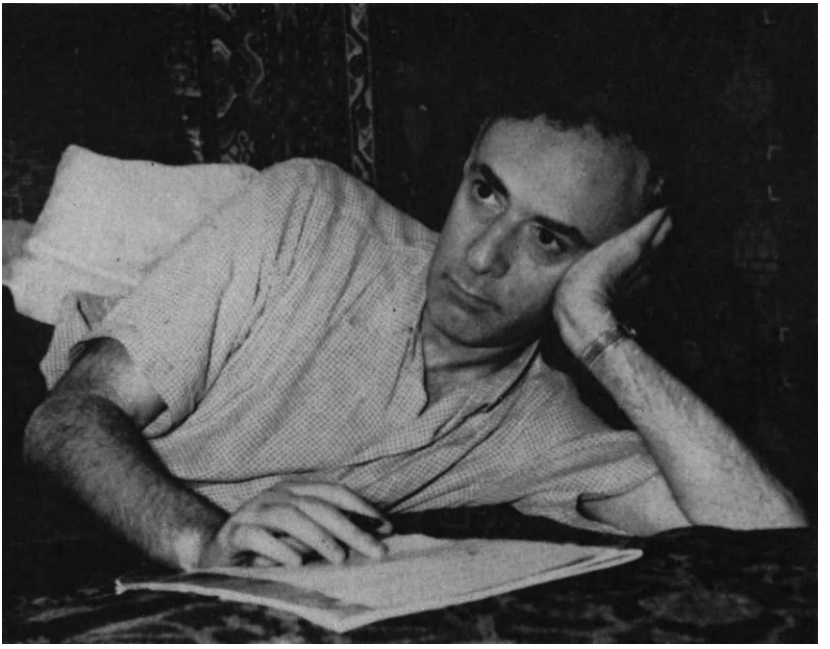

L. D. LANDAU.

(*to face p. iii*)

MEN OF PHYSICS

L. D. LANDAU

VOLUME 2: *Thermodynamics, Plasma Physics and Quantum Mechanics*

BY

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Preface

THE Nobel Prize in Physics for 1962 was awarded to L. D. Landau for his work on the theory of condensed media, and especially for his work on the theory of liquid helium. Recently, the *Collected Papers* of L. D. Landau have been published, and it is clear from those that the Nobel Prize might equally well have been awarded for Landau's work in plasma physics, in high energy physics, in quantum mechanics, or in the theory of magnetism. Unfortunately, it is unlikely that the *Collected Papers* will be widely available to undergraduates except in libraries, and it was therefore felt desirable to issue some of the most important papers in the form of two paperbacks and provide them with an introduction intended to make the undergraduates familiar with at least some of Landau's work, apart from his textbooks.

The first volume of the paperbacks contained eight papers: two on the theory of helium II; two on the theory of Fermi liquids; two on superconductivity; one on electron diamagnetism; and one on ferromagnetism. The second volume contains ten papers: one on the theory of phase transitions; two on plasma physics; one on the statistical model of nuclei; one on stellar energy; one on the multiple production of particles in cosmic rays; one on the uncertainty principle in relativistic quantum mechanics; one on the quantum theory of collisions; and two on field theory.

I should like to express my thanks to Professor E. M. Lifshitz for his assistance in obtaining a complete list of Landau's papers and his help in selecting from this *embarras de richesses* those contained in the two paperbacks, and to Dr. W. E. Parry and Professor R. E. Peierls for critical comments on the introduction. My gratitude is also due to Springer Verlag, the American Institute of Physics, the North Holland Publishing Company, and Wiley—

Interscience, for permission to reprint papers which first appeared in the *Zeitschrift für Physik*, *Soviet Physics-JETP*, *Nuclear Physics*, and the *Pauli Memorial Volume*.

Oxford D. TER HAAR

Introduction

IN THE first volume of the set of two paperbacks (ter Haar, 1965) dealing with Landau's greatest contributions to physics, we discussed his contributions to low temperature physics and solid state physics. In the present volume we shall discuss some of Landau's contributions to thermodynamics, plasma physics, and quantum theory. The introductory part of this volume is divided into five chapters. In Chapter I we discuss Landau's theory of second-order phase transitions, a theory which has found many applications and is of great general value. Chapter II is devoted to plasma physics. We discuss his papers on the transport equation for ionized systems and on plasma oscillations. In Chapter III Landau's contributions to nuclear physics, cosmic ray physics, and astrophysics are discussed; Chapter IV deals with his work in pure quantum mechanics, while in Chapter V we briefly consider his contributions to field theory. Even though the two volumes will have covered many branches of physics, there are still large areas which we do not have the space to explore. For his contributions to hydrodynamics or theoretical chemistry, for instance, we must refer the reader to his *Collected Papers*.†

† Just before the reprint section, on p. 54 of the present volume, we have given a complete list of Landau's scientific papers as they appear in the *Collected Papers* volume published by Gordon & Breach, New York, and Pergamon Press. When referring to papers by Landau we shall always quote the appropriate number from the list (for instance, Landau 10 refers to a paper on polarons).

I

Second-order Phase Transitions

LANDAU'S main original contribution to thermodynamics has been his theory of second-order phase transitions. Of the other papers in this field we mention first of all a general discussion with Bronstein on the irreversibility of thermodynamic phenomena (Landau 11) in which they come out against Boltzmann's idea that the universe as a whole is in statistical equilibrium and argue that there are regions in the universe which are not covered by the classical and quantum mechanical theories which lead to thermodynamics (compare his discussion of stellar energy, considered in Chapter III). He also considered the theory of the accommodation coefficient (Landau 20) and the limitations imposed by thermodynamics upon the yield of photoluminescence and on the distribution of the radiation intensity in such processes (Landau 62).

In 1935 Landau (17) started a discussion on second-order phase transitions, paying special attention to the specific heat anomaly at the transition point (or λ-point), and 2 years later he developed this into a more general theory (Landau 29: this paper appeared in two parts, the first part is reprinted as paper 1 at the end of this volume: see also Landau and Lifshitz, 1958, chap. XIV). This theory has been the starting point of many investigations, and in the present chapter we shall attempt to sketch an outline of it. We should mention here that an essential feature of Landau's theory is his assumption that all thermodynamic functions would allow regular series expansions near the transition point. Theoretical work on soluble models—such as the two-dimensional Ising model —as well as experimental data, have shown this assumption to be

erroneous, but Landau's papers focused attention on this problem, and most of his arguments about symmetries, and so on, remain valid.

Landau himself applied his theory to the problem of X-ray scattering by crystals in the neighbourhood of the Curie point (Landau 32) and with Khalatnikov to the problem of sound absorption near the λ-point of liquid helium (Landau 82).

There are various kinds of phase transitions. On the one hand, there are such transitions as those between a liquid and a gas. We know that this is a first-order phase transition below the critical point, involving a latent heat, but that above the critical point the transition can take place without involving a discontinuity in some of the thermodynamic functions. On the other hand, the transition between a crystal and a liquid or between different crystal modifications can never take place without a discontinuity in some of the thermodynamic functions. The two phases have different symmetries, and in any given state the system has either the one symmetry or the other. On the other hand, while in a first-order transition the energy and often quantities such as the volume change discontinuously, in a second-order phase transition, the symmetry changes discontinuously, but the energy and the volume change continuously. This change in symmetry we shall call a change in order, and we shall, with Landau, introduce a parameter η which characterizes the order which disappears at the transition point T_c so that for temperatures above T_c, $\eta = 0$. As a typical example—and one to which we shall refer several times—we mention the Ising model of a ferromagnet in which there is a spin on each lattice site which can be in one of only two possible orientations (see, for instance, ter Haar, 1954, chap. XII, for a discussion of this model and its relation to other order–disorder problems). The two-dimensional Ising model was solved exactly by Onsager (1944) for the square lattice, and he found that the specific heat at the transition temperature diverged as $\ln | T - T_c |$. There are at the moment strong indications (Fisher, 1963) that such a logarithmic divergence is the general behaviour of the specific heat at the transition temperature for all second-

order phase transitions, and later in this chapter we shall see how this can be understood.

In Landau's theory one assumes that the thermodynamic potential Φ can be expanded in the neighbourhood of the transition point (and below it, as above T_c: $\eta = 0$) in a power series in η. From symmetry considerations, the term linear in η does not occur, and from the usual thermodynamic formulae (see Landau and Lifshitz, 1958, for details) one can calculate the specific heat except at the so-called critical points. We saw a moment ago that we cannot have a continuous transition between two states of different symmetry such as can take place between a gas and a liquid; the second-order phase transition curve in a PT diagram cannot simply stop at a point. It may, however, change at some point into a first-order transition: such a point is called a critical point, and if we use a power series expansion in η for Φ, we find a $(T_c - T)^{-\frac{1}{2}}$ divergence.

Levanyuk (1963) has slightly extended Landau's theory, but possibly the most interesting progress has been made by Vaks and Larkin (1966)—two theorists from the Landau school—who show that under quite general conditions second-order transitions are mathematically equivalent to the Ising model; in the following we shall sketch their arguments. Consider the case where the change in symmetry is caused by the displacement of one of the atoms in the unit cell from a symmetry point to a position of lower symmetry. Assume that the potential energy of this atom can be expressed as the sum of the average periodic potential $U(\boldsymbol{r}_s)$ due to the other atoms and the interaction potential $V(\boldsymbol{r}_s - \boldsymbol{r}_{s'})$, of the atom considered with similar atoms in other cells. To simplify the considerations we shall assume that $U(\boldsymbol{r}_s)$ has equal minima at two points placed symmetrically with respect to the centre of the cell, at distances $\pm\boldsymbol{b}$ from it. If the temperature is sufficiently high, so that $\beta V \ll 1$ ($\beta = 1/k_B T$, k_B: Boltzmann constant), we can neglect V and the probability to be at $+\boldsymbol{b}$ or $-\boldsymbol{b}$ in any one cell is the same, and the crystal will be symmetric about the cell centre. On the other hand, at low temperatures when $\beta V \gg 1$, the interaction V will cause an ordering, practically all atoms being at

$+\boldsymbol{b}$ (or at $-\boldsymbol{b}$). This is a state of lower symmetry, and the change in symmetry will occur at a temperature for which $\beta_c V$ will be of the order of magnitude of unity.

Consider now the partition function Z of the system:

$$Z = \int \Pi_s \left\{ d^3 p_s \exp\left(-\beta \boldsymbol{p}_s^2/2m \right) \right\} \int \Pi_s d^3 \boldsymbol{r}_s \exp\left[-\beta \left\{ \sum_s U(\boldsymbol{l}r_s) + \sum_{s,s'} V(\boldsymbol{r}_s - \boldsymbol{r}_{s'}) \right\} \right]. \tag{1.1}$$

We shall assume that $\beta_c U \gg 1$ (that is, we assume that U is much larger than V) and that V has a sufficiently short range so that we need only consider in the last sum in (1.1) interactions between atoms in adjacent cells.

We now write for $\boldsymbol{r}_s$:

$$\boldsymbol{r}_s = \boldsymbol{r}_{os} + \xi_s \quad = \quad \Sigma\, s_\alpha \boldsymbol{a}_\alpha \pm \boldsymbol{b} + \xi_s, \tag{1.2}$$

where the s_α determine the location of the centre of the cell, where the $\boldsymbol{a}_\alpha$ are the basic lattice vectors, and where the ξ_s measure the displacements of the atoms from their equilibrium position.

Under our assumptions that U is much larger than V and than β^{-1}, it is sufficient to expand U up to second-order terms in the ξ_s and to neglect the dependence of V on the ξ_s. We then have

$$\sum_s U(\boldsymbol{r}_s) + \sum_{s,s'} V(\boldsymbol{r}_s - \boldsymbol{r}_{s'}) =$$

$$\sum_s \left[U(\boldsymbol{r}_{os}) + \frac{1}{2} \sum_{\mu,\nu=1}^{3} U_{\mu\nu} \xi_s^\mu \xi_s^\nu \right] + \sum_{s,s'} V(\boldsymbol{r}_{os} - \boldsymbol{r}_{os'}), \tag{1.3}$$

where

$$U_{\mu\nu} = \left(\frac{\partial^2 U}{\partial x_\mu \, \partial x_\nu} \right)_{r_s = r_{os}}. \tag{1.4}$$

Introduce now an operator σ_r such that $\sigma_r = +1$ corresponds to $\boldsymbol{r}_{os} = \Sigma s_\alpha \boldsymbol{a}_\alpha + \boldsymbol{b}$ and $\sigma_r = -1$ to $\boldsymbol{r}_{os} = \Sigma s_\alpha \boldsymbol{a}_\alpha - \boldsymbol{b}$, or

$$\boldsymbol{r}_{os} = \Sigma s_\alpha \boldsymbol{a}_\alpha + \boldsymbol{b}\sigma_r \tag{1.5}$$

The last term on the right-hand side of (1.3) can then be written as

$$C + \tfrac{1}{2} \sum_{r\alpha} J_\alpha \sigma_r \sigma_{r+a_\alpha}, \tag{1.6}$$

where C is a constant and

$$J_\alpha = \tfrac{1}{2} [2V(a_\alpha) - V(a_\alpha + 2b) - V(a_\alpha - 2b)]. \tag{1.7}$$

Substituting (1.6) into (1.3) and then into (1.1) and integrating over all p_s and ξ_s, we get

$$Z = Z_0(T) \sum_{\text{all } \sigma r = \pm 1} \exp [- \tfrac{1}{2} \beta \sum_{r,\alpha} J_\alpha \sigma_r \sigma_{r+a_\alpha}]. \tag{1.8}$$

Let us now consider the partition function of a three-dimensional Ising model. If K_α is the exchange interaction between a spin on a site r and a site $r + a_\alpha$, and if μ_r is $+1$ if the spin at the site r is up and -1 if the same spin is down, the energy of the spin system will be

$$\tfrac{1}{2} \sum_{a,\alpha} K_{\alpha r} \mu r \mu r_{r + a\alpha}$$

and the Ising partition function Z_{Ising} will be

$$Z_{\text{Ising}} = \sum_{\text{all } \mu r = \pm 1} \exp [- \tfrac{1}{2} \beta \sum_{r,\alpha} K_\alpha \mu_r \mu_{r + a_\alpha}]. \tag{1.9}$$

We see thus that in our present approximation the specific heat near T_c for the case of a second-order phase transition in crystals behaves like the specific heat near the Curie point of an Ising ferromagnet—which is probably a $\ln | T - T_c |$-kind behaviour.

Vaks and Larkin make it plausible that the influence of atomic vibrations and of quantum effects will not alter the above conclusions.

In conclusion we should once again mention that although Landau's analytical considerations of the behaviour of the thermodynamic quantities near the transition point are incorrect, his general discussion about the relation between the symmetry changes and the kind of phase transitions to be expected remain valid and have been applied successfully by many authors.

II

Plasma Physics

A STUDY of the behaviour of ionized gases is important in many branches of physics and engineering, and since the Second World War such a study has been made in many cases in connection with thermonuclear research. Among the properties of an ionized gas one studies are the transport coefficients such as the electrical and thermal conductivity and the damping of waves (for a general account of these problems see Spitzer, 1962). The model we shall be mainly concerned with in the present chapter is one of (negatively charged) electrons moving in a compensating, uniform positively charged background. Strictly speaking, one should take the motion of the positive ions into account, but to a first approximation the influence of the ions can be taken into account by assuming the positive charge to be uniformly distributed over space. Such a system is called a *plasma*, a term first introduced by Langmuir (1928). We shall also assume that the system as a whole is electrically neutral so that the positive charge density is equal to the product of the electron density multiplied by the elementary charge.

To evaluate transport coefficients of a gas, one often uses a transport or kinetic equation (see, for instance, Chapman and Cowling, 1953), and we shall return to this approach presently, but we shall first discuss why the case of a system of particles interacting through Coulomb forces (or gravitational forces) needs special treatment. To see this, we shall consider collisions between charged particles. The dynamics of a collision are determined not only by the momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ of the colliding particles, but also by the impact parameter ρ that is, the closest distance at which

particle 1 would pass particle 2, if there were no interactions (Fig. 2.1). This is the problem of Rutherford scattering and one can evaluate the scattering cross-section exactly (see, for instance, ter Haar 1967, Ch. III). Here, however, we shall follow the treatment given by Landau in his paper on the kinetic equation for a system of charged particles (Landau 24; reprinted as paper 2 at the

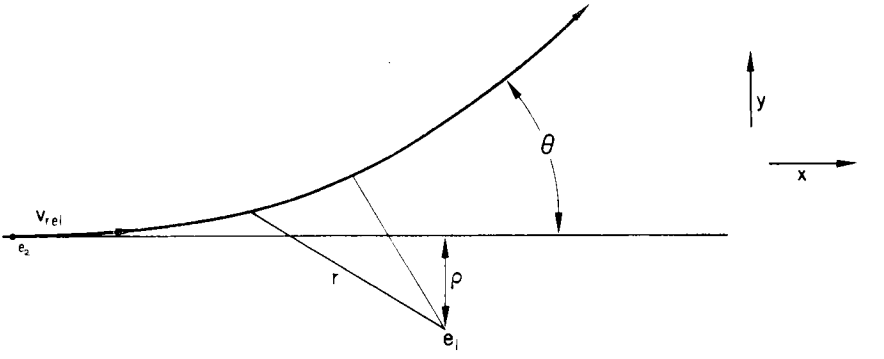

FIG. 2.1. Collision of particles 1 and 2 in frame of reference in which e_1 is at rest.

end of this volume). Landau's treatment is valid when the deflection of the particles is small. Let v_{rel} be the relative velocity of particle 2 with respect to particle 1 (see Fig. 2.1, where the collision is described in the frame of reference in which particle 1 is at rest), and let us take the x-axis along v_{rel} at $t = -\infty$. If the deflection is small, we may assume that p_{2x} does not change during the collision, and the only change in p_2 is in a direction at right angles to the x-axis, say the y-direction. The change Δp_{2y} is given by the equation

$$\Delta p_{2y} = -\int_{-\infty}^{+\infty} F_y dt, \tag{2.1}$$

where the force in the y-direction F_y, is given by the equation

$$F_y = -\frac{e_1 e_2}{4\pi\epsilon_0 r^3} y. \tag{2.2}$$

In the approximation of small deflections we can replace the motion by a rectilinear one and replace r^2 by $\rho^2 + v_{rel}^2 t^2$ and y by ρ, and we then get from (2.1) and (2.2)

$$\Delta p_{2y} = \int_{-\infty}^{+\infty} \frac{e_1 e_2 \rho dt}{4\pi\epsilon_0 (\rho^2 + V_{rel}^2 t^2)^{3/2}} = \frac{e_1 e_2}{2\pi\epsilon_0 \rho v_{rel}}. \tag{2.3}$$

This change in p_2 produces in the same approximation a change in energy ΔE_2 given by (we have put $e_1 = e_2 = e$)

$$\Delta E_2 = \frac{\Delta p_{2y}^2}{2m} = \frac{e^4}{8\pi^2\epsilon_0^2 m\rho^2 v_{rel}^2}. \tag{2.4}$$

The average energy† transferred per unit time, $<\Delta E>$, by collisions will be obtained from (2.4) by averaging over all possible impact parameters. If n is the number of particles per unit volume, there will be $nv_{rel}.2\pi\rho d\rho$ collisions per unit time with values of ρ between ρ and $\rho + d\rho$. For $<\Delta E>$ we thus get

$$<\Delta E> = \frac{n}{v_{rel}} \frac{e^4}{4\pi m\epsilon_0^2} \int_{\rho_1}^{\rho_2} \frac{d\rho}{\rho} = \frac{ne^4}{4\pi\epsilon_0^2 m v_{rel}} \ln \frac{\rho_2}{\rho_1}, \tag{2.5}$$

where ρ_1 and ρ_2 have to be determined.

We note here the problem peculiar to the inverse square force law: the range of the forces is so long that it is impossible to speak of the "duration" of a collision as can be done for the case of short-range forces. As a result any particle will be concerned with collisions with many other particles. This makes it difficult to write down a kinetic equation for the distribution function. Luckily, the dependence on ρ_1 and ρ_2 is only logarithmic so that we should get a reasonable estimate for $<\Delta E>$ from qualitative arguments. The lower limit ρ_1 can be taken to be the impact parameter for which the angle of deflection θ which is given by the equation [see Fig. 2.1. and eqn. (2.3)]

$$\theta \sim \tan\theta \sim \frac{\Delta p_{2y}}{p_{2x}} = \frac{e^2}{2\pi\epsilon_0 \rho m v_{rel}^2}, \tag{2.6}$$

† Note that $<\Delta E>$ and ΔE_2 do not have the same dimensions.

becomes of the order of magnitude unity, or

$$\rho_1 \sim \frac{e^2}{2\pi\epsilon_0 m v_{rel}^2}. \tag{2.7}$$

To estimate ρ_2 we bear in mind that our system consisted of electrons moving in a positive background. This means that the influence of a particular negative charge will be screened. This was used by Debye and Hückel (1923) in their theory of electrolytes. We can find an expression for the so-called Debye radius r_D as follows. Consider a test charge q put into the system at the origin. The charge density $\rho(r)$ will then change from $\rho_0(r) = ne$ to $\rho_0 + \delta\rho$. The electrostatic potential $\phi(r)$ will satisfy the Poisson equation

$$\nabla^2\phi = \frac{\delta\rho + q}{\epsilon_0}. \tag{2.8}$$

On the other hand, at a temperature T, we expect $\delta\rho$ to be given by the equation

$$\delta\rho = \rho - \rho_0 = \rho_0 e^{\beta e} - \rho_0 \approx \beta e\phi\rho_0 = \beta ne^2\phi, \tag{2.9}$$

where, again,

$$\beta = \frac{1}{k_B T}. \tag{2.10}$$

Combining (2.8) and (2.9) we find

$$\left(\nabla^2 + \frac{\beta ne^2}{\epsilon_0}\right)\phi = \frac{q}{\epsilon_0}, \tag{2.11}$$

with the solution

$$\phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-r/r_D}}{r}, \tag{2.12}$$

showing a screened Coulomb field with a screening radius—the Debye radius—given by the equation

$$r_D = \sqrt{\left(\frac{\epsilon_0}{\beta ne^2}\right)}. \tag{2.13}$$

If there is no electrical neutrality, it is more difficult to fix ρ_2, and in the case of gravitational forces where the same problem arises—for instance in discussing the statistical mechanics of star clusters (Chandrasekhar, 1942)—there is no easy solution.

Having found an expression for $<\Delta E>$ we can now estimate the mean free path λ of a particle which we shall define as the path length over which $\lambda<\Delta E>/v_{rel}$ becomes of the order of magnitude of mv^2_{rel}. We have already indicated why in the present case we cannot define the mean free path as the distance between successive collisions. From our definition and equation we get

$$\lambda^{-1} = \pi n \rho_1^2 \ln \frac{\rho_2}{\rho_1}. \tag{2.14}$$

We note that λ is smaller than one might have expected on a simple single-collision model by a factor $\ln(\rho_2/\rho_1)$.

Once one knows λ one can use simple kinetic theory to obtain estimates for relation transport coefficients. The thermal conductivity κ, for instance, is given by the relation

$$\kappa = nv\lambda c_v \tag{2.15}$$

or

$$\kappa \sim \frac{\epsilon_0^2 \, k_B^{7/2} \, T^{3/2}}{m^{1/2} \, e^4} \left[\ln \frac{\rho_2}{\rho_1}\right]^{-1}, \tag{2.16}$$

where we have used the fact that mv_{rel}^2 is of the order of k_BT for a classical system, where we have omitted all numerical constants, and where c_v is the specific heat per particle.

To see the importance of the factor $\ln(\rho_2/\rho_1)$ we note that we can use (2.7) and (2.13) to write for it

$$\ln \frac{\rho_2}{\rho_1} \sim \tfrac{1}{2} \ln \left[\frac{\epsilon_0^3 \, k_B^3 \, T^3}{ne^6}\right], \tag{2.17}$$

which for $T \sim 10^4$ °K and $n \sim 10^9 \text{cm}^{-3}$, is of the order of magnitude of 20 (Spitzer, 1962, gives a table of values of $\ln(\rho_2/\rho_1)$ for temperatures between 10^2 and 10^8 °K and n between 1 and 10^{24} cm^{-3}).

Let us conclude the first half of this chapter with a very brief discussion of the derivation of the kinetic equation. For details of the derivation for the case of a Coulomb gas we refer to Landau's paper (24) mentioned earlier. Let $f(\boldsymbol{r}, \boldsymbol{p}, t)d^3\boldsymbol{r}d^3\boldsymbol{p}$ be the number of particles with positions within a volume element $d^3\boldsymbol{r}$ in coordinate space around $\boldsymbol{r}$ and momenta within a volume element $d^3\boldsymbol{p}$ in momentum space around $\boldsymbol{p}$ at time t. At time $t + dt$ there will be $f(\boldsymbol{r}, \boldsymbol{p}, t + dt)\, d^3\boldsymbol{r}\, d^3\boldsymbol{p}$ particles within the same volume elements and the difference $(\partial f/\partial t)\, d^3\boldsymbol{r}\, d^3\boldsymbol{p}\, dt$ is due to two causes: (i) there may not be a balance between the particles entering $d^3\boldsymbol{r}\, d^3\boldsymbol{p}$ during the time interval t, $t + dt$ because of collisions and those particles which leave $d^3\boldsymbol{r}\, d^3\boldsymbol{p}$ because of collisions, and (ii) the number of particles acquiring positions and momenta within $d^3\boldsymbol{r}\, d^3\boldsymbol{p}$ during the interval t, $t + dt$ because of changes in their positions due to velocities and in their momenta due to accelerations, may not equal the number of particles which left $d^3\boldsymbol{r}\, d^3\boldsymbol{p}$ during the same interval because of drift. We can express this by writing

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_{\text{collisions}} + \left(\frac{\partial f}{\partial t}\right)_{\text{drift}}. \tag{2.18}$$

By considering the flow in the six-dimensional (coordinate-momentum) phase space, one finds in a straightforward way that $(\partial f/\partial t)_{\text{drift}}$ equals $-(\boldsymbol{v}^{(6)} \nabla^{(6)} f)$, where $\boldsymbol{v}^{(6)}$ is the six-dimensional velocity vector and $\nabla^{(6)}$ the six-dimensional gradient vector, or (compare ter Haar, 1966a, § 2.5)

$$-\left(\frac{\partial f}{\partial t}\right)_{\text{drift}} = \left(\frac{\boldsymbol{p}}{m}.\nabla\right) f + \left(\boldsymbol{F}.\frac{\partial}{\partial \boldsymbol{p}}\right) f, \tag{2.19}$$

where we have used Newton's second law of motion to equate the rate of change of $\boldsymbol{p}$ to the force $\boldsymbol{F}$ acting upon the particle.

To find $(\partial f/\partial t)_{\text{collisions}}$ we must study collisions between the charged particles. The difficulty is that—as we have emphasized earlier—any particle is simultaneously colliding with many other particles. This means that the expression for $(\partial f/\partial t)_{\text{collisions}}$ will contain products of many f's. Landau assumes that to a first

approximation one can still treat the collisions as binary encounters, and he evaluates $(\partial f/\partial t)_{\text{collisions}}$ under that assumption, using (2.3) for the change in the momentum of one of the particles, and assuming that the most important collisions are those for which (2.3) holds and in which the change in momentum is so small that we can expand the transition probabilities in a power series in Δp and retain only the first non-trivial term. We refer to Landau's paper for details.

The second topic to be discussed in the present chapter is the so-called Landau damping (Landau 61; reprinted as paper 3 at the end of this volume). This is the damping of longitudinal plasma waves in a collisionless plasma where at first sight, therefore, no dissipation mechanism is present. We shall discuss this phenomenon following closely a recent paper by Hopman (1965).

We consider first of all the case of a zero-temperature plasma in which all collisions are neglected. In that case, the electrons would at equilibrium be stationary in classical statistics which we assume to be applicable and there would be no influencing of one part of the plasma by another part. Consider now what would happen if some of the electrons were displaced. To simplify our discussion we shall assume that our plasma is one-dimensional, that is we assume that all quantities depend on x only. At equilibrium, the charge density is a constant, and there is no electric field. A displacement of the electrons will result in an electric field $\boldsymbol{E}$ which tends to counteract the displacement, and we may expect that the electrons will oscillate around their equilibrium positions. We can use the Maxwell equations:

$$\left.\begin{aligned} (\nabla.\boldsymbol{E}) = \frac{\rho}{\epsilon_0}, \quad (\nabla.\boldsymbol{H}) = 0, \\ [\nabla \wedge \boldsymbol{E}] = -\mu_0 \frac{\partial \boldsymbol{H}}{\partial t}, \quad [\nabla \wedge \boldsymbol{H}] = \boldsymbol{j} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \end{aligned}\right\} \quad (2.20)$$

where we have assumed that $\boldsymbol{B} = \mu_0 \boldsymbol{H}$ and $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$, and where the symbols have their usual meaning. We shall assume that we may

neglect the term $[\nabla \wedge \boldsymbol{H}]$ in the last of eqns. (2.20) which then becomes

$$\boldsymbol{j} + \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} = 0. \tag{2.21}$$

If $\boldsymbol{v}$ is the electron velocity, we have, on the one hand (e = charge of an electron),

$$\dot{\boldsymbol{v}} = \frac{e}{m} \boldsymbol{E}, \tag{2.22}$$

and, on the other hand,

$$\boldsymbol{j} = n_0 e \boldsymbol{v}, \tag{2.23}$$

where n_0 is the equilibrium electron density. Equation (2.21) now reduces to

$$\ddot{\boldsymbol{E}} + \omega_p^2 \boldsymbol{E} = 0, \tag{2.24}$$

where ω_p is the so-called plasma frequency,

$$\omega_p = \sqrt{\left(\frac{n_0 e^2}{m \epsilon_0}\right)}. \tag{2.25}$$

These plasma oscillations were first considered by Tonks and Langmuir (1929; for a derivation using so-called collective co-ordinates see, for instance, ter Haar, 1958).

We note that so far we have found oscillations with a frequency which is independent of the wavelength, which means that the group velocity v_{gr}, given by the equation

$$v_{gr} = \frac{d\omega}{dk} \tag{2.26}$$

(k = wave number) vanishes. This is not surprising as we have already noted that there is in our model no way for one part of the plasma to influence another part.

However, if we look at a plasma at a finite temperature, electrons can move from one part to another, and we would expect

that the group velocity will no longer vanish; that is we expect dispersion—dependence of ω on k—to appear. To find the dispersion we follow Bohm and Gross (1949a, b) and divide the plasma in a number of mono-energetic beams. Let $F(v_i)\Delta v_i$ be the number of electrons per unit volume with velocities between v_i and $v_i + \Delta v_i$. If a plasma wave propagates through the plasma the density in the beam will change from $F\Delta v$ to $(F + \delta F)\Delta v$ and its velocity from v_i to $v_i + \delta v_i$. While $F(v_i)$ will be the equilibrium distribution and thus independent of x and t, as will be v_i, δF and δv_i will depend both on x and on t. We shall assume that the deviations from equilibrium are sufficiently small that we may neglect all second-order terms. Due to the deviation from a uniform equilibrium distribution an electric field will be set up and this will also be a small quantity. The unknown quantities are thus δF, δv_i, and the electric field E, and, taking into account that for a moving mass element $d/dt = \partial/\partial t + (\boldsymbol{v}\cdot\nabla)$, we have for them the following equation of motion:

$$v_i \frac{\partial \delta v_i}{\partial x} + \frac{\partial \delta v_i}{\partial t} = \frac{e}{M} E, \tag{2.27}$$

as well as the Maxwell equation, in the form (2.21), or

$$\epsilon_0 \frac{\partial E}{\partial t} = e \sum_i (F\delta v_i + v_i \delta F)\Delta v, \tag{2.28}$$

and the equation of continuity

$$\frac{\partial \delta F}{\partial t} + v_i \frac{\partial \delta F}{\partial x} + F \frac{\partial \delta v_i}{\partial x} = 0. \tag{2.29}$$

We note that if $v_i = 0$, we get back to eqns. (2.21) to (2.24). We now look for a plane wave solution

$$E(x, t) = E_0 e^{i(\omega t - kx)}, \tag{2.30}$$

and find from (2.27) and (2.29)

$$\delta v_i = \frac{-ieE_0}{m(\omega - kv_i)} e^{i(\omega t - kx)}, \tag{2.31}$$

$$\delta F = \frac{-ieE_0kF}{m(\omega - kv_l)^2} e^{i(\omega t - kx)}. \tag{2.32}$$

Substituting expressions (2.30) to (2.32) into (2.28) we get the following dispersion relation:

$$\frac{m\epsilon_0}{e^2} = \sum_l \frac{F\Delta v}{(\omega - kv_l)^2} = \int \frac{Fdv}{(\omega - kv)^2}. \tag{2.33}$$

We shall assume that in the sum only terms occur for which $kv_l \ll \omega$, so that we can expand $(\omega - kv_l)^{-2}$ in powers of kv_l/ω. Using (2.25) and the relations

$$n_0 = \int Fdv, \tag{2.34}$$

$$< v^2 > = \frac{1}{n_0} \int Fv^2 dv, \qquad < v > = \frac{1}{n_0} \int Fvdv, \tag{2.35}$$

we get from (2.33)

$$\omega^2 = \omega_p^2 + 3k^2 < v^2 >, \tag{2.36}$$

where we have used the fact that if we use for $F(v)$ the Maxwell distribution, $<v>$ vanishes (and $<v^2> = k_BT/m$).

We must now discuss the assumptions used in our derivation. The basic one was that we do not need to worry about the fact that the integral in (2.33) diverges even though the Maxwell distribution is non-vanishing for all values of v. We must thus study in somewhat more detail what happens to those electrons which have velocities in the neighbourhood of ω/k. These are electrons which occupy the shaded area in Fig. 2.2 and are called the trapped electrons for reasons to be explained in a moment. In Fig. 2.2 we have also indicated the root mean square velocity. We may expect that equation (2.36) is not too bad an approximation as long as $\omega/k \gg \sqrt{<v^2>}$, or, as long as

$$\lambda > r_D, \tag{2.37}$$

where we have used (2.36), the relation $<v^2> = k_BT/m$ and equations (2.13) and (2.25).

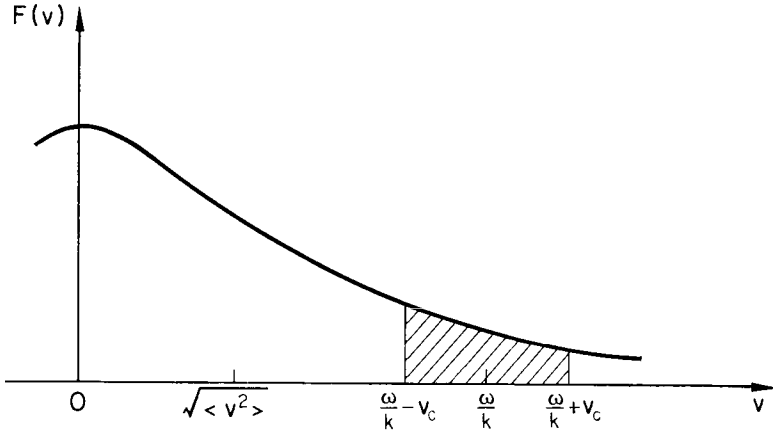

FIG. 2.2. Electron Maxwell distributions. The electrons in the shaded area are the so-called trapped electrons.

Consider now a wave with frequency ω and wave number k propagating through the plasma. The phase velocity is ω/k. Let us consider the wave in a frame of reference moving with a velocity ω/k—in which the wave is at rest (Fig. 2.3). The electron beams are moving through this periodic field.

From the requirement of continuity we know that j_i in each beam must be constant. If the beam moves through the periodic field, the electrons will be accelerated and decelerated, and as $j_i = n_i e v_i$, we see that the electron beam will get a periodic structure. From Fig. 2.3 we see that electrons with $v_i < \omega/k$ which move from right to left are accelerated between O and B and from D to G and decelerated between B and D and from G to I. This means that at B and G the electrons have their maximum velocity and thus their minimum density (see Fig. 2.3). The electrons in the plasma themselves, by their space charge distribution, thus stabilize the wave. It is left to the reader to verify that the situation is the same for the electrons with $v_i > \omega/k$.

What happens to electrons with $v_i \sim \omega/k$? Their velocity in the moving frame of reference is so small that they cannot get over

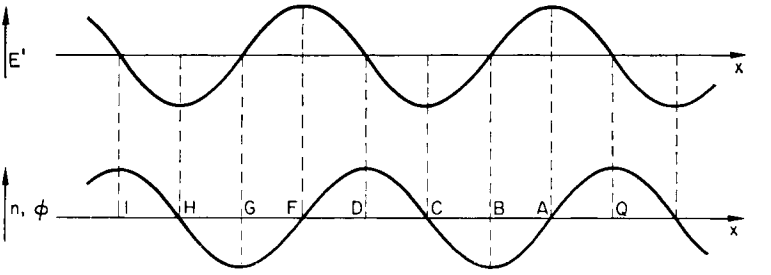

FIG. 2.3. The electrical field, $E' = -E_0 \sin kx$, of a sine wave, $E = E_0 \sin(\omega t - kx)$, in a frame of reference moving with a velocity ω/k. The lower curve is the density distribution n in the electron beams and is proportional to the electrical potential ϕ.

the potential barriers produced by the electrical potential ϕ, which is related to E by the relation $E = d\phi/dx$. This means that the electrons will be captured in the potential wells at B, G. . . . We can see that all electrons with velocities between $\omega/k - v_c$ and $\omega/k + v_c$ will be trapped, where v_c is determined by the condition that $\frac{1}{2}mv_c^2 = e\phi_0$ is the amplitude of the potential. From this condition we get

$$v_c = \sqrt{\left(\frac{2e\phi_0}{m}\right)} = \sqrt{\left(\frac{2eE_0}{mk}\right)}, \tag{2.38}$$

where E_0 is the amplitude of the electrical field.

Consider now an unperturbed plasma upon which suddenly a periodic electrical field is imposed. The electrons with velocities in the shaded area in Fig. 2.2 will get trapped and will be forced to move all with a velocity ω/k. Electrons which originally were moving faster are slowed down, and those which originally were slower are speeded up. As in the case of a Maxwell distribution $dF/dv < 0$ for $v = \omega/k$, more electrons are speeded up than slowed down and the wave loses energy to the electrons: damping is present.

The theory used here is a linearized one and uses plane waves. Recently, Wong, Motley, and d'Angelo (1964; See also Alexeff, Jones, and Montgomery, 1967) have measured the phase velocity

v_{ph} and damping decrement δ for waves in a plasma consisting of a moving Cs^+ beam. Their results are given in Table 2.1 and it is seen that the experimental data are in good agreement with the theoretical predictions. It was also found that the results were independent of E_0—as should be the case for a linear theory.

