$$\Delta p\,\Delta x \geq \frac{1}{2}\hbar$$

$$S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \quad S_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$$

$$\Delta E\,\Delta t \geq \frac{1}{2}\hbar \quad S_z = \frac{\hbar}{2}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

Probing the Meaning of

Quantum Mechanics

Physical, Philosophical and Logical Perspectives

$$H(t)\,|\,\psi(t)\rangle = i\hbar\frac{d}{dt}\,|\,\psi(t)\rangle$$

$$\int_{-\infty}^{\infty} dx\,|\Psi(x,t)|^2 = 1$$

Editors

Diederik Aerts

Sven Aerts

Christian de Ronde

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Quantum Mechanics
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Proceedings of the Young Quantum Meetings
CLEA, Vrije Universiteit Brussel — 8 – 9 October, 2009

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PROBING THE MEANING OF QUANTUM MECHANICS
Physical, Philosophical and Logical Perspectives
Proceedings of the Young Quantum Meetings

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DEDICATION

We want to dedicate this volume to the memory of Bart D'Hooghe, our dear friend and collaborator, who is an author of one of the articles, and who died suddenly in August 2012. Bart, born on July 7th, 1971, was theoretical physicists and coordinator of the interdisciplinary Center Leo Apostel. Those who have known Bart personally have experienced him as an exceptionally friendly, helpful and warm-hearted person. And those had the luxury to collaborate with him, know how extremely intelligent and talented he was as a researcher. Bart was also someone who played a very binding and unifying role in the group at the Center Leo Apostel, always ready to help out where needed or to join in the brainstorm necessary to solve an ongoing problem. Bart was gentle and full of sweet life force, and often his spiritual and dry humor, never hurtful but always clever, brought good-natured laughter during stressful joint work for deadlines. He was also the person who always had an eye for special details overlooked by others, so that his last review of a project or article text, was our warranty for perfection. His departure is a great loss for the Center Leo Apostel. We miss him very much.

Diederik Aerts, Sven Aerts & Christian de Ronde

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The gatherings that took place in Brussels since 2004 were possible mainly because of the support of the *Centre Leo Apostel* and in particular his director, Diederik Aerts, who has always encouraged the members of the centre to engage in many activities which exceed the exclusive writing of papers, implying our compromise to reflect about the world and try to change it.

As one of the organizers of the *Young Quantum Meetings*, I would like to thank the participation in the gatherings of some researchers who though, by different reasons, were not able to contribute a paper in this book, are still present in it: Karin Verlest, Michiel Seevinck, Sonja Smets and Giovanni Valente. I would also like to thank the support of Prof. Dr. Miklos Rédei, Prof. Dr. Dennis Dieks and Dr. Michel Bitbol. Last but not least our colleagues at Centre Leo Apostel who were always happy to help during our meetings.

Buenos Aires, December 10th, 2013
Christian de Ronde

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PREFACE

Quantum theory was born in the year 1900 when Max Planck found an answer to the well-known "black body problem." He did so by postulating a discrete character in the function which governed the emission of radiation of a black body. This bold step would eventually transform the knowledge and description of nature in an unimaginable way, transcending the limits of classical thought. Quantum mechanics is perhaps our best confirmed physical theory. However, in spite of its great empirical effectiveness and the subsequent technological developments that it gave rise to in the 20th century, from the interpretation of the periodic table of elements to CD players, holograms and quantum state teleportation, it stands even today without a universally accepted interpretation. Despite the fact that the majority of physicists are self-proclaimed subscribers to the so called Copenhagen interpretation, there is no such majority among researchers working in the foundations of quantum physics. The problem of finding a viable interpretation of quantum mechanics has long been sidestepped as being "misguided." Yet, no matter how authoritative the voice proclaiming the irrelevance of such an undertaking, the number of people drawn towards this problem has only increased over time. The new born theory of quantum information together with the persistent problems in quantum cosmology have in no small way contributed to a renewed interest in the problem. Much progress has been made in the issue of interpreting quantum theory since the 1927 Solvay congress in which, as John Bell put it, Niels Bohr managed to convince everybody the job was done. We now know that quantum mechanics does not allow for a local-realistic interpretation; that not all observables may be attributed definite values simultaneously, that single unknown states cannot be cloned, but may be teleported. Yet, despite all the efforts, no single interpretational framework has emerged that provides consistent answers to the many questions that are raised when we attempt to understand the meaning and implications of quantum theory. This is why the phrase of Richard Feynman remains still today an accurate description of the state of affairs in the field: *"I think it is safe to say that no one understands quantum mechanics."*[1]

Convinced that this should not stop us from trying, a small but enthusiast group of young researchers working in the foundations of quantum physics started to gather on a regular basis to discuss interpretational issues. The idea was to provide an agreeable context for discussion following the European tradition of Natural Philosophy, which goes back to Kepler, Newton, Leibniz and Kant, continues to Ernst Mach, Henri Poincare and Hendrik Lorentz and which was followed by the main characters of the quantum revolution: Max Planck, Albert Einstein, Niels Bohr, Max Born, Louis de Broglie, Werner Heisenberg, Erwin Schrödinger, Wolfgang Pauli, Pascual Jordan and many others.

Every year since 2004 and up to the present, young quantum physicists and philosophers from Europe (Belgium, France, Germany, Italy, the Netherlands, Spain, Hungary and Greece) and South America (Argentina and Brasil) meet in Brussels under the auspices of the Center Leo Apostel (CLEA) to discuss and go still further in their attempt to *understand* one of the most beautiful theories ever created by man.[2] The meetings were called *Young Quantum Meetings*. Even though no member of this group seems convinced that a simple framework will explain it all, they all shared the idea that progress is indeed possible through constructive *dialogue* between the various critical proponents of the existing interpretations. As Bohr put it, quoted in Heisenberg's autobiography, *Der Teil und das Ganze*, "It is only when we talk without rest with different concepts about the marvelous relations between the formal laws of quantum theory and the observed phenomena, that these relations become illuminated in all their aspects. Their apparent inherent contradictions acquire strength in our consciousness, and it is possible to transform the structure of thought, which is a necessary presupposition to understand quantum theory."

Each member of the group presented ideas concerning the interpretation of quantum mechanics. We had discussions ranging from the philosophical underpinnings of local realism and holism, information and decision theoretic approaches to quantum theory all the way to the many-worlds interpretation. Strikingly, in much the same way as different, and indeed incompatible observations are needed to fully describe the physical state of

[1] R.P. Feynman (1967, p. 129), "The character of physical law."
[2] YQM: http://www.vub.ac.be/CLEA/pub/yqm/ and
Workshops: http://www.vub.ac.be/CLEA/workshop/index.shtml.

affairs in quantum mechanics, the various interpretations of the theory also seem to shed viable, but not necessarily compatible, perspectives on different aspects of the same grand framework. The discussions that followed were both technical and lively, but perhaps their most remarkable quality, was the absence of rigid points of view that unfortunately seems to paralyze so much of the discussion in this area. It was decided in one of these meetings we should make an attempt to crystallize some of the essence of these gatherings in a published volume. One decade after many of us, PhD students at the time have found permanent positions in Europe, South and North America proving our research valuable to the international academic community. We feel that, among the enormous amount of research that is being published these days, this volume could stand out in two ways:

(1) The constitution of the group was mainly PhD students in Europe — who by now have found permanent positions in Europe, North and South America — working in the physics, philosophy and logic of quantum theory. The group, though young, is technically skilled both in the formalism as well as in the traditional and contemporary philosophical discussions regarding the interpretation of quantum mechanics. It is such a constitution which can provide the conditions for a "fresh look" at the field of foundations of quantum mechanics.

(2) Quantum mechanics is simply fascinating and remains even today an open problem for those who wish to seek for answers. Within the book there are different paths which are an expression of the personal aspect in this quest. It is all these different views which give richness to the dialogue that is being produced not only in the articles but also in the discussions. The book attempts to be a reflection not only of this dialogue, but also of our personal choices and believes.

In conclusion, the book is a single *unity*, as it is directed by "seeking understanding of quantum mechanics," but it is also *wide and diverse* in scope of topics and personal in choice and motivation of the topics handled. We believe that this is what makes the enterprise unique.

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CONTENTS

DO QUANTUM DICE REMEMBER?

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We shall present certain experiments aimed at testing the Markovian nature of the quantum statistical distributions and comment their results, which confirmed the standard quantum interpretation. We shall also show how certain sophisticated experiments that were realized in the framework of quantum optics during the last decade in order to test fundamental effects such as quantum non-locality also lead us to eliminate certain (non-Markovian and non-local) alternatives to the standard quantum theory.

Keywords: Markovian; memory; quantum.

0. Introduction

It is commonly accepted that the fundamental laws of nature are Markovian in the sense that the future of a physical system depends on its present state (and of the influences occuring in the future) only. In contrast to biological systems, it seems that material systems "do not remember".

Historically it was not always universally taken for granted that the future of a system depended on its state at only a time. Variational principles allow us to believe that systems "choose" the optimal future among all possible futures, a vision shared by Leibnitz and Maupertuis for instance. In principle it is possible to reformulate the dynamics of many physical systems with two constraints on the boundaries (one in the past and one in the future), even in the quantum regime. If we think to the emergence in time of an interference pattern, when particles arrive one by one, the temptation is strong to believe in the existence of a preestablished program (the interferometric pattern) that the particles would obey. This "finalist" vision of the world has disappeared from modern physics since it was realised that variational principles are equivalent to local evolution laws. Nevertheless it is still present in biology, in particular for what concerns the theory of evolution.

Now, L. de Broglie wrote in[14] "Nouvelles Perspectives en Microphysique" that

"L'histoire des Sciences montre que les progrès de la Science ont constamment été entravés par l'influence tyrannique de certaines conceptions que l'on avait fini par considérer comme des dogmes. Pour cette raison, il convient de soumettre périodiquement à un examen très approfondi les principes que l'on a fini par admettre sans plus les discuter."

In other words, commonly accepted truths should be questioned too. Therefore it is interesting to investigate to which extent fundamental laws of nature (here we have in mind quantum mechanics) are Markovian theories.

Beside these very general philosophical motivations, it is also interesting to study whether quantum probabilities can be simulated by Markovian (thus without memory) processes in the framework of standard quantum mechanics itself. This question has been addressed in several publications during the last decade.[23,24,41] More specifically, non-Markovian effects were predicted in the framework of stochastic quantum mechanics. Essentially, when one modellizes the interaction between a genuine quantum system (an excited atom that behaves as a radiating dipole for instance) and its surroundings (the electro-magnetic field treated as a heat bath for instance), memory effects are likely to occur, simply due to the temporal inertia of the quantities that characterise the physical state of the surroundings.[15,43]

More pragmatically, it is worth considering in detail the role played by measuring apparatuses in quantum experiments. Usually, one minimizes the influence of the measurement process by considering it as an instantaneous, memory-free process. Obviously, this is nothing else than a convenient idealisation, because every measurement requires a macroscopic amplification process that is never instantaneous. One should always be aware of such effects because any collection of experimental data is necessarily realised through a detection device. From this point of view, it is interesting and useful to analyse experimental data with the goal of obtaining information about the measurement process itself. This is true for instance for all tomographic[22] experiments that were realised during the last decade in order to determine the density matrix of identically prepared quantum systems (light, trapped ions, atoms[39] and so on). The data obtained during this kind of experiments can provide us information about the state of the system as it is produced by the source, or about the evolution undergone between the source and the detector (inside an interferometer for instance), but we should not forget that the functioning of the measuring apparatus also plays a role in the elaboration of the data.

The experiments described in the present work can be considered as attempts, in very specific situations, to quantify the departure from the Markovian paradigm.

The paper is organised as follows. In Section 1 we show that the property that we call Markovian or memory-free is in a sense already present at the level of classical (non-relativistic as well as relativistic) physics. Actually, whenever a physical system is described by integro-differential equations, its future is determined by the initial data, that is, by the present. That this present is "now" or "here and now" depending on the description that we give of the system (Newtonian or Einsteinian) does not alter the fact that the knowledge of only the present is enough in order to determine the future. In the same section we also show that quantum unitary evolutions are compatible with the "now" picture (in the case of Schrödinger's equation) and with the "here and now" picture (in the case of Dirac's equation or in quantum field theory). We also explain how, even in the case of non-unitary evolution processes (like the collapse process), a peaceful coexistence is possible between both pictures (which is close to Poincaré's views on relativity[36]).

In Section 2 we describe measurements aimed at testing the Markovian nature of quantum probabilities in the non-relativistic regime, that were realized in Paris-Nord. We describe certain statistical tests that were realized by us recently on the basis of the data accumulated during these experiments. Associated plots can be found in Appendix.

In Section 3 we describe what could be called quantum Michelson-Morley experiments in the sense that they are aimed at revealing an hypothetical "quantum etheric memory" (in other words non-local hidden memory effects).

All these experiments confirmed standard quantum predictions (in the sense that they bring us to conclude to the non-existence of measurable "hidden" mesoscopic or macroscopic transient memory effect).

1. The Markovian Paradigm: From Newton to Einstein and Beyond

1.1. *About relativistic and non-relativistic memory times in classical physics: Newton versus Einstein*

In the Newtonian formulation of the principles of mechanics, space and time were considered to be independent entities. Time was assumed to flow uniformly everywhere and gravitational influences were assumed to propagate instantaneously through space (action at a distance). One of the most remarkable features of special relativity is that it unifies space and time in one description. Absolute simultaneity disappears but the relativistic invariance of the speed of light makes it possible to give a new sense to the concept of causality. Mutual influences are now assumed to propagate with, at most, the speed of light. The relativistically invariant light cones make it possible to establish a clear topological distinction between events that could influence each other (they are separated by a timelike or null distance) and the other ones (separated by a spacelike distance). Events correspond to elementary physical processes (absorbtion or emission of light by an atom, passage of an atom at a given time and place). One assumes usually that the spatio-temporal extension of the events is sufficiently small (atoms are microscopic, absorption or emission of light is a nearly instantaneous process) so that they can be considered to be pointlike. Furthermore, in the special relativistic formulation of the supposedly fundamental laws (this is the case for instance with Maxwell equations), the variation of physical quantities (locally defined fields and/or positions of pointlike particles) in some place is assumed to depend only on the values of these quantities at that place because evolution laws are continuous and described by local differential equations. This does not mean that quantities at one place cannot influence another, because these influences could well propagate from place to place, but the nature of the evolution laws is such that this propagation never occurs faster than light. This makes it possible to reduce the description of the universe to the description of events, that can be considered to be the ultimate constitutive elements of the physical world. One should note that such a Cartesian reduction of the whole to its constitutive parts also occurs when matter is described as a collection of atoms (or fundamental particles). These atoms are assumed to be localised in small regions of space, and to be independent in the sense that their trajectories are assumed to obey local laws (the acceleration of Lorentzian electrons for instance depends on the local values of the electric and magnetic fields only). Actually, Einsteinian events of special relativity (SR) viewed as ultimate constitutive parts of space-time and atoms (viewed as point particles) share several properties. Both are assumed to be localized in small regions of space. Although events are furthermore assumed to be localised in time, while particles exhibit some persistence in time, this persistence is generally considered to reflect merely the validity of conservation laws (of mass-energy, charge...) that in turn reveal the existence of symmetries at a fundamental level.

At this level, we can already indicate a fundamental difference between Newtonian and Einsteinian mechanics: in the former, time is an external and absolute parameter, and action at a distance is possible; in the latter, space and time are not independent and influences may not propagate faster than light. This difference is manifest at the level of the Cauchy problem: in the Newtonian approach, the evolution in present time of local physical quantities is conditioned by the past history of the whole universe; in special relativity it depends only on the interior and on the surface of the backwards light cone.

In either case the future of the world is contained in its initial conditions. Moreover, in each approach, fundamental laws are reversible under an inversion of the time arrow. Now, our everyday experience shows that such an arrow exists, and that time flows along one direction only. It is possible to express consistently the difference between past and future in each formalism and to formulate unambiguously the concept of causality: the future cannot influence the past. In fact, one can say more, at a fundamental level, the present is, roughly speaking, influenced by the present only. Indeed, Newtonian gravitation is instantaneous, while, in virtue of the Huygens principle, electro-magnetic influences propagate at the surface of the light cone which is at the edge of past and future and, considered so, can be viewed as the present in a relativistic description of space-time. Here again, our everyday experience shows that complex systems are more sophisticated: they remember the past, and are also influenced by past events. In biology and human sciences in general (psychology, social sciences and so on), the past history of the system is essential. In the framework of these disciplines, one can simply not neglect the past influences accumulated by the system under study if one wants to analyze, understand or simply describe its properties. It is then not sufficient (at first sight) to know the initial conditions of a system in order to determine its future; the knowledge of its global history is necessary. In fundamental physics, strong correlations also exist between temporally successive states of the same system. This persistence reflects the inertia of the physical quantities which in turn is due to the continuous nature of evolution laws and/or to certain conservation laws. No global memory effect is present at a fundamental level. Even the conservation laws can be shown to reflect merely the existence of some fundamental symmetries. It is worth noting that even when physicists study irreversible and dissipative systems, they usually assume that the present is influenced by the present only. For instance, in the physics of dissipative systems, the stochastic processes that are usually invoked in order to establish the (irreversible under time inversion) evolution laws of such systems on the basis of fundamental (reversible) laws are always assumed to be Markovian (no memory effect). Although such memory effects are likely to occur, one considers in general that they are due essentially to the temporal persistence (inertia) of the subparts of the system under study, and that they rapidly vanish after a transient time short in comparison to the typical evolution time of the system. To conclude, the Newtonian and Einsteinian formulations of physical laws give us the image of a world in which physical influences occur "now" (in the Newtonian approach) , and "here and now" (in the Einsteinian one).

1.2. *About relativistic and non-relativistic memory times in quantum physics*

It is interesting to analyze the relevance of these ideas in the light of the teaching of quantum mechanics. Curiously, quantum mechanics offers, as we shall now show, a combination of Newtonian and Einsteinian features that presents strong analogies with Poincaré's views on relativity[36] as we shall discuss further. In quantum mechanics, the state of a system (expressed by a wave-function in the most simple cases) undergoes two distinct types of temporal evolutions. When no measurement is performed, the state of the system obeys a Schrödinger-like equation of evolution that is unitary, deterministic and continuous in time. It is possible to express such a law in a relativistically covariant form (this is the case for instance with Dirac's equation that describes the evolution of the wave-function of one electron in the presence of external potentials). When a measurement is performed, the system is assumed to perform a sudden transition (quantum jump or collapse of the wave-function) that is supposedly instantaneous in time, non-deterministic and non-unitary. Note that at the present time we do not know exactly when one or the other regime of evolution will be followed. In other words, we cannot define exactly what is the border line between a (supposedly classical) measurement device and a quantum system.[6] It is not our purpose here to devote much time to this question that lies at the core of the measurement problem and has motivated several interpretations of quantum mechanics. All that we need to say at this level is that some of the proposals for solving the measurement problem (or interpretative schemes of quantum mechanics) are mutually exclusive and can be discriminated experimentally as we shall show in what follows. Let us firstly consider what does happen in the absence of measurement. The Dirac equation, which provides the perfect paradigm of a quantum relativistically covariant evolution law is, like Maxwell's equations, a partial differential equation, local and continuous in space and time. Relevant physical quantities such as the charge density and the charge current vector behave on average as relativistically covariant quantities (as a 4-vector in this case) that obey local evolution laws. From this point of view, the Dirac equation describes a physics of the "here and now". In quantum field theory, the situation is more complex but relativistic causality is still guaranteed in the sense that the quantum propagators that express the influence due to local fluctuations of the quantum fields can be shown to have a null effect at the level of events that are not causally related to these fluctuations, that is, of events that lie outside the causal cone. Quite naturally, we have that in the non-relativistic limit, the Schrödinger equation describes a physics of the "now". Remark that it is impossible to consider the quantum particle as a material point endowed with a well defined trajectory because the average values of the measurable observables associated to the system will in general depend, as in the two-slit experiment, or in the Bohm-Aharonov experiment on the values of the disturbances (phase shifts, potentials, and so on) that correspond to different, mutually disconnected from a causal point of view, trajectories or classical histories. The best illustration of this property can be found in the numerous examples of delayed choice experiments that were realised during the last decades. From a theoretical point of view, one can better understand the puzzling nature of these non-local properties by considering the Feynman's formulation of quantum mechanics which assigns an amplitude to every possible path and sums all these amplitudes in order to obtain the quantum amplitude of transition.

Whenever measurement occurs, the tension between special relativity and quantum mechanics becomes more acute because the collapse process is generally assumed to be instantaneous, and instantaneous influences (actions at a distance) are forbidden in special relativity in the sense that when the events A and B are separated by a spacelike distance, A could belong to the future of B or vice versa depending on which is the inertial frame of reference from which they are observed. A physical retroaction is thus possible whenever we assume that action at a distance and special relativistic transformation laws are mutually valid. Furthermore, we do not know relative to which frame this action at a distance occurs. Fortunately, quantum mechanics itself allows us to discharge partially this tension in the sense that whenever we assume such a frame of reference to exist and that we compute the average values of physical observables, it can be shown that, although the collapse process instantaneously propagates correlations between the results of measurements performed in distant regions, it does not change the average mean values of local observables. More technically, this is expressed by the fact that the reduced density matrix in a finite region A (that contains all the statistical information regarding measurements performed in A) does not change, in average, when a measurement is performed in a distant (finite) region B. In this sense one commonly talks in the literature about a present state of peaceful coexistence between quantum mechanics and special relativity. In other words, the action at a distance that occurs during the collapse is very peculiar in the sense that its influence is null in average. Besides, we do not control quantum stochasticity, and information can only be encrypted through averaged values of quantum observables, but these values cannot be influenced at a distance so that no supraluminal transmission of information is possible in quantum mechanics (a property called the no-signalling condition) . It is worth noting that such a description is closer to Poincaré's interpretation of the theory of special relativity than to Einstein's one. Summarizing, one could say that Einstein considered that no ether exists, although Poincaré and also Lorentz considered that such an ether exists but that it is not possible to reveal its existence because whenever we try to do so, mechanisms of compensation (the Lorentz contraction) occur in such a way that we cannot distinguish the "true" and "absolute" frame of reference from the other inertial frames.[6,36] Similarly, a peaceful coexistence between special relativity and quantum mechanics is maintained because mechanisms of compensation annihilate in average the non-local influence that is assumed to occur during the quantum collapse. At this level, three attitudes are possible.

(i) One could argue that, because the collapse process has no directly measurable effect, one can consistently forget its existence and do as if the problem did not exist. Similarly, Einstein built the theory of special relativity by postulating that no ether (or preferred, absolute and true frame of reference) existed. This pragmatic interpretation is standard and is shared by the majority of physicists.[8,9]

(ii) One could negate the validity of the formalism of quantum mechanics, precisely because it requires such supraluminal influences (for well chosen quantum states, the entangled states, the correlations do not vanish when the distance increases-considered so, quantum action at a distance is more disturbing than Newtonian gravitational forces or than Coulombian electrical forces). The correlations would then preexist prior to the measurement, and the quantum theory would be incomplete because it does not include the description of such preexisting correlations. This was EPR's attitude,[21,38] an attitude that was later defined under the words "local realism". The tension between local realism and quantum mechanics reached an ultimate paroxysm when John Bell[5]showed that EPR's local realistic view is contradicted by experimental predictions of quantum mechanics. Since then, these predictions have been confirmed in several experiments which can be considered as a clear experimental condemnation of local realism.[1]

(iii) One could choose to believe that "something propagates faster than light" in some frame of reference, an attitude that could be qualified by the words "non-local realism".

In the next sections, we shall comment on some models, experimental proposals and experimental realisations aimed at testing the extent of validity of the general properties that we have been discussing so far. Some of them deal with the question of the existence of non-standard memory effects in quantum mechanics. They aim at testing whether quantum systems only live in the "now" or whether they "remember".[17] The others are the quantum analogous of the Michelson-Morley experiment in the sense that they aim at revealing the existence of a "wind of ether" during the collapse process. They aim at discriminating between the standard interpretation and non-local realistic interpretations.

2. Do Quantum Dice Remember? — Experimental Tests

As it was pointed out before, physicists are not able presently to define exactly what is the border line between a (supposedly classical) measurement device and a quantum system. This kind of problem constitutes the so-called measurement problem, which motivated numerous alternative interpretations, among which it is worthwhile distinguishing the hidden variables approach.[4] The basic hypothesis made in hidden variable theories is the existence of "extra" degrees of freedom, the so-called hidden variables, which, if they were accurately specified, would allow us to predetermine exactly the result of any kind of measurement performed on a quantum system. In analogy with classical statistical mechanics, the probabilistic features of the quantum theory would then be explainable as a consequence of the statistical distribution of the hidden variables. Viewed in this way, the hidden variable approach is not totally satisfying, because it does not explain the origin of the probabilistic distribution of the hidden variables. To provide a serious alternative to the standard interpretation, it is necessary to explain the emergence of the probability distribution of the hidden variables by a convenient randomization process. If we pursue the analogy with classical statistical mechanics, where several attempts were made (in ergodic theory for instance) in order to describe such processes, we come to the conclusion that a major difference must exist between hidden variable theories and the standard quantum theory: in the first approach, the average distribution is not instantaneously reached, an effect that we could measure through the careful observation of the temporal statistics of quantum phenomena. In the standard quantum theory, the measurement process is usually considered to be instantaneous, an hypothesis that is indirectly confirmed by several experimental tests. Until now, few attempts were made in order to test whether successive quantum measurements are really independent, Markovian (memory-free) processes. In a recent paper,[19] we described three of them that were motivated by the hypothesis of hidden variables ((i) the Papaliolos experiment[35] performed in order to test Bohm-Bub's model,[7] (ii) the Summhammer experiment[11,45] performed in order to test Buonomano's model[10] and (iii) an experiment realised in 1999 and 2000 in Paris-Nord in order to test the possibility of a memory effect at the level of the measuring apparatus[3]). In Section 2.1 and in Appendix (Section 4), we shall describe in detail certain statistical tests that were recently performed on the data accumulated in the interferometer of Paris-Nord.

Finally (Section 2.2), we shall mention certain recent tests aimed at checking the true random nature of the quantum signal that were performed in the framework of quantum information science[20] and quantum cryptography.[26]

2.1. *The Paris-Nord experiment*

The Paris-Nord experiment was realised by sending 2.6 million of atoms successively into an atomic interferometer and establishing the statistical distribution of the delay times (times series) between them. The device consists of a source that emits metastable excited atoms.[2] These atoms are collimated and arrive in the region of detection in which an electric field induces a stimulated decay of the excited atomic states that is accompanied by the emission of a photon. This photon is then detected through photoelectric effect in a channel electron amplifier (channeltron).

2.1.1. *Departures from the Poissonian paradigm*

It could happen that successive detections are no longer independent, for one or another reason, in which case departures from the Poissonian distribution could be observed. A very convenient parameter for measuring the departure from the Poissonian statistics is provided[30] by the normalised root mean square deviation g, that is, the ratio between the root mean square deviation and the mean value of the distribution of the time-delays between the detections of two successive photons. This parameter was measured several times in different contexts in quantum optics and delivered precise information. When no memory effect is present, the distribution is Poissonian and this parameter g is equal to unity. When detections tend to arrive together, g gets larger than 1 (bunching[27]), and when a detection is followed by a dead time, it gets smaller than 1 (antibunching[29]). In other papers we described how the existence of short range memory effects inside the detector itself could also be revealed by the departure of g from unity.[3] We evaluated $g_{\overline{N}}$ after having accumulated $N = 2.6 \cdot 10^6$ measured values of the time delay between successive detections. We get that $g_{\overline{N}} = 1.0001$. According to the law of large numbers, the extent of the interval of values of $g_{\overline{N}}$ centered around 1 which ought to be occupied in 95% of the cases is equal to $\sqrt{\frac{1{,}96}{N}} \cdot \sigma = \sqrt{\frac{1{,}96}{N}} \cdot 6.83$ so to say, to $5.93 \cdot 10^{-3}$ when $N = 2.6 \cdot 10^6$, in agreement with the observed value (as shown in Fig. 1 in Appendix). Now, it could occur that $\frac{\tau}{T}$ is not equal to zero but is so small that the departure from 1 of g_N belongs to the margin of statistical fluctuations. This is possible provided $\tau < 5.93 \cdot 10^{-3} \cdot T$. The average time between two detections being equal to 0.5 ms, the conclusion of this analysis was thus that, even if some memory time (or eventually some dead-time) characterises the detection process, such a time may not exceed $5.93 \cdot 10^{-3} \cdot 0.5$ ms, that is, $\sim 3 \cdot 10^{-6}$ seconds. Our analysis confirmed similar results previously obtained with electrons[46] and neutrons.[37] All these experiments confirm that the hypothesis according to which the detection process is Markovian is valid in the sense that no departure from the Poissonian distribution was observed until now in the temporal distribution of "clicks" obtained during the detection of interferometric properties of massive quantum systems such as electrons, neutrons and atoms. In the rest of this section, we shall present the results of other types of statistical tests that we implemented in order to test the Markovian nature of the statistical data accumulated in Paris-Nord interferometer.

2.1.2. *Law of large numbers*

Another way of observing the validity of the law of large numbers is to estimate the dispersion of the mean detection time for large samples of length N. We did it for hundred successive samples of length equal to 1000. Their dispersion is obviously in accordance with the law of large numbers (that is they are distributed around the mean value within an interval of breadth comparable to $\frac{\sigma}{\sqrt{N}}$, 494 in this case) as shown in the Fig. 1. We also confirmed that the variance of a sample of length N obeys the law in 1 over N by computing successive values of it for samples of lengths that varied from 8000 to 100000 by steps 1000. The agreement with the law of large numbers was obvious as shown in the Fig. 2.

2.1.3. *The Hurst parameter*

In the previous reasonings, we made use of the law of large numbers. This law deals with independently distributed values. It is not valid in principle when memory effects are present because then successively measured variables are no longer independently distributed. Now, it is easy to show that we can consistently make use of the law of large numbers even when a memory effect is present provided it is a short range memory effect, a condition that we imposed by requiring the constraint $\tau < T$ to be fulfilled from the beginning. Considered so, the measurement of g does not reveal anything about the existence of long range memory effects. Remarkably, the existence of long range memory effects can be revealed by another parameter, the Hurst parameter. This parameter was initially applied by Hurst when he measured the fluctuations of the level of the river Nile. It is equal to the difference between the maximum and the minimum of the cumulated sum of a normalised sample divided by a normalisation factor which is equal to the squared root of the product of the variance of the distribution with the length of the sample: $H_N = \frac{Max_{j=1...N}(\Sigma_{i=1}^{j}(t_i - t_{\overline{N}})) - Min_{j=1...N}(\Sigma_{i=1}^{j}(t_i - t_{\overline{N}}))}{\sigma \cdot \sqrt{N}}$. It can be shown that Hurst's parameter must be of the order of unity for large values of N in the case of a Markovian probability model (or eventually in the case of a non-Markovian model with short memory times comparatively to the length of the samples used in the estimation of the Hurst parameter). In the case of the river Nile, the study of Hurst's parameter significantly differed from unity, which revealed that no Markovian model and even no non-Markovian model with short range memory effects could explain the observed fluctuations. In our case, Hurst's parameter is of the order of unity, so that nothing allows us to conclude that long-range correlation effects exist inside the signal. In the Fig. 3, we represent the evolution of the Hurst parameter when N goes from 1000 to 10000 by steps 1000. We also compared the behavior of the cumulated sum of the time delays with the cumulated sum obtained from the computer's generator of pseudo-random numbers. They are shown in Figs. 4 and 5. Obviously, these figures are qualitatively the same.

The fluctuations of the level of the river Nile exhibited another effect called the Joseph effect that consists of persistence in the time-series. The word Joseph refers to the bible where the appearance of a periodicity of 6 years in dry and fertile years is mentioned. Such an effect occurs when the sign of the departure from the average value of the time-delay persists over M successive measures ($M = 1, 2, ...$). The strength of this effect is measured by the persistence parameter which is equal, for a sample that contains N delay-times, to $\frac{1}{N-2M+1} \cdot \{\Sigma_{j=M...N-M}(\frac{1}{M \cdot \sigma^2}(\Sigma_{i=j-M+1}^{j}(t_i - t_{\overline{N}})) \cdot (\Sigma_{i=j+1}^{j+M}(t_i - t_{\overline{N}}))\}$. When the persistence parameter is positive (negative), persistence (anti-persistence) occurs. When the signal is Markovian, the persistence parameter does not significatively differ from zero.[a] In Fig. 6 shows the value of this parameter for successive values of M between 0 and 80, for a sample of length $N = 10000$. The observations are in agreement with the predictions that one makes for white noise using the law of large numbers. They do not allow us to infer that significant persistent or anti-persistent correlations would exist between successive detection times. We also investigated the auto-correlation function of order M of the signal that consists of the in-product between the series of time delays and the same series shifted by an amount $M - 1$, normalised by the in-product of the series of time-delays with itself. In Fig. 7 represents this auto-correlation function when N varies from 2 to 20, when the sample contains 100000 time-delays. It does not differ significatively from zero.

2.2. *Tests of randomness of quantum random number generators*

Finally, it is worth noting that tests of randomness were also realised in the framework of quantum information science. The ultimate goal was to test whether quantum random number generators were really random in order to create a quantum random number generator.[20,20,26,28,42,49]

Such statistical tests were performed on time series extracted from single-atom quantum telegraphs and also from polarised photons. The pur-

[a]Note that the first satisfying model that could simulate the fluctuations of the river Nile, peculiarly the non-convergence of the Hurst parameter and the persistence of the fluctuations as well was a model of fractal[31] noise. Incidentally, our observations indirectly confirm Nottale's hypothesis[34] that the fractal dimension of quantum fluctuations is 2, which corresponds to a Markovian, memory-free, behavior.

poses of the experimentators were more pragmatic, they aimed at establishing the independence of the data obtained from a quantum random generator in order to commercialise it. A battery of statistical tests[49] was performed onto the data obtained from various quantum systems. Some of them are the same tests that make it possible to distinguish a true random signal from a pseudo-random one. They all confirmed the independence of successive measurements.[20]

3. Test of the Existence of "Non-Local Memory Times"

As we previously mentioned, the existence of an absolute frame of reference relative to which all the collapse processes would occur instantaneously is compatible with all experimental data collected till now. This is related to the fact that the time ordering of two observables performed on distant components of an entangled system does not influence the statistics of the results. Nevertheless, this peaceful coexistence is very fragile, and, provided we formulate extra-hypotheses relative to the collapse process, one could predict the appearance of non-standard effects that would in fact reveal the existence and the properties of this hypothetical quantum ether. The experimental quest of such effects would present strong analogies with the famous Michelson-Morley experiments in the sense that they are aimed at revealing the existence of a "quantum ether wind". We shall now briefly present an experiment proposed by us in the past in order to detect non-local memory times.[16,18] Then we shall present an experiment, realised in Geneva by Gisin's group,[50] that tests an hypothesis formulated by Suarez and Scarani in 1997.[44]

3.1. *Memory at a distance: One absolute time*

The Bohm-Bub hidden variable theory that we mentioned previously[7] is formulated in Newtonian space-time, and is an attempt to describe the measurement process as a continuous, extended in time, process, which is more physically appealing than the standard quantum jump picture: could an instantaneous process be anything else than a convenient abstraction? Furthermore, Bohm-Bub theory is based on the realistic assumption that hidden variables characterise the measurement process, and it describes a non-unitary dynamics. As can be shown,[16,18] the Bohm-Bub theory makes it possible to break the peaceful coexistence between quantum mechanics and relativity (see also Refs. 32, 47, 48), and, in certain circumstances, allows us to realise a supraluminal telegraph. Furthermore, we showed that these circumstances are very rare and difficult to produce experimentally, which could explain why, if they existed, their non-standard effects were not noticed. Three ingredients present in the theory are essential in order to obtain faster than light propagation of information. They are: (a) the evolution law is non-linear, thus non-unitary (the connections between faster than light transmission of information and non-linear generalisations of Schrödinger's equation were clarified in several articles[13,25]), (b) the whole formalism is not Lorentz invariant, and (c) there are hidden variables which provide "realistic" features to the measurement process through the persistence in time of the values taken by the hidden variables (mesoscopic randomization time) and/or the active role played by these variables which influence directly the dynamics of the wave-function. These ingredients allowed us to show that memory effects and/or interference effects[16,18] between spatially separated measurement processes were likely to occur because there is a short but non-instantaneous collapse process. These effects are expected to occur i) when the measurements performed in the distant regions A and B are simultaneous (in the hypothetical absolute quantum frame) in which case they interfere, or ii) when they are separated by a time shorter than the randomization time of the hidden variables in which case non-standard memory effects are likely to occur. The non-unitarity of the collapse process implies, per se, the possibility of supraluminal transmission of information if we consider the collapse process to be a realistic and thus a process continuous in time that it is described by non-linear equations. In order to test the existence of such effects experimentally, it is necessary to guess correctly which is the absolute frame of reference associated to the collapse process. If one makes the natural hypothesis that it is the frame of reference associated to the center of mass of the regions A and B, then some of the Franson-like interferometric experiments realised by Gisin's team[50] make it possible in principle to test the existence of a collapse time and of a memory time. Although they were not aimed at testing such effects, they could serve to test our hypothesis because during some of these experiments two components of an entangled system are measured nearly simultaneously (relative to the center of mass of the regions A and B) by detectors at rest relative to the ground. These experiments confirmed standard predictions[50] so that one can infer that no non-standard memory effect was present, which can be considered once more as a negative result regarding the existence of hidden variables.

3.2. *Memory at a distance: Different absolute times*

In 1997, Suarez and Scarani proposed[44] that the collapse process associated with the measurement in the region A (B) occurs instantaneously relative to the proper time associated to the detector in the region A (B), an hypothesis dictated by the relativity principle. Then, it is possible in certain circumstances that the collapse process that occured in A (B) is still ignored by the component of the system in B (A) when it is measured in B (A). Discrepancies to the standard correlations are then likely to occur. In order to be able to observe these discrepancies it is necessary that the detector in the region B is sufficiently distant from the detector in the region A and that it moves relative to it. Remark that the paradoxical features of a situation in which entanglement is revealed by two detectors in relative movement was firstly emphasised in Ref. 33. An experiment similar to the one proposed by Suarez and Scarani was performed in Geneva.[50] The entangled state consisted of pairs of light pulses produced by a process of parametric down conversion. They were sent in opposite directions along optical fibers in which entanglement can be preserved over distances of some kilometers. One of the detectors at one side was moving relative to the other one that was bound to the ground. The experimental results confirmed the standard predictions once more, providing thereby a new negative result regarding the existence of an hidden ether. Note that the existence of such an ether did not necessarily imply in the previously described schemes the possibility of supraluminal transmission of information.

4. Conclusions

Experimental violations of Bell like inequalities[1,5] are not the only tests that can be performed in order to discriminate between hidden variable theories and standard quantum mechanics. Well-chosen tests can be realised in order to reveal whether hidden elements of reality with a mesoscospic memory time would exist. Two classes of experiments can be conceived, depending on the local or non-local features of the hidden variables.

So far, all experiments confirmed the standard features of commonly accepted theories such as special relativity and quantum mechanics: the physical world "lives here and now", it does not remember, and supraluminal transmission of information is not possible. Note that strictly speaking the existence of such effects is not absolutely impossible. One could always argue that a memory time exists but that it is too short to be observed experimentally. Therefore, rather than showing definitively the non-existence of hidden times in quantum mechanics, the experiments mentioned in the present paper provide upper bounds on the typical memory times and/ or on the speed of transmission of the collapse process. These constraints are the following:

- the Paris-Nord experiment (with an atomic interferometer) showed that if the detectors of individual photons emitted by atoms at the outcome of the atomic interferometer exhibit a randomization time, this time is certainly less than 1.5 microseconds ($1.5 \cdot 10^{-6}$ s).
- the Gisin experiment (with entangled photons in energy-time Bell states) showed the non-existence of the effect predicted by Suarez and Scarani[44] because this effect was likely to occur within a period equal to the product of the distance between both detection devices times the velocity of the moving one divided by the square of the speed of light. During their experiment, the velocity of the moving detector was equal to 100 m/s and the distance between the detectors was more or less equal to 10 km. This gives a period close to 10 ps (10^{-11}s). The standard quantum correlations were observed for quantum systems whose temporal extension was closer to 5 ps which shows the non-existence of the effect. Undirectly, this experiment shows that if the collapse instantaneously propagates relative to the surface of the earth, a randomization time or collapse time of the type predicted in Refs. 16 and 18 ought to be shorter than 5 ps. It also shows that if the collapse propagates at finite speed relative to the surface of the earth, its speed is certainly superior to 10 km divided by 5 ps, that is, to more or less 7 millions times the speed of light[50]!

We invite the interested reader to consult the references in order to learn more about the experimental set-ups of any of the previously mentioned experiments.[b]

Incidentally, to some extent, this series of results also constituted a negative result[19] for the hypothesis of the shape wave theory of Rupert Sheldrake[40] according to which the evolution of living organisms on the surface of the earth would be accelerated by a kind of cosmic memory that would gather and disseminate, non-locally at the scale of the planet, the

[b]For information, the Papaliolos experiment (with polarized photons[35]) showed[19] that if the polarization of individual photons exhibits a randomization time, this time is certainly less than $2.4 \cdot 10^{-14}$ seconds; the Summhammer experiment (with a neutron interferometer[45]) showed the non-existence of Buonomano's memory effect, independent of its memory time, because this effect was assumed to be a long range effect from the beginning.[19]

learning obtained through the lived experiences of individual organisms. That such a wave exists at the level of living organisms is still an open question. R. Sheldrake[40] also admitted the hypothesis according to which the de Broglie frequency that is equal to the mass-energy of a quantum object divided by the Planck constant expresses the level of self-memory of this object. Inertia, or resistance to a change of position, that is also proportional to the mass-energy would reveal the tendency of an object to "repeat itself", or to stay equal to itself. In the framework of Bohm-Bub theory, Cerofolini[12] showed independently that it is impossible to measure such memory effects at the level of quantum systems, which implies that Sheldrake's hypothesis is unfalsifiable. Our results show that no such global memory effect seems to exist at the level of individual quantum systems, as far as we can measure them. This does not invalidate Sheldrake's hypothesis because it is essentially unfalsifiable but nevertheless our results can be considered as another negative result.

In a sense, the quest of hidden memory effects in the quantum regime is an endless quest, for the same reason that it is impossible to decide whether a finite series of numbers is randomly distributed.[19] The best that we can do in order to check whether a series of numbers (say, bits) is randomly distributed is to perform as many statistical tests as we can, and to check that their results agree with the predictions made in the case of "truly" random series.[20] We face the same limitation in front of quantum statistical data. We are free to invent as many non-standard models as we want, and to check whether or not their predictions provide a better fit with experimental data compared to standard quantum predictions. We presented here several attempts of that kind, that always resulted in a confirmation of the quantum theory. It is always possible that surprises occur in the future but this is another story.

References

1. A. Aspect, P. Dalibard and G. Roger, *Phys. Rev. Lett.*, **47**, 460 (1981).
2. J. Baudon, J. Robert, Ch. Miniatura, O. Gorceix, B.-J. Lawson-Daku, K. Brodsky, R. Mathevet and F. Perales, *Comments At. Mol. Phys.*, **34**, Nos. 3–6, 161 (1999).
3. J. Baudon, T. Durt, R. Mathevet, J. Robert and B. Viaris de Lesegno, Memory effects in atomic interferometry: A negative result, in *Entanglement, Non-linearity, Quantum Structures and New Experiments* (World Scientific, Singapore, 2002).
4. F. J. Belinfante, *A Survey of Hidden Variable Theories* (Pergamon, Oxford, 1973).

5. J. S. Bell, *Physics*, **1**, 195 (1964).
6. J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, 1987).
7. D. Bohm and J. Bub, *Rev. Mod. Phys.*, **18**, No. 1, 453 (1966).
8. N. Bohr, *Phys. Rev.*, **48**, 696 (1935).
9. N. Bohr, Discussion with Einstein on epistemological problems in atomic physics, in *Albert Einstein: Philosopher-Scientist* (ed. P. A. Schilpp, The Library of Living Philosophers, Evanston, 1949).
10. V. Buonomano and F. Bartmann, *Nuov. Cim.*, **95** B, No. 2, 99 (1986).
11. V. Buonomano, *Founds. Phys. Lett.*, **2**, No. 6, 565 (1989).
12. G. Cerofolini, *Lettere al Nuovo Cimento*, **35**, No. 15, 457 (1982).
13. M. Czachor, *Found. Phys. Lett.*, **4**, 351 (1991).
14. L. de Broglie, *Perspectives en Microphysique* (Champs, Flammarion, Paris, 1993).
15. L. Diosi, N. Gisin and W. T. Strunz, *Phys. Rev. A*, **58**, No. 3, 1699 (1998).
16. T. Durt, Non-local effects in the Bohm-Bub theory, in *Proceedings of the Conference "Physical Interpretations of Relativity Theory"* (London, 1998).
17. T. Durt, *Int. J. Theor. Phys.*, **38**, 457 (1999).
18. T. Durt, *Helv. Phys. Acta*, **72**, 356, (1999).
19. T. Durt, *Int. Journ. of Mod. Phys. B*, **26**, Nos. 27–28, 1243005 (2012).
20. T. Durt, C. Belmonte, L.-P. Lamoureux, K. Panajotov, F. Van den Berghe and H. Thienpont, *Phys. Rev. A*, **87**, 022339 (2013).
21. A. Einstein, B. Podolsky and N. Rosen, *Phys. Rev.*, **47**, 777 (1935).
22. M. Freyberger, P. Bardroff, C. Leichtle, G. Schrade and W. Schleich, *Phys. World*, 41 (1977).
23. D. T. Gillespie, *Am. J. Phys.* **54**, No. 10, 889 (1986).
24. D. T. Gillespie, *Phys. Rev. A* **49**, No. 3, 1607 (1994).
25. N. Gisin, *Phys. Lett. A*, **143**, No. 1, 1 (1990).
26. N. Gisin, G. Ribordy, W. Tittel and H. Zbinden, *Review of Modern Physics*, **74**, 145 (2002).
27. R. Hanbury-Brown and R. Q. Twiss, Interferometry of the Intensity Fluctuations in Light I and II, in *Proc. Roy. Soc. A* (London) **242**, 300 and **243**, 291 (1957 and 1958).
28. T. Jennewein, U. Achleitner, G. Weihs, H. Weinfurter and A. Zeilinger, "A fast and compact quantum random number generator", 1999, quant-ph/9912118.
29. L. Mandel and R. Short, *Phys. Rev. Lett.*, **51**, No. 5, 384 (1983).
30. L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, 1995).
31. B. Mandelbrot, *The Fractal Geometry of Nature* (eds. WH Freeman, 1982).
32. R. D. Mattuck, *Phys. Lett. A*, **81**, 331 (1981).
33. B. Mielnik, *Founds. Phys.*, **20**, No. 6, 745 (1990).
34. L. Nottale, *Fractal Space-Time and Microphysics* (World Scientific, Singapore, 1993).
35. C. Papaliolos, *Phys. Rev. Lett.*, **18**, No. 15, 622 (1997).

36. Y. Pierseaux, La Structure Fine de la Relativité Restreinte (L' Harmattan, Paris, 1999).
37. H. Rauch, *J. Phys. Soc. Jpn.*, **65**, Suppl. A., 45, (1996) and H. Rauch, J. Summhammer and M. Zawisky, *Phys. Rev. A*, **50**, 5000 (1994).
38. E. Schrödinger, Discussion of probability relations between separated systems, *Proc. of the Cambridge Phil. Soc.*, **32**, 446 (1936).
39. R. A. Rubenstein, D. A. Kokorowski, A. Dhirani, T. Roberts, S. Gupta, J. Lehner, W. W. Smith, E. T. Smith, H. J. Bernstein and D. E. Pritchard, *Phys. Rev. Lett.*, **12**, No. 83, 2285 (1999).
40. R. Sheldrake, Une nouvelle science de la vie (Eds. du Rocher, Monaco, 1985).
41. G. Skorobogatov and R. Svertilov, *Phys. Rev. A*, **58**, No. 5, 3426 (1998).
42. A. Stefanov, N. Gisin, O. Guinnard, L. Guinnard and H. Zbinden, Optical quantum random nuber generators, 1999, quant-ph/9907006.
43. W. T. Strunz, L. Diosi and N. Gisin, *Phys. Rev. Lett.*, **82**, No. 9, 1801 (1999).
44. A. Suarez and V. Scarani, *Phys. Lett. A*, 232, 9 (1997).
45. J. Summhammer, *Nuov. Cim. B*, **103**, No. 3, 265 (1985).
46. A. Tonomura, J. Endo, T. Matsuda and T. Kawasaki, *Am. J. Phys.*, **57**, 117 (1989).
47. J. H. Tutsch, *Rev. of Mod. Phys.*, **40**, 232 (1968).
48. J. H. Tutsch, *Phys. Rev.*, **183**, No. 5, 1116 (1969).
49. F. Vanden Berghe, Quantum aspects of cryptography: From qutrit symmetric informationally complete projective operator valued measure key encryption to randomness quality control (PhD thesis, Vrije Universiteit Brussels, 2011).
50. H. Zbinden, J. Brendel, W. Tittel and N. Gisin, Experimental test of Relativistic Quantum state Collapse with Moving Reference Frames, 2000, quant-ph/0002031.

5. Appendix: Plots Representing Experimental Results

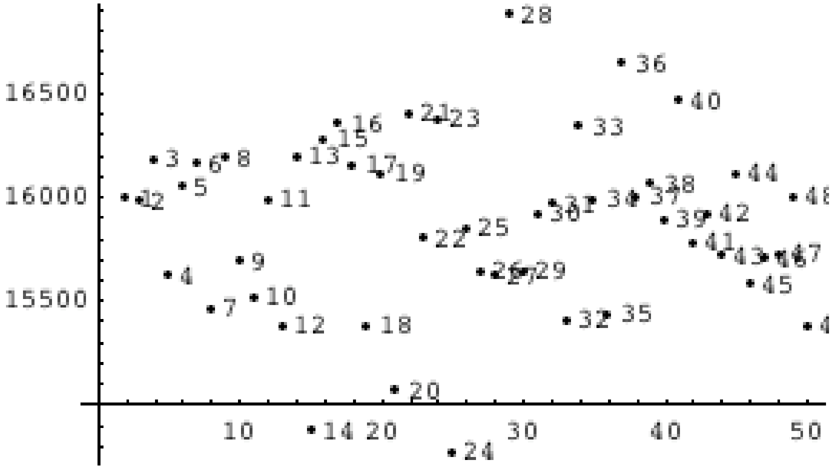

Fig. 1. Dispersion of the average values of the time-delay.

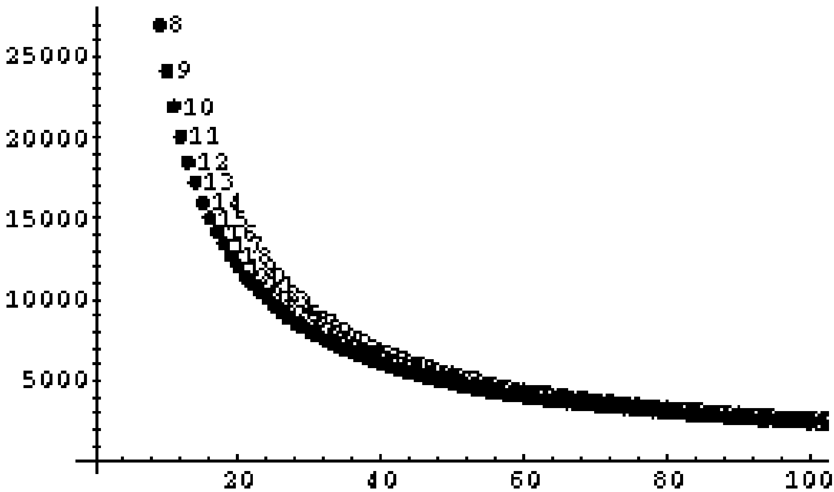

Fig. 2. Law in $1/N$.

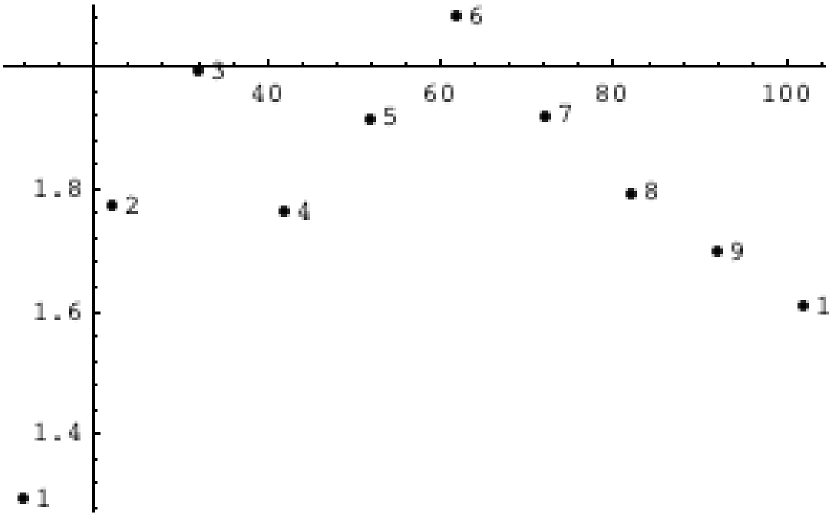

Fig. 3. The Hurst parameters for several values of N between 10000 and 100000.

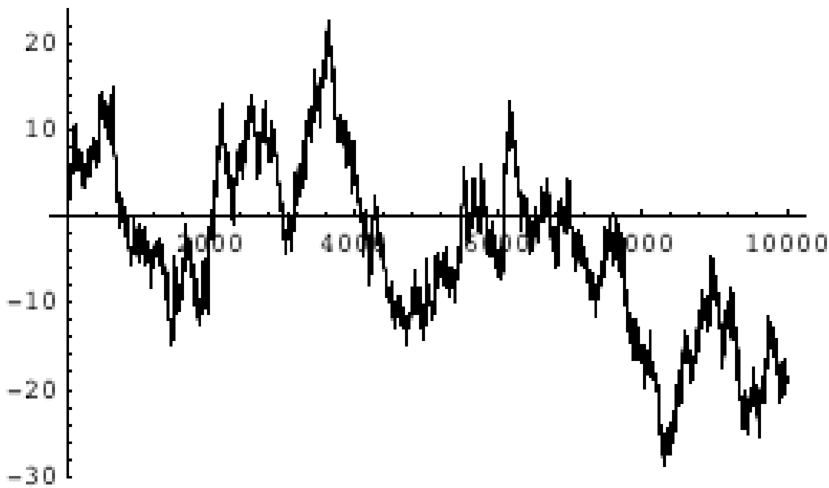

Fig. 4. Cumulated sum of 10000 normalised time delays.

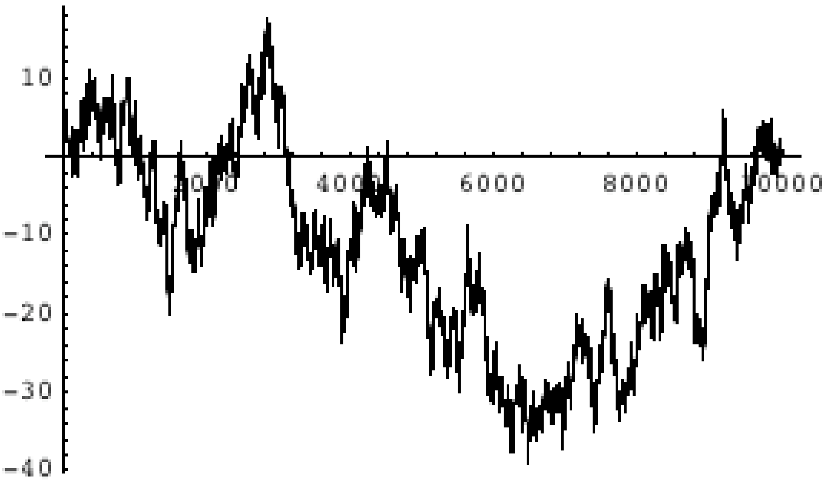

Fig. 5. Cumulated sum of 10000 similarly normalised pseudo-random numbers.

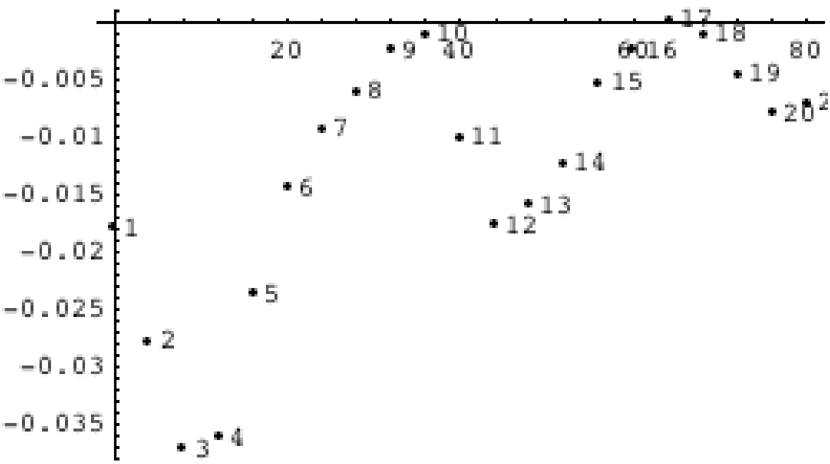

Fig. 6. The persistence parameter for several values of N between 1 and 81.

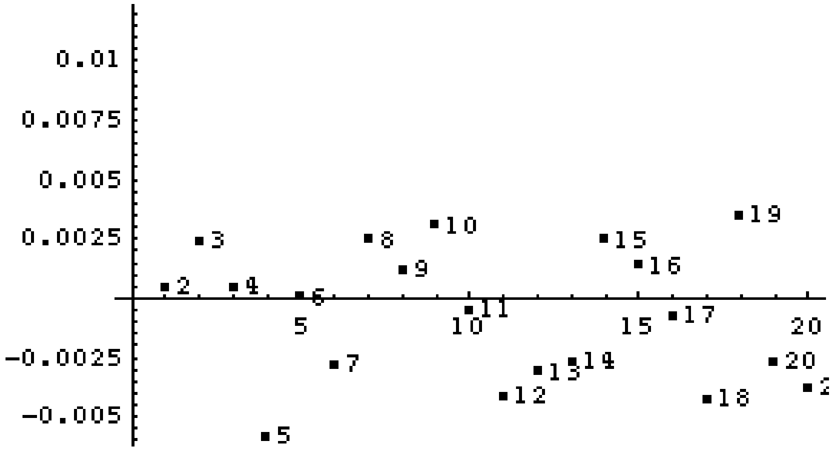

Fig. 7. The auto-correlation function for several values of N between 1 and 20.

QUANTUM ONTOLOGY IN THE LIGHT OF GAUGE THEORIES

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By using the conceptual framework provided by the theory of constrained Hamiltonian systems, we propose a quantum ontology based on two independent postulates, namely the *phase postulate* and the *quantum postulate*. The phase postulate generalizes the gauge correspondence between first-class constraints and gauge transformations to the observables of *unconstrained* Hamiltonian systems. The quantum postulate establishes a faithful correspondence between the observables that allow us to identify the states and the operators that act on these states. According to this quantum ontology, quantum states provide a *complete* description of all the *objective properties* of quantum systems.

Keywords: Quantum mechanics; gauge theories; geometric quantization; symplectic geometry.

1. Introduction

In this article, we consider quantum mechanics in the light of a fundamental postulate (proposed by Dirac) coming from gauge theories, namely that *first-class constraints* induce *gauge transformations* [16,21]. In gauge theories, the symmetries defined by gauge transformations reduce the amount of *observable* (i.e. gauge invariant) information that is necessary to completely describe a physical system. More precisely, the *symplectic reduction* defined by the first-class constraints of the theory permits us to pass from the original constrained phase space to the unconstrained *reduced phase space*. Whereas the former contains both physical and non-physical degrees of freedom, the latter only describes gauge invariant quantities [8,21,26]. In this way, a constrained Hamiltonian system of $2n$ canonical variables and k first-class constraints can be reduced to an unconstrained Hamiltonian system of $2(n-k)$ *physical* canonical variables. The heuristic conjecture of the present article is that a similar idea can be used for explaining the fact that both members of a complementary pair of observables cannot define (what we shall call) *objective properties* of the same quantum system in accordance with Heisenberg indeterminacy principle. In order to unpack this conjecture, we generalize the "gauge correspondence" between first-class constraints and gauge transformations to unconstrained Hamiltonian systems. To do so, we argue that there exists a universal "phase symmetry" acting on the phase space of *every* Hamiltonian system, *be it constrained or not*. The action of this universal symmetry defines a "projection" from the $2n$ classical observables q *and* p to the n quantum observables q *or* p (or a combination of both in accordance with Heisenberg indeterminacy principle).

In Section 2, we argue that the notions of *phase symmetry* and *phase transformations* stem from a group-theoretical model of physical systems. We then argue that a satisfactory *quantum ontology* can be obtained by supplementing this preliminary group-theoretical model with two postulates, namely the *phase postulate* and the *quantum postulate*. On the one hand, the *phase postulate* (introduced in Section 3) states that the operators associated to the observables that define the *objective* (i.e. phase-invariant) *properties* of a physical system generate the phase transformations of the latter. We argue that Heisenberg indeterminacy principle for the extreme cases of an objective property and its (completely undetermined) conjugate variable is a direct consequence of this postulate. It results from this interpretation of Heisenberg indeterminacy principle that quantum states, far from being states of incomplete knowledge, provide a *complete* description of all the *objective properties* of the corresponding system. On the other hand, the *quantum postulate* (introduced in Section 4) establishes an *injective* correspondence between the numerical value of an objective property and the operator associated to this property that acts on the corresponding state. This postulate allows us to propose a group-theoretical definition of physical observables according to which the numerical values obtained by evaluating the observables on the states are "*indices characterizing representations of groups*" [36, p. xxi]. In Section 5, we consider the proposed postulates in the light of the standard Hilbert space formulation of quantum mechanics. We then use the quantum ontology to analyze the Heisenberg indeterminacy principle for the general case of two partially undetermined properties associated to non-commuting operators. In Section 6, we argue that the phase postulate entails a natural interpretation of the polarization condition used in the framework of geometric quantization.[a] In the final section, we summarize the obtained results.

In what follows, we shall assume that the reader is familiar with the basics of *symplectic geometry* [1,2,8], the *theory of constrained Hamiltonian systems* [16,17,21] and *geometric quantization* [6,9,22,27,33,38].

2. Preliminary Group-Theoretical Model for Mechanical Systems

The preliminary model that we shall now introduce is intended to provide a notion of symmetry–that we shall call *phase symmetry*–capable of explaining the fact that the *complete* description of a finite quantum system only requires n canonical variables (instead of the $2n$ canonical variables required in classical mechanics).

The preliminary group-theoretical model in question is based on the idea according to which a physical system must be understood as a *multifaceted structure* characterized by a set of *invariant objective properties*. According to this model, a physical system, far from being a *point-like* or *structureless* entity, is a structure-endowed entity that has different "components" (that we shall call *phases*). Roughly speaking, we could say that a physical system is a multifaceted "superposition" of phases. We will call *phase transformations* the transformations connecting these phases and *phase orbit* a set of phases connected by means of a one-parameter family of phase transformations. The set of phase transformations that generate all the system's phase orbits defines (what we shall call) the *phase group* of the system.

Besides having different phases, a physical system is, according to the proposed model, a configuration completely characterized by a set of *objective properties*. More precisely, a physical system is completely defined by the numerical values $\left\{f^1_\alpha, ..., f^n_\rho\right\}$, that we call *objective properties*, defined by a set of observables $\left\{f^1, ..., f^n\right\}$, where f^i_μ denotes the possible

[a] It is worth stressing that the proposed quantum ontology is indeed based on an analysis of this particular mathematical formalization of canonical quantization, namely the *geometric quantization* formalism (see Ref. [9]). In the present context, the importance of this formalism relies on the fact that it shows that quantum operators can be obtained by means of a suitable extension of the Hamiltonian vector fields (or *classical operators* as we call them in what follows) that appear in the symplectic formulation of classical mechanics. This extension is performed in such a way that the resulting quantum operator Lie algebra is isomorphic to the Poisson algebra of classical observables. This also justifies the importance that we attach in what follows to the analysis of the relationships between the (classical and quantum) operators algebras and the Poisson algebra of observables.

numerical values (indexed by μ) of the observable f^i. Since phase transformations interchange different phases of the same system, the objective properties $\{f^1_\alpha, ..., f^n_\rho\}$ that characterize the system as such must be *invariant under phase transformations.* In fact, the term *objective* means here *phase-invariant.* In this way, this preliminary model for physical systems provides a particular realization of the standard group-theoretical relationship between *objectivity* and *invariance* under symmetry transformations (different analysis of this correspondence can be found in Refs. [3,5,14,15,20,28,29,35]). To sum up we can say that, according to the proposed model, *there is no physical entity without an invariant identity,*[31] *nor without different variant phases.*

However, this preliminary model does not convey a satisfactory ontology of physical systems in the absence of further specifications. Indeed, the phase group of a given system–and therefore the corresponding objective invariants–remains for the moment unspecified.[b] In order to bypass this objection, one could try to establish a relation–in the wake of Wigner's seminal 1939 paper [37] (see also Refs. [4,13,14,23,28,34])–between the symmetries of the mechanical description of a physical system and the kinematical group of symmetries of the spatiotemporal background in which the system is embedded (e.g. Galilei group, Poincaré group, etc.). However, there is no reason to presuppose that every possible Hamiltonian system necessarily describes a dynamical system embedded in a homogeneous spatiotemporal background. In particular, we can describe the Hamiltonian dynamics of a system embedded in a non-homogeneous (fixed or dynamical) spatiotemporal background. Since the final objective of the present analysis is to construct a formal quantum ontology valid for every possible Hamiltonian system, the only geometric setting that we presuppose is the symplectic manifold (P, ω) that parameterizes the possible states of the classical system.

We could still reply that a background independent theory such as general relativity (which describes dynamical space-times that are not necessarily homogeneous) is still characterized by the invariance of the physical observables under a symmetry group, namely the group of general diffeomorphisms of space-time. However, the action of this symmetry group does not induce the transition from a classical system to a quantum one, but

[b]This problem was clearly stated by Nozick in the following terms: 'The notion of invariance under transformations cannot (without further supplementation) be a *complete* criterion of the objectivity of facts, for its application depends upon a selection of *which* transformations something is to be invariant under.' (Ref. [29], p. 79).

rather the transition from a constrained Hamiltonian system to an unconstrained one. Indeed, the gauge symmetries of a classical gauge system can be eliminated (at least in principle) by passing to the unconstrained reduced phase space description. However, the universal phase symmetry that we want to define must explain why the *classical* description provided by this reduced phase space is still overdetermined, *even if the gauge group action has already been quotiented out.* Hence, the phase group that generates the phase symmetry cannot be identified with the symmetry group of a constrained Hamiltonian system. In the case of a system with a finite number of degrees of freedom, the gauge symmetries generated by a set of k first-class constraints explain the symplectic reduction from the $2n$ *constrained* degrees of freedom to the $2(n-k)$ *physical* degrees of freedom. On the other hand, the phase symmetry must explain the reduction from the resulting $2(n-k)$ *classical* degrees of freedom to the $n-k$ *quantum* degrees of freedom.

