mathcal{E}\}$ is obviously the probability of observing in $\mathcal{E}$ the property $q^{(k)}$, on test, this information is of no use since the original ensemble ($\mathcal{E}$) no longer exists. (Of course the systems remain; $\mathcal{E}$ has been changed into an ensemble with different properties.)

However, these difficulties are avoided easily. A sample, $\mathcal{E}_s$, consisting of a small fraction of the ensemble $\mathcal{E}$, is selected at random. Even though $\mathcal{E}_s$ is only a small part of $\mathcal{E}$ we assume that the number of systems, N_s, in $\mathcal{E}_s$ is "large" (the meaning of "large" will be explained a little later). By test for q on $\mathcal{E}_s$, the number, $N_s{}^{(k)}$, of the systems in $\mathcal{E}_s$ that exhibit property $q^{(k)}$ can be counted and the fractions $N_s{}^{(k)}/N_s \equiv \{q^{(k)}|\mathcal{E}_s\}$ computed. When the number, N_s, is "large," an increase in the size of the tested sample is unlikely to change materially the distribution function that is obtained. Stated another way, if several samples of "sufficiently large" size are used, – say $\mathcal{E}_{s1}$, $\mathcal{E}_{s2}$, etc – distribution functions, $\{q|\mathcal{E}_{s1}\}$, $\{q|\mathcal{E}_{s2}\}$, etc., differ so little that they may be considered for practical purposes as equal (i.e., the differences constitute tolerable errors). Since we are now concerned with matters of principle, we are free to assume that the ensemble is so large that satisfactory distribution functions can be measured on samples (containing "large" numbers of systems) that are negligibly small by comparison with the parent ensemble. (This would be true, for example, if the sample contained $\sim 10^{-8}N$ systems, where N is the number of systems in $\mathcal{E}$.)

The number of systems affected by the measurement is so small compared with the total number in $\mathcal{E}$ (by assumption) that the ensemble directly after the test is still essentially the same as the ensemble just prior to the test (the interval Δt, between "just prior" and "just after," is assumed to be zero). The distribution on the sample, then, is a distribution function for the ensemble $\mathscr{E}$. Its meaning is best expressed in terms of probabilities. The value of the distribution function at $q^{(k)}$, i.e., $\{q^{(k)}|\mathscr{E}\}$, is the probability of finding, directly after the tests that measure $\{q^{(k)}|\mathscr{E}\}$, the property $q^{(k)}$, in a test for q on a system selected at random from $\mathscr{E}$. Note that if $q^{(k)}$ is measured when an observable, q, is tested on a single system, no inference can be drawn about the property that will be exhibited in a second test of q immediately afterwards. (A test may even destroy the system tested; e.g., a test for photon energy may involve the absorption of the photon.) Tests generally modify the system tested in an unpredictable way. But a "distribution of q" in an ensemble may be measured without changing the "distribution of q." Hence distribution functions (but not system properties) may be considered to have an "objective" meaning.

If two observables, q, r, are incompatible, it is impossible to test for both observables simultaneously. However, it is easy to see that the distributions in the observables, q, r, can be measured simultaneously. Simply make the measurements for $\{q|\mathscr{E}\}$ and for $\{r|\mathscr{E}\}$ on different samples drawn from $\mathscr{E}$. Clearly it is possible, in principle, to measure the distributions of any set of observables on $\mathscr{E}$ at the same time. Note that "simultaneous distributions in incompatible observables" is meaningful, but that "distribution in two incompatible observables" is not.

If the systems in ensemble $\mathscr{E}$ have property $q^{(k)}$, i.e., $\mathscr{E}$ has been prepared by a $\mathscr{P}(q^{(k)})$ instrument, the distribution in q is given by

$$\{q^{(k)}\mid\mathscr{E}\} = 1 \quad \text{(all systems in } \mathscr{E} \text{ exhibit } q^{(k)} \text{ on test)}$$

$$\{q^{(i)}\mid\mathscr{E}\} = 0 \quad (i \neq k) \text{ (no system in } \mathscr{E} \text{ exhibits } q^{(i)} \text{ on test)} \tag{7-3}$$

Such a distribution is said to be "sharp" or "dispersion-free" (see Fig. ~~7~~-9b). In other words, if the distribution of observable q is sharp in $\mathscr{E}$, then the systems in $\mathscr{E}$ all have one property of the set q. We need not speak separately of "ensembles with specific properties" on the one hand and distributions in observables on the other. That an ensemble has a specific property may be indicated by stating that the distribution in an observable containing that property is sharp. Consequently all the measurable features of an ensemble are known if the distributions in all the observables measurable on the systems in the ensemble are known. Therefore, two ensembles, $\mathscr{E}_1\,\mathscr{E}_2$, such that

$$\{q\mid\mathscr{E}_1\} = \{q\mid\mathscr{E}_2\}$$

for all observables q are said to be equivalent.

The time dependence of the distribution of an observable can be measured on an ensemble that changes its properties in time. Select distinct samples $\mathscr{E}'$, $\mathscr{E}''$, $\mathscr{E}'''$, etc., from the ensemble given at $t = t^0$; measure the distribution of q at t' on $\mathscr{E}'$, the distribution of q at t'' on $\mathscr{E}''$, etc. Notice that no system is subjected to more than one measurement ($\mathscr{E}'$, $\mathscr{E}''$, etc., contain no system in common). Consequently the distribution at t'' (say) is a characteristic of the ensemble initially prepared at $t = t^0$ after the time interval $t'' - t^0$ (during which the ensemble was undisturbed). The measurements at t' do not influence the results of measurements at later times[15]. If the samples $\mathscr{E}'$, $\mathscr{E}''$, etc., together compromise only a small fraction of the total number of systems in the ensemble, the time dependence of the distribution may be measured without affecting the normal development of the ensemble.

The distributions in microphysical observables have many of the pleasing characteristics that are enjoyed by classical observables. Classical observables can be measured without changing the properties of a system, they are simultaneously measurable (compatible), and the motion of an observable can be measured without changing it in the course of measurement. None of these characteristics applies to microphysical observables; all apply to distributions of microphysical observables. These considerations suggest that distributions of observables will play, in microphysical theory, the important roles played by observables in classical theory.

Average Value; Uncertainty. Just how much information we may need about a distribution of an observable depends on the character of

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[15] Compare the measurement of the time dependence of a distribution with the measurement of the time dependence of an observable (i.e., the "motion of an observable") on a system (Subsection 7.4). In the latter case, successive measurement of q at t'' reflects not only the properties of the system prepared at t^0 but also the disturbances introduced by measurements at t', t''.

the questions we seek to answer. Often it is sufficient to know merely that tests are effectively certain to discover positions in a definite range about some particular position. For a rough and often useful characterization of a distribution, the "distribution parameters," "average value" and "uncertainty" (defined below) are employed.

(1) Average value (mean value, expectation value) of observable q in $\mathscr{E}$.

This is a physical magnitude denoted by $\{\bar{q}|\mathscr{E}\}$, or by the abbreviation $\bar{q}$ (when there is no ambiguity about the $\mathscr{E}$ to which it refers); it is defined by

$$\{\bar{q} \mid \mathscr{E}\} = \sum_k q^{(k)} \{q^{(k)} \mid \mathscr{E}\} \qquad (7\text{-}4)$$

(the sum is over the properties in q). The $q^{(k)}$ in the product $q^{(k)}\{q^{(k)}|\mathscr{E}\}$ is a physical magnitude – a quantitative measure of the property, $q^{(k)}$, in some specific set of units. The average value is expressed in the units used in the measure of $q^{(k)}$ (angstroms, electron volts, etc.). Since $\{q^{(k)}|\mathscr{E}\} = N^{(k)}/N$ (see Eq. 7-1) we have from Eq. 7-4

$$\{\bar{q} \mid \mathscr{E}\} = \frac{1}{N}\sum_k q^{(k)} N^{(k)}. \qquad (7\text{-}4a)$$

(2) Uncertainty (dispersion, root mean square deviation) of q in $\mathscr{E}$.

This is a physical magnitude with the units of $q^{(k)}$ (or $\bar{q}$) which is denoted by $\{\Delta q|\mathscr{E}\}$, or by the abbreviation Δq; it is defined by

$$\{\Delta q \mid \mathscr{E}\} = \Big(\sum_k (q^{(k)} - \bar{q})^2 \{q^{(k)} \mid \mathscr{E}\}\Big)^{1/2} \qquad (7\text{-}5)$$

Substitution of $N^{(k)}/N$ for $\{q^{(k)}|\mathscr{E}\}$ leads to

$$\{\Delta q \mid \mathscr{E}\} = \frac{1}{N^{1/2}}\Big(\sum_k (q^{(k)} - \bar{q})^2 N^{(k)}\Big)^{1/2} \qquad (7\text{-}5a)$$

For distributions of the sort shown in Fig. 7-10a, the probability of finding observable q (on test) with values in the range $q \pm \Delta q$ is of the order of one. The parameters, average value and uncertainty, provide a rough picture of the distribution. The average value and uncertainty provide a very poor characterization, however, of the distribution shown in Fig. 7-10b.

The distributions in all observables can be measured simultaneously; consequently the distribution parameters (average value, uncertainty)

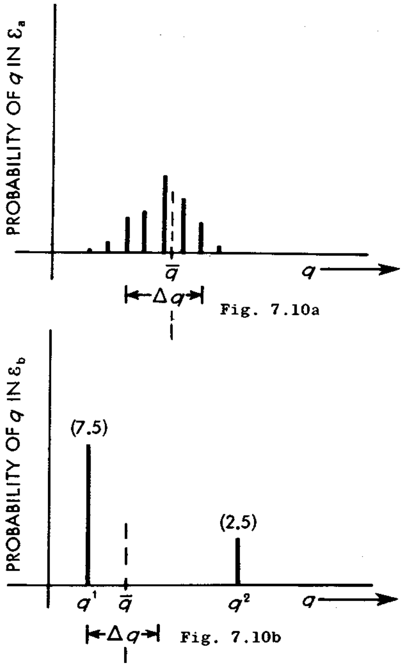

Fig. 7-10. Examples of the relation of the parameters "average value" $\bar{q}$ and "dispersion" Δq to distributions. In the distribution of q in $\mathscr{E}_a$ (Fig. 7-10a) the parameters provide a rough description of the distribution; in the case of $\mathscr{E}_b$ (Fig. 7-10b), however, the general character of the distribution is not described by $\bar{q}$, Δq.

may also be measured simultaneously. Though x and p are not simultaneously measurable on an electron with precision (i.e., simultaneous x and p is not an electron property), the average values $\bar{x}$, $\bar{p}$ in any ensemble may be known simultaneously and precisely. In fact, the time dependences of the average values of all observables $\bar{x}(t)$, $\bar{p}(t)$, etc., in a given ensemble are measurable, objective properties of ensembles. These average values have, relative to ensembles of systems, many of the characteristics the functions $x(t)$, $p(t)$, etc., of classical physics have relative to a single system.

If the distribution of q in $\mathscr{E}$ is sharp, i.e., if $\mathscr{E}$ has one of the properties $q^{(k)}$ in q, it follows from Eqs. 7-4 and 7-5, with the use of Eq. 7-3, that $\bar{q} = q_k$, $\Delta q = 0$ ($\mathscr{E}$ has property q_k).

It is not difficult to see that if the uncertainty of q in an ensemble is zero, $\mathscr{E}$ must have one of the properties in q. Thus an ensemble has a property contained in observable, q, if, and only if, $\Delta q = 0$.

Suppose that measurements of the properties in observable q are themselves inaccurate to within, roughly, a range of δ. The results of measurements on an ensemble with property $q^{(k)}$ will be distributed, because of the inaccuracy of observation, within the interval between $q^k - \delta$ and $q^k + \delta$. On the other hand, if a second ensemble (of the sort shown in Fig. 7-10a) has an average value close to $q^{(k)}$ (say in the interval $q^k - \delta/2$ and $q_k + \delta/2$) and has an uncertainty, Δq, less than δ, the distribution in the results of measurements will differ little from the distribution obtained for the ensemble with property $q^{(k)}$. Thus, by measurements of q with "error" δ, ensembles with an exact property (i.e., ensembles with $\Delta q = 0$), and ensembles with uncertainties $\Delta q < \delta$ are indistinguishable.

In our discussions of the Heisenberg principle, the symbols Δx, Δp_x were used to indicate roughly the ranges of position and momentum over which the position or the momentum of an electron, prepared by some given method, would be found on observation. These symbols correspond (again roughly) to the uncertainties in x and p_x (as defined by Eq. 7-5) in an ensemble of electrons prepared by the given method. The Heisenberg principle can be given an exact formulation in terms of uncertainties defined by Eq. 7-5 (see Section 9.5).

Intermediate Measurements on Classical and Microphysical Ensembles. The following considerations illustrate an error that is frequently made in the analysis of microphysical observations.

Let $\mathscr{E}$ be an ensemble[16] prepared in some specific way. Our problem is to determine, if possible, the distribution $\{y|\mathscr{E}\}$ with the aid of the following information. In measurement of the q observable on ensembles of the kind $\mathscr{E}$ (i.e., ensembles prepared by the same procedures used in the preparation of $\mathscr{E}$), only two properties $q^{(1)}$, $q^{(2)}$ are observed with the given probabilities $\{q^{(1)}|\mathscr{E}\}$, $\{q^{(2)}|\mathscr{E}\}$. The two ensembles into which $\mathscr{E}$ is divided on measurement of q with properties $q^{(1)}$, $q^{(2)}$ have known distributions in y which are designated by $\{y|q^{(1)}\}$, $\{y|q^{(2)}\}$.

The problem seems almost trivially simple. The ensemble $\mathscr{E}$ contains two subensembles with properties $q^{(1)}, q^{(2)}$; in these subensembles the distributions in y are known. The probability of y in $\mathscr{E}$ is therefore the probability of $q^{(1)}$ in $\mathscr{E}$ multiplied by the probability of y in the $q^{(1)}$ ensemble plus a similar probability relating to the $q^{(2)}$ ensemble;

$$\{y \mid \mathscr{E}\} = \{y \mid q^{(1)}\}\{q^{(1)} \mid \mathscr{E}\} + \{y \mid q^{(2)}\}\{q^{(2)} \mid \mathscr{E}\} \text{ (classical).} \tag{7-6}$$

This argument is correct for classical ensembles; it is false (in general) for microphysical ensembles. Its falsity may be established by experiment (see example below). The error consists in the assumption that $\mathscr{E}$ is composed of subensembles with properties $q^{(1)}, q^{(2)}$. But, in general, measurement on systems result in changes of their properties. The measurement of $q^{(1)}$, $q^{(2)}$ on $\mathscr{E}$ does not imply[17] that, prior to the measurement, the systems in the ensemble have either property $q^{(1)}$ or $q^{(2)}$. The given information is essentially irrelevant for the determination of the distribution of y in $\mathscr{E}$.

