

CLASSICAL AND QUANTUM MECHANICS OF THE DAMPED HARMONIC OSCILLATOR

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Abstract:

The relations between various treatments of the classical linearly damped harmonic oscillator and its quantization are investigated. In the course of a historical survey typical features of the problem are discussed on the basis of Havas' classical Hamiltonian and the quantum mechanical Süssmann-Hasse-Albrecht models as coined by the München/Garching nuclear physics group. It is then shown how by imposing a restriction on the classical trajectories in order to connect the Hamiltonian with the energy, the time-independent Bateman-Morse-Feshbach-Bopp Hamiltonian leads to the time-dependent Caldirola-Kanai Hamiltonian. Canonical quantization of either formulation entails a violation of Heisenberg's principle. By means of a unified treatment of both the electrical and mechanical semi-infinite transmission line, this defect is related to the disregard of additional quantum fluctuations that are intrinsically connected with the dissipation. The difficulties of these models are discussed. Then it is proved that the Bateman dual Hamiltonian is connected to a recently developed complex symplectic formulation by a simple canonical transformation. The fundamental commutator is still in error. Therefore it is demonstrated how, either separating the dual oscillators according to a modified version of Bopp's original treatment or reducing classical complex phase space by an integration over the mirror image subspace, a quantum continuity equation is obtained that leads to Dekker's master equation following the usual operator algebra. The dissipation induces additional fluctuations. The same density operator equation is shown to arise in quantum optics in the weak coupling limit. Next, for weak friction, Hasse's pure state condition is used to derive an equivalent nonlinear but normconserving frictional Schrödinger equation. It involves a particular non-Hermitian Hamiltonian which, if rewritten in terms of Dekker's complex variables, reveals an elegant extension of the classical Hamilton-Jacobi theory. Finally, this formalism is used to make contact with Kostin's fluid dynamical Schrödinger-Langevin equation.

1. Introduction

The obviously irreversible, dissipative behaviour of the vast majority of physical phenomena in the every-day world, in particular when contrasted with the reversible nature of our basic models, has saddled science with an intriguing and apparently enduring problem. In fact, it was not until the times of Newton before it was clearly formulated that the fundamental laws of mechanics indeed were of the reversible type. Dissipation then arises in principle from interactions between the actually observed system and another one (or more) into which energy flows in an irreversible manner. As it usually turns out, the details of the structure of the other system (often called the reservoir or bath) are quite immaterial especially in the weak damping limit. However, the calculations from first principles are cumbersome and little transparent as a rule, and the ultimate transition from reversibility to irreversibility remains a difficult problem the solutions of which can often barely stand a serious critical examination.

Luckily, it is not so difficult to account for dissipative forces in classical mechanics in a phenomenological manner, at least not in the Newtonian formulation [1, 2]. Noteworthy examples are Stokes' linear frictional force proportional to the velocity v , Coulomb's friction $\sim v/v$ [3], Dirac's radiation damping $\sim \ddot{v}$ [4], and the viscous force $\sim \nabla^2 v$ [5, 6]. The phenomenological approach applied to the quantum domain will be the main topic of this article.

Unfortunately, as opposed to the Newtonian (vector) formulation, the situation is much less comfortable within the framework of the Hamiltonian (scalar) formalism by which theoretical physics has advanced so much. The deeper reason for this difference can be found in the derivation of the Lagrange-Hamilton variational formulations from D'Alembert's principle [1, 2, 7]. Apart from the fact that this derivation can be carried through fruitfully only for systems with holonomic constraints, the latter principle (which in fact is the dynamical generalization of the static principle of virtual work) is at best equivalent to Newtonian mechanics only if the virtual work of the forces of constraint is zero. And this is clearly not true for friction forces.

The troubles are immediately manifest for one dimensional systems with Hamiltonians that do not explicitly depend on time, at least if these Hamiltonians are required to represent the physical energy of the system [8–10]. However, the time-independence of the forces and the auxiliary conditions is not required for the Hamiltonian theory to be valid. Evidently then, one could have resort to in principle

three alternatives. First, one could proceed to multi-dimensional systems (see e.g. [11, 12]). Second, the Hamiltonian could be allowed to have an explicit time-dependence (e.g. [13, 14]). Third, one could attempt Hamiltonians that have no relation to the energy [8, 15], so-called mathematical Hamiltonians or merely generators of the motion. Usually, these possibilities (all of which have indeed been discussed in the literature) are intertwined. In the present text several examples will be encountered in the course of an attempt to elucidate the relations between quite different historical approaches to the problem of the classical damped oscillator.

The difficulties with the Hamiltonian description of frictional phenomena become the more prominent at the microscopic quantum level as quantum mechanics is an essentially Hamiltonian theory. Of course, and with an even deeper physical significance than in classical mechanics, the fundamental quantum dynamical laws are of the reversible type. A closed system, say the universe, is supposedly described by a Schrödinger equation. The system's dynamics is governed by the Hamiltonian that represents its total energy, and which is a constant of the motion. Principally, as in classical mechanics, dissipation is merely observed for open systems, being only a part of the universe. The dissipation arises from the subsystem's interactions with the rest of the universe, again often referred to as the reservoir or bath, whether it be thermal or not. Confining oneself to the subsystem of interest, the explicit microscopic reservoir variables are eliminated from the description by means of projection operator techniques or tracing procedures (see e.g. [16–25]). Usually this extracts from the Liouville–von Neumann equation a (generalized, or Nakajima–Zwanzig) master equation for the reduced density operator. That is, in effect one reduces the considered phase space by means of an appropriate integration procedure over the irrelevant actual dynamical variables of the reservoir. As a result the reservoir can only be recognized in the description through a few parameters, such as the friction constant. These features are familiar, for example, from the theory of Nyquist's formulae for the thermal noise in an electric resistor [6, 26–30]. Once more, the essential point is the observation that the detailed microscopic structure of the bath is mostly irrelevant (see also e.g. [31–33]).

The latter recognition has given impetus to the search for a so-called mesoscopic description of reduced systems [34]. One might approach this idea either philosophically or technically from two points of view. First, physics has often advanced by looking for the simplest possible description that can be considered as comprehensible from an intelligent anthropomorphic position, and that conforms to observation. Why should one carry along part of the way a tremendous amount of information that in fact will be eliminated in due course? Actually, what is the physical significance of such effectively hidden information? Of course, the prime significance could very well be the intellectually satisfying achievement of unification, but otherwise the answers to these questions are definitely nontrivial (see also [35–37]). Second, the ab-initio calculations are not only cumbersome and the approximations involved at the various stages often hard to control, but they are also afflicted with particular problems that have not yet been resolved entirely satisfactory. As a typical example serve the divergencies occurring in certain correlation functions (e.g. [38, 39] and section 5 of the present text), which may have either physical or mathematical origins (e.g. [40–42]). In any case, these deficiencies seem to be intimately connected with the infinite number of degrees of freedom which the reservoir must possess in order for the dissipated energy never to return to the considered subsystem.

The existence of a regime where the concept of phenomenological friction applies is easily conceived in the realm of classical physics where the particle under investigation is much more massive than the constituents of the reservoir (see also [43]). Also in Brownian motion the concept is certainly legitimate, albeit that in general there exist at this scale in between macroscopic and purely microscopic phenomena so-called fluctuation–dissipation relations (or generalized Einstein relations; see e.g.

[6, 21, 24, 28, 30, 34–38, 44–64]). The earlier mentioned Nyquist relation is a typical example. In classical thermal systems these relations are of the general type

$$2\lambda k_B T = S_\xi(0), \quad (1.1)$$

where $S_\xi(0)$ is the dc spectral density of the fluctuations and where λ is the relaxation constant. By way of simple illustration, consider the Langevin equations for a particle of unit mass [21, 34, 37, 38, 56–60, 65–67]:

$$\dot{x} = p, \quad (1.2)$$

$$\dot{p} = -\lambda p + \xi(t). \quad (1.3)$$

The noise term, which here is a real physical fluctuating Newtonian force, is assumed to represent Gaussian white noise, with

$$\langle \xi(t) \rangle = 0; \quad \langle \xi(t + \tau) \xi(t) \rangle = 2D \delta(\tau). \quad (1.4)$$

This generates a time-homogeneous Wiener process [29, 48, 56, 62, 68–76]; D is the diffusion coefficient. The spectral density of the fluctuations in this case reads

$$S_\xi(\omega) = 2 \int_0^\infty \langle \xi(\tau) \xi(0) \rangle \cos \omega \tau \, d\tau = 2D, \quad (1.5)$$

so that $S_\xi(0) = 2D$. Further, the formal solution for the velocity is

$$p(t) = e^{-\lambda t} p(0) + \int_0^t e^{-\lambda(t-t')} \xi(t') \, dt'. \quad (1.6)$$

Rather than treating the adiabatic limit $\lambda \rightarrow \infty$ (so that $p(t) \approx \xi(t)/\lambda$, and where the problem is readily reduced to one of position only, leading to the common diffusion equation [37, 44, 68, 77–84]), we calculate the velocity autocorrelation function directly from (1.6). By means of (1.4) one obtains in the long time limit $t \rightarrow \infty$,

$$\langle p(t + \tau) p(t) \rangle = (D/\lambda) e^{-\lambda \tau}. \quad (1.7)$$

Thus one finds for the mean kinetic energy

$$E = \frac{1}{2} \langle p^2 \rangle = D/2\lambda. \quad (1.8)$$

On the other hand, we know from different reasoning (e.g. [28, 30]) that in thermal equilibrium the equipartition law yields $E = \frac{1}{2} k_B T$ per particle per degree of freedom. Combining these results, one indeed arrives at the relation (1.1).

In the classical regime, where $k_B T \gg \hbar\omega$, $k_B T$ is the average energy per unit bandwidth. According to quantum theory (see e.g. [38, 41, 50, 57, 85–94]), when $k_B T \approx \hbar\omega$, this quantity should be replaced by Planck's function,

$$\mathcal{P}(\omega) = (N_\omega + \tfrac{1}{2})\hbar\omega, \quad (1.9)$$

$$N_\omega = (\exp(\hbar\omega/k_B T) - 1)^{-1}, \quad (1.10)$$

which can also be cast in the form

$$\mathcal{P}(\omega) = \tfrac{1}{2}\hbar\omega \coth(\tfrac{1}{2}\hbar\omega/k_B T). \quad (1.11)$$

These formulae present the energy available from the reservoir per unit bandwidth. In connection with the Brownian motion, (1.2) and (1.3), the precise significance of the frequencies ω is not so obvious as there is no resonance phenomenon involved here at some characteristic frequency, in contrast with the harmonic oscillator case. These questions will be further touched upon in the sequel of this paper. For the moment being, it is important to observe that in the pure quantum regime, where $k_B T \ll \hbar\omega$, the so-called vacuum or zero-point fluctuations will become the prominent source of noise. In that limit the fluctuation–dissipation relation (1.1) should be changed into

$$\lambda\hbar\omega = S_\xi(0). \quad (1.12)$$

Recently, van der Ziel and coworkers [94] have seriously considered the possibility of an explicit experimental test of the “thermal” zero-point energy at easily achievable temperatures ($T \approx 100$ K) and frequencies ($\omega/2\pi \approx 100$ GHz). Evidently, the general expectation is that a connection must exist between dissipation and quantal vacuum fluctuations. As, according to the general theory of the hierarchical elimination of dynamical variables (e.g. [19, 21, 23, 35–37, 57, 58]), an increased reduction of phase space usually introduces additional fluctuations in the remaining variables, one should not be surprised if the damped subsystem will be endowed with zero-point noise of greater intensity than that of the free universe (per unit bandwidth).

Aside from some of the historic examples mentioned in the beginning of this section (among which Dirac's is a most interesting one; see also [95–98]), the relevance of frictional forces in the microscopic domain has become especially apparent in the fields of nuclear fission (the fluid dynamical liquid drop model; see e.g. [99, 100]), giant resonances (e.g. [101]), and heavy-ion collisions (e.g. [102–107]). For an extensive review, in particular of the latter subject, see e.g. [108, 109]. Further, one obviously expects the quantum fluctuations to dominate over the in essence classical thermal fluctuations in particular at very low temperatures. Therefore, in cryogenic investigations (say below 1 mK) using NMR-techniques (at e.g. $\omega/2\pi \approx 20$ MHz) and doing, for instance, ultrasound attenuation measurements (say between 10 and 100 MHz), the primary noise sources become quantum mechanical, provided of course that the electronics of the measuring device is sensitive enough and has sufficiently low noise levels. For example, squid circuits have been proposed to test theoretical predictions for damped quantum systems [110] (see also [111, 112]).

Of course, questions can be raised concerning the very existence of a microscopic regime where the mass of the particle under investigation is of the same order as that of the constituents of the interacting reservoir and where the notion of classical friction has any meaning. Nonetheless, in view of the above

given considerations and examples, such a regime will be tacitly presumed in the sequel. Thus, in close analogy with Brownian motion, we shall consider dissipative systems where the damping itself is described macroscopically from the outset by a phenomenological friction parameter, but where the associated microscopic quantum fluctuations can not be ignored. It is precisely for such an approach that the term mesoscopic has been coined within the framework of classical stochastic processes [34]. In view of the actual (and, perhaps, unexpected) complexity of the problem we shall confine ourselves to the simple linearly damped harmonic oscillator in one dimension, and often only at zero temperature.

2. Historical survey

One of the earliest investigations to date has been presented by Bateman in 1931 within a purely classical context [10, 11]. Bateman discussed two different Hamiltonians (or, to be historically correct, at least the associated Lagrangians). One Hamiltonian was time-independent and describes the damped oscillator by virtue of the addition of the so-called mirror image system (also typified sometimes as the dual, adjoint or complementary system). The energy dissipated by the actual oscillator of interest is absorbed at the same pace by this artificial device that takes the place here of the unknown physical reservoir. Therefore the energy of the total system is a constant of the motion, as of course is the Hamiltonian. However, the Hamiltonian represents the energy only for a restricted set of dynamical solutions, a feature that is relevant to quantum mechanics as will be seen furtheron. The Bateman dual Hamiltonian has been rediscovered subsequently by Morse and Feshbach [113], and Bopp [114]. The former treated the model only classically, while the latter also considered its quantization. Apparently independent of Bopp, a rather detailed alternative quantum mechanical treatment has been given by Feshbach and Tikochinsky at the occasion of Rabi's birthday in 1977 [115, 116].

The other Bateman Hamiltonian was time-dependent, although it turns out to be a constant of the secular motion (i.e. averaged over an oscillator period). Therefore, it can in effect not be considered as the energy representative of a dissipating system. This peculiarity was most clearly noted by Ray [10]. If viewed as a classical mechanical Hamiltonian, it rather describes a system with time-dependent mass. An initially microscopic mass then grows exponentially to macroscopic figures, which explains the ultimate killing of vacuum fluctuations (i.e. of Heisenberg's uncertainty principle) in the usual quantum theory of this model. This time-dependent Hamiltonian has been, so to speak, reinvented by several authors, and is commonly known under the name of Caldirola and Kanai [13, 14]. Its quantum mechanics has been explored for instance in [12, 96, 116–138]. Related work may be found in [139–155], while for other surveys reference is made to Hasse [108, 109] and Messer [156].

Of particular interest, both historically and in relation to the present text, is the work of Stevens [120, 121]. In his 1958-report Stevens in fact reinvents and considers the quantization of the Bateman–Caldirola–Kanai Hamiltonian. Physically, he approaches the problem from the point of view of tuned electric circuits. In the paper cowritten with Josephson [121], the coupling of spin systems to dissipative cavity modes has been studied. Considerable misgivings about the validity of the usual quantization of the Bateman–Caldirola–Kanai Hamiltonian, expressed by M.H.L. Pryce in private conversations, prompted Stevens to investigate the behaviour of a resonant circuit (as the subsystem) coupled to a semi-infinite electric transmission line (as the reservoir) in more detail [157, 158]. In this manner he succeeded in making contact between the conservative Hamiltonian dynamics of the entire system and the time-dependent Hamiltonian description of the effectively damped oscillator (the subsystem) on its own. In brief, the conclusion was that the latter description is valid quantum mechanically only for the

oscillator variables *including* the noise from the transmission line. The more profound reason is, of course (one can easily say a posteriori), that quantum mechanically absorption and emission processes (of the oscillators in the bath) can not be really separated [159–167]. It should be remarked that Stevens' model is quite akin to works of Ford, Kac and Mazur [38], Ullersma [53] and others (e.g. [16–23, 41, 42, 63, 168–179]). A seemingly different, though *au fond* closely related treatment of a mechanical model (the semi-infinite string) can be found in [39]. In the present paper it will be attempted to outline these models in as unified a manner as possible. At this moment it is further worth to note that the actual noise from the (thermal) quantum mechanical bath is by no means white. Therefore, subsequent attempts to remedy the Caldirola–Kanai model in its violation of quantum principles (in particular the earlier mentioned decay of the ground state, i.e. the violation of the fundamental commutator) by means of the ad-hoc introduction of a simple delta-correlated c-number Langevin noise source like (1.4) seems to be lacking a microscopic fundament [180, 181] (see also [59, 156, 182–188]). On the other hand, as noted earlier, the above reservoir models suffer from certain defects (the infinities) that give them in some sense the characteristics of quicksand, which is worth remembering.

In 1957 Havas [8] considered the range of application of the classical Langrange formalism in considerable detail, presenting a further investigation of Helmholtz's condition [189]. By means of integrating factors new formal mathematical Lagrangians and Hamiltonians may be obtained without introducing additional dynamical variables (see also [15, 190]). The typical example for the damped harmonic oscillator, as given by Havas, reads

$$H = \ln x - \ln \cos(\omega x p_x) - \lambda x p_x, \quad (2.1)$$

where p_x is the canonical momentum (and which here is quite different from the mechanical momentum $p = \dot{x}$). Havas' Hamiltonian (2.1) generates the equations of motion

$$\dot{x} = H_{,p_x} = [\omega \operatorname{tg}(\omega x p_x) - \lambda] x, \quad (2.2)$$

$$\dot{p}_x = -H_{,x} = -[\omega \operatorname{tg}(\omega x p_x) - \lambda] p_x - 1/x. \quad (2.3)$$

Combining these equations in order to eliminate the tangent readily gives

$$x p_x = -t + c, \quad (2.4)$$

c being an integration constant. Then (2.2) is easily integrated:

$$x(t) = a e^{-\lambda t} \cos(\omega t + b), \quad (2.5)$$

with $b = -\omega c$, and where a is a second constant of integration. Indeed, (2.5) precisely represents the motion of the damped harmonic oscillator. However, as is obviously the case with (2.1), not any of these formally equivalent Hamiltonians allows for its interpretation as the system's energy, usually not even in the limit of vanishing friction $\lambda = 0$. Thus, the choice of coordinates in quantizable systems appears to be a nontrivial matter (see also [119, 120, 184, 191]). This rather crucial point has been observed by Havas, was noted in passing by Stevens [120, 157] and is given further attention in recent explorations by Cawley [192–194]. The present author has developed a formalism, using complex dynamical variables instead of the real ones, wherein this aspect was accounted for, at least in the limit of zero damping [24,

59, 182–184, 190, 195, 196], and that will be further reviewed in the course of the present article. At this place it suffices to mention its connection with a complex symplectic formulation that has been proposed recently [197].

In 1972 Kostin proposed a nonlinear Schrödinger equation to describe one dimensional dissipative systems [198, 199]. The nonlinearity spoils, of course, the superposition principle. On the other hand, the effective Hamiltonian in this approach does represent the actual energy of the system. The Kostin–Schrödinger equation, originally derived from a Heisenberg–Langevin noise-operator equation (see also e.g. [21, 24, 57–59, 92, 134, 184, 200]), has been revisited from the closely interrelated points of view of the fluid dynamical interpretation of wave mechanics [201–204], stochastic quantization [205–213] (see also, in a broader context, [214–222]), and Hamiltonian–Jacobi–Schrödinger theory [1, 2, 9, 43, 190, 223–227] (of related interest may also be e.g. [228–231]; ref. [229] also contains a reprint of [89]). Information theoretical aspects have been touched upon in [207] (in addition, see for instance [5, 31, 37, 52, 232–246]). The usually presented wave packet solutions for this dissipating oscillator turn out to have time-independent widths [129, 136, 156, 207, 203, 247] (related papers are [248–250]). A slightly more general, dynamical solution has been considered in [136]. In any case, all solutions exhibit an approach to the undamped oscillator eigenstates. Nevertheless, the persisting existence of these states, against friction, need not be a very serious objection against the model as in these cases the expectation value of the momentum is zero and, hence, there is no effective frictional force [180, 109]. At the roots of this feature one finds the (fluid dynamical) quantum-classical correspondence used in the Kostin theory. In that respect, isolated quantum mechanical stationary states differ a good deal from wave packet solutions that implicitly involve Ehrenfest’s theorem [251]. Further, as already mentioned by Kostin in his original derivation, the possible presence of Langevin noise sources leads to a stochastic potential in the Schrödinger equation and will destroy the steady states.

In general, noise sources in the Heisenberg representation are operators in a space different from the Hilbert space of the genuine oscillator [19, 21, 23, 37, 57, 91, 200] (actually, a corresponding remark applies to the classical case, see e.g. [67, 252]). In Kostin’s mesoscopic approach to quantum friction, however, the noise sources are assumed to be c-number functions of time only. Actually, the noisy potential in the Schrödinger–Langevin equation is usually taken to be identically zero. Even in that case the damped oscillator does not decay below its free ground state, in contrast with the Caldirola–Kanai model. Therefore, Heisenberg’s uncertainty relation is preserved for the Kostin oscillator. Finally, let us note Messer’s paper [156] wherein he did carry along the classical noisy Kostin potential, assuming delta-correlated forces. For one particular initial condition (the free oscillator ground state), and with a thermally excited oscillator at $t = \infty$, his result (a Gaussian distribution) is identical to that of Svin’in’s [180] upshot for the Caldirola–Kanai Hamiltonian. Of course, this technique applied to Kostin’s model could also be added the earlier made remarks concerning the correlations of quantal noise sources (see also [24]).

Heuristic arguments led Süssmann in 1973 to the formulation of another nonlinear frictional Schrödinger equation [253]. In this case, the nonlinearity is apparent through the explicit occurrence of expectation values in the effective Hamiltonian. Soon after its discovery, the model was extended by Hasse [129] and further generalized by Albrecht [254]. The Hamiltonian is closely related to the energy. Its general form may be given as

$$H = H_0 + \lambda W, \quad (2.7)$$

where H_0 represents the free oscillator Hamiltonian,

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2 x^2, \quad (2.8)$$

and where W may be interpreted as a frictional potential:

$$W = 2\langle p \rangle(x - \langle x \rangle) + c[x - \langle x \rangle, p - \langle p \rangle]_+. \quad (2.9)$$

λ is again the friction constant, c is a real parameter and $[\cdot, \cdot]_+$ denotes anticommutation. In fact, Süssmann discovered the case $c = 1$, Hasse originally discussed $c = \pm \frac{1}{2}$, while Albrecht considered general c with emphasis on $c = 0$. Wave packet solutions of the Schrödinger equation with the Hamiltonian (2.7)–(2.9) were studied in considerable detail by Hasse [129]. See also [107–109, 136, 254–257]. Only certain cases of the general version (2.9) turn out to be physically relevant. In brief, the analysis leads to the following conclusions. All potentials (2.9) allow for undamped, stationary solutions of the Schrödinger equation

$$i\hbar\psi_t = (H_0 + \lambda W)\psi \quad (2.10)$$

of the type

$$\psi_n = N_n \exp[-i(n + \frac{1}{2})\omega_c t - (\omega_c + 2i\lambda c)x^2/2\hbar] H_n(x\sqrt{\omega_c/\hbar}), \quad (2.11)$$

where $N_n = (\omega_c/\pi\hbar)^{1/4} (2^n n!)^{-1/2}$, with $n = 0, 1, \dots$, and where

$$\omega_c = [\Omega^2 - 4c^2\lambda^2]^{1/2}. \quad (2.12)$$

In order for (2.11) to belong to Hilbert space, these functions should be square integrable. Hence, one is led to require in general that $|c| < \Omega/2\lambda$. Confining ourselves to the underdamped case $\Omega > \lambda$, we have $|c| \leq \frac{1}{2}$, which still includes the Hasse ($c = \pm \frac{1}{2}$) and the Albrecht ($c = 0$) Hamiltonians.*

The expectation values of the position and momentum operators may be obtained directly from the Schrödinger equation (2.10). They obey the correct Newtonian equations following Ehrenfest's theorem:

$$\langle x \rangle' = \langle p \rangle, \quad (2.13)$$

$$\langle p \rangle' = -2\lambda\langle p \rangle - \Omega^2\langle x \rangle. \quad (2.14)$$

Their general solution is given by (2.5), with $\omega = (\Omega^2 - \lambda^2)^{1/2}$. However, the Hamiltonian (2.7) equals the energy of the oscillator only if $\langle W \rangle = 0$. For nonzero λ , this is the case only if either $\sigma_{px} = \frac{1}{2}\langle px + xp \rangle - \langle p \rangle\langle x \rangle = 0$ or if $c = 0$, which seems to be an argument in favour of Albrecht's case.** Defining further $\sigma_{xx} = \langle x^2 \rangle - \langle x \rangle^2$ and $\sigma_{pp} = \langle p^2 \rangle - \langle p \rangle^2$, one may obtain from (2.10) the equations of motion for the second moments:

* Compare this with our recent work [258], where the cases $|c| > \frac{1}{2}$ (or at least their close counterparts) are intrinsically suppressed. Further, note that throughout the present text Ω represents the free oscillator frequency, while we use $\omega = \omega_{1,2} = (\Omega^2 - \lambda^2)^{1/2}$ for the classical reduced damped oscillator frequency. This convention is also followed e.g. in [136], but it is the reverse of Hasse's [108, 109, 129].

** If one considers force free motion for the general Süssmann-Hasse-Albrecht model (i.e. $\Omega = 0$), the range $c < 0$ is immediately ruled out physically as it leads to an exponential increase of the kinetic energy. This is easily seen in (2.16).

$$\dot{\sigma}_{px} = \sigma_{pp} - \Omega^2 \sigma_{xx}, \quad (2.15)$$

$$\dot{\sigma}_{pp} = -4\lambda c \sigma_{pp} - 2\Omega^2 \sigma_{px}, \quad (2.16)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px} + 4\lambda c \sigma_{xx}. \quad (2.17)$$

Considering Gaussian wave packets, the standard treatment of Heisenberg's uncertainty relation provides us with the additional relation [136, 164]

$$\sigma_{pp} \sigma_{xx} = \frac{1}{4} \hbar^2 + \sigma_{px}^2, \quad (2.18)$$

which allows the construction of a second order equation for σ_{xx} separately. Introducing the width w_x of the Gaussian wave packet by means of

$$|\psi|^2 = [\pi w_x^2]^{-1/2} \exp[-(x - \langle x \rangle)^2 / w_x^2], \quad (2.19)$$

one obtains (see also [136])

$$\ddot{w}_x + \omega_c^2 w_x = \hbar^2 / w_x^3, \quad (2.20)$$

which immediately reveals the existence of a stationary solution $\sigma_{xx} = w_x^2/2 = \hbar/2\omega_c$. In view of (2.15) the associated uncertainty product reads

$$\sigma_{pp} \sigma_{xx} = \frac{1}{4} \hbar^2 \Omega^2 / \omega_c^2 \geq \frac{1}{4} \hbar^2. \quad (2.21)$$

As it should, regarding (2.18), the quantum mechanical fluctuations are not only preserved but even enlarged by the dissipation, in line with earlier made more general remarks (see e.g. the Introduction, below (1.2)). Unfortunately, the above stationary solutions for the Hasse-Albrecht model are not (asymptotically) stable [34, 37, 136, 238, 259–269]. For that reason we have purposely refrained from adding them the argument $t = \infty$. It is easy to see from (2.20), that this damped quantum oscillator will exhibit persistent, undamped oscillations in its width if the system is initially not exactly in the above found steady state. This feature is not observed in any of the other known models, and it is quite unlikely to be physically realistic [136]. In closing the story on the München-model, it may be noted that it was recently indicated by Stocker and Albrecht [43] how these nonlinear frictional potentials can be generated formally within the framework of the fluid dynamical interpretation of the classical real space Hamilton-Jacobi theory and Schrödinger's wave mechanics.

Each of the above mentioned theories is essentially concerned with a Schrödinger equation representation of the dissipative system and the associated effective Hamiltonian. However, in view of the experience with fully quantum mechanical treatments of open systems (see e.g. [16–25, 50, 55–58, 61, 87–89, 200, 270–275]), actually mainly concerned with oscillators and their interactions with oscillator-reservoirs, it seems more natural to seek a proper description in terms of mixed states rather than pure states. Anyway, a quantum mechanical phase space description by means of a density operator is fundamentally more general than by means of a Schrödinger state vector. In the case of a closed system (the universe) it can always be cast into a pure state representation, but not so vice versa (see in this context also for instance [270, 276–287]). For that reason, already in 1975 [59] we have introduced in fact the density operator from the very beginning in our mesoscopic approach to quantum friction. The theory was further developed successively in [184, 24, 195, 196, 288, 258]. Trains of thought

along similar lines were revealed independently by Burzlaff [289], Huguenin [290] and Balazs [291]. In his mesoscopic approach, the present author has always emphasized the relation between Heisenberg–Langevin quantal noise sources (operators, not classical functions) and additional diffusive-like contributions in the master equation for the density operator.^{*} This notion gives rise to typical quantum-mechanical fluctuation-dissipation relations. For the linearly damped harmonic oscillator the characteristic example is (see (7.36) and [24]):

$$D_{xp} - D_{px} = i \hbar \lambda, \quad (2.22)$$

where the D 's are diffusion coefficients. Dekker's theory employs complex dynamical variables that are directly connected with the usual quantum mechanical commutator algebra (see also [57, 58, 160–164, 293–303], and more specifically: [114, 115, 137, 181, 197, 289, 290, 304–309]). It ensures that in the limit of vanishing friction the known undamped oscillator results emerge, and it also preserves the uncertainty principle for nonzero dissipation.

At the end of this historical survey it is interesting to note that Hasse in 1979 [107, 310, 311] succeeded in making contact between the master equation and nonlinear Schrödinger equations in the limit of weak damping (radiation damping limit). As it turns out, this relation implies a constraint on the possible diffusion coefficients. As expected, and as will be shown in the sequel, this condition guarantees the survival of the uncertainty principle and the vacuum fluctuations for the oscillator under the action of dissipating forces. In agreement again with the earlier expressed general ideas about dissipation and fluctuations, the quantal vacuum fluctuations appear to be increased by the damping^{**} (see also (2.21)). Hasse's nonlinear frictional Hamiltonian looks somewhat similar to the heuristic Süssmann–Hasse–Albrecht species (2.9), but is definitely different. For example, Hasse's model does not show the peculiarity of the undamped, persistent oscillations in the width of the wavepacket. It is further interesting to observe that Dekker's theory of the linearly damped harmonic oscillator leads to a class of specific diffusion coefficients all satisfying Hasse's pure state Schrödinger representation condition if considered in the weak damping and long time limit, which seems to add to the credibility of that phase space approach [288].

As the outline of the present paper should be clear from the table of contents and from the abstract, we now start our comparison of the various approaches to the damped oscillator problem in more detail by considering the time-independent Bateman–Morse–Feshbach–Bopp Hamiltonian.

3. Bateman's dual Hamiltonian

3.1. Classical mechanics

The classical deterministic equation of motion for the simple one-dimensional damped oscillator with unit mass in Newtonian mechanics reads

^{*}The significance of the noise operators in the original paper [59] (see also in particular [24]) has not always been properly recognized [190, 290]. But see for the contrary e.g. [292].

^{**}To be precise, the increase is measured with respect to the "absolute vacuum", i.e. the isolated undamped oscillator case (which is physically unrealistic). The increase of the zero-point fluctuations shows that the oscillator is not placed in the "absolute vacuum", but interacts with a reservoir (e.g. the electromagnetic or gravitational fields) which may be in its own ground state. Semantically, the term ground state is perhaps more appropriate than vacuum state.

$$\ddot{x} + 2\lambda\dot{x} + \Omega^2 x = 0. \quad (3.1)$$

Its general solution has already been given in fact in (2.5). The equation (3.1) can be obtained as the Euler–Lagrange equation

$$(L_{,\dot{x}}) - L_{,x} = 0 \quad (3.2)$$

from the Lagrangian

$$L = \dot{x}\dot{\bar{x}} - \Omega^2 x\bar{x} + \lambda(x\dot{\bar{x}} - \dot{x}\bar{x}), \quad (3.3)$$

that is by a variation of the auxillary variable $\bar{x}$ [1, 2, 11, 113–116]. On the other hand, varying x leads to

$$\ddot{\bar{x}} - 2\lambda\dot{\bar{x}} + \Omega^2 \bar{x} = 0, \quad (3.4)$$

which clearly represents the time reversed process of (3.1). The system (3.4) is called the mirror-image oscillator of (3.1). The canonical momenta for the above dual system of oscillators become

$$p = L_{,\dot{x}} = \dot{\bar{x}} - \lambda\bar{x}, \quad \bar{p} = L_{,\dot{\bar{x}}} = \dot{x} + \lambda x. \quad (3.5)$$

Obviously, these differ essentially from the oscillator's mechanical momenta. However, this is in no sense an obstruction to the classical theory and the Hamiltonian is easily found to be

$$H = p\bar{p} - \lambda(xp - \bar{x}\bar{p}) + \omega^2 x\bar{x}, \quad (3.6)$$

with $\omega = (\Omega^2 - \lambda^2)^{1/2}$ representing the reduced actual frequency. If not indicated otherwise, this reduced frequency will be assumed to be real throughout the present text. That is, we shall mostly be concerned with the underdamped case although occasionally results can be taken over to the overdamped case.

Hamilton's equations,

$$\begin{aligned} \dot{x} &= H_{,p}, & \dot{p} &= -H_{,x}, \\ \dot{\bar{x}} &= H_{,\bar{p}}, & \dot{\bar{p}} &= -H_{,\bar{x}}, \end{aligned} \quad (3.7)$$

of course lead back to (3.1) and (3.4). Almost trivially, the Hamiltonian (3.6) is a constant of the motion: the energy dissipated by the original oscillator is completely absorbed at the same pace by the mirror-image system. Nevertheless, in general this H is not equal to the energy of the total closed system, not even in the limit of vanishing friction $\lambda \downarrow 0$. This presents a first hindrance to the canonical quantization of the physical system, as noted before. It can be remedied, however, by restricting the solutions to the Hamiltonian equations to those for which both oscillators start off from the same initial state. Yet there exists another difficulty. Namely, the only nonzero Poisson brackets of the above classical theory would lead in quantum mechanics to the commutators^a

^a For typographical reasons, and if there can be no misunderstanding, we denote commutators by $[\cdot]$ rather than $[\cdot]_-$. Anticommutators will always be written as $[\cdot]_+$.

$$[x, p] = i\hbar, \quad [\bar{x}, \bar{p}] = i\hbar. \quad (3.8)$$

Evidently, in view of (3.5) this does never imply a nonzero commutator between for instance the position x and the mechanical momentum $p = \dot{x}$, not even in the limit $\lambda \downarrow 0$ where we know it should exist. Thus, unless bold measures are taken, the quantum theory based on the Hamiltonian (3.6) can be expected to be endowed with difficulties.

Nonetheless, it is rather interesting to survey the attempts at a quantum theory based on the Bateman dual Hamiltonian. To the author's knowledge, there are only two papers to date on the subject. Bopp's treatment [114] (see also [108]) is closely related to a symplectic formulation [197] that will be discussed furtheron; his presentation is somewhat loose and confusing.[¶] The canonical quantization procedure devised by Feshbach and Tikochinsky [115] is of a higher subtlety. It is this more sophisticated approach that will be sketched here.

3.2. Quantum mechanical spectrum

As usual, the dynamical variables x , p and $\bar{x}$, $\bar{p}$ are considered as operators in a linear space. They obey the commutation relations (3.8); all other commutators are zero. The basic Hamiltonian is the apparently Hermitian form (3.6). Then introduce the following annihilation and creation operators:

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2\hbar\omega}} (p - i\omega x), & \bar{\alpha} &= \frac{1}{\sqrt{2\hbar\omega}} (\bar{p} - i\omega \bar{x}), \\ \alpha^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} (p + i\omega x), & \bar{\alpha}^\dagger &= \frac{1}{\sqrt{2\hbar\omega}} (\bar{p} + i\omega \bar{x}), \end{aligned} \quad (3.9)$$

where $\alpha^\dagger$ is the customary Hermitian conjugate of α . Note that the bar can be considered as a formal operation that will be called mirror conjugation.^{¶¶} For the sake of completeness, we also list the inverse transformations:

$$\begin{aligned} x &= i\sqrt{\frac{\hbar}{2\omega}} (\alpha - \alpha^\dagger), & p &= \sqrt{\frac{\hbar\omega}{2}} (\alpha + \alpha^\dagger), \\ \bar{x} &= i\sqrt{\frac{\hbar}{2\omega}} (\bar{\alpha} - \bar{\alpha}^\dagger), & \bar{p} &= \sqrt{\frac{\hbar\omega}{2}} (\bar{\alpha} + \bar{\alpha}^\dagger). \end{aligned} \quad (3.10)$$

The only nonzero commutators, after the above canonical transformation, are

$$[\alpha, \alpha^\dagger] = [\bar{\alpha}, \bar{\alpha}^\dagger] = 1. \quad (3.11)$$

It turns out to be useful to define

$$\mathcal{A} = \frac{1}{\sqrt{2}} (\alpha + \bar{\alpha}), \quad \mathcal{B} = \frac{1}{\sqrt{2}} (\alpha - \bar{\alpha}), \quad (3.12)$$

[¶] For example, his a and $a^\dagger$ are not the usual Hermitian adjoints. See section 6.2.

^{¶¶} To be explicit: in view of the equations of motion (3.1) and (3.4), mirror conjugation is defined as $\{\lambda \leftrightarrow -\lambda; x, p \leftrightarrow \bar{x}, \bar{p}\}$.

and their conjugates, so that in effect

$$\mathcal{A} = \frac{1}{2\sqrt{\hbar\omega}} [(\hat{p} + \bar{\hat{p}}) - i\omega(x + \bar{x})], \quad (3.13)$$

$$\mathcal{B} = \frac{1}{2\sqrt{\hbar\omega}} [(\hat{p} - \bar{\hat{p}}) - i\omega(x - \bar{x})]. \quad (3.14)$$

These new operators obey the same algebra as in (3.11), that is

$$[\mathcal{A}, \mathcal{A}^\dagger] = [\mathcal{B}, \mathcal{B}^\dagger] = 1, \quad (3.15)$$

all other commutators being zero. The transformed Hamiltonian (3.6) reads

$$H = H_0 + H_1, \quad (3.16)$$

$$H_0 = \hbar\omega(\mathcal{A}^\dagger \mathcal{A} - \mathcal{B}^\dagger \mathcal{B}), \quad (3.17)$$

$$H_1 = i\hbar\lambda(\mathcal{A}^\dagger \mathcal{B}^\dagger - \mathcal{A}\mathcal{B}). \quad (3.18)$$

The eigenvalues of $\mathcal{A}^\dagger \mathcal{A}$ and $\mathcal{B}^\dagger \mathcal{B}$ are the common $n_A, n_B = 0, 1, 2, \dots$, so that the eigenvalues of H_0 are $\hbar\omega(n_A - n_B)$ with eigenstates $|n_A, n_B\rangle$, belonging to Hilbert space. Notice that H_0 represents the difference rather than the sum of two free oscillator Hamiltonians (see also [116]). In the limit $\lambda \downarrow 0$, the known harmonic oscillator results emerge intact only if the $\mathcal{B}$ -oscillator is kept in its ground state, for which $\mathcal{B}|n_A, 0\rangle = 0$.