In order to specify the phase group that defines the objective invariants of a physical system, we shall supplement the proposed preliminary model for mechanical systems with an additional postulate, namely the *phase postulate.*

3. Phase Postulate

The identification of physical states in gauge theories depends on the existence of two kinds of properties: *first-class constraints* and *gauge observables* (i.e. the observables on the reduced phase space). On the one hand, the first-class constraints $G_a \in \mathcal{C}^{\infty}(P)$ allow us to define the *reduced phase space* (P_{red}, ω_{red}) of the theory. On the other hand, the gauge observables $f \in \mathcal{C}^{\infty}(P_{red})$ allow us to identify the different physical states in P_{red}. We could say that the first-class constraints define the *kind* of possible physical states considered by the theory.[c] In turn, the gauge observables can be used to identify the different physical states of the corresponding kind. We shall now argue that these two classes of properties—the first-class constraints and the observables—are differently treated in classical mechanics.

[c]For instance, the presence of first-class constraints in the Hamiltonian formulation of general relativity implies that the possible states described by the theory do not represent 4-dimensional manifolds M endowed with Lorentzian metrics g_{ab}, but rather equivalence classes $[(M, g_{ab})]$ of Lorentzian manifolds under general diffeomorphisms of M [25,32]. Far from being a mere epistemic requirement, these constraints encode the ontological commitment of the theory, i.e. the kind of its possible physical states (see Ref. [12] for a detailed analysis of this statement in the framework of classical Yang-Mills theory).

On the one hand, the first-class constraints $G_a(q,p) \in \mathcal{C}^\infty(P)$ ($a = 1,...,k$) "strike" twice. Firstly, they define a restriction to the *constraint surface* $\Sigma \subset P$ defined by the constraint equations $G_a(q,p) = 0$ for $a = 1,...,k$. Secondly, the Hamiltonian vector fields v_a associated to G_a generate *gauge transformations* that define a *projection* $\Sigma \to P_{red}$ to the space of *gauge orbits* (or *reduced phase space*).[d] We could then say that the *restriction* to a constraint surface Σ defined by a set of first-class constraints $\{G_a\}$ entails a *projection* to the orbit space defined by the action of the associated Hamiltonian vector fields $\{v_a\}$.

On the other hand, the gauge observables on the reduced phase space $f \in \mathcal{C}^\infty(P_{red})$ (or, in general, the observables on any unconstrained phase space) "strike" only once. Whereas a classical observable $f(q,p)$ induces *restrictions* to the surfaces defined by its possible values $f = f_0$, the action of the associated Hamiltonian vector field v_f is not interpreted as a symmetry transformation. Let's consider for instance the subspace of classical states characterized by the property $p = p_0$. According to the usual understanding of classical mechanics, the restriction to the surface $p = p_0$ is not followed by a projection defined by the action of the Hamiltonian vector field $v_p = \frac{\partial}{\partial q}$ (which generates translations along the coordinate q). In other terms, the fact that the momentum p of a physical system has the value p_0 does not imply that the transformations between the different values of q (generated by v_p) are mere symmetry transformations. Hence, the restriction to the surface defined by a particular value p_0 of p does not forbid us from identifying the points (q, p_0) for all q with *different* physical states. In fact, it is necessary to fix the values of both q and p in order to identify a classical physical state (see Ref. [11] for a detailed discussion).

Let's summarize what we have just said. A classical observable $f(q,p)$ only defines *restrictions* to its possible values $f = f_0$. On the contrary, first-class constraints G_a define both a *restriction* to the constraint surface Σ (defined by the equations $G_a = 0$) and a *projection* $\Sigma \to P_{red}$ to the reduced phase space (defined by the action of the Hamiltonian vector fields v_a). The following postulate removes this difference between observables and first-class constraints:

[d] The Hamiltonian vector field $v_f \in \Gamma(TP)$ associated to an observable f is defined by means of the expression $i_{v_f}\omega = df$, where $i_{v_f}\omega$ denotes the contraction of the symplectic 2-form ω with v_f and $d : \Omega^p(P) \to \Omega^{p+1}(P)$ is the exterior derivative on P. In what follows, the Hamiltonian vector field v_f will be called *classical operator*. This terminology is justified by the fact that, in the framework of the geometric quantization formalism, the quantum operator $\hat{v}_f$ associated to an observable f can be obtained by means of a suitable extension of the Hamiltonian vector field v_f.

Phase Postulate: the transformations generated by the operators associated to the objective properties of a state must be interpreted as phase transformations.

This postulate generalizes to unconstrained Hamiltonian systems the (Dirac) postulate according to which the first-class constraints of a gauge theory induce gauge transformations [16,21]. In order to abridge the formulation of this postulate, we shall use the following terminology. The operator associated to an observable defining an objective property of a state will be called *eigenoperator* of the state. Hence, the phase postulate states that *the transformations generated by the eigenoperators of a state must be interpreted as phase transformations.* According to this postulate, *a restriction*–be it defined by a first-class constraint or by an observable–*must always be followed by a projection.* Whereas the first-class constraints G_a induce–by means of theirs associated operators v_a–gauge transformations, the observables defining the objective properties of a state induce phase transformations.

In Refs. [10,11], we revisited the extreme cases of Heisenberg indeterminacy principle–namely the cases of a completely determined observable and the completely undetermined conjugate variable–in the light of the phase postulate. We argued that the indeterminacy principle, far from resulting from an epistemic restriction to the amount of information an observer can have about a physical system, can be understood as a consequence of the circular imbrication between objective properties and non-objective phases established by the phase postulate. Indeed, this postulate implies that physical states cannot be sharply localized in both members of a complementary pair (such as q and p). According to the proposed definition, a property is *objective* if it is phase-invariant. Now, the phase postulate states that the phase transformations are induced by the objective properties themselves. Therefore, *an objective property must be invariant under the phase transformations induced by all the other objective properties of the same state*. This condition of compatibility between the objective properties of a state reduces the number of possible objective properties from $2n$ to n. In order to show this, we must take into account that the transformations of an observable g induced by an observable f (i.e. generated by the Hamiltonian vector field v_f associated to f) is given by the Lie derivative $\mathcal{L}_{v_f} g = v_f(g) = \{g, f\}$. Two observables f and g will be said to be *compatible* if g is invariant under the transformations induced by f and viceversa, i.e. if $\mathcal{L}_{v_f} g = \{g, f\} = 0$. Let's consider now two objective properties of the same state defined by the observables f and g. Since these properties are assumed to be objective, they must be both phase-invariant. In particular, they must be invariant under the phase transformations induced by v_f and v_g. In other terms, f and g must satisfy $\{g, f\} = 0$, i.e. they must be compatible. Hence, *two observables can define objective properties of the same state if and only if they have vanishing Poisson bracket.*[e]

Consider again the example of a physical state characterized by the objective property $p = p_0$. According to the phase postulate, the transformations generated by the operators associated to p should be considered phase transformations of the corresponding state. Now, q is not invariant under the transformations induced by p since the Hamiltonian vector field $v_p = \frac{\partial}{\partial q}$ generates translations in q. Hence, the phase postulate implies that q cannot be also an objective property of a such a state. Indeed, $\mathcal{L}_{v_p} q = \{q, p\} \neq 0$. In other terms, the phase transformations induced by p "*phase out*" the coordinate q. This means that the different values of q should be considered "*pure phase*". In general, the sharp localization on one canonical variable entails that the conjugate variable is completely "phased out" by the phase transformations induced by the former. Since classical states are defined by specifying the values of both q and p, the phase postulate is not fulfilled in classical mechanics.

The phase postulate generalizes the gauge correspondence between first-class constraints and symmetry transformations to the case of unconstrained Hamiltonian systems. Hence, we could compare the consequences of the phase postulate with what we know from gauge theories. The first thing to remark is that the points in the constraint surface Σ of a gauge system provide an *overdetermined* description of physical states. Indeed, a point in Σ defines not only a gauge orbit (i.e. a physical state), but also a particular coordinate representative in the orbit. Now, the phase postulate implies that the points in the phase space of an *unconstrained* Hamiltonian system also provide an *overdetermined* description of physical states. Exactly as the Dirac observables do not distinguish between the different representatives of the same gauge orbit, the objective properties of a state

[e]The utilization of the classical notions of *Hamiltonian vector fields* and *classical observables* for understanding the Heisenberg indeterminacy principle in quantum mechanics is not straightforward. In order to justify this procedure we must take into account that Heisenberg indeterminacy principle results from the non-commutativity of quantum operators associated to conjugate canonical variables. In turn, this quantum non-commutativity reflects the Poisson non-commutativity of conjugate classical observables. Hence, it seems reasonable to try to shed some light on Heisenberg indeterminacy principle by analyzing the conceptual meaning of the Poisson algebra of classical observables.

do not distinguish between elements belonging to the same phase orbit. It is also worth noting that the fact that Dirac observables do not single out a particular representative in each gauge orbit does not mean that the theory is "incomplete", i.e. that some hypothetical "*hidden variables*" could establish a *physical* distinction between gauge equivalent states. Gauge observables cannot distinguish between states belonging to the same gauge orbit because these states are just different coordinate representations of the same physical state.[f] Analogously, the fact that the objective properties of a state cannot separate elements in a phase orbit does not mean that the corresponding (quantum) theory is "incomplete".

The second thing to note is that first-class constraints obey by definition a compatibility condition given by the following involutivity condition:

$$\{G_a, G_b\} = f^c_{ab} G_c \approx 0, \tag{1}$$

where "$\approx$" means equal to zero only on the constraint surface Σ defined by the equations $G_a(q,p) = 0$. This condition guarantees that the gauge orbits induced by one constraint remain in the constraint surface defined by the others, i.e. that the gauge transformations preserve the constraint surface. For instance, the gauge transformation of G_a induced by G_b is given by the Poisson bracket $\mathcal{L}_{v_b} G_a = \{G_a, G_b\}$. The involutivity condition (1) guarantees that G_a is invariant, on the constraint surface Σ, under the gauge transformation generated by G_b. Therefore, the gauge orbits generated by G_b remain in the surface $G_a = 0$. We can then say that two constraints G_a and G_b are *compatible* if they have a (weakly) vanishing Poisson bracket. Now, according to the phase postulate two observables can define objective properties of the same physical state if and only if they obey a similar compatibility condition, namely the condition of having a vanishing Poisson bracket. In this way, this compatibility condition between the objective properties of the same state plays an analogous role in the framework of unconstrained Hamiltonian systems than the involutivity condition (1) in the framework of gauge theories.

4. Quantum Postulate

The phase postulate states that the objective properties of a physical state induce, by means of the action generated by theirs associated operators,

[f] If we did not assume that states in a gauge orbit are physically equivalent, i.e. if we did not assume that the transformations induced by first-class constraints are symmetry transformations, then the theory would be indeterministic [16,18,21].

the phase transformations between its non-objective phases. We shall now go one step further in the analysis of the relationship between the observables that define the objective properties of a system and the corresponding operators. To do so, we shall consider the relationship between the numerical value assigned by an observable f to a state $x \in P$ and the vector $v_f(x) \in T_xP$ defined at x by the Hamiltonian vector field v_f.

An observable f associates to any state $x \in P$ two different kinds of entities: the evaluation of f on x yields a real number $f(x)$, whereas the Hamiltonian vector field associated to f defines a vector $v_f(x) \in T_xP$ on x. Now, is there any relationship between the number $f(x)$ and the vector $v_f(x)$? To investigate this, let's consider for instance the observable $f = p$ and a classical state $x_0 = (q_0, p_0)$. Let's analyze how the Hamiltonian vector field v_p changes when we modify the momentum of the state. In other terms, let's compare the vectors $v_p(q_0, p_0)$ and $v_p(q_0, p_0 + \epsilon)$ defined at states that slightly differ in the value of their momenta. Now, the translation in p that takes the state (q_0, p_0) to the state $(q_0, p_0 + \epsilon)$ is generated by the Hamiltonian vector field $v_q = -\frac{\partial}{\partial p}$. In other terms, the states (q_0, p_0) and $(q_0, p_0+\epsilon)$ are placed on an integral curve of the Hamiltonian vector field v_q associated to the observable q. Now, the variation of the vector field v_p along the integral curves of the vector field v_q is given by the Lie derivative of v_p along v_q evaluated at (q_0, p_0), that is $(\mathcal{L}ie_{v_q} v_p)(q_0, p_0) = [v_p, v_q](q_0, p_0)$. In this way, the Lie derivative $\mathcal{L}ie_{v_q} v_p = [v_p, v_q]$ measures how the vectors defined by the vector field v_p change under a transformation of p generated by v_q. Now, the application $f \mapsto v_f$ is a Lie algebra homomorphism, i.e. it satisfies $[v_f, v_g] = v_{\{g,f\}}$. Hence, we have:

$$\mathcal{L}ie_{v_q} v_p = [v_p, v_q] = v_{\{q,p\}} = v_1 = 0, \tag{2}$$

where we have also used that the kernel of the application $f \mapsto v_f$ is given by the set of constant functions:

$$f(q,p) = k \in \mathbb{R} \mapsto v_k = \frac{\partial k}{\partial p}\frac{\partial}{\partial q} - \frac{\partial k}{\partial q}\frac{\partial}{\partial p} = 0.$$

We can thus conclude that it is precisely the non-injectivity of the Lie algebra homomorphism between observables and Hamiltonian vector fields that entails the commutativity of v_p and v_q. In turn, this implies that the vector field v_p does not change along the integral curves of v_q. In other words, the non-injectivity of the application $f \mapsto v_f$ implies that the vectors $v_p(x)$ do not depend on the value of the momentum of x.

We shall now argue that if there were a faithful relationship between the numerical value $f(x)$ and the vector $v_f(x)$, then we could propose a group-theoretical definition of physical observables. More precisely, if such a relationship were faithful, then we could interpret the numerical values assigned by physical observables to states as "*indices characterizing representations of groups.*" (see Ref. [36], p. xxi). In order to unpack this statement, we shall appeal to the so-called *moment map formalism* [24,30].

Let's consider a Hamiltonian action $\Phi : G \times P \to P$ of a Lie group G on a symplectic manifold P (see Ref. [11] for details). Such an action defines a map $\iota : \xi \mapsto v_\xi(x) = \frac{d}{d\lambda}(exp(-\lambda\xi) \cdot x)|_{\lambda=0}$ (for $x \in P$) between Lie algebra elements $\xi \in \mathfrak{g}$ and *fundamental vector fields* v_ξ on P. We shall say that the vector field v_ξ is the *concrete classical operator* that *realizes* on the symplectic manifold P the abstract Lie algebra element ξ. Since we assumed that the action is Hamiltonian, it is possible to obtain the fundamental vector field v_ξ as a Hamiltonian vector field associated to an observable $h_\xi(x) \in \mathcal{C}^\infty(P)$ defined by ξ. In other terms, there exists a *co-moment map*:

$$\begin{aligned} \tilde{\mu} : \mathfrak{g} &\to \mathcal{C}^\infty(P) \\ \xi &\mapsto h_\xi(x) \end{aligned} \tag{3}$$

such that the fundamental vector field v_ξ that realizes the element $\xi \in \mathfrak{g}$ as a vector field on P is the Hamiltonian vector field associated to the observable $h_\xi(x) \in \mathcal{C}^\infty(P)$. The corresponding *moment map*

$$\mu : P \to \mathfrak{g}^*$$

is defined by the expression

$$\langle \mu(x), \xi \rangle = \tilde{\mu}(\xi)(x) = h_\xi(x), \tag{4}$$

where $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ is the natural duality pairing. Thanks to the existence of a moment map μ, each state x can be interpreted as a "weight"

$$\begin{aligned} x : \mathfrak{g} &\to \mathbb{R} \\ \xi &\mapsto \langle \mu(x), \xi \rangle \end{aligned}$$

that assigns a real number $x(\xi)$ to each abstract Lie algebra element $\mathfrak{g}$. By fixing a Lie algebra element $\xi \in \mathfrak{g}$, we obtain an observable $h_\xi(x) \in \mathcal{C}^\infty(P)$ that encodes the numbers assigned by the different "weights" x to ξ. It is worth noting that the co-moment map (3) can be interpreted as a *Gelfand transform* that realizes an abstract Lie algebra element ξ as an observable h_ξ on the space of "weights" P. In turn, expression (4) can be recast in terms of the usual *state-observable duality*:

$$h_\xi(x) = x(\xi).$$

The existence of a (co-)moment map means that the map ι between Lie algebra elements $\xi \in \mathfrak{g}$ and fundamental vector fields v_ξ on P is "factorized" by the observable h_ξ, that it to say by the composition of $\pi : f \mapsto v_f$ and $\tilde{\mu}$:

$$\xi \xrightarrow{\tilde{\mu}} h_\xi = \langle \mu, \xi \rangle \xrightarrow{\pi} v_\xi, \qquad \iota = \pi \circ \tilde{\mu} : \xi \mapsto v_\xi .$$

This factorization suggests that we could try to interpret the number $h_\xi(x_0)$ obtained by evaluating the observable h_ξ on the "weight" x_0 as a "representation index" that defines the "weighted" concrete operator $v_\xi(x_0)$. Now, if this interpretation were correct, the "weighted" concrete operators associated to classical states in M that differ in the value of the observable h_ξ should be different. However, we have seen that the non-injectivity of the application $f \mapsto v_f$ implies (for instance) that the vector field $v_p(x)$ does not depend on the value of the momentum of x. Therefore, the value $p(x_0)$ taken by the observable p at the state x_0 cannot be interpreted as a "representation index" associated to the "weighted" concrete operator $v_p(x_0)$. All in all, the proposed group-theoretical interpretation of the numerical values obtained by evaluating observables on classical states as "representation indices" is not valid in the framework of classical mechanics.

If the application $f \mapsto v_f$ between observables and concrete classical operators were injective, then the numerical value obtained by evaluating an observable h_ξ on a state could be interpreted as a *representation index* characterizing a "weighted" concrete operator. In order to force the validity of such an interpretation of physical observables, we shall introduce the second postulate of the quantum ontology:

Quantum Postulate: an objective property of a state given by the numerical value of an observable h_ξ must be a *representation index* faithfully defining a "weighted" concrete operator.

According to the previous arguments, the implementation of this postulate requires to force the injectivity of the Lie algebra homomorphism between observables and operators. In turn, this amounts to force the non-commutativity of the concrete operators associated to complementary pairs of observables. As we shall see in the next section, this can be done by extending classical operators v_f to quantum operators $\hat{v}_f$.

5. From the Quantum Ontology to the Quantum Formalism

In this section, we shall establish the dictionary between the conceptual framework provided by the quantum ontology on the one hand and the Hilbert space formulation of quantum mechanics on the other. This can be done by identifying the *objective properties* of a quantum system with the *eigenvalues* (or *quantum numbers*) that define the corresponding quantum vector. In other terms, we assume the validity of the so-called *eigenvalue-eigenstate link* [7,19]. More precisely, we assume that the numerical value f_α of the observable f is an objective property of the system represented by the pure state ψ if and only if ψ is an eigenstate of the quantum operator $\hat{v}_f$ with eigenvalue f_α, i.e. if and only if $\hat{v}_f\psi = f_\alpha\psi$. In what follows, $\{f^1, ..., f^n\}$ denotes a complete set of commuting observables and f^j_α denotes the α eigenvalue of $\hat{v}_{f^j}$ (which will be written, for the sake of simplicity, as $\hat{v}_j$).

Let's analyze now whether quantum states satisfy the postulates of the quantum ontology. According to the quantum postulate, the objective properties of a physical state must be representation indices characterizing how the abstract Lie algebra elements are realized as "weighted" concrete operators. The term "weighted" means here that the action of the operator on the state faithfully depends on the value that the observable associated to the operator takes on the state. As we have argued in the previous section, this will be the case if the application between observables and operators is injective. Now, the classical application $f \mapsto v_f$ between observables and Hamiltonian vector fields is not injective. In fact, the *prequantization construction* of the geometric quantization formalism shows that the classical operators v_f can be *extended* to quantum operators in such a way that the correspondence between observables and operators is now injective (see Refs. [1,6,9,22,27,33,38]).[g] Hence, we could now expect the quantum postulate to be valid. More precisely, we expect each quantum number f^j_α of a state $|..., f^j_\alpha, ...\rangle$ to be a representation index characterizing in a faithful manner how the corresponding abstract element in the central extension $\mathfrak{g}'$ is realized as a concrete "weighted" operator acting on the state $|..., f^j_\alpha, ...\rangle$. In order to see that this is indeed the case, let's consider the one-parameter

[g]The quantum extension of classical operators has a "universal" counterpart in the central extension $\mathfrak{g}' = \mathfrak{g} + \mathbb{R}$ (called *Heisenberg algebra*) of the Lie algebra $\mathfrak{g} = \mathbb{R}^2$ that generates the translations in (q, p). Whereas the first extension bypasses the non-injectivity of the application $f \mapsto v_f$, the central extension of $\mathfrak{g}$ bypasses the fact that the moment map $\mu : P \to \mathfrak{g}^*$ is not infinitesimally equivariant. The connected and simply-connected Lie group associated to the central extension $\mathfrak{g}'$ is the so-called *Heisenberg group* H (see Refs. [11,24,30] for details).

subgroup $g = e^{i\xi^j \phi}$ of the Heisenberg group whose tangent vector at the identity is equal to $\xi^j \in \mathfrak{g}'$. The action of the abstract element ξ^j on the quantum state $|..., f^j_\alpha, ...\rangle$ is implemented by means of the *quantum* operator $\hat{v}_j$. Now, the action of $\hat{v}_j$ on the state $|..., f^j_\alpha, ...\rangle$ is given by the following expression:

$$e^{i\hat{v}_j \phi}|..., f^j_\alpha, ...\rangle = e^{i f^j_\alpha \phi}|..., f^j_\alpha, ...\rangle. \tag{5}$$

This expression explicitly shows that the eigenvalues f^j_α are representation indices that define the "weighted" concrete operators $e^{i f^j_\alpha \phi}$ that realize the abstract Lie algebra element ξ^j on $|..., f^j_\alpha, ...\rangle$. Indeed, two different eigenvalues f^j_α and $f^j_{\alpha'}$ define different "weighted" concrete operators $e^{i f^j_\alpha \phi}$ and $e^{i f^j_{\alpha'} \phi}$ of the same abstract group element $e^{i\xi^j \phi}$. Hence, quantum numbers can indeed be consider as "*indices characterizing representations of groups.*" ([36], p. xxi).

Let's summarize the previous arguments. In order to force the injectivity of the application between observables and operators it is necessary to pass from classical operators to quantum operators. By construction, the quantum operators $\hat{v}_q$ and $\hat{v}_p$ satisfy the commutation relation $[\hat{v}_p, \hat{v}_q] = -i\hbar$. This means that $\hat{v}_p$ is not invariant under the transformations generated by $\hat{v}_q$. Since the latter generates translations in p, the non-zero commutator $[\hat{v}_p, \hat{v}_q]$ measures the non-trivial variation of $\hat{v}_p$ under a translation in p. This means that the concrete operator that realizes the Lie algebra element associated to the group of translations in q is given by a vector field $\hat{v}_p$ that changes along the integral lines (parameterized by p) of $\hat{v}_q$. In other terms, the vector field $\hat{v}_p$ changes when the representation index given by the value of the observable p changes. Hence, the group of translations in q is realized by concrete operators "weighted" by the value of the observable p.

Let's analyze now whether quantum states satisfy the phase postulate. According to this postulate, the transformations generated by the eigenoperators of a state (i.e. by the operators associated to the objective properties that define the state) must be phase transformations that do not modify the state as such. Now, expression (5) shows that the finite action of the eigenoperator associated to the observable f^j on the state $|..., f^j_\alpha, ...\rangle$ just multiplies the state by the phase $e^{i f^j_\alpha \phi}$. According to the phase postulate, the multiplication of the state by such a phase must be considered a mere phase transformation that does not modify the state as such. Hence, the phase postulate forces us to assume that quantum states are defined modulo an overall phase factor. In this way, the phase postulate allows us to recover the usual definition of quantum states as elements of the unit sphere of the Hilbert space modulo a phase factor.[h]

The phase postulate implies that the position q of a quantum state characterized by the objective property $p = p_0$ is completely "phased out" by the phase transformations generated by the quantum operator associated to p. We could then say that the phase postulate allows us to understand from a conceptual viewpoint the extreme cases of Heisenberg indeterminacy principle, i.e. the cases given by a completely determined observable and the completely undetermined conjugate variable. According to the proposed arguments, we could say that the extreme cases of Heisenberg indeterminacy principle are already implemented in the framework of classical gauge theories. Indeed, the usual understanding of gauge theories is based on the interpretation according to which the variable canonically conjugate to a first-class constraint $G_a(q, p)$–i.e. the variable acted upon by the classical operator v_a–is completely "gauged out" by the gauge transformations generated by v_a (see Refs. [16,18,21]). In other terms, the variable conjugate to the completely determined variable G_a (whose numerical value is 0) is completely "undetermined".

We shall now argue that the indeterminacy principle for the general case of two coexisting partially indeterminate properties associated to non-commuting operators results from the conjunction of the phase postulate and the quantum postulate. To do so, let's analyze what happens when we superpose two states $|p_1\rangle$ and $|p_2\rangle$ characterized by different objective values of p. The fact that the objective properties $p = p_1$ and $p = p_2$ of these states are representation indices that faithfully specify how these states transform under the translations in q generated by $\hat{v}_p$ implies that the superposition $\psi = |p_1\rangle + |p_2\rangle$ is not invariant under such a translation. Indeed, while the position q of the states $|p_1\rangle$ and $|p_2\rangle$ is pure phase, the position of the superposed state $|p_1\rangle + |p_2\rangle$ is no longer a pure phase. In other terms, a translation in q of the superposed state $|p_1\rangle + |p_2\rangle$, far from producing an overall phase factor, modifies the relative phase between the two components. This is a direct consequence of the fact that states that

[h]It is worth stressing that the invariance of quantum states under phase transformations does not mean that the latter are trivial transformations that cannot produce observable effects. Even if the overall phase of a quantum state ψ has no physical significance, phase transformations have observable effects when ψ is one of the terms of a quantum superposition. Indeed, relative phases account for the quantum interference phenomena. Hence, the fact that the action of a quantum operator on one of its eigenstates depends on the corresponding eigenvalue is far from being physically trivial, even if the corresponding phase transformations do not modify the state as such.

differ in the value of p transform under the action of $\hat{v}_p$ by means of different "weights". Therefore, such a translation transforms ψ into a new state. The important point is that *the superposition of* $\hat{v}_p$*-invariant states defined by different eigenvalues of* $\hat{v}_p$ *is not itself* $\hat{v}_p$*-invariant*, i.e. invariant under translations in q. In this way, by introducing an indeterminacy in the value of p (between p_1 and p_2), we partially break the phase symmetry along the variable q, i.e. we partially remove the complete indeterminacy in the value of q. We could say that the *infinitesimal* $\hat{v}_p$-symmetry of each term of the superposed state $|p_1\rangle + |p_2\rangle$ is not a *global* symmetry of the whole state.

We can now summarize the proposed quantum ontology in the following terms. The referent of a pure state $\psi = |f^1_\alpha, ..., f^n_\rho\rangle$ is a physical system defined by a set of *objective* (i.e. phase-invariant) *properties* $\{f^1_\alpha, ..., f^n_\rho\}$ associated to a complete set of *compatible* observables. According to the quantum postulate, these objective properties are *representation indices* that faithfully define the realization of the corresponding Lie algebra elements as "weighted" concrete operators. According to the phase postulate, the transformations generated by these operators are *phase transformations* that do not modify the state as such, i.e. transformations under which the objective properties of the state remain invariant.

6. Polarization of Quantum States

As we have said before, in gauge theories the *restriction* to the constraint surface defined by a set of first-class constraints entails a *projection* to the reduced phase phase. We shall now argue that in quantum mechanics the (generalized) eigenvalue equations that define the "*restriction*" of the space of states to the states characterized by the corresponding eigenvalues entails a "*projection*" to the corresponding "phase orbits". To do so, we shall use the quantum operators (obtained by means of the so-called *prequantization construction* of the geometric quantization formalism)

$$\hat{v}_f = -i\hbar\nabla_{v_f} + f, \tag{6}$$

where the covariant derivative ∇_{v_f} is given by the expression

$$\nabla_{v_f} = v_f + \frac{i}{\hbar}\theta(v_f), \tag{7}$$

being θ the symplectic potential (see Refs. [6,9,22,27,33,38]). These quantum operators act on the so-called *pre-quantum states*, i.e. on the sections $\psi(q,p)$ of the bundle $\tilde{L} = L \times_{U(1)} \mathbb{C} \xrightarrow{\pi} P$ associated to a $U(1)$-principal fiber bundle $L \xrightarrow{\pi} P$ over the phase space (P, ω).[i] In geometric quantization, an irreducible representation space for the quantum operators associated to a complete set of classical observables can be obtained by *polarizing* the pre-quantum states. This means that the quantum states have to satisfy the following polarization condition:

$$\nabla_{\mathcal{P}} \psi = 0, \tag{8}$$

where $\mathcal{P}$ is a polarization of the symplectic manifold(P, ω).[j] Roughly speaking, the polarization condition (8) "cuts in half" the Hilbert space of pre-quantum states in such a way that the resulting quantum states satisfy the Heisenberg indeterminacy principle. Now, if the quantum states are polarized with respect to a polarization $\mathcal{P}$, the quantum operators acting on these states adopt a particular form that we shall call *polarized operators* and denote $\hat{v}_f^{\mathcal{P}}$.[k]

Let's consider now what we shall call the *generalized eigenvalue equation* defined by an observable f:

$$\hat{v}_f \psi_\alpha(q, p) = f_\alpha \psi_\alpha(q, p), \tag{9}$$

where $\psi_\alpha(q, p)$ denotes the pre-quantum states defined by the possible eigenvalues f_α of the quantum operator $\hat{v}_f$ (we omitted the other possible quantum numbers that define the states). In this equation, we do not presuppose–as it is the case in the usual formulation of quantum mechanics–that the states ψ_α are already polarized. Since no polarization is

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[i]In order to define an isomorphism between the Poisson algebra $\mathcal{C}^\infty(P)$ of classical observables and the Lie algebra of quantum operators, the $U(1)$-principal fiber bundle $L \xrightarrow{\pi} P$ has to be endowed with a Hermitian connection of curvature defined by the symplectic form ω. It is possible to show that this geometric construction exists if and only if ω satisfies the so-called *integrality condition*, i.e. if and only if $(2\pi\hbar)^{-1}\omega$ defines an integral cohomology class in $H^2(P, \mathbb{Z})$ (see Refs. [6,9,27,33,38] for details).

[j]A polarization of a $2n$-dimensional symplectic manifold (P, ω) is a foliation of P by Lagrangian (i.e. maximally isotropic) submanifolds, i.e. by n-dimensional submanifolds $K \subset P$ such that ω vanishes on $T_x K \times T_x K$.

[k]For instance, the prequantization of the cotangent bundle $M = T^*\mathbb{R}$ yields the quantum operators $\hat{v}_q = i\hbar \frac{\partial}{\partial p} + q$ and $\hat{v}_p = -i\hbar \frac{\partial}{\partial q}$ associated to the complete set of observables q and p. Let's consider now the polarized states of the form $\psi(q) \in \mathcal{C}^\infty(\mathbb{R}) \subset \mathcal{C}^\infty(M)$. These states are polarized with respect to the so-called *vertical polarization* $\mathcal{P}_V$ of the symplectic manifold $T^*\mathbb{R}$, i.e. the polarization defined by the Hamiltonian vector field $\frac{\partial}{\partial p}$. The point that we want to stress here is that the quantum operators associated to q and p take, *on this subset of polarized states*, the usual forms $\hat{v}_q^{\mathcal{P}_V} = q$ and $\hat{v}_p^{\mathcal{P}_V} = -i\hbar \frac{\partial}{\partial q}$. This means that these usual expressions for the quantum operators are only valid with respect to the vertical polarization. The most general form of the quantum operators is given by the expression (6) obtained by means of the prequantization construction.

presupposed, we do not use polarized operators, but rather the quantum operators given by the whole expression (6).

We shall now interpret the generalized eigenvalue equation (9) as a "constraint equation" that selects the states in the "constraint surfaces" $f = f_\alpha$ defined by the eigenvalues of $\hat{v}_f$. By using expression (6), equation (9) yields

$$-i\hbar\nabla_{v_f}\psi_\alpha = (f_\alpha - f)\psi_\alpha, \tag{10}$$

$$\approx_{|f=f_\alpha} 0, \tag{11}$$

where the *weak equality* "≈" means that the left hand side is equal to zero only on the "constraint surfaces" $f = f_\alpha$. This equation means that a state ψ_α characterized by a sharp value f_α of the observable f must be weakly invariant along the foliation defined by the Hamiltonian vector field v_f associated to f. In other terms, *the very "constraint equation" that fixes the value of the observable f entails the "weak polarization" of ψ_α along the variable acted upon by v_f*. In this way, the polarization of states, far from being imposed by hand as it is the case in the framework of geometric quantization, is a direct consequence of this generalized eigenvalue equation.

Let's specialize the generalized eigenvalue equation for the particular case of the observable $f = p$. By using the adapted connection potential $\theta = -pdq$, equation (10) yields the weak polarization condition:

$$\begin{aligned} -i\hbar\nabla_{v_p}\psi &= (p_0 - p)\psi \\ (-i\hbar v_p - pdq(v_p))\psi &\approx_{|p=p_0} 0 \\ (-i\hbar\frac{\partial}{\partial q} - p_0)\psi &= 0, \end{aligned} \tag{12}$$

whose solution is the pure phase state $\psi_{p_0}(q) = e^{\frac{i}{\hbar}p_0 q}$ that only depends on q. This quantum state is v_p-invariant, i.e. weakly invariant with respect to the polarization defined by the Hamiltonian vector field $v_p = \frac{\partial}{\partial q}$ associated to p. Indeed, a translation in q just multiplies the state by an overall phase factor. In the conceptual framework provided by the quantum ontology, the weak polarization condition (13) simply means that the position q of a state characterized by the objective property $p = p_0$ is completely "phased out" by the phase transformations generated by the quantum operator associated to p. In other terms, the coordinate q of a state characterized by a completely determined value of p is completely undetermined, i.e. a pure phase. In this way, the polarization of quantum states, and the concomitant fact that they satisfy Heisenberg undeterminacy principle, far from being imposed by hand, naturally stems from the generalized eigenvalue equation (10).

7. Conclusion

In gauge theories, the existence of gauge symmetries reduce the number of degrees of freedom that are necessary to completely describe the states of a physical system. The heuristic idea of the present article is that the reduction in the number of observables that are necessary to describe a physical system from $2n$ classical observables (q *and* p) to n quantum observables (q *or* p) can be explained in an analogous way. Since this "quantum reduction" should be valid for both constrained and unconstrained Hamiltonian systems, it is necessary to introduce a *universal symmetry principle* different from that of gauge theories. In order to define such a symmetry principle, we proposed a group-theoretical quantum ontology for Hamiltonian systems defined by two postulates, namely the *phase postulate* and the *quantum postulate*.

On the one hand, the quantum postulate establishes an injective correspondence between the two essential roles played by physical observables in mechanics, namely (a) to assign numerical values to physical states, and (b) to define operators that generate transformations of the states. As we have argued, this postulate allows us to claim that the objective properties that define a state are *representation indices* that characterize how the corresponding elements of a Lie algebra are realized as "weighted" concrete operators acting on the states. On the other hand, the phase postulates complements the standard correspondence between *objectivity* and *invariance* under a group action by stating that *the phase transformations under which the objective properties of a state must be invariant are induced by (the operators associated to) the objective properties themselves.* As we have argued, this postulate extends the gauge correspondence between first-class constraints and gauge transformations to the observables of unconstrained Hamiltonian systems. We have then argued that, far from producing a vicious circle, the virtuous circular relationship between objective properties and phase transformations (established by the phase postulate) provides a conceptual explanation of Heisenberg indeterminacy principle. Indeed, the phase postulate entails a compatibility condition among the objective properties of the same state that reduces in half the number of observables that are necessary to completely describe the state. It is worth stressing that this explanation of Heisenberg indeterminacy principle does not appeal to any kind of epistemic restriction to the amount of information an observer can have about a system. Indeed, it follows from the phase postulate that the classical description of physical reality is overdetermined, since it does not take into account the difference between phase-invariant (or objective) properties and phase-dependent properties. On the contrary, quantum vectors, far from being states of incomplete knowledge, convey a *complete* description of all the *objective* properties of physical systems.

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References

1. Abraham, R., Marsden, J.E.: Foundations of Mechanics, 2nd edn. Addison-Wesley Publishing Company, Massachusetts (1978).
2. Arnold, V.I.: Mathematical Methods of Classical Mechanics. Springer, New York (1989).
3. Auyang, S.Y.: How is Quantum Field Theory Possible? Oxford University Press, Oxford-New York (1995).
4. Bargmann, V.: On Unitary Ray Representations of Continuous Groups. Annals of Mathematics 59, 1–46 (1954).
5. Born, M.: Physical Reality. In: Castellani, E. (ed.) Interpreting Bodies. Classical and Quantum Objects in Modern Physics, pp. 155–167. Princeton University Press, New Jersey (1998).
6. Brylinski, J.-L.: Loop Spaces, Characteristic Classes, and Geometric Quantization. Progr. Math. 107. Birkhäuser Boston Inc, Boston (1993).
7. Busch, P., Lahti, P.J., Mittelsteadt, P.: The Quantum Theory of Measurement. Springer-Verlag, Heidelberg (1996).
8. Butterfield, J.: On Symplectic Reduction in Classical Mechanics. In: Butterfield, J., Earman, J. (eds.) Philosophy of Physics, Handbook of the Philosophy of Science, Part A, pp. 1–131. North-Holland, Amsterdam (2007).
9. Catren, G.: Fibras cuánticas para sistemas clásicos: Introducción a la cuantificación geométrica. Scientiae Studia, Vol. 11, No. 1 (2013).
10. Catren, G.: Can Classical Description of Physical Reality be Considered Complete?. In: Bitbol, M., Kerszberg, P., Petitot, J. (eds.) Constituting Objectivity: Transcendental Approaches of Modern Physics, pp. 375–386. The Western Ontario Series in the Philosophy of Science, Vol. 74. Springer-Verlag, Berlin (2009).
11. Catren, G.: On Classical and Quantum Objectivity. Foundations of Physics 38, 470–487 (2008).
12. Catren, G.: Geometric foundations of classical Yang-Mills theory. Studies in History and Philosophy of Modern Physics 39, 511–531 (2008).

13. Castellani, E.: Galilean Particles: An Example of Constitution of Objects. In: Castellani, E. (ed.) Interpreting Bodies. Classical and Quantum Objects in Modern Physics, pp. 181–194. Princeton University Press, New Jersey (1998).
14. Castellani, E.: Quantum Mechanics, Objects and Objectivity. In: Garola, C., Rossi, A. (eds.), The Foundations of Quantum Mechanics. Historical Analysis and Open Questions. Kluwer, Dordrecht (1995).
15. Debs, T., Redhead, M.: Objectivity, Invariance, and Convention: Symmetry in Physical Science. Harvard University Press, Cambridge MA (2007).
16. Dirac, P.M.: Lectures on Quantum Mechanics. Dover Publications, New York, 2001 for the Dover edition (1964).
17. Earman, J.: Tracking down gauge: an ode to the constrained Hamiltonian formalism. In Brading, K., Castellani, E. (eds.) Symmetries in Physics. Philosophical Reflections, pp. 140–162. Cambridge University Press, Cambridge (2003).
18. Earman, J., Norton, J.: What price spacetime substantivalism? The hole story. The British Journal for the Philosophy of Science 38, 515–525 (1987).
19. Fine, A.: Probability and the Interpretation of Quantum Mechanics. The British Journal for the Philosophy of Science, Vol. 24, 1, 1–37 (1973).
20. French, S.: Symmetry, Structure and the Constitution of Objects. Available in http://philsci-archive.pitt.edu/archive/00000327/ (2001).
21. Henneaux, M., Teitelboim, C.: Quantization of gauge systems. Princeton University Press, New Jersey (1994).
22. Landsman, N.P.: Between Classical and Quantum. In Butterfield, J., Earman, J. (eds.) Philosophy of Physics, Handbook of the Philosophy of Science, Part A, pp. 417–553. North-Holland, Amsterdam (2007).
23. Levy-Leblond, J.-M.: Nonrelativistic Particles and Wave Equations. Communications in Mathematical Physics 6, 286–311 (1967).
24. Libermann, P., Marle, C.-M.: Symplectic Geometry and Analytical Mechanics. Reidel, Dordrecht (1987).
25. Misner C., Thorne K., Wheeler J.A: Gravitation. Freeman, San Francisco (1973).
26. Kostant, B., Sternberg, S.: Symplectic Reduction, BRS Cohomology and Infinite-Dimensional Clifford Algebras. Annals of Physics 176, 49–113 (1987).
27. Kostant, B.: Quantization and Unitary representations. Lecture Notes in Mathematics 170, Springer-Verlag, Berlin, Heidelberg, New York (1970).
28. Mittelstaedt, P., Weingartner, P.: Laws of nature. Springer-Verlag, Berlin, Heidelberg (2005).
29. Nozick, R.: Invariances: The Structure of the Objective World. Belknap Press of Harvard University Press, Cambridge MA (2001).
30. Ortega, J.-P., Ratiu, T.S.: Momentum Maps and Hamiltonian Reduction. Birkhäuser, Boston (2004).
31. Quine, W.V.: Ontological Relativity and Other Essays. Columbia University Press, New York (1969).
32. Rovelli, C.: Quantum Gravity. Cambridge University Press, Cambridge MA (2004).

33. Souriau, J.-M.: Structure of Dynamical Systems. A Symplectic View of Physics. Birkhäuser Boston, Cambridge MA (1997).
34. Sternberg, S.: Group Theory and Physics. Cambridge University Press, Cambridge (1995).
35. Weyl, H.: Symmetry. Princeton University Press, New Jersey (1952).
36. Weyl, H.: The Theory of Groups and Quantum Mechanics. Trans. by H.P. Robertson. Dover Publications, Inc, USA (1931).
37. Wigner, E.: On Unitary Representations of the Inhomogeneous Lorentz Group. Annals of Mathematics, Vol. 40, No. 1, 149–204 (1939).
38. Woodhouse, N.: Geometric Quantization. 2nd ed., Oxford University Press, Oxford (1939).

THE PROBABILISTIC STRUCTURE OF QUANTUM THEORY AS ORIGINATING FROM OPTIMAL OBSERVATION IN THE FACE OF THE OBSERVER'S LACK OF KNOWLEDGE OF HIS OWN STATE

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One of the problems facing any attempt to understand quantum theory is that the theory does not seem to offer an explanation of the way the probabilities arise. Moreover, it is a commonly held view that no such explanation is compatible with the mathematical structure of quantum theory, i.e. that the theory is inherently indeterministic, simply because nature is like that. We propose an abstract formalisation of the observation of a system in which the interaction between the system and the observer deterministically produces one of n possible outcomes. If the observer consistently manages to realize the outcome which maximizes the likelihood ratio that the outcome was inferred from the state of the system under study (and not from his own state), he will be called optimal. The probability for a repeated measurement on an ensemble of identical system states, is then derived as a measure over observer states. If the state of the system is a statistical mixture, the optimal observer produces an unbiased estimate of the components of the mixture. In case the state space is a complex Hilbert space, the resulting probability is equal to the one given by the Born rule. The proposal offers a concise interpretation for the meaning of the occurrence of a specific outcome as the unique outcome that, relative to the state of the system, is least dependent on the state of the observer. We note that a similar paradigm is used in the literature of perception to explain optical illusions in human visual perception. We argue that the result strengthens Helmholtz's view that all observation, is in fact a form a inference.

1. Introduction

As early as 1935, Schrödinger[29] wrote: "*The rejection of realism has logical consequences. In general, a variable has no definite value before I measure* *it; then measuring it does not mean ascertaining the value that it has. But then what does it mean?*". As the advent of quantum mechanics solved the long standing problem of providing an adequate description for several important and unexplained experiments, the problem of realism in quantum mechanics was initially perceived mainly as a challenge to the construction of a new philosophy of natural science. In support of this perception, is the fact that almost all later theoretical advances with experimental consequences came about without any serious progress with this very basic problem. Yet at the same time, there was a growing recognition that progress in this problem would likely have deep consequences for the quantum-classical transition, the attempt to produce a successful unification of quantum mechanics and relativity theory, and the related problem of quantum cosmology. Halfway the sixties two important advances were made. In 1964, John Bell showed that any local hidden variable theory will yield predictions that are at odds with quantum mechanics. A few years later Kochen and Specker[24] presented an explicit set of measurements for which the simultaneous attribution of values for each of these measurements, leads to a logical contradiction. The two results can be regarded as opposite faces of the same coin. Whereas Bell's result can be verified (or refuted) by experiment, Kochen and Specker's argument shows the problem also to be a deeply-rooted theoretical one. These two results have been of such importance that the notion of realism in quantum physics is usually considered automatically as having either the meaning of 'locally realistic' (Bell), or that of 'the impossibility of attributing predetermined outcome values to the set of observables' (Kochen and Specker). The apparent lack of realism in quantum mechanics has been illustrated again and again by clever theoretical constructions ranging from Bell-type arguments to impossible coloring games and the countless attempts to produce an as loophole free as possible experimental verification of these arguments.[a]

But if "measuring a variable does not mean ascertaining the value that it has" then it is not surprising that (in general) quantum mechanics does

[a]Because local theories, by Bell's theorem, cannot give rise to some of the experimentally verifiable predictions of quantum mechanics, the requirement of locality, or so-called "local-realism" takes a prominent role. However, realism seems more fundamental than locality, in the sense that the latter is only well-defined if we can attribute some form of reality with respect to the whereabouts of the system. Moreover, the derivation of the quantum correlation for most Bell-type experiments do not, at any point, invoke spatial coordinates. As far as concerns the actual application of quantum theory, it is quite immaterial whether we calculate the correlations between various outcomes that are obtained in a single location or at space-like separated locations. Of course, for a locally realistic theory, the difference is huge.

not predict the particular outcome of an experiment, only the probability for the occurrence of that outcome. It is however unjustified to conclude from this inability that the answer to Schrödinger's question is that the occurrence of a particular outcome has *no* meaning. Every proper quantum experiment is a testimony to the contrary, for if a single outcome has nothing to say about the system at all, then how are we to derive anything at all from the sum of a great number of informationally empty statements? Whether we perform a tomographic state reconstruction or experimentally estimate the value of a physical quantity of a system, we accept that in a well constructed experiment every outcome presents a piece of information, a piece of evidence, that brings us closer to the true state of affairs. To give a more detailed answer to the question, we are in need of a model that shows *how* a single outcome is obtained. We will provide such a model[b] in an attempt to understand the meaning of the occurrence of an outcome in a quantum mechanical experiment.

2. The Process of Observation

2.1. *The deterministic observer*

The approach we present here centers crucially around the concept of observer. By an observer we mean a physical system M that interacts with a system S to produce an outcome that is relevant to a physical quantity. Every outcome will be assumed to be a member of a discrete set. This outcome can be freely copied and communicated to other observers. This is necessary as scientific results need to be communicated to have their validity checked (or refuted) by other scientists. In general, this definition of an observer will include the experimental setup, apparata, sensors, and the human operator. It is however quite irrelevant to our purposes whether we consider an apparatus or a detector, an animal or a human being as observer, as long as we agree that it is this system that has produced the outcome. We will furthermore assume the observer comes to this outcome through a *deterministic* interaction. Thus if we have perfect knowledge of the initial state of the system and of the observer, the outcome is predetermined. Besides the fact that all fundamental theories of physics (even classical chaotic systems and quantum dynamics) postulate deterministic evolution laws, the requirement of determinism allows to regard probability as a derived concept. So let us assume that the outcome of an observation

[b]A first, very similar version appeared in S. Aerts[8] in 2006.

is the result of a deterministic interaction τ:

$$\tau : \Sigma_S \times \Sigma_M \to X, \qquad \tau(\psi^s, \psi^m) = x \tag{1}$$

Here τ is the interaction rule, Σ_S is the set of states of the observed system, Σ_M the set of states of the observing system and X the set of outcomes that our observable quantity can have. We will deal only with the observation of outcomes that pertain to a *single* observable,which is why we have chosen to have no notational reference to the particular observable. The mapping τ encodes how an observer in a state $\psi^m \in \Sigma_M$, observing a system in the state $\psi^s \in \Sigma_S$, comes to the outcome $x \in X$. Because our observer is deterministic and because every outcome is assumed to be the result of such an interaction, we assume τ is a single-valued surjective mapping or a surjection. We will later consider the preimage of τ, (also called the fibre of $x \in X$ under τ, or the level set) defined in the standard way: $\tau^{-1}(x) = \{(\psi^s, \psi^m) \in \Sigma_S \times \Sigma_M : \tau(\psi^s, \psi^m) = x\}$. We assume the set of states of S and M are appropriately chosen so that the interaction of every couple (ψ^s, ψ^m) leads to an outcome in X, hence we have $\tau^{-1}(X) = \Sigma_S \times \Sigma_M$. For the observation to be meaningful, the observer faces the task of selecting an outcome from the set X that tells something about the system under observation. But the outcome is always formulated by the observer, it has to be encoded somehow in the state of the observer after the observation. Hence the outcome itself is also an observable quantity of the post-measurement state of the observer. The outcome will then have to share its story among the two participating systems that gave rise to its existence: it will always have something to say about both the observer *and* the system under study. In S. Aerts[6] it was shown by a diagonal argument, that even in the most simple case of a perfect observer, observing only classical properties,[c] there exist classical properties pertaining to himself that he cannot perfectly observe. The observer cannot obtain logical certainty with respect to correctness of a single, deterministic observation that he performed. On the other hand, observation is an absolutely indispensable part of doing science, hence it is only natural that every scientist

[c]We say the property **a** of a system S in the state s is actual, iff testing property **a** for S in the state s, yields an affirmation with certainty. A property is called classical when the outcome of the observation to test that property, was predetermined by the state of the sytem (whatever that state was) prior to the test. For a classical property we can define a negation in the lattice of properties that is simply the Boolean NOT. A property **a** is then classical for S iff for each state of S the property, *or* its negation, is actual. For details, see Ref. 6.

believes that truthful observation can and does indeed occur. Living in the real world, somewhere between the extremes of the ideal and the impossible, we wonder whether there is a strategy for the observer so that he is guaranteed that each outcome he picks uses his observational powers to the best of his ability.

2.2. *Probability in a run of experiments*

To increase knowledge (or decrease our uncertainty) about a system, the observer performs a repeated experiment. For each trial in the experiment, the interaction rule τ will determine the outcome of the experiment. Because τ is single-valued, the outcome will always be the same when the state of both the system under study and the observer are the same. But in a real life situation, no two experiments will be exactly the same. Suppose then that we have a lack of knowledge about the precise state of both the system and apparatus. With $\mathcal{B}(\Sigma_S)$ (and $\mathcal{B}(\Sigma_M)$) the Borel field of σ-additive subsets of Σ_S (and Σ_M), our repeated experiment is in its most general form characterized by two probability measures: μ_{Σ_S} as a probability measure from $\mathcal{B}(\Sigma_S) \to [0,1]$ and μ_{Σ_M} as probability measure $\mathcal{B}(\Sigma_M) \to [0,1]$. So we postulate two probability spaces:

$$\mathcal{P}_{\Sigma_S} = (\Sigma_S, \mathcal{B}(\Sigma_S), \mu_{\Sigma_S})$$
$$\mathcal{P}_{\Sigma_M} = (\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M})$$

The way the system and the apparatus interact is governed solely by τ: the measures on each set are assumed to be independent. In general, it is impossible to give a direct operational meaning to $(\Sigma_S, \mathcal{B}(\Sigma_S), \mu_{\Sigma_S})$ and $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M})$ separately, as the final probability a combination of both. To define the probability of the occurrence of an outcome we assume τ is a measurable, independent random variable from $\Sigma_S \times \Sigma_M$ onto X. The measures μ_{Σ} and μ_{Σ_M} induce the unique product probability measure $\rho : \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M) \to [0,1]$, such that for every $\sigma_s \in \mathcal{B}(\Sigma_S)$ and $\sigma_m \in \mathcal{B}(\Sigma_M)$, we define the product measure as $\rho(\sigma_s \times \sigma_m) = \mu_{\Sigma_S}(\sigma_s)\mu_{\Sigma_M}(\sigma_m)$. Because $\mu_{\Sigma_S}(\Sigma_S) = \mu_{\Sigma_M}(\Sigma_M) = 1$, we have $\rho(\Sigma_S \times \Sigma_M) = 1$. We can then construct the probability space $\mathcal{P}_{\Sigma_S \times \Sigma_M} = (\Sigma_S \times \Sigma_M, \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M), \rho)$. For an experiment to have a meaningful outcome, the experimentalist narrows down the possible states of the systems he wants to study and he designs, constructs and fine tunes his apparatus to measure the quantity of interest. As we are able to improve our capability in filtering the set of states of the system we are studying, as well as the set of detector states that we are studying the system with, we increase the precision of the questions that can be asked to nature. The final product of the efforts of the experimentalist are modelled here as the measures μ_{Σ_S} and μ_{Σ_M} and can in some instances be thought of as the preparations of both the system and the apparatus. However, because detailed control and knowledge of these measures may be impossible, we prefer to think of them as mathematical representatives of a very general kind of lack of knowledge situation. The probability of an outcome x in this setting is then defined as the probability (given μ_{Σ_S} and μ_{Σ_M}) of picking a couple $\psi^s \in \Sigma_S$ and $\psi^m \in \Sigma_M$ for which $\tau(\psi^s, \psi^m) = x$. More precisely, the probability of an outcome is the measure ρ of $\tau^{-1}(x) \in \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M)$, the subset of the states for system and observer that give rise to the outcome x. This leads to the following definition:

Definition 2.1. Given an ensemble of systems S described by the measure μ_{Σ_S} over $\mathcal{B}(\Sigma_S)$ and an ensemble of observers M described by the measure μ_{Σ_M} over $\mathcal{B}(\Sigma_M)$. Let ρ be the unique product measure $\rho = \mu_{\Sigma_S}\mu_{\Sigma_M}$ with $\rho(\Sigma_S \times \Sigma_M) = 1$. The probability $p : X \times \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M) \to [0,1]$ of the occurrence of the outcome x in the setting described by ρ, is defined as:

$$p_\rho(x) = \rho(\tau^{-1}(x))$$

Because τ is a surjective function, we have that $\cup_{x_i \in X}\tau^{-1}(x_i) = \Sigma_S \times \Sigma_M$, and thus $\sum_{x_i \in X} p_\rho(x_i) = 1$. This is our description of the most general[d] lack of knowledge situation: fluctuations in the occurrence of outcomes is a consequence of the inability to prepare identical states for the system, the apparatus, or both. The quantity $p_\rho(x)$ is what we actually measure in every experiment. The problem is that ρ contains μ_{Σ_M} and we would rather eradicate the observers' influence and this information may not be directly accessible. If we have an irreducible uncertainty about the way we study nature, it is problematic to give an *operational* meaning to both $(\Sigma_S, \mathcal{B}(\Sigma_S), \mu_\Sigma)$ and $(\Sigma_M, \mathcal{B}(\Sigma_M), \mu_{\Sigma_M})$ separately and we have to look for other strategies to obtain a probability that depends only on the system state.

[d] The definition could be made even more general by defining the probability as a map $p : \mathcal{P}(X) \times \mathcal{B}(\Sigma_S) \times \mathcal{B}(\Sigma_M) \to [0,1]$. However, as X is assumed to be a finite, discrete set, we have a trivial counting measure on the set of outcomes. Therefore it is sufficient to consider only the probability for the occurrence of a single outcome.

2.3. *Repeated measurement on identical system states*

Assume the observer is given a large number of identically prepared system states and he asked to find the state of the system. To do so, our observer will interact with each of the members of this ensemble in turn. For each and every single interaction, he will pick the outcome that somehow ‘has the largest likelihood’ of pertaining to the system. By randomizing his probe state and picking the outcomes in this way, the observer hopes to restore objectivity, so that he will eventually obtain information that pertains solely to the system under observation. To calculate the probability of an outcome if the system is in the given state ψ^s within the deterministic setting of the previous section Eq. (1) is in principle straightforward. The experiment our observer will perform is a repeated one, in which the measure μ_{Σ_S} on the set of states of the system under study is reduced to a point measure δ on the singleton ψ^s, and the measure μ_{Σ_M} on the set of states for the observer: $\rho = \delta(\psi - \psi^s)\mu_{\Sigma_M}$. (We will later assume μ is the Lebesgue measure on Σ_M, but for this section this is not required.) The probabilistic setting defined in the previous section reduces to one in which there is only a lack of knowledge situation about the state of the observer, so we will seize the opportunity to simplify the notation a bit and write $\mu = \mu_{\Sigma_M}$. With $\mathcal{B}(\Sigma_M)$ the $\sigma-$algebra of Borel subsets of Σ_M, (which we assume includes $eig(x_i, \psi^s)$ for every i), we have that the probability measure μ acts on the measure space $(\Sigma_M, \mathcal{B}(\Sigma_M))$. For any two disjoint σ_i, σ_j in $\mathcal{B}(\Sigma_M)$, we have

$$\begin{aligned} \mu : \mathcal{B}(\Sigma_M) &\to [0,1] \\ \mu(\sigma_i \cup \sigma_j) &= \mu(\sigma_i) + \mu(\sigma_j) \\ \mu(\Sigma_M) &= 1 \end{aligned} \tag{2}$$

Two observer states are equivalent if they both produce outcome x when observing $\psi^s : \psi_1^m \approx \psi_2^m$ iff $\tau(\psi^s, \psi_1^m) = \tau(\psi^s, \psi_2^m)$. Hence, for any given system state ψ^s, the mapping τ defines in a natural way a partition of the state space of the observer with each member $eig(x_i, \psi^s)$ in the partition belonging to exactly one outcome x_i. This leads to the following important definition:

Definition 2.2. The set of states of the observer that (for the interaction τ) give the outcome $x \in X$ when the system under observation is in the state ψ^s, will be denoted as $eig(x, \psi^s)$:

$$eig(x, \psi^s) = \{\psi^m \in \Sigma_M : \tau(\psi^s, \psi^m) = x\} \tag{3}$$

From the single-valuedness of τ in Eq. (1), it follows that different outcomes (for the same ψ^s) necessarily correspond to different states of the observer, so that for $x_i \neq x_j$:

$$eig(x_i, \psi^s) \cap eig(x_j, \psi^s) = 0 \tag{4}$$

If we assume that the act of observation of an observable leads to an outcome for every state of the system investigated, we have

$$\cup_{i=1}^n eig(x_i, \psi^s) = \Sigma_M \tag{5}$$

In fact, in what follows we will no longer use the mapping τ; all we need from τ are the eigensets. The probability that a state $\psi^m \in \Sigma_M$ is picked from $eig(x_i, \psi^s)$, is governed by the measure μ and given by

$$p(\psi^m \in eig(x_i, \psi^s)) = \mu(eig(x, \psi^s) \cap \Sigma_M)/\mu(\Sigma_M) \tag{6}$$

If we denote by $x|\psi^s$ the event that x is the outcome for observation if the state of the system is ψ^s, then the physical occurrence of the event $x|\psi^s$ coincides with the occurrence $\psi^m \in eig(x_i, \psi^s)$. Hence if we denote by $p(x|\psi^s)$ the probability of the outcome x if the system is in the state ψ^s, we can calculate $p(x|\psi^s)$ from μ using Eqs. (6) and (2) to obtain:

$$p(x|\psi^s) = p(\psi^m \in eig(x_i, \psi^s)) \tag{7}$$

$$= \mu(eig(x, \psi^s)) \tag{8}$$

This last formula is fundamental to this paper. It says that for a repeated experiment on a collection of identically prepared pure system states, the probability $p(x|\psi^s)$ is the ratio of observer states that, given the system state ψ^s tell the outcome is x, to the total number of observer states. For our normalized measure μ, this is simply $\mu(eig(x, \psi^s))$.

Three remarks are in order here. First, the notation $p(x|\psi^s)$ should be interpreted with care. It denotes the probability that x obtains when the system is in the state ψ^s. In this sense it corresponds to our intuitive notion of a conditional probability which explains the notation. However, we do not propose this is a mathematical conditioning on ψ^s. It will be understood that whenever we write $p(x|\psi^s)$ we simply take it to have the meaning given by Eq. (7). Second, the sets $eig(x_i, \psi^s)$ are *not* sets of eigenvectors in the algebraic sense of the word[e] but a generalization of these. Indeed, if it

[e]The sets (3) are called in eigensets in accordance with Ref. 4.

happens to be the case that for a given ψ^s and for (almost) every $\psi \in \Sigma_M$, we have $\tau(\psi^s, \psi) = x_k$ so that

$$\mu(eig(x_k, \psi^s)) = \mu(\Sigma_M) \tag{9}$$

then, for that particular ψ^s, we obviously have $p(x_k|\psi^s) = 1$. The vector ψ^s thus defined, will evidently coincide with a regular eigenvector that corresponds to the eigenvalue x_k if the state space is a Hilbert space. Our last remark concerns the outcome x. It is obvious that Eq. (2) is additive in X:

$$\mu(eig(x_i, \psi^s) \cup eig(x_j, \psi^s)) = \mu(eig(x_i, \psi^s)) + \mu(eig(x_j, \psi^s)) \tag{10}$$

because of Eq. (4). Therefore we can (and will) restrict our discussion to the probability of the occurrence of a single outcome. The success of the program to model the probabilities in quantum mechanics as coming from a lack of knowledge about the precise state of the observer stands or falls with the question of defining a natural mapping τ (which determines the outcome and hence $eig(x, \psi^s)$) such that the measure μ of the eigenset $eig(x_i, \psi^s)$ pertaining to outcome x_i is identical with the probability obtained by the Born rule.

2.4. *The optimal observer*

We can see from Eq. (7) that the system state ψ^s can be associated with a probability in a fairly trivial way: the probability of an outcome x when the system is in a pure state ψ^s, is the probability the observer attributes x to the outcome of the experiment. In a repeated trial this equals the relative proportion of observer states that attribute outcome x to that state. Even for a repeated measurement on a set of identical pure states, probability can arise from a lack of knowledge concerning the precise state of the observer. Suppose now the observer, considered as a system in its own right, is in a state ψ^m. Then in exactly the same way we can associate probabilities with that state too. The operational meaning of this association is given either by a secondary observer observing an ensemble of observers in the state ψ^m, or by the observer consistently (mis)identifying his own state ψ^m for a state of the system ψ^s. We have argued that every outcome will say something about the observer, (that is, about ψ^m), and something about the system (that is, about ψ^s). The problem is that this information is mixed up in a single outcome. Some outcomes might contain more information about the state of the system, and some more about the state of the apparatus.[f] Eventually, we, as operators of our detection apparatus, will have to decide whether we will retain a given outcome or reject it. Such decisions are a vital part of experimental science. For example, an outcome that is deemed too far off the expected value (so-called outliers) is rejected and excluded from subsequent analysis. The usual rationale for this exclusion is that an outlier does not contain information about the system we seek to investigate, but rather that it represents a peculiarity of that particular measurement. Rejection or acceptance of an outcome is in its most rudimentary form based on the following binary hypotheses:

H_0 : The observer takes ψ^s as the state of the system under investigation
H_1 : The observer takes ψ^m as the state of the system under investigation

The observer chooses the outcome that maximizes the likelihood that H_0 prevails, which is the optimal strategy if the outcome he delivers would be judged for acceptance or rejection by one with absolute knowledge about ψ^s and ψ^m. If it is possible (with non-vanishing probability) to get an outcome x_i in an experiment under either hypothesis, then the factual occurrence of x_i supports *both* hypotheses simultaneously. What really matters in deciding between H_0 and H_1 on the basis of a single outcome, is not the probability of the correctness of each hypothesis itself, but rather whether one hypothesis has become *more likely* than the other as a result of getting outcome x_i. From decision theory (see, for example, Jaynes[22]), we have that all the information in the data that is relevant for deciding between H_0 and H_1, is contained in the so-called likelihood ratios or, in the binary case, the *odds:* $\Lambda_i = \frac{p(x_i|H_0)}{p(x_i|H_1)}$. If we equate $p(x_i|H_0) = p(x_i|\psi^s)$ and $p(x_i|H_1) = p(x_i|\psi^m)$, we get:

$$\Lambda_i = \frac{p(x_i|\psi^s)}{p(x_i|\psi^m)}, i = 1, \ldots, n \tag{11}$$

In this last formula, the numerator and denominator are given by Eq. (7). We are now in position to state our proposed strategy for the optimal observer.

Definition 2.3. We call a system M in the state ψ^m an *optimal observer* iff, after an interaction with a system in the state ψ^s, the state of M

[f]A typical example arises when it is known that the noise level fluctuates, but the particular noise level for a given outcome is unknown.

expresses the outcome x_j which maximizes the likelihood ratio Λ_i (11):

$$x_j = \arg\max_{x_i} \frac{p(x_i|\psi^s)}{p(x_i|\psi^m)}$$

Picking the outcome x_j from X that maximizes the corresponding likelihood ratio Λ_j, is simply optimizing the odds for H_0, given the states ψ^s and ψ^m. This concludes our description of the observer. To calculate the probability for a repeated experiment when an observer is optimal, we first need a state space. We are especially interested in complex Hilbert space, but it is instructive to look at statistical mixtures first.