The argument leading to Eq. 7-6 (classical) was used in Chapter 4 in an effort to analyze the two-slit diffraction pattern (see Fig. 4.7). If the properties, α = "location behind slit a", β = "location behind slit b" (α and β are exclusive properties), are measured on the ensemble of electrons that pass the slits, equal numbers of electrons are found with the properties α and β, i.e., $\{\alpha|\mathscr{E}\} = \{\beta|\mathscr{E}\} = \frac{1}{2}$. If slit a is open and b is closed, all electrons that pass the slit system have the property α; if a is closed and b is open, they all have property β. The distributions $\{y|\alpha\}$ or $\{y|\beta\}$, where y is a position component on the photographic plate, i.e., the one-slit patterns when the electrons *have* property α, or property β, may be observed. By the argument leading to Eq. 7-6 (classical) it is concluded that the two-slit distribution should be ($\mathscr{E}$ is the ensemble in two-slit case)

$$\{y \mid \mathscr{E}\} = \{y \mid \alpha\}\frac{1}{2} + \{y \mid \beta\}\frac{1}{2};$$

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[16] For simplicity, assume that the ensemble is independent of time.

[17] These objections do not apply to classical ensembles. In classical physics a measurement reveals the property possessed by a system.

i.e., the two-slit pattern, according to Eq. 7-6, is a sum of single-slit patterns. Observation shows that this conclusion is false.

Clearly the distinction that must be made between "a system has a property $q^{(k)}$" and "a system exhibits, on test the property $q^{(k)}$" is of great importance in microphysics. Because the two phrases have the same meaning in classical physics the difference between them is easily overlooked; the consequences of this oversight are often catastrophic.

8 Determinism and State; Statistical Determinism

"He is no wise man that will quit a certainty for an uncertainty."

Samuel Johnson

A clear appreciation of the significance of the classical concepts of "determinism" and "state" is essential for an understanding of the modifications required to adapt them to the needs of microphysics.

8.1 The Principle of Determinism and the Classical Concept of State. Classical physics tacitly assumes that physical systems obey the principle of determinism; in effect, "like conditions produce like consequences and unlike consequences can only follow from unlike conditions." It is the principle of determinism that leads to the expectation that acorns will always grow into oaks and not, on occasion, into maple trees, mulberry bushes, or tulips, and to the inference that the oaks and tulips in the park did not all develop from acorns. In the technical language of physics, the principle asserts that when two physical systems have identical properties[1] at some time, t_0, and are subject to the same external conditions as viewed from the individual systems, they will be alike in all properties at any later time, t_1; also, if two systems (under the given conditions) are different at t_1, they must have been different at t_0. Two ideal classical pendulums, e.g., of the same length, and started at rest with the same deflections θ' from the equilibrium position, will have the same displacement at any later time. (If, however, the two pendulums have different deflections at t, they cannot both have been at rest with the same deflection at any previous time.) In classical physics, an isolated system, i.e., a system shielded from all variable external influences[2] always has a definite set of properties at an instant (even if they are unknown), and its development in time is completely determinate. An isolated system cannot choose different courses of development, just as the acorn cannot choose to become a giant redwood. This feature of deterministic development enables astronomers to predict the occurrence of eclipses in the distant future from a knowledge of the present properties of the solar system.

Different systems of the same kind (same "fixed properties") may have different "motions." The word "state" is used to specify a particular motion from among all the possible motions of a system. For example, the two pendulums just referred to are in the same state of motion. A third pendulum started with the same displacement but with an initial velocity different from zero has a different state of motion. The word "state" is also used, not as a direct description of the whole motion, but to specify the particular set of dynamical properties possessed by a physical system at some instant. Thus the state of a pendulum is specified at an instant by giving the deflection and the velocity of its bob at that instant. By the principle of determinism the specification of the state of a system at some instant is equivalent to the specification of the motion of the system in time; two systems in the same state at t_0 must have identical motions, i.e., they have the same state of motion[3].

[1] Naturally, the identical properties must be internal to the two systems. Different external properties, such as location in different spatial regions, are assumed to be without effect. Examples of internal properties are: the distance between two mass points in the system, the relative velocity of two mass points, the internal energy (as against the kinetic energy of the center of the mass motion – an external property – which depends on the velocity relative to the frame of reference), etc.

[2] Perhaps no system except possibly the whole universe is completely isolated. But a high degree of isolation of systems can often be achieved in the laboratory and sometimes occurs naturally (e.g., the solar system).

Clearly two systems (of the same kind) are in the same state if they are alike in all internal dynamical properties. But how many is "all"? How, in practice, can we be certain whether two systems are alike in all their properties without a theory and without, perhaps, even a knowledge of the constitution of the systems?

An example will clarify these questions and indicate how they may be answered. Suppose I am given a collection of simple harmonic oscillators (one-dimensional) all of which have a moving mass, m, and spring constant, k. Let's pretend I know nothing whatever about these systems to begin with, except that they are all of the same kind. Suppose that after many investigations I have discovered various observables – in particular, an energy observable, a location observable (the distance of the oscillating mass relative to the equilibrium position of the oscillator), and a momentum observable (the momentum of the moving mass in a frame of reference in which the equilibrium position of the oscillator is at rest). My object is to discover what properties my systems need have in common at some instant to assert that they are "in the same state."

The investigation may be started many different ways. I begin, say, with the energy observable. Since systems in the same state must be alike in all properties, they must certainly have the same energy. Therefore I select, by an energy measurement, a large number of systems with the same energy property, E', and observe whether a second property – location, for instance – is the same for all the selected systems. Tests show that the systems that are alike (uniform, homogeneous) in the energy property are variable in location. Systems with the same energy E', then, are not in the same state. To reduce the observed variability, I select a set of systems with the same energy, E', and the same location, x' (this is always possible according to classical

[3] A motion is described completely by the set of functions $q(t)$, $r(t)$, etc., which specify the dependence of all system-observables on time. The equivalence of "state at an instant" and "state of motion" means that the system properties $q(t')$, $r(t')$, etc., at instant t' uniquely determine the functions $q(t)$, $r(t)$, etc.

theory) and observe whether all these systems have the same momentum[4].

Only two momentum values are found and these differ only in sign. In other words, when systems have the same energy and location properties, they also have the same "magnitude-of-momentum" property; the magnitude of momentum is uniquely determined by the energy and location properties already measured[5].

My general strategy is easily summarized (Fig. 8-1). Whenever variability in an observable q is discovered (i.e., measurements reveal the properties q', q'', etc., belonging to the observable q) in a set of systems with certain properties r', s', I seek a new set in which the variability in q is also eliminated. Since, for example, oscillators with energy, E', location, x', vary in the sign (±) of the momentum (let κ designate the observable "sign of momentum"), I proceed by selecting a set of systems homogeneous in E', x', and κ'[6].

I can no longer go on quite as before since I only know how to measure energy, location, and momentum. Therefore I test whether systems vary in some property – again, say, location–at a later time, t_1 (i.e., I investigate whether the systems are in the same state by testing whether the given properties uniquely determine the motion). I find that all systems have the same location property at t_1. Similar tests made at various times of both location and momentum properties indicate that systems with properties E', x', κ' at t_0 are the same in any property measured at any time (see Fig. 8-1). I conclude, therefore, that systems with the same energy, location, and sign of momentum, at some time, are in the same state; i.e., *all* properties at t_0 are the same in a set of systems if each system has the properties E', x' and κ'. Similar investigations would show that a set of systems homogeneous in x' (location) and p' (momentum) are in the same state and that, if $p' = \kappa' \; [2mE' - mk(x')^2]^{1/2}$, systems with the properties x', p', and systems with properties E', x', κ', are in the same state.

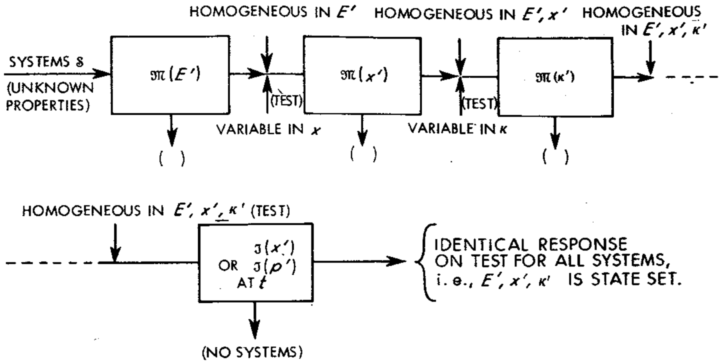

Fig. 8-1. Diagram of procedure for discovering state sets of properties (classical).

Suppose that in the initial investigations of the system I had failed to discover a momentum property. After selecting a set of systems homogeneous in energy and position at t_0, I could test whether they are in the same state by making location measurements at later times. I would discover that location at t_1 is not the same for all systems. Using the principle of determinism, I would then infer that the systems differed in some property at t_0. Note that the principle of determinism directs a search for additional properties. Unless a group of properties are found such that systems homogeneous in these properties are the same in all known properties at all times, the principle of determinism asserts that more properties are discoverable.

For many natural systems, sets of properties which uniquely determine system states may be found by procedures similar to those sketched above. The number of state variables required for a system

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[4] It must be imagined that the momentum measurement is made immediately after a system is prepared with the chosen energy, E', and location, x'.

[5] All these results are obvious from the theory of the oscillator. The energy, E, is related to the displacement, x, and momentum, p, by

$$E = p^2/2m + kx^2/2$$

Hence,

$$x = \pm\sqrt{2E/k - p^2/mk}, \qquad p = \pm\sqrt{2mE - mkx^2}.$$

Thus x is not determined by E alone, but p is determined to within a sign by E and x.

[6] Note that κ is a nonquantitative observable with only two properties (designated by the symbols + and –). The symobls E', x' designate specific energy, location properties; similarly κ', which may be either + or –, indicates a particular property of the sign of momentum observable. E' or x' may be specified by numbers related to appropriate units; e.g., $E' = 3.0$ joules, $x' = 5$ cm. Obviously κ' has no corresponding numerical expression.

may depend on the nature of the properties that are investigated, the accuracy of measurement, and the length of the time period over which measurements are made. An increase in the level of accuracy or of the time period under consideration may require an extension of the set of properties required for the specification of state.

Two useful technical definitions may be understood by reference to the foregoing example. A set of properties, u', such as x', p' or E', x', κ', is called a set of state properties ("state set"), for systems of kind S , if, whenever different S systems have the same properties, u',[7] at some instant, all properties of the S systems are the same at all times; i.e., different systems with the same state properties, u', are in the same state. A set of properties, v', is called an independent set of state properties if v' is a set of state properties, and no property in the set v' is uniquely determined by the remaining properties in the set[8].

Suppose someone came to you and said, "You're a student of physics. Perhaps you can help me. In the next room I have a simple harmonic oscillator with a mass, m, and spring constant, k. Exactly how far will the mass be from its equilibrium position at two o'clock this afternoon?" You could not, of course, give an answer. You might explain that physics never gives answers to questions such as the one proposed; a knowledge of system constitution is not sufficient to permit a prediction of a dynamical property of the system. "What else," he may then ask, "must I tell you to enable you to answer my question?" A sufficient answer is "any set of state properties of the oscillator at some definite time."

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[7] The symbols u', v', w' will be used to denote sets of properties.

[8] For oscillators, the properties energy, location, and sign of momentum form an independent set of state properties. No two of these properties uniquely determines the third. Location and momentum is a second independent set of state properties. Energy and location properties are independent (the energy property does not uniquely determine the location property of an oscillator), but this pair is not a set of state properties. Energy, location, and momentum magnitude are dependent properties since momentum magnitude is uniquely determined by the energy and location. A set of state properties for the simple harmonic oscillator is "energy, location, momentum magnitude, momentum sign"; these properties are not independent since momentum magnitude is uniquely determined by the energy and momentum. Note that if any property is omitted from an independent set the remaining group is not a set of state properties.

This answer illustrates the central significance of the state concept. The determinate connections among phenomena that are discoverable by experiment or predictable by theory are:

(1) The connections between a set of independent state properties at some time and the properties dependent upon the state set at the same time (e.g., the relation between x, p for the oscillator and the energy of the oscillator; see footnote 8).

(2) The connection between state properties at some time and any property at another time (e.g., given x_0 and p_0 for the oscillator at t_0 there is a unique x_1 at t_1 which may be known experimentally or calculated from the laws of motion).

The exact predictions of physics are essentially of this form: If the set of independent state properties of system S at t_0 is u', then, at t_1, property r' will be found[9]. In short, prediction of all system properties is possible only if a set of state properties (i.e., the state of the system) is known.

8.2 The Principle of Statistical Determinism. It is easy to see that the classical concept of state cannot play a role in microphysics. The concept is wholly dependent on the principle of determinism, and microphysical phenomena simply are not governed by this principle. A number of examples of the indeterminate behavior of microphysical entities were cited in previous sections. The location at which an electron in a diffraction experiment will arrive at a detecting screen cannot be controlled. If we set up two single-slit diffraction experiments, it is impossible to prepare an electron in each apparatus so as to be certain that both will arrive at their respective detecting screens at the same relative position (Fig. 8-2). It is impossible, similarly, to prepare two neutrons to ensure that both will turn into protons at the same time. In other words, we cannot prepare two electrons or two neutrons in the same state (in the classical sense).

If we insist on the absolute validity of classical determinism, we must conclude that the electrons and neutrons in the foregoing

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[9] Note, however, that a complete set of state properties is not always required for prediction. For example, the properties E', x', of an oscillator – which do not form a set of state properties – uniquely determine the momentum magnitude, p'.

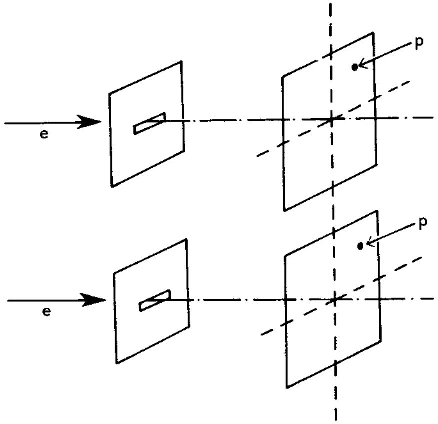

Fig. 8-2. An impossible experiment. Electrons behind the entrance slit cannot be so prepared as to determine their positions at the detector.

examples had different properties in their initial preparations, since differences in final behavior imply differences in initial properties. In short, the principle of classical determinism leads back to the hypothesis of hidden variables. Reasons for rejecting this hypothesis were considered in Chapter 6.

From the existence of incompatibility, one may show that classical determinism cannot hold in microphysics; if classical determinism were true, no system properties could be incompatible. Suppose it were possible to prepare a set of electrons so that they were in the same state (in the classical sense). Let x (the x component of position) be tested on half the ensemble, and p_x (the x component of momentum) be tested on the other half. Since all systems are, by assumption, in the same state, all systems in the half tested for x must exhibit the same position – say x' – and all systems in the other half must exhibit the same momentum – say $p_x{}'$. (Systems are in the same classical state only if responses to identical tests are identical.) But these results imply that both x and p_x have "sharp" distributions in the original ensemble (Section 7.5); i.e., the ensemble was prepared with both an exact position and an exact momentum property. The existence of such a preparation implies further that the measured properties are compatible. Since the same argument may be applied to any set of observables, it is clear that if it is possible to prepare systems in classical states, then all the properties of the systems are compatible.