TABLE 2.1

	δ/λ		ω/k in 10^5 cm sec^{-1}	
	Theoretical	Experimental	Theoretical	Experimental
Plasma waves moving in the direction of the Cs^+ beam	0·51	0·55	1·2	1·3
Plasma waves moving in the opposite direction	0·27	0·25	0·8	0·9

These experimental results demonstrate the essential correctness of the concepts involved in Landau damping, and especially the idea that collisions may be neglected in describing this phenomenon.

III

Nuclear Physics, Astrophysics, and Cosmic Ray Physics

As ONE would naturally expect, Landau has considered many aspects of nuclear physics. We have chosen a paper on the statistical theory of nuclei (Landau 31; reprinted at the end of this volume as paper 4) to represent his work in this field, and we shall discuss this paper presently. Before doing this, however, we shall discuss a few of his other contributions to nuclear physics. In a short note (Landau, 1935) he points out that some nuclear reactions such as α-decay of certain compound nuclei will not take place because of the forbiddenness of transitions between states of different parity. We just mention his papers on meson theory and the theory of nuclear forces (Landau 40, 42, and 45).

We finally mention three papers on scattering theory. With Smorodinskii, Landau (55) discussed proton–proton scattering, using the fact that for relative energies small compared to the interaction energy, and for de Broglie wavelengths long compared to the range of the forces, the shape of the interaction potential is immaterial. This paper essentially introduces the effective-range method for the scattering of charged particles, in particular for proton–proton scattering. Landau and Lifshitz (68) extended earlier work by Landau (7; see Chapter IV for a discussion of this paper) to discuss deuteron fission in a Coulomb field. It is interesting to note that they carefully give this paper the same title as the earlier ones, because at the time of publication nuclear physics papers seldom obtained permission to be published. The title is very uninformative, but a title such as "On the theory of the (*d*, *pn*) reaction" might have meant that the paper could not have

been published. The final paper on scattering is one with Pomeranchuk (Landau 77) in which the scattering of fast pions by nucleons is discussed. In this case one can assume the nucleon (or nucleus) to be a "black" sphere so that only the wave function of the pion outside the nucleon need be considered. Landau and Pomeranchuk evaluated the γ-ray spectrum emitted in such scattering processes.

In his paper on the statistical theory of nuclei, Landau shows how one can use quite general considerations to find expressions for nuclear energy level densities. Bethe (1936) had derived such expressions treating the nucleus essentially as a system of independent particles, but Landau showed that one can derive similar expressions under much less stringent restrictions. He also used his results to discuss nuclear reactions on a statistical basis. We shall indicate here how one can use statistical methods to find the nuclear energy level density as a function of energy (see ter Haar, 1954, chap. XIII, for a more extensive discussion and for literature references).

In statistical mechanics one shows that if the number of energy levels between E and $E + dE$ is given by $\rho(E)dE$, we can calculate for a given temperature $T(=1/k_B\beta$:k_B Boltzmann constant) the average energy $<E>$ as function of T (or β) and of the numbers of protons Z and of neutrons N†. We know from statistical mechanics that the Helmholtz free energy of a system is given by the equation

$$e^{-\beta F} = \int \rho(E)dE\, e^{-\beta E}. \tag{3.1}$$

As the nucleus consists of a relatively large number of particles, $\rho(E)$ will be a steeply increasing function of E and the integrand in (3.1) will have a relatively steep maximum at $E = <E>$ so that we can write‡

$$e^{-\beta F} = C\rho(<E>)\, e^{-\beta <E>}, \tag{3.2}$$

† We assume here that the numbers of protons and of neutrons are fixed. If this is not the case, one uses the method of the so-called grand ensembles (see, for example, ter Haar, 1954, 1966a).

‡ Equation (3.2) is obtained by writing $\rho(E) = e^{\psi(E)}$ and expanding the index $\psi(E) - \beta E$ around its maximum $<E>$; retaining only the first non-trivial terms, one is left with a Gaussian integral which gives us C.

where the constant C is only weakly dependent on $<E>$. On the other hand, we have for the entropy S the two relations

$$F = <E> - TS_\lambda \tag{3.3}$$

and

$$\frac{\partial S/k_B}{\partial <E>} = \beta, \quad \text{or} \quad \frac{S}{k_B} = \int \frac{dE}{d\beta}\beta \, d\beta. \tag{3.4}$$

The idea is now that if a nucleus is excited to a certain energy, this will correspond to a certain nuclear temperature, such that the average energy at that temperature is just the excitation energy. Equations (3.2), (3.3), and (3.4) are not sufficient by themselves to determine for a given E: T, S, F, and $\rho(E)$. However, one can introduce models for the nucleus, which will provide us with an additional relation between the various quantities. For instance, if we assume the nucleons to move as free particles inside the nucleus, we are dealing with two systems of fermions, each of which is a degenerate perfect gas. In that case, we have, using the fact that for a perfect fermion gas near total degeneracy the energy is a quadratic function of T (see, for example, ter Haar, 1966a, p. 133),

$$<E> = E - E_0 = C'T^2, \tag{3.5}$$

where C' is a constant and E_0 is the ground state energy, and where we have reckoned $<E>$ from the ground state energy as zero. In fact, (3.5) holds for any system of independent fermions. We now easily find from (3.2) to (3.5) that $\rho(E)$ is given by the expression

$$\rho(E) = \phi_1(E)e^{\sqrt{E/A}}, \tag{3.6}$$

where $\phi_1(E)$ is a smooth function of E and where Δ is an energy of the order of the single-particle level distances.

In the case of a liquid drop, the E-dependence of the index is slightly different (see Bethe, 1937) and we find

$$\rho(E) = \phi_2(E) \exp\left[(E/\Delta')^{4/7}\right] \tag{3.7}$$

It is interesting to note the order of magnitude of the quantities involved. The excitation energies will be of the order of MeV, as will be the single particle level splitting,

$$\Delta \sim \frac{\hbar^2}{md^2} \sim 1 \text{ MeV}, \tag{3.8}$$

if we put the size of the nucleus $d \sim 10^{-12}$ cm. This means that S/k_B will be of the order of magnitude of unity [cf. (3.2) and (3.6)] and combining (3.5) and (3.4) we find for the order of magnitude of the nuclear temperature

$$T \sim 1 \text{ MeV}. \tag{3.9}$$

We refer to the literature for a discussion of the statistical theory of nuclear reactions (see especially Weisskopf, 1937; Weisskopf and Ewing, 1940).

Landau's contributions to astrophysics were all concerned with stellar structure and energy and, as could be expected in a fast-moving field, are now rather out of date. In his first paper in this field (Landau 8) he discussed stellar equilibrium in general terms. His first point is that theoretical physics should be able to discuss the physical nature of stellar equilibrium. We can first neglect energy radiation and look for the minimum of the Helmholtz free energy. This free energy consists of two parts: the gravitational free energy, which depends on the average density, ρ, as $\rho^{1/3}$ and is negative, and the free energy connected with the equation of state. If the gas, which makes up the star, is a classical one, this second contribution is positive and varies as ln ρ, and there is thus no equilibrium: the minimum occurs for $\rho = \infty$: the star collapses to a point. If the gas is a non-relativistic fermion gas, the second contribution varies as $\rho^{2/3}$ and there would be a stable equilibrium. However, if the gas is a relativistic fermion gas, the second contribution varies as $\rho^{1/3}$, and it now depends on the actual coefficients whether there exists a stable configuration. Landau shows that it depends on the stellar mass whether or not there is a stable configuration, that is whether or not the star will collapse to a point. The critical mass is of the order of one and a half solar masses. As larger stars

do exist, Landau suggests that each star has a core "in which the laws of quantum mechanics (and therefore the quantum statistics) are violated", and that this is what makes a star a star. As he explicitly discards the idea that "some mysterious process of mutual annihilation of protons and electrons" would supply the stellar radiation, Landau then suggests that in that pathological core the energy conservation law is violated—at that time some people believed that there was other evidence for such a violation (compare the remarks at the end of Landau 6, to be discussed in Chapter IV)—and that this would occur when the density becomes of the order of the nuclear density. We should emphasize that in the early 1930's there was a great deal of discussion of the limitations of quantum mechanics and of laws such as the energy conservation law. It is now known that these conclusions have to be modified when radiation is present, but a more detailed analysis (Chandrasekhar, 1935, 1939) shows that the relativistic zero-temperature theory of a completely degenerate configuration predicts that its radius tends to zero as its mass tends to a finite limiting value and this limiting mass is of the order of a few solar masses.

In a short note Gamow and Landau (13) look into some of the consequences of Atkinson and Houterman's (1929) suggestion that processes of thermal transformation of light elements occur in stars. They considered the reaction $^{7}Li + {}^{1}H \rightarrow 2{}^{4}He$, and they showed that either practically no lithium is present at the stellar surface or there is a limit to the temperature in the interior of the star.

In his last paper on astrophysics Landau (27; reprinted as paper 5 at the end of this volume) suggests that stellar energy might derive from the combination of electrons and nuclei into neutronic matter. Under atmospheric pressure such a combination is energetically unfavourable, but for sufficiently massive stars the pressure in the interior of the star will be so large that the gravitational energy gained in going over to the neutronic state will overcompensate the loss in binding energy. To see when this will happen, we find for the gravitational energy of the neutronic

matter, assuming a density of 10^{14} g cm^{-3}, $3 \times 10^{-3}\, M^{5/3}$ erg. However, we must also take into account the fact that the zero-point pressure of a fermion gas depends on the mass of the particles and going over from the electronic to the neutronic state we gain $7 \times 10^{-22}\, M^{7/3}$ erg. As the average energy lost by going over to the neutronic state is about 8 MeV per neutron or 7×10^{18} erg g^{-1}, we lose $7 \times 10^{18}\, M$ erg in going over to the neutronic state. Comparing the losses and gains, we find that as soon as M is larger than about one-thousandth of a solar mass, energy is gained by going over to the neutronic state. For a star such as the sun, the energy radiated during several times 10^9 years requires only a fraction of the sun to have gone into the neutronic state.

We must, however, emphasize that nowadays, following the work of Bethe (1939) and von Weizsäcker (1937), it is generally accepted that the following nuclear reactions are the main sources for the radiative energy of the stars:

$$\left.\begin{aligned} &{}^{12}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + \mathrm{e}^{+},\ {}^{13}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{14}\mathrm{N}, \\ &{}^{14}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{15}\mathrm{O} \rightarrow {}^{15}\mathrm{N} + \mathrm{e}^{+},\ {}^{15}\mathrm{N} + {}^{1}\mathrm{H} \rightarrow {}^{12}\mathrm{C} + {}^{4}\mathrm{He}, \end{aligned}\right\} \tag{3.10}$$

and

$$\begin{aligned} &{}^{1}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{2}\mathrm{H} + \mathrm{e}^{+},\ {}^{2}\mathrm{H} + {}^{1}\mathrm{H} \rightarrow {}^{3}\mathrm{He}, \\ &{}^{3}\mathrm{He} + {}^{3}\mathrm{He} \rightarrow {}^{4}\mathrm{He} + 2\,{}^{1}\mathrm{H}. \end{aligned} \tag{3.11}$$

To conclude this chapter we shall discuss Landau's contributions to cosmic ray physics. In the late 1930's and early 1940's Landau, often in collaboration with Rumer, considered the production of electronic showers by energetic particles (Landau 34, 36), a field to which many people contributed. In their theory they consider the penetration of a fast β-particle through matter. It produces photons through Bremsstrahlung, and these photons may be of sufficiently high energy to produce pairs which in turn can produce photons, and so on, until there is no longer enough energy available. Landau (43) returned to this problem later on, refining some of the results and at the same time finding an expression for the width of a shower, and he also (Landau 44) extended the theory to a consideration of showers produced by mesons rather than by β-particles.

The energy distribution function of fast particles which are losing energy through ionization collisions was considered by Landau (56) as was the general theory of Bremsstrahlung and pair production for high-energy electrons (Landau 75). In this last paper, written in collaboration with Pomeranchuk, he found that the Bethe–Heitler formula ceases to be valid for energies above about 2000 GeV if the electron is moving through heavy matter. The processes to take the place of the ordinary Bremsstrahlung and pair production which involve higher-order processes were discussed by the same authors in a subsequent paper (Landau 76), and it was shown that at sufficiently high energies the soft component of cosmic rays (electrons and positrons) has similar properties as the hard component (nucleons), which is hardly surprising as at those energies the rest mass energy is negligible compared to the kinetic energy of the particles.

Perhaps Landau's most interesting contribution to cosmic ray physics was his development of Fermi's ideas (1950, 1951) of a statistical or hydrodynamic theory of the multiple production of particles during collisions of fast particles—that is, the production of many-pronged stars (Landau 74, 88; the first of these papers is reprinted as paper 6 at the end of this volume). We should mention that with the advent of large accelerators these processes no longer only are produced by cosmic rays, but also by man-made projectiles. Fermi's basic point was that perturbation theory is inapplicable to this problem as there are too many elementary processes involved, but that one can make a virtue out of this necessity and use statistical methods.

We shall sketch Landau's theory, but for a discussion of details we must refer the reader to the two papers quoted a moment ago. Let us consider the collision of two nucleons, due to the impact of a nucleon with an energy of 1000 GeV or more on a stationary nucleon. In Fermi's theory the following assumptions were made:

(a) It is assumed that in the collision, energy is released in a very small volume V in the centre of mass system. As the interactions are strong and the volume—which will be

Lorentz-contracted—small, we can use for the energy-distribution statistical laws and can thus treat this collision of high-energy nucleons without recourse to any specific theories of nuclear interaction.

(b) The volume V will be determined by the dimensions of the pion cloud surrounding the nucleons, whose radius is $\hbar/m_\pi c$, m_π being the pion mass, and by the Lorentz contraction in the direction of motion. The Lorentz contraction is given by (see, for instance, Leighton, 1959) $2Mc^2/E'$, where M is the nucleon mass and E' the nucleon energy in the centre of mass systems. We have thus

$$V \sim \left(\frac{\hbar}{m_\pi c}\right)^3 \frac{Mc^2}{E'} \,. \tag{3.12}$$

(c) Fermi assumes that the particles are formed at the instant of collision in the volume V, and do not interact further with one another, but leave the volume in a "frozen" state.

This last assumption is unjustified because at the instant of collision the energy density, and thus the particle density is so large and the interactions between them so strong that one cannot legitimately speak about a fixed number of particles and certainly cannot assume that the interaction between the particles stops as soon as they leave the volume V. In actual fact the system expands and the number of particles does not become definite until the interaction between the particles becomes small and they can move away from one another as free particles.

Thermodynamics can be applied to the system when the mean free path λ is much smaller than the dimensions L of the system, but in that case one can also apply hydrodynamics, and Landau therefore suggested that one should apply both thermodynamics and hydrodynamics to the system until it is expanded sufficiently for the particles to be treated as being independent. Because of the very high energies involved one must apply relativistic thermodynamics and hydrodynamics.

The collision process can now be described as follows.

(1) The first stage is much as envisaged by Fermi. At the instant of collision a large number of particles are formed, their mean free path is small compared to the dimensions of the system, and statistical equilibrium is set up.

(2) The second stage is the expansion of the system. Now one can use a hydrodynamics approach, and one can regard the expansion as the motion of an ideal fluid with zero viscosity and zero thermal conductivity. This is justified because the Reynolds number,

$$\mathrm{Re} = \frac{\rho V L}{\eta} \sim \frac{VL}{v\lambda}, \tag{3.13}$$

is much larger than unity. In (3.13) V is the "macroscopic" and v the "molecular" velocity which both are of the order of the velocity of light.

(3) Finally, the interaction will become weaker and now one can characterize the state of the system by giving the number of particles. "Break-up" will occur when λ becomes of the order of L, and this will happen at a temperature T_3 which is given by

$$k_B T_3 \approx m_\pi c^2 . \tag{3.14}$$

This can be seen from the fact that the equilibrium number of particles will depend on T as $\exp[-m_\pi c^2/k_B T]$ so that for temperatures just below T_3 the number of particles will become small and the mean free path long.

The equation of state for the matter in the relativistic region will probably not be very different from the black-body equation of state (see, for instance, ter Haar 1966a, chap. VII)

$$P = \tfrac{1}{3}\epsilon, \tag{3.15}$$

where P is the pressure and ϵ the energy density. This is, of course, just the equation of state for photons and for ultra-relativistic free particles—independent of their statistics.

As the number of particles in the system is not fixed, the chemical potential (or Gibbs free energy) vanishes, or

$$\epsilon - Ts + P = 0, \tag{3.16}$$

where s is the entropy density. It thus follows that

$$Ts = \tfrac{4}{3}\epsilon, \tag{3.17}$$

and as long as the expansion is adiabatic we find (see, for instance, ter Haar, 1966b, or ter Haar and Wergeland, 1966) the same relations as for black-body radiation, that is

$$\epsilon = c_1 T^4, \; s = c_2 \epsilon^{3/4}, \tag{3.18}$$

where c_1 and c_2 are constants.

During the break-up different parts of the system will start free expansion at different moments. In each of these regions it follows from relativistic thermodynamics (or thermostatistics) that although the entropy and number of particles vary strongly with temperature, their ratio is not very sensitive to a change in temperature around T_3. This means that for each of the regions we have (i numbers the regions)

$$n_i = c_3 S_i, \tag{3.19}$$

and thus

$$N = c_3 S, \tag{3.20}$$

where S is the total entropy and N the total number of particles produced.

In the case of a head-on collision of two nucleons, one at rest and the other incident with energy E, they each have in the centre of mass system an energy E' given by the equation

$$2E'^2 = Mc^2 E. \tag{3.21}$$

The entropy S is proportional to $\epsilon^{3/4}V$, or as $\epsilon = 2E'/V$,

$$N \propto S \propto \epsilon^{3/4} V \propto E'^{3/4} V^{1/4}. \tag{3.22}$$

The volume is given by (3.12) so that we get from (3.22), (3.12), and (3.21)

$$N \propto E^{1/4}, \tag{3.23}$$

and as $m_\pi \sim M$ and using dimensional arguments, we find

$$N = c_4 \left(\frac{E}{Mc^2}\right)^{1/4}, \tag{3.24}$$

where c_4 should be of order of magnitude unity and experimentally turns out to be about 2.

Consider now a head-on collision of two identical nuclei of atomic weight A. We have already indicated that perturbation theory is not applicable. This entails that we cannot consider such a collision as a series of collisions between separate nucleons. If v/c is about the same as in the case of the nucleon–nucleon collision, the energy density and the initial Lorentz contraction will be the same, but the volume will be A times larger so that the number of particles produced will be A times larger for E/A the same as a moment ago. We thus get

$$N = c_5 A \left(\frac{E}{AMc^2}\right)^{1/4} = c_5 A^{3/4} \left(\frac{E}{Mc^2}\right)^{1/4} \tag{3.25}$$

and we see that two nuclei of atomic weight A with energy E produce as many particles as two nucleons with energy EA^3.

IV

Quantum Mechanics

APART from solving—so to speak *en-passant*—any problem which was drawn to his attention and which was a sufficient challenge, Landau throughout his life has always been working on those branches of physics which constituted at any particular time the frontier of physics. In the 1950's this was quantum electrodynamics and elementary particle physics, and in the next chapter we shall discuss some of his contributions to that part of physics, while in the later 1920's and the 1930's it was the new quantum mechanics developed by Schrödinger, Heisenberg, and the Copenhagen school around Bohr—of which Landau considered himself to be a member. In fact, his earliest papers published when he was still in his teens dealt with various problems in wave mechanics.

Some of Landau's early papers deal with spectra. In the first paper he wrote alone (Landau 1†) a discussion is given of the rotational spectrum of diatomic molecules and the frequencies corresponding to the *P*-, *Q*-, and *R*-branches of the Fortrat diagram are derived. The splitting of the rotational energy levels due to magnetic or electric fields (Zeeman and Stark effects) is also considered. The role of sum-rules is pointed out in a brief note (Landau 5) while, with Placzek, Landau (14) considered the structure of the so-called undisplaced line occurring when light is scattered by liquids or not-too-dilute gases. This line can be

† Although this is the first paper in the list of papers on p. 54 and also the first paper in the *Collected Papers* volume, it is not Landau's first paper. In Appendix B of the *Collected Papers* volume a list is given of some fifteen papers which were not included in full because they either were out of date or contained errors.

shown to be a triplet, the scattering being due to density fluctuations in the scattering system. With the advent of lasers, strong light beams are now available and the Landau–Placzek triplet has been studied experimentally. Among other papers on scattering is one which deals with the slowing down of electrons through Bremsstrahlung emission (Landau 15) where a simple—to a large extent dimensional—analysis leads to the same result as one derived more rigorously by Heitler (1933) while at the same time pointing out the limits of applicability of the Heitler formula. With Akhiezer and Pomeranchuk, Landau (26) evaluated the cross-section for the scattering of light by light for the case of high frequencies, while in a short note (Landau 39) he points out that the fact that electrons do not show polarization on scattering, as predicted by theory, can be explained by the fact that most electrons will be multiply scattered.

The electron–positron system was considered by Landau from various angles. The problem of pair-production was studied with Lifshitz (Landau 16) for the case of high-energy collisions of charged particles. In the opposite process, that of the mutual annihilation of electrons and positrons, usually two photons are emitted. Landau (65) showed that this two-photon annihilation of positronium is strictly forbidden in all states with odd orbital and total angular momenta and also in all states with total angular momentum equal to unity—an example of which is the case of a stationary electron and a stationary positron with their spins parallel. With Berestetskii, Landau (71) discussed the wave equation for the electron–positron system up to terms of order v^2/c^2, and he also (Landau 19) gave a simple derivation of the v^2/c^2 correction term to the Schrödinger equation [the so-called Breit term (Breit, 1929)].

It is well known that quantum mechanics is intimately connected with probability statements: the wave function can be used for a statistical treatment and for evaluating the statistical distribution of possible values of various observables. If we now consider quantum statistics, we must add the statistical treatment of statistical mechanics to that of quantum mechanics, and the

average value of any physical quantity is obtained by first averaging over the state of each system in a statistical ensemble and then averaging over the ensemble. A similar situation arises when the system considered is coupled to another system. Again we have a twofold average, this time firstly over the state of the system and secondly over all possible states. To see how this works, let us consider the second case. Let at $t = 0$, ψ be the wave function of the system considered and ψ' that of the system to which it is coupled at $t = 0$, and let ϕ_n and ϕ'_k be complete orthonormal sets in terms of which ψ and ψ' can be expanded:

$$\psi = \Sigma a_n \phi_n \tag{4.1}$$

and

$$\psi' = \Sigma b_k \phi'_k. \tag{4.2}$$

The wavefunction of the combined system ψ can be written in the form

$$\psi = \psi\psi' = \Sigma a_n b_k \phi_n \phi'_k = \Sigma_{n,k} c_{nk} \phi_n \phi'_k, \tag{4.3}$$

and the c_{nk} completely describe the state of the combined system. At $t = 0$ the c_{nk} are given by the equation

$$c_{nk} = a_n b_k, \tag{4.4}$$

but there is no reason to assume that at a later time ψ can still be written simply as a product of two wavefunctions pertaining to the two parts of the combined system. Although (4.4) will no longer hold, the c_{nk} still determine ψ.

Let us now consider an operator $\hat{\Omega}$ operating upon the coordinates of the original system. If this system is in a state ψ, the average value $<\Omega>$ is given by the equation

$$<\Omega> = \int \psi^* \hat{\Omega} \psi d\tau = \sum_{nm} v^*_n a_m \Omega_{nm}, \tag{4.5}$$

where

$$\Omega_{nm} = \int \phi^*_n \hat{\Omega} \phi_m d\tau. \tag{4.6}$$

However, if we only know that the combined system is given by ψ we have

$$\begin{aligned} <\Omega> &= \int \psi^* \hat{\Omega} \psi d\tau d\tau' \\ &= \Sigma_{nmkl} c_{nk}^* c_{ml} \int d\tau' \phi_k'^* \phi_l' \int d\tau \phi_n^* \hat{\Omega} \phi_m \\ &= \Sigma_{nm} \Omega_{nm} \Sigma_k c_{nk}^* c_{mk}, \end{aligned} \tag{4.7}$$

and we see that instead of the $a_n^* a_m$ in (4.5) we now have an average $\Sigma_k c_{nk}^* c_{mk}$. If we introduce now a *density matrix* or *statistical operator* $\hat{\rho}$ by its matrix elements in the ϕ-representation

$$\rho_{mn} = \Sigma_k c_{nk}^* c_{mk}, \tag{4.8}$$

we can write (4.7) in the form

$$<\Omega> = \Sigma_{nm} \rho_{nm} \Omega_{nm} = \mathrm{Tr} \hat{\rho} \hat{\Omega}, \tag{4.9}$$

and we see that once $\hat{\rho}$ is known we can evaluate all averages. We note that if the c_{nk} are given by (4.4), (4.7) reduces to (4.5).

The density matrix in the form discussed a moment ago was introduced by Landau (2) in a paper discussing damping of radiation. The system described by the density matrix is in this case an atom and the coupling is with the radiation field. A more general discussion of the density matrix was given almost at the same time by von Neumann (1927; for a general discussion of the density matrix see, for instance, ter Haar, 1961, 1966a).

We now come to the two papers reprinted in this volume. In the first paper (Landau 6; reprinted as paper 7 at the end of this volume) Landau and Peierls discussed how relativity theory impinged on the theory of measurement in quantum mechanics. We do not have the space here to discuss the difficult question of the meaning of a measurement on a quantum system. This is discussed in section 2 of the paper by Landau and Peierls (see also London and Bauer, 1939; ter Haar, 1961). The remainder of the paper is concerned with a discussion of the Heisenberg relations in relativistic quantum theory. It is of interest to note that the authors repeat Bohr's suggestion that conservation of energy breaks down for the electrons in the nucleus which are emitted in

β-decay. We shall only briefly discuss one aspect of the paper, namely the consideration of the measurement of electromagnetic fields.

If an electric field $\mathscr{E}$ is to be measured, one uses a charged particle and determines the change in its momentum p. If the charge of the particle is e and the measurement takes a time Δt, we have

$$e\Delta\mathscr{E}\Delta t > \Delta p, \tag{4.10}$$

where Δp is the uncertainty in the momentum after the measurement. The value of Δp arises from two sources: first of all, from the uncertainty in energy ΔE connected with a finite duration Δt of the measuring process one finds

$$(\Delta p)_1 > \frac{\hbar}{\Delta t \mid v - v' \mid}, \tag{4.11}$$

where $\mid v - v' \mid$ is the change in the velocity of the particle, and where we have used the relation

$$\Delta E = v\Delta p; \tag{4.12}$$

secondly, a charged particle will emit energy when accelerated and we have, therefore, a second source of energy uncertainty

$$\Delta E_2 = \frac{e^2}{6\pi\epsilon_0 c^3} \int a^2 \, dt, \tag{4.13}$$

where a is the acceleration of the particle. Minimizing ΔE_2 by putting $\mid a \mid = \mid v - v' \mid / \Delta t$, and using (4.12), we have

$$(\Delta p)_2 > \frac{e^2}{4\pi\epsilon_0 c^3} \frac{\mid v - v' \mid}{\Delta t}, \tag{4.14}$$

and combining (4.11) and (4.14) we find

$$\Delta p > \frac{\hbar}{c} \sqrt{\left(\frac{e^2}{4\pi\epsilon_0 \hbar c}\right)}. \tag{4.15}$$

From (4.10) and (4.15) we then get

$$\Delta\mathscr{E} > \frac{\sqrt{(\hbar c/\epsilon_0)}}{(c\Delta t)^2}. \tag{4.16}$$

Similarly, we find for the magnetic field by considering the action on a magnetic dipole

$$\Delta\mathscr{H} > \frac{\sqrt{(\hbar c/\mu)_0}}{(c\Delta t)^2}. \tag{4.17}$$

If we wish to measure $\mathscr{E}$ and $\mathscr{H}$ simultaneously, we must take into account the magnetic field produced by the particle, and if Δr is the distance between it and the magnetic dipole, we find an additional uncertainty in $\mathscr{H}$ of the order of

$$\Delta\mathscr{H}' > \frac{e(v - v')}{(\Delta r)^2}. \tag{4.18}$$

Combining (4.18), (4.10), and (4.11), we find

$$\Delta\mathscr{E}\Delta\mathscr{H} > \frac{\hbar}{(\Delta t)^2} \quad \frac{1}{(\Delta r)^2}, \tag{4.19}$$

where the right-hand side is similar to the product of the right-hand sides of (4.16) and (4.17), but with $(c\Delta t)^2$ replaced by $c\Delta t\Delta r$. We note that for static fields the simultaneous measurement can be made as small as we want by choosing Δt sufficiently large.

Let us now consider a radiation field. If E is the energy of the radiation considered, we need to measure the field strength in a time Δt such that the field strength can be considered to be essentially constant, because in that case our earlier considerations hold, and this means clearly that we must satisfy the inequality

$$\Delta t < \frac{\hbar}{E}. \tag{4.20}$$

In that case inequality (4.16) will hold so that a measurable field strength $\mathscr{E}$ must satisfy the relation

$$\mathscr{E} > \frac{\sqrt{(\hbar c/\epsilon_0)}}{(c\Delta t)^2}. \tag{4.21}$$

On the other hand, the wavelength of the radiation must be at least $\hbar c/E$ and the field strength will be essentially constant over a volume of at least the size $(\hbar c/E)^3$. The total energy must therefore satisfy

$$E > \epsilon_0 \mathscr{E}^2 \left(\frac{\hbar c}{E}\right)^3 > \frac{(\hbar c)^4}{E^3 (c\Delta t)^4}, \tag{4.22}$$

or

$$\Delta t > \frac{\hbar}{E} \tag{4.23}$$

in contradiction to (4.20). It looks thus as if the radiation field can not be properly defined in relativistic quantum theory. However, Bohr and Rosenfeld (1933) in their thorough analysis of the measurability of electromagnetic fields pointed out that Landau and Peierls' analysis mixed up the problems of field theory and of atoms. It is well known that even three decades later there is no really satisfactory theory incorporating both radiation and atoms. However, if we are interested only in electromagnetic quantities, there is no elementary length, as only $\hbar$ and c occur in the theory and they are insufficient to determine a quantity of the dimensions of a length for which the electron mass, for instance, is necessary. Bohr and Rosenfeld point out that the difficulties encountered by Landau and Peierls are a consequence of their introduction of point particles—which are used as test particles—and that they will disappear if one uses test particles, the dimensions of which are large compared to atomic dimensions.

The other paper on quantum theory reprinted here is one dealing with collision theory (Landau 7, 9; together reprinted as paper 8 at the end of this volume). In this paper Landau considers systems which can partly be described by semi-classical wave-functions.† We shall discuss here the application of this theory to the predissociation of a diatomic molecule (see Landau and Lifshitz, 1965, § 90).

† For a description of the semi-classical (or W-K-B) approach to quantum mechanics, we refer to text books (for instance, Landau and Lifshitz 1965 or Davydov 1965).

In the usual treatment of diatomic molecules, one uses the fact that the nuclei are so much heavier than the electrons to separate the electronic motion from that of the nuclei (for a discussion see, for instance, Davydov, 1965, chap. 12). This means that we can—to a first approximation—treat the nuclei as being stationary when discussing the electronic energies and in turn this means that we can plot the electronic energies $U(R)$ as a function of the distance apart R of the nuclei. These energies in turn can be used as

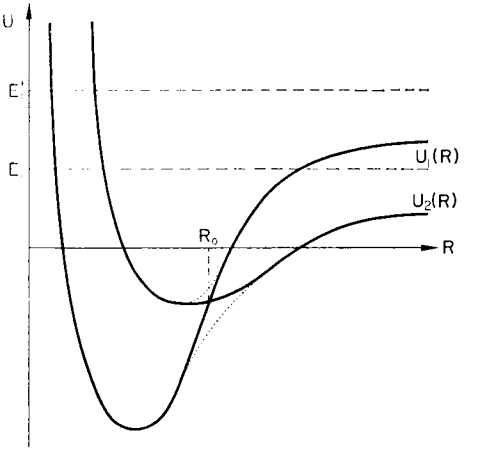

FIG. 4.1. Two possible molecular energy curves.

potential energies for the nuclear motion and the rotational and vibrational motion of the nuclei can be determined from them. In Fig. 4.1 we have sketched two such energy curves, $U_1(R)$ and $U_2(R)$, that is, two curves corresponding to the energy of a diatomic molecule as function of R corresponding to different electron configurations. Let E_1 be the energy of some vibrational state corresponding to the electronic state $U_1(R)$. If the molecule could make a transition to the state $U_2(R)$, it would get into a state corresponding to the continuous spectrum of that electron state, and the molecule would fall apart: this is called *pre-dissociation.*

Another possibility is a so-called collision of the second kind, where the molecule originally has an energy E_1' corresponding to a state in the continuum of $U_1(R)$ and makes a transition to a state in the continuum of $U_2(R)$. This corresponds to a collision between two atoms with a decrease in their kinetic energy corresponding to the difference in the excitation energies. If transitions between $U_1(R)$ and $U_2(R)$ can take place, the energy curves will, of course, not be the full drawn ones, but where otherwise they would intersect, they must be replaced by the dotted curves† (repulsion of levels; see the discussion in Davydov, 1965, § 49).

The probability that a transition $1 \to 2$ will take place will be proportional to the matrix element of the perturbing energy (which in this case is that part of the energy which so far has been neglected), and after integrating over the electron coordinates we find that we must evaluate an integral of the form

$$I = \int \psi_2^* \hat{V} \psi_1 d^3\boldsymbol{R}, \tag{4.24}$$

where ψ_1 and ψ_2 are nuclear wavefunctions and $\hat{V}$ is a matrix element as far as the electronic coordinates are concerned and an operator as far as the nuclear coordinates are concerned. The nuclear motion can be considered to be quasi-classical so that $\hat{V}$ will simply be a function of R.

We can take the ψ_i in integral (4.24) in their semi-classical form, and as long as we are in the classical region we have

$$\Psi_i = \frac{1}{R} \frac{1}{\sqrt{p_i}} \exp\left[\frac{i}{\hbar} \int_a^R p_i dR + \alpha_i\right], \tag{4.25}$$

where a is some value of R in the classical region, the α_i are constant phases, and the p_i are given by the equation (m is here the reduced mass)

$$p_i(R) = \sqrt{[2m\{E - U_i(R)\}]}. \tag{4.26}$$

† If a transfer of angular momentum from the electron system to the nuclear motion can take place, one is concerned with two electronic energy curves corresponding to different angular momenta, which can in fact cross.

We have made use here of the fact that the problem has spherical symmetry so that it can be reduced to a one-dimensional problem, depending on R only.

We can now write for I

$$I = \int dR f(R) \exp \frac{i}{\hbar} \left\{ \int_a^R \langle \sqrt{[2m\{E - U_1(R')\}]} - \sqrt{[2m\{E - U_2(R')\}]} \rangle dR' \right\}, \tag{4.27}$$

and we see that the main contribution to the integral will come from the region where

$$E - U_1(R) = E - U_2(R) \tag{4.28}$$

or

$$U_1(R) = U_2(R), \tag{4.29}$$

that is the region where the two potential curves intersect ($R = R_0$, see Fig. 4.1). From this argument, we can say loosely that the transition takes place when

$$R_1 = R_2 \quad \text{and} \quad p_1 = p_2 \tag{4.30}$$

or during the transition the distance apart of the nuclei and their relative momentum remain unchanged (Franck–Condon principle). We can understand this as during an electron transition the nuclei cannot appreciably change their momentum or position.

The further evaluation of the matrix element I is straightforward but not without interest. We refer the reader to Landau's paper or to Landau and Lifshitz's textbook (1965, § 90) for details and for a discussion of the role of selection rules.

V

Field Theory†

LANDAU'S contributions to quantum electrodynamics and in general to quantum field theory have been many and important. In the early days, soon after the development of quantum electrodynamics by Heisenberg and Pauli (1929, 1930), Landau (3) with Peierls constructed a Schrödinger equation to describe in configuration space the electromagnetic field and its interaction with matter and showed the equivalence of their approach with that of Heisenberg and Pauli. In an interesting paper which uses to a large extent dimensional arguments, Landau (41) discusses the limiting length, below which electrodynamics must be modified.

In the late forties and early fifties quantum electrodynamics became one of the focal points of theoretical research. This was started off by the Lamb–Retherford experiment (1947) which—using microwave techniques developed during the war for radar—showed conclusively that there was, indeed, a difference in energy of the hydrogen $2s_{\frac{1}{2}}$ and $2p_{\frac{1}{2}}$ levels. This difference, the so-called Lamb shift, had been suspected to exist, but it was too small to be determinable without reasonable doubt by ordinary spectroscopic methods. It was in contradiction to Dirac's relativistic theory of the hydrogen spectrum, but it was immediately pointed out by Kramers that the Lamb shift could be explained on the basis of an interaction of the electron with the electromagnetic field produced by itself. Subsequent more refined calculations have shown the correctness of Kramers' point of view. The inter-

† We have had to limit the selection of papers for the reprint section to those which were not too technical. As a result we did not include paper 84 as we had originally intended and we included paper 100 rather than paper 98.

action between the electron and its own field can be expressed in terms of a *self-energy* of the electron, and if one ascribes a mass to this self-energy, one is led to a *renormalization* of the electron mass: part of it corresponds to the so-called *bare* mass—the mass the electron would have, if it had no interaction with the electromagnetic field—and part to the self-energy. The total mass is the mass observed experimentally and there is no way to determine experimentally the bare mass. Apart from a renormalization of the electron mass, one is also led to a renormalization of its charge and its magnetic moment.

Tomonaga, Schwinger, Feynman, Dyson, and a host of other theorists developed a relativistically invariant perturbation theory and were able to find theoretical expressions for such quantities as the Lamb shift which were in excellent agreement with experiment. Moreover, many of the infinities originally occurring in the theory were removed in a consistent way. In fact, the Lamb shift is the difference between the infinite self-energies in the $2s_{\frac{1}{2}}$ and the $2p_{\frac{1}{2}}$ states. In classical theory, a spherical electron of radius a with its charge e uniformly distributed over its surface will produce, when it is at rest with its centre at the origin, an electric field which is radially directed and the magnitude of which $\mathscr{E}$ is equal to

$$\mathscr{E} = \frac{e}{4\pi\epsilon_0 r}, \quad r > a; \quad \mathscr{E} = 0, \quad r < a, \tag{5.1}$$

where r is the distance from the origin. The total energy E of the electric field is given by

$$E = \int \frac{1}{2}\,\epsilon_0 \mathscr{E}^2\, d^3r$$

$$= \frac{e^2}{8\,\pi\epsilon_0 a}, \tag{5.2}$$

and we see that $E \to \infty$ as $a \to 0$. If we put

$$E = mc^2, \tag{5.3}$$

that is, ascribe the whole of the electron's mass to the electromagnetic self-energy, we find that we must assign a radius a to the electron given by

$$a = \frac{e^2}{8\,\pi\epsilon_0 mc^2}\,, \tag{5.4}$$

which is one-half of the so-called *classical electron radius.* In quantum electrodynamics, most expressions diverge not as $1/a$, but as $\ln a$. However, a consistent application of renormalization theory shows that all quantities of physical interest are independent of the value of a.