2.5. *The optimal observer for statistical mixtures*

First we define the convex closure of a number of elements $a_1, \ldots, a_n \in A$:

$$[a_1, \ldots, a_n] = \left\{ a \in \mathbb{R}^n : a = \sum \lambda_i a_i, 0 \leq \lambda_i \in \mathbb{R}, \sum \lambda_i = 1 \right\} \tag{12}$$

With $[C]$, we will denote the convex closure of the elements in C. The standard $(n-1)$ simplex Δ_{n-1} generated by the outcome set X is:

$$\Delta_{n-1}(X) = [x_1, \ldots, x_n] \tag{13}$$

By identification of the axes of $\mathbb{R}^n$ with the members of X, we have $\Delta_{n-1}(X) \subset \mathbb{R}^n(X)$, the free vector space generated by the outcome set X. If the conditional probabilities $p(x_i|\psi^s)$ are well defined (which we will just accept for now), we can make a summary of them in a single vector $\mathbf{x}(\psi^s)$:

$$\mathbf{x}(\psi^s) = \sum_{i=1}^{n} p(x_i|\psi^s) x_i \tag{14}$$

We see from Eqs. (14) and (9) that $\mathbf{x}(\psi^s)$ is in $\Delta_{n-1}(X)$. Vectors like $\mathbf{x}(\psi^s)$ are often called 'statistical states' or 'mixtures' in the literature. Suppose now that all we can or care to know about the system S and the observer M, are the statistical states, i.e. the probabilities related to the outcomes of a single experiment. Within this constraint, the vector $\mathbf{x}(\psi^s)$ represents all there is to know about S and the state spaces Σ_S and Σ_M reduce to $\Delta_{n-1}(X)$:

$$\Sigma_S = \Sigma_M = \Delta_{n-1}(X) \tag{15}$$

Having identified ψ^s with $\mathbf{x}(\psi^s)$ in this particular case, the conditional probability $p(x_1|\mathbf{x}(\psi^s))$ denotes the probability that outcome x_1 occurs when our knowledge about the system is encoded in the statistical state $\mathbf{x}(\psi^s)$:

$$\mathbf{x}(\psi^s) = p(x_1|\mathbf{x}(\psi^s))x_1 + \ldots + p(x_n|\mathbf{x}(\psi^s))x_n \tag{16}$$

In this section $\langle , \rangle$ denotes the standard real inner product in a finite dimensional vector space, and with $\langle x_i, x_j \rangle = \delta_{ij}$, we have from this last equation

$$p(x_k|\mathbf{x}(\psi^s)) = \langle \mathbf{x}(\psi^s), x_k \rangle \tag{17}$$

What this last equation tells us is that for a statistical state, the magnitude of the i^{th} coordinate equals the probability of outcome x_i. We have a state space Eq. (13), and we have a rule to extract a probability from a state Eq. (17), so we are in position that allows us to characterize the sets $eig(x_k, \mathbf{x}(\psi^s))$. Let $\mathbf{x}(\psi^s)$ and $\mathbf{x}(\psi^m)$ be states in $\Delta_{n-1}(X)$, written as:

$$\begin{aligned} \mathbf{x}(\psi^s) &= \sum_{i=1}^{n} t_i x_i \\ \mathbf{x}(\psi^m) &= \sum_{i=1}^{n} r_i x_i \end{aligned} \tag{18}$$

By the definition of optimal observation, we have that the outcome x_k is chosen, if for all $j \neq k$, the corresponding likelihood ratio's satisfy $\Lambda_k > \Lambda_j$. By Eq. (11) and Eq. (14), x_k is chosen, iff for all $j = 1, \ldots, n \quad (j \neq k)$, we have:

$$\frac{p(x_k|\mathbf{x}(\psi^s))}{p(x_k|\mathbf{x}(\psi^m))} > \frac{p(x_j|\mathbf{x}(\psi^s))}{p(x_j|\mathbf{x}(\psi^m))} \tag{19}$$

The regions $eig(x_k, \mathbf{x}(\psi^s))$, are found by substitution of Eq. (18) in Eq. (17) and then into Eq. (19). With $j = 1, \ldots, n; \quad j \neq k$, we obtain:

$$eig(x_k, \mathbf{x}(\psi^s)) = \left\{ \mathbf{x}(\psi^m) \in \Delta_{n-1} : \frac{t_k}{r_k} > \frac{t_j}{r_j} \right\} \tag{20}$$

According to Eq. (7), the probability of the outcome x for the repeated experiment on a set of identical system states, is the ratio of observer states that tell the outcome is x, to the total number of observer states. Because the state space is Euclidean, it is natural to take for μ the $(n-1)$-Lebesgue measure in Δ_{n-1}, assumed to be normalized: $\mu(\Delta_{n-1}(X)) = 1$. Because of Eq. (7), we have:

$$p(x_k|\mathbf{x}(\psi^s)) = \mu(eig(x_k, \mathbf{x}(\psi^s))) \tag{21}$$

However, because of the way we defined the statistical state, the probability should also be given directly by the components of the state. So the question is whether the optimal observer Eq. (21) can recover that probability, i.e. is it true that Eq. (21) equals Eq. (17):

$$\mu(eig(x_k, \mathbf{x}(\psi^s))) = \langle \mathbf{x}(\psi^s), x_k \rangle \tag{22}$$

To see if this is the case, we need to characterize the eigensets for this particular state space. To do so, we first define the open convex closure of a number of elements $x_1, \ldots, x_n \in \mathbb{R}^n$ as

$$]x_1, \ldots, x_n[= \left\{ x \in \mathbb{R}^n : x = \sum \lambda_i x_i, 0 < \lambda_i \in \mathbb{R}, \sum \lambda_i = 1 \right\} \tag{23}$$

With this definition, we can characterize the $eig(x_k, \mathbf{x}(\psi^s))$ for the statistical state as being 'almost equal' to the set of open simplices in a simplicial subdivision of $\Delta_{n-1}(X)$ that one obtains when one replaces one vertex of the original simplex $\Delta_{n-1}(X)$ with the vector representing the state:

$$C_k^s =]x_1, \ldots, x_{k-1}, \mathbf{x}(\psi^s), x_{k+1}, \ldots, x_n[\tag{24}$$

Lemma 2.1. *Let C_k^s be defined as in Eq. (24), $[C_k^s]$ be the convex closure of C_k^s, and $eig(x_k, \mathbf{x}(\psi^s))$ by Eq. (20), then:*

$$C_k^s \subset \ eig(x_k, \mathbf{x}(\psi^s)) \subset [C_k^s]$$

Proof. We start with the first inclusion. Suppose $\mathbf{x}(\psi^m)$ is in one of the open $(n-1)-$simplices C_k^s, then, by definition, there exist λ_i such that, with $0 < \lambda_i < 1, \sum \lambda_i = 1$,

$$\mathbf{x}(\psi^m) = \sum_{i \neq k}^{n} \lambda_i x_i + \lambda_k \mathbf{x}(\psi^s) \tag{25}$$

On the other hand we have that $\mathbf{x}(\psi^s) \in \Delta_{n-1}$ and hence there exist $t_l \geq 0, \sum_{l=1}^{n} t_l = 1$ such that Eq. (14) holds:

$$\mathbf{x}(\psi^s) = \sum_{l=1}^{n} t_l x_l \tag{26}$$

Substitution of Eq. (25) into Eq. (26) yields

$$\mathbf{x}(\psi^m) = \lambda_k t_k x_k + \sum_{i \neq k}^{n} (\lambda_i + \lambda_k t_i) x_i$$

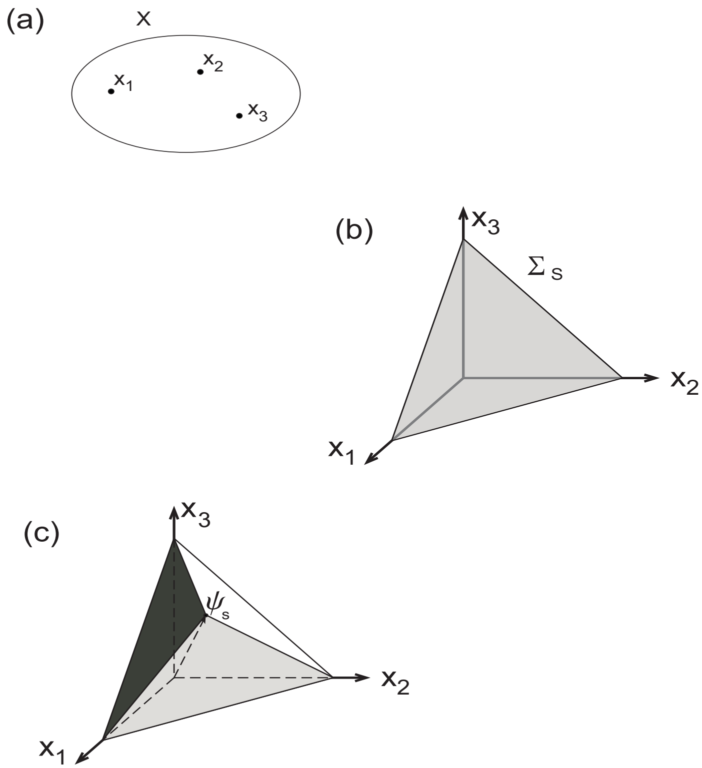

Fig. 1. Illustration of the scheme in the simplex state space. We start with the discrete outcome set, depicted in (a). The state space for an outcome set with three outcomes is the standard 2-simplex in the free vector space generated by the outcome set over the field of real numbers as depicted in (b). In (c) we see the regions of the simplex that show what outcome will be obtained from a Bayes-optimal measurement. An apparatus state picked from the darkest shaded triangle (which represents $eig(x_2, \mathbf{x}(\psi^S))$ will lead to the outcome x_2, in the lightest region to x_1, and the intermediately shaded region leads to the outcome x_3. The probability is the Lebesgue measure over the depicted eigensets. I.e., the probability of obtaining the outcome x_2, is the normalized area of the darkest triangle.

Calculating the likelihood ratios Eq. (11), we obtain $\Lambda_k = \frac{1}{\lambda_k}$, and for $i \neq k$ we have:

$$\Lambda_i = \frac{t_i}{\lambda_i + \lambda_k t_i}$$

We easily see that $\Lambda_k > \Lambda_i$ iff $\lambda_i > 0$, which is satisfied by assumption. Hence, by Eq. (20) every $\mathbf{x}(\psi^m) \in C_k^s$ gives an outcome x_k, establishing the result. For the second inclusion, suppose there exists some $\mathbf{x}(\psi^m) \in \Delta_{n-1}$

with $\mathbf{x}(\psi^m) \notin [C_k^s]$. The sets C_k^s in our theorem, as can be seen from the definition of Eq. (24), are disjoint open $(n-1)$-simplices. If we had defined them by means of the *closed* convex closure, they would maximally share the $(n-2)$-simplex $\Delta_{n-2}^s(j,k) = [\mathbf{x}(\psi^s), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]$:

$$[C_j^s] \cap [C_k^s] = \Delta_{n-2}^s(j,k)$$

Assume first a is not in the boundary of $[C_k^s]$, i.e. not in one of the lower dimensional sub-simplices $\Delta_{n-2}^s(j,k)$. Then $\mathbf{x}(\psi^m) \in C_i^s$ with $i \neq k$. Because of the above demonstrated first inclusion we have $\mathbf{x}(\psi^m) \in eig_{\mathbf{x}(\psi^s)}(x_i)$ and hence $\mathbf{x}(\psi^m) \notin eig(x_k, \mathbf{x}(\psi^s))$. If on the other hand $\mathbf{x}(\psi^m) \in \Delta_{n-2}^s(j,k)$, our outcome assignment on the basis of the maximum likelihood principle is ambiguous, as there will be two equal maxima, and even more when $\mathbf{x}(\psi^m)$ is chosen in a still lower dimensional subsimplex. However, we are free to choose whatever outcome we like as long as it is one of the maxima. Because the maxima coincide, these points lie in the boundary and hence the conclusion remains $eig(x_k, \mathbf{x}(\psi^s)) \subset [C_k^s]$. A graphical representation of the eigensets for this state space is depicted in Fig. 1. To obtain the probability of Eq. (21), we calculate the $\mu-$measure of $[C_k^s]$, which is simply the $(n-1)$-dimensional volume of the simplex $[C_k^s]$. □

Lemma 2.2. *If μ is a (probability) measure such that $\mu(\Delta_{n-1}(X)) = 1$, and C_k^s is defined by the convex closure of Eq. (24), then we have $\mu([C_k^s]) = \langle \mathbf{x}(\psi^s), x_k \rangle$.*

Proof. One can calculate of the volume of a simplex straightforwardly by determinant calculus, as was done by D. Aerts.[2] For completeness, we include an alternative in the form of a simple geometric argument. Let ρ_{n-1} be the (not necessarily normalized) $(n-1)$-Lebesgue measure in $\Delta_{n-1}(X)$. Then we have

$$\begin{aligned}
\mu([C_k^s]) &= \frac{\rho_{n-1}([C_k^s])}{\rho_{n-1}(\Delta_{n-1})} \\
&= \frac{\rho_{n-1}([x_1, \ldots, x_{k-1}, \mathbf{x}(\psi^s), x_{k+1}, \ldots, x_n])}{\rho_{n-1}([x_1, \ldots, x_n])} \\
&= \frac{\rho_{n-2}\,(B)\, d(B, \mathbf{x}(\psi^s))}{\rho_{n-2}\,(B)\, d(B, x_k)}
\end{aligned}$$

In this last equation, $B = [x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n]$ is the face shared by the two simplices, and $d(B,a)$ the smallest Euclidean distance between point a and each point of face B, which is proportional to the norm of the orthogonal projection of a onto a unit vector b perpendicular to B. In

$\mathbb{R}^n$ many vectors are perpendicular to B (which only has affine dimension $n-2$), but as long as we stick to the same vector b for both simplices, the same constant of proportionality will apply and the ratio will eliminate that constant. Pick the x_k base vector as b, which is obviously unit-norm and perpendicular to B. The orthogonal projection P_b of the top of C_k^s to b is given by: $P_b(\mathbf{x}(\psi^s)) = \langle \mathbf{x}(\psi^s), x_k \rangle x_k = t_k x_k$. For Δ_{n-1}, the top is the vector x_k itself and its projection $P_b(x_k) = \langle x_k, x_k \rangle x_k = x_k$. Hence we have

$$\begin{aligned} \frac{d(B, \mathbf{x}(\psi^s))}{d(B, x_k)} &= \frac{||P_b((\mathbf{x}(\psi^s))||}{||P_b((x_k))||} \\ &= t_k x_k / x_k = t_k \\ &= \langle \mathbf{x}(\psi^s), x_k \rangle \end{aligned} \tag{27}$$

□

Theorem 2.1. $\mu(eig(x_k, \psi^s)) = t_k$.

Proof. By the first lemma, we have $C_k^s \subset eig(x_k, \mathbf{x}(\psi^s)) \subset [C_k^s]$. Because $A \subset B \implies \mu(a) \leq \mu(B)$ we have

$$\mu(C_k^s) \ \leq \mu(eig(x_k, \mathbf{x}(\psi^s))) \leq \mu([C_k^s])$$

By the second lemma we have $\mu([C_k^s]) = t_k$. To calculate $\mu(C_k^s)$, we note that $\mu(C_k^s) = \mu([C_k^s]) - \mu([C_k^s] \cap C_k^s)$. Because $[C_k^s] \cap C_k^s$ is the collection of faces of C_k^s, a set of finite cardinality whose members have an affine dimension maximally equal to $n-2$, it is $\mu-$negligible, hence we also have $\mu(C_k^s) = t_k$, establishing the result. □

Note that we did not specify what happens on the boundary of $[C_k^s]$. In this case, the principle of optimal observation does not allow us to pick a unique outcome because several of the Λ_i will have the same magnitude. As the above theorems show, this is of no consequence with respect to the magnitude of the probability for each outcome. We see that indeed Eq. (22) holds and the optimal observer recovers the probability that was encoded in the statistical state:

$$p(x_k|\mathbf{x}(\psi^s)) = t_k = \mu(eig(x_k, \mathbf{x}(\psi^s)))$$

In this way the observer succeeds in obtaining a quantity that, in the limit of an infinite number of measurements, depends only on the state of the system under investigation, and not on his own state. The idea of obtaining the probability in the simplex state space as a uniform measure over the eigensets was first presented in 1986 by Diederik Aerts,[2] where the scheme was proposed under the name “hidden measurements” to indicate the origin of the lack of knowledge. We feel this paper extends these ideas in two important ways. Firstly, in publications on hidden measurements, the eigensets are postulated *ad hoc* (i.e. because they yield the correct probability), whereas we have here derived their simplicial shape from the principle of optimal observation. Secondly, in D. Aerts[2] the result is only presented in a real vector space, with the exception of the two dimensional case where the Bloch sphere representation is used. The generality of the principle of optimal observation and the more abstract approach used here, allows to motivate the shape of the eigensets and extend the results of D. Aerts[2] to systems with a complex state space of arbitrary countable dimension.

Before we investigate the complex state space, two remarks are in order. First, we did not specify whether the state $\mathbf{x}(\psi^s)$ is the result of mixing 'pure' components with appropriate weights, as indicated by the components of the state, or whether it represents a statistical tendency of an ensemble of identical 'pure' states to reveal itself in the different outcomes. If all we are allowed to do is perform a single experiment on each member of the ensemble, then from the statistics of a single observable, we cannot distinguish between these two situations. Say we have an urn filled with coins and we are allowed to inspect the coin only after a single throw of the coin, for every coin in the urn, then we cannot know whether it is a tendency of the coin to show heads with probability 1/2, or whether half of the coins have both sides heads and half of them have both sides tails (or indeed a mixture of these two situations). Secondly, it is interesting that, even for the conceptually simple statistical mixtures, the outcome assignment given by the optimal observer is *contextual* in the following sense: given a state for the observer and system that lead to the outcome x_l, then the mere interchanging of the coefficients t_j and t_k (equal to the probability for the outcomes x_j and x_k) can easily result in a different outcome than x_l, even if *neither* x_j, *nor* x_k is equal to x_l! This can readily be verified in Fig. 2. However, the probability $p(x_i|\mathbf{x}(\psi^s))$ of the outcome is a function of t_i only, hence the probability itself is non-contextual because the effect is cancelled by the uniform distribution of the observer states.

Conversely, given a state of an observer ψ^m and a system state ψ^s that interact to yield the outcome x_k, it is often possible to change the outcome of the optimal observer to a different outcome by interchanging suitable coefficients of the observer, leaving r_k untouched. This means that changing the observer's preferences over the outcomes x_j and x_l, may let the optimal observer decide x_m is a more optimal outcome than x_k, for

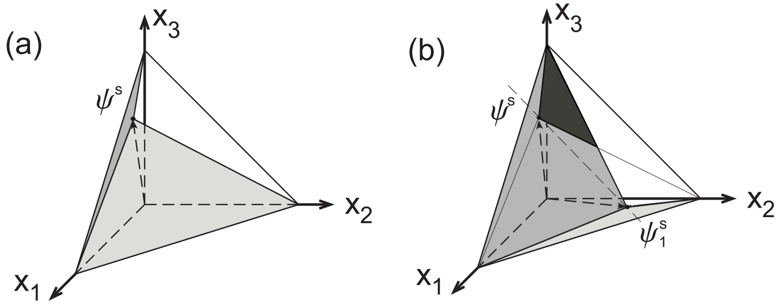

Fig. 2. (a) Suppose a measurement of the state ψ_s yields outcome x_1. This means the state of the observer was somewhere in the white triangle. (b) If we interchange the second and third component of ψ^s , we obtain ψ_1^s. The probability of obtaining outcome x_1 is the same as in (a), because the two triangles have the same area as we did not change the x_1 component in the state ψ^s to obtain ψ_1^s. However, an observer state choosen from the black shaded region would yield outcome x_1 in (a), whereas it would yield x_2 in (b).

j, l, m, k all different! This contextual aspect of the outcome assignment is here understood as a result of the bias of the observer, for the coefficients of his state represent his tendencies for each outcome when he is looked upon by a second optimal observer. From Fig. 2 we see that the contextuality of the outcome assignment depends on the classical entropy of the state. According to a well-known theorem due to Shannon, the higher the entropy of the state ψ^s, the closer the coefficients of ψ^s in Eq. (14) are to $1/n$ and the closer this state will reside near the centre of the simplex, effectively limiting the possibilities for producing a contextual outcome change by interchanging coefficients.

2.6. *Optimal observation in complex Hilbert space*

Complex Hilbert spaces are of considerable interest as they arise naturally in many prominent scientific areas including quantum theory, signal analysis (both in time-frequency and in wavelet analysis), electromagnetism and electronic networks,[g] and the more recently founded shape theory.[23] We are here interested especially in probability for quantum states in a finite dimensional Hilbert space, hence the probability of outcomes relating to an

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[g]Interestingly, the name probability *amplitude*, and indeed the Born interpretation of the wave vector in quantum mechanics, were conceived by Born in analogy with electromagnetic waves. In that case, the norm of the vector is not equal to unity, but equal to the energy in the wave, as probability conservation is replaced by conservation of energy.

observable with a non-degenerate discrete spectrum. The natural setting for the discrete state space is the space of square integrable functions on a Hilbert space $\mathcal{H}_n(\mathbb{C})$ over the field of complex numbers. A general state of the system $\psi^s \in \Sigma_S = \mathcal{H}_n(\mathbb{C})$ can then be written as:

$$\psi^s = \sum_{i=1}^{n} q_i x_i \tag{28}$$

where $q_i \in \mathbb{C}$ and $|\psi^s| = 1$. In this case the outcome set X consists of an orthonormal frame of complex vectors $\{x_i\}$, thought of as the set of (distinct) eigenvectors of the Hermitian operator A that represents the observable $\mathcal{A}$ that we are interested in. An observer (or a detector, which is quite the same for our purposes) usually has a very large number of internal degrees of freedom. Accordingly it lives in a Hilbert space of appropriately high dimensionality. However, by the Schmidt bi-orthogonal decomposition theorem, we know we can model every possible interaction between two systems, one living in a Hilbert space of dimension n and one in a Hilbert space of dimension m with $m > n$, by an interaction of two systems, each one living in a Hilbert space of dimension n. With this in mind, we model the set of states of the observer as unit vectors in $\mathcal{H}_n$:

$$\Sigma_M = \{\psi \in \mathcal{H}_n(\mathbb{C}) : |\psi| = 1\}$$

The state of the observer, to us, always means the subset of the state space that is of relevance to the production of outcomes. This is especially relevant for the interpretation of sentences such as “uniform distribution of initial observer states”, which taken too literally, would indicate the observer is perhaps doing something completely different than observing. The state of an observer with respect to an experiment with outcome set X can be written as ($r_i \in \mathbb{C}$)

$$\psi^m = \sum_{i=1}^{n} r_i x_i \tag{29}$$

Because the coefficients now assume complex values, they cannot be interpreted as probabilities because, for one thing, we do not have a total order relation in the field of complex numbers. This difference also affects the deeper, deterministic level of the description in a profound way. Let us briefly explain why. For the statistical states of the former section, each eigenset is a subsimplex of the state space. Because the eigensets share at most a lower dimensional face, any two different eigensets (for a fixed system state) can be separated[h] by a single hyperplane. But in a complex space a hyperplane does not separate that space in two half-spaces. To apply the criterion of optimality, one needs to decomplexify the space to restore the order relation, but this can be done in a variety of ways. It seems the principle of optimality alone cannot resolve this ambiguity. On the other hand, this plurality of decomplexifications need not bother us too much. Just as in the case of the statistical states of the former section, the observer can check the statistical validity of his outcome assignment by verifying that the probability (in the sense of a relative frequency) that results from repeated application of his outcome assignment, equals the *assumed* probability. In the same way, we can simply postulate, or even guess, a specific form of the probability assignment and justify it *a posteriori*: If the relative frequency of an outcome (as a result of the observers' outcome assignment, based on the optimal condition), converges to a limit that yields (a monotone function of) the very probability assignment he used to obtain those outcomes, the optimal observer knows a posteriori he was optimal. If he wasn't, he can always restart with another guess. Let us attempt a minimal generalization of the real case Eq. (20), with ψ^s and ψ^m defined as in Eqs. (28) and (29) and $j = 1, \dots, n; \quad j \neq k$:

$$eig^{\mathbb{C}}(x_k, \psi^s) = \left\{ \psi^m \in \Sigma_M : \frac{|q_k|}{|r_k|} > \frac{|q_j|}{|r_j|} \right\} \tag{30}$$

The only difference with Eq. (20), is that we take the *modulus* of the coefficients and that the eigenset contains complex vectors, which is why we have given it the superscript $\mathbb{C}$. To check the consistency of our optimal observer in the complex state space, we evaluate the Lebesgue measure $\nu(eig^{\mathbb{C}}(x_k, \psi^s))$. Therefore we regard measure ν in $\mathbb{C}^n$ as the Lebesgue measure μ over $\mathbb{R}^{2n}$. The calculation of the measure by direct integration can be avoided by use of a mapping ω that preserves the measure. Recall that a measurable mapping $\omega : X \to Y$ is called *measure-preserving* between two measure spaces $(X, \mathcal{B}(X), \nu)$ and $(Y, \mathcal{B}(Y), \mu)$ if, for every $A \in \mathcal{B}(Y)$, we have $\nu(\omega^{-1}(A)) = \mu(A)$. In our following lemma we show that the Haddamard product preserves our measures up to a constant of proportionality.

[h]If C_1 and C_2 are two sets in $\mathbb{R}^n$, then a hyperplane H is said to *separate* C_1 and C_2 iff C_1 is contained in one of the closed halfspaces associated with H and C_2 lies in the opposite closed half-space. Two convex sets in $\mathbb{R}^n$ that share at most an affine set of dimension $n-1$, can be separated by a hyperplane.

Lemma 2.3. *The mapping* ω

$$\omega : S_n \to \Delta_{n-1}$$
$$\omega(z) = (z_1 z_1^*, z_2 z_2^*, \ldots, z_n z_n^*)$$

is measure-preserving up to a constant of proportionality, i.e. for the two measure spaces $(S_n, \mathcal{B}(S_n), \nu)$ *and* $(\Delta_{n-1}, \mathcal{B}(\Delta_{n-1}), \mu)$ *with* ν *and* μ *Lebesgue measures and* $A \in \mathcal{B}(\Delta_{n-1})$ *and* $\omega^{-1}(A) \in \mathcal{B}(S_n)$*, we have:*

$$\nu(\omega^{-1}(A)) = \frac{2\pi^n}{\sqrt{n}} \mu(A)$$

Proof. [i]Let A be an arbitrary open convex set in $\Delta_1 : A = \{(x_1, x_2) : a < x_1 < b,\ x_2 = 1 - x_1\}$. Evidently, $\mu(A) = \sqrt{2}(b-a)$. Consider the set B

$$B = \{(z_1, z_2) \in Z_1 \times Z_2 \subset \mathbb{C}^2 : Z_1 = \{z_1 : a < |z_1|^2 < b\},$$
$$Z_2 = \{z_2 :\ z_2 = \sqrt{1 - |z_1|^2} e^{i\theta}, \theta \in [0, 2\pi[\}\}$$

Clearly, the members in B are unit norm, hence $B \subset S_n$. Using the definition of ω it is a matter of straightforward verification to see B is the pull-back of A under ω, i.e. $\omega(B) = A$. Its measure factorizes

$$\nu(B) = \nu(Z_1)\nu(Z_2) = \pi(b-a).2\pi = \frac{2\pi^2}{\sqrt{2}} \mu(A)$$

Where we have used $\nu(Z_1) = \pi(b-a)$ because, as can be seen in Fig. 3, Z_1 is bounded between two concentric cylinders, one of radius $\sqrt{a}$ and one of radius $\sqrt{b}$. Hence the theorem holds for open convex sets in Δ_2. This conclusion can readily be extended to an arbitrary open $(n-1)$-dimensional rectangle set A in Δ_{n-1}:

$$A = \left\{ \left(x_1, \ldots, x_{n-1}, 1 - \sum_{i=1}^{n-1} x_i \right) : \forall i = 1, \ldots, n-1 : a_i < x_i < b_i;\ a_i, b_i \in [0,1] \right\}$$

Its measure factorizes into:

$$\mu(A) = \sqrt{n} \prod_{i=1}^{n-1} (b_i - a_i)$$

[i]This proof was first presented in Ref. 5, but as it is central in our argument, we include it here.

Next consider n-tuples of complex numbers:

$$\begin{aligned} B &= \{(z_1, z_2, \ldots, z_n) \in Z_1 \times \ldots \times Z_n\} \\ Z_i &= \{z_i \in \mathbb{C} : a_i < |z_i|^2 < b_i, i \neq n\}, \\ z_n &= \sqrt{1 - |z_1|^2 - \ldots - |z_{n-1}|^2} e^{i\theta_n}, \theta_n \in [0, 2\pi[\}\} \end{aligned}$$

Clearly we have $\omega(B) = A$. The measure of B can be factorized as:

$$\begin{aligned} \nu(B) &= \nu(Z_1)\nu(Z_2)\ldots\nu(Z_n) \\ &= 2\pi \prod_{i=1}^{n-1} \pi(b_i - a_i) = \frac{2\pi^n}{\sqrt{n}} \mu(A) \end{aligned}$$

Hence the theorem holds for an arbitrary rectangle set $A \subset \Delta_{n-1}$. But every open set in Δ_{n-1} can be written as a pair-wise disjoint countable union of rectangular sets. It follows that $\nu(\omega^{-1}(\cdot)) = \frac{2\pi^n}{\sqrt{n}}\mu(\cdot)$ for all open sets in Δ_{n-1}. Both ν and μ are finite Borel measures because Δ_{n-1} and S_n are both compact subsets of a vector space of countable dimension. Therefore they must be regular measures (see Rudin,[27] p. 47), which are completely defined by their behavior on open sets. Hence ω is measure preserving for Borel sets. □

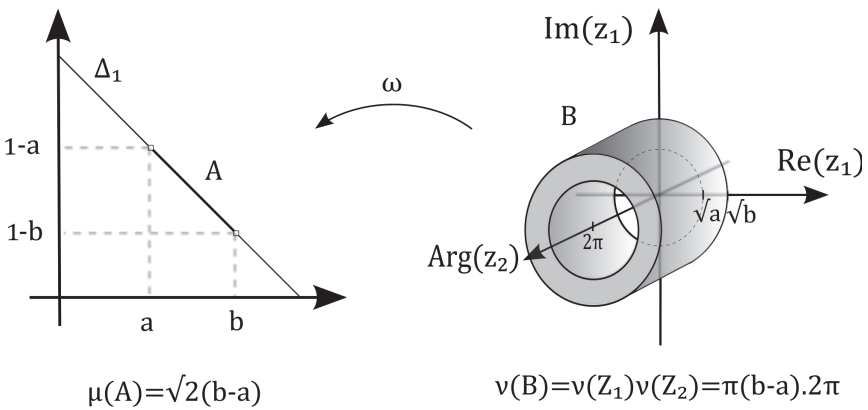

Fig. 3. A graphical representation of the calculation of the measure preserving properties of ω.

We have given a graphic representation of the action of ω in Fig. 4. We are now in a position to prove our main result.

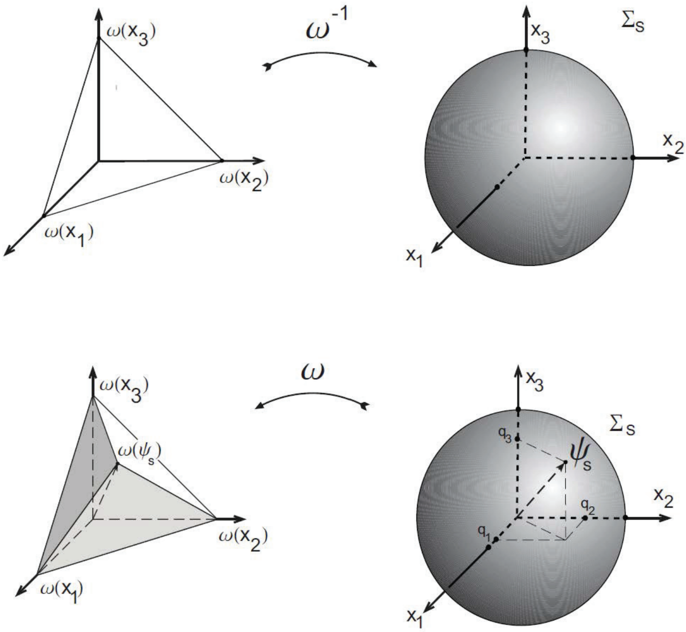

Fig. 4. The action of the mapping ω sends elements of the unit sphere to the standard simplex (upper figure). The probability for the occurrence of outcome x_k is the measure of the eigenset corresponding to outcome x_k and is calculated in the simplex using the measure preserving mapping ω. The eigensets are depicted in the lower figure for the simplex; it is not possible to show graphically what these sets look like in the complex unit sphere.

Theorem 2.2. *Let M be an optimal observer, observing a quantity of a system S such that the observation can have n possible outcomes. Let both have as states unit norm vectors in an n-dimensional complex Hilbert space. Then the probability for the occurrence of outcome x_k if the system state is ψ^s, is given by the Born rule:*

$$p(x_k|\psi^s) = |\langle x_k, \psi^s\rangle|^2$$

Proof. With $eig^{\mathbb{C}}(x_k, \psi^s)$ defined by Eq. (30), and

$$C_k^s =]\omega(x_1), \ldots, \omega(x_{k-1}), \omega(\psi^s), \omega(x_{k+1}), \ldots, \omega(x_n)[,$$

it is straightforward to show that (for more details, see Ref. 5) we have:

$$C_k^s \subset \omega(eig^{\mathbb{C}}(x_k, \psi^s)) \subset [C_k^s]$$

Let $\tilde{\mu}$ and $\tilde{\nu}$ stand for the normalized versions of the measures μ and ν in the proof of our last lemma: $\tilde{\nu}(S_n) = \tilde{\mu}(\Delta_{n-1}) = 1$, so that their constant of proportionality equals one: $\tilde{\nu}(\omega^{-1}(A)) = \tilde{\mu}(A)$. By definition $p(x_k|\psi^s) = \tilde{\nu}(eig^{\mathbb{C}}(x_k, \psi^s))$, and by the previous lemma, we have

$$\begin{aligned} \tilde{\nu}(eig^{\mathbb{C}}(x_k, \psi^s)) &= \tilde{\nu}(\omega^{-1}(C_k^s)) \\ &= \tilde{\mu}(C_k^s) \end{aligned}$$

The normalized measure $\tilde{\mu}(C_k^s)$ of the real simplex C_k^s was calculated for the real state space. We can repeat the same calculation we did to obtain Eq. (27), which in this case gives us:

$$\tilde{\mu}(C_k^s) = \langle \omega(x_k), \omega(\psi^s) \rangle$$

But $\langle \omega(x_k), \omega(\psi^s) \rangle = |q_k|^2$ by the definitions of the complex state Eq. (28) and the Haddamard product. As $|q_k|^2 = |\langle x_k, \psi^s \rangle|^2$, we have proved the result. □

We see that indeed the optimal observer recovers the Born rule as a result of maximizing the odds with respect to the outcome that pertains to the system. To be precise, we did not maximize the odds because substitution of the Born rule in Eq. (11) gives:

$$\Lambda_k = \frac{|\langle x_k, \psi^s \rangle|^2}{|\langle x_k, \psi^m \rangle|^2} = \frac{|q_k|^2}{|r_k|^2} \tag{31}$$

Whereas our observer, by Eq. (30), used the ratio's:

$$\tilde{\Lambda}_k = \frac{|q_k|}{|r_k|} \tag{32}$$

where the tilde denotes the fact that strictly speaking this is not a likelihood because $|q_k|$ and $|r_k|$ aren't probabilities (they are square roots of probabilities). Yet it is obvious that the value of k for which Eqs. (31) and (32) are maximal, is the same because one is the square of the other which is clearly a monotone function. As a consequence, it does not matter if the optimal observer works with Eq. (31) or with Eq. (32): repeated application of either strategy on the same pure state will make the relative frequency converge to the Born rule in exactly the same way in both cases.

3. Consequences of Optimal Observation

The last few sections have mainly been used to derive the mathematical result. In the remainder we will briefly sketch further avenues worthy of investigation that have more of an interpretative character.

3.1. *Decision invariance and unitarity*

The outcome chosen by a optimal observers, is the one that maximizes the corresponding likelihood ratio Λ_i. Any monotonously increasing function of the likelihood ratio's preserves their relative order and hence their maximum. By Eqs. (20) and (30), this carries over to the coefficients of the state vectors in both the real and the complex state space. The same is true for multiplication by a phase factor, which is cancelled by taking the moduli in Eq. (30). As a result, the state space is a *projective* vector space: if the vectors in the state space are multiplied by $z \in \mathbb{C}$, $0 < |z| < \infty$, this does not change the result of the decision procedure adopted by the optimal observer. But there is a much bigger class of transformations that leaves the optimal decision unaltered. For any ψ^s, the probability of x_k is defined as:

$$p(x_k|\psi^s) = \mu(eig^{\mathbb{C}}(x_k, \psi^s))$$

Because $\omega(eig^{\mathbb{C}}(x_k, \psi^s)) \subset [C_k^s]$; ω is continuous, and because the elements of $[C_k^s]$ have finite norm, the norm of the vectors in $eig^{\mathbb{C}}(x_k, \psi^s)$ is finite too. We can then apply a linear transformation to the base vectors of the state space:

$$\begin{aligned} T \ &: \ \Sigma_S \to \Sigma_S \\ T(x_j) &= \sum_i^n \sigma_{ij} x_j \end{aligned} \tag{33}$$

The eigenset $eig^{\mathbb{C}}(x_k, \psi^s)$ will accordingly be transformed by applying T to x_k and ψ^s. By Lebesgue measure theory, the volume of the transformed set is proportional to the volume of the original set, the constant of proportionality being the determinant of the transformation:

$$\mu(T(eig^{\mathbb{C}}(x_k, \psi^s))) = |\det(T)|\mu(eig^{\mathbb{C}}(x_k, \psi^s))$$

for all $eig^{\mathbb{C}}(x_k, \psi^s) \in \mathcal{B}(\Sigma)$. This is a classic result,[j] and we refer the interested reader to Rudin[27] page 54 for a proof. Note that this would typically be untrue for a nonlinear transformation. As a result, all transformations with $|\det(T)| = 1$ leave the probabilities invariant, which means we have invariance under *unitary* transformations. Intuitively this is obvious: if the probabilities have their origin in a measure on state space, then unitarily

[j] As before, we regard the complex $n-$space as a real $2n-$space, for which the theorem is applicable.

transforming the entire state space, does not alter the relative proportions of the eigensets, hence the invariance. Of course, once we have the Born rule, it is easy to derive that the probabilities are invariant under unitary transformations, because the Born rule is the square modulus of an inner product and a unitary transformation can be defined as a linear operator that leaves the inner product invariant. However, our invariance principle tells us the same story at a deeper level, for not only the probabilities are invariant under unitary transformation, but also each individually obtained outcome will be the same whether or not we unitarily transform the eigensets.

3.2. *The elusive quantum to classical transition*

Suppose we have a particular statistical mixture

$$\varphi = \xi\psi_1 + (1-\xi)\psi_2 \tag{34}$$

of two (pure) states ψ_1 and ψ_2. Suppose furthermore that

$$p(x_i|\psi_1) = q_1$$
$$p(x_i|\psi_2) = q_2$$

Then an observing system is said to satisfy the *linear mixture property* iff

$$p(x_i|\varphi) = \xi q_1 + (1-\xi)q_2 \tag{35}$$

In words: the probability of a mixture equals the mixture of the probabilities. Does the optimal observer satisfy the linear mixture property? Well, φ is a statistical mixture, as defined in the section on optimal observation of statistical mixtures, and each of the constituents in the mixture is a pure state, as defined in the section optimal observation in Hilbert space. So clearly, our optimal observer satisfies the linear mixture property. In essence, this stems from his initial states being uniformly random (almost everywhere). Indeed, suppose the distribution of the initial states of the observer is *not* uniform a.e.. Then one can always find a convex region Ω in state space with surface measure A, for which the density of observer states is not equal to $1/A$. Without giving a formal proof, one can see that, it is always possible to find two states ψ_1, $\psi_2 \notin \Omega$ and a real number $\xi \in]0,1[$, such that $\xi\psi_1 + (1-\xi)\psi_2 \in \Omega$ and for which the linear mixture property will be violated.

The linear mixture property may be essential to experimental observation: no experimenter would put his faith in the hands of a detection apparatus that manifestly fails this most basic requirement. From this perspective, the difficulty of finding an intermediate region between the classical and the quantum originates from the lack of a principle that determines *how* the observer should behave in order to objectively observe the intermediate region in absence of the linear mixture property. As an example, suppose we want to determine the length of a linearly extended system. In a classical setting we are in principle free to choose the number of outcomes and we are allowed to make many observations before we settle on the result of a single measurement. For example, we can align the zero of the measuring rod with one end point of the system and read the outcome at the other end point as many times as we want to. If we are not satisfied with the precision that the measuring rod affords we can pick a better one or improve it by adding a nonius (or vernier) system to it. As long as we are able to do this we are still in a classical regime of observation. In the classical regime of observation, the distribution of observer states will be highly non-uniform. Ideally, of all possible measurements, the only uncertainty we have about the state of the observer that is assumed to be of relevance to the measurement outcome, is an uncertainty of the order of the smallest number the measuring rod can represent. To decrease the uncertainty about the result, even beyond the precision offered by the smallest number the measuring rod can represent, it is common scientific practice to repeat the measurement many times. Assuming identical, independent observations, one can apply standard error theory. In the beginning of the eighties Wootters has shown,[33,34] using standard error theory, that the distance (angle) between two states on the unit sphere in (real) Hilbert space, is proportional to the number of maximally discriminating observations along the geodesic between those two points. This fascinating result gains in richness when considered from the point of view that the probabilities arise in a optimal way. In our search for ever more precise measurements or measurements on ever smaller constituents of nature, we eventually reach a region where we cannot repeat measurements without absorbing the system or altering its state. We may not even be able to choose freely the set of outcomes for a particular measurement as is the case in the quantum regime. It is then no longer possible to directly obtain the “true” value of a physical quantity because the eigenstate of the observing system may not (and in general will not) coincide with the state of the system under investigation. We cannot attempt the same measurement (or one with altered eigenstates) on the same system because the state of the system has been altered or even destroyed. In view of this impossibility, we are led to statistical observation on ensembles. We have shown it is possible to recover an objective probability *if* the distribution of observer states is uniform. We see that the best possible observation scheme in the classical regime entails a *minimal* uncertainty (i.e. about the interpretation of the last digit only) in the state of the observer, and in the quantum regime a *maximal* uncertainty (any outcome is in principle possible) about the state of the observer. The consequence of such an interpretation is, that we will only be able to identify intermediate regions when we allow for a more complete description of the observing system. In essence, we need to describe how to go from this minimal to this maximal uncertainty state. There are good reasons for cautiously entering this intermediate region. Some of the beautiful properties of the classical and the quantum regime will not hold. For example, the linear mixture property cannot be universally satisfied. Moreover, we will obtain probability distributions that depend not only on the system, but at least partially on the dynamics of the observing system. It is possible to construct explicit models that show[4] one can identify an intermediate region where the probabilities satisfy neither the classical statistical Bonferroni inequalities[k] indicating the absence of a straightforward Kolmogorovian model, nor the Accardi-Fedullo inequalities[1] that constrain the set of probabilities that are derivable from a Hilbert space model. This opens up a whole new area of investigation, but only if we are willing to abandon the full generality of the linear mixture property.

3.3. *Is the optimal observer objective?*

The purpose of objective observation is to obtain a probability for the outcome that depends only on the system under study. How fast the sequence of outcomes converges to this probability, depends on how well the observer manages to distinguish his state from the state of the system under study. This aspect was neglected in the previous discussion. If we apply the Born rule to calculate the quantities $p(x_i|H_0)$ and $p(x_i|H_1)$, we imply that $\sum_i p(x_i|H_0) = \sum_i p(x_i|H_1) = 1$. However, if the choice between H_0 and H_1 is indeed a binary decision problem, we should have:

$$\sum_i p(x_i|H_0) = \alpha \text{ and } \sum_i p(x_i|H_1) = 1 - \alpha \tag{36}$$

[k] The Bonferroni inqualities indicate when a set of (joint) probabilities can be derived from a Kolmogorovian probability model. The best known example of a Bonferroni inequality to the physics community, is the Bell inequality.

The reason why this apparent contradiction leads to no difficulty, is because the observer chooses his outcome, *as if* the outcome will be judged afterwards as a binary decision problem. The observer himself has no clue what the value of α might be. But even if he knew the value of α, this would make no difference. To him, this would merely have the effect of scaling the odds in Eq. (11) by $\frac{1-\alpha}{\alpha}$. The choice of the outcome for the optimal observer is based on the maximal likelihood and a monotone function of the likelihoods will not change the maximum. Thus we see that the specific value of α has no influence on the actual choice. If μ is truly uniform, then, the resulting relative frequency will converge to the optimal probability that only depends on the state of the system, whatever value α happens to have in practice. However, a small value of α implies that the probability of the outcome depending on the state of the system is small. So the expected increase in information about the system as a result of obtaining that outcome, is small. Evidently this will extend the number of measurements needed for the relative frequency of an outcome to converge to a limit. If $\alpha = 0$, then observation is impossible. If on the other hand $\alpha = 1$, then every observation is perfect, which is impossible[6] in general. Therefore we constrain α to $]0,1[$. We see that α is a crude statistical measure for the objectivity of the observer. It represents his ability to separate interior from exterior influences. It turns out we can always pick an outcome that supports H_0 more than it supports H_1 iff $\alpha > 1/2$. To see this, we proceed ad absurdum. If no outcome supports H_0 more than it supports H_1, then for all x_j,we have $\frac{p(x_j|H_0)}{p(x_j|H_1)} \leq 1$. But then[1] we also have:

$$\frac{\sum_j^n p(x_j|H_0)}{\sum_j^n p(x_j|H_1)} \leq 1, \tag{37}$$

which implies $\alpha \leq 1-\alpha$. We obtain the contradiction iff $\alpha > 1/2$. In words: if we can do only slightly better than completely arbitrary in letting the outcome probability depend on the system, we can guarantee the existence of an outcome that maximizes the odds and is greater than unity. In fact,

[1]This specific condition is known in the literature as majorization. It plays an important role in the investigation of bipartite state conversions by local operations and classical communications (LOCC). This may seem relevant in connection to our problem, as the basic scheme we present can be described as a bipartite state conversion problem. However, we cannot use the many interesting results in the literature on bipartite state conversion because LOCC's in this particular problem are operationally defined by means of local unitary transformations and a local measurement, and it is the local measurement that we seek to understand!

for any value of α we can find an (almost always unique) outcome that maximizes the odds, but when $\alpha > 1/2$, the maximal likelihood ratio enjoys the property of being greater than one.

3.4. *The optimal observer as a paradigm for observation?*

The proposed principle of observation is based on a maximum likelihood criterion, which would by some be considered a Bayesian treatment of a binary decision problem, but even so, it is not used in its decision-theoretic form. In decision theory we seek to establish which of the hypotheses enjoys the strongest support in evidence of the data. In our case, there is no data to feed the likelihood with, because we produce the data by means of the odds. The way we employ the principle is rather like an inverse decision problem, as if anticipating that the result will be judged afterwards by a decision procedure performed by one with absolute knowledge of the system and observer states prior to the measurement. A somewhat similar paradigm (without reference to quantum mechanics) is proposed in several papers that deal with visual perception by humans. The idea that the visual system is rooted in inference, can be traced back to the work of Helmholtz,[21] who proposed the notion of *unconscious inference*. It was only in the last decade that it was accepted and translated into a mathematical framework, not in the least because computer scientists who want to model the human vision system are faced with the apparent complexity that underlies human perception. The Bayesian framework provides the tools necessary to understand and explain a wide variety of sometimes baffling visual illusions that occur in human perception.[20] The names that are given in this literature to the observer are *ideal* and *bayes-optimal*. There are however some differences in the application of the principle with respect to our proposal. In the literature on visual perception, the prior distributions are often derived from real world statistics. Of course, this begs the question how these prior distributions were obtained in the first place. There are two basic possibilities to obtain a prior: either a prior distribution is based on some theoretical assumption, or it is established by looking at the relative frequency of actual recordings. The first option is the one we pursued in this article, where we assumed a uniform distribution of observer states.[m] In the second case, which is the one adopted in the literature on perception, one has the

[m] The absence of a more informative prior distribution effectively reduces the criterion of Bayes-optimality to a Neyman-Pearson maximum-likelihood criterion, so we could have called our observer a maximum likelihood observer!

advantage of being able to explain a wide variety of visual effects in human perception and how the priors can be adapted through the use of Bayesian updating, but we cannot explain observation itself. The relative frequency needed to obtain the prior is rooted in the observation of data, which requires another prior and so on ad infinitum. One can break from this loop by reconsideration of what a state is. In the literature on perception states are considered only as (real) statistical mixtures, severely limiting both the applicability and the philosophical scope of the paradigm. The state, as we have defined it here, can be a complex vector, not obtainable as a mixture in principle, and yet give rise to probabilities if we attempt to observe it as good as possible. So the state is simultaneously a description of the 'mode of being' (the pure state that physically interacts), and a 'catalogue of information' (the probabilities the optimal observer obtains). The possibility that the same principle governs human perception and quantum mechanical observation strengthens the optimal paradigm. Measurement apparata and human perception can be rooted in the same principle: the attempt to relate the outcome to the object under investigation as unambiguously as possible by choosing the outcome that has the largest odds Eq. (11). By repeating the observation many times, each time randomizing the internal state of the sensor, we obtain an invariant of the observation that pertains solely to the system.

Another interesting link with the existing literature was pointed out to me by Thomas Durt.[16] The regions of the Bohm-Bub model[12] seem to coincide with our definition of the eigensets in the complex case Eq. (30). Moreover, Bohm and Bub propose a uniform measure of states that they interpret as apparatus states. They perform the integration directly for the two dimensional case and indicate the integration scheme can be extended to the more dimensional case. Their main result, in spite of using a mathematically very different approach, is also the reproduction of the Born rule. From the perspective of this paper, optimal observation yields an interpretation for the regions employed by Bohm and Bub.

4. Concluding Remarks

If one postulates all the regular axioms of quantum theory, but replaces the probabilistic Born postulate with the assumption that outcome of a measurement is the result of a deterministic interaction with an observing system that obeys the principle of optimality, one recovers the Born rule. In this sense it is an alternative for the Born rule. It shows the primitive machinery of quantum probability can be recovered from a statistically optimal choice for the outcome in the process of observation. Not in the least as a result of the fast growing field of quantum information, the search for a Bayesian framework for quantum probability has recently been subject of number of interesting publications.[14,17,19,25,26,28,30,32] One important motivation for seeking such an interpretation is that it allows for a subjective interpretation of quantum probability by regarding the state vector as a mathematical representation of the knowledge an agent has about a system. The principle of optimal observation as a paradigm for quantum probability allows for an epistemic interpretation of the state vector too and in this respect it is in agreement with some of the lines of thought found in the references given above. An often heard critique of Bayesian interpretations of quantum probability is that from a Bayesian point of view the state vector represents *only* the knowledge available to the agent that deals with it. Many physicists reject this notion, perhaps mainly because they feel the relative frequencies obtained in actual experiments are objective features of the system and not of the knowledge of the agent. It is nevertheless true that what can be inferred about a system depends on one's prior knowledge of the system. The problem we face in the construction of a theory that seeks to understand observation as a primitive concept, is that one cannot assume to be in possession of *a priori* knowledge about the world. This is translated here as the uniform distribution of initial observer states and optimal observation of an ensemble of identical states will then result in an unbiased probability, i.e. a quantity that pertains only to the state of the system under investigation. In this sense, the state vector (and hence also the probabilities derived from it) can be truly assigned in an objective way to a system. This also follows from assumption Eq. (1); that the state is a realistic description of the system and it is the state of the system and the observer that physically and deterministically interact to produce the measurement outcome. Systems are in a state and that state uniquely determines every possible interaction. The state vector represents complete information about a system but not merely as a collection of objective attributes, but as a representation of the possible deterministic interactions with any other system, in particular observing systems. A classical, objective attribute is only a limiting case where the same outcome follows for the vast majority of states of observing systems that the system can interact with. Of course, whether this point of view is scientific or rather philosophic depends ultimately on its falsifiability. The proposed interpretation is falsifiable in principle if we succeed in tailoring the probe states of the apparatus to our needs. If we can produce a non-uniform distribution for the initial states, we would be able to distinguish some pairs of states better, and some pairs of states worse than the usual Born rule allows. This implies such a probe can only be used to our advantage if we posses *some* information about the state prior to the measurement. It also means that the probability for the occurrence of an outcome when we measure a mixture of states, depends nonlinearly on the probabilities for each component of the mixture; a failure of what we have called the linear mixture property.

There may be physical and perhaps even logical reasons why it is not possible for the observer to completely control the state of the measuring apparatus. The source of probability in observation, the randomness in the state of the observer immediately prior to an observation, may very well at some point in our scientific explorations become fundamentally incontrollable. Breuer[13] has shown by an elegant construction that for every observer there will be different states of himself that he cannot distinguish. In Ref. 6 it is shown on logical grounds that it is not possible for an observer to determine his own state with certainty.[n] It is argued in Refs. 31 and 9, that any gain in information about a system is accompanied by an equal increase of entropy in the state of the observing system. It seems that for every single measurement outcome there is a trade-off between the information an observer can choose to extract about himself and about the system he is observing. If this is indeed the underlying reason for the occurrence of the quantum probabilistic structure, then the probabilities in quantum mechanics are simultaneously ontic and epistemic. From an omniscient perspective probability is epistemic. It arises only because there is a lack of knowledge situation; it can be represented as a measure over deterministic interactions between the observer and the observed. But to the one who observes within the universe, being part of what he aims to describe, this lack of knowledge may be fundamentally irreducible.

References

1. L. Accardi, A. Fedullo, On the statistical meaning of complex numbers in quantum mechanics, *Nuovo Cimento* **34**, 161-173 (1982).
2. D. Aerts, A possible explanation for the probabilities of quantum mechanics, *J. Math. Phys.* **27**, 202-210 (1986).

[n] To the best of our knowledge, the relation to such logical arguments and the quantum measurement problem was pointed out for the first time in 1977 in a remarkable pioneering paper by Dalla Chiara in Ref. 15.

3. D. Aerts, The origin of the non-classical character of the quantum probability model, in *Information, Complexity, and Control in Quantum Physics*, A. Blanquiere, *et al.*, (eds.), Springer-Verlag (1987).
4. D. Aerts, S. Aerts, T. Durt, O. Lévêque, Quantum and classical probability in the epsilon model. *Int. J. Theor. Phys.* **38**, 407-429 (2002).
5. S. Aerts, The Born rule from a consistency requirement on hidden measurements in complex Hilbert space, *Int. J. Theor. Phys.* **44**, 7 (2005).
6. S. Aerts, Undecidable classical properties of observers, *Int. J. Theor. Phys.* **44**, 12 (2005).
7. S. Aerts, Hidden measurements from contextual axiomatics, in *Probing the Structure of Quantum Mechanics*, D. Aerts, M. Czachor, T. Durt (eds.), World Scientific Publishers, 149-164 (2002).
8. S. Aerts, Quantum and classical probability as Bayes-optimal observation, arxiv.org/abs/quant-ph/0601138 (2006).
9. R. Alicki, M. Horodecki, P. Horodecki, R. Horodecki, Thermodynamics of quantum informational systems — Hamiltonian description, quant-ph/0402012 (2004).
10. T. Bayes, An essay towards solving a problem in the doctrine of chances. *Phil. Trans. R. Soc.* **53**, 370-418 (1763).
11. J.S. Bell, On the Einstein-Podolsky-Rosen Paradox, *Physics* **1**, 195-200 (1964).
12. D. Bohm, J. Bub, A proposed solution of the measurement problem in quantum mechanics by a hidden variable model, *Rev. Mod. Phys.* **38**, 453-469 (1966).
13. T. Breuer, The impossibility of accurate self-measurements, *Philosophy of Science* **62**, 197-214 (1995).
14. C.M. Caves, C.A. Fuchs, R. Schack, Quantum probabilities as Bayesian probabilities. *Physical Review A.* **65**, 022305 (2002).
15. M. Dalla Chiara, Logical self-reference, set theoretical paradoxes and the measurement problem in quantum mechanics, *Journal of Philosophical Logic* **6**, 331-347 (1977).
16. T. Durt, About the possibility of supraluminal transmission of information in the Bohm-Bub theory, *Helv. Phys. Acta* **72**, 356-376 (1999).
17. D. Deutsch, Quantum theory of probability and decisions, *Proceedings of the Royal Society of London A* **455**, 3129 (1999).
18. D. Deutsch, Quantum Theory, the Church-Turing principle and the universal quantum computer, *Proceedings Royal Society London Ser. A* **400**, 96-117 (1985).
19. C.A. Fuchs, Quantum Mechanics as Quantum Information (and only a little more), quant-ph/0205039 (2002).
20. W.S. Geisler, D. Kersten, Illusion, perception and Bayes, *Current Opinion in Neurobiology* **13**,150-158 (2003).
21. H. Helmholtz, *Handbuch der Physiologischen Optik*, Leipzig, Voss L., Germany (1867).
22. E.T. Jaynes, *Probability theory, The Logic of Science*, Cambridge University Press (2003).

23. D.G. Kendall, D. Barden, T.K. Carne, H. Le, *Shape and Shape Theory*, Wiley series in probability and statistics (1999).
24. S. Kochen, E.P. Specker, The problem of hidden variables in Quantum Mechanics. *J. Math. Mech.* **17**, 59-87 (1967).
25. E. Lehrer, E. Shmaya, A subjective approach to quantum probability, arXiv:quant-ph/0503066 (2005).
26. I. Pitowsky, Betting on the outcomes of measurements. *Studies In History and Philosophy of Modern Physics* **34**, 395-414 (2003).
27. W. Rudin, *Real and Complex Analysis*, third edition, McGraw-Hill (1987).
28. S. Saunders, What is probability? in *Quo Vadis Quantum Mechanics*, A. Elitzur, S. Dolev, N. Kolenda (eds.), Springer Verlag (2005).
29. E. Schrödinger, Die gegenwärtige Situation in der Quantenmechanik, *Naturwissenschaften* **23** (1935).
30. R.W. Spekkens, In defense of the epistemic view of quantum states: a toy theory, quant-ph/0401052v1 (2004).
31. L. Szilard, *Z. Phys.* **53**, 840, (1929), reprint of the english translation can be found in *Quantum Theory and Measurement*, J. A. Wheeler and W. H. Zurek (eds.), Princeton University Press, Princeton (1983).
32. D. Wallace, Everettian Rationality: defending Deutsch's approach to probability in the Everett interpretation, in *Studies in the History and Philosophy of Modern Physics* **34** , 415-438 (2003).
33. W.K. Wootters, The acquisition of information from quantum measurements, Ph.D. dissertation, University of Texas at Austin (1980).
34. W.K. Wootters, Statistical distance and Hilbert space, *Phys. Rev. D* **23**, 357–362 (1981).

QUANTUM REALISM, INFORMATION, AND EPISTEMOLOGICAL MODESTY

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It is usually asserted that physical theories, in particular quantum mechanics, support a certain view of what the world really is. To such claims I oppose an attitude of epistemological modesty. Ontological statements on the nature of reality, when made on the basis of quantum mechanics, appear unwarranted. I suggest that an epistemic loop connects physical theory grounded in informational notions, and a theory of information developed through a theoretical account of the physical support of information.

Keywords: Information; observer; epistemic loop; realism.

1. Against Ontology

Our brains are wired to perform a variety of operations. They are wired to look for a cause for every event that they register, so that the cause-effect relations appear to us to be fundamental elements of reality. They are wired to build mental constructions which we perceive as representations of the external world, hence the mental difficulty that we experience if one tries to question the very existence of the external world. This difficulty, among other things, helps to understand why solipsism is such a remarkable achievement of the human mind, for solipsistic worldview seems to proceed against the natural way of operation of the human brain.

Further down this road, I find it interesting to ask what remains of the philosophy of physics when one removes the prejudice of believing that some premises are fundamental just because our brains are wired to hold corresponding beliefs. Human habits, conventions, natural inclinations, and neuronal connections must all depart. But can they? Take the problem of foundations of a physical theory. What remains of a theory if we remove its human author, user, and observer from the picture? Probably not much, or at least it is so for some physical theories. Quantum mechanics is one example. It relies on the assumption that one can perform multiple runs of an experiment on identical systems, or one can identify different measurements made over time as referring to the same experimental situation, or some other such convention.[1] However, identical systems and ideal experimental conditions are axiomatized illusions. Other examples tend to support the need for a priori axioms in the foundations of physical theories. General relativity, for instance, uses Riemannian manifolds, but it is an open question whether this geometric toolkit is adequate. Poincaré famously tried to motivate the inevitability of Riemannian and even Euclidian geometry,[2] but any a priori argument, as it seems from the history of science, ends up being refuted.

Can one find suitable material for building the foundations of a physical theory? One who searches for such ontic bricks seems doomed to pessimism. Many varieties of realism based on different ontological commitments about the world have lived and died under the pressure of new phenomena. Admitting any such commitment means taking a risk of being subdued to the wirings of one's own brain: here and now we may simply lack the imagination which is needed in order to free oneself from a well-wired assumption.

If we clear the foundational warehouse of the stocks of ontology, what remains? The mathematical structure of physical theories, plus the relations between theories that based on their mathematical formalisms. Such foundations do not involve any notion of entities that exist in the world or stand behind the observable phenomena. Such entities can only arise as elements of the mathematical description of these phenomena by physical theories.

2. Epistemological Modesty

Writing about the measurement problem in quantum mechanics as early as 1939, London and Bauer emphasized that physics makes an impact on philosophy insofar as it permits "negative philosophical discoveries",[3] i.e., some old philosophical views cannot be maintained following the advent of new physical knowledge. It follows from their discussion that physical science does not warrant positive claims that the correct ontology of the world is so-and-so. But each physical theory, for sure, relies on a set of first principles. Aren't these axioms our best candidates for being fundamental truths about reality?

Not necessarily. The first principles may be given a minimal epistemic status of being postulated for the purpose of building up a specific theory. One's ontological commitments, whether profound or superficial, are completely irrelevant and remain personal beliefs for which science has no need.

As in the 19th-century mathematics, the axiomatic method in theoretical physics is to be separated from the Greek attitude that axioms represented truths about reality. Much of the progress of mathematics is due to understanding that an axiom may no longer be considered an ultimate truth, but merely a fundamental structural element, i.e., an assumption that lies at the basis of a certain theoretical structure. In mathematics, after departing from the Greek concept of axiom, "not only geometry, but many other, even very abstract, theories have been axiomatized, and the axiomatic method has become a powerful tool for mathematical research, as well as a means of organizing the immense field of mathematical knowledge which thereby can be made more surveyable".[4] A similar attitude is to be taken with respect to axioms used for reconstructing a physical theory. The methodological precept that gives a minimal status to the first principles in a reconstruction program, runs as follows:[5]

Criterion 2.1. If the theory itself does not tell you that the states of the system (or any other variables) are ontic, then do not take them to be ontic.

I call this attitude *epistemological modesty.* It is more economical to treat the foundational principles as axioms *hic et nunc*, i.e., in a given theoretical description. Epistemological modesty requires that one brackets his or her personal motives for the choice of first principles, which merely become axiomatic statements in the reconstruction of a given theory. An unambiguous derivation of the theory's formalism is therefore detached from the question of reality of a world that the theory describes. Unless contradicted by a "negative philosophical discovery", one is free to hold personal beliefs of any kind about what constitutes reality.

3. Observers as Informational Agents

Historically quantum physics has been predominantly conceived as a theory of non-classic waves and particles, while special relativity was thought of as a theory of moving rods and clocks. Fock argues that such views have only been well-motivated at an early historic stage, when only a few experimental results were available and the dominant philosophy was still couched in the physics that preceded the creation of these new theories. Today a minimal description of quantum mechanics presents it as a mathematical formalism, such that, when applied to physical setups, it provides an accurate prediction of experimental results. This standard formalism relies on a cut between the observer and the system being observed.[6,7] No ruse can remove such a "shifty split"[8] of the world into two parts: the formalism only applies if the observer and the system are demarcated as two separate entities. Physical properties of the system, on one side of the split, do not exist independently of the observer, on the other side of the split, and can only be instantiated during the observation, or 'measurement', of some dynamical variable of the system chosen by the observer. Standard quantum mechanics says nothing about the physical composition of the observer, who is an abstract notion having no physical description from within quantum theory. One cannot infer from the formalism if the observer is a human being, a machine, a stone, a Martian, or the whole Universe. As emphasized by Wheeler, this makes it extraordinarily difficult to state clearly where "the community of observer-participators" begins and where it ends.[9]

As a part of his relative-state interpretation, Everett argued that observers are physical systems with memory, i.e., "parts... whose states are in correspondence with past experience of the observers".[10] We call this a *universal observer* hypothesis: any system with certain information-theoretic properties can serve as quantum mechanical observer, independently of its physical constituency, size, presence or absence of conscious awareness and so forth. In this vein, Rovelli claimed that observers are merely systems whose degrees of freedom are correlated with some property of the observed system: "Any system can play the role of observed system and the role of observing system. . . . The fact that observer O has information about system S (has measured S) is expressed by the existence of a correlation. . . "[11] However, the universal observer hypothesis has remained a controversial statement to this day. For example, Peres claims in the way exactly opposite to Rovelli's, that "the two electrons in the ground state of the helium atom are correlated, but no one would say that each electron 'measures' its partner".[12]

What characterizes an observer is that it has information about some physical system. This information fully or partially describes the state of the system. The observer then measures the system, obtains further information and updates his description accordingly. Physical processes listed here: the measurement, updating of the information, ascribing a state, happen in many ways depending on the physical constituency of the observer. The memory of a computer acting as an observer, for instance, is not the same as human memory, and measurement devices vary in their design and functioning. Still one feature unites all observers: that whatever they do, they do it to a *system*. In quantum mechanics, defining an observer goes hand in hand with defining a system under observation. An observer without a system is a meaningless nametag, a system without an observer who measures it is a mathematical abstraction. What remains constant throughout measurement is the identification, by the observer, of the quantum system. So, whatever else he might happen to be physically, the observer is first of all a machine that identifies physical systems whose states it will then observe and record. Different observers having different features (clock hands, eyes, optical memory devices, internal cavities, etc.) all share this central characteristic.

Definition 3.1. An observer is a system identification algorithm (SIA).

Particular observers can be made of flesh or, perhaps, of silicon. 'Hardware' and 'low-level programming' are different for such observers, yet they all perform the task of system identification. This task can be defined as an algorithm on a universal computer, e.g., the Turing machine: take a tape containing the list of all degrees of freedom, send a Turing machine along this tape so that it puts a mark against the degrees of freedom that belong to the quantum system under consideration. Any concrete SIA may proceed in a very different manner, yet all can be modelled with the help of this abstract construction.[13]

4. Epistemic Loops

For a long time many physicists have lacked understanding of the epistemological lesson coming from the necessity of the cut between the observer and the observed. Einstein, for instance, believed all his life that the postulate of the existence of a particle or a quantum is a basic axiom of the physics. In a letter to Born as late as 1948 he writes:[14]

> We all of us have some idea of what the basic axioms in physics will turn out to be. The quantum or the particle will surely be one amongst them; the field, in Faraday's or Maxwell's sense, could possibly be, but it is not certain.