The indeterminate character of the behavior of microphysical systems has been emphasized repeatedly earlier in this volume. The concept of incompatibility is, of course, closely related to indeterminacy. The incompatibility of two properties means that when an ensemble of systems has one of the properties of the incompatible pair, the results of tests for the second property are necessarily indeterminate. This microphysical indeterminacy implies naturally that the classical concept of state, depending as it does on the principle of determinism, is inapplicable in microphysics.

The behavior of microphysical systems, though clearly indeterminate in some features, is far from chaotic. In complete chaos, experiment could discover no regular relations among phenomena, and all theory could tell us is "Your guess is as good as mine." Many regular and determinate relations have been described in the preceding sections. We have noted that definite diffraction patterns can be produced repeatably despite the indeterminacy in the behavior of individual electrons; radioactive substances exhibit definite half-lives. Many other regularities are obvious. The spectrum of hydrogen observed today contains the same lines Balmer measured many years ago. A gas of hydrogen molecules has well-defined, stable properties. And so on. Obviously a great deal of regularity may be found in microphysical behavior despite the appearance of indeterminacy; theory has much to "explain."

In all physics (macro- and micro-), observable regularities are of the form: "If the object (entity, ensemble) of interest has certain properties, then some other properties of the object will be observed," or, more explicitly, "certain directly observable macroscopic phenomena will be produced when tests are made with suitable apparatus on objects with the specified properties." Or, put in a different way: "If physical apparatus is arranged in some definite way, then certain other manifestations will be observed when tests are made."

If, for example, a classical pendulum is started from rest with a deflection θ', then we shall find that deflections larger than θ' never occur. If the electron momentum of the incident beam in a diffraction experiment is sufficiently well defined in magnitude and direction, then a characteristic diffraction pattern is produced. In short, all observable regularities have an "if_____, then_____" structure.

It has been shown that in classical physics the blank after "if" must (in general) be filled in with a set of state properties: If a certain state is prepared, then some other property will be found. Is it possible to give a similar characterization for the contents of the blank after "if" for microphysical systems? The answer is yes; the necessary content of the blank after "if" may be discovered by examining more closely the implications of the existence of determinate relations in microphysics.

In classical physics, the concept of state depends essentially on the principle of determinism. Because this principle is invalid in microphysics, so too is the concept. Now the principle of determinism expresses, in effect, the existence of precise, determinate relations among observations. But such relations also occur in microphysics. This suggests that it should be possible to formulate a principle for microphysical processes which plays the same role relative to the regularities of microphysics that the classical principle of determinism plays in classical physics.

Although we cannot predict where a single electron will appear on a detecting screen in a diffraction experiment, we can certainly predict the diffraction patterns that are produced in experiments such as the one described in Fig. 4-1. Two diffraction experiments with different apparatus (but both with the arrangement shown in Fig. 4-1), and different ensembles of electrons (but with the same large total number) produce indistinguishable patterns. Similarly, the exact time at which a radioactive nucleus will experience a transformation cannot be controlled or predicted, but the fraction of the initial number of nuclei which remain after a lapse of time, t, is the same for different ensembles of radioactive nuclei of the same kind. These observations are typical; along with the indeterminate behavior of single systems, determinate (regular, repeatable) behavior of ensembles of these systems is observed. This suggests that a principle of determinism holds in microphysics, not for single systems, but for suitable ensembles of systems.

Diffraction experiments reveal the determinate relation: If an ensemble of electrons is formed (e.g., by the use of the source and accelerating plates as in Fig. 4-1) with a well-defined momentum, then the distribution in position of the electrons over the plane of the detecting screen forms a definite and reproducible pattern. Similarly, from observations on neutron decay, we can conclude: If an ensemble of free neutrons is prepared, then the fraction of the initial number of neutrons that remain after interval, t, is a definite and reproducible exponential function of t. These regular relations have the general structure: If certain conditions are true of an ensemble of microphysical systems, then the distribution function of some observable has a definite form.

It is now possible to discern the content of the microphysical principle for which we are searching. To distinguish it from its classical counterpart and to suggest its general character it is called the Principle of Statistical Determinism. The classical and microphysical principles may be given parallel formulations.

Classical Principle of Determinism: It is always possible to find a set of properties – generally many different sets – such that whenever two systems are alike (at t_0) in all the properties of the set, they are also alike in any property (at t_0 or any time t_1). If two systems differ in any property at t_1, then they must have been different in some property (or properties) at a previous time t_0.

Microphysical Principle of Statistical Determinism: It is always possible to find a set of compatible properties – generally many different sets – such that whenever two *ensembles* of systems are alike (at t_0) in all the properties of the compatible set, then the distribution in any observable is the same in both ensembles (at t_0 or any later time t_1); if two *ensembles* exhibit different distributions in any observable at t_1, then the ensembles must have been different in some property at a previous time t_0[10].

[10] For brevity, a number of obvious but necessary stipulations have been omitted. The systems are of the same kind; between t_0 and t_1, such external influences as may affect the system are presumed to be identical for both systems (or all the systems in the two ensembles). Observation is not permitted on the ensembles in the interval between t_0 and t_1 since observations produce, in general, large and uncertain property changes. If the same observations are made on both ensembles (in the t_0, t_1 interval), the distributions observed at t_1 are again the same for both ensembles but, in general, the distributions so obtained are different from those found if the intermediate observation is omitted (see the discussion of the intermediate measurement in Section 7.5).

A very long and possibly dangerous inductive leap is necessary to get from the few examples of determinate statistical relations mentioned in previous paragraphs to the grand generalities of the principle of statistical determinism. But a theory must presuppose a deterministic principle of some kind, and the one formulated above is at least consistent with the microphysical observations we have analyzed. The new principle may be considered as a generalization of the classical principle of complete determinism; if we replace "the distribution of any observable is the same . . . ," with "the distribution of any observable is sharp, and is the same . . . ," the resulting statement has exactly the meaning of classical determinism. Thus, in accepting the new principle, the possibility of complete determinism is not ruled out since no restriction is imposed on the character of the distributions in the ensemble to which the microphysical principle refers.

In any event, a principle of determinism does not so much state an obvious truth as specify an open program. The crucial question is: Can laboratory observations be fitted naturally, i.e., without the constant addition of ad hoc assumptions or new and as yet unobserved properties, into a theoretical structure based on statistical determinism? One can only say that the theory of quantum mechanics, which incorporates the principle of statistical determinism, has been highly successful in the realm of microphysics. As yet, there is no evidence that suggests the invalidity of the principle. (But remember that after two centuries of successful application of classical determinism, experience in microphysics forced its abandonment.)

8.3 The Microphysical Concept of State. It is readily seen that the classical state concept defined in Section 8.1 is closely related to the classical principle of determinism (D). The set of properties referred to in D is a state set of properties; according to D, whenever systems are alike at some instant in the properties of a state set, they are alike in all properties at any time – i.e., the systems are in the same state (or state of motion).

It is natural to define the state concept for microphysics so that it relates to the principle of statistical determinism, S.D., exactly as the classical concept relates to D. Any set of properties, then, of the kind referred to in S.D., will be called a "state set" of properties for the systems composing the ensemble[11].

According to the S.D. principle, all ensembles that have the same state sets at some instant are equivalent at any later time (the distributions in these ensembles are alike in all observables). Just as the "motion of a system" in classical physics means simply the time dependence of its observables, the "motion of an ensemble" means the time dependence of the distributions of its observables. Consequently the principle of statistical determinism may be expressed: Ensembles with the same state sets of properties have the same motions.

An ensemble with a state set of properties is called a state ensemble. Different ensembles with the same state sets are said to be in the same state[12]; therefore, ensembles in the same state are equivalent. The term "state," clearly, has been defined for ensembles and not for single systems. When, however, an ensemble is known to be in some particular state, no complications arise if each system in the ensemble is said to be in that state. Note that for such systems, a known state set of properties is possessed by the system, i.e., these properties are certain to be found on test.

[11] The number of properties required to make up a state set for systems of some kind depends on what properties are chosen. For an ensemble of hydrogen atoms, the property "energy = −13.6 eV" is, by itself, a state set (note that a state set may be composed of a single property); a particular set of components of the position of the electron, x', y', z', is also a state set (subject to the limitations of footnote 13 of this chapter). However, no state set x', y', z' defines the same state as the property "$E' = -13.6$ eV"; the two state sets are incompatible.

Clearly the properties contained in a state set are compatible. If observables q, r, . . . , are such that all the compatible sets of properties q', r', . . . , formed from these observables are state sets, the observables q, r, . . . , are said to constitute a "complete set of compatible observables." The sets of observables (for hydrogen atoms): "The components x, y, z of the vector from proton to electron," or "energy, angular momentum magnitude, z component of angular momentum," are complete (subject to the conditions referred to above).

[12] In the literature of quantum mechanics, the term "pure state" is used instead of the unmodified word "state." The adjective, "pure," adds no content to the term; it is employed for emphasis as is the word "honest" in "honest truth." Generally, phrases such as "honest truth" are used when the "truth" is doubtful or when one expects doubt from a listener. I imagine that the word "pure" in the term "pure state" implies that the ensemble really is in a state even if (on classical principles) it does not much look like one.

It is useful to have special notations for state sets and state ensembles. That an ensemble $\mathscr{E}$ is a state ensemble will be indicated by writing $|\mathscr{E}]$; similarly, if a set of properties u' is a state set, we denote the set by the symbol $[u']$ (or simply by u' when no confusion is possible). Thus, after it is established that a pair of properties q', r', e.g., constitute a state set, we shall write $[q', r']$ instead of q', r'. All ensembles with the same state set of properties $[u']$ are equivalent (they differ, as ensembles, only in the numbers of systems they contain); consequently, they may be denoted by the same symbol. We shall use $|u']$ to denote any ensemble with the state set $[u']$. The distribution function of an observable, s, in a state ensemble, $|u']$[13], will be indicated by the symbol $\{s|u']$.

This distribution, it will be recalled, is a function defined over the domain of the properties in the observable s. The symbols introduced above and a few others, together with their meanings, are collected in Table 8-1.

TABLE 8-1

(a)	A "state set of properties u'" (also "state properties")	$[u']$
(b)	A "state ensemble $\mathscr{E}$"	$\|\mathscr{E}]$
(c)	An "ensemble with the state set $[u']$"	$\|u']$
(d)	"Distribution of observable s in $\|u']$"	$\{s\|u']$
(e)	"Probability of property s' in $\|u']$"	$\{s'\|u']$
(f)	"Probability of state set $[v']$ in $\|u']$"	$[v'\|u']$
(g)	"Ensemble prepared initially in state $\|u']$ after interval t."	$\|u';t]$

With the help of the definitions of Table 8-1, the principle of statistical determinism may be expressed as follows: "The time-dependent distribution $\{s|u'; t]$ (where s is any observable) is uniquely determined by the state set $[u']$ prepared at $t = 0$." More simply, the motion of an ensemble is uniquely determined by the initial state set $[u']$. All statistical properties testable on a state ensemble are predictable from a knowledge of the state properties with which the ensemble is prepared. Note in particular that the history of an ensemble (or of its component systems), before its preparation as a state ensemble, has no influence on its statistical properties after the preparation. By the S.D. principle, all ensembles with the same state properties are necessarily equivalent (see Fig. 8-3 for an example). From these considerations it is obvious that the blank after "if" in the "if_____, then_____" statements of microphysics must contain a state set of properties and the blank after "then" may contain any testable statistical property.

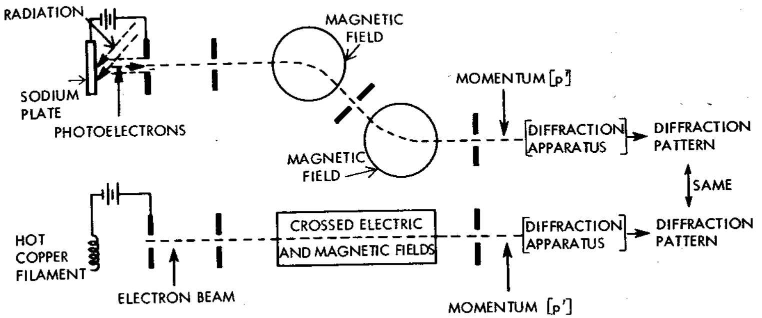

Fig. 8-3. Example of equivalent state ensembles $|p']$ with different prior histories.

It is known, for example, that the set of properties of hydrogen atoms, "energy E', angular momentum magnitude l', z component of angular momentum m_z'" constitute a set of state properties[14]. These properties are "constants of motion – i.e., they do not change with time. Consequently, a state ensemble with properties $[E', l', m_z']$ remains a state ensemble with the same properties at any later time. Thus the distributions in all other observables are independent of time and no reference need be made to the time at which a distribution is measured. Such ensembles, or states, are called "stationary."

In the state ensemble (of hydrogen atoms) $|E', l', m_z']$, observables such as r, the electron proton distance, m_x, the x component of the angular momentum have specific distributions that depend parametrically on the state properties E', l', m_z'. In the case of the diffraction experiments of Chapter 4, the incident ensemble is

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[13] Often "state $[u']$" is used; also somewhat inconsistently the phrase, "ensemble in state $|u']$."

[14] Spin properties and the interaction of the atom with the electromagnetic field are neglected in these statements. Also, only the "internal" properties of the hydrogen atom are considered.

characterized by the properties, “$p_x' = 0$, $p_y' = 0$, $p_z' = p_0$; “these constitute a state set of properties for electrons. This state set determines a unique distribution function $\{y \mid p']$ of the observable y (distance from axis of apparatus transverse to the beam and perpendicular to the slits – see Fig. 4-7), which describes the observed diffraction pattern.

Our last example illustrates time dependence. Consider an ensemble at $t = 0$ of N “normal neutrons” at rest. The characterization of a system as a neutron at rest specifies a state set of properties. The neutrons spontaneously undergo transformation. The probability that any neutron retains its initial state properties at time t is found by observation to be a definite exponential function of the time.

In principle, state sets of properties for systems of some kind may be discovered experimentally by procedures similar to those used to find classical state sets (Section 8.1; see Fig. 8-1). Naturally, as in the classical case, it is necessary to know how to measure a variety of observables on the systems under consideration (Chapter 7). Suppose that instruments that measure observables $l, m, q, r, s \ldots$ are available, i.e., we have instruments, $\mathscr{M}(l)$, $\mathscr{M}(m)$, etc. We choose an observable, say q, and prepare a collection of ensembles with different prior histories but all with the property q'. The different ensembles are tested for equivalence by measuring the distributions in all other known observables and the time dependence of these distributions. If the distributions in all ensembles prepared with property q' are the same at all times, we conclude that property q' is a state set – since q' determines all statistical properties – and that an ensemble with property q' is in state $\mid q']$. If, however, in different ensembles with property q', the distributions of some observable s are not all identical, q' is not a state set of properties. The principle of statistical determinism then assures us that there exist properties (or at lease one property) compatible with and independent of q'[15].