In a series of papers with Abrikosov, Khalatnikov, Pomeranchuk, Galanin, Gorkov, and Ter Martirosyan, Landau (78, 79, 80, 81, 84, 86, 87, 89, 96) discusses the various problems connected with the infinities arising in quantum electrodynamics and in the quantum theory of strongly interacting particles. In this approach to electrodynamics they replace the point-interactions occurring in the theory by "smeared-out" interactions with a finite range a. Later on one must take the limit as $a \to 0$. This approach has the advantage over the usual perturbation theory that it may be applicable even in those cases where perturbation theory is not. A well-known example of the inapplicability of ordinary perturbation theory is the BCS theory of superconductivity (Bardeen, Cooper, and Schrieffer 1957) where the important quantities behave as $\exp(-c/\lambda)$ which while vanishing as $\lambda \to 0$ cannot be expanded in a power series in λ. In taking the limit as $a \to 0$, Landau and his coworkers came up against the problem that in this limit both the renormalized charge and thus also the interactions between particles vanish.

We should mention here first of all, that other people shared Landau's view that perturbation theory had only limited applicability. Specific of the work of Landau and coworkers was their conclusion that present field theory is nonsense in so far as it will necessarily lead to the vanishing of the renormalized, that is, the physical quantities, such as the charge of the electron. This conclusion they claim to be true independently of whether or not

perturbation theory is involved. Although other theorists have questioned the mathematical rigour of the arguments, the Landau school maintains that these are pedantic quibbles and that their argument can be seen to hold, provided one uses a certain amount of physical intuition.

We should mention secondly that Landau in his last years of activity felt strongly that Feynman graphs—although usually derived from conventional field theory—have an independent basic importance, and hence that it is important to study the analytic singularities corresponding to various graphs (see below).

At the end of the fifties and the beginning of the sixties a new approach to quantum field theory was developed. It was to a large extent built upon the remark by Heisenberg that one should as far as possible introduce only observable quantities into the theory. As the wavefunctions themselves cannot be observed and as the Hamiltonian formalism is intimately connected with the wavefunctions, this formalism must be abandoned and the fundamental quantities to be studied are the scattering amplitudes which directly determine the cross-section for various physical processes, involving those in which particles are created or annihilated. This means that we must study processes where one set of particles with given energies, momenta, angular momenta, go in and another set come out. In a paper contributed to the *Pauli Memorial Volume*, Landau (100; reprinted as paper 9 at the end of this volume) discusses this situation and points out the necessity to find the analytical properties of the various quantities which occur in this new theory. Some of these properties are described by the so-called dispersion relations, and other properties concern the so-called vertex parts. Landau (98; see also 99; his results were found independently by J. C. Taylor (1960)) discussed in a general way the singularities which can occur in the vertex parts.

Finally, we must mention Landau's contribution to the discussion of parity non-conservation (Landau 92; reprinted as paper 10 at the end of this volume). We shall first mention the crucial experiments which led to the postulate that sometimes parity—that is, left- or right-handedness—may not be conserved. We

shall then discuss some aspects of conservation laws and conclude the chapter with a discussion of a few of the consequences of parity non-conservation. In the 1950's it was found that there were two mesons (nowadays called kaons) which were then called the θ- and τ-meson. Their modes of decay were

$$\theta \to \pi + \pi, \tag{5.5}$$

$$\tau \to \pi + \pi + \pi, \tag{5.6}$$

and it was found that their masses and lifetimes were identical. A pion is a so-called pseudoscalar particle (cf. Davydov 1965, § 54); it has zero spin and its wavefunction changes sign under inversion, i.e. under the transformation $x, y, z \to -x, -y, -z$. A pseudoscalar particle is said to have odd parity and a particle described by a scalar wavefunction (which does not change under inversion) to have even parity. From (5.5) it follows—if we may neglect the relative motions of the pions—that the θ-meson should have even parity, and similarly from (5.6) that the τ-meson should have odd parity. Careful analysis of the decay data confirmed this conclusion and hence that the τ- and θ-mesons could not be the same particle, although other evidence, such as that about masses and lifetimes pointed clearly to the identity of the two particles. In 1956 Yang (1957) stated: "However, it will not do to jump to hasty conclusions. This is because experimentally the K-mesons seem all to have the same masses and the same lifetimes. Since particles which have different spin and parity values, and which have strong interactions with the nucleons and pions are not expected to have identical masses, and lifetimes, one is forced to keep the question open whether the inference mentioned above that the τ and θ are not the same particle is conclusive. Parenthetically, I might add that the inference would certainly have been regarded as conclusive, and in fact better founded than many inferences in physics, had it not been for the anomaly of mass and lifetime degeneracies." Lee and Yang (1956) then studied carefully the evidence for parity conservation in nuclear reactions. They came to the conclusion that although for reactions which involve nuclear or electromagnetic forces (strong and electromagnetic interactions) there

were many experiments establishing parity conservation to a high degree of accuracy, data on reactions involving weak (or decay) interactions, were not available to establish parity conservation in such reactions. If parity is not conserved in the reactions (5.5) and (5.6) we cannot reach any conclusion about the parity of the τ- and θ-mesons and there is no reason why they should not be the same particle. Once the idea of non-conservation of parity in decay processes is accepted, other consequences follow and many of those were predicted by Lee and Yang and observed experimentally practically immediately. We shall return to those at the end of our discussion. The 1957 Nobel Prize of Physics was awarded to Lee and Yang for their suggestion of a possible violation of the parity conservation law. For an extensive discussion of the whole problem of parity conservation we refer to the Nobel lectures of Lee (1958) and Yang (1958) or to expositions by Wilkinson (1959; we refer also to this reference for a general discussion of conservation laws) or—at a much more advanced level —by Okun' (1965).

Both in classical mechanics (see, for instance, ter Haar, 1964) and in quantum mechanics (see, for instance, Davydov 1965, § 19) the fact that physical systems possess certain symmetries leads to conservation laws. For instance, in a system possessing translational symmetry, linear momentum is conserved, and if there is rotational symmetry, angular momentum is conserved. In nuclear physics and elementary particle physics one meets other conservation laws—not all of which are simply related to symmetry problems. For instance, as far as we can ascertain the total electrical charge of a system is conserved (charge conservation) and so are the total number of baryons, that is of all elementary particles which are at least as heavy as the proton (baryon conservation). We must remark here that just as in charge conservation the total charge is the algebraic sum of positive and negative charges, so in baryon conservation we must take the algebraic sum of the number of baryons (reckoned positive) and the number of antibaryons (reckoned negative). Apart from the baryons there is another group of elementary particles to which a conservation

law applies. These are the leptons, that is, the electrons, neutrinos, and mesons, and their antiparticles. As far as experimental evidence goes lepton conservation is always true, provided, of course, that we again count the number of antiparticles as negative. On the other hand, the number of photons, pions, or kaons is not conserved [see, for instance, reactions (5.5) and (5.6)]. It is of interest to note that the particles for which particle conservation laws hold are all fermions, while the other particles are all bosons.

We come now to the problem of mirror symmetry. In classical physics if a given situation is possible, so also is the situation obtained from the first one by reflection in a mirror. This would mean that the laws of physics do not depend on whether we describe them in a left-handed or right-handed system of coordinates. The mirror symmetry entails a conservation law—conservation of parity. As we have just seen that in decay processes such as (5.5) or (5.6) parity is not conserved, we have found a class of phenomena which do not possess mirror symmetry. This preference of nature for a particular handedness shocked many people and it is interesting to see Pauli's reaction in a letter to Weisskopf (reproduced in his *Collected Papers* (Pauli, 1964)) where he writes: "What shocks me is not the fact that 'God is just

TABLE 5.1

	Strong	Electromagnetic	Weak
Translational symmetry (conservation of linear momentum)	Yes	Yes	Yes
Rotational symmetry (conservation of angular momentum)	Yes	Yes	Yes
Charge conservation	Yes	Yes	Yes
Baryon conservation	Yes	Yes	Yes
Lepton conservation	Not applicable	Yes	Yes
Mirror symmetry (conservation of parity)	Yes	Yes	No
Particle–antiparticle symmetry	Yes	Yes	No
Conservation of combined parity	Yes	Yes	Yes

left-handed' but the fact that in spite of this He exhibits Himself as left/right symmetric when He expresses Himself strongly." A symmetry related to the mirror symmetry, as we shall see presently, is the particle–antiparticle symmetry. If this symmetry exists, a given situation is transformed into another physically possible situation by changing every particle into its own antiparticle in the same state of motion. This symmetry appears to hold for processes involving strong (nuclear) and electromagnetic interactions, but not for those involving weak (decay) interactions. We summarize the results of our considerations in Table 5.1, where we list in the first column the various symmetry or conservation properties, in the second (third, fourth) column whether they are valid or not for processes involving strong (electromagnetic, weak) interactions.

In the last row we have inserted the conservation of combined parity, or combined inversion, a concept introduced independently by Landau (92; reprinted as paper 10 at the end of this volume), Lee and Yang (1957), and Salam (1957). It involves taking the mirror image at the same time as changing all particles into their antiparticles.

Before concluding this part of our discussion we must emphasize that like all laws of physics conservation laws can only be said to hold as long as there is no experimental evidence to the contrary. We may remind ourselves here of earlier discussions querying the firmly established energy conservation law. We mentioned that Landau at various times questioned its validity, both in connection with the problem of stellar energy and in connection with relativistic quantum mechanics, while until the introduction of a neutrino β-decay seemed to violate energy conservation. We must also mention the famous paper by Bohr, Kramers, and Slater (1924) in which the energy conservation law is assumed not to be universally valid. Since the discovery of the non-conservation of parity in weak interactions, a systematic search has been—and is being—conducted to find possible violations of other conservation laws. We refer to Okun' (1965) and Wilkinson (1959) for a discussion of various conservation laws, including several which we have not mentioned here.

We shall now discuss a few consequences of parity non-conservation. One of the earliest experiments to show non-conservation was the one where one measures the angular correlation between the momenta of the electron (e) and muon (μ) involved in a $\pi \to \mu \to e$ decay process (υ: neutrino)

$$\pi^{\pm} \to \mu^{\pm} + \upsilon, \tag{5.7}$$

$$\mu^{\pm} \to e^{\pm} + \upsilon + \upsilon'. \tag{5.8}$$

From the angular correlation measured by Garwin, Lederman and Weinrich (1957; for details see also Lee, 1958) it followed conclusively that parity was not conserved. The first experiment to show the effect in β-decay was the one by Wu, Ambler, Hayward, Hoppes, and Hudson (1957). In this experiment ^{60}Co nuclei are polarized by an applied magnetic field and if parity were conserved, the electrons from the decay process

$$^{60}\text{Co} \to {}^{60}\text{Ni} + e + \upsilon \tag{5.9}$$

would be emitted preferentially in the direction of the magnetic field, and there would be as many in the parallel as in the anti-parallel direction. However, it was found that there is a strong forward–backward asymmetry, showing also in this case that parity is not conserved in β-decay. (For details we refer to standard textbooks of nuclear physics such as those by Burcham, 1963, or Smith, 1965.)

To conclude the discussion we want to consider the properties of the neutrino. In quantum mechanics one is led to consider a multicomponent wavefunction to describe the relativistic behaviour of an electron. The wave equation satisfied by this function is the Dirac equation and for an electron one needs at least four components. However, if one considers a zero-mass particle one finds for its free motion the wave equation

$$c\,(\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}})\,\Psi = i\hbar\frac{\partial\psi}{\partial t}, \tag{5.10}$$

where the components of the operator $\frac{\boldsymbol{\sigma}}{2}$ are the Pauli matrices,

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.11}$$

and where ψ is a two-component wavefunction. One proves in the usual way (cf. Davydov, 1965, § 64) that (5.10) describes a spin-$\frac{1}{2}$ particle. Moreover, as the energy of a state with a well-defined momentum p is either cp (positive energy state or particle state) or $-cp$ (negative energy or anti-particle state), it follows from (5.10) that the *helicity* of a particle described by (5.10), which is defined as the expectation value of the operator $(\hat{\boldsymbol{\sigma}}.\hat{\boldsymbol{p}}_0)$, where $\boldsymbol{p}_0$ is the unit vector in the direction of $\boldsymbol{p}$, is $+1$ or -1 according to whether we are dealing with a positive or a negative energy state. If we describe the neutrino by (5.10), we are dealing with what Landau calls a *longitudinal neutrino*: its spin is either parallel or anti-parallel to its momentum. Many experiments (see Smith, 1965, § 14.8, for a discussion) have shown that the neutrino produced in β^--decay has a helicity -1. This means that it can probably—as was suggested by Landau, Salam, and Lee, and Yang—be described by a two-component theory.

A two-component theory was discussed as early as 1929 by Weyl (1929), but as one can show that the helicity of any system possessing well-defined parity must vanish, this theory was rejected, for instance by Pauli (1933). One can put this slightly differently. One obtains (5.10) from the four-component Dirac equation by letting the rest-mass of the particle go to zero. In that case the four equations split into two independent pairs of equations. As, however, the one pair transforms into the other under inversion, as long as one sticks to the principle of conservation of parity, one cannot consider only one pair of equations [such as (5.10)], and one is led to a four-component theory of the neutrino. If one accepts non-conservation of parity, the two-component theory can be resurrected. It is interesting to note that the theory conserves combined parity. We cannot enter here into a discussion of the complications introduced by the apparent existence of two kinds of neutrino: the electron neutrino which appears in β-decay and the muon neutrino which is the decay product of the muon [compare the decay reactions (5.8) and (5.9): as μ takes the place of ^{60}Co, so ν' takes the place of ^{60}Ni].

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† The numbers of this list correspond to those in the *Collected Papers* volume.

‡ *JETP* = *Journal of Experimental and Theoretical Physics of the USSR.*

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1

On the Theory of Phase Transitions†

The question of continuous phase transitions (without latent heat) have been investigated from the general thermodynamical point of view. In doing this it becomes clear that such transitions can take place when the symmetry of the lattice changes. There are two possible types of transition, namely: (1) Curie points with a discontinuity in the specific heat, which lie on a curve in the p–T diagram, (2) isolated points in the p–T diagram which lie in a certain way on intersections of curves of normal phase transitions.

Up to the present time, among all phase transitions, Curie points, and so on, only the transition between a liquid and a gas has been fully investigated. It is known that the liquid–gas equilibrium curve in the p–T diagram has an end point, and that a continuous transition between liquid and gas can be realised by going round it. As for transitions between a liquid and a crystal, or between different crystal modifications, the question about them has not been fully clarified. In a number of cases people talk about transitions connected with rotations of molecules; however it is not at all clear how rotations can lead to phase transitions, and in particular to discontinuities in the specific heat.

One even finds strange statements that there is no essential difference at all between liquids and crystals, and that continuous transitions between them are possible. However, liquids differ essentially from crystals in that they are isotropic in contrast to anisotropic crystals. Every transition from a crystal to a liquid or to a crystal of a different symmetry is associated with the disappearance or appearance of some elements of symmetry. But elements of symmetry are either present or absent; no intermediate

† *Phys. Z. Soviet Un.*, **11**, 26, 1937.

case is possible. And so continuous transitions (in the sense that transitions between liquid and gas are continuous) connected with changes of the symmetry of the body are absolutely impossible.

Until recently the exact formulation of the very idea of the crystal lattice was lacking. Only quite recently Bethe and Peierls[1] have stressed the role of correlations at infinity in the crystal lattice.

Note that normal phase transitions between liquid and crystal or between different modifications where the state of the body, particularly the energy, changes discontinuously are not the main interest of the present investigation. Our main interest is in such transitions where the state of the body (particularly the energy) changes continuously even although the symmetry changes discontinuously. (See below for details.) We shall call these transitions the continuous ones. Let us emphasise once again that they are not continuous in the sense that transitions between liquid and gas are. At every moment we can tell that we have a body of this or that symmetry.

Usually the approach to this question is made difficult by the use of an idealised model of the lattice in which all atoms are placed in their positions and thermal motion is ignored.

These difficulties can be avoided if a distribution probability $\varrho(x\,y\,z)$ is used, where $\varrho(x, y, z)\,dx\,dy\,dz$ determines the probability for finding an atom in the given volume element of the body. If the body consists of different kinds of atoms then it would be possible to introduce several functions $\varrho_1, \varrho_2, \ldots$, which would determine the probabilities for each kind of atom. Even in that

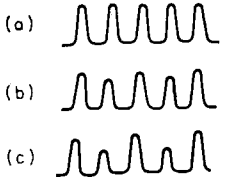

FIG. 1.

case it would instead be possible to use only one distribution function. For instance we can determine that function as one which gives the mean charge density at every point of the body (multiplied by $dx\,dy\,dz$ it would give the charge in that volume). In the following, we shall talk simply about the "density" $\varrho(x, y, z)$, meaning by that some function which determines the distribution of atoms in the body under consideration. Note that such a method based on the function ϱ also has the advantage that it is possible in quantum mechanics as well.

The important feature of the function ϱ is its symmetry, i.e. that group of co-ordinate transformations with respect to which ϱ is invariant. The same group also determines the symmetry of the body. It is known that there are in all 230 possible different groups of transformations, i.e. types of symmetry. In isotropic bodies (liquids) obviously $\varrho = \text{const}$.

As already mentioned we shall consider here those transitions where, regardless of a discontinuity in the symmetry, the state of the body changes continuously. In other words the density ϱ changes continuously. It is easy to see that such transitions are possible because even a very small change in the distribution of the atoms in the lattice is enough to change its symmetry. If for instance ϱ is represented by the curve Fig. 1a (schematically drawn in one dimension) and some of the maxima decrease (Figs. 1b and 1c), then the symmetry changes as soon as the decreasing starts (the translational period of the lattice increases).

Let us consider a crystal with some density ϱ_0 which has a certain symmetry (we shall talk about the totality of symmetry transformations of ϱ_0 as the group ϱ_0). At the transition point the density starts to change and becomes $\varrho = \varrho_0 + \delta\varrho$, where $\delta\varrho$ is small compared with ϱ_0. $\delta\varrho$ also has some symmetry (group $\delta\varrho$) which is lower than that of ϱ_0 (i.e. not all elements, that is symmetry transformations of ϱ_0, are elements of symmetry of $\delta\varrho$; the group $\delta\varrho$ is a subgroup of the group ϱ_0). Then $\varrho = \varrho_0 + \delta\varrho$ has the same symmetry, because the sum of two functions has the same symmetry as the less symmetric term. We can therefore neglect the case where $\delta\varrho$ has a higher symmetry than ϱ_0, since

then $\varrho_0 + \delta\varrho$ would have the same symmetry as ϱ_0, so that no change in the symmetry of the body would take place.

Symmetry transformations from the group ϱ_0 which do not belong to the group $\delta\varrho$ change $\delta\varrho$ into some other function. It is known from group theory that the function $\delta\varrho$ can be broken into a sum of functions the number of which is equal to the number of elements of the group ϱ_0, in such a way that under every transformation of that group all these functions transform among themselves, i.e. become linear combinations of themselves.

Matrices of these linear transformations form the so-called "representation" of the group ϱ_0. Further, all these functions into which $\delta\varrho$ is broken, can be separated into groups or "races",† where all functions composing them again transform among themselves. So we can write:

$$\delta\varrho = \sum_n \sum_i c_i^{(n)} \varphi_i^{(n)}, \tag{1}$$

where n is the number of the race and i is the number of the function in the race.

Each of these races of functions can be used as a basis for the representation of the group. That representation is realised by the transformation matrices of the functions of that race. It is known that there exists an expansion of $\delta\varrho$, into $\varphi_i^{(n)}$ where every race consists of the smallest possible number of functions (i.e. an irreducible partition, thus realising the "irreducible representation").

In (1) we shall suppose just such a partition. We could after all simply write it as $\delta\varrho = \Sigma_n \Sigma_i \varphi_i^{(n)}$, because the functions $\varphi_i^{(n)}$ are not determined beforehand; in the following it will be convenient to consider the functions $\varphi_i^{(n)}$ somehow normalised.

Among all $\varphi_i^{(n)}$ there is always one function (which forms a "race" by itself) which is invariant with respect to all transformations of the group ϱ_0. In the sum $\varrho_0 + \delta\varrho$ we shall consider this function to belong to ϱ_0 so that $\delta\varrho$ has no such function.

† The term "race" is used here both for an irreducible representation and for the basis of the irreducible representation. The theory presented here is given in modern terms by Landau and Lifshitz (1958, § 136) (note by Editor).

The thermodynamic potential Φ, of the body, is determined by the density ϱ, i.e. depends on the form of the function ϱ. In other words Φ is a functional of ϱ: $\Phi = \Phi\{\varrho\}$. Φ depends also on the temperature T and the pressure p of the body as parameters. When p and T are given the form of the function ϱ is determined from the condition that Φ should have a minimum.

Let us expand the thermodynamic potential $\Phi\{\varrho_0 + \delta\varrho\}$ in the state with density $\varrho = \varrho_0 + \delta\varrho$ in powers of $\delta\varrho$ (of course this expansion is not a normal power series; individual terms in the expansion are integral operators of $\delta\varrho$). Saying this in another way, we have an expansion in powers of $\varphi_i^{(n)}$ and $c_i^{(n)}$.

It can be seen that first order terms in the expansion are zero. The potential Φ as the quantity which characterises the physical properties of the body obviously should not change under any movements of the body, i.e. should be invariant under all possible co-ordinate transformations. If such a transformation changes ϱ_0 into ϱ_0' and $\delta\varrho$ into $\delta\varrho'$, then

$$\Phi\{\varrho_0 + \delta\varrho\} = \Phi\{\varrho_0' + \delta\varrho'\}.$$

From this it can be seen that if Φ is considered as a function only of $\delta\varrho$, then Φ is invariant only with respect to those transformations which do not change ϱ_0, i.e. the transformation group ϱ_0. Since the functions $\varphi_i^{(n)}$ under transformations of this group transform among themselves we can consider only the coefficients $c_i^{(n)}$ to change under these transformations, because the expression for Φ should be invariant with respect to transformations of these coefficients. In particular the coefficients of the powers of the $c_i^{(n)}$ in the expression of Φ will be invariants of the relevant degree. It is known that it is impossible to construct linear invariants from quantities transforming as an irreducible representation.

As to the terms of second order, they are known to separate into a sum of groups of terms consisting only of the quantities $c_i^{(n)}$ (consequently of the functions $\Phi_i^{(n)}$) belonging to one race.

The transition point is thus characterised by the fact that for a small change in T and p, an extra term $\delta\varrho$ appears in the density ϱ_0. On one side of the transition point (which we shall call the

"upper" side) terms of second order in the expansion are obviously essentially positive for all T and p. Thus the minimum is at $\delta\varrho = 0$ i.e. the state of the body corresponds to $\varrho = \varrho_0$, that is, the body has a higher symmetry. On the other ("lower") side of the transition point terms of the second order are not essentially positive and thus to the minimum of Φ corresponds some $\delta\varrho$ different from zero, which really determines the symmetry of the body. Consequently at the transition point itself the sum of all terms of the second order should be zero for any fixed $\delta\varrho$.

For that it is obviously sufficient that any group of terms of second order belonging to one race becomes zero at the transition point. On the other hand the $\delta\varrho$ which make the sum zero are just those $\delta\varrho$ which can appear at the transition point.

After the functions $\varphi_i^{(n)}$ belonging to one of the races have been chosen such that the corresponding second order terms are equal to zero, then the rest of the $\varphi_i^{(n)}$ can be taken to be equal to zero. Then $\delta\varrho = \sum_i c_i^{(n)}\varphi_i^{(n)}$ (summation only over functions of one race) is just that change of the density which makes the term of the second order vanish at the transition point, and is consequently physically realised. Therefore in future we shall only be concerned with that one race and shall drop the superscript (n), specifying the race.

Because the functions φ_i are determined by the condition that they should make the terms of the second order vanish at the transition point, Φ can now be considered as a function only of the c_i, and the expansion in $\delta\varrho$ as an expansion in c_i where there are no terms of first order. As has been already said, the terms of the second order should form an invariant (with respect to all transitions of the group ϱ_0). In accordance with group theory such an invariant (in an irreducible representation) is a positive definite quadratic form, which, by suitable choice of the normalisation of the c_i, can always be written as the sum of squares. In this way terms of the second order (of the given race) have the form:

$$A\sum c_i^2. \tag{2}$$

At the transition point this expression need not be zero i.e. at that point $A = 0$ (A is of course a function of p and T).

In an analogous way terms of the third, fourth, . . ., order are formed correspondingly from invariants of the third, fourth, . . ., order. Terms of the third order can in some cases be absent. If, for instance, in a given race only one function φ enters, then by acting with transformations of the group ϱ_0 the coefficient c can change sign. Therefore, in that case, all invariants and consequently all terms of odd orders are equal to zero.

If at a certain point (i.e. at specified p and T) $A(p, T)$ should vanish, then, in order that this point really be a point of a continuous transition, it is necessary that the terms of third order are zero. Otherwise Φ cannot have a minimum (as a function of c_i) at that point, because that point would not correspond to a stable state of the body.

Two cases are possible:

1. Terms of the third order are identically zero (there are no invariants of third order). Transition points are determined from one condition:

$$A(p, T) = 0; \tag{3}$$

besides this terms of the fourth order should be positive definite. In that case transition points lie thus on a certain curve, which is determined by (3). This is the case of Curie points.

A physical state is realised and is determined by the coefficients c_i which correspond to the minimum of Φ (at given p and T). Define

$$\sum c_i^2 = \eta^2 \tag{4}$$

and

$$\frac{c_i}{\eta} = \gamma_i .$$

Then the expansion of Φ is written in the form

$$\Phi = \Phi_0 + A\eta^2 + B(\gamma_i)\eta^4 + \ldots,$$

where all coefficients are also functions of p and T.

Because the term of the second order does not depend on γ_i the values of γ_i can be obtained by finding the minimum of $B(\gamma_i)$. Having found these values and substituting them into $B(\gamma_i)$ we get

$$\Phi = \Phi_0 + A\eta^2 + B\eta^4 + \ldots, \tag{5}$$

where $B = B(p, T)$ is the minimum value of $B(\gamma_i)$.

According to the above

$$B(p, T) > 0. \tag{6}$$

Above the Curie point $A > 0$; to the minimum of Φ corresponds $\eta = 0$, i.e. the body has the symmetry ϱ_0. At the Curie point $A = 0$, and below it $A < 0$. From the minimisation of Φ, i.e., from $\partial\Phi/\partial\eta = 0$, we find $A + 2B\eta^2 = 0$ or

$$\eta^2 = -\frac{A}{2B}. \tag{7}$$

Then

$$\Phi = \Phi_0 - \frac{A^2}{4B}.$$

The specific heat of the body is

$$C = -T\frac{\partial^2\Phi}{\partial T^2} = C_0 + T\frac{(\partial A/\partial T)^2}{2B}. \tag{8}$$

Terms which vanish at the Curie point are omitted. C_0 is the specific heat of the body with the symmetry ϱ_0, i.e. above the Curie point. Because of (8) we see that at the Curie point $C > C_0$. In this way at the Curie point the heat capacity has a discontinuity and it increases in going from a more to a less symmetric body (note, that one body is less symmetric than the other if its symmetry transformation group is a sub-group of the symmetry group of the other).

As was pointed out at the beginning of this case the coefficients γ_i are determined from $B(\gamma_i)$, i.e. they depend on the form of the terms of fourth order.

But all these terms depend also on p and T; because of that the γ_i depend on p and T too. But the quantities γ_i determine the

symmetry of $\delta\varrho$, i.e. the symmetry of the crystal. Because of that it may happen that at different parts of the Curie point curve a transition takes place from a more symmetric crystal (where $\delta\varrho = 0$) to less symmetric crystals of different symmetries (i.e. where $\delta\varrho$ has a different symmetry).

In that case in the phase diagram there is a point of intersection of the Curie curve (curve 1) with the phase transition curve (curve 2, Fig. 2); I is the most symmetric phase ($\delta\varrho = 0$); along curves *AB* and *BC* at Curie points it goes over into less symmetric phases II and III, where $\delta\varrho_{II} \neq 0$, $\delta\varrho_{III} \neq 0$.

Symmetry groups $\delta\varrho_{II}$ and $\delta\varrho_{III}$ are sub-groups of the symmetry group of the first phase. However, they are not generally sub-groups of each other. Because of this the difference $\delta\varrho_{II} - \delta\varrho_{III}$

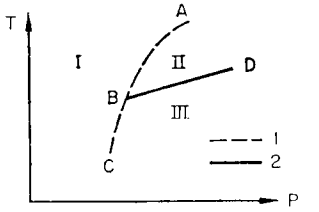

FIG. 2.

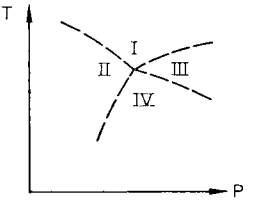

FIG. 3.

cannot become zero; consequently between phases II and III there should not be a Curie line, but a phase transition line. At the point *B* all three phases are identical; along the line *AB* the phases I and II are identical ($\delta\varrho_{II} = 0$); along *BC*: $\delta\varrho_{III} = 0$.

It can further be shown, that the intersection of one Curie line with another can happen only at a point of the type shown in Fig. 3. If I is the most symmetric phase then the phases II and III have lower symmetries; their symmetry groups are sub-groups of the symmetry group of the phase I. Phase IV has even lower symmetry than II or III. Its symmetry group is simultaneously a sub-group of the symmetry groups of the phases II and III.

Finally, let us consider those cases where terms of fourth order in the expansion of Φ also become zero at the transition point.

For this it is necessary that the terms of the fourth order have only *one* coefficient which depends on p and T, together with which they would become zero. Otherwise the vanishing of fourth-order terms together with the condition $A(p, T) = 0$ would give more than two equations with two unknowns (p and T), which would generally have no solutions. For this it is required that only one invariant of the fourth order (formed from the c_l) exists, i.e. the terms of the fourth order are identically equal to $B(p, T)\eta^4$ for arbitrary c_l.

If terms of the fourth order are equal to zero, then for the stability of the state (i.e. for Φ to be a minimum) it is necessary for the term of the fifth order to be identically zero and the term of the sixth order to be positive. Two conditions, $A = B = 0$ then determine an isolated point. That point is a λ-point whose properties have already been investigated by the author.† There it has been pointed out that λ-points are the points where the Curie curve goes over into the phase transition curve. Here I shall only consider an additional intersection of the Curie curve with the phase transition curve in bodies which are mixtures of two substances. In that case it appears that the specific heat does not become infinite but, as in pure substances, experiences only a finite jump.

The fact that the body is a mixture does not introduce anything essentially new into our considerations. The symmetry of the crystal is, as before, determined by the density ϱ, and the expansion of Φ in the vicinity of a point of a continuous transition is

$$\Phi = \Phi_0 + A\eta^2 + B\eta^4 + \ldots;$$

but now Φ_0, A, B depend not only on p and T but on the concentration x of the mixture.

Let us prove that at the transition point of the Curie line into the phase transition line for mixtures (we shall in this case also call such a point a λ-point) the coefficient B in the expansion of Φ should be zero. And indeed from this it will follow that the specific heat does not become infinite at that point (see equation (8)).

† In this earlier paper[2] the quantity ξ corresponds to η^2.

Let us investigate the neighbourhood of the λ-point. First we shall write conditions for the equilibrium of two phases on the transition curve (either a phase or a continuous transition). It is known that the thermodynamic potential Φ is an additive quantity and because of that in mixtures it should be a homogeneous function of the first order of the number of particles of each kind. In particular for the mixture of two materials $\Phi = Nf(n/N)$, where n and N are the numbers of both kinds of particles. The chemical potentials of each kind of particles are

$$\frac{\partial \Phi}{\partial N} = f - x \frac{\partial f}{\partial x}, \quad \frac{\partial \Phi}{\partial n} = \frac{\partial f}{\partial x}$$

(where $x = n/N$). The equilibrium conditions are equality of the chemical potentials of both phases. In our case on one side of the transition point (where $\eta = 0$, i.e. in the more symmetric phase) $\Phi = \Phi_0$; on the other side $\Phi = \Phi_0 + A\eta^2 + B\eta^4$. If x_0 and x are the concentrations of both phases then the equilibrium conditions are

$$\frac{\partial \Phi_0}{\partial x_0} = \frac{\partial \Phi}{\partial x}$$

and

$$\Phi_0(x_0) - x_0 \frac{\partial \Phi_0}{\partial x_0} = \Phi - x \frac{\partial \Phi}{\partial x}.$$

Substituting $\Phi = \Phi_0 + A\eta^2 + B\eta^4$, we find from the first condition

$$\frac{\partial \Phi_0}{\partial x_0} = \frac{\partial \Phi_0}{\partial x} + \frac{\partial A}{\partial x} \eta^2$$

($\partial A/\partial x$ is not generally zero at the transition point and because of that it is possible to limit ourselves to the term in η^2) or, expanding $\partial \Phi_0/\partial x$ in a series:

$$\frac{\partial \Phi_0}{\partial x} = \frac{\partial \Phi_0}{\partial x_0} + (x - x_0) \frac{\partial^2 \Phi_0}{\partial x_0^2} + \cdots,$$

$$- \frac{\partial^2 \Phi_0}{\partial x_0^2} (x - x_0) = \frac{\partial A}{\partial x} \eta^2. \tag{9}$$

In the second condition to the same accuracy we put

$$\frac{\partial \Phi_0}{\partial x} \cong \frac{\partial \Phi_0}{\partial x_0}$$

and get

$$\Phi = \Phi_0(x_0) + \frac{\partial \Phi_0}{\partial x_0}(x - x_0).$$

Substituting here the expression for Φ_0 we find

$$A\eta^2 + B\eta^4 = \Phi_0(x_0) - \Phi_0(x) + (x - x_0)\frac{\partial \Phi_0}{\partial x_0}$$

and expanding $\Phi_0(x_0) - \Phi_0(x)$ in a series:

$$A\eta^2 + B\eta^2 = -\frac{(x - x_0)^2}{2}\frac{\partial^2 \Phi_0}{\partial x_0^2}.$$

Further substituting $x - x_0$ from equation (9), then

$$A\eta^2 + B\eta^4 = \frac{(x - x_0)}{2}\frac{\partial A}{\partial x}\eta^2,$$

or

$$A + B\eta^2 = \frac{(x - x_0)}{2}\frac{\partial A}{\partial x}. \tag{10}$$

Also remember that one of the conditions for the stability of the state of the body, i.e. the condition that Φ is a minimum, is $\partial\Phi/\partial\eta = 0$ (in that phase where $\eta \neq 0$). From this we get from (7):

$$\eta^2 = -\frac{A}{2B}.$$

Substituting this into (10), we find

$$A - (x - x_0)\frac{\partial A}{\partial x} = 0.$$

Substituting from here $x - x_0 = A/(\partial A/\partial x)$ and $\eta^2 = -A/2B$ in equation (9), we find

$$\frac{(\partial^2\Phi_0/\partial x_0^2)A}{\partial A/\partial x} = \frac{\partial A}{\partial x}\frac{A}{2B}$$

or

$$B = \frac{(\partial A/\partial x)^2}{2(\partial^2\Phi_0/\partial x_0^2)} \tag{11}$$

From this it is obvious that at a λ-point B never becomes zero and that always $B > 0$. The last statement follows from (11) because $\partial^2\Phi_0/\partial x_0^2 > 0$ according to the known thermodynamical inequalities for solutions. Further the equation

$$A(x) + (x_0 - x)\frac{\partial A}{\partial x} = 0$$

can be written in the form $A(x_0) = 0$ to the accepted approximation, i.e. the phase transition points for the more symmetric phase satisfy the same equation as the Curie curve.

In this way the neighbourhood of a λ-point has thus for mixtures the form shown in Fig. 4 (plotted along the co-ordinate axes are concentration and temperature). The dotted line is the continuous transition curve, i.e. the Curie curve. I is the more, and II the less symmetric phase. The line 10 goes continuously into the line 03; the line 02 branches away from it. The line 302 is the phase transition line; the shaded region 302 is the region of separation into two phases I and II, the concentrations of which are determined by the lines 03 and 02.

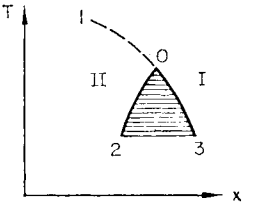

FIG. 4.

2. Let the term of third order in the expansion of Φ now not be identically zero. The continuous transition in this case is only possible where terms of the second and third order are equal to zero. The first of these conditions gives again $A(p, T) = 0$. For the second condition to hold it is necessary for only one invariant of the third order to exist, i.e. the terms of third order should only possess one coefficient depending on p and T. Otherwise we would have too many equations which would not be possible to satisfy simultaneously.

Let us again introduce the quantities $\gamma_i = c_i/\eta$.

The term of third order should have the form

$$B(p, T)\, b(\gamma_i)\eta^3$$

(it is assumed that there is only one invariant of the third order) and the expansion is

$$\Phi = \Phi_0 + A(p, T)\eta^2 + B(p, T)\, b(\gamma_i)\eta^3 + C(p, T, \gamma)\eta^4 + \ldots \quad (12)$$

At a continuous transition point

$$A = B = 0.$$

Consequently the continuous transition points are in this case isolated, i.e. there is no Curie line. Therefore, such points should in some way lie on the phase transition lines. Accordingly it is necessary to investigate the character of the phase transition line in the vicinity of such points.

In the neighbourhood of a continuous transition point of the type under consideration A and B are close to zero (but $C > 0$). On the equilibrium curves of the more and less symmetric phase their thermodynamic potentials are equal, i.e. $\Phi = \Phi_0$, or

$$A\eta^2 + Bb\eta^3 + C\eta^4 = 0. \quad (13)$$

Besides that $\partial\Phi/\partial\eta$ should be zero, as it should be for all possible equilibrium states, i.e.

$$\eta(2A + 3Bb\eta + 4C\eta^2) = 0. \quad (14)$$

These two equations should have a common solution different from zero (different from zero because the solution $\eta = 0$ would

mean that at the transition points $\delta\varrho = 0$, i.e. a Curie line would exist and that as has already been mentioned is impossible).

It is easy to see that for this it is necessary that

$$B^2b^2 = 4AC \tag{15}$$

and

$$\eta = -\frac{Bb}{2C}. \tag{16}$$

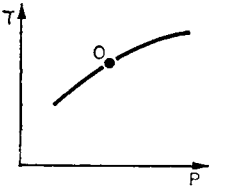

FIG. 5.

It could be thought that the continuous transition points considered simply lie on a phase transition curve like the point O in Fig. 5. However that is not so, but instead we shall now show that the point O should lie on the intersection of several phase transition curves.

Let us investigate points in the neighbourhood of O but not lying on phase transition curves. For them (as in every stable state) $\partial\Phi/\partial\eta = 0$. This equation has solutions $\eta = 0$ and also solutions of the quadratic equation (14).