In order to investigate the eigenspectrum of the full Hamiltonian (3.16), it is of advantage to define

$$\phi_0 = \frac{1}{2}(\mathcal{A}^\dagger \mathcal{A} - \mathcal{B}^\dagger \mathcal{B}), \quad (3.19)$$

and to introduce

$$\phi_x = \frac{1}{2}(\mathcal{A}^\dagger \mathcal{B}^\dagger + \mathcal{A}\mathcal{B}), \quad (3.20)$$

$$\phi_y = \frac{1}{2}i(\mathcal{A}^\dagger \mathcal{B}^\dagger - \mathcal{A}\mathcal{B}), \quad (3.21)$$

$$\phi_z = \frac{1}{2}(\mathcal{A}^\dagger \mathcal{A} + \mathcal{B}^\dagger \mathcal{B}). \quad (3.22)$$

Notice that

$$H_0 = 2\hbar\omega\phi_0, \quad H_1 = 2\hbar\lambda\phi_y. \quad (3.23)$$

The eigenvalues of ϕ_z are $\frac{1}{2}(n_A + n_B + 1)$, the eigenstates being the same as those of the free Hamiltonian H_0 , i.e. of ϕ_0 . Indeed, ϕ_0 commutes with ϕ_z , as well as with ϕ_x and ϕ_y . The algebra of these spin-like operators is given by

$$[\phi_x, \phi_y] = i\phi_z, \quad (3.24)$$

$$[\phi_z, \phi_y] = i \phi_x, \quad (3.25)$$

$$[\phi_x, \phi_z] = i \phi_y. \quad (3.26)$$

Be aware of the difference in sign in (3.25) and (3.26) with respect to the usual spin algebra (see e.g. [164, 300, 312–314]). One easily computes now that

$$\phi_z^2 - (\phi_x^2 + \phi_y^2) = \phi_0^2 - \frac{1}{4}. \quad (3.27)$$

Of course, ϕ_0^2 commutes with ϕ_x , ϕ_y and ϕ_z . It is the only Casimir operator for the present algebra (as the rank is unity; see e.g. [163, 300, 315–318]). Further, let us conventionally introduce

$$j = \frac{1}{2}(n_A - n_B), \quad m = \frac{1}{2}(n_A + n_B), \quad (3.28)$$

and label the eigenstates of H_0 , ϕ_0 and ϕ_z as $|jm\rangle$ rather than $|n_A, n_B\rangle$.

To relate the eigenproperties of ϕ_y (in which we are interested in view of (3.23)) to those of ϕ_z (which are known), the Baker–Hausdorff relation [57, 301, 319–323] for the present algebra,

$$\exp(\mu \phi_x) \phi_y \exp(-\mu \phi_x) = \phi_y \cos \mu + i \phi_z \sin \mu, \quad (3.29)$$

is particularly useful (see also [164] p. 167). Namely, with $\mu = \pm\pi/2$ one obtains

$$\phi_y = \pm i \exp\{\mp(\pi/2)\phi_x\} \phi_z \exp\{\pm(\pi/2)\phi_x\}. \quad (3.30)$$

Therefore, the eigenvalue equation may be written as

$$\phi_y \Psi_{jm}^{(\pm)} = \pm i(m + \frac{1}{2}) \Psi_{jm}^{(\pm)}, \quad (3.31)$$

where

$$\Psi_{jm}^{(\pm)} = \exp\{\mp(\pi/2)\phi_x\} |jm\rangle. \quad (3.32)$$

The eigenfunctions $\Psi_{jm}^{(\pm)}$ do not belong to ordinary Hilbert space because they can not be normalized in the usual manner. In fact, their conventional norm can be calculated, similar to the Wigner D -functions, by means of Racah's methods (related to the better known Clebsch–Gordan coefficients; see e.g. [163, 164, 300, 303, 317, 318, 324–331]). The result is [115]:

$$\langle jm | \exp(2\mu \phi_x) | jm \rangle = \frac{(\operatorname{tg} \mu)^{2m+1}}{(\sin \mu)^{2j+1}} \sum_l \binom{m-j}{l} \binom{2m-l}{m+j} (\cotg \mu)^{2l}, \quad (3.33)$$

where the sum runs over all integers l . Evidently, the norm (3.33) becomes infinite if $\mu = \pm\pi/2$. This may be remedied, however, by redefining the length of a vector as the inner product

$$(\Psi, \Psi') = S_\nu \Psi_{(\nu)}^T \Psi'_{(\nu)}, \quad (3.34)$$

where Ψ^T represents the time reverse of Ψ , and where S indicates summation or integration (with the

appropriate measure) over vector components. In the same way matrix elements, and in particular expectation values, can be defined as

$$(\Psi, F\Psi) = S_\nu \Psi_{(\nu)}^\dagger F \Psi_{(\nu)}, \quad (3.35)$$

F being an arbitrary operator working to the right. The absence of a finite norm in the usual Hermitian sense invalidates the standard proof that eigenvalues of self-adjoint operators are real. Therefore, the eigenvalue equation (3.31) can be accepted as a proper result.

Since ϕ_0 and ϕ_ν commute, they may have simultaneous eigenstates. Indeed, the eigenstates $\Psi_{jm}^{(\pm)}$ are also eigenstates of ϕ_0 as ϕ_0 commutes with $\phi_\pm$ too. So, the complete eigenspectrum of H will be known if we determine the allowed values of m for given j . Defining

$$\phi_\pm = \phi_\nu \mp \phi_z, \quad (3.36)$$

one easily derives from the commutation relations (3.24)–(3.26) that

$$[\phi_\nu, \phi_\pm] = \pm i \phi_\pm, \quad (3.37)$$

so that ϕ_+ resp. ϕ_- are the raising resp. lowering operators for the eigenstates of ϕ_ν . In view of (3.31), m can be taken to be nonnegative without loss of generality. Let m_0 be the smallest value of m . Then

$$\phi_- \Psi_{jm_0}^{(+)} = 0, \quad (3.38)$$

where the vector is nonzero. Multiplication of (3.38) from the left with ϕ_+ and using (3.36) leads to

$$(\phi_+^2 \pm [\phi_+, \phi_\pm] - \phi_\pm^2) \Psi_{jm_0}^{(\pm)} = 0. \quad (3.39)$$

Invoking the commutator (3.26), the eigenvalue equation (3.31), the Casimir relation (3.27) and the known eigenvalues of ϕ_0 , one gets $m_0^2 = j^2$, so that $m = |j|, |j| + \frac{1}{2}, |j| + 1, |j| + \frac{3}{2}, \dots$. In summary, the eigenstates of the Bateman–Feshbach–Tikochinsky Hamiltonian are

$$\Psi_{jm}^{(\pm)}(t) = \exp\{-2i\omega j t \pm \lambda(2m+1)t \mp (\pi/2)\phi_\nu\} |jm\rangle, \quad (3.40)$$

the corresponding eigenvalues being

$$H_{jm}^{(\pm)} = 2\hbar\omega j \pm i\hbar\lambda(2m+1); \quad m = |j|, |j| + \frac{1}{2}, \dots \quad (3.41)$$

If, in view of the earlier noted limit $\lambda \downarrow 0$, we keep the $\mathcal{B}$ -oscillator in its ground state with $n_B = 0$, so that $2j = 2m = n_A = n$, the eigenvalues become

$$H_n^{(\pm)} = n\hbar\omega \pm i(n+1)\hbar\lambda; \quad n = 0, 1, 2, \dots \quad (3.42)$$

This is almost identical to the spectrum found by Bopp [114]. In fact, Bopp's procedure is somewhat ambiguous in the separation of the dual oscillators. As will be seen more precisely furtheron, (3.42) is, however, one of the possibilities allowed for by Bopp's theory.

3.3. Wave packets

The result (3.42) for the eigenvalues indicates that the dissipation also afflicts the lowest state $n = 0$, as the imaginary part of the eigenvalues remains nonzero.[¶] In this respect, it should be recalled that the Feshbach–Tikochinsky quantum mechanics is based on the Bateman Hamiltonian for the dual $x, \bar{x}$ -system, but that the mirror image oscillator in fact is nothing but a mathematical device invented in order to remain within the framework of conservative Hamiltonian classical mechanics. It should be emphasized that the only physically observable system is the damped x -oscillator for which, however, the correct commutator $[x, p] = i\hbar$ is absent from the theory (see (3.5) and (3.8)). Therefore, it is particularly relevant to investigate in more detail the behaviour of the damped oscillator on its own. In principle the eigenfunctions can be used to calculate the propagator (see e.g. [72]) and more general wavepackets. But since the relation of the above obtained eigensolutions with the expectation values of the original dynamical variables for the physical oscillator is not that straightforward (note for instance that the $\mathcal{A}$ -oscillator in fact contains a mixture of both the damped and the adjoint oscillator; and e.g. ϕ_x and x do not commute), it makes sense to turn to the time-dependent Schrödinger equation itself. The solutions for the mean values and higher moments can be calculated directly from it, without knowing the wave function explicitly, because of the linearity of the present system (see also e.g. [34, 332, 333]).

The Schrödinger equation may be written as

$$i\hbar \Psi_{,t} = [\hbar\bar{\hbar} - \lambda(x\hbar - \bar{x}\bar{\hbar}) + \omega^2 x\bar{x}] \Psi. \quad (3.43)$$

The time reverse reads^{¶¶}

$$-i\hbar \Psi_{,t}^T = [\hbar\bar{\hbar} + \lambda(x\hbar - \bar{x}\bar{\hbar}) + \omega^2 x\bar{x}] \Psi^T. \quad (3.44)$$

The equation of motion of the mean value (3.35) is easily obtained as

$$\langle F \rangle' = -\frac{i}{\hbar} \langle FH - (H^T)^a F \rangle, \quad (3.45)$$

where the superscript a indicates the real adjoint operator (e.g. [334]). It is not difficult to see that for the Bateman Hamiltonian $(H^T)^a = H$, so that (3.45) reduces to its usual commutator form

$$\langle F \rangle' = -(i/\hbar) \langle [F, H] \rangle. \quad (3.46)$$

By means of the commutation relations (3.8) one now easily computes the mean value equations

$$\langle x \rangle' = \langle \bar{\hbar} \rangle - \lambda \langle x \rangle, \quad (3.47)$$

$$\langle \bar{\hbar} \rangle' = -\lambda \langle \bar{\hbar} \rangle - \omega^2 \langle x \rangle, \quad (3.48)$$

which, in line with Ehrenfest's theorem, agree with the classical results and lead back to

[¶] It is precisely this feature that was eliminated in [114] in a rather ad-hoc manner.

^{¶¶} To be clear, time reversal is effectuated by means of $T = \{t \rightarrow -t; x \rightarrow x, \hbar \rightarrow -\hbar; \bar{x} \rightarrow \bar{x}, \bar{\hbar} \rightarrow -\bar{\hbar}\}$.

$$\langle x \rangle'' + 2\lambda \langle x \rangle' + \Omega^2 \langle x \rangle = 0, \quad (3.49)$$

the quantal version of (3.1). The mirror image oscillator is obtained from (3.47) and (3.48) by interchanging x , $\hbar$ and $\bar{x}$, $\bar{\hbar}$ and letting $\lambda \rightarrow -\lambda$. For the second moments one finds

$$\langle \bar{\hbar}x \rangle' = -2\lambda \langle \bar{\hbar}x \rangle + \langle \bar{\hbar}^2 \rangle - \omega^2 \langle x^2 \rangle, \quad (3.50)$$

$$\langle \bar{\hbar}^2 \rangle' = -2\lambda \langle \bar{\hbar}^2 \rangle - 2\omega^2 \langle \bar{\hbar}x \rangle, \quad (3.51)$$

$$\langle x^2 \rangle' = -2\lambda \langle x^2 \rangle + 2\langle \bar{\hbar}x \rangle, \quad (3.52)$$

which form a closed set of equations pertaining in effect precisely to the physical damped oscillator. Note that, in view of the algebra of this model, $\langle \bar{\hbar}x \rangle = \langle x \bar{\hbar} \rangle$. Introducing the mechanical momentum $p = \dot{x} = \bar{\hbar} - \lambda x$ and the variances (see also the discussion of the Süssmann–Hasse–Albrecht theory, in the Introduction)

$$\sigma_{px} = \frac{1}{2} \langle px + xp \rangle - \langle p \rangle \langle x \rangle, \quad (3.53)$$

$$\sigma_{pp} = \langle p^2 \rangle - \langle p \rangle^2, \quad (3.54)$$

$$\sigma_{xx} = \langle x^2 \rangle - \langle x \rangle^2, \quad (3.55)$$

one obtains

$$\dot{\sigma}_{px} = -2\lambda \sigma_{px} + \sigma_{pp} - \Omega^2 \sigma_{xx}, \quad (3.56)$$

$$\dot{\sigma}_{pp} = -4\lambda \sigma_{pp} - 2\Omega^2 \sigma_{px}, \quad (3.57)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px}. \quad (3.58)$$

Note that presently $\langle px \rangle = \langle xp \rangle$. Contrary to the Süssmann–Hasse–Albrecht model (2.15)–(2.17), the present Bateman model apparently allows only for the steady state $\sigma_{px} = \sigma_{pp} = \sigma_{xx} = 0$. Actually, the time-dependent solution for σ_{xx} with $\sigma_{xx}(0) = \hbar/2\Omega$ and $\dot{\sigma}_{xx}(0) = 0$ (see also [136, 156, 180]) is easily found to be

$$\sigma_{xx}(t) = \frac{\hbar}{2\Omega} e^{-2\lambda t} \left[1 + \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right]. \quad (3.59)$$

The associated momentum spread is

$$\sigma_{pp}(t) = \frac{\hbar\Omega}{2} e^{-2\lambda t} \left[1 - \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right]. \quad (3.60)$$

Clearly, in the end the uncertainty product

$$\sigma_{pp}\sigma_{xx} = \left(\frac{\hbar}{2}\right)^2 e^{-4\lambda t} \left[1 + 4 \left(\frac{\lambda}{\omega}\right)^2 \left(\frac{\Omega}{\omega}\right)^2 \sin^4 \omega t \right] \quad (3.61)$$

inevitably tends to zero, violating Heisenberg's principle. Notice that this violation occurs with certainty once $\lambda \neq 0$, no matter how small the friction coefficient may be. If $\lambda = 0$, the model (3.56)–(3.58) does seem to reproduce the free oscillator equations, and allows for a steady state $\sigma_{px} = 0$, $\sigma_{pp} = \Omega^2 \sigma_{xx}$. However, as in the dual model p and x commute, even if $\lambda = 0$, the relation (2.18) reduces in this case to $\sigma_{pp}\sigma_{xx} = \sigma_{px}^2 = 0$, which is manifestly at variance with well-established quantum physics.[¶]

3.4. Summary

Although Bateman's time-independent dual Hamiltonian is correct classically, it has its problems in quantum mechanics. It can not readily be associated with the system's energy and it does not lead to the correct fundamental commutator. Therefore, it violates Heisenberg's uncertainty principle for the physical oscillator, even in the limit of vanishing friction $\lambda \downarrow 0$. This feature cannot be swept under the carpet by a sophisticated treatment of eigensolutions.

4. Kanai's time-dependent Hamiltonian

4.1. Classical mechanics

The explicit solutions to the classical Hamiltonian equations (3.7) for the Bateman Hamiltonian (3.6) are

$$x(t) = e^{-\lambda t} [x_0 \cos \omega t + (1/\omega) \dot{x}_0 \sin \omega t], \quad (4.1)$$

$$\dot{x}(t) = e^{\lambda t} [\dot{x}_0 \cos \omega t - \omega x_0 \sin \omega t], \quad (4.2)$$

and similar results for $\bar{x}(t)$ and $\bar{\dot{x}}(t)$. Let us first transform the irreversible part of the motion away by means of the canonical generator^{¶¶}

$$F = x\mathcal{P} e^{\lambda t} + \bar{x}\bar{\mathcal{P}} e^{-\lambda t}, \quad (4.3)$$

which contains both the old coordinates and the new canonical momenta $\mathcal{P}$, $\bar{\mathcal{P}}$. One finds

$$\begin{aligned} \dot{x} &= F_{,x} = \mathcal{P} e^{\lambda t}, & X &= F_{,\mathcal{P}} = x e^{\lambda t}, \\ \bar{\dot{x}} &= F_{,\bar{x}} = \bar{\mathcal{P}} e^{-\lambda t}, & \bar{X} &= F_{,\bar{\mathcal{P}}} = \bar{x} e^{-\lambda t}. \end{aligned} \quad (4.4)$$

The new Hamiltonian $H' = H + F_{,t}$ becomes

$$H' = \mathcal{P}\bar{\mathcal{P}} + \omega^2 X\bar{X}, \quad (4.5)$$

[¶] Note that (2.18) can be derived within the framework of the Feshbach–Tikochinsky theory if $\lambda = 0$ following the conventional reasoning as in that case time reversal is identical to complex conjugation in the quantum theory.

^{¶¶} This generator $F(x, \mathcal{P}; \bar{x}, \bar{\mathcal{P}})$ is the F_2 of Goldstein [1].

which describes undamped oscillations at the reduced frequency ω , as expected from comparing (4.1), (4.2) and (4.4). Note that the new dynamical variables fulfil the same commutation relations (3.8) as the old ones. Of course, H' is a constant of the motion, equal to its initial value $H'(0) = \mathcal{P}_0 \bar{\mathcal{P}}_0 + \omega^2 X_0 \bar{X}_0$. For this to coalesce with the energy (at least in the weak damping limit) it suffices to consider $\bar{\mathcal{P}}_0 = \mathcal{P}_0$ and $X_0 = \bar{X}_0$. Because of the time-reversible nature of the new solutions, this entails $\mathcal{P}(t) = \mathcal{P}(t)$ and $\bar{X}(t) = X(t)$.

It is now convenient to extend the real coordinates and momenta into the complex plane, and to introduce a canonical transformation from $X, \mathcal{P}, \bar{X}, \bar{\mathcal{P}}$ to $Q, P, \bar{Q}, \bar{P}$ by means of

$$\begin{aligned} X &= \frac{1}{2}(Q + \bar{Q}) + (i/2\omega)(P - \bar{P}), & \bar{X} &= \frac{1}{2}(Q + \bar{Q}) - (i/2\omega)(P - \bar{P}), \\ \mathcal{P} &= \frac{1}{2}(P + \bar{P}) + (i\omega/2)(Q - \bar{Q}), & \bar{\mathcal{P}} &= \frac{1}{2}(P + \bar{P}) - (i\omega/2)(Q - \bar{Q}), \end{aligned} \quad (4.6)$$

where the new variables can be taken to be real. In the end one only considers those solutions for which $\bar{Q}_0 = Q_0$ and $\bar{P}_0 = P_0$, which confines the original variables again to real phase space. Note once more that the bar can be formally considered as a mirror conjugation. The inverse transformations may be written as

$$P = \frac{1}{2}[(\mathcal{P} + \bar{\mathcal{P}}) - i\omega(X - \bar{X})], \quad Q = -(i/2\omega)[(\mathcal{P} - \bar{\mathcal{P}}) + i\omega(X + \bar{X})], \quad (4.7)$$

the other two following by mirror conjugation. Note the differences with the Feshbach–Tikochinsky transformations (3.13) and (3.14). The only nonzero commutators (Poisson brackets) for the new variables, of course, are

$$[Q, P] = i\hbar, \quad [\bar{Q}, \bar{P}] = i\hbar. \quad (4.8)$$

Inserting (4.6) into (4.5) yields

$$H' = \frac{1}{2}(P^2 + \omega^2 Q^2) + \frac{1}{2}(\bar{P}^2 + \omega^2 \bar{Q}^2). \quad (4.9)$$

This Hamiltonian evidently represents the energy of two independent identical oscillators. For that reason we shall focus our attention on one oscillator only, say the Q, P -system. The canonical generator

$$F = \Pi Q + \frac{1}{2}\lambda Q^2 \quad (4.10)$$

then transforms Q, P into Y, Π as (see also [119])

$$P = F_{,Q} = \Pi + \lambda Q, \quad Y = F_{,\Pi} = Q. \quad (4.11)$$

Hence, the pertinent oscillator from (4.9) is described by (see also [119, 120, 137, 181])

$$H' = \frac{1}{2}\Pi^2 + \lambda \Pi Y + \frac{1}{2}\Omega^2 Y^2. \quad (4.12)$$

One further transformation (from Y, Π to y, π), generated by

$$F = \pi Y e^{-\lambda' Y}, \quad (4.13)$$

so that

$$\Pi = F_{,y} = \pi e^{-\lambda t}, \quad y = F_{,\pi} = Y e^{-\lambda t}, \quad (4.14)$$

while $H'' = H' + F_{,t}$, finally leads to

$$H'' = \frac{1}{2} e^{-2\lambda t} \pi^2 + \frac{1}{2} e^{2\lambda t} \Omega^2 y^2. \quad (4.15)$$

This is the well-known Caldirola–Kanai Hamiltonian [13, 14]. As noted before, historically, the corresponding Lagrangian was already presented in Bateman's original paper [11]. See further e.g. [10, 12, 96, 116–156, 180, 181, 275, 335]. In view of (4.14), (4.11), (4.6) with the constraint $\bar{Q} = Q$ and $\bar{P} = P$, and (4.4), one readily infers that $x = X e^{-\lambda t} = Q e^{-\lambda t} = Y e^{-\lambda t} = y$. In the remainder of this section we replace therefore y again by x .

4.2. Quantum mechanics

The Hamiltonian (4.15) has in fact been constructed purposely so as to represent the energy of the system, at least in the weak friction limit. As noted earlier, this seems to be one of the prerequisites for the construction of a quantum theory. However, from the equation of motion for x , that is (3.1), it is known that the irreversible part of $x(t)$ behaves like $e^{-\lambda t}$, and so does $\dot{x}(t)$. Further, according to the Hamilton equation

$$\dot{x} = H'_{,\pi} = e^{-\lambda t} \pi, \quad (4.16)$$

the canonical momentum behaves like $e^{\lambda t}$. But then (4.15) would imply that there is no secular change in the energy of the oscillator and, hence, there is no genuine dissipation at all [10, 275]. To be explicit, if one considers the solution for $x(t)$ from (4.1) with $\bar{p}_0 = 0$ for convenience, makes use of (4.16), substitutes into (4.15) and time-averages the Hamiltonian over any of its oscillation periods π/ω , one obtains the time-independent result $H''_{av} = \frac{1}{2} \Omega^2 x_0^2$. This feature is most easily clarified by looking at the Lagrangian belonging to (4.15). It may be written as (see Bateman [11] and e.g. [10, 128, 136])

$$L'' = \frac{1}{2} m(t) (\dot{x}^2 - \Omega^2 x^2), \quad (4.17)$$

with mass $m(t) = m_0 \exp(\lambda t)$, our choice being $m_0 = 1$. This Lagrangian immediately allows for an interpretation in terms of kinetic and potential energies. Physically, the situation described by (4.17) might be realized for example by a frictionless pendulum consisting of a pail collecting rain [10]. See fig. 4.1.

Notwithstanding these considerations, the Caldirola–Kanai Hamiltonian (4.15) has been used to study dissipation in quantum mechanics (e.g. [12–14, 96, 116–138]). Turning the only nonzero classical Poisson bracket into a commutator gives

$$[x, \pi] = i \hbar. \quad (4.18)$$

Using again the Hamilton equation (4.16), this implies in terms of the damped oscillator's mechanical

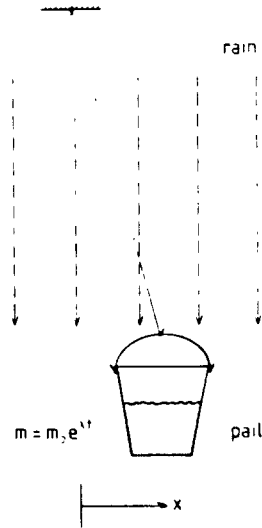


Fig. 4.1. Model for a simple harmonic oscillator with a time dependent mass (after [10]).

momentum $p = \dot{x}$, that

$$[x, p] = i \hbar e^{-2\lambda t}. \quad (4.19)$$

Hence, as is indeed found in known explicit quantum mechanical solutions, Heisenberg's uncertainty principle will be violated in the course of time.^{*} This forbidding feature was already noticed by Kanai [14], and most clearly formulated by Brittin [12]. It should be clear, however, from the preceding discussion, that this can be considered as an inappropriate physical interpretation. If one agrees to really equate the model's Hamiltonian to the mechanical energy, the solution of the problem becomes simple indeed: the quantal features of the system become increasingly irrelevant in the course of time as the growing mass takes the particle from the initial microscopic domain to the ultimate macroscopic, classical regime.

Thus, as far as the damped oscillator is concerned, it does not seem that we have much gained beyond the time-independent Bateman-model (3.6). Nevertheless, before taking another view of the matter, let us briefly survey the results from the quantum theory based on the time-dependent Bateman-Caldirola-Kanai Hamiltonian (4.15). The Schrödinger equation in the coordinate representation reads

$$i \hbar \psi_{,t} = -\frac{1}{2} \hbar^2 e^{-2\lambda t} \psi_{,xx} + \frac{1}{2} e^{2\lambda t} \Omega^2 x^2 \psi. \quad (4.20)$$

Although the Hamiltonian is nonstationary, probability is conserved as the continuity equation

$$\rho_{,t} + j_{,x} = 0 \quad (4.21)$$

^{*} A recent claim that this can be remedied by simply considering the quantization in phase space as done in [137] is untenable. See also furtheron.

holds with the probability and current densities respectively given by

$$\rho = \psi^* \psi, \quad j = e^{-2\lambda t} (\hbar/2i) (\psi^* \psi_x - \psi_x^* \psi). \quad (4.22)$$

It is easily verified that (4.20) allows for the following set of so-called pseudo-stationary states (see e.g. [96, 119, 129, 133]):^{*}

$$\psi_n = \mathcal{N}_n \exp[-i(n + \frac{1}{2})\omega t + \frac{1}{2}\lambda t - (\omega + i\lambda) e^{2\lambda t} x^2 / 2\hbar] H_n(e^{\lambda t} x \sqrt{\omega/\hbar}), \quad (4.23)$$

where $\mathcal{N}_n = (\omega/\pi\hbar)^{1/4} (2^n n!)^{-1/2}$, with $n = 0, 1, \dots$. For curiosity, compare this with (2.11) for the Süssmann–Hasse–Albrecht model. And for the sake of completeness it could be mentioned that Dodonov and Man'ko (e.g. [133], see also [10]) denote (4.23) as loss-energy-states.^{**}

4.3. Quantum fluctuations

Since (4.20) is a linear partial differential equation, the pseudo-eigenstates (4.23) can be used to construct more general solutions by superposition, for example Gaussian wavepackets (see e.g. [116, 128, 129, 131, 135, 136, 180]). See also (2.19). The mean values, of course, follow the classical path obtained from (2.13) and (2.14); the widths of the packets require further investigation. From the Schrödinger equation one obtains the usual equation of motion (see e.g. (3.46)) for the expectation value of an arbitrary function of the canonical operators x and π . For the second moments one readily finds

$$\langle \pi x \rangle' = e^{-2\lambda t} \langle \pi^2 \rangle - e^{2\lambda t} \Omega^2 \langle x^2 \rangle, \quad (4.24)$$

$$\langle \pi^2 \rangle' = -e^{2\lambda t} \Omega^2 \langle \pi x + x \pi \rangle, \quad (4.25)$$

$$\langle x^2 \rangle' = e^{-2\lambda t} \langle \pi x + x \pi \rangle. \quad (4.26)$$

It may be noted *en passant* that the Gaussian Heisenberg relation (2.18) presently holds *not* for p and x , but in terms of π and x ; that is

$$\sigma_{\pi\pi} \sigma_{xx} = \frac{1}{4} \hbar^2 + \sigma_{\pi x}^2, \quad (4.27)$$

where the variances are defined by (3.53)–(3.55) with p replaced by π . Rewriting (4.24)–(4.26) in terms of the variances for the mechanical variables $p = \dot{x}$ and x , by means of (4.16), leads to

$$\dot{\sigma}_{px} = -2\lambda \sigma_{px} + \sigma_{pp} - \Omega^2 \sigma_{xx}, \quad (4.28)$$

$$\dot{\sigma}_{pp} = -4\lambda \sigma_{pp} - 2\Omega^2 \sigma_{px}, \quad (4.29)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px}, \quad (4.30)$$

^{*} Be aware of some differences in notation. For instance, in [133] Ω and ω are precisely interchanged.

^{**} In [133] the authors consider (in analogy with real time periodic Hamiltonians) the periodicity of the Caldirola–Kanai Hamiltonian over displacements π/λ along the imaginary time axis, i.e. $H^*(t + i\pi/\lambda) = H^*(t)$. These considerations directly lead to a pseudo-eigenspectrum with complex eigenvalues $H_n^* = \hbar(\omega + i\lambda)(n + \frac{1}{2})$. The significance of this spectrum, however, remains in vain.

which are readily observed to be identical to the corresponding equations (3.56)–(3.58) for the Bateman–Feshbach–Tikochinsky model. The only improvement that has been obtained is the recovering of the correct quantum mechanics if $\lambda = 0$. The solutions to (4.28)–(4.30) with $\sigma_{xx}(0) = \hbar/2\Omega$ and $\dot{\sigma}_{xx}(0) = 0$ have already been explicated in (3.59)–(3.60), while the consequences for the uncertainty principle are shown in (3.61). Of course, other initial conditions are possible too. As an example, one might investigate the behaviour of the first and second moments for the pseudo-stationary states (4.23). Note that the pseudo-groundstate ψ_0 is Gaussian. It is not difficult to see that $\langle x \rangle$ is strictly zero for any n . Hence $\langle x \rangle' = 0$, so that $\langle \pi \rangle = 0$ and $\langle p \rangle = 0$. Further, using the standard properties of Hermite polynomials, one finds

$$\langle x^2 \rangle = e^{-2\lambda t} \frac{\hbar}{\omega} \left(n + \frac{1}{2}\right). \quad (4.31)$$

Substituting this into (4.26) learns that $\langle \pi x + x \pi \rangle$ is time-independent for all states ψ_n . Therefore the l.h.s. of (4.24) is zero, which leads to

$$\langle \pi^2 \rangle = e^{2\lambda t} \frac{\hbar \Omega^2}{\omega} \left(n + \frac{1}{2}\right). \quad (4.32)$$

Hence, the product

$$\sigma_{\pi\pi} \sigma_{xx} = \frac{\hbar^2 \Omega^2}{\omega^2} \left(n + \frac{1}{2}\right)^2 \quad (4.33)$$

is conserved in the course of time. This formula is the generalization of a result from [137] to arbitrary n . However, it must be emphasized that (4.33) applies to the canonical momentum in stead of the proper mechanical momentum.* In terms of the latter one finds

$$\sigma_{pp} \sigma_{xx} = e^{-4\lambda t} \frac{\hbar^2 \Omega^2}{\omega^2} \left(n + \frac{1}{2}\right)^2, \quad (4.34)$$

which, like (3.61), inevitably tends to zero even for the lowest state $n = 0$, thus violating the uncertainty principle whenever $\lambda > 0$.

4.4. Summary

In the course of improving the energy representation, the time-independent Bateman Hamiltonian has been related to the time-dependent Caldirola–Kanai Hamiltonian by a succession of canonical transformations. However, if this Hamiltonian is indeed taken to be equal to the mechanical energy, there can be no energy dissipation and the described system rather is a mass-accreting pendulum. The alternative consideration of the Hamiltonian as a mere generator of the motion, leads to a violation of the fundamental quantum mechanical commutator in the course of time. In fact, the uncertainty

* It should also be noted, therefore, that (4.33) – especially with $n = 0$ for the pseudo-groundstate – essentially differs from Dekker's result [195]. See also further on in the present article e.g. eq. (7.51).

product is identical to the exponentially decaying form (3.61) connected with Bateman's dual Hamiltonian.

5. The semi-infinite transmission line

5.1. Preliminaries

In this section we will attempt to shed further light on the deficiencies of the above discussed theories, in particular that of the time-dependent Hamiltonian, by considering an explicit physical model. In fact we shall treat a mechanical model and an electrical model in a unified manner. The essence of these models is the coupling of a privileged, tagged oscillator to a reservoir consisting of an infinity of tiny other oscillators, and to let the latter system tend to have infinite size. In that case, simply classically speaking, signals emitted by the tagged oscillator, which is the one under actual investigation i.e. the subsystem of interest, can never (that means, within any finite time interval) return to it by being reflected from the boundaries of the reservoir.

The electrical network model that will be discussed in the sequel is an adapted version of that of Stevens, who's explorations were originally based on a suggestion by M.H.L. Pryce [157]. The mechanical analogue of the electrical semi-infinite transmission line is a semi-infinite string. The present formulation is akin to recent work of B. Yurke and O. Yurke [39]. Within the model's framework the results are exact. More or less similar approaches have been presented for instance by Senitzky [50] and by Ford, Kac and Mazur [38]. More general reference can be made e.g. to [16–23, 41, 42, 53–58, 63, 92, 93, 168–179, 200, 275, 336–338]. Further, it is interesting to note that historically the infinite harmonic chain was already discussed in the early days by Hamilton [223] and in the beginning of this century for example by Schrödinger [339].

Let us first consider the mechanical model, depicted in fig. 5.1. A finite mass m_0 has been attached to a chain of identical masses $m_k = m$ ($k = 1, 2, \dots$), located at positions z_k and bound to each other by harmonic forces. The mass m_0 is further attached to a special harmonic spring with Hooke constant b_0 , while it is constrained by a frictionless guide to move up and down the x -axis. Clearly, if the tagged mass m_0 were uncoupled from the harmonic chain, it would itself just be a simple undamped harmonic oscillator with eigenfrequency $\Omega = (b_0/m_0)^{1/2}$. But if connected to the chain, it will excite the transverse modes of this reservoir. In that way the oscillator may dissipate some of its energy into the chain, at least for some time.

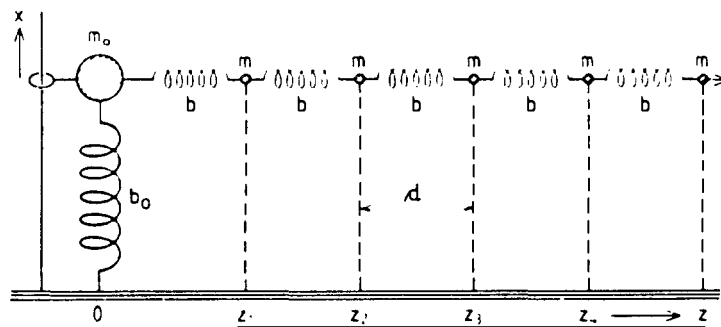


Fig. 5.1. Model for a mechanical transmission line.

The latter *proviso* takes care of the following. If there were only a small number of elements in the chain, the signal travelling away from the tagged oscillator would return to it almost immediately. If the number of elements would tend to infinity while keeping the physical length Λ of the system finite, the chain becomes an elastic string as depicted in fig. 5.2. But even in that case, by reflection from the far end of the string, emitted signals return to the actual oscillator after a time period of the order of $2\Lambda/c$, c being the average propagation velocity. The details of the dynamics would, of course, involve the particular nature of the boundary condition at the far end. Clearly, only if Λ becomes arbitrarily large there will be the possibility of dissipation for the tagged oscillator over an arbitrarily long time interval. The infinite length is most easily incorporated by using periodic boundary conditions, which are analogous to tying the transmission line to a characteristic impedance in order to prevent reflections. The characteristic impedance, attached to the finite line at $z = \Lambda$, is simply equivalent to “an infinity of more line” [72].

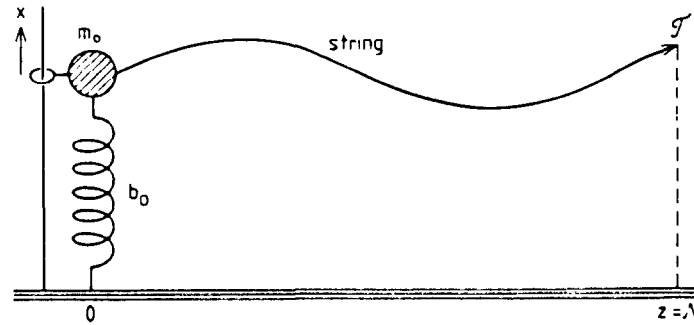


Fig. 5.2. The mechanical transmission line of fig. 5.1 in the limit $\epsilon \rightarrow 0$, but m/ϵ and the length Λ kept finite. $\mathcal{T}$ is the tension of the string (after [39]).

It can hardly be overemphasized that there are thus two essential steps in reaching irreversible behaviour if starting from reversible mechanics. First, the reservoir must have an infinite number of degrees of freedom over which the energy drained from the low dimensional subsystem of interest can be distributed. This is effectuated by taking the continuum or *field limit*. In terms of the normal (Fourier) modes of the reservoir, however, this still means a discrete spectrum entailing a finite recursion time. That is, the system will exhibit periodicity, which is a typically reversible phenomenon. Therefore, second, the system must have infinite size. This is sometimes called the thermodynamic limit (see e.g. [28, 34, 37, 52, 238, 332, 340, 342]). In terms of the normal modes this means a *continuous spectrum*. It implies an infinite recursion time. As will become evident, both steps must be taken for the following program to be carried through.