This is to say that Einstein believed that a proper physical theory must be based on the ontology of certain physical systems, such as particles or fields, and will build upon the known facts about these elementary systems in order to provide an account of all physical phenomena. In another illuminating piece of his late writing, Einstein at the same time acknowledges the necessity of the epistemological cut but fails to recognize its implications for the way new physical theories must be thought of:

> One is struck [by the fact] that the theory [of special relativity] ... introduces two kinds of physical things, i.e., (1) measuring rods and clocks, (2) all other things, e.g., the electromagnetic field, the material point, etc. This, in a certain sense, is inconsistent; strictly speaking measuring rods and clocks would have to be represented as solutions of the basic equations (objects consisting of moving atomic configurations), not, as it were, as theoretically self-sufficient entities. However, the procedure justifies itself because it was clear from the very beginning that the postulates of the theory are not strong enough to deduce from them sufficiently complete equations ... in order to base upon such a foundation a theory of measuring rods and clocks.[15]

Epistemologically, it is unreasonable to expect, as Einstein did, that the theory of measuring rods and clocks could be based on a set of yet stronger postulates that would, at the same time, provide also an account of all physical phenomena measured by means of these rods and clocks. To see why Einstein found himself at an impasse, albeit an unnecessary one, consider the following schematic representation of physical theories. Assume that phenomena are best described by theories that are interconnected in the form of loop. Any particular theory is represented by cutting the loop at some point and thus separating the target object of the theory from the theory's presuppositions. Due to the necessity of the cut, it is impossible to give a theoretical description of the loop as a whole. Now, when the position of the cut is fixed, some elements of the loop are treated as objects of the theory, while other elements fall into the domain of meta-theory. At another loop cut, those elements exchange roles: the ones that had been *explanans* become *explanandum*, and those that had previously been *explanandum* become *explanans*. Different theories do not form a pyramid which is reduced to yet more and more fundamental theories with "stronger postulates"; on the contrary, for the purposes of each theory, a part of the loop must be taken as a given, and the relation between theories is the one of mutual illumination rather than that of reduction. Metatheory of a given theory, i.e., the part of the loop kept fixed in the task of reconstructing the theory in question, is no more and no less than the theory that explains the functioning of measuring devices of the theory that is being reconstructed. For the purposes of the reconstruction of a given theory, the loop view demonstrates how measuring devices of the theory can be assumed to be meta-theoretic and abstract in Fock's sense while driving the reconstruction; but in a different loop cut, the same measuring devices become themselves objects of another theory that would explain their functioning.

Consider the loop between physical theory and information. Physics and information mutually constrain each other, and every theory will give an account of but a part of the loop, leaving the other part for meta-theoretic assumptions. In the cut shown on Fig. 1 information lies in the meta-theory of the physical theory, and physics is therefore based on information. In a different loop cut (Fig. 2), informational agents are physical beings, and one can describe their storage of, and operation with, information, by means of effective theories that are reduced, or reducible in principle, to physical theory.

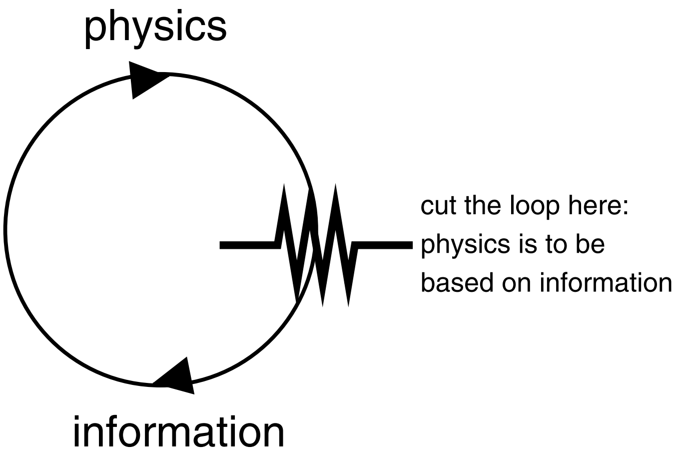

Fig. 1. Loop cut: physics is informational.

Without recognizing the importance of the cut one cannot fully appreciate the unbridgeable (within a given theory) separation between the observer and the observed. The loop view allows one to make sense of assertions that mark a no small change in the conception of physics, e.g., of Bub's idea that information must be recognized as "a new sort of physical entity, not reducible to the motion of particles and fields".[16] In the loop epistemology, however, information is an entity, but not a physical entity or object of physical theory like particles or fields are. Were information a basic

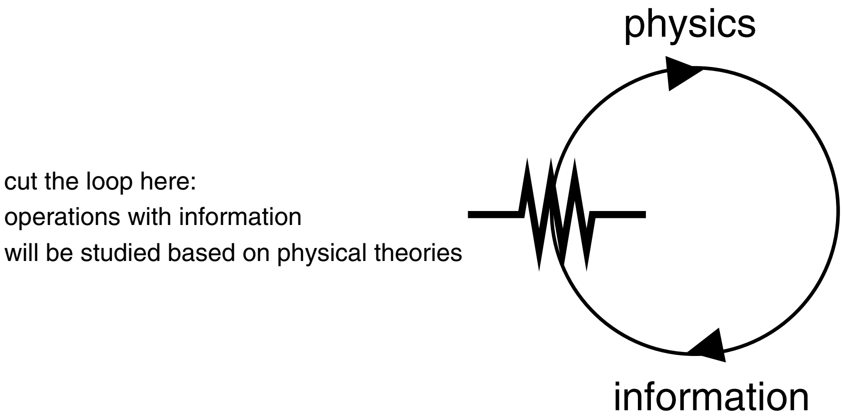

Fig. 2. Loop cut: information is physical.

physical entity, the information-theoretic viewpoint would then do nothing to approach the problem of giving quantum *physics* a foundation, for it would need to provide a inevitably circular argument taking, at the same time, information to be a fundamental irreducible concept, and explaining how information can be stored, or operated with, physically. The only way to avoid circularity is through removing information into metatheory. When one does so consistently, the conventional physical concepts such as particles and fields are reduced to information, not put along with it on equal grounds; but information itself, in another loop cut, can be too treated as a derivative notion, in another theory that would itself take particles and fields as givens.

In the loop cut of Fig. 2 the question of reconstruction of physical theory is meaningless, because physical theory is taken for granted. Indeed, once a particular loop cut is assumed, it is a *logical error* to ask questions that only make sense in a different loop cut. For instance, the critique of information-theoretic reconstructions motivated among some physicists mainly by Landauer's "information is physical"[17] can be avoided by adopting the loop view. Einstein's plea for "stronger postulates" is then a mere epistemological illusion. Fervent defender of such view would be led to a logical impasse, one to consider that it is possible to have a physical theory that would include within itself a description of the physical structure of its own measurement devices.

References

1. A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic Publishers, 1993).
2. H. Poincaré, *La Science et l'Hypothèse* (Flammarion, Paris, 1902).
3. F. London and E. Bauer, *La théorie de l'observation en mécanique quantique* (Hermann, Paris, 1939). English translation in [18, p. 218–259].
4. A. Heyting, *Axiomatic Projective Geometry* (North-Holland, Amsterdam, 1963).
5. A. Grinbaum, *British Journal for the Philosophy of Science* **58**, 387 (2007).
6. P. Dirac, *The Principles of Quantum Mechanics* (Clarendon, Oxford, 1930).
7. J. von Neumann, *Mathematische Gründlagen der Quantenmechanik* (Springer, Berlin, 1932).
8. J. Bell, *Physics World* **8**, 33 (1990).
9. J. Wheeler, Law without law Published in [18, pp. 182–213].
10. H. Everett, *Rev. Mod. Phys.* **29**, 454 (1957).
11. C. Rovelli, *Int. J. of Theor. Phys.* **35**, p. 1637 (1996).
12. A. Peres, *Am. J. Phys.* **54**, 688 (1986).
13. A. Grinbaum, Quantum observer and kolmogorov complexity, in *Proceedings of Current Interpretational Problems of Quantum Theory*, eds. A. Wüthrich and T. Sauer (Max Planck Research Library for the history and development of knowledge, Berlin, 2013).
14. M. Born, *The Born-Einstein letters* (Walker and Co., London, 1971).
15. A. Einstein, Autobiographical notes, in *Albert Einstein: Philosopher-Scientist*, ed. P. Schlipp (Open Court, Illinois, 1969) pp. 1–94. Quoted in Ref. 19.
16. J. Bub, *Studies in the History and Philosophy of Modern Physics* **35**, 241 (2004).
17. R. Landauer, *Phys. Scripta* **35**, p. 88 (1987).
18. J. Wheeler and W. Zurek (eds.), *Quantum Theory and Measurement* (Princeton University Press, 1983).
19. H. Brown and C. Timpson, Why special relativity should not be a template for a fundamental reformulation of quantum mechanics, in *Physical Theory and Its Interpretation: Essays in Honor of Jeffrey Bub*, eds. W. Demopoulos and I. Pitowsky (Springer, 2006) pp. 29–41.

THE PROBLEM OF REPRESENTATION AND EXPERIENCE IN QUANTUM MECHANICS

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In this paper we discuss the problem of representation and experience in quantum mechanics. We analyze the importance of metaphysics in physical thought and its relation to empiricism and analytic philosophy. We argue against both instrumentalism and scientific realism and claim that both perspectives tend to bypass the problem of representation and justify a "common sense" type experience. Finally, we present our expressionist conception of physics.

Keywords: Representation; physical reality; quantum mechanics.

1. Introduction: The Metaphysical Quest

In the literature the term "metaphysics" comprises a series of many different definitions. For some, it can be considered as a supreme form of knowledge, while for others, it remains an occupation constituted by unfruitful discussions. As noticed by Gilles Gaston Granger the development of metaphysics is intrinsically related to the different questions which have determined the path of Western thought:

> "le problème moderne: qu'est-ce que le réel ? n'est pas posé en ces tremes par les deux grands philosophes classiques Platon et Aristote. La question, héritée sans doute de Parménide, est alors très généralement exprimée par eux sous la forme : qu'est-ce que l'Être ?" [13, p. 13]

Metaphysics designates a controversial field which has accompanied the development of Western thought since Plato and Aristotle. However, there is already a huge distance between these two main figures and creators of metaphysical thought. Within the analysis proposed by Granger in his book *Sciences et réalité*, Plato proposes an interpretation of the real as participation of the absolute Being [13, p. 22]. "Cet Être absolu est 'Idée', c'est-à-dire non point le produit d'une pensée mais une entité accessible seulement par un acte de 'vision' et non par un acte discursif de l'entendement, à travers un langage. L'Être absolu est normatif, en ce sens qu'il est le *modèle* et la *source* de toutes ses *réalisations* imparfaites; c'est pourquoi l'Idée par excellence est l'Idée du bien et du beau en soi." The proposal of Aristotle already takes distance from Plato and proposes in a very different style an interpretation of the real which can be already considered as much closer to ours.

"En ce qui concerne la question qui nous ocupe, la grande innovation aristotélicienne est double. D'une part, Aristote découvre et systématise *à travers les formes du langage* une structure et une organisation de l'Être ; d'autre part, il réintroduit de plein droit dans l'Être, et decrit comme type d'objet connaissable, un *individuel* qui n'est pas celui très abstrait des idées platoniciennes." [13, pp. 23-24]

Aristotle defines metaphysics as a theory of "being *qua* being" [Aristotle, Met. 1003a20] a theory about what it means or implies to "be" in its different senses. Since then, it has become clear that the importance of metaphysical thought within physical theories can be hardly underestimated. As noticed by Edwin Arthur Burtt [5, p.224]: "[...] there is no escape from metaphysics, that is, from the final implications of any proposition or set of propositions. The only way to avoid becoming a metaphysician is to say nothing." It is Grager who remarks that, apart from both Plato and Aristotle, it is Leibniz who, through a new idea of the possible, is capable of introducing the specifical sense of the question about reality and existence [13, p. 14]. According to Wolff — who gave, following Leibniz, a classic definition of what is metaphysics from the perspective of 17th century rationalism — *metaphysica specialis* is divided in four main regions (see for example[33] and references therein). Rational theology, which discusses the existence and attributes of God, rational psychology, which studies the soul as a simple non-extended substance, rational cosmology, which discusses the world as a whole, and metaphysical generalis or ontology, which discusses traits of the existent in general, the being qua being. This characterization of metaphysics was severely criticized by Immanuel Kant who reconfigured the discussions regarding metaphysical thought and the limits of knowledge.

According to Kant — in the "Preface" to his *Critique of Pure Reason*[19] — Wolff is "the greatest of all dogmatic philosophers". Kant wanted to escape metaphysics, which meant for him the possibility to go beyond dogma and belief, to understand the *finite* access with which every human being is confronted.[a] By understanding the limits of human knowledge metaphysics would finally follow the secure path of science and show how (scientific) knowledge is possible.

Within philosophy of science, as a part of the analytic tradition, the tension between metaphysical and anti-metaphysical positions did not disappear but remained at the center of gravity of many discussions. As remarked by van Fraassen, although analytic philosophy had begun as a revolt against metaphysics, this movement was very soon subverted. Quantum mechanics (QM), as we shall argue, has become during the 20th Century, not only an external witness, but the field of battle itself were the confrontation between metaphysicians and anti-metaphysicians took place.

2. Empiricism and Analytic Philosophy: Against Metaphysics?

In order to understand oneself one needs to know one's own history and traditions. Where do we come from? Who was our father? What religion did he profess? Who was our grandfather? In which war did he fight? The same applies to philosophical stances which are always connected to traditions of thought, to lineages, to fights and battles which go back in time. In particular, one could interpret the history of occidental philosophy as a confrontation between two main forces. On the one side the metaphysical or ontological force, which seeks to answer the question of *Being qua Being*; and on the other side, an anti-metaphysical or epistemological force, which focus its attention on the limits and constrains of such a question. Analytic philosophy has been clearly, not only from an historical perspective but also methodologically, part of this second force.

Empiricism and logicism are two of the main sources of the origin of analytic philosophy. An idea often found in empiricism is that science should use theories as an instrument and should renounce the quest for explanation. The search for such explanations is a metaphysical enterprise. As noticed by van Fraassen [29, p. xviii] "Empiricist philosophers have always concentrated on epistemology, the study of knowledge, belief, and opinion,

[a]See in this respect the very interesting paper by Michel Bitbol[1] where he discusses the importance of the notion of metaphysics within Kant and its relation to QM.

with a distinct tendency to advocate the importance of opinion." Against the ontological concerns of the metaphysicians, analytic philosophers engaged in epistemological issues. Escaping from the true statements of the metaphysicians, of *episteme*, analytic philosophy remained closer to opinion and *doxa*.[6] True knowledge was regarded with suspicion, as a dogma of the past, as a metaphysical idol with no proper fundament. This is what Burtt calls the "central position of positivism itself", the idea that it is possible to "acquire truths about things without presupposing any theory of their ultimate nature; or more simply, it is possible to have a correct knowledge of the part without knowing the nature of the whole." According to van Fraassen, the history of analytic philosophy is also directly connected to a criticism of and reaction against the dominant metaphysical attitude in Continental Europe in the 17th century.

> "The story of empiricism is a story of recurrent rebellion against a certain systematizing and theorizing tendency in philosophy: a recurrent rebellion against the metaphysicians." [29, p. 36]

However, even though analytic philosophy started from a revolution against metaphysics,[b] the introduction of metaphysical questions reappeared very soon within analytic philosophy itself.

> "As I see it, analytic philosophy — which is the strand to which I belong — began with a revolution that was subverted by reactionary forces. I am speaking here of reversion to a seventeenth-century style of metaphysics. I do not reject all metaphysics, but this reversion I see as disastrous. Paradoxically, this disaster seems to be worst in two areas that scarcely relate to each other at all. I mean, on one hand, the area loosely characterized as "science and religion" studies and, on the other, academic analytic philosophy. Both suffer from unacknowledged as well as explicit metaphysics." [29, p. xviii]

As noticed by van Fraassen, one of the most interesting and subversive starting points of analytic philosophy was very soon turned upsidedown.

> "[...] with the rise of analytic philosophy something paradoxical happened. This movement began in a series of revolts, across Europe and America, against all forms of metaphysics. And lo, even before mid-century, some of its ablest

[b]One could argue that it was this same revolution against dogmatic metaphysics which set the conditions of possibility for QM and Relativity to be developed.

adherents began to make the world safe for metaphysics again. Since then we have seen the growth of analytic ontology, analytic metaphysics, and it thrives today.

Or so it seems. I say that metaphysics is dead. What I see is false consciousness, a philosophy that has genuinely advanced beyond the past, but a philosophy that misunderstands itself." [29, pp. 3-4]

3. Philosophy of Quantum Mechanics Today

QM has played a major role in the development of such philosophical debate bringing up new questions regarding the possibility of a metaphysics of physics. Since the second world war the philosophical analysis of science, and of quantum theory in particular, has been an almost exclusive field owned by analytic philosophy — in distinction to the so called "continental" philosophical tradition of thought which "has discussed the large spiritual problems that are concern of every thinking person: the meaning of life, the nature of humanity, the character of a good society" [11, p. 9]. Although the analytical tradition was the inheritor — via logical positivism and logical empiricism — of a deep criticism to metaphysics, the 20th Century has been witness to the return of metaphysics in its most dogmatic form both in the analysis of physics in general and of the interpretation of QM in particular. The criticism by van Fraassen to analytic philosophy can be clearly explained within the philosophy of QM today.

It is interesting to point out that, within this context, something very similar to the history of analytic philosophy itself happened in relation to the metaphysical presuppositions very soon imposed on the formal structure of QM. The position of Bohr against metaphysics, which can be very well regarded in close continuation to analytic concerns, was soon replaced by much more metaphysical approaches, such as, for example, Bohmian mechanics and DeWitt's MWI. While Bohr attempted to analyze the logical structure of the theory and concentrated on the analysis of phenomena, such attempts as those of MWI and Bohmian mechanics intended to recover the metaphysical conditions under which one could talk, for example, about classical properties and trajectories. It seems in this case a bit ironic that the aversion professed by many philosophers of physics within the analytic tradition to Bohr's ideas does not recognize the profound connection of his thought to analytic philosophy itself. These same philosophers choose — knowingly or not — for metaphysical schemes going very much against their own tradition. In the case of MWI the metaphysical step goes as far as to propose non-observable entities in order to explain the formal aspects of QM. Also, from a metaphysical point of view, the many worlds attempt seems to end up in an extreme violation of Ockham's principle: "Entities are not to be multiplied beyond necessity".[c] In the case of Bohmian mechanics the metaphysical dogma relates to particles with trajectories. Bitbol notices in this respect [1, p. 8] that: "Bohm's original theory of 1952 is likely to be the most metaphysical (in the strongest, speculative, sense) of all readings of QM. It posits free particle trajectories in space-time, that are unobservable in virtue of the theory itself." Furthermore, that which should play the role of space-time in the mathematical formalism varies its dimension with the addition or substraction of particles breaking down the initial attempt to recover trajectories in space-time. It is not at all clear that these kind of attempts bring more solutions than problems.

From this analysis, it might seem obvious why van Fraassen has chosen for Bohr rather than these new lines of thought, which to great extent, go against many of the analytical *a priori* concerns and methodology [31, p. 280]. The Danish physicist remained agnostic regarding the metaphysical concerns raised strongly by Einstein, but also by Heisenberg and Pauli [22, chap. 4]. He tried by all means to restrict his analysis to the empirical data as exposed by classical physical theories and language, and not go beyond the interpretation of the formalism in terms of a new conceptual scheme — an aspect which is shared by the semantical approach to theories. Contrary to this analysis, Bohmian mechanics and MWI, two of the most important interpretational lines of investigation today — especially in the United States and the United Kingdom, there where analytic philosophy is strongest — compose their analysis with heavy metaphysical commitments. Rather than starting from the analysis of the logical formal structure of the theory, the metaphysical presuppositions constitute the very foundation and center of gravity of such interpretations. Even attempting in some cases to change the formalism in order to recover — at least some of — our classical (metaphysical) conception of the world.

[c] Although Lev Vaidman[27] claims that: "in judging physical theories one could reasonably argue that one should not multiply physical laws beyond necessity either (such a verion of Ockham's Razor has been applied in the past), and in this respect the many worlds interpretations (MWI) is the most economical theory. Indeed, it has all the laws of the standard quantum theory, but without the collapse postulate, the most problematic of physical laws." One could argue however, that due to the existence of modal interpretations, which are also no-collapse interpretations and share the same formal structure as many worlds, there is no clear argument why one should be forced into this expensive metaphysical extension.

4. Observation and Representation in Physics

In the first half of the twentieth century, the logical positivists and their successors, the logical empiricists, approached the issue of scientific realism reflecting on the role of observation. Within their scheme, as remarked by Curd and Cover [7, p. 1227], "it was natural for the logical empiricists to emphasize a distinction between the observational components of a theory, which refer to objects and properties that are directly observable, and the theoretical components which apparently refer to objects and properties that are not directly observable." Philosophers of science in the second half of the twentieth century based themselves, in great measure, on these same grounds.

> "Logical positivism is dead and logical empiricism is no longer an avowed school of philosophical thought. But despite our historical and philosophical distance from logical positivism and empiricism, their influence can be felt. An important part of their legacy is observational-theoretical distinction itself, which continues to play a central role in debates about scientific realism." [7, p. 1228]

The realism-antirealism debate stands in close relation to the observational-theory distinction. As noticed by Alan Musgrave:

> "As usually understood, the realism-antirealism issue centers precisely on the question of truth. Positivists deny the existence of 'theoretical entities' of science, and think that any theory which asserts the existence of such entities is *false*. Instrumentalists think that scientific theories are tools or rules which are *neither true nor false*. Empistemological antirealists like van Fraassen or Laudan concede that theories have truth-values, even that some of them might be true, but insist that no theory should be *accepted as true*." [7, pp. 1209-1210]

Regardless of the different positions it is clear that the center of gravity of these discussions is the notion of truth. The relevant conception of truth is a version of the common-sense correspondence theory of truth. As remarked by Musgrave [7, p. 1221]: "In traditional discussions of scientific realism, common sense realism regarding tables and chairs (or the moon) is accepted as unproblematic by both sides. Attention is focused on the difficulties of scientific realism regarding 'unobservables' like electrons."

One of the most important antirealist positions has been developed by van Fraassen with his constructive empiricism. According to him: "science aims to give us theories which are empirically adequate: an acceptance of a theory involves as belief only that it is empirically adequate." A theory is empirically adequate when it 'saves the phenomena' — when what it says about *observables* objects, events, and properties is true. "The respect in which van Fraassen's antirealism departs both from logical empiricism and from scientific realism is thus apparent. To accept (hold) a theory is to claim that it accurately describes observable phenomena; this does not entail that talk of theoretical entities is meaningless, nor does it entail that such entities are fictional or real. By distinguishing in this way between accepting a theory and believing it to be true, the constructive empiricist recommends a position of agnosticism about the theoretical." Van Fraassen agrees at the same time that all language is theory-infected, but he denies that this shows anything about scientific realism. At the same time he claims that:

> "To be an empiricist is to withhold belief in anything that goes beyond the actual, observable phenomena, and to recognize no objective modality in nature. To develop an empiricist account of science is to depict it as involving a search for truth only about the empirical world, about what is actual and observable." [28, pp. 202-203]

The role of observation in this account has been criticized by Musgrave and others (see for example, Ref. 7).

After the revolution brought by positivism, one can hardly deny the importance of empirical observation in physics. Einstein [8, p. 175] was very clear regarding this point: "[...] the distinction between 'direct observable' and 'not directly observable' has no ontological significance [...] the only decisive factor for the question whether or not to accept a particular physical theory is its empirical success." At the same time, Einstein was conscious that the uncritical consideration of observation was out of the question. This can be seen from the very interesting discussion between Heisenberg and Einstein were the latter explains: "I have no wish to appear as an advocate of a naive form of realism; I know that these are very difficult questions, but then I consider Mach's concept of observation also much too naive. He pretends that we know perfectly well what the word 'observe' means, and thinks this exempts him from having to discriminate between 'objective' and 'subjective' phenomena. No wonder his principle has so suspiciously commercial a name: 'thought economy.' His idea of simplicity is much too subjective for me. In reality, the simplicity of natural laws is an objective fact as well, and the correct conceptual scheme must balance the subjective side of this simplicity with the objective. But that is a very difficult task." (A. Einstein quoted by W. Heisenberg in [14, p. 66]).[d] The interrelation between metaphysics and the description of physical reality seemed to remain a central problem for Einstein, whom in a letter to Schrödinger in the summer of 1935 wrote that:

> "The problem is that physics is a kind of metaphysics; physics describes 'reality'. But we do not know what 'reality' is. We know it only through physical description..." [7, p. 1196]

It is clear that, even from an empiricist account, we must recognize the importance of metaphysical schemes.[e] But independently of the importance of metaphysics as related to empirical theories and stepping outside the realist-antirealist debate, we claim that we must also consider the dominant role played by metaphysical schemes *within* physical experience itself, for a world of pure sensation remains outside the limits of language and expression.[f]

From our metaphysical constructive stance[g] we stress the need of considering the conceptual scheme which relates to the mathematical structure

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[d]Einstein was part, willingly or not, of the neo-Kantian tradition (see Ref. 17) and, as noticed by Howard: "he was not the friend of any simple realism" [16, p. 206].

[e]As it has been stressed already by Feyerabend [10, pp. 943-944]: "A good empiricist will not rest content with the theory that is in the center of attention and with those tests of the theory which can be carried out in a direct manner. Knowing that the most fundamental and the most general criticism is the criticism produced with the help of alternatives, he will try to invent such alternatives. [...] His first step will therefore be the formulation of fairly general assumptions which are not yet directly connected with observations; this means that his first step will be invention of a new *metaphysics*. This metaphysics must then be elaborated in sufficient detail in order to be able to compete with the theory to be investigated as regards generality, details of prediction, precision of formulation. We may sum up both activities by saying that a good empiricist must be a critical metaphysician. Elimination of all metaphysics, far from increasing the empirical content of the remaining theories, is liable to turn these theories into dogmas."

[f]This has been expressed by Jorge Luis Borges in a beautiful story called 'Funes el memorioso'.[4]

[g]According to this stance physical theories are necessarily related to a *formal* and *conceptual* scheme, a physical representation which allows, and is at the same time a precondition, to consider *physical experience*. There is no experment nor meaning of a 'physical situation' — or even a 'physical property' — without the presupposition of a conceptual scheme which provides a representation of physical reality. Instead of the fixed *a priories* present in the Kantian architectonic, our stance proposes to reflect about the possibility of considering constructive *a priories*; i.e. metaphysical conditions which are developed in physical theories in order to access reality. Rather than concentrating in the question of truth — which is mainly addressed in the realism-antirealism debate in philosophy of science — we are interested in discussing the role played by metaphysical presuppositions within ontological interpretations of quantum theory. See [22, chaps. 2 and 15].

and physical phenomena. According to this position there are no 'naked facts'. Physics relates and is necessarily involved with metaphysical schemes which constitute and configure physical experience. Remaining in the limits of empirical evidence one cannot access physical representation for, in order to describe any 'observed actuality' we are necessarily committed to a conceptual scheme. The conceptual representation of the actually given, the *hic et nunc*, is always — implicitly or explicitly — needed in order to describe a state of affairs and remains still today maybe the most problematic issue within physics itself. The metaphysical choices that one introduces for such a description configure and constitute the possibility of a particular physical experience. From this standpoint, we consider concepts as creations, creations through which the physicist relates to reality and physical experience. As remarked by Einstein:

"Concepts that have proven useful in ordering things easily achieve such an authority over us that we forget their earthly origins and accept them as unalterable givens. Thus they come to be stamped as 'necessities of thought,' 'a priori givens,' etc. The path of scientific advance is often made impossible for a long time through such errors. For that reason, it is by no means an idle game if we become practiced in analyzing the long commonplace concepts and exhibiting those circumstances upon which their justification and usefulness depend, how they have grown up, individually, out of the givens of experience. By this means, their all-too-great authority will be broken. They will be removed if they cannot be properly legitimated, corrected if their correlation with given things be far too superfluous, replaced by others if a new system can be established that we prefer for whatever reason." [9, p. 102]

Following this line of thought, the concept of 'physical entity' must be also considered as a creation, a conceptual representation which has played a major role in the history of Western thought. But quite independently of the unquestionable development which this concept has undergone through more than twenty centuries, it is not self evident whether this notion is also well suited to account for what QM is telling us about the world.[h]

"In one of his lectures on the development of physics Max Planck said: 'In the history of science a new concept never springs up in complete and final form as in the ancient Greek myth, Pallas Athene sprang up from the head of Zeus.' The

[h]For a discussion regarding the limits of the notion of entity see Ref. 23.

history of physics is not only a sequence of experimental discoveries and observations, followed by their mathematical description; it is also a history of concepts. *For an understanding of the phenomena the first condition is the introduction of adequate concepts. Only with the help of correct concepts can we really know what has been observed.*" [15, p. 264] (emphasis added)

The problem, from this perspective, becomes the justification of the relation between conceptual schemes and reality. Intimately connected with this problematic is the problem of representation. What is a physical theory representing? Is it possible for a theory to represent? And also, what is the meaning of representation? These questions have been addressed in the context of philosophy of science in the last decades. Unfortunately, as stressed by Mauricio Suárez, the community has not been able even to achieve agreement with respect to what is exactly meant by 'representation':

"Many philosophers of science would agree that a primary aim of science is to represent the world (Cartwright (2000), Giere (1988, 2000), Friedman (1982, chapter VI), Kitcher (1983), Morrison (2001, chapter II), Morrison and Morgan (1999), Van Fraassen (1981, 1987); a well known dissenter is Ian Hacking (1983)). What those philosophers understand by 'represent' is however a lot less clear. No account of representation in science is well-established." [25, p. 1]

We believe that an understanding of this important notion within the philosophy of science could shed new light on the problem of interpretating QM, for if representation through classical concepts is at stake and not just uncritically accepted there would be no need to "restore a classical way of thinking about *what there is*" and new conceptual schemes could be developed without the resistance of present physicists and philosophers of science — who either turn their back to the question of interpretation or attempt to return to a classical metaphysical scheme.

5. Instrumentalism and Scientific Realism: The End of Representation

Kant introduced the problem of representation together with his philosophy when he founded a justification of the relation between subject and object. Representation was then provided via a fixed set of *a priori* categories and forms of intuition which constituted and configured not only the object but also objective knowledge itself. The problem, which has haunted philosophy ever since, is that 'objective knowledge', (i.e., the knowledge provided by the transcendental subject) does not refer to the world *as it is*. Metaphysical questions of the type: 'what is the world *in itself*?' are regarded, form this perspective, as completely meaningless. But, completely neglecting the problems raised by the Kantian scheme regarding the possible knowledge about the world, and after the positivistic critic to Kantian metaphysics, the XX Century left us with two main lines of thought regarding the justification of scientific knowledge: instriumentalism and scientific realism.

The debate regarding quantum theory and its development, had an unquestionable influence in science and its philosophy. In this respect, maybe the most important figure regarding the development and fate of XX Century physics and philosophy is the Danish Niels Bohr.

> "These instrumentalist moves, away from a realist construal of the emerging quantum theory, were given particular force by Bohr's so-called 'philosophy of complementarity'; and this nonrealist position was consolidated at the time of the famous Solvay conference, in October of 1927, and is firmly in place today. Such quantum nonrealism is part of what every graduate physicist learns and practices. It is the conceptual backdrop to all the brilliant success in atomic, nuclear, and particle physics over the past fifty years. Physicists have learned to think about their theory in a highly nonrealist way, and doing just that has brought about the most marvelous predictive success in the history of science." [7, p. 1195]

Although Bohr recognized the importance of representation within phenomena he restricted representation to classical physics and language, for according to him [32, p. 7] "[...] the unambiguous interpretation of any measurement must be essentially framed in terms of classical physical theories, and we may say that in this sense the language of Newton and Maxwell will remain the language of physicists for all time." More importantly, "it would be a misconception to believe that the difficulties of the atomic theory may be evaded by eventually replacing the concepts of classical physics by new conceptual forms." Bohr's philosophical scheme is based, on the one side, on the recognition of classical discourse to account for phenomena, and on the other side, on the necessity of an intersubjective account of physical experience. The interpretational gap in between the quantum formalism and the empirical substructures explained in classical terms is resolved by evading the question about the manner in which the world — and not the measurement outcomes — is represented according to the quantum formalism. While holding fast to classical representations,[i] Bohr was forced to abandon representation in the quantum realm. In order not to step outside his original (classical) conceptual scheme he was forced to grant that the quantum wave function Ψ is only an *algorithm.* The notion of *complementarity*, taken as a *regulative principle*, is able to dissolve the contradiction between the same quantum wave function Ψ and its possible representations in terms of particles or waves.[j] The price to pay is that the quantum wave function must be left without a conceptual scheme that supports it. Detached from the classical world it must stand as an algorithm outside physical (quantum) reality.[k]

An algorithm is a set of finite instructions or steps which allows to execute or resolve a problem, a calculus machine through which one obtains results. Bohr's ideas mistaken in this radical form end up in the statement phrased explicitly by Fuchs and Peres that: "[...] quantum theory does not describe physical reality. What it does is provide an algorithm for computing probabilities for the macroscopic events ("detector clicks") that are the consequences of experimental interventions. This strict definition of the scope of quantum theory is the only interpretation ever needed, whether by experimenters or theorists." [12, p. 1] The danger of this instrumentalist position lies not only in the complete denial of the problems regarding physical experience and representation but also in the complete obturation of any possible creative solution to the problem of interpreting QM in relation to physical reality.

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[i]See also the very interesting discussion related to the importance of classical mechanics in Bohr's philosophy of physics in Ref. 3.

[j]I am grateful to Hernan Pringe for the many discussions regarding this subject. See also Ref. 20.

[k]Feyerabend was again very critical of Bohr's attempt to close this discussion using as an argument that QM is able to "account for an immense body of experience": "the semblance of absolute truth *is nothing but the result of an absolute conformism.* For how can we possibly test, or improve upon, the truth of a theory if it is built in such a manner that any conceivable event can be described, and explained, in terms of its principles? The *only* way of investigating such all embracing principles is to compare them with a different set of *equally all-embracing* principles — but this way has been excluded from the very beginning. The myth is therefore of no objective relevance, it continues to exist solely as the result of the effort of the community of believers and of their leaders, be these now priests or Nobel prize winners. *Its 'success' is entirely manmade.* This I think, is the most decisive argument against any method that encourages uniformity, be it now empirical or not. Any such method is in the last resort a method of deception. It enforces an unenlightened conformism, and speaks of truth; it leads to a deterioration of intellectual capabilities, of the power of imagination, and speaks of deep insight; it destroys the most precious gift of the young, their tremendous power of imagination, and speaks of education."

Was this revolution subverted by reactionary forces? ‘Scientific realism’ can be characterized in first place as a stance which considers physical theories as being able to describe the world independently of consciousness and observers. More importantly, scientific realism considers there is a *true* story of how the world *is*, and that science — physics above all — is able to find out what exactly is this story about. At the end of the road, there is a true and final representation provided exclusively through scientific research which describes reality as it is — a direct *correspondence* between our final theory and the world. Implicitly, scientific realism, takes for granted the fact that ‘experience’ is already well accounted by science. There is no problem with experience as exposed by experimental outcomes. Almost in the same way as instrumentalism, scientific realism deals with the problem of experience and representation in a very simple way: simply neglecting it.

6. Realism After Kant: Physical Theories as Expressions of the World

But today, after Kant, in the XXI Century, it is simply anacronic to neglect the importance of representation within both experience and the physical description of the world. The problem still stands, shaking the foundations of reference. How can we make reference to the world after Kant? This is the question that interests us.

Physical experiences are not suddenly discovered, we do not get hit by new physical experience if we are not prepared. There is a creative aspect involved within representation which allows us to set up the conditions of possibility for new physical experience to be found. A subtle interplay between creation and discovery which allows physical representation to expose an expression of the world.[1] In this same sense, a statement such as ‘the world is deterministic’ must be regarded today as meaningless or at least incomplete, for it is only through representation that one can discuss about ‘the world’. The notion of ‘determinism’ is a concept — and as such, part of a representation — not something we find in the world. One can say instead: ‘according to classical physics the world is deterministic’. In this case, the notion refers to the world only indirectly, through a specific formal and con-

[1]In line with this remark about the importance of creation, as noticed by Granger [13, p. 10]: “Un autre aspect de la connaissance scientifique est qu’elle consiste pour une bonne part à *suivre des règles*. Elle est en ce sens encore pensée. Mias on en manquerait un caractère essentiel si l’on ne remarquait qu’elle consiste aussi à *échapper* à des règles préétablies, à en créer de nouvelles, à figurer des exceptions.”

ceptual representation. It is exactly this distance, between representation and experience, which needs to be brought again into contact.

What are the conceptual presuppositions involved in classical physics? That the world is constituted by objects, that these objects are logically founded on the **principles of existence**, **non-contradiction** and **identity**, that they exist in space-time, etc. These particular presuppositions are a way to configure phenomena and not something which can be inferred from phenomena. For example, the notion of 'identity' is a presupposition to talk about objects, but we never find 'identity' in the world. We use identity as a presupposition to deal with phenomena and to constitute the notion of object, and so when we see a chair, we presuppose it remains the same through time; and even though we might change our perspective and not see the same face of the chair we keep in mind we are seeing the same chair.[m]

The need to resume and unify different theories comes implicitly from the idea — present in scientific realism — there is a single true description of the world which, in turn, implies there is a set of true concepts which describe the world *as it is.* Classical physics talks about a particular metaphysical world built up from classical objects. Does quantum physics makes reference to the same representation of the world as classical physics? Only if that would be the case, we would arrive at a contradiction by comparing the statements: 'according to quantum physics the world is indeterministic' and 'according to classical physics the world is deterministic'. The contradiction arises when presuposing there is a single answer to the question of representation of the world, that the questions have meaning independently of their reference frame. There is however a different path: Heisenberg's *closed theories.* Heisenberg understands closed theories as a relation of tight interconnected concepts, definitions and laws whereby a large field of phenomena can be described. Every physical theory needs to develop its own conceptual scheme — conceptual schemes which are independent of another different (closed) theory. As remarked by Heisenberg in an interview by Thomas Kuhn (quoted from [2, p. 98]): "The decisive step is always a rather discontinuous step. You can never hope to go by small steps nearer and nearer to the real theory; at one point you are bound to jump, you

[m]Contrary to Funes the memorious,[4] who could not see the connection between different perspectives: "It was not only difficult for him to understand that the genesic term *dog* embraced so many unlike specimens of differing sizes and different forms; he was disturbed by the fact that a dog at three-fourteen (seen in profile) should have the same name as the dog at three-fifteen (seen from the front)."

must really leave the old concepts and try something new... in any case you cant keep the old concepts. The only important aspect to consider a physical theory as closed is the internal coherency between the formal mathematical elements, the conceptual structure and the physical experience involved." We cannot access the world without representation, new theories determine intrinsically new phenomena, phenomena which cannot be seen from a different theory — and thus, cannot be translated. A fundamentally new theory is one that arrives at completely new phenomena through intrinsically different formal and conceptual presuppositions. Thus, from this perspective, the problem is not to find a bridge between quantum mechanics and classical mechanics — what is known today to be the 'quantum to classical limit' and was firstly proposed by Bohr in terms of his 'correspondence principle' — nor to find an encompassing theory which allows to unify them. We need not justify why the world is classical. The problem is to find what QM is talking about. What we need to do is to find a conceptual structure which allows to expose quantum experience in all its strength; find the metaphysical presuppositions which coherently — namely, without any *ad hoc* moves or unobservable metaphysical objects — relate to the mathematical formalism and are able to explain quantum phenomena. Each concept must find a meaning from *within* the theory itself.

The correspondence between concepts and things — accepting the common-sense correspondence theory of truth — takes us down the road of naive realism, but the plurality of incommensurable representations threatens us with both relativism and solipsism. At this point we need to explain how, from a realist perspective, we can consider physics as a creative and productive enterprise which relates to the world. The problem, from this perspective, is how to retain the univocity of the world and nature, while at the same time allowing for the multiplicity of representations.

The realism that we propose is based on the metaphysics of Spinoza who conceived Being as a singular substance with infinite modes. Modes are the way in which the attributes of the one, infinite substance manifests its essence; in other words, everything we know is a mode of the eternal substance manifesting in itself. Each mode is capable of expressing an attribute. While the attribute 'extension' is expressed by 'physical bodies'; the attribute 'thought' is expressed by 'ideas'. There are other modes or expressions of Being but we, as human beings, know only these two. Because extension and thought have nothing in common, the two realms of matter and mind are causally closed systems. However, both are expressions of the same substance. Although being different modes, the attributes are independent and equal, it is the same modification of the substance which is expressed in one mode or the other. In other words, the attributes are *parallel* expressions of Being.[n] They are the same! But it is exactly this *principle of univocity* which we need in order to bring into unity the multiple representation of physics — given by classical mechanics, relativity theory, QM, etc. In analogous fashion to the way in which Spinoza claims that the attribute extension and the attribute thought express univocally one and the same substance, we could think — using Spinoza's metaphysical scheme — that each closed physical representation provides an expression of one and the same world.[21] Within this scheme, an *adequate physical experience* is one which can be coherently configured from a particular theory and exposes the world through a specific set of phenomena. Every phenomenon is *local* in the sense that the presuppositions involved can be only applied to the specific designed physical experience, but never to the world in itself. To believe that such presuppositions talk about the world in a "correspondence" manner implies once again a non-representational scheme of thought — which is exactly the jump we want to avoid. Although every phenomena is perspectival in the sense that it depends directly on the theory from which it is observed, there is no relativism involved in our scheme simply because every statement arising from an empirically adequate, closed and coherent theory expresses the world adequately — just in the same way the attribute extension and the attribute thought express the substance. Statements which pertain to intrinsically different theories cannot be compared not even in principle, for such comparison would imply a translation, the presupposition that one can find, for any two representations of the world, an encompassing representation or theory which takes them both into account — analogously: thought cannot be translated into extension nor extension into thought. Escaping relativism, it cannot be claimed that "anything works!" for the final judge of physical expression is always physical experience and its relation to the *hic et nunc*. Classical physics is just a particular metaphysical scheme which expresses the world and through which we have found an amazing range of empirical findings. We know what classical physics talks about. Quantum physics stands still today

[n]It is important to remark that the term 'parallelism' does not appear explicitly in Spinoza. As explained by Diego Tatian [26, p. 50]: "Perhaps the notion of 'parallelism' is not the best choice in order to interpret what, within Spinoza, makes reference to an identity. The metaphor of parallelism restores plurality and dualism, precisely what Spinoza seeks to overcome."

incoherently related to a still too classical physical experience, still awaiting a metaphysical scheme which allows us to explain what the theory is talking about. Only when we answer this question in a coherent manner, we will be able to say that we have "understood" QM.

7. Final Remarks: What is a 'Click' in a Detector?

It is only the theory which can tell you what can be observed.[o] From our stance, a phenomenon is not independent of the particular theory which lies down the conditions of possibility to account for physical experience. Each particular physical phenomenon stands within the limits of the particular physical theory which contains it as a possibility. Thus, to point out there is a 'click' in a detector is not enough to provide an adequate account of phenomena. Contrary to radical empiricist ideal, we do not agree — after Kant — that a 'click' in a detector can be regarded as an observation voided of theoretical content. And this counts of course not only for electrons but also for chairs and tables. A 'click' can be understood from within different, mutually incompatible, physical theories. One cannot presuppose that physical experience appears itself naked. Thus, a 'click' in a detector must not necessarily be considered as limited by classical physical concepts. We leave open the possibility that new concepts can allow us to configure a conceptual scheme which closes the circle connecting the orthodox mathematical formulation of quantum mechanics to physical experience. Unlike Bohr, we do not agree that, the observable quantum phenomena must be necessarily considered as "classical phenomena" simply because there is a 'click' in a detector and thus a space-time event. It is the specific representation of a 'click' which configures or not physical experience in classical terms — by considering the detector as a classical object in space-time, etc. As a matter of fact, in QM, it is the 'click' itself which cannot be configured under such classical presuppositions in a closed and coherent manner, in other words, the 'click' does not seem to come from a 'classical object'! The Bell inequality provides statistical limits to the outcomes of classical physical experience and proves that QM cannot be subsumed under such

[o]This phrase has been quoted in several occasions by Heisenberg [15, p. 269]: "Could it be that we had asked the wrong question? I remember Einstein telling me, 'it is always the theory which decides what can be observed.' And that meant, if it was taken seriously, that we should not ask: 'How can we represent the path of the electron in the cloud chamber?' We should ask instead: 'Is it not perhaps true that in nature only such situations occur which can be represented in quantum mechanics or wave mechanics?"

classical presuppositions. From our standpoint, the analysis of a 'click' in a photographic plate presupposes the conditions under which such 'click' arises, it is only then that we can talk about phenomena. Only in the case a classical representation would be able to account for the result and explain the 'click' we could say we would be talking about classical phenomena. To say it differently, not every set of 'clicks' can be seen as arising from a classical theory. The question is: what are the conditions under which we can explain, in both conceptual and formal terms, the 'click' of which quantum theory is talking about? We need to be able to close the gap in between the formalism — which provides a mathematical representation of the theory — and the concepts — which provide a conceptual representation of the theory — in order to properly account for phenomena. Thus, in the same way we use a point in phase space to describe the trajectory of an object in classical space-time — both of which, namely, 'the point in phase space' and 'the object in space-time', are part of the mathematical and conceptual representation of classical physics —, we need to find out in a coherent manner what is QM describing in conceptual terms.

According to our expressionist stance, there is no 'physical world' nor 'physical context', what there is instead is a 'physical representation' given by a particular theory which allows to configure and consider a particular physical experience. Physical experiences in the *hic et nunc* express *singularities*. We take a singularity to be the meeting point between the *hic et nunc* and *physical representation*. It is in this point were Being is expressed through physical representation itself. The problem of such a realist stance remains to build up a representational conceptual scheme which would allow us to relate coherently the quantum formalism to the empirical structure predicted by the theory in a closed and coherent manner. The concepts must be internally defined by the theory itself and not presupposed as self-evident extensions of a different theory. This means, for example, that it is not obvious to us that the notion of possibility used in classical physics is, or should be considered as the same notion of possibility used in quantum theory.[p] Neither we accept that the only type of effectuation can be that of actuality. Potential effectuations can open the door to a new experience. The concepts which allow us to provide a coherent account of quantum phenomena must be able to provide a story about how the world *is* according to the representation provided by QM. In turn, these same concepts might be even capable of developing new physical experiences.

[p]See in the respect the discussion presented in Ref. 24.

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References

1. Bitbol, M., 2010, "Reflective Metaphysics: Understanding Quantum Mechancis from a Kantian Standpoint", *Philosophica*, **83**, 53-83.
2. Bokulich, A., 2006, "Heisenberg Meets Kuhn: Closed Theories and Paradigms", *Philosophy of Science*, **73**, 90-107.
3. Bokulich, P., and Bokulich, A., 2005, "Niels Bohr's Generalization of Classical Mechanics", *Foundations of Physics*, **35**, 347-371.
4. Borges, J.L., 1989, *Obras completas: Tomo I*, María Kodama y Emecé (Eds.), Barcelona.
5. Burtt, E. A., 2003, *The Metaphysical Foundations of Modern Science*, Dover, New York.
6. Carnap, H., Hahn, H. and Neurath, O., 1929, "The Scientific Conception of the World: The Vienna Circle", *Wissendchaftliche Weltausffassung*.
7. Curd, M. and Cover, J. A., 1998, *Philosophy of Science. The central issues*, Norton and Company (Eds.), Cambridge University Press, Cambridge.
8. Dieks, D., 1988, "The Formalism of Quantum Theory: An Objective Description of Reality", *Annalen der Physik*, **7**, 174-190.
9. Einstein, A., 1916, "Ernst Mach." *Physikalische Zeitschrift*, **17**, 101-104.
10. Feyerabend, P., 1963, "How to Be a Good Empiricist — A Plea for Tolerance in Matters Epistemological", in *Philosophy of Science*, 922-949, M. Curd and J. A. Cover (Eds.), 1998.
11. Friedman, M., 2000, *A Parting of the Ways: Carnap, Cassirer, and Heidegger*, Open Court, Chicago.
12. Fuchs, C. and Peres, A., 2000, "Quantum theory needs no 'interpretation' ", *Physics Today*, **53**, 70.
13. Granger, G. G., 2001, *Sciences et réalité*, Editions Odile Jacob, Paris.
14. Heisenberg, W., 1971, *Physics and Beyond*, Harper & Row.
15. Heisenberg, W., 1973, "Development of Concepts in the History of Quantum Theory", In *The Physicist's Conception of Nature*, 264-275, J. Mehra (Ed.), Reidel, Dordrecht.
16. Howard, D., 1993, "Was Einstein Really a Realist?", *Perspectives on Science*, **1**, 204-251.
17. Howard, D., 1994, "Einstein, Kant, and the Origins of Logical Empiricism", In *Logic, Language, and the Structure of Scientific Theories: Proceedings of the Carnap-Reichenbach Centennial, University of Konstanz, 21-24 May 1991*, 45-105, W. Salmon and G. Wolters (Eds.), University of Pittsburgh Press, Pittsburgh.
18. Howard, D, 2010, "Einstein's Philosophy of Science", *The Stanford Encyclopedia of Philosophy (Summier 2010 Edition)*, E. N. Zalta (Ed.), URL: http://plato.stanford.edu/archives/sum2010/entries/einstein-philscience/.

19. Kant, I., 1998, *The Cambridge Edition of the Works of Immanuel Kant: The Critique of Pure Reason*, Trans. and ed. P. Guyer and A. Wood, Cambridge University Press, Cambridge.
20. Pringe, H., 2007, *Critique of the Quantum Power of Judgement. A Transcendental Foundation of Quantum Objectivity*, Kantstudien Ergnzungshefte 154, Berlin, 2007.
21. de Ronde, C., 2009, "El enfoque de descripciones complementarias: en búsqueda de un desarrollo expresivo de la realidad física", *Perspectivas Metodológicas*, **9**, 126-138.
22. de Ronde, C., 2011, *The Contextual and Modal Character of Quantum Mechanics: A Formal and Philosophical Analysis in the Foundations of Physics*, PhD dissertation, Utrecht University.
23. de Ronde, C. and Bontems, V., 2011, "La notion d'entité en tant qu'obstacle épistémologique: Bachelard, la mécanique quantique et la logique", *Bulletin des Amis de Gaston Bachelard*, **13**, 12-38.
24. de Ronde, C., Freytes, H. and Domenech, G., 2013, "Interpreting the Modal Kochen-Specker Theorem: Possibility and Many Worlds in Quantum Mechanics", *Studies in History and Philosophy of Modern Physics*, in press.
25. Suárez, M., 2003, "Scientific representation: Against similarity and isomorphism", *International Studies in the Philosophy of Science*, **17**, 225-244.
26. Tatián, D., 2009, *Spinoza. Una introducción*, Editorial Quadrata-Biblioteca Nacional, Buenos Aires.
27. Vaidman, L., 2008, "Many-Worlds Interpretation of Quantum Mechanics", In *The Stanford Encyclopedia of Philosophy (Fall 2008 Edition)*, E. N. Zalta (Ed.), URL: http://plato.stanford.edu/archives/fall2008/entries/qm-manyworlds/.
28. Van Fraassen, B. C., 1980, *The Scientific Image*, Clarendon, Oxford.
29. Van Fraassen, B. C., 2002, *The Empirical Stance*, Yale University Press, New Haven.
30. Van Fraassen, B. C., 2008, *Scientific Representation: Paradoxes of Perspective*, Oxford University Press, Oxford.
31. Van Fraassen, B. C., 1991, *Quantum Mechanics: An Empiricist View*, Clarendon, Oxford.
32. Wheeler, J. A. and Zurek, W. H. (Eds.) 1983, *Theory and Measurement*, Princeton University Press, Princeton.
33. Hettche, M., "Christian Wolff", *The Stanford Encyclopedia of Philosophy (Fall 2008 Edition)*, E. N. Zalta (Ed.), URL: http://plato.stanford.edu/archives/fall2008/entries/wolff-christian/.

BOHRIAN COMPLEMENTARITY IN THE LIGHT OF KANTIAN TELEOLOGY

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The Kantian influences on Bohr's thought and the relationship between the perspective of complementarity in physics and in biology seem at first sight completely unrelated issues. However, the goal of this work is to show their intimate connection. We shall see that Bohr's views on biology shed light on Kantian elements of his thought, which enables a better understanding of his complementary interpretation of quantum theory. For this purpose, we shall begin by discussing Bohr's views on the analogies concerning the epistemological situation in biology and in physics. Later, we shall compare the Bohrian and the Kantian approaches to the science of life in order to show their close connection. On this basis, we shall finally turn to the issue of complementarity in quantum theory in order to assess what we can learn about the epistemological problems in the quantum realm from a consideration of Kant's views on teleology.

Keywords: Bohr; complementarity; Kant; teleology.

1. Bohr on Biology and Atomic Physics

Bohr maintains that the limits of mechanical explanations of atomic phenomena are analogous to those of physical or chemical accounts of biological phenomena:[a]

> "Indeed, the essential non-analyzability of atomic stability in mechanical terms presents a close analogy to the impossibility of a

[a]Some of the ideas put forward in this paper were first discussed in Ref. 1, specially in [1, pp. 189ff].

> physical or chemical explanation of the peculiar functions characteristic of life." Bohr [2, p. 9]

Neither atomic nor biological phenomena can be accounted for in a mere mechanical way. In the case of atomic physics it is necessary to introduce a postulate which goes beyond any classical-mechanical description of nature: the quantum postulate. This postulate

> "attributes to any atomic process an essential discontinuity, or rather individuality, completely foreign to the classical theories and symbolised by Planck's quantum of action." Bohr [3, p. 53]

The assumption of the quantum postulate entails that atomic processes do not obey the law of *continuity* of causality, according to which a changing thing passes through all the infinite spatio-temporal states that lie between the initial and the final state. More precisely, the continuity of *efficient* causality is thereby rejected, the limits of classical-mechanical explanation of phenomena being thus transposed.

In the case of biology, one must include in the analysis of living organisms a teleological point of view, strange to any mechanical conception of the problem:

> "Indeed, in this sense teleological argumentation may be regarded as a legitimate feature of physiological description which takes due regard to the characteristics of life in a way analogous to the recognition of the quantum of action in the correspondence argument of atomic physics." Bohr [2, p. 10]

The point of view of teleology transcends a mere mechanical account, which considers only *efficient* causes, by moreover appealing to *final* ones.

But even though atomic and life phenomena go beyond mechanical considerations, they must be explained in those very terms which are insufficient for their full understanding. While *quantum* phenomena must be described by means of *classical* concepts, *biological* phenomena demand the use of *physical* and *chemical* terms:

> "[N]o result of biological investigation can be unambiguously described otherwise than in terms of physics and chemistry, just as any account of experience even in atomic physics must ultimately rest on the use of the concepts indispensable for a conscious recording of sense impressions." Bohr [2, p. 21]

From this point of view, Bohr rejects as dogmatic the objective *denial* of teleology (i.e., mechanicism) as well as its objective *affirmation* (i.e., vitalism):[b]

> "It will be seen that such a viewpoint is equally removed from the extreme doctrines of mechanicism and vitalism. On the one hand, it condemns as irrelevant any comparison of living organism with machines, be these the relatively simple constructions contemplated by the old iatro-physicists, or the most refined modern amplifier devices, the uncritical emphasis of which would expose us to deserve the nickname of 'iatro-quantists'. On the other hand, it rejects as irrational all such attempts at introducing some kind of special biological laws inconsistent with the well-established physical and chemical regularities, as we have in our days been revived under the impression of the wonderful revelations of embryology regarding cell growth and division." Bohr [2, p. 21]

Mechanicists claim that *final* causes play no role in the account of life, because organisms are nothing but complex machines, the operation of which can be totally understood by means of *efficient* causes. By contrast, vitalists see in organisms final causes at work, which are irreducible to the kind of causes involved in any physical or chemical explanation. Bohr's position aims at overcoming this dispute. On the one hand, since teleological arguments are *necessary* for the science of life, mechanicism cannot be accepted. On the other hand, Bohr claims against vitalism that the *explanation* of biological phenomena cannot appeal to any non-efficient causality, i.e., it *must* rest solely on the use of physical and chemical terms. Thus, Bohr states:

> "Though the view point of complementarity rejects every compromise with any anti-rationalistic vitalism, it ought at the same time to be suited for revealing certain prejudices in the so-called mechanicism." Bohr [5, p. 90]

In summary, teleological and mechanical accounts exclude each other but both are nevertheless necessary for an appropriate understanding of biological phenomena. In Bohr's terms, teleology and mechanicism stand to each other in a relation of *complementarity*:

[b]On Bohr's relation to the mechanicism-vitalism dispute see: [4, pp. 211ff].

> "[I]t must even be realized that mechanistic and finalistic argumentation, each within its proper limits, present inherently complementary approaches to the objective description of the phenomena of organic life." Bohr [6, p. 159][c]

Bohr delineates his views on biology by drawing a quite far-reaching parallel between the science of life and atomic physics. We shall now take this result as the point of departure of two successive analyses. Firstly, we shall show the close connection between the Bohrian ideas on biology and those put forward by Kant on the problem of living entities. This will allow us in a second instance to shed light on Bohr's ideas on atomic physics from a Kantian point of view, precisely by considering the analogies between biology and quantum theory.

2. The Parallelism between Kant and Bohr on the Problem of Life

As well as Bohr, Kant claims that a mere mechanical account of life is insufficient. On this issue, Kant states in a very well-known passage:

> "It is indeed quite certain that we cannot adequately cognise, much less explain, organised beings and their internal possibility, according to mere mechanical principles of nature; and we can say boldly it is alike certain that it is absurd for men to make any such attempt or to hope that another Newton will arise in the future, who shall make comprehensible by us the production of a blade of grass according to natural laws which no design has ordered. We must absolutely deny this insight to men." Kant [8, AA V p. 400]

No proper account of the phenomena of life can be gained by mere mechanical considerations. Rather, a teleological perspective is indispensable for the study of organisms:

[c]Also: "In fact, we are led to conceive the proper biological regularities as representing laws of nature complementary to those appropriate to the account of the properties of inanimate bodies, in analogy with the complementarity relationship between the stability properties of the atoms themselves and such behaviour of their constituents particles as allows of a description in terms of space-time coordination." Bohr [2, p. 21]. In the same sense: "[T]he essence of the analogy considered is the typical relation of complementarity existing between the subdivision required by a physical analysis and such characteristic biological phenomena as the self-preservation and the propagation of individuals." Bohr [7, p. 458]

> "It is an acknowledged fact that the dissectors of plants and animals, in order to investigate their structure and to find out the reasons, why and for what end such parts, such a disposition and combination of parts, and just such an internal form have been given them, assume as indisputably necessary the maxim that nothing in such a creature is *vain*. . . . In fact, they can [not] . . . free themselves from this teleological proposition . . .; for . . . without [it] . . . we should have no guiding thread for the observation of a species of natural things which we have thought teleologically under the concept of natural purposes." Kant [8, AA V p. 376]

Yet the Kantian defence of teleology, just as the Bohrian one, far from putting biological phenomena completely beyond the limits of any mechanical account, entails the necessity of this kind of explanation for an appropriate understanding of life. Thus, Kant states that, unless we investigate organisms in accordance with mechanical laws, we will not obtain any knowledge of them, for without a physical explanation "there can be no proper knowledge of nature at all." Kant [8, AA V p. 387]. In other words, only if we make use of efficient causes to account for the structure and functioning of organisms may we have "experience at all" of them, Kant [8, AA V p. 376].

On the one hand, teleology and mechanicism stand to each other in an exclusion relation:

> "If I choose, e.g. to regard a maggot as the product of the mere mechanism of nature (of the new formation that it produces of itself, when its elements are set free by corruption), I cannot derive the same product from the same matter as from a causality that acts according to purposes. Conversely, if I regard the same product as a natural purpose, I cannot count on any mechanical mode of its production and regard this as the constitutive principle of my judgement upon its possibility, and so unite both principles. One method of explanation excludes the other." Kant [8, AA V pp. 411ff]

On the other hand, both perspectives are indispensable for our knowledge of organisms:

> "We should explain all products and occurrences in nature, even the most purposive, by mechanism as far as is in our power (the limits of which we cannot specify in this kind of investigation).

> But at the same time we are not to lose sight of the fact that those things which we cannot even state for investigation except under the concept of a purpose of Reason, must, in conformity with the essential constitution of our Reason, mechanical causes notwithstanding, be subordinated by us finally to causality in accordance with purposes." Kant [8, AA V p. 415]

Just as Bohr, Kant maintains that both mechanical and teleological perspectives on organisms are *necessary*, *even though* they exclude one another. We may thus say, using Bohrian terminology, that Kant conceives these perspectives as *complementary*.[d]

In a purposive object, the relationships between the efficient causes which explain its existence are such that all the constituent parts of the object have a common ground for their possibility. This common ground is precisely the representation of the whole, which therefore precedes the representation of the parts. The relationship between the parts of purposive products of nature must be that of an efficient causality. While the representation of the whole must play the role of a *final* cause of the existence of the parts, the reciprocal causality of the parts must be the *efficient* cause of the whole. But thereby no intentional cause is attributed to nature. Rather, the object is thought of *as if* its concept *contained* the cause of the existence of the object, i.e., *as if* the object *were* an end, [8, AA V p. 180]. This *symbolic* analogy provides us with a *heuristics* that guides our empirical research:

> "E.g., by saying that the crystalline lens in the eye has the end of reuniting, by means of a second refraction of the light rays, the rays emanating from one point at one point on the retina, one says only that the representation of an end in the causality of nature is conceived in the production of the eye because such an idea serves as a principle for guiding the investigation of the eye as far as the part that has been mentioned is concerned, with regard to the means that one can think up to promote that effect." Kant [11, AA XX p. 236]

The concept of a natural end is therefore not a rule for the *determination* of nature, but a rule for our *reflection* upon her. The *objective* character of the representation of an organism, i.e., its independence from any contingent state of the cognitive subject, does not rely on its subsumption

[d] Favrholdt makes the same point in [9, vol. 10 p. 4]. See also [10, p. 18].

under the concept of a natural end. Rather, this subsumption brings about the *systematic unity* of the mechanical laws, which explains the organism's existence and the harmony among its constituent parts. The concept of a natural end plays no *constitutive*, but solely a *regulative* role in experience:

> "The concept of a thing as in itself a natural end is therefore not a constitutive concept of the understanding or of reason, but it can still be a regulative concept for the reflecting power of judgment, for guiding research into objects of this kind and thinking over their highest ground in accordance with a remote analogy with our own causality in accordance with ends." Kant [8, AA V p. 376][e]

Kant argues that his conception of teleology is the only one which can overcome the antinomy between those who claim that one should completely dispense with final causes in nature (the *idealists* of final causes) and those who affirm them *objectively* (the *realists* of final causes), [8, AA V pp. 386ff]. As we have seen, a new version of this old dispute can be solved, according to Bohr, from the point of view of complementarity.

This profound coincidence between the Kantian and the Bohrian analysis of the problem of life reflects the influence that the epistemological ideas of Harald Høffding and Bohr's father, the physiologist Christian Bohr, exercised on Niels Bohr.[f] In fact, in the introduction to an article in which he discusses the complementarity between teleological and mechanical considerations, Bohr quotes a long passage from his father:

> As far as physiology can be characterized as a special branch of natural sciences, its specific task is to investigate the phenomena peculiar to the organism as a given empirical object in order to obtain an understanding of the various parts in the self-regulation and how they are balanced against each other and brought into harmony with variations in external influences and inner processes. It is thus in the very nature of this task to refer the word purpose to the maintenance of the organism and consider as purposive the regulation mechanism which serve this maintenance.

[e]The "remote" character of this analogy must be stressed. A natural end is a self-organizing being, i.e., each part does not just exist for the sake of the others but produces the others according to the concept of the whole. Thus, the formative power is to be thought of as an internal principle. To the contrary, in the case of a work of art or a machine, the producing cause is to be found outside of the product, in a rational being acting in view of the concept of the whole.

[f]See [12, pp. 12ff and pp. 157ff].

> . . .
> The *a priori* assumption of the purposiveness of the organic process is, however, in itself quite natural as a heuristic principle and can, due to the extreme complication and difficult comprehension of the conditions in the organism, prove not only useful, but even indispensable for the formulation of the special problem of the investigation and the search of ways for its solution." Bohr [2, p. 96]

Bohr acknowledges that these remarks express the attitude in the circle in which he grew up and to whose discussions he listened in his youth. He sees in these ideas a suitable framework for his reflections on complementarity, since in both biology and atomic physics the challenge is how to cope with a *limitation* in principle of the so-called mechanical conception of nature. In the next section we shall turn to the systematic consequences for our understanding of atomic phenomena that may be obtained by considering such a Kantian perspective on the problem of life.

3. Bohrian Complementarity in the Light of Kantian Teleology

Scholars have hitherto analysed Bohr's views on biology, trying to establish how and to what extent complementarity may be applied in biology, given the analogies Bohr draws between the latter and atomic physics.[g] In contrast, the reciprocal question, i.e., how the epistemological problems of quantum theory may be highlighted by the consideration of principles used for the investigation of organisms, has hardly been studied. However, the analogies already discussed not only suggest the adoption of a complementary point of view in order to face biological problems, but also enable us to conceive the novel situation in atomic physics in the light of teleology.[h]

Just as it is the case in biology, atomic physics discovers a limitation of the mechanical conception of nature. To cope with such limitation, a *discontinuous efficient* causality is introduced in the quantum realm by means of the quantum postulate. In this situation, one nevertheless ought to make extensive use of classical images, i.e., spatio-temporal descriptions of phenomena where a continuous efficient causality takes place. This demand

[g]See Refs. 4,13.