The solution $\eta = 0$ corresponds to points which represent the state of the more symmetric phase ($\delta\varrho = 0$). In the second phase η is determined by equation (14). But quadratic equations have in general two solutions. At the point O: $A(p, T) = B(p, T) = 0$; in the neighbourhood of the point O the equation $B(p, T) = 0$ determines a line. On that line (14) has two solutions with opposite signs

$$\eta = \pm\sqrt{-\frac{A}{2C}}. \tag{17}$$

That means that in the neighbourhood of the point O near to the line $B = 0$ (14) has solutions with different signs, almost equal to each other in absolute value (because close to the line $B = 0$, B is small). On one side of the line $B = 0$, B is positive; there the negative solution of (14) corresponds to the stable state, otherwise by changing the sign of η it would be possible to decrease Φ, i.e. Φ would not have a minimum. By the same reason on the other side of the line $B = 0$ (where $B < 0$) the other solution of (14) becomes valid. Consequently the line $B(p, T) = O$ is also a phase transition line, where η changes sign discontinuously.

In this way the neighbourhood of the point O has the appearance shown in Fig. 6, i.e. at the point O the other phase transition line

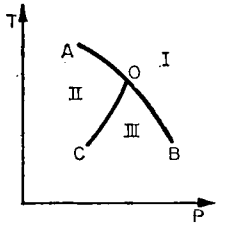

FIG. 6.

ends. The phase I is the more symmetric phase (in it $\eta = 0$, $A > 0$). On the phase transition line AB, $A = 0$. The less symmetric phases II and III (where $A < 0$) have the same symmetry (in them η differs only in sign, but this does not influence the symmetry of ϱ). On the phase line CO, $B(p, T) = 0$. At the point O all three phases become identical.

Let us determine the latent heat on the curves CO and AB. For the entropy we have

$$S = -\left(\frac{\partial \Phi}{\partial T}\right)_p = -\left(\frac{\partial \Phi}{\partial T}\right)_{p,\eta} - \left(\frac{\partial \Phi}{\partial \eta}\right)_{p,T} \frac{d\eta}{dT}.$$

But in all stable states $\partial\Phi/\partial\eta = 0$. Therefore

$$S = -\left(\frac{\partial \Phi}{\partial T}\right)_{p,\eta}.$$

Substituting (12), we find in the neighbourhood of the point O (i.e. for small η):

$$S = S_0 - \frac{\partial A}{\partial T}\eta^2 \tag{18}$$

$S_0 = -\partial\Phi_0/\partial T$ is the entropy of the phase I. Terms of higher orders can be neglected because unlike A, $\partial A/\partial T$ does not become zero.

Let us find the latent heat on the curve AB. On it $\eta = -Bb/2c$ (see equation (16)) and the latent heat of transition from the less symmetric to the more symmetric phase is

$$Q = T(S_0 - S) = \frac{\partial A}{\partial T}T\eta^2 = \frac{\partial A}{\partial T}\frac{Tb^2}{4C^2}B^2. \tag{19}$$

Near the point O the quantity B is a linear function of the distance along the curve from the point O (because at the point O, $B = 0$). In this way on the curve AB, near the point O, the latent heat is proportional to the square of the distance from O.

In order to find Q on the curve OC close to O, write down the next term in the entropy S:

$$S = -\left(\frac{\partial\Phi}{\partial T}\right)_{p,\eta} = S_0 - \frac{\partial A}{\partial T}\eta^2 - \frac{\partial B}{\partial T}b\eta^3. \tag{20}$$

Since on the curve CO the quantity η is equal in absolute value in both phases, then the difference in entropy between phases II and III is $2b\eta^3\partial B/\partial T$, where η is determined from (17). The latent heat is

$$Q = 2T\frac{\partial B}{\partial T}b\eta^3. \tag{21}$$

From (21) and (17) it can be seen that Q is proportional to $(-A)^{3/2}$, i.e. proportional to the distance from O to the power 3/2.

Finally, it can be shown that when terms of fourth order have a complex structure new phase transition lines can appear. The neighbourhood of the point O then does not look as shown in Fig. 6, but as in Fig. 7.

Phase I has the highest symmetry. Phases II and III have the same symmetry; the same applies to phases IV and V. At the point O all phases become identical, that is indeed the point of continuous transition. At the point O two of the phase transition curves have a common tangent and the third ends. Here we have assumed that two curves of phase transitions touch at the point O. In the general case there may be several of them.

In a subsequent paper it will be shown that in the case of transitions between liquids (i.e. isotropic bodies) and crystals terms of the third order are not identically zero. Therefore continuous transitions between liquids and crystals are only possible at

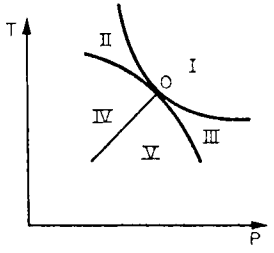

FIG. 7.

isolated points of the type shown in Figs. 6 and 7. In particular Curie lines are impossible.

In the whole of the preceding part of the paper we have assumed that the symmetry properties of crystals are determined by the symmetry of the mean density function ϱ. But the moving charges (electrons) in the body can create in the crystal a mean current density $\boldsymbol{j}$ as well. Then the properties of the crystal will depend not only on the symmetry of the density ϱ but also on the symmetry of $\boldsymbol{j}$. Note that $\int \boldsymbol{j} dV$ over the whole volume of the crystal should be equal to zero. Otherwise that current would create a magnetic field and the crystal would possess some magnetic energy. That energy would very rapidly increase with an increase in the dimensions of the crystal and this would be energetically disadvantageous.

In the majority of bodies $j = 0$. In particular $j \neq 0$ in ferromagnetic bodies. In the latter, in addition to this, the magnetic moment is not equal to zero in every part, i.e. $\int [r \wedge j] dV \neq 0$ over an elementary cell. However not every body with $j \neq 0$ is ferromagnetic, because although $j \neq 0$, $\int [r \wedge^{\bullet} j] dV$ can be zero.

If $j = 0$, then the symmetry properties of the crystal are determined by the density ϱ. It is known that there exists a limit to the number (230) of possible types of symmetry, i.e. space groups. If besides that $j \neq 0$ then the classification of the types of symmetry follows from the properties of ϱ and j; then it is possible for there to be more than 230 space groups.

The presence of $j \neq 0$ (crystals with $j \neq 0$ we can call magnetic) does not introduce anything essentially new into the preceding discussion about transition points. At transition points the change in symmetry is then determined by $\delta\varrho$ and δj. As before only the transition points discussed above are possible.

Let us concentrate for a while on transitions connected with the appearance (or disappearance) of j, i.e. on transitions between magnetic and non-magnetic crystals. Since on one side of these points $j = 0$, then $\delta j = j$. As before we shall consider only the continuous transition points of this type, i.e. points where $\delta j = j = 0$, in the neighbourhood of which (on one side) j is small. Instead of expanding the thermodynamic potential Φ in powers of $\delta\varrho$ we shall now have an analogous expansion in powers of j. In view of the symmetry of all the properties of the body in relation to the exchange of the future with the past the potential Φ, in particular, cannot change when the sign of time is reversed. When such a change is made the density ϱ does not change, but the current j has its sign reversed. From this it follows that in the expansion of Φ in powers of j all terms with odd powers of j should be identically zero. It means that transitions connected with the appearance of j always belong to the case 1, i.e. Curie points are possible which form Curie lines, and under suitable conditions λ-points also. Such are the Curie points in ferromagnetic bodies. The discontinuities in the specific heats in chlorides of Fe, Cr, Ni at low temperatures are apparently of the same nature, there is

also a λ-point in MnO. All these materials have $j \neq 0$ below the transition point, and at the transition point j becomes zero (above that point j remains equal to zero).

Until now we have been talking about transitions with a change in the symmetry of the crystal, but we have not discussed the physical nature of such changes which take place. Atoms in a crystal usually perform small oscillations about their equilibrium positions, i.e. the lattice points. In view of their smallness these oscillations cannot cause changes in the lattice symmetry. This does not apply, of course, to the jump-like transitions when the atoms start oscillating around new equilibrium positions.

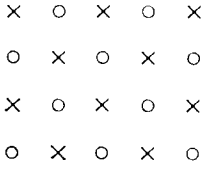

FIG. 8.

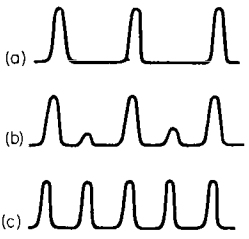

FIG. 9.

The continuous transitions with a change in the symmetry are always connected with a change in the ordering of the crystals, which follows when the number of places in the lattice where atoms of a given kind can reside is larger than the number of such atoms. There exists one particular distribution of the atoms in the lattice which is energetically most favourable. This is realised at sufficiently low temperatures. At higher temperatures the distribution of atoms deviates from this. As an example, let us consider a crystal formed from two kinds of atoms (binary mixture). The ideal configuration is that in which the atoms of different kinds are placed at lattice points in a definite order one relative to another (this is schematically shown in Fig. 8).

Such a crystal is said to be completely ordered. But every atom can in principle be found at any lattice point, i.e. there are more

possible places for atoms of a given kind than there are atoms of that kind. Therefore, the crystal can also be incompletely ordered if some atoms are in "foreign" places, i.e. places at which, in the completely ordered crystal, there should be atoms of the other kind. The probability, i.e. the density function ϱ of finding atoms of one kind at lattice points in the completely ordered crystal can be represented schematically (in one dimension) by the curve in Fig. 9a, where the probability has a sharp maxima at every second lattice point. In the incompletely ordered crystal there appears some probability of finding atoms of a given kind at other (foreign) lattice points (Fig. 9b).

Finally the number of atoms of a given kind in the lattice residing at "foreign" points can be equal to the number of these atoms residing at their "own" places. This means that the probability of finding atoms of a given kind becomes equal at all lattice points (Fig. 9c). The crystal is then called disordered. It is easy to see that at the moment when this disorder appears the symmetry of the crystal changes (namely: the symmetry increases). That can be seen, for instance, in Fig. 9c; the curve c has, in comparison with curves a and b, an extra translational period equal to the distance between two neighbouring lattice points (the curves a and b have only a period equal to twice the distance between lattice points).

A second example is the crystal of NH_4Cl. This crystal has a lattice of the type NaCl, where at the lattice points are Cl and NH_4. The NH_4 groups have the form of tetrahedra and in the NH_4Cl crystal they can be orientated in two directions. If all NH_4 groups are pointing in the same direction the crystal is completely ordered; if some of the groups NH_4 are pointing in the opposite direction the crystal is incompletely ordered. Finally if the numbers of NH_4 groups pointing in each direction are equal the crystal is disordered. Its symmetry has then changed, namely: the ordered crystal has the symmetry of a tetrahedron and the disordered the symmetry of a cube.

We can introduce the quantity "degree of order", which would characterise the deviation of the crystal from its ordered state; it is equal to 1 in the completely ordered crystal, decreases as a

function of the deviation from the ordered state, and becomes zero in the disordered crystal. In our preceding discussions the transition from the function ϱ to $\varrho + \delta\varrho$ corresponded to the continuous transition from the more to the less symmetric body, i.e. from the disordered crystal to the appearance of the beginning of orderliness. In this way $\delta\varrho$ just determines how close the crystal is to complete disorder; $\delta\varrho = 0$ in the disordered crystal. But we have seen that $\delta\varrho$ is determined by the quantities c_l which are moreover proportional to η. Obviously η can be chosen as the degree of order. In the above mentioned paper[2] we used as the degree of order always the positive quantity $\xi = \eta^2$.

At a continuous transition (for instance at a Curie point) ξ as a function of T has the form as shown in Fig. 10a. At the phase transition it becomes zero abruptly (Fig. 10b).

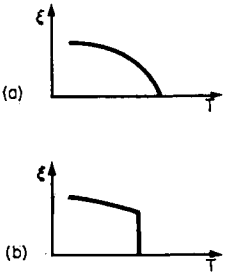

FIG. 10.

In the case of a binary mixture discussed above, the degree of order can be chosen in the following way. Let N_1 be the number of atoms of a given kind residing at their places, and N_2 at foreign places. In a disordered crystal $N_1 = N_2$. The probability of finding an atom in its place is proportional to $N_1/(N_1 + N_2)$, and in a foreign place $N_2/(N_1 + N_2)$. In a disordered crystal each of those fractions is equal to 1/2. Therefore the deviations of the probabilities from their values in the disordered crystal are proportional to

$$\frac{N_1}{N_1 + N_2} - \frac{1}{2} = \frac{N_1 - N_2}{2(N_1 + N_2)}, \quad \frac{N_2}{N_1 + N_2} - \frac{1}{2} = \frac{N_2 - N_1}{2(N_1 + N_2)}.$$

In this way $\delta\varrho$ is proportional to the quotient $(N_1 - N_2)/(N_1 + N_2)$, which can indeed be chosen to be η.

In the case of transitions between magnetic and non-magnetic crystals atoms with differently orientated magnetic moments play the role of atoms of different kinds. To a disordered crystal corresponds the case where the probabilities for an atom to have differently orientated moments are equal for every atom. In the case of the ferromagnetic state these probabilities cease to be equal, since the crystal as a whole has a magnetic moment. Finally in the case of magnetic, but not ferromagnetic, bodies the probabilities for different orientations of the moment for a single atom are also not equal, but in different atoms of the lattice the opposite orientation of the moments are more probable. In that way in this case the mean magnetic moments of different atoms have the opposite directions and the crystal as a whole does not have a magnetic moment.

Conclusions

1. The transitions between bodies of different symmetry (in particular between a liquid and a crystal) cannot happen continuously, in the same sense as the transition between a liquid and a gas above the critical point; at every moment the body has this or that symmetry.

2. Besides phase transitions the only other possible transitions are those which are continuous in the sense that at the transition point no abrupt change in the state of the body occurs (in particular there is no latent heat), but the symmetry changes suddenly. Such transitions are inevitably followed by a jump in the specific heat. These transitions are connected with a crystal becoming disordered.

3. The following types of continuous transitions with a change of symmetry are possible: (a) Curie points lying on a curve in the (p, T) diagram. These curves can intersect each other or the phase transition line in points of the kind shown in Figs. 2 and 3. The Curie line can go continuously into a phase transition line. The

point where this happens is a λ-point. At the λ-point of a pure substance the specific heat becomes infinite; if the body is a mixture the specific heat only experiences a finite jump. (b) Isolated continuous transition points. These points lie on the intersections of several phase transition lines (Figs. 6 and 7).

4. Continuous transitions are possible which are connected with the appearance or disappearance of the mean magnetic moments of every atom in the crystal (in particular such is the Curie point of ferromagnetic bodies). For such transitions case (a) is appropriate.

References

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2

The Transport Equation in the Case of Coulomb Interactions†

A transport equation is derived for a system consisting of charged particles taking their interactions into account. The order of magnitude of the mean free path of the particles in such a system is determined. The rate at which the temperatures of the ions and electrons in the plasma become equal is evaluated.

In the case of Coulomb interactions there appear, in the formulae for the kinetic theory of gases, integrals which are divergent when the distances between the particles are large. This means that an important role is played by those collisions in which the distances between the colliding particles are large. But at large distances the particles are only scattered through small angles with small changes in velocity. Thus collisions in which the velocity vector is only slightly changed are important.

Let $n(p_i)$ be the distribution function in momentum space. It is a function of the three components of the momentum of the particle ($i = x, y, z$). The change in the momentum during a collision we shall denote by Δ_i where $\Delta_i \ll p_i$ in all the collisions. Further, let dW be the probability (per unit time) of a collision between particles with momentum p_i and a particle with momentum p_i', such that p_i is changed to $p_i + \Delta_i$ and p_i' to $p_i' + \Delta_i'$. Because of momentum conservation $\Delta_i = -\Delta_i'$. We shall not, however, use this fact for the moment, in order that we may obtain formulae which are valid in the general case. The number of such collisions will then be $dWn(p)n'(p')$ (for simplicity we shall omit the indices on p_i and Δ_i in $n(p_i)$ and so on).

† *Phys. Z. Soviet Un.*, **10**, 154, 1936.

The number of collisions changing particle momenta $p_i + \Delta_i$ and $p_i' + \Delta_i'$ back to p_i and p_i' will equal $dWn(p + \Delta)n(p' + \Delta')$, since according to the Liouville theorem the probabilities of forward and reverse transitions are equal.

Let us express the probability dW as a function of the half-sum and half-difference of the momenta in the initial and final states. Then the probability of a forward transition will be

$$dW(p + \tfrac{1}{2}\Delta, p' + \tfrac{1}{2}\Delta', \Delta, \Delta'),$$

and for the reverse transition $dW(p + \frac{1}{2}\Delta, p' + \frac{1}{2}\Delta', -\Delta, -\Delta')$. Since these probabilities are equal, $dW(p, p', \Delta, \Delta')$ is an even function of Δ_i and Δ_i'.

Hence the number of particles with momentum p_i is changed, due to collisions, in unit time by

$$\int dW(p + \tfrac{1}{2}\Delta, p' + \tfrac{1}{2}\Delta', \Delta, \Delta')\{n(p)n'(p') - n(p + \Delta)n'(p' + \Delta')\}.$$

The probability dW we write in the form $dW = w\,(p + \frac{1}{2}\Delta, p' + \frac{1}{2}\Delta', \Delta, \Delta')d\tau'\,d\tau_\Delta$, where $d\tau' = dp_x'\,dp_y'\,dp_z'$ and $d\tau_\Delta$ is the product of the differentials of the parameters which define the collision.

Thus the change in the number of particles with momentum p_i' is:

$$\int d\tau'\,d\tau_\Delta w(p + \tfrac{1}{2}\Delta, p' + \tfrac{1}{2}\Delta', \Delta, \Delta')\{n(p)n'(p') - n(p + \Delta)\,n'(p' + \Delta')\}. \tag{1}$$

Let us expand the expression under the integral in a series in powers of Δ_i and Δ_i' (w should of course, be expanded only with respect to Δ_i, appearing in $p_i + \Delta_i/2$ and $p_i' + \Delta_i'/2$). The zero order terms cancel each other and the terms of the first order are

$$-\int d\tau'\,d\tau_\Delta \left(w n' \frac{\partial n}{\partial p_i}\,\Delta_i + w n \frac{\partial n'}{\partial p_i'}\,\Delta_i'\right),$$

where $w = w(p, p', \Delta, \Delta')$ (summation is everywhere implied over indices which are repeated twice). But w is an even function of Δ_i and Δ_i'. Therefore the integral written above is equal to zero.

The second order terms are the following

$$-\int d\tau'\, d\tau_\Delta w \left\{\frac{\Delta_i \Delta_k}{2} n' \frac{\partial^2 n}{\partial p_i\, \partial p_k} + \Delta_i \Delta_k' \frac{\partial n'}{\partial p_k'} \frac{\partial n}{\partial p_i} + \frac{\Delta_i' \Delta_k'}{2} n \frac{\partial^2 n'}{\partial p_i' \partial p_k'}\right\}$$

$$-\int d\tau'\, d\tau_\Delta w \frac{1}{2}\left(\Delta_i \frac{\partial w}{\partial p_i} + \Delta_i' \frac{\partial w}{\partial p_i'}\right)\left(\Delta_k n' \frac{\partial n}{\partial p_k} + n \Delta_k' \frac{\partial n'}{\partial p_k'}\right). \quad (2)$$

Let us integrate two of these terms by parts over $d\tau'$, namely:

$$-\frac{1}{2}\int d\tau'\, d\tau_\Delta \Delta_i' \Delta_k n' \frac{\partial w}{\partial p_i'} \frac{\partial n}{\partial p_k'} = \frac{1}{2}\int d\tau'\, d\tau_\Delta \Delta_i' \Delta_k w \frac{\partial n'}{\partial p_i'} \frac{\partial n}{\partial p_k},$$

$$-\frac{1}{2}\int d\tau'\, d\tau_\Delta \Delta_i' \Delta_k' \frac{\partial w}{\partial p_i'} \frac{\partial n'}{\partial p_k'} n = \frac{1}{2}\int d\tau'\, d\tau_\Delta \Delta_i' \Delta_k' w \frac{\partial^2 n'}{\partial p_i' \partial p_k'} n.$$

Since the integration is performed over the whole of p' space, the surface integral is equal to zero, because $n' = 0$ at infinity.

As a result the second-order terms give

$$-\int d\tau'\, d\tau_\Delta w \left\{\frac{\Delta_i \Delta_k}{2} n' \frac{\partial^2 n}{\partial p_i\, \partial p_k} + \frac{\Delta_i \Delta_k'}{2} \frac{\partial n}{\partial p_i} \frac{\partial n'}{\partial p_k'}\right\}$$

$$-\int d\tau'\, d\tau_\Delta w \left\{\frac{\Delta_i\, \Delta_k}{2} \frac{\partial w}{\partial p_i} \frac{\partial n}{\partial p_k} n' + \frac{\Delta_i \Delta_k'}{2} n \frac{\partial w}{\partial p_i} \frac{\partial n'}{\partial p_k'}\right\}.$$

This can be re-written in the form

$$-\frac{\partial}{\partial p_i}\int d\tau'\, d\tau_\Delta w \left\{\frac{\Delta_i \Delta_k}{2} n' \frac{\partial n}{\partial p_k} + \frac{\Delta_i \Delta_k'}{2} n \frac{\partial n'}{\partial p_k'}\right\}.$$

Thus the integral (1), defining the change due to collisions in the number of particles with given momentum is expressed, as it should be, as the divergence $\partial j_i/\partial p_i$ in momentum space, of the flow vector j_i in momentum space. The components of this flow equal

$$j_i = -\int d\tau'\, d\tau_\Delta w \left\{\frac{\Delta_i \Delta_k}{2} n' \frac{\partial n}{\partial p_k} + \frac{\Delta_i \Delta_k'}{2} n \frac{\partial n'}{\partial p_k'}\right\}.$$

As was already noted at the beginning, $\Delta_i = -\Delta'_i$. Therefore in our case the flow is

$$j_i = \int d\tau' \, d\tau_\Delta \left\{ \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \int \frac{\Delta_i \Delta_k}{2} w d\tau_\Delta \right\}.$$

If the system consists of different types of particles, then the flow j_i for a given type of particle is equal to

$$j_i = \sum \int d\tau' \left\{ \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \int \frac{\Delta_i \Delta_k}{2} w d\tau_\Delta \right\}, \tag{3}$$

where the summation is performed over all the kinds of particles in the system, unprimed variables being related to the given type of particle and primed variables to each type of particle in turn (in this number, of course, is included the given type).

Let us apply the formulae thus obtained to the case of a system of particles with Coulomb interactions, which we are considering. For this system let us determine the change in the momenta of two particles with charges, e and e' and momenta p_i and p'_i moving at some distance from one another. Let ϱ be the impact parameter, i.e. the distance at which the two particles would pass each other if there were no interaction between them, and u_i their relative velocity. Let us consider this collision in the co-ordinate system in which the particle e' is at rest, with the x-axis along the direction of motion of the particle e, which has velocity u. We consider the scattering angle to be small. Because of this the momentum along the x-axis does not change to this approximation, and only the momentum in a direction perpendicular to the x-axis (along the y-axis) changes. This change equals

$$\Delta_y = \int_{-\infty}^{+\infty} - (\partial U / \partial y) dt,$$

where $U = e\,e'/r$ is the energy of interaction between the particles.

Since the scattering is considered to be small it is possible to

consider, in the integral, that the motion is unperturbed, i.e. directed along the x-axis. Then

$$\Delta_y = \int_{-\infty}^{+\infty} \frac{e\, e'\, \varrho\, dt}{(\varrho^2 + u^2\, t^2)^{3/2}} = \frac{2e\, e'}{\varrho\, u}.$$

Going back to an arbitrary coordinate system, and noticing that the vector of the change in momentum is directed along the direction of $\boldsymbol{\varrho}_i$ we find

$$\Delta_i = \frac{2e\, e'}{u} \frac{\varrho_i}{\varrho^2}. \tag{4}$$

Let us now calculate the integrals

$$a_{ik} = \int \frac{\Delta_i\, \Delta_k}{2}\, w\, d\tau_\Delta = \int \frac{2e^2\, e'^2}{u^2} \frac{\varrho_i\, \varrho_k}{\varrho^4}\, w\, d\tau_\Delta,$$

appearing in (3). $n' dW = w\, n'\, d\tau'\, d\tau_\Delta$ is the number of collisions per unit time with particles e', undergone by the particle e with momentum p_i, in which its momentum changes by the given value Δ_i. In other words this is the number of collisions in which particles e and e' pass a definite distance ϱ_i apart, the particles e' having definite momentum p'_i (Δ_i is completely determined for given p'_i and ϱ_i). Denote by v_i and v'_i the velocities of the particles e and e'. Their relative velocity $u_i = v_i - v'_i$ has absolute value u. The number of collisions of the particle e which take place at a given distance ϱ_i with the given relative velocity u_i is obviously $u\varrho d\varrho d\varphi u' d\tau'$, where φ is the angle determining the direction of ϱ_i (at the given velocity $\boldsymbol{u}_i$ all the possible $\boldsymbol{\varrho}_i$ lie in one plane which is perpendicular to $\boldsymbol{u}_i$; φ is the angle in that plane).

Hence we can change $w d\tau_\Delta$ to $u\varrho d\varrho d\varphi$ in the integrals a_{ik}

$$a_{ik} = \frac{2e^2\, e'^2}{u} \int \frac{\varrho_i\, \varrho_k}{\varrho^3}\, d\varrho\, d\varphi.$$

In order to perform the integration, introduce, temporarily, coordinate axes with the x-axis directed along $\boldsymbol{u}_i$. Then $\varrho_x = 0$ since

$\varrho_l \perp \boldsymbol{u}_l$. Because of this $\alpha_{xx} = \alpha_{xy} = \alpha_{xz} = 0$. Also $\alpha_{yz} = 0$ since the integral of $\varrho_y \varrho_z = \varrho^2 \sin\varphi \cos\varphi$ over all angles φ vanishes.

Thus for α_{yy} and α_{zz}, which are not equal to zero, we find (substituting $\varrho_z = \varrho \sin\varphi$, $\varrho_y = \varrho \cos\varphi$ and integrating with respect to $d\varphi$)

$$\alpha_{yy} = \alpha_{zz} = \frac{2\pi e^2 e'^2}{u} \int_{\varrho_1}^{\varrho_2} \frac{d\varrho}{\varrho}. \tag{5}$$

The integral appearing here diverges logarithmically. The divergence at small ϱ is due to the fact that for small ϱ the scattering angle of the particles in the collision is large, and hence all the previous formulae are no longer valid. If the exact formulae are used then there would, of course, be no divergence (at small ϱ).

Since a logarithm is insensitive to small changes in its argument, we can take in (5), as the lower limit ϱ_1, that value ϱ at which the scattering angle becomes of the order of unity, i.e. the interaction energy ee'/ϱ becomes of the order of the mean kinetic energy $\bar{\epsilon}$ of the particles:

$$\varrho = \frac{e\,e'}{\bar{\epsilon}}.$$

As far as the upper limit ϱ_2 in (5) is concerned, two cases must be distinguished. If the total charge on the particles in the system is not equal to zero, then as the upper limit one must take the linear dimension R of the region in which these particles lie. In the most interesting case, when the total charge of the system is zero, the charges are screened and as ϱ_2 one should take the Debye–Hückel screening radius. This radius is $1/\kappa$ where κ is the coefficient in the screened Coulomb law $e^{-\kappa r}/r$ and is determined by the well-known equation

$$\kappa^2 = \sum \frac{N_i e_i^2}{kT}.$$

Here the summation is taken over all types of particles in the system and N_i is the number of particles of the ith kind in 1 cm^3.

To an order of magnitude $\kappa \cong \sqrt{\bar{N}\, e^2/kT}$ where $\bar{N}$ is the number of particles in 1 cm^3. But $kT \cong \bar{\epsilon}$ so that $\kappa = \sqrt{\bar{N}\, e^2/\bar{\epsilon}}$. Thus we can take for the upper limit in (5),

$$\varrho_2 = \sqrt{\frac{\bar{\epsilon}}{\bar{N}\, e^2}}.$$

Substituting ϱ_1 and ϱ_2 in (5) we find

$$\alpha_{yy} = \alpha_{zz} = \frac{\pi\, e^2\, e'^2}{u} L,$$

where

$$L = \ln[(\bar{\epsilon}/e^2)^3/\bar{N}]. \tag{6}$$

Returning now to an arbitrary coordinate system we can write, in tensor form,

$$\alpha_{ik} = \pi\, e^2\, e'^2\, L\, \frac{(u^2\, \delta_{ik} - u_i\, u_k)}{u^3},$$

where

$$\delta_{ik} = 1,\ i = k;\quad \delta_{ik} = 0,\ i \neq k.$$

Substituting this expression into (3) we find the flow of particles e in momentum space in the form

$$j_i = \pi\, e^2\, L \sum e'^2 \int \left\{ n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right\} \frac{u^2\, \delta_{ik} - u_i\, u_k}{u^3}\, d\tau'. \tag{7}$$

The transport equation in the presence of a temperature gradient and an external electric field E_i has the form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial T} v_i \frac{\partial T}{\partial x_i} + e\, E_i \frac{\partial n}{\partial p_i} + \frac{\partial j_i}{\partial v_i} = 0. \tag{8}$$

The Maxwellian distribution makes j_i zero, as it should do.

It would, in principle, be possible to determine from this equation the electrical and thermal conductivity of the gas consisting of the charged particles. This, however, meets considerable mathematical difficulties. We restrict ourselves to a qualitative determination of the conductivities, namely, we determine, to within

an order of magnitude, the mean free path l of the particles, from which it is possible to find the electrical and thermal conductivities by the use of well-known formulae.

Let $\bar{N}$ be (to an order of magnitude) the number of particles in 1 cm^3, e the charge of the particles and T the temperature of the gas. As is seen from (7), when it is substituted into (8), $\bar{N}$ and e appear in the formulae only in the combination $\bar{N} L e^4$. Therefore, the mean free path of the particles should be determined only in terms of the quantities $e^4 L\bar{N}$, kT and the mass of the particles. From these one can construct only one combination having the dimensions of a length, namely $(kT)^2/(e^4 L\bar{N})$. To within an order of magnitude the mean free path will be equal to just this ratio

$$l \cong \frac{k^2 T^2}{e^4 L\bar{N}}. \tag{9}$$

This result disagrees with Gabor's formulae[1], which points to the incorrectness of his assumptions.

Let us consider a gas consisting of electrons and ions. Because of the large difference in masses between the electrons and ions, the exchange of energy by the electrons amongst themselves and the ions amongst themselves will take place much more rapidly than the exchange of energy between the electrons and ions (in a collision between a very heavy particle and a very light one, the energy of each of them is almost unchanged). Because of this the equilibrium in the energies of the electrons amongst themselves and the ions amongst themselves will be established much sooner than the equilibrium between the unlike groups. Let us consider that such an equilibrium is already established, i.e. the electrons and the ions both have a Maxwellian distribution, but the temperatures of these distributions, T' and T, are different. Let us find the rate at which the equilibrium between the electrons and ions is established, i.e. the rate of equalisation of the temperatures T' and T.

Let us work out the energy transmitted by the electrons to the ions in unit time (in 1 cm^3) by collisions between them. Let e, m

and e', m' be the charges and masses of the ions and electrons and n and n' their distributions:

$$n = N(2\pi mkT)^{-3/2} e^{-\epsilon/kT},\ n' = N'(2\pi m'kT)^{-\epsilon/2} e^{-\epsilon'/kT}; \quad (10)$$

N and N' are the numbers of ions and electrons in 1 cm^3 and ϵ and ϵ' are their energies. The flow of ions in momentum space is, according to (7):

$$j_i = \pi e^2 e'^2 L \int \left(n \frac{\partial n'}{\partial p'_k} - n' \frac{\partial n}{\partial p_k} \right) \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau' \quad (11)$$

(all primed variables correspond to the electrons, unprimed variables to the ions). In the sum in (7) only one term remains, since the term which corresponds to the collisions of ions one with another vanishes, because the distribution of the ions is Maxwellian.

The change per unit time in the number of ions with given momenta due to collisions with electrons is $-\partial j_i/\partial p_i$. Thus the change in their energy is

$$-\int \epsilon \frac{\partial j_i}{\partial p_i} d\tau,$$

or, integrating by parts

$$-\int \epsilon \frac{\partial j_i}{\partial p_i} d\tau = \int j_i \frac{\partial \epsilon}{\partial p_i} d\tau = \int j_i v_i d\tau$$

($\partial\epsilon/\partial p_i = v_i$). Since the integration is taken over all momentum space, the surface integral disappears.

Substitute the distributions (10) into (11). We have

$$\frac{\partial n}{\partial p_k} = -\frac{n}{kT} \frac{\partial \epsilon}{\partial p_k} = -\frac{n v_k}{kT}, \quad \frac{\partial n'}{\partial p'_k} = -\frac{n' v'_k}{kT}.$$

Then we find

$$j_i = \pi e^2 e'^2 L \int n n' \left(\frac{v_k}{kT} - \frac{v'_k}{kT'} \right) \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau'$$

$$= \pi e^2 e'^2 L \int n n' \left[v_k \left(\frac{1}{kT} - \frac{1}{kT'} \right) + \frac{u_k}{kT'} \right] \frac{u^2 \delta_{ik} - u_i u_k}{u^3} d\tau'.$$

But

$$u_k \frac{u^2 \delta_{ik} - u_i u_k}{u^3} = 0$$

and therefore

$$j_i = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'}\right) \int n n' \frac{u^2 v_i - (u_i v_i^2)^2}{u^3} d\tau d\tau'.$$

The change in energy, which we are seeking, is then equal to

$$\int j_i v_i d\tau = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'}\right) \int\int n n' \frac{u_i^2 v_i^2 - (v_i u_i)^2}{u^3} d\tau d\tau'.$$

Since the mass of the electrons is much less than the mass of the nuclei, their velocity v_k' is much larger than the velocity of the ions v_k. Because of this one may consider that $u_i \cong v_i'$. Then

$$\int j_i v_i d\tau = \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'}\right) \int\int n n' \frac{v_i^2 v_i'^2 - (v_i v_i')^2}{v'^3} d\tau d\tau'.$$

Averaging over the angles between v_i and v_i' we find

$$\int j_i v_i d\tau = \frac{2}{3} \pi e^2 e'^2 L \left(\frac{1}{kT} - \frac{1}{kT'}\right) \int n v^2 d\tau \int \frac{n'}{v'} d\tau'.$$

Substituting (10) we have:

$$\int n v^2 d\tau = N \frac{3kT}{m},$$

$$\int \frac{n'}{v'} d\tau' = 4\pi N' \left(\frac{m'}{2\pi k T'}\right)^{3/2} \int_0^\infty e^{-(m'v'^2/2kT')} v' dv'$$

$$= 2N' \sqrt{(m'/2\pi k T')}.$$

As a result we find:

$$\int v_i j_i d\tau = \frac{2N N' e^2 e'^2 (2\pi m')^{1/2} L}{m k^{1/2} T'^{3/2}} (T' - T).$$

If there are ions of different types in the gas, the total energy transmitted by the electrons to the ions per unit time is

$$\frac{2N'\,e'^2\,(2\pi\,m')^{1/2}\,L}{k^{1/2}\,T'^{3/2}}\,(T'-T)\sum\frac{N\,e^2}{m} \tag{12}$$

(Σ is over all types of ions).

The energy of the electrons in 1 cm^3 is equal to $3N'\,kT'/2$. Dividing the energy (12), lost by the electrons in unit time, by $3N'\,k/2$, we obtain the rate of change of the electron temperature T':

$$\frac{dT'}{dt} = -\frac{4}{3}\,\frac{e'^2(2\pi\,m')^{1/2}\,(T'-T)}{(k\,T')^{3/2}}\,L\sum\frac{N\,e^2}{m}\,. \tag{13}$$

Reference

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3

On the Vibrations of the Electronic Plasma†

The vibrations of the electronic plasma are considered, which arise as a result of an arbitrary initial non-equilibrium distribution in it. It is shown that the vibrations of the field in the plasma are always damped, and the dependence of the frequency and of the damping decrement on the wave vector is determined for small and for large values of the latter.

The penetration of a periodical external electric field into the plasma is considered. The case of the frequency of the external field being almost at resonance with the proper frequency of plasma is considered separately.

The high frequency vibrations of the electronic plasma are described by comparatively simple equations. If the frequency is high enough, the collisions of the electrons with the ions and with each other are inessential, and in the kinetic equation the collision integral can be neglected. The distribution function of ions can be considered as invariable, and only the distribution of electrons vibrates. Let $F(\boldsymbol{v}, \boldsymbol{r}, t)$ be the electronic distribution function; if $f_0(v)$ is the equilibrium function (the Maxwell distribution), then

$$F = f_0(v) + f(\boldsymbol{v}, \boldsymbol{r}, t) \tag{1}$$

f being a quantity small as compared with f_0. The kinetic equation (without the collision integral) is

$$\frac{\partial f}{\partial t} + (\boldsymbol{v} \cdot \nabla) f - \frac{e}{m}\left(\nabla\varphi \cdot \frac{\partial f_0}{\partial \boldsymbol{v}}\right) = 0 \tag{2}$$

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(φ—the electric field potential). The Poisson equation is

$$\nabla^2\varphi = -4\pi e\int f d\tau \quad (d\tau = dv_x\, dv_y\, dv_z) \tag{3}$$

(the equilibrium electronic charge $e\int f_0 d\tau$ is of course compensated by the positive charge of the ions). Equations (2) and (3) form a complete set of equations.

These equations were used by A. A. Vlasov[1, 2] for an investigation of the vibrations of plasma. However, most of his results turn out to be incorrect. Vlasov looked for solutions of the form const $e^{-i\omega t + i(\boldsymbol{k}\cdot\boldsymbol{r})}$ and determined the dependence of the frequency ω on the wave vector $\boldsymbol{k}$. The equation, which he obtained for this dependence contains a divergent integral; this already indicates the mathematical incorrectness of his method. Vlasov[2] (and also Adirovich[3]) tries to escape from this difficulty by taking the principal value of the integral involved, however, without any foundation. Actually there exists no definite dependence of ω on $\boldsymbol{k}$ at all, and for a given value of $\boldsymbol{k}$ arbitrary values of ω are possible. The fact that solutions of the form of $e^{-i\omega t + i(\boldsymbol{k}\cdot\boldsymbol{r})}$ are insufficient can be seen already by observing that they give only a ∞^3 multiple of solutions (according to three independent parameters k_x, k_y, k_z), whereas there must actually exist a ∞^6 multitude of solutions (the equations contain six independent variables x, y, z, v_x, v_y, v_z).

1. The Vibrations with a Given Initial Distribution

In order to obtain a correct solution of equations (2) and (3), it is necessary to consider the problem concretely stated; we shall discuss here two of such problems.

Let us assume, that a definite (non-equilibrium) electronic distribution in a plasma is given in the initial moment. The problem is to determine the resulting vibrations. As equations (2) and (3) are linear and do not contain the coordinates explicitly, the function $f(\boldsymbol{r}, \boldsymbol{v}, t)$ can be expanded into a Fourier integral with respect to coordinates, and the equation can be written for every Fourier component separately. This means, that it is sufficient to consider solutions of the form $f_{\boldsymbol{k}}(\boldsymbol{v}, t)e^{i(\boldsymbol{k}\cdot\boldsymbol{r})}$.