5.2. The mechanical model

The classical mechanical Lagrangian for the harmonic system shown in fig. 5.1 reads

$$L = \frac{1}{2}m_0\dot{x}_0^2 - \frac{1}{2}b_0x_0^2 + \sum_{k=1}^N \frac{1}{2}m\dot{x}_k^2 - \sum_{k=1}^N \frac{1}{2}\frac{\mathcal{T}}{\epsilon}(x_k - x_{k-1})^2, \quad (5.1)$$

where $\mathcal{T} = b(\epsilon - \epsilon_0)$ is the bias force applied to the little springs, which possess a rest length ϵ_0 and Hooke constant $b_k = b$. The quantity $\mathcal{T}$ is usually called the tension of the string (e.g. [343]). The index

$k \in [0, N]$ measures the distance along the z -axis, which is along the chain, according to $z_k = k\mathcal{A}$. The tagged oscillator is at $z = 0$. The canonical momenta following from (5.1) are

$$p_k = L_{,\dot{x}_k} = m_k \dot{x}_k, \quad (5.2)$$

so that the Hamiltonian becomes

$$H = \frac{1}{2m_0} p_0^2 + \frac{1}{2} b_0 x_0^2 + \frac{1}{2m} \sum_{k=1}^N p_k^2 + \frac{1}{2} \frac{\mathcal{T}}{\mathcal{A}} \sum_{k=1}^N (x_k - x_{k-1})^2. \quad (5.3)$$

The first Hamilton equation, as usual, reproduces (5.2). The second equation leads to

$$\dot{p}_k = -H_{,x_k} = (\mathcal{T}/\mathcal{A})(x_{k+1} - 2x_k + x_{k-1}); \quad k = 1, 2, \dots, \quad (5.4)$$

$$\dot{p}_0 = -H_{,x_0} = -b_0 x_0 + (\mathcal{T}/\mathcal{A})(x_1 - x_0), \quad (5.5)$$

so that the (Newton–Lagrange) equations of motion become

$$m \ddot{x}_k = (\mathcal{T}/\mathcal{A})(x_{k+1} - 2x_k + x_{k-1}); \quad k = 1, 2, \dots, \quad (5.6)$$

$$m_0 \ddot{x}_0 + b_0 x_0 = (\mathcal{T}/\mathcal{A})(x_1 - x_0). \quad (5.7)$$

Letting now the number N of chain oscillators tend to infinity, keeping the length of the string fixed at $\Lambda = N\mathcal{A}$, so that $\mathcal{A} \downarrow 0$, and confining ourselves to differentiable physical solutions $x(z, t)$, the above set of coupled differential-difference equations leads to the following two coupled partial differential equations:

$$x_{,tt}(z, t) = c^2 x_{,zz}(z, t); \quad z > 0, \quad (5.8)$$

$$x_{,tt}(0, t) + \Omega^2 x(0, t) = \kappa x_{,z}(0+, t). \quad (5.9)$$

Equation (5.8) is just the usual wave equation (e.g. [83, 113]) for the transverse vibrations of the string. The speed of propagation is given by $c = (\mathcal{T}/\rho)^{1/2}$, with ρ being the mass density $\rho = m/\mathcal{A}$, kept finite while $\mathcal{A} \downarrow 0$. Further $\Omega = (b_0/m_0)^{1/2}$, while $\kappa = \mathcal{T}/m_0$ can be considered as the coupling constant of oscillator and reservoir. In terms of the fields, the Hamiltonian may be written as

$$H = \frac{1}{2} [p(0, t)]^2 + \frac{1}{2} \Omega^2 [x(0, t)]^2 + \int_0^\Lambda \left\{ \frac{1}{2} [p(z, t)]^2 + \frac{1}{2} c^2 [x_{,z}(z, t)]^2 \right\} dz, \quad (5.10)$$

where the canonical momentum density is conveniently defined as $p(z_k) = p_k/\mathcal{A}$ if $k \geq 1$, while $p(0) = p_0$, and where we have chosen $\rho = m_0 = 1$, so that

$$p(z, t) = x_{,t}(z, t) \quad (5.11)$$

for all z . With these choices^{*} the only nonzero Poisson brackets lead to the commutators

$$[x(z, t), p(z', t)] = i\hbar \delta(z - z'); \quad z \vee z' \neq 0, \quad (5.12)$$

$$[x(0, t), p(0, t)] = i\hbar. \quad (5.13)$$

5.3. The electrical model

The electrical analogue of the above mechanical model has been sketched in fig. 5.3. The tagged resonant circuit, with selfinductance l_0 and capacity c_0 , is coupled to a chain of identical tuned circuits with selfinductance $l_k = l$ and capacity $c_k = c$ ($k = 1, 2, \dots$). As indicated in the figure, all transverse currents are taken positive into the positive x -direction, all longitudinal currents are similarly defined along the positive z -axis. Going around any circuit, Kirchhoff's voltage law (usually called his second law; see e.g. [37, 344, 345]) yields

$$q_k/c_k + l_k \ddot{q}_k - q_{k+1}/c_{k+1} = 0, \quad (5.14)$$

where q_k represents the charge at the k th condensor, and where I_k is the longitudinal current through the k th circuit located at z_k . Differentiating (5.14) with respect to time and using the relation $\dot{q}_{k+1} = I_{k,k+1}$ for the transverse currents, one obtains

$$I_{k-1,k}/c_k + l_k \ddot{I}_k - I_{k,k+1}/c_{k+1} = 0, \quad (5.15)$$

with the understanding that $I_{-1,0} = I_0$. The other Kirchhoff law, expressing current conservation at the vertices, relates the transverse currents to the longitudinal currents as

$$I_{k,k+1} = I_{k+1} - I_k. \quad (5.16)$$

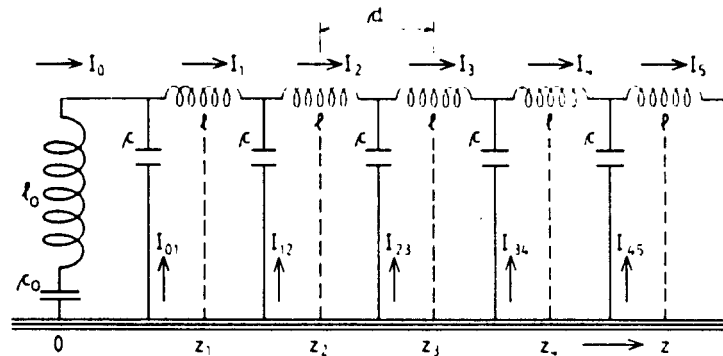


Fig. 5.3. Model for an electrical transmission line (after [157]).

^{*} An alternative, used in [39], is to introduce a spatially dependent mass density $\rho(z) = \rho + 2m_0\delta(z)$. In that case, (5.12) and (5.13) can be presented in a single formula, in fact (5.12).

Inserting this into (5.15) one finds

$$l_k \ddot{I}_k = (1/c)(I_{k+1} - 2I_k + I_{k-1}); \quad k = 1, 2, \dots, \quad (5.17)$$

$$l_0 \ddot{I}_0 + (1/c_0)I_0 = (1/c)(I_1 - I_0), \quad (5.18)$$

which are immediately seen to be completely identical to their mechanical counterparts (5.6) and (5.7). In the limit of an infinite number of circuits in the chain, keeping the length of the transmission line fixed again at $\Lambda = N\ell$, one obtains the partial differential equations (5.8) and (5.9) with the appropriate definitions: the propagation velocity becomes $c = (1/l'c')^{1/2}$, where $l' = \ell/\ell$ and $c' = c/\ell$ are the specific selfinductance and capacity respectively; the characteristic frequency of the tagged circuit becomes $\Omega = (1/l_0 c_0)^{1/2}$, while the coupling constant is $\kappa = 1/l_0 c'$. The model, be it mechanical or electrical, will now be further explored on the basis of (5.8) and (5.9).

5.4. The classical damped oscillator

Following d'Alembert the general solution of the wave equation (5.8) may be written as (see e.g. [83])

$$x(z, t) = x_A(t - z/c) + x_B(t + z/c). \quad (5.19)$$

The x_A , respectively x_B represent waves travelling to the right (outgoing, so to speak, from the point of view of the tagged oscillator) respectively to the left (incoming from the bath). Using (5.19) one readily verifies the identity

$$x_{,z}(z, t) = -(1/c)x_{,t}(z, t) + (2/c)x_{B,t}(z, t), \quad (5.20)$$

so that (5.8) can be recast into the form

$$\ddot{x}(0, t) + 2\lambda \dot{x}(0, t) + \Omega^2 x(0, t) = 4\lambda x_{B,t}(0, t), \quad (5.21)$$

where $\lambda = \kappa/2c$. It should be noted that d'Alembert's solution typically applies to the wave equation. That is, the exact and complete separation (5.19) into left- and right travelling waves would in principle not have been feasible if we had not already taken the limit to continuous fields (i.e. the first step mentioned earlier: $N \rightarrow \infty$, $\ell \downarrow 0$, $\Lambda = N\ell$ finite). Hence, the equation of motion for the tagged oscillator could not have been given the appearance of (5.21) (compare especially with [157]).

Now in prequantal classical theory one may conceive of the situation that there are no incoming waves at all present in the transmission line at the initial time, say $t = 0$. Exciting at that instant the tagged system at $z = 0$ will, of course, produce waves travelling to the right. But, as noted earlier, if the transmission line were semi-infinite, these outgoing waves would never be reflected back, i.e. they would never be transformed into incoming waves. In other words, if $x_B(z, 0) = 0$ for all $z \geq 0$ and $\Lambda = \infty$, one has classically that $x_B(z, t) \equiv 0$. Hence in that case (5.21) reduces to

$$\ddot{x}(0, t) + 2\lambda \dot{x}(0, t) + \Omega^2 x(0, t) = 0, \quad (5.22)$$

which is exactly identical to the classical equation of motion (3.1) for the linearly damped harmonic oscillator. See also fig. 5.4.

However, in general the very special initial condition of no incoming wave excitations being present in the reservoir field will not be encountered in real physical systems. Normally there are at least thermal excitations, and these do not discriminate between left- and right travelling waves. And in quantum mechanics, even at zero temperature there would still exist the zero-point excitations of the oscillators in the transmission line. The basic reason for this, of course, is that if the line is capable of transmitting signals, its microscopic elements must be emitters as well as absorbers. Hence, in quantum theory the inevitable excitation/de-excitation processes in the reservoir can not be circumvented. For that reason, in constructing the model's quantum mechanics we must carry along the incoming waves.

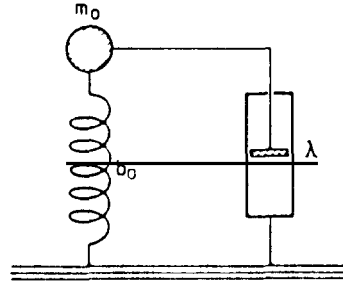


Fig. 5.4. The mechanical damped oscillator.

5.5. Normal mode expansion

The total field in the transmission line can be written in terms of its normal modes as

$$x(z, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k \Lambda}} [A_k \exp\{-i \omega_k (t - z/c)\} + B_k \exp\{-i \omega_k (t + z/c)\} + \text{conj.}], \quad (5.23)$$

where "conj." means complex (Hermitian) conjugation in classical (quantal) theory. However, for a finite transmission line the left- and right travelling modes are not independent of each other as a consequence of the boundary conditions. The most general boundary conditions pertaining to the wave equation are of the so-called mixed type (see e.g. [83, 113, 346, 347]), but it suffices to illustrate the point by considering for convenience the Dirichlet condition $x(\Lambda, t) = 0$. This leads to

$$\begin{aligned} A_k \exp(i \omega_k \Lambda/c) + B_k \exp(-i \omega_k \Lambda/c) &= 0, \\ A_k^* \exp(-i \omega_k \Lambda/c) + B_k^* \exp(i \omega_k \Lambda/c) &= 0, \end{aligned} \quad (5.24)$$

so that

$$A_k^* A_k = B_k^* B_k. \quad (5.25)$$

Hence, we infer the relation

$$A_k = B_k \exp(-2i \varphi_k), \quad (5.26)$$

φ_k as yet being an arbitrary scalar. The boundary condition at the near end $z = 0$ of the transmission line is simply given by the equation of motion (5.9) for the tagged oscillator. It gives in principle two constraints involving the A_k , B_k and their conjugates. Inserting the relation (5.26), these constraints are found to be identical and one is left with

$$\operatorname{tg} \varphi_k = (\Omega^2 - \omega_k^2)/2\lambda\omega_k, \quad (5.27)$$

where we have used the relation $\kappa/c = 2\lambda$. Finally, inserting (5.26) into (5.24), the eigenvalues follow from

$$\omega_k \Lambda/c - \varphi_k = (k - \frac{1}{2})\pi; \quad k = 1, 2, \dots \quad (5.28)$$

In fact, the above procedure is precisely the one used by Yurke and Yurke [39].^{*} Since their ultimate results are identical to those obtained presently,^{**} we here face an explicit confirmation of the general expectation that the precise form of the boundary conditions at the far ends of a system should somehow become irrelevant in the thermodynamic limit of infinite system size $\Lambda \rightarrow \infty$ (see also [348–350]). Therefore, it is a useful common practice to introduce the most appropriate, though physically somewhat artificial, boundary conditions for large systems, namely periodic boundary conditions, from the outset. So we return to the normal mode decomposition (5.23) and require $x(z, t)$ to be periodic over Λ . This leads to the two conditions

$$A_k \pm B_k = A_k \exp(i \omega_k \Lambda/c) \pm B_k \exp(-i \omega_k \Lambda/c), \quad (5.29)$$

and their conjugates. Either adding or subtracting the two equations contained in (5.29) yields in stead of (5.28)

$$\omega_k \Lambda/c = 2k\pi; \quad k = 1, 2, \dots, \quad (5.30)$$

which fixes the eigenvalues without requiring any relation between the mode amplitudes.

5.6. The quantal damped oscillator

The normal mode amplitudes are now determined by the initial conditions as

$$A_k = (2\hbar\omega_k\Lambda)^{-1/2} \int_0^\Lambda dz \exp(-i \omega_k z/c) [\omega_k x(z, 0) + i p(z, 0)], \quad (5.31)$$

$$B_k = (2\hbar\omega_k\Lambda)^{-1/2} \int_0^\Lambda dz \exp(i \omega_k z/c) [\omega_k x(z, 0) + i p(z, 0)], \quad (5.32)$$

plus their conjugates. In these formulae we have used the canonical relation (5.11). By means of these

^{*} Let $k \rightarrow n$ and $\varphi_k = \pi/2 - \theta_n$ for complete equivalence.

^{**} However, there is a subtle problem involved as their eigenfunctions are not strictly orthogonal. This facet requires further study, as it may be related to problems of divergencies. See e.g. eq. (5.80).

transformations one easily verifies that the only nonzero commutator for the reservoir field variables, that is (5.12), leads to

$$[A_k, A_l^\dagger] = [B_k, B_l^\dagger] = \delta_{kl}, \quad (5.33)$$

any other conceivable commutator being zero. Evidently, the A_k , B_k and $A_k^\dagger$, $B_k^\dagger$ are the annihilation respectively creation operators for the normal mode excitations. We have now prepared the system adequately for the limit $\lambda \rightarrow \infty$ (i.e. for the second step to be taken), essentially implying the (linear) independence of the incoming and outgoing waves.

Now substitute the pertinent part from the normal mode expansion (5.23) for the incoming driving field on the r.h.s. of the equation of motion (5.21) for the actual oscillator. Then define

$$y(t) = x(0, t) - \eta(t), \quad (5.34)$$

$$\eta(t) = \sum_k \eta_k(t), \quad (5.35)$$

such that $\eta_k(t)$ must be solved from

$$\ddot{\eta}_k + 2\lambda\dot{\eta}_k + \Omega^2\eta_k = i\lambda \sqrt{\frac{8\hbar\omega_k}{\lambda}} (B_k^\dagger \exp(i\omega_k t) - B_k \exp(-i\omega_k t)), \quad (5.36)$$

with the initial conditions $\dot{\eta}_k(0) = \eta_k(0) = 0$. As a result one obtains

$$\ddot{y} + 2\lambda\dot{y} + \Omega^2 y = 0, \quad (5.37)$$

with $\dot{y}(0) = \dot{x}(0, 0) = p(0, 0)$ and $y(0) = x(0, 0)$. The required solution from (5.36) is easily found by means of elementary analysis [351, 352] to be

$$\eta_k(t) = i\lambda \sqrt{\frac{8\hbar\omega_k}{\lambda}} [B_k \phi_k(t) - \text{conj.}], \quad (5.38)$$

$$\phi_k(t) = \frac{1}{\omega_k^2 - \Omega^2 + 2i\lambda\omega_k} [e^{-i\omega_k t} - \gamma_k e^{-\lambda t - i\omega t} - (1 - \gamma_k) e^{-\lambda t + i\omega t}], \quad (5.39)$$

$$\gamma_k = \frac{1}{2}(1 + \omega_k/\omega) + \frac{1}{2}i\lambda/\omega, \quad (5.40)$$

where $\omega = (\Omega^2 - \lambda^2)^{1/2}$ is the reduced classical frequency as before. What has been achieved is in fact quite interesting. The result (5.37) shows that $y(t)$ represents the position of a classical damped oscillator, even in the case that there do exist incoming signals from the semi-infinite transmission line. In other words, the dynamics of a classical damped oscillator can be obtained in an exact manner if one subtracts the incoming reservoir signals (thermal noise, for instance) from the actual oscillator variables. This represents a well-defined example of so-called smoothing (see e.g. [29, 66, 353]) by noise-subtraction [157]. The explicit solution for the smoothed variable $y(t)$, satisfying (5.37) and the required initial conditions, reads

$$y(t) = e^{-\lambda t} \{x(0, 0) \cos \omega t + (1/\omega)[p(0, 0) + \lambda x(0, 0)] \sin \omega t\}. \quad (5.41)$$

5.7. Connection with Caldirola–Kanai model

Following Stevens, we now conjecture that a time-dependent canonical transformation exists from $x(0, t)$, $p(0, t)$ to $y(t)$, $\pi(t)$ with $y(t)$ given by (5.34) and the new, noise-subtracted momentum defined as

$$\pi(t) = e^{2\lambda t} \{p(0, t) - \eta(t)\}. \quad (5.42)$$

In view of the initial conditions on $\eta(t)$, one immediately concludes from (5.13) that

$$[y(0), \pi(0)] = i\hbar, \quad (5.43)$$

while any other commutator at $t = 0$ will be zero. This gives reason to calculate the time derivative^{*}

$$[y(t), \pi(t)]' = 2i\hbar\lambda e^{2\lambda t} \left\{ 1 - \sum_k \sqrt{\frac{2\omega_k}{\hbar\Lambda}} [x(0, t), B_k^\dagger e^{i\omega_k t} - B_k e^{-i\omega_k t}] \right\}, \quad (5.44)$$

where we have used (5.11), (5.21) and (5.36). According to (5.32), the normal mode amplitudes are expressible in terms of the initial values $x(z, 0)$ and $p(z, 0)$. For the further evaluation of the commutators on the r.h.s. of (5.44), it is therefore useful to observe that (5.41) expresses the smoothed position $y(t)$ for all times entirely in terms of the initial values $x(0, 0)$ and $p(0, 0)$ at $z = 0$. As these commute with any of the field variables on^{**} $z \in (0, \Lambda)$, it can be concluded that $y(t) = x(0, t) - \eta(t)$ commutes with the driving force of the bath at any time. Therefore, (5.44) is equal to

$$[y(t), \pi(t)]' = 2i\hbar\lambda e^{2\lambda t} \left\{ 1 - \sum_k \sqrt{\frac{2\omega_k}{\hbar\Lambda}} [\eta(t), B_k^\dagger e^{i\omega_k t} - B_k e^{-i\omega_k t}] \right\}. \quad (5.45)$$

Since $\eta(t)$ has been expressed in terms of the $B_k^\dagger$, B_k by means of (5.35) and (5.38)–(5.40), the commutator can now be calculated explicitly. Using the algebra (5.33), one easily obtains

$$[y(t), \pi(t)]' = 2i\hbar\lambda e^{2\lambda t} \left\{ 1 - \frac{16\lambda^2}{\Lambda} \sum_k \frac{\omega_k^2}{(\omega_k^2 - \Omega^2)^2 + 4\lambda^2\omega_k^2} \right\}. \quad (5.46)$$

We now take the actual infinite system size limit $\Lambda \rightarrow \infty$, so that in view of (5.30)

$$(2\pi/\Lambda) \sum_k \rightarrow (1/c) \int d\omega', \quad (5.47)$$

which yields

$$[y(t), \pi(t)]' = 2i\hbar\lambda e^{2\lambda t} \left\{ 1 - \frac{4\lambda}{\pi} \int_0^\infty \frac{\omega'^2 d\omega'}{(\omega'^2 - \Omega^2)^2 + 4\lambda^2\omega'^2} \right\}. \quad (5.48)$$

^{*}One may of course compute the commutator itself, but the present method turns out to be much simpler.

^{**}Note, for the sake of detail, that $z = 0$ is contained in (5.31) and (5.32) only with measure zero, while the nonzero commutator (5.13) at $z = 0$ is nonsingular.

where we have further used the relation $2\lambda/c = \kappa/c^2 = \kappa\rho/\mathcal{T} = \rho/m_0 = 1$ (see (5.8)–(5.11)).²⁷ The integral can be evaluated analytically as

$$\begin{aligned} \mathcal{J}_\lambda &= \frac{1}{\pi} \int_0^\infty \frac{\omega'^2 d\omega'}{(\omega'^2 - \Omega^2)^2 + 4\lambda^2 \omega'^2} = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\omega'^2 d\omega'}{(\omega'^2 + \Omega^2 + 2\omega\omega')(\omega'^2 + \Omega^2 - 2\omega\omega')} \\ &= \frac{1}{8\pi\omega} \int_{-\infty}^\infty \left[\frac{1}{(\omega' - \omega)^2 + \lambda^2} - \frac{1}{(\omega' + \omega)^2 + \lambda^2} \right] \omega' d\omega' = \frac{1}{4\pi} \int_{-\infty}^\infty \frac{dx}{x^2 + \lambda^2} = \frac{1}{4\lambda}. \end{aligned} \quad (5.49)$$

Hence,

$$[y(t), \pi(t)]' = 0, \quad (5.50)$$

so that, in view of its initial value (5.43),

$$[y(t), \pi(t)] = i\hbar \quad (5.51)$$

at any time $t \geq 0$. Other commutators can be handled similarly to show that $y(t)$ and $\pi(t)$ are indeed proper canonical variables. Note that in proving (5.51) we have used the nature of the solution for $y(t)$ in going from (5.44) to (5.45). In fact, the precise explicit form (5.41) of that solution, together with the corresponding expression for $\pi(t)$, was used by Stevens [157] to arrive at the same conclusions. Thus, to be clear, whereas Stevens' proof involves the explicit formulae for the smooth (noise-subtracted) oscillator variables, the present treatment rather makes use of the explicit expressions for the noise $\eta(t)$. The results are the same. Further note that the transformation replaces $x(0, t)$ and $p(0, t)$ by $y(t)$ and $\pi(t)$, but that it does not affect the normal mode amplitudes of the reservoir $z \in (0, \infty)$.²⁸

From (5.34), (5.11), (5.42) and (5.37) it is clear that the canonical equations for the noise-subtracted tagged oscillator variables are

$$\dot{y} = H'_{,\pi} = e^{-2\lambda t} \pi, \quad (5.52)$$

$$\dot{\pi} = -H'_{,y} = -e^{2\lambda t} \Omega^2 y. \quad (5.53)$$

Evidently, the total Hamiltonian H' will therefore consist of two commuting parts H'' and H''_R , where H'' generates the dynamics of the smoothed oscillator and where H''_R merely pertains to the reservoir. That is,

$$H' = H'' + H''_R(A_k, A_k^\dagger; B_k, B_k^\dagger; t), \quad (5.54)$$

$$H'' = \frac{1}{2} e^{-2\lambda t} \pi^2 + \frac{1}{2} e^{2\lambda t} \Omega^2 y^2. \quad (5.55)$$

The latter effective Hamiltonian for the smoothed oscillator is an exact isomorph of the original

²⁷ For the electrical model the choice $\rho = m_0 = 1$ translates into $l' = l_0 = 1$, so that again $2\lambda/c = \kappa/c^2 = l'/l_0 = 1$ (see below (5.18)).

²⁸ Recall that $y(t)$, and similarly $\pi(t)$, commutes with the B_k (see (5.44)–(5.45)). The same argument of course applies to the A_k .

Bateman–Caldirola–Kanai Hamiltonian (4.15). However, the different physical significance of the canonical variables in (5.55) and (4.15) must not be forgotten. The present model does not violate the fundamental commutator $[x, p] = i\hbar$ in the course of time, in contrast with the standard Caldirola–Kanai theory. As we have seen, the single reason for this is the absence of noise (i.e. the usually thermal, incoming signals for the same subsystem, the reservoir, that accounts for the dissipation) in the latter phenomenological classical approach.

These findings, originally most clearly formulated within the present context by Stevens in 1961 [157], were in fact corroborated at about the same time by Senitzky [50, 354]. However, the latter did not discuss the connection between the harmonic oscillation-reservoir model and the Bateman–Caldirola–Kanai model. His treatment is limited to the weak damping approximation (first order in λ), whereas the present formulae are still exact within the model's framework. Actually, it is somewhat hard to see good reasons for restricting the treatment to the weak damping limit. It is very well conceivable that the single privileged oscillator, which is under investigation, is strongly damped while the vast majority of the oscillators in the essentially infinite reservoir still is in the same equilibrium state as in the weak damping case. The amount of energy dissipated by the tagged oscillator will always be negligible compared to the total energy content of the reservoir. See further section 8.

5.8. Svin'in's treatment

Consider the classical Hamiltonian (5.55). The canonical generator

$$F = \pi_x y + \pi_x \eta(t) - y e^{2\lambda t} \dot{\eta}(t) \quad (5.56)$$

transforms y, π into x, π_x as follows:

$$\pi = F_{,y} = \pi_x - e^{2\lambda t} \dot{\eta}(t), \quad x = F_{,\pi_x} = y + \eta(t). \quad (5.57)$$

Compare these formulae with (5.34) and (5.42). The new Hamiltonian $H''' = H'' + F_{,t}$ becomes

$$H''' = \frac{1}{2} e^{-2\lambda t} \pi_x^2 + \frac{1}{2} e^{2\lambda t} \Omega^2 x^2 - e^{2\lambda t} B(t) x, \quad (5.58)$$

where an irrelevant term independent of π_x, x has been disregarded, and where we have used (5.35) and (5.36), so that

$$B(t) = i\lambda \sum_k \sqrt{\frac{8\hbar\omega_k}{\Lambda}} (B_k^\dagger \exp(i\omega_k t) - B_k \exp(-i\omega_k t)). \quad (5.59)$$

The Hamiltonian (5.58) has precisely the form of the Hamiltonian introduced by Svin'in in 1976 [180]. It was recently also used by Brinati and Mizrahi [181], investigating its quantum mechanics in complex phase space (see also [156, 195]). Svin'in treated $B(t)$ as a classical, Gaussian delta-correlated thermal Langevin noise source in order to remedie the observed violation of Heisenberg's principle in the original Caldirola–Kanai quantum theory (see refs. quoted below (4.15)). That is, Svin'in's Ansatz is quite akin to the Langevin–Wiener assumptions (1.4) for the process $\xi(t)$ as discussed in the Introduction.

Let us presently calculate the correlation function of $B(t)$, as given in (5.59), explicitly. Considering

the noise in principle to be of a quantum mechanical origin, we determine the symmetrized Weyl–Wigner [276, 277] correlation function in order to obtain a real outcome. Then, using the properties of a thermal bath (see e.g. [19, 23, 50, 57, 164]),

$$\langle B_k^\dagger B_l \rangle = (\exp(\hbar\omega_k/k_B T) - 1)^{-1} \delta_{kl}, \quad (5.60)$$

one easily obtains in the infinite system limit:

$$\frac{1}{2} \langle [B(t+\tau), B(t)]_+ \rangle = \frac{4\lambda}{\pi} \int_0^\infty \mathcal{P}(\omega') \cos \omega' \tau \, d\omega', \quad (5.61)$$

where $\mathcal{P}$ represents the modified Planck function (1.11). Evidently, the noise from the transmission line is quite different from being delta-correlated. Actually, although the thermal part converges, the integral in (5.61) does not even exist because of the diverging zero-point contribution. Only in the extreme classical limit $k_B T \rightarrow \infty$, (5.61) can be reduced to

$$\langle B(t+\tau) B(t) \rangle = \langle \xi(t+\tau) \xi(t) \rangle = 4\lambda k_B T \delta(\tau). \quad (5.62)$$

From comparison with its definition (1.4), we see that this would imply a diffusion coefficient $D = 2\lambda k_B T$.⁸ Of course, the integral in (5.61) does converge if we drop the zero-point fluctuations, but that would spoil the cure of the deficiencies of the damped oscillator as observed in the Caldirola–Kanai theory. On the other hand, treating the bath (semi-) classically and ascribing the vacuum fluctuations in fact merely to the tagged oscillator must be considered rather unsatisfactory, to say the least.

Notwithstanding that the correlation function (5.61) can not be calculated in general, the spectral density defined by (1.5) does exist. If $\omega > 0$, one obtains

$$S_B(\omega) = 4\lambda \mathcal{P}(\omega), \quad (5.63)$$

while $S_B(-\omega) = S(\omega)$. Defining a frequency dependent diffusion coefficient by means of $S_B(\omega) = 2D(\omega)$, one has $D(\omega) = 2\lambda \mathcal{P}(\omega)$. Svin'in's Ansatz amounts to the assumption that only the actual value of the spectral density at the free oscillator frequency $\omega = \Omega$ will be relevant to dynamical quantities of the damped oscillator. Intuitively, this will at least require weak damping. But, as should be understood from the preceding, this is by no means sufficient.

For complete equivalence with Svin'in's treatment [180] we can simply replace in the Hamiltonian (5.58) $B(t)$ by $\xi(t)$ with the properties (1.4) and the diffusion coefficient given by $D = 2\lambda \mathcal{P}(\Omega)$. Hence,⁹

$$\begin{aligned} \langle \xi(t) \rangle &= 0; & \langle \xi(t+\tau) \xi(t) \rangle &= 2D \delta(\tau), \\ D &= \lambda \hbar \Omega \coth(\frac{1}{2} \hbar \Omega / k_B T). \end{aligned} \quad (5.64)$$

⁸ This is twice as large as the diffusion coefficient in the Introduction. See especially (1.1), (1.5) and (1.8). Both here and there 2λ has been chosen as the relaxation constant of the energy. However, presently not only the momentum but also the position carries energy. Therefore, in thermal equilibrium now $E = k_B T$ instead of $\frac{1}{2} k_B T$ (see below (1.8)), which explains the factor two in the diffusion coefficient.

⁹ Note that Svin'in's $\alpha = 2\lambda$, while his ω is the present Ω and the present ω is his ω_0 .

As usual, one may compute the equations of motion for expectation values from (3.46) with $H \rightarrow H'''$, or equivalently directly from the Schrödinger equation

$$i \hbar \psi_t = H'''(\pi_x, x, \xi(t), t) \psi. \quad (5.65)$$

Note that the averaging procedure will consist in fact of two parts: first, the common quantal averaging with ψ , and second, the thermal averaging of $\xi(t)$. In terms of the mechanical variables the result for the variances reads

$$\dot{\sigma}_{px} = -2\lambda\sigma_{px} + \sigma_{pp} - \Omega^2\sigma_{xx} + \langle\langle x \rangle \xi(t) \rangle, \quad (5.66)$$

$$\dot{\sigma}_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} + 2\langle\langle p \rangle \xi(t) \rangle, \quad (5.67)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px}. \quad (5.68)$$

The quantities $\langle\langle p \rangle \xi \rangle$ and $\langle\langle x \rangle \xi \rangle$ are easily calculated following the lines of standard Markov–Langevin theory (see e.g. [19, 21, 23, 24, 37, 57–59, 65, 66, 92] and the Introduction). That is, formally integrating the equation of motion of the as yet merely quantum mechanically averaged momentum,

$$\langle p \rangle' = -2\lambda\langle p \rangle - \Omega^2\langle x \rangle + \xi(t), \quad (5.69)$$

and using (5.64) and $\langle\langle x(t') \rangle \xi(t) \rangle = 0$ if $t' < t$, one finds $\langle\langle p \rangle \xi \rangle = D$. Similarly, by means of $\langle x \rangle' = \langle p \rangle$, one obtains $\langle\langle x \rangle \xi \rangle = 0$. Hence,

$$\dot{\sigma}_{px} = -2\lambda\sigma_{px} + \sigma_{pp} - \Omega^2\sigma_{xx}, \quad (5.70)$$

$$\dot{\sigma}_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} + 2D, \quad (5.71)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px}, \quad (5.72)$$

D given in (5.64). The only, but essential difference between (5.70)–(5.72) and the original Caldirola–Kanai model (4.28)–(4.30) is the present additional time-independent term in the equation for σ_{pp} , which is due to the noise. It is a matter of straightforward calculation to obtain the solution with $\dot{\sigma}_{xx}(0) = 0$ and $\sigma_{xx}(0) = \hbar/2\Omega$. It reads

$$\sigma_{xx}(t) = \frac{\hbar}{2\Omega} \coth\left(\frac{\hbar\Omega}{2k_B T}\right) + \frac{\hbar}{2\Omega} e^{-2\lambda t} \left[1 - \coth\left(\frac{\hbar\Omega}{2k_B T}\right) \right] \left[1 + \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right], \quad (5.73)$$

the associated result for the momentum spread being

$$\sigma_{pp}(t) = \frac{\hbar\Omega}{2} \coth\left(\frac{\hbar\Omega}{2k_B T}\right) + \frac{\hbar\Omega}{2} e^{-2\lambda t} \left[1 - \coth\left(\frac{\hbar\Omega}{2k_B T}\right) \right] \left[1 - \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right]. \quad (5.74)$$

The ultimate value of the uncertainty product becomes

$$\sigma_{pp}(\infty) \sigma_{xx}(\infty) = \frac{\hbar^2}{4} \coth^2\left(\frac{\hbar\Omega}{2k_B T}\right) \geq \frac{\hbar^2}{4}. \quad (5.75)$$

so that the uncertainty principle is preserved, albeit – as discussed – somewhat artificially. As they should, (5.73) and (5.74) reduce to (3.59) and (3.60) in absence of the noise.^{*} Finally, it is worth mentioning that the above formulae can also be obtained entirely classically from the Fokker–Planck equation (see e.g. [21, 32, 34–37, 44–47, 51, 56–58, 62, 66, 73, 76, 90, 92, 231, 242, 332, 333]) for the probability density^{**} $W(p, x, t)$ associated with the Langevin equation (5.69),

$$W_{,t} = -pW_{,x} + \Omega^2 x W_{,p} + 2\lambda(pW)_{,p} + DW_{,pp}. \quad (5.76)$$

This connection will be made more precise in the sequel. See section 7.

5.9. Transmission line fluctuations

Returning now to the genuine transmission line model, it should be noted that in principle it can be solved exactly, without enforcing the Markov hypothesis. Namely, the dynamics of the well-defined noise-subtracted variables is governed by the simple linearly damped harmonic oscillator equation (5.37), that can be solved easily. In fact, the evolution of the position is given explicitly by (5.41). Let us just consider the final state in some more detail. Clearly $y(\infty) = 0$, so that the actual oscillator position acquires the noisy value $x(0, \infty) = \eta(\infty)$. Hence, denoting $x(0, t)$ by $x(t)$ for convenience, $\langle x(\infty) \rangle = 0$. More generally $y^n(\infty) = 0$ with $n = 1, 2, \dots$. For the second moment one thus gets, using the above result for the first moment, $\langle x^2(\infty) \rangle = \langle \eta^2(\infty) \rangle$. Invoking (5.35) and (5.38) for $\eta(t)$, and using the properties (5.60) of the thermal bath, one finds^{***}

$$\sigma_{xx}(\infty) = \frac{4\lambda}{\pi} \int_0^\infty \mathcal{P}(\omega') \frac{d\omega'}{(\omega'^2 - \Omega^2)^2 + 4\lambda^2 \omega'^2}. \quad (5.77)$$

Confining ourselves to the zero temperature case $T = 0$, where the modified Planck distribution (1.11) reduces to $\mathcal{P}(\omega') \rightarrow \frac{1}{2}\hbar\omega'$, the integral in (5.77) is most easily evaluated setting $\omega'^2 = (\omega^2 - \lambda^2) + 2\lambda\omega x$, with $\omega' d\omega' = \lambda\omega dx$ and $\omega = (\Omega^2 - \lambda^2)^{1/2}$. The upshot is

$$\sigma_{xx}(\infty) = \frac{\hbar}{2\pi\omega} \operatorname{arccotg} \left(-\frac{\omega^2 - \lambda^2}{2\lambda\omega} \right). \quad (5.78)$$

This result agrees with Yurke and Yurke's [39] and is also consistent with the formula of Ford, Kac and Mazur [38].^{****} In the weak friction limit (5.78) can be expanded in a Taylor series as

$$\sigma_{xx}(\infty) = \frac{\hbar}{2\Omega} \left[1 - \frac{2\lambda}{\pi\Omega} + o\left(\frac{\lambda^2}{\Omega^2}\right) \right], \quad (5.79)$$

where, for future comparison,^{*****} one should note the occurrence of a term linear in λ .

^{*} Recall that (4.28)–(4.30) for the Caldirola–Kanai model are identical to (3.56)–(3.58) for the dual Bateman-model.

^{**} Since in (5.66)–(5.68) σ_{pi} is the Weyl–Wigner symmetrized expression, the diffusive master equation (5.76) is in fact satisfied by Wigner's distribution function [277]. See also, especially in view of (5.76) at $T = 0$, [24, 256].

^{***} Recall the relation $2\lambda/c = 1$. See (5.46)–(5.48).

^{****} In [39], $\gamma = 2\lambda$ and $Q = \Omega/2\lambda$. In [38], replace the Planck function P by the modified $\mathcal{P}$, set $T = 0$ and calculate their (82), with $f = 2\lambda$ and $\kappa = \Omega$.

^{*****} See (7.51).

Finally it is not difficult to see that the formula pertinent to $\sigma_{pp}(x)$ will be $\langle p^2(x) \rangle = \langle \dot{\eta}^2(x) \rangle$. At $T = 0$ one obtains

$$\sigma_{pp}(x) = \frac{2\hbar\lambda}{\pi} \int_0^\infty \frac{\omega'^3 d\omega'}{(\omega'^2 - \Omega^2)^2 + 4\lambda^2 \omega'^2}. \quad (5.80)$$

As it stands,* this expression cannot be evaluated as the integral diverges logarithmically. Hence, the semi-infinite transmission line model does not allow the evaluation of the uncertainty relation, unless we would disregard the vacuum fluctuations from the oscillators in the reservoir. That, however, would imply the violation of the uncertainty principle for the tagged oscillator at low enough temperatures.

5.10. Summary

The semi-infinite transmission line model shows that two steps must be taken in order to arrive at irreversible behaviour of a subsystem (the tagged oscillator at the near end of the line) if starting from reversible dynamics for the system as a whole. First, for finite system size, the number of degrees of freedom must become infinite (spatial continuum or field limit). Second, the size of the system must become infinite (spectral continuum or thermodynamic limit). Then left and right travelling waves in the line can be completely separated. In quantum mechanics there will always be fluctuations present in the incoming waves at the subsystem. This noise source remedies the violation of the fundamental commutator for the tagged oscillator. The noise is non-white, and unfortunately the uncertainty product for the damped oscillator cannot be evaluated as it becomes infinite, save for the limit of vanishing friction $\lambda \downarrow 0$.

6. The dual Hamiltonian and complex calculus

6.1. Complex symplectic model

6.1.1. Classical mechanics

Recently a complex symplectic formulation for the damped harmonic oscillator has been proposed by Dedene [197], touching the problem from the point of view of geometric quantization (see e.g. [355–369]). Apparently, Dedene was stimulated in his investigations by earlier attempts of the present author to describe the damped oscillator in terms of complex dynamical variables [59, 24, 195, 258]. The relation between these complex variables theories will be studied in some more detail furtheron. Currently we will show that Dedene's Hamiltonian is connected with the time-independent dual Bateman Hamiltonian (3.6) by a simple complex canonical transformation. In fact, it is precisely the transformation used previously by the present author [59].

*We do not consider a frequency cut-off [38], nor any nonlinear or relativistic modification of the model. Nor do we consider the extreme limit $\lambda \downarrow 0$ separately. In that case (5.80) can be evaluated as usual introducing Dirac's delta-distribution. Using (5.49) one then obtains $\sigma_{pp}(x) = \hbar\Omega/2$, the standard result for the undamped harmonic oscillator groundstate. The extension of distribution theory beyond the extreme limit $\lambda \downarrow 0$ is currently under investigation. The basic ideas are closely related to the "partie finie" of Hadamard [518] and the modern theory of generalized functions developed by Schwartz [519], Temple [520] and Lighthill [521]. See also [41, 491, 522, 523]. The results will be published elsewhere. See also the footnote below (5.28).