[h]This is a possibility explicitly left open by Bohr when he indicates that "however unfamiliar the aspects of the observation problem met with in atomic theory may appear on the background of classical physics, they are by no means new in other fields of science." Bohr [14, p. 108]. See also [15, p. 134].

is contained in the principle of correspondence:

> "[T]he so-called correspondence argument ... expresses the endeavour of utilizing to the outmost extent the concepts of the classical theories of mechanics and electrodynamics, in spite of the contrast between these theories and the quantum of action." Bohr [2, p. 5]

Only if described in classical terms may experimental devices and results count as *objective*, i.e., as free from any contingent dependence on the state of an individual observer:[i]

> "*However far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms.* The argument is simply that by the word 'experiment' we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics." Bohr [17, p. 39]

But this application of classical concepts does not provide us with a *single* space-time *and* causal picture. On the contrary, incompatible classical images, e.g., wave- and particle-pictures, are required for a proper interpretation of empirical data:

> "Very striking illustrations are afforded by the well-known *dilemmas* regarding the properties of electromagnetic radiation as well as of material corpuscles, evidenced by the circumstances that in both cases contrasting pictures as waves and particles appear equally indispensable for the full account of experimental evidence." Bohr [18, p. 87]

The classical pictures corresponding to different experimental conditions exclude each other. However, only if one takes all of them into consideration may the empirical evidence be exhausted. Therefore, these pictures are *complementary*:

[i] "It is also essential to remember that all unambiguous information concerning atomic objects is derived from the permanent marks ... left on the bodies which define the experimental conditions ... The description of atomic phenomena has in these respects a perfectly objective character, in the sense that no explicit reference is made to any individual observer ... As regards all such points, the observation problem of quantum physics in no way differs from the classical physical approach." Bohr [16, p. 3]

> "[E]vidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as *complementary* in the sense that only the totality of the phenomena exhausts the possible information about the objects." Bohr [17, p. 40]

At this point, the problem of *how* these different pictures may be unified arises. One may argue that, e.g., wave- and particle-phenomena are phenomena *of* a quantum object, and *in this way* they obtain their *systematic unity*. But then one should explain how the relation between the object and its complementary phenomena is to be conceived.

A quantum object *grounds* the complementary phenomena appearing in different experimental situations, thereby bringing about systematic unity among them. However, a *direct* intuitive representation of this grounding relation cannot be given, for this kind of exhibition in intuition demands that the requisites of spatio-temporal co-ordination and (continuous) causality be *both* fulfilled, while these exclude each other:

> "The very nature of the quantum theory thus forces us to regard the space-time co-ordination and the claim of causality ... as complementary but exclusive features of the description." Bohr [3, p. 34-35]

Thus, the grounding relation of the quantum object to its phenomena can only be *indirectly* exhibited in intuition, i.e., by means of symbols. Accordingly, *complementary* pictures are used as *symbols* of a quantum *object*.[j] Thereby, one affirms, e.g., that the object behaves in certain situations *as if* it *were* a particle and in certain others *as if* it *were* a wave:

> "[W]e symbolize [the quantum object] by the abstractions of isolated particles and radiation." Bohr [3, p. 69]

When asserting that a quantum object behaves in certain situations as if it were a wave and in certain others as if it were a particle, one conceives the corresponding wave- and particle-pictures as phenomena *of* the quantum object. In this way, they become connected and subsumed under a common general concept.[k]

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[j]In a similar sense, Pauli asks: "Wäre es [das Atom] kein Symbol, wie könnte es 'sowohl Welle als auch Teilchen' sein?" Pauli, quoted in [19, p. 21].

[k]Faye posed the question: "how can the concept of a cause ... be applied to the atomic object if is only cognitively meaningful to use it for the description of the phenomenal

By this, quantum physics achieves an essential new way of interpreting empirical data. A distinction is made between the way in which data are represented as *objective* results of a measurement and the way in which they acquire *systematic unity*. In the first case, a *direct* exhibition of *classical* concepts in intuition is required. In the second case, an *indirect* or *symbolic* exhibition of *quantum* concepts:

> "While ... in classical physics the distinction between object and measuring agencies does not entail any difference in the character of the description of the phenomena concerned, its fundamental importance in quantum theory ... has its roots in the indispensable use of classical concepts in the interpretation of all proper measurements, even though the classical theories do not suffice in accounting for the new types of regularities with which we are concerned in atomic physics." Bohr [1, p. 701]

The framework of complementarity shares with the Kantian account of organisms the double-layer structure of the epistemological analysis. On the one hand, the *objectivity* of experience is gained in the quantum realm by means of the application of classical concepts, which generates mechanical images of nature. On the other hand, the *systematic unity* of those pictures results from a *reflection* by which one does *not* claim to *determine* nature. Rather, the concepts thereby used, i.e., the concepts of quantum objects, are *regulative* concepts applied through their *symbolic* exhibition in intuition. In this way, the point of view of complementarity provides a heuristic framework for the unification of the classical descriptions obtained under the guidance of the correspondence principle.

Just as in the case of the Kantian views on teleology, the complementarity approach enables us to overcome the dispute between those who affirm and those who deny *objectively* a heuristic assumption. While in the case of biology this assumption is that of a final causality, in atomic physics the quantum postulate entails the discontinuity of efficient causality.[l] Those who ascribe this discontinuity to quantum objects as to objects of possible experience are *realists* regarding quantum objects.[m] By contrast,

object?" Faye [12, p. 207]. One may answer that the application of the concept of cause is performed by means of a *symbolic* analogy: the quantum object causes its phenomena *as if* it *were* a wave or a particle.

[l]Nonetheless, this discontinuity of efficient causality does bear a relationship to final causality. For a discussion of this point, see: [20, pp. 156ff].

[m]See, e.g. Ref. 21.

instrumentalists reject to give the concepts of quantum objects objective validity, so that they are considered at most useful instruments for the economy of thought.[n] From a Bohrian-Kantian perspective one may argue that this debate rests on the premise that the concept of a quantum object is a rule for the *determination* of nature rather than one guiding our *reflection* on her. But, as we have seen, one may consider the concept of a quantum object as a *regulative* representation, necessary not for the objectivity but for the systematic unity of experience. In this case, one may argue against realism that quantum objects are not objects of possible experience themselves. However, in contradistinction to the views of instrumentalism, the *necessary* (and not just useful or convenient) role the concept of a quantum object plays does provide it with objective (albeit regulative) *validity*. Briefly, even though quantum objects are not objects of experience in the Kantian sense, their concepts are necessary for the systematic unity of experience under the assumption of the quantum postulate.

By this, no transcendent character is ascribed to quantum objects, for they do play a role in experience and must be assumed on *empirical grounds*, in order to connect complementary phenomena. These are then conceived as phenomena *of the object*, and the latter is nothing "over and above its possible manifestations." Faye [12, p. 229]. The way in which the latter manifests itself, i.e., the kind of relationship it maintains to its phenomena, is, however, *symbolic*.

In this way, an appropriate epistemological distinction between quantum objects and classical objects can be gained. In contradistinction to quantum objects, classical objects are *themselves* objects of experience in the Kantian sense. This means that their concepts perform a *constitutive* task: when a sensible manifold is subsumed under such a concept, *objective* knowledge is first obtained.[o] In Kantian terminology, these concepts are rules for the *determining* power of judgment. By contrast, the regulative task of the concepts of quantum objects is to bring about systematic unity

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[n]This position, which paradoxically keeps on being attributed to Bohr –e.g. in [22, p. 120]–, seems to be the most widespread among the physics community.
The fact that Bohr's philosophy of physics overcomes the instrumentalist-realist antagonism is seen by Murdoch, who calls Bohr's position "instrumentalistic realism." Murdoch, see [23, pp. 222 ff]. Beller nevertheless criticizes this expression, considering it a "philosophical hybrid." Beller, see [24, p. 185]. On Bohrian complementarity and the realism debate see also Ref. 25.

[o]This is of course (for the argument's sake) a very simplified account of a complex problem in which the categories of understanding and their corresponding schemata play a central role.

among *already objective* phenomena, constituted as such by the application of classical concepts. The concepts of quantum objects are thus rules for the *reflecting* power of judgment.

4. Conclusions

Let us summarize the analysis carried out in this paper. We have seen that Bohr draws a far-reaching parallelism between the epistemological problems in biology and atomic physics, delineating his views on the science of life on a Kantian basis. By making use of the analogy between biology and quantum physics and considering the Kantian understanding of teleology, we were able to trace some important aspects of the point of view of complementarity.

Whereas in the Kantian account of teleology the constitution of organisms as objects of experience is carried out by subsuming their constituent parts under mechanical laws, in the Bohrian account of atomic physics the constitution of quantum phenomena as objective phenomena is achieved by their description in classical terms. Moreover, while the systematic unity of the laws explaining the existence of an organism is thought of as guided by final causes, the systematic unity of complementary phenomena is obtained under the assumption of a discontinuous causality. In both cases, firstly, objectivity is achieved only when the demands of both causality and spatio-temporal representation are fulfilled. Secondly, a heuristic principle is assumed for the sake of the systematic unity of the so constituted phenomena: the principle of teleology in the case of organisms, the quantum postulate in atomic physics.

The concept of a quantum object, therefore, plays a role analogous to that played by the concept of an organism as a natural end in the Kantian sense. Both are regulative, i.e., they are rules according to which systematic unity in experience is brought about. In the case of organisms, this unity is established among their components parts, while, in the case of quantum objects, among their complementary phenomena.

Moreover, *the way in which* this task is fulfilled in both cases is similar: symbols should be used in order to *indirectly* exhibit in intuition a causal relation that cannot be directly exhibited. The constituent parts of an organism are related to each other *as if* the very organism *were* a natural end. In turn, the complementary phenomena get connected when we think of the quantum object *as if* it *were* in certain situations a wave or a particle. Whilst the final causality in organisms is represented by means of an analogy with artefacts, for the intuitive representation of the discontinuous causality in the quantum realm we make symbolic use of classical concepts.

Briefly, teleology and complementarity provide us with heuristic frameworks to cope with different kinds of limitation of the mechanical image of nature. As Bohr stated:

> "Physicists may derive some help and encouragement from the recognition that the novel situation as regards the causality problem in which they find themselves is not unique." Bohr [14, p. 108]

Understanding the parallel between the epistemological problems in biology and quantum physics shows that Bohr's piece of advice seems in fact to be useful.

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References

1. N. Bohr, *Phys. Rev.* **48**, (reprinted in [9, p. 291–298, vol. 7]), (1935).
2. N. Bohr, *Atomic Physics and Human Knowledge* (John Wiley & Sons, New York, 1958).
3. N. Bohr, *Atomic Theory and the Description of Nature* (Cambridge University Press, Cambridge, 1934).
4. H. Folse, *Bio. and Phil.* **5**, (1990).
5. N. Bohr, *Phil. of Sci.* **4**, (reprinted in [26, pp. 83–91]), (1937).
6. N. Bohr, Physical science and the study of religions, reprinted in [26, pp. 155–160]), (1953).
7. N. Bohr, *Nature* **131**, (1933).
8. I. Kant, *Critique of Judgement*, translated with Introduction and Notes by J.H. Bernard (2nd ed. revised), (Macmillan, London, 1914). (AA V, pp. 165–486[27]).
9. L. Rosenfeld, J. Rud Nielsen, E. Rüdinger and F. Aaserud (eds.), *Bohr's Collected Works* (North-Holland, Amsterdam; American Elsevier, New York, 1972 ff).
10. R. Gómez, *Revista Latinoamericana de Filosofía* **XIII**, (1987).
11. I. Kant, First introduction to the critique of the power of judgment, in *Critique of the Power of Judgment*, I. Kant, trans. P. Guyer and E. Matthews, ed. P. Guyer, (Cambridge University Press, Cambridge, 2000) (AA XX, pp. 193–251[27]).
12. J. Faye, *Niels Bohr: His Heritage and Legacy. An Anti-Realist View of Quantum Mechanics* (Kluwer, Dordrecht, 1991).

13. P. Hoyningen-Huene, Niels Bohr's argument for the irreducibility of biology to physics, in *Niels Bohr and Contemporary Philosophy*, eds. J. Faye and H. Folse (Kluwer, Dordrecht, 1994) pp. 231–255.
14. N. Bohr, The causality problem in atomic physics, (reprinted in [26, pp. 94–121]), (1938).
15. N. Bohr, Atomic physics and international cooperation, (reprinted in [26, pp. 132–135], (1947).
16. N. Bohr, *Essays 1958–1962 on Atomic Physics and Human Knowledge* (Wiley, London, 1963).
17. N. Bohr, Discussion with Einstein on epistemological problems in atomic physics, in *Albert Einstein. Philosopher-Scientist*, ed. P. A. Schilpp (Library of Living Philosophers, Evanston, 1949) (Reprinted in [2, pp. 32–66]), (1949).
18. N. Bohr, *Scientific Monthly* **82**, (1956). (Reprinted in [26, pp. 164–169], (1956).
19. K. Laurikainen, *Beyond the Atom. The Philosophical Thought of Wolfgang Pauli* (Springer, Berlin, 1988).
20. H. Pringe, *Critique of the Quantum Power of Judgment. A Transcendental Foundation of Quantum Objectivity* (Gruyter, Berlin, 2007).
21. P. Mittelstaedt, *Philosophia Naturalis* **41**, (2004).
22. P. Mittelstaedt, The constitution of objects in Kant's philosophy and in modern physics, in *Kant and Comtemporary Epistemology*, ed. P. Parrini (Kluwer Academic Publishers, Dordrecht, 1994) pp. 115–129.
23. D. Murdoch, *Niels Bohr's Philosophy of Physics* (Cambridge University Press, Cambridge, 1987).
24. M. Beller, *Quantum Dialogue, The Making of a Revolution* (University of Chicago Press, Chicago & London, 1999).
25. H. Folse, Bohr's framework of complementary and the realism debate, in *Niels Bohr and Contemporary Philosophy*, eds. J. Faye and H. Folse (Kluwer, Dordrecht, 1994) pp. 119–139.
26. N. Bohr, *Causality and Complementarity. The Philosophical Writings of Niels Bohr, vol. IV*, eds. J. Faye and H. Folse (Ox Bow, Woodbridge, Connecticut, 1998).
27. Kants gesammelte Schriften (AA), (Königlichen Preußischen (Deutschen) Akademie der Wissenschaften, Berlin, 1902 ff).

HOW UNDERSTANDING MATTERS — OR NOT

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The aim of the paper is to show that understandability is not a basis for choosing between Bohm's theory and the Many-Worlds interpretation of standard quantum mechanics. Advocates on both sides assert that their preferred account is more understandable than the other. On that score, they are both right. The seeming inconsistency involved in this claim is dissolved when one realizes that they employ different notions of understandability. Moreover, understandability, on either notion, is not an overriding criterion of choice between competing accounts if our aim in developing physical theories is truth.

Keywords: Quantum mechanics; Bohm; Many-Worlds; understanding.

0. Introduction

It is well known that there are several ways to account for quantum phenomena. Bohm's theory (in the equilibrium case) and the Many-Worlds interpretation of standard quantum mechanics (no-collapse, no extra values) seem to be among the most competitive on the market. As it stands, we do not seem to possess any good criterion to choose between them. There are several criteria on the basis of which we usually choose between theoretical accounts of a set of physical phenomena. Internal consistency, explanatory and unificatory power are among such criteria. Arguably though, the most important is empirical adequacy. The main problem, when it comes to choosing between Bohm's theory and the Many-Worlds interpretation of standard quantum mechanics (SQM) is that they are empirically equivalent.

Defendants on each side try to argue for the superiority of their account of quantum phenomena. Some have defended accounts on the basis of its higher degree of "understandability". Such a defense begins with the following premises:

(a) One of the accounts is more "understandable" than the other;
(b) Understandability is a good basis for a criterion of choice.

They conclude that their favored interpretation is the one we ought to adopt. In this paper, I would like to show that this is a poor argument. Sections 1 and 2 highlight problems with premise 1. Section 3 highlights problems with premise 2.

Claims about the higher degree of understandability of each account over the other typically are along the following lines. Bohmians claim that Bohm's theory is more understandable because it has a "clear ontology". To this however, Many-Worlders have answered that their account is more understandable because it is closer to the standard formalism and has a "minimal" ontology. We would like to show that such claims involve two different meanings of understandability, and the Bohmians and Many-Worlders are talking past one another. With the analysis in hand, we shall assess to what extent Bohm's theory and the Many-Worlds interpretation can respectively be said to be "more understandable" than the other.

A notion of understandability is famously very difficult to define. The analytic tradition since Hempel [3, p. 413] has rejected it from the domain of philosophy for want of an objective definition. It should be noted from the outset that we shall not attack the notion of understanding and understandability as psychological or subjective. We shall partly rely on the work of de Regt and Dieks[1] who give a general account of scientific understanding. We hope to contribute to a completion of their analysis.

The confusion in the Bohm vs. Many-Worlds debate over understandability stems in part from the failure to recognize that Bohm's theory and the Many-Worlds interpretation are not competing interpretations of the same formalism but competing interpreted theories. We shall propose that the understandability of an interpreted theory consists in the combination of the understandability of its theory (T-understandability) and the understandability of its interpretation (I-understandability). We shall further claim that, in the Bohm vs. Many-Worlds debate, the sense of understandability that the Many-Worlder uses is the former, while Bohmians use the latter. We will argue that these notions of understandability do not provide a means deciding the Bohm vs. Many-Worlds debate over understandability. Moreover, we will argue that understandability, generally speaking, is a weak criterion of theory choice because it has nothing to do with the truth of a theory.

In Section 1, we will first introduce some terminological clarifications that are useful for getting a handle on what is at issue in the Bohm vs. Many-Worlds debate. We will proceed to define two different notions of understandability and discuss how they are related. In Section 2 we will discuss the understandability of Bohm's theory and the Many-Worlds theory in relation to the notions defined. In Section 3 we will argue that understandability is a poor criterion of theory choice.

1. Notions of Understandability

1.1. *Terminological clarifications*

One of the confusions that pervades the Bohm vs. Many-Worlds debate is to think of them as two competing interpretations for the same formalism. It should be noted right from the outset that Bohm's theory is not an interpretation of standard quantum mechanics because it utilizes a different formalism. The following clarifications in our basic terminology will prove useful for the upcoming analysis of understandability:

Physical theory – T – A physical theory consists primarily in a mathematical apparatus, the so-called formalism, F. That said, the formalism alone does not suffice to constitute a *physical* theory. The formalism comes with some basic correspondence rules, CR, which tell us how to apply the formalism to certain *physical* situations, thus allowing us to make some empirical predictions.

Thus, we have $T = F + CR$.

We hold throughout that theories are chosen primarily for their empirical adequacy.[a]

Interpretation – I – Whereas physical theories usually come with some basic correspondence rules, they do not come with a complete interpretation. To provide a physical theory with an interpretation is to give a story of what the world could be like if the theory were true. An interpretation I is primarily an ontological framework for the formalism. An interpretation tells us what entities and what physical quantities correspond to the mathematical constructions of the formalism. A constraint on acceptable interpretations is that they are able to recover the appearance of the world to us, even when it may seem strikingly at odds with the proposed fundamental ontology.

[a] "Empirical adequacy" here is not intended in the sense of van Fraassen.[4] We simply mean that the theory has made successful predictions in its intended domain of application.

Different interpretations of the same formalism are, by definition, empirically equivalent. We do not have definitive criteria for choosing between consistent interpretations which are compatible with all the empirical predictions of a theory.

Interpreted theory – *IT* – An interpreted theory is a physical theory provided with an interpretation. This is to say, an interpreted theory consists in (1) a formalism, (2) some basic correspondence rules, and (3) an interpretation.

Thus, we have: $IT = F + CR + I$.

Bohm's theory and the Many-Worlds interpretation, instead of being competing interpretations of the same theory, are really competing interpreted theories. They differ not only as far as their ontological and epistemological assumptions are concerned, but also formally. In order to be clear at all times, I shall use the following acronyms in the remaining of the paper:

- BIT: the full package of Bohm's formalism, correspondence rules, plus a Bohmian type interpretation
- BT: Bohm's formalism and correspondence rules
- BI: a Bohmian interpretation
- MWIT: the full package of standard quantum mechanics (no collapse, no extra values) plus a Many-Worlds-type interpretation
- MWT: the physical theory used in MWIT, that is, standard quantum mechanics (without collapse)
- MWI: a Many-Worlds-type interpretation

These clarifications form the foundation of our analysis of notions of understandability and how they relate to Bohm's theory and of the Many-Worlds interpretation.

1.2. *Two different notions of understandability*

Our main point in this section is that the notion of understandability of physical theory, T-understandability, is different than the notion of understandability of an interpretation, I-understandability. We shall maintain that each of these two kinds of understandability seem to independently contribute to the understandability of the interpreted theory.

IT-understandability The understandability of an interpreted theory consists in the interpreted theory being both I-understandable and T-understandable.

The issue of how the understandability of the theory and of the interpretation respectively contribute to the understandability of the interpreted theory will be addressed later. We shall now try to contrast the two kinds of understandability.

1.2.1. *Understandability of a physical theory*

Concerning what it is to understand a physical theory, one might be tempted to say that it amounts to mastering the mathematics involved in the formalism. According to such a rather simplistic view, physicists understand a theory whenever they know well the rules for manipulating the equations. What is required to acquire such understanding is simply to learn the math. Such an account of understanding is rather unsatisfactory. Arguably, understanding involves more than computing abilities. For example, computers compute, and often compute very well, though it is highly doubtful that they "understand" anything.

De Regt and Dieks[1] have given a much more interesting analysis of the kind of understanding scientists can have of a physical theory. Note that they actually give 1) a necessary and sufficient condition for a phenomena to be understood in terms of having an intelligible theory and 2) a sufficient condition for a theory to be intelligible. Our condition for T-understandability is based on what they call intelligibility of a theory:

T-understandability A theory is *T-understandable* if and only if there is a person at some time (past, present, or future) who could predict how a system in a given theoretical context is going to evolve, without any need for explicit computation.[b] Moreover, this ability is not due to the person's T-understanding of another theory of the domain.[c]

Someone has actual *T-understanding* of a theory if and only if they can predict how a system in a given theoretical context is going to evolve, without any need for explicit computation and this ability has developed in virtue of the person's familiarity with the theory in question.

[b]We render de Regt's and Dieks' sufficient condition necessary in adding an element of modality in the condition: they hold that it is sufficient for a theory to be intelligible that some scientist(s) is (are) able to predict without computation; we add to this that it is necessary for a theory to be intelligible that someone *could* be able to predict without computation.

[c]This last clause is not included in de Regt's and Dieks' condition for intelligibility, but is required in the case of empirically equivalent theories. One does not have T-understanding of theory A if one makes predictions in accordance with A in virtue of one's T-understanding of B, an empirically equivalent theory.

> The T-understandability of a theory for a person comes in degrees; it depends on how well they can predict how a system can evolve without the need for explicit computation.

History tells us that it takes one to two generations of physicists before a new theory to be understood in this sense. It has been the case for Newton's theory, for which, for example, the Cartesian model of contact action had to be given up in favor of action at a distance. It has been also the case for quantum theory where the twentieth century idea that physical entities are either waves or particles had to be given up. It is important to note that, if to understand quantum theory is just this, to be able to use the theory in predicting a given system's behavior without explicit computation, then physicists have certainly come to understand quantum theory by now.

T-understanding is certainly an important part of scientific activity. As de Regt and Dieks points out, it allows the physicist to understand the observable phenomena in a very specific way. Such T-understanding is sufficient for a physicist to make his way through the experiment and do most of his work. The impressive progress made during the twentieth century in quantum physics is due to the possibility and efficiency of T-understanding. The majority of quantum physicists, who are not interested in foundational issues, have proved able to work with SQM without further need for a coherent interpretation.

That said, T-understandability is not all there is in the understandability of a fully interpreted theory. The understandability of the interpretation also matters. For example, the orthodox interpretation (misnamed as it is really a separate theory form standard quantum mechanics because of the added collapse dynamics) is perfectly T-understandable, but very few physicists think it has any understandable interpretation. Many classes on quantum physics start with some warnings about the weirdness of the theory as interpreted in the orthodox way or even go so far to say that quantum theory is not something that should be attempted to be understood. These warnings seem inexplicable if T-understandability exhausted understandability. We also must consider I-understandability, i.e. the notion of understandability of an interpretation.

1.2.2. *Understandability of an interpretation*

What is it for an interpretation to be understandable? We might first ask what an interpretation does. An interpretation indicates what the world might be like if a theory is true. In doing so it describes at least some of the ontology of the world (that pertaining to the domain of the theory) that is compatible with the theory. Moreover, it also indicates how the ontology proposed can account for the macroscopic appearances, especially if it is far at odds from a classical ontology of definite-valued properties evolving in space and time.

This latter feature of interpretations is a criterion of sufficiency. It sorts potentially acceptable interpretations from unacceptable ones. Note however that acceptability and understandability of an interpretation do not necessarily go together. Of course, not all acceptable interpretations, that is, that recover the appearances, are understandable. On the other hand, it is reasonable to expect that if a person understands an interpretation, he or she be able to indicate how it does *or does not* recover the appearances. It is important that an interpretation is understandable even when it cannot recover the appearances, that is, when it is not acceptable. Bell's theorem provides a perfect example. Bell's theorem shows that quantum probabilities cannot be interpreted in terms of our ignorance of underlying, definite valued properties of physical systems evolving according to local, non-contextual laws. Such an interpretation is, however, clearly understandable. So, the understandability of an interpretation must consist in more than just saying how the appearances are recovered, or fail to be recovered by the interpretation. The understandability of an interpretation must have something to do with the understandability of the ontology associated with the interpretation as well.

We propose the following rough characterization of the understandability of an interpretation:[d]

I-understandability An interpretation is *I-understandable* if and only if there is a person at some time who could define the ontology employed by the interpretation in terms of concepts they possess and indicate how the interpretation succeeds or fails at recovering the appearances.

A person has actual *I-understanding* of an interpretation if and only if they can define the ontology employed by the interpretation in terms of concepts they possess and indicate how the interpretation succeeds or fails at recovering the appearances.

The I-understandability of a theory comes in degrees; it depends on how completely one can define the ontology in terms of concepts they

[d]Note that we do not pretend in this paper to provide a full analysis of the understandability of an interpretation. We only seek to establish rough notions of understandability to bring some clarity to the debate between Bohmians and Many-Worlders.

possess and how completely they can indicate how the interpretation succeeds or fails at recovering the appearances.

So, understanding an interpretation consists in having the right kind of concepts with which to make sense of the world. A few examples will make the analysis plausible. In order to understand a Newtonian world, the concept of instantaneous velocity is required, as is instantaneous action at a distance, and inertia as well. To understand the world of classical electrodynamics, one requires the concept of a field. Having the right concepts goes hand in hand with the understandability of the interpretation of a theory. Then one needs to know whether or not the ontology is compatible with the appearances.

Our proposal here is less controversial than it might seem. As emphasized above, we do not pretend to give here a full and definitive theory of our categories of understanding. I-understanding is only supposed to characterize our understanding of an interpretation of a theory. We would not maintain that such categories of understanding, the concepts we possess to define the fundamental ontology, are fixed, quite the contrary. Also, we strictly avoid the issue of what the actual ontology of the world is.

1.2.3. *The relationship between I and T-understandability*

There are good reasons to believe that I-understandability and T-understandability are independent notions, but not definitive ones. This is because I and T understandability are threshold notions: theories achieve a sort of understandability when there exists a person that could develop certain abilities. So, simply giving examples of people who have I-understanding without T-understanding and vice versa will not be sufficient. It would have to be shown that there are people who could achieve I-understanding while no other person could achieve T-understanding of a theory, and vice versa to establish complete independence.

It is certainly plausible that I-understandability is independent of T-understandability. We only need to imagine an interpreted theory with a simple ontology which can recover the appearances, but with a convoluted dynamics that is beyond the practical capabilities of human computational abilities. It is not clear that T-understandability is independent of I-understandability. A similar thought experiment in this case is not convincing. It's not clear that there could be a person who could develop T-understanding of an interpreted theory while it being somehow impossible for anyone anywhere at any tine to develop I-understanding of the interpreted theory.

What can be said with certainty is that *for a particular person*, T-understanding is independent of I-understanding. It is a fact that many twentieth century quantum physicists have reached T-understanding without having I-understanding. Many quantum physicists who have a T-understanding of quantum mechanics and use the theory without thinking about a coherent interpretation. Only a minority of physicists have gotten interested in foundational issues. It is striking that the majority of working physicist have actually reached T-understanding through use of the orthodox interpretation (theory), even if it is inconsistent, which makes any kind of I-understanding impossible.

I-understanding is also independent of T-understanding for a particular person. Indeed, many philosophers know the details of the possible ontologies corresponding to the different interpretations and can indicate how the ontology can succeed or fail in accounting for the appearances. That said, few would pretend to be able to use the theory as professional physicists do, i.e. given a system in a certain theoretical context, to be able to predict how the system is going to evolve without explicit computation. This seeming independence of I and T understandability allows us to make sense of claims regarding relative understandability made by Bohmians and Many-Worlders.

2. The Bohm vs. Many-Worlds Debate Over Understandability

In this section we argue that BIT is more easily I-understandable than MWIT. We also argue that MWIT is more easily T-understandable than BIT. We will argue that these facts do not allow us to say definitively whether BIT or MWIT is more IT-understandable. Moreover, these facts regarding I and T-understandability alone do not allow one to decide the debate.

2.1. *Bohm vs. Many-Worlds: T-understandability*

MWIT contains standard quantum mechanics. It does not add anything to the standard formalism and basic correspondence rules. Since SQM is T-understandable, MWIT is T-understandable. Now, BIT is not restricted to the standard formalism. It contains the guiding equation in addition. Though the introduction of the guiding equation does not necessarily render BIT unT-understandable, it certainly makes it more difficult for people to develop a T-understanding of the theory. Explicit calculations are typically more difficult with Bohm's theory. It is no great leap of faith that this hinders the T-understandability of the theory. So, the MWIT is more easily T-understandable than BIT.

2.2. *Bohm vs. Many-Worlds: I-understandability*

Bohmians are well known to claim that their interpretation is best because it has a "clear ontology": particles with definite trajectories in space and time. A typical example of such a claim is found in the abstract of a famous paper by Dürr, Goldstein, and Zanghi:

> We argue that the quantum formalism can be regarded, and best be understood, as arising from Bohmian mechanics, which is what emerges from Shrödinger's equation from a system of particles when we merely insist that "particles" means particles. While distinctly non-Newtonian, Bohmian mechanics is a fully deterministic theory of particles in motion, a motion choreographed by the wave function. [...] When the quantum formalism is regarded as arising in this way, the paradoxes and perplexities so often associated with (nonrelativistic) quantum theory simply evaporate. [2, abstract]

In what sense can an ontology of particles moving in space and time be said to be more easily understandable than an ontology of worlds with internal observers? In the sense of I-understanding as defined above. The concepts in which the ontology that BI employs is definable are widely possessed. The BI employs an ontology consisting of particles and a field which acts non-locally. We all know what particles and fields are. The fact that the field is non-local presents no special conceptual challenge as we are perfectly familiar with non-locality from Newton's theory. Moreover, probabilities present no special problem for the BIT. The theory is deterministic and probabilities receive an ignorance interpretation, just as they do in statistical mechanics. Indeed, BI is easily I-understandable.[e]

The strength of the BIT here is the MWIT's weakness. The appearances are Newtonian. The world appears to consist in definite valued properties that evolve deterministically in space and time. The difficulty for Many-

[e]One might point out that contextuality is an interpretational oddity of Bohm's theory as an objection to the claim that it is I-understandable. While it may be an ontological oddity in relation to what we come to expect from the appearances, contextuality itself is perfectly commonplace. E.g. The probability that a certain horse will win a race will depend on the other horses in the race. So, in terms of the analysis of I-understandability offered above, it is not a challenge for the BIT.

Worlders is that they give up definite values for properties. In taking the wave function as a complete representation of the world, the Many-Worlders accept superpositions, and hence indefinite properties — spin and position for instance, in their ontology, and this both at the fundamental and phenomenological levels. Moreover, they have to account for probabilities, but cannot do so in terms of an ignorance account because generally all values associated with the measurement of a property are realized in some sense, but not in the sense of possessing sharp values. The Many-Worlder faces the task of reconstructing the appearances with tools that could hardly seem more unsuitable.

That the ontology proposed seems so at odds with the appearances entails nothing about the understandability of the interpretation. The problem lies in the fact that no one seems to be able (for now) to define the ontology in terms of concepts that are possessed widely, or even at all. Many-Worlders will often claim that the only ontological features of the world are those that correspond to the wave function, but that existential claim does nothing to define what the ontology is. One can say that the world is composed of non-definite properties, but simply because we possess the concepts in which "definite property" can be defined does not mean that those same concepts can be used to define non-definite properties. It is this fact that challenges the I-understandability of the MWIT. The fact that arguably no one I-understands the MWIT is, of course, no guarantee that it is not I-understandable. That said, on this matter, clearly the BIT is more easily I-understandable than the MWIT.

2.3. *Bohm vs. Many-Worlds: Minimality*

No argument in favor of MWIT can be made on the basis that the MWIT is easily I-understandable. However, one often encounters the argument that the *minimality* of the MWI *vis à vis* the formalism makes it preferable to Bohm's theory. It will be argued that minimality has nothing to do with the I-understandability of an interpretation.

The theory, because it works wonderfully as far as empirical adequacy is concerned, is good enough for what physicists do. Further, they know by now so well how to use it, that they do their job without being puzzled by the quantum behavior anymore. Given such T-understandability of the theory, it is often argued that the most easily understandable interpretation would be one that takes the formalism at face value; namely, that which is *minimal* vis à vis *the formalism*. An interpretation is *minimal* vis à vis *the formalism* only if it only postulates as fundamental entities the counterparts of some abstract mathematical objects inhabiting the formalism (there always may be mathematical artifacts in the formalism). The Many-Worlds interpretation is one such interpretation. It does not postulate any extra elements in the ontology besides the ontological correspondent of the universal wave function. By contrast, the argument goes, Bohmians postulate particles with definite trajectories, hence at least one non-reducible definite valued property, which is not needed for an empirically adequate theory. Instead, Many-Worlds interpretations thus only juxtapose the simplest ontology there is to the formalism. To adopt a Bohmian-type interpretation thus involves an inflated ontology.

Minimality may be a virtue of an interpretation. That said, it is clearly a separate virtue from I-understandability as defined above. The MWI is not easily I-understandable, despite it's minimality. Minimality seems to respect a principle of ontological parsimony that is a guiding rule in metaphysics. Postulate no more than needed to explain what needs to be explained (the appearances). However, whether an ontology is minimal or not has nothing to do with whether it is easily understandable. A world in which every physical quantity has a definite value is less minimal than a world in which many physical quantities have contextually defined values that are reducible to other physical quantities, as in Bohm's theory, despite the fact that the latter is less easily understandable. So, minimality has nothing to do with the debate over the understandability of the MWIT or BIT.

2.4. *Bohm vs. Many-Worlds: IT-understandability*

What we have argued so far is the the MWIT is more easily T-understandable than BIT, but BIT is more easily I-understandable than MWIT. Claims regarding the understandability of MWIT and BIT are not that specific, but general. What we can tell regarding the comparative IT-understandability of these interpreted theories from those facts? What weighting does one assign to I and T-understandability to determine IT-understandability? We will argue that T-understandability is more important than I-understandability, but even so, it doesn't give us a reason to prefer BIT over MWIT.

The T-understandability of a theory seems quite important. If physicists cannot T-understand a theory, it is likely that their creative abilities and insights will be hampered from always having to do explicit calculations. Moreover, as an eminently practical matter, the more easily T-understandable a theory is for students, the better, as it will quite literally give them more time for research.

The I-understandability of a theory seems less important. To frame the issue, one might ask, “What can a physicist *do* with I-understanding that can’t be done with T-understanding?” After all, the physicist with T-understanding of a theory will be able to employ a theory to empirical effect in a way a physicist with only an I-understanding of a theory could not. If the aim of physics is to effectively interact with the world, it would seem that T-understanding is more important than I-understanding.

One thing that can be said of I-understandability is that it is helpful for developing a T-understanding of an interpreted theory. When one has a sense of what the abstract formalism of a theory corresponds to, one has an easier time learning the formalism (There is a reason why we learn arithmetic by thinking about apples and oranges.). Moreover, when one has a representation of what happens in the world during certain kinds of physical interactions, perhaps a visual representation, one can more easily acquire the ability to make predictions without explicit calculations.

Note that we are not suggesting that I-understandability isn’t important. Obviously, reflections on the I-understandability of interpreted theories can lead to new research programs and even new theories. Perhaps Bohm was dissatisfied with the understandability of the Copenhagen interpretation and this motivated him to develop his theory of quantum phenomena.

At this juncture, philosophers might want to invoke Einstein’s reflections on the conceptual foundations of electrodynamics and it’s importance for the development to the theories of relativity, to emphasize the importance of I-understandability. It is important to keep in mind that the understandability of an interpretation is not correlated to the acceptability of an interpretation, e.g. modal interpretations of quantum mechanics (which have severe problems but are I-understandable). Einstein was not satisfied with the interpretation of electrodynamics, and finding the flaws in that theory spurred great breakthroughs. That said, electrodynamics was perfectly I-understandable and understood by a great many physicists.

To sum up, T-understandability is more important than I-understandability and should contribute more to IT-understandability than I-understandability does. One might think that this decides the BIT v. MWIT debate in favor of MWIT. The problem here is that the difference in T-understandability is slight, and the difference in I-understandability is more substantial. So, even if T-understandability is weighted more heavily than I-understandability, the IT-understandability of MWIT and BIT seem to be roughly equal. Understandability does not appear to give us a means of preferring one interpreted theory over another in this case.

3. Understandability vs. Truth

The above analysis of I-, T-, and IT-understandability of physical theories provides a reasonable framework for assessing and comparing the understandability of theories. Even though the determination of the IT-understandability of BIT and MWIT failed to be decisively in favor of one intepreted theory or another, it would appear that the IT-understandability could be one of many criteria of choice for interpreted theories along with accuracy, simplicity, scope, etc. In this section, we will actually suggest that understandability of a theory or of an interpretation is a poor criterion of choice, especially if our aim is truth.

In formulating physical theories, we minimally aim at true observable predictions. A physical theory must save the phenomena of its domain. T-understandability of the physical theory does not weigh very much in comparison to empirical adequacy. For example, a theory that implies false predictions cannot be accepted on the argument that it is perfectly T-understandable. Conversely, were we given, say by some god, the true theory of the world, that is, as we defined it, a formalism with the basic correspondence rules that allows us to apply the formalism to make empirical predictions, there is little doubt that we would accept and use it, even if, due to our cognitive insufficiencies, we were doomed to never T-understand it and to go through painstaking computations forever. In such a case we might also want to utilize a more easily T-understandable theory for practical matters, much as we use Newton's theory for many calculations where relativistic effects are negligible. The point is that we would certainly never reject the true theory of the world in virtue of it's T-understandability.

Many would further maintain that, in designing physical theories, we ultimately aim at finding the true laws that are behind the observed regularities in the world, not only finding empirically adequate laws. Of course, we can never be assured that we possess the true laws in hand because several sets of laws are compatible with a set of phenomena. That said, it does not follow from this that the true laws are not what we are aiming at as an ideal. If true, whether the dynamical laws a theory contains be T-understandable or not should not weigh much when it comes to accepting it.

The same argument holds in the case of interpretations. In formulating interpretations, we hope to grasp something of the fundamental ontology of the world. Granted again, we do not have any empirical access to such ontology. Neither do we have the means to test an interpretation besides its consistency with the observable predictions of the theory. This does not imply that we are not trying to get as close to the true story of the world as we can. For example again, we cannot see anyone adopting an interpretation which we would know is false, on the basis that it is highly understandable. Conversely, were we given, by some goddess this time, the true ontology of the world, there is little doubt that we would use it, be it in terms of definite-valued properties distributed in space and time or not. Moreover, such knowledge would likely contribute to the development of our theories.

In the case of theories as in the case of interpretations, understandability thus is neither necessary nor sufficient as a criterion of choice. This is because theories and interpretations are, ideally, designed to be true of the world, and that understandability is neither necessary nor sufficient to warrant anything like truth. That it is not sufficient is clear enough. All our false but understandable theories are obvious examples. It is not necessary either, for it could be well the case that the fundamental laws, if any, and the fundamental ontology cannot fit into the framework we happen to use to understand the interactions between physical systems at the observable level.

The understandability of an interpreted theory obviously depends on our cognitive abilities, as twenty-first century adult human beings. Such cognitive abilities are highly contingent. They arguably depend on the average scale of the objects with which we usually interact, and on the evolution of our sense apparatus and neural connections, which make us interact in a particular ways with such objects. The relative understandability of theories and interpretations could be ranked completely differently by an organism which was equipped differently than we are, either physically or experientially. If true, then it seems besides the point to consider understandability as a criterion of choice for theories and interpretations. In short, the type of environment in which we happened to evolve does not have much to do with the truth of the matter about what the world is ultimately like.

4. Conclusion

We have tried to clarify the debate over the respective understandability of the Many-Worlds interpretation of the standard formalism and Bohm's theory. We have understood claims on each side of the debate regarding superior understandability as reconcilable on the notions of I- and T-understandability developed in this paper. BIT is more easily

I-understandable, and the MWIT is more easily T-understandable. In terms of IT-understandability, the MWIT and BIT seem to be on roughly equal footing. So, we cannot advocate one of these two interpreted theories over the other on the basis of its higher degree of understandability. Moreover, even if there was a great difference in IT-understandability between them, it would be a weak reason to prefer one over the other, as IT-understandability is independent of truth, the ultimate aim in accepting theories for most of us. That understandability is a poor criterion for theory choice does not imply that it is not useful for philosophy of science in general. The notions of understanding and of understandability of scientific theories are still useful for any descriptive account of science.

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References

1. de Regt, H.W. and Dieks, D., A contextual approach to scientific understanding, *Synthese*, 144:1, 137-170 (2005).
2. Dürn, D., Goldstein, S. and Zanghi, N., Quantum equilibrium and the origin of absolute uncertainty, *Journal of Statistical Physics*, 67: 5-6, 843-907 (1992).
3. Hempel, C.G., *Aspects of Scientific Explanation and Other Essays in the Philosophy of Science* (Free Press, 1965).
4. van Fraassen B.C., *The Scientific Image* (Oxford University Press, 1980).

ON THE ORTHOCOMPLEMENTATION OF STATE-PROPERTY-SYSTEMS OF CONTEXTUAL SYSTEMS

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We describe a model in which the maximal change of state of the system due to interaction with the measurement context is controlled by a parameter which corresponds with the number N of possible outcomes in an experiment. In the limit $N = 2$ the system reduces to a model for the spin measurements on a quantum spin-$\frac{1}{2}$ particle. This model fits in the hidden measurement approach to quantum mechanics in which quantum probabilities are explained as due to an uncontrollable fluctuation in the measurement process. In the limit $N \to \infty$ the system is classical, i.e. the experiments are deterministic and its set of properties is a Boolean lattice. For intermediate situations the change of state due to measurement is neither 'maximal' (i.e. quantum) nor 'zero' (i.e. classical). We show that two of the axioms used in Piron's representation theorem for quantum mechanics are violated, namely the covering law and weak modularity. Next, we discuss a modified version of the model for which it is even impossible to define an orthocomplementation on the set of properties. Another interesting feature for the intermediate situations of this model is that the probability of a state transition in general not only depends on the angular distance between the two states but also on the measurement context which induces the state transition. Therefore our models also shed new light on Gleason's theorem and suggest that transition probability maybe is not a secondary concept which can be derived from the structure on the set of states and properties, but instead should be regarded as a primitive concept by its own right for which the measurement context is crucial.

Keywords: Quantum logic; orthomodular lattices; Piron's representation theorem.

1. Introduction

In the last century two fundamental new physical theories have been introduced in science, which force us to abandon the age-old classical Newtonian intuition about the nature of reality and how this reality can be represented in a mathematical–physical framework. On the one hand, the theory of general relativity (GR) puts forward a model of a universe populated by material objects moving along world-lines on a 4-dimensional manifold, bending and curving this space–time manifold by their mass. On the other hand, quantum mechanics (QM) describes the reality of microscopic particles on the (sub)atomic level by representing their states and properties in an infinite dimensional complex Hilbert space. Unfortunately, these two basic scientific theories are not fully compatible with each other, such that in their current formulations they cannot be combined to provide a global, integrative view on physical reality. For instance, it is not clear how quantum mechanics can be elaborated to include gravitational effects, neither how quantum effects should be incorporated into a theory based on general relativistic principles. At this point in the history of science one is forced to accept that each of these two fundamental theories is useful to represent only those elements of reality within its own scale of application, i.e. for microscopic particles, QM should be used and for large-scale macroscopic systems, GR is applicable. Therefore QM and GR can be considered 'preliminary theories' only, i.e. steps-in-between towards the development of a more general theory which should describe phenomena featuring both gravitational and quantum effects. In this sense one could also state that 'understanding QM' contributes only to an 'in-between' understanding of reality as a whole. Nevertheless, in our view a constructive approach towards the understanding of QM is useful to develop new and more general theories, possibly including gravitational effects as well.

It is also worthwhile to note that even within the foundations of QM itself the need for a better understanding and further generalization of QM theory is necessary. For instance, already more than 25 years ago it was shown that a compound system consisting of two separated quantum systems does not fit structurally within the mathematical framework of standard quantum mechanics.[1] This follows from the fact that the lattice of properties of such a compound system does not obey the axioms of standard QM, suggesting that more general mathematical structures than Hilbert spaces need to be explored in order to represent such systems. So already to derive a valid general description of a compound system of two quantum systems, we need to explore more general mathematical structures than those used in standard QM. This is not only useful to solve some of the important theoretical and interpretational problems discussed in the foundations of standard QM, but could also contribute to building a bridge to the theory of GR.

In this paper we explore a generalized framework for quantum mechanics following an operational approach in which a physical system is determined by the mathematical structure of its set of states and properties. In this State–Property–System formalism (**SPS**) entities are assumed at each instant of time to be in a definite state (known or not known to the observer, pure or mixed state) such that some properties are actual in a specific state, while other properties are only potential, i.e. not actual. Only by a change of state (by measurement interaction or free evolution) can potential properties become actual, i.e. are they 'realized' by evolution of the (state of the) system. To illustrate this approach, we consider a model in which the maximal change of state of the system due to interaction with the measurement context is controlled by a parameter, reflecting the 'robustness' of the system under measurement. In one limit the system reduces to a model for the spin measurements on a quantum spin-$\frac{1}{2}$ particle, while in the other limit the system behaves like a classical model, i.e. its set of properties is a Boolean lattice. For intermediate values of the robustness parameter, the change of state due to measurement is neither 'maximal' (i.e. quantum) nor 'zero' (i.e. classical). To deal with these issues in a rigorous mathematical way, we study the State–Property–System of this model for different values of the contextuality parameter.

2. An Operational Approach to Quantum Mechanics

2.1. *The SPS formalism*

We adopt an operational approach to quantum mechanics in which a physical entity is described by its set of states, its set of properties and a relation of 'actuality' between these two sets which expresses which properties are actual when the system is in a specific state, as follows.[2] First, we consider that at any moment the entity S is in a (known or unknown to the observer) state $p \in \Sigma$. Also, S has a set of properties $\mathcal{L}$, defined by the set of available experiments which can be performed on S. A property a is either 'actual' or 'potential' for the entity S, which means that if the property a is actual in the state p, then each time the corresponding experiment is performed, the positive outcome is found with certainty. Between the set of states and (power)set of properties is a relation $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ of 'actuality' that maps each state $p \in \Sigma$ onto the set $\xi(p)$ of those properties that are actual in this state. Dually, one can consider the Cartan map $\kappa : \mathcal{L} \to \mathcal{P}(\Sigma)$, which maps a property $a \in \mathcal{L}$ onto the set of states $\kappa(a)$ that make this property actual. Depending on the nature of the entity S, one obtains a different structure on the set of states Σ, the set of properties $\mathcal{L}$ and the relation between these two sets. Hence, if we are only concerned with the structural behavior of an entity, we can focus on the triple $(\Sigma, \mathcal{L}, \xi)$. More abstractly, even without an underlying physical entity S, we can consider any two sets Σ and $\mathcal{L}$ and a function $\xi : \Sigma \to \mathcal{L} : p \to \xi(p)$ and study the emerging structure. We call $(\Sigma, \mathcal{L}, \xi)$ a State–Property–System (**SPS**). If one considers the **SPS** of a quantum entity, one observes that certain 'quantum axioms' hold, i.e. the mathematical structure of the **SPS** needs to obey certain specific rules. Conversely, one could start from an abstract **SPS**, and by imposing a suitable set of axioms one can derive a structure on the set of properties which is isomorphic with the Hilbert space representation of quantum mechanics (with superselection rules). In the next subsection we briefly present these quantum axioms to give an idea of how the operational approach works and how quantum structure arises. In the operational **SPS** approach, a physical system is determined by the structure on its set of states and properties, and the actuality map ξ between these two. This means that it is not necessary (or even meaningful) to make a distinction between quantum and classical systems on the basis of their size (classical macroscopic versus microscopic quantum) but rather on the different structure on their **SPS**. Following this operational approach, it is a natural step to consider models which exhibit 'quantum behavior' on a structural level, even if they are not microscopic in size. This has led to a 'quantum-like' model representing the spin properties of a quantum spin-$\frac{1}{2}$ particle,[3] which will be discussed in subsequent (sub)sections. The **SPS**-formalism is an elaboration of the original Geneva-Brussels approach,[1,4–11] in which the set of experiments defines the set of properties by yes–no tests.

2.2. *Quantum axioms and Piron's representation theorem*

In the Geneva-Brussels approach[1,4–11] the set of experiments defines the set of properties. Each physical property is identified by its set of eigenstates, i.e. if the system S is in an eigenstate of this property, the measurement yields the corresponding outcome with certainty. Therefore, the set of properties can be identified with a subset of the powerset of the state space Σ such that statements (i.e. axioms) about the set of physical properties of a system can be translated into statements about the corresponding **SPS** structure (i.e. the eigenstate sets). In this subsection we briefly present (generalized) Piron's representation axioms and show how quantum structure arises and at what stage the orthocomplementation enters the discussion. Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. For properties $a, b \in \mathcal{L}$ we introduce an implication relation: $a < b \iff \kappa(a) \subset \kappa(b)$. Similarly, for states $p, q \in \Sigma$ we introduce $p < q \iff \xi(q) \subset \xi(p)$. It can be shown that for a **SPS** $(\Sigma, \mathcal{L}, \xi)$: $(\Sigma, \leq)$ and $(\mathcal{L}, \leq)$ are pre-ordered sets.

Axiom 2.1 (Property determination). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The axiom of property determination is satisfied iff for $a, b \in \mathcal{L}$:

$$\kappa(a) = \kappa(b) \Rightarrow a = b$$

If the axiom of property determination is satisfied, then $\mathcal{L}, \leq$ is a partial order relation on $\mathcal{L}$.

Axiom 2.2 (Property completeness). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The axiom of property completeness is satisfied iff $\exists$ generating subset $\mathcal{T} \subseteq \mathcal{L}$ such that $\forall (a_i)_i \subseteq \mathcal{T}, \exists a \in \mathcal{L}$:

$$\kappa(a) = \bigcap_i \kappa(a_i) \tag{1}$$

and $\forall a \in \mathcal{L} : \exists (a_i)_i \subseteq \mathcal{T}$ such that (1) is satisfied.

The property a is called the meet of $(a_i)_i$, denoted as $a = \bigwedge_i a_i$. Consider a State–Property–System **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and property completeness are satisfied. Then $(\mathcal{L}, <, \wedge, \vee)$ is a complete lattice. Hence alternatively, one could consider Piron's axiom of completeness of the lattice of properties as a basic axiom instead of the axioms of property determination and property completeness.

Definition 2.1 (Property state). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and completeness are satisfied. $\forall p \in \Sigma$: the 'property state' is the property $s(p) = \bigwedge_{a \in \xi(p)} a$.

Definition 2.2 (Atom). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The element $a \in \mathcal{L}$ is called an atom iff $\forall x \in \mathcal{L} : 0 < x < a \Rightarrow x = 0$ or $x = a$, i.e. a covers 0.

Axiom 2.3 (Atomicity). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and completeness are satisfied. The axiom of atomicity is satisfied iff $\forall p \in \Sigma : s(p)$ is an atom of $\mathcal{L}$ and $\forall a \in \mathcal{L} : a = \vee_{a \in \xi(p)} s(p)$.

Axiom 2.4 (Orthocomplementation). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination and property completeness are satisfied. The axiom of orthocomplementation is satisfied iff there exists a function $^\perp : \mathcal{L} \to \mathcal{L}$ such that for $a, b \in \mathcal{L}$:

$$(a^\perp)^\perp = a \tag{2}$$

$$a \leq b \Rightarrow b^\perp \leq a^\perp \tag{3}$$

$$a \wedge a^\perp = 0 \text{ and } a \vee a^\perp = 1 \tag{4}$$

We call $^\perp$ the orthocomplement and for $a \in \mathcal{L}$ we call $a^\perp$ the orthocomplement of a.

Theorem 2.1 (Orthocomplemented lattice). *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination, property completeness and orthocomplementation are satisfied. Then $(\mathcal{L}, \leq, \wedge, \vee)$ is a complete orthocomplemented lattice.*

Axiom 2.5 (Covering Law). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The covering law is satisfied iff for $a, b \in \mathcal{L}$ and $p \in \Sigma : a \wedge s(p) = 0$:

$$a < b < a \vee s(p) \Rightarrow b = a \text{ or } b = a \vee s(p)$$

Axiom 2.6 (Weak Modularity). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The lattice $\mathcal{L}$ is weakly modular, i.e. $\forall a, b \in \mathcal{L} : a < b \Rightarrow (b \wedge a^\perp) \vee a = b$.

Axiom 2.7 (Plane Transitivity). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$. The orthocomplemented lattice $\mathcal{L}$ is plane transitive. This means that for atoms $s, t \in \mathcal{L}$, there are two distinct atoms $s_1 \neq s_2$ and a symmetry f such that $f\big|_{[0, s_1 \vee s_2]}$ is the identity and $f(s) = t$.

The **SPS** $(\Sigma, \mathcal{L}, \xi)$ of quantum and classical mechanical entities is such that $\mathcal{L}$ is a complete orthocomplemented lattice that satisfies the covering law, and is weakly modular and plane transitive. This can be checked straightforward from the Hilbert space representation of $\mathcal{L}$ by the lattice of closed subspaces in Hilbert space for quantum systems, or the phase space representation for classical mechanical entities. The reverse statement is less trivial to prove. In fact, the (re)formulation of a decisive set of axioms which forces a quantum structure on the set of properties (i.e. an isomorphism with the closed subspaces in a complex Hilbert space) has been at the heart of on-going scientific research over the last decades.

In the representation theorem of Piron[5,6] it is shown that if the set of properties $\mathcal{L}$ contains at least 4 atoms and is a *complete*, *orthocomplemented*, *atomistic*, *weakly modular* lattice that satisfies the *covering law*, and such that the states are represented by the atoms of this lattice, in the infinite dimensional case the standard quantum formalism with classical superselection rules is recovered, but over a 'generalized Hilbert space' (also called modular space). At that time it was mistakenly believed that Piron's axiomatics yielded a complete quantum axiomatics because the only existing examples of orthomodular spaces were given by the traditional Hilbert spaces (over the field of reals, complex numbers or quaternions). In 1980 Keller constructed a counterexample[12] which reopened the problem of a completion of quantum axiomatics. The orthomodular spaces were studied systematically but their classification turned out to be filled with very hard mathematical problems. Finally, a young Slovak mathematician, Maria Pia Solèr, proved a theorem which meant a breakthrough: each infinite dimensional modular space that has an orthonormal basis is isomorph with one of the three standard Hilbert spaces.[13] In the research team FUND at the VUB an alternative new axiom has been put forward, inspired by the theorem of Solèr, namely the operational defined axiom of *(6) plane transitivity*,[14] which, together with the previous five axioms, *(1) completeness*, *(2) atomicity*, *(3) orthocomplementation*, *(4) weak modularity*, *(5) covering law*, yields a full axiomatization of standard quantum mechanics. This leads to the following statement:

Theorem 2.2 ((generalized) Representation theorem of Piron). *Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ such that $\mathcal{L}$ contains at least 4 atoms and is a complete orthocomplemented atomistic lattice that satisfies the covering law, is weakly modular and plane transitive. Then $\mathcal{L}$ is isomorphic with a family of lattices (with superselection rules) of closed subspaces in a Hilbert space over the field of reals, complex numbers or the division ring of quaternions.*

Under the conditions of Piron's (generalized) Representation theorem the **SPS** of a physical system has to be isomorphic with a quantum description in Hilbert space (possibly with superselection rules). In order to better understand the role and meaning of these quantum axioms, we shall explore in the next sections some examples of **SPS** which violate some of these quantum axioms. As was already known, the axiom of covering law and weak modularity encounters problems when considering separated quantum entities and 'between quantum and classical systems'. As we shall see later on, also the axiom of orthocomplementation itself is not trivially satisfied for a general **SPS**.

3. When Quantum Axioms Break Down I: Separated Quantum Entities

We consider the situation of a physical entity S that consists of two physical entities S_1 and S_2. Let us consider two experiments e_1 and e_2 for the entities

S_1, S_2 respectively, from which we can define the joint experiment $e_1 \times e_2$ for the compound entity S, which consists of performing the experiment e_1 on entity S_1 and experiment e_2 on entity S_2. Experiments e_1 and e_2 are called 'separated experiments' iff for an arbitrary state p of S holds that (x_1, x_2) is a possible outcome for the joint experiment $e_1 \times e_2$ iff x_1 is a possible outcome for e_1 and x_2 is a possible outcome for e_2. The entities S_1 and S_2 are 'separated entities' iff all experiments e_1 on S_1 are separated from all experiments e_2 on S_2. It was an unexpected and also non-trivial result that standard quantum mechanics can *not* describe the situation of separated quantum systems. In fact, if the compound system obeys the axioms of quantum mechanics, then one of the subsystems needs to be classical, i.e. all its experiments are deterministic. This was proven within an operational approach which was a predecessor of the **SPS** formalism (Aerts' theorem[1]), and these results can be translated into the framework of **SPS** as follows:

Theorem 3.1. *Suppose the axioms of completeness, atomicity and orthocomplementation are satisfied and* $(\Sigma, \mathcal{L}, \xi)$ *is the **SPS** describing* S *and* $(\Sigma_1, \mathcal{L}_1, \xi_1)$ *and* $(\Sigma_2, \mathcal{L}_2, \xi_2)$ *the **SPS** describing the separated entities* S_1 *and* S_2 *then:*

- *If Axiom* 4 *(covering law) is satisfied then one of the two separated entities* S_1 *or* S_2 *is a classical entity.*
- *If Axiom* 5 *(weak modularity) is satisfied then one of the two separated entities* S_1 *or* S_2 *is a classical entity.*

It is important to notice that the result of this theorem points at a failure of standard quantum mechanics, and is not just a characteristic of the axiomatic approach itself. Indeed, one can show that the structure on the set of states and set of properties of a compound system of two separated systems does not fit in Hilbert space, i.e. the quantum axioms are violated.[15,16] This also sheds new light on the claims of EPR, namely that quantum mechanics is incomplete. This is true, but in a slightly different way: if separated quantum systems do exist, then the corresponding compound system cannot be described by quantum mechanics.

What is especially interesting for us, is that if separated entities exist in reality, then this part of reality cannot be fully described within standard QM, i.e. (at least) one of the quantum axioms needs to be violated. In the case of separated entities it is the axioms of covering law and weak modularity which need to be dropped. In later sections we will consider other examples of entities violating some of the quantum axioms and try to understand why this is the case.

4. SPS and Eigenclosure Structure

To tackle the problem of orthocomplementation on the set of properties, we consider an operational approach in which the set of properties is constructed by defining an eigenclosure operation on the set of states. By introducing an orthogonality relation on the set of states, we construct an orthoclosure structure which is orthocomplemented under this orthogonality relation. Hence if these two closure structures coincide, one automatically obtains that the set of properties is orthocomplemented. Let us start by showing how we can define a **SPS** for a physical system in a natural way by a closure operation on the set of states, defined by eigenstate sets of experiments. We refer to Ref. 17 for a more detailed discussion of these topics, including most of the proofs of theorems mentioned in this section.

4.1. *Eigenstate map for a primitive experiment*

The first step is to consider all eigenstate sets for a single experiment. Let us consider an experiment e and a subset A of the set of outcomes O_e. This defines a property a_e^A which is actual whenever the measurement e yields with certainty an outcome in A. By $eig_e(A)$ we denote the set of states for which the experiment e would yield with certainty an outcome in A, and call these the 'eigenstates' of property a_e^A. Obviously, the set $eig_e(A)$ coincides with the Cartan image of the property $a_e^A : \kappa\left(a_e^A\right) = eig_e(A)$. The Cartan map is an isomorphism between the lattice of properties $\mathcal{L}$ and the collection of eigenstate sets: a state for which the property a_e^A is actual has to be in $eig_e(A)$ and vice versa (see Ref. 17 for a more detailed discussion on this isomorphism).

This means that results obtained within the framework of eigenstate sets can be translated directly by the inverse Cartan map into statements about the structure of the lattice of properties. This isomorphism also justifies identifying an eigenstate set with its corresponding property. Therefore, we can study the structure of the set of properties in the state space of the entity. The following trivial properties are defined: the property $a_e^{O_e}$, which is actual in any state, and the property $a_e^{\emptyset}$, which is never actual. For the eigenstate map $eig_e : P(O_e) \to P(\Sigma)$, the following holds:

(1) $eig_e(\emptyset) = \emptyset$
(2) $eig_e(O_e) = \Sigma$
(3) $\forall A_i \subset O_e : eig_e\left(\bigcap_i A_i\right) = \bigcap_i eig_e(A_i)$

4.2. *Eigenstate map for a union experiment* e_E

Let us consider a set E of experiments. A natural way to 'combine' the experiments of E is the union experiment:

Definition 4.1. The union experiment e_E consists in choosing at random an element of E and performing that experiment, and attributing the observed outcome to the experiment e_E.

The outcome set of the experiment e_E is given by $O_E = \bigcup_{e \in E} O_e$. The experiments in the set E are called 'primitive experiments' and the 'combination' experiment e_E is called a 'union experiment'. One could even consider the entire set $\mathcal{E}$ containing all 'primitive' experiments to construct a union experiment $e_\mathcal{E}$ with outcome set $O_\mathcal{E} = \bigcup_{e \in \mathcal{E}} O_e$. Clearly, the outcome set O_e of every experiment e, including the primitive and the union experiments, is contained in this outcome set $O_\mathcal{E}$, which contains all possible experimental outcomes for the entity.

The eigenstate set of a union experiment is defined in complete analogy with the definition of eigenstate set for a primitive experiment. Let us consider a union experiment e_E with outcome set O_E. The eigenstate set for $A \subseteq O_E$ is defined as the set of states for which every experiment $e \in E$ would yield an outcome in A with certainty. The collection of all eigenstate sets for a primitive experiment e is denoted by $\mathcal{F}_e$ and for a union experiment e_E by $\mathcal{F}_E$. The eigenstate set of a union experiment is completely defined by the eigenstate sets of the primitive experiments of which it is constructed, as the following theorem shows:[17]

Theorem 4.1. $eig_E(A) = \bigcap_{e \in E} eig_e(A \cap O_e)$.

This means that the eigenstate set of any union experiment is completely defined by the eigenstate sets of the primitive experiments in the union. The concept of union experiments is useful for two reasons. Firstly, by making intersections of eigenstate sets we can define the conjunction (meet) of properties. Secondly, the eigenstate sets for all experiments (primitive and union) can be generated by a suitable class of eigenstate sets, as the following section shows.

4.3. *Eigenstate map for the union experiment* $e_\mathcal{E}$

Let us denote the collection of all eigenstate sets for the union experiment $e_\mathcal{E}$ by $\mathcal{F}_\mathcal{E}$.

Theorem 4.2. *The collection $\mathcal{F}_E$ of eigenstate sets for a union experiment E is contained in the collection of eigenstate sets $\mathcal{F}_\mathcal{E}$ of the union experiment $e_\mathcal{E}$.*

Proof. See e.g. Ref. 17. As an illustration we present a brief outline of the proof here. For every element $F \in \mathcal{F}_E$ there exists a subset $A \subseteq O_E$ such that $F = eig_E(A)$. Let us consider $A' = A \cup \left(\underset{e \in E^C}{\cup} O_e \right)$, $E^C = \mathcal{E} \setminus E$, then $eig_\mathcal{E}(A') = \underset{e \in \mathcal{E}}{\cap} eig_e(A' \cap O_e) = \underset{e \in E}{\cap} eig_e(A \cap O_e) = eig_E(A)$, which shows that $\mathcal{F}_E \subseteq \mathcal{F}_\mathcal{E}$. □

This theorem states that every eigenstate set is contained in $\mathcal{F}_\mathcal{E}$. Therefore one can study the structure of the set of properties of the system on the basis of the structure generated by the elements of $\mathcal{F}_\mathcal{E}$. The following theorem holds:

Theorem 4.3. $\emptyset, \Sigma \in \mathcal{F}_\mathcal{E}$ *and* $F_i \in \mathcal{F}_\mathcal{E}, \forall i \Rightarrow \underset{i}{\cap} F_i \in \mathcal{F}_\mathcal{E}$.

The interpretation of this theorem follows immediately from the Cartan isomorphism between the lattice of properties and the collection of eigenstate sets: the 'never actual property' $0 = eig_\mathcal{E}(\emptyset)$ and 'always actual property' $1 = eig_\mathcal{E}(O_\mathcal{E})$ are properties of the system, and every conjunction of properties $eig_\mathcal{E}(A_i) = F_i$ is again a property of the entity, namely the meet $\wedge F_i$ for which it holds that $\kappa(\wedge F_i) = eig_\mathcal{E}(\cap A_i) = \cap eig_\mathcal{E}(A_i)$.

4.4. *Closure structures*

In the previous subsection it was shown that the collection of eigenstate sets of the union experiment $e_\mathcal{E}$ generate all other eigenstate sets by intersection. These eigenstate sets generate a 'closure', i.e. they satisfy the following definition:

Definition 4.2. A closure structure is a couple (Σ, cl) with Σ a set and cl a mapping of $P(\Sigma)$ onto itself with the following four properties:

(1) $K \subseteq cl(K)$
(2) $K \subseteq L \Rightarrow cl(K) \subseteq cl(L)$
(3) $cl(cl(K)) = cl(K)$
(4) $cl(\emptyset) = \emptyset$

Definition 4.3. If (X, cl) is a closure structure, then a subset F of X is called closed iff $cl(F) = F$.

In the case that the set X is fixed, we can identify the closure structure (X, cl) by its set of closed sets, i.e. $\mathcal{F}_{cl} = \{F \subset P(X) \mid cl(F) = F\}$, and call $\mathcal{F}_{cl}$ the closure structure.

This definition of a closure operation is less restrictive than the usual topological definition, for which also the (set-theoretic) union of two closed sets has to be closed. This is not necessarily the case for the closure as we define it here. The following general theorem holds:

Theorem 4.4. *For a closure structure (X, cl), the family $\mathcal{F}$ of closed subsets has the following properties:*

$$\emptyset \in \mathcal{F}, X \in \mathcal{F} \tag{5}$$

$$F_i \in \mathcal{F} \Rightarrow \bigcap_i F_i \in \mathcal{F} \tag{6}$$

This theorem implies that every closure structure defines a set of closed subsets, for which (5) and (6) are satisfied. On the other hand:

Theorem 4.5. *If a family $\mathcal{F}$ of subsets of a set X is such that (5) and (6) are satisfied, then the map cl:*

$$cl : P(X) \to P(X) : K \to cl(K) = \bigcap_{K \subseteq F_i, F_i \in \mathcal{F}} F_i$$

defines a closure structure (X, cl).