Further we shall, for the sake of convenience, omit the index $\boldsymbol{k}$ in f_k so that $f(\boldsymbol{v}, t)$ will denote the Fourier component of the distribution function in question. By $g(\boldsymbol{v})$ we denote the Fourier component of the initial distribution $f(\boldsymbol{r}, \boldsymbol{v}, 0)$; we shall write simply $g(\boldsymbol{v})$ for $g_k(\boldsymbol{v})$. Finally, we choose the x-axis along the direction of the considered value of the vector $\boldsymbol{k}$.

Taking the Fourier components of equations (2) and (3), we obtain

$$\frac{\partial f}{\partial t} + ik\, v_x f - ik \frac{e}{m} \varphi \frac{\partial f_0}{\partial v_x} = 0, \tag{4}$$

$$k^2 \varphi(t) = 4\pi e \int f d\tau, \tag{5}$$

where $\varphi(t)$ is the Fourier component of the potential $\varphi(\boldsymbol{r}, t)$. These equations can be solved by using the operational method. Following this method, we introduce the function $f_p(\boldsymbol{v})$ defined by means of

$$f_p(\boldsymbol{v}) = \int_0^\infty f(\boldsymbol{v}, t)\, e^{-pt}\, dt; \tag{6}$$

then

$$f(\boldsymbol{v}, t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} f_p(\boldsymbol{v}) e^{pt} dp, \tag{7}$$

the integration being performed here in the plane of the complex variable p along a straight line parallel to the imaginary axis and passing to the right of it ($\sigma > 0$).

We multiply both sides of equation (4) by e^{-pt} and integrate over dt. Noting that

$$\int_0^\infty \frac{\partial f}{\partial t} e^{-pt} dt = f e^{-pt}\Big|_0^\infty + p \int_0^\infty f e^{-pt} dt = p f_p - g$$

(we insert $f(v, 0) \equiv g(v)$) we obtain

$$(p + i k v_x) f_p - i k \frac{e}{m} \varphi_p \frac{\partial f_0}{\partial v_x} = g.$$

In the same way (5) gives

$$k^2 \varphi_p = 4\pi e \int f_p \, d\tau.$$

The first of these equations yields

$$f_p(v) = \frac{1}{p + i k v_p} \left\{ g(v) + i k \frac{e}{m} \varphi_p \frac{\partial f_0(v)}{\partial v_x} \right\}, \tag{8}$$

and inserting this into the second one, we obtain for φ_p:

$$\varphi_p = \frac{4\pi e}{k^2} \frac{\int \frac{g(v)}{p + i k v_x} d\tau}{1 - \frac{4\pi i e^2}{km} \int \frac{\partial f_0}{\partial v_x} \frac{d\tau}{(p + i k v_x)}}. \tag{9}$$

These formulae solve, in principle, the problem considered. They determine the electronic distribution and the electric field for an arbitrarily given initial distribution.

Before proceeding to the investigation of the formulae obtained, we note that in (9) the integration over $dv_y \, dv_z$ can be performed directly. Introducing for the following the notation $v_x \equiv u$ and

$$g(u) = \int g(v) dv_y dv_z,$$

we write

$$\varphi_p = = \frac{4\pi e}{k^2} \frac{\int_{-\infty}^{+\infty} \frac{g(u)}{p + iku} du}{1 - \frac{4\pi i e^2}{km} \int_{-\infty}^{+\infty} \frac{df_0(u)}{du} \frac{du}{(p + iku)}}, \tag{10}$$

the equilibrium function being

$$f_0(u) = n\sqrt{\frac{m}{2\pi\kappa T}}\, e^{-mu^2/2\kappa T} \tag{11}$$

(κ—the Boltzmann constant, n—the equilibrium number of electrons per unit volume of the plasma).

An expression of the type of

$$\varphi_p = \int_0^\infty \varphi(t)\, e^{-pt}\, dt,$$

considered as a function of the complex variable p has a sense only in the right half-plane, i.e. for $\mathrm{Re}(p) > 0$. The same refers correspondingly to the expression (10). However, we can define φ on the left half-plane as the analytical continuation of expression (10).

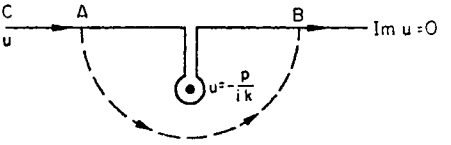

FIG. 1.

It is easy to see, that if $g(u)$ (considered as a function of the complex variable u) is an entire function of u (i.e. it has no singularities at finite u), then the integral

$$\int_{-\infty}^{+\infty} \frac{g(u)\, du}{p + iku},$$

continued analytically into the left half-plane of p also defines an entire function of p. Actually, to perform the analytical continuation of the function, defined by this integral, from the right half-plane to the left one, we displace the integration path in the complex plane of u far enough into the lower half-plane, so that the point $u = -p/ik$ would lie above it. In this way we shall obtain an

analytical function, defined by the integral which for $\mathrm{Re}(p) > 0$, is taken along the real axis, and for $\mathrm{Re}(p) < 0$ along the path, which is drawn in Fig. 1 by a full-drawn line. This function has no singularities at finite values of p, i.e. it is an entire function.

The same refers also to the integral in the denominator of expression (10), for $df_0(u)/du$ is an entire function. Thus, in the whole plane an analytical function φ_p is (if $g(u)$ is entire) a ratio of two entire functions. Hence the only singularities (poles) of the function φ_p are the zeros of the denominator in (10); all of these poles lie in the left half-plane.

These considerations allow us to determine the asymptotical form of the potential $\varphi(t)$ for large values of the time t. In the inversion formula

$$\varphi(t) = \frac{1}{2\pi i} \int_{-i\infty+\sigma}^{+i\infty+\sigma} \varphi_p e^{pt}\, dp \tag{12}$$

the integration is performed along a vertical line in the right half-plane. However, if φ_p is defined in the manner indicated above as a function which is analytical in the whole plane of p, we can displace the integration path into the left half-plane going around all the poles of φ_p it meets. Let p_k be that of the poles of φ_p, i.e. that of the roots of the equation

$$\frac{4\pi i e^2}{k m} \int_c \frac{df_0}{du} \frac{du}{(p + i k u)} = 1 \tag{13}$$

(integration along the path shown in Fig. 1), which has the least absolute value of its real part (i.e. which is the nearest to the imaginary axis). Let us perform the integration in (12) along the path, which is displaced far enough to the left and goes around the point $p = p_k$ in the manner shown in Fig. 2. Then in the integral (12) (with large values of t) only the residue relative to the pole p_k will be of importance. All other parts of the integral (among them

the integral along the vertical line) will be exponentially small in comparison with the residue due to the presence of the factor e^{pt} in the integrated expression, which decreases rapidly with increasing $|\operatorname{Re}(p)|$.

Thus, for large values of t the potential of the field $\varphi(t)$ is proportional to $e^{p_k t}$. With complex p_k this factor splits into a periodical part and a decreasing ($\operatorname{Re}(p) < 0$) one. We arrive, consequently, at an essential result, that the field is damped with time, the damping decrement being equal to $-\operatorname{Re}(p_k)$.

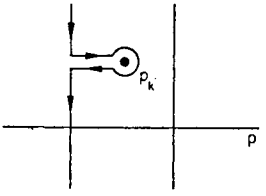

FIG. 2.

Equation (13) determines p_k, i.e. the frequency and the damping decrement of the vibrations. It coincides formally with Vlasov's equations, the difference being that here the integration is performed along the path C, whilst Vlasov integrates simply along the real axis. This difference leads, as we shall see, to qualitatively new results, namely to the presence of damping.

Consider the limiting case of long waves, $k \to 0$. The point $u = -p/ik$ (Fig. 1) is displaced to very large $|u|$ and as the function $f_0(u)$ decreases rapidly with increasing $|u|$, we can integrate in (13), in the first approximation, only along the real axis. We expand the integrand in powers of k. The first term of the expansion disappears because

$$\int_{-\infty}^{+\infty} \frac{df_0}{du}\, du = f_0\Big|_{-\infty}^{+\infty} = 0.$$

The second term gives

$$\frac{4\pi e^2}{p^2 m} \int_{-\infty}^{+\infty} u \frac{df_0}{du}\, du = 1$$

Taking into account that

$$\int_{-\infty}^{+\infty} u \frac{df_0}{du}\, du = uf_0\Big| - \int_{-\infty}^{+\infty} f_0 du = -n_1 \tag{14}$$

we find

$$p_k = -i\omega, \quad \omega = \sqrt{\frac{4\pi n e^2}{m}} \equiv \omega_0 \tag{15}$$

(we have chosen here the sign of ω which corresponds to a wave, propagating in the positive direction of the x-axis). This expression corresponds to the ordinary proper frequency of plasma; we denote it by ω_0. In the next approximation the calculation leads to the following dependence of the frequency on the wave vector:

$$\omega = \omega_0 \left(1 + \frac{3}{2} a^2 k^2\right) \tag{16}$$

$a = \sqrt{\kappa T/4\pi n e^2}$ being the electronic Debye–Hückel radius. We omit here the detailed calculation because it coincides with that of Vlasov done in his first paper.[1] This part of his calculations turns out to be correct due to the fact, that in calculating the frequency for small values of k, we can approximately integrate in (13) only along the real axis.

However, the vibrations are actually damped, although the damping coefficient is small for small k. To calculate this decrement we start from an assumption (which is verified by the result), that for $k \to 0$ the real part of p_k tends to zero, the imaginary part remaining finite. Hence for small k the point $u = -p_k/ik$ (Fig. 1)

is situated at a finite distance from the imaginary axis and very near to the real one (under the latter). Let

$$p_k = -i\omega - \gamma,$$

γ is the damping coefficient in question ($0 < \gamma \ll \omega$). We choose a point A on the real axis (Fig. 1), situated not far from the point $u = -p_k/ik$, but so, that its distance from this point is still large as compared with $|\operatorname{Im}(u)|$. Then we draw a semicircle AB through this point (shown by a dotted line in Fig. 1) and use it instead of the corresponding part of the integration path C.

The integral along the straight parts of the integration path is real in the limiting case of $\operatorname{Re}(p) = 0$; in the approximation considered we can put it equal to $-4\pi ne^2/mp^2$. As to the integral along the semicircle, it equals the residue relative to the pole, multiplied by πi (a half of the total circle!). In this way we obtain equation (13) in the form

$$-\frac{4\pi ne^2}{mp^2} + i\frac{4\pi^2 e^2}{mk^2}\frac{df_0(-p/ik)}{du} = 1.$$

Putting here $p = -i\omega - \gamma$ and solving the equation by means of successive approximations, we get finally the following expression for the damping decrement:

$$\gamma = \omega_0\sqrt{\frac{\pi}{8}}\frac{1}{(ka)^3}e^{-(ka)^2/2}. \tag{17}$$

Thus, the damping decrement decreases exponentially with decreasing k.

Formulae (15-17) are valid for $\gamma \ll \omega$. This condition leads to the inequality

$$ka \gg 1.$$

Consider now the opposite limiting case of large k. We put again $p = -i\omega - \gamma$. It will be verified by the result, that both ω and γ increase indefinitely with $k \to \infty$ but in such a way, that for large k, $\omega \ll \gamma$ and the ratios ω/k, γ/k tend to zero and infinity

respectively. Then the pole $u = -p/ik$ is situated relatively near to the imaginary, but far from the real axis ($\mathrm{Re}(u)$ is small, $\mathrm{Im}(u)$ is large). As the function f_0 increases exponentially for large imaginary values of u, we can integrate in (13) only along the circle around the pole, neglecting the integral along the real axis. In this way we obtain from (13)

$$\frac{4\pi e^2}{m k^2} 2\pi i \frac{df_0(-p/i k)}{d u} = 1.$$

or, using expression (11), for $f_0(u)$

$$\sqrt{2\pi} \frac{p}{\omega_0 (ka)^3} e^{\frac{p^2}{2\omega_0^2 (ka)^2}} = 1. \tag{18}$$

By taking the moduli of the expression on the both sides of the equation, and using the suggested inequality $\gamma \gg \omega$, we get

$$\xi e^{\xi^2/2} = \frac{1}{\sqrt{2\pi}} (a k)^2 \tag{19}$$

with

$$\xi = \gamma/\omega_0 ka.$$

The phase factor of the expression in the left side of equation (18) is equal, in the same approximation, to

$$-\exp\left(\frac{i\gamma\omega}{\omega_0^2 a^2 k^2}\right).$$

As on the right of the equation stands a real positive quantity, this factor must be equal to $+1$. Hence we find:

$$\frac{\gamma\omega}{\omega_0^2 a^2 k^2} = \pi$$

(it can be shown, that by equating it to 3π, 5π, we would get a root of the equation (13), which is not the nearest one to the imaginary axis). Together with the definition of the quantity ξ this gives

$$\omega = \pi \sqrt{\left(\frac{\kappa T}{m}\right)} \frac{k}{\xi}, \quad \gamma = \sqrt{\left(\frac{\kappa T}{m}\right)} k\xi. \tag{20}$$

These formulae determine the frequency and the damping decrement of the vibrations, the function $\xi(k)$ being defined implicitly by equation (19). $\xi(k)$ is a slowly increasing function of k (it goes approximately as $\sqrt{\ln ka}$). The ratio γ/ω increases with k as ξ^2, i.e. approximately as $\ln ka$.

In the preceding calculations we supposed that the given function $g(u)$ is an entire function. If this function has singularities, then φ_p will also possess singularities apart from the poles, which are zeros of the denominator in (10). The point p_k in Fig. 2, which determines the behaviour of the potential $\varphi(t)$ for large t, must be chosen as the nearest to the imaginary axis of all the roots of equation (13) and of the singularities, which arise from the singular points of $g(u)$.

In particular, if $g(u)$ is (on the real axis) a continuous function with a discontinuous derivative, then φ_p will have purely imaginary singular points $p = -iku_s$; u_s being the discontinuity points of $g(u)$. Thus, the behaviour of $\varphi(t)$ for large t will be determined by purely imaginary values of p_k, i.e. there will be no damping of the field. Hence it follows, that it is by no means possible to use a curve with angles (e.g. composed of straight pieces) for $g(u)$ instead of a smooth one in order to get an approximate solution of a given problem. Such a substitution will lead to a qualitatively incorrect picture with an undamped field vibration.

Finally, it is necessary to discuss the electronic distribution function itself. For the distribution function, integrated over $dv_y\, dv_z$ we have, according to (8):

$$f_p(u) = \frac{1}{p + iku}\left\{g(u) + \frac{ike}{m}\varphi_p \frac{df_0(u)}{du}\right\},$$

$$f(u, t) = \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{+i\infty+\sigma} f_p(u)e^{pt}\, dt.$$

The behaviour of the function $f(u, t)$ for large t is determined by the purely imaginary singular point $p = -iku$ of the function $f_p(u)$. Thus, the distribution function turns out to be proportional

(for large t) to a periodical factor $e^{-iku t}$, i.e. it performs undamped vibrations with a frequency ku which depends on the velocity u.

2. The Vibrations of a Plasma in a External Electric Field

Suppose the plasma is placed into an external periodical electric field. The problem is to find the law of the penetration of the field inside the plasma. The external field can be expanded into a Fourier integral with respect to time; therefore, we can confine ourselves to the consideration of a monochromatic field of a frequency ω. We suppose that the plasma is bounded by a plane wall; the distribution is a function of only one coordinate, say x, along the axis, perpendicular to the wall.

The electric field can be split into a longitudinal part, directed along the x-axis, and a transversal part p which is parallel to the plane of the wall. There is no need to consider the transversal field, because the behaviour of a plasma in a transverse electromagnetic wave is described by well-known formulae. Therefore, we confine ourselves to the case of a longitudinal field.

As in section 1, we use the distribution function, integrated over the essential variables v_y, v_z. We can look for this function $f(u, x, t)$ in the form of $f(u, x)e^{-i\omega t}$ (u denotes, as above v_x).

The kinetic equation (2) becomes now

$$-i\omega f + u\frac{\partial f}{\partial x} + \frac{e\,E(x)}{m}\frac{df_0(u)}{du} = 0 \tag{21}$$

(we write the electric field in the form $E(x)e^{-i\omega t}$). As a second equation it is convenient to use here, instead of the Poisson equation (3), the equation, which expresses the absence of sources for the total current (the real current j and the displacement current):

$$\operatorname{div}\left(j - \frac{i\omega}{4\pi}E\right) = \frac{d}{dx}\left(j - \frac{i\omega}{4\pi}E\right) = 0.$$

Hence we find that $4\pi j - i\omega E$ is a constant. Outside the plasma $j = 0$; therefore, this constant equals $-i\omega E_0$ where $E_0 e^{-i\omega t}$ is the external field. Thus, we have an equation

$$-i\omega E(x) + 4\pi j(x) = -i\omega E_0. \tag{22}$$

The current density $j(x)$ can be expressed through the distribution function by means of

$$j = e \int_{-\infty}^{+\infty} uf(u, x)\, du. \tag{23}$$

At large distances from the wall the field E in the plasma is determined directly by the condition of the constancy of the longitudinal component of the induction $D = \epsilon E$, the electric constant ϵ of the plasma being equal to the well-known expression

$$\epsilon = 1 - \frac{4\pi n e^2}{m\omega^2}. \tag{24}$$

Outside the plasma $D = E_0$; hence the boundary condition at infinity is

$$E = \frac{E_0}{\epsilon} \quad \text{for} \quad x = +\infty \tag{25}$$

(the positive direction of the x-axis is into the plasma).

As to the properties at the wall, we shall suppose (as is usually done in analogous cases), that it has an ideal reflection power. This means that an electron, colliding with the wall, is reflected at an angle, equal to that of its incidence, and with an unchanged absolute value of its velocity (so that v_y, v_z remain unchanged, and $v_x = u$ changes its sign). Then the distribution function must satisfy on the wall ($x = 0$) the boundary condition

$$f(u, 0) = f(-u, 0). \tag{26}$$

We formally integrate equation (21) and find:

$$f(u, x) = -e^{i\omega x/u} \int \frac{e\, E(x)}{m\, u} \frac{df_0}{du} e^{-i\omega x/u}\, dx.$$

In order to determine the integration constant, we proceed in the following way. Consider ω as a complex parameter with a small positive imaginary part (which tends in the following to zero). Then the external field $E_0 e^{-i\omega t}$ increases with time, but as it is finite for every finite value of t, the distribution function must also be everywhere (for all $x = \infty$) finite.

If $u < 0$ then the factor $e^{i\omega x/u}$ increases indefinitely with x, and in order that $f(u, \infty)$ remains finite we must write for $u < 0$:

$$f(u, x) = e^{i\omega x/u} \int_x^\infty \frac{e\,E(\xi)}{m\,u} \frac{df_0(u)}{du} e^{-i\omega\xi/u}\, d\xi. \tag{27}$$

As to the function $f(u, x)$ for $u > 0$ it must be written so as to fulfil the condition (26). This gives for $u > 0$:

$$f(u,x) = e^{i\omega x/u}\left[\int_0^\infty \frac{e\,E(\xi)}{m\,u} \frac{df_0(u)}{d\,u} e^{i\omega\xi/u}\, d\xi - \int_0^x \frac{e\,E(\xi)}{m\,u} \frac{df_0}{d\,u} e^{-i\omega\xi/u} d\xi\right] \tag{28}$$

(it is to be remembered, that $f_0(u)$ is an even function of u, hence df_0/du is an odd function).

Using the obtained expressions, we calculate the current density (23):

$$j = \frac{i\omega}{4\pi}\left\{\int_0^x E(\xi)\,K(x-\xi)\,d\xi + \int_x^\infty E(\xi)\,K(\xi - x)\,d\xi - \int_0^\infty E(\xi)\,K(x+\xi)\,d\xi\right\}, \tag{29}$$

where the function $K(\xi)$ is defined by means of

$$K(\xi) = \frac{4\pi i e^2}{m\,\omega} \int_0^\infty \frac{df_0}{du} e^{i\omega\xi/u} du, \quad \xi > 0 \tag{30}$$

((29) contains $K(\xi)$ only for positive values of the argument).

In the following it is convenient to split $E(x)$ into two terms, separating the value of the field for $x \to +\infty$:

$$E(x) = \frac{E_0}{\epsilon} + E_1(x). \tag{31}$$

According to (25), $E_1(x)$ satisfies the boundary condition $E_1(\infty) = 0$. Inserting (31) into (29), we obtain easily:

$$j = j_1(x) + \frac{i\omega}{2\pi\epsilon} E_0 \int_0^x K(\xi)\, d\xi, \tag{32}$$

with $j_1(x)$ defined by (29) with $E_1(x)$ standing instead of $E(x)$.

Inserting (31), (32) into (22) and performing some elementary transformations, we obtain the following integral equation for the function $E_1(x)$

$$E_1(x) - \int_0^x K(x-\xi)\, E_1(\xi)\, d\xi - \int_x^\infty K(\xi - x)\, E_1(\xi)\, d\xi + \int_0^\infty K(\xi + x)\, E_1(\xi)\, d\xi = -\frac{2E_0}{\epsilon} \int_0^\infty K(\xi)\, d\xi. \tag{33}$$

In calculations we used here expression (24) for ϵ and the expression for the integral $\int_0^\infty K(\xi)d\xi$ which can be obtained in the following way. Consider again ω as a complex parameter with $\operatorname{Im}\omega > 0$. Then $e^{i\omega\xi/u}$ is zero for $\xi = \infty$, and integrating over $d\xi$ under the integral sign in (30), we get

$$\int_0^\infty K(\xi)\, d\xi = -\frac{4\pi e^2}{m\omega^2} \int u \frac{df_0}{du}\, du.$$

The integrand $u df_0/du$ is an even function of u hence this integral is one half of the integral (14). Finally,

$$\int_0^\infty K(\xi)\, d\xi = \frac{2\pi e^2 n}{m\omega^2}. \tag{34}$$

The integral equation (33) can be solved in the following way. The function $E_1(x)$ has a physical meaning only inside the plasma, i.e. for $x > 0$. We continue this function, and also the function $K(\xi)$ into the region of negative values of the argument by means of the definitions:

$$K(-\xi) = K(\xi),\ E_1(-x) = -E_1(x) \tag{35}$$

(the function $E_1(x)$, thus defined, has a discontinuity at $x = 0$). Then equation (33) after a simple transformation is reduced to a simpler form:

$$E_1(x) - \int_{-\infty}^{+\infty} K(x-\xi)\, E_1(\xi)\, d\xi = \begin{cases} -\dfrac{2E_0}{\epsilon} \displaystyle\int_x^{\infty} K(\xi)\, d\xi & \text{for } x > 0, \\ \dfrac{2E_0}{\epsilon} \displaystyle\int_{-x}^{\infty} K(\xi)\, d\xi & \text{for } x < 0. \end{cases} \tag{36}$$

In this form it can be solved by using the Fourier method. Multiplying both sides of the equation by e^{-ikx} and integrating over dx within the limits between $-\infty$ and $+\infty$, we obtain:

$$E_{1k}(1 - K_k) = \frac{2i\,E_0}{\epsilon}\,\frac{K_0 - K_k}{k},$$

E_{1k}, K_k being Fourier components:

$$E_{1k} = \int_{-\infty}^{+\infty} E_1(x)\, e^{-ikx} dx, \qquad K_k = \int_{-\infty}^{+\infty} K(\xi)\, e^{-ik\xi}\, d\xi$$

(K_0 is the value of K_k for $k = 0$). By means of the inverse transformation

$$E_1(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{1k}\, e^{ikx}\, dk$$

we get the function $E_1(x)$ in question in the form of an integral:

$$E_1(x) = \frac{iE_0}{\pi\epsilon} \int_{-\infty}^{+\infty} \frac{K_0 - K_k}{K(1 - K_k)} e^{ikx}\, dk. \tag{37}$$

The function K_k can be presented in the following form:

$$K_k = \frac{4\pi e^2}{m\omega} \int_{-\infty}^{+\infty} \frac{u(df_0/du)}{ku - \omega}\, du \tag{38}$$

(we used the definitions (30), (35) and the integration over $d\xi$ is performed under the sign of the integral over du with ω considered again as complex with Im $\omega > 0$). If this integral is taken simply along the real axis, it diverges at the point $u = \omega/k$. However, it is easy to see which must be actually the path of integration. In deducing (38) we assmed that Im $\omega > 0$ and the integral was taken along the real axis, i.e. along a path, passing below (if $k > 0$), or above (if $k < 0$) the singular point $u = \omega/k$. Therefore, after putting Im ω equal to zero, the integral (38) must be taken (if $k > 0$) along the path C_1 (Fig. 3), which proceeds along the real axis and goes around the singular point below it, or (if $k < 0$) along the path C_2 which goes around the singular point above it.

We introduce the notations:

$$K_k = K_1(k) \text{ for } k > 0;\ K_k = K_2(k) \text{ for } k < 0. \tag{39}$$

The functions $K_1(k)$ and $K_2(k)$, defined formally by (38) with the integral taken along the path C_1 or C_2, are analytical functions in the whole plane of the complex variable k.

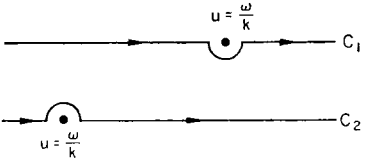

FIG. 3.

Expression (37) is inconvenient for calculations. Introducing the functions K_1, K_2 we can represent it, after a simple transformation, in the form of

$$E_1(x) = \frac{iE_0}{\pi\epsilon}\int_{-\infty}^{+\infty}\frac{K_0 - K_2(k)}{k[1 - K_2(k)]}e^{ikx}dk$$

$$+\frac{iE_0}{\pi\epsilon}\int_{-\infty}^{+\infty}\frac{K_2(k) - K_1(k)}{k[1 - K_1(k)][1 - K_2(k)]}e^{ikx}\,dk. \tag{40}$$

In this transformation we used the fact that according to (24), (38) and (14) we have

$$K_0 = 1 - \epsilon. \tag{41}$$

The difference $K_2(k) - K_1(k)$ is evidently expressed by the same formula (38), the integration being performed simply along a closed contour enclosing the pole (in the negative direction). According to the theorem of the residues, we have, consequently,

$$K_2(k) - K_1(k) = -\frac{4\pi e^2}{m\omega k}2\pi i\left(u\frac{df_0}{du}\right)_{u=\omega/k} \tag{42}$$

or

$$K_2(k) - K_1(k) = \frac{i\sqrt{2\pi\omega}}{\omega_0 a^3 k^3,}\ e^{-\frac{\omega^2}{2\omega_0^2 a^2 k^2}}.$$

It is easy to see, that the functions K_1, K_2 are connected with each other by means of the following relations:

$$[K_2(k)]^* = K_1(k^*),\ K_1^*(-k^*) = K_1(k),\ K_2^*(-k^*) = K_2(k). \tag{43}$$

At infinity both functions K_1, K_2, vanish. An investigation which we omit here, shows, that the functions $K_1(k)$, $K_2(k)$ have in the whole plane of the complex variable k only one singular point—

namely, an essential singularity at $k = 0$. The quantity K_0 is the limit to which K_1, K_2 tend when k tends to zero along the real axis. It can also be shown, that $K_1(k)$ tends to the same limit K_0 when k tends to zero along an arbitrary path, passing outside a right-angled sector in the upper half-plane, bounded by two straight lines, which intersect at the coordinate origin and make an angle of 45° with the imaginary axis. The same holds for $K_2(k)$ outside an analogous sector in the lower half-plane.

In the integrals (40) those points are of importance, at which K_1, K_2 are equal to unity. It can be shown, that the equation $K_1(k) = 1$ has an infinite number of roots in the upper half-plane, which converge to a condensation point at $k = 0$. In the lower half-plane there are no roots at all if $\epsilon > 0$ (i.e. if $K_0 < 1$), or there is one root on the imaginary axis if $\epsilon < 0$ (i.e. if $K_0 > 1$). Analogous results for the function $K_2(k)$ follow directly from the relations (43): the equation $K_2(k) = 1$ has an infinite number of roots in the lower half-plane, and has no roots at all (if $\epsilon > 0$), or has one root on the imaginary axis (if $\epsilon < 0$) in the upper one.

Consequently, if $\epsilon > 0$ the integrand of the first integral in (40) has no poles in the upper half-plane and by pushing the path of integration to infinity in this half-plane, we find, that the integral vanishes. If, on the other hand, $\epsilon < 0$, there is a pole in the upper half-plane and the integral is reduced to the residue relative to this pole. Its dependence on x is, consequently, given by an exponentially decreasing factor $e^{-\alpha x}$, $\alpha \geqq 0$.

A complete evaluation of the integrals in (40) can be performed only numerically. It is, however, possible to obtain an asymptotical expression, which determines $E_1(x)$ for large values of $x (x \gg a)$. We shall see, that in this region the second integral in (40) is larger as compared with the first one and we must calculate only it. We shall do it with the aid of the well-known "method of steepest descent". Inserting (42) into (40) we obtain in the integrand an exponential factor

$$\exp\left\{-\frac{1}{2}\left(\frac{\omega}{\omega_0 ak}\right)^2 + ikx\right\}.$$

Following the method of steepest descent we expand the exponent in powers of $\delta k = k - k_0$ where

$$k_0 = e\left(\frac{\omega^2}{\omega_0^2 a^2 x}\right)^{\frac{1}{3}} e^{i\pi/6}$$

is the extremum point of the exponent, and then integrate along the path of "the steepest descent". In the non-exponential factor we can put $k = k_0$ and take it out of the integration sign. In the denominator we can put $1 - K_2(k_0) = 1 - K_1(k_0) \cong 1 - K_0 = \epsilon$ (k_0 is small for large x). After a simple calculation we obtain the following final result

$$E_1(x) = \frac{2E_0}{\sqrt{3}\epsilon^2}\left(\frac{\omega}{\omega_0}\right)^{4/3}\left(\frac{x}{a}\right)^{2/3} \exp\left[-\frac{3}{4}\left(\frac{\omega x}{\omega_0 a}\right)^{2/3}\right]$$

$$\exp\left\{i\left[\frac{3\sqrt{3}}{4}\left(\frac{\omega x}{\omega_0 a}\right)^{2/3} + \frac{2\pi}{3}\right]\right\}. \tag{44}$$

Thus, the field $E_1(x)$ decreases according to an exponential law with $x^{2/3}$ in the exponent (as to the first term in (40), we have seen that it decreases according to a stronger law $e^{-\alpha x}$ and is, consequently, insignificant for large x). Expression (44) contains also a periodical factor.

The case of the frequency ω, being nearly at resonance with the proper frequency of the plasma, needs a separate consideration. The dielectric constant is here small, $|\epsilon| \ll 1$ (and is connected with the frequency by means of a simple relation $\epsilon = 2[(\omega-\omega_0)/\omega_0]$. The calculations proceed differently for $\epsilon < 0$ and for $\epsilon > 0$.

Suppose first that ϵ is small and negative. We have seen, that for $\epsilon < 0$ the first term in (40) decreases as $e^{-\alpha x}$, i.e., faster than the second one. But with $|\epsilon| \ll 1$ the coefficient α turns out to be small, and therefore, the second term becomes predominant only for very large x; for smaller values of x the first term prevails.

We shall see, that the integrand of the first term has (for small $|\epsilon|$) a pole, lying on the imaginary axis near to the coordinate

origin (we are speaking of the only root of the equation $K_2(k) = 1$ in the upper half-plane). To calculate this root we can, therefore, expand $K_2(k)$ in powers of k. As to the path of integration C_2 in the integral (38), which defines $K_2(k)$, it is reduced simply to the whole real axis—this path passes above the singular point $u = \omega/k$ (which lies now on the negative half of the imaginary axis). A simple calculation gives in the second approximation

$$K_2(k) = 1 - \epsilon + 3(ka)^2.$$

Hence we find for the root of the equation $K_2(k) = 1$:

$$k = \frac{i}{a}\sqrt{\frac{|\epsilon|}{3}}.$$

Evaluating the first integral (40) as the residue relative to this pole, we find, finally, the following expression for the total field $E(x)$

$$E(x) = \frac{E_0}{\epsilon}\left(1 - \exp\left[-\frac{x}{a}\sqrt{\frac{|\epsilon|}{3}}\right]\right). \tag{45}$$

Thus, if ϵ is small and negative, the field increases monotonically, according to a simple exponential law, tending to the limit E_0/ϵ. For $x = 0$ (45) gives $E(x) = 0$ instead of the correct value E_0, this is connected with the fact that in the adopted approximation the quantities of the order of ϵ are neglected.

Consider, finally, the case of small positive values of ϵ. For $\epsilon > 0$ the first term in (40) vanishes. However, the second integral contains, except the expression (44), also a term, which decreases according to a law $e^{-\alpha x}$. For very small ϵ this term becomes predominant for all values of x, except the largest. This term is due to the residue relative to the integrand, which lies in the upper half-plane near the real axis. It turns out, that among the infinite number of the roots of the equation $K_1(k) = 1$ in the upper half-plane there exists one, which lies (for small ϵ) very near to the real axis. Expanding the $K_1(k)$ in powers of k, it is easy to obtain the following expression for the root in question:

$$k = \frac{1}{a}\left[\sqrt{\frac{\epsilon}{3}} + i\frac{3}{2\epsilon^2}\sqrt{\frac{\pi}{2}}\, e^{-3\epsilon/2}\right].$$

Calculating the residue relative to this pole, we obtain, finally, the following expression for the field:

$$E(x) = \frac{E_0}{\epsilon}\left[1 - \exp\left\{\frac{i}{a}\sqrt{\frac{\epsilon}{3}}\,x - \frac{3}{2a}\sqrt{\frac{\pi}{2\epsilon}}\,e^{-3x/2\epsilon}\right\}\right]. \quad (46)$$

Thus, in this case we find that the amplitude of the field increases, first, from zero (actually from E_0) up to $2E_0/\epsilon$, and then it performs damped oscillations (with a very small damping decrement) around the value E_0/ϵ to which it tends at large distances.

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4

On the Statistical Theory of Nuclei†

It is shown that the formulae for the distribution of nuclear energy levels which Bethe has obtained by considering the nucleus as an ideal gas, can also result from quite general considerations. A formula is derived for determining the width of the nuclear energy levels which are associated with scattering. Both inelastic scattering of neutrons, and emission of γ-rays are considered.

1. Distribution of Nuclear Levels

Niels Bohr has shown in his fundamental work that the nuclei of elements of sufficiently large atomic weight can be treated by statistical methods.

Developing these ideas, Bethe[1] has investigated the distribution of nuclear energy levels as function of the energy. In this treatment he has made assumptions which are equivalent to considering the nucleus as an ideal gas. In reality the interaction of the particles within the nucleus is definitely very strong, so that the nucleus should not be treated as a gas. It turns out, however, that Bethe's results do not depend on his assumptions.

If we allow for the interaction of the particles in the nucleus, there is naturally very little reason to consider the nucleus as a "solid body". That is to say we should imagine it as a "liquid drop" of protons and neutrons, and not as a "crystal". In contrast to normal liquids quantum effects are extremely important in this liquid. The uncertainty of the coordinates of the particles is distinctly larger than their mutual separation. Notwithstanding that we have no methods available for the theoretical treatment of

† *Phys. Z. Soviet Un.*, **11**, 556, 1937.

"quantum liquids", we can still derive a few properties of the nuclei.

We set the energy of the nucleus in the ground state equal to zero. In this state the "temperature" of the nucleus is zero (T will be measured in units of energy). Since the excitation energy of the nucleus in the cases which are of interest to us is small by comparison with the total nuclear binding energy, we can also assume the temperature of the excited nucleus to be small. Thus the free energy of the excited nucleus can be expressed in powers of T. Taking only the first term, we have:

$$F = -\tfrac{1}{2}aT^2, \tag{1}$$

where a is a constant. The linear term is equal to zero in accordance with the Nernst theorem. That the expansion for liquids begins with the quadratic term can be seen either from the analogous conditions for gases, or directly from the measurements of Keesom, which show that the specific heat of an "electron liquid"† in metals is proportional to T.

From (1) we find for the energy E of the nucleus, and the entropy S:

$$S = -\frac{\partial F}{\partial T} = aT, \quad E = F + TS = \frac{aT^2}{2}. \tag{2}$$

Expressing S as a function of E, we have

$$S = \sqrt{2aE}. \tag{3}$$

It is well known that we can write the number, N, of states, i.e. of energy levels, with an energy less than E as:

$$N = e^{S(E)} = e^{\sqrt{2aE}} \tag{4}$$

The number of levels per unit energy interval is:

$$\frac{dN}{dE} = e^S \frac{dS}{dE} = \frac{1}{T} e^S = \frac{1}{T} e^{\sqrt{2aE}}. \tag{5}$$

† The term "electron gas" is purely conventional, the interactions between electrons being very large.

Equation (5) gives the distribution of the energy levels, where all states, i.e. all possible angular momenta are included.

We now consider states with a specific total angular momentum j. Since the nucleus may be considered as a spherical drop, the energy of the nucleus, "rotating" with angular momentum j, is given by

$$\frac{\hbar^2}{2J} j(j+1),$$

where J is the moment of inertia of the nucleus. The corresponding number of states for this energy we know from the theory of the spherical top to be $(2j+1)^2$.[$A\,(2j+1)$-fold degeneracy is due to the $(2j+1)$ different orientations of the vector $\boldsymbol{j}$ in space, and a further $(2j+1)$ due to the different possible orientations of the same vector with respect to the top.]