In the canonical transformations

$$\begin{aligned} z &= \frac{1}{\sqrt{2\omega}} (\tilde{h} - i\omega x), & \tilde{z} &= \frac{1}{\sqrt{2\omega}} (h + i\omega \tilde{x}), \\ z^* &= \frac{1}{\sqrt{2\omega}} (\tilde{h} + i\omega x), & \tilde{z}^* &= \frac{1}{\sqrt{2\omega}} (h - i\omega \tilde{x}), \end{aligned} \quad (6.1)$$

the dynamical variables z and $i\tilde{z} = \pi_z$ (as well as their complex (or Hermitian) conjugates) play the role of a canonically paired coordinate and momentum. The wiggle on z indicates a formal complex (Hermitian) mirror conjugation. That is, one must both interchange the real system variables by means of the earlier defined mirror conjugation $\{\lambda \leftrightarrow -\lambda; x, h \leftrightarrow \tilde{x}, \tilde{h}\}$, see (3.9)–(3.10), and take complex (Hermitian) conjugates. Hence, $\tilde{z} = \tilde{z}^*$ (or $\tilde{z}^*$). The inverse formulae of (6.1) are easily found to be

$$\begin{aligned} x &= \frac{i}{\sqrt{2\omega}} (z - z^*), & h &= \sqrt{\frac{\omega}{2}} (\tilde{z} + \tilde{z}^*), \\ \tilde{x} &= \frac{-i}{\sqrt{2\omega}} (\tilde{z} - \tilde{z}^*), & \tilde{h} &= \sqrt{\frac{\omega}{2}} (z + z^*). \end{aligned} \quad (6.2)$$

One should very clearly realize the difference between (6.1) respectively (6.2) and (3.9) respectively (3.10) (aside, of course, from the factor $\sqrt{\hbar}$). The complex variables introduced in the Feshbach–Tikochinsky theory each involve a mixture of both the original and the artificial mirror image oscillator in terms of their mechanical variables, whereas the above definitions instead involve mixtures in terms of the canonical variables. In fact, writing (6.1) in terms of the mechanical variables one obtains (consult (3.5))

$$\begin{aligned} z &= \frac{1}{\sqrt{2\omega}} [p + (\lambda - i\omega)x], & \tilde{z} &= \frac{1}{\sqrt{2\omega}} [\tilde{p} - (\lambda - i\omega)\tilde{x}], \\ z^* &= \frac{1}{\sqrt{2\omega}} [p + (\lambda + i\omega)x], & \tilde{z}^* &= \frac{1}{\sqrt{2\omega}} [\tilde{p} - (\lambda + i\omega)\tilde{x}], \end{aligned} \quad (6.3)$$

while (6.2) leads to

$$\begin{aligned} x &= \frac{i}{\sqrt{2\omega}} (z - z^*), & \tilde{x} &= \frac{-i}{\sqrt{2\omega}} (\tilde{z} - \tilde{z}^*), \\ p &= \frac{1}{\sqrt{2\omega}} [(\omega - i\lambda)z + (\omega + i\lambda)z^*], & \tilde{p} &= \frac{1}{\sqrt{2\omega}} [(\omega - i\lambda)\tilde{z} + (\omega + i\lambda)\tilde{z}^*]. \end{aligned} \quad (6.4)$$

The formulae in the first column of (6.3) and (6.4) are exactly identical⁴ to those discussed earlier by the author in connection with the real physical damped oscillator [59–24–195]. The fundamental

$$[z, \bar{z}] = \hbar, \quad [\bar{z}^\dagger, z^\dagger] = \hbar. \quad (6.5)$$

As the transformation does not involve time explicitly, straightforward substitution of (6.2) into the dual Hamiltonian (3.6) leads to

$$H = \mathcal{H} + \mathcal{H}^*, \quad (6.6)$$

where

$$\mathcal{H} = (\omega - i\lambda)z\bar{z}. \quad (6.7)$$

This is precisely Dedene's Hamiltonian,^{*} which proves our assertion. Recalling that $\bar{z}^* = \bar{z}$, the canonical equations are conveniently written as^{**}

$$\begin{aligned} \dot{z} &= H_{,\pi_z} = \mathcal{H}_{,\pi_z}, & \dot{\pi}_z &= -H_{,z} = -\mathcal{H}_{,z}, \\ \dot{\bar{z}} &= H_{,\pi_{\bar{z}}} = \mathcal{H}_{,\pi_{\bar{z}}}, & \dot{\pi}_{\bar{z}} &= -H_{,\bar{z}} = -\mathcal{H}_{,\bar{z}}^*. \end{aligned} \quad (6.8)$$

For instance, with $\pi_z = i\bar{z}$ (i.e. $i\bar{z}^*$) and the algebra (6.5) one readily gets

$$\dot{z} = \mathcal{H}_{,i\bar{z}} = -i\omega z - \lambda z, \quad (6.9)$$

$$\dot{\bar{z}} = i\mathcal{H}_{,z} = i\omega\bar{z} + \lambda\bar{z}. \quad (6.10)$$

These equations must be supplemented with their complex conjugates, which stem from the other part of H , i.e. $\mathcal{H}^*$. Clearly, (6.9) describes the damped oscillator, while (6.10) pertains to the mirror image system. Using the transformations (6.4) one obtains [59, 195]

$$\dot{x} = p, \quad (6.11)$$

$$\dot{p} = -2\lambda p - \Omega^2 x, \quad (6.12)$$

and

$$\dot{\hat{x}} = \bar{p}, \quad (6.13)$$

$$\dot{\hat{p}} = 2\lambda\bar{p} - \Omega^2\bar{x}, \quad (6.14)$$

which are equivalent to (3.1) and (3.4) respectively, i.e. describe the damped oscillator and its mirror image, as it should.

^{*} There is a minor difference in sign, not in λ (as it might seem at first glance) but in ω . See for instance Dedene's equation of motion for z . Note further, that in [197] the bar denotes complex conjugation instead of real mirror conjugation.

^{**} In the symplectic formulation [197] the equations of motion arise from the Hamiltonian string given by the vector field $\xi_{\mathcal{H}} = -i(\omega - i\lambda)(z\partial_z - \bar{z}\partial_{\bar{z}})$, operating on a function $F = z$, resp. $\bar{z}$, as $\dot{F} = \xi_{\mathcal{H}}(F) = \xi_{\mathcal{H}} \lrcorner dF$; and similarly for $\mathcal{H}^*$.

6.1.2. Quantum mechanical spectrum

Let us briefly consider the quantum mechanics based on the Hamiltonian (6.6)–(6.7) with the algebra (6.5). Obviously, (6.5) is very similar (apart from a trivial scale factor $\sqrt{\hbar}$) to the commutation relations for the usual oscillator creation and annihilation operators. However, it should be observed that $\tilde{z}$ is not the usual Hermitian conjugate of z , but requires an additional (real) mirror conjugation. Following Dedene [197], we shall here call this formal conjugation generalized Hermitian, for short G-Hermitian.⁴ If we now define a G-Hilbert space with the G-inner product (compare e.g. with (3.34))

$$\langle \psi | \psi' \rangle = S_\nu \tilde{\psi}_{(\nu)} \psi'_{(\nu)} = S_\nu \tilde{\psi}_{(\nu)}^* \psi'_{(\nu)}, \quad (6.15)$$

the norm conventionally chosen as $\langle \psi | \psi \rangle = 1$, then the evaluation of the eigenvalues of the operators $N_- = \tilde{z}z/\hbar$ and $N_+ = z^\dagger \tilde{z}^\dagger/\hbar$ can be carried through in the standard manner (see e.g. [57, 160–164, 300, 313, 314]). In brief, using (6.5) for instance for z and $\tilde{z}$, one readily shows that these operators indeed are the annihilation respectively creation operator on the elements of the G-set $\{|n_-\rangle\}$, for which

$$N_- |n_-\rangle = n_- |n_-\rangle. \quad (6.16)$$

That is, one finds

$$\frac{z}{\sqrt{\hbar}} |n_-\rangle = c_n |n_- - 1\rangle, \quad \frac{\tilde{z}}{\sqrt{\hbar}} |n_-\rangle = c'_n |n_- + 1\rangle, \quad (6.17)$$

where c and c' are c -numbers depending on the value of n_- only. Then

$$n_- = \langle n_- | N_- | n_- \rangle = \left(\frac{z}{\sqrt{\hbar}} |n_-\rangle \right)^\sim \frac{z}{\sqrt{\hbar}} |n_-\rangle = |c_n|^2 \geq 0, \quad (6.18)$$

where we have used the normalization of the eigenvectors and $\tilde{c}_{n_-} = \tilde{c}_{n_-}^* = c_{n_-}^*$ because $\tilde{n}_- = n_-$. Hence, the lowest eigenvalue of N_- must be zero, and the spectrum is given by $n_- = 0, 1, 2, \dots$. Further, as usual $c_n = n^{1/2}$. In the same way one finds $c'_n = (n_- + 1)^{1/2}$. A similar analysis for the G-set $\{|n_+\rangle\}$ with $N_+ |n_+\rangle = n_+ |n_+\rangle$ shows that $n_+ = 0, 1, 2, \dots$. It should be noted that these sets of eigensolutions of H are completely independent of each other. Clearly, the Hamiltonians $\mathcal{H}$ and $\mathcal{H}^\dagger$, defined in (6.6), are defined on separate G-Hilbert spaces, say $G_\pm$. Notice, however, that it is not the physical oscillator and its mirror image that have been separated.

The separation of H into $\mathcal{H}$ and $\mathcal{H}^\dagger$ deserves a more careful consideration. Let us first allow for the usual Weyl symmetrization (see also [370]) for both the N_- and the N_+ oscillator. Hence,

$$H = \omega(\tilde{z}z + z^\dagger \tilde{z}^\dagger + \hbar\epsilon) - i\lambda(\tilde{z}z - z^\dagger \tilde{z}^\dagger), \quad (6.19)$$

where $\epsilon = 1$ for Weyl ordering, and $\epsilon = 0$ for so-called normal ordering [57, 299].⁵ Evidently, the ordering (if identical for both subsystems) does not show up in the damping part of the Hamiltonian.

⁴ Note e.g. the ρ is not G-Hermitian, but that as usual $\rho^\dagger = (\tilde{\rho})^\sim = \rho$, as is easily verified by explicit evaluation.

⁵ Any conceivable ordering is possible in principle. See esp. [57, 371].

This latter part, on the other hand, obviously allows for a nonzero separation constant, say δ , as follows:

$$H = \mathcal{H}_- + \mathcal{H}_+, \quad (6.20)$$

$$\mathcal{H}_- = \omega(\bar{z}z + \tfrac{1}{2}\hbar\epsilon) - i\lambda(\bar{z}z + \tfrac{1}{2}\hbar\delta), \quad (6.21)$$

$$\mathcal{H}_+ = \omega(z^\dagger \bar{z}^\dagger + \tfrac{1}{2}\hbar\epsilon) + i\lambda(z^\dagger \bar{z}^\dagger + \tfrac{1}{2}\hbar\delta). \quad (6.22)$$

The eigenspectra may be given conveniently in a single formula, setting $n_+ = n_- = n$, as

$$\mathcal{H}_n^{(\pm)} = (n + \tfrac{1}{2}\epsilon)\hbar\omega \pm i(n + \tfrac{1}{2}\delta)\hbar\lambda; \quad n = 0, 1, 2, \dots \quad (6.23)$$

Leaving aside here the precise observable physical significance of these spectra, it may be noted that the choice $\epsilon = 0$, $\delta = 2$, makes (6.23) an isomorphism of the Feshbach–Tikochinsky formula (3.42). On the other hand, with the choice $\epsilon = \delta = 0$ one obtains the Bohr spectrum as discussed by Dedene [197]. Further, the choice $\epsilon = \delta = 1$ reproduces the so-called Bohr–Sommerfeld–Maslow spectrum (see also [360]), another possibility discussed in [197]. Finally (but see furtheron), the choice $\epsilon = 1$, $\delta = 0$ leads to Bopp's spectrum [114].

6.1.3. Quantum fluctuations

The dynamics of the total system is described by the Schrödinger equation, that we presently write formally as

$$i\hbar|\dot{\rangle} = H|\rangle. \quad (6.24)$$

If the two subsystems, the dynamics of which is generated by $\mathcal{H}_\pm$, are independent (i.e. uncorrelated) at the initial time, they remain uncorrelated for ever in this complex-calculus-Bateman model. This is readily shown by setting $|\rangle = |+\rangle|-\rangle$ at any time, leading to

$$i\hbar|-\rangle' = \mathcal{H}_-|-\rangle, \quad i\hbar|+\rangle' = \mathcal{H}_+|+\rangle. \quad (6.25)$$

To be explicit, the G-Hermitian adjoint equations read

$$-i\hbar\langle -|' = \langle -|\tilde{\mathcal{H}}_-, \quad -i\hbar\langle +|' = \langle +|\tilde{\mathcal{H}}_+. \quad (6.26)$$

From (6.21) and (6.22) it is readily verified that the $\mathcal{H}_\pm$ are G-self adjoint, i.e. $\tilde{\mathcal{H}}_\pm = \mathcal{H}_\pm$. Let us consider now the equation of motion for an arbitrary function $F_- = F_-(z, \bar{z})$. Since usually $F_-|-\rangle$ is again a vector in G_- -space, using the G-inner product definition (6.15) together with (6.25) and (6.26) one obtains the equation of motion in the conventional commutator form. As the analogous conclusion, of course, applies to an $F_+ = F_+(z^\dagger, \bar{z}^\dagger)$, we have

$$\langle F_\pm \rangle' = -\frac{i}{\hbar} \langle [F_\pm, \mathcal{H}_\pm] \rangle. \quad (6.27)$$

More general functions $F(z, \bar{z}, z^\dagger, \bar{z}^\dagger)$ can almost always be written as sums of factorizing terms F_+F_- .

As an immediate application of (6.27) one obtains the correct equations

$$\langle z \rangle' = -i\omega\langle z \rangle - \lambda\langle z \rangle, \quad \langle \bar{z} \rangle' = i\omega\langle \bar{z} \rangle + \lambda\langle \bar{z} \rangle, \quad (6.28)$$

on G_- , and their complex conjugates independently on G_+ . As to be expected in view of (6.27), notice that the equations of motion are entirely independent of a particular choice of ϵ and δ . Considering the model's quantum mechanical fluctuations for the physical oscillator, however, the consequences of the above performed separation of the total system H into two noninteracting subsystems $\mathcal{H}_\pm$ are more or less fatal. Namely, since $F = z^\dagger z$ factorizes as $F = F_+ F_-$, one obtains $\langle z^\dagger z \rangle \equiv \langle z z^\dagger \rangle \equiv \langle z^\dagger \rangle \langle z \rangle$ at any time, or $\sigma_{z^\dagger z} \equiv \sigma_{zz^\dagger} \equiv 0$ (which implicitly defines the variances in the usual manner). By means of the transformation formulae (6.3) one easily infers the consequences for the variances in terms of the proper mechanical variables of the real physical damped oscillator. One finds:

$$\sigma_{pp}(t) + 2\lambda\sigma_{px}(t) + \Omega^2\sigma_{xx}(t) = 0. \quad (6.29)$$

Evidently, we are in fact not even able to let the system start off at $t = 0$ from the free oscillator ground state for which $\sigma_{px} = 0$ and $\sigma_{pp} = \Omega^2\sigma_{xx} = \hbar\Omega/2$. Of course, the fundamental reason for the failure of the formalism is the fact that neither of the separated Hilbert spaces $G_\pm$ contains the physical oscillator as an entity. And the separation became feasible in view of the fact that both z and $\bar{z}$ commute with both $z^\dagger$ and $\bar{z}^\dagger$, as the only nonzero commutators are given in (6.5). Since this algebra is connected with the basic Bateman–Feshbach–Tikochinsky commutators (3.8) only by means of the simple complex canonical transformation (6.1), we are back again at the origin of the problems that have already been discussed in section 3. Presently it is even more clear that the dual Hamiltonian on its own can never describe the correct oscillator quantum mechanics, as it can not even do so if $\lambda = 0$.

A possible obvious improvement of the situation could be obtained assuming that, before turning on the dissipation at $t = 0$, the system has been endowed on $t < 0$ with the appropriate quantum correlations by some other means (most likely the correct free oscillator Hamiltonian). In that case the factorization $|\pm\rangle = |+\rangle|-\rangle$ is not valid at $t = 0$, and, switching on the dual Hamiltonian at that time, the subsystems $|\pm\rangle$ remain correlated as they were, since from then on they are noninteracting again. With these initial assumptions the separation of the dynamics with $\mathcal{H}_-$ and $\mathcal{H}_+$ is in fact not allowed and one should consider the dynamics of the system as a whole, generated according to (6.24). Actually, this must be the situation that has been considered intuitively in the discussion of the Feshbach–Tikochinsky model in section 3. From (6.24), implying in fact (3.46), one again obtains (6.28) for the first moments, while the equations of motion for the variances now become^{*}

$$\dot{\sigma}_{z^\dagger z^\dagger} = 2i\omega\sigma_{z^\dagger z^\dagger} - 2\lambda\sigma_{z^\dagger z^\dagger}, \quad (6.30)$$

$$\dot{\sigma}_{zz} = -2i\omega\sigma_{zz} - 2\lambda\sigma_{zz}, \quad (6.31)$$

$$\dot{\sigma}_{z^\dagger z} = -2\lambda\sigma_{z^\dagger z}, \quad (6.32)$$

so that

^{*} It should be recalled that in the present formalism $\sigma_{z^\dagger z} \equiv \sigma_{zz^\dagger}$.

$$\sigma_{z'z'}(t) = \sigma_{z'z'}(0) e^{-2\lambda t}, \quad (6.33)$$

$$\sigma_{zz}(t) = \sigma_{zz}(0) e^{-2\lambda t}, \quad (6.34)$$

$$\sigma_{z'z}(t) = \sigma_{z'z}(0) e^{-2\lambda t}. \quad (6.35)$$

Using the transformation formulae (6.4) one readily obtains for the position and momentum spread of the damped oscillator the expressions

$$\sigma_{xx} = -(1/2\omega)[\sigma_{zz} - 2\sigma_{z'z'} + \sigma_{z'z'}], \quad (6.36)$$

$$\sigma_{pp} = (1/2\omega)[(\omega - i\lambda)^2\sigma_{zz} + 2\Omega^2\sigma_{z'z'} + (\omega + i\lambda)^2\sigma_{z'z'}], \quad (6.37)$$

Letting the system start at $t = 0$ from the proper free oscillator ground state, where $\sigma_{pp}(0) = \Omega^2\sigma_{xx}(0) = \hbar\Omega/2$ and $\sigma_{p'z'} = 0$, amounts to $\sigma_{zz}(0) = \hbar\lambda(\lambda - i\omega)/2\omega\Omega$, $\sigma_{z'z'}(0) = \hbar\lambda(\lambda + i\omega)/2\omega\Omega$ and $\sigma_{z'z}(0) = \hbar\Omega/2\omega$. Inserting (6.33)–(6.35) with these initial values into (6.36) and (6.37) finally yields

$$\sigma_{xx}(t) = \frac{\hbar}{2\Omega} e^{-2\lambda t} \left[1 + \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right], \quad (6.38)$$

$$\sigma_{pp}(t) = \frac{\hbar\Omega}{2} e^{-2\lambda t} \left[1 - \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right], \quad (6.39)$$

which are easily seen to be identical to (3.59) and (3.60). Hence, the thus obtained solutions from the complex calculus for the dual Bateman model are the same as those from the real calculus of section 3. Clearly, a prerequisite to obtain (3.59)–(3.60) or (6.38)–(6.39) is the preparation of the actual oscillator, by an expedient other than the dual Hamiltonian, initially in a proper quantum state: the Bateman Hamiltonian does not in any way have the power to do so. Therefore, we again face the troubles of the killing of the vacuum fluctuations.

6.2. Modified Bopp-theory

6.2.1. Separation of the dual oscillators

The above analysis has oncemore shown that the roots of the troubles are at the incorrect fundamental commutator (3.8). That is, the physical oscillator and its artificial, purely mathematical mirror adjoint do not commute, whereas the dynamical variables of the physical oscillator on its own do all commute with each other, even in the limit of vanishing friction. No doubt, rather radical measures must be taken in order to improve the case. An interesting hint in the proper direction may be found in Bopp's largely overlooked 1973-paper [114]. In the sequel we shall present a modified version of Bopp's treatment.

Bopp's Hamiltonian, closely related indeed to Bateman's dual model,^{*} is quite akin to the complex

^{*}It is curious to note that Bopp [114] apparently was unaware of the work of Bateman [11] and Morse and Feshbach [113] in this matter. Incidentally, probably due to the limited accessibility of [114], Feshbach and Tikochinsky [115, 116] in their turn were unaware of Bopp's paper. The present author's attention was drawn to [114] and [115] during a visit to Garching/München in 1979 [372]. See also [108, 109, 289].

symplectic form (6.6)–(6.7), or alternatively (6.20)–(6.22).^{***} As shown above, however, a separation of the total Hamiltonian into two commuting parts one of which solely pertains to the physical damped oscillator will be a highly nontrivial matter. Now, note that in the complex formulation, as in the real case in fact, the equations of motion for the damped oscillator and the mirror adjoint system are actually fully separated. See for instance (6.28). In the complex calculus the dynamics of the physical system on it own may be profitably written as

$$\dot{z} = -i \mathcal{H}_{,z^*}, \quad \dot{z}^* = i \mathcal{H}_{,z}, \quad (6.40)$$

for which the term improper (incomplete^{****}) Hamilton pair has been coined previously [196]. The only remnants of the artificial mirror adjoint system in these expressions are $\bar{z}$ and its complex (Hermitian) conjugate, although they do not appear in the resulting equations. The essential notion oncemore is that the physical oscillator and its mirror image are fundamentally independent in Newtonian mechanics. Evidently, the physical content of (6.40) remains unaltered if one simply disregards the bars. Admittedly, this may be considered as a bold leap.^{*****} Indeed, its substantiation must arise from the resulting physics, in particular the ensuing quantum mechanics, to be discussed hereafter.

The fundamental commutator (6.5) from the complex dual calculus now leads to

$$[a, a^\dagger] = 1, \quad (6.41)$$

where for convenience we have introduced the conventional variables $a = z/\sqrt{\hbar}$ and $a^\dagger = z^*/\sqrt{\hbar}$. The pertinent transformations back to the real coordinate and momentum for the damped oscillator are immediately obtained from the first columns of (6.3) and (6.4) as [59, 24, 195]

$$a = \frac{1}{\sqrt{2\hbar\omega}} [p + (\lambda - i\omega)x], \quad a^\dagger = \frac{1}{\sqrt{2\hbar\omega}} [p + (\lambda + i\omega)x], \quad (6.42)$$

and

$$x = i \sqrt{\frac{\hbar}{2\omega}} (a - a^\dagger), \quad p = \sqrt{\frac{\hbar}{2\omega}} [(\omega - i\lambda)a + (\omega + i\lambda)a^\dagger]. \quad (6.43)$$

By means of these formulae it is readily verified that (6.41) transforms into^{*****}

$$[x, p] = i\hbar. \quad (6.44)$$

^{***} Complete formal equivalence may be obtained by replacing $z/\sqrt{\hbar}$, $\bar{z}/\sqrt{\hbar}$ respectively $\bar{z}'/\sqrt{\hbar}$, $z'/\sqrt{\hbar}$ by Bopp's operators, a , $a^\dagger$ respectively b , $b^\dagger$. Note, however, that these operators are not intrinsically guaranteed to behave as ordinary Hermitian conjugates. This facet is only vaguely touched upon in [114]. See furtheron in the present text. Parenthetically, Bopp uses Weyl-ordering i.e. $\epsilon = 1$, but as it can be easily verified that the parameter ϵ will never appear in observable dynamics (because the pertinent part of the Hamiltonian will always only occur in commutator expressions), this possibility can be disregarded from the outset for convenience. The insignificance of the separation constant δ (see (6.21)–(6.22)) is less obvious, but will be pointed out in due course. See the footnote following (6.67).

^{****} The pair of equations (6.40) may be considered incomplete in the sense that, although merely describing the dynamics of the damped oscillator variables z and z^* , as it stands (6.40) still makes reference to the mirror image system through the occurrence of $\bar{z}$ and $\bar{z}^*$.

^{*****} It was precisely this bold leap that was not clearly uncovered as such in ref. [114].

^{*****} Incidentally, (6.42)–(6.43) are in fact only special numbers of a much broader group of transformations connecting (6.41) and (6.44). See esp. [258].

Within the present context, this is an important upshot. Namely, it means that henceforth it must be possible to construct a formulation for the damped oscillator which intrinsically includes the usual quantum mechanics in the limit of vanishing friction. The price that has to be paid for the correct operator algebra is the Hermiticity of the Hamiltonian, together with a somewhat unusual form of the equations of motion. From (6.40) it is seen at once that the latter become [59]

$$\dot{a} = -(i/\hbar)\mathcal{H}_{,a^*}, \quad \dot{a}^* = (i/\hbar)\mathcal{H}_{,a}^*, \quad (6.45)$$

with the generator for the damped oscillator now obviously given by

$$\mathcal{H} = \hbar(\omega - i\lambda)a^*a. \quad (6.46)$$

Clearly, $\mathcal{H}^* \neq \mathcal{H}$. The complex (non-Hermitian) generator (6.46) is in fact precisely Bopp's Hamiltonian H' (apart from the unobservable Weyl ordering constant). For more on non-Hermitian Hamiltonians, often in connection with the so-called optical model potential, one should consult e.g. [10, 24, 59, 102, 164, 218, 300, 303, 373–386]. It should be noticed that (6.46) indeed reduces to the well-known harmonic oscillator Hamiltonian if $\lambda \downarrow 0$ [160–164, 300, 303] (see also [190, 192–194]). Finally, for the sake of completeness, let us explicitly state that for the mirror image oscillator one now separately has

$$\dot{\bar{a}} = -(i/\hbar)\bar{\mathcal{H}}_{,\bar{a}^*}, \quad \dot{\bar{a}}^* = (i/\hbar)\bar{\mathcal{H}}_{,\bar{a}}^*, \quad (6.47)$$

$$\bar{\mathcal{H}} = \hbar(\omega + i\lambda)\bar{a}^*\bar{a}, \quad (6.48)$$

which indeed can be considered as the formal (real) mirror conjugates of (6.45)–(6.46). Note that in effect we have really separated now the physical system from its artificial adjoint.

6.2.2. Quantization

In order to investigate the quantum mechanics of the Bopp model for the damped oscillator, one constructs an ordinary Hilbert space (in lieu of a G-space), wherein the quantum dynamics is given by the Schrödinger equation

$$i\hbar|\dot{\rangle} = \hbar(\omega - i\lambda)a^\dagger a|\rangle, \quad (6.49)$$

its usual Hermitian adjoint given by

$$-i\hbar\langle\dot{|} = \langle|\hbar(\omega + i\lambda)a^\dagger a. \quad (6.50)$$

Evidently, the eigenvalues of the Hamiltonian become

$$\mathcal{H}_n = n\hbar(\omega - i\lambda); \quad n = 0, 1, 2, \dots \quad (6.51)$$

Compare this with (6.23). Note, however, the differing physical meanings here and there, connected with the differing Hilbert spaces. A first thing to observe from (6.49)–(6.50) is that, in consequence of the non-Hermiticity of the Hamiltonian, the initial norm of the quantum states will not be conserved in the course of time. Hence, in order to remain within the framework of the conventional

probabilistic interpretation, expectation values will be defined in the general sense

$$\langle F \rangle = \langle |F| \rangle / \langle | \rangle, \quad (6.52)$$

F being an arbitrary operator. Let us now introduce the pseudo-density operator w as the projection operator $w = | \rangle \langle |$, the state vectors obeying (6.49)–(6.50). The equation of motion for w becomes (see esp. [270])

$$i \hbar \dot{w} = \mathcal{H}w - w\mathcal{H}^\dagger. \quad (6.53)$$

It is interesting, not in the least for comparison with Bopp's original treatment [114], to write (6.53) in the number of representation[¶] where (see e.g. [57, 58, 270, 298, 299, 302, 387])

$$w = \sum_{n,m} |n\rangle w_{nm} \langle m|, \quad (6.54)$$

$$w_{nm} = \langle n|w|m\rangle. \quad (6.55)$$

One easily obtains

$$\dot{w}_{nm} = -i\omega(n-m)w_{nm} - \lambda(n+m)w_{nm}. \quad (6.56)$$

Notice that, as expected, the trace of the pseudo-density matrix $\{w_{nm}\}$ decays and, secondly, that diagonal and off-diagonal elements do not mix as time proceeds. Even stronger, in their evolution the elements w_{nm} do not interact at all. For the latter reason, (6.56) is almost trivially solved by

$$w_{nm}(t) = \rho_{nm}(0) e^{-i\omega(n-m)t - \lambda(n+m)t}, \quad (6.57)$$

where we have chosen to set $w_{nm}(0) = \rho_{nm}(0)$, which for the moment being just defines $\rho_{nm}(0)$. At arbitrary times $t \geq 0$, the appropriate density matrix $\{\rho_{nm}\}$ is defined as

$$\rho_{nm}(t) = w_{nm}(t) / \text{trace } w(t), \quad (6.58)$$

so that $\text{trace } \rho(t) = 1$, while

$$\langle F \rangle = \text{trace } \rho F = \sum_{n,m} \rho_{nm} F_{mn}, \quad (6.59)$$

as usual. Note, however, that the present theory provides us with w rather than ρ . In order to obtain $\rho(t)$ one must specify particular initial conditions, i.e. $\rho_{nm}(0) = w_{nm}(0)$. Therefore, in principle, no equation of motion for the proper density operator of the damped oscillator exists within the present context that does not involve the system's initial state.^{¶¶}

[¶] Be aware of the difference between the present and the usual (free oscillator) number states. In principle, the latter are a linear combination of the former, and vice versa.

^{¶¶} A possibly more precise relation between this facet and the observed non-Markovian behaviour of the quantum mechanical transmission line (section 5) remains as yet unexplored.

It is illustrative (and rather unexpected) that the above description allows for the paradoxical stationary states, quite similar to those observed in the Süssmann–Hasse–Albrecht model [129, 253, 254] (see (2.7)–(2.12)) and in Kostin's theory [198, 199] (see section 11).^{*} Namely, if one sets $w_{nm}(0) = \rho_{nm}(0) = \delta_{n,m} \delta_{n,n_0}$, one obtains by (6.57) $w(t) = \exp(-2\lambda n_0 t)$, so that (6.58) leads to

$$\rho_{nm}(t) = \delta_{n,m} \delta_{n,n_0}. \quad (6.60)$$

Evidently, $\dot{\rho}_{nm}(t) = 0$ represents an equation of motion that essentially involves the special pure state initial condition. Moreover, as such it is not unambiguous as, for instance, the equation $i\hbar \dot{\rho}_{nm} = (n - m)\rho_{nm}$, with completely arbitrary time scale, would do as well.

6.2.3. The density operator

Having started in fact from a classically damped oscillator (see e.g. (6.11)–(6.12) and (2.13)–(2.14)) one should, in view of Ehrenfest's theorem, pay special attention to classical initial conditions. Stationary quantum states as such do not behave very classical. For instance, $\langle p \rangle = 0$ for any n since $\langle n|a|n \rangle = \langle n|a^\dagger|n \rangle = 0$. The latter formulae immediately indicate the way out. In order to obtain states for which $\langle p \rangle \neq 0$, which in regard of (2.13)–(2.14) is a prerequisite for observing classical dissipation [108, 109], one should consider the most classical states of quantum theory [184, 302], namely the coherent^{***} states $|\alpha\rangle$. By definition, the states $|\alpha\rangle$ have the eigenproperty

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad (6.61)$$

α being a complex continuous valued number. These eigenstates of the annihilation operator have been extensively studied (see esp. [57, 298, 299, 301, 302, 371]), notably in relation to coherent optical phenomena in laser theory (see e.g. [20–22, 35–37, 58, 92, 184, 388]). They form a (over-)complete set of not strictly orthogonal states, properties expressed by

$$(1/\pi) \int |\alpha\rangle \langle \alpha| d^2\alpha = 1, \quad (6.62)$$

$$|\langle \alpha|\beta \rangle| = \exp(-\frac{1}{2}|\alpha - \beta|^2), \quad (6.63)$$

where $d^2\alpha = d \operatorname{Re} \alpha d \operatorname{Im} \alpha$, and where the integration covers the entire complex α -plane. In analogy with (6.54), the density operator ρ can be expressed in the coherent state representation, for example, as^{****}

$$\rho = \int P(\alpha) |\alpha\rangle \langle \alpha| d^2\alpha, \quad (6.64)$$

where the diagonal, P - or Glauber distribution $P(\alpha)$ is a so-called quasi-probability density.^{*****} The

^{*} It should be noted that at the end of the semi-infinite transmission line (section 5) shows a similar feature, as one might in principle prepare the reservoir's incoming modes system in an arbitrary pure number state so that $\langle B_1^\dagger B_1 \rangle = n_k$ (see e.g. (5.60)). Of course, the notion of a temperature becomes moot in that case.

^{**} These minimum uncertainty states [293, 296] are also frequently denoted as Glauber states (see e.g. [302]).

^{***} The function $P(\alpha)$ is the antinormally ordered associated density $\rho^{(A)}$ of Louisell [57], who also discusses extensively different orderings (see also [371]). As such, $P(\alpha)$ is one-to-one related to normally ordered averages.

^{****} The reader should not get confused with the notions "pseudo-density" (used in connection with w , see (6.53)) and "quasi-density" (used for P). The latter is meant to say that P may take on negative values, in contrast with a proper classical density.

coherent states are a specific linear combination of the number states, namely

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha a^\dagger)|0\rangle = \exp(-\frac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \quad (6.65)$$

Let us now assume that the damped oscillator is at $t = 0$ prepared in a pure coherent state, say $|\alpha_0\rangle$. Hence, corresponding to $P(\alpha) = \delta(\alpha - \alpha_0) \delta(\alpha^* - \alpha_0^*)$ in (6.64), one has $\rho^{(0)}(0) = |\alpha_0\rangle \langle \alpha_0|$, so that using (6.65)

$$\rho_{nm}^{(0)}(0) = \exp(-|\alpha_0|^2) \alpha_0^n \alpha_0^{*m} / \sqrt{n!m!}. \quad (6.66)$$

Substituting this expression for $\rho_{nm}(0) = w_{nm}(0)$ into (6.57), one easily computes the sum of diagonal elements of $\{w_{nm}(t)\}$, with the result

$$\text{trace } w(t) = \exp[-|\alpha_0|^2(1 - e^{-2\lambda t})]. \quad (6.67)$$

Inserting (6.67) into (6.58) one obtains^{**}

$$\rho_{nm}^{(0)}(t) = \frac{\alpha_0^n \alpha_0^{*m}}{\sqrt{n!m!}} \exp[-|\alpha_0|^2 e^{-2\lambda t} - i\omega(n-m)t - \lambda(n+m)t]. \quad (6.68)$$

It should, however, be pointed out that (6.68) as it stands is not the most useful density matrix one might wish. Namely, by means of the transformations (6.43) it is not difficult to verify that a pure damped oscillator coherent state implies $\sigma_{xx} = \hbar/2\omega$ and $\sigma_{pp} = \hbar\Omega^2/2\omega$. Since both of these variances are greater than their free oscillator ground state values, we face the difficulty (which is quite akin to one earlier noted in the symplectic treatment along with (6.29): but be aware of the differences) that the system can not even start off with the usual minimum uncertainty.

6.2.4. The master equation

Of course, more general initial conditions than the damped pure coherent state are possible, but they would spoil the derivation of (6.68). Even worse, they would spoil Ehrenfest's quantum-classical correspondence theorem. For instance, it is a matter of somewhat detailed but in principle straightforward calculations to show that with initial Gaussian conditions, the distribution $P(\alpha, t)$ remains a Gaussian (see also [62]), but leads to mean values, e.g. for $\langle x(t) \rangle$, that deviate from the correct classical ones by contributions that do in general not vanish in the limit $\hbar \downarrow 0$. To be more explicit,^{***} the analysis leads to the conclusion that these unwanted terms are zero only in three cases. First, if $\lambda = 0$ (for arbitrary initial conditions). Second, if $x(0) = p(0) = 0$ (for any λ and arbitrary initial variances). Third, if the system is initially in a pure damped oscillator coherent state (for all λ and arbitrary $p(0)$ and $x(0)$). Hence, on the one hand we see that more general initial states than the coherent ones, if incorporated in Bopp's model from the outset, generally violate Ehrenfest's principle and prohibit the derivation of

^{**} This seems the appropriate place (see also the footnote preceding (6.40)) to point out that the ultimate results for $\rho_{nm}(t)$, i.e. (6.69), is independent of any additional "separation" constant (δ) in the anti-Hermitian part of the Hamiltonian (6.46). It would merely multiply both the numerator and the denominator in (6.58) with the same extra real exponentially decaying (or growing, if one wishes) factor, which thus drops out.

^{***} Generally speaking the additional terms are proportional to $\lambda t e^{-\lambda t} \cdot \{x(0); p(0)\} \cdot \{\Delta\sigma_{\mu\nu}(0)/\hbar\}$, where $\Delta\sigma_{\mu\nu} = \sigma_{\mu\nu} - \sigma_{\mu\nu}^{(0)}$; $\mu, \nu = x, p$; $\sigma_{\mu\nu}^{(0)}$ represents the damped oscillator coherent state value. Further, also the relation $\langle x \rangle = \langle p \rangle$ is altered by terms of a similar structure.

(6.68). On the other hand, considering more general conditions by averaging in (6.68) over α_0 preserves the required quantum-classical correspondence, but in effect questions the validity of (6.49) as a proper Schrödinger equation for the damped oscillator. The latter point of view will be given more emphasis further on (section 7).

Let us then introduce the rather general density matrix

$$\rho_{nm} = \int P(\alpha_0) \rho_{nm}^{(0)} d^2\alpha_0, \quad (6.69)$$

$\rho_{nm}^{(0)}$ given by (6.68), which is easily shown to satisfy the master equation [16–24, 34–37, 55–58, 62, 82, 114, 195, 196, 200, 270, 271, 274, 289, 332, 333, 340, 389–393]

$$\dot{\rho}_{nm} = -i\omega(n-m)\rho_{nm} - \lambda(n+m)\rho_{nm} + 2\lambda\sqrt{(n+1)(m+1)}\rho_{n+1,m+1}. \quad (6.70)$$

Remarkably, the result (6.70) does not explicitly depend on the initial conditions. Nevertheless, it does involve the particular initial states used in its derivation in an implicit way. For instance, this facet is rather well illustrated noticing that (6.70) does not allow for the pure number steady states (6.60), contrary to (6.56) on which it is based.^{*} The ground state $n=0$ (which is a so-called random phase coherent state [57]) is a noteworthy exception. It is also worth noting that Bopp [114], in fact, gave a somewhat heuristic derivation of the diagonal part of (6.70).^{**}

$$\dot{\rho}_{nn} = -2\lambda n\rho_{nn} + 2\lambda(n+1)\rho_{n+1,n+1}, \quad (6.71)$$

and solved it subject to the initial condition $\rho_{nn}(0) = \delta_{n,2}$. Notice that, as in (6.56), diagonal and off-diagonal elements do not mix in (6.70) and (6.71). Further, (6.71) is the well-known classical master equation for radioactive nuclear decay (see e.g. [34] p. 272).