Let r' be such a property. Ensembles with both q' and r' properties are prepared and tested for equivalence. If ensembles with properties q', r' are equivalent whenever, wherever, and however they are found, then $[q', r']$ is a state set and an ensemble with this pair of properties is in the state $\mid q', r']$. If the properties q', r' do not constitute a state set – i.e., if ensembles with this pair of properties are not necessarily equivalent – then further independent and compatible properties exist. The general form of the process is clear. Until equivalent ensembles are achieved, properties independent of and compatible with those already employed are added. The principle of statistical determinism assures us that this procedure, if sufficiently extended, will always lead to a state set of properties. (The procedure outlined above is summarized in Fig. 8-4.)

The concept of a state ensemble may be illuminated through a consideration of the related ideas of “homogeneity” and “indecomposability” as applied to ensembles. An ensemble is said to be homogeneous or, equivalently, indecomposable if no matter how it is divided into subensembles (but without in the process changing the statistical properties of the total ensemble), the subensembles are all equivalent (same statistical properties) to each other and to the original ensemble. Homogeneity, then, implies that all ensemble parts are equivalent with the whole.

Obviously an ensemble of men half 5 feet in height and half 6 feet in height is inhomogeneous. We can divide it into two subensembles of 5-foot men and 6-foot men that are clearly inequivalent with each other and with the parent ensemble. It is easy to see that in classical ensembles, homogeneity implies complete identity of the systems composing the parent ensemble. For if some observable q is distributed in the parent ensemble, we can divide it into subensembles with

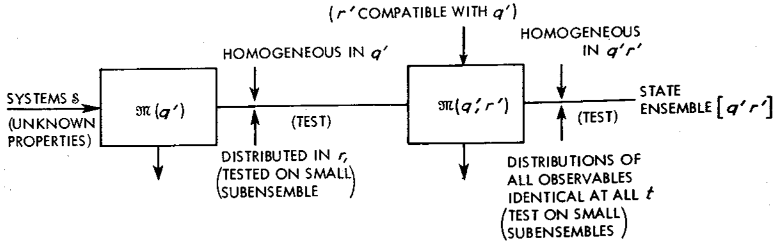

Fig. 8-4. Diagram of procedure for discovering state sets of properties (microphysical). Compare with Fig. 8-1.

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[15] The terms “dependent” and “independent,” as applied to compatible microphysical properties have the same meanings as in classical physics. However, for classical properties, there is no need to add the adjective “compatible.”

different properties q', q'', etc. without change in any of the individual system properties. The subensembles and the parent ensemble are certainly inequivalent since they differ in the distribution of the q observable. Thus homogeneity and system identity are equivalent characterizations for classical ensembles. It follows that classical systems in a homogeneous ensemble are all in the same state.

Microphysical ensembles cannot be prepared with the property of identity, i.e., so that the results of arbitrary test measurements are the same for all systems in the ensemble. This again expresses the indeterminacy manifested in the behavior of microphysical systems or, equivalently, the existence of incompatible properties. Nevertheless, as in classical theory, the concepts of state and ensemble homogeneity have the same effective meaning: State ensembles are homogeneous and conversely. Suppose we have a state ensemble prepared with the state properties q', r'. Since the q', r' properties are a state set, all ensembles with these properties are equivalent. Consider now any decomposition of the ensemble in which the total ensemble (collection of subensembles) after decomposition remains equivalent to the original state ensemble. Since the distributions of the properties q', r' in the original ensemble are sharp, they must be sharp also in the subensembles. Otherwise, the distributions in q', r' in the subensembles could not be the same as the distributions in the parent ensemble. Hence, each subensemble is a state ensemble characterized by the q', r' properties, and they are all equivalent and also equivalent to the parent ensemble. State ensembles, therefore, are homogeneous or indecomposable.

We shall not now try to extend the argument further to show also that homogeneous ensembles are state ensembles. In both classical physics and microphysics it is a fact that the concept of state may be defined in terms of the homogeneity or indecomposability of ensembles. For classical theory in which all properties are compatible, the homogeneous ensemble consists of a collection of systems identical in all properties. In microphysics, which is characterized by the presence of incompatible properties, the tested properties of systems in a homogeneous (state) ensemble are distributed in general.

In the sense that there is no way in which they may be discriminated, the systems in a homogeneous ensemble may be considered to be alike. In this sense the indeterminacy in system behavior of which we have repeatedly spoken is irreducible; like systems in a homogeneous ensemble produce, in general, unlike results on test[16].

An ensemble that is not a state ensemble is called a "mixture." Mixtures are inhomogeneous – i.e., decomposable. If $|u']$ and $|u'']$ are two state ensembles containing, e.g., N' and N'' systems, the collection of $N' + N''$ systems in the two ensembles forms a new ensemble which is a mixture of the $|u']$ and $|u'']$ ensembles. By measurement of the observable u on the mixture, it can be decomposed into the component state ensembles. Naturally all mixtures are decomposable.

A number of the concepts that have been introduced are illustrated in the following example. Consider an ensemble, $\mathscr{E}$, of photons in a monochromatic unpolarized beam of radiation. This ensemble is decomposable and therefore a mixture. By passing the radiation through suitable polarizing materials, we can divide the original ensemble into subensembles linearly polarized along the x and y axes; let these subensembles be indicated by $\mathscr{E}_x$ and $\mathscr{E}_y$ and the ensemble consisting of these two subensembles taken together by $\mathscr{E}+$. After the decomposition, $\mathscr{E}+$ remains equivalent to $\mathscr{E}$. The probability of any specific state of polarization in both $\mathscr{E}$ and $\mathscr{E}+$ is one half. However, $\mathscr{E}_x$ and $\mathscr{E}_y$ are clearly inequivalent and inequivalent to $\mathscr{E}$. The probability of linear polarization along x, for example, is 1 in $\mathscr{E}_x$, 0 in $\mathscr{E}_y$ and ½ in $\mathscr{E}$ or $\mathscr{E}+$. Thus the original ensemble is decomposable and a mixture.

Attempts to subdivide $\mathscr{E}_x$ by splitting the linearly polarized beam into parts with different polarization do not lead to decomposition; after splitting into different states of polarization, the total ensemble is not equivalent to $\mathscr{E}_x$. The $\mathscr{E}_x$ ensemble as it happens is indecomposable and hence a state ensemble.

The microphysical concept of state is subtle and easily misinterpreted. Clear differences between the classical and microphysical state

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[16] If one takes the position that unlikeness in test implies prior unlikeness in the systems tested and that, therefore, the microphysical state ensembles are not truly homogeneous, one appeals effectively to the principle of complete determinsim. On this view, microphysical processes are incompletely controlled but are otherwise deterministic and classical. Some of the consequences of this position were examined in Sections 6.1 and 6.2.

concepts are easily distinguished. Since, in classical physics, system properties have objective meaning, any system is always in a state whether or not its properties are known. It is not true, however, that any ensemble of microphysical systems with the same fixed properties is a state ensemble. Moreover, the statement, "A single microphysical system is in some state although its state properties are unknown" is without testable meaning or, more bluntly, is meaningless[17].

It may be shown that any mixture ensemble (i.e., not a state ensemble) is equivalent to an ensemble obtained by putting together, with appropriate weights, a number of different state subensembles. It is possible, however, to produce equivalent mixtures (ensembles with the same statistical properties) by composing quite different state subensembles. For example, unpolarized radiation may be formed from an equal mixture of orthogonal linear polarizations or by the equal mixture of right and left circular polarizations. Consequently, it cannot be said that every mixture is formed of some unique combination of states (this may be said of "classical mixtures," i.e., ensembles of classical systems in different "objective" states). Classically, if two systems are alike in all their properties they are in the same state. However, if two microphysical ensembles are equivalent, they certainly need not be in the same state. Equivalent mixtures are easily produced. If two ensembles are in the same state they are necessarily equivalent; but the equivalence of two ensembles does not even imply that they have been formed by the same mixtures of state ensembles.

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[17] The sentence quoted has exactly as much meaning as: "There are numerous physical bodies all about us that experience no interactions whatever with the entities of which we are capable of having any knowledge."

9 Probability Amplitudes; The Superposition Principle

"Errors, like straws, upon the surface flow.
He who would search for pearls must dive below."

John Dryden

9.1 Introduction. It is the object of physical theory to formulate laws which give explicit content to the general statements of the principle of determinism on which the theory is based. According to the principle of statistical determinism the statistical properties of an ensemble are uniquely defined by the state properties with which the systems in the ensemble have been prepared. Thus if an ensemble is prepared in state $|u']$, the probability of finding any other state property v' on test is a uniquely defined number which is designated by $[v' | u']$ (see Table 8-1). Put another way, the state $|u']$ uniquely defines the distributions in all observables at the time of preparation of $|u']$ or at any later time at which observation is made. It is the function of theory to provide general rules by means of which the determinate probabilities or distributions may be computed from a knowledge of the state properties that characterize the ensemble.

The differences between the requirements on microphysical theory and classical theory are worth noting even at the risk of some repetition. In classical physics, the state properties at some instant determine all other properties. In microphysics, state properties at an instant do not determine uniquely the incompatible properties at that instant but only the probabilities that these properties will be observed on test. In classical dynamics, we require the connections only between state properties at different times. These connections are established through the ordinary differential equations of classical mechanics and the partial differential equations of electromagnetism. Microphysical theory, however, must connect the state properties of an ensemble at some instant with the distributions in all observables (or the probability of any specific property) at the same instant or at any later time.

From the work of the preceding paragraphs and Chapter 8, it is natural to expect probability distributions to play a role in microphysical theory similar to the role of state variables in classical theory. We might reasonably guess that the basic equations of the theory would connect probability distributions rather than state variables. Like many other "reasonable" guesses, this one is wrong. The quantum mechanics formulates equations for intermediate mathematical elements, the so-called "probability amplitudes," rather than for the probabilities directly.

Now this is not the place nor is there space available for even a cursory sketch of the structure of the theory of quantum mechanics. However, the probability amplitude concept is of overwhelming importance for the form and structure of the theory and cannot be omitted from even a brief consideration of its conceptual foundations.

A more extended analysis of our much overworked interference experiment of Section 4 will serve to introduce the concept of probability amplitude.

9.2 The Interference of Electrons. The form of the distribution of the electrons at the detecting plane (Fig. 4-7) in the electron interference experiment considered in Chapter 4 can be calculated by the use of a "wave model." The electrons of incident momentum p_0 are replaced in imagination by a plane wave of wavelength $\lambda = h/p_0$ (the nature of the wave is immaterial) and the resulting intensity (squared amplitude) of the wave at the plane of the detector is computed by the use of Huygens' principle. The distribution of the wave intensity obtained this way is of the same form as the observed distribution of the electrons. It is perhaps worth repeating that the wave picture does not, despite this relation between the wave intensity and the observed distribution, provide an adequate theory of the diffraction and interference phenomena. Individual electrons appear at specific but indeterminate positions at the detector and are not spread out as is the intensity of the diffracted wave.

Although it is unnecessary in the calculation of the wave intensity to specify what it is that is waving, it is essential that the wave theory employed be linear and that the wave intensity be proportional to the square of the amplitude. By virtue of the linearity, the solution of the wave equation for the two slit system can be obtained by adding the solutions for two simpler cases: slit a open and b closed, and slit b open and a closed. In consequence, it is natural to consider the wave amplitude at Q (Fig. 4-11) as the sum of amplitudes arising from slit a and slit b separately, and to express the wave amplitude at Q (denoted by $\psi(Q)$) in the form

$$\psi(\mathrm{Q}) = C_\mathrm{a}\chi_\mathrm{a}(\mathrm{Q}) + C_\mathrm{b}\chi_\mathrm{b}(\mathrm{Q}) \qquad (9\text{-}1)$$

where C_a, C_b are the amplitude of the wave at the two slits and $\chi_\mathrm{a}(\mathrm{Q})$, $\chi_\mathrm{b}(\mathrm{Q})$ are the amplitudes for the solutions representing the single slit cases (with unit amplitude at each slit). The intensity at Q, $I(\mathrm{Q})$, is then[1]

$$\begin{aligned} I(\mathrm{Q}) \propto |\psi(\mathrm{Q})|^2 &= |C_\mathrm{a}\chi_\mathrm{a} + C_\mathrm{b}\chi_\mathrm{b}|^2 \\ &= |C_\mathrm{a}|^2|\chi_\mathrm{a}|^2 + |C_\mathrm{b}|^2|\chi_\mathrm{b}|^2 + 2\mathrm{Re}\, C_\mathrm{a}^*\chi_\mathrm{a}^*C_\mathrm{b}\chi_\mathrm{b}. \qquad (9\text{-}2) \end{aligned}$$

Note that $|C_\mathrm{a}|^2 \propto I_\mathrm{a}$ is the intensity at slit a; $|\chi_\mathrm{a}|^2 \propto I_\mathrm{a}(\mathrm{Q})$ is the intensity at Q for unit amplitude at a (with b closed) etc. With these notations Eq. 9-1 becomes

$$I(\mathrm{Q}) = I_\mathrm{a} \cdot I_\mathrm{a}(\mathrm{Q}) + I_\mathrm{b} \cdot I_\mathrm{b}(\mathrm{Q}) + 2\mathrm{Re}\, C_\mathrm{a}^*\chi_\mathrm{a}^*C_\mathrm{b}\chi_\mathrm{b}. \qquad (9\text{-}2\mathrm{a})$$

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[1] Complex amplitudes are used to describe both the magnitudes and phases of wave oscillations. The real intensities are the absolute squares of the amplitudes. The symbol * means "complex conjugate of."

The last term is the "interference" term which is characteristic of wave theories.

The interpretation of the interference experiment in terms of a "particle" picture leads, it will be recalled, to the following argument. The electrons pass either slit a or slit b. (It is this seemingly obvious thought which is false and invalidates the argument on which it is based.) If an electron passes a there is a probability $P_a(Q)$ that it will arrive at Q; there is a corresponding probability $P_b(Q)$ that it will get to Q if it comes through b. By this argument we reach the false conclusion that the probability of reaching Q is simply the sum of the probabilities of arriving at Q via the two slits separately:

$$P(Q) = P_a \cdot P_a(Q) + P_b \cdot P_b(Q) \quad \text{(classical)}, \qquad (9\text{-}3)$$

where P_a, P_b are the probabilities of passage through a, b. Eqs. 9-3 and 9-2a are alike, if probabilities and intensities are identified, except for the interference term. But note that, most remarkably, Eqs. 9-3 and 9-1 have the same structure; they are identical in form and differ only in the interchange of probabilities and amplitudes.

Now there are no physical waves (so far as anyone knows) associated with the phenomena of electron diffraction and interference. The wave picture must be considered as a mathematical fiction by which the probability distribution of the electrons at the detector may be computed. Neither do the amplitudes used in the theory define the amplitude of any physical oscillation nor does the square of the amplitude determine an intensity flux or an energy density of some physical activity. The square of the amplitude determines only a probability; hence the amplitude of the wave can be described legitimately only as a *probability amplitude.* In our example, the relevant amplitudes combine in a simple manner; the structure of their combination as given by Eq. 9-1 is just of the form expected, but unrealized, for classical probabilities (Eq. 9-3). The amplitudes have simple additive properties that are unshared by the probabilities: compare Eq. 9-1 with Eq. 9-2a (with intensities interpreted as probabilities).