We can now write down the number of states for given j:

$$\frac{dN_j}{dE} = A\,(2j+1)^2 \exp\left[-\frac{\hbar^2}{2JT} j(j+1)\right],$$

where A is a quantity which does not depend on j, and which is defined by the relation:

$$\sum_j \frac{dN_j}{dE} = A \sum_j (2j+1)^2 \exp\left[-\frac{\hbar^2}{2JT} j(j+1)\right] = \frac{dN}{dE}.$$

The exponential expression decreases rapidly from $j = 7$ to 8 onwards. Thus we may consider j as large compared with unity, and replace the summation by integration:

$$\frac{dN}{dE} = A \int_0^\infty e^{-\frac{\hbar^2}{2JT} j^2}\, 4j^2\, dj = A \left(\frac{2JT}{\hbar^2}\right)^{3/2} \sqrt{\pi}.$$

Comparison with (5) gives the value of A, from which:

$$\frac{dN_j}{dE} = \frac{1}{\sqrt{\pi}} \left(\frac{\hbar^2}{2J}\right)^{3/2} \frac{1}{T^{5/2}} (2j+1)^2 \exp\left[-\frac{\hbar^2}{2JT} j(j+1)\right] e^S, \qquad (6)$$

or, for the number of energy levels with angular momentum j:

$$\frac{1}{2j+1}\frac{dN_j}{dE} = \frac{1}{\sqrt{\pi}}\left(\frac{\hbar^2}{2J}\right)^{3/2}\frac{1}{T^{5/2}}(2j+1)\exp\left[S - \frac{\hbar^2}{2JT}j(j+1)\right] \tag{7}$$

When a nucleus with angular momentum j_0, comes into collision with a neutron, the resonance levels correspond to the states of the compound nucleus, i.e. the system, nucleus + neutron, with an angular momentum $j_0 \pm \frac{1}{2}$. If we add the two expressions (7) for $j = j_0 \pm \frac{1}{2}$, we find for the number, $(2j_0+1)^{-1} \times dN_0/dE$:

$$\frac{1}{2j_0+1}\frac{dN_0}{dE} = \frac{1}{\sqrt{2\pi}}\left(\frac{\hbar^2}{J}\right)^{3/2}\frac{1}{T^{5/2}}(2j_0+1)\,e^S \tag{8}$$

(Since the value of j_0 is small, we can neglect the second term in the exponential.) As this expression is in any case not very accurate, we may put $j_0 = 0$ in what follows. The reciprocal of dN/dE is the spacing D between the levels. From (8) we obtain for the spacing of the resonance levels:

$$D = \sqrt{2\pi}\left(\frac{J}{\hbar^2}\right)^{3/2}T^{5/2}e^{-S}. \tag{9}$$

From (2) and (3) we have $T = 2E/S$. Putting this in (9) gives:

$$S^{5/2}\,e^S = \frac{\sqrt{2\pi}}{D}\left(\frac{J}{\hbar^2}\right)^{3/2}(2E)^{5/2}. \tag{10}$$

Using the experimentally known values of E and D, we can calculate S from this formula. For heavy nuclei, $E \sim 8$ MeV. For the spacing of the energy levels, we put $D \sim 5$ eV. The moment of inertia, J, of a sphere of mass, m, and radius r, is $J = 2mr^2/5$. For a nuclear radius $r = 7 \times 10^{-13}$ cm and an atomic weight of 100, we find $J = 3 \times 10^{-47}$ g cm^2. Finally we obtain from (10):

$$S = 18,$$

and thus for the temperature, $T\,(= 2E/S)$:

$$T \sim 1 \text{ MeV}.$$

Let us now determine the order of magnitude of the energies of the lowest nuclear energy levels. We are thus specifying that the number, N (equation 4), of states belonging to the corresponding energy is of the order one. From (4) follows for $N \sim 1$ that S is also ~ 1. Since from (3) the energy is proportional to the square of the entropy, we can write:

$$E_{\min} \sim \frac{E}{[S(E)]^2}. \tag{11}$$

We put $S \sim 20$ and obtain for $E_{\min}$ a value of the order 2×10^4 eV.

We should note that whereas the formula (5) is valid for an energy of a few ten thousand eV onwards, formula (9) is only applicable for a range with a lower limit of about 2 MeV. For smaller energies the condition $E \gg D$ is not satisfied.

Since E is roughly equal for the various elements, we can deduce from the relation $S = \sqrt{2aE}$ that S is proportional to $\sqrt{a}$. Since the free energies are additive, we can say further that a increases with the number of nuclear particles, that is it is proportional to $\sqrt{A}$ (A is the atomic weight). Since the number of particles in the nucleus is usually not very large, we do not expect this formula to be particularly useful.

2. The Width of Resonance Peaks for Neutrons

The neutron can be captured on collision by the nucleus, forming the so-called compound nucleus of neutron + atomic nucleus. The neutron may subsequently be expelled from the nucleus, with or without loss of energy (inelastic or elastic scattering, respectively). Alternatively, it may be absorbed finally by the nucleus, and the compound nucleus will change to its ground state with the emission of light quanta (capture of neutrons).

Let us consider the elastic scattering. Let $2\Gamma_e/\hbar$ be the probability per unit time that the neutron, having been captured into a level, E_n, of the compound nucleus, is then re-emitted. Γ_e is the width of the energy peak for this process (for brevity the index n of Γ_e is omitted).

In order to determine the approximate magnitude of Γ_e we treat the problem as follows: we consider the nucleus in thermodynamic equilibrium with the neutron gas. The probability that the nucleus will be in the compound state with energy E_n, is exp $[(\mu - E_n)/T]$ (μ is the chemical potential). The nucleus can only capture neutrons with angular momentum $l = 0$ (neutrons with $l = 1, 2 \ldots$ pass by at greater distances). The number of such neutrons whose momentum lies in the range dp and whose distance from the nucleus lies in the range dr, is:†

$$\frac{dp\,dr}{2\pi\hbar}\, e^{(\mu - E)/T},$$

where E is the energy of the neutron. Instead of dp we can put dE/v where v is the speed of the neutron. Then the number of neutrons which strike the nucleus per sec is

$$e^{(\mu - E)/T}\,\frac{dE}{2\pi\hbar}.$$

Now let us consider a small energy interval which is nonetheless large enough to contain many energy levels of the compound nucleus. We use $\bar{w}$ for the mean probability that a compound nucleus is formed on the collision of the neutron with the nucleus. Then per sec there are:

$$e^{(\mu - E)/T}\,\frac{\Delta E\,\bar{w}}{2\pi\hbar} \tag{12}$$

compound nuclei formed in some state or other in the energy interval ΔE. This number must be equal to the number of neutrons which leave the compound nucleus on its decay. Since the number of compound nuclei with energy E is exp $[(\mu - E)/T]$, the mean probability of decay, $2\bar{\Gamma}_e/\hbar$, and the number of levels in the energy interval ΔE is $\Delta E/D$, where D is the mean spacing between levels, then the number of nuclei which decay per second is:

$$e^{(\mu - E)/T}\,\frac{\Delta E}{D}\,\frac{2\bar{\Gamma}_e}{\hbar}. \tag{13}$$

† Both the spin of the neutron and the angular momentum of the nucleus have been ignored, since the results are anyway only approximate.

From (12) and (13) we have:

$$\bar{\Gamma}_e = \frac{D\bar{w}}{4\pi}. \tag{14}$$

Since the normalised wave function of the neutron in the region of the nucleus is proportional to the square root of the speed, then $\bar{\Gamma}_e$ and $\bar{w}$ are proportional to the speed itself, i.e. to the square root of the energy, and in particular:

$$\bar{w} = a\sqrt{\overline{E}}.$$

When we consider the nucleus as a liquid drop, on the other hand, we can afford to disregard the spacing of the energy levels, just as for a real liquid. We can now argue that if the energy of the neutron were of the same order as the binding energy E_0 of the nucleus, then the probability of neutron capture would be of the order one. We can thus write:

$$\bar{w} = \sqrt{\frac{\overline{E}}{E_0}}. \tag{15}$$

Comparing (14) and (15) gives:

$$\frac{\bar{\Gamma}_e}{D} = \frac{1}{4\pi}\sqrt{\frac{\overline{E}}{E_0}}. \tag{16}$$

Putting, for example, $E \sim 1$ eV, $E_0 \sim 8$ MeV, $D \sim 5$ eV we find:

$$\bar{\Gamma}_e \cong 2 \times 10^{-4}\,\text{eV}$$

which is apparently in agreement with experimental data.

Formula (14) remains obviously valid when a proton or an α-particle is emitted on decay of the nucleus instead of a neutron. In place of (15) we now have:

$$\bar{w} = e^{-\gamma},$$

where γ is the Gamow penetrability of the potential barrier for a proton or an α-particle. We then have for the decay constant of a radioactive nucleus:

$$\lambda = \frac{2\bar{\Gamma}_e}{\hbar} = \frac{D}{2\pi\hbar}\, e^{-\gamma}. \tag{17}$$

For D we must here substitute the difference between the first excited nuclear states. According to the above calculations, D is $\sim 2 \times 10^4$ eV. In place of this Bethe[2] has taken 1 eV.$e^{-\gamma}$ for Γ_e. Thus he has obtained the value of 13×10^{-13} cm for the radii of the heavy nuclei, which is too large. By means of (17) we obtain the value of $10{\cdot}5 \times 10^{-13}$ cm for the radius of the ThA nucleus.

3. Inelastic Scattering

In the case of inelastic scattering the neutron leaves the compound nucleus with a reduced energy, and the nucleus remains in an excited state. We denote the probability of such a decay of the compound nucleus by $2\Gamma_i/\hbar$: Γ_i is the corresponding part of the level width. Since there is no fundamental difference between the ground state and the excited states, it is apparent that a formula for Γ_i is valid, analogous to formula (16) for elastic scattering:

$$\bar{\Gamma}_i = \frac{D}{4\pi}\sqrt{\frac{\epsilon}{E_0}}. \tag{18}$$

Here ϵ is the energy of the emitted neutron. Since ϵ is smaller than the energy E of the incoming neutron, we can see by comparing (16) and (18) that $\bar{\Gamma}_i < \bar{\Gamma}_e$.

At the same time the number of energy levels increases very sharply with increasing energy, and the nucleus can remain in any one of the lower excited states for inelastic scattering. For this reason the width of the highly excited nuclear levels is defined principally by the inelastic scattering. It should not be concluded from this that the scattering will be principally inelastic since, as Bethe and Placzek[3]† have shown, the cross section for elastic scattering has an extra term, $4\pi r^2$ (r is the nuclear radius), which corresponds to elastic scattering without the formation of a compound nucleus. In fact the resonance at highly excited levels will be almost entirely due to inelastic scattering.

† The author would like to express his thanks to G. Placzek for communicating the results of this work prior to publication.

Owing to the rapid broadening of the levels for inelastic scattering, the tendency for resonance must decrease sharply at sufficiently high energy. We should expect that at an energy of the order of 1 MeV it should vanish.

When the energy of the incident neutron is sufficiently large to enable us to speak of the temperature of the nucleus left in an excited state, then the mean energy of the emitted neutron will clearly be of the order:

$$\epsilon \sim T, \tag{19}$$

where T is the temperature of the excited nucleus. A more exact calculation shows that the mean value of ϵ is about $2T$. From (19) and (2) it can be seen that the energy of the neutron on leaving the nucleus varies with the square root of its incident energy. Thus when the incident energy is large, the neutron leaves with appreciably less energy. We can estimate, for example, that when a neutron of energy 10 MeV is captured by the nucleus, the latter retains about 8 MeV in its excited state.

4. Nuclear Radiation

Taking a statistical standpoint, we can consider the emission from the highly excited nucleus as thermal radiation. In particular, we can apply Kirchhoff's law. Unfortunately, the calculation of the absorption coefficient is very difficult. It can be assumed, however, that whatever form the absorption coefficient may take, its dependence on frequency will be basically defined from Planck's formula; i.e. principally quanta of energy $\hbar\omega \sim T$ will be emitted, where T is the temperature of the nucleus. We should thus expect that the nucleus will radiate its energy not in one, but in several stages. In the particular case of neutron capture, γ-quanta with energy 1–3 MeV will be emitted, rather than quanta of 8 MeV.

Finally, I should like to thank G. Placzek for the very interesting discussions of this problem.

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5

On the Origin of Stellar Energy†

IT IS well known that matter consists of nuclei and electrons. Nevertheless it can be shown that in bodies of very large mass this usual "electronic" state of matter can become unstable. The reason for this lies in the fact that the "electronic" state of matter does not lead to extremely great densities, because at such densities electrons form a Fermi gas having an immense pressure. On the other hand, it is easy to see that matter can go into another state which is much more compressible—the state where all the nuclei and electrons have combined to form neutrons.[1] Even if we assume that neutrons repel each other, this repulsion can become appreciable only at densities of the order of magnitude of nuclear densities, i.e. 10^{14} g/cm^3, and the pressure of a Fermi gas consisting of neutrons is much less than an electronic gas of the same density, because of the greater mass of the neutrons.

Therefore in spite of the fact that the "neutronic" state of matter is, in usual conditions, energetically less favourable, since the reaction of neutron formation is strongly endothermic, this state must nevertheless become stable when the mass of the body is large enough. In this case the gravitational energy gained in going over to the neutronic state with its greater density, compensates the losses of internal energy.

It is easy to compute the critical mass of the body for which the "neutronic" state begins to be more stable than the "electronic" state. First of all we must calculate the energy necessary to form one neutron. For instance in the reaction ${}^{8}_{16}O + 8\,e^{-} = 16\,{}^{0}_{1}n$

† *C. R. Acad. Sci. URSS*, **17**, 305, 1937.

we find from the mass defects that to form one neutron the energy required is 0·008 mass units or $1{\cdot}2 \times 10^{-5}$ erg (7·5 MeV). To transform one gram of matter into neutrons we thus need 7×10^{18} erg/g.

Now we must calculate the gain in gravitational energy. The gravitational energy of the much less dense "electronic" state can, of course, be neglected. Let us assume first of all that the neutronic state has a constant density 10^{14} g/cm^3. The gravitational energy of a homogeneous sphere of mass M is then

$$3 \times 10^{-3} M^{5/3} \text{ erg.}$$

For the stability of the neutronic phase we must then have

$$3 \times 10^{-3} M^{5/3} > 7 \times 10^{18} M$$

or

$$M > 10^{32} \text{ g} = 0{\cdot}05\odot$$

On the other hand, if we assume that the neutrons behave like a Fermi gas we find for the energy

$$4 \times 10^{-22} M^{7/3} \text{ erg}$$

and, hence,

$$M > 1{\cdot}5 \times 10^{30} \text{ g} = 10^{-3}\odot$$

which critical value is even less than on the first assumption.

When the mass of the body is greater than the critical mass, then in the formation of the "neutronic" phase an enormous amount of energy is liberated, and we see that the conception of a "neutronic" state of matter gives an immediate answer to the question of the sources of stellar energy. The sun during its probable time of radiation (about 2×10^9 years according to general relativity theory) must have emitted something of the order of magnitude of 3×10^{50} erg. The liberation of this amount of energy requires the transition of only about 2 per cent of the mass of the sun (with the assumption of constant density) or even only $3 \times 10^{-3}\odot$ (with the Fermi gas model) into the "neutronic" phase. Even for such a

bright star as β-Orionis we find for the mass of the neutronic core only about $0 \cdot 1 \odot$ (with the Fermi gas model).

Thus we can regard a star as a body which has a neutronic core whose steady growth liberates the energy which maintains the star at its high temperature; the condition at the boundary between the two phases is as usual the equality of chemical potentials. The detailed investigation of such a model should make possible the construction of a consistent theory of stars.

As regards the question of how the initial core is formed, the author has shown in a previous article[2] that the formation of a core must certainly take place in a body with a mass greater than $1 \cdot 5 \odot$. In stars with smaller mass the conditions which could make possible the formation of the initial core have yet to be made clear.

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6

On the Multiple Production of Particles during Collisions of Fast Particles†

1. General Relations

Collisions of ultra-fast nuclear particles can be accompanied by the appearance of a large number of new particles (many-pronged stars in cosmic radiation). Fermi[1] propounded the ingenious idea of the possibility of applying statistical methods for studying this process. However, the quantitative calculation given by him appears unconvincing to us and incorrect at several points (in particular, in regard to distribution in energy and angle).

Qualitatively the whole process of collision has the following appearance. At the moment of collision there appear a large number of particles‡ concentrated in a volume whose linear dimensions are determined by the range of the nuclear forces and by the energies of the colliding particles (concerning this, see below); it must be emphasised that we can speak of the number of particles at this moment only in a limited sense, since for a system with such a high density of strongly interacting particles (mesons and nucleons) the concept of the number of particles has in general no precise meaning. The "mean free path" of particles in such a system is clearly very small compared to its dimensions. In the course of time, the system expands, but the aforementioned property of the mean free path must be valid also for a significant

† *Izv. Akad. Nauk SSSR, ser. fiz.*, **17**, 51, 1953.

‡ In fact, the appearance of a large number of particles is the condition for the applicability of the method for treating the problem which is presented below, and of the associated formulae.

part of the process of expansion. This part of the expansion process must have a hydrodynamic character, since the smallness of the mean free path permits us to consider the motion of the matter in the system in a macroscopic hydrodynamical fashion as the motion of an ideal (non-viscous and non-heat-conducting) liquid. Since the velocities in the system are comparable to the velocity of light, we are dealing, not with ordinary, but rather with relativistic hydrodynamics.

The total "number of particles" in the system is not at all constant during the course of the hydrodynamic stage of the expansion. Therefore, the number of particles in the resulting star is determined, not by the number of particles which appear at the very moment of collision (as Fermi mistakenly assumes) but rather by the number of particles in the system at the moment of transition to the second stage of the expansion—the stage of free separation of the particles. This essential point was first made by I. Ya. Pomeranchuk.[2]

The transition from the first stage to the second occurs when the mean free path of particles in the system becomes equal to its linear dimensions. A very essential point is that at that moment the order of magnitude of the temperature of the system is

$$T_c \sim \mu c^2 \tag{1}$$

(μ is the meson mass; the temperature is always given in energy units), practically independent of the properties of the system, i.e. of the energy of the colliding particles. In fact, for values of the temperature substantially lower than μc^2, the density of the equilibrium number of particles falls exponentially with cooling (as $e^{-\mu c^2/T}$) so that the mean free path rapidly becomes equal to the dimensions of the expanding system, even when the latter are relatively large. Formula (1) for T_c (with the π-meson mass substituted for μ) is also valid when, in addition to mesons, other heavier particles are formed, since in order for the free path of all particles to be small, it is already sufficient that there be a high density of π-mesons in the system.

For the hydrodynamic considerations, it is necessary to have an equation of state for the matter in the system. As equation of state of highly compressed matter for temperatures $T \gg \mu c^2$ we use

$$p = \epsilon/3, \tag{2}$$

where p is the pressure and ϵ is the energy density. Although we have not at present any rigorous proof that this must be the equation of state for arbitrary matter in the ultrarelativistic case, nevertheless in our opinion this assumption is highly plausible.

Since the number of particles in the system is not fixed, but is rather determined from the conditions of statistical equilibrium, its chemical potential (just as for black-body radiation) is $\zeta = \epsilon - Ts + p = 0$ (s is the entropy per unit volume). Then $Ts = \epsilon + p = 4\epsilon/3$, so that if we take into account also that for fixed volume (equal to unity) $d\epsilon = Tds$ we find the relations:

$$s \sim \epsilon^{3/4}; \; T \sim \epsilon^{1/4} \tag{3}$$

which, as expected, coincide with the relations for black-body radiation.

The computation of the total number of particles appearing during the break-up is greatly simplified if we consider the motion of the ideal fluid to be adiabatic. The only thing that could destroy the adiabaticity would be shock waves, and it is hard to imagine how they could be formed during the expansion process. Therefore, the entropy of each of the individual regions of the system remains unchanged during the expansion.

Let us break up the system into a set of regions which are macroscopically small, i.e. practically uniform, but which still contain a sufficiently large number of particles; let s_α be the entropies of these regions. Also let n_α be the number of particles in the α-th region which have been produced at the time of the start of its free separation. This time may not be the same for the various regions, since the system as a whole is highly non-uniform. The quantities s_α and n_α individually depend strongly on the temperature (for $T \ll \mu c^2$, they vary as $e^{-\mu c^2/T}$), but the ratio

s_α/n_α depends only slightly on temperature, so that, since T_c in turn depends little on the properties of the system, we may consider that $n_\alpha = \text{const} \cdot s_\alpha$, where the constant ratio is a universal constant [if we measure entropy in dimensionless units, dimensional arguments show that the constant is of order $(\mu c/\hbar)^3$]. Summing this equality over all domains, we find that

$$N = \text{const} \cdot S, \tag{4}$$

where N is the total number of particles in the star, and S is the total entropy of the system. Since the entropy stays constant during the whole course of the hydrodynamic stage of the expansion, we may consider S to be the entropy of the system at the initial time—the time of the collision.† Formula (4) enables us to determine the total number of particles appearing during the collision, without a detailed examination of the motion of the system.

2. Total Number of Particles

Let us first consider "head-on" collisions in which the particles pass each other at distances comparable to the range of interaction, as distinguished from peripheral collisions where the impact parameter is large compared to the range of force.

We start with head-on collisions of two protons, and determine the energy dependence of the total number of particles formed. Let E' be the energy of each of the protons in the centre of mass system (c.m.s.) The total entropy of the system, S, is proportional to $\epsilon^{3/4} V$ where V is the volume over which the energy is distributed. In the c.m.s. the matter is at rest at the moment immediately following the collision. Therefore, $\epsilon = E'/V$, and so the entropy, and consequently the number of particles, is proportional to $E'^{3/4} V^{1/4}$.

† More precisely, after the passage of shock waves, which can arise at the moment of collision; the passage of a shock wave is accompanied by a compression of the matter, after which the expansion stage begins and proceeds adiabatically from then on.

The transverse dimension of the system, a, is of order of magnitude of the range of nuclear force, i.e. $a \sim \hbar/\mu c$. The longitudinal dimension (in the c.m.s.) is shortened by the Lorentz contraction in the ratio $\sim Mc^2/E'$ (M is the proton mass). Thus the system is in the form of a highly flattened disk, and its volume is $V \sim a^3 Mc^2/E'$. Therefore the number of particles is $N \sim E'^{3/4} V^{1/4} \sim \sqrt{E'}$, or, going over to the energy E in the laboratory system in which one of the protons is at rest, using the formula $EMc^2 = 2E'^2$, we finally get: $N \sim E^{1/4}$. This formula coincides with the one obtained by Fermi, but his reasoning appears to us to be completely unconvincing. From dimensional arguments (and taking account of the fact that the ratio of masses of proton and π-meson is fairly close to the unity) we may write:

$$N = K\left(\frac{E}{2Mc^2}\right)^{1/4} \tag{5}$$

where K is a constant of the order of unity.

Now let us consider the collision of two identical nuclei of atomic weight A. It would be completely erroneous to treat such a collision as a series of collisions of nuclear protons and neutrons. In fact, since the distance between nucleons in the nuclei is precisely of the order of their range of interaction, we must look upon the result of the collision as a process of meson formation involving as a unit the whole space occupied by the nuclei.

Suppose that the speed of the incident nucleus is equal to that of the proton in the preceding problem. Then its energy will be A times as large. Since the mass density in a nucleus is approximately the same as that of the proton, referred to its sphere of interaction, the energy density immediately after collision is the same as in the previous case. Since the Lorentz contraction is unchanged, the number of particles formed is simply proportional to the volume of the nucleus, i.e. to A. Thus we finally obtain:

$$N = KA\left(\frac{E}{2AMc^2}\right)^{1/4} = KA^{3/4}\left(\frac{E}{2Mc^2}\right)^{1/4}. \tag{6}$$

For a given energy, the number of particles is proportional to $A^{3/4}$. We note that according to this formula, heavy nuclei are much more effective in particle formation than protons: two nuclei with energy E give as many particles as two protons with energy EA^3.

When the two nuclei have different weights the problem becomes more complicated, but elementary considerations related to the fact that in a collision the lighter nucleus pulls out only a part of the heavier one, show that the number of particles is determined essentially by the mass of the lighter nucleus, and depends only slightly on the mass of the heavier one.

If we are dealing with collisions of a meson with a nucleon or nucleus, it follows that we should expect relatively little difference from the case of a nucleon.

Determination of the constant K from existing experimental data gives the value $K \sim 2$.

As for peripheral collisions of the nucleons, at first glance one might conclude that the average number of particles produced should decrease rapidly with increasing impact parameter. A basis for this conclusion might be the fact that the rest energy of the matter concentrated in each individual region of the meson field of the colliding nucleons decreases rapidly (exponentially) with increasing distance from their "centre". However, the incorrectness of this derivation is clear from the fact that it leads to a contradiction with the quantum uncertainty relations; the rest energy of a portion of the system would turn out to be small compared to the uncertainty $\Delta E \sim \hbar c/\Delta$, where Δ is the thickness of the region, compressed by the Lorentz contraction just as for central collisions. In fact, this relation means only that the quantity which is small is not the actual energy of the system (in those cases where such a system occurs at all) but rather its mathematical expectation. In other words, it is not the number of particles appearing that decreases, but only the probability that such a collision shall occur.

Thus for collisions of two nucleons it is in general meaningless to distinguish between central and peripheral collisions; the

effective cross-section for collision with production of a many-pronged star is determined by the "radius" of the nucleon, $\hbar/\mu c$†. The picture is somewhat changed in the case of a collision of two nuclei. It is clear that as we vary the impact parameter from zero to the sum of the radii of the nuclei, the number of particles formed must decrease from the maximum value given by formula (6) to the value given by (5) and corresponding to the collision of two nucleons.

3. Distribution of Particles Produced in Energy and Direction

A study of the angular distribution of the particles formed, and their distribution in energy, requires a detailed consideration of the hydrodynamical motion of the matter in the system.

The relativistic hydrodynamic equations are contained in the relations

$$\frac{\partial T^{ik}}{\partial x^k} = 0, \tag{7}$$

where T^{ik} is the energy-momentum tensor of the matter:

$$T^{ik} = pg^{ik} + (\epsilon + p)u^i u^k \tag{8}$$

(u^i is the four-velocity; $g^{11} = g^{22} = g^{33} = 1$, $g^{00} = -1$; from now on we set $c = 1$).

As we have already indicated, at the moment of collision the system has the form of a highly flattened disk. This shape is maintained throughout a significant part of the hydrodynamical stage of the expansion. During this stage, the motion of the matter can be considered to be one-dimensional, along the short axis of the disk (x-axis). Then the equations of motion are:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{01}}{\partial x} = 0, \quad \frac{\partial T^{01}}{\partial t} + \frac{\partial T^{11}}{\partial x} = 0, \tag{9}$$

† This result was clarified in discussions with E. L. Feinberg.

where

$$T^{00} = \epsilon(u^0)^2 + p(u^1)^2,\ T^{01} = (\epsilon + p)u^0u^1,\ T^{11} = \epsilon(u^1)^2 + p(u^0)^2, \tag{10}$$

and u^0 and u^1 are related by the equation

$$(u^0)^2 - (u^1)^2 = 1. \tag{11}$$

In the c.m.s. the "disk" expands symmetrically to both sides. We choose our co-ordinate origin in the median plane and shall consider the motion in the half-space expanding along the positive x-axis (so that $x > 0$, $u^1 > 0$).

Let us call the initial thickness of the "disk" Δ. We consider some instant of time $t \gg \Delta$, when the expansion has already progressed significantly. Neglecting the initial thickness of the disk we can assert that all the matter will be in the region $0 < x < t$, since the velocity cannot exceed that of light. Most of this space will contain matter which, though moving with a speed comparable to the light velocity, is not ultra-relativistic; only in a thin layer $t - x \ll t$ will there be matter moving with a velocity close to that of light. As we shall see later, in this last region there is concentrated only a *small* part of the entropy, but a *large* part of the energy of the system. Therefore, the examination of this small-sized ultra-relativistic region is very essential. To do this we replace the variable x by $\xi = t - x$. Then the first of equations (9) takes the form:

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial (T^{00} - T^{11})}{\partial \xi} = 0, \tag{12}$$

and, subtracting (12) from the second equation of (9) we find:

$$\frac{\partial}{\partial t}(T^{00} - T^{11}) + \frac{\partial}{\partial \xi}(T^{00} - 2T^{01} + T^{11}) = 0. \tag{13}$$

In the ultra-relativistic case both components u^0, u^1 of the four-velocity are large compared to unity and almost equal (we recall that $u^0 = 1/\sqrt{1 - v^2}$, $u^1 = v/\sqrt{1 - v^2}$, where v is the ordinary

velocity (in units $c = 1$). Later we shall denote by u (in first approximation) either of the quantities u^0 and u^1.

According to (10):

$$u^0 \approx u^1 = u \gg 1, \quad u^0 - u^1 \approx \frac{1}{2u}$$

Using these equalities and the equation of state (2), we obtain from (10):

$$\left.\begin{aligned} T^{00} &\approx (\epsilon + p)\, u^2 = \frac{4}{3}\,\epsilon\, u^2, \\ T^{00} - T^{01} &= (\epsilon\, u^0 - p\, u^1)(u^0 - u^1) \approx \frac{\epsilon}{3}, \\ T^{00} - 2T^{01} + T^{11} &= (\epsilon + p)(u^0 - u^1)^2 \approx \frac{\epsilon}{3\,u^2}, \end{aligned}\right\} \tag{14}$$

after which equations (12)–(13) take the form:

$$\left.\begin{aligned} \frac{\partial}{\partial t}(\epsilon\, u^2) &= -\frac{1}{4}\frac{\partial \epsilon}{\partial \xi}, \\ \frac{\partial \epsilon}{\partial t} &= -\frac{\partial}{\partial \xi}\left(\frac{\epsilon}{u^2}\right) \end{aligned}\right\} \tag{15}$$

We shall look for solutions of these equations in the domain of values $t \gg \xi \gg \Delta$.

A solution satisfying all the necessary requirements can be obtained as follows. Let us make the assumption, which we later show to be valid, that the function $u(\xi, t)$ is such that

$$u^2 = f\frac{t}{\xi}, \tag{16}$$

where f is a slowly (logarithmically) varying function of ξ and t. Neglecting the derivatives of f, we then obtain from (15)

$$f\frac{\partial}{\partial t}(\epsilon\, t) = -\frac{\xi}{4}\frac{\partial \epsilon}{\partial \xi}, f\, t\frac{\partial \epsilon}{\partial t} = -\frac{\partial}{\partial \xi}(\epsilon\, \xi)$$

Next we introduce the new variables

$$\tau = \ln \frac{t}{\Delta}, \eta = \ln \frac{\xi}{\Delta}, \tag{17}$$

and in place of ϵ, a new unknown function φ according to the relation

$$\epsilon = e^{\varphi} \tag{18}$$

From the two equations thus obtained,

$$f\left(1 + \frac{\partial \varphi}{\partial \tau}\right) = -\frac{1}{4} \cdot \frac{\partial \varphi}{\partial \eta} \text{ and } f\frac{\partial \varphi}{\partial \tau} = -\left(1 + \frac{\partial \varphi}{\partial \eta}\right), \tag{19}$$

we eliminate f and get:

$$1 + \frac{\partial \varphi}{\partial \tau} + \frac{\partial \varphi}{\partial \eta} + \frac{3}{4} \cdot \frac{\partial \varphi}{\partial \tau} \cdot \frac{\partial \varphi}{\partial \eta} = 0 \tag{20}$$

Following the general procedure for obtaining the general integral of a partial differential equation of first order, we first form the complete integral:

$$\varphi = A\,\eta - \frac{4(1 + A)}{4 + 3A}\,\tau + B, \tag{21}$$

containing two constants A and B. The general integral (containing one arbitrary function) is obtained from the complete integral if we consider B to be a function of A, determined by eq. (21) and the equation

$$\eta - \frac{4}{(4 + 3A)^2}\,\tau + \frac{dB}{dA} = 0 \tag{22}$$

obtained by setting the derivative $\partial\varphi/\partial A$ equal to zero.

Since we are looking for a solution of the equation of motion in the region of values $t \gg \Delta$, $\xi \gg \Delta$, the "initial moment" of the motion corresponds to values $\tau \sim 1$, $\eta \sim 1$. At this "moment", the system in the domain under consideration can be regarded as still uniform, so that the function φ is practically constant and

equal to some value φ_0 (the logarithm of the initial energy density ϵ_0). Thus the initial condition for our problem, to within the logarithmic accuracy we are using, is:

$$\varphi - \varphi_0 \sim 1 \text{ for } \eta \sim 1, \tau \sim 1 \tag{23}$$

A solution satisfying this condition is obtained from (21)–(22), if the arbitrary function $B(A)$ is chosen so that

$$B - \varphi_0 \sim 1, B' \sim 1$$

Then we can omit B' altogether in (22), (since $\eta \gg 1, \tau \gg 1$), and can set $B \approx \varphi_0$ in (21). We then have from (22)

$$\frac{4 + 3A}{2} = \sqrt{\frac{\tau}{\eta}}$$

(we choose the positive root, since in the other case the function f in (16) would turn out to be negative, which is clearly impossible), after which (21) gives:

$$\left.\begin{aligned} \varphi &= \varphi_0 - \frac{4}{3}(\eta + \tau - \sqrt{\tau\eta}), \\ \epsilon &= \epsilon_0 \exp\left[-\frac{4}{3}(\eta + \tau - \sqrt{\tau\eta})\right] \end{aligned}\right\} \tag{24}$$

When ξ becomes of the order of t, formula (16) as expected gives $u \sim 1$. From formula (24) it follows that in this region ($\eta \approx t$):

$$\epsilon = \epsilon_0 e^{-4\tau/3} = \epsilon_0 \left(\frac{\Delta}{t}\right)^{4/3}.$$

Even though the domain $\xi \sim t$ is outside the region of ultra-relativistic motion, this result should be correct as to order of magnitude.

The function f is found from φ using either of the equations (19): $f = \frac{1}{2}\sqrt{(\tau/\eta)}$. In accordance with our assumptions, it is a slowly varying function of t and ξ, of order unity.

Using the formulae we have obtained, let us see how the energy and entropy are distributed throughout the thickness of the "disk".

The energy density is given by the component $T^{00} \sim \epsilon u^2$ of the energy-momentum tensor (we recall that for each element of the matter ϵ is the energy density in the "proper" frame of reference, in which that element is at rest). So for the energy dE located in a slab of thickness $d\xi$ we have: $dE \sim \epsilon a^2 u^2 d\xi = \epsilon a^2 u^2 \xi d\eta$, where a is the radius of the disk. Setting $u^2 \sim t/\xi$, in accord with (16) and using equation (24) we obtain:

$$dE \sim \exp\left[-\frac{1}{3}(\sqrt{\tau} - 2\sqrt{\eta})^2\right] d\eta. \tag{25}$$

From this it is clear that the energy distribution has a maximum at $\eta = \tau/4$; this means that the energy is concentrated mainly in the region $\xi \sim \sqrt[4]{t\Delta^3}$. For $t \gg \Delta$ we get $\xi \ll t$, so that this region is at the limit of applicability of the one-dimensional solution we are considering.

The entropy density is given by the fourth component s^0 of the four-vector of entropy current density $s^i = su^i$. Since $s \sim \epsilon^{3/4}$ (according to (4)), $s^0 \sim u\epsilon^{3/4}$, and we find for the entropy associated with a slab of thickness $d\xi$: $dS \sim sua^2 d\xi \sim a^2 su\xi d\eta$, or, using formula (24):

$$dS \sim \exp\left[-\frac{1}{2}(\sqrt{\tau} - \sqrt{\eta})^2\right] d\eta. \tag{26}$$

This distribution has a maximum for $\eta = t$; i.e. the entropy unlike the energy is concentrated mainly in the region $\xi \sim t$.

The solution of the equation of motion which we have obtained is applicable so long as the angle of flight θ (the angle which the trajectory of a given element of the matter makes with the x-axis) is sufficiently small. This is necessary in order that the distance $t\theta$, which the element travels during the time t in the transverse direction, be small compared to the transverse dimensions of the system, a:

$$t\theta \ll a. \tag{27}$$

To evaluate the small angle θ, we use the transverse components of equation (7), which we have as yet not considered. Thus we get: $\partial T^{02}/\partial t \sim \partial T^{22}/\partial y$, or, to order of magnitude, $T^{02}/t \sim T^{22}/a$; so

that substituting $T^{02} \sim \epsilon u^2 \theta$ and $T^{22} \sim \epsilon$ (the transverse component of the four-velocity is $u^r \sim u\theta$), we get: $u^2\theta \sim t/a$. Finally, noting that $u^2 \sim t/\xi$, we find:

$$\theta \sim \frac{\xi}{a}. \tag{28}$$

Combining this formula (27) we see that the condition for applicability of the one-dimensional solution is:

$$t\,\xi \ll a^2. \tag{29}$$

We note that the limiting time for the one-dimensional solution is the greater, the smaller the value of ξ. For the central region, $\xi \sim t$, and the limiting time is $t \sim a$.

Starting at the moment

$$t_1 = \frac{a^2}{\xi}, \tag{30}$$

a significant sideways motion appears in the hydrodynamical motion; we shall call the resulting motion of the matter conical hydrodynamic flight. As we shall see later, in this stage of the motion the velocity approaches that of light so quickly, that for each element of matter the quantity ξ remains practically constant in time. In addition, one can show that all derivatives of hydrodynamic quantities, both with respect to the direction of ξ as well as with respect to the transverse direction, can be neglected in the equations. Thus, in particular, it follows that, because of the smallness of the sidewise forces, the direction of motion will remain unchanged, i.e. the flight will proceed radially (conically).

Furthermore, in view of the smallness of the forces during conical flight, the energy flux travelling within any cone, $\theta = \text{const}$, must remain constant; the same is true for the entropy flow. The cross sectional area of such a cone is proportional to t^2, so the conditions of constancy of flow of energy and entropy are:

$$\epsilon u^2 t^2 = \text{const}, \quad sut^2 \sim \epsilon^{3/4} u t^2 = \text{const}. \tag{31}$$

From these two relations we find:

$$u \sim t, \quad \epsilon \sim \frac{1}{t^4}, \tag{32}$$

which give the law of variation with time of u and ϵ during conical flight. From (32) we see that in this case the velocity actually approaches that of light faster than during the preceding stage. The change in the coordinate ξ of the moving element of matter is given by the formula:

$$\frac{d\xi}{dt} = 1 - v_x = (u^0 - u^1)\sqrt{1 - v^2} \approx \frac{1}{2u^2} \sim \frac{1}{t^2},$$

from which it is clear that during this stage of the flight, the quantity ξ approaches a constant value more rapidly.

For $t \sim t_1$, the solution (32) must agree to order of magnitude with the one-dimensional solution considered earlier. For the "joining" of the two solutions, it is convenient to introduce the symbols λ and L, according to the equations:

$$\frac{\xi}{a} = e^{-\lambda}, \frac{\Delta}{a} = e^{-L}. \tag{33}$$

Then

$$\eta = \ln \frac{\xi}{\Delta} = L - \lambda, \tag{34}$$

while the value of the variable τ corresponding to the moment t_1 is

$$\tau_1 = \ln \frac{t_1}{\Delta} = \ln \frac{a^2}{\xi\Delta} = L + \lambda. \tag{35}$$

Substituting this value in (26), we find that the entropy distribution is given by $dS \sim e^{\sqrt{L^2 - \lambda^2}}\, d\lambda$. Since each element of the matter now moves with $\xi = \text{const}$ while its entropy, by virtue of the adiabaticity of the motion, remains constant, the same formula gives the entropy at the moment of break-up of the matter into individual freely moving particles. The number of particles produced will be distributed according to this same law:

$$dN = Ce^{\sqrt{L^2 - \lambda^2}}\, \lambda^2 d\lambda, \tag{36}$$

where C is a normalising factor. The angle of flight

$$\theta \sim \frac{\xi}{a} = e^{-\lambda} \tag{37}$$

remains constant along with ξ for each element of the matter, and consequently for each particle. Consequently, formulae (36) and (37) determine in parametric form (parameter λ) the angular distribution of the produced particles (in the c.m.s.). The constant L which appears in the formula is related simply to the energy of the colliding particles. In fact, the ratio Δ/a is the Lorentz contraction of the system and is equal, in the notation of section 2 to

$$MAc^2/E' = \sqrt{(2MAc^2/E)}$$

(where MA is the mass of the particles). Therefore

$$L = \frac{1}{2} \ln \frac{E}{2MAc^2}. \tag{38}$$

The distribution (36) shows that, although the angle of departure in the c.m.s. is of the order of unity for most of the particles, there also occur much smaller angles. It is easy to see that the angular distribution does not at all show spherical symmetry, as Fermi assumed, but that $dN/d\theta$, referred to unit solid angle, increases rapidly with decreasing θ.