6.2.5. Quantum fluctuations

From (6.69) it is easy to obtain explicit formulae for the moments. The first moments immediately lead to the correct classical expectation values $\langle x(t) \rangle$ and $\langle p(t) \rangle$. For the variances one finds (use (6.59))

$$\sigma_{a'a'}(t) = \sigma_{a'a'}(0) e^{2i\omega t - 2\lambda t}, \quad (6.72)$$

$$\sigma_{aa}(t) = \sigma_{aa}(0) e^{-2i\omega t - 2\lambda t}, \quad (6.73)$$

$$\sigma_{aa'}(t) = \sigma_{aa'}(0) e^{-2\lambda t} + (1 - e^{-2\lambda t}), \quad (6.74)$$

$$\sigma_{a'a}(t) = \sigma_{a'a}(0) e^{-2\lambda t}, \quad (6.75)$$

where for instance $\sigma_{aa}(0) = \langle \alpha_0^2 \rangle - \langle \alpha_0 \rangle^2$, etc. Using (6.43) one obtains the relations

$$\sigma_{xx} = -(\hbar/2\omega)[\sigma_{aa} - (\sigma_{a'a} + \sigma_{aa'}) + \sigma_{a'a'}], \quad (6.76)$$

^{*} This is in line with the known nonexistence (within the trace class) of the P -distribution for pure number states [211, 212, 371].

^{**} Bopp's w_n has the same significance as the present ρ_{nn} .

$$\sigma_{pp} = (\hbar/2\omega)[(\omega - i\lambda)^2 \sigma_{aa} + \Omega^2(\sigma_{a^*a} + \sigma_{aa^*}) + (\omega + i\lambda)^2 \sigma_{a^*a^*}], \quad (6.77)$$

$$\sigma_{px} = (\hbar/2\omega)[(\lambda + i\omega)\sigma_{aa} - \lambda(\sigma_{a^*a} + \sigma_{aa^*}) + (\lambda - i\omega)\sigma_{a^*a^*}], \quad (6.78)$$

where σ_{px} is again the Weyl form (see (3.53)). Notice the essential difference between (6.76)–(6.77) and (6.36)–(6.37), which principally stems from the fact that presently a^* and a do not commute. By means of (6.42) it is further observed that the usual free oscillator ground state initial uncertainty presently amounts to $\sigma_{aa}(0) = \lambda(\lambda - i\omega)/2\omega\Omega$, $\sigma_{a^*a}(0) = \lambda(\lambda + i\omega)/2\omega\Omega$, $\sigma_{a^*a^*}(0) = (\Omega/2\omega) - \frac{1}{2}$, $\sigma_{aa^*}(0) = (\Omega/2\omega) + \frac{1}{2}$. Inserting (6.72)–(6.75) with these initial values into (6.76)–(6.78) finally yields^{*}

$$\sigma_{xx}(t) = \frac{\hbar}{2\Omega} e^{-2\lambda t} \left[1 + \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right] + \frac{\hbar}{2\omega} (1 - e^{-2\lambda t}), \quad (6.79)$$

$$\sigma_{pp}(t) = \frac{\hbar\Omega}{2} e^{-2\lambda t} \left[1 - \frac{\lambda}{\omega} \sin 2\omega t + 2 \frac{\lambda^2}{\omega^2} \sin^2 \omega t \right] + \frac{\hbar\Omega^2}{2\omega} (1 - e^{-2\lambda t}), \quad (6.80)$$

$$\sigma_{px}(t) = -\frac{\hbar\lambda\Omega}{\omega^2} e^{-2\lambda t} \sin^2 \omega t - \frac{\hbar\lambda}{2\omega} (1 - e^{-2\lambda t}). \quad (6.81)$$

Comparison of these results, in particular of (6.79)–(6.80), with (6.38)–(6.39) for the (complex) dual model, shows the important upshot that has been achieved by imposing the correct quantum mechanical algebra (6.41)–(6.44): the quantal vacuum obviously has not been turned off by the frictional forces [24], i.e. Heisenberg's principle is not violated here.

6.3. Summary

The time-independent Bateman Hamiltonian has been related to the symplectic damped oscillator Hamiltonian by means of a complex canonical transformation. The symplectic Hamiltonian can be separated, but the two uncorrelated parts are mixtures of the physical oscillator and its mirror image. Heisenberg's principle is obviously violated. Then, following Bopp, the separation of the physical system and its artificial adjoint has been enforced. The resulting quantum mechanics has the correct algebra. Unfortunately, the derivation of the density matrix equation (6.70) from the postulated Schrödinger equation (6.49) involves somewhat unsatisfactory subtleties. As it stands, (6.70) guarantees the validity of both Ehrenfest's and Heisenberg's principle.

7. Complex phase space quantization

7.1. Preliminaries

As noted already in the Historical survey (section 2), each of the theories discussed so far intrinsically involves the postulate of the existence of a Schrödinger equation for the dissipative system. In essence, four types of dissipative Schrödinger equations have been encountered. First, a nonlinear Schrödinger

^{*} Note that presently $\dot{\sigma}_{aa}(0) \neq 0$. See e.g. (3.59) and [136, 156, 180].

equation (of the Süssmann–Hasse–Albrecht type:^{***} see section 2). Second, a multidimensional equation (two dimensional for the Bateman–Feshbach–Tikochinsky dual model, as well as for Dedene’s complex symplectic formulation: see sections 3 resp. 6; infinite dimensional for the Stevens–Yurke semi-infinite transmission line model: see section 5). Third, an explicitly time-dependent Hamiltonian model (namely that of Bateman, Caldirola and Kanai^{***}: see section 4). And fourth, a Schrödinger equation with a noteworthy non-Hermitian Hamiltonian (a modified presentation of Bopp’s model: see section 6). However, it is in fact well-known that the general state of an interacting subsystem will not be a pure quantum state,^{****} but rather a statistical mixture to be described by a density operator [16–25, 50, 55–58, 61, 87–89, 200, 270–287, 298–303, 310–314, 375, 387–393]. For that reason, the significance of the assumed dissipative Schrödinger equations has been questioned, in particular by Burzlaff^{*****} [289] and the author [184, 24, 195].

In the preceding section it has been shown that Bopp’s Schrödinger equation (6.49) has certain undesirable implications (notably the possible violation of the correspondence principle), whereas the density matrix equation (6.70) as such is much more satisfying. Of course, in questioning (6.49) one should not overlook the evident importance of the correct fundamental commutator in Bopp’s (modified) theory. As should be clear, the latter (i.e. (6.44)) is intimately connected with the separation of the damped oscillator and its mirror image, which are (quantum mechanically uncomfortably) intertwined in the complex (symplectic) dual model. On the other hand, the symplectic formulation is intentionally on good terms with the notions of classical and quantal phase spaces (symplectic manifolds and Hilbert spaces), employing the methods of geometrical quantization [197, 355–369]. Let us therefore return to Dedene’s complex symplectic calculus for the damped oscillator and introduce the phase space description from the very beginning (i.e. already in the classical formulation) in an explicit manner. Then, imposing the correct algebra (as in Bopp’s theory) and eliminating the mirror image system from the description, yields a continuity (or: master) equation in the reduced phase space of the physical oscillator. Only after these (*classical*) preparations, the system will be quantized. That is, rather than seeking a pure state Schrödinger representation (which need not necessarily exist), one allows from the outset for arbitrary mixed states represented by a density operator. Of course, the latter representation contains the former in special cases (e.g. if $\lambda = 0$).

7.2. Reduction of the dual phase space

Consider the Liouville equation for the closed complex dual system of oscillator and mirror image, in the phase space spanned by $z, \bar{z}, \pi_z = i\bar{z}^*$ and $\pi_{\bar{z}} = iz^*$. It may be written as (e.g. [1])^{*****}

$$R_t = -i[H, R], \quad (7.1)$$

where R is the total phase space density, the Hamiltonian $H = \mathcal{H} + \mathcal{H}^*$ is given in (6.6)–(6.7) and where $-i[\cdot, \cdot]$ can be identified as classical Poisson brackets.^{*****} More explicitly, (7.1), reads in view of (6.5):

^{***} Two other types of nonlinear Schrödinger equations will be discussed further on: Hasse’s non-Hermitian model (section 9) and Kostin’s Schrödinger–Langevin or fluid dynamical equation (section 11).

^{****} One could also catalogue here the Stevens transmission line model after the time-dependent canonical transformation.

^{*****} Within the present context we shall define a pure quantum state as a state obeying any sort of Schrödinger equation.

^{*****} It is amusing to note that Bopp promoted Burzlaff’s thesis [289].

^{*****} In another notation [197, 355–369] it reads $R_t = -\xi_H(R)$, with $\xi_H = \xi_x + \xi_{x^*}$; ξ_x is the vector field specified in the footnote at (6.8).

^{*****} Recall that H with the algebra (6.5) is quantum mechanically wrong, not classically. Therefore, one should notice that the preliminary manipulations here are in classical phase space.

$$R_{,t} = i H_{,z} \cdot R_{,z} - i R_{,z} \cdot H_{,z} + \text{conj.} , \quad (7.2)$$

where "conj." denotes complex conjugation. Note that $H^* = H$. As it stands, (7.2) is not convenient for the intended reduction of phase space. Therefore, we rather cast it into the form of a continuity equation. Eq. (7.2) is evidently equivalent to*

$$R_{,t} = i(H_{,z} \cdot R)_{,z} - i(RH_{,z})_{,z} + \text{conj.} . \quad (7.3)$$

Recalling e.g. (6.8), noticing that for the classical Dedene Hamiltonian $\mathcal{H}^* = \bar{\mathcal{H}}$, and trivially interchanging the second term on the r.h.s. of (7.3) with its complex conjugate, one readily casts (7.3) into the form

$$R_{,t} = i(\mathcal{H}_{,z} \cdot R)_{,z} + i(\bar{\mathcal{H}}_{,z} \cdot R)_{,\bar{z}} + \text{conj.} , \quad (7.4)$$

which clearly has the structure $R_{,t} = (R_{,t}^{(1)} + \text{mirror conj.}) + \text{complex conj.} = (R_{,t}^{(1)} + \text{complex conj.}) + \text{mirror conj.}$. Recalling (6.40), one again notes that the first term (plus its complex conj.) on the r.h.s. of (7.4) completely describes the phase space flow for the damped oscillator on its own, whereas the remaining terms just pertain to the mirror image. Therefore, the elimination of the artificial mirror system from (7.4) is most easily accomplished by an integration with respect to $\bar{z}$ and $\bar{z}^*$, with the appropriate measure. The latter need not be specified (but see [195]) as it suffices to indicate this operation by a classical projector S_z , so that

$$R = S_z R + (1 - S_z) R , \quad (7.5)$$

$$S_z R = \rho , \quad (7.6)$$

where ρ represents the reduced phase space density, also called the relevant part of R (see e.g. [16–25]). However, since the integration is in fact not over a genuine canonical pair, one must redefine the remaining subspace algebra. As in Bopp's theory (section 6), one naturally imposes the conventional algebra on the physical oscillator's subspace. That is, *en passant* rescaling the dynamical variables again as $z/\sqrt{\hbar} = a$ and $z^*/\sqrt{\hbar} = a^*$, one is concerned henceforth with the fundamental commutator (6.41), or (6.44), and the transformation formulae (6.42)–(6.43). Eq. (7.4) becomes

$$\rho_{,t} = \frac{i}{\hbar} (\mathcal{H}_{,a} \cdot \rho)_{,a} - \frac{i}{\hbar} (\rho \mathcal{H}_{,a}^*)_{,a^*} , \quad (7.7)$$

with

$$\mathcal{H} = \hbar(\omega - i\lambda)a^*a , \quad (7.8)$$

the latter in fact given earlier in (6.46). It should be noted that as yet (7.7) still is a classical, c-number equation so that the ordering of quantities is actually immaterial. However, it will turn out that considering the quantal analogue of (7.7), one should specify the ordering of $\mathcal{H}$, ρ and $\mathcal{H}^*$ even for

* Previous footnote.

linear systems if $\mathcal{H}^* \neq \mathcal{H}$. As is well known, in principle such ambiguities cannot be circumvented in quantizing classical systems (i.e. in adding quantum fluctuations^{***}), although in many cases the appropriate choice can be inferred from intuitive arguments or imposing additional constraints (such as general covariance; see e.g. [228, 231, 394, 395]). In [195] we have given a somewhat intuitive reasoning concerning the detailed structure of the quantal version of (7.7). Presently, we shall rather leave the ordering undecided until we have extracted some more definite consequences from the theory.^{****} The final conclusion will be the same as in [195].

The so-called complex Hamiltonian $\mathcal{H}$ featuring in (7.7) generates the classical, deterministic dynamics according to the equations of motion (6.45), which are obviously akin to the usual canonical equations. But it should be noted that, as a consequence of (6.45), this complex variable theory differs in a number of respects essentially from the usual canonical theory, although it naturally reduces to the latter if $\mathcal{H}^* = \mathcal{H}$ (i.e. for the damped oscillator if $\lambda \downarrow 0$). For instance, although $\mathcal{H}$ does not explicitly depend on time it need not be a constant of the motion. It also need not represent the proper mechanical energy of the system. Further, $\mathcal{H}$ is in general not invariant under a complex canonical transformation.^{*****} Therefore, the theory should rather be called quasi-Hamiltonian. Nevertheless, $\mathcal{H}$ does play the role of a kind of complex potential determining the trajectories in complex phase space. Moreover, it should be noted that the continuity equation in phase space is always more general than the standard density equation for the incompressible phase space fluid, which expresses Liouville's theorem (compare e.g. (7.3) and (7.2)). The latter is contained in the former in the case of closed, nondissipative Hamiltonian systems only (see esp. [28, 37]). Hence, if one is not starting from a complete, fundamentally reversible Hamiltonian description of, so to speak, the entire universe (and one is even almost never interested in anything of that sort [35–37, 402]), then the Liouville-continuity equation^{*****} is a sound starting point. In particular this seems relevant if one is aiming at a probabilistic description of the system. And, to be clear, we shall here entirely conform to the conventional, essentially probabilistic interpretation of quantum mechanics. Incidentally, other approaches to the (quantum) dynamics of open (non-Hamiltonian) systems which are closely related to the above basic ideas, may be found for instance in [61, 271, 289–291, 403–416]. In many cases a (Markovian) semi-group property (i.e. in fact a master equation; see also e.g. [73, 417]) is introduced as the fundamental dynamical postulate (see esp. [271], and references contained therein).

^{***} This is quite analogous to adding Langevin noise sources to deterministic equations in the classical theory of stochastic processes (e.g. [21, 34, 37, 56, 59, 396]); the observable results depend for instance on the choice of coordinates. See esp. [34], and also [62, 397]. In general, the ambiguities can be cleared up if one has a well-defined microscopic physical model for the process.

^{****} At the risk of overburdening, let us note that from (7.7) it should be clear that neither an ordering constant (ϵ) nor a separation constant (δ ; see, for both ϵ and δ , (6.23) and the footnote preceding (6.68)) will influence the results arising from the present formulation.

^{*****} Such a transformation is conventionally defined as preserving the basic form of the equations of motion (6.45). See also e.g. [1]. In the one dimensional complex case this is almost trivial, and those transformations that leave the Poisson brackets (6.41) invariant rather form a subgroup (in the standard real variable theory this invariance is deductive). Note that (6.43) in fact is not a proper complex canonical transformation in the classical theory (but see [24, 190, 398]). The complex character of $\mathcal{H}$ generally breaks the invariance of the classical Poisson brackets under the particular time-dependent (canonical) transformation that solves the dynamical problem. Of course, there is no fundamental need to formulate classical mechanics in terms of brackets rather than in differential notation, not even in order to quantize the system (see e.g. [164], esp. p. 355; of course, the importance of the bracket representation in the historical development of (conservative) quantum mechanics should not be depreciated [160, 399–401]). For some more details on the above aspects of the classical complex variable theory, see the section on the Hamilton–Jacobi formalism.

^{*****} The nomenclature in the literature is not unique. Although somewhat sloppy, if there can be no misunderstanding, the continuity equation will sometimes also be called conventionally Liouville equation, namely if the dynamics is intrinsically deterministic (see e.g. [333]). The master equation will always be of the more general continuity type. In quantum mechanics it is now and then denoted as Liouville–von Neumann equation, or more appropriately for open systems (with mixed states) as Nakajima–Zwanzig equation (e.g. [16–19, 23]).

7.3. Quantization

Let us then consider the quantization of the Liouville-continuity equation (7.7), as it stands.^{*} To this end we let the classical dynamical variables become operators in a conventional Hilbert space. The only nonzero fundamental commutator is given in (6.41), i.e. $[a, a^\dagger] = 1$, which fixes the algebra (see for a particularly clear treatment Louisell's book [57]). Since the ordering of $\mathcal{H}$, ρ and $\mathcal{H}^\dagger$ is not irrelevant in quantum mechanics, we introduce a real continuous ordering parameter $0 \leq c \leq 1$ and write the quantum version of (7.7) as

$$\rho_t = \frac{i}{\hbar} \{ (1-c) \mathcal{H}_{,a^\dagger} \rho + c \rho \mathcal{H}_{,a^\dagger} \}_{,a} - \frac{i}{\hbar} \{ c \mathcal{H}_{,a}^\dagger \rho + (1-c) \rho \mathcal{H}_{,a}^\dagger \}_{,a^\dagger}. \quad (7.9)$$

By means of the general operator relations (e.g. [57])

$$[a, F] = F_{,a^\dagger}, \quad [a^\dagger, F] = -F_{,a}, \quad (7.10)$$

F being an arbitrary well-behaved operator function, one readily casts (7.9) into commutator form, with the result^{**}

$$\dot{\rho} = -\frac{i}{\hbar} [a^\dagger, (1-c)[a, \mathcal{H}]\rho + c\rho[a, \mathcal{H}]] + \text{conj.} \quad (7.11)$$

First of all, notice that by using the fundamental commutator (6.41) in going from (7.9) to (7.11) one simply endows the mechanics with the usual algebra. This is a *conditio sine qua non* for obtaining the well-known nondissipative results. Further, note that (7.11) by construction guarantees the conservation of probability, i.e. $(\text{trace } \rho)^\cdot = 0$. In [59] it has been proved that the classical $\mathcal{H}$ can always be taken real whenever there is no dissipation, that is if the process is time-reversible (see also [300, 331, 418]). Consider then (7.11) with Hermitian $\mathcal{H}$, i.e. $\mathcal{H}^\dagger = \mathcal{H} = H$. It is not difficult to show that for Hamiltonians of the form (see e.g. [57, 164, 195, 302])

$$H = \hbar \omega(t) a^\dagger a + a^\dagger s(t) + a s^\dagger(t), \quad (7.12)$$

$\omega(t)$ and $s(t)$ being arbitrary scalar functions of time, (7.11) reduces to the usual commutator equation

$$\dot{\rho} = -(i/\hbar)[H, \rho], \quad (7.13)$$

independent of a specific choice for the ordering parameter c . The Hamiltonian (7.12) generates the most general dynamics of a driven linear oscillator. For more general Hamiltonians the above reduction

^{*}Requiring the usual oscillator quantum mechanics in the limit $\lambda = 0$, the continuity equation in terms of x and p leads to a violation of the uncertainty principle whenever $\lambda \neq 0$. Moreover, in terms of the real variables the ordering arguments of [195] cannot be used as there is no Hamiltonian whatsoever.

^{**}We have chosen to write $\dot{\rho}$ rather ρ_t in expressions where the explicit differential representation of the operators is not shown. See also section 6.

usually depends on c or ceases to be valid (in principle for nonlinear systems).^{*} It is quite interesting, however, to observe that for the rather basic Hamiltonian

$$H = \hbar \omega_a(t) a^\dagger a + \hbar \omega_b(t) b^\dagger b + g(t) a^\dagger b + g^*(t) ab^\dagger, \quad (7.14)$$

which describes two interacting oscillators, the reduction to (7.13) is still valid, even if one generalizes (7.14) straightforwardly to an arbitrary number of oscillators. See also the section on the quantum optics model.

Let us now return to (7.11) with the Hamiltonian (7.8). The Hermitian part of this Hamiltonian clearly falls into the class (7.12)–(7.14). Inserting (7.8) into (7.11) one obtains

$$\dot{\rho} = -i\omega[a^\dagger a, \rho] - \lambda(1-c)([a^\dagger, a\rho] + [\rho a^\dagger, a]) + \lambda c([a, a^\dagger \rho] + [\rho a, a^\dagger]). \quad (7.15)$$

Although, of course, any averaged quantity can be calculated directly from (7.15), it is convenient and instructive to represent it in terms of a quasi-probability density. This can be done in different ways (see e.g. [20–22, 57, 182–184, 298–302]). Here we choose the diagonal P -distribution, defined earlier in (6.64), which implies c -number averaging of normally ordered products $(a^\dagger)^k a^l$. Following Louisell [57], one then assumes ρ to be in antinormally ordered form and by means of (7.10) arranges all terms on the r.h.s. of (7.15) into that order, whereupon one may simply replace the operators by their coherent state eigenvalues (see e.g. (6.61)). This leads to

$$P_{,\alpha} = (\lambda + i\omega)(\alpha P)_{,\alpha} + (\lambda - i\omega)(\alpha^* P)_{,\alpha^*} - 2\lambda c P_{,\alpha\alpha^*}, \quad (7.16)$$

which is a Fokker–Planck equation with a nonpositive diffusion coefficient. The mean values $\langle \alpha(t) \rangle$ and $\langle \alpha^*(t) \rangle$ immediately lead back to the correct results in terms of the real dynamical variables x and p , satisfying Ehrenfest's theorem. The equations of motion for the second moments are easily found from (7.16) to be

$$\langle \alpha^2 \rangle' = -2(\lambda + i\omega)\langle \alpha^2 \rangle, \quad (7.17)$$

$$\langle \alpha^{*2} \rangle' = -2(\lambda - i\omega)\langle \alpha^{*2} \rangle, \quad (7.18)$$

$$\langle \alpha^* \alpha \rangle' = -2\lambda \langle \alpha^* \alpha \rangle - 2\lambda c. \quad (7.19)$$

Hence, in the long time limit $\langle \alpha^2 \rangle \rightarrow 0$, $\langle \alpha^{*2} \rangle \rightarrow 0$, while $\langle \alpha^* \alpha \rangle \rightarrow -c$. Since further $\langle \alpha \rangle \rightarrow 0$ and $\langle \alpha^* \rangle \rightarrow 0$, one obtains for the variances (compare with (6.72)–(6.75)):

$$\sigma_{a^\dagger a}(\infty) = \sigma_{aa}(\infty) = 0, \quad (7.20)$$

$$\sigma_{a^\dagger a}(\infty) = -c, \quad \sigma_{aa}(\infty) = 1 - c. \quad (7.21)$$

^{*} Quite generally, Hamiltonians with more complicated contributions in terms of the creation and annihilation operators can be considered as effective Hamiltonians, which are actually inferred from perturbational considerations based on a more fundamental, elementary interaction. For bosons, the latter has the basic structure $b^\dagger a$ and $a^\dagger b$. See for instance the quantum optics model [19–23, 35–37, 41, 50, 55–58, 87–93, 200, 275, 297, 340, 354, 388–392] and, mutatis mutandis (electrons are fermions), the theory of superconductivity [387, 419–421]. See also [195], the present section on the quantum optics model and eq. (7.14).

By means of (6.76)–(6.77) the variances in terms of the real space variables are then easily calculated, with the result

$$\sigma_{pp}(\infty) = \Omega^2 \sigma_{xx}(\infty) = (1 - 2c)\hbar\Omega^2/2\omega. \quad (7.22)$$

Evidently, in order for these results to make sense at least $c \leq \frac{1}{2}$, that is $0 \leq c \leq \frac{1}{2}$. Moreover, forming the uncertainty product one obtains

$$\sigma_{pp}(\infty) \sigma_{xx}(\infty) = (1 - 2c)^2 \hbar^2 \Omega^2 / 4\omega^2. \quad (7.23)$$

If $\lambda \downarrow 0$, this uncertainty product approaches the value $(1 - 2c)^2 \hbar^2 / 4 \geq \hbar^2 / 4$. The latter inequality only holds if $c \geq 1$ or $c \leq 0$. Hence, in order to fulfil the well-established principles of conservative quantum mechanics in the limit of vanishing friction, one must conclude that $c = 0$.

With this value for c , the basic master equation (7.11) reads

$$\dot{\rho} = -\frac{i}{\hbar} [a^\dagger, [a, \mathcal{H}]\rho] + \frac{i}{\hbar} [\rho[\mathcal{H}^\dagger, a^\dagger], a]. \quad (7.24)$$

which precisely is the master equation invented previously in [195]. In that paper, in fact with an eye at the pseudo-density operator equation (6.53), it was argued that structurally one expects $\mathcal{H}$ respectively $\mathcal{H}^\dagger$ to stand on the left respectively right hand side of the density operator. This argument immediately leads to (7.24). See also the footnote preceding (7.9).

7.4. The master equation

Setting $c = 0$ in (7.15) gives

$$\dot{\rho} = -i\omega[a^\dagger a, \rho] - \lambda([a^\dagger, a\rho] + [\rho a^\dagger, a]). \quad (7.25)$$

This upshot is in fact well-known, notably in quantum optics (the laser well below threshold; see esp. [19–22, 57, 92]). It is further in line with Lindblad's general form for the generator of the quantum dynamical semi-group in the Markovian approximation^{*} [271]. And it also agrees with the results of Burzlaff [289] and of Huguenin [290].^{**}

Consulting (6.54)–(6.55), with w replaced by the proper density operator ρ , one readily writes (7.25) in the number representation. It yields

$$\dot{\rho}_{nm} = -i\omega(n - m)\rho_{nm} - \lambda(n + m)\rho_{nm} + 2\lambda\sqrt{(n+1)(m+1)}\rho_{n+1, m+1}. \quad (7.26)$$

Comparison with (6.70) immediately shows that (7.26) is identical to the result from the modified Bopp-theory (see also [290]). Hence, the formulae (6.72)–(6.81) for the dynamical variances can be carried over without any modification.

^{*} Notice that presently we have not separately postulated this approximation. Compare e.g. also with [19, 24]. Remember that there is evidence that the Markov approximation is not valid in the quantum domain (i.e. if $k_B T \ll \hbar\omega$). See section 5. See also e.g. [38, 274, 281].

^{**} In the transformation to real coordinates, note that their $p = \dot{x} + \lambda x$ differs somewhat from our mechanical $p = \dot{x}$. Further, it is worth observing that Burzlaff explicitly invokes Lindblad's theorem.

Let us now transform back from the complex variables to the physical real ones. Inserting the transformation (6.42) for a and $a^\dagger$ into (7.25) one obtains

$$\dot{\rho} = -\frac{i}{\hbar} [H, \rho] - i \frac{\lambda}{\hbar} ([x, \rho p] - [p, \rho x]) - \frac{\lambda}{2\hbar\omega} [p, [p, \rho]] - \frac{\lambda\Omega^2}{2\hbar\omega} [x, [x, \rho]] - \frac{\lambda^2}{2\hbar\omega} ([x, [p, \rho]] + [p, [x, \rho]]) , \quad (7.27)$$

where (see also [24, 190, 195, 398])

$$H = \frac{1}{2}p^2 + \frac{1}{2}\lambda[p, x]_+ + \frac{1}{2}\Omega^2 x^2 . \quad (7.28)$$

The master equation (7.27) is comfortably cast into the more general form [196]

$$\dot{\rho} = -\frac{i}{\hbar} [H_0, \rho] - i \frac{\lambda}{\hbar} [x, [p, \rho]_+] + \frac{1}{\hbar^2} (D_{px} + D_{xp})[p, [x, \rho]] - \frac{1}{\hbar^2} D_{xx}[p, [p, \rho]] - \frac{1}{\hbar^2} D_{pp}[x, [x, \rho]] , \quad (7.29)$$

where

$$H_0 = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2 x^2 \quad (7.30)$$

represents the free oscillator Hamiltonian, and where the diffusion coefficients

$$D_{xx} = \frac{\hbar\lambda}{2\omega} , \quad D_{pp} = \frac{\hbar\lambda\Omega^2}{2\omega} , \quad D_{px} + D_{xp} = -\frac{\hbar\lambda^2}{\omega} \quad (7.31)$$

can be defined by means of the so-called generalized Einstein relations (see e.g. [57, 58, 184]). If we define as usual the drift operators according to

$$\langle x \rangle' = \langle D_x \rangle , \quad \langle p \rangle' = \langle D_p \rangle , \quad (7.32)$$

we infer from (7.29) that

$$D_x = p , \quad D_p = -2\lambda p - \Omega^2 x , \quad (7.33)$$

which once more confirms the validity of Ehrenfest's theorem. The generalized Einstein relation defining for instance D_{px} can now be explicated as [57, 58, 184]

$$2D_{px} = -\langle D_p x \rangle - \langle p D_x \rangle + \langle p x \rangle' , \quad (7.34)$$

which using (7.29) and (7.33) yields

$$D_{px} = \frac{1}{2}(D_{px} + D_{xp}) - \frac{1}{2}i\hbar\lambda , \quad (7.35)$$

while $D_{xp} = D_{px}^*$. Hence,

$$D_{xp} - D_{px} = i\hbar\lambda . \quad (7.36)$$

This formula, already mentioned in the Historical survey as (2.22), was reported originally in [24], where it played a crucial role in the derivation of the master equation. It represents a typical example of a quantum mechanical fluctuation–dissipation relation (see e.g. [21, 24, 57–59, 92] and the Introduction), connecting the (operator) noise (expressed in the diffusion constants) with both the fundamental commutator (expressed in Planck’s constant) and the damping (expressed in the friction constant).^a

The density operator equation (7.29) will considerably gain in clarity if it is represented in terms of the Wigner-distribution function [24, 57, 195, 277, 278, 299]. In the coordinate representation this quasi-probability density may be given as

$$W(p, x) = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipy/\hbar} \langle x - \frac{1}{2}y | \rho | x + \frac{1}{2}y \rangle dy. \quad (7.37)$$

In two previous papers [24, 195] we have transformed (7.29) into its Weyl-ordered c-number equivalent directly by means of (7.37), invoking the fundamental eigenvalue relations in the coordinate representation. Let us presently rather do the calculations in the coherent state representation. Actually, we can then start from (7.16), with $c = 0$, for the P -distribution. In this representation, the latter is related to the Weyl–Wigner function by the Gaussian convolution (see e.g. [21, 56, 57, 92, 182–184, 298, 299, 371])

$$W(\alpha) = \frac{2}{\pi} \int \exp(-2|\alpha - \beta|^2) P(\beta) d^2\beta. \quad (7.38)$$

From this integral relation one easily deduces the transformation

$$P_{,t} = \mathcal{D}(\partial/\partial\alpha, \alpha)P \rightarrow W_{,t} = \mathcal{D}(\partial/\partial\alpha, \alpha + \frac{1}{2}\partial/\partial\alpha^*)W, \quad (7.39)$$

where $\mathcal{D}$ represents an arbitrary well-behaved differential operator function. By means of (7.39) one transforms (7.16) into

$$W_{,t} = (\lambda + i\omega)(\alpha W)_{,\alpha} + (\lambda - i\omega)(\alpha^* W)_{,\alpha^*} + \lambda W_{, \alpha\alpha^*}. \quad (7.40)$$

Reintroducing now the mechanical variables according to (6.42)–(6.43) by means of

$$\alpha = \frac{1}{\sqrt{2\hbar\omega}} [p + (\lambda - i\omega)x], \quad \alpha^* = \frac{1}{\sqrt{2\hbar\omega}} [p + (\lambda + i\omega)x], \quad (7.41)$$

it is an elementary exercise to show that (7.40) becomes [195]

$$W_{,t} = -pW_{,x} + \Omega^2 xW_{,p} + 2\lambda(pW)_{,p} + \frac{\hbar\lambda}{2\omega} W_{,xx} - \frac{\hbar\lambda^2}{\omega} W_{,px} + \frac{\hbar\lambda\Omega^2}{2\omega} W_{,pp}. \quad (7.42)$$

This is a linear Fokker–Planck equation with diffusion coefficients precisely given by (7.31), while the

^a If, following for instance the Introduction, we relate the diffusion coefficients to Gaussian white noise operators, (7.36) leads to $\langle \xi_i(t + \tau), \xi_p(t) \rangle = 2i\hbar\lambda \delta(\tau)$. In fact, in such a form the fluctuation–dissipation relation was mentioned earlier in the present context in [59], while it is well-known in quantum optics in the theory of quantum mechanical Langevin equations (e.g. [21, 50, 54, 57, 58, 87, 88, 92, 200, 388, 422, 423]).

drift functions correspond with (7.33). It is interesting to compare this equation with (5.76), that has been associated heuristically with Svin'in's theory, and where $D = \hbar\lambda\Omega$ at zero temperature. See also [24, 258, 288]. The result (7.42) was first obtained in 1977 [424] and published later in [195].

7.5. Quantum fluctuations

From (7.42) one easily computes the equations of motion for the variances (3.53)–(3.55). They are

$$\dot{\sigma}_{px} = -2\lambda\sigma_{px} + \sigma_{pp} - \Omega^2\sigma_{xx} + D_{px} + D_{xp}, \quad (7.43)$$

$$\dot{\sigma}_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} + 2D_{pp}, \quad (7.44)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px} + 2D_{xx}, \quad (7.45)$$

the diffusion coefficients being given by (7.31). It is worthwhile to briefly compare (7.43)–(7.45) with the corresponding formulae of the other theories treated so far in the present article. First, (2.15)–(2.17) of the Süssmann–Hasse–Albrecht nonlinear frictional potential models can not be cast into the form (7.43)–(7.45) whatsoever. This fact is closely related to the observed absence of asymptotic stability about the steady state (section 2). Next, (3.56)–(3.58) of the Bateman–Feshbach–Tikochinsky time-independent dual Hamiltonian model correspond to the general form of (7.43)–(7.45) with all diffusion coefficients identically zero.[¶] This fact is intimately connected with the recognized violation of Heisenberg's uncertainty relation (section 3). Exactly the same comments serve (4.28)–(4.30) of the Bateman–Kanai–Caldirola time dependent Hamiltonian model^{¶¶} (section 4). Further, in the Stevens–Yurke semi-infinite transmission line model σ_{pp} could not be evaluated explicitly, so that the comparison becomes moot in that case (section 5). As already noted above, Svin'in's modified noise-added Kanai model, i.e. (5.70)–(5.72), corresponds to zero diffusion coefficients save $D_{pp} = \hbar\lambda\Omega$, which has been seen to suffice in order to remedy the violation of the uncertainty principle (section 5). Dedene's complex symplectic formulation is basically equivalent to the Bateman–Feshbach–Tikochinsky theory and, hence, can be given a like commentary (section 6). Finally, coming to the modified version of Bopp's complex Hamiltonian theory (section 6), which has been purposely endowed with the proper commutator, it is easily verified that the explicit solutions (6.79)–(6.81) exactly satisfy (7.43)–(7.45) with Dekker's diffusion coefficients (7.31).^{¶¶¶}

The steady state solutions of (7.43)–(7.45) read

$$\sigma_{xx}(\infty) = [D_{pp} + (\Omega^2 + 4\lambda^2)D_{xx} + 2\lambda(D_{px} + D_{xp})]/2\lambda\Omega^2, \quad (7.46)$$

$$\sigma_{pp}(\infty) = (D_{pp} + \Omega^2 D_{xx})/2\lambda, \quad (7.47)$$

$$\sigma_{px}(\infty) = -D_{xx}, \quad (7.48)$$

which on substituting the specific values (7.31) result in

[¶] This model in fact also implies $D_{xp} - D_{px} = 0$, which evidently violates the fundamental-fluctuation dissipation relation (7.36).

^{¶¶} Here one will find $D_{xp} - D_{px} = i\hbar\lambda e^{-2\lambda t}$. See e.g. (4.19). Compare with the previous footnote.

^{¶¶¶} Actually, this is to be expected in consequence of the identity of (7.26) – and, hence, of (7.42) – and (6.70).

$$\sigma_{xx}(\infty) = \frac{\hbar}{2\omega}, \quad \sigma_{pp}(\infty) = \frac{\hbar\Omega^2}{2\omega}, \quad \sigma_{px}(\infty) = -\frac{\hbar\lambda}{2\omega}. \quad (7.49)$$

These expressions are, of course, in line with (6.79)–(6.81). The mechanical energy in the steady state becomes

$$E(\infty) = \frac{\hbar\Omega^2}{2\omega} \geq \frac{1}{2}\hbar\Omega. \quad (7.50)$$

Note the equipartition over the two degrees of freedom in real phase space. The equilibrium uncertainty product takes the value⁴⁹

$$\sigma_{pp}(\infty) \sigma_{xx}(\infty) = \frac{\hbar^2\Omega^2}{4\omega^2} \geq \frac{1}{4}\hbar^2, \quad (7.51)$$

and the Gaussian uncertainty relation (2.18) is easily verified. Manifestly, the present theory ensures the survival of the uncertainty principle under the action of dissipative forces. The results (7.49)–(7.51) confirm the earlier expressed general insight (see e.g. the Introduction) that dissipation goes hand in hand with additional fluctuations.

Expanding $\sigma_{xx}(\infty)$ from (7.49) for small λ in a Taylor series, one finds

$$\sigma_{xx}(\infty) = \frac{\hbar}{2\Omega} \left[1 + \frac{\lambda^2}{2\Omega^2} + o\left(\frac{\lambda^4}{\Omega^4}\right) \right]. \quad (7.52)$$

It is interesting to compare this with the semi-infinite transmission line result (5.79), from which it differs essentially in the absence of a term linear in λ [110]. Leaving the small friction regime, the above formulae (7.49)–(7.51) show a catastrophe if the system approaches critical damping $\lambda = \Omega$. It should, therefore, be expected that the present model is invalidated for strongly damped oscillators. See, ⁵⁰ [258] and section 8. In the end, let us note that the present approach seems easily extensible to more general systems than the simple linear oscillator.

7.6. Summary

The complex symplectic Hamiltonian has been used to construct the classical Liouville-continuity equation in the complex dual phase space. The separation of the physical system and its artificial adjoint has been accomplished by an integration over the classical phase space of the latter. The quantization of the reduced continuity equation results in a quantum mechanical master equation with the correct

⁴⁹ Using the explicit dynamical solutions (6.79)–(6.80) it is straightforward to show that $\sigma_{pp}(t) \sigma_{xx}(t) \geq \frac{1}{4}\hbar^2$ for any $t \geq 0$. If $\lambda \neq 0$, the equality sign applies only at $t = 0$, due to the chosen initial condition.

⁵⁰ Actually, it may well be that in the strong damping limit the linearity of the macroscopic oscillator model cannot in fact be substantiated by microscopic ab initio calculations and a simple application of Ehrenfest's theorem. Note, for instance, that in the mechanical transmission line model (section 5) the friction constant λ is proportional to the square root of the tension $\mathcal{F}$ of the string. An increasing bias force generally implies increasing nonlinearities. On the other hand, it is known [24, 258] that quantizing the damped oscillator by means of other complex variables than the ones used here, leads to a variety of diffusion coefficients, among which nonsingular ones. Incidentally, a similar ambiguity is also observed in ab initio calculations depending on the nature of certain approximations (in quantum optics, for example, the rotating wave approximation), or on the basic model. See esp. [55] and also the next section.

algebra. In the specific case of the linearly damped harmonic oscillator this master equation turns out to be identical to the one obtained from Bopp's modified theory. The intimate relation between the dissipation, quantal fluctuations and diffusion coefficients in the Wigner equation has been emphasized. The diffusion coefficients guarantee the validity of Heisenberg's uncertainty relation.

8. The quantum optics oscillator

8.1. Introductory remarks

This section will be devoted to the discussion of an explicit physical model in order to shed some further light on the results of the complex phase space quantization. To this end we will start *ab initio* with the description of a closed system, consisting of a tagged, perfect oscillator as the open system of interest and a (thermal) reservoir. It is essential to note that either subsystem is considered *quantum mechanically* from the outset.