Often a closure structure can be generated by a subset of all closed sets, as follows:

Definition 4.4. Let (X, cl) be a closure structure and $\mathcal{F}$ the set containing all closed subsets with respect to this closure cl. The collection $\mathcal{B} \subset \mathcal{F}$ is called a *generating set* for $\mathcal{F}$ iff for every $F \in \mathcal{F}$ there exists a family $\{B_i\} \in \mathcal{B}$ such that $F = \bigcap_i B_i$.

Theorem 4.6. *Let (X, cl) be a closure structure and $\mathcal{B}$ the generating set for $\mathcal{F}$. Then for every $K \subset X$ it holds that*

$$cl(K) = \bigcap_{K \subseteq B_i, B_i \in \mathcal{B}} B_i$$

4.5. *Eigenclosure structure*

The collection of eigenstate sets of all primitive experiments generates the full set of eigenstate sets by intersection. Theorem 4.3 states that the set $\mathcal{F}_{\mathcal{E}}$ of eigenstate sets satisfies the conditions (5) and (6) such that Theorem 4.5

can be applied, showing that this defines a closure structure on Σ. The associated closure is called 'eigenclosure', i.e. this closure maps a set $A \subset \Sigma$ of states to the set $cl_{eig}(A)$ of eigenstates for the property which is the meet of all properties which are actual in all states of A. Therefore the structure of the set of properties can be translated into a topological structure on the set of states, such that each property corresponds with its set of eigenstates.

Definition 4.5. For an entity with state space Σ, a set of experiments $\mathcal{E}$ and a collection of eigenstate sets $\mathcal{F}_{\mathcal{E}}$, the eigenclosure cl_{eig} is a map $P(\Sigma) \to P(\Sigma)$ such that for $K \subseteq \Sigma : cl_{eig}(K) = \bigcap_{K \subseteq F_i, F_i \in \mathcal{F}_{\mathcal{E}}} F_i$.

From Theorem 4.5 it follows immediately that (Σ, cl_{eig}) is a closure structure, called the eigenclosure. Obviously, all eigenstate sets are closed with respect to the eigenclosure.

4.6. *The generating set of the eigenclosure structure*

Let us recall that the eigenstate set for a union experiment is given by the intersection of the eigenstate sets of the experiments in the union (Theorem 4.1). The set of all available experiments (primitive and union experiments) contains thus a subset (the set of all primitive experiments $\mathcal{E}$) of which the eigenstate sets generate all the other eigenstate sets in $\mathcal{F}_{\mathcal{E}}$ by intersection.

Theorems 4.2 and 4.6 lead to:

Theorem 4.7. *The collection* $\bigcup_{e \in \mathcal{E}} \mathcal{F}_e$ *of eigenstate sets of the primitive experiments is a generating set for the collection of all eigenstate sets for all the experiments of the entity, i.e. all primitive and union experiments.*

The physical interpretation of the eigenclosure is as follows. Consider a set of states A, then $cl_{eig}(A)$ is the intersection of all eigenstate sets containing the set A. This corresponds with the meet of the corresponding properties. It is the smallest possible eigenstate set containing A which corresponds with the 'smallest' possible property which is actual in all states in set A.

4.7. *A primitive generating set of* $\mathcal{F}_{\mathcal{E}}$

As shown above, the collection $\bigcup_{e \in \mathcal{E}} \mathcal{F}_e$ of eigenstate sets of the primitive experiments $\mathcal{E}$ is a generating set for the collection $\mathcal{F}_{\mathcal{E}}$ of all eigenstate sets for all experiments of the entity. It is interesting to note that one could even go a step further, namely for each $e \in \mathcal{E}$ only a subset of $\mathcal{F}_e$ is required to reconstruct the whole set of eigenclosed sets $\mathcal{F}_e$ and hence also $\mathcal{F}_\mathcal{E}$.

Theorem 4.8. *If the axiom of property determination is satisfied, the collection* $\underset{e\in\mathcal{E},x_e^i\in O_e}{\cup} eig_e\left(\left\{x_e^i\right\}^C\right)$ *is a generating set for* $\mathcal{F}_\mathcal{E}$.

Proof. Let $F \in \mathcal{F}_e$, i.e. $\exists A \subseteq O_e : F = eig_e(A) = \kappa\left(a_e^A\right)$. Next, let us consider an outcome $x_e^i \in A^C$ for the experiment e. Let us denote by $\left\{x_e^i\right\}^C$ the set $O_e \backslash \left\{x_e^i\right\}$. Then the property $a_e^{\left\{x_e^i\right\}^C}$ is actual in state p iff the corresponding measurement e yields an outcome in $O_e \backslash \left\{x_e^i\right\}$ with certainty. Since $A^C = \cup_{x_e^i \in A^C}\left\{x_e^i\right\}$ it follows that $A = A^{CC} = \cap_{x_e^i\in A^C}\left(O_e\backslash\left\{x_e^i\right\}\right)$. Hence $eig_e(A) = eig_e\left(\underset{x_e^i\in A^C}{\cap}\left(O_e\backslash\left\{x_e^i\right\}\right)\right) = \underset{x_e^i\in A^C}{\cap} eig_e\left(O_e\backslash\left\{x_e^i\right\}\right) = \underset{x_e^i\in A^C}{\cap}\kappa\left(a_e^{\left\{x_e^i\right\}^C}\right) = \kappa\left(\underset{x_e^i\in A^C}{\wedge} a_e^{\left\{x_e^i\right\}^C}\right)$. From the axiom of property determination it follows that $a_e^A = \underset{x_e^i\in A^C}{\wedge} a_e^{\left\{x_e^i\right\}^C}$. Therefore the family $\left\{a_e^{\left(x_e^i\right)^C} \mid x_e^i \in A^C\right\}$ generates the property a_e^A by conjunction, i.e. all eigenstate sets of primitive experiments e are generated by the eigenstate sets of the 'primary' primitive properties $a_e^{\left(x_e^i\right)^C}$. Since these generate the eigenclosure by means of the construction of the union experiment $e_\mathcal{E}$, it follows that the set of 'primary' primitive properties $a_e^{\left(x_e^i\right)^C}$ is a generating set for the whole eigenclosure structure $\mathcal{F}_\mathcal{E}$. □

4.8. *SPS and eigenclosure structure*

To conclude this section, let us briefly show how the eigenclosure structure defines a **SPS**. Clearly, between the set of states and (power)set of properties we have the relation $\xi : \Sigma \to \mathcal{P}(\mathcal{L})$ of 'actuality', which maps each state $p \in \Sigma$ onto the set $\xi(p)$ of those properties that are actual in this state, i.e. $p \in eig_\mathcal{E}(A) = \kappa\left(a_\mathcal{E}^A\right) \Leftrightarrow a_\mathcal{E}^A \in \xi(p)$; therefore the Cartan map on the set of properties can be identified with the eigenclosure operation $eig_\mathcal{E}$. Hence our discussion of eigenclosure structure fits completely within the operational **SPS** approach.

5. Orthogonality Relations and Orthoclosure

Another way to introduce a closure operation is by defining an orthogonality relation on the set of states, and consider the sets which are closed under the bi-orthogonal operation. Orthogonality relations can be introduced in many ways, using operationally defined concepts such as the possible measurement outcomes and state transitions due to measurement. In this section we show how an orthogonality relation on the set of states generates a closure operation on the set of states by means of the bi-orthogonal construction following Birkhoff.[18,19]

5.1. *Orthogonality relations and orthoclosure*

An orthogonality relation $\perp: \Sigma \to \Sigma$ is defined as a relation on the set of states which is anti-reflexive:

$$\nexists p \in \Sigma : p \perp p$$

and symmetric:

$$\forall p, q \in \Sigma : p \perp q \Rightarrow q \perp p$$

The orthogonal $K^\perp$ of an arbitrary set of states $K \subset \Sigma$ is defined as:

$$K^\perp = \{p \in \Sigma \mid p \perp q, \forall q \in K\}$$

If the set K is a singleton, we can abbreviate the notation $\{p\}^\perp$ by $p^\perp$. A set K which is equal to its bi-orthogonal is called an orthoclosed set, i.e. K is orthoclosed iff $K = K^{\perp\perp}$. It can be shown that the bi-orthogonal operation indeed defines a closure operation,[17,20] which justifies using the name 'orthoclosure'. Let us denote the set of orthoclosed sets by $\mathcal{F}_\perp$.

5.2. *The generating set of $\mathcal{F}_\perp$*

The set of state orthogonals generates the whole set of orthoclosed sets, which can be seen as follows. Let us consider the set $F^\perp$. It is the set-theoretic union of its elements:

$$F^\perp = \bigcup_{p \in F^\perp} \{p\}$$

Following the definition of the orthogonal of a set, the orthogonal of this set is given by the intersection of the corresponding state orthogonals:

$$\left(F^\perp\right)^\perp = \bigcap_{p \in F^\perp} \{p\}^\perp = \bigcap_{p \in F^\perp} p^\perp$$

(abbreviating $\{p\}^\perp$ by $p^\perp$) such that for an orthoclosed set F holds that:

$$F = F^{\perp\perp} = \bigcap_{p \in F^\perp} p^\perp$$

which shows that the orthoclosed sets are indeed generated by making an intersection of state orthogonals.

5.3. *Orthoclosure versus eigenclosure*

One can verify easily that $\perp$ defines an orthocomplementation on the set of orthoclosed sets. Therefore, if orthoclosure and eigenclosure coincide then the set of properties is orthocomplemented under $\perp$, which is one of the axioms in the (generalized) Piron representation theorem. Since an orthoclosed set is given by an intersection of state orthogonals, the orthoclosure is contained within the eigenclosure iff the state orthogonals are eigenclosed: $\mathcal{F}_\perp \subset \mathcal{F}_{eig}$ iff $cl_{eig}\left(p^\perp\right) = p^\perp, \forall p \in \Sigma$. On the other hand, the eigenclosure is generated by 'primary' primitive properties $a_e^{\{x_e^i\}^C}$, such that a necessary and also sufficient condition for $\mathcal{F}_{eig} \subset \mathcal{F}_\perp$ is that $\forall e \in \mathcal{E}, \forall x_e^i \in O_e : a_e^{\{x_e^i\}^C} = \left(a_e^{\{x_e^i\}^C}\right)^{\perp\perp}$, i.e. $a_e^{\{x_e^i\}^C}$ is given by an intersection of state orthogonals. If these two conditions are satisfied, the set of eigenclosed sets coincides with the set of orthoclosed sets, such that the **SPS** is orthocomplemented under the $\perp$ relation.

6. The Quantum Sphere Model

The sphere model is a generalization of the Bloch sphere representation[a] such that also the measurements are represented.[3,21] The main differences with the standard use of the Bloch representation are as follows. First, in our approach all points of the Bloch sphere represent states of the spin, such that points on the surface correspond to pure states, while interior points correspond to density states. This is because an arbitrary point $u(r, \theta, \phi)$, $r \in [0, 1], \theta \in [0, \pi], \phi \in [0, 2\pi)$, of the Bloch sphere can be expressed as a convex linear combination

$$u(r, \theta, \phi) = ru(1, \theta, \phi) + (1 - r)u(0, \theta, \phi)$$

from which follows the corresponding density state

$$\begin{aligned} D(r, \theta, \phi) &= rD(1, \theta, \phi) + (1 - r)D(0, \theta, \phi) \\ &= \frac{1}{2}\begin{pmatrix} 1 + r\cos\theta & r\sin\theta e^{-i\phi} \\ r\sin\theta e^{i\phi} & 1 - r\cos\theta \end{pmatrix} \end{aligned}$$

In this expression $D(1, \theta, \phi) = |\psi\rangle\langle\psi|$ is the usual density state representation of a pure state, while $D(0, \theta, \phi)$ is the density matrix representing

[a]In this representation, a spin-$\frac{1}{2}$ state $|\psi\rangle = \cos\frac{\theta}{2}e^{\frac{-i\phi}{2}}|0\rangle + \sin\frac{\theta}{2}e^{\frac{i\phi}{2}}|1\rangle$ is represented by the point $u(1, \theta, \phi) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ on the surface of a 3-dimensional sphere, called the Bloch sphere.

the center of the sphere. The spin up state (the pure state $|0\rangle$) corresponds with the 'North pole' $D(1, 0, \phi) = |0\rangle\langle 0|$, while the spin down state (hence pure state $|1\rangle$) is represented by the 'South pole' $D(1, \pi, \phi) = |1\rangle\langle 1|$.

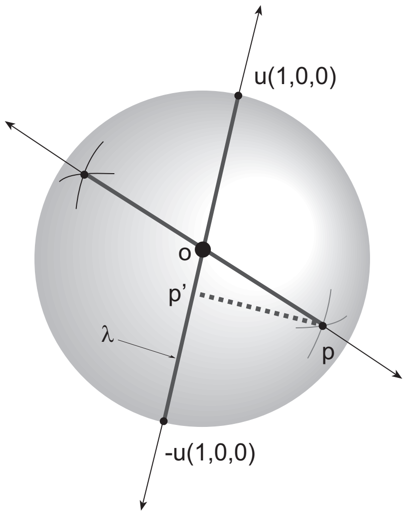

Fig. 1. The macroscopic spin-$\frac{1}{2}$ model.

Next to this, the sphere model allows a representation of measurements (see Fig. 1). The experiments e_u are defined as follows. We put an elastic of length 2 centered in the origin o of the sphere S^2 between the point u and its antipode $-u$. Let us denote the segment between u and $-u$ with the interval $[-u, u]$. Next, the particle falls from its initial position p orthogonally onto the interval $[-u, u]$ in the point p', where it stays attached to the elastic. Then the elastic breaks randomly at some point $\lambda \in (-u, u)$ such that two possibilities can occur. If the elastic breaks between p' and $-u$, then it will pull the point particle towards u where it stays attached and the experiment is said to yield the outcome 'spin up'. If on the other hand the elastic breaks between u and p', then it will pull the particle towards $-u$, where it stays attached, and the measurement is said to yield outcome 'spin down'. To make the description of the experiment e_u complete, one could specify that in the event that the elastic breaks at exactly the point p' where the particle is attached, we assume that the measurement always yields the outcome 'spin up'. Notice, however, that this event has measure zero to occur, and in such a sense it is physically irrelevant with respect to the resulting probability distribution over the set of outcomes. Let θ denote the angle between the state p of the system and the direction u of the measurement device. The probability for outcome 'spin up' is given by the length of the elastic between the projection point p' and the point $-u$, normalized by dividing by the total length of the elastic. This yields the following probability $P(u \mid p)$ for the 'spin up' outcome and corresponding state transition from initial state p towards final state u, eigenstate of the 'spin up' outcome:

$$P(u \mid p) = \frac{\cos\theta + 1}{2} = \cos^2\frac{\theta}{2}$$

Similarly, we can calculate the probability for the outcome 'spin down' as

$$P(-u \mid p) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2}$$

These probabilities coincide with the quantum probabilities for a spin experiment on a spin-$\frac{1}{2}$ particle. Note that if one had knowledge about where the elastic breaks, the measurement procedure would be deterministic. If we call e_u^λ the measurement that consists in performing e_u and such that the elastic breaks at the point λ for some $\lambda \in (-u, u)$, then, each time e_u is performed, it is actually one of the e_u^λ that takes place. We do not control this, in the sense that the e_u^λ are really 'hidden measurements' that we cannot choose to perform. The probability $\mu(e_u, p, o_1)$ that the experiment e_u gives the outcome o_1 if the entity is in state p is a randomization over the different situations where the hidden measurements e_u^λ give the outcome o_1 with the entity in state p. This is exactly the way we have calculated the probability for the quantum sphere model. Remark that if we have a physical entity S and we have a lack of knowledge about the state p of the physical entity S, then a theory describing this situation is necessarily a classical statistical theory with a Kolmogorovian probability model.[22] However, if we have a physical entity S and a lack of knowledge about the measurement e_u to be performed on S and changing the state of S, then we cannot describe this situation by a classical statistical theory, because the probability model that arises is non-Kolmogorovian. For the sphere model it can be shown that Bayes' axiom is violated, rendering the probability model non-Kolmogorovian. Clearly, this is due to a lack of knowledge about the measurement e_u, which can be regarded as a 'non-pure' measurement in the sense that there are hidden properties of the measurement, such that the performance of e_u introduces the performance of a 'hidden' measurement, denoted e_u^λ, with the same set of outcomes as the measurement e_u. The measurement e_u consists then in fact in choosing one way or another between one of the hidden measurements e_u^λ and then performing the chosen measurement e_u^λ. More generally one could regard any quantum experiment as a class of 'hidden measurements' such that each hidden measurement by itself is deterministic but a lack of knowledge about which hidden measurement is actually going on leads to a lack of knowledge on the level of the measurement outcomes, i.e. quantum probability can be explained as due to an uncontrollable (and irreducible) lack of knowledge on the interaction between the measurement device and the system. This interpretation of quantum mechanics is called the 'hidden measurement interpretation' of quantum mechanics.[3] Since it locates the lack of knowledge in the measurement interaction, it goes beyond the reach of standard 'no go theorems' against hidden variable theories for quantum mechanics. Indeed, by its very nature the hidden measurement approach is highly contextual and non-local from the very start. In the next section we briefly discuss this 'hidden variable' issue in some more detail.

7. Hidden Variable Theories and the Hidden Measurement Approach to QM

One of the new features in quantum mechanics is the presence of irreducible probabilities (e.g., see Ref. 23), even if the system is in a pure state. In classical theories, probabilities can always be explained as due to an uncertainty about the state of the system, since for each pure state a (classical) experiment yields a predetermined outcome. The probabilities appearing in quantum theory are of a new, non-classical kind and cannot simply be explained as due to a lack of knowledge about the state of the entity. Indeed, attempts have been made to construct so-called 'hidden variable' theories (for an overview, see e.g. Ref. 24) in which quantum probabilities are explained by a lack of knowledge about a 'hidden' variable in the state of the system. The impossibility of such hidden variable theories was the content of several 'no-go theorems',[25,26] but actually these theorems only imply that in order to be a successful hidden variable theory (i.e. predicting the same results as quantum theory), it has to be contextual. An example of a contextual hidden variable theory was given by David Bohm,[27] who proposed a hydrodynamic-like reformulation of quantum mechanics in which the hidden variable is given by the 'location' of the particle in Hilbert space and contextuality is realized through a non-linear potential for which a 'natural physical explanation' is hard to find. Another example of a contextual hidden variable model for a quantum mechanical entity was given by Kochen and Specker,[26] who proposed a hidden variable model for the spin measurements on a quantum spin-$\frac{1}{2}$ particle. Although the hidden variable in the Kochen–Specker model represents a 'hidden state' of the system, the model is still able to reproduce the quantum mechanical results due to the peculiar condition that the hidden variable has to be reset after each measurement. Finally, the theoretical debate was more or less closed by powerful mathematical results, e.g. Gudder showed that it is always possible to construct contextual hidden variable theories which produce the same results as quantum mechanics.[28] However, these *mathematical* proofs of the *possibility* of contextual hidden variable theories are not necessarily sufficient to give the hidden variable theories a *physical meaning or interpretation*, such that these results could be regarded as only mathematically relevant.

From the Kochen–Specker model it is a small — but physically and philosophically important — step to shift the hidden variable from the state of the system to the measurement process. This approach is known as the 'hidden measurement approach'.[3] In this approach the hidden variable is not a 'hidden state' or 'hidden location' of the system, nor is it hidden exclusively in the measurement environment. Instead, the hidden variable is part of the measurement process, which automatically endows the hidden variable with a physical meaning. The examples of Bohm[27] and Kochen–Specker[26] show that the hidden measurement approach does not provide the only possible contextual hidden variable interpretation of quantum mechanics, but in our view the hidden measurement approach has the advantage of introducing contextuality in a natural way by the measurement process.

8. When Quantum Axioms Break Down II: The Epsilon Model

8.1. *The epsilon model*

The macroscopic model for the spin properties of a quantum spin-$\frac{1}{2}$ particle can be modified to obtain 'intermediate situations' between quantum and classical as follows.[30–32] Let us consider the sphere model for a spin-$\frac{1}{2}$ entity with experiments in which the lack of knowledge about which submeasurement is occurring, is controlled by a couple of parameters (ϵ, d) such that ϵ controls the fluctuations on the measurement interaction and the parameter d controls the 'symmetry' of the experiments. Again, the physical entity S that we consider is a point particle P on the Bloch sphere. Hence the set of pure states is given by $\Sigma = \{p_v \mid v \in S^2\}$. The set of experiments

$\mathcal{E}(\epsilon, d)$ is given by $\mathcal{E}(\epsilon, d) = \left\{e_u^{\epsilon,d} \mid u \in S^2\right\}$, with experiments $e_u^{\epsilon,d}$ defined as follows. We put an elastic of length 2ϵ centered around the point d on the interval $[-u, u]$, and attach the end points of the elastic to the points u and $-u$ with 'unbreakable' cords. Obviously, this restricts the parameter d to the interval $[-1+\epsilon, 1-\epsilon]$. Next, the particle falls from its position p orthogonally onto the interval $[-u, u]$ in the point p' and stays attached there. The measurement process continues as the elastic breaks randomly at some point, represented by a uniformly distributed random variable λ in the interval $(d-\epsilon, d+\epsilon)$. If the elastic breaks between p' and $-u$, it will pull the point particle towards u, where it stays attached, and the experiment is said to yield the outcome $+1$. If on the other hand the elastic breaks between u and p', then it will pull the particle towards $-u$, where it stays attached, and the measurement is said to yield outcome -1. If the string breaks at exactly the point where the particle is attached, then we assume that in such a case the measurement always yields the outcome $+1$. We remark that these events are physically irrelevant since they have measure zero, such that we can adopt the mathematically most suitable choice for the model. This completes the description of the experiment $e_u^{\epsilon,d}$.

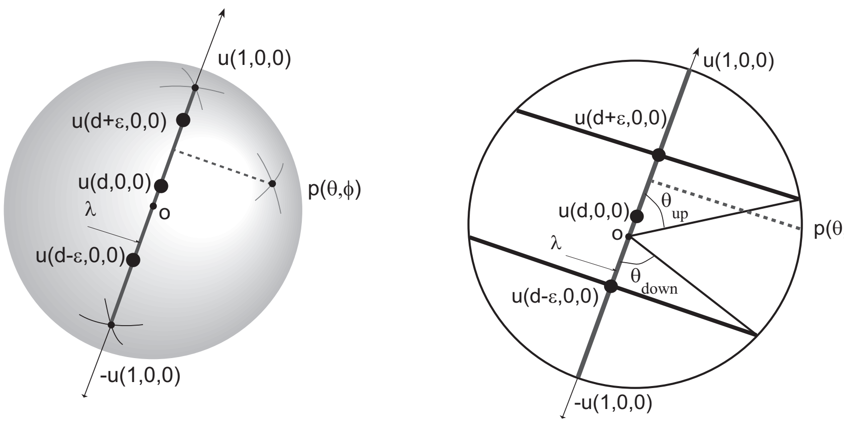

Fig. 2. The epsilon model.

Let $\theta(p, u)$ denote the angle between the state p of the entity and the direction u of the measurement device. If $\cos\theta(p, u) \geq d + \epsilon := \cos\theta_{\text{up}}(\epsilon, d)$, then the elastic will always pull the particle towards u, resulting in an outcome '+1' with certainty. Analogously, if $\cos\theta(p, u) \leq d - \epsilon :=$

$\cos(\pi - \theta_{\text{down}}(\epsilon, d))$, the measurement always yields '-1'. If p is such that $d - \epsilon < \cos\theta(p, u) < d + \epsilon$, the measurement yields one of the two possible outcomes '+1' or '-1'. According to the definition of the experiment e_u^ϵ the probabilities of the respective outcomes for this situation are as follows. The probability for outcome '+1' is given by the length of the elastic between the projection point p' and the point $d - \epsilon$, normalized by the total length of the elastic. This is

$$P(u \mid p) = \frac{\cos\theta(p, u) - d + \epsilon}{2\epsilon} \tag{7}$$

Similarly, we can calculate the probability for the outcome '-1' as

$$P(-u \mid p) = \frac{d + \epsilon - \cos\theta(p, u)}{2\epsilon} \tag{8}$$

Hence the set of states is divided into three regions, defined by closed spherical caps (omitting (ϵ, d) in expressions for $\theta_{\text{up}}, \theta_{\text{down}}$)

$$cap(u, \theta_{\text{up}}) = \left\{q \in S^2 \mid \theta(u, q) \leq \theta_{\text{up}}\right\}$$
$$cap(-u, \theta_{\text{down}}) = \left\{q \in S^2 \mid \theta(-u, q) \leq \theta_{\text{down}}\right\}$$

and the intermediate region

$$S^2 \backslash \{cap(u, \theta_{\text{up}}) \cup cap(-u, \theta_{\text{down}})\}$$

If $q \in cap(u, \theta_{\text{up}})$ the measurement $e_u^{\epsilon,d}$ always yields outcome '+1', if $q \in cap(-u, \theta_{\text{down}})$ the measurement $e_u^{\epsilon,d}$ always yields outcome '-1'. Hence $\kappa\left(a_{e_u^{\epsilon,d}}^{\{+1\}}\right) = \left\{p \in S^2 \mid p \in cap(u, \theta_{\text{up}})\right\}$ and $\kappa\left(a_{e_u^{\epsilon,d}}^{\{-1\}}\right) = \left\{p \in S^2 \mid p \in cap(-u, \theta_{\text{down}})\right\}$. It is easy to verify that $a_{e_u^{\epsilon,d}}^{\{+1\}} = a_{e_{-u}^{\epsilon,-d}}^{\{-1\}}$ since $\cos\theta_{\text{down}}(\epsilon, -d) = \epsilon + d = \cos\theta_{\text{up}}(\epsilon, d)$.

Let us consider the special cases $\epsilon = 1$ and $\epsilon = 0$. Firstly, if $\epsilon = 1$, necessarily $d = 0$ and there is a maximal fluctuation on the measurement interaction. The probabilities are given by

$$P(u \mid p) = \frac{1 + \cos\theta}{2} = \cos^2\frac{\theta}{2} \tag{9}$$
$$P(-u \mid p) = \frac{1 - \cos\theta}{2} = \sin^2\frac{\theta}{2} \tag{10}$$

which coincides with the probability distribution of a spin measurement of a quantum spin-$\frac{1}{2}$ particle. Secondly, if $\epsilon = 0$, there are no fluctuations on the measurement interaction and the state p determines completely which outcome occurs in a measurement. Therefore, this can be called the deterministic limit of the epsilon model. Hence, by varying the parameter $\epsilon \in [0, 1]$ — i.e. the lack of knowledge about the fluctuations in the measurement interaction — from maximal to minimal, one obtains in a continuous way a family of models ranging from a pure quantum mechanical spin-$\frac{1}{2}$ model to a deterministic model. For $\epsilon \in (0,1)$ 'intermediate' situations are found, which are neither classical (since the experiments are not deterministic) nor quantum (since the structure on the set of states and properties does not allow a representation in a Hilbert space[30,31]). This latter result can be shown by the violation of two of Piron's quantum axioms, namely the axiom of weak modularity and the covering law, which only hold in the quantum limit ($\epsilon = 1$) and the deterministic limit ($\epsilon = 0$) of the epsilon model. In the intermediate cases ($0 < \epsilon < 1$) these two axioms are violated. This shows how it is possible within an operational approach to go 'beyond the reach of standard quantum mechanics'. Interestingly, it is still possible to define an orthocomplementation on the set of properties, which is necessary to even express the law of weak modularity.

8.2. *The SPS of the epsilon model*

The **SPS** $(\Sigma, \mathcal{L}, \xi)$ of the epsilon model is as follows. First, the set of states $\Sigma = \left\{p_v \mid v \in S^2\right\}$. The set of properties is given by the set of primitive properties $\left\{a^{\{+1\}}_{e^{\epsilon,d}_u}, a^{\{-1\}}_{e^{\epsilon,d}_u} \mid u \in S^2\right\}$ and their meets, of which the Cartan maps are given by spherical caps, $cap\,(u, \theta_{\rm up})$ and $cap\,(-u, \theta_{\rm down})$, respectively, and all their intersections, together with the trivial properties 0 and 1, with the Cartan map being the empty set $\emptyset$ and set of states Σ, respectively. For $p \in \Sigma : a^{\{+1\}}_{e^{\epsilon,d}_u} \in \xi\,(p) \Leftrightarrow p \in cap\,(u, \theta_{\rm up})$ and $a^{\{-1\}}_{e^{\epsilon,d}_u} \in \xi\,(p) \Leftrightarrow p \in cap\,(-u, \theta_{\rm down})$.

If $\theta_{\rm up} \leq \theta_{\rm down}$ the spherical cap $cap\,(u, \theta_{\rm up})$ can be written as an intersection of spherical caps with angle $\theta_{\rm down}$:

$$cap\,(u, \theta_{\rm up}) = \bigcap_{\theta(u,v)=\theta_{\rm down}-\theta_{\rm up}} cap\,(v, \theta_{\rm down})$$

which means that

$$\kappa\left(a^{\{+1\}}_{e^{\epsilon,d}_u}\right) = \bigcap_{\theta(u,v)=\theta_{\rm down}-\theta_{\rm up}} \kappa\left(a^{\{-1\}}_{e^{\epsilon,d}_{-v}}\right) = \kappa\left(\bigwedge_{\theta(u,v)=\theta_{\rm down}-\theta_{\rm up}} a^{\{-1\}}_{e^{\epsilon,d}_{-v}}\right)$$

such that $a^{\{+1\}}_{e^{\epsilon,d}_u} = \bigwedge_{\theta(u,v)=\theta_{\rm down}-\theta_{\rm up}} a^{\{-1\}}_{e^{\epsilon,d}_{-v}}$, i.e. $\left\{a^{\{+1\}}_{e^{\epsilon,d}_u} \mid u \in S^2\right\}$ is generated by taking meet of subsets of $\left\{a^{\{-1\}}_{e^{\epsilon,d}_v} \mid v \in S^2\right\}$. Similarly, if $\theta_{\rm up} \geq \theta_{\rm down}$ then the spherical cap $cap\,(u, \theta_{\rm down})$ can be written as an intersection of spherical caps with angle $\theta_{\rm up}$:

$$cap\,(u, \theta_{\rm down}) = \bigcap_{\theta(u,v)=\theta_{\rm up}-\theta_{\rm down}} cap\,(v, \theta_{\rm up})$$

such that $a^{\{-1\}}_{e^{\epsilon,d}_{-u}} = \bigwedge_{\theta(u,v)=\theta_{\rm up}-\theta_{\rm down}} a^{\{+1\}}_{e^{\epsilon,d}_{v}}$, i.e. $\left\{a^{\{-1\}}_{e^{\epsilon,d}_{u}} \mid u \in S^2\right\}$ is generated by taking meet of subsets of $\left\{a^{\{+1\}}_{e^{\epsilon,d}_{v}} \mid v \in S^2\right\}$. Hence the set of properties $\mathcal{L}$ is generated by spherical caps $cap\left(u, \theta_M = \max\left(\theta_{\rm up}, \theta_{\rm down}\right)\right)$. In contrast with the 'quantum' sphere model, the meet of two non-trivial properties does not necessarily result in the trivial 'never actual' property 0, but can result in a new non-zero property, of which the Cartan map is given by the intersection of the respective spherical caps.

First, let us notice that it is easy to check that the axioms of property determination and property completeness are satisfied for the epsilon model. For state $p \in \Sigma$ the 'property state' $s(p) = \wedge_{a\in\xi(p)} a$. Since $\kappa(\wedge_{a\in\xi(p)} a) = \wedge_{a\in\xi(p)}\kappa(a)$ we can see easily that $\kappa(\wedge_{a\in\xi(p)} a) = \{p\}$ if e.g. we consider $a^{\{+1\}}_{e^{\epsilon,d}_{u}}, a^{\{+1\}}_{e^{\epsilon,d}_{v}}$ such that $\theta\left(u,v\right) = 2\theta_M, \theta\left(u,p\right) = \theta\left(v,p\right) = \theta_M$. Then $p \in \kappa(\wedge_{a\in\xi(p)} a) \subset \kappa(a^{\{+1\}}_{e^{\epsilon,d}_{u}}) \cap \kappa(a^{\{+1\}}_{e^{\epsilon,d}_{v}}) = \{p\}$. Hence $\forall p \in \Sigma : s\left(p\right)$ is an atom of $\mathcal{L}$ and atomicity follows straightforwardly. In the next section we discuss the problem of orthocomplementation of the epsilon model.

8.3. *Orthogonality relations and orthocomplementation of the epsilon model*

In this section we consider two specific orthogonality relations defined by operational concepts of eigenstate and measurement-induced state transition and we study under which conditions they define an orthocomplementation on the set of properties. These two orthogonality relations have been introduced by Aerts and Durt, respectively.[1,20]

8.3.1. *Aerts orthogonality relation*

Let us consider an entity S with a set of states Σ and a set of experiments $\mathcal{E}$. Aerts introduced the following orthogonality relation on the set of states:[1]

Definition 8.1. The states p and q are Aerts orthogonal (denoted as $p \perp_A q$) iff $\exists e \in \mathcal{E}, A, B \subset O_e, A \cap B = \emptyset : p \in eig_e\left(A\right), q \in eig_e\left(B\right)$.

In other words, there exists at least one experiment such that the states p and q are eigenstates of mutually exclusive outcome sets. Since for the ϵ-model every experiment has only two possible outcomes, the states p and q are Aerts orthogonal iff there exists an experiment such that p and q are eigenstates of different outcomes. Let us show that only for the extreme situation $d = -1 + \epsilon$ this orthogonality relation defines an orthocomplementation on the set of properties. First, two states p and q are orthogonal iff they are at least separated by angle $\theta_{\text{sup}} = \pi - \theta_{\text{up}} - \theta_{\text{down}}$. Indeed, let $\theta(p,q) \geq \theta_{\text{sup}}$, then we can choose $a^{\{+1\}}_{e^{\epsilon,d}_u}$ such that u, p and q are coplanar, and $\theta(p,u) = \theta_{\text{up}}$. Hence $p \in \kappa\left(a^{\{+1\}}_{e^{\epsilon,d}_u}\right) = eig_e(+1)$. Also, $\theta(q,-u) = \pi - \theta_{\text{up}} - \theta(p,q) \leq \pi - \theta_{\text{up}} - \theta_{\text{sup}} = \theta_{\text{down}}$ such that $q \in \kappa\left(a^{\{-1\}}_{e^{\epsilon,d}_u}\right) = eig_e(-1)$; therefore $p \perp_A q$. Conversely, let $p \perp_A q$. Then $\exists e \in \mathcal{E}, A, B \subset O_e, A \cap B = \emptyset : p \in eig_e(A), q \in eig_e(B)$. Since any property a is generated by taking meets over $\left\{a^{\{\pm 1\}}_{e^{\epsilon,d}_u} \mid u \in S^2\right\}$, it follows that the angular distance between $eig_e(A)$ and $eig_e(B)$ is at least θ_{sup}, from which the necessary condition $\theta(p,q) \geq \theta_{\text{sup}}$ follows immediately.

Therefore, a property state orthogonal has as Cartan map a closed spherical cap with opening angle $\pi - \theta_{\text{sup}} : \kappa\left(s(p)^{\perp_A}\right) = cap(-p, \pi - \theta_{\text{sup}})$. On the other hand, we have already shown in a previous section that the lattice of properties $\mathcal{L}$ is generated by taking meets of properties with as Cartan map the closed spherical cap $cap(u, \theta_M = \max(\theta_{\text{up}}, \theta_{\text{down}}))$. So in order that the property state orthogonals are contained within $\mathcal{L}$ and also generate $\mathcal{L}$ by intersection, we need that $\pi - \theta_{\text{sup}} = \max(\theta_{\text{up}}, \theta_{\text{down}})$. Since $\pi - \theta_{\text{sup}} = \theta_{\text{up}} + \theta_{\text{down}}$ it follows that either $\theta_{\text{up}} = 0$ or $\theta_{\text{down}} = 0$. Hence only in the case of ‘extreme distribution’ $d = -1 + \epsilon$ or $d = 1 - \epsilon$, we have that the lattice of properties is orthocomplemented under this orthogonality relation.

8.3.2. *Epsilon orthocomplementation*

Although the standard way of introducing an orthocomplementation by means of the Aerts orthogonality relation fails for the epsilon model in the non-extreme situations, it is possible to define an orthogonality relation on the set of states which yields an orthocomplementation on the lattice of properties for all intermediate situations of the epsilon model.[20] The orthogonality relation on the set of states is defined as follows: $p \perp_\epsilon q \Leftrightarrow$ there exists at least one experiment that does not induce a transition from one of the states to the other.[b] Let us denote the angle

$$\theta_\epsilon(\epsilon,d) = \min\{\theta_{\text{sup}}(\epsilon,d) + \theta_{\text{up}}(\epsilon,d), \theta_{\text{sup}}(\epsilon,d) + \theta_{\text{down}}(\epsilon,d)\}$$

The following theorem holds for the ϵ-model:

[b]Of course assuming that at least one of the states is a possible final state of the system after the measurement.

- For $\epsilon \neq 0 : p \perp_\epsilon q \Leftrightarrow \theta\,(p,q) \geq \theta_\epsilon\,(\epsilon, d)$.
- For $\epsilon = 0$:
  1) if $\theta_{\text{up}}\,(0,d) < \theta_{\text{down}}\,(0,d) : p \perp_\epsilon q \Leftrightarrow \theta\,(p,q) > \theta_\epsilon\,(0,d)$,
  2) if $\theta_{\text{up}}\,(0,d) \geq \theta_{\text{down}}\,(0,d) : p \perp_\epsilon q \Leftrightarrow \theta\,(p,q) \geq \theta_\epsilon\,(0,d)$.

For $\epsilon \neq 0$, the state orthogonal $p^\perp$ is the closed spherical cap with angle $\theta_M = \max(\theta_{\text{up}}, \theta_{\text{down}})$ around the antipode $-p : p^\perp = cap\,(-p, \theta_M = \max(\theta_{\text{up}}, \theta_{\text{down}}))$. For $\epsilon = 0$ we have to consider two cases: (i) $\theta_{\text{down}} \leq \theta_{\text{up}}$, $p^\perp = cap\,(-p, \theta_M = \max(\theta_{\text{up}}, \theta_{\text{down}})) = cap\,(-p, \theta_{\text{up}})$; (ii) $\theta_{\text{down}} > \theta_{\text{up}}, p^\perp = cap^o\,(-p, \theta_M = \max(\theta_{\text{up}}, \theta_{\text{down}})) = \left\{q \in S^2 \mid \theta\,(q, -p) < \theta_{\text{down}}\right\}$, i.e. the open spherical cap around $-p$ with opening θ_{down}. These are exactly the spherical caps which by intersection generate the Cartan maps of the properties of the epsilon model, i.e. the property state complements $s(p)^{\perp_\epsilon}$ generate the lattice of properties. Therefore it follows that $\mathcal{L}$ is orthocomplemented.

8.4. *Failing quantum axioms for the epsilon model*

Let us now focus on the axioms of weak modularity and the covering law and check their validity for the different cases of the epsilon model. It was shown that both axioms hold in the quantum ($\epsilon = 1$) and the classical limit ($\epsilon = 0$), but they are violated in the intermediate cases of the epsilon model ($0 \neq \epsilon \neq 1$).[30,31] These two axioms are used in the representation theorem to obtain a vector space structure on the set of states of the system. The violation of these two axioms also explains why standard quantum mechanics — for which these two axioms hold — is not capable of representing the ϵ-model in its continuous transition from quantum to classical. Since the proofs are relatively short, we will present them here for a better understanding of how the violation of these axioms occurs.

8.4.1. *Weak modularity*

Recall that $p \perp_\epsilon q \Leftrightarrow \theta\,(p,q) \geq \pi - \theta_M$. Since $\mathcal{L}$ is orthocomplemented, each property can be obtained by taking meet of state property orthogonals $s(p)^{\perp_\epsilon}$ such that $\kappa\left(s(p)^{\perp_\epsilon}\right) = cap\,(-p, \theta_M)$. Let us choose the property a such that $\kappa\,(a) = cap\,(p, \theta_1)$ with $0 < \theta_1$ and $2\theta_M - \pi < \theta_1 < \theta_M$, which is always possible for $\epsilon \notin \{0,1\}$ (since then $0 < \theta_M < \pi$). Property b is chosen such that $\kappa\,(b) = cap\,(p, \theta_M)$. Clearly $a < b$ and $a \neq b$, such that if the axiom of weak modularity is fulfilled, there should exist a property $c \in \mathcal{L}$ orthogonal to a such that $b = a \vee c$, to be more specific $c = a^{\perp_\epsilon} \wedge b$. Since

$c < a^{\perp_\epsilon}$, any state $q \in \kappa(c)$ lies in a spherical cap $cap(-p, \theta_M - \theta_1) = \bigcap_{r \in cap(p,\theta_1)} cap(-r, \theta_M)$. Hence $\theta(p,q) > \pi - (\theta_M - \theta_1) > \theta_M$ such that $q \notin cap(p, \theta_M) = \kappa(b)$, which shows that it is impossible that $\kappa(c) \subset \kappa(b)$ and therefore $c \not< b$. Hence the axiom of weak modularity cannot hold.

8.4.2. *Covering law*

Let us recall that the property lattice $\mathcal{L}$ of a State–Property–System $(\Sigma, \mathcal{L}, \xi)$ satisfies the 'covering law' iff for $p \in \Sigma$ and $a, b \in \mathcal{L}$ such that $a \wedge s(p) = 0$ and $a < b < a \vee s(p)$ we have that either $a = b$ or $a \vee s(p) = b$. Let us choose the property a such that $\kappa(a) = cap(q, \theta_M - \delta)$ with δ such that $0 < \delta < \min(\pi - \theta_M, \theta_M)$. Next, let p be a state such that $\theta(p,q) = \theta_M + \delta$. Because the angle between p and q is greater than θ_M the only property of which the Cartan image contains $\kappa(a)$ and $\kappa(s(p))$ is the maximal property 1 (with $\kappa(1) = \Sigma$), which shows that $a \vee s(p) = 1$. Next, let us consider b such that $\kappa(b) = cap(q, \theta_M)$. Then $a < b < a \vee s(p)$ but neither $a = b$ nor $b = a \vee s(p)$, which shows that the covering law does not hold.

8.5. *Towards experiments with more than two outcomes: the N-model(s)*

In the following sections we consider sphere models with measurements having more than two outcomes.[33,34] The number of possible outcomes (and corresponding eigenstate sets) is given by a parameter N which also controls the maximal possible change of state due to measurement. For $N = 2$ the possible change of state is maximal and the system reduces to the sphere model for a quantum spin-$\frac{1}{2}$ particle. In the 'classical limit' $N \to \infty$ the measurement induces no state transition at all and all experiments are deterministic. We consider various orthoclosure structures generated by an orthogonality relation on the set of states and check whether these coincide with the eigenclosure structure or not. These models illustrate clearly that it is not always possible to define an orthocomplementation on the set of properties with this procedure. Therefore, it is the structure on the set of properties itself which determines whether Piron's axiom of orthocomplementation holds or not.

9. The Symmetric N-Model

The physical entity S that we consider is a point particle P on the Bloch sphere. Hence the set of pure states is given by $\Sigma = \{p_v \mid v \in S^2\}$. The set of experiments is $\mathcal{E}(N) = \left\{e_u^N \mid u \in S^2\right\}, N : 2 \to \infty$ with e_u^N defined as follows. We consider the point u on the Poincaré sphere and its antipode $-u$, and divide the angular interval $[\theta_u, \theta_{-u}] = [0, \pi]$ into N equidistant angles $\theta_k = \frac{\pi k}{N-1},\ k = 0, \ldots, N-1$. The circle C_k is defined as the border of the spherical cap $cap\,(u, \theta_k)$, i.e. $C_k = \left\{q \in S^2 \mid \theta(u,q) = \theta_k\right\}$ and corresponds with the set of eigenstates of outcome $o_u^k = \cos\left(\frac{\pi k}{N-1}\right),\ k = 0, \ldots, N-1$. Two consecutive circles C_k and C_{k+1} define a band $B_k,\ k = 0, \ldots, N-2:$ $B_k = \left\{v \in S^2 \mid \theta_k \leq \theta(u,v) \leq \theta_{k+1}\right\}$. For $k = 0$ and $k = N-1$, the circles C_k reduce to points u and $-u$ on the sphere, respectively. The result of the measurement e_u^N is defined as follows: we consider the great circle $C_{\{p,u,-u\}}$ on S^2 through the triplet $\{p, u, -u\}$. The intersections of $C_{\{p,u,-u\}}$ with the circles C_k and C_{k+1} are denoted by p_k and p_{k+1} respectively, with orthogonal projections p'_k and p'_{k+1} onto the line segment between u and $-u$. Let us assume that the initial state p of the entity lies in the band B_k. By analogy with the ϵ-model, we put an elastic piece along the great circle $C_{\{p,u,-u\}}$ on S^2 between the points p_k and p_{k+1} and attach the point particle P to this elastic. The measurement process continues as the elastic breaks randomly at some point, denoted by p_λ. If $p_\lambda \in (p_{k+1}, p]$ the elastic breaks and pulls the particle P towards the point p_k, where the point particle stays attached and we assign the outcome o_u^k to the experiment e_u^N. If $p_\lambda \in (p, p_k)$ the point particle is pulled by the piece of elastic towards the point p_{k+1}, where it stays attached, and we assign the outcome o_u^{k+1} to the experiment e_u^N. Again, the event that the elastic band breaks at exactly the point where the point particle is situated, i.e. $p_\lambda = p$, has measure zero, so our choice for the measurement procedure in this case does not affect the overall probabilities of the model.

In a sense, for the discussion of orthocomplementation of the set of properties only the structure on the set of eigenstates is relevant and the probability distribution over non-eigenstates is only of secondary importance. Nevertheless, we can always choose a procedure of ‘breaking the elastic’ which fits the N-model with the quantum sphere model for $N = 2$. For instance, one way of breaking the elastic in a suitable way is by considering a ‘quantum probability-compatible gun’ moving in the interval $[-u, u]$ and shooting bullets straight at the circle segment $C_{\{p,p_k,p_{k+1}\}}$. If the gun fires when it is at a point $p'_\lambda \in \left[p'_k, p'_{k+1}\right]$, the elastic breaks at the corresponding point p_λ between p_k and p_{k+1}. If the probability that the gun fires at point p'_λ is uniformly distributed over the interval $\left[p'_k, p'_{k+1}\right]$, the resulting probability distribution over the set of outcomes coincides with the quantum probability distribution for $N = 2$, as shown in the next section.

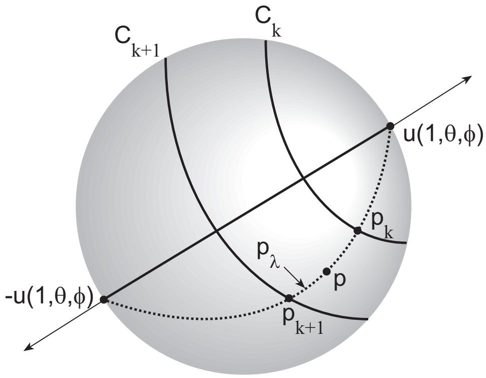

Fig. 3. The symmetric N-model.

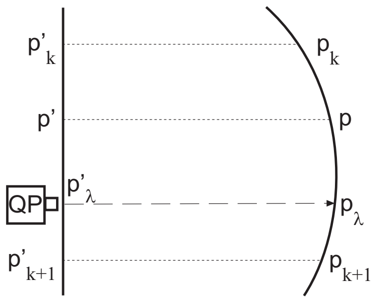

Fig. 4. The measurement procedure with a 'quantum probability-compatible gun'.

9.1. *The transition probabilities*

Let us calculate the probability of each outcome for a 'superposition' state. The probability to obtain the outcome o_u^k if the system is in a state p is denoted by $P\left(o_u^k \mid p\right)$ and the probability for an outcome o_u^{k+1} by $P\left(o_u^{k+1} \mid p\right)$. Let us denote the orthogonal projection of p onto the line segment between u and $-u$ as p'. By definition of the 'quantum probability-compatible gun', the probability $P\left(o_u^k \mid p\right)$ is given by the length of the interval between the points p' and p'_{k+1}, normalized by the total length of the interval, i.e. $\cos\left(\frac{\pi k}{N-1}\right) - \cos\left(\frac{\pi(k+1)}{N-1}\right)$. Hence the probabilities are given by:

$$P\left(o_u^k \mid p\right) = \frac{\cos\theta\left(u,p\right) - \cos\left(\frac{\pi(k+1)}{N-1}\right)}{\cos\left(\frac{\pi k}{N-1}\right) - \cos\left(\frac{\pi(k+1)}{N-1}\right)}$$

and

$$P\left(o_u^{k+1} \mid p\right) = \frac{\cos\left(\frac{\pi k}{N-1}\right) - \cos\theta\,(u,p)}{\cos\left(\frac{\pi k}{N-1}\right) - \cos\left(\frac{\pi(k+1)}{N-1}\right)}$$

For $N = 2$ the probabilities are given by:

$$P\left(o_u^0 \mid p\right) = \frac{\cos\theta\,(u,p) + 1}{2} = \cos^2\left(\frac{\theta\,(u,p)}{2}\right)$$

$$P\left(o_u^1 \mid p\right) = \frac{1 - \cos\theta\,(u,p)}{2} = \sin^2\left(\frac{\theta\,(u,p)}{2}\right)$$

which are the transition probabilities for a spin measurement on a quantum spin-$\frac{1}{2}$ particle, similarly as the ϵ-model in its quantum limit ($\epsilon = 1$). Therefore it is a model for the spin properties of a quantum spin-$\frac{1}{2}$ particle. See Figs. 5 and 6 for $N = 5$ and $N = 6$.

9.2. *Orthogonality relations and orthocomplementation of the N-model*

9.2.1. *Aerts orthogonality relation*

By definition, two states p and q are Aerts orthogonal iff there exists a measurement $e_u \in \mathcal{E}$ such that p and q are eigenstates of different out-

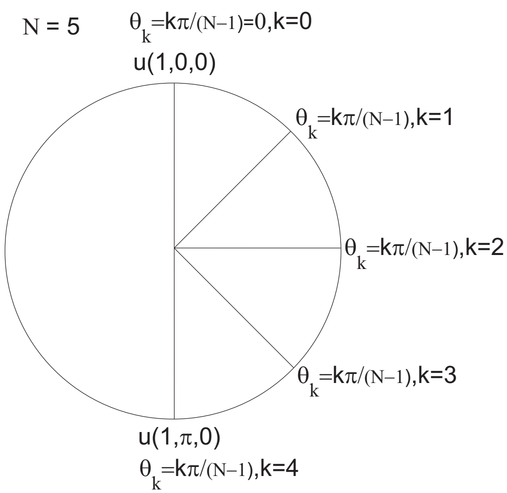

Fig. 5. The symmetric N-model for $N = 5$.

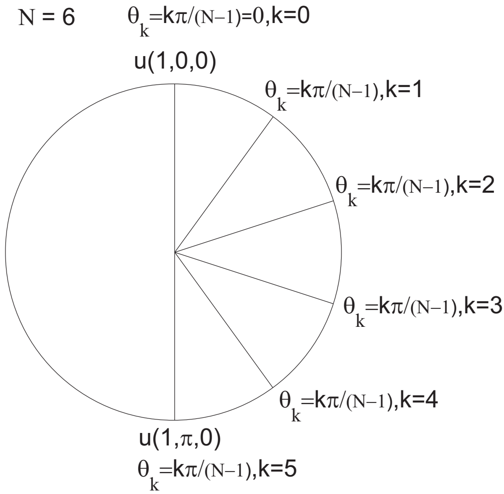

Fig. 6. The symmetric N-model for $N = 6$.

comes. For the N-model, two states are Aerts orthogonal iff there exists an experiment such that p and q are in different eigenstate sets. A necessary condition for two states to be Aerts orthogonal is thus that they are separated by an angle that is greater than the smallest superposition angle, i.e. the smallest angle between two consecutive eigenstate sets. For the symmetric N-model, these superposition angles are independent of k, since $\Delta\theta_k = \frac{\pi}{N-1}$. Hence, denoting the angle between the states p and q by $\theta(p,q)$ the necessary condition for $p \perp_A q$ is that $\theta(p,q) \geq \frac{\pi}{N-1}$. It can be shown that this necessary condition is also sufficient. Let us consider states p and q such that $\theta(p,q) \geq \frac{\pi}{N-1}$. It suffices to consider the experiment e_p^N for which $p \in eig\left(o_u^0 = 1\right)$ while $q \in eig\left(\left\{o_u^0\right\}^C\right)$, which shows that states p and q are Aerts orthogonal. To summarize, if $\theta_{N,\perp_A} \equiv \frac{\pi}{N-1}$ then the following holds: two states p and q are Aerts orthogonal iff $\theta(p,q) \geq \theta_{N,\perp_A}$.

9.2.2. *Durt orthogonality relation*

Two states p and q are 'epsilon orthogonal' iff there exists at least one experiment with p or q as possible final states that does not induce a transition from one of the states to the other.[20] This original formulation of an orthogonality relation has to be modified to be useful for the N-model.[33] Indeed, by definition a measurement e_u^N induces a state transition along the great circle containing the measurement direction and the initial state of the entity. So a state p can only make a transition to the state q if the measurement direction lies in the plane defined by the great circle through p and q. For an experiment whose measurement direction lies outside this plane, a state transition from p to q is never possible. If these experiments were taken into account, two different states would always be orthogonal for $N \gneqq 2$. This would not be very interesting, and therefore the definition was modified for the N-model as follows:[33]

Definition 9.1. Two states p and q are Durt orthogonal, denoted as $p \perp_D q$, iff there exists at least one experiment with p or q as one of its final states, and with the measurement direction in the plane of p and q, such that this experiment does not induce a transition from one of the states to the other.

For the ϵ-model each measurement has only two possible final states, given by the (opposite) endpoints of the elastic. Therefore the measurement direction is completely defined by *one* final state. If we want to study the Durt orthogonality relation on the N-model for $N \gneqq 2$, it has to be taken into account that the N-model contains more than two eigenstate sets. The set of eigenstates of the N-model does not only contain the two endpoints of the measurement axis, but also the 'eigencircles' $C_k, k = 1, \ldots, N-2$, for which $\theta_k = \frac{\pi k}{N-1}$. Let us consider two states p and q. By definition we only take into account the measurements e_u^N such that the measurement direction u lies in the plane defined by p and q, and such that q is a possible final state. Obviously, then $\theta\left(u,q\right) \in \left\{\frac{k\pi}{N-1} \mid k = 0, \ldots, N-1\right\}$. It is easy to show that a sufficient and necessary condition for $p \perp_D q$ is that $\theta\left(p,q\right) \geq \frac{\pi}{N-1}$. Hence the Durt orthogonality angle $\theta_{N,\perp_D}$ is given by $\theta_{N,\perp_D} = \frac{\pi}{N-1}$, which is equal to the Aerts orthogonality angle $\theta_{N,\perp_A}$, which can therefore be abbreviated as $\theta_{N,\perp}$. Hence the generated Aerts and Durt orthoclosure structures coincide.

9.2.3. *Orthocomplementation of the symmetric N-model*

The eigenclosure structure of the N-model is generated by eigenstate sets $eig_{e_u^N}\left(\left\{o_u^i\right\}^C\right), i = 0, \ldots, N-1$. Clearly, $p^\perp = \left\{q \in \Sigma \mid \theta\left(p,q\right) \geq \theta_{N,\perp}\right\} = eig_{e_p^N}\left(\left\{o_p^0\right\}^C\right)$ and therefore $\mathcal{F}_\perp \subset \mathcal{F}_{eig}$. Also, $eig_{e_u^N}\left(\left\{o_u^i\right\}^C\right) = \cap_{q \in C_i} eig_{e_q^N}\left(\left\{o_q^0\right\}^C\right) = \cap_{q\in C_i} q^\perp$ and hence $\mathcal{F}_{eig} \subset \mathcal{F}_\perp$. This shows that both Aerts and Durt orthoclosure structures coincide with the eigenclosure structure. Since an orthoclosure structure is orthocomplemented it follows trivially that also the lattice $\mathcal{L}$ of properties is orthocomplemented.

9.3. *Failing quantum axioms for the symmetric N-model*

Let us now show that the axioms of weak modularity and the covering law only hold in the quantum ($N = 2$) and the classical limit ($N \to \infty$) and that they are violated in the intermediate cases of the symmetric N-model ($2 \neq N \neq \infty$), which could have been expected by analogy with the violation of these axioms in the epsilon model. Hence, also the proofs are very similar to those in Refs. 20 and 34.

9.3.1. *Weak modularity*

Recall that $p \perp_N q \Leftrightarrow \theta(p,q) \geq \theta_{N,\perp} = \frac{\pi}{N-1}$. Since $\mathcal{L}$ is orthocomplemented, each property can be obtained by taking meet of state property orthogonals $s(p)^{\perp_N}$ such that $\kappa\left(s(p)^{\perp_N}\right) = cap\left(-p, \frac{N-2}{N-1}\pi\right)$. Let us choose the property a such that $\kappa(a) = cap(p, \theta_1)$ with $\frac{N-3}{N-1}\pi < \theta_1 < \frac{N-2}{N-1}\pi$, which is always possible for $N \notin \{2, \infty\}$. Property b is chosen such that $\kappa(b) = cap\left(p, \frac{N-2}{N-1}\pi\right)$. Clearly $a < b$ and $a \neq b$, such that if the axiom of weak modularity is fulfilled, there should exist a property $c \in \mathcal{L}$ orthogonal to a such that $b = a \vee c$, to be more specific $c = a^{\perp_N} \wedge b$. Since $c < a^{\perp_N}$, any state $q \in \kappa(c)$ lies in a spherical cap $cap\left(-p, \pi - \frac{\pi}{N-1} - \theta_1\right) = \bigcap_{r \in cap(p,\theta_1)} cap(-r, \frac{N-2}{N-1}\pi)$. Hence $\theta(p,q) > \pi - \left(\frac{N-2}{N-1}\pi - \theta_1\right) = \frac{\pi}{N-1} + \theta_1 > \frac{N-2}{N-1}\pi$ such that $q \notin cap\left(p, \frac{N-2}{N-1}\pi\right) = \kappa(b)$, which shows that it is impossible that $\kappa(c) \subset \kappa(b)$ and therefore $c \not< b$. Hence the axiom of weak modularity does not hold.

9.3.2. *Covering law*

Let us show that the covering law does not hold for the N-model. Let us choose the property a such that $\kappa(a) = cap\left(q, \frac{N-2}{N-1}\pi - \delta\right)$ with δ such that $0 < \delta < \frac{\pi}{N-1}$. Next, let p be a state such that $\theta(p,q) = \frac{N-2}{N-1}\pi + \delta$. Because the angle between p and q is greater than $\frac{N-2}{N-1}\pi$, the only property of which the Cartan image contains $\kappa(a)$ and $\kappa(s(p))$ is the maximal property 1 (with $\kappa(1) = \Sigma$), which shows that $a \vee s(p) = 1$. Next, let us consider b such that $\kappa(b) = cap\left(q, \frac{N-2}{N-1}\pi\right)$. Then $a < b < a \vee s(p)$ but neither $a = b$ nor $b = a \vee s(p)$, which shows that the covering law does not hold.

10. The Asymmetric N-Model

Let us now present an asymmetric N-model in which again a parameter N controls the maximal change of state due to measurement, but with a different eigenclosure structure and different probability structure. The state of the entity is represented by a point p on the surface of the Poincaré sphere S^2. To define the experiment e_u^N we proceed as follows. We consider the point u on the Poincaré sphere and its antipode $-u$, and divide the $[u,-u]$-axis into N intervals of equal length, $I_k = \left[1 - \frac{2(k+1)}{N-1}, 1 - \frac{2k}{N-1}\right], \; k = 0, \ldots, N-2$. The border points of the interval I_k are denoted by i_k and i_{k+1}, such that $i_k = 1 - \frac{2k}{N-1}$ and $I_k = [i_{k+1}, i_k], \; k = 0, \ldots, N-2$. The angle θ_k is defined by $\cos\theta_k = i_k = 1 - \frac{2k}{N-1}$. The circle C_k is defined as the border of the spherical cap $cap\,(u, \theta_k)$, i.e. $C_k = \left\{q \in S^2 \mid \theta\,(u, q) = \theta_k\right\}$ and corresponds with the set of eigenstates of outcome $o_u^k = \cos\theta_k = 1 - \frac{2k}{N-1}, \; k = 0, \ldots, N-1$. Two consecutive circles C_k and C_{k+1} define a band $B_k, \; k = 0, \ldots, N-2 : B_k = \left\{v \in S^2 \mid \theta_k \leq \theta\,(u, v) \leq \theta_{k+1}\right\}$. The result of the measurement e_u^N is defined as follows: we consider the great circle $C_{\{p,u,-u\}}$ on S^2 through the triplet $\{p, u, -u\}$. The intersections of $C_{\{p,u,-u\}}$ with the circles C_k and C_{k+1} are denoted by p_k and p_{k+1}, respectively. Let us assume that the initial state of the entity is in a band B_k. The interval $[i_{k+1}, i_k]$ is divided into three pieces, by an interval $I_{k,\mathrm{sup}}$ of length $\frac{2}{(N-1)^2}$ centered around $m_k = \frac{i_{k+1}+i_k}{2}$, i.e. the middle of the interval I_k. Similarly to the symmetric N-model, we put an elastic piece along the great circle $C_{\{p,u,-u\}}$ on S^2 between the points p_k and p_{k+1} and attach the point particle P to this elastic. The measurement process continues as the elastic breaks randomly at some point, denoted by p_λ. If $p_\lambda \in (p_{k+1}, p]$ the elastic breaks and pulls the particle P towards the point p_k where the point particle stays attached and we assign the outcome o_u^k to the experiment e_u^N. If $p_\lambda \in (p, p_k)$ the point particle is pulled by the piece of elastic towards the point p_{k+1}, where it stays attached, and we assign the outcome o_u^{k+1} to the experiment e_u^N. Again, the event that the elastic band breaks at exactly the point where the point particle is situated, i.e. $p_\lambda = p$, has measure zero, so our choice for the measurement procedure in this case does not affect the overall probabilities of our model. We choose a procedure of 'breaking the elastic' as follows. The 'quantum probability-compatible gun' shoots bullets straight at the circle segment $C_{\{p,p_k,p_{k+1}\}}$, but is restricted to the interval $I_{k,\mathrm{sup}}$. If the gun fires when it is at a point $p'_\lambda \in \left[p'_k, p'_{k+1}\right]$, the elastic breaks at the corresponding point p_λ between p_k and p_{k+1}. If the probability that the gun fires at point p'_λ is uniformly distributed over the interval $I_{k,\text{sup}}$, the resulting probability distribution over the set of outcomes coincides with the quantum probability distribution for $N = 2$, as shown in the next section (which in fact is the reason why we construct the model this way). To conclude, there are three cases:

(1) $p' \in I_{k,\text{up}} = \left[1 - \frac{2k}{N-1} - \frac{N-2}{(N-1)^2}, 1 - \frac{2k}{N-1}\right]$ and the elastic pulls the point particle towards p_k, where it stays attached and the measurement e_u^N yields the outcome $o_u^k = 1 - \frac{2k}{N-1}$,

(2) $p' \in I_{k,\text{down}} = \left[1 - \frac{2(k+1)}{N-1}, 1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}\right]$ and the elastic pulls the point particle towards p_{k+1}, where it stays attached and the measurement e_u^N yields the outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$,

(3) $p' \in I_{k,\text{sup}} = \left]1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}, 1 - \frac{2k}{N-1} - \frac{N-2}{(N-1)^2}\right[$ we consider a random variable λ in the interval $I_{k,\text{sup}}$ by employing the 'quantum probability-compatible gun'. If $\lambda \leq p'$ the experiment yields the outcome $o_u^k = 1 - \frac{2k}{N-1}$; if $\lambda > p'$ the experiment yields the outcome outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$. If the outcome $o_u^k = 1 - \frac{2k}{N-1}$ occurs, then the measurement induces a transition of state from p to p_k and if the outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$ occurs, then the measurement induces a transition of state from p to p_{k+1}.

See Figs. 7 and 8 for $N = 5$ and $N = 6$.

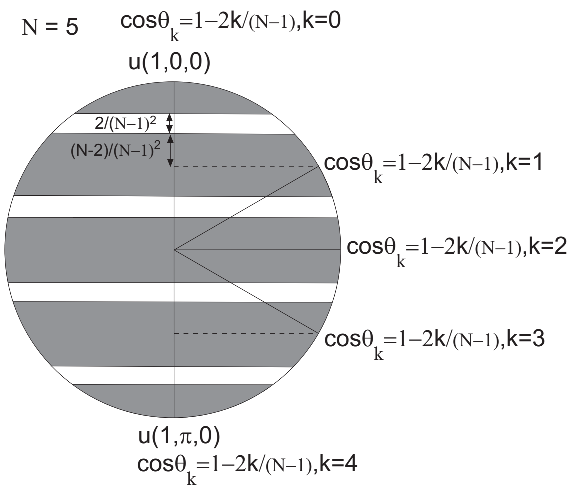

Fig. 7. The asymmetric N-model for $N = 5$, the eigenstate sets of the respective outcomes are represented by the spherical sectors coloured in gray.

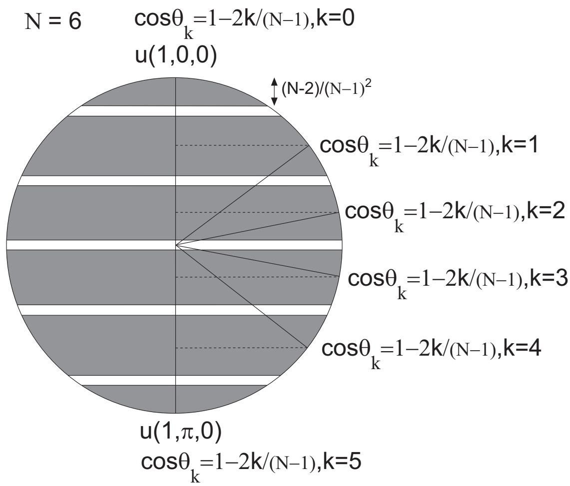

Fig. 8. The asymmetric N-model for $N = 6$.