Formula (36) is easily written in explicit form. In order to take into account angles of the order of unity, we define λ as

$$\lambda = -\ln \tan (\theta/2). \tag{39}$$

With this definition, the smallest value $\lambda = 0$, corresponds to the largest possible value $\theta = \pi/2$. Formula (36) then becomes

$$dN \sim \exp \left[\sqrt{L^2 - \ln^2 \tan \frac{\theta}{2}} \right] \frac{d\theta}{\sin^2 \theta}. \tag{40}$$

This formula agrees well with the experimental data[3]. For practical purposes, formula (36) can be written to sufficient accuracy in the form:

$$dN \sim e^{-\lambda^2/2L} \, d\lambda. \tag{41}$$

Thus the angular distribution can be written as a Gaussian distribution, if we choose as variable the quantity $\lambda = -\ln \tan (\theta/2)$. In view of the logarithmic dependence of λ on θ,

the actual distribution curve of the particles with respect to the angle θ itself must have relatively very straight tails on both sides of the maximum.

We note that the largest value of λ which it is still meaningful to consider must correspond to the condition:

$$\int_{\lambda=\lambda_{\max}}^{\lambda=L} dN \sim 1,$$

or, substituting (36), to logarithmic accuracy, $C \exp \sqrt{L^2-\lambda_{\max}^2} \sim 1$. According to formulae (5) and (36) the total number of particles is $N \sim e^{L/2}$; therefore

$$\int_{\lambda=0}^{\lambda=L} dN \sim ce^{L} \sim e^{L/2}.$$

Thus $C \sim e^{L/2}$, and we obtain for $\lambda_{\max}$,

$$\lambda_{\max} = \tfrac{1}{2}\sqrt{3}\, L. \tag{42}$$

For determining the energy distribution of the particles, we consider the quantity u, which is proportional to the energy of the particles (the energy of a particle is the time component of the four-vector μu^i: $\mu u^0 \approx \mu u$). During the stage of one-dimensional motion $u \sim \sqrt{t/\xi}$, and at the moment $t = t_1$ it reaches the value $u \sim \sqrt{t_1/\xi}$. Therefore, "tacking on" the one-dimensional motion to the solution (32), we find that during the stage of conical motion:

$$u \sim \sqrt{\frac{t_1}{\xi}} \cdot \frac{t}{t_1} \sim \frac{t}{a}. \tag{43}$$

In similar fashion we match the laws (24) and (32) of variation of the "proper" energy density ϵ. For $t \sim t_1$ the quantity ϵ reaches the value:

$$\epsilon = \epsilon_0 \exp\left[-\frac{4}{3}(2L - \sqrt{L^2-\lambda^2})\right].$$

Determining from this the coefficient of proportionality in (32), we find:

$$\epsilon = \epsilon_0 \left(\frac{t_1}{t}\right)^4 \exp\left[-\frac{4}{3}(2L - \sqrt{L^2 - \lambda^2})\right]. \tag{44}$$

The start of the free separation of the particles corresponds to the moment t_0, when ϵ, decreasing, reaches the value ϵ_c corresponding to the criterion (1). From (44) we find:

$$t_c \sim t_1 \left(\frac{\epsilon_0}{\epsilon_c}\right)^{1/4} \exp\left[-\frac{1}{3}(2L - \sqrt{L^2 - \lambda^2})\right].$$

Setting $t \sim t_c$ in (43) and substituting for t_1 from (35), we find the following expression for the energy $\mu u'$ of the particles at the moment of their free separation:

$$\mu u' \sim \mu \frac{t_c}{a} = \text{const} \exp\left[\lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right].$$

We note that the energy of the outgoing particles is measured by the ratio of the time (or the distance from the origin) at the moment of decay into particles to the characteristic time a/c of the system. The constant coefficient in the expression for $\mu u'$ is determined from the obvious relation:

$$\int \mu u' dN = E' \sim \sqrt{EMA} \sim MAe^L,$$

and we get finally:

$$\mu u' \sim M \exp\left\{-\frac{L}{6} + \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}. \tag{45}$$

Formulae (36) and (45) give in parametric form the energy distribution (in the c.m.s.) of the particles produced. From (45) we see that most of the particles ($\lambda \sim 0$) have energies $\mu u' \sim Me^{L/6} \sim M(E'/AM^{1/6})$ only slightly exceeding M.

We must still go over from the c.m.s. to the original laboratory frame of reference in which one of the nucleons was at rest before the collision. The angle χ of the outgoing particle in the laboratory

system is related to the angle θ in the c.m.s. by the transformation formula:

$$\tan\chi \approx \chi = \frac{v'\sqrt{1-V^2}\sin\theta}{v'\cos\theta + V},$$

where v' is the velocity of the particle in the c.m.s., and V is the velocity of the c.m. relative to the laboratory system. We may immediately write $v' = 1$ in the numerator, and in the denominator write:

$$v'\cos\theta + V \approx v'(1+\cos\theta) + \frac{1}{2}(V^2 - v'^2),$$

or, since V is closer to unity than v':

$$v'\cos\theta + V \approx 1 + \cos\theta + \frac{1}{2}(1 - v'^2) = 1 + \cos\theta + \frac{1}{2u'^2}.$$

The last term on the right can be neglected for all cases except when θ is too close to π. However, it is easy to see that the angles we have found satisfy the inequality $\theta,\ \pi - \theta \gg 1/u'$; this is equivalent to the inequality:

$$\exp\left\{\frac{L}{6} - \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\} \ll 1$$

(according to (39) and (45)), which is actually satisfied for all $\lambda < \lambda_{max}$. Thus we can set $v'\cos\theta + V \approx 1 + \cos\theta$, and the formula for transforming angles to the laboratory system takes the form:

$$\chi = \sqrt{1-V^2}\tan\frac{1}{2}\theta. \tag{46}$$

In this connection we note the following curious fact. Independently of any detailed computations, the distribution of outgoing particles, for a collision of two identical particles, is symmetric in the c.m.s., i.e. angles θ occur just as often as $\pi - \theta$. Since $\tan(\pi - \theta)/2 = 1/\tan(\theta/2)$, it follows automatically that, upon averaging over all particles,

$$\overline{\ln\chi} = \ln\sqrt{1-V^2} = -L. \tag{47}$$

In other words, the geometrical mean of all the angles of separation gives just the value of the velocity of the c.m. and, consequently, the velocities of the incident particles (for a collision of two identical particles).

Substituting the value $\tan(\theta/2) = e^{-\lambda}$ and $\tan(\pi - \theta)/2 = e^{-\lambda}$ in (46) for particles moving in opposite directions in the c.m.s., we obtain: $\chi = e^{-L \mp \lambda}$. This formula has the special property that when we change from particles going to the right in the c.m.s. to particles travelling to the left, there is merely a change in sign of the quantity λ. We can therefore write

$$\chi = e^{-L-\lambda}, \tag{48}$$

and consider formulae (36) and (48) as giving the angular distribution of all particles in the laboratory system, where λ can take both positive and negative values.

For the transformation of the energy of particles moving to the right, we have

$$u \sim \frac{u'}{\sqrt{1 - V^2}} = e^L u',$$

and for particles moving to the left we get (noting that $\theta \gg 1/u$):

$$u \sim \frac{\theta^2 u'}{\sqrt{1 - V^2}} = u' e^{L-2\lambda}.$$

Substituting (45), this gives:

$$\mu u \sim M \exp\left\{\frac{5L}{6} \pm \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}.$$

This formula too has the property that it describes particles moving to both the right and the left in the c.m.s., if we write

$$\mu u \sim M \exp\left\{\frac{5L}{6} + \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}, \tag{49}$$

and give λ both signs.

Formulae (36) and (49) give the energy distribution of the particles in the laboratory system. The coefficients in these formulae can be made more precise if we use the obvious relations:

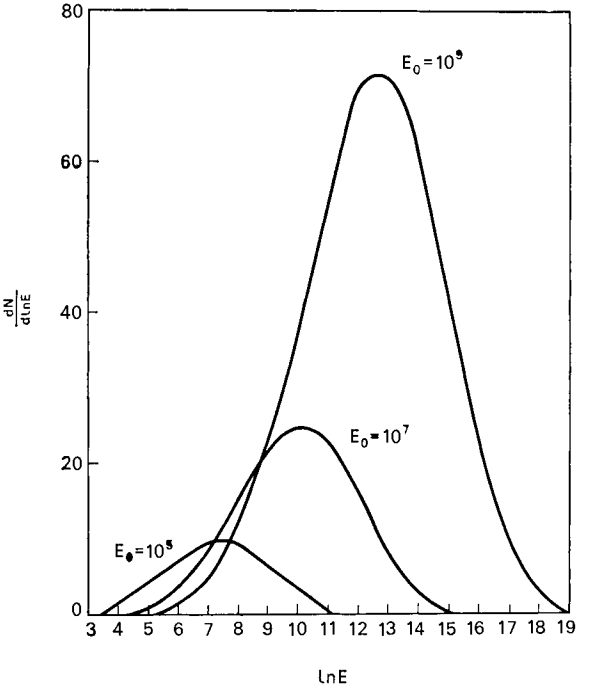

FIG. 1. Differential energy spectra of secondary particles during nuclear interactions at high energy (for varying energies E_0 of the initial particles). The areas under the curves are proportional to the total number of secondary particles (mesons and nucleons).

$\int dN = N$, $\int \mu u dN = E$. In the integrations we can, to the accuracy we are considering, expand the exponent in a series in the neighbourhood of the maximum. We then get

$$dN = \frac{N}{\sqrt{2\pi L}} \exp\left[\sqrt{L^2 - \lambda^2}\right] d\lambda,$$

or, taking account of (6) and (38),

$$dN = \frac{KA}{\sqrt{2\pi L}} \exp\left[-\tfrac{1}{2}L + \sqrt{L^2 - \lambda^2}\right] d\lambda. \qquad (50)$$

It is understood that the coefficient in this formula is actually a slowly varying (non-exponential) function of the ratio λ/L. For the energy we get:

$$\mu u = \frac{5\sqrt{5}}{2\sqrt{3}} \cdot \frac{M}{K} \exp\left\{\frac{5L}{6} + \lambda + \frac{1}{3}\sqrt{L^2 - \lambda^2}\right\}. \qquad (51)$$

Here the same remarks apply to the coefficient as were made in the last case. From formula (51) it is clear that most of the particles have an energy of order $Mc^2(E/2MAc^2)^{7/12}$ in the laboratory system.

We note that both the angular and energy distributions of the particles are close to Gaussian if we use the logarithm of these quantities as variables; consequently, they have quite straight tails on both sides of the maximum. The results of a computation based on (51) are shown in Fig. 1.

In conclusion, I should like to thank E. M. Lifshitz, I. Ya. Pomeranchuk, and E. L. Feinberg for discussion of the questions touched upon here, and also L. I. Saruchev for permission to use the drawing of the spectra which he calculated from formula (51).

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7

Extension of the Uncertainty Principle to Relativistic Quantum Theory†

It is shown by considering possible methods of measurement that all the physical quantities occurring in wave mechanics can in general no longer be defined in the relativistic range. This is related to the well-known failure of the methods of wave mechanics in that range.

1. Introduction

It is known that the application of the methods of wave mechanics to problems in which the speed of light cannot be regarded as infinite leads to absurd results. In the first place, states with negative mass appear in Dirac's relativistic equation[1]. This difficulty arises because the relation between momentum and energy in relativity theory is quadratic, so that two energy states are possible for a given momentum. In contrast to classical ($\hbar = 0$) relativity theory, where the continuous change of all quantities means that transitions between the two kinds of state are impossible, such transitions cannot reasonably be forbidden in quantum theory.

In the second place, the interaction of a charged particle with the field produced by itself is inevitably divergent[2].

The infinite zero-point energy of the radiation field which occurs on quantisation of the field[3] can be avoided by the use of suitable variables[4], but it still has the effect that the energy-density matrix elements become infinite. This is very closely related to the self-energy difficulty mentioned above (see also a paper by Rosenfeld[5]).

† *Z. Phys.*, **69**, 56, 1931; with R. Peierls.

This complete failure of the theory suggests that in the range considered the physical requirements for the applicability of the methods of wave mechanics are no longer satisfied. The present paper investigates this problem.†

2. The Concept of Measurement in Wave Mechanics‡

The significance of any physical theory is to derive from the result of an experiment conclusions regarding the results of subsequent experiments. Thus the relations between measurements and the physical states of a system are of two kinds. Firstly, the measurement determines the state of the system *after* the measurement is made, and secondly it examines the state of the system which existed *before* the measurement. In classical ($\hbar = 0$) mechanics this distinction is of no importance, since the states of the system before and after the measurement can be regarded as identical.

In wave mechanics, however, the situation is quite different, since the measurement always causes a change in the state of the system, and this change is in principle impossible to determine. If the measurement had no other property, the wave-mechanical description would be neither possible nor meaningful. It is necessary to make use of another physical property of measurements, which is usually described as repeatability. This signifies that, when the same measurement is immediately repeated, the same result is certainly obtained. In this form, however, the hypothesis is physically incorrect in most cases, as will be shown in more detail below; and in this strict form it is not necessary for wave mechanics. The important point is that for any system there should exist *predictable* measurements. This means measurements such that for every result there is a state of the system in which this measurement *certainly* gives that result. For, if this requirement

† The uncertainty relations on which our conclusions are based are derived mainly from discussions in Copenhagen. Professor Bohr's attitude to these relations will be described in an article to appear shortly in *Nature*.

‡ This section is essentially a development of ideas put forward by N. Bohr in his lecture at Como[6].

were not fulfilled, the state of the system after a measurement could not be described by a ψ function. This may be seen as follows. We can describe the state of the system and the measuring apparatus together by a wave function which, before the measurement, consists of a product $\psi\varphi_0$. Here ψ is the initially arbitrary wave function of the system, and φ_0 the known wave function of the measuring apparatus. After the interaction, the wave function will in general no longer be a product. If we expand it in terms of the eigenfunctions of the measuring apparatus, in the form $\Sigma\psi_n\varphi_n$, then ψ_n describes the state in which the system remains after the measurement. In general, the form of ψ_n depends on that of ψ. If the wave function of the system is to be deduced from an observation of the measuring apparatus, ψ_n must be independent of ψ apart from a constant factor, i.e. $\psi_n = a_n u_n$, with u_n normalised to unity. From the linearity of the wave equation it follows that a_n depends linearly on ψ, i.e. can be written in the form $\int \psi v_n^* d\tau$, with v_n any function dependent on the process of measurement. Then $|a_n|^2$ is the probability that the measurement gives the nth result. The sum of all these probabilities must be equal to unity, i.e. $\Sigma |a_n|^2 = 1$ independent of ψ (provided that ψ is normalised):

$$\sum a_n a_n^* = \int \sum a_n v_n \psi^* \, d\tau.$$

This expression must therefore always equal unity if $\int \psi\psi^* d\tau = 1$, i.e. we must have

$$\sum a_n v_n = \psi.$$

(The v_n form a complete orthogonal system.) From this, however, it follows that the measurement is predictable if we take ψ to have the particular value of one of the v_n; then only one of the a_n is not zero. The repeatability of the measurement would signify that the v_n and the u_n are identical, and this is not in general true.†

† In a measurement which occupies a short time it can easily be shown that the u_n are identical with the v_n only when the corresponding operator commutes with the energy of interaction between the system and the apparatus. In wave mechanics (neglecting relativity) this interaction energy is always a

If, however, the wave function of the system cannot be determined by any measurement, it can have no physical meaning. The use of wave functions would then be as pointless as, for example, the use of the concept of paths in quantum mechanics. Thus the existence of predictable measurements is an absolutely necessary condition for the validity of wave mechanics.

The condition of repeatability cannot in general be satisfied. This is particularly seen if the time necessary for the measurement is taken into account. This time is restricted by the relation $\Delta E \Delta t > \hbar$, which has very often been stated, but which has been correctly interpreted only by Bohr[6]. Clearly it does *not* signify that the energy can not be known exactly at a given time (for in that case the concept of energy would have no meaning), nor does it mean that the energy can not be measured with arbitrary accuracy within a short time. We must take into account the change caused by the process of measurement even in the case of a predictable measurement, i.e. of the difference between the result of the measurement (v_n) and the state after the measurement (u_n). The relation then signifies that this difference causes an energy uncertainty of the order of $\hbar/\Delta t$, so that in a time Δt no measurement can be performed for which the energy uncertainty in *both* states is less than $\hbar/\Delta t$.

This follows from a consideration of the time evolution of the interaction process. The method of the variation of constants shows that transitions within short times occur not only between states which satisfy the condition $E + \epsilon = E' + \epsilon'$ (E and E' being the energy of the system before and after the transition, ϵ and ϵ' that of the apparatus). These states are given preference by resonance only after a long time, the corresponding transition probabilities increasing greatly with time. In practice, after a time

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function of the coordinate. The only quantity for which a repeatable measurement is possible is therefore the coordinate. Measurements of the coordinate always actually have this property. It is also seen that the u_n need not in general be orthogonal, i.e. the measurement does not in general diagonalise an operator. This physical circumstance also is usually overlooked in the presentation of transformation theory.

Δt, only transitions for which $| E + \epsilon - E' - \epsilon' | \lesssim \hbar/\Delta t$ are of importance. This fact does not, of course, contradict the strict validity of the law of conservation of energy in wave mechanics, but the energy of interaction between the system and the apparatus is also indeterminate by the same amount. In the most favourable case, where ϵ and ϵ' are precisely known, the uncertainty must be $\Delta(E - E') > \hbar/\Delta t$.

This relation has important consequences as regards the measurement of momentum. Any measurement of momentum is made by allowing the body to collide with another. In measuring a component of momentum (most simply done by collision with a plane mirror) the law of conservation of momentum is to be applied rigorously, but that of conservation of energy applies only to within $\hbar/\Delta t$, because of the unknown interaction energy. Thus to determine the particle momentum $\boldsymbol{P}$ we have the equations

$$\boldsymbol{p} + \boldsymbol{P} - \boldsymbol{p}' - \boldsymbol{P}' = 0,$$

$$| \epsilon + E - \epsilon' - E' | \gtrsim \hbar/\Delta t;$$

$\boldsymbol{p}, \boldsymbol{p}', \epsilon, \epsilon'$, i.e. the motion of the measuring apparatus before and after the collision, may be regarded as known. Then $\Delta P = \Delta P'$ and, since $\Delta E = v\Delta P$,

$$(v - v')\Delta P > \hbar/\Delta t. \qquad (1)$$

Thus any measurement of momentum necessarily involves a *definite* change of momentum (in addition to the unknown change which restricts the accuracy of the measurement).† This fact was first recognised by Bohr[6]. The non-repeatability of the momentum measurement in a short time is thus shown with particular clarity. Momentum measurements which last a long time, on the other hand, are meaningful only for free particles.

† Here an important point is that not every Hamiltonian can actually occur in nature; as already mentioned, the interaction function is always a function of the coordinates and so does not commute with the momentum. If the form of the Hamiltonian could be chosen arbitrarily, the momentum could be measured in an arbitrarily short time without change of velocity; this is a trivial deduction from the fact that coordinates and momenta are then equivalent.

3. Momentum Measurement in the Relativistic Case

We now wish to make use of relativity, i.e. of the finite speed of propagation. There exists as yet no satisfactory relativistic quantum theory, but it is clear that here also we certainly cannot go beyond the limits imposed on the accuracy of measurement by the general principles of wave mechanics.

The scope of the relation just derived for a momentum measurement is considerably extended by relativity. In the non-relativistic theory, the definite change of velocity could be made arbitrarily large, and so the momentum could be measured with arbitrary accuracy even in a short time. If, however, we take into account the fact that the velocity cannot exceed c, then $v - v'$ can be at most of the order of c, so that equation (1) gives

$$\Delta P \Delta t > \hbar/c. \tag{2}$$

The inequality (2) is particularly easy to derive for the state *after* the measurements. If we assume that the particle had a definite position before the measurement, then after a time Δt, on account of the finite velocity limit, the position is still known with accuracy $c\Delta t$. If the momentum after this time were determined more accurately than as given by (2), this would contradict the result $\Delta P \Delta q > \hbar$.

On account of (2) the concept of momentum has a precise significance only over long times. Thus, in cases where the momentum changes appreciably within such times, the use of the concept of momentum is purposeless.

In the measurement of momentum of a charged body, in addition to the above-mentioned inaccuracy, a further perturbation of the measurement arises because the body will emit radiation in the necessary change of velocity. We shall consider only the case where the velocity of the body before the measurement is certainly small compared with c. In this case it is favourable to conduct the measurement so that after the measurement the velocity is again considerably less than c. For, if the velocity approaches c, the relation (1) gives very little benefit, while the accuracy is greatly

reduced by the emission of radiation. Thus the non-relativistic formula for radiation damping can be used. The energy emitted is then

$$\frac{e^2}{c^3}\int \dot{v}^2\, dt$$

where e is the charge on the body. This energy evidently has its least value for uniform acceleration, i.e. for $\dot{v} = (v' - v)/\Delta t$, so that the energy emitted is

$$\frac{e^2\,(v' - v)^2}{c^3\,\Delta t}.$$

This unknown change of energy has to be taken into account in the energy balance, and there thus arises in the momentum a further inaccuracy:

$$(v' - v)\,\Delta P > \frac{e^2}{c^3}\frac{(v' - v)^2}{\Delta t},$$

or

$$\Delta P\,\Delta t > \frac{e^2}{c^3}(v' - v). \tag{3}$$

For electrons this inequality gives no new information, since even in the most unfavourable case where $v - v' \sim c$ it gives only $\Delta P \Delta t > e^2/c^2$, and this is weaker than (2), since $e^2 < \hbar c$. For macroscopic bodies, however, the relation (3) is significant. Multiplication by (1) gives

$$\Delta P\,\Delta t > \frac{\hbar}{c}\sqrt{\frac{e^2}{\hbar c}}, \tag{4}$$

and in this form we shall make use of it later. The inequality (4) is, of course, valid independently of the method of measurement used, and in particular when the measurement is made by means of the charge on the body, as in the case of the Compton effect, where, in addition to the Compton scattered radiation used in the

measurement, there is a further radiation corresponding to that discussed above, obtained when higher approximations are taken into account in the perturbation theory calculation for the interaction between the radiation and the particle.[7] (In the ordinary Compton effect with electrons this effect is of no importance, on account of the smallness of $e^2/\hbar c$.)

4. Field Measurement

The simplest method of measuring an electric field is to observe the acceleration of a charged test body. In order to avoid interference by magnetic fields, we use a body of very large mass and very small velocity. Let the momentum of the body before the measurement be known, and let the momentum afterwards be measured, again with accuracy ΔP. From this we can deduce the electric field strength with an accuracy such that

$$e\Delta\mathscr{E}\ \Delta t > \Delta P. \tag{5}$$

In addition, however, the condition (4) must be satisfied in the momentum measurement. Multiplication of equations (4) and (5) gives

$$\Delta\mathscr{E} > \frac{\sqrt{\hbar c}}{(c\ \Delta t)^2}. \tag{6}$$

For the magnetic field strength we easily obtain the same result by considering the motion of a magnetic needle:

$$\Delta\mathscr{H} > \frac{\sqrt{\hbar c}}{(c\ \Delta t)^2}. \tag{6a}$$

If it is desired to measure the electric and magnetic field strengths simultaneously, then, in addition to the effects already discussed, we have to take into account the effect on the needle of the magnetic field due to the charged body and vice versa. This magnetic field is, in order of magnitude,

$$\Delta\mathscr{H} > \frac{e}{(\Delta l)^2}\frac{v'}{c}, \tag{7}$$

where Δl is the distance between the test body and the needle. If we multiply this inequality by equations (5) and (1), then (with $(v = 0)$ we have

$$\Delta \mathscr{E} \, \Delta \mathscr{H} > \frac{\hbar c}{(c \, \Delta t)^2} \, \frac{1}{(\Delta l)^2}. \tag{6b}$$

This condition differs from the product of (6) and (6a) in that $c\Delta t$ in the denominator is partly replaced by Δl.

If follows from (6), (6a) and (6b) that for $\Delta t = \infty$ the measurement can be made arbitrarily accurate for both $\mathscr{E}$ and $\mathscr{H}$. Thus static fields can be completely defined in the classical sense.†

In wave fields (that is, fields which are further than $c/v = \lambda$ from the bodies which produce them), it is sufficient to use (6) and (6a), because as a result of the coupling of the space and time variation nothing is discovered about the field if the region of measurement has an extent less than $c\Delta t$ for a given Δt. Thus here also the measurements of $\mathscr{E}$ and $\mathscr{H}$ do not interfere, and to the extent that the field strengths can be measured in accordance with (6) and (6a) they can be measured simultaneously. Thus the field strengths are in accordance with the classical theory in so far as they can be defined at all. In the quantum range, on the other hand, the field strengths are not measurable quantities.‡

5. Measurements on Light Quanta

We shall now show that in a radiation field no measurements can be carried out with certainty within a short time, i.e. measurements for which every possible result gives information about the state of the system. (Thus we do not consider such measurements as, for example, a measurement of position by means of a collision which does not occur with probability unity within the period of observation, so that, although a deflection of the test body shows

† Our thanks are due to Professor Bohr for pointing out this situation and the significance of the time in general.

‡ The inaccuracy for the field measurement with an electron found by Jordan and Fock[8] is greater than (6) and therefore proves only that the electron is not a suitable means of measuring the field.

that the body under measurement was at the point considered, the absence of such a deflection shows nothing.) The time necessary for the measurement depends on the state of the system. If the energy of the radiation field is approximately determined as E, we shall show that this time is greater than $\hbar/E$. Since the field consists of light quanta, the greatest frequency occurring in the Fourier resolution of the field can be at most $E/\hbar$; if we carry out the measurements in times small compared with $\hbar/E$, we remain within the period of oscillation, and so the field strength may be regarded as constant during the measurements. All measurements in such short times are therefore field measurements and are subject to the inaccuracy (6). Thus, in order that an effect should be detectable, the field strength must considerably exceed $\sqrt{(\hbar c)}/(c\Delta t)^2$ The smallest wavelength occurring, on the other hand, is $\hbar c/E$, and so the field strength, if non-zero at any point, must be non-zero in a region of at least this extent. Consequently, the total field energy must be at least of the order $E > \mathfrak{E}^2 (\hbar c/E)^3 > (\hbar c)^4/E^3(c\Delta t)^4$, i.e.

$$\Delta t > \hbar/E, \tag{8}$$

in contradiction with our hypothesis. Thus measurements which do not satisfy equation (8) are impossible.

This result applies in particular, of course, when the radiation field consists of a single quantum of light. Within the time given by (8), a quantum of light is therefore undetectable, and in particular its position cannot be determined with any accuracy. In a measurement of position, the time to which that position refers is therefore indeterminate by more than $\hbar/E$. If the measurement of position is to be used to investigate a state, as discussed in section 2, we are interested in the position at a time up to which the state under investigation (i.e. the state whose energy was of the order of E) existed. The measurement of position determines it with an inaccuracy; at best we have $\Delta q > \hbar c/E$.

It might be thought that the accuracy could be improved by measuring momentum at the same time as position (within the limits given by $\Delta P \Delta q > \hbar$, of course), and seeking to deduce how far and in which direction the light quantum has meanwhile

travelled. A closer examination, however, shows that the resulting accuracy can be no better than $\hbar c/E$. Thus in every state it is meaningful to give the probability of presence of the light quantum only for regions large compared with the wavelength, and the position of the light quantum is a meaningful concept only in geometrical optics.

If the number of light quanta varies appreciably within the period of oscillation, the concept of light quanta itself is meaningless.

6. Measurements on Material Particles

Let us now investigate the corresponding relations for material particles. (We shall always speak of electrons, but the arguments of course apply for any kind of material particle.) Such particles can best be detected by means of collision processes, for example using the Compton effect. Thus the presence of an electron is demonstrated by making two momentum measurements on the light quantum and deducing from the change in momentum that a collision has occurred in between. Here, however, the course of the process depends considerably on the length of the time interval between the two measurements. For a long time interval the Compton effect is obtained, i.e. the momentum of the light quantum changes either not at all or by an amount determined by the initial momenta, which can be made arbitrarily large by using very hard radiation. For very short intervals however, any changes of momentum may occur, provided that the law of conservation of momentum remains valid; the sum of the energies of the electron and the light quantum need be conserved only to within $\hbar/\Delta t$, as shown in section 2. For the same reasons as in measurements on light quanta, the small momentum changes have much the greatest probability. An elementary calculation shows that the second behaviour begins when the time interval is no longer large compared with $\hbar/E$, where E is the approximate energy of the electron before the measurement.

Thus, if the duration of the process of measurement is made

shorter than $\hbar/E$, the momentum of the light quantum (and therefore also that of the electron) changes by an arbitrary amount. Hence, from the fact that no measurable change of momentum has occurred, we cannot conclude that no collision has taken place. Physically this means that the measurement of the momentum of the light quantum destroys the initial state of the electron. We cannot ensure that the electron is found with probability unity at the first measurement: if it was in a volume δq before the measurement, a time $\delta q/c$ is necessary before we can be sure that the light has reached the electron. Since $\delta q/c > \hbar/c\delta P > \hbar/c\, P > \hbar/E$, we should therefore have to make several measurements before being able to detect the electron, and thus completely destroy its state before we find it. Measurements in times less than $\hbar/E$ are therefore useless.

We now again ask how accurately this measurement can be used to derive the position of the electron at a time up to which it was in its former state. To do this, of course, only the knowledge about the velocity which is compatible with the position measurement can be used, and not, for example, the velocity in the state before the measurement. If an exact measurement of position is made, no information is obtained as regards the velocity, which thus remains indeterminate within c. The coordinate can therefore be derived only with an error $\Delta q > \hbar c/E$. Elementary considerations show that no higher accuracy can be achieved by measuring the momentum and coordinate simultaneously with any accuracies compatible with $\Delta P \Delta q > \hbar$. Thus

$$\Delta q > \hbar c/E \tag{9}$$

represents the limit of the accuracy with which the position of the electron can reasonably be defined. In particular, if the velocity of the electron is not very close to c, this becomes

$$\Delta q > \hbar/mc. \tag{10}$$

The derivation of (10) shows that it is valid only for electrons which are not moving too rapidly. The statement frequently found

in the literature that $\hbar/mc$ is a general limit for the accuracy of measurements of position is based on incorrect arguments.

A superficial consideration might suggest that the uncertainty relations derived above are not relativistically invariant. In reality, of course, there can be no contradiction with relativity, which has been taken into account throughout the derivations. The explanation is that the inequalities themselves need not transform in a relativistically invariant manner, since the most favourable possible measurements of a quantity need not be so when they are viewed from a moving system of coordinates. Thus we have only to require that the limit of accuracy should not be exceeded when such a measurement is viewed from a moving system of coordinates. This requirement is, of course, always satisfied.

Particular care is needed in this respect with position measurements, for here the statement of the problem is itself not relativistically invariant, but distinguishes a time axis, since we ask for the coordinate at the moment up to which the unperturbed state existed.

7. Mathematical Failure of the Methods of Wave Mechanics

The above-stated unmeasurability of all wave-mechanical quantities also appears, of course, in the formalism which results when we attempt to apply the methods of wave mechanics to the relativistic case.

The most fundamental quantity in the theory, both for electrons and for light quanta, is the momentum; this is of course due to the fact that if it remains constant in time it can be defined with arbitrary accuracy, although very long times are needed for its measurement. This latter fact does not, of course, appear in the wave-mechanical formalism, and in consequence the statements of the theory regarding short times have no meaning.

The unmeasurability of the position, on the other hand, is directly expressed in the formalism. For electrons this is because the Dirac equation also allows the physically meaningless solu-

tions with negative energy. The result of a measurement can, of course, in reality only be a wave function composed only of states with positive energy. Such states cannot, however, form an arbitrary wave packet. It is easily seen that the dimensions of a wave packet in general cannot be less than $\hbar/mc$. There are, it is true, special wave packets of smaller size (namely those whose centre moves at almost the speed of light†), but the corresponding wave functions do not form a complete system, and the state before the measurement cannot in general be expanded in terms of them. This corresponds to the result shown earlier that in short times a determination of position may sometimes happen to be possible, but a measurement cannot be carried out with certainty.

The conditions for light quanta are still more extreme in that no mathematical expression can be given for the probability density. This is seen from the fact that, on account of polarisation properties, the wave function for a light quantum must be a tensor of rank two[4]. The probability density and current must form a four-vector, which is impossible, because they depend quadratically on the wave function. In geometrical optics it is, of course, possible to construct wave packets in which all effects vanish outside a certain region. But here also these wave functions do not form a complete system.

The unmeasurability of the field strength is shown by the fact that in empty space (no light quanta) the field strength operator[2, 4] is not zero, but even the expectation of the square of the field strength is infinite. This is related to the fact that for $\Delta t = 0$ we have from (6) an infinite indeterminacy of the field strength.

8. Conclusions

We have seen that no predictable measurements can exist for the fundamental quantities of wave mechanics (except when these quantities are constant in time, and then an infinitely long time is needed for an exactly predictable measurement). It cannot, of course, be formally demonstrated that in nature there are not some

† Our thanks are due to Professor O. Klein for pointing this out.

particularly complicated quantities for which predictable measurements are possible, but such a speculation need not be discussed. The assumptions of wave mechanics which have been shown to be necessary in section 2 are therefore not fulfilled in the relativistic range, and the application of wave mechanics methods to this range goes beyond their scope. It is therefore not surprising that the formalism leads to various infinities; it would be surprising if the formalism bore any resemblance to reality.

The applicability of wave mechanics is restricted to processes where the state of the system varies sufficiently slowly. In cases where the ordinary Schrödinger equation is applicable this is of course not always true. For radiation alone wave mechanics is never meaningful, since the limit $c = \infty$ has no meaning.

In the correct relativistic quantum theory (which does not yet exist), there will therefore be no physical quantities and no measurements in the sense of wave mechanics. One can, of course, cause the system to interact with some apparatus and ask what happens to the latter. The theory will give a probability for the result of this experiment, but this cannot be interpreted as the probability of a parameter of the system under investigation, since it can in no way be ensured that the probability of a given result is unity and that of all other results is zero. In addition, it is in principle impossible to make the duration of such an experiment arbitrarily short.

This view is confirmed by the known fact that the β-spectra of radioactive nuclei are continuous, although the uniform lifetime indicates that the nuclei are not in different states. For, if all the β-particles had the same energy, the process could be regarded as a predictable measurement.

This fact presents an insuperable difficulty in wave mechanics because, as Bohr has emphasised, it means that the law of conservation of energy is probably invalid for nuclear electrons. This law is indissolubly connected with the foundations of wave mechanics. In relativistic quantum theory, however, the preceding discussion shows that the concept of energy need not be mechanically definable. It is of course definable in a certain sense in terms

of the total mass of the nucleus, because the nucleus in its motion as a whole satisfies wave mechanics, but this does not imply a predictable measurement of quantities related to the internal state of the nucleus.

If the law of conservation of energy is not valid, then in radioactive processes the mass of the whole system will of course change, but this change cannot be followed in the course of time, since the mass cannot be measured in an arbitrarily short time. If we consider the process of measurement of the mass as in section 3, the time needed for the measurement is such that $\Delta m \Delta t > \hbar/c^2$.

The preceding discussion is not contradicted by the fact that the spectra of protons and α-particles are discrete. On account of their large mass (low velocity) these particles obey wave mechanics even in the nucleus, rather as the nuclei in a molecule can be essentially described in classical terms despite their strong interaction with the electrons, for which classical mechanics fails completely.

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8

A Theory of Energy Transfer in Collisions

A theory of "adiabatic", inelastic collisions is developed. The application of this theory to atomic collisions of the second kind leads to the result that the azimuthal quantum number of the entire system always changes by ± 1 on collision, forbidding a transition from two S-states to two other S-states. The corresponding reaction cross-section is proportional to $(E - U)^{3/2}/E$ where U is the energy at the intersection of the two terms, at which the transition always occurs. The case of polyatomic molecules is discussed generally. The process of nuclear excitation and splitting without alpha capture is considered. The results are applied to: 1. Predissociation; 2. Excitation of oscillations during an optical transition; 3. The case of closely approaching potential curves. The question of the cross-section is discussed.

Part I†

1. Electron collisions for the theoretically most important case of fast incident electrons were treated by Bethe. The problem of collisions of the second kind has, however, till now given rise merely to false results. Kallmann and London[1] for instance, treat the excitation of electrons on collision when the motion of the nucleus is fixed. This is, however, contradicted by known wave-mechanical treatment, and leads to a strange vanishing of the energy. In a paper by Morse and Stückelberg this error is avoided, but only by taking the opposite case and treating the interactions between the atom as a perturbation, which has still less bearing on reality. Only recently has attention been drawn to the fundamental role of the intersection of the energy curves[2]. However, no quantitative theory has been developed. The basis of the difficulties seems

† *Phys. Z. Sowjet.*, **1**, 88, 1932.

to be that we have here a kind of "quasi-classical" process; the strong periodicity of the wave function makes a direct estimate of the perturbation integral impossible. G. Beck[3] has attempted to solve this for the case of a Coulomb field, but he has not really answered the problem as he has substituted for the continuous field an infinite potential barrier. This allows no evaluation of the correct order of magnitude of the effect. The aim of this article is to estimate this value.