Aside from a general reference to the theory of open systems (e.g. [16–25, 35–37, 92, 93]), it is worthwhile to mention the following literature in particular. The earlier investigations of linear damping phenomena (e.g. [425]) followed the lines set out by Pauli [426] with emphasis on the assumption of repeatedly random phases [18, 19] (see also [427–429] and [524–529]). The damped harmonic oscillator has been treated in this spirit for instance in [430], although later work (e.g. [427–429]) has shown that the random phase assumption and Pauli's treatment of the master equation can be replaced by the more general Nakajima–Zwanzig projection operator techniques [16–19]. For a rather clear outline of this method the reader is referred to e.g. [19] (esp. section 3b), from which it can be concluded that we should anticipate at least three essential approximations. First, the large reservoir limit. Second, a Markovian approximation. And third, a weak coupling limit. Apart from using the density operator approach in the Schrödinger picture one can of course also perform the calculations on the dynamical variables themselves in the basically equivalent Heisenberg picture (see e.g. [24, 200]). Eliminating the observables of the reservoir from the Heisenberg equations of motion, quantum mechanical Langevin equations for the observables of the considered open subsystem are obtained (see esp. [24, 38, 50, 57, 87, 88, 92, 200, 297, 354, 431]). Presently, however, we will rather use a somewhat different technique, in essence following the theory of laser operation as given by Scully and Lamb, dealing with the density operator in the interaction picture [389–392].

The system we wish to investigate is an adapted version of the laser model depicted in fig. 8.1. See also [20, 22]. The radiation oscillator (the system of interest) consists of an electromagnetic field in an optical resonator (see e.g. [57, 58, 375, 432–439]), which on its own is perfectly lossless (in engineering terms, it has an infinite quality factor). Cavity losses are introduced by the inescapable coupling of the

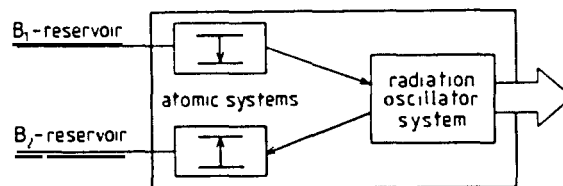


Fig. 8.1. Model for the quantum optics oscillator. The radiation system is coupled to the thermal atomic bath, containing both upper (B_1) and lower (B_2) state atoms (after [20]).

radiation mode to the material constituents of the resonator. Since there will be a variety of atoms present in the walls, the mirrors and the medium inside the cavity, the losses are broadband in the frequency domain. In the present treatment we disregard a possible continuous, nonthermal pumping of a subset of the atoms within the cavity into particular excited states, which might result in actual laser operation of the device. The atomic system will be considered to be a thermal bath. To be precise, the atomic reservoir will be assumed to be in thermal equilibrium on its own, at temperature T , before the interaction with the tagged radiation oscillator is switched on. Of course, the nonthermally excited electromagnetic mode will in principle disturb the thermal equilibrium of the bath. Nevertheless, we will assume the atomic system to remain stationary in its equilibrium. This assumption requires, qualitatively: (i) the tagged oscillator's initial excitation to be not too high above thermal, (ii) the interactions to be weak, (iii) the atomic system's relaxation to equilibrium to be fairly rapid and (iv) the number of available atoms to be very large. Since quantum mechanical atoms, unlike the classical point-"atoms" in the mechanical string of section 5, basically have a finite size, an infinite number of atoms necessarily implies an infinite size of the system. In order to place the present model in proper perspective with the classical semi-infinite transmission line, notice that the field limit is presently implied by starting with quantal atoms, i.e. wave functions, from the outset."

8.2. The model Hamiltonian

Consider the total Hamiltonian

$$H = H_r + H_R + H_I, \quad (8.1)$$

where r = radiation, R = atomic reservoirs, I = interaction, and [22, 300]

$$H_r = \frac{1}{2} \epsilon_0 \int d^3r [E^2 + (c \nabla \times A)^2], \quad (8.2)$$

$$H_R = \int d^3r \psi^\dagger \left(\frac{1}{2m} p^2 + V \right) \psi, \quad (8.3)$$

$$H_I = -(e/m) \int d^3r \psi^\dagger A \cdot p \psi. \quad (8.4)$$

This Hamiltonian can be derived from a slightly more general one [22, 300] in the Coulomb gauge $\nabla \cdot A = 0$, neglecting the interelectronic Coulomb energy (certainly allowed in a one-electron atom) and disregarding a term quadratic in the vector potential"" (see for discussions on that point e.g. [440–459]).

Inserting the usual expansion for the vector potential operator""

"To be very clear, one quantum mechanical bound state (an atom) implies a field, the wave function, that can be expanded in terms of a discrete, infinite set of eigenfunctions. This can be compared with the normal mode expansion (5.23) for the continuous, finite length transmission line. Next, a dense spectrum can be obtained by considering an infinite ensemble of atoms with continuously varying eigenspectra. This is equivalent to taking the infinite length limit for the transmission line.

"Since, essentially due to the weak coupling assumption, in the end only those atoms contribute that are at resonance with the radiation field, the interaction Hamiltonian (8.4) is presently equivalent to $\sim eE \cdot r$, even without making the dipole approximation.

"In [22] we have used Ω for volume. However, throughout the present text, Ω represents the free oscillator frequency. Recalling that L denotes the length of the transmission line in section 5, A^3 is taken as the volume.

$$A = \sum_k \sqrt{\frac{\hbar}{2\epsilon_0 \Lambda^3 \omega_k}} \epsilon_k [a_k e^{i\mathbf{k} \cdot \mathbf{r}} + a_k^\dagger e^{-i\mathbf{k} \cdot \mathbf{r}}], \quad (8.5)$$

where $\Lambda^3 = \text{volume}$ and ϵ_k represents the polarization vector, together with the corresponding formula for the electric field operator (see e.g. [300]) into (3.2), one finds the well-known expression for the energy of the free radiation system in terms of the photon creation and annihilation operators. Since by assumption the free radiation system can oscillate only at its resonant frequency, say $\omega_0 = \Omega$, the general expression reduces to⁸

$$H_r = \hbar \Omega a_0^\dagger a_0, \quad (8.6)$$

while the interesting fundamental commutator reads

$$[a_0, a_0^\dagger] = 1. \quad (8.7)$$

Now the atomic system must be considered. The field operator ψ will be assumed to be the sum of field operators ϕ_μ , which refer to a single nucleus. In line with the omission of the interelectronic Coulomb energy from the Hamiltonian, each separate atom is assumed to contain only one single active electron. The one-electron one-atom field operators ϕ_μ are further expanded in terms of the eigenstates $u_{\mu j}$. That is,

$$\psi(r) = \sum_{\mu j} c_{\mu j} u_{\mu j}(r). \quad (8.8)$$

The electronic annihilation and creation operators $c_{\mu j}$ and $c_{\mu j}^\dagger$ obey the usual anticommutator fermion algebra (see e.g. [92, 160–166, 300, 303, 313, 421, 463]):

$$[c_{\mu i}, c_{\nu j}^\dagger]_\pm = \delta_{\mu\nu} \delta_{ij}, \quad (8.9)$$

all other anticommutators being zero. Inserting (8.8) into (8.3) leads to

$$H_R = \sum_{\mu j} \mathcal{E}_{\mu j} c_{\mu j}^\dagger c_{\mu j}, \quad (8.10)$$

where $\mathcal{E}_{\mu j}$ is the j th energy eigenvalue of the unperturbed Hamiltonian $H_0 = p^2/2m + V$ for the μ th atom.

The model is considerably simplified by confining ourselves to two-level atoms.⁹ The levels are labelled by $j = 1, 2$ for the upper respectively lower level (see also fig. 8.1), and it is convenient to introduce the operators

$$b_\mu = c_{\mu 2}^\dagger c_{\mu 1}, \quad b_\mu^\dagger = c_{\mu 1}^\dagger c_{\mu 2}. \quad (8.11)$$

⁸ A minute's thought will show that this assumption is in fact highly nontrivial. It will be taken here for granted, as usual. See for further reading e.g. [22, 58, 92, 460–462].

⁹ This assumption does not seem to be very dramatic regarding the ultimate results, especially not since the dense spectrum in the relevant frequency domain can still be constructed for the essentially infinite atomic ensemble.

The raising operator $b_\mu^\dagger$ lifts the electron of the μ th atom from the lower to the upper level, the lowering operator b_μ does the reverse. In the one electron subspace one has

$$c_{\mu 1}^\dagger c_{\mu 1} + c_{\mu 2}^\dagger c_{\mu 2} = 1, \quad (8.12)$$

and one easily infers that the newly defined raising and lowering operators again behave as simple fermion operators. That is, the only nonzero anticommutator reads

$$[b_\mu, b_\nu^\dagger]_+ = \delta_{\mu\nu}. \quad (8.13)$$

One further computes the commutator

$$[b_\mu, b_\mu^\dagger] = c_{\mu 2}^\dagger c_{\mu 2} - c_{\mu 1}^\dagger c_{\mu 1}. \quad (8.14)$$

Using (8.11)–(8.14) and setting $\omega_\mu = (\mathcal{E}_{\mu 1} - \mathcal{E}_{\mu 2})/\hbar$ into (8.10), one finds the free reservoir Hamiltonian

$$H_R = \sum_\mu \hbar \omega_\mu b_\mu^\dagger b_\mu. \quad (8.15)$$

Now inserting the electromagnetic field representation (8.5), again with only $\omega_k = \omega_{k_0} = \omega_0 = \Omega$, and the matter field expansion (8.8) into the interaction Hamiltonian (8.4), one obtains

$$H_I = \sum_\mu (b_\mu^\dagger + b_\mu)(g_{\mu 0} a_0^\dagger + g_{\mu 0}^* a_0), \quad (8.16)$$

where we have further assumed that the wave functions of separate atoms do not have spatial overlap and use has been made of the well-known commutation relation $i(\hbar/m)p = [r, H_0]$. In the dipole approximation the coupling factors in (8.16) can be written as

$$g_{\mu 0} = i \sqrt{\frac{\hbar}{2\epsilon_0 A^3 \Omega}} \omega_\mu \exp(-i \mathbf{k}_0 \cdot \mathbf{r}_\mu) \mathcal{P}_\mu, \quad (8.17)$$

wherein

$$\mathcal{P}_\mu = \int d^3r u_{\mu 2}^* (\boldsymbol{\epsilon}_0 \cdot \mathbf{e} \mathbf{r}) u_{\mu 1} \quad (8.18)$$

represents the dipole matrix element. Finally, making the usual quantum optical rotating wave approximation (i.e. neglecting processes like $b_\mu^\dagger a_0^\dagger$ and $b_\mu a_0$; see e.g. [55, 58, 92]) the interaction Hamiltonian (8.16) becomes

$$H_I = \sum_\mu (g_{\mu 0} a_0^\dagger b_\mu + g_{\mu 0}^* a_0 b_\mu^\dagger). \quad (8.19)$$

The rotating wave approximation typically disregards nonresonant processes. Therefore, in general it

does not significantly influence the balance between the resonant emission and absorption processes, that is the energy dissipation. However, as most clearly shown for instance by Agarwal [55, 464], omitting the antiresonant interactions entails a neglect of the proper classical frequency shift from Ω to $\omega = (\Omega^2 - \lambda^2)^{1/2}$.^{*} This defect is most easily remedied by expressing the Hamiltonian from the outset in terms of the damped oscillator creation and annihilation operators $a^\dagger$, a rather than the free $a_0^\dagger$, a_0 and making in fact the rotating wave approximation in terms of the reduced frequency ω rather than Ω . This procedure has the additional advantage that we can readily compare the outcome with the previous section. Moreover, it clearly explains why the results will fail on approaching critical damping $\lambda = \Omega$: in that case the rotating wave approximation in terms of ω ($\downarrow 0$) becomes moot. From (6.42) one finds the relations^{**}

$$\begin{aligned} a_0 &= \frac{1}{2\sqrt{\omega\Omega}} [(\omega + \Omega - i\lambda)a + (\omega - \Omega + i\lambda)a^\dagger], \\ a_0^\dagger &= \frac{1}{2\sqrt{\omega\Omega}} [(\omega + \Omega + i\lambda)a^\dagger + (\omega - \Omega - i\lambda)a]. \end{aligned} \quad (8.20)$$

Inserting (8.20) into (8.16) and (8.6), again making the rotating wave approximation (i.e. neglecting processes like $b_\mu^\dagger a^\dagger$, $b_\mu a$, $a^{\dagger 2}$ and a^2), and finally adding (8.15), one obtains the total model-Hamiltonian^{***}

$$H = \hbar\omega a^\dagger a + \sum_\mu \hbar\omega_\mu b_\mu^\dagger b_\mu + \sum_\mu (g_\mu a^\dagger b_\mu + g_\mu^* a b_\mu^\dagger). \quad (8.21)$$

Notice that this Hamiltonian is of the form (7.14). The coupling factors in (8.21) are easily expressed in terms of the $g_{\mu 0}$ and its conjugate:

$$g_\mu = \frac{i}{2} \sqrt{\frac{\hbar}{2\varepsilon_0 A^3 \omega}} \omega_\mu \left[\left(1 + \frac{\omega}{\Omega} + i\frac{\lambda}{\Omega}\right) \exp(-i\mathbf{k}_0 \cdot \mathbf{r}_\mu) \mathcal{P}_\mu + \left(1 - \frac{\omega}{\Omega} - i\frac{\lambda}{\Omega}\right) \exp(i\mathbf{k}_0 \cdot \mathbf{r}_\mu) \mathcal{P}_\mu^* \right]. \quad (8.22)$$

Since we shall be confined to second order perturbation theory,^{****} only $|g_\mu|^2$ will occur in the resulting formulae. Summing over the atoms implies an integration over space.^{*****} As the atoms will be considered to be spatially homogeneously distributed throughout the volume of the radiation system,

^{*} Notice, however, that if the spatial exponential in $g_{\mu 0}$, (8.17), cannot be disregarded, the contributions from the nonresonant interactions will vanish in second order perturbation theory after summing over all (presumably spatially homogeneously distributed) atoms, because these contributions are proportional to $g_{\mu 0}^2$ instead of $|g_{\mu 0}|^2$. Indeed, Agarwal's model concerns a smeared out effective interaction for all processes. Since we do make the rotating wave approximation in the present treatment, we will not dwell upon this problem any further.

^{**} The a_0 , $a_0^\dagger$ are simply obtained setting $\lambda = 0$.

^{***} Of course, the Hamiltonian (8.21) contains the as yet unknown linear damping constant λ . Therefore, in the final formulae (8.58)–(8.61), λ is not expressed explicitly as usual, but it can readily be determined selfconsistently.

^{****} Higher orders necessarily introduce nonlinear terms and, hence, should not be relevant to the linearly damped oscillator. See further on, especially below (8.34).

^{*****} The sum over μ in effect is threefold. First, it implies a summing of the subensembles B_1 and B_2 (see also fig. 8.1), containing those atoms in a spatial cell $\Delta\mathbf{r}_\mu$ that are initially in their upper respectively lower state, with a certain ω_μ . Second, it amounts to summing over the frequencies ω_μ , per cell $\Delta\mathbf{r}_\mu$. It should be noted that the present model does not basically specify the distribution over the frequencies. Third, one must sum over all spatial cells. See e.g. [22] for some more details. In the perfect spatial homogeneous case this latter summation commutes with the other two. In thermal equilibrium, at a finite temperature above zero, the first and second summation can not be interchanged since the relative weights of B_1 and B_2 depend on $\hbar\omega_\mu$ through the Boltzmann factor.

the rapidly oscillating exponential terms in $|g_\mu|^2$ vanish and one easily determines the effective coupling factor in (8.21) to be

$$g_\mu = i \sqrt{\frac{\hbar}{2\epsilon_0 V \omega}} \omega_\mu \mathcal{P}_\mu. \quad (8.23)$$

The important alteration with respect to (8.17) is the replacement of the free oscillator frequency Ω by the classical reduced frequency ω .

8.3. Reduction of the Hilbert space

Equation (8.21) represents the most frequently used model Hamiltonian in quantum optics (see e.g. [19–22, 41, 42, 50, 57, 58, 92, 200, 302, 389–391, 465–468]). Since, as already noted, it is of the form (7.14), it is consistent with the master equation (7.24) to start directly from the conservative Liouville–von Neumann equation. We then transform from the Schrödinger picture to the interaction picture in the usual manner by means of the unitary operator

$$U = \exp[-(i/\hbar)(H_r + H_R)], \quad (8.24)$$

$$H_r = \hbar\omega a^\dagger a, \quad H_R = \sum_\mu \hbar\omega_\mu b_\mu^\dagger b_\mu, \quad (8.25)$$

so that

$$\dot{R} = -(i/\hbar)[H'_I, R], \quad (8.26)$$

with

$$H'_I = \sum_\mu g_\mu a^\dagger b_\mu \exp\{i(\omega - \omega_\mu)t\} + \text{conj.}, \quad (8.27)$$

R being the total density operator for the combined radiation-reservoir system. The reduced density operator ρ for the radiation oscillator alone, is obtained by tracing over the reservoir variables:

$$\rho = \text{tr}_R R. \quad (8.28)$$

It will suit our purpose, in view of the Markovian approximation that will be made, to calculate a so-called secular or coarse-grained equation of motion for ρ , in close analogy with the outlines of Langevin reservoir theory.^{*} That means, employing in fact the basic ideas of the theory of Brownian motion, we presume the existence of a tiny time lapse Δt , which is intermediate in length between the characteristic (dephasing) time τ_R of the reservoir and the relaxation time $\tau_r = 1/\lambda$ of the radiation oscillator.^{**} That is, $\tau_R \ll \Delta t \ll \tau_r$. In other words, if we define the effective bandwidth of the atomic

^{*} One should notice some subtle differences with our earlier treatments [20, 22], as well as with that given e.g. by Scully and Whitney [200]. For a detailed account of the Langevin theory see e.g. [57, 58]. See also [24].

^{**} For a criticism of the Markovian assumption, especially in the extreme quantal regime $T \downarrow 0$, see section 5.

system as $\Delta\omega_R = 1/\tau_R$, then $\Delta\omega_R \Delta t \gg 1$, while on the other hand $\lambda \Delta t \ll 1$. The assumption is meant to imply that, in the interaction picture, ρ varies only slowly over Δt . Furthermore, the rapid relaxation of its variables and its additionally assumed infinite size, are taken to imply that the reservoir, described by ρ_R , preserves its statistical properties in the course of time. Indeed, the reservoir will be considered throughout to be in its own thermal equilibrium state at temperature T .

The coarse-grained time rate of change for $\rho(t)$ reads

$$\dot{\rho} = \text{tr}_R \frac{R(t + \Delta t) - R(t)}{\Delta t}. \quad (8.29)$$

The general solution for $\Delta R(t) = R(t + \Delta t) - R(t)$ from the Liouville-von Neumann equation (8.26) may be written iteratively as (e.g. [20, 22, 24])

$$R(t + \Delta t) = R(t) + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 \cdots \int_t^{t_{n-1}} dt_n [H'_1(t_1), [H'_1(t_2), \cdots [H'_1(t_n), R(t)] \cdots]]. \quad (8.30)$$

Having worked one's way back in time from $t + \Delta t$ to t , with $\Delta t \gg \tau_R$, one sets $R = \rho\rho_R$ on the right hand side of (8.30), with

$$\rho_R = \prod_{\mu} \rho_{\mu}, \quad (8.31)$$

$$\rho_{\mu} = \sum_j |\mu j\rangle \rho_{\mu j} \langle \mu j|, \quad (8.32)$$

$$\rho_{\mu j} = Z_{\mu}^{-1} \exp(-\mathcal{E}_{\mu j}/k_B T), \quad (8.33)$$

where $j = 1$ or 2 for the two level atom, and where Z_{μ} is the canonical partition function for the species μ . From (8.30) one has through second order:

$$\begin{aligned} R(t + \Delta t) = R(t) &- \frac{i}{\hbar} \sum_{\mu} \int_t^{t+\Delta t} dt_1 [g_{\mu} a^{\dagger} b_{\mu} e^{i(\omega - \omega_{\mu})t_1} + \text{conj.}, \rho(t) \rho_R] \\ &- \frac{1}{\hbar^2} \sum_{\mu} \sum_{\nu} \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 [g_{\mu} a^{\dagger} b_{\mu} e^{i(\omega - \omega_{\mu})t_1} + \text{conj.}, [g_{\nu} a^{\dagger} b_{\nu} e^{i(\omega - \omega_{\nu})t_2} + \text{conj.}, \rho(t) \rho_R]]. \end{aligned} \quad (8.34)$$

Terms in (8.30) with $n \geq 3$ necessarily introduce nonlinear terms into the equations of motion and, hence, should not be relevant to the linearly damped oscillator. Actually, the higher order, nonlinear terms are difficult to estimate but finally (i.e. after tracing over the reservoir etc.) turn out to be roughly of relative order $\lambda \Delta t \langle a^{\dagger} a \rangle$. Since in the Markovian approximation $\lambda \Delta t \ll 1$, the mean number of photons should not be too high, i.e. $\langle a^{\dagger} a \rangle \ll (1)$.^{*} Inserting now (8.31)–(8.33) for ρ_R into (8.34), and performing in

^{*} In the purely classical regime the perfect Markovian limit is possible in principle (see section 5), so that $\Delta t \ll (\hbar^2) \downarrow 0$. See e.g. (5.61). Even though $\langle a^{\dagger} a \rangle \propto (1/\hbar)$ in that case, the linear approximation can be made exact then. See also [38, 43]. Further, anticipating the final formulae of this section, λ will be proportional to $|g(\omega)|^2$, i.e. to $|P|^2$, and to N_{μ}/V . In summary, the nonlinear terms can be disregarded provided (1) the Markov assumption holds good, (2) the excitation of the tagged oscillator is not too high, (3) the interaction matrix element per relevant atom is small, and (4) the atomic density is only moderate.

view of (8.29) the trace over the reservoir, the first order term on the r.h.s. vanishes because it contains only single fermion operators. The second order term yields

$$\rho(t + \Delta t) = \rho(t) - \frac{1}{\hbar^2} \sum_{\mu} \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 \operatorname{tr}_{\mathbf{R}} Q_{\mu}(t_1, t_2), \quad (8.35)$$

with

$$Q_{\mu}(t_1, t_2) = (U_{\mu} + V_{\mu}) |g_{\mu}|^2 \exp\{i(\omega - \omega_{\mu})t_1 - i(\omega - \omega_{\mu})t_2\} + \text{conj.}, \quad (8.36)$$

$$U_{\mu} = [a^{\dagger} b_{\mu}, ab_{\mu}^{\dagger} \rho_{\mathbf{R}}], \quad (8.37)$$

$$V_{\mu} = [\rho_{\mathbf{R}} ab_{\mu}^{\dagger}, a^{\dagger} b_{\mu}], \quad (8.38)$$

and where we have already used the standard properties of thermal fermion systems, namely that $\operatorname{tr}_{\mathbf{R}}(\rho_{\mathbf{R}} b_{\mu} b_{\nu})$ is always zero, while $\operatorname{tr}_{\mathbf{R}}(\rho_{\mathbf{R}} b_{\mu}^{\dagger} b_{\nu})$ is nonzero only if $\mu = \nu$. Let us perform the trace of one term in (8.35) in some detail. For instance,

$$\operatorname{tr}_{\mathbf{R}} U_{\mu} = (a^{\dagger} a \rho - a \rho a^{\dagger}) \operatorname{tr}_{\mathbf{R}}(\rho_{\mathbf{R}} b_{\mu} b_{\mu}^{\dagger}). \quad (8.39)$$

By means of (8.31)–(8.33) one easily evaluates the trace:

$$\begin{aligned} \operatorname{tr}_{\mathbf{R}}(\rho_{\mathbf{R}} b_{\mu} b_{\mu}^{\dagger}) &= \operatorname{tr}(\rho_{\mu} b_{\mu} b_{\mu}^{\dagger}) \prod_{\nu \neq \mu} \operatorname{tr} \rho_{\nu} = \sum_{ij} \langle \mu i | \mu j \rangle \rho_{\mu ij} \langle \mu j | b_{\mu} b_{\mu}^{\dagger} | \mu i \rangle \\ &= \sum_j \rho_{\mu jj} \langle \mu j | b_{\mu} b_{\mu}^{\dagger} | \mu j \rangle = \rho_{\mu 22} = Z_{\mu}^{-1} \exp(-\mathcal{E}_{\mu 2}/k_{\mathbf{B}} T), \end{aligned} \quad (8.40)$$

so that, evidently, only those atoms that are initially in their lower state contribute to this interaction term. Similarly one obtains

$$\operatorname{tr}_{\mathbf{R}} V_{\mu} = (\rho a a^{\dagger} - a^{\dagger} \rho a) Z_{\mu}^{-1} \exp(-\mathcal{E}_{\mu 1}/k_{\mathbf{B}} T). \quad (8.41)$$

Having thus in effect calculated $\operatorname{tr}_{\mathbf{R}} Q_{\mu}$, we next concentrate on the time integrals occurring in (8.35). It will be useful to define the spectral function

$$J(\omega - \omega_{\mu}) = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt_1 \exp\{i(\omega - \omega_{\mu})t_1\} \int_t^{t_1} dt_2 \exp\{-i(\omega - \omega_{\mu})t_2\}, \quad (8.42)$$

which is easily seen to be independent of the global time t . Interchanging the orders of integration one is left with

$$J(\omega - \omega_{\mu}) = \frac{1}{\Delta t} \int_0^{\Delta t} dt_1 (\Delta t - t_1) \exp\{i(\omega - \omega_{\mu})t_1\}. \quad (8.43)$$

8.4. The master equation

With the simple, provisional result (8.43) at hand the reduced secular master equation (8.29) is written as

$$\dot{\rho} = \kappa_1[a^\dagger, \rho a] - \kappa_2[a^\dagger, a\rho] + \text{conj.}, \quad (8.44)$$

where $\kappa_j = \kappa_j(\omega)$, with $j = 1, 2$ and

$$\kappa_j(\omega) = \frac{1}{\hbar^2} \sum_{\mu} Z_{\mu}^{-1} \exp(-\mathcal{E}_{\mu}/k_B T) |g_{\mu}|^2 \mathcal{J}(\omega - \omega_{\mu}). \quad (8.45)$$

The atomic filter function $\mathcal{J}(\omega - \omega_{\mu})$ clearly picks out the relevant portion of the total reservoir spectrum (see also [299, 338, 469]). If in that relevant range the reservoir spectrum is densely distributed, which requires the number of atoms to be very large, the sum over μ in (8.45) can be replaced by an integral over ω_{μ} , such that (compare with (5.47) of the transmission line model)

$$\sum_{\mu} \rightarrow \int dN(\omega') = \int (dN/d\omega') d\omega' = N \int \rho(\omega') d\omega', \quad (8.46)$$

which defines $\rho(\omega')$ and where N represents the total number of atoms in the system's volume Λ^3 which interact with the radiation oscillator. The coefficients κ_j , (8.45), can now be written as

$$\kappa_j(\omega) = \int_0^{\infty} \mathcal{J}(\omega - \omega') \sigma(\omega') [1 + \exp(\pm \hbar \omega' / k_B T)]^{-1} d\omega', \quad (8.47)$$

where the plus and minus signs belong to $j = 1$ and $j = 2$ respectively, where we have invoked the explicit result for the partition function

$$Z_{\mu} = \sum_{j=1}^2 \exp(-\mathcal{E}_{\mu}/k_B T), \quad (8.48)$$

with $\hbar \omega' = \mathcal{E}_1 - \mathcal{E}_2 > 0$, and where we have further introduced the function⁴

$$\sigma(\omega') = N |g(\omega')/\hbar|^2 \rho(\omega'). \quad (8.49)$$

Notice, in view of (8.23) for $g(\omega') = g_{\mu}$, that $\sigma(\omega')$ will be independent of Planck's constant and that it is proportional to the spatial density N/Λ^3 of the atoms (see also the footnote below (8.34)).

The atomic filter function (8.43) has both a real and an imaginary part, which can be easily calculated exactly. Defining

$$\mathcal{J}(\omega - \omega') = \mathcal{J}'(\omega - \omega') + i \mathcal{J}''(\omega - \omega') \quad (8.50)$$

⁴ Effectively, combining (8.46) and (8.49), one has $\sum_{\mu} |g_{\mu}/\hbar|^2 \rightarrow \int d\omega' \sigma(\omega')$.

one obtains

$$j'(\omega - \omega') = \frac{1}{(\omega - \omega')^2 \Delta t} [1 - \cos(\omega - \omega') \Delta t], \quad (8.51)$$

$$j''(\omega - \omega') = \frac{1}{\omega - \omega'} \left[1 - \frac{\sin(\omega - \omega') \Delta t}{(\omega - \omega') \Delta t} \right]. \quad (8.52)$$

Here we come across well-known subtleties. Although Δt is quite short compared to the characteristic time constant $\tau_r = 1/\lambda$ of the tagged oscillator, it must be considered as quite long compared to the decorrelation time of the reservoir in order for the Markov assumption to make any sense. That is, as noted before, $1/\Delta t \ll \Delta\omega_R$, $\Delta\omega_R$ being the effective bandwidth of the bath as now expressed in terms of $\sigma(\omega')$ in (8.49). See fig. 8.2. Evidently, $j'(\omega' - \omega')$ picks out only a small portion of the broadband reservoir spectrum, namely near the actual oscillator frequency ω . Indeed, if we formally let Δt tend to infinity, (8.51) is recognized as one of the standard representations of the Dirac delta function (see e.g. [163]). Analogously, in that limit (8.52) leads to Cauchy's principal value of the pertinent integrals over ω' . Hence, in the Markovian limit one has^a

$$j(\omega - \omega') = \pi \delta(\omega - \omega') + \mathfrak{P} \left(\frac{i}{\omega - \omega'} \right). \quad (8.53)$$

Also separating the κ_j , (8.47), into their real and imaginary parts according to $\kappa_j(\omega) = \kappa'_j(\omega) + i \kappa''_j(\omega)$, one now obtains

$$\kappa'_j(\omega) = \pi \sigma(\omega) [1 + \exp(\pm \hbar \omega / k_B T)]^{-1}, \quad (8.54)$$

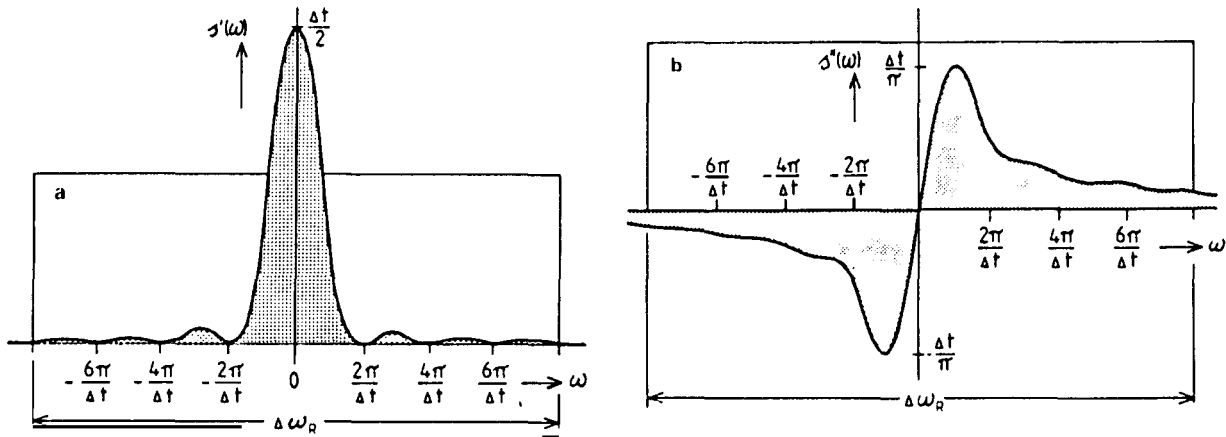


Fig. 8.2. Real (a) and imaginary (b) part of the reservoir spectrum filter function $\sigma(\omega)$, according to (8.50)–(8.52). $\Delta\omega_R$ is the reservoir bandwidth.

^a This upshot is consistent with introducing a complex frequency $\omega + i\epsilon$ and letting $\Delta t \rightarrow \infty$ in (8.43), and in the end considering $\epsilon \downarrow 0$. See e.g. [72, 163, 164].

$$\kappa_j''(\omega) = (\mathfrak{P}/\pi) \int_0^\infty \frac{\kappa_j'(\omega')}{\omega - \omega'} d\omega'. \quad (8.55)$$

The result (8.55) is a clear-cut example of a Kramers–Kronig dispersion relation (see e.g. [166, 303, 314, 465, 470–480] and also [113] p. 371 and [481] chapter 5).

In the extreme classical regime $\hbar\omega \ll k_B T$, (8.54) yields $\kappa_j' = \frac{1}{2}\pi\sigma(\omega)$. On the other hand, in the pure quantal regime $\hbar\omega \gg k_B T$, one sees that $\kappa_1' = 0$ while $\kappa_2' = \pi\sigma(\omega)$. Recalling that to a very good approximation the broad reservoir spectrum $\sigma(\omega)$ can be assumed to be constant over the entire relevant frequency range, (8.55) is usually taken to show that the frequency shifts κ_j'' are effectively nullified in both limits. Only in the transition regime $\hbar\omega \approx k_B T$ some subtle dispersive phenomena are expected. Anticipating our particular interest in the case $T = 0$, we indeed take κ_j real, i.e. $\kappa_j'' = 0$ in the sequel.[¶]

As in section 7, it will be useful to represent the master equation (8.44) in terms of coherent states. Following the procedure outlined below (7.15), one arrives at the Fokker–Planck equation

$$P_{,t} = (\lambda + i\omega)(\alpha P)_{,\alpha} + (\lambda - i\omega)(\alpha^* P)_{,\alpha^*} + 2\lambda N_\omega P_{,\alpha\alpha^*} \quad (8.56)$$

for Glauber's P -function.^{¶¶} In going from (8.44) to (8.56) the opportunity has been taken to transform back from the interaction picture to the Schrödinger picture. Furthermore, we have identified

$$\lambda = \kappa_2' - \kappa_1', \quad (8.57)$$

$$N_\omega = \kappa_1'/\lambda. \quad (8.58)$$

By means of (8.54) for κ_j' and comparison with (1.10), it is readily verified that

$$N_\omega = (\exp(\hbar\omega/k_B T) - 1)^{-1} \quad (8.59)$$

does represent the standard thermal Planck function. Evidently, at $T = 0$ (8.56) is identical to (7.16) with $c = 0$. Indeed, in the Schrödinger picture the presently derived Markovian master equation (8.44) for the reduced density operator reduces at $T = 0$ to

$$\dot{\rho} = -i\omega[a^\dagger a, \rho] - \lambda([a^\dagger, a\rho] + [\rho a^\dagger, a]), \quad (8.60)$$

with $\lambda = \pi\sigma(\omega)$, which is exactly identical to the basic result (7.25) of the mesoscopic complex phase space quantization.^{¶¶¶}

[¶]The sloppiness of these considerations is acknowledged. However, since they depend on the details of the actually unknown reservoir spectrum $\sigma(\omega)$, they are difficult to improve and, as usual, will be taken for granted.

^{¶¶}In order to update and more or less complete the list of references: see also [482–486].

^{¶¶¶}Hence, the present upshot also agrees with the result (6.70) from the modified Bopp-theory. See also a recent preprint by Dedene [487], which however only contains the diagonal dynamics (as in Bopp's original paper [114]); see further the remarks in section 6. Finally, it should be noticed that in both sections 7 and 8 the calculations have been performed using the frictional creation and annihilation operators rather than the free ones. In fact, it is not so easy to compare the complex phase space quantization using the free operators in the effective complex Hamiltonian (see e.g. [258, 288]) with ab-initio model calculations. Namely, in that case, according to Agarwal [55, 464], the antiresonant interactions in the Hamiltonian (8.16) must be carried along in order for the classical reduced frequency ω to be reproduced in the results. However, as noted in section 7, for the full Hamiltonian (8.16) the master equation (7.24) does not exactly reduce to the standard Schrödinger–Liouville equation, so that the comparison is not internally consistent. Rather, one should also do the model calculations from (7.24).

8.5. Summary

Starting from the well-known microscopic Hamiltonian for an atomic reservoir interacting with a single quantized radiation oscillator, the reduced master equation for the latter subsystem has been derived. It describes linear damping if the usual sequence of rather subtle steps is taken. The rotating wave approximation is made and the Hamiltonian is written in terms of the creation and annihilation operators of section 7 in order to guarantee the radiation field to oscillate at the proper classical frequency ω . For the model Hamiltonian (8.21) the master equation (7.24) is equivalent to the conventional Schrödinger–Liouville–von Neumann equation, from which the further calculations are made along standard lines. The interaction between atoms and radiation is assumed to be weak, and neither the density of atoms nor the initial excitation of the oscillator should be too high, otherwise the effective equations of motion would become nonlinear. A Markovian assumption is needed for the thermal reservoir, that is taken to possess a dense and broadly distributed frequency spectrum, which implies an infinite number of atoms and, hence, an infinite size of the system. The resulting coarse-grained master equation reduces in the zero temperature limit precisely to the essential upshot (7.25) of the mesoscopic complex phase space quantization of the previous section.