These considerations raise the following general question: Do there exist, below the level of the probabilities that must be computed by microphysical theory, associated probability amplitudes which satisfy

linear equations (as in the wave theory used in the analysis of the interference experiment) and for which relations similar to Eq. 9-1 are true? The quantum mechanics makes the wide ranging assumption that the answer to this question is yes.

The significance of this assumption cannot be overemphasized; it effectively defines the mathematical form of the quantum mechanics. But we are overreaching ourselves. The general assumptions involved in the concept of probability amplitude must be stated before we can go on to more detailed considerations of its consequences.

9.3 Probability Amplitudes. To perceive the general meaning behind the special relations that apply to the case of electron interference, it is useful to summarize again the arguments leading to Eqs. 9-1 and 9-3, and to express the ideas involved with the help of a more general (also, unfortunately, a more cumbersome) notation.

In the derivation of Eq. 9-3, it was incorrectly assumed that the incident electrons go through either slit a or slit b. In consequence, the ensemble of electrons employed in the experiment – which we now designate by $|p_0]$ (see Table 8-1) – may be considered as a mixture of two subensembles, designated by $|a]$, $|b]$, consisting of electrons that pass through slit a or slit b. The relative weights of the $|a]$, $|b]$ properties are given, naturally, by the probabilities, $[a|p_0]$, $[b|p_0]$, of passage through slits a, b for the electrons that reach the detector. (Obviously $[a|p_0] + [b|p_0] = 1$; this expresses the assumption that every electron that reaches the detector gets through one slit or the other.) The relation implied by our classical picture of electrons may be expressed symbolically by[2]

$$|p_0] = |a]\,[a|p_0] + |b]\,[b|p_0]. \qquad (9\text{-}4)$$

This may be read: The ensemble $|p_0]$ is a mixture of $|a]$, $|b]$ ensembles with relative weights $[a|p_0]$ and $[b|p_0]$. It follows – and this feature completely defines the significance of Eq. 9-4 – that the

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[2] The ensembles indicated by $|p_0]$, $|a]$, and $|b]$ are assumed to have the same number N of systems. Then $|a]\,[a|p_0]$ describes an ensemble with the property a (i.e., passage through slit a) containing $N[a|p_0]$ systems, etc.

probability of position Q in $|p_0]$ can be obtained from the probabilities of Q in $|a]$ and $|b]$:

$$[Q \mid p_0] = [Q \mid a]\,[a \mid p_0] + [Q \mid b]\,[b \mid p_0] \quad \text{(classical)} \qquad (9\text{-}5)$$

(this is just Eq. 9-3 in our new notation). It has often been emphasized both in this and in foregoing chapters that this relation is false as is also the classical picture of the electron on which it is based.

Application of the wave theory to the interference experiment led to the determination of probabilities (proportional to wave intensities) as the absolute squares of underlying wave amplitudes. We shall, in what follows, designate the amplitude associated with any probability of the form $[v' \mid u']$ by the symbol $\langle v' \mid u' \rangle$; thus the amplitude for the probability $[Q \mid p_0]$ is $\langle Q \mid p_0 \rangle$. Since the absolute squares of amplitudes[3] are probabilities, we have

$$|\langle v' \mid u' \rangle|^2 = [v' \mid u']. \qquad (9\text{-}6)$$

In the wave description of the electron interference experiment, the amplitude describing the system can be expressed as the sum of two amplitudes relating respectively to the contributions of slit a and slit b. If we denote the wave characterizing the whole system by $|p_0\rangle$, and the waves for the case with only slit a, slit b open by $|a\rangle$, $|b\rangle$,[4] the relation between $|p_0\rangle$, $|a\rangle$, $|b\rangle$ may be expressed by

$$|p_0\rangle = |a\rangle\langle a \mid p_0\rangle + |b\rangle\langle b \mid p_0\rangle. \qquad (9\text{-}7)$$

The full wave $|p_0\rangle$ is the superposition of partial waves with certain amplitudes $\langle a \mid p_0\rangle$, $\langle b \mid p_0\rangle$; $|\langle a \mid p_0\rangle|^2$ and $|\langle b \mid p_0\rangle|^2$ determine the probabilities $[a \mid p_0]$, $[b \mid p_0]$ that the electrons in the incident beam pass slits a, b. The superposition of waves described by Eq. 9-7 implies that the amplitude for the property Q in $|p_0\rangle$ can be expressed in terms of the amplitudes for Q in the $|a\rangle$, $|b\rangle$ partial waves:

$$\langle Q \mid p_0\rangle = \langle Q \mid a\rangle\langle a \mid p_0\rangle + \langle Q \mid b\rangle\langle b \mid p_0\rangle. \qquad (9\text{-}8)$$

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[3] Questions concerning the normalization of the amplitudes so that they provide true probabilities rather than quantities proportional to probabilities have been neglected. To consider them would add complicating details and perhaps obscure the central ideas.

[4] Just as the ensembles $|p_0]$, $|a]$, $|b]$ have the same numbers of systems the waves $|p_0\rangle$, $|a\rangle$, $|b\rangle$ are assumed of equal integrated intensity.

The remarkable formal similarity between the classical expressions (Eqs. 9-4 and 9-5) and the wave-theoretic expressions (Eqs. 9-7 and 9-8) cries out for attention. The latter relations, which provide a correct picture of the electron distribution (given by $|\langle Q \mid p_0\rangle|^2$), are obtained from the former simply by replacing probabilities with the associated amplitudes.

These special relations for the case of electron diffraction are easily generalized. Consider an arbitrary ensemble $|u']$ (in place of $|p_0]$). Let w be some state-observable[5], incompatible with the state property u', where the w-observable consists of the set of exclusive state properties $w_1, w_2, \ldots$. (The w properties, in the example of electron interference, correspond to location at slit a and slit b. The relative simplicity of this example arises from the fact that the w observable contains only two exclusive properties). The probability of observation of w_k in the u' state is $[w_k \mid u']$ and since on observation every system must exhibit some w property,

$$\sum_k [w_k \mid u] = 1.$$

On a classical (and false) interpretation of the property concept, in which properties that are observable are assumed to be objectively existent, the $|u']$ ensemble must be considered as a mixture of $|w_k]$ subensembles with the relative weights $[w_k \mid u']$. This relation may be expressed symbolically by

$$|u'] = \sum_k |w_k]\,[w_k \mid u'] \quad \text{(classical).} \qquad (9\text{-}9)$$

It follows from this incorrect relation that the probability of any property v' in the $|u']$ ensemble is related to the probabilities of v' in the various $|w_k]$ subensembles:

$$[v' \mid u'] = \sum_k [v' \mid w_k]\,[w_k \mid u'] \quad \text{(classical).} \qquad (9\text{-}10)$$

Clearly Eqs. 9-9, 9-10 correspond to a generalization of Eqs. 9-4, 9-5.

The classical results 9-9, 9-10 are false. In the case of the interference experiment we saw that a probability amplitude could be

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[5] Any complete set of compatible observables may be used to define the basis states. For the case of the hydrogen atom, the w-observables may be chosen as the position of the electron relative to the proton or the set of compatible observables, energy, angular momentum magnitude, and the z component of the angular momentum. (See Section 8.3.)

associated with each probability and that correct relations between amplitudes are obtained from Eqs. 9-9 and 9-10 by substituting amplitudes for probabilities throughout. Quantum mechanics makes the extraordinarily fecund assumption that this feature of the interference experiment holds generally. The theory makes the following *Assumption on Probability Amplitudes*: Associated with each pair of state properties u', v', there exists not only a determinate probability, $[v' \mid u']$, but also an associated probability amplitude $\langle v' \mid u'\rangle$ such that

$$[v'|u'] = |\langle v'|u'\rangle|^2 \tag{9-11}$$

and that these amplitudes satisfy the system of relations

$$\langle v'|u'\rangle = \sum_k \langle v'|w_k\rangle\langle w_k|u'\rangle \tag{9-12}$$

This result is obtained by substituting amplitudes for probabilities in Eq. 9-10 for any choice of state observable w. Since Eq. 9-12 holds for any state property v' it may be expressed in a "symbolic" form similar to Eq. 9-9:

$$|u'\rangle = \Sigma|w_k\rangle\langle w_k|u'\rangle \tag{9-13}$$

This "symbolic" equation has no other meaning than that the amplitude of an arbitrary state property v' in the state with property u' can be formed from a combination of the amplitudes of v' in w_k states, $\langle v' \mid w_k\rangle$, as in Eq. 9-12. Thus the expression given by Eq. 9-13 is not so much an equation as a rule for creating equations.

On test for w in a u'-ensemble, the various w_k properties are found with probabilities $[w_k \mid u']$. This does not imply, however, that the u'-ensemble can be considered as a collection or mixture of these w_k-ensembles. But relations 9-12, 9-13 show that the properties of the u' state are in fact closely and simply related to the properties of the w_k states that are found by measurement of w on the u' ensemble. These relations are governed, however, by the probability amplitudes $\langle w_k \mid u'\rangle$ rather than, as classically expected, by the probabilities $[w_k \mid u']$. All probability amplitudes for the u' state can be obtained, according to Eq. 9-13, from a fixed linear combination of the amplitudes of the w_k states where the coefficients in the linear combination are the probability amplitudes for w_k in the u' state, $\langle w_k \mid u'\rangle$. The "latency" rather than the objectivity of the w_k properties in the u'-state results, it appears, in the change of the classical Eq. 9-10 into the quantum mechanical Eq. 9-12, where Eq. 9-12 derives from Eq. 9-10 by the substitution of amplitudes for probabilities.

Because of Eq. 9-13, it is natural to speak of a state with property u' as consisting of a linear combination of w_k states with the coefficients $\langle w_k \mid u'\rangle$. This language is useful if the sharp restrictions on its meaning are clearly understood. Of course it does not mean that the state ensemble $\mid u']$ is a "linear combination of state ensembles" $\mid w_k]$. No meaning has been or will be assigned to the phrase "linear combination of ensembles." Nor does it mean, as has been repeatedly emphasized, that the state $\mid u']$ is some mixture of $\mid w_k]$ ensembles (which is simply false). All that is meant by the assertion that a u' state is a linear combination of the w_k states with certain coefficients is that all the amplitudes of the u' state are linear combinations of the amplitudes of the w_k states with fixed coefficients. The meaning of Eq. 9-13 is completely covered by Eq. 9-12, which holds for any property v'.

These considerations indicate that it is useful to refer to states in two different ways. State ensembles have been designated by symbols such as $\mid u']$. But the linear relations between states expressed by Eq. 9-13 refer not to ensembles but to the description of states in terms of amplitudes. For this reason it is convenient to use the new symbol, $\mid u'\rangle$, for the designation of a state with property u'; in a wholly formal sense, the relation of the state $\mid u'\rangle$ to the amplitudes $\langle v'|u'\rangle$ is similar to the relation of the ensemble $\mid u']$ to the probabilities $[v'|u']$. Whenever state amplitudes and their relations are under consideration, states will be designated (somewhat abstractly) by the symbols $\mid u'\rangle$, $\mid w_k\rangle$, etc.

It seems natural to assume that states may not only be analyzed into combinations of w_k-states as in Eq. 9-13 but also (as in classical wave theory) that all linear combinations of given states are also possible states of the system. This somewhat obscure statement may be clarified by the following considerations. Suppose $\mid \psi_1\rangle$ and $\mid \psi_2\rangle$ are states of the system. These states are completely defined if the amplitudes $\langle u' \mid \psi_1\rangle$, $\langle u' \mid \psi_2\rangle$ are given for arbitrary state properties u'. Consider now the set of numbers obtained by linearly combining the amplitudes $\langle u' \mid \psi_1\rangle$, $\langle u' \mid \psi_2\rangle$ with the fixed coefficients c_1, c_2, for all u' properties, i.e., the numbers $\langle u' \mid \psi_1\rangle c_1 + \langle u' \mid \psi_2\rangle c_2$. Our assumption is that these numbers (for arbitrary $c_1, c_2, |\psi_1\rangle\ |\psi_2\rangle$) are the u' amplitudes for some state $|\psi\rangle$, i.e., there exists a state $|\psi\rangle$ with amplitudes $\langle u' | \psi\rangle$ for which

$$\langle u' | \psi\rangle = \langle u' | \psi_1\rangle c_1 + \langle u' | \psi_2\rangle c_2. \tag{9-14a}$$

In consequence, it is natural to express the relation between $|\psi\rangle$ and $|\psi_1\rangle, |\psi_2\rangle$ by

$$|\psi\rangle = |\psi_1\rangle c_1 + |\psi_2\rangle c_2, \tag{9-14}$$

and to say that the state $|\psi\rangle$ is formed of a linear combination of states $|\psi_1\rangle, |\psi_2\rangle$.

The assumption that linear combinations of states are also states of the system constitutes a *principle of superposition*. In linear wave theory a similar principle applies: Any linear combination of two wave motions determines a possible wave motion. But in quantum mechanics the principle applies to probability amplitudes rather than to the physical displacements considered in classical theories of wave motion. As a result, the conclusions that may be drawn from the superposition principle in quantum mechanics are quite different from those of classical theory. If, for example, there are two motions of a string with energies E_1, E_2, then a specific linear combination of these motions describes another motion of the string with some different energy, say E. In the quantum mechanics, however, the superposition of two states with the energy properties E_1, E_2 produces a state in which the energy property does not have any definite value. The state so formed is characterized by a property which is incompatible with the energy observable. In this superposed state an energy measurement will yield E_1 or E_2 with probabilities given by the absolute squares of the amplitudes in the linear combination.

It may be worth interpolating here that although the principle of statistical determinism implies the existence of a definite probability $[v' | u']$ of finding v' on test of the ensemble $|u']$ (for any choice of v', u'), neither this principle nor any of the work of prior chapters provides methods by which the probability associated with any particular pair of state properties can be computed. Similarly the work of this chapter, which posits the existence of probability amplitudes, introduces no method by which the amplitudes associated with a particular pair of

states may be computed. The amplitude concept carries us considerably forward, however, by virtue of the relations given by Eqs. 9-12 and 9-14; from a knowledge of certain amplitudes, others may be calculated. But these relations alone remain insufficient for the determination of amplitudes and hence of probabilities.