2. Let us consider a system which is comprised of a "quantised" part as well as a "quasi-classical" part. This is to say a part for which the wavelength is small compared to the dimensions of the system. Then to a first approximation the eigenfunction of the system is made up of the product of the eigenfunction of the quantised part in the fixed quasi-classical part and the eigenfunction of the classical part in the external field produced by the state of the quantised part. In this first approximation there is no energy exchange between the two parts of the system, and the quasi-classical part behaves "adiabatically". If we wish to estimate the probability of energy transfer we must go to the next order of magnitude and evaluate the perturbation integral. This we know is given by the expression $\int \psi'^* D\psi d\tau$ where D is the Schrödinger operator for the entire system. (For an exact solution $D\psi = 0$.) The integration for the quantised variables is carried out in the normal manner, but for the quasi-classical variables this is not possible. The corresponding parts of ψ are strongly periodic functions, which does not allow an evaluation in the normal manner. However, in order to solve the integral we can make use of just these classical properties by applying a canonical transformation. We introduce as a variable the action variable of the motion of the quasi-classical part in one of the two states. It is well known that we can perform this canonical transformation by subtracting from the differential of the action another total differential of the action as function of the coordinates and of the action variables:

$$dS = \Sigma p dq - dS_1(I_1, q). \tag{1}$$

If we calculate the change of this action function with regard to the action variable of the first state for the motion of the second state, that is for given I_2, we have by integration:

$$S = S_2(I_2, q) - S_1(I_1, q). \tag{2}$$

Here I_1 is considered as an independent variable and I_2 as a known quantity and the q as functions of I_1 and I_2. The corresponding ψ function for I_1 has the form $\psi_2(I_1) = a \exp(iS/\hbar)$. The integral $\int \psi_2 \eta \psi_1 d\tau$ (η is the perturbation energy) is equal to $\psi_2(I_1)\eta(I_1)$ since ψ_1 is a δ-function of I_1. Here I_1 is not a variable, but denotes the value of this variable in the first state. The essential part of this expression is of the form

$$\exp\left\{\frac{i}{\hbar}[S_2(I_2, q) - S_1(I_1, q)]\right\}. \tag{3}$$

As we should expect, we have an equation which is symmetrical in the two systems. The coordinates q can be obtained as functions of I_1 and I_2 by elimination of the momentum in the expressions $I_1(q, p)$ and $I_2(q, p)$ or, equivalently, from the equations:

$$p = f_1(I_1, q) = f_2(I_2, q). \tag{4}$$

The physical meaning of this is that the important points for the transition are those at which both q and p remain unaltered. This implies more here than merely the derivation of the so-called Franck–Condon rule, as it is valid not only for the real, but also for the imaginary part.

If the condition (4) is not applicable to the real part then the expression (3) defines the probability when we take the imaginary solution for which the expression $i(S_2 - S_1)/\hbar$ corresponds to the smallest absolute value of the real part which must naturally always be taken to be negative. If only the energies of the two states are specified, then the condition

$$E_1 - U_1(q) = E_2 - U_2(q) \tag{5}$$

must be fulfilled. As we are dealing with radiationless transitions, we set $E_1 = E_2$, giving:

$$U_1(q) = U_2(q) \tag{6}$$

The intersections of the potential energy curves or (more generally, of the multi-dimensional surfaces) are important.

As far as the order of magnitude of the perturbation integral is concerned we can thus say that it is relatively large if the curves $U_1 = U_2$ cut one another in the real region and thus in the region of real kinetic energy. On the other hand it becomes exponentially small if this condition is not fulfilled. It is also easy to see what a "near miss" for the intersecting curves means: nothing more than that the curves just cut in the imaginary region. In certain circumstances it can happen that the intersection lies so close to the real axis that the real part of S becomes very small, and that we no longer have an exponentially vanishing expression.

3. Now let us consider more exactly the collision of two atoms. We write the Hamiltonian for the whole system in the well-known form:

$$H = H_e + \frac{\boldsymbol{p}^2}{2m}, \tag{7}$$

where H_e is the Hamiltonian of the electrons, m the reduced mass ($1/m = 1/m_1 + 1/m_2$) of the nuclei, and $\boldsymbol{p}$ is the nuclear momentum. We can split the kinetic energy of the nuclei into its radial and transverse components. We can now write H in the form:

$$H = H_{e,r} + \frac{\boldsymbol{m}^2}{2m\,r^2}, \tag{8}$$

where $H_{e,r}$ now contains the radial kinetic energy as well as H_e and $\boldsymbol{m}$ is the nuclear angular momentum. Finally, we introduce the total angular momentum, $\boldsymbol{M}$, of the system, and the electronic angular momentum $\boldsymbol{\theta}$ instead of the nuclear angular momentum:

$$H = H_{e,r} + \frac{1}{2\,m\,r^2}(\boldsymbol{M} - \boldsymbol{\theta})^2. \tag{9}$$

The operator $H_{e,r}$ contains only radial operators, and it thus clearly has only matrix elements which give rise to terms with the same electronic angular momentum. But by a well-known rule[4] such terms may only cross terms with different total spin. But

owing to the small transition probability these terms are unimportant for us, and the corresponding perturbation elements can be neglected. The operator $\boldsymbol{M}$ is, of course, a diagonal matrix. There exist transitions for $\boldsymbol{\theta}$ where the electronic angular momentum quantum number changes by 0, ± 1. Thus $\boldsymbol{\theta}^2$ is also diagonal with respect to k. The remaining perturbation energy is thus merely

$$\eta = -\frac{1}{m r^2}(\boldsymbol{\theta} \cdot \boldsymbol{M}), \tag{10}$$

and we can thus formulate the law that for atomic collisions the transitions with $\Delta k = \pm 1$ (thus from Σ to Π, Π to Σ and Δ, and so on) are by far the most probable. Since, for example, from an S and a P term both a Σ and a Π term can arise, this selection rule, in fact, says little. However, a transition from two S states to two S states is expressly forbidden since two S states always give rise only to a Σ term. It must be emphasised here that this condition is indeed necessary, but by no means sufficient, since it only states which terms can cross, but not which must cross. Thus not all $\Delta k = \pm 1$ correspond to a transition on collision.

We next evaluate the matrix element for η with respect to all coordinates except r. We can then consider the angular momentum $\boldsymbol{M}$ as a normal vector in our quasi-classical treatment. This gives:

$$\eta(r) = -\frac{M}{m r^2} D(r), \tag{11}$$

where M is the absolute value of the angular momentum of the system, and D the matrix element of the electronic angular momentum corresponding to the transition $\Delta k = \pm 1$. This is, of course, also a function of the coordinates r.

The general formula of perturbation theory gives for the probability of a collision transition:

$$\omega = \left(\frac{|\eta|}{\hbar}\right)^2, \tag{12}$$

where $|\eta|$ is the matrix element of η in which the ψ functions are normalised for a flux of 4π atoms per unit time. In the quasi-classical approximation we have:

$$\psi = \frac{1}{\sqrt{\pi\, v(r)}} \frac{\cos[S(r)/\hbar]}{r}. \tag{13}$$

Here v is the speed expressed as function of the coordinates. Substitution of this expression gives:

$$\sqrt{\omega} = \int \frac{\eta}{\hbar} \psi_1 \psi_2^* 4\pi r^2 dr = -\int \frac{MD(r) \cos[S_1(r)/\hbar] \cos[S_2(r)/\hbar] 4\pi\, dr}{m r^2 \hbar \sqrt{\pi\, v_1(r)}\sqrt{\pi\, v_2(r)}}$$

$$= -\int \frac{4MD(r) \cos[S_1(r)/\hbar] \cos[S_2(r)/\hbar]}{\hbar m r^2 \sqrt{v_1(r)\, v_2(r)}}\, dr. \tag{14}$$

In this integral, as we have already said, we are considering the term involving the difference of the action functions, or:

$$\sqrt{\omega} = -\int \frac{2MD(r) \cos\{[S_1(r) - S_2(r)]/\hbar\}}{\hbar m r^2 \sqrt{v_1(r)\, v_2(r)}}\, dr. \tag{15}$$

We must take only that part of this integral coming from the neighbourhood of the intersection of the two curves. Here we take all quantities except the rapidly changing cosine to be constant. We also expand the action function in powers of the distance from the intersection. We take into account that $dS/dr = p$ and we determine $d^2S/dr^2 = dp/dr$ from $(p^2/2M) + V = E$ (V, of course, also contains the centrifugal energy). From this, we have

$$v \frac{dp}{dr} = F, \tag{16}$$

where $F = -dv/dr$ is the force (including the centrifugal force) at the point r. Since at the intersection $p_1 = p_2$ and $V_1 = V_2$, we have

$$S_1 - S_2 = A + \frac{F_1 - F_2}{2v}(r - r_0)^2, \tag{17}$$

where A denotes the values of $S_1 - S_2$ at the intersection. The integral is now of the form:

$$\sqrt{\omega} = -\int_{-\infty}^{+\infty} \frac{2MD}{\hbar m r^2 v} \cos\left[\frac{A}{\hbar} + \frac{F_1 - F_2}{2\hbar v}\xi^2\right] d\xi, \qquad (18)$$

where $\xi = r - r_0$ is the new integration variable. All variables are evaluated relative to the point of intersection. We have chosen $\pm\infty$ as integration limits since the region important for the integration lies anyway near the intersection. We have

$$\int_{-\infty}^{+\infty} \cos(\alpha + \beta\xi^2)\, d\xi = \sqrt{\frac{\pi}{\beta}} \cos\left(\alpha + \frac{\pi}{4}\right), \qquad (19)$$

and thus

$$\sqrt{\omega} = -\frac{2MD}{\hbar m v r^2}\sqrt{\frac{2\pi\hbar v}{F_1 - F_2}} \cos\left(\frac{A}{\hbar} + \frac{\pi}{4}\right), \qquad (20)$$

or

$$\omega = \frac{8\pi M^2 D^2}{\hbar m^2 v r^4 (F_1 - F_2)} \cos^2\left(\frac{A}{\hbar} + \frac{\pi}{4}\right). \qquad (21)$$

The phase $A/\hbar$ changes very quickly with different M, corresponding to the quasi-classical character of the system. Thus we can replace the expression, $\cos^2[(A/\hbar) + (\pi/4)]$ by its mean value $\frac{1}{2}$ even over a small interval. This gives finally:

$$\omega = \frac{4\pi M^2 D^2}{\hbar v(F_1 - F_2) m^2 r^4}.$$

We now insert the value of the speed v. We have

$$\frac{1}{2} mv^2 + V = E, \qquad (22)$$

or, when we write out the centrifugal energy explicitly,

$$\frac{1}{2} m v^2 + U + \frac{M^2}{2m r^2} = E. \qquad (23)$$

For the force we have

$$F = -\frac{dV}{dr} = -\frac{d}{dr}\left(U + \frac{M^2}{2mr^2}\right),$$

or, since the two states are only distinguishable by the form of U, $F_1 - F_2 = d(U_2 - U_1)/dr$. We thus have for ω:

$$\omega = \frac{4\pi D^2 M^2}{\hbar m r^4 (F_1 - F_2)\sqrt{2m(E - U) - (M^2/r^2)}} \tag{24}$$

To obtain the total cross-section we need only to take into account that $M = mv_0\varrho$ where v_0 is the speed at infinity, and ϱ is related to the largest possible cross-section by $d\lambda = 2\pi\varrho d\varrho$. We then have:

$$d\lambda = \frac{2\pi M\, dM}{m^2 v_0^2} = \frac{\pi M\, dM}{m E} \tag{25}$$

The total cross-section is then clearly:

$$\sigma = \int \omega\, d\lambda = \int \frac{4\pi^2 D^2 M^3\, dM}{\hbar m^2 E r^4 (F_1 - F_2)\sqrt{2m(E - U) - (M^2/r^2)}}. \tag{26}$$

The integration is to be taken from $M = 0$ to that M for which the speed at the intersection is still real, i.e. up to $M^2/r^2 = 2m(E - U)$. If we choose $M/r\sqrt{2m(E - U)}$ as variable, we get:

$$\sigma = \frac{4\pi D^2 [2m(E - U)]^{3/2}}{\hbar m^2 E (F_1 - F_2)} \int_0^1 \frac{x^3\, dx}{\sqrt{1 - x^2}} = \frac{2}{3}\frac{4\pi D^2 [2m(E - U)]^{3/2}}{\hbar m^2 E (F_1 - F_2)}$$

$$= \frac{2^{9/2}\pi D^2}{3\hbar\sqrt{m}\,(d/dr)(U_2 - U_1)} \frac{(E - U)^{3/2}}{E}. \tag{27}$$

This final formula describes the cross-section as a function of the collision velocity. We can estimate its order of magnitude from the fact that the denominator contains $\sqrt{m}$. We should thus expect that when E is equal to a few volts, the cross-section is

about 100 times smaller than the so-called kinetic cross-section of the atom, in other words that an effective collision only occurs about once every few hundred times. If U is negative effective collisions can also result at smaller speed with relatively great probability. For small E, σ is inversely proportional to E and σ can then become appreciable for small E. The cross-section reaches a minimum at $E = -2U$ and then increases as $\sqrt{E}$ for large E, i.e. proportional to the speed. For positive U, a noticeable effect begins at $E = U$ and increases steadily with increasing E.

To calculate the angular distribution we must know not only the value of U at the intersection, but also the precise form of the function. If we know this, we can obtain the deflection, Θ, as a function of M from the classical equations of motion. Here we would assume that the transition always takes place from the surface of the "sphere of intersection" so that the coordinates and momentum remain unaltered.

4. The relations are much more complicated for polyatomic molecules. In any case, there is in general no crossing of the terms for complicated molecules except when the transition corresponds to different spin values. This case can thus be significant, although the probability for it is relatively small (reduced by the order of the relativistic correction). In the case of complicated molecules an approximate intersecting may occur. The available experimental material is too limited to enable a thorough investigation of the various cases which occur. For collisions between atoms and diatomic molecules there are generally terms having different mirror symmetry with respect to the nuclear plane which can cross. Then we should have conditions analogous to those for atomic collisions. On the other hand, for two diatomic molecules there are no crossing points in the general case, and an intersection is only possible for certain symmetrical conditions. For this reason two diatomic molecules do not in general pass through an intersection during their motion. There are, however, certain directions of motion (in multi-dimensional coordinate space) which do possess this property. Thus, the probability of a transition here depends on the product of the expression considered above with

the probability that the collision occurs in a particular direction, and is thus distinctly smaller.

5. A quite different application of the theory developed above arises in the case of nuclear excitations or reactions due to a passing α-particle. In this case we always have a Coulomb field at large distances from the nucleus, both for the incoming and for the outgoing α-particle. In other words, at large distances $U_2 - U_1$ is constant and equal to the energy given up by the α-particle. An intersection of the energy curves is not possible at large distances. This must always happen somewhere in the region of the nuclear radius (whether the intersection is real or imaginary is here of no interest to us). The important part of the probability is thus determined by the value of the exponential function $\exp [2i(S_2 - S)\hbar]$ in the region of the nucleus. But for small r_0:

$$2iS = -2\int_{r_0}^{r} \sqrt{\frac{2m}{\hbar^2}\left(\frac{2Ze^2}{r} - E\right)}\, dr \approx -\frac{4\pi Ze^2}{\hbar v} + \frac{8e\sqrt{mZr_0}}{\hbar}, \tag{28}$$

so that we have for the difference

$$2i(S_2 - S_1) = -\frac{4\pi Ze^2}{\hbar}\left(\frac{1}{v_2} - \frac{1}{v_1}\right), \tag{29}$$

which is independent of r_0. We obtain an expression for the cross-section which is entirely independent of the nature of the potential energy near the nucleus

$$\sigma = A \exp\left[-\frac{4\pi Ze^2}{\hbar}\left(\frac{1}{v_2} - \frac{1}{v_1}\right)\right], \tag{30}$$

where A is a factor which only varies slowly with the speed. We obtain an entirely analogous expression for the case of the emission of a proton from the nucleus. The only difference is that in the case where there is no resonance, the proton has another potential barrier to pass through, which leads to an expression:

$$\sigma = B \exp\left\{-\left[\frac{4\pi Ze^2}{\hbar}\left(\frac{1}{v_2} - \frac{1}{v_1}\right) - \frac{2\pi(Z-1)\,e^2}{\hbar v_p}\right]\right\} \tag{31}$$

(v_p is the speed of the proton). In the case of a resonance reaction (with regard to the proton) the previous expression is valid. Resonance as far as the α-particle is concerned should be considered as a normal α-particle capture. The formula (30) was applied by G. Gamow at my suggestion to the experimental results of reactions. It was, however, established that in all cases known so far we are dealing with the capture of an α-particle. The same seems to be the case for the observed artificial gamma radiation.

Part II†

THE results obtained in Part I can be applied to a number of effects. It must be added, however, that during the collision of similar atoms it is necessary to add to the selection rules developed in Part I, the postulate of the change in parity which is evident from the fact that the electron-spin components which are normal to the line connecting the nuclei change their sign when mirrored.

1. The most direct generalisation is predissociation. For the dissociation probability during an oscillation we can use here expression (21) directly. This can also be written in the following form:

$$\omega = \frac{4\pi \hbar D^2}{(F_1 - F_2) m^2 r^4} \frac{j^2}{v}. \tag{1}$$

The proportionality to j^2/v can be found in a paper by Kronig[5] where, however, v designated the velocity at infinity instead of the actual radial velocity at the intersection, the reason lying in the estimates used by Kronig.

The treatment given in Part I can also be applied to the case of an intersection made possible by the difference of the total spin. Elementary considerations lead to the following selection rules: variation of the total spin by ± 1, in addition to the optical selection rules for the azimuthal quantum number and possibly the parity. Now, since the matrix element no longer depends on the

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nuclear velocity one obtains instead of (1) a result simply proportional to $1/v$ as might be deduced from the time spent at the intersection.

2. A very special case of the application of the theory is the excitation of oscillations during an optical transition when the temperature corresponds to a non-oscillating molecule. In this case, the energy of the most highly excited state is determined by the Franck–Condon rule and the co-ordinates of the points of lowest energy of the normal state. When calculating the drop in intensity when moving away from this most highly excited state we have, according to the theory developed here, to determine the intersections of the following curves:

$$E_1 - U_1 = E_2 - U_2. \tag{2}$$

When denoting the distance of the intersection from the normal state by ξ and considering that the parabola of the lower state can for this purpose be replaced by a constant value of U and the upper curve by a straight line, one obtains approximately:

$$\epsilon = F\,\xi, \quad \text{or,} \quad \xi = \frac{\epsilon}{F}, \tag{3}$$

where F is the force acting on the upper curve, and ϵ the energy difference from the most highly excited state. Now, we only have to determine the action integrals up to the point ξ. The integral along the parabola is obviously the decisive one. One hence obtains for the exponent:

$$2\int \frac{m\,\omega\,\xi\,d\xi}{\hbar} = \frac{m\,\omega}{\hbar}\,\xi^2 = \frac{m\,\omega}{\hbar}\left(\frac{\epsilon}{F}\right)^2 \tag{4}$$

($\omega = 2\pi \cdot$ frequency of normal oscillation). It is seen that the intensity distribution is Gaussian. The width of the intensity distribution is given by

$$\Delta\epsilon = \sqrt{\frac{\hbar}{2m\,\omega}}\,F. \tag{5}$$

It thus usually includes many vibrational levels.

3. It is interesting to determine the probability when the energy curves approach closely to one another. This is invariably the case when two ions have at infinity an energy which is not very different from the energy of two atoms. Then the attractive Coulomb force gives an intersection in a region where all the other interactions, which might destroy the intersection, are still very small.

When the curves intersect in the near complex plane, we must extend our action integral into a region where the two potential energies approach one another very closely. We can write accordingly

$$p_1 - p_2 = \frac{\partial p}{\partial U}(U_1 - U_2) = -\frac{\partial p}{\partial E}(U_1 - U_2) = -\frac{U_1 - U_2}{v}, \quad (6)$$

where v is the velocity in the given region. The value of the probability is consequently represented by the quantity

$$\exp\left[-\frac{2i}{\hbar v}\int (U_1 - U_2)\, dx\right]. \quad (7)$$

According to a well-known formula we have, however:

$$U_1 - U_2 = \sqrt{(U_{10} - U_{20})^2 + 4a^2},$$

where U_{10} and U_{20} are the eigenvalues in zeroth approximation and a the "exchange energy" of the two terms. The width of the smallest distance is obviously equal to $2a$.

In the vicinity of the intersection we can write, except for an unimportant constant,

$$U_{10} = F_1 x, \; U_{20} = F_2 x. \quad (8)$$

In the next approximation, the position of the intersection is given by $(F_1 - F_2)^2 x^2 + 4a^2 = 0$ whence follows

$$x = \frac{2i\,a}{F_1 - F_2}. \quad (9)$$

The exponent of (7) is hence equal to

$$-\frac{2i}{\hbar v}\int_0^{2ia/(F_1-F_2)}\sqrt{(F_1-F_2)^2x^2+4a^2}\,dx$$

$$=-\frac{2}{\hbar v}\int_0^{2a/(F_1-F_2)}\sqrt{4a^2-(F_1-F_2)^2\xi^2}\,d\xi. \qquad (10)$$

The probability is therefore proportional to

$$\exp\left[\frac{-2\pi a^2}{\hbar v\,(F_1-F_2)}\right]. \qquad (11)$$

To estimate the exponent in the above-mentioned Coulomb case, we put

$$F=\frac{e^2}{r^2}=\frac{\Delta^2}{e^2}, \qquad (12)$$

where Δ is the energy difference at infinity. In the case of an ion collision v is simply determined by Δ. If $\Delta = 1$ eV, the condition:

$$\frac{4\pi a^2}{\hbar v}\,\frac{1}{F_1-F_2}=1$$

leads to $2a = 0{\cdot}006$ eV.

4. When the exponent in (11) is small, we can neglect to a first approximation the splitting of the curves.

The probability of the electron transition is therefore given by the probability of the transition in the case of two intersecting curves. This case can be treated by the method used in Part I with the sole difference that the place of the matrix elements of $\vartheta M/mr^2$ is taken by a which hence gives for the probability.

$$\frac{4\pi a^2}{\hbar v\,(F_1-F_2)}. \qquad (13)$$

In the intermediate case the probability will be of the order of magnitude 1. To express this probability in terms of the cross

section, we introduce the following relations. The cross section in the case $\omega = 1$ is $2\pi\, \varrho d\varrho$, where ϱ is the minimum distance without interaction. Because of the conservation of angular momentum $\varrho v_0 = r v_\Theta$ (r being the coordinate of the intersection, and v_Θ the angular velocity at the intersection) we have:

$$d\sigma = \frac{2\pi r^2}{v_0^2}\, v_\Theta\, dv_\Theta, \tag{14}$$

or, since $v_r^2 + v_\Theta^2 = v_0'^2$ has a fixed value for a given energy,

$$d\sigma = \frac{2\pi r^2}{v_0^2}\, v_r dv_r. \tag{15}$$

Now, when multiplying by ω and integrating (v in (13) means of course v_r) over all values of v_r from zero to v', we obtain

$$\sigma = \frac{8\pi^2 a^2 r^2 v_0'}{\hbar F v_0^2}, \tag{16}$$

or

$$\sigma = \frac{4\pi^2 a^2 r^2 \sqrt{2m(E-U)}}{\hbar F E}. \tag{17}$$

5. The statements made in Part I require, on the other hand, considerable modification when the two energy states are situated close together at infinity (resonance case). This may be brought about either by transitions of fine structure levels into one another or by an accidental approach of the levels of the two atoms. To determine the range of validity of the earlier discussion, we calculate the action difference for the two curves which do not intersect in the real region. When denoting the order of magnitude of the complex value of r at the intersection by R we can write

$$S_1 - S_2 = \frac{R\Delta}{v}, \tag{18}$$

where Δ is the energy difference of the two terms at infinity (R is of the order of magnitude of r at which the interaction energy is of the order of the energy difference).

For the exponent we have

$$\frac{R\Delta}{\hbar v}. \tag{19}$$

Here, too, we can therefore differentiate between two cases depending on whether the exponent is small or large compared with 1. In the first case the energy difference between the two states can be neglected. In this case we may apply for the calculation the method of Kallmann and London.[1] When estimating the exponent it will be convenient to put $R = 10^{-7}$ cm, since the interaction decreases rapidly for larger distances. When assuming for v the thermal velocity of about 3×10^4 cm/sec, one obtains Δ of the order of 0·1 meV. On the other hand, if the exponent is large, the intersection of the curves again plays an important part and we must use (I.27).

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9

Fundamental Problems†

It is with the deepest sorrow that I send this article written in the honour of the sixtieth birthday of Wolfgang Pauli to a volume dedicated to his memory, which will always be cherished by those who had the good luck of knowing him personally.

It will be impossible now to know his opinion about the ideas expressed in this article but I am still encouraged by the thought that his views on the subject would not be very different.

It is well known that theoretical physics is at present almost helpless in dealing with the problem of strong interactions. For this reason, any remarks on the subject must necessarily be of the nature of forecasts, and their authors are peculiarly apt to find themselves barking up the wrong tree.

It was long thought that the main difficulty of the theory lies in the occurrence of infinities which can be avoided only by the use of perturbation theory. The habit of using the device of re-normalisation, which had achieved brilliant success in perturbation theory, was carried so far that the concept of re-normalisation acquired a certain mystical aura. The situation, however, becomes clear if, as is usual in theoretical physics, point interaction is regarded as the limit of some "distributed" interaction. This treatment, although it assumes weak interaction, goes considerably beyond the scope of perturbation theory, and makes possible the derivation of asymptotic expressions for the energy dependence of the basic physical quantities[1]. These expressions show that the effective

† From *Theoretical Physics in the Twentieth Century, a Memorial Volume to Wolfgang Pauli*, ed. by M. Fierz and V. F. Weisskopf (Interscience, New York 1960), p. 245.

interaction always diminishes with increasing energy, so that the physical interaction at finite energies is always less than the interaction at energies of the order of the cut-off limit which is given by the bare coupling constant appearing in the Hamiltonian.

Since the magnitude of the re-normalisation increases indefinitely with the cut-off limit, it follows that even an extremely weak interaction implies a large bare coupling constant when the cut-off radius is sufficiently small. Thus it was supposed that the main problem is to devise a theory of very strong interactions.

Further investigation showed, however, that the matter was by no means so easily dealt with. It was demonstrated by Pomeranchuk in a series of papers[2] that, as the cut-off limit is increased, the physical interaction tends to zero, no matter how large the bare coupling constant is. At about the same time Pauli and Källén[3] obtained the same result for the so-called Lee model.

The correctness of "nullifying" the theory has often been called in question. The Lee model is a very special one, considerably differing in several respects from physical interactions; and the validity of Pomeranchuk's proofs has been doubted. In my opinion such doubts are unfounded. For example, Källén has several times put forward the view that unusual properties of the series to be summed are involved, but he has never given reasons to support this view. By now, the "nullification" of the theory is tacitly accepted even by theoretical physicists who profess to dispute it. This is evident from the almost complete disappearance of papers on meson theory, and particularly from Dyson's assertion[4] that the correct theory will not be found in the next hundred years—a piece of pessimism which would be impossible to understand if one supposed that the present meson theory leads to finite results which we are yet unable to derive from it. It therefore seems to me inopportune to attempt an improvement in the rigour of Pomeranchuk's proofs, especially as the brevity of life does not allow us the luxury of spending time on problems which will lead to no new results.

The vanishing of the point interaction in the present-day theory leads to the idea that it is necessary to consider "distributed", non-

local, interactions. Unfortunately, the non-local nature of the interaction renders completely useless the technique of the present existing theory. Of course the undesirability of this occurrence is a poor argument against the non-local nature of the interaction; but there are stronger arguments against it. All the conclusions derived by means of the quantum theory of fields without the use of particular Hamiltonians seem to get confirmed experimentally. They include, in the first place, dispersion relations. Moreover, the number of mesons formed in high-energy collisions is in agreement with Fermi's formula,[6] which involves the use of the ideas of statistical thermodynamics for dimensions very much less than any possible radius of interaction.

The idea of the possibility of a marked modification of the existing theory without abandoning local interaction, was first suggested by Heisenberg[7]. Besides this general idea, Heisenberg has also suggested a number of further assumptions, which to me appear dubious. I shall therefore attempt to describe the general situation in what seems to me the most convincing form.

Almost 30 years ago Peierls and I had noticed that in the region of relativistic quantum theory no quantities concerning interacting particles can be measured and the only observable quantities are the momenta and polarisations of freely moving particles. Therefore if we do not want to introduce unobservables we may introduce in the theory as fundamental quantities only the scattering amplitudes.

The ψ operators which contain unobservable information must disappear from the theory and, since a Hamiltonian can be built only from ψ operators, we are driven to the conclusion that the Hamiltonian method for strong interaction is dead and must be buried, although of course with deserved honour.

The foundation of the new theory must consist of a new diagrammatic technique which must deal only with diagrams with "free" ends i.e. with scattering amplitudes and their analytic continuation. The physical basis of this technique is the unitarity conditions and the principle of locality of interaction which expresses itself in the analytic properties of the fundamental

quantities of the theory, such as the different kinds of dispersion relations.

As such a new diagrammatic theory is not as yet constructed, we are obliged to derive the analytical properties of the vertex parts from a Hamiltonian formalism, but it requires much naivety to try to make such derivations "rigorous", forgetting that we derive existing equations from Hamiltonians which do not really exist.

One of the conclusions of such an approach to the theory is that the old problem of the elementarity of particles finally loses its meaning, as it cannot be formulated without considering interactions between particles.

I think that the development of a theory on such lines has progressed very much in recent times and the time is not far away when the equations of the new theory will be finally written down.

But one must remember that in this case, contrary to what has happened in all the previous stages of theoretical physics, the wiriting down of the equations will mark not the end but only the beginning of the construction of the theory. The equations of the theory will be an infinite system of integral equations, each of which has the form of an infinite series, and it will be a hard task to learn how to work with such equations.

It is, of course, impossible to predict now how many constants in the theory may be chosen arbitrarily. We cannot even exclude the possibility that the equations will have no solutions at all, i.e. that the theory will also be nullified. This might be regarded as the rigorous proof of the non-locality of nature, but it might also mean that a theory of strong interactions alone cannot exist by itself and that weak interactions, and especially electrodynamics, must also be included in the picture. The infrared "catastrophe" would then make the situation infinitely more complicated.

But even in the best case we have still a great struggle before us, a struggle which has now become much more difficult without the brilliant unerring light of the mind of Wolfgang Pauli.

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10

On the Conservation Laws for Weak Interactions†

A variant of the theory is proposed in which non-conservation of parity can be introduced without assuming asymmetry of space with respect to inversion.

Various possible consequences of non-conservation of parity are considered which pertain to the properties of the neutrino and in this connection some processes involving neutrinos are examined on the assumption that the neutrino mass is exactly zero.

1. Combined Parity

As is well known, the unusual properties of K-mesons have created a perplexing situation in modern physics. The correlation between π-mesons in τ-decay ($K^+ \to 2\pi^+ + \pi^-$) leads to the necessity of assigning a 0^- state to K^+-mesons. This kind of system, however, cannot decay into two π-mesons ($K^+ \to \pi^+ + \pi^0$) We are thus faced with the dilemma of either assuming that two different K-mesons exist or that the conservation laws are violated in K-meson decay. In the first case one must then explain the identity of masses (which are equal to within two electron masses) and the near coincidence in lifetime of the θ and τ-decays. One may attempt to explain the equality of K-meson masses by postulating, as Lee and Yang[1] have done, the existence of some hitherto unknown symmetry property of nuclear forces which transforms the τ-meson into a θ-meson. If, however, decay involving a neutrino ($K^+ \to \mu^+ + \nu$, $K^+ \to \mu^+ + \nu + \pi^0$, $K^+ \to e^+ + \pi^0 + \nu$) is considered to be essentially the same for particles of various parity a difference in lifetime related to the different rate of τ and

† *Nuclear Physics*, 3, 127, 1957.

θ-decay ($\approx$ 8 per cent and $\approx$ 25 per cent) should be anticipated. This discrepancy should be not less than 30–40 per cent, a result which seems to be inconsistent with experiment[2].

Thus we come to the conclusion that the hypothesis of the existence of two different K^+-mesons is contrary to the experimental facts and the only alternative is to assume that the generally accepted conservation laws are violated in K-decay. Since there is no reason to think that the law of conservation of angular momentum is untenable, we are apparently dealing here with a direct violation of the law of conservation of parity.

It might seem at first glance that non-conservation of parity implies asymmetry of space with respect to inversion. If however, complete isotropy of space (conservation of angular momentum) is taken into account this type of asymmetry would seem to be extremely strange and in my opinion a simple rejection of parity conservation would create a difficult situation in theoretical physics. I would like to point out a solution of this problem which consists in the following. As is well known, both the law of conservation of parity and charge conjugation invariance undoubtedly hold in strong interactions. Let us now assume that each of these conservation laws does not hold separately in weak interactions. However, invariance with respect to the set of both operations (which we shall call combined inversion) will be assumed to exist. In combined inversion, space inversion and transformation of a particle into an antiparticle occur simultaneously.

It is easy to see that invariance of the interactions with respect to combined inversion leaves space completely symmetrical, and only the electrical charges will be asymmetrical. The effect of this asymmetry on the symmetry of space is no greater than that due to chemical stereo-isomerism.

On the other hand the law of conservation of parity of charged particles will not hold as the operator of combined inversion does not transform charged particles into themselves.

Furthermore, it is easy to see that the constants characterising the particles and anti-particles (masses, lifetimes) should be identical since, as a result of invariance with respect to combined in-

version, all processes involving particles and antiparticles should differ from each other only in regard to space inversion. Graphically speaking, a K^--meson is a mirror reflected K^+-meson.

Truly neutral particles, that is, particles which are identical to their anti-particles, transform into themselves in combined inversion. Consequently, with respect to these particles combined inversion leads to a law of conservation of combined parity. It should be emphasised that conservable parity is the product of ordinary parity and charge parity of the particles. Evidently, in this sense the π^0-meson is an odd particle: the K_1^0 (θ^0)-meson which decays into 2π-mesons is an even particle and the K_2^0-meson predicted by Gell-Mann and Pais[3] and recently discovered experimentally[4] is an odd particle. Combined inversion changes the sign of the magnetic field of a photon but does not change that of the electric field. The ordinary parities of electric and magnetic multipoles are reversed for combined inversion.

It is easy to show from the foregoing that despite the absence of ordinary parity the particles cannot possess dipole moments. Indeed, the only vector which can be constructed from ψ-operators for a particle at rest is its spin vector which is even with respect to inversion and odd with respect to charge. It is consequently odd with respect to combined inversion and, in accord with the foregoing regarding the electromagnetic field, it defines only a magnetic but not an electric moment.

Lee and Yang[5]† have shown that non-conservation of parity leads to correlations in a number of hyperon production and decay processes. It can be shown that a consequence of invariance with respect to combined inversion is that the weak interaction operators in the Lagrangian contain real coefficients. This circumstance, however, does not appreciably modify the qualitative picture which is obtained in the general case of non-conservation of parity. Therefore asymmetry of hyperon decay with respect to the plane of their creation, which has been predicted by Lee and Yang[5], will also hold in this case.

† I would like to thank sincerely the authors for sending me a preprint of their paper.

I would like to express my deep appreciation to L. Okun', B. Ioffe and A. Rudik for discussions from which the idea of this part of the present paper emerged.

2. Properties of the Neutrino

Rejection of the law of conservation of parity entails the possibility of the existence of new properties of the neutrino. The Dirac equation for the case of zero mass splits into two independent pairs of equations. It will be recalled that in the usual theory one cannot confine oneself to a single pair of equations since both pairs transform into each other as a result of space inversion. If, however, we restrict our attention to combined inversion we arrive at the possibility of describing the neutrino by a single pair of equations. In the sense of the usual scheme this would signify that the neutrino is always polarised in the direction of its motion (or in the opposite direction). The polarisation of the antineutrino is correspondingly reversed. According to this model the neutrino is not a truly neutral particle and this agrees with the fact that double β-decay has not been observed experimentally and especially with the results of experiments on induced β-decay. We shall call this kind of neutrino a longitudinally polarised neutrino or briefly a longitudinal neutrino.

In the usual theory the neutrino mass is zero, so to say, accidentally. Thus, account of neutrino interactions automatically leads to the appearance of a definite, albeit vanishingly small, rest mass. The mass of the longitudinal neutrino, on the other hand, vanishes automatically and this situation cannot be altered by the existence of any type of interaction.

The longitudinal neutrino concept appreciably reduces the possible number of types of weak interaction operators. Consider, for example, the decay of a μ-meson into an electron and two neutrinos. In the usual manner we represent the interaction operator as the product of operators consisting of μ-meson and electron ψ-operators on the one hand and ψ-operators of the two neutrinos on the other. For the longitudinal neutrino only one

combination can be made from the two neutrino operators (a scalar with respect to rotation; the operation of ordinary inversion is not applicable), as it is well known that the tensor combination of two identical operators obeying Fermi statistics is equal to zero. In this case two combinations, scalar and pseudo-scalar (in the usual sense of the word), can be constructed for a μ-meson and electron.

If a neutrino and anti-neutrino are emitted in μ-meson decay the situation changes. Only a four-dimensional vector can then be constructed from the longitudinal neutrino and antineutrino operators. In this case two combinations—vector and pseudo-vector—can be made from the μ-meson and electron operators. Thus, despite the absence of invariance with respect to inversion in each of the two cases only two interaction operators are possible.

It is easy to calculate the energy spectrum of the μ-meson decay electrons. It is found to be exactly the same as that calculated by Michel[6]. The two-neutrino case thus yields for Michel's constant, ϱ, the value $\varrho = 0$ and for a neutrino and antineutrino $\varrho = 0{\cdot}75$. The former case is apparently inconsistent with the experiments whereas the latter agrees with the results obtained in refs. 7 and 8 which yield $\varrho = 0{\cdot}64 \pm 0{\cdot}10$ and $\varrho = 0{\cdot}57 \pm 0{\cdot}14$. Thus μ meson disintegration experiments do not contradict the longitudinal neutrino concept and in this case lead to a unique result, namely, that a neutrino and antineutrino are involved in μ-meson decay.

Consider now the reaction $\pi \to \mu + \nu$. Since the π-meson is spinless we are obliged to set up a scalar expression for the μ-meson and neutrino ψ-operators in the $\pi \to \mu + \nu$ decay operator. This automatically yields that if the neutrino is longitudinal the μ-mesons produced in $\pi \to \mu + \nu$ decays will be completely polarised in the direction of their motion (or in the opposite direction).

As Lee and Yang[5] have noted, a possible consequence of non-conservation of parity is the correlation between the directions of the μ-meson and electron involved in the $\pi \to \mu \to$ e decay. Simple

calculations based on our scheme give the following energy and angular distribution for the emitted electrons:

$$\frac{dN}{N} = 2\epsilon^2[(3 - 2\epsilon) + \lambda \cos \vartheta(2\epsilon - 1)]\, d\epsilon. \qquad (1)$$

Here ϵ is the ratio of electron energy to the largest possible energy, ϑ is the angle between the directions of motion of the μ-meson and electron and λ is a constant which depends on the relation between the vector and pseudo-vector parts in the combination of the μ-meson and electron ψ-operators,

$$\lambda = \frac{2ab}{a^2 + b^2}, \qquad (2)$$

where a and b are coefficients of the respective terms and, according to the foregoing, are real. Evidently, λ varies between -1 and $+1$. It is possible that λ is in fact equal to zero. The integral electron distribution is obviously proportional to $(1 + \frac{1}{3}\lambda \cos \vartheta)$ and this means that the largest possible value of the forward–backward asymmetry is 2. It should be noted that even if λ appreciably differs from zero it may be difficult to observe $\mu \rightarrow$ e correlation because of depolarisation of the slowed down mesons and in particular for μ^+-mesons because of formation of mesonium ($\mu^+ + e^-$ system).

Consider now the effect of longitudinality of the neutrino on β-decay. According to experiment the decay operator should be represented as the sum of the scalar and tensor variants. It can be shown that in either case the same electron polarisation in the direction of motion will arise, which is equal to v/c (or $-v/c$), the ratio of the electron velocity to that of light. Thus high energy electrons will be totally polarised in the direction of their motion.

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