9. Hasse's pure state representation

9.1. The Schrödinger equation

Let us return to the complex phase space quantization of section 7. The basic outcome (7.25) of that procedure has been represented in terms of the original Hermitian coordinate and momentum in (7.27). For convenience, we repeat it here in the somewhat more general form (7.29). That is,

$$\dot{\rho} = -\frac{i}{\hbar} [H_0, \rho] - i\frac{\lambda}{\hbar} [x, [p, \rho]] + \frac{1}{\hbar^2} (D_{px} + D_{xp}) [p, [x, \rho]] - \frac{1}{\hbar^2} D_{xx} [p, [p, \rho]] - \frac{1}{\hbar^2} D_{pp} [x, [x, \rho]], \quad (9.1)$$

where H_0 is the free oscillator Hamiltonian, and where the diffusion coefficients have been specified in (7.31). The master equation (9.1) cannot be written in pure commutator form. In other words, there is no ordinary Schrödinger wave equation corresponding to it, save for the case $\lambda = 0$ (i.e. also all D 's equal to zero). Nevertheless, it turns out to be possible to find a special, stochastically equivalent Schrödinger equation in the limit of weak damping $\lambda \downarrow 0$. However, as will be shown in the sequel, this is possible only if the diffusion coefficients in (9.1) obey Hasse's pure state representation condition. See [107, 310, 311] for Hasse's original derivation, and [196] for a slightly generalized formulation.*

One may trivially write the density operator ρ in (9.1) as

$$\rho(t) = \rho_0(t) + \rho_1(t), \quad (9.2)$$

where $\rho_1(t)$ is of the order λ and represents a small correction to $\rho_0(t)$. As $\rho_0(t)$ is taken to describe a nondissipative system that can be represented by a pure Schrödinger state vector, say $|\psi\rangle$, one has at any time

* See also [292] for related considerations.

$$\rho_0^2(t) = \rho_0(t) = |\rangle\langle|. \quad (9.3)$$

An arbitrary density operator possesses the property $\text{tr } \rho^2(t) \leq 1$. In order that it represents a pure state, it is both necessary and sufficient (see e.g. [163] p. 336) that

$$\text{tr } \rho^2(t) = 1. \quad (9.4)$$

If the system starts off from an initial pure state, then $\text{tr } \rho^2(0) = \text{tr } \rho(0) = 1$, and (9.4) is guaranteed if the time derivative of its l.h.s. is identically zero. Therefore, since $(\rho^2)' = \rho\dot{\rho} + \dot{\rho}\rho$, we multiply (9.1) respectively from the left and from the right with ρ , and add up the resulting expressions. In a first order (Born [55, 163, 164, 303, 314, 464, 488–490]) approximation one then replaces $\rho(t)$, in terms that are already of order λ , by $\rho_0(t)$. This leads to

$$\begin{aligned} (\rho^2)' = & -\frac{i}{\hbar} [H_0, \rho^2] - i\frac{\lambda}{\hbar} [\frac{1}{2}(px + xp), \rho_0^2] - i\frac{\lambda}{\hbar} [x, \rho_0 p \rho_0] + i\frac{\lambda}{\hbar} [p, \rho_0 x \rho_0] + 2\lambda \rho_0^2 \\ & + \frac{1}{\hbar^2} (D_{px} + D_{xp}) \{ \rho_0 (px + xp) \rho_0 + [\frac{1}{2}(px + xp), \rho_0^2]_+ - [x, \rho_0 p \rho_0]_+ - [p, \rho_0 x \rho_0]_+ \} \\ & - \frac{1}{\hbar^2} D_{xx} \{ 2\rho_0 p^2 \rho_0 - 2[p, \rho_0 p \rho_0]_+ + [p^2, \rho_0^2]_+ \} - \frac{1}{\hbar^2} D_{pp} \{ 2\rho_0 x^2 \rho_0 - 2[x, \rho_0 x \rho_0]_+ + [x^2, \rho_0^2]_+ \}. \end{aligned} \quad (9.5)$$

Notice the anticommutators in the dissipative terms. Now using the idempotency property $\rho_0^2 = \rho_0$ from (9.3) and noticing that for an arbitrary operator function F

$$\rho_0 F \rho_0 = \langle F \rangle \rho_0, \quad (9.6)$$

one easily evaluates (9.5) further as:

$$\begin{aligned} (\rho^2)' = & -\frac{i}{\hbar} [H_0, \rho^2] - i\frac{\lambda}{\hbar} [W, \rho_0] + 2\lambda \rho_0 + \frac{1}{\hbar^2} (D_{px} + D_{xp}) [\frac{1}{2}\langle px + xp \rangle + \frac{1}{2}(px + xp) - \langle p \rangle x - \langle x \rangle p, \rho_0]_+ \\ & - \frac{1}{\hbar^2} D_{xx} [\langle p^2 \rangle - 2\langle p \rangle p + p^2, \rho_0]_+ - \frac{1}{\hbar^2} D_{pp} [\langle x^2 \rangle - 2\langle x \rangle x + x^2, \rho_0]_+, \end{aligned} \quad (9.7)$$

where

$$W = \frac{1}{2}\langle px + xp \rangle - \frac{1}{2}(px + xp) + \langle p \rangle x - \langle x \rangle p. \quad (9.8)$$

Note that the convenient choice $\langle W \rangle = 0$ only involves a nonobservable time dependent change in the phase of the wave function. As discussed below (9.4), the propagation of a pure state is guaranteed if the trace of (9.7) is equal to zero. This requirement immediately leads to Hasse's condition* [196, 258]

$$D_{xx}\sigma_{pp} + D_{pp}\sigma_{xx} - (D_{px} + D_{xp})\sigma_{px} = \frac{1}{2}\hbar^2\lambda, \quad (9.9)$$

* In Hasse's original notation [311]: $\lambda \rightarrow \frac{1}{2}\gamma$, $D_{xx} = 0$, $D_{pp} \rightarrow D$, and $D_{px} + D_{xp} \rightarrow -d$.

where the usual definitions (3.53)–(3.55) for the variances have been used. Since under the condition (9.9) for all times $\rho^2(t) = \rho(t)$, and since within the Born-approximation we may again replace $\rho_0(t)$ by $\rho(t)$ in (9.7), the latter formula can be written as (see also e.g. [59])

$$\dot{\rho} = -(i/\hbar)[H_0 + \lambda W, \rho] + (1/\hbar)[\mathfrak{D}, \rho]_+, \quad (9.10)$$

where we have introduced the operator [196]

$$\begin{aligned} \mathfrak{D} = & \hbar\lambda + (1/\hbar)(D_{px} + D_{xp}) \left\{ \frac{1}{2}(px + xp) + \frac{1}{2}\langle px + xp \rangle - \langle p \rangle x - \langle x \rangle p \right\} \\ & - (1/\hbar)D_{xx} \{ \langle p^2 \rangle - 2\langle p \rangle p + p^2 \} - (1/\hbar)D_{pp} \{ \langle x^2 \rangle - 2\langle x \rangle x + x^2 \}. \end{aligned} \quad (9.11)$$

Now defining the nonlinear, non-Hermitian Hamiltonian

$$\mathfrak{H} = H_0 + \lambda W + i \mathfrak{D}, \quad (9.12)$$

the result (9.10) can also be presented as (compare with (6.53))

$$\dot{\rho} = -(i/\hbar)(\mathfrak{H}\rho - \rho\mathfrak{H}^*), \quad (9.13)$$

which manifestly shows that the wave function obeys the Schrödinger equation

$$i \hbar \psi_{,t} = \mathfrak{H} \psi. \quad (9.14)$$

It is amusing to compare (9.14), with the Hamiltonian (9.12), with the Süssmann–Hasse–Albrecht Schrödinger equation (2.10). The presently derived frictional potential (9.8) appears to be identical* to the heuristically proposed W in (2.9) with $c = \frac{1}{2}$, being one of Hasse's choices [129]. However, the present model has quite different features in view of the additional operator $\mathfrak{D}$.

9.2. Wave packet solutions

The equations of motion for the moments are easily calculated from the Schrödinger equation (9.14), or alternatively from the equivalent density operator equations (9.10) or (9.13). For the zeroth moment, the norm, one has

$$\langle 1 \rangle' = (2/\hbar)\langle \mathfrak{D} \rangle = 0 \quad (9.15)$$

by virtue of (9.11) and the pure state condition (9.9). In fact, (9.15) just re-expresses the latter. For the first moment $\langle x \rangle$ one finds

$$\langle x \rangle' = -(i/\hbar)\langle [x, H_0 + \lambda W] \rangle + (1/\hbar)\langle [x, \mathfrak{D}]_+ \rangle. \quad (9.16)$$

The contribution from the potential W vanishes and one obtains

* Apart from an unimportant phase factor.

$$\begin{aligned} \langle x \rangle' &= \langle p \rangle + 2\lambda \langle x \rangle + (2/\hbar^2)(D_{px} + D_{xp})\{\langle xpx \rangle - \langle x^2 \rangle \langle p \rangle\} \\ &\quad - (2/\hbar^2)D_{xx}\{\langle pxp \rangle - \langle px + xp \rangle \langle p \rangle + \langle x \rangle \langle p^2 \rangle\} - (2/\hbar^2)D_{pp}\{\langle x^3 \rangle - \langle x \rangle \langle x^2 \rangle\}. \end{aligned} \quad (9.17)$$

Defining the third deviation-moments (i.e. from the mean)

$$\sigma_{\mu\nu\kappa} = \langle (\mu - \langle \mu \rangle)(\nu - \langle \nu \rangle)(\kappa - \langle \kappa \rangle) \rangle, \quad (9.18)$$

where μ , ν and κ each stand for x or p , (9.17) is rewritten as

$$\langle x \rangle' = \langle p \rangle + (2/\hbar^2)\{(D_{px} + D_{xp})\sigma_{xpx} - D_{xx}\sigma_{pxp} - D_{pp}\sigma_{xxx}\} + (2/\hbar)\langle x \rangle \langle \mathfrak{D} \rangle. \quad (9.19)$$

Since $\langle \mathfrak{D} \rangle = 0$ in view of (9.15), one is left with*

$$\langle x \rangle' = \langle p \rangle + (2/\hbar^2)\{(D_{px} + D_{xp})\sigma_{xpx} - D_{xx}\sigma_{pxp} - D_{pp}\sigma_{xxx}\}, \quad (9.20)$$

which reduces to the correct result $\langle x \rangle' = \langle p \rangle$ as obtained from the original master equation (9.1) for wave packets with zero skewness. As in view of the linearity of the oscillator problem the propagator of (9.14) will be a Gaussian, zero skewness is guaranteed if the system is initially in a Gaussian state. This is, indeed, the usual assumption. Under the same conditions

$$\langle p \rangle' = -2\lambda \langle p \rangle - \Omega^2 \langle x \rangle + (2/\hbar^2)\{(D_{px} + D_{xp})\sigma_{pxp} - D_{xx}\sigma_{ppp} - D_{pp}\sigma_{xpx}\} \quad (9.21)$$

also reduces to the correct equation for $\langle p \rangle$. Confining ourselves henceforth to Gaussian wave packets, it is a matter of straightforward application of the Gaussian theorem to obtain for the second moments, for instance,

$$\dot{\sigma}_{xx} = 2\sigma_{px} + 2D_{xx} + (4/\hbar)\langle x^2 \rangle \langle \mathfrak{D} \rangle - (2/\hbar)\langle x \rangle^2 \langle \mathfrak{D} \rangle + (4/\hbar^2)D_{xx}(\sigma_{pp}\sigma_{xx} - \frac{1}{4}\hbar^2 - \sigma_{px}^2). \quad (9.22)$$

Once more, $\langle \mathfrak{D} \rangle = 0$ in view of (9.15). Further, the expression between brackets is exactly zero on account of the uncertainty relation (2.18) for Gaussian wave packets. Similarly calculating the equations for the other variances, one obtains altogether

$$\dot{\sigma}_{px} = -2\lambda\sigma_{px} + \sigma_{pp} - \Omega^2\sigma_{xx} + D_{px} + D_{xp}, \quad (9.23)$$

$$\dot{\sigma}_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} + 2D_{pp}, \quad (9.24)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px} + 2D_{xx}. \quad (9.25)$$

Comparison with (7.43)–(7.45) readily proves that, albeit only for Gaussian wave functions, the frictional Schrödinger equation (9.14) indeed represents the same dynamical state as the original master equation (9.1).

The essential *proviso* on the above results is that the pure state condition (9.9) must be satisfied. In

* Notice a difference in sign with respect to [311].

general, this will not be the case in the dynamical state. But, even in the final steady state, if it exists, the fulfilment of Hasse's condition is nontrivial. Nevertheless, at least we can show that the above theory is *selfconsistent* in the long time limit. Namely, since the stationary solution of the Schrödinger equation (9.14) with the Hamiltonian (9.12) will be a Gaussian, the special form (2.18) of the uncertainty relation holds in that case. It then remains to show that the steady state solutions of (9.23)–(9.25) do imply $\langle \mathfrak{D} \rangle = 0$. The pertinent steady state fluctuations have already been given in (7.46)–(7.48), from which we first of all learn that (2.18) can be specialized to

$$\sigma_{pp}(\infty) \sigma_{xx}(\infty) = \frac{1}{2} \hbar^2 - D_{xx} \sigma_{px}(\infty). \quad (9.26)$$

Inserting (7.47) for $\sigma_{pp}(\infty)$, one has

$$D_{xx} [\Omega^2 \sigma_{xx}(\infty) + 2\lambda \sigma_{px}(\infty)] + D_{pp} \sigma_{xx}(\infty) = \frac{1}{2} \hbar^2 \lambda. \quad (9.27)$$

Most easily using (7.43), with $\dot{\sigma}_{px} = 0$, for the quantity within the square brackets in (9.27), one finds

$$D_{xx} \sigma_{pp}(\infty) + D_{pp} \sigma_{xx}(\infty) + (D_{px} + D_{xp}) D_{xx} = \frac{1}{2} \hbar^2 \lambda. \quad (9.28)$$

Finally, once more using $D_{xx} = -\sigma_{px}(\infty)$ in the last term on the l.h.s. of the equality (9.28), it is immediately seen to represent the steady state version of the pure state condition (9.9), which is what we set out to prove. Vice versa, *if* the steady state fluctuations follow from (9.23)–(9.25), and moreover, *if* (and only if) they satisfy Hasse's condition (9.9), *then* the Gaussian wave packet uncertainty relation (2.18) holds. It is quite interesting to see, as is most easily verified by mere inspection of (7.31), (7.49) and (9.9), that the specific fluctuations emerging from the complex phase space quantization of section 7 precisely fall into this category (this was originally noted in [196]). In other words, the dissipative ground state of the damped oscillator of section 7 has the special property that it fulfils Hasse's pure state condition and that, hence, it can be described in terms of a Gaussian Schrödinger wave function.^{***}

9.3. Summary

It has been discussed how the quantum mechanical master equation for the linearly damped oscillator can be approximated by a nonlinear frictional Schrödinger wave equation with an essentially non-Hermitian Hamiltonian. Two requirements must be met: first, only Gaussian wave packets should be considered, and second, Hasse's pure state representation condition must be fulfilled. Unfortunately, the latter is generally violated. It has been shown, however, that the specific steady state fluctuations predicted by the complex phase space quantization of section 7 do satisfy Hasse's condition, such that the dissipative ground state can indeed be described exactly by means of a Gaussian wave function.

^{*}This really is a nontrivial additional condition. Counter examples are easily constructed.

^{**}Three remarks seem in place. First, the wave packet in question, of course, is not a common, free oscillator minimum uncertainty packet. Second, in transient phenomena the pure state condition will not be met in general, and, hence, the spread of Hasse's Gaussian wave packet in the dynamical state will definitely be different from that calculated by means of the original master equation. And thirdly, the connection between Hasse's and the author's theory is even more remarkable as we have recently shown. One can easily construct a whole group of diffusion coefficients, parametrized by a real scalar degree of freedom, that fulfil Hasse's condition exactly in the long time limit. See [258] for details.

10. Elements of complex Hamilton–Jacobi theory

10.1. Hasse's classical dynamics

10.1.1. Complex calculus

As noted in the previous section, below the Schrödinger equation (9.14), there is a close similarity between Hasse's potential (9.8) and the Süssmann–Hasse–Albrecht-potential (2.9) with $c = \frac{1}{2}$, although the additionally occurring non-Hermiticity of Hasse's Hamiltonian (9.12) should not be forgotten. In a recent paper [43] Stocker and Albrecht indicated how the nonlinear frictional potentials (2.9) of Süssmann, Hasse and Albrecht might be considered as formally generated within the framework of classical real space Hamilton–Jacobi theory (see e.g. [1, 2, 7, 9, 223, 224]) and the closely related fluid dynamical interpretation of Schrödinger's wave mechanics (see e.g. [190, 201–204, 216, 225]). Let us therefore consider the classical Hamiltonian analogue of Hasse's theory.

In the classical formalism it is convenient to set $\langle F(x, p) \rangle = F(x(t), p(t))$, where $F(x, p)$ is an arbitrary function of the intrinsically prescribed functions of time $x(t)$ and $p(t)$. The latter are the solutions of the dynamical problem. The complex Hamiltonian (9.12), where H_0 is the free oscillator Hamiltonian (7.30), W is given in (9.8) and $\mathfrak{D}$ follows from (9.11), is now comfortably written as^{*}

$$\begin{aligned} \mathfrak{H} = & H_0 + \lambda(p + p(t))(x - x(t)) + \frac{i}{\hbar}(D_{px} + D_{xp})(p - p(t))(x - x(t)) \\ & - \frac{i}{\hbar}D_{xx}(p - p(t))^2 - \frac{i}{\hbar}D_{pp}(x - x(t))^2. \end{aligned} \quad (10.1)$$

Evidently, by choice in fact, along the classical trajectories $\mathfrak{H}(t) = H_0(t)$, i.e. the classical mechanical energy.^{**} It is particularly interesting to investigate the usual Hamiltonian equations of motion with (10.1). One obtains

$$\dot{x} = \mathfrak{H}_{,p} = p + \lambda(x - x(t)) + (i/\hbar)(D_{px} + D_{xp})(x - x(t)) - 2(i/\hbar)D_{xx}(p - p(t)), \quad (10.2)$$

$$\dot{p} = -\mathfrak{H}_{,x} = -\Omega^2 x - \lambda(p + p(t)) - (i/\hbar)(D_{px} + D_{xp})(p - p(t)) + 2(i/\hbar)D_{pp}(x - x(t)). \quad (10.3)$$

Since by definition we are looking for the solutions $x = x(t)$ and $p = p(t)$, these equations reduce to (6.11)–(6.12),

$$\dot{x} = p; \quad \dot{p} = -2\lambda p - \Omega^2 x, \quad (10.4)$$

which correctly describe the classical linearly damped harmonic oscillator. Notice that this result is obtained independent of a particular value of the diffusion coefficients. Of course, quite generally it would be rather problematic for imaginary terms to enter the classical real variables equations of motion.

Although as noted, the classical dynamical theory obviously holds for arbitrary diffusion coefficients,

^{*}Note that the diffusion coefficients (7.31) are of the order $\hbar$, so that the coefficients in (10.1) do have proper classical limits.

^{**}Remember that in the quantum theory we have $\langle \mathfrak{H} \rangle = \langle H_0 \rangle$.

we will confine ourselves in the sequel to the specific case (7.31),

$$D_{xx} = \frac{\hbar\lambda}{2\omega}, \quad D_{pp} = \frac{\hbar\lambda\Omega^2}{2\omega}, \quad D_{px} + D_{xp} = -\frac{\hbar\lambda^2}{\omega}, \quad (10.5)$$

which is quantum mechanically closely related to the canonical transformation (6.42)–(6.43), or equivalently to the first columns of (6.3) and (6.4). In the classical formalism it is preferred to eliminate Planck's constant and, hence, to use the complex variables z, z^* rather than $a = z/\sqrt{\hbar}$, $a^* = z^*/\sqrt{\hbar}$. Inserting

$$x = \frac{i}{\sqrt{2\omega}}(z - z^*), \quad p = \frac{1}{\sqrt{2\omega}}[(\omega - i\lambda)z + (\omega + i\lambda)z^*] \quad (10.6)$$

and the diffusion coefficients (10.5) into (10.1), one obtains

$$\dot{\Phi} = (\omega - i\lambda)z^*z + 2i\lambda z^*(t)z + \varphi(t), \quad (10.7)$$

where $\varphi(t)$ is an in principle dynamically arbitrary function of time that can be used, for instance, to make $\dot{\Phi}(t)$ real (and even represent the true mechanical energy) along the classical trajectory. It is well-known that the Hamiltonian (10.7) can be obtained from the Lagrangian (see e.g. [24, 182, 195, 300, 331, 418])

$$\mathcal{L} = i z^* \dot{z} - \dot{\Phi}(z^*, z, t), \quad (10.8)$$

so that the momentum conjugate to the (chosen) coordinate z reads $\pi_z = i z^*$, which is line with the modified Bopp-theory (section 6) and the complex phase space quantization of section 7. The canonical equations of motion become

$$\dot{z} = -i\dot{\Phi}_{,z} = -i\omega z - \lambda z, \quad (10.9)$$

$$\dot{z}^* = i\dot{\Phi}_{,z^*} = i\omega z^* - \lambda z^* + 2\lambda(z^* - z^*(t)), \quad (10.10)$$

which along the proper trajectories $z = z(t)$ indeed reduce to the correct results. Compare e.g. with (6.9); see further [24, 182, 195, 196, 258, 288] and [197]. Notice for example, following Bopp [114], that

$$\ddot{z} + 2\lambda\dot{z} + \Omega^2 z = \ddot{z} + 2\lambda\dot{z} - (\lambda - i\omega)\dot{z} = (\dot{z} + \lambda z + i\omega z)' = 0. \quad (10.11)$$

In the preceding we have used the time independent canonical transformation (10.6) which preserves the basic Poisson-brackets of coordinate and conjugate momentum. That is*

$$\{x, p\} = \{z, \pi_z\}_{x,p} = 1. \quad (10.12)$$

* Although the quantum mechanical literature (e.g. [163, 164, 300]) is reasonably consistent in denoting commutator brackets by $[,]$ and Poisson brackets by $\{ , \}$, in the realm of classical mechanics various notations for the latter are floating around (see e.g. [1, 2, 7, 9]). For definiteness, (10.12) generalized to higher dimensional systems amounts to $\{f, g\}_{x,p} = \sum_k (f_{,x_k} g_{,p_k} - g_{,x_k} f_{,p_k})$. For convenience, we further set e.g. $\{x_k, p_k\}_{x,p} = \{x_k, p_k\}$.

Of particular interest, however, are time dependent transformations. Of course, especially that transformation is of value that solves the dynamics or, within the present context, at least removes the irreversible part. It must then immediately be pointed out, that such transformations need not be canonical in the usual sense as contained in (10.12). Consider in view of the linearly damped oscillator problem, for example, the dynamical transformation from z to z' according to

$$z = z' e^{-\lambda t}, \quad \pi_z = \pi_{z'} e^{-\lambda t}. \quad (10.13)$$

Remember that $\pi_z = i z^*$, so that $\pi_{z'} = i z'^*$. Substitution of (10.13) into (10.12) yields

$$\{z', \pi_{z'}\}_{z,p} = e^{2\lambda t}, \quad (10.14)$$

so that (10.13) is certainly not canonical in the sense of leaving the fundamental bracket invariant. However, on the other hand, it is obvious that the remaining reversible dynamics can be obtained from the Hamiltonian

$$\mathfrak{H}' = \omega z'^* z', \quad (10.15)$$

with

$$\{z', \pi_{z'}\} = 1 \quad (10.16)$$

in lieu of (10.14). The other way around, if we had started from the well-known conservative dynamics (10.15)–(10.16), the transformation (10.13) would lead to an exponentially decaying bracket in (10.12), i.e. for instance,

$$\{x, p\}_{z', \pi_{z'}} = e^{-2\lambda t}, \quad (10.17)$$

which is reminiscent of the unwanted feature encountered earlier in (4.9) within the Caldirola–Kanai theory, and which has been seen to be closely related to the violation of Heisenberg's principle in quantum mechanics. Of course, in classical theory (10.17) has no basic physical, but only mathematical implications.

It is instructive to briefly investigate the time dependent transformation (10.13) in slightly more detail. Introducing, for convenience, the notation $z' = \tilde{\mathfrak{z}}(z, t)$, one determines the equation of motion

$$\dot{z}' = -i(\tilde{\mathfrak{z}}_{,z} \tilde{\mathfrak{z}}_{,z}^* - \tilde{\mathfrak{z}}_{,z}^* \tilde{\mathfrak{z}}_{,z}) \tilde{\mathfrak{z}}_{,z} + \tilde{\mathfrak{z}}_{,t}, \quad (10.18)$$

making use of the general form of the original Hamiltonian equations (10.9)–(10.10). The expression in brackets on the r.h.s. of (10.18) is easily identified with the Poisson bracket (10.14). Noticing further that $\tilde{\mathfrak{z}}_{,t}$ does not depend on $\pi_{z'} = i z'^*$, one obtains the transformed Hamiltonian

$$\mathfrak{H}' = e^{2\lambda t} \mathfrak{H} + \pi_{z'} \tilde{\mathfrak{z}}_{,t} + z' \kappa(t), \quad (10.19)$$

where $\kappa(t)$ is an as yet arbitrary function of time. The latter is fixed by the “complex dynamical consistency requirement”, demanding that

$$\dot{z}^{**} = i \mathfrak{H}'_{,z^*} = i \mathfrak{H}'_{,z^*}|_{z^*=z'(t)}. \quad (10.20)$$

In words, (10.20) requires the operations of complex conjugation and time differentiation to commute with each other, at least along the true classical trajectory. This condition is both necessary and sufficient in order for the complex dynamical formalism to be selfconsistent. In the present case, using (10.19) and recalling that $\mathfrak{H}(z, t) = z'$, (10.20) readily leads to

$$\kappa(t) = -i \mathfrak{H}'_{,t} - \pi_{z^*}(\mathfrak{H}_{,t})_{,z^*}|_{z^*=z'(t)} = -2i \lambda z'^*(t). \quad (10.21)$$

Inserting this result for $\kappa(t)$ into (10.19) finally shows that the original Hamiltonian (10.7) indeed transforms into (10.15).^{*}

Transformations like (10.19) are not obtained within the standard treatment of the theory of canonical transformations^{**} (see e.g. [1, 2, 7, 9]). At least, this is consistent with our earlier observations concerning the conservation of Poisson brackets under canonical transformations. Indeed, their invariance can be derived once the usual form of the canonical generators is given (see especially Goldstein's [1] treatment, proceeding via the Poincaré integral invariants and the Lagrange brackets). However, the basic definition of a canonical transformation is that it preserves the form of the canonical equations of motion. This definition, almost trivially, implies that the most interesting time dependent transformation, namely the one that solves the dynamics, is *always* canonical. No doubt, transforming to the initial state, the new variables are time independent by choice and, hence, can always be derived from the zero Hamiltonian.^{***} In other words, the dynamical Hamilton–Jacobi transformation is canonical *per se*, whether it preserves Poisson brackets or not.

10.1.2. The Hamilton–Jacobi equation

Let us now recall the complex canonical momentum relation $\pi_z = i z^*$ and extend for a while the formulation to an N -dimensional problem.^{****} Suppose then that we were given a function $\mathfrak{S}(z, t)$, with the property

$$i z_k^* = \mathfrak{S}_{,z_k}; \quad k = 1, 2, \dots, N. \quad (10.22)$$

Note that $z = \{z_k\} = \{z_1, z_2, \dots, z_N\}$, and that the r.h.s. of (10.22) solely depends on z , *not* on z^* . Separating (10.22) into real and imaginary parts, one obtains $2N$ relations connecting $2N$ unknowns. Clearly, (10.22) suffices in principle to find a function $z(t)$, and, of course, also $z^*(t)$.^{*****} It will be our task

^{*} The Hamiltonian $\mathfrak{H}'$ as given in (10.15) is not unique. For instance, if in (10.18) one starts with the equation of motion for z^{**} rather than z' , and imposes the complex dynamical constraint (10.20) then on $\dot{z}'$ instead of $\dot{z}^{**}$, one obtains $\mathfrak{H}' = \omega z^{**} z' - 2i \lambda (z^{**} - z^{**}(t))(z' - z'(t))$, which is obviously dynamically equivalent to (10.15). The latter is just the simplest member of a group of dynamically indistinguishable generators. This ambiguity is intimately connected with the arbitrariness of the diffusion coefficients in the original real variable Hamiltonian (10.1) with respect to the classical dynamics (10.2)–(10.4).

^{**} A way out in certain cases may be to add the time t to the mechanical variables (its conjugate variable will be minus the Hamiltonian) and transform it simultaneously, considering the problem in the so-called “extended phase space”. See especially Lanczos’ treatise [7].

^{***} Of course, it may also be a constant, i.e. independent of the new “dynamical” variables.

^{****} To be clear, N is arbitrary and the phase space will have $2N$ dimensions.

^{*****} Notice here a difference with the usual theory of canonical transformations in terms of real space mechanical variables. The standard analogue of (10.22) is $p = S_{,x}$, which yields a relation between p , x and t . Obviously, this is not sufficient to obtain the solution of the dynamical problem in integrated form. For that purpose one needs the second half of the transformation formula, for example $x' = S_{,p}$, where the primed variables are the new ones, say the initial values. In the complex variable theory the second set is, in fact, supplied by the operation of complex conjugation itself.

to determine that particular generator $\mathfrak{S}(z, t)$ that makes this function $z(t)$ identical to the solution of the dynamical problem.

To achieve this, consider the total differential of z_k^* . On one hand, by (10.22), it can be considered as a function of z and t , so that

$$dz_k^* = \sum_i z_{k,i}^* dz_i + z_{k,t}^* dt. \quad (10.23)$$

Using the general form of the canonical equation (10.9) for z_i in the sum, and (10.22) for z_k^* in the last term on the r.h.s. of (10.23), one obtains

$$dz_k^* = -i \sum_i z_{k,i}^* \mathfrak{H}_{,z_i}^* dt - i(\mathfrak{S}_{,t})_{,z_k} dt. \quad (10.24)$$

On the other hand, from the second canonical equation (10.10) one has

$$dz_k^* = i \mathfrak{H}_{,z_k} dt. \quad (10.25)$$

Equating (10.24) and (10.25) leads to

$$\mathfrak{H}_{,z_k} + \sum_i z_{k,i}^* \mathfrak{H}_{,z_i}^* + (\mathfrak{S}_{,t})_{,z_k} = 0. \quad (10.26)$$

Now notice that $\mathfrak{S}$ is a function only of z and t , but that in (10.26) the Hamiltonian $\mathfrak{H}$ still depends on both z and z^* (and possibly on t , of course). Substituting (10.22) for z^* into $\mathfrak{H}$, we define^{*} $\mathfrak{H}(z^*(z, t), z, t) \equiv \mathfrak{H}'(z, t)$, and consider

$$\mathfrak{H}'_{,z_k} = \mathfrak{H}_{,z_k} + \sum_i \mathfrak{H}_{,z_i}^* z_{i,z_k}^*. \quad (10.27)$$

Substituting (10.27) for $\mathfrak{H}_{,z_k}$ into (10.26), the result may be written as

$$(\mathfrak{H}' + \mathfrak{S}_{,t})_{,z_k} = \sum_i (z_{i,z_k}^* - z_{k,i}^*) \mathfrak{H}_{,z_i}^* \quad (10.28)$$

In view of (10.22), the r.h.s. of (10.28) vanishes termwise identically. Hence, $\mathfrak{H}'(z, t) + \mathfrak{S}_{,t}(z, t)$ is a function of time only. The latter dependence reflects the well-known possibility to choose a different zero reference point for the Hamiltonian at each instant t , without affecting the classical dynamics. This has been expressed earlier, for example, in (10.7) by the arbitrary function $\varphi(t)$ in the Hamiltonian $\mathfrak{H}$. In conclusion, we may write

$$\mathfrak{H}'(z, t) + \mathfrak{S}_{,t}(z, t) = 0, \quad (10.29)$$

^{*} One should, of course, not confuse this $\mathfrak{H}'(z)$ with the $\mathfrak{H}'(z^*, z)$ in (10.15)–(10.20).

or, in extenso, in an obvious notation:

$$\dot{\mathfrak{S}}(-i \mathfrak{S}_{,z}, z, t) + \mathfrak{S}_{,t} = 0. \quad (10.30)$$

This is the complex Hamilton–Jacobi equation relevant to Hasse’s classical dynamics. As usual, it has the structure of a first-order partial differential equation in $N + 1$ variables and, consequently, a complete solution must involve $N + 1$ independent constants of integration.

In the particular case of the damped oscillator Hamiltonian (10.7), the Hamilton–Jacobi equation (10.30) specializes to^a

$$2\lambda z \mathfrak{S}_{,z}(t) - (\lambda + i\omega)z \mathfrak{S}_{,z} + \mathfrak{S}_{,t} = 0. \quad (10.31)$$

Although (10.31) can, of course, be solved by general techniques (e.g. the method of characteristics [78, 83, 113, 346, 491, 492]), mere inspection of this special case immediately reveals that it is satisfied by

$$\mathfrak{S}(z, t) = \alpha_1 z e^{i\omega t - \lambda t} + \alpha_2, \quad (10.32)$$

which does contain two constants of integration α_1, α_2 . As usual, the additive constant α_2 arises because (10.31) does not involve the generator $\mathfrak{S}$ itself, but only its partial derivatives with respect to z or t . But, by the same argument, the additive constant is irrelevant for the solution of the dynamical problem as contained in the transformation formula (10.22). Applying the latter to (10.32), one readily concludes that $\alpha_1 = i z^*(0)$. It is finally easily verified that the currently obtained solutions for $z(t)$ and $z^*(t)$ indeed satisfy the original canonical equations (10.9)–(10.10). That is, in view of the time independent complex canonical transformation (10.6), they properly represent the classical linearly damped harmonic oscillator, (3.1) or (6.11)–(6.12), in real space.

10.2. Complex phase space dynamics

10.2.1. The Hamilton–Jacobi equation

So far certain elements have been given of the complex Hamilton–Jacobi formalism associated with Hasse’s classical dynamics. Let us now recall that Hasse’s Hamiltonian, (10.7), (10.1) or (9.12), has been derived in section 9 in the weak friction limit from the complex phase space quantum mechanical master equation (9.1), or (7.29). It will be interesting to investigate the underlying classical mechanics of the latter in a little more detail. Thereto we return to the beginning of section 7, or to Bopp’s modified theory in the second part of section 6. From either (7.7)–(7.8) or (6.45)–(6.46) we infer the so-called improper or quasi-Hamilton equations [59, 195, 196, 258, 288], which are repeated here in terms of the rescaled classical variable $z = a\sqrt{\hbar}$:

$$\dot{z} = -i \mathcal{H}_{,z^*}, \quad \dot{z}^* = i \mathcal{H}_{,z}, \quad (10.33)$$

$$\mathcal{H} = (\omega - i\lambda) z^* z. \quad (10.34)$$

As before, $\pi_z = i z^*$ can formally be considered as a conjugate momentum. Of particular importance,

^a Note the difference between $\mathfrak{S}_{,z} = \mathfrak{S}(z, t)_{,z}$ and $\mathfrak{S}_{,z}(t) = \mathfrak{S}_{,z}(z(t), t)$, although it turns out to be of no importance in the present case.

once again, will be to find the generator of the canonical transformation that is equivalent to solving (10.33). Suppose therefore, as in (10.22), that we are given a function $\mathcal{F}(z, t)$, with the property^{*}

$$i z^* = \mathcal{F}_{,z}. \quad (10.35)$$

According to this relation one then has available a complex function $z^* = z^*(z, t)$, which in principle suffices to obtain functions $z(t)$ and $z^*(t)$. The total differential of z^* , considered first as a function of z and t , and using the first equation from (10.33) and also (10.35), leads to

$$dz^* = -i z^*_{,z} \mathcal{H}_{,z} \cdot dt - i(\mathcal{F}_{,t})_{,z} dt. \quad (10.36)$$

This is the immediate analogue of (10.24). However, unlike (10.25) one presently has, on the other hand, from the second equation of (10.33), that

$$dz^* = i \mathcal{H}_{,z}^* dt. \quad (10.37)$$

Equating (10.36) and (10.37), and defining [24, 59, 195]

$$\mathcal{H} = H + i \Gamma, \quad (10.38)$$

H and Γ being real, one obtains

$$\mathcal{H}_{,z} + z^*_{,z} \mathcal{H}_{,z}^* + (\mathcal{F}_{,t})_{,z} = 2i \Gamma_{,z}, \quad (10.39)$$

in lieu of (10.26). Introducing the function $\mathcal{H}'(z, t) \equiv \mathcal{H}(z^*(z), z, t)$, the equation (10.39) reduces to

$$(\mathcal{H}' + \mathcal{F}_{,t})_{,z} = 2i \Gamma_{,z}. \quad (10.40)$$

One must be aware of the fact, that on the l.h.s. of (10.40) the partial derivative with respect to z concerns a function of t and z only, but that on the r.h.s. it applies to a function of (possibly) t , and both z and z^* . Keeping this in mind, (10.40) leads to the quasi-Hamilton–Jacobi equation

$$\mathcal{H}'(z, t) + \mathcal{F}_{,t}(z, t) = \mathcal{J}(z, t), \quad (10.41)$$

where the dissipative “inhomogeneity” $\mathcal{J}$ reads^{**}

$$\mathcal{J} = 2i \int_{\gamma} \Gamma_{,z}(\mathcal{F}_{,z}, \zeta) d\zeta. \quad (10.42)$$

^{*} We confine ourselves for the moment being to a one-dimensional complex valued process. The generalization of the present Hamilton–Jacobi formalism to higher dimensional systems is nontrivial, both classically and within a quantum mechanical context, as will be further clarified by the ultimate results and in section 11.

^{**} Clearly, this integration procedure may get in difficulty in higher dimensional systems. See also section 11, concerning Kostin’s nonlinear frictional Schrödinger equation [129, 136, 198, 199, 203, 207, 254].

Since in the absence of damping $\mathcal{H}^* = \mathcal{H} = H$ is real [59], so that $\Gamma = 0$, the “inhomogeneity” $\mathcal{J}$ vanishes identically in that case and (10.41) takes on its common structure.

10.2.2. The effective Hamiltonian

In the special case of the linearly damped oscillator, comparison of (10.34) and (10.38) shows that $\Gamma = -\lambda z^* z$. Hence, $\Gamma_{,z} = -\lambda z^*$ and by (10.35) the integrand in (10.42) becomes $i\lambda \mathcal{S}_{,z}$. Therefore, $\mathcal{J} = -2\lambda \mathcal{S}(z, t)$, apart from an irrelevant arbitrary function of time, and the quasi-Hamilton–Jacobi equation (10.41) yields:

$$2\lambda \mathcal{S} - (\lambda + i\omega)z \mathcal{S}_{,z} + \mathcal{S}_{,t} = 0. \quad (10.43)$$

It is interesting to compare this upshot with (10.31). According to the theory of characteristics (see e.g. [78, 83, 113, 346, 491, 492]), the general solution of (10.43) is given by

$$\mathcal{S} = e^{-2\lambda t} \mathcal{F}(y), \quad (10.44)$$

where $\mathcal{F}(y)$ is an arbitrary function of $y = z \exp(i\omega t + \lambda t)$. Applying the transformation relation (10.35) to (10.44) at the initial time $t = 0$ gives $\mathcal{F}_{,y} = i z^*(0)$, so that $\mathcal{F}(y) = i y z^*(0)$.^{*} Hence,

$$\mathcal{S}(z, t) = i z^*(0) e^{i\omega t - \lambda t}, \quad (10.45)$$

which is identical to the generator $\mathcal{S}(z, t)$ found in (10.32) for Hasse’s model.

Finally, it is quite interesting to note that (10.43) suggests the possibility of defining a proper, effective Hamiltonian, say $\mathcal{H}$, looking at it as if it were a genuine, “common” Hamilton–Jacobi equation. Then

$$\mathcal{H} = (\omega - i\lambda)z^* z + 2\lambda \mathcal{S}(z, t), \quad (10.46)$$

which generates the following proper canonical equations:

$$\dot{z} = -i \mathcal{H}_{,z^*} = -i\omega z - \lambda z, \quad (10.47)$$

$$\dot{z}^* = i \mathcal{H}_{,z} = i\omega z^* + \lambda z^* + 2i\lambda \mathcal{S}_{,z}. \quad (10.48)$$

Since, by (10.35), $\mathcal{S}_{,z} = i z^*(t)$ along the true trajectory, (10.48) is in fact identical to (10.10). So, (10.47)–(10.48) do correctly describe the damped oscillator and $\mathcal{H}$ can indeed be considered as a genuine classical Hamiltonian. It will be further exploited in the next section.

10.3. Summary

In the first part of this section the classical analogue of Hasse’s pure state Hamiltonian dynamics (section 9) has been investigated. The classical dynamics is independent of the diffusion coefficients.

^{*}The integration constant can be set equal to zero for it does not affect the transformation, as encountered earlier. For the same reason an additive constant does not appear in (10.44). Remember that (10.43) is unique, save for an arbitrary additive function of time.

Hasse's Hamiltonian has been rewritten in terms of Dekker's complex canonical variables. The generator of that particular time-dependent canonical transformation that solves the dynamics (and which need not leave the Poisson brackets invariant), has then been shown to obey a complex Hamilton–Jacobi equation. In the second part of this section similar techniques have been applied to the classical dynamics underlying both Bopp's modified theory (section 6) and the complex phase space quantization of section 7. A complex quasi-Hamilton–Jacobi equation has been derived, from which a new proper Hamiltonian could be identified. This Hamiltonian will be discussed further in section 11.