9.4 State Representatives. Experience indicates, in many particular instances, that a remarkable symmetry exists between the pair of properties u', v' that specify the probability $[v' | u']$: the probability of measuring v' on a u'-ensemble is found to be equal to the probability of measuring u' on a v'-ensemble. In the quantum mechanics, this symmetry is assumed to be universally true: For arbitrary pairs of state properties u', v',

$$[u' | v'] = [v' | u']. \tag{9-15}$$

This, together with the previous amplitude assumptions, implies a corresponding symmetry property for amplitudes (which we state without proof):

$$\langle u' | v'\rangle = \langle v' | u'\rangle^*. \tag{9-16}$$

The interchange of the state properties converts an amplitude number to its conjugate complex. With the help of Eq. 9-16, Eq. 9-12 may be expressed as

$$\langle v' | u'\rangle = \sum_k \langle w_k | v'\rangle^* \langle w_k | u'\rangle, \tag{9-17}$$

which indicates that the amplitudes $\langle v' | u'\rangle$ for any pair of states may be calculated if the w_k-amplitudes for the states in question are known. In consequence, the problem of finding the amplitudes for all pairs of states is reduced to that of finding, for an arbitrarily chosen w-observable, the w_k amplitudes for all states. The remaining amplitudes may then be obtained from Eq. 9-17.

All amplitudes and hence all statistical statements concerning measurements are determined, according to Eq. 9-17, once the w_k-amplitudes of all states are known. The w_k-amplitudes for a state, therefore, may be considered as a numerical characterization or "representation" of that state, the set of numbers $\langle w_1 | u'\rangle, \langle w_2 | u'\rangle \ldots,$

which defines a function $\langle w|u'\rangle$ over the domain of the exclusive properties $w_1, w_2, \ldots$ in state observable w, is called the "w-representative of the u' state" or the "representative of the u' state in the w-basis."

The principle of superposition implies, moreover, that any function of w (more accurately, a function defined on the domain of the properties of w) may be considered as the representative of some state. If a function f has the value f_k at state property w_k, a state $|\phi\rangle$ (say) may be formed as a linear combination of $|w_k\rangle$ states with coefficients f_k; for $|\phi\rangle$, then, the w-representative is the function f ($f = \langle w|\phi\rangle$).[6]

The absolute square of the value of the w-representative of the u' state at the property w_k ($|\langle w_k|u'\rangle|^2$) is the probability of finding w_k on test in a u' ensemble. From the w-representatives of states u' and v', the probability of the measurement of v' in a u'-ensemble can be obtained; it is the absolute square of the amplitude given by Eq. 9-17.

Before we go on, it is worth noting that all probability amplitudes can be obtained, at least in principle, from laboratory measurements of probabilities. Clearly the measurement of the probability of finding v' on test of a u'-ensemble determines the absolute value of $\langle v'|u'\rangle$; by Eq. 9-12:

$$|\langle v'|u'\rangle| = [v'|u']^{1/2}. \qquad (9\text{-}18a)$$

By physical measurements on ensembles, therefore, the absolute values of all the complex amplitudes are easily found. However, for the complete determination of an amplitude, its complex "phase" as well as its absolute value is required. The phases may be obtained from the solutions of the large system of equations obtained by taking the absolute squares of Eq. 9-17 for all pairs of states:

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[6] An important (if secondary) restriction on the choice of f has been omitted. Since the absolute squares of amplitudes are probabilities and since some property in w must be found on test (some location, or some energy, etc., is certain to be observed), it follows that for any state $|\phi\rangle$

$$1 = \Sigma[w_k|\phi] = \Sigma|\langle w_k|\phi\rangle|^2 = \Sigma|f_k|^2$$

Subject to this normalization condition any function f may be considered as the w-representative of some state.

Suppose that an f does not satisfy this property (is not normalized) and that $\Sigma f_k^2 = K$. Then a properly normalized f' is easily obtained from f: $f' = (K)^{-1/2}f$. It is sometimes convenient to work with representatives that are not normalized; these provide "relative" rather than "absolute" probabilities.

$$|\langle v'|u'\rangle|^2 = [v'|u'] = |\sum_k \langle w_k|v'\rangle^* \langle w_k|u'\rangle|^2. \qquad (9\text{-}18)$$

In these equations the left sides are determined by observation and we may assume that the absolute values of all the amplitudes have already been determined by use of Eq. 9-18a; the remaining unknowns are the phases of the numbers $\langle w_k|v'\rangle$, $\langle w_k|u'\rangle$. Now the number of equations defined by Eq. 9-18 (for all u', v') far exceeds the number of the unknown phases. In general, a system having more equations than unknowns has no solution unless the number of independent equations in the set is less than or equal to the number of unknowns. Essentially, the assumption that amplitudes exist with the properties given in Subsection 9.3 is equivalent to the assumption that the equations in the system specified by Eq. 9-18 are sufficiently interdependent to guarantee that they are consistent and therefore solvable for the amplitudes.

The sketch above indicates how amplitudes may be obtained by means of measurements of probabilities. The laws of quantum mechanics provide equations for the calculation of amplitudes and, at least for simple systems, it is far easier to calculate amplitudes than to measure them. But for systems whose properties are not completely known (nuclei, elementary particles), or systems of such complexity that the equations of theory cannot be solved with sufficient accuracy, the determination of amplitudes by means of probability measurements is often the object of laboratory experiments.

At the beginning of this chapter, it was remarked that it is "reasonable" to expect that microphysical theory would formulate equations for the determination of probabilities of the type $[v'|u']$, but that this expectation was unrealized. It will come as no surprise now that the theory of quantum mechanics formulates equations for amplitudes rather than for probabilities. In a sense, the descriptions of the relations between states provided by the amplitudes and by the probabilities are equivalent; the amplitude determines the probabilities and, as shown in the previous paragraphs, the probabilities determine amplitudes. But by means of the amplitudes, the relations which exist between states can be expressed by means of simple linear combinations. The amplitudes exhibit the presence of a powerful principle of superposition and indicate that the basic equations that are to be formulated by the theory have that most desirable of properties, linearity. For these good and sufficient reasons the equations governing amplitudes are vastly simpler than the equations for probabilities. Unfortunately it will not be possible, in this short introduction to underlying concepts, to describe the general laws of quantum mechanics which permit the theoretical derivation of the probability amplitudes in specific cases.

It has been shown that all amplitudes are effectively known if the representatives of all states in some particular basis are known. For this reason the equations of quantum mechanics are often formulated to provide representatives of states in a basis which is convenient for the problem in hand. The basis most frequently employed is that first used by Schrödinger in his papers on "wave mechanics"; this basis consists, for an electron in one dimension (x), of a set of states each with a definite location property (if electron spin is neglected). For each location, x', there is a state which we may designate by $|x'\rangle$ in which a test measurement is certain to discover the position x'. The representative of a u' state in this basis is a function of position $\langle x|u'\rangle$, the so-called "Schrödinger wave function" for the state $|u'\rangle$. The fact that the basis states form a continuum rather than a discrete set (as was assumed in the basis $w_1, w_2, \ldots w_k \ldots$) forces a conceptually unimportant modification in the formalism of the preceding pages; sums are merely replaced by integrals so that in place of Eqs. 9-13, 9-17 we have

$$|u'\rangle = \int |x\rangle\langle x|u'\rangle dx. \tag{9-13a}$$

$$\langle v'|u'\rangle = \int \langle x|v'\rangle^* \langle x|u'\rangle dx. \tag{9-17a}$$

Also, the quantity $|\langle w'|u'\rangle|^2$ is not the probability of the precise location x' in state u', but the "probability density" at x' (so that for a sufficiently small Δx the probability of finding the electron within the internal Δx in the neighborhood of $x'v$ is $|\langle x'|u'\rangle|^2 \Delta x$). Schrödinger's equations for the x-amplitudes have the form of linear differential equations. One simple example must suffice; the equation for the amplitude $\langle x|p'\rangle$ – i.e., for the amplitudes of a state with the definite momentum p' – is

$$-i\hbar \frac{d}{dx}(\langle x|p'\rangle) = p'(\langle x|p'\rangle).^{7} \tag{9-19}$$

In Heisenberg's original paper, where the mathematical structure of quantum mechanics was first presented, equations were formulated in terms of representatives in a discrete set of basis states defined by a quantized energy observable. In this basis the equations for amplitudes constitute a system of linear algebraic equations rather than Schrödinger's differential equations. The large difference in the forms of the theories of Heisenberg and Schrödinger prevented for only a brief period a recognition of their basic identity. The superificial differences in the two formulations were seen to reflect merely a difference in the choice of an underlying basis.

9.5 Motions of Ensembles and States. The number $[v'|u']$, it will be recalled, is the probability of finding property v' on test in an ensemble which, at the time of test, has the property u'. However these are not the only probabilities in which we have an interest. Ensembles prepared with some specific property will, in general, undergo changes in time. Electrons will move from place to place (more accurately, the distribution in position will change in time), atoms will emit radiation, nuclei will undergo transformation, etc. We are interested naturally in the dynamical laws which govern how a state ensemble of systems of some kind, prepared initially with a property u', changes its statistical properties in time. Effectively, this means that we wish to determine probabilities such as $[v'|u';t]$ (see Table 8-1), the probability of testing for v' in an ensemble prepared initially with property u' after a lapse of time t. We have seen that the amplitudes play a most important role in relation to the determination of the quantities $[v'|u']$. It is natural to ask what implications the concepts of probability amplitude and superposition have for the general dynamical problem posed above.

The concept of amplitudes was formed by consideration of the

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[7] Integration of this differential equation yields $\langle x|p'\rangle = C \exp(ip'x/\hbar)$ where C is an arbitrary integration constant.

classical theory of wave motions. It is natural to follow this model still further for the considerations of time-dependent motions. The classical theory is deterministic; consequently, a specific state of the system (wave) develops in time to another state of the system. It will be assumed that this relation is also true in microphysics: In the course of time states change into states and not into mixtures. The state $|u'\rangle$ after a lapse of time t will be denoted by $|u';t\rangle$ and the w-representative of this state, a numerical time-dependent function of w, by $\langle w|u';t\rangle$. In the Schrödinger representation (with $w = x$), this amplitude becomes a time-dependent function of position.

The principle of superposition for classical linear wave theory asserts that arbitrary linear combinations of wave motions are also possible motions. For a string along the x axis capable of displacement along the y axis, the possibility of the motions (solutions of the wave equation) $y_1(x,t)$ and $y_2(x,t)$ implies that the string can also have the class of motions defined by $y(x,t) = c_1 y_1(x,t) + c_2 y_2(x,t)$ for all values of c_1, c_2. In the quantum mechanics a similar principle is assumed. If, therefore, two states, initially specified by properties u', u'', for instance, which separately develop in time to states $|u';t\rangle$, $|u'';t\rangle$, then the state given initially by $|u'\rangle c' + |u''\rangle c''$ (with arbitrary coefficients c', c'') will develop in time to $|u';t\rangle c' + |u'';t\rangle c''$

In classical wave theory it is generally the case that special motions exist in which the spatial and time dependence "separate." In these motions there is a fixed amplitude at each position modulated by a harmonic time dependence of definite frequency (standing vibrations in a string, for example). For the analysis of a wave motion with prescribed initial conditions, it is often useful to express the motion as a combination of these special motions.

We cannot now go into the proof of the fact that our hypotheses on time-dependence imply that, for closed systems, a set of special states that have a harmonic dependence on time also exist in quantum mechanics no matter what the nature of the system. In particular, the following result may be established: For any system there must exist an observable E with properties E_1, E_2 . . . such that the states with properties E_k must depend on time according to the simple law,

$$|E_k;t\rangle = |E_k\rangle e^{-2\pi i\nu_k t}, \tag{9-20}$$

where ν_k is the frequency of the oscillation in the state $|E_k\rangle$. In these very special and remarkable states, the probability of finding any property v' on test is totally independent of time. This probability is, by use of Eq. 9-20,

$$\begin{aligned}[v'|E_k;t] &= |\langle v'|E_k;t\rangle|^2 = |\langle v'|E_k\rangle e^{-2\pi i\nu_k t}|^2 \\ &= |\langle v'|E_k\rangle|^2 = [v'|E_k].\end{aligned}$$

The probability of observing v' in the E_k state after any time t is equal to the probability of observing v' immediately after the preparation of the E_k state. Because of this constancy in the properties of E_k states (or ensembles) these states are said to be *stationary*. It is not difficult to show (but the proof is omitted) that the distribution of this special observable E in any state (or mixture) does not change with time, although distributions in other observables are time-dependent. This constancy in time obviously expresses a conservation law of great generality.

It would be hard at this stage to avoid the guess that the E observable can be identified with the total system energy. This identification is suggested in the first instance by the de Broglie relation $E = h\nu$, which connects frequency and energy. By further extensions of the theory it may be shown that E is indeed the energy of the system and that the frequency ν_k is precisely that given by the de Broglie relation: $\nu_k = E_k/h$.

From the foregoing it follows that the dynamical problem of quantum mechanics is completely solved once methods for the determination of the representatives of states in the E basis, $\langle E|u'\rangle$, and the possible values of the energy of the system E_1, E_2, etc. are known. Suppose that an ensemble has been prepared initially in state u'. This can be expressed as a linear combination of E_k states:

$$|u'\rangle = \Sigma\,|E_k\rangle\langle E_k|u'\rangle.$$

From the time-dependent principle of superposition it follows that

$$|u';t\rangle = \Sigma\,|E_k\rangle e^{-iE_k t/\hbar}\langle E_k|u'\rangle$$

so that the amplitude of v' in $|u';t\rangle$ is

$$\langle v'|u';t\rangle = \Sigma\,\langle E_k|v'\rangle^* e^{-iE_k t/\hbar}\langle E_k|u'\rangle. \tag{9-21}$$

The absolute square of $\langle v'|u';t\rangle$ is the probability $[v'|u';t]$ of finding v' in the ensemble prepared with property u' after an interval t. The right side of the equation above contains only the representatives of u', v' in the E-basis and the possible energy values of the system.

9.6 The Motion of Free Electrons. Classical linear wave theory has been used extensively as a guide in the formulation of our assumptions about probability amplitudes. Consequently, it is perhaps not very surprising that probability amplitudes provide a description of the "wavelike" properties of microphysical entities (electrons, protons, etc.). Indeed the considerations of this section developed from an analysis of the phenomenon of electron interference which consisted of little more than the interpretation of classical wave amplitudes as probability amplitudes. That the concepts developed above are also applicable to the "particlelike" properties of electrons is by no means obvious. Their applicability may be illustrated by an analysis of the simplest of all microphysical systems – a free electron in one dimension.

The observables of major interest for our purposes are the position x and momentum p of the electron[8]. The Heisenberg principle implies that x and p are incompatible. Each of these observables is complete by itself – i.e. there is no other classical property independent of x (or of p) which is compatible with x (or with p). This fact may be derived from the general laws of quantum mechanics which endow the open mathematical framework constructed in this section with explicit content. Since electrons may be found at any specific location x' or with any momentum p', each of these observables specifies a continuum of basis states; members of these continua, for specific values x', p' of position, momentum, will be denoted by $|x'\rangle$, $|p'\rangle$.

For reasons that will be given later it is convenient to work in the p-basis. By our assumptions on probability amplitudes, every state of a free electron can be represented by a p-amplitude, and every function of p, considered as a p-amplitude, determines uniquely some state of the electron. Therefore, let $\psi(p)$ be some function of p chosen at will and let $|\psi\rangle$ designate that state for which the p-amplitude is $\psi(p)$; all properties of the state designated by $|\psi\rangle$ are completely determined by the assumption $\langle p|\psi\rangle = \psi(p)$. The statistical properties of all state ensembles of free electrons may be therefore surveyed by investigating how these properties depend on the choice of the function $\psi(p)$. Fortunately, we need not be concerned here with the difficult problem of determining what state property is implied by a given $\psi(p)$ or how an ensemble in the state with p-amplitude $\psi(p)$ may be prepared.