10.1. *The transition probabilities*

Let us calculate the probability for each outcome for a 'superposition' state. The probability of obtaining the outcome $o_u^k = 1 - \frac{2k}{N-1}$ if the system is in a state $p \in B_k$ is denoted by $P\left(o_u^k \mid p\right)$ and the probability for an outcome $o_u^{k+1} = 1 - \frac{2(k+1)}{N-1}$ by $P\left(o_u^{k+1} \mid p\right)$. By definition, the probability $P\left(o_u^k \mid p\right)$ is given by the length of the interval between the projection point p' and $1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}$, normalized by the total length of the superposition interval, i.e. $\frac{2}{(N-1)^2}$. Denoting by $\theta'_{k,\text{up}}$ the angle for which $\cos\theta'_{k,\text{up}} = 1 - \frac{2k}{N-1} - \frac{N-2}{(N-1)^2}$, and $\theta'_{k,\text{down}}$ for which $\cos\theta'_{k,\text{down}} = 1 - \frac{2(k+1)}{N-1} + \frac{N-2}{(N-1)^2}$, the probabilities are given by:

$$P\left(o_u^k \mid p\right) = \frac{\cos\theta\left(u,p\right) - \cos\theta'_{k,\text{down}}}{\cos\theta'_{k,\text{up}} - \cos\theta'_{k,\text{down}}} = \frac{(N-1)^2}{2}\left(\cos\theta\left(u,p\right) - \cos\theta'_{k,\text{down}}\right)$$

and

$$P\left(o_u^{k+1} \mid p\right) = \frac{\cos\theta'_{k,\text{up}} - \cos\theta\left(u,p\right)}{\cos\theta'_{k,\text{up}} - \cos\theta'_{k,\text{down}}} = \frac{(N-1)^2}{2}\left(\cos\theta'_{k,\text{up}} - \cos\theta\left(u,p\right)\right)$$

since $\cos\theta'_{k,\text{up}} - \cos\theta'_{k,\text{down}} = \frac{2}{(N-1)^2}$, i.e. the total length of $I_{k,\text{sup}}$.

For $N = 2$ the probabilities are given by:

$$P\left(o_u^0 \mid p\right) = \frac{1}{2}\left(\cos\theta\left(u,p\right)+1\right) = \cos^2\left(\frac{\theta\left(u,p\right)}{2}\right)$$
$$P\left(o_u^1 \mid p\right) = \frac{1}{2}\left(1-\cos\theta\left(u,p\right)\right) = \sin^2\left(\frac{\theta\left(u,p\right)}{2}\right)$$

which are again the transition probabilities for a spin measurement on a quantum spin-$\frac{1}{2}$ particle.

In the other limit $N \to \infty$ the system has a classical structure, i.e. its set of properties is represented by a Boolean lattice. The orthogonal of a state p is given by the set-theoretic complement $\{p\}^C$, and the eigenstate set of property $eig_{e_p^{N\to\infty}}\left(\left\{o_p^0\right\}^C\right)$ is $\{p\}^C$. Therefore the eigenclosure coincides with the identity map such that each subset $A \in P\left(\Sigma\right)$ corresponds to an eigenclosed set. Hence the set of properties is isomorphic with $P\left(\Sigma\right)$ and it is not difficult to show that this is indeed a Boolean (i.e. distributive) lattice. In the classical limit each measurement can be regarded as an *observation* only such that the state transition due to measurement is zero.

10.2. *Orthogonality relations of the asymmetric N-model*

10.2.1. *Aerts orthogonality relation*

For the N-model, two states p and q are Aerts orthogonal iff there exists an experiment such that p and q are in different eigenstate sets. A necessary condition for two states to be Aerts orthogonal is thus that they are separated by an angle greater than the smallest superposition angle, i.e. the smallest angle between two consecutive eigenstate sets. These superposition angles can be explicitly calculated, as a function of k, i.e. the index of the superposition zone. We find that the minimal superposition angle is obtained for $k = \frac{N}{2}$ for even N, and $k = \frac{N-1}{2}$ for odd N. Let us consider the two different cases for odd and even N. The smallest superposition angle, denoted as $\theta_{N,e}$, for even N is given by:

$$\theta_{N,e} = 2\arcsin\left(\frac{1}{\left(N-1\right)^2}\right) \tag{11}$$

Hence the necessary condition for $p\perp_A q$ is that $\theta\left(p,q\right) \geq \theta_{N,e}$. It can be shown that this necessary condition is also sufficient,[33] so the following holds:

$$\text{For even } N: p\perp_A q \Leftrightarrow \theta\left(p,q\right) \geq \theta_{N,e}.$$

For odd N, an analogous reasoning can be followed. The smallest superposition angle is given by:

$$\theta_{N,o} = \arcsin\left(\frac{N}{(N-1)^2}\right) - \arcsin\left(\frac{N-2}{(N-1)^2}\right) \tag{12}$$

Again, this is not only a necessary condition, but a sufficient condition as well since for any two states p and q for which $\theta(p,q) \geq \theta_{N,o}$ holds, a measurement e_u^N can be found such that p and q are in two different eigenstate sets, which implies that the states p and q are Aerts orthogonal. To summarize, the following holds:

$$\text{For odd } N : p \perp_A q \Leftrightarrow \theta(p,q) \geq \theta_{N,o}.$$

Let us denote by $\theta_{N,\perp_A}$ the angle given by (11) for even N and (12) for odd N. Hence two states p and q are Aerts orthogonal iff $\theta(p,q) \geq \theta_{N,\perp_A}$:

$$p \perp_A q \Leftrightarrow \theta(p,q) \geq \theta_{N,\perp_A} \text{ defined by}$$

$$\begin{cases} \text{for even } N : \theta_{N,\perp_A} = 2\arcsin\left(\frac{1}{(N-1)^2}\right) \\ \text{for odd } N : \theta_{N,\perp_A} = \arcsin\left(\frac{N}{(N-1)^2}\right) - \arcsin\left(\frac{N-2}{(N-1)^2}\right) \end{cases}$$

10.2.2. *Durt orthogonality relation*

We recall that two states p and q are Durt orthogonal iff there exists at least one experiment with p or q as one of its final states, and with measurement direction in the plane of p and q, such that this experiment does not induce a transition from one of the states to the other. The set of final eigenstates of the N-model does not only contain the two endpoints of the measurement axis, but also the 'eigencircles' $C_k, k = 1, \ldots, N-2$, for which $\cos\theta_k = 1 - \frac{2k}{N-1}$.

Let us consider two states p and q. By definition we only take into account the measurements such that the measurement direction lies in the plane defined by p and q, and such that q is a possible final state. Let us study when we can find at least one experiment which does not induce a state transition from p to q. It can be shown[33] that for odd N the Durt orthogonality angle is given by:

$$\theta_{N,\perp_D} = \arcsin\left(\frac{N}{(N-1)^2}\right) \tag{13}$$

and for even N by:

$$\theta_{N,\perp_D} = \arcsin\left(\frac{1}{(N-1)^2}\right) + \arcsin\left(\frac{1}{N-1}\right) \qquad (14)$$

Combining the results for odd and even N, the following holds:

$p\perp_D q \Leftrightarrow \theta(p,q) \geq \theta_{N,\perp_D}$ defined by

$$\begin{cases} \text{for even } N: \theta_{N,\perp_D} = \arcsin\left(\frac{1}{N-1}\right) + \arcsin\left(\frac{1}{(N-1)^2}\right) \\ \text{for odd } N: \theta_{N,\perp_D} = \arcsin\left(\frac{N}{(N-1)^2}\right) \end{cases}$$

10.3. *The orthocomplementation of the N-model*

10.3.1. *Inclusions between the Aerts and Durt orthoclosure structures*

It can be shown easily for $2 \neq N \neq \infty$ that $\theta_{N,\perp_A} < \theta_{N,\perp_D}$ such that $p^{\perp_D} \subsetneq p^{\perp_A}, \forall p \in \Sigma$. Since the state orthogonals generate the orthoclosed sets by intersection, i.e. $F \in \mathcal{F}_\perp \Leftrightarrow F = \bigcap_{q\in F^\perp} q^\perp$, it follows immediately that the strict inclusion $\mathcal{F}_{\perp_D} \subsetneq \mathcal{F}_{\perp_A}$ holds. Next, for the N-model, the 'largest' non-trivial eigenstate set for an experiment e_u^N is given by $eig_{e_u^N}\left(\left\{o_u^0\right\}^C\right)$. This is a spherical cap $cap\left(-u, \pi - \theta_{eig}\right)$ with θ_{eig} given by $\cos\theta_{eig} = 1 - \frac{N}{(N-1)^2}$ (see Fig. 9).

For $2 \neq N \neq \infty$ one finds that $\theta_{eig} > \theta_{N,\perp_D} > \theta_{N,\perp_A}$, which means that the state orthogonals $p^{\perp_A}$ and $p^{\perp_D}$ are not contained in a non-trivial eigenstate set, i.e. they are not eigenclosed. Therefore neither of the two inclusions $\mathcal{F}_{\perp_A} \subseteq \mathcal{F}_{eig}$ and $\mathcal{F}_{\perp_D} \subseteq \mathcal{F}_{eig}$ can hold. The possible reverse inclusion $\mathcal{F}_{eig} \subseteq \mathcal{F}_\perp$ is discussed in the next subsection.

10.3.2. *Problem of orthocomplementation on the set of properties*

It was shown in Ref. 20 how to define an orthogonality relation on the set of states of the ϵ-model, such that the lattice of properties is orthocomplemented under this orthogonality relation. Let us now show under which conditions it is possible to define an orthogonality relation $\perp_N$ on the set of states of the N-model such that $\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$, i.e. such that the set of eigenstate sets is orthocomplemented under this orthogonality relation. Let us assume first that the orthogonality relation $\perp_N$ depends on the angle between the states only, i.e. there exists $\theta_{\perp_N}$ such that $p \perp_N q \Leftrightarrow \theta(p,q) \geq \theta_{\perp_N}$. Let us formulate the necessary and sufficient conditions on $\theta_{\perp_N}$ such that

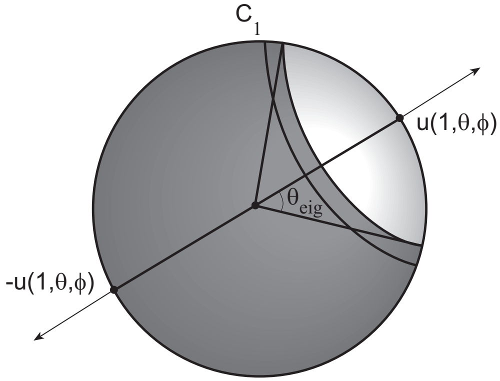

Fig. 9. The 'largest' non-trivial eigenstate set for an experiment e_u^N, namely $eig_{e_u^N}\left(\left\{o_u^0\right\}^C\right)$, which is the spherical cap $cap\,(-u, \pi - \theta_{eig})$ with $\cos\theta_{eig} = 1 - \frac{N}{(N-1)^2}$.

$\mathcal{F}_{eig} = \mathcal{F}_{\perp_N}$ holds and sketch briefly some of the proofs to obtain the main results.

Theorem 10.1. *$\mathcal{F}_{\perp_N} \subseteq \mathcal{F}_{eig}$ iff $\theta_{\perp_N} \geq \theta_{eig}$.*

Proof. For $\mathcal{F}_{\perp_N} \subseteq \mathcal{F}_{eig}$ to hold, it is necessary and sufficient that each state orthogonal is eigenclosed. Since a state orthogonal is given by $cap\,(-p, \pi - \theta_{\perp_N})$ the opening angle of the 'largest' non-trivial eigenstate set (i.e. $eig_{e_p^N}\left(\left\{o_p^0\right\}^C\right)$) should be at least as large as $\pi - \theta_{\perp_N}$. Since $eig_{e_p^N}\left(\left\{o_p^0\right\}^C\right) = cap\,(-p, \pi - \theta_{eig})$ it is necessary that $\theta_{\perp_N} \geq \theta_{eig}$. This is also a sufficient condition: if $\theta_{\perp_N} \geq \theta_{eig}$ then each state orthogonal can be written as an intersection of eigenstate sets. This can be seen by considering the complement of the state orthogonal, i.e. $\left(p^{\perp_N}\right)^C$. This is an open spherical cap $cap^\circ\,(p, \theta_{\perp_N})$. Since $\theta_{\perp_N} \geq \theta_{eig}$ this set can be covered by open caps $cap^\circ\,(q, \theta_{eig})$ such that $cap^\circ\,(p, \theta_{\perp_N}) = \underset{q\in Q}{\cup} cap^\circ\,(q, \theta_{eig})$ for some suitable set Q. Therefore,

$$\begin{aligned} p^{\perp_N} &= \left(\left(p^{\perp_N}\right)^C\right)^C = \left(cap^\circ\,(p, \theta_{\perp_N})\right)^C \\ &= \left(\underset{Q}{\cup} cap^\circ\,(q, \theta_{eig})\right)^C = \underset{Q}{\cap} cap^\circ\,(q, \theta_{eig})^C \\ &= \underset{Q}{\cap} cap\,(-q, \pi - \theta_{eig}) = \underset{Q}{\cap} eig_{e_q^N}\left(\left\{o_q^0\right\}^C\right) \end{aligned}$$

which shows that $p^{\perp_N}$ is eigenclosed. □

For $N > 2$, let us define θ_m as the smallest angular distance between the consecutive eigenstate sets $eig_{e_u^N}\left(\left\{o_u^l \mid l = 0, \ldots, k-1\right\}\right)$ and $eig_{e_u^N}\left(\left\{o_u^l \mid l = k+1, \ldots, N-1\right\}\right)$. It can be shown that θ_m is the following function of N:

$$\theta_m\,(N \text{ odd}) = 2\theta_{N,\perp_D} = 2\arcsin\left(\frac{N}{(N-1)^2}\right) \tag{15}$$

$$\theta_m\,(N \text{ even}) = \arcsin\left(\frac{1}{(N-1)^2}\right) + \arcsin\left(\frac{2N-1}{(N-1)^2}\right) \tag{16}$$

The following theorem holds, see Fig. 10:

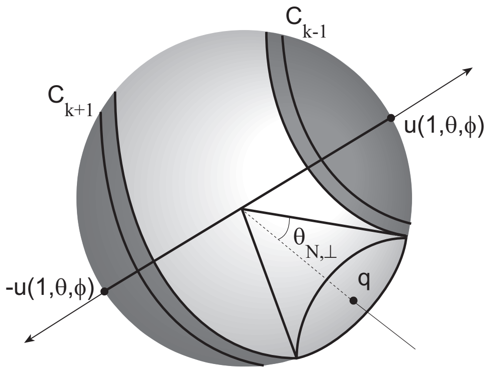

Fig. 10. Problem of orthocomplementation on the set of properties: Illustration why $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ is necessary for $\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$.

Theorem 10.2. *$\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$ iff $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ and $\theta_{\perp_N} \leq \theta_{eig}$.*

Proof. If $\mathcal{F}_{eig} \subseteq \mathcal{F}_{\perp_N}$ then it is necessary that each eigenstate set can be written as an intersection of state orthogonals. More concretely, it is necessary and sufficient that the eigenstate sets $eig_e\left(O_e \backslash \left\{1 - \frac{2k}{N-1}\right\}\right), k = 0, \ldots, N-1$ are orthoclosed, since they generate the lattice of properties $\mathcal{F}_{eig}$. Considering that the eigenstate set $eig_e\left(O_e \backslash \{1\}\right)$ should be contained within a state orthogonal, one obtains immediately a necessary condition $\theta_{\perp_N} \leq \theta_{eig}$. Next, following the same procedure as in the proof of the previous theorem, we can show that the set $eig_e\left(O_e \backslash \left\{1 - \frac{2k}{N-1}\right\}\right), k = 1, \ldots, N-1$ is orthoclosed iff $\theta_{\perp_N} \leq \frac{\theta_m}{2}$. Indeed, in such a case the set-theoretic complement of $eig_e\left(O_e \backslash \left\{1-\frac{2k}{N-1}\right\}\right)$ can be written as a union of open spherical caps $cap^\circ\left(q,\theta_{\perp_N}\right)$:

$$\left(eig_e\left(O_e \backslash \left\{1-\frac{2k}{N-1}\right\}\right)\right)^C = \underset{Q}{\cup} cap^\circ\left(q,\theta_{\perp_N}\right).$$

After taking the set-theoretic complement a second time, we obtain:

$$eig_e\left(O_e \backslash \left\{1-\frac{2k}{N-1}\right\}\right) = \underset{Q}{\cap} cap\left(-q,\pi-\theta_{\perp_N}\right)$$

which means that $eig_e\left(O_e \backslash \left\{1-\frac{2k}{N-1}\right\}\right)$ is orthoclosed. This procedure can be repeated for all values of k, from which the necessary condition follows: $\theta_{\perp_N} \leq \frac{\theta_m}{2}$. Following a similar reasoning one can show that $\theta_{\perp_N} \leq \frac{\theta_m}{2}$ is also a sufficient condition for $eig_e\left(O_e \backslash \left\{1-\frac{2k}{N-1}\right\}\right), k=1,\ldots,N-1$ to be orthoclosed. □

Combining the last two theorems, the necessary and sufficient condition on $\theta_{\perp_N}$ to obtain $\mathcal{F}_{eig}=\mathcal{F}_{\perp_N}$ is given by:

Theorem 10.3. *$\mathcal{F}_{eig}=\mathcal{F}_{\perp_N}$ iff $\theta_{\perp_N}=\theta_{eig}$ and $\theta_{\perp_N} \leq \frac{\theta_m}{2}$.*

Using expressions (15) and (16) for θ_m, one can check whether $\theta_{eig} \leq \frac{\theta_m}{2}$ — i.e. $\mathcal{F}_{eig}=\mathcal{F}_{\perp_N}$— is possible *at all*. It turns out that for all finite values of $N>2$ the inequality $\theta_{eig} \leq \frac{\theta_m}{2}$ is violated. This shows that there cannot exist an orthogonality relation $\perp_N$ such that $\mathcal{F}_{eig}=\mathcal{F}_{\perp_N}$.

Due to the latter theorem, the asymmetric N-model with $N>2$ has no orthocomplementation on the set of properties which can be defined by means of an orthogonality relation on the set of states such that $p \perp q \Leftrightarrow \theta(p,q) \geq \theta_\perp$. Of course, one could still imagine some other kind of 'non-standard' way of defining an orthocomplementation on the set of properties. For instance, one could consider 'strange' orthogonality relations which do not simply depend on the angle between the two states. However, in the next subsection we show that if one assumes a kind of 'minimal state-invariance' then also this kind of orthocomplementation is in fact impossible for the asymmetric N-model.

10.4. *Problem of state-invariant orthocomplementation*

Let us assume that the N-model would allow an orthocomplementation for $N>2$, denoted by $'$. One can show that the set of states of the N-model is atomistic, such that each property a can be written as the join of its property states: $a = \vee_{p<a} s_p$. Also, the axiom of state determination holds, such that in fact $\kappa(s_p) = \{p\}$. Hence one can abbreviate the previous expression as $a = \vee_{p<a} p$. Then analogously $a' = \vee_{p<a'} p$, such that, using De Morgan laws, one has $a = a'' = (\vee_{p<a'} p)' = \wedge_{p<a'} p'$, which shows that the properties are generated by taking meets of (property) state orthocomplements (similar as the orthoclosure is generated by state orthogonals). On the other hand, the eigenclosure structure is generated by the eigenstate sets $eig_e\left(O_e \setminus \left\{1 - \frac{2k}{N-1}\right\}\right)$, which — as we have seen — do not have a single family of generators, e.g., the eigenstate set $eig_e\left(O_e \setminus \left\{1 - \frac{2k}{N-1}\right\}\right), k = \frac{N}{2}$ (or $k = \frac{N-1}{2}$ for odd N) cannot be generated by making intersections of eigenstate sets of the form $eig_e(O_e \setminus \{1\})$ and vice versa. This means that there exist at least two 'incompatible' generating subsets for the eigenclosure structure of the asymmetric N-model. Therefore an orthocomplemented structure which is generated by the 'single' family of generators $\{p' \mid p \in \Sigma\}$ cannot reproduce the eigenclosure structure. Let us introduce the following definition to state this result in more formal wording:

Definition 10.1 (State-invariant orthocomplementation). Consider a **SPS** $(\Sigma, \mathcal{L}, \xi)$ for which axioms of property determination, property completeness and atomicity are satisfied. An orthocomplementation $'$ on $\mathcal{L}$ is called state-invariant iff $\forall p, q \in \Sigma, p'$ is congruent with q', i.e. there exists an isometry between p' and q'.

Then it follows that such state-invariant orthocomplementation is generated by a single family of congruent sets, the (property) state orthocomplements. It immediately follows that for $N > 2$ the N-model does not allow such a state-invariant orthocomplementation on its set of properties, since it requires at least two generating families, namely the eigenstate set $eig_e(O_e \setminus \{1\})$, which corresponds with the spherical cap $cap(-p, \pi - \theta_{eig})$ and the eigenstate set $eig_e\left(O_e \setminus \left\{1 - \frac{2k}{N-1}\right\}\right), k = \frac{N-1}{2}$ for odd N (respectively $k = \frac{N}{2}$ for even N).

11. Conclusions

We have presented a model in which the change of state of the system induced by interaction with the measurement context is controlled by a parameter N reflecting the number of outcomes (and corresponding eigenstate sets). In the limit $N = 2$ the system reduces to a model for the spin measurements on a quantum spin-$\frac{1}{2}$ particle. In the other limit $N \to \infty$

the system has a classical structure, i.e. its set of properties is represented by a Boolean lattice. For intermediate values of the parameter, the change of state under measurement is neither 'maximal' (i.e. quantum) nor 'zero' (i.e. classical), and the system does not fit within a quantum Hilbert space nor a classical phase space description. To deal with these issues in a rigorous mathematical way, we have constructed the **SPS** of these models for different values of the contextuality parameter N and studied the problem of orthocomplementation on the set of properties. We have put forward an 'asymmetric' modification of the sphere model for which it is impossible to define an orthocomplementation on the set of properties by a suitable orthogonality relation on the set of states. Therefore the validity of the axiom of orthocomplementation as a general and 'trivial valid' axiom is at stake.

Another interesting feature (not discussed in detail here) for the intermediate situations of the asymmetric N-model is that the probability of a state transition in general depends not only on the angular distance between the two states but also on the measurement context which induces the state transition. In the axiomatic foundations of quantum theory, the theorem of Gleason dictates the uniqueness of the transition probability function between two states in Hilbert space.[35] Moreover, this probability function only depends on (the angle in Hilbert space between) the initial and the final state. Therefore Gleason's theorem does not apply to the asymmetric N-model, implying that the probability distribution over the set of outcomes cannot be derived from the structure of the set of properties in the same way as for quantum entities. Therefore our model also sheds new light on Gleason's theorem and suggests that the transition probability maybe is not a secondary concept which can be derived from other more basic concepts such as states and properties as in Gleason's theorem, but instead should be regarded as a primitive concept by its own right.

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References

1. Aerts, D. (1981), The one and the many: towards a unification of the quantum and the classical description of one and many physical entities. Doctoral dissertation, Vrije Universiteit Brussel (VUB).
2. Aerts, D., Colebunders, E., Van der Voorde, A. and Van Steirteghem, B.

(1999), State property systems and closure spaces: a study of categorical equivalence. *International Journal of Theoretical Physics* **38**, 359–385.
3. Aerts, D. (1986), A possible explanation for the probabilities of quantum mechanics. *Journal of Mathematical Physics* **27**, 202–210.
4. Jauch, J.M. (1968), *Foundations of Quantum Mechanics.* Reading, Mass.: Addison–Wesley.
5. Piron, C. (1964), Axiomatique quantique. *Helvetica Physica Acta* **37**, 439–468.
6. Piron, C. (1976), *Foundations of quantum physics.* Reading, Mass.: W. A. Benjamin.
7. Piron, C. (1989), Recent developments on quantum physics. *Helvetica Physica Acta* **62**, 82–90.
8. Piron, C. (1990), *Mécanique quantique, bases et applications.* Presses Polytechniques et Universitaire Romandes.
9. Aerts, D. (1982), Description of many separated physical entities without the paradoxes encountered in quantum mechanics. *Foundations of Physics* **12** (12), 1131–1170.
10. Aerts, D. (1983), Classical theories and non-classical theories as a special case of a more general theory. *Journal of Mathematical Physics* **24**, 2441–2453.
11. Aerts, D. (1983), The description of one and many physical systems. In C. Gruber (Ed.), *Foundations of Quantum Mechanics* (pp. 63–148). Lausanne: AVCP.
12. Keller, H. (1980), Ein nicht-klassischer Hilbertscher Raum. *Mathematische Zeitschrift* **172** (1), 41–49.
13. Solèr, M.P. (1995), Characterization of Hilbert spaces by orthomodular spaces. *Communications in Algebra* **23**, 219–243.
14. Aerts, D. and Van Steirteghem, B. (2000), Quantum axiomatics and a theorem of M.P. Solèr. *International Journal of Theoretical Physics* **39**, 497–502.
15. Aerts, D. and Valckenborgh, F. (2002), The linearity of quantum mechanics at stake: the description of separated quantum entities. In D. Aerts et al. (Eds.), *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Probability and Axiomatics.* Singapore: World Scientific, 20–46.
16. Aerts, D. and Valckenborgh, F. (2004), Failure of standard quantum mechanics for the description of compound quantum entities. *International Journal of Theoretical Physics* **43**, 251–264.
17. Aerts, D. (1994), Quantum structures, separated physical entities and probability. *Foundations of Physics* **24** (9), 1227–1259.
18. Birkhoff, G. (1967), *Lattice Theory.* Providence, Rhode Island: AMS.
19. Birkhoff, G. and von Neumann, J. (1936), The logic of quantum mechanics. *Annals of Mathematics* **37** (4), 823–843.
20. Durt, T. (1996), From quantum to classical: a toy model. Doctoral dissertation, Vrije Universiteit Brussel (VUB).
21. Czachor, M. (1992), On classical models of spin. *Foundations of Physics Letters* **5**, 249–264.
22. Kolmogorov, A.N. (1933), *Grundbegriffe der Wahrscheinlichkeitsrechnung.* Berlin: Springer.

23. Hughes, R.I.G. (1989), *The Structure and Interpretation of Quantum Mechanics*. Cambridge: Harvard University Press.
24. Belinfante, F.J. (1973), *A Survey of Hidden Variable Theories*. Oxford: Pergamon.
25. Jauch, J.M. and Piron, C. (1963), Can hidden variables be excluded in quantum mechanics? *Helvetica Physica Acta* **36**, 827–837.
26. Kochen, S. and Specker, E.P. (1967), The problem of hidden variables in quantum mechanics. *Journal of Mathematics and Mechanics* **17**, 59–87. Reprinted in: C. A. Hooker (Ed.), *The Logico-Algebraic Approach to Quantum Mechanics, II.* Reidel Publishing Company, 293–328.
27. Bohm, D. (1952), A suggested interpretation of quantum theory in terms of hidden variables, I and II. *Physical Review* **85**, 166–193.
28. Gudder, S.P. (1970), On hidden-variable theories. *Journal of Mathematical Physics* **11** (2), 431–436.
29. Aerts, D. (1985), A possible explanation for the probabilities of quantum mechanics and a macroscopic situation that violates Bell inequalities. In P. Mittelstaedt and E. W. Stachow (Eds.), *Recent Developments in Quantum Logic,* Grundlagen der Exacten Naturwissenschaften **6**, Wissenschaftverlag. Mannheim: Bibliographisches Institut, 235–251.
30. Aerts, D. and Durt, T. (1994), Quantum, classical and intermediate, an illustrative example. *Foundations of Physics* **24**, 1353–1369.
31. Aerts, D. and Durt, T. (1994), Quantum, classical and intermediate: A measurement model. In K. V. Laurikainen, C. Montonen and K. Sunnaborg (Eds.), *Symposium on the Foundations of Modern Physics 1994,* Helsinki, Finland. Gives Sur Yvettes, France: Editions Frontieres.
32. Aerts, D., Aerts, S., Coecke, B., D'Hooghe, B., Durt T. and Valckenborgh, F. (1997), A model with varying fluctuations in the measurement context. In M. Ferrero and A. van der Merwe (Eds.), *New Developments on Fundamental Problems in Quantum Physics*. Kluwer, 7–9.
33. D'Hooghe, B. (2000), From quantum to classical: A study of the effect of varying fluctuations in the measurement context and state transitions due to experiments. Doctoral dissertation, Vrije Universiteit Brussel (VUB).
34. Durt, T. and D'Hooghe, B. (2002), The classical limit of the lattice-theoretical orthocomplementation in the framework of the hidden-measurement approach. In D. Aerts, M. Czachor and T. Durt (Eds.), *Probing the Structure of Quantum Mechanics, Nonlinearity, Nonlocality, Computation and Axiomatics*. Singapore: World Scientific, 111–129.
35. Gleason, A.M. (1957), Measures on the closed subspaces of a Hilbert space, *Journal of Mathematics and Mechanics* **6**, 885–893.

THE DELEUZIAN CONCEPT OF STRUCTURE AND QUANTUM MECHANICS

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Gilles Deleuze wanted a philosophy of nature in a pre-kantian almost archaic sense. A central concept in his philosophy is ‘multiplicity’. Although the concept is philosophical through and through, it has roots in the mathematical notion of manifold, specifically the state spaces for dynamical systems exhibiting non-linear behaviour. Deleuze was attracted to such mathematical structures because he believed they indicated a break with the dogmatic image of thought (the kind of thought that constrains itself into producing representations of reality conceived as particular things with strict borders, behaving and interacting according to invariant covering laws within space). However, even though it is true that a phase space representation of a physical entity is not a typical materialist picture of reality, it derives from a normal Euclidean representation, and can in principle be reduced to it. We want to argue that the real break happens with the quantum state space, and that Deleuze's typical description of a multiplicity fits even better with the quantum state space.

Keywords: Quantum mechanics; Deleuze; quantum structures.

1. Introduction

The concept of *multiplicity*, which is central to Deleuze's philosophy of difference, derives, in part, from the notion of manifold, itself basically a concept of mathematical structure applied in physics. Some mathematical structures of physics capture space-time (M^{cs}), others are first and foremost state spaces (M^{ss}). Deleuze is especially interested in the latter: clearly he believes the dynamical structure of non-linear, and especially chaotic, M^{ss} tells us something about multiplicities.

The opposite of an *ontology of difference* would be one of substance, where, for example, being is a Newtonian or Euclidean M^{cs} — a spatial container enveloping particulars that behave and interact according to invariant covering laws —, but which also lies behind the exotic M^{ss} of twentieth century physics. However, it is not just Euclidean space (or some other M^{cs}) that is recalcitrant towards multiplicity; mathematics as such is, at first sight, a major obstacle. Mathematical structures are invariants, typically with the following characteristics: (i) they are propositional, (ii) they are axiomatizable (at a determinate level), (iii) they form a homogeneous system;[a] furthermore they are characterized by extrinsic variables, and intrinsic constants (which, I suppose, means that there there is a clear distinction between identity and difference, and that the structure remains invariant under transformations); not only that: set theory pushes us to think in terms of individuals, particulars (yes, set theory underlies most of mathematics); and on a more mundane level: numbers carve up the world, make it an order of definite things. In short, not just M^{cs}, but all of the concepts of mathematics reveal, at first sight, a tendency to prefer "being" over "becoming". Why would a Deleuze believe that the mathematics of certain M^{ss} subvert this? Why would deleuzian believe he can, through them, turn mathematics on its head, "de–substantialize" it, define being *as* becoming, instead of opposing them.

Some M^{ss} are promising candidates to express multiplicity, because a multiplicity is supposed to subtract, in principle, from classical ontology; admittedly, the least one can say about a $6n$–dimensional "space" is that it breaks with our familiar representation of things, Euclidean reality. Furthermore, the rupture in the edifice was, according to Deleuze, already made by the concept of limit; moreover basic operations and structures from calculus and differential topology function so well within an ontology of pure difference that Manuel de Landa could describe a multiplicity as a "nested set of vector fields related to each other by symmetry–breaking bifurcations, together with the distributions of attractors which define each of its embedded levels" [2, p. 32] – a typical M^{ss}. Both deterministic and indeterministic physical systems can be described with state spaces: Deleuzians are, of course, interested in the latter: they *force* one to turn away from the equations that rule the state transitions (the covering law), and instead concentrate on the structure of the set of states M^{ss}.

We have no qualms with this. But we do believe, however, that one can get even closer to the desired notion of multiplicity by taking a different route. All of the above is from a dynamical point of view. But one can classify state spaces by looking at the structure of the state space from a non-dynamical, specifically logico-mathematical perspective, point of view. This

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[a]See for example Ref. 1.

gives us two kinds: those that do not allow superposition states (M^{ss}_C) and state spaces that describe quantum entities, in which superposition states proliferate (M^{ss}_Q). (From now on *classical* will mean absence of superposition states, and *non-classical* the structural possibility of superposition states.) We believe that Deleuzians have not payed enough attention to the following fact: no matter how indeterministic the behaviour of a classical physical system is — i.e., no matter how rich and intricate M^{ss} becomes: a nested set of vector fields related to each other by symmetry–breaking bifurcations, together with the distributions of attractors which define each of its embedded levels etc. —, in principle that system can again be represented in a Euclidean R^3–space (or some other M^{cs}). In general, any kind of state space for an entity that does not allow for superposition states in the sense of quantum mechanics (QM), is describing something that can in principle also be described using the more familiar R^3–space. This is not the case for QM in the usual Dirac–Von Neuman Hilbert space formulation. If the latter has a distinguishing characteristic, it is its defiance to representation. Which is already apparent when physicists tell us that QM is "a very mathematical theory".[b] In Newtonian physics, despite the extensive use of mathematics, a correspondence is preserved between those mathematical concepts and everyday notions, familiar Euclidean reality, substance and identity; in QM the intervention of the central mathematical structure (Hilbert space) is much more direct and enigmatic: things like the uncertainty relations, the jump of the state vector and the non-separability of compound entities in a superposition state, are impossible to represent in everyday physical terms of Euclidean R^3.

From a dynamical perspective quantum mechanics (QM) is anything but expressive of a multiplicity (because its dynamic equation is a fully deterministic covering law), while certain M^{ss}_C — the ones where one is forced to consider the structure of the state space instead of the covering law if one is going to describe the physical system — have many of their characteristics coincide with those of a multiplicity; from a logico-mathematical (non-dynamical) point of view, it is the other way around. (See the table.)

It is the logico-mathematical point of view that we want to develop in this paper.

In Section 2 we define the concept of multiplicity. Next, suppose we describe a M^{ss} as $\langle A, R_1, \ldots, R_n \rangle$ where A is the set of states and the R_i are the mathematical structures defined on A. In Section 3 we argue two

[b]See for example [3, p. 5].

	M_C^{ss}	M_Q^{ss}
Dynamical perspective	ontology of difference (non-linear dynamics)	classical ontology
Logico-algebraic perspective	classical ontology	ontology of difference (linear algebra)

things: first, not only does the classical nature of A in any M_C^{ss} subvert a complete Deleuzian construal of M_C^{ss}, also its indeterminism is basically more epistemological than ontological (and a multiplicity requires ontological indeterminism). In Section 4 we discuss QM, argue that the behaviour of the state vector in a Hilbert space shows how the Hilbert space instantiates those requirements that define a multiplicity, (more specifically, how, in a quantum state space, just the A in $\langle A, R_1, \ldots, R_n \rangle$ already realizes the requirements).

2. Multiplicities: Structures of Chaos

Classical ontology is the determination of being in terms of presence, unity, identity. Not just space, time, physical objects, but the very notions of of structure, set, number ... express an age-old idea: "That alone is truly real which abides unchanged"(Augustine in his Confessions, book vii, quoted in [4, p. 29]). By conceiving of being as difference one is to undercut classical ontology, subtract from self-identical, invariant, stable substance. To think is not to refer or to identify but to differ, to move within multiplicities. One grasps difference by divergence with oneself: its only invariant, the only thing that returns, is difference; repetition is the "differenciator of difference" [5, p. 74]. If classical thought obtains structure by insulating a set of elements against change, a multiplicity is the paradoxical notion of the structure of *pure* change. Which is not a night where all the cows are grey, deleuzians ensure us: being is highly and intricately structured, only it cannot be accessed if one fits it within the premises–conclusion structure of logic, the subject–predicate structure of language, the terms–relations structure of mathematics or marxian dialectics. "Being, or Time, is a multiplicity" [6, p. 85]: actuality is classical ontology (the actual is logically consistent and spatial), the past is virtuality, i.e., structures of pure difference or multiplicities, and the future is the return of difference. "There is no present that does not actualize the past. It is all of the past that is actualized at every moment. The past that is actualized exists" [7, p. 52].

Time is not determined by space (in the sense of being represented by a time line), nor does it accord to the consistency requirement of logic; it has neither identity nor logical consistency — "a chronic non-chronological time" [8, p. 129]. The past can never be recomposed with instants or intervals since this would be to negate its specific mode of being. "We might as well look for darkness beneath the light" [9, p. 181]. The future can never be the same as the past, because the past is not even identical with itself.

Very tricky therefore if one is going to list the characteristics that define a multiplicity. Nevertheless, here we go.

M(1) A multiplicity consists of varying variations. Variations are not variables. Variables have a steady identity within which variation happens; the only invariant of a multiplicity is difference.[c] In *Difference and Repetition* Deleuze uses the term series, which we will, for the sake of brevity, associate with variations. Series not only diverge internally, but also externally. What returns within series and between series is difference: "only differences are alike" [5, p.116].[d] Suppose we write a series $\ldots - E - E' - E'' - \ldots$ as $[E]$. A multiplicity will then consist of $[E], [\varepsilon], [e], \ldots$ These are "first-order" differences. There is also a second-order difference $\ldots - [E] - [\varepsilon] - [e] - \ldots$; a difference traversing the series: something that is continually changing jumps from one series into another. We will denote it with $[M]$.[e] So we can write our multiplicity as

$$\begin{aligned} \ldots - E - E' - E'' - \ldots \\ \ldots - \varepsilon - \varepsilon' - \varepsilon'' - \ldots \\ \ldots - e - e' - e'' - \ldots \\ \ldots - [E] - [\varepsilon] - [e] - \ldots \end{aligned}$$

which we will refer to as M. Another way to think of it would be that the $\ldots$, $[E]$, $[\varepsilon]$, $[e]$, $\ldots$ are different levels of contraction of the whole past: each contraction repeats the whole of the past, but in radically, i.e., incomparably, new configurations.

M(2) The continual reciprocal determination of incompatible varying variations, makes the emergence of a stable element or individual within the virtual structure impossible. In a strict sense M has no elements. When

[c] We have to careful with the term variation, because Deleuze also uses variety as a synonym for multiplicity. We will only use it to refer to the "components" of multiplicities.
[d] It "has the strange power to affirm simultaneously fragments which do not constitute a whole in space, any more than they form a whole by succession within time. Time is precisely the transversal of all possible space, including the space of time" [10, p. 115].
[e] What Deleuze calls the dark precursor (le précurseur sombre) [5, p. 119].

Deleuze refers to an element of an M what he means is the cross roads of different series; an element *is* the relation between varying variations.

M(3) Multiplicities have no sensible form, no conceptual meaning, no practical purpose, no actual existence, and no founding or overarching identity. Since the most obvious (logically) stable identity is space, the virtual is not spatial, and M^{cs} exists "within" it, namely as the most contracted state of virtuality, as an aspect of actuality.

M(4) The communication between series results in resonances, a "mouvement forcé", an overflowing ("un débordement": [5, pp. 117-118]). Deleuze distinguishes between differen*t*iation (the structures of the virtual: multiplicities) and differen*c*iation (the structures of the actual: M^{cs}, spatialized time, ...), where the latter is an ultimate contraction within differentiation; actuality is an aspect of virtuality. The "ratio" different/ciation represents the moment of transition from virtuality into actuality. Normally one would like the possible to be in the image of the real; which, among other things, involves limitation: some possibilities are realized, others aren't, and one can see a clear, continuous identity between the possible and the actual. However, for Deleuze the virtual does not resemble the actual. First, position is an acquired characteristic, marking the difference in nature between the virtual and the actual. Spatial definiteness/uniqueness and logical consistency is what defines something as an individual. Individuals with properties, individuals–with–properties acting according to covering laws, individuals–with–properties acting according to covering laws within space .... — all of this is differenciation, actualized differentiation. Second, the "rules" of actualization are those of divergence and creation; actualization involves an indeterministic scattering. Actualizations of the differences of a multiplicity will diverge. Different/ciation, the actualization of the virtual, involves a 'leap' from a difference in degree, to a difference in kind [6, p. 62]. Note that there is always actualization going on.

M(5) Multiplicities cohere in a non-local way, i.e., the communications between the differential relations are not limited by space. Space is simply an aspect of actuality, differenciation just an aspect of differentiation.

A familiar definition of the limit of a function $f(x) = y$ is of course

$$lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

which is usually quickly rewritten in the intuitive notation dy/dx invented by Leibniz. h on the X-axis, and $f(x+h) - f(x)$ on the Y–axis, go to zero. For Deleuze the fraction dy/dx already realizes different aspects of a multiplicity. He writes: "the notion of limit founds a new static and purely ideal definition of continuity" [5, p. 172]. Static here means that the limit is not something that happens in classical time: the limit exists in a point and in that point a process is perpetually happening that is itself not happening on a time-line.[f] Furthermore it is ideal, that is, ideal in Deleuze's sense: virtual. dy/dx is not a number, but a process expressing a tendency. If numbers stand for actual reality, then the linear approximation to f at x is "the other side of the mirror" [5, p. 171], virtuality. Any further consideration based on calculus, like differentiable topology for example, can be construed as a further exploration of multiplicity. (It is quite remarkable that the mathematics developed to make that science possible which supports classical ontology, introduces the concepts that subvert classical ontology.)

3. Classical Dynamical State Spaces and Multiplicites

An M^{ss} is something like $\langle A, R_1, \ldots, R_n \rangle$. One starts with a set A; all mathematical structure needed to obtain a specific physical description, $\{R_i\}$, is defined *on* A: a topology, a coordinatisation, an affine connection, and so on.[g] Both M^{cs} and M^{ss}_C are manifolds: to obtain the former the elements of A are made into space-time points, when they are constructed as states we have M^{ss}. In the case of M^{ss}_C (classical state space) there is a dimension for each variable needed to specify the state of the system, which means that for a single physical system a point x in its phase space has six coordinates: $(x_1, x_2, x_3, p_1, p_2, p_3))$. For a detailed discussion of how the different aspects of the M^{ss}_C for physical systems with a non-linear dynamics can be made to work for an ontology of multiplicities, we refer the reader to Ref. 2. Here we are mostly interested in showing certain limitations of that approach.

Classical state spaces are limited in two ways: first, they accord to classical logic (while M does not accord to the law of the excluded third), and second, they are mostly epistemologically indeterministic (even though M is ontologically indeterministic).

3.1. *Classical logic*

It is in the domain of Newtonian physics that classical ontology finds an exemplary expression: "... the individuals are regarded as spatio-temporally

[f] This also called a "point–fold" [11, p. 14].
[g] For details concerning differentiable manifolds we refer the reader to for example [12, p. 118] and [13, p. 27].

localized, impenetrable and (relatively) indestructible substances. Complex objects can be analyzed ultimately into collections — it would be almost more illuminating to say sets — of non-complex or irreducible atomic objects related spatio-temporally to one another in a particular way" [14, p. 203]. We will look at: (1) the nature of classical logic, (2) the role of classical logic in M^{cs}, (3) the role of M^{cs} in M_C^{ss}; and explain why there is no irreversable break with classical ontology for a M_C^{ss}, even when it is characterized by the indeterminism of non-linearity and chaos.

For Deleuze difference is not the difference between this and that, rather difference is something in itself, it is a process; behind everything there is difference, behind difference there is nothing. Clearly one cannot define or describe ontological difference logically. Within classical ontology difference is defined in terms of classical negation: there is a difference between an individual a and an individual b, when a has a property F and b does not have this property; the world can be partitioned in two sets: the set with all the individuals that have F, and the complementary set of individuals that do not have F. Classical ontology distinguishes two things quite clearly: (a) on the one hand "the indeterminate, the indifferent, the undifferentiated", which is actually nothing, because it is no–thing, no determinate object; (b) and on the other hand "a difference already determined as negation" which actually determines definite objects [10, p. 552].[h] In mathematics, logic and physics, negation is generally classical, or, more precisely, Boolean. A classical negation has two aspects: the principle of non-contradiction (PNCa) and the principle of negation–completeness (PNCb).

PNCa In any theory T, if $A, \sim A \in T$, then A and $\sim A$ cannot both be true.

PNCb In any theory T, if $A, \sim A \in T$, then either A is true or $\sim A$ is true, but they cannot both be false.

The most obvious example of classical thought is M^{cs}, specifically Euclidean space: if something exists in a certain place, it cannot be in any other place (PNCa); if something exists it has to exist somewhere in space (PNCb). Any structure is modeled after real space. The obvious example is the way we represent time by places on a time-line. But one could also think of temperature measured by a thermometer: intensities become places on

[h]Deleuze reacts against the world view of classical physics and classical negation, *and* against Hegelian negation, which is obviously different from Boolean negation. We only focus on the discussion regarding classical negation.

a number continuum. M^{cs} is *the* basic element of classical ontology: something exists if is uniquely and determinately localized in M^{cs}.

With this we have answered questions (1) and (2). What about the third question: the link between the classicality of M^{cs} and the classicality of M_C^{ss}? In its most general form M_C^{ss} is just A, i.e., a set.[i] This might not seem so important, but it serves to indicate what M_C^{ss} does not have: "The structural transposition of this metaphysics [i.e., classical ontology] is the fact that no superposition principle can be operative for phase space or function space" [14, p. 184]. The statements 'M_C^{ss} basically has the mathematical structure of a set' and 'there are no superpositions of states in a M_C^{ss}' are two sides of the same coin.

So what does it mean to have a superposition state in a state space? When in a superposition state a quantum system can have a property for which PNCb is not true: neither the property nor the negation of that property is the case.[j] Having a certain position in space is a property. Let us denote the entity being in the position (a_1, a_2, a_3) (where the a_i are coordinates within a XYZ–frame) with $\sim A$. An entity in a superposition state can be neither in that position nor in any other position of space. In other words we have that neither A nor $\sim A$ are true. No matter what else one can say about a classical dynamical system, because in the end it still exists in a Euclidean space, it can never be in a state where for certain properties A, B, ..., neither A, B, ..., nor $\sim A$, $\sim B$, ... are the case. The state is always a definite and unique $x_i \in A$ in M_C^{ss} (a trajectory links points $x_i, x_{i+1}, \ldots \in A$).

In a Deleuzian approach towards state spaces it is emphasized that one has to clearly distinguish between the trajectories on the one hand, and the vector field, the attractors on the other hand; the latter describe tendencies inherent in singularities in the vector field.[k] It is the decision

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[i] "... a complex Hilbert space is a very specific mathematical structure, in a certain sense more specific than the state space of a classical system, which in its most general form is just a set" [3, p. 69].

[j] The relation between the states and the properties is determined by the relation between the state space and the property lattice, and this is a subtle and complicated story we cannot go into here. We refer to the literature, especially to the work of Piron, Aerts, Coecke and others, the so-called Geneva approach.

[k] In fact, the scientist does this too, and the philosopher is just following him here: "... we are often less concerned to find an explicit solution for a given set of initial conditions than to derive some general results about solutions" [15, p. 3], and to this end: "It is extraordinarily helpful to look at things geometrically" [15, p. 3]. This seems to be in tune with the modeltheoretic view on scientific theories in analytical philosophy of science: a theory is basically a family of models, which are defined in such a way that they make certain equations — the laws of the theory — true.[16]

of the philosopher to concentrate on $\{R_i\}$; and we agree that they can be construed in terms of tendencies, potentialities, that they come pretty close to the notion of virtuality/multiplicity, and yes, up to a point, given the right amount of interpretation, this constitutes a break with classical ontology. But from our non–dynamical, logico-mathematical, point of view, the structure of a classical phase space is classical: however uncertain it becomes where the state is in A, the state is always definitely somewhere; hence for all its properties we have either A or $\sim A$, B or $\sim B$, C or $\sim C$ . . .

3.2. *Epistemological indeterminism*

Although the subject of determinism is complicated, we will be fairly brief. One straightforward definition of determinism would be:[17] a physical system S is deterministic iff (i) from the theory describing S (the theory from which we obtain the state space description) one can also obtain a function that describes a relevant aspect of S in function of time; (ii) if f is determined for a specific t_0, then f is determined for all $t \geq t_0$. One can then distinguish ontological and epistemological determinism. *Epistemological indeterminism* is the fact that one cannot calculate what is determined by the theory. The Baker transformation is a much cited example.[1] *Ontological indeterminism* is the fact that the theory does not give us a precise, definite state and state transition; it is not just difficult or impossible to calculate, there simply is nothing one can calculate. If radical creation is the goal of the Deleuzian, if irreducible novelty is to be the essence of virtuality, then the minimal requirement for engaging a mathematical structure in the specification of a multiplicity should be ontological indeterminism; nothing less will do. A first remark one can make here is that just like the equation, the topology and the vector fields that determine the behaviour of the state, are completely determined by the theory. Therefore the behavior of the system is completely determined by the theory. Secondly (and more importantly), the fact that one cannot calculate the equation, the uncertainty about where the state of an entity is in A, does not change anything about the fact that the state always definitely is somewhere in A: neither PNCa nor PNCb are ever ontologically violated. Therefore the only thing we will ever have on the classical level for the kinds of cases Deleuzians consider, is epistemological indeterminism.

[1]See [18, p. 269].

The Deleuzian is only interested in the $\{R_i\}$; but he is also very aware that one needs philosophical interventions and constructions not belonging to the physical theories themselves, to single them out: "... Deleuze's description of the virtual topology goes beyond the resources available from those formal theories and may therefore seem much too speculative and complicated" [2, p. 109]. In QM on the other hand, there is no need to make this separation, since the state itself comes very close to what a multiplicity is supposed to be: "a quantum state is a network of potentialities" [19, p. 374]. Furthermore, the state space of QM is not obtained from a M^{cs}, and in principle it cannot be reconverted into one. M^{ss}_Q constitutes a definitive break with classical ontology because A is no longer simply a set; in other words the break is already the premise from which the philosophizing starts, not a result of external philosophical interventions or constructions.

4. The Quantum State Space and Multiplicities

We distinguished two kinds of mathematical structures used in physics: M^{cs} and M^{ss}. The former one can further classify into Euclidean and non–Euclidean (both classical ontology); M^{ss} are state spaces, and, if one takes into account only their instantaneous logico-mathematical structure (a non-dynamical point of view), they can be classified into: classical (M^{ss}_C) and non–classical or quantum (M^{ss}_Q, Hilbert spaces). Our aim was to show that M^{ss}_Q is, with respect to its logico-mathematical structure, not its dynamics, a Deleuzian multiplicity.

The proof is in the pudding: we will look at M(1)–M(5) one by one and cite the relevant aspects of QM that instantiate them.

M(1) M^{ss}_Q is basically a set on which is defined a linear algebra. In other words, it is a vector space. First, physical states of a physical system P are represented by state vectors: $|\psi\rangle$, $|\chi\rangle$, ... Typically a state vector $|\psi\rangle$ is a superposition: $\sum_i c_i \, |q_i\rangle$. Measurable properties are represented by linear operators: for example Q with discrete values, no degeneracy and a set of eigenstates $\{|q_i\rangle\}$; similarly for R, S ... each with their own set of eigenstates. P can be in superposition for Q and in an eigenstate for another variable, say T. In that case, P has a sharp value for T, say t_4. While a state vector necessarily makes a value of one variable actual, in this case t_4, it is in a superposition state for many other variables Q,R, S, ... and $|\psi\rangle$ has to be written as :

$$\ldots = \sum_i c_i \, |q_i\rangle = \sum_i c_i^{'} \, |r_i\rangle = \sum_i c_i^{''} \, |s_i\rangle = \ldots = |t_4\rangle = \ldots \qquad (1)$$

Even without any change of state (without the dynamical change of state and in the absence of a jump of the state vector), the state vector can be decomposed into an endless number of orthonormal bases, — each of which decisively does and does not capture the complete reality of the physical system.[m] A big problem for classical ontology; not when one thinks of superposition as a variation that constitutes a multiplicity.[n] Remember that a multiplicity not only consists of variations $[e]$, $[\epsilon]$, ... , but that it is "held together" by a variation $[M]$ that traverses through $[e]$, $[\epsilon]$, ... The state vector $|\psi\rangle$ is $[M]$, the orthonormal bases $\{|q_i\rangle\}$, $\{|r_i\rangle\}$, ... correspond to $[e]$, $[\epsilon]$, ... In formula (1) we see how the state vector is a linear combination of each orthonormal base, without *being* any of the vectors in the base specifically, and neither being none of them, nor being all of them at once: "electrons are not black and not white and not both and not neither" [21, p. 38].[o] M diverges with itself, constantly, intrinsically,variations diverge with respect to each other? Similarly, in M_Q^{ss} the superpositions $\sum_i c_i |q_i\rangle$, $\sum_i c_i^{'} |r_i\rangle$, ... are, for the most part, and most of the time, incompatible among each other: "there must necessarily be a continuous infinity of mutually incompatible complete measurable properties" [21, p. 41]. In a multiplicity there is a maximal level of contraction, actuality, — not another level of reality, but just one specific level of contraction of the multiplicity, intrinsic to any multiplicity. In the same way, even though the state vector $|\psi\rangle$ can be infinitely spectrally decomposed, at the same time, P "actualizes" one orthonormal base, i.e., an eigenvector for the variable T is sharp, certain, classical (in formula (1) $|t_4\rangle$).

M(2) A multiplicity is multiple without being constituted by individuals, implicating individuals or containing individuals; classically, when one thinks of multiple entities or a composite entity, one tends to imagine collections of things — billiard balls, planets ... —, uniquely and determinately

[m] "The procedure of expressing a state as the result of superposition of a number of other states is a mathematical procedure that is *always* permissible, independent of any reference to physical conditions ... Whether it is useful in any particular case, though, depends on the special physical conditions of the problem under consideration" [20, p. 12] (our italics).

[n] "The superposition principle might lead to identifications with other classical phenomena. The fact that QM is sometimes called wave mechanics is a consequence of this. It is important to remember, however, that the superposition that occurs in quantum mechanics is of an essentially different nature from any occurring in the classical theory, as is shown by the fact that the quantum superposition principle demands indeterminacy in the results of observations in order to be capable of a sensible physical interpretation. The analogies are thus liable to be misleading" [20, p. 14].

[o] Obviously the color talk refers to spin properties, and not actual colors!

localized individuals that one can distinguish clearly and unambiguously. This is reflected by the fact that one can simply count them and order them. (a) In QM countability is something of a problem.[p] For a run of the mill quantum compound entity S, the result of joining quantum entities $S1$ and $S2$ is: "We may ask what state 1 is in by itself, and what state 2 is in by itself, when 1+2 is in state Ψ. The answer is remarkable. Neither 1 nor 2 is in a definite state!" [19, p. 379]. Such loss of individuality is a consequence of entangled states (superpositions of states from orthonormal bases for the compound state space). (In a Fock space — in quantum field theory — classical individuality comes under even more strain).[q] Entangled quantum entities can be compared with amounts of water that are added together.[3] Or to money in the bank. Suppose you deposit 5 euro in an empty bank account. Then you deposit another 5 euro in the account. Next, you withdraw 5 euro. It doesn't make much sense to ask if this is the first or second euro you deposited earlier. One is inclined to speak of the *loss* of individuality in QM; but: QM is physics' most fundamental theory, and: one can expect all quantum entities to flow into one entity,

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[p] This is usually illustrated with the following thought experiment. Suppose you have a box with two compartments. In the box there are two entities: A and B. (A) A classical model of this situation will be built from two state spaces: $M^{ss}_{C,1} = \{p_1, p_2\}$ where p_1 means that A is in the left compartment and p_2 means that A is in the right compartment and $M^{ss}_{C,2} = \{q_1, q_2\}$ where q_1 means that B is in the left compartment and q_2 means that B is in the right compartment. The compound entity has state space: $M^{ss}_{C,1} \times M^{ss}_{C,2} = \{(p_1, q_1), (p_1, q_2), (p_2, q_1), (p_2, q_2)\}$. Probabilities are distributed over the four cases as $\frac{1}{4}$ for each possible arrangement. In this situation the entities are treated as particulars, i.e. they are ordinally countable and they are uniquely and determinately located in space at all times. (B) A QM–model where A and B are two electrons in the box is much more difficult to interpret. Electrons have no distinguishing fixed properties. Qualitatively they are indistinguishable, although they are numerically different. At least that is what the Hilbert–space description seems to indicate. A QM–model of this situation will be built from two state spaces: $M^{ss}_{Q,1}$ with orthonormal base $\{|p_1\rangle, |p_2\rangle\}$ where $|p_1\rangle$ means that A is in the left compartment and $|p_2\rangle$ means that A is in the right compartment and $M^{ss}_{Q,2}$ with orthonormal base $\{|q_1\rangle, |q_2\rangle\}$ where $|q_1\rangle$ means that B is in the left compartment and $|q_2\rangle$ means that B is in the right compartment. The compound entity has state space: $M^{ss}_{Q,1} \otimes M^{ss}_{Q,2}$ with orthonormal base $\{|p_1\rangle|q_1\rangle, |p_1\rangle|q_2\rangle, |p_2\rangle|q_1\rangle, |p_2\rangle|q_2\rangle\}$. Suppose we remain with Bose–statistics. There are only three compound states available for the electrons: $|p_1\rangle|q_1\rangle, |p_2\rangle|q_2\rangle, \frac{1}{\sqrt{2}}(|p_1\rangle|q_2\rangle + |p_2\rangle|q_1\rangle)$.

[q] A model of quantum field theory will have a Fock–space at its heart.[22] Quantum field theory (of a free quantum field) leads to the introduction of non–countables at the base of the physical world. Let us look at the two electrons in the box from the previous footnote. A ket–vector that describes our physical situation is written as $|n_1, n_2\rangle_Q$. There are three possible ket–vectors: $|1,1\rangle_Q, |2,0\rangle_Q$ and $|0,2\rangle_Q$. A ket–vector that describes our physical situation is written as $|n_1, n_2\rangle_Q$. There are three possible ket–vectors: $|1,1\rangle_Q, |2,0\rangle_Q$ and $|0,2\rangle_Q$.

— it is the individual things of classical ontology that are emergent. Even though, in much of the literature (which is dominated by analytical philosophy of science) there is a tendency to start from the classical paradigm (billiard balls, planets ...), only to gradually eliminate aspects that don't work anymore,[23] quantum descriptions actually need a different conceptual framework altogether, one in which there are no individuals to start with. No priori's in ontology.— In any case, the peripheral nature of countability in QM instantiates a key aspect of a multiplicity, namely, that an M "divides up and does so constantly: That is why it is a multiplicity $\cdots$ it changes in kind in the process of dividing up $\cdots$ There is other without there being several; *number exists only potentially.*" [6, p. 42](our italics). (b) Entangled states for compound entities allow us to derive correlations that make individuals wither away: "*the only proper subject of physics are correlations among different parts of the physical world.* Correlations are fundamental, irreducible, and objective. They constitute the full content of physical reality. There is no absolute state of being; there are only correlations between subsystems" (there are only possible correlations, we should say, and no subsystems, we might add).[24] Much like a multiplicity we have a varying relation between variations, reciprocal determination of relations without relata.[r]

M(3) No aspect of a Deleuzian multiplicity has a clear conceptual or empirical meaning. Is this true for the Hilbert space? — The connection of a description in terms of a state vector with experiment (with actual classically determinate measurement results) is prescribed by the Born rule. The Born rule gives the probability that a quantum entity described by $|\psi\rangle$, will be in a state $|q_i\rangle$ after measurement for the variable Q: $|c_i|^2$ where $c_i = \langle q_i \, |\psi\rangle$. The state vector by itself only gives us amplitudes, complex numbers. Without this rule there is no connection with M^{cs}, classical reality. Such quantum "potentialities" are like nothing we can find in the classical world view, because of the possibility of superposition: "... it is vacuous to assign objective probability to the possible outcomes if definite outcomes do not occur" [19, p. 389]. In other words, one cannot simply substitute a mixture $|c_i^2|P_{|q_i\rangle}$ for a pure state $P_{c_i|q_i\rangle}$. Classical ontology — classical physics — needs real numbers, ones that can pop up as the

[r] Aerts distinguishes between two kinds of correlations: correlations of the first kind are correlations that were already present before an experiment and are only detected by the experiment; correlations of the second kind are correlations that were not present before the experiment and are created by the experiment [28, p. 16];[26] Aerts' distinction seems to us a crucial complement to Mermin's claim, but we cannot go into this here.

result of measurements. In QM it is only when measurement comes into view that one can apply the Born rule, start talking about probabilities. From a prima facie deleuzian point of view, the fact that we have amplitudes instead of probabilities, complex numbers instead of real ones, can be construed in terms of the difference between virtuality and actuality: probability amplitudes are virtual, probabilities are actual (they belong to a definite empirical context of measurement, which is always classical).

M(4) The following question haunts physics: "Why (and when) do squared moduli of amplitudes 'become probabilities'?" [27, p. 325]. Do the mathematical procedures of projection and the Born rule correspond to real processes? From a deleuzian perspective, the answer is yes. We can construe them in terms of the actualization of virtuality. — Even though QM has a dynamical equation (the Schrödinger–equation) that exhibits classical determinism, in the quantum domain, one also has

$$P_{|q_i\rangle} |\psi\rangle = |q_i\rangle \langle q_i|\psi\rangle = c_i |q_i\rangle \tag{2}$$

which projects $|\psi\rangle$ onto $|q_i\rangle$ telling us "how much of $|q_i\rangle$ is in $|\psi\rangle$" (which immediately gives us the amplitude c_i that $|q_i\rangle$ would occur). Contrary to a deterministic evolution, these jumps of the state vector do not seem to involve a passage of time that one can picture with a real line. Is the transition from $|\psi\rangle = \sum_i c_i |q_i\rangle$ to $|q_i\rangle$ also a description of what happens during a measurement? Most physicists would agree with the following:

> "... " this is just a mathematical rule that indicates what will be the possible state of the entity after the measurement has been performed if we know the state of the system before the measurement. There is no time involved in this rule, neither gives this rule a description of what happens during the measuring process. One must not confuse between the physics that takes place during the measurement and the mathematical picture of the formalism that consists of projecting a vector of a Hilbert space onto a closed subspace. [3, p. 114]

Is change really only conceivable in terms of a differential equation (Newton, Schrödinger)? Does physics only have room for dynamical change?[s] Remember that (i) for a multiplicity the actualization of virtuality cannot be described in terms of the structures of actuality, (ii) time is in the first

[s]On the whole Aerts' creation–discovery view comes close to a Deleuzian interpretation of QM, except for one thing: his insistence that the transition from $\sum_i c_i |q_i\rangle$ to $|q_i\rangle \in \{|q_i\rangle\}$ can be described as a classical dynamical process.

place defined in terms of a multiplicity and not a real number line. Within such a view the actualization of virtuality — expressed with the "ratio" t/c — is the natural counterpart for the jump of the state vector. The latter can be construed as physical change that does not involve a change in position and cannot be measured with a time line of real numbers. Within a Deleuzian ontology, classical reality (M^{cs}) is the most contracted state of virtuality, a special case of multiplicity; difference is basic.[t] When we look at the behaviour of a quantum entity, we see that the moment it has a definite value for a specific variable, say Q in the equation in this section, then another variable with which it is incompatible, for example T, will become indeterminate, even when T had a determinate value before the "actualization" of Q. In terms of multiplicities: can never exhaustively contract virtuality into actuality; difference is the canvas, the lack of "container" is the only container. Furthermore, according to at least one physicist (Aerts), the "jump of the state vector" *creates* (part of) a quantum entities' classical properties: for example, when one measures the position variable, a bit of space is created, emerges; since quantum variables of one quantum entity are always, on the whole, incompatible among each other, like M, M_Q^{ss} is intrinsically and basically creative. Because of the way compound entities are described, superposition immediately spreads to anything that comes into contact with quantum entities. The compound entity of a quantum entity and a measurement system would entail superposition for the measurement system. Which would then spread to the physicist looking at the machine ... Such a "quantum catastrophe" is stopped by an actual jump of the state vector: the physicist sees that a specific measurement result is the case. How can physical reality be so intimately dependent on measurement? Some physicists have gone so far as to make physical reality physically dependent on human consciousness.[u] What could be a Deleuzian interpretation of the measurement problem? The ontology of difference is fundamentally anti–anthropocentric. It rejects the humanism which has been part and parcel of the Western world–view, and which also underlies the epistemology of science. A Deleuzian we believe, should welcome the "absurdity" of a physicist in superposition. The subversion of humanism could hardly be more radical.

[t] "Mathematically speaking, one can make millions of tensor products of Hilbert spaces, but such a huge tensor product will never deliver a classical phase space" [28, p. 11].
[u] Even Prigogine and Stengers see this as a major obstacle for scientific realism: see chapter six of Ref. 29.

M(5) QM vindicates a key idea that follows from the Deleuzian scheme: that M^{cs}, and more generally every classical structure, is just an aspect of being, not its overarching principle. — Locality is a basic principle of classical ontology: "Without such an assumption of the mutually independent existence (the "being–thus") of spatially distant things, an assumption which originated in everyday thought, physical thought in the sense familiar to us would not be possible. Nor does one see how physical laws could be formulated and tested without such a clean separation" (Einstein quoted in [30, p. 241]. A quantum compound of two entities, S_1 and S_2, is described by $M^{ss}_{Q,1} \otimes M^{ss}_{Q,2}$, which is itself again *one* vector space, because, like in formula (1), infinitely many spectral decompositions of the state of S can be performed on the system, many of them non-local (non-separable); the most straightforward reading: this does not describe a things composed of two other things, but one entity.[v] That is why, even when S_1 and S_2 are located in different regions of space (separated by several meters or even kilometers), coincidence measurements in these locations reveal correlations between the measurement results. When a measurement is performed on one of the subsystems, does this instantaneously and non–locally "influence" the probabilities for the possible measurements on the other? Apparently. However, the measurement performed, the correlation is lost: before one has $M^{ss}_{Q,1} \otimes M^{ss}_{Q,2}$ which, on a straightforward reading of the mathematics, is one entity, not two; afterwards one has $M^{ss}_{Q,1} \times M^{ss}_{Q,2}$,[26] i.e., two (separate) entities, in accord with locality. Classical ontology (specifically, M^{cs}) would induce us to attribute $M^{ss}_{Q,1} \times M^{ss}_{Q,2}$ to the system, even before the measurement: according to M^{cs} the quantum entity should be composed of things S_1 and S_2, separated in space; but the behavior of S is determined by $M^{ss}_{Q,1} \otimes M^{ss}_{Q,2}$ alone, which tells us: of course the parts influence each other, there are no parts. When the idea is raised that quantum entities are not spatial, this usually leads to the conclusion that they do not exist outside of our minds, or, that they do not exist independently of our measuring apparatuses. Slowly people are beginning to acknowledge the most

[v]There is a theorem that says that QM cannot describe separated entities.[3,26,31–33] For the claims about non–spatiality and the EPR–Bell–experiments, see Refs. 28 and 34. We are aware that the discussion is far from over; we are also aware of the infinite complexity of the discussion, and the enormous amount of literature on the subject since 1935, the year that Einstein, Podolsky and Rosen published their "paradox". Having only limited space here, we have to draw the line somewhere; we decided to center our own presentation around the work of Aerts and co.: it has the merit that it is, first of all, supported by interesting and strong mathematical results, and second, it is transparent and intuitive.

straightforward reading of QM–models: "Indeed, it is easy to show ... that if a particle's location is at a given moment in a well-defined spatial region, that condition will not last very long. So the electron is, as it were, going in and out of space" [35, p. 53].[w] And Aerts again: "A structured space is not considered as a necessary a priori theater for the motion and interactions of the physical entities" [34, p. 545]. Properties are created during a measurement. Only, sometimes the creation aspect is minimal to non-existent (classical ontology), at other times there is a lot of creation involved (quantum domain). Classical reality is an emergent characteristic of $M_{Q,}^{ss}$; in general, for every variable, specifically for space, position. Differenciation exists against a background of differentiation. Philosophers of physics are quick to point out that one cannot use the quantum correlations to manipulate instantaneously at a distance: having performed the measurement, the connection is severed. That makes sense from a Deleuzian perspective; the possibility of manipulation in a billiard ball fashion within virtuality would actually deny the virtual character of virtuality; it would resemble the actual, while the whole point is exactly that the virtual is different in nature from the actual, that they do not resemble. $M_{Q,1}^{ss} \otimes M_{Q,2}^{ss}$ (differentiation) is a wholly different structure than $M_{Q,1}^{ss} \times M_{Q,2}^{ss}$ (differenciation).