11. Kostin's nonlinear Schrödinger equation

11.1. Classical mechanics

In the previous section the ideas of a complex Hamilton–Jacobi formalism have been applied to both the Hasse and the modified-Bopp–Dekker dynamics for the damped oscillator. From the latter a novel proper classical Hamiltonian $\mathcal{H}$ of a peculiar nature emerged. In principle, the availability of a true Hamiltonian offers new vista's on the application of conventional quantization procedures. In view of the unconventional nature^{*} of $\mathcal{H}$, however, it seems wise to at least transform back to the real classical coordinate and momentum, x and p . Consult (10.6), (6.42)–(6.43), or the first columns of (6.3)–(6.4). Inserting

$$z = \frac{1}{\sqrt{2\omega}} [p + (\lambda - i\omega)x], \quad z^* = \frac{1}{\sqrt{2\omega}} [p + (\lambda + i\omega)x] \quad (11.1)$$

into (10.46), readily yields

$$\mathcal{H} = \frac{1}{2}(1 - i\lambda/\omega)(p^2 + 2\lambda px + \Omega^2 x^2) + 2\lambda \mathcal{S}(z(p, x), t). \quad (11.2)$$

Anticipating the real space Hamilton–Jacobi theory, a function $S(x, t)$ can be introduced, with the usual property

$$p = S_{,x}, \quad (11.3)$$

which is the real mechanical analogue of (10.22) or (10.35). Using (11.3) we define the function $\mathcal{S}'(x, t)$ according to

$$\mathcal{S}(z(p, x), t) = \mathcal{S}(S_{,x} + (\lambda - i\omega)x, t) = \mathcal{S}'(x, t), \quad (11.4)$$

and consider $\mathcal{S}'_{,x} = \mathcal{S}_{,x} z_{,x} = i z^* z_{,x}$. In the latter we have invoked (10.35). With the aid of (11.1) and (11.3) one then calculates

$$\mathcal{S}'_{,x} = \left[S - \frac{1}{2}px + \frac{i}{4\omega} (p^2 + 2\lambda px + \Omega^2 x^2) \right]_{,x}. \quad (11.5)$$

^{*} For instance, it is by no means clear what the quantum analogue of the complex generator $\mathcal{S}$ itself should be.

Hence,

$$\mathcal{S}(z(p, x), t) = S(x, t) - \frac{1}{2}px + \frac{i}{4\omega} (p^2 + 2\lambda px + \Omega^2 x^2), \quad (11.6)$$

plus an arbitrary function of time that may be set equal to zero by the same token as usual, i.e. dynamical invariance. Inserting (11.6) into (11.2), one obtains the upshot

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}\Omega^2 x^2 + 2\lambda S. \quad (11.7)$$

If, considering again p and x as independent conjugate variables, this result is used as the Hamiltonian in the canonical equations of motion, it leads to

$$\dot{x} = \mathcal{H}_{,p} = p, \quad (11.8)$$

$$\dot{p} = -\mathcal{H}_{,x} = -\Omega^2 x - 2\lambda S_{,x}. \quad (11.9)$$

Because, in view of (11.3), $S_{,x} = p(t)$ along the true trajectory, (11.8)–(11.9) are identical to (10.4), and, hence, indeed properly represent the original Newtonian damped oscillator (3.1).

The real classical Hamilton–Jacobi equation

$$\mathcal{H}(x, S_{,x}) + S_{,t} = 0, \quad (11.10)$$

associated with (11.7), becomes

$$2\lambda S + \frac{1}{2}\Omega^2 x^2 + \frac{1}{2}(S_{,x})^2 + S_{,t} = 0. \quad (11.11)$$

In this explicit form, (11.11) has been proposed earlier by Razavy [190, 226]. It is also mentioned in [43] within the framework of extended fluid dynamics.

11.2. Quantum mechanics

11.2.1. The variational principle

The Hamilton–Jacobi equation (11.11) appears to be amenable to quantization following a method originally due to Schrödinger [225]. In fact, the method is still usually cited in the textbooks on quantum mechanics as giving the exact solution to the (Rayleigh–Ritz [493–495]) variational problem (see e.g. [162–164, 300, 303, 314, 331, 496]). Its application to dissipative quantum physics stems from Razavy's work [190, 226]. The method, however, is not free of subtleties.^{*} Although we have not much to say concerning improvements at present, at least an attempt will be made at a viable formulation. In the

^{*} It is amusing to note Schrödinger's footnote [225]: "Es entgeht mir nicht daß diese Formulierung nicht ganz eindeutig ist". Notice further, comparing [225] and [190, 226], that Schrödinger introduces a real valued wave function ϕ (which certainly is allowed in the stationary states he considers), according to $S = \hbar \ln \phi$. However, in this way, the wave function is connected with the classical principal function S in a rather unusual way (consult e.g. [72, 113, 160, 163, 164, 201–231, 300, 303, 313, 314, 417, 497–501]). On the other hand, Razavy follows the common technique of introducing a complex valued wave function ψ (which certainly is required in the dynamical case), according to $S = (\hbar/i) \ln \psi$, $S = (i\hbar) \ln \psi^*$, or $S = (\hbar/2i) \ln(\psi/\psi^*)$. The ambiguity is that each of these expressions must be introduced at its "appropriate" place in the classical formulae. In fact, the two different connections of the wave function with the classical principal function hardly seem compatible. Finally, note that these problems are basically related to the undamped system, rather than to the dissipation. See further the present text.

first place it will be shown that the dynamical variational principle basically is a Lagrangian recipe. It reduces to the usual variation of the Hamiltonian (the Rayleigh–Ritz method for obtaining the eigenvalues and eigenfunctions) only at stationary states. Secondly, it is noted that in the classical theory there exists apparently only one *real* field $S(x, t)$. The subsequent introduction of a *complex* field $\psi(x, t)$, leading to a two dimensional variational problem (i.e. with $\delta\psi$ and $\delta\psi^*$ as independent alterations), will be given a little more attention than usual (see e.g. [190, 226]). Actually starting with the latter aspect, let us return to the Hamiltonian $\mathcal{H}(x, S_x)$ that can be read from (11.10) and (11.11). It is

$$\mathcal{H} = \frac{1}{2}(S_x)^2 + \frac{1}{2}\Omega^2 x^2 + 2\lambda S + \varphi(t), \quad (11.12)$$

where we have once more explicitly accounted for the invariance of the classical dynamics under the addition of an arbitrary time dependent function $\varphi(t)$ to the Hamiltonian.[•] Compare e.g. with (10.7). Next, a complex function is introduced according to

$$\psi(x, t|x_0) = A(t) \exp\left[\frac{i}{\hbar} S(x, t|x_0)\right], \quad (11.13)$$

where both S and the amplitude A are taken to be real. In the definition (11.13) it has been carefully explicated that the classical principal function, besides depending on x and t , also contains an integration constant. For convenience, this parameter has been identified with the initial value $x(0) = x_0$. For a classical, deterministic process the actual value of x_0 is a sure quantity, i.e. it is specified with probability one. However, in view of the probabilistic nature of quantum mechanics, it will be important to recognize x_0 in the wave function (11.13) as a basically free parameter. Actually, $\psi(x, t|x_0)$ is not yet a true wave function. For instance, it does not conserve total probability in the usual sense. Rather, it represents a quantum mechanical propagator (Green's function, or transition amplitude).^{••} See e.g. [70, 72, 116, 228, 231, 387, 417, 500, 501, 505]. In the form (11.13) it is known to be exact for the undamped (possibly driven) harmonic oscillator^{•••} [72]. From (11.13) one has

$$S = (\hbar/2i) \ln(\psi/\psi^*). \quad (11.14)$$

Schrödinger's quantum mechanical postulate now amounts to defining the expectation value of the Hamiltonian as the functional

$$K = \int \psi^* \mathcal{H} \psi \, dx, \quad (11.15)$$

so that $|\psi|^2$ attaches the meaning of a (relative [72]) probability density, and considering $K(\psi^*, \psi, t)$ as the

[•] This notion takes the place here of the additionally imposed normalization condition in [190, 226]. Actually, that subsidiary condition would lead to nonsensical results in the case of the semi-classical wave function (11.13), which, on the other hand, must be introduced in order to obtain a selfconsistent variational procedure leading to the standard linear Schrödinger equation if $\lambda = 0$. See also [502–504] and the previous footnote.

^{••} For the nondissipative linear (in ψ) Schrödinger equation the important superposition principle holds, so that a general wave function can be obtained as a linear weighted combination of propagators with different x_0 , i.e. $\psi(x, t) = \int \psi(x, t|x_0) \phi(x_0) dx_0$. Unfortunately, the superposition principle is invalidated for the nonlinear frictional Schrödinger equations, including Kostin's. See further on.

^{•••} The expression (11.13) is also the first step in the standard WKB-analysis. Recall that the WKB-eigenvalues are exact (in the first order analysis) for the harmonic oscillator. See e.g. [162–164, 230, 300, 303, 313, 314, 481, 496, 498, 499, 506–516]. Further, (11.13) is also exact within the Caldirola–Kanai model of the damped oscillator (section 4).

canonical generator of quantum dynamics. Notice, that (11.15) involves *both* S and A . Because there are no additional constraints in the present formulation, these functions can indeed be altered independently. Hence, alternatively, we may consider $\delta\psi$ and $\psi\psi^*$ as independent variations in the following. Inserting (11.12) with (11.14) into (11.15) straightforwardly leads to

$$K = -\frac{1}{8}\hbar^2 \int [(\psi^*/\psi)(\psi_{,x})^2 - 2\psi_{,x}^*\psi_{,x} + (\psi/\psi^*)(\psi_{,x}^*)^2] dx + \frac{1}{2}\Omega^2 \int x^2 \psi^*\psi dx - i\hbar\lambda \int \psi^*\psi \ln(\psi/\psi^*) dx + \varphi(t) \int \psi^*\psi dx. \quad (11.16)$$

The combination of the first and last term within the square (kinetic) brackets is easily further evaluated using the identity^{*}

$$(\psi^*/\psi)(\psi_{,x})^2 + (\psi/\psi^*)(\psi_{,x}^*)^2 = -2\psi_{,x}^*\psi_{,x} + (\psi^*\psi)_{,x}[\ln(\psi^*\psi)]_{,x}. \quad (11.17)$$

Evidently, in view of (11.13), the second term on the r.h.s. of (11.17) is definitely zero. Therefore, (11.16) reduces to

$$K = \int [\frac{1}{2}\hbar^2 \psi_{,x}^*\psi_{,x} + \psi^*\psi(\frac{1}{2}\Omega^2 x^2 - i\hbar\lambda \ln(\psi/\psi^*) + \varphi)] dx. \quad (11.18)$$

With $\lambda = 0$ this Hamiltonian is well-known in the standard theory of "second quantization" (see e.g. [22, 300, 331]). By means of one partial integration in the kinetic term, and putting $\lambda = \varphi = 0$ for convenience, (11.18) can be written as

$$K_0 = \int \psi^*(\frac{1}{2}p^2 + V)\psi dx, \quad (11.19)$$

with $p = -i\hbar\partial_x$ and $V = \frac{1}{2}\Omega^2 x^2$. The Hamiltonian K_0 is immediately recognized as the harmonic potential c-number analogue of the atomic reservoir Hamiltonian (8.3), that has been employed in the quantum optics model of section 8. Following the usual procedure, the Hamiltonian (11.18) can be obtained from the Lagrangian

$$L = i\hbar \int \psi^*\psi_{,t} dx - K(\psi^*, \psi, t). \quad (11.20)$$

It is instructive to compare the functional (11.20) with the function (10.8). Presently, the canonical momentum $\pi(x, t)$ conjugate to the field $\psi(x, t)$ is $\pi = i\hbar\psi^*$. The Lagrange variational principle $\delta \int L dt = 0$, taken over from classical mechanics, now becomes

$$\delta \int \int [i\hbar\psi^*\psi_{,t} - \frac{1}{2}\hbar^2\psi_{,x}^*\psi_{,x} - \psi^*\psi(\frac{1}{2}\Omega^2 x^2 - i\hbar\lambda \ln(\psi/\psi^*) + \varphi)] dx dt = 0, \quad (11.21)$$

which yields the Euler-Lagrange equation (see e.g. [1, 113, 300, 331])

^{*}This procedure is not unique. It aims, however, at a nondissipative Schrödinger equation linear in and containing only ψ .

$$i \hbar \psi_t = -\frac{1}{2} \hbar^2 \psi_{xx} + \frac{1}{2} \Omega^2 x^2 \psi - i \hbar \lambda \psi \ln(\psi/\psi^*) + (i \hbar \lambda + \varphi) \psi \quad (11.22)$$

in consequence of a variation $\delta\psi^* \neq 0$, and the complex conjugate equation by the independent variation $\delta\psi \neq 0$.

11.2.2. The Schrödinger equation

Clearly, in absence of dissipation (11.22) reduces to the usual Schrödinger equation, which is linear in ψ . Hence, in that case more general solutions than (11.13), which forms an essential element in the derivation of (11.22), can be readily constructed by superposition of propagators with different initial states. See also again the footnote following (11.13). Unfortunately, as will be obvious from (11.22), the superposition principle ceases to be valid as soon as $\lambda \neq 0$. The bold leap of the present theory is the postulate that, nevertheless, (11.22) does have physical significance for solutions other than (11.13), even if $\lambda \neq 0$.^{*} Of course, of particular interest are those solutions that can be normalized in the conventional way, thus providing a proper probability density $\rho(x, t) = |\psi|^2$. From (11.22) and its complex conjugate one finds

$$\rho_t + j_x = 2 \left(\lambda + \frac{1}{\hbar} \text{Im } \varphi \right) \rho, \quad (11.23)$$

where $j(x, t)$ represents the usual current density

$$j = (\hbar/2i)(\psi^* \psi_x - \psi \psi_x^*). \quad (11.24)$$

Compare this e.g. with (4.21)–(4.22). Clearly, total probability will be conserved in the course of time if $\text{Im } \varphi = -\hbar\lambda$. The real part of $\varphi(t)$ produces an irrelevant scalar phase shift in the wavefunction, as usual, and can be set equal to zero. The Schrödinger equation (11.22) can now finally be written as

$$i \hbar \psi_t = (H_0 + \lambda W_K) \psi, \quad (11.25)$$

H_0 again representing the free oscillator (2.8), and where

$$W_K = -i \hbar [\ln(\psi/\psi^*) - \langle \ln(\psi/\psi^*) \rangle]. \quad (11.26)$$

As before (see e.g. (2.9) for the Süssmann–Hase–Albrecht models, and (9.8) for Hasse's pure state model), we have added in (11.26) a non-observable time dependent function in order to let $\langle W_K \rangle = 0$. The result (11.25)–(11.26) represents Kostin's Schrödinger–Langevin^{**} equation [198, 199]. It has subsequently been discussed within the frameworks of the fluid dynamical interpretation of the

^{*} Intuitively, this may restrict the validity of (11.22) to the weak friction limit.

^{**} In fact, (11.25)–(11.26) represent the $T = 0$ case. At elevated temperatures one could add a random stochastic potential $V_R = x\xi(t)$, which is due to a classical thermal reservoir. Although this procedure is quite close to the Svin'in–Langevin treatment ([180, 181], and section 5) of the Caldirola–Kanai model (section 4), the present version is more satisfying. Namely, the pure classical Kostin–Langevin noise source $\xi(t)$ need not generate the zeropoint fluctuations of the tagged quantum mechanical oscillator. That is, here $\langle \xi(t + \tau)\xi(t) \rangle = 2D_K \delta(\tau)$ with $D_K = 2\lambda N_0 \hbar \Omega$, in lieu of $D = 2\lambda(N_0 + \frac{1}{2})\hbar\Omega$. See e.g. (1.9)–(1.11) and (5.64). Kostin's original papers did include V_R . It's implications have been further discussed by Messer [156].

Schrödinger equation [201–204], stochastic quantization [205–213] and Hamilton–Jacobi theory [43, 190, 226]. See further [129, 136, 156, 247–250] and the Historical survey.

The Kostin model (11.25)–(11.26) has the remarkable property that every stationary solution of the free oscillator also solves the damped problem.^{***} In general there is a regression towards these stationary states [203, 207, 517], according to^{***}

$$\psi_n = \phi_n(y) \exp \left[\frac{i}{\hbar} (y \dot{x}(t) + S_0) \right], \quad (11.27)$$

where $y = x - x(t)$, and where $x(t)$ represents the classical trajectory of the damped oscillator, i.e. (2.5) or (5.41).^{***} Further, the ϕ_n represent the undamped stationary state eigenfunctions:

$$\phi_n(y) = \mathcal{N}_n \exp[-i(n + \frac{1}{2})\Omega t - \Omega y^2/2\hbar] H_n(y\sqrt{\Omega/\hbar}), \quad (11.28)$$

with $\mathcal{N}_n = (\Omega/\pi\hbar)^{1/4} (2^n n!)^{-1/2}$. Finally, $S_0 = S_0(x(t), t|x_0)$ in (11.27) stands for the free oscillator action calculated along the dissipative trajectory, i.e.

$$S_0 = \int_0^t L_0(x(t'), \dot{x}(t')) dt', \quad (11.29)$$

$$L_0 = \frac{1}{2}\dot{x}^2(t) - \frac{1}{2}\Omega^2 x^2(t). \quad (11.30)$$

11.2.3. Wave packet solutions

At the ground state $n = 0$, the wave function (11.27) is a Gaussian. Compare with section 2, especially (2.11) and (2.19). In that case one obtains, with $\langle x \rangle = x(t)$:

$$|\psi|^2 = [2\pi\sigma_{xx}]^{-1/2} \exp \left[-\frac{(x - \langle x \rangle)^2}{2\sigma_{xx}} \right], \quad (11.31)$$

which compares with the wave packet (2.19) of the Süssmann–Hasse–Albrecht models at the stationary width $2\sigma_{xx} = w_x^2 = \hbar/\Omega$, i.e. for $c = 0$ (Albrecht's choice; see (2.12)). However, as has been shown by Remaud and Hernandez [136], the stability properties of the Kostin fluctuations are essentially different from the München-models. Computing the first moments directly from the nonlinear Schrödinger equation (11.25)–(11.26), one finds $\langle x \rangle' = \langle p \rangle$ and, using $p = -i\hbar\partial_x$:

$$\langle p \rangle' = -\Omega^2 \langle x \rangle + i\hbar\lambda \langle (\ln(\psi/\psi^*))_{,x} \rangle = -\Omega^2 \langle x \rangle - 2\lambda \langle p \rangle, \quad (11.32)$$

invoking the usual definition of expectation values, as contained for example in (11.15). Similarly calculating the equations of motion for the variances, the results for $\dot{\sigma}_{px}$ and $\dot{\sigma}_{xx}$ emerge quite

^{*} This property is destroyed if the random potential V_R is present i.e. at nonzero temperature (see previous footnote). Consult section 2 and [108, 109] for further comments.

^{**} The solution (11.27) differs somewhat from that given in [156, 207], but agrees (at least at $n = 0$) with that presented in [129, 136]. The difference is in the phase factor only.

^{***} Let in (5.41) $y(t)$ be $x(t)$.

straightforwardly and appear to be identical to (3.56) and (3.58) of the Bateman–Feshbach–Tikochinsky theory, (4.28) and (4.30) of the Caldirola–Kanai model, (5.70) and (5.72) of the Kanai–Svin’in treatment; and to (7.43) and (7.45), or to (9.23) and (9.25), if in the latter two cases one sets the diffusion coefficients $D_{px} + D_{xp} = D_{xx} = 0$. The equation for σ_{pp} is more involved and, as will be shown, leads to a somewhat unusual result. Obviously,

$$\langle p^2 \rangle' = -\Omega^2 \langle px + xp \rangle - \lambda \langle [p^2, \ln(\psi/\psi^*)] \rangle. \quad (11.33)$$

Writing the dissipative term on the r.h.s. explicitly in the coordinate representation, one obtains

$$\begin{aligned} \langle p^2 \rangle' = & -\Omega^2 \langle px + xp \rangle + \lambda \hbar^2 \int (\psi^* \psi_{,xx} - \psi_{,xx}^* \psi) dx - 2\lambda \hbar^2 \int \psi_{,x}^* \psi_{,x} dx \\ & + \lambda \hbar^2 \int [(\psi^*/\psi)(\psi_{,x})^2 + (\psi/\psi^*)(\psi_{,x}^*)^2] dx. \end{aligned} \quad (11.34)$$

By partial integration the first dissipative integral in (11.34) is noted to vanish, while the second leads to $-2\lambda \langle p^2 \rangle$. Moreover, applying (11.17) to the last integral on the r.h.s. of (11.34) leads to

$$\langle p^2 \rangle' = -\Omega^2 \langle px + xp \rangle - 4\lambda \langle p^2 \rangle + \lambda \hbar^2 \int (|\psi|^2)_{,x} (\psi_{,x}^*/\psi^* + \psi_{,x}/\psi) dx. \quad (11.35)$$

Substituting now the general form (11.31) for the density $|\psi|^2$ of a Gaussian wave packet into the integral in (11.35), gives

$$\langle p^2 \rangle' = -\Omega^2 \langle px + xp \rangle - 4\lambda \langle p^2 \rangle - (\lambda \hbar^2 / \sigma_{xx}) \int (x - \langle x \rangle) (|\psi|^2)_{,x} dx. \quad (11.36)$$

Either performing a partial integration, or once more using (11.31) and the very definition of σ_{xx} , yields the final result. In conclusion, the Gaussian variances for the Kostin model obey the following set of equations:

$$\dot{\sigma}_{px} = -2\lambda \sigma_{px} + \sigma_{pp} - \Omega^2 \sigma_{xx}, \quad (11.37)$$

$$\dot{\sigma}_{pp} = -4\lambda \sigma_{pp} - 2\Omega^2 \sigma_{px} + \hbar^2 \lambda / \sigma_{xx}, \quad (11.38)$$

$$\dot{\sigma}_{xx} = 2\sigma_{px}. \quad (11.39)$$

This result agrees with [136]. The stationary solution, contained in fact in the state $\psi = \psi_0$ of (11.27), is easily found: $\sigma_{px}(\infty) = 0$, $\sigma_{pp}(\infty) = \Omega^2 \sigma_{xx}(\infty)$ and $\sigma_{pp}(\infty) \sigma_{xx}(\infty) = \hbar^2/4$. That is, there is minimum uncertainty with $\sigma_{xx}(\infty) = \hbar/2\Omega$ and $\sigma_{pp}(\infty) = \hbar\Omega/2$, representing indeed the free oscillator ground state. Notice that, due to (11.38), the dynamics of the quantum mechanical fluctuations for the Gaussian Kostin-oscillator can not be brought into the form of any of the other models treated in the previous sections. Actually, the general dynamical solution of (11.37)–(11.39) is not known in closed form [136]. However, if $\sigma_{xx} \gg \hbar^2\lambda$ the Kostin equations (11.37)–(11.39) reduce to (3.56)–(3.58), so that for large

coordinate spread the solutions are given by functions of the type (3.59) and (3.60).^{*} What will happen if the variances come close to the oscillator ground state values, is clarified by means of a linear stability analysis of (11.37)–(11.39). Setting $\sigma_{xx}(t) = \sigma_{xx}(\infty)[1 + \epsilon(t)]$, (11.38) can be linearized to

$$\dot{\sigma}_{pp} = -4\lambda\sigma_{pp} - 2\Omega^2\sigma_{px} - 4\lambda\Omega^2\sigma_{xx} + 4\hbar\lambda\Omega. \quad (11.40)$$

Note in passing, that (11.40) cannot be obtained from a diffusion equation with constant (i.e. x -independent) diffusion coefficient D_{pp} . The eigenvalues of (11.37), (11.39) and (11.40) are -4λ and $-\lambda \pm 2i(\Omega^2 - \lambda^2/4)^{1/2}$. Hence, the Kostin-oscillator widths (i.e. for instance, the energy) are asymptotically stable about the ground state. Compare this with the remarks below (2.21), concerning the München-models.

Let us finally note, more or less for the sake of completeness, that because (i) the Kostin-oscillator is described by means of a Schrödinger wave function, because (ii) the canonical momentum $p = -i\hbar\partial_x$ is identical to the mechanical momentum, and because (iii) a Gaussian wave packet solution exists, one may conclude that the “normal” uncertainty relation (2.18) can be applied to (11.37)–(11.39). Hence, as for the Süssmann–Hasse–Albrecht oscillators, one can find a second order equation for σ_{xx} separately.^{**} Introducing once more the width w_x according to $w_x^2 = 2\sigma_{xx}$, one obtains

$$\ddot{w}_x + 2\lambda\dot{w}_x + \Omega^2 w_x = \hbar^2/w_x^3. \quad (11.41)$$

Compare this with (2.20), and notice the presently occurring damping term on the l.h.s. of (11.41), which explains the gross differences between the Kostin and the original München-models. Analysis of (11.41) completely confirms the earlier conclusions concerning the stability of the Kostin-width about its ground state value $w_x(\infty) = (\hbar/\Omega)^{1/2}$, as it should [136].

11.3. Summary

The classical Hamiltonian, obtained in section 10 on the basis of the complex quasi-Hamilton–Jacobi formalism associated with the modified Bopp–Dekker dynamics (sections 6 and 7), has been written in terms of the real space canonical variables. It has subsequently been quantized using an adapted dynamical version of the original Schrödinger–Razavy theory. The result is Kostin’s nonlinear frictional Schrödinger equation. It violates the superposition principle and is difficult to generalize to higher dimensional systems. Its one-dimensional stationary state and Gaussian wave packet solutions have been discussed in some detail. The latter always decays into the free oscillator ground state. The Kostin damped oscillator respects Heisenberg’s uncertainty principle.

12. Summary and final remarks

Dissipation in classical and quantum mechanics has been discussed from different point of views, both microscopically and phenomenologically. In fig. 12.1 the general interrelations between the various

^{*} The eigenvalues of (3.56)–(3.58) are -2λ and $-2(\lambda \pm i\omega)$.

^{**} This is most easily done taking the derivative $\dot{\sigma}_{xx} = 2\dot{\sigma}_{px}$ from (11.39), and using (11.37) in order to express $\dot{\sigma}_{px}$ in terms of σ_{px} , σ_{pp} and σ_{xx} . Multiplying the resulting equation on both sides with σ_{xx} , then invoking (2.18) to eliminate the product $\sigma_{pp}\sigma_{xx}$, and finally re-expressing σ_{px} in terms of $\dot{\sigma}_{xx}$ by means of (11.39), leads to the required equation of motion. Notice that (11.38) has not been used.

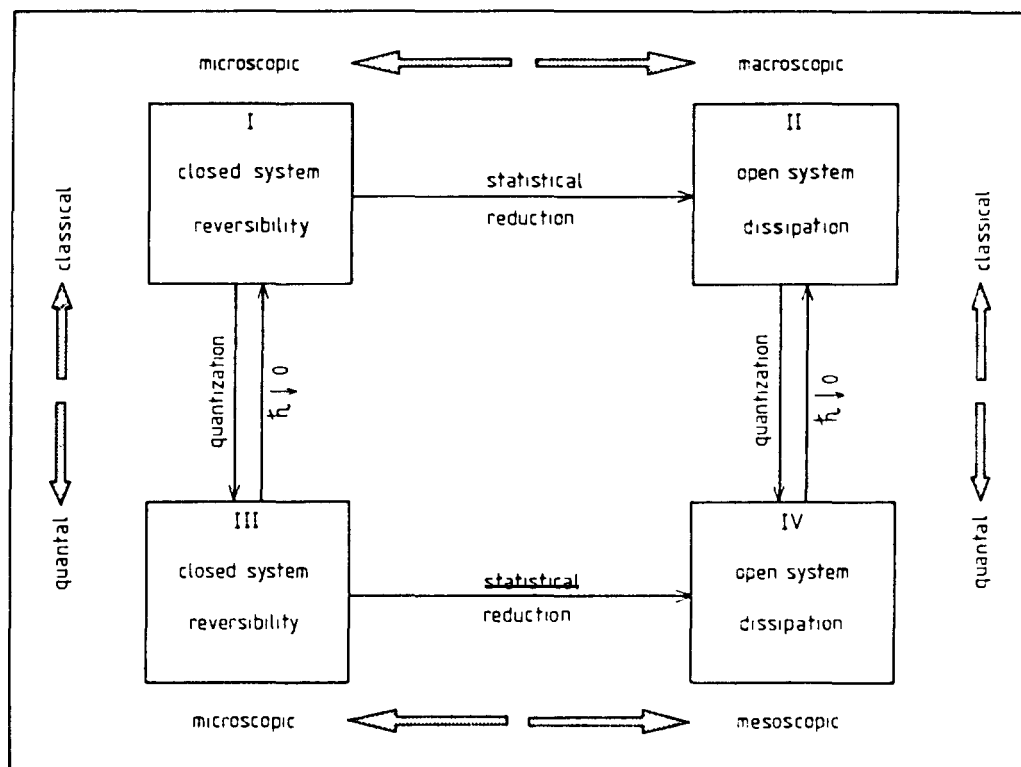


Fig. 12.1. Interrelations between various levels of description of physical systems.

descriptions are depicted. The present article has been written with the intention to clarify the more precise connections between several specific, seemingly disjunct models for the simplest conceivable nontrivial system: the linearly damped harmonic oscillator. In that sense, the text aims at a unified treatment. On the other hand, it attempts to reveal the differences between the various approaches as well as their intrinsic difficulties, in particular in the quantum domain. The treatment is a physicist's and has not been aimed at mathematical rigor.

As usual, the logical context overrules the historical developments. After the general introduction in section 1 on reversible and irreversible phenomena, and on the relation between dissipation and fluctuations, a historical survey has been given in section 2. With reference to that survey for more information, fig. 12.2 shows a rough sketch of the historical advent of the theories discussed in the subsequent sections, as well as their position within fig. 12.1.

Aside from some remarks concerning Havas' example of a typically unphysical Hamiltonian [8], section 2 also contains a brief discussion of the mesoscopic Süssman–Hasse–Albrecht or München-models for the damped quantum oscillator [129]. Since the canonical momentum and the mechanical momentum are equal, while the system is described by means of a Schrödinger wave function, these models respect Heisenberg's uncertainty principle. On the other hand, however, they lead to the unexpected feature of non(asymptotically) stable fluctuations.

In section 3 Bateman's classical dual Hamiltonian model, comprising the damped oscillator plus mirror image, and its quantization following Feshbach and Tikochinsky is considered [11, 115]. The basic commutator is incorrect and, hence, Heisenberg's principle is violated.

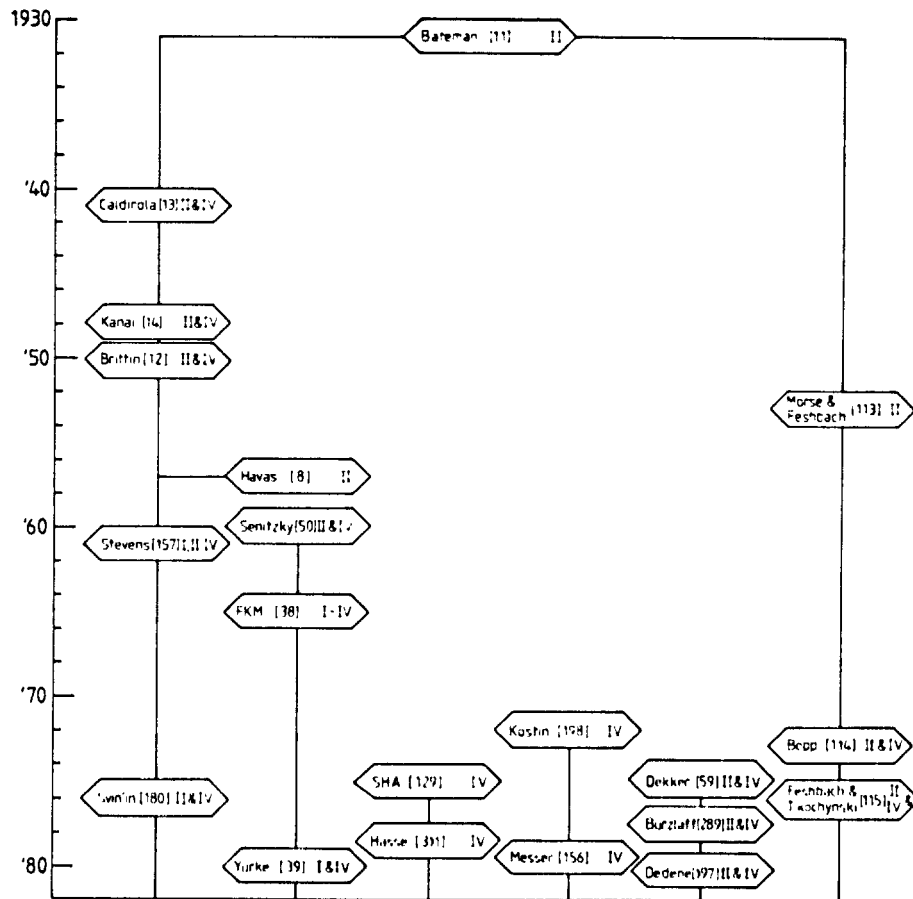


Fig. 12.2. Historical survey of publications on the damped harmonic oscillator. Roman numbers refer to fig. 12.1.

Section 4 has been devoted to Bateman's time-dependent Hamiltonian model. It is related to the dual model by extending the dynamical variables into the complex plane and performing a canonical transformation. The quantum mechanics of this model (related to a mass-accreting oscillator rather than an energy dissipating one) following Caldirola and Kanai [13, 14], involves an exponentially decaying fundamental commutator and thus kills the quantum fluctuations in the course of time.

The classical semi-infinite transmission line models of Stevens [157] and Yurke–Yurke [39] (which are closely related to the Ford–Kac–Mazur model [38]) are investigated in section 5. It is shown that the Caldirola–Kanai theory applies to the smoothed (or: noise-subtracted) dynamical variables rather than the actual ones. Unfortunately, the required infinite number of degrees of freedom in the model prohibits the calculation of all observable quantities. This can, in a sense, be remedied by Svin'in's mesoscopic quantum mechanical treatment of the Caldirola–Kanai model [180], as is also outlined in section 5.

In section 6 Dekker's complex dynamical variables are introduced into Bateman's dual Hamiltonian by means of a canonical transformation and Dedene's complex symplectic formulation of the damped oscillator is examined [197]. Its quantum mechanics violates the uncertainty principle for the same reasons as with the original Bateman model. The second part of section 6 is concerned with a modified version of

Bopp's theory [114]. In terms of the complex variables the correct quantum mechanical commutator is imposed on the physical system, which is separated from its artificial mirror image. The resulting density operator (or: quantum mechanical master) equation describes mixed rather than pure states. The uncertainty principle is respected. The dissipation increases the uncertainty product over its vacuum value.

Section 7 presents a discussion of Dekker's complex phase space quantization [195], starting from Dedene's symplectic formulation. It leads to the same quantal master equation as the modified Bopp-theory, and, hence, also defers to Heisenberg's principle.

Section 8 describes the quantum optics oscillator [22, 55]. The quantum mechanical model-Hamiltonian yields the modified Bopp–Dekker master equation for the reduced density operator in a weak damping limit.

In that limit, and in a Gaussian approximation, this density operator equation can be replaced by a nonlinear frictional Schrödinger equation with a non-Hermitian but normconserving Hamiltonian, according to a theory due to Hasse [311]. It is studied in section 9.

In section 10 a (quasi-) Hamilton–Jacobi formalism has been outlined for the classical complex variables dynamics associated with Hasse's and the modified Bopp–Dekker model. In the case of the latter a new dissipative Hamiltonian emerged.

This novel Hamiltonian has been quantized in section 11 using a modified dynamical version of the Schrödinger–Razavy variational procedure [190]. It leads to Kostin's nonlinear Schrödinger equation, which guarantees the validity of Heisenberg's principle [198].

The essentials of the relations between the models discussed in this article can be seen from the diagram in fig. 12.3.

Although completeness is certainly not claimed, it is felt that the present text covers a substantial portion of the relevant work done during the last half century. All models agree on the classical

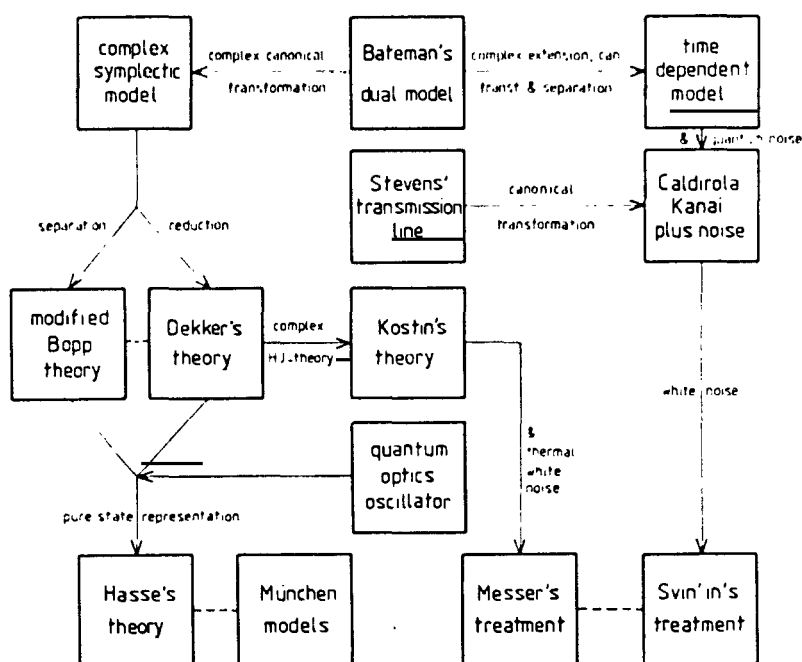


Fig. 12.3. Block diagram showing specific relations between various theories of the damped harmonic oscillator.

dynamics. From the quantal point of view this is to say that they all satisfy Ehrenfest's principle. However, the actual quantum mechanics of the various models reveals a considerable variety in fluctuation behaviour.

Closer inspection of the models further shows that none of them, neither the microscopic nor the mesoscopic ones, are completely satisfactory in all respects. For instance, the Bateman–Feshbach–Tikochinsky theory is at variance with the uncertainty principle, even in the limit of vanishing dissipation; the transmission line model involves infinities; Svin'in's classical reservoir requires quantum mechanical properties; Bopp's modelling is, in fact, in line with Ehrenfest's principle only for certain special, nearly classical initial conditions; for arbitrary nondissipative Hamiltonians, Dekker's theory is not always exactly equivalent to the usual Schrödinger description; the evaluation of the quantum optics oscillator involves quite a number of delicate steps; Hasse's pure state representation condition is usually not satisfied in the dynamical state; and Kostin's Schrödinger equation can hardly be generalized to more than one single oscillator. This list should suffice here. For further comments the reader is referred to the pertinent sections.

Let us in the end again remember that dissipation basically arises from microscopic time-reversible interactions, and that it is observed essentially looking at a subsystem of the universe. Such problems are intrinsically difficult, although the observed subsystem dynamics is often relatively simple. In conclusion, there can be no doubt that dissipative phenomena – being in their modelling often at the borderline between microscopy and macroscopy – will continue to be a challenging subject [431, 530–563].*

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* Ref. [530] seems to be the earliest reference to the problem of quantization of a damped system. Ref. [531] presents a first derivation of the quantum mechanical operator equation for a radiating electron [532]. See also [533].

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