The distribution of p in state $|\psi\rangle$ is easily obtained since this state has been defined by its amplitudes in the p-basis: $[p|\psi] = |\langle p|\psi\rangle|^2 = |\psi(p)|^2$. To obtain the distribution in position it is necessary to calculate $\langle x|\psi\rangle$. Eq. 9-17, after replacement of u' by ψ, v' by x, w by p and (because of the continuous nature of the p observable) the sum by an integral, yields

$$\langle x|\psi\rangle = \int \langle x|p\rangle\langle p|\psi\rangle\,dp. \tag{9-22}$$

Hence the x-amplitude for $|\psi\rangle$ can be obtained from its p-amplitude if the x-amplitudes of all p states are known.

Unfortunately our assumptions on amplitudes do not suffice for their explicit determination. The laws which permit the calculation of amplitudes are outside the scope of the considerations of the monograph. However, the $\langle x|p\rangle$ amplitudes can be all but guessed by a consideration of de Broglie's assumption that a wave of wavelength h/p is to be associated with a particle of momentum p. Surely the amplitude of this wave, to which no direct physical meaning is attached, can be no more than a probability amplitude. This idea (along with others of lesser importance) suggests that the x-amplitude describing a state of momentum p should be

$$\langle x|p\rangle = (2\pi)^{-1/2}e^{ipx/\hbar} \tag{9-23}$$

a harmonic wave with wavelength h/p. The factor $(2\pi)^{-1/2}$ is a normalizing factor which is chosen to make the resulting probability distribution in x "true" rather than "relative." Naturally this result may be derived, not merely guessed, from the laws of quantum mechanics. (The amplitude of (9-23) is in fact a solution of Schrödinger's Eq (9-19).) Eq. 9-22, with the substitution of the result 9-23, now permits the calculation of the x-amplitude and, naturally, the distribution in position for any state with a given p-amplitude. It is worth noting that Eqs. 9-22 and 9-23 indicate that the x-amplitude is a linear combination of the simple harmonic waves given by Eq. 9-23 with the coefficients

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[8] Electron spin and other relativistic effects are neglected.

$\langle p|\psi\rangle$. From this fact it is not difficult to see that the range of the distribution in x is inversely correlated with the range of the distribution in p. If $\langle p|\psi\rangle$ is different from zero over a "narrow" range of p, then $\langle x|\psi\rangle$ is a superposition of a group of waves with little difference in wavelength; consequently these waves will superpose constructively over a "broad" range of x. A broad range in p, on the other hand, implies a correspondingly large range in λ; in this case constructive interference of the component waves can be managed over only a small range in x.

From the amplitude $\langle p|\psi\rangle = \psi(p)$ that defines state $|\psi\rangle$ and the calculated x-amplitude $\langle x|\psi\rangle$, the distributions in x and p, $[x|\psi]$, $[p|\psi]$, are easily obtained. From these the average values of momentum and position $\bar{p},\bar{x}$ and the uncertainties $\Delta p,\Delta x$ for a ψ-ensemble may be derived (Chapter 7). A detailed analysis of the connection established by Eq. 9-22 between the x and p distributions shows that for any state – i.e., any choice of $\psi(p)$ – the uncertainties in the x and p distributions are related by the inequality

$$\Delta x \Delta p \geqslant \hbar/2;$$

this provides a quantitative expression of the uncertainty principle.

It is not difficult to determine the time-dependence of the amplitudes and consequently the time-dependence of the distributions in any state. For free electrons, the energy and momentum are related by $E = p^2/2m$ (where m is the electron mass). Consequently, states with a definite momentum are also states with definite energy and, therefore, have a simple harmonic dependence on time (it is for this reason that the p-basis is more convenient than the x-basis); by Eq. 9-20

$$|p';t\rangle = |p'\rangle e^{-iE't/\hbar} \qquad \text{with } E' = (p')^2/2m.$$

After a lapse of time the state

$$|\psi\rangle = \int |p\rangle\langle p|\psi\rangle dp$$

becomes

$$|\psi;t\rangle = \int |p\rangle e^{-iEt/\hbar}\langle p|\psi\rangle dp \qquad (E = p^2/2m),$$

so that

$$\langle x|\psi;t\rangle = \int \langle x|p\rangle e^{-iEt/\hbar}\langle p|\psi\rangle dp. \qquad (9\text{-}24)$$

This integral may be evaluated (in principle) once $\langle p|\psi\rangle$ is given.

Because the momentum states are also energy states, the distribution in momentum as well as of the energy is constant in time (see earlier in this section). In particular, the average value $\bar{p}$ of the momentum and the uncertainty Δp are constant (momentum is conserved). This is not the case for the distribution in position. It follows from Eq. 9-24 that the dependence of the average value of position on time is given by

$$\bar{x}(t) = \bar{x} + (\bar{p}/m)t,$$

($\bar{x}$ is the avergae value of position in the initially prepared state and $\bar{p}$ is the constant average value of momentum) and that the uncertainty Δx is a quadratic function of time. Note that the average value of position moves with constant velocity $\bar{p}/m$; this velocity is related to the average value of momentum exactly as the classical velocity of a particle is related to its momentum. As the average value moves with constant velocity the distribution spreads, and Δx becomes arbitrarily large for sufficiently large t.

It is possible to choose states, by suitable choice of $\psi(p)$, with $\Delta p/\bar{p}$ small (say $\sim 10^{-3}$) but yet with Δp large enough (i.e. with $\bar{p}$ sufficiently large) to permit a Δx which is macroscopically small (say $\sim 10^{-2}$ cm). Within macroscopic errors of observation electrons described by such a state will behave, over time intervals customary in laboratory experiments, like free classical particles. If observations are not too accurate they will be consistent with the assumption that the electrons in the ensemble are prepared with position $\bar{x}$ and momentum $\bar{p}$, that the momentum is constant, and that the electrons move with velocity $\bar{p}/m$. The spread of the distribution of x over the course of a normal laboratory experiment is insufficient to upset this picture. Thus, despite the radical incompatibility of position and momentum, it is possible to describe the particlelike properties of electrons within a formalism constructed on the basis of the concept of probability amplitudes.

10 Summary and Comment

"And, no matter what sort of Hell hath popped,
Let not the constant h *be dropped."*

Leonard Bacon

10.1 Epitome. A first reading of a set of new ideas often results in confusion, especially if the organization of that set is complex. Each new concept requires detailed study; the successive focusing of attention on small parts makes it difficult to perceive the design of the whole. But clearly, an approximation to an understanding of the whole is what is desired. A summary may help clarify the structure and intent of the material that has been presented.

Since classical theory fails so completely in the realm of microphysics, it is obvious that the conceptual tools it provides for the study of microphysical entities cannot be trusted. However, before the inadequacy of classical conceptions became evident, many microphysical observations were interpreted classically. It is important, therefore, in a

fundamental review of microphysical observations and concepts, to differentiate clearly between classical interpretation of microphysical phenomena and the brute facts of observation (Chapter 2).

Obviously re-examination of all classical interpretations of experiments in microphysics cannot be undertaken. Fortunately, a number of the properties of electrons and photons are obtained by methods that are hardly subject to doubt (for the most part by the application of general conservation laws that are verified in the realm of microphysics); some of these properties are listed in Chapter 3. Of particular importance for the work of this monograph are the de Broglie relations; their experimental confirmation and the observational significance of their terms are examined in Section 3.1.

The central argument of the book begins in Chapter 4 with a study of the so-called "wave-particle dualism." Examination of "diffraction" and "interference" experiments located the deficiencies of both wave and particle pictures of electrons and photons, and exhibited a number of curious consequences of the de Broglie relations:

(1) The motions of single electrons cannot be observed in detail; even the simple observation of whether an electron passes one or another of two slits cannot be effected without marked changes in the phenomena manifested when observation is not made.

(2) The regularities in observed microphysical processes apply to ensembles rather than single systems; it is an ensemble of electrons rather than a single electron that produces a diffraction pattern.

(3) There exists a basic incompatibility between the observables, "electron position" and "electron momentum," expressed by the Heisenberg uncertainty principle (Chapter 5). The degree or measure of this incompatibility is given, roughly, by Planck's constant, h. For processes in which h may be considered "small," microphysical behavior may be described by classical methods.

In Chapter 6, alternative interpretations of the consequences of the de Broglie relations are examined. Two classically based suggestions – the use of classical statistical methods, and the assumption of the existence of "hidden variables" that might restore classical determinism – are rejected as unsatisfactory. A continued analysis of further consequences of the de Broglie relations leads to the realization that there is hardly a feature of the classical picture of motion that is observable in those microphysical processes for which h is not negligible. Velocities, accelerations, forces, and paths, all concepts of primary importance in classical physics, simply cannot be observed in the microphysical realm when the limitations of the Heisenberg principle are significant. It is in this realm that microphysical behavior exhibits a paradoxical character. Our incapacity to observe classical properties seems to make the achievement of an understanding of microphysics very difficult, if not impossible.

This hardship stimulated a perception of fundamental significance. The deepening gloom caused by the recognition of so broad an incapacity to learn more about microphysical processes gave way to renewed optimism with the realization that what is truly unobservable requires no explanation and, indeed, may not even exist. This thought frees us from the prison of the classical conceptual system; while we struggled ineffectually to bend the bars at the prison window, the door was opening to a new and revolutionary interpretation of the Heisenberg principle. The Heisenberg principle does not represent a limitation on our capacity to observe but rather a limitation on the validity of the classical conceptual scheme! In this interpretation the concepts derived from macrophysical experience are useful, and appear to have precise content only because in macroprocesses the quantity, h, is wholly negligible.

Revolutions, no matter how soundly based, well intentioned, and essential to continued progress, are highly destructive in their initial phases. If a revolution is to succeed, it must look quickly to the replacement of those institutions that performed essential services in the old regime with new ones that are consistent with revolutionary goals. Unless the vitally needed reconstruction is possible within the new system, the revolution must fail. The new interpretation of the Heisenberg principle undermined a number of the concepts on which every scientific structure was believed to rest (Chapter 6).

After the chains that bound thought to classical ideas were broken, it was found that the classical concepts of "system property" and "state," on which classical scientific structures had been based, were in need of either extensive repair or complete replacement. Unless concepts were created to do the work normally required of the "property" and "state" concepts in classical physics, a consistent theory of microphysics, founded on the new interpretation of the Heisenberg principle, could not be formulated. Chapters 7 and 8 constitute a sketch of the required creations. They indicate that meanings, closely allied to but essentially different from classical meanings, can be assigned to the concepts of "property" and "state," and that with these new meanings there remain no apparent internal contradictions in the conceptions required for a reconstruction of theory.

In the course of redefining the "property" and "state" concepts, two ideas of central significance for the character and content of microphysical theory were singled out for close examination: "incompatible properties" and "statistical determinism." The complex of features related to the words "indeterminacy" and "unobservability," so characteristic of microphysical phenomena, may be traced to the relation of incompatibility between pairs of microphysical properties (Chapter 7). This new relation of incompatibility forms a bridge between the sharply differentiated relations of "compatibility" and "exclusiveness" that are familiar in classical physics. It is the incompatibility of observable properties that causes classical interpretation to fail so completely in microphysics.

The principle of statistical determinism (like the corresponding classical principle of determinism), is an inductive generalization based on the observation of the regular and lawful behavior of ensembles (for the classical case, single systems) of suitably prepared microphysical systems (Chapter 8). Although the behavior of individual microsystems is indeterminate, relations between the properties of ensembles may be definite and reproducible. The principle of statistical determinism makes possible the construction of consistent ideas of "state" and "state properties," and characterizes the general form of law and prediction in the realm of microphysics.

The measurement of a state observable w on a state ensemble $|u']$ converts the ensemble into a mixture of subensembles, each characterized by one of the state properties in w. The fraction of systems in the w_k subensemble is just the probability, $[w_k|u']$, of finding w_k in $|u']$. This consideration suggests the existence of a relation of some kind between the statistical properties of the u' state and the statistical properties of the w_k states which result on measurement of the observable w.

On a classical (and incorrect) interpretation of the property concept, which assumes the objective existence of any measurable property, the relation between the u' and w_k ensembles is obvious: The u' ensemble is simply a mixture of w_k ensembles with the relative weights $[w_k|u']$. In this view (Subsection 9.2), the probability of measuring v' in $|u']$ is linearly related to the probabilities of the measurement of v' in the separate w_k states:

$$[v' | u'] = \sum_k [v' | w_k] [w_k | u'] \qquad \text{(classical) (9-10)}$$

The falsity of this classically based result has been stressed repeatedly (Chapters 4, 7, and 9).

The relation between states postulated in quantum mechanics is quite different from Eq. 9-10. The structure of this new relation is suggested by the application of the classical theory of linear fields to the analysis of microphysical experiments. In this theory a field motion is described by amplitudes of some kind (e.g., the displacement of a string). Because of the linearity of the theory, particular field amplitudes can be expressed as linear combinations of component field amplitudes. Such a classical theory may be used for the interpretation of certain microphysical experiments if the squared amplitudes are interpreted not as classical intensities but as probabilities. (For this reason the amplitudes when used for microphysical considerations are called probability amplitudes.) These considerations suggest the possibility of a connection between microphysical states in terms of probability amplitudes of the same form as that which occurs in the classical theory of fields. The theory of quantum mechanics assumes that a probability amplitude exists for each pair of state properties – for the v', u' properties the amplitude (a number) is denoted by $\langle v'|u'\rangle$ (read: the amplitude of property v' in state u') – such that $|\langle v'|u'\rangle|^2 = [v'|u']$ and

$$\langle v' | u' \rangle = \sum_k \langle v' | w_k \rangle \langle w_k | u' \rangle. \qquad (9\text{-}12)$$

This relation states that the amplitudes of any property in the u' state can be expressed as a linear combination of the amplitudes of that property in the w_k states into which u' is decomposed by observation of w. The coefficients in this combination, however, are not the probabilities $[w_k|u']$ but the probability amplitudes $\langle w_k|u'\rangle$. It is of interest to note that Eq. 9-12 can be obtained from the incorrect classical result (Eq. 9-10) by substituting amplitudes for probabilities throughout.

The assumption of the general validity of (Eq. 9-12) is of great importance for the mathematical structure of quantum mechanics. In Chapter 9, various consequences of the existence of probability amplitudes and of the relations Eq. 9-12 are examined.

10.2 The Laws of Quantum Mechanics. This book is concerned not with the content but with the underlying conceptions of quantum mechanics. Our work specifies the questions that microphysical theory must answer but does not describe the theory from which explicit answers may be derived. A detailed presentation and analysis of this theory may be found in the many texts on quantum mechanics; space limitations prevent a full treatment here. It seems unreasonable, however, to close our development of the subject without at least a brief sketch of the remarkable and beautiful theory that has been built on the foundations examined in Chapters 7, 8, and 9.