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References

1. G. Deleuze, *Foucault* (University of Minnesota Press, Minneapolis, 1988).

[w]Similarly Albert writes: "And of course the space those sorts of objects *live* in, and (therefore) the space *we* live in, the space in which any realistic understanding of quantum mechanics is necessarily going to depict the history of the world as *playing itself out* (if space is the right name for it ...) is *configuration space*. And whatever impression we have to the contrary (whatever impression we have, say, of living in three-dimensional space, or in a four-dimensional space–time) is somehow flatly illusory" [36, p. 277]. Still, it is not obvious to say this out loud. As Albert writes this seemed so "straightforward and so ineluctable" to him that it did not merit any further discussion. "But it turns out not to have seemed that way to everybody. It turns out (as a matter of fact) that this sort of talk still frequently manages to surprise people, even to *appal* them" [36, p. 277].

2. M. DeLanda, *Intensive Science and Virtual Philosophy* (Continuum, London, 2002).
3. D. Aerts, The one and the many: towards a unification of the quantum a classical description of one and many physical entities (Doctoral dissertation, Brussels Free University, 1981).
4. E. Underhill, *Mysticism. The Nature and Development of Spiritual Consciousness* (Oneworld, Oxford, 1999).
5. G. Deleuze, *Difference and Repetition* (Colombia University Press, New York, 2001).
6. G. Deleuze, *Bergsonism* (Zone Books, New York, 1988a).
7. T. May, *Deleuze. A General Introduction* (Cambridge University Press, Cambridge, 2005).
8. G. Deleuze, *Cinema 2: The Time Image* (University of Minnesota Press, Minnesota, 2003).
9. K. A. Pearson, *Philosophy and the Adventure of the Virtual. Bergson and the Time of Life* (Routledge, 2002).
10. G. Deleuze, *Proust and Signs* (University of Minnesota Press, Minneapolis, 2000).
11. G. Deleuze, *The Fold* (University of Minnesota Press, Minnesota, 1993).
12. D. R. J. Chillingworth, *Differential Topology with a View to Applications* (Pitman, Brooklyn, 1976).
13. S. Auyang, *How is Quantum Field Theory Possible?* (Oxford University Press, New York, 1995).
14. C. A. Hooker, Metaphysics and modern physics. A prolegomenon to the understanding of quantum theory, in *Contemporary Research in the Foundations and Philosophy of Quantum Theory*, ed. C. A. Hooker (Dordrecht, Reidel, 1973).
15. P. Smith, *Explaining Chaos* (Cambridge University Press, Cambridge, 1998).
16. R. Giere, *Explaining Science: A Cognitive Approach* (University of Chicago Press, Chicago, 1988).
17. J.-P. Van Bendegem, The logical analysis of time and the problem of indeterminism, *Communication and Cognition* **26**.
18. I. Prigogine and I. Stengers, *Order out of Chaos. Man's New Dialogue with Nature* (Bantam Books, 1984).
19. A. Shimony, Conceptual foundations of quantum mechanics, in *The New Physics*, ed. Paul Davies (Cambridge University Press, Cambridge, 1989).
20. P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947).
21. D. Albert, *Quantum Mechanics and Experience* (Harvard University Press, Cambridge, 1992).
22. P. Teller, *An Interpretative Introduction to Quantum Field Theory* (Princeton University Press, Princeton, 1995).
23. J. Seibt, Quanta, tropes, or processes: On ontologies for QFT beyond the myth of substance, in *Ontological Aspects of Quantum Field Theory*, eds. Kuhlmann, M. et al. (World Scientific, Singapore, 2002).

24. N. D. Mermin, The Ithaca interpretation of quantum mechanics, *Pramana* **51**, (1998), arXiv: quant-ph/9609013.
25. D. Aerts, Quantum mechanics: Structures, axioms and paradoxes, in *Quantum Mechanics and the Nature of Reality*, eds. D. Aerts and J. Pykacz, (Kluwer, Dordrecht, 1999).
26. B. Coecke, A representation for compound quantum systems as individual entities: Hard acts of creation and hidden correlations, *Foundations of Physics* **28**, (1998).
27. R. Penrose, *The Emperor's New Mind: Concerning Computers, Minds, and The Laws of Physics* (Oxford University Press, Oxford, 1989).
28. D. Aerts, An attempt to imagine parts of the reality of the micro-world, in *Problems in Quantum Physics*, eds. J. Mizerski, A. Posiewnik, J. Pykacz and M. Zukowski, (World Scientific, Singapore, 1990) pp. 3–25.
29. I. Prigogine and I. Stengers, *Entre le temps et l'éternité* (Flammarion, 1992).
30. D. Howard, Einstein on locality and separability, *Studies in the History and Philosophy of Science* **16**, (1985).
31. D. Aerts, Description of many physical entities without the paradoxes encountered in quantum mechanics, *Foundations of Physics* **12**, (1982).
32. D. Aerts, Quantum structures, separated physical entities and probability, *Foundations of Physics* **24**, (1994).
33. F. Valckenborgh, Operational axiomatics and compound systems, in *Current Research in Operational Quantum Logic*, eds. Bob Coecke, David Moore and Alexander Wilce (Kluwer, Dordrecht, 1999).
34. D. Aerts and J. Reignier, On the problem of non-locality in quantum mechanics, *Helvetica Physica Acta* **64**, (1991).
35. B. Van Fraassen, *The Empirical Stance* (Yale University Press, 2002).
36. D. Albert, Bohmian mechanics and quantum theory: An appraisal, in *Bohmian Mechanics and Quantum Theory: An Appraisal*, eds. J. T. Cushing, A. Fine and S. Goldstein (Kluwer, Dordrecht, 1996).

UNDERSTANDING PROBABILITIES IN THE EVERETT INTERPRETATION OF QUANTUM MECHANICS

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I focus here on a specific aspect of the "problem of probabilities" in the Everett interpretation of quantum mechanics, namely the problem of understanding probabilities. Such a task would imply to answer three questions: "What are subjective probabilities?", "What are objective probabilities?", and "Why should a rational agent match her subjective probabilities with her estimation of the values of the objective probabilities?" I show how a conception of subjective probabilities as degrees of care (inspired by Hilary Greaves) would answer the first question; and how interpreting quantum weights as proportions of worlds would answer the second question and help answering the third one.

Keywords: Probability; Everett interpretation; understanding.

1. Introduction

Everett's[2] interpretation of quantum mechanics, better known as the "many-worlds" interpretation,[6] has been revived recently by the "Oxford school."[5,8,17,23] It is now generally considered to raise two problems. First, the "preferred basis problem": according to this interpretation, the universe is made of multiple worlds constantly spltitting; but how is it that we observe the universe according to a specific basis, rather than another one? Second, the "problem of probabilities": how can one make sense of probabilities in a universe in which every experimental result will occur? The theory of decoherence has been used to solve the problem of preferred basis.[17,22] Therefore, most of the ongoing debate has focused on the "problem of probabilities." This article will focus on a particular aspect of this problem, namely: What does it imply to understand[a] probabilities in the context of the Everett interpretation? And which view (amongst the ones that have been developed in the last years) enables the best understanding of these probabilities?

The word "probability" can have different meanings; therefore, there are several faces in the problem of "understanding probabilities." More specifically, two families of probabilities have commonly been identified:[7] *objective* and *subjective* probabilities.[b] Whereas the objective probabilities characterize our objective world, the subjective probabilities are guides for our rational actions.[c] These two types of probabilities are related by Lewis' "Principal Principle,"[12] which states that the subjective probabilities of a rational agent must match her evaluation of the objective probabilities, or more formally, $Cr(A|P(A) = x) = x$ where $P(A)$ refers to the objective probability of A and $Cr(A)$ refers to the subjective probability of the agent in A. For example, according to the Born rule of textbook quantum mechanics, the objective probabilities are given by the quantum weights, that is the mod-squared amplitudes. By application of the Principal Principle, the subjective probabilities must also match these values.

Arguably, understanding probabilities in a probabilistic physical theory would imply answering at least three questions. The first one is Q_1: "What are subjective probabilities?" In statistical physics for example, one can argue that the subjective probabilities represent degrees of belief in the different possible values of the specific state of a system, and that they arise because of our ignorance of this exact state. The second question

[a]The notion of scientific understanding has been relatively little studied in philosophy of science.[4] Here, I will only deal with the question of understanding in the context the Everett interpretation of quantum mechanics, and I will not try to elaborate on works concerning the general notion of scientific understanding. I will also not try to analyse whether the task of "understanding probabilities" could be formulated more specifically, like e.g. "understanding probability-related statements", "understanding the nature and the role of probabilities", etc.

[b]Some philosophers (e.g. Popper,[15] with his propensity interpretation of probability) have tried to reduce all probabilities to objective ones. But it seems difficult to do without the notion of subjective probabilities nowadays, given the major role of degrees of belief in contemporary epistemology. On the other hand, some extreme subjectivists, such as de Finetti, have rejected any kind of objective probabilities. However, this position seems difficultly to hold in the framework of quantum mechanics, as stated by Wallace (2006): "How they can maintain with a straight face that the half-life of uranium is not an objective propriety of the world is beyond me."

[c]Subjective probabilities are generally called "degrees of belief." There are also purely epistemic approaches of degrees of belief,[10] according to which subjective probabilities do not need to guide our rational actions, that I will not consider here.

Q_2 is: "What are objective probabilities?" In Popper's view of quantum mechanics, objective probabilities are propensities, namely, dispositions of nature to produce a given result. Frequentism or Lewisian best-system analysis also aim at defining precisely objective probabilities. However, there is no clear consensus on a specific theory of objective probabilities. Finally, the third question Q_3 is the following: "Why must a rational agent match his subjective probabilities on his estimation of the objective probabilities?" Answering to this question would amount to justify the Principal Principle.

I do not claim that answering these three questions would guarantee a perfect understanding of the probabilities in the Everett interpretation (other relevant questions may have to be answered), but that this is a minimal requirement for a good understanding of these probabilities. In this article, I will investigate some recent analyses of probabilities in the Everett interpretation of quantum mechanics, and how they can help to answer these questions Q_1–Q_3.

2. The Origin of Subjective Probabilities

The question of the origin of subjective probabilities is one of the most heavily debated in the context of the philosophy of the contemporary Everett interpretation.[5,8,18,24] As a matter of fact, the Everett interpretation is a deterministic one. Therefore, a rational agent who would know perfectly the state of the universe could predict with certainty (given enough computing resources[d]) its future state. This seems to leave no room to subjective probabilities measuring degrees of belief of a rational agent concerning the state of a system she is partially ignorant about. However, two different ways to answer this problem have been proposed these last years, called "subjective uncertainty" and "fission program."[8,26]

2.1. *Subjective uncertainty*

Let us first consider the "subjective uncertainty" point of view. Imagine a rational agent who is going to undergo a personal fission — that is, she is going to be divided into multiple successors (similarly to what happens constantly according to the Everett interpretation of quantum mechanics). Proponents of the "subjective uncertainty" approach argue that this agent is uncertain about what she is going to observe, even if she knows perfectly the state of the future universe. Wallace[25] holds such a position,

[d]That may actually exceed the resources available in the universe.

grounded on the principle of charity advocated by philosophers of language like Quine and Davidson: charity considerations compel us to choose a semantics (named "branching-time semantics" by Wallace) that can maximize the truth of the statements used by the people of the linguistic community. In particular, this semantic would imply that when scientists say that they are uncertain about which result they are going to observe after a quantum experiment, they are right, even if they in fact have some successors who will observe one result, and some successors who will observe another result. However, this approach requires an elaborated position in philosophy of language, and remaining agnostic about such matters when interpreting quantum mechanics would surely be preferable, if possible. Fortunately, there is another approach that can be accepted less controversially, and that is also compatible with the "subjective uncertainty" approach: the "fission program."

2.2. *Fission program: Degrees of care*

According to the fission program, "uncertainty is not after all an indispensable prerequisite to the application of decision theory":[8] the fact that a rational agent knows (with certainty) that she is going to undergo a fission is sufficient for applying decision theory. Indeed, a rational agent must consider the wellbeing of all her future successors. An action that will be "good" for some of her successors can be "bad" for some other successors; therefore, the agent must weigh the consequences of her action on all her successors. Greaves[8] has introduced "degrees of care" that would enable such a weighing. However, some questions concerning these "degrees of care" are left open: how can we define them precisely? And why should they obey the axioms of probability? Several theories have attempted to define precisely the notion of subjective probability, De Finetti's[3] betting coefficients and Savage's[19] decision theory being two of the most influential ones. I will show here how De Finetti's theory can be adapted to define "degrees of care" in personal fission situations, and how this will help to answer questions Q_1–Q_3; I will also quickly mention how this approach can be adapted to a decision-theoretical approach like Wallace's.[27]

2.3. *De Finetti's subjective probabilities*

Let us start by reminding De Finetti's conception of subjective probabilities or "degrees of belief"[e] — we'll show after how it can be adapted to define

[e] De Finetti did not use the phrase "degree of belief" in his article of 1937, but the subsequent tradition has done so.

degrees of care in a fission situation. For de Finetti, subjective probabilities are defined by our disposition to bet. For example, if the maximum odds I am ready to bet on the occurrence of an event E is "two-to-one" (that is, I win 2 euros if E occurs, and I lose 1 euro if E does not occur), then my subjective probability in E would be 1/3. More generally, if E is an uncertain event, one can define the subjective probability of a rational agent B in E by the following protocol, where A refers to an external bookie:

(1) B chooses a coefficient p
(2) A chooses an amount of money S, that can be positive or negative
(3) Depending on whether E occurs or not, B exchanges the following amount of money with A:
    (a) If E occurs, then B receives the money $S - pS$ ("receive" should be understood in the algebraic sense: this amount can be positive or negative)
    (b) If E does not occur, then B receives the amount $-pS$

Said differently, agent B must 'pay' the amount pS to participate to the bet, and 'receives' the amount S if E occurs. One can then define the probability of B in E as the quantity p that B considers as fair in such a situation[f] — it will be written $p(E)$. Such a quantity is traditionally named "degree of belief of B in E".

Such betting coefficients do indeed fit with our intuitive conception of degrees of belief. For example, $p(E) = 1$ implies that B considers as fair a bet in which he would not win anything if E would occur, and in which he would lose money if E would not occur. This fits the pre-theoretical idea that B is certain that E will happen: he considers he has nothing to fear about E not occurring. If $p(E) = 1/2$, B considers as fair a bet in which the amount of money he receives if E occurs is equal to the amount he loses if E does not occur; this fits the intuitive scenario in which he believes with the same intensity that E will and will not occur. If $p(E) = 2/3$, B considers as fair a bet involving him losing twice as much money if E would not occur as he would win if E would occur; this fits the intuition of an observer believing "twice more" that E will occur rather than E will not occur. Therefore, this formalization of betting coefficients fits well our pre-theoretical intuition of degrees of beliefs.

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[f] De Finetti (1937) defined probability as the quantity p that B would be ready to accept, rather than the one he would consider as fair. However, such a definition raises serious difficulties.[1,21] I use therefore here a de-pragmatized definition, defended in particular by Howson and Urbach.[9]

We can now investigate the rational constraints that bear on such betting coefficients. De Finetti names "Dutch Book" a bet in which an agent is sure to lose, whatever happens (that is, a bet in which all the possible gains of the external bookie A are positive). This is obviously a bad situation for the agent, and rationality would require to consider as unfair any Dutch Book. Ramsey,[16] De Finetti[3] and Kemeny[11] have shown that an agent is not vulnerable to any Dutch Book if and only if her betting coefficients follow the axioms of probability (the derivation of this result will be adapted later to degrees of care).

2.4. *Adaptation of de Finetti's approach to degrees of care*

Let us now consider how we can adapt this subjectivist interpretation of probabilities to the context of a classical (that is, non-quantum) fission of a single observer — this result will then generalize easily to the case of a quantum fission in an Everettian universe. Imagine that a "fission machine" has been invented: a mechanism that can duplicate people cell by cell — with an import of matter and energy that ensures that the copies after the fission are identical to the original before fission. A rational agent B is put to sleep and placed into the device. At the end of the operation, several successors of B, all identical and perfectly functional, will go out of the machine. This division is made in a totally symmetrical way, such that B (before the fission) has no privileged successor (after the fission): all of them are her successors to the same degree. Imagine that each successor is put into her own colored room: a green room for some of the successors ("G" will refer to this set of successors) and a red room for the others ("R" will refer to these successors).

We will show that one can define degrees of care of B towards her successors as rational betting coefficients, similarly to the definition of degrees of belief as rational betting coefficients by De Finetti.[3] The adapted definition would involve the following situation, that we will call "distribution situation" (here again, A will refer to an external bookie):

(1) B chooses a coefficient p
(2) A chooses an amount of money S, that can be positive or negative
(3) B's successors receive the following amount of money, depending on whether they belong to G or not:
    (a) Every successor of B in G receives $S - pS$ (the word "receive" must here also be understood in an algebraic sense: one can receive a positive or negative amount of money)

(b) Every successor of B in R receives $-pS$

We can now define degrees of care of B the following way: B has a degree of care p towards her successors belonging to G (that we will write $p(G)$) if and only if B would consider as fair a coefficient p in such a distribution situation (this mirrors the definition of degrees of belief as betting coefficients in a classical uncertainty situation).

We can now investigate what are the rational constraints that bear on the choice of the degrees of care. We can adapt De Finetti's concept of "Dutch Book" and define a "Dutch Book in a fission situation", as a distribution situation at the end of which all (actual) successors of B lose money (as a reminder, a classical Dutch Book was defined as a bet at the end of which all the *possible* future successors will lose money). This is a bad situation for the agent, and here again rationality would require to consider as unfair any Dutch Book in a fission situation.

Similarly to what happened to degrees of belief, this simple constraint will imply that the degrees of care are constrained by the following probability axioms (Ω will refer to the set of all successors of B):

Axiom 1: $p(\Omega) = 1$
Axiom 2: $0 \leq p(G) \leq 1$
Axiom 3: $p(G) + p(R) = 1$

Here is the proof of the three axioms, modeled after Gillies' (2000) proof for degrees of belief.

Axiom 1: $p(\Omega) = 1$

Proof. All successors of B belong to Ω. Therefore, in a distribution situation involving the group Ω, all successors of B win $(1 - p(\Omega))S_\Omega$. Thus, if B chooses a coefficient $p(\Omega) < 1$, A can make a Dutch Book by choosing $S_\Omega < 0$; and if B chooses a coefficient $p(\Omega) > 1$, A can make a Dutch Book by choosing $S_\Omega > 0$. Reciprocally, if $p(\Omega) = 1$, then $(1 - p(\Omega))S_\Omega$ is always zero, and therefore, no Dutch Book is possible. In conclusion, B must choose a coefficient $p(\Omega) = 1$ to avoid a Dutch Book. □

Axiom 2: $0 \leq p(G) \leq 1$

Proof. In a distribution situation, each successor of B in G wins $(1 - p(G))S$; and each successor of B in R wins $-p(G)S$. Consequently, if B chooses a coefficient $p(G) < 0$, A can make a Dutch Book by choosing $S < 0$; and if B chooses a coefficient $p(G) > 1$, A can make a Dutch Book by choosing $S > 0$. Reciprocally, if $0 \leq p(G) \leq 1$, $(1-p(G))S$ and $-p(G)S$ never have the same sign, and therefore, no Dutch Book is possible: if the successors of G lose money, then the successors of R will earn money, and vice versa. Therefore, B must choose a coefficient $0 \leq p(G) \leq 1$ to avoid a Dutch Book. □

Axiom 3: $p(G) + p(R) = 1$

Proof. Imagine that two distribution situations are proposed to B: one involving G and the other involving R, with the respective fees S_G and S_R. Then, each successor of B in G wins $M_G = S_G - [p(G)S_G + p(R)S_R]$; and each successor in R wins $M_R = S_R - [p(G)S_G + p(R)S_R]$. If A chooses the same fee S for the two bets (i.e., $S_G = S_R = S$), then all the successors of B (both G-successors and R-successors) will win $[1 - [p(G)) + p(R))]S$. Consequently, if B chooses betting coefficients such that $p(G)) + p(R)) > 1$, then A can make a Dutch Book by choosing $S > 0$; and if B chooses betting coefficients such that $p(G)) + p(R)) < 1$, then A can make a Dutch Book by choosing $S < 0$. Therefore, B must choose coefficients such that $p(G) + p(R) = 1$. □

Reciprocally, if $p(G) + p(R) = 1$, then $p(G)M_G + p(R)M_R = 0$. Consequently, M_G and M_R are not both negative (otherwise, we would have $p(G)M_G + p(R)M_R < 0$). Thus, no Dutch Book is possible.

This approach enables to define degrees of care as clearly as de Finetti's degrees of belief. It can therefore make sense of probabilities in the Everett interpretation, which also involves a fission of the observer. This answers question Q_1: in the fission program, subjective probabilities are degrees of care.[g] Moreover, such an interpretation is also compatible with a "subjective uncertainty" point of view: if one commits to the view that the agent is in a state of subjective uncertainty before an Everettian split, if she has a degree of belief 1/2 in the proposition "I'm going to observe spin-up",

[g]It could be objected that in some circumstances, or given some preferences, an agent would not consider as fair some of her successors "suffering" (e.g. losing money) for the sake of others (cf. Price, 2009). "Fair" should been taken here in a restricted sense that assumes that such a compensation is possible. I agree however that in some very specific cases, this reasoning would not hold, and an agent should not have degrees of care the way described here; this kind of problem also appears in some circumstances for classical degrees of belief: an agent could consider as unfair a situation in which some of her possible successors would suffer for the sake of others possible successors. But the approach I presented can define adequately degrees of care in most of the situations we are interested in.

then she should also have a degree of care of 1/2 towards her successors who are going to observe spin-up. However, the fission program does not require the strong commitment to a specific view in philosophy of language that the “subjective uncertainty” framework needs.

2.5. *Quantum weights and proportions*

We can now turn to question Q_2: what are objective probabilities? First, let us remind that mod-squared amplitudes are mathematical objects in the theory of quantum mechanics. In order to interpret this theory, one needs to put in correspondence these mathematical objects and some physical objects. In a Popperian view of sciencec, interpretations can then be experimentally tested, and corroborated or falsified.

What I propose here is to interpret the quantum weight attached to a kind of world in the universal quantum state as the proportion of this kind of world among all the worlds. This idea is not entirely new, and has been proposed in other forms in the past. For example, Greaves[8] writes that one could interpret the quantum weight by speaking of “how much” successors are going to see a given result (without detailing her proposition further). Vaidman[20] “measure of existence” may also be related to this idea; however, this concept should be made specific, and its link with subjective probabilities would need to be justified.

On the contrary, the notion of proportion is a familiar concept that we can grasp intuitively. Admittedly, we generally do not use it to consider proportion of worlds; but we are also not familiar (whether in daily life or in science) with the concept of multiple worlds. The Everett interpretation leads us to change our ontology in a radical way, by accepting many worlds; after such a major ontological change, considering that each kind of world exists in a given proportion is arguably a quite minor addition.

This view gives an answer to question Q_2, namely: objective probabilities are proportion of worlds. Let us consider how it deals with question Q_3. Consider a world W in an Everettian universe which is going to split into multiple worlds; then an agent belonging to W is also going to split into multiple successors.[h] In order to answer Q_3, one should justify why the agent should have a degree of care toward a given group of successors of hers which matches with her estimation of the proportion of this group amongst all her successors. I will justify this rationality principle in the case of a “classical” situation of personal fission. Let us consider again the example

[h] I will not consider the case in which the agent does not survive in some of these worlds.

of personal fission presented earlier, in which some successors wake up in a green room, and some in a red room. If m is the number of successors of B in G (i.e., those who are going to wake up in a green room) and n is the number of successors of B in R (i.e., those who are going to wake up in a red room), then I will show that B should hold a degree of care of $m/(m+n)$ to G, and a degree of care of $n/(m+n)$ to R. To show this result, I will use a Dutch Book argument that requires to modify slightly the procedure of betting, by adding the following fourth step, after the steps 1–3 that have been described earlier:

All B's successors put the money they have "won" (either positive or negative) on a common bank account, and they share this money equally.

For example, if there are three successors whose gains are -6 euros, -6 euros, and 33 euros, then the three successors will share 21 euros, and therefore, each of them will win 7 euros. In such a case, it can be shown that B must adopt degrees of care that match the proportions of her successors to avoid a Dutch Book.

Proof. Each successor of B in G will win $S - pS$, and each successor in R will win $-pS$. During the fourth step of the bet, all the money $m(S-pS)+n(-pS)$ is put on the common bank account, and each successor receives the money, $S(m(1-p)-np)/(m+n))$. To avoid a Dutch Book, this sum must be positive in every case (S positive or S negative). Consequently, we must have $m(1-p)-np \leq 0$ and $m(1-p)-np \geq 0$, which means $m(1-p)-np = 0$. Therefore, $p = m/(m+n)$. □

I claim that the addition of the fourth step in the betting protocol is innocuous, namely that if a rational agent would consider as fair a bet involving the steps 1–3, then he would consider as fair a similar bet (on the same event, with the same odds) involving the steps 1, 2, 3, and the fourth step. As a matter of fact, before the fission, B must rationally care about all her successors, as they are all "equally actual" (there is no degree in the notion of actuality), and none of them is privileged above the others (as the fission has been supposed to be totally symmetric).[i] It is therefore rational for B before the fission to accept this common bank account system, in order to treat equally well all her successors. Thus, I have shown that in a situation of classical fission, an agent's degrees of care must track (his

[i]On the opposite, in a classical uncertainty situation, we cannot justify that all the successors are "equally possible" (as there is a kind of degree for the notion of possibility, which is — according to some views — given by the concept of objective probability).

estimation of) the proportions of its successors; this can be adapted to the Everettian case,[j] and thus, answers question Q_3.

3. Generalization to Decision Theory

I have proposed here an answer to questions Q_1–Q_3 in the context of a de Finetti-inspired approach of subjective probabilities. This approach may also be adapted to a decision-theoretical approach, like Wallace's. Wallace proposed a remarkable derivation of the Born rule on the basis of Deutsch's analysis:[5] he applies decision theory in an Everettian framework to demonstrate that the subjective probabilities must track the quantum weights. He proposed several versions of his derivation,[23,27,28] a full discussion of which largely exceeds the scope of this article. Let us just state two points, concerning the status of subjective probabilities and his derivation of the Born rule.

First, let us consider question Q_1 about subjective probabilities. In one of his derivation, Wallace[27] starts with a qualitative relation of likelihood and deduces a quantitative probability, using some principles of rationality. He interprets this relation of qualitative likelihood in a "subjective uncertainty" framework: therefore, his proof amounts to derive a quantitative notion of degrees of belief (following the axioms of probability) from a qualitative one. However, one could also interpret his derivation in the fission program;[26] in this context, his proof would amount to derive a quantitative notion of degrees of care from a qualitative one. In summary, both the "subjective uncertainty" and "fission program" frameworks are compatible with his derivation.

Now, let us turn to his derivation of the Born rule. Here, interpreting quantum weights as proportions may help to justify some controversial principles.[8] For example, Wallace uses an "equivalence" principle, which states that if the sum of the weights of the branches corresponding to the event E is equal to the sum of the weights of the branches corresponding to the event F, then E and F have the same qualitative likelihood. In one defense of this equivalence principle, Wallace[27] lists some possible differences between different physical situations that may be relevant for an agent:

[j]It could be objected that in an Everettian situation, such a common bank account is physically impossible, because the different worlds cannot communicate with each other. However, the point here is that *if* such a common bank account would be possible, then an agent who would consider as fair a bet formed by steps 1, 2 and 3 would also consider as fair to add step 4. This is all what is required by the argument.

(1) A change to a given branch which an agent cares about;
(2) A change to a given branch which an agent doesn't care about;
(3) A change to the relative weight of branches;
(4) A splitting of one branch into many, all of which are qualitatively identical for the agent's descendant in that branch

Wallace argues that the changes 1, 2, and 4 cannot bear rationally any consequences on the relation of qualitative likelihood, and deduces that only change 3 can be relevant for qualitative likelihood. But if one interprets quantum weights as proportion of worlds, one can understand better why change 3 is relevant: one should care more about a first group of successors than about a second one (all successors being identical) if there are more of them in the first one. It also becomes clear that change 4 is irrelevant, because it does not change the proportion of successors in the different branches. Therefore, the interpretation of quantum weights as proportions of worlds may help in understanding and justifying some principles needed to derive the Born rule.

4. Conclusion

We can now propose an answer to the three questions we had identified as necessary for a good understanding of probabilities in the context of the Everett interpretation. First, subjective probabilities can be seen as degrees of care towards our successors,[k] as was explained by Hilary Greaves; here, we showed how to adapt de Finetti's subjective probabilities to build a theory of degrees of care. Second, quantum weights can be interpreted as proportions of worlds, which play the role of objective probabilities. And third, it can be shown that a rational agent must match her degrees of care about a group of successors with (her estimation of) the proportions of this group amongst all her successors[l] by a Dutch Book argument.[m] Interpreting

[k] As said earlier, we can interpret these subjective probabilities as degrees of belief, if we accept the subjective uncertainty approach. However, as this last conception rests on some strong commitment in philosophy of language, I stick here with the less controversial approach.

[l] There are, however, some cases where the degrees of care must be attached to worlds rather than to successors. For example, I can care about the state of the world even after my death (a world in which I will have no successor at all) if, for example, my relatives and friends still exist in it — or even just because I care about the future of humanity. The adaptation of the reasoning to this kind of cases does not seem to raise serious conceptual difficulties.

[m] Or, for a more general justification, by Wallace's derivation of the Born rule, that is compatible with subjective probabilities being interpreted as degrees of care, and that may be easier to justify when interpreting quantum weights as proportions of worlds.

quantum weights as proportions give us an account of a mysterious entity, namely objective probability, as a much more familiar concept, namely proportion, thus enabling a better understanding of objective probabilities; and it helps us to understand better why they are related to degrees of care: we should care more about our successors if there are more of them. Therefore, the view sketched here arguably contributes to a better understanding of probabilities in the Everett interpretation of quantum mechanics.

References

1. D. Christensen, Dutch-Book arguments depragmatized: Epistemic consistency for partial believers, *The Journal of Philosophy*, **93**(9), 450-479 (1996).
2. H. Everett, "Relative State" formulation of quantum mechanics. *Reviews of Modern Physics*, **29**(3), 454-462 (1957).
3. B. De Finetti, *La Prévision: Ses Lois Logiques, Ses Sources Subjectives, Annales de l'Institut Henri Poincaré* **7**, 1-68; translated as *Foresight. Its Logical Laws, Its Subjective Sources, in Studies in Subjective Probability*, H. E. Kyburg, Jr. and H. E. Smokler (eds.), Robert E. Krieger Publishing Company, (1980).
4. H. W. De Regt and D. Dieks, A contextual approach to scientific understanding, *Synthese*, **144**(1), 137-170 (2005).
5. D. Deutsch, Quantum theory of probability and decisions, *Proceedings of the Royal Society of London* **A 455**, 3129-3137 (1999).
6. B. S. DeWitt, H. Everett and N. Graham, The many-worlds interpretation of quantum mechanics, Princeton: Princeton University Press (1973).
7. D. Gillies, *Philosophical Theories of Probability*, London; New York: Routledge (2000).
8. H. Greaves, Understanding Deutsch's probability in a deterministic universe, *Studies in History and Philosophy of Modern Physics* **35**(3), 423-256 (2004).
9. C. Howson and P. Urbach, *Scientific Reasoning: The Bayesian Approach* (3rd ed.) Chicago: Open Court (2006).
10. J. M. Joyce, A nonpragmatic vindication of probabilism, *Philosophy of Science*, **65**(4), 575-603 (1998).
11. J. G. Kemeny, Fair bets and inductive probabilities, *The Journal of Symbolic Logic*, **20**(3), 263-273 (1955).
12. D. Lewis, A subjectivist's guide to objective chance, in R. C. Jeffrey (Ed.), *Studies in Inductive Logic and Probability*, Vol II. (pp. 263-293). Berkeley, Los Angeles and London: University of California Press (1980).
13. D. Papineau, Many minds are no worse than one, *British Journal for the Philosophy of Science* **47**, 233-241 (1996).
14. D. Papineau, Why you don't want to get in the box with Schrödinger's cat. *Analysis*, **63** (277), 51-58 (2003).
15. K. R. Popper, *A World of Propensities*, Bristol: Thoemmes (1990).
16. F. P. Ramsey, in Braithwaite R. B. (Ed.), *The Foundations of Mathematics and Other Logical Essays*, London: K. Paul, Trench, Trubner and Co. Ltd. (1931).

17. S. Saunders, Time, quantum mechanics, and decoherence, *Synthese*, **102**(2), 235-266 (1995).
18. S. Saunders, Time, quantum mechanics, and probability, *Synthese* **114**, 373-404 (1998).
19. L. J. Savage, *The Foundations of Statistics*, New York: Wiley (1954).
20. L. Vaidman, On schizophrenic experiences of the neutron or why we should believe in the many-worlds interpretation of quantum theory, *International Studies in the Philosophy of Science*, **12**(3), 245-261 (1998).
21. S. Vineberg, Dutch books, Dutch strategies and what they show about rationality, *Philosophical Studies*, **86**(2), 185-201 (1997).
22. D. Wallace, Everett and structure, *Studies in History and Philosophy of Modern Physics*, **34**(1), 87-105 (2003).
23. D. Wallace, Everettian rationality: Defending Deutsch's approach to probability in the Everett interpretation, *Studies in the History and Philosophy of Modern Physics*, **34**, 415-438 (2003).
24. D. Wallace, Quantum probability from subjective likelihood: Improving on Deutsch's proof of the probability rule (2005a).
25. D. Wallace, Language use in a branching Universe (2005b).
26. D. Wallace, Epistemology quantized: Circumstances in which we should come to believe in the Everett interpretation, *Brit. J. Phil. Sci.*, **57**, 655-689 (2006).
27. D. Wallace, Quantum probability from subjective likelihood: Improving on Deutsch's proof of the probability rule, *Studies in History and Philosophy of Modern Physics*, **38**(2), 311-332 (2007).
28. D. Wallace, How to prove the Born rule, in S. Saunders, J. Barrett, A. Kent and D. Wallace (ed.), *Many Worlds? Everett, Quantum Theory, and Reality, Oxford University Press* (2012).

METAPHYSICAL UNDERDETERMINATION AND LOGICAL DETERMINATION: THE CASE OF QUANTUM MECHANICS

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The 'underdetermination of metaphysics by the physics' is the thesis that our best scientific theories do not uniquely determine their ontologies. Non-relativistic quantum mechanics is famously thought to exemplify this kind of underdetermination: it may be seen as compatible with both an ontology of individual objects and with an ontology of non-individual objects. A possible way out of the dilema thus created consists in adopting some version of Ontic Structural Realism (OSR), a view according to which the metaphysically relevant aspect of the theory is its structure, not the nature of the objects dealt with. According to OSR, particular objects may be dispensed with (eliminated or re-conceptualized) in favor of the structure of the theory. In this paper we shall argue that the underdetermination of metaphysics by the physics is a consequence of a too strict naturalism in ontology. As a result, when a mitigated ontological naturalism is taken into account, underdetermination does not appear to have such dark consequences for object-oriented ontologies in quantum mechanics.

Keywords: Metaphysical underdetermination; ontological naturalism; non-individuality; ontic structural realism.

1. The Quandary on Quantum Ontology

Quantum mechanics has given rise to many difficult conceptual problems. In fact, most of the questions to which the theory gave birth remain as controversial as ever. To mention just a few of them, we could recall the problem of measurement, the nature of entanglement and the interpretation of probability. Furthermore, in a more speculative side of the issue, there is the infamous problem of identity and individuality of quantum particles; according to a tradition that originates in the first days of the theory, quantum particles (roughly speaking) do not have neither identity nor individuality: they are *non-individuals.* That view of quantum particles was called the *Received View* of quantum non-individuality.

The good news brought by the Received View – at least as long as it lasted as the mainstream view on the nature of particles – was that some metaphysics could be deemed as being "derived" from quantum mechanics. The problem of individuality, a problem that for centuries haunted metaphysicians with its never ending controversies, seemed now to have a negative answer provided by quantum mechanics; that is, there seemed to be no principle of individuality for quantum particles, since it could be claimed that *it followed from the theory* that they were not individuals at all. Of course, as is now well-known, that particular understanding of the situation became untenable when it was finally shown that quantum mechanics is also compatible with the interpretation of quantum particles as being individuals (see French,[1] for instance). An impasse was created: are quantum particles individuals or non-individuals?

Due to that impasse, it was soon recognized that the discussion on the ontological nature of quantum particles should deal more rigorously with the notions of identity, indistinguishability and individuality. If we agree that individuality and identity are closely related notions, and that absolute qualitative indistinguishability somehow prevents particles from having identity, then it is clear that quantum particles should be regarded as non-individuals. This was the proposal advanced by the Received View, roughly stated. However, the gap in that reasoning appears if we accept that things (in the plural) may be qualitatively indistinguishable but still retain their identity; then some notion of individuation may be retained. In that case, the principle grounding individuality should simply employ something "over and above" the distinguishing qualities to ensure individuality (there are some such principles available in the literature; see the discussions in Lowe[2]). So, to escape from non-individuality one has to locate individuality not in qualities, but somewhere else above qualitative differences.

The very possibility of understanding quantum mechanics as compatible with both an ontology of individuals and an ontology of non-individuals is strange enough to call for some further thinking: really, does it mean that something has gone wrong somewhere? It seems that our feelings concerning the possibility of *reading our ontology off from quantum mechanics* were, as mentioned, only ungrounded optimism. The hope that quantum mechanics could provide an answer to a typical philosophical problem, it seems, soured the naturalists' strategy. Really, quantum mechanics is *compatible* with both kinds of ontology, but, it is said, it does not force upon us any one of those ontologies, in the sense that there are no purely *quantum mechanical* constraints that could be called on to help us proving that one of those ontologies is better than the other (and for our purposes here, we should emphasize: there are no purely *quantum mechanical* constraints). That situation was baptized as the *underdetermination of metaphysics by the physics* (see specially French and Krause [3, chap. 4], and also French[4]).

Due to the underdetermination of metaphysics, if we are interested on ontological matters, a look at the literature on the subject will reveal that besides the purely instrumental approach (unattractive to the metaphysician) we have at least two options: i) forget about trying to read our ontology directly from quantum mechanics and adopt one of the metaphysical packages mentioned before based on other reasons than those provided solely by quantum mechanics, or ii) forget about particular objects and shift to an ontology of structures. As is well-known, the second option is called *Ontic Structural Realism* (OSR). According to the friends of OSR, not only in the face of the metaphysical underdetermination, but as a general revisionary metaphysical stance, particular objects should be eliminated in favor of relations and structures; the bearers of relations are only secondary, in some metaphysical sense (see for instance French and Ladyman,[5] Ladyman,[6,7] there are other versions of ontic structural realism endowing objects with some kind of ontological dignity too, but we shall not discuss them here).

However, it seems for us that those points only feed for more controversy. Indeed, those controversial issues are not as clearly put as they should. First of all, it should be recalled that OSR is motivated by two distinct issues (see the historical discussion in Ladyman[7]): the first one concerns the desire to be a realist in the face of the dispute concerning basically the no-miracles argument and the pessimistic meta-induction in recent debates about realism in philosophy of science. In that case, it is said, one can accommodate accumulation of knowledge through radical ontological revisions by being a realist only about the structure that is preserved in theory change, not about the nature of the entities dealt with. The second motivation comes, as we are discussing here, from the underdetermination of metaphysics by the physics. If one wishes to be a realist in philosophy of science and *cannot account for the nature of quantum entities* (due to metaphysical underdetermination), then, it seems, it is reasonable to be realist about the structure, not about the particular unobservable objects posited by the theory. So, according to the structuralist the best thing to do is to leave particular objects only as secondary characters, with no proper ontological dignity.

It is precisely in this second point that the issue is not as clearly put as it should. As we have mentioned before, the idea is that metaphysicians cannot uniquely read (some part of) the ontology of this world from quantum mechanics. The positive statement of that position, i.e. that we can univocally read our ontology from quantum mechanics, amounts obviously to some kind of *ontological naturalism*, the doctrine that science should provide us with all the answers to questions concerning the ultimate furniture of the world. Metaphysical underdetermination is an obstacle to such strong forms of *naturalism*, which we shall call *strict ontological naturalism*. But now, and that is the relevant point, what is the relationship between ontological naturalism and realism? Considering the motivations proposed by the friends of OSR, it seems that realism and naturalism are very close conceptions, so that underdetermination is a problem also for the realist. But is that necessarily so? What if the realist is not an adherent of such a strong form of naturalism? The same thing put differently: underdetermination is seen as a threat for the realist because the realist should be able to explain the metaphysical nature of the entities posited; however, underdetermination appears where a strictly naturalistic position concerning ontology is endorsed. The question then is: what is the relation between being a realist and being a strict naturalist? If one cannot be a realist without being a strict naturalist, then it seems we are doomed to be ontic structural realists.

In this paper we shall discuss one aspect of that problem, *viz.* the one concerning the relationship between metaphysical underdetermination in quantum mechanics and strict naturalism. That will be done with an eye in the debate about adopting an ontology of structures. We shall propose in Section 2 that the troubles the naturalist faces in ontology are a consequence of the (deliberate) conflation of two distinct kinds of problem: one of a quantificational nature, in which the theory may have active voice, and one of a metaphysical nature, in which the theory may have only some kind of negative influence. Having made that distinction, in Section 3 we argue that the particular problem of ontology concerning individuality manifests itself at a deeper level, in the underlying logic. The problem of whether the entities dealt with by a theory are individuals or non-individuals should be discussed in relation to the underlying logic. In Section 4 we propose that once those distinctions are made, one can consider being a realist of entities in some sense even in the face of metaphysical underdetermination; that is, underdetermination arises only for those holding too strict naturalistic constraints. We conclude in Section 5 with some remarks concerning the consequences of those arguments for the most radical version of ontic structural realism, *i.e.* the one proposing total eliminativism concerning particular objects. Our proposal is that even though OSR may be a plausible metaphysical position, it is not necessarily better equipped to deal with the metaphysical problems concerning the ontological nature of quantum particles, so that the possibility it provides for overcoming underdetermination should not be advanced as a further reason for us to become ontic structuralists of the eliminativist kind.

2. Ontology and Naturalism

The words "ontology" and "metaphysics" are sometimes used interchangeably. In this sense, one speaks sometimes about the ontology of a scientific theory, or, if it is taken as a synonym of metaphysics, about the metaphysics of that theory. In this paper, we distinguish both terms; we follow Lowe[8] in claiming that ontology is a branch of metaphysics. That is, there is a bunch of problems that are dealt with by metaphysicians, problems which constitute the core of known metaphysical subjects such as the studies concerning the nature of space and time, causality, possible worlds and the like. Ontology is one of such subjects, and it is concerned with the most general features of what there is. In that sense – as we shall discuss in what follows – ontologists are worried about whether there are particular individuals, properties and relations, abstract objects, and the like, and also on the precise characterization of those notions and their relations. It must also be remarked that this distinction does not mean that metaphysical subjects are not related to each other; on the contrary, they seem to benefit from their mutual relations, witness the weight that the discussion about possible worlds and modality bears on most metaphysical subjects nowadays, including the problem of individuality and identification.

But what are the reasons for making such a distinction? The whole point about ontology taken in the above sense, as concerned with the most general categories of being, is that it can be safely seen as a purely metaphysical investigation. That is, it seems to be a common ground of discussion for two kinds of philosophers: the ones taking science as having privilege in claiming what is the furniture of the world, and the ones looking for an extra room to allow the possibility of some kind of *a priori* investigation concerning those matters. Indeed, both will have to argue in favor of their views, but there will not be privilege for one of those sides as would happen, for instance, if we accepted that ontology concerns what there is according to a scientific theory. In that sense, it seems, there is only room for a naturalistic approach, and we would hardly be able to discuss traditional philosophical themes, as for instance the problem of individuality or the existence of possible worlds, since those are not subjects of scientific theories (but may, and indeed are, used to explain some metaphysical features of the entities inhabiting the world according to scientific theories).

The proper study of ontology as the investigation of the most general characteristics of being, as we mentioned, allows us to see where lies the problem with a purely naturalistic investigation of ontology. First of all, as we mentioned before, by a *naturalistic approach to ontology* we understand here the idea that science is the sole responsible for telling us what the furniture of the world is and which are its most general features. That is, to determine the components of reality, one must ask our most mature and successful scientific theories. Also, there is nothing further than what scientific theories say there is. So, besides leaving no room for things such as the Hegelian Absolute or Leibnizian monads (at least in this moment, 03/2013), naturalism also proposes that it is science itself that furnishes the methods, whichever those may be, to answer questions concerning the existence and ontological nature of the items populating this world.

It is precisely in this point, we believe, that the naturalist will have to face some difficulties in his treatment of ontological problems. By its very character, the naturalist interested in ontological problems will try to determine whether, for instance, quantum entities are individuals or non-individuals. To restrict ourselves to quantum mechanics (since that is the subject of this paper anyway), the investigation of that problem would demand, at first, that the naturalist make clear what is the interpretation of quantum mechanics being adopted. In general, for the sake of generality the interpretation adopted for those discussions is the one that is generally called the "minimal interpretation", according to which, roughly speaking, the theory is seen as furnishing only a mathematical instrument to make probabilistic predictions about measurements on physical systems, without further assumptions about the precise nature of those systems. That happens because it is expected by the naturalist that the theory itself should provide such further information concerning the nature of the entities in question. However, other interpretations, such as Bohm's, for instance, provide more substance to the theory (and some slight mathematical adjustments too, but that is not relevant for our purposes here), allowing that quantum particles always have well defined trajectories, among other features making the theory look a little more similar to classical mechanics, in strong contrast to the minimal interpretation (even though the trajectories are hidden variables, not known before measurement, and there is a mysterious pilot wave included in the ontology).

Just to have a name for future reference, we shall call the further details concerning the entities dealt with by the particular interpretation being held as the commitments at a *quantificational* level. In the case of non-relativistic quantum mechanics, an interpretation gives us the first answer to tell what the world looks like according to the theory. So, once an interpretation is adopted, some simple questions concerning the entities quantum mechanics deals with may be answered (even if in the negative sense, by denying meaning to those questions). We may positively say that the entities are waves, particles, pilot waves and particles, or that they have hidden well defined trajectories, and so on. We have called that one the quantificational level because it seems that providing an interpretation allows us some straightforward answers to those kinds of questions. That is, an interpretation furnishes some of those details; we may say that modulo one particular interpretation there are particles, that there are quantum events, that there are entities bearing well defined spatial positions every moment of their existence, and so on. However, an interpretation still does not provide answers to questions about the ontological nature of those entities, in the sense of an ontology of categories. We may reasonably ask about the particles, for instance (when the interpretation concerns particles) whether they are individuals or non-individuals, and which among the many distinct senses of the concept "individuality" is being employed in this particular case.

In that sense, having chosen one particular interpretation, can the naturalist solve the question once and for all, as would be expected? Having chosen, for instance, the minimal interpretation, can the naturalist clearly derive from quantum mechanics a unequivocal answer to the problem of individuality? Or can the naturalist at least solve the problem restricted to the actual state of the art scientific theory? Well, it seems that she cannot; there are some further questions that must be previously answered. The main problem here is: what is the concept of individual that is being assumed? The naturalist cannot approach quantum mechanics and simply derive from it that quantum entities are individuals or non-individuals without first having clearly determined that individuals are such and such, or that non-individuals are things such that ... (and there comes a definition or at least an explanatory note). That question would also plague the Bohmian account, even if only to a minor extent, in the sense that to grant that in such an interpretation the entities dealt with are individuals one must first accept (and argue for) the thesis that having well-defined trajectories in the particular sense proposed by this interpretation (as a hidden variable) is enough to grant individuation. Indeed, alternative accounts on individuality may impose some stronger constraints on what individuality entails and presupposes.

Once again, to have a proper name for this second problem in particular, we call the determination of the ontological nature of the entities quantum mechanics deals with the *ontological problem*, or, *the ontological level*. In this level we deal with the most general features of the entities, such as the ontological nature of particulars, the nature of properties and relations, and so on. It is at this level that the problem of understanding the meaning of individuality and non-individuality manifests itself. Notice, the quantificational level provides us with some of the features of the entities dealt with (for instance: they behave like waves, like particles, they bear properties in some rather unusual or even bizarre way, and so on). However, no interpretation fixes whether the entities it deals with are individuals, for the precise determination of what an individual consists in belongs to the ontological level, to the determination of what are the ontological categories we are willing to commit ourselves to. Those issues bear distinct relations to quantum mechanics.

In the face of the previous discussion, there is a problem that must be faced by anyone trying to account for the ontological nature of quantum entities: what does individuality consist in? Or, as the problem have been traditionally posed: what is the principle of individuality? That is the same old problem of individuality, the one that medieval philosophers have so much battled with. One cannot simply expect that quantum mechanics could answer those kinds of questions. That is, there is a plethora of options concerning the principle of individuality that could in principle be applied to quantum mechanics, and the theory cannot choose one of those options and claim that the entities it deals with satisfy that condition exclusively. The most discussed options are the bundle theories, transcendental individuality and also primitive individuality. Also, of course, there is the option of adopting some kind of non-individuality, which was first preferred by some of the founding fathers of the theory. Quantum mechanics, as we mentioned before, underdetermines those options, in the sense that the theory does not compel us to accept only one of them.

But what is the relation between the quantificational level and the ontological level in quantum mechanics? Are those issues completely unrelated? Not really. Given an interpretation of quantum mechanics, such as the minimal one (or some more contentious one, such as those committing themselves exclusively with particles), and an understanding of what do we mean by individuality (provided by the proper investigation in ontology), one may properly inquire whether quantum mechanics does support or reject such an approach to individuality. As is well known from the literature, it seems that quantum mechanics (in the minimal interpretation) is incompatible with versions of the bundle theory committed with a strong form of the Principle of Identity of Indiscernibles (PII), according to which items sharing all their monadic properties are the same. That is, quantum mechanics with the minimal interpretation does not support strong bundle theories that are attached to strong versions of PII, but it is an open question whether it supports other versions of the bundle theory.

Recently, to keep with the debate concerning the validity of PII, it was proposed that quantum mechanics could be seen as endorsing a weaker version of that principle. In a series of papers, Saunders, Muller and Seevinck have proposed that a version of PII now known as weak PII is compatible with quantum mechanics, and that this principle may be seen as somehow derivable from the theory (see Saunders,[9] Muller and Saunders,[10] Muller and Seevinck[11]). Those authors claim that every argument proposed until now concerning the failure of PII in the context of quantum mechanics has ignored some fine distinctions that should be made for the proper understanding of *discernibility*. That is, authors arguing that PII fails in quantum mechanics are right in claiming that strong versions of PII do not work in the theory, such as the version stating that items sharing all their monadic properties are one and the same (known as *Absolute PII*). However, that is not the end of it: one may discern things by employing relations too. For instance, the relation "having opposite spin to" in some given direction may be employed to discern two electrons, for no electron has spin opposite to itself. The general idea is that since quantum particles cannot be discerned by properties, the relations making the distinctions should not discern particles by the particular order of its relata, so that the only options left are those relations that are symmetric and irreflexive. Items discernible by this kind of relation are called *weakly discernible*, and items discernible exclusively by this kind of relation are called *relationals*.

Now, to point to the discussion that is being carried out in this section, the problem concerns whether relationals are individuals. That is, let us suppose that quantum particles are indeed relationals (as Saunders, Muller and Seevinck argue). Does that mean that it follows from quantum mechanics that quantum entities are individuals? There comes the problem of individuality at the ontological level, prior to the quantificational one. That is, to properly address that question we must first of all have made our minds as to whether weakly discerning relations do individuate, and whether discernibility by relations is allowed to count as discernibility at all. The idea, then, is that even though quantum mechanics may be compatible with a weak version of PII, it is a previous question of ontological nature to determine whether this compatibility is enough to grant individuality. Just to mention how that kind of controversy appeared already in the nest of the weak discernibility defenders, Saunders[9] seems to believe that discernibility through relations grants objecthood, and consequently, it also grants individuality (since he equates individuality with objecthood). Muller and Saunders,[10] on the other hand, argue that relationals are not individuals, but are objects nonetheless. They propose that individuals should be absolutely discernible, but objects may satisfy only the weaker condition of being relationals. Finally, Muller[12] believes that relationals are neither individuals nor objects, but are relationals nonetheless. Obviously, that controversy on the metaphysical nature of those entities cannot be decided on purely quantum mechanical grounds, since the facts at the quantificational level are the same for those authors, but it must rather be decided on metaphysical grounds (which are not the same for those authors, at least at first sight), once some ontological issues are advanced and made clear (see also Arenhart[13] for a discussion of how weak discernibility does not afford neither individuality nor discernibility so straightforwardly).

So, our general thesis is that what we have called the ontological level may gather indirect evidence in favor of its correction from quantum mechanics. That is, once we choose some principle of individuation, it is easier to show that quantum mechanics violates it, as is the case of strong forms of the bundle theory, rather than to have a prove that quantum mechanics approves that principle in some positive and definitive fashion. Really, even if we could agree that quantum mechanics provides us the reasons to believe that the entities it deals with are relationals, we still have not settled the ontological question of whether quantum entities are individuals or non-individuals. To determine the individuality question we must have previously decided ourselves about the individuality problem, *i.e.* we would have to decide ourselves as to the status of relations concerning the individuality and discernibility of particular items. Witness the difference of opinion concerning ontological matters between Saunders and Muller mentioned before in the face of the same quantum mechanical facts.

The consequences of those facts for the ontological naturalist (which illustrate the general thesis) seem clear enough. Once ontological issues seem to be out of the reach of a direct definitive confirmation by a scientific theory, it seems that ontological arguments gain some prominence and must be dealt with on other grounds (or, at least, should not be expected to be decided on purely scientific grounds). That is, some metaphysics will have already to be taken into account which cannot be taken from quantum mechanics. But that does not necessarily mean that all the accounts of individuality are on the same footing, as ontic structural realists would have it. On the contrary, that gives us the indication of where to look at in case of underdetermination, *i.e.* we must simply acknowledge that some issues must be decided on more speculative grounds, taken together with quantum mechanical facts: there must be some collaborative work between science and metaphysics. Before we discuss how that point is related to the realism problem that is at stake in the case concerning the motivations for OSR we have to make a brief detour through logic. It is our suggestion that since the problem of individuality does not appear to be solved positively at the level of quantum mechanics, so that there are no naturalistic grounds to prefer one answer to the other, it does manifest itself at the level of the underlying logic of the theory. It is the logic underlying the theory that somehow encodes some "hidden" information and assumptions concerning that issue. An investigation on these grounds may uncover important information. Let us see how.

3. Logic and Individuality

We have proposed that a convenient distinction between two levels in ontological research may be helpful in providing a better understanding of the underdetermination dilemma in which we find ourselves when pursuing the ontology of quantum mechanics. However, some further issues still have to be made clear before we can say more about how the distinction will improve our understanding of the problem. As we mentioned in the previous section, in the particular case of non-relativistic quantum mechanics half of the relevant information will be provided by the theory with an interpretation. The other half will rely more heavily on what would be called properly a metaphysical investigation, and concerns what we have called a category ontology (see also Lowe,[8,14] for a defense of one particular category ontology). Both the individuality of particular concrete objects as well as the presupposed acceptance of particular objects as ontologically primary in the ontology are matters to be discussed by this area, in connection with some of the demands posed by the theory.

Before we go on, it will be helpful to exemplify what we have in mind when we speak about such interplay between ontology and quantum mechanics, since it is relevant for the steps to be taken in this section. Suppose that grounded on philosophical reasons one adopts a one category ontology comprising only universals. In that case, particulars are generally understood as bundles of universals, tied together by a compresence relation, or something to that same effect, depending on who is proposing the theory. It is usual to try to explain the individuality of particulars in bundle theories by requiring uniqueness of each bundle composing a particular entity; that is, for any *two* bundles there is some universal that is instantiated by one of them, but not by the other. That assumption guarantees that numerical difference is always grounded in some form of qualitative difference, in some property possessed by one of the particulars but not by the other. In those versions of the bundle theory in which individuality and numerical difference are grounded in qualitative differences, one generally adopts some form of the Principle of the Identity of Indiscernibles (PII) to grant that the required fact really obtains. According to the PII, if items a and b share all of their properties, then, they are one and the same, $a = b$. That is a very sketchy description of how a bundle theory works, and obviously distinct versions of it are available accordingly as to how we fill the details, but what we have mentioned is enough for our purposes for this moment. Until now, we were concerned with what we have previously called the ontological level; there are however important details that must be filled by the particular scientific theory with which we are dealing.

The important details in this particular example concern mainly the way properties and property instantiation are to be understood. Those points must be furnished by our metaphysical theory of instantiation in connection with quantum mechanics if the mentioned cooperative approach between science and metaphysics is to work. The main problem for bundle theories adopting PII is that there is general consensus among philosophers that PII fails in quantum mechanics. Really, according to the usual understanding of quantum theory, properties are conceived as represented by Hermitean operators in the Hilbert space describing the possible states of the system. In that case, two or more entities of the same kind may be completely indiscernible, sharing all of their properties and nonetheless count as *two* entities (see French and Krause [3, chap. 4]). That is an instance of the negative influence of scientific theories over the ontological level we mentioned before. In this case, quantum entities seem to refuse to be understood as bundles of universals when bundle theory assumes some version of PII to grant the uniqueness of each bundle. Other possibilities to adopt bundle theory concern assuming a weaker form of PII (in connection with the proposals of Saunders, Muller and Seevinck mentioned before) or else adopt some strategy to grant numerical diversity in the face of qualitative identity other than PII, such as distinguishing a bundle of universals, which is unique, and its instances, which may be many and numerically distinct (see for instance Rodriguez-Pereyra[15]).

Other kinds of possible configurations of primitive categories are available, of course, and they may be equally applied in conjunction with our two level ontological scheme. What really matters for us here is both the case that a theory (with an interpretation) may be seen as rejecting a particular set of ontological categories and also that an interpretation of the theory helps us furnishing some of the details about how some of the metaphysical notions are to be understood, provided that those notions are already given. That second point is fundamental for our purposes here; one does not "read off" metaphysical notions from the theory, such as how property instantiation works and which individuality principle should be adopted. The theory only affords us an answer to such kind of problems as "is x an individual?" provided that a particular theory of individuality is given and a bunch of other notions is made precise enough. Now, it is time for us to provide an answer for the question of how those notions play a role in the theory; we must also make clear how the distinction between the quantificational and the ontological level may help us in overcoming the limitations of a too strict naturalistic approach to ontology. It is our proposal that the strict naturalist tried to derive the answers to problems at the ontological level by examining the theory without taking into account the proper role of the ontological level. But since the ontological level cannot be correctly approached by such a naturalistic method, how should we understand its proper study?

One possible suggestion of how the ontological level may be approached so that it may influence the quantificational level and be influenced by it concerns the behavior of the underlying logic of the theory. Really, one may propose that notions such as identity, property instantiation, quantity and unity, which are to be furnished by the ontological level of inquiry, are represented in the theory not only by the particular vocabulary of the theory, but also be influenced in great measure by the underlying logic. Classical first-order logic already assumes some kind of distinction between objects on the one side, and properties and relations on the other. That distinction is good for those willing to adopt at least two ontological categories, such as particulars and universals, but may also be adapted for a one category ontology comprising only properties and relations in some sort of bundle theory (for one such proposal, see for instance McGinn [16, pp. 55-56]). On the other hand, classical first-order logic does not distinguish between sortal predicates and adjectival predicates, or between simple accidental properties and natural kinds; consequently, there is no distinction between an item exemplifying a quality (such as being bald) and the same item instantiating a given kind (such as being a human being). Those distinctions must be introduced somehow if one is to adopt a category ontology comprising kinds, universals and particular items as distinct primitive ontological categories (see Lowe[14] for a possible extension of classical logic performing that kind of ontological work). Also, in classical first-order logic self-identity is valid for every item dealt with, because the reflexive law of identity is an axiom for identity in this logic, and this fact will be relevant in connection with individuality.

Still concerning property instantiation, one could adopt some kind of paraconsistent logic and allow that contradictory properties be instantiated by some items. Really, whether there are such cases is something to be decided by an examination of the theory in question, but property instantiation itself is something to be elicited in a great measure by the workings of the underlying logic. In the case of contradictory properties, a paraconsistent logic will allow us to do the required job. An instance could be Bohr's theory of the atom, which used tools and concepts from theories not compatible among themselves, or else Cantor's intuitive set theory and Frege's system in *Grundgesetze*, in both of which Russell's paradox may be derived (see da Costa[17] for paraconsistency in Cantor's system, and da Costa and Krause[18] for the case of Bohr's ideas). Notice, the possibility of an ontology comprising contradictory objects in this case is something that must be adjusted already in the ontological level of discussion, not readily read off from the scientific theory in question. Really, it is the correct adjustments in the underlying logic that will allow us to have such objects instead of a trivial theory. On the other hand, those not willing to concede that the logic must be changed will have to provide for changes in the theory or explain why the contradiction is only apparent (as it was done in the examples of the theories of Frege and Cantor mentioned before).

The problem of identity and individuality is the most important one when it comes to discuss individuality in quantum mechanics and the related issue of metaphysical underdetermination. Just to put things on a more rigorous basis, let us concede that when we speak of individuals we are talking about concrete particulars, and that any characterization of an individual must satisfy a *minimal requirement*: an individual is self-identical, and numerically distinct from every other individual. That may sound just too liberal a requirement for some philosophers, but it is purposely so in order to include some versions of primitive individuality. According to some views on individuality, this is a primitive notion; to clarify the idea of primitive individuality, it is said that what is meant by it is that every individual is self-identical and numerically distinct from every other individual. Numerical distinction taken in the sense of primitive individuality also grants that there is always a matter of the fact as to how many individuals there are in any circumstance, and that matter of fact is precisely the primitive identities of the individuals in that circumstance. There is no further explanation of individuality, since it is primitive and irreducible to any other concept (see also Dorato and Morganti[19] for the proposal that individuality is a primitive notion for quantum entities).

Self-identity and numerical difference are the minimal requirements. If something is an individual it satisfies the minimal requirements. The other way around may not be true for every account of individuality, but the sufficient condition for individuality is stated in such a liberal fashion to encompass primitive individuality. Other metaphysical approaches to individuality add to that minimal clause their own specific requirements, such as that numerical distinctness should be grounded by a qualitative difference in the case of some versions of bundle theory, or that there is some bare particular as an extra ingredient composing concrete particulars and working to the effect of endowing every item with individuality in the case of substratum theory, and so on for other approaches to individuality as well. However that may be, the minimal requirement is always present and satisfied by individuals, and it links identity with individuality in such a way that one may make clear our previous claim that the underlying logic of a scientific theory is responsible for attributing individuality in the minimal sense to every item it deals with when this logic happens to have an identity relation symbol and the reflexive law of identity holds universally.

Recall that the usual approach to first-order logic consists in assuming a primitive relation symbol "=" for the identity relation and adding to it the usual two postulates of reflexivity and the substitution law:

(1) For every x, $x = x$.
(2) $x = y \to (Fx \to Fy)$,

where Fx is a formula in which y does not appear free and Fy is the result of a substitution of one or more occurrences of x by y.

As is well-known, those axioms are not enough to characterize identity as the relation every object bears with itself and not with anything else. That is, if only those axioms are provided, it is not the case that in every interpretation of the first-order language with identity the denotation of the symbol = will be the diagonal of the domain (*i.e.* the set $\{\langle x, x\rangle : x \in D\}$, with D the domain of interpretation). To solve that problem, it is common to postulate that identity is a logical symbol and that it has a fixed interpretation given by the diagonal of the domain of the structure in which the interpretation is performed. Obviously, that move is performed in the metalanguage, and it is enough to grant that every item is identical to itself.

Identity is also present in higher-order languages. In those cases, it is usual to define identity through Leibniz' Law using quantification over higher-order variables for properties:

$$x = y \leftrightarrow \forall F(Fx \leftrightarrow Fy) \tag{1}$$

In that case, it is enough that we allow the property "to be identical with x" in the scope of the higher-order variable F to grant that the defined symbol is really identity, and that condition surely obtains in the standard semantics for higher-order languages. It was presupposed that x and y are individual variables, ranging over objects, but the definition easily generalizes itself to an identity relation between items of higher-orders, such as properties and relations, according to the level of the language.

Now, those discussions are relevant for our purposes because they indicate that irrespective of how we decide to ground our scientific theory, if the underlying logic is classical logic it introduces identity and its reflexive property for everything dealt with. It does not matter whether we choose to regiment the theory in a higher-order language, or in a set theory with first-order language as the underlying logic: identity is always present when the logic is classical. The same holds for theories understood according to the semantic approach, since most of its formulations are based on set theoretical structures; given that the set theory usually employed has first-order classical logic as its underlying logic (plus extensionality axiom), it follows that the semantic approach endorses this kind of view (see Krause, Arenhart and Moraes[20] for a rigorous development of the semantic view