To describe the additional assumptions or laws that, together with the assumptions on amplitudes of Chapter 9, complete the theory of quantum mechanics, an important new concept, the "linear operator associated with an observable," must be introduced. An observable, s, it will be recalled, is characterized by the set of its possible values or the "spectrum of the observable," and the states[1] in which the observable has a definite value (the "proper states of the observable"). From the spectrum of an observable and the associated proper states, a linear operator S belonging to or associated with the observable s may be constructed[2] For our present purposes the technical procedures for the

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[1] The state concept used here is not that of the state ensemble but the mathematical state of Chapter 9. A state $|\psi\rangle$, it will be recalled, is defined by its various amplitudes or by its representation in some basis. Any state may be expressed as a linear combination of other states. In particular, it is meaningful to multiply states by numbers and to add them.

[2] A function f associates to each number x in some domain another number $f(x)$. Similarly an operator S associates with each state $|\psi\rangle$ of a system another state of the system which may be designated by $S|\psi\rangle$. The operator S is completely defined if the state $S|\psi\rangle$ is specified for each state $|\psi\rangle$. A function f is linear if $f(ax) = af(x)$ and $f(x+y) = f(x) + f(y)$. Similarly an operator S is linear

construction of operators from observables (which, incidentally, depend critically on the amplitude concept of Chapter 9) and the many ways that operators may be mathematically formulated need not concern us. Of importance is simply the fact that the spectrum and proper states of an observable uniquely determine the associated operator and, conversely, that the operator determines the spectrum and proper states of the associated observable. The introduction of operators involves no new physical concepts. But operators provide more than merely compact instruments for the description of observables; they form the microphysical analogs of the "dynamical variables" of classical theory and it is to this fact that they owe their theoretical importance.

Definitions may be given for the multiplication of an operator by a number and the addition of operators and products of operators. In the resulting algebra of operators all the rules of ordinary algebra hold except the commutative rule: the products SR and RS of two operators need not be equal – i.e., R and S need not "commute."

The way in which operators are defined in terms of the proper states and spectra of observables directly implies that operators constructed from compatible observables commute and that operators constructed from incompatible observables do not. Consequently, if the operators R, S belonging to observables r, s are given, it is a simple matter to determine whether the observables are compatible or incompatible. If the operators commute they are compatible; if not they are incompatible. However, once the operators for a pair of observables are supplied, much more can be derived than such qualitative features as compatibility or incompatibility. The possible values of the observables r, s and the amplitudes (representatives) of the proper states may be obtained. In particular the amplitudes $\langle s_j | r_k \rangle$ for all pairs of possible values of s and r and, hence, the associated probabilities $[s_j | r_k]$ can be

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if $S\{c\,|\,\psi\rangle\} = c\{S\,|\,\psi\rangle\}$ and $S\{|\,\psi\rangle + |\,\phi\rangle\} = S\,|\,\psi\rangle + S\,|\,\phi\rangle$. Thus if the states $S\,|\,\psi\rangle$, $S\,|\,\chi\rangle$, $S\,|\,\phi\rangle$ are given the state associated by S with any linear combination of $|\,\psi\rangle$, $|\,\chi\rangle$, $|\,\phi\rangle$ can be determined. Since any state is a linear combination of a set of basis states a linear operator is completely defined when it is defined for a set of basis states. The operator S associated with observable s is defined by the associations $S\,|\,s_k\rangle = s_k\,|\,s_k\rangle$, ($k=1, 2, \ldots$) where s_k is a possible value of the observable s and the $|\,s_k\rangle$ are the states in which s is certain to have the value s_k. Since any state can be expressed as a linear combination of the $|\,s_k\rangle$ states these associations completely define S.

computed. It follows that all the properties of a microphysical system can be derived once the operators that are associated with the observables measurable on that system are known.

In the theory of quantum mechanics, *laws are formulated which specify the operators* that are to be associated with observables that have classical descriptions. Consider, to begin with, the observables x, p, the components of position, momentum along some axis, of an entity (electrons, protons, neutrons, etc.) on which these observables are measurable. Let the unknown associated operators be X, P. By the Heisenberg principle, x and p are incompatible; consequently, the X, P operators do not commute. Since the "degree" of incompatibility (this "degree" is measured by the precision with which x and p are simultaneously measurable) is determined by Planck's constant, one must expect that the difference between XP and PX will involve this constant. The theory of quantum mechanics assumes the operators for the position and momentum observables are related by:

$$XP - PX = i\hbar. \tag{10-1}$$

(Other arguments into which we cannot go here may be adduced to suggest the truth of Eq. 10-1, but ultimately this relation must be taken as a primary assumption of the theory.) It is a remarkable fact that this relation, together with the natural assumption that both x and p have infinite continuous spectra of possible values, effectively determines the structures of both the X and P operators. From Eq. 10-1 the amplitudes $\langle x\,|\,p\rangle$ and the expression $\Delta x\ \Delta p \geqslant \hbar/2$ for the uncertainty principle may be derived. It is through the law expressed by Eq. 10-1, perhaps the most important of all the laws of quantum mechanics, that Heisenberg's uncertainty principle is inserted into the structure of the theory.

Any two components of position (or of momentum) are assumed to be compatible. Compatibility is also assumed for any component of position and an orthogonal component of momentum. Finally all observables belonging to one "particle" are assumed to be compatible with observables relating to another. The operators associated with these compatible pairs of observables therefore commute. The operator relations implied by these statements together with Eq. 10-1 specify the operators for all the Cartesian components of the position and momentum observables.

Any classical observable may be expressed as a function of the position and momentum variables. Thus, the third component of the angular momentum is given by $m_3 = x_1p_2 - x_2p_1$, the energy of a one-dimensional oscillator by $e = p^2/2m + kx^2/2$, etc. What operators are to be associated with such observables? The answer to this question requires an additional assumption: Effectively, it is assumed that if observables r, s are described by operators R, S, then the observable classically defined by $r + s$ is associated with the operator $R + S$. By the use of this assumption, the operators for classical observables such as m_3 and e can be expressed in terms of the previously specified position and momentum operators. In most simple cases the operators are formed by the substitution of the known X, P operators for the corresponding variables in the classical expressions for the observables. Thus the operator associated with the m_3 observable is $M_3 = X_1P_2 - X_2P_1$, and that for the energy of a one-dimensional oscillator is $E = P^2/2m + \frac{1}{2}kX\ 2$.

Note that the scheme developed above provides operators only for observables that have a classical expression in terms of position and momentum variables. Many nonclassical observables have been discovered in microphysical studies; the most familiar example is that of spin. For nonclassical observables no general rules for the theoretical formation of associated operators is known. Operators must be designed or invented for such observables to provide a fit between theory and observation.

Once the operator for e, for instance, has been specified in terms of the X, P operators ($E = P^2/2m + kX^2/2$), it is possible to determine the spectrum of e and to calculate amplitudes such as $\langle x \mid e_k \rangle$, $\langle p \mid e_k \rangle$ and, hence, the distribution of x or p in states with the energy value e_k, etc. By means of the work of Section 9.5, moreover, it is possible to determine the time dependence of arbitrary states of the oscillator.

The mathematical form of the theory of quantum mechanics may now be summarized. The existence of the linear relations between states formulated in Eqs. 9-12, 9-13 permit the description of observables by associated linear operators. The operators that describe classical observables can be constructed theoretically by the use of the fundamental law expressed by Eq. 10-1 and subsidiary rules. No similar rules exist for nonclassical observables; the associated operators must be obtained by other means. Once the mathematical operators for all the observables of a microphysical system are known, the measurable properties of the system may be deduced.

10.3 Consequences of a Successful Theory of Microphysics. The development of quantum mechanics and the associated conceptual shift from classical thought constitutes one of the most exciting of the many intellectual adventures recorded in the history of civilization. The construction of a theory that describes the properties of the constituents of bulk matter represents, however, much more than an achievement in pure thought. The importance of the theory for chemistry is obvious; the quantum mechanics provides chemistry with a complete theoretical framework. Developments in the chemistry of large molecules has made possible the recent exciting discoveries in biology relating to the structure of the gene. It would seem that biologists, equipped now with a powerful theory of atomic and molecular processes, are only just short of a physical understanding of living organisms.

The numerous technological, and hence social and political, consequences of quantum mechanics are rapidly changing the very character of our lives. The "bomb" is no doubt the most widely known technical achievement made possible by a knowledge of microphysical theory, but, hopefully, by no means the most important[3].

Although the work in microphysics during the 45 years since the discovery of quantum mechanics far exceeds (in volume) the work of all previous physics, it is likely that we are still far from a full appreciation of its content and significance.

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[3] The probability of establishing communication with beings on planets belonging to other solar systems in our galaxy (it is believed that many solar systems have planets which enjoy conditions favorable to the formation and evolution of life) depends critically on what estimate is made of the durability of technical civilizations with scientific capacities as good or better than our own. If societies with a knowledge of quantum mechanics and its many consequences are viable and persist, without loss of technical capacity, for millions of years, then it is likely that there are many technical civilizations in our galaxy with the instruments needed to send us meaningful signals. If, however, societies with

10.4 Critical Notes. Some physicists find quantum mechanics psychologically unsatisfactory despite its great success in explaining observed microphysical relations and in predicting new ones. Quantum mechanics does not "explain," and we appear to be unable to observe, what happens to an electron in the course of passing a slit in a diffraction experiment. Instead, the theory specifies merely the distribution in position of an ensemble of electrons at a detecting screen. In accepting the successful and revolutionary concepts of indeterminacy and incompatibility, physics gives up the effort to provide a "picture" of "objective reality" and limits itself to what some regard a relatively menial task – the prediction of the results of observations.

Consider, for example, the question, "What is an electron?" In the early sections of this chapter both the classical "wave" and "particle" models of an electron were rejected, for neither provides a satisfactory description of all features of diffraction experiments. A fair number of words have been devoted to describing what the electron is *not*, but the reader will search unsuccessfully through the constructive considerations of Chapters 6, 7, and 8 for the sentence: "Thus we see that the electron is . . . ." The observable phenomena produced by electrons are described by quantum mechanics in great detail, but to a request for a short description of an electron the theory can only reply: "The electron is an entity with a certain set of observable properties; among these there exist well-defined statistical relations." This answer is not likely to excite the response, "Oh! So that's what an electron is."

Because of the indefiniteness of the statistical descriptions in the new theory and because of the failure of quantum mechanics to describe phenomena in terms of the objective characteristics of microsystems, a few of the great contributors to the theory, including Einstein, Schrödinger, and de Broglie, remained dissatisfied. As yet, however, no successful alternative to quantum mechanics has been discovered and most physicists believe that further advances in theory will not lead back to classical concepts. In fact, there is a general tendency to examine critically various classical presumptions that are still retained in quantum mechanics. The meaningfulness, e.g., of arbitrarily small space-time intervals has been questioned. Are distances of the order of 10^{-16} cm observable, or time intervals of 10^{-25} sec (in 10^{-25} sec a photon travels about 10^{-15} cm)? If not, are there new constants that determine limitations on the meaning of small intervals in the same sense that h measures the limitations on the concept of simultaneous position and momentum?

The future of physics cannot, of course, be predicted. However, the character of current theoretical investigations suggests that, as penetration into the microphysical domain goes on – with investigations into the interiors of "elementary" particles, and with the study of processes involving energy transformations of billions of electron volts – new revolutions will carry physics ever further from classical concepts.

10.5 Conceptual Revolutions in Physics. The history of physics includes three major conceptual revolutions. The work of the seventeenth century which, with the publication of Newton's monumental *Principia*, gave to physics its distinctive character, revised earlier concepts of "motion," "force," and "mass." Einstein's theory led to radical changes in concepts of "space," "time," and "gravitation." Quantum mechanics revised concepts of "system property" and "determinism."

These revolutions have several characteristics in common. In each case, certain observations appear to face existing theory with insuperable difficulties; within the reigning system of concepts the problems set by these observations appear to have no solution. Instead of planning a painful and diligent search for new features of nature which might make solutions possible within old patterns of thought, the new theory lightly bypasses the problems and frames its concepts and axioms so that the offending observations are introduced into its very foundations. Instead of continuing to lose the game to nature, the physicist tries changing the rules of play. It takes great genius, however,

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developed technologies are inherently unstable and are likely to destroy themselves, either by the powerful weapons that an understanding of microphysics makes possible or perhaps merely through a decay of spirit after a period of stressless living in technological abundance, then the chance that two such societies should coexist in the galaxy is negligible. During the period we on this planet are ready to communicate, there would be no one out there to whom we could successfully talk.

The detection of intelligent signals from interstellar space would not only have enormous intrinsic interest but would be very reassuring as regards the future existence of our own civilization.

to invent new rules that make for playable, profitable, and interesting games.

Before the time of Galileo and Newton, men had struggled for centuries to "explain" the motion of an arrow after it leaves a bow, or the motion of a pebble after it leaves a sling. Accepted theory, largely Aristotelian, claimed that motion without a mover is unthinkable. What, then, pushed the arrow along in its flight? Newton does not answer this question; he simply assumes, in his first law, that objects "naturally" maintain motion. The feature of motion that is significant for Newton is not constant velocity, but changes in velocity – i.e., acceleration.

Within classical conceptions of space and time, the implications of the Michelson, Morley experiment (and many others) are paradoxical. How can the speed of light be independent of the motion of an observer? Einstein does not answer this question; instead, he takes the constancy of the velocity of light as one of the axioms of his theory. His great achievement is his proof that with this axiom a self-consistent theory that preserves the principle of relativity is possible. To build the constancy of the velocity of light into the theory required, however, highly significant revisions in classical concepts of space and time.

The application of classical theory to microphysics appeared to be blocked by the complex of observations associated with the words "wave-particle dualism." How could electrons appear to behave sometimes like waves and sometimes like particles? To classical theory, electron behavior seemed simply schizophrenic. Quantum mechanics does not "explain" the apparent duality. Instead, the dualism is interpreted as reflecting the inadequacy of the conceptual scheme underlying classical theory, and a new conceptual framework, within which the apparent dualism fits in a natural manner, is sought. By means of changes in the concepts of "determinism" and "property," together with important theoretical inventions, it is possible to construct a consistent theory into which the wave-particle characteristics of microphysical entities can be fitted. The construction has been costly; to achieve it many cherished habits of thought had to be sacrificed. "But," as the saying goes, "the recompense is ample."

GENERAL REFERENCES

The experiments that reveal the nonclassical characteristics of microphysical phenomena are described, at various levels of sophistication and detail, in a large and rapidly growing number of books. The following is a partial list.

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NOTE:

References 5 and 7 are on a somewhat more advanced level than the others. Reference 5 gives considerable detail about the physical arrangements and observational results of experiments; it contains many references to original papers. Reference 7 is especially recommended for the acuity of the physical insights it provides into the significance of various microphysical observations; its discussion of blackbody radiation is most instructive.

All of the general references listed above contain sections or chapters devoted to the photoelectric and Compton effects. See reference 5 for references to the literature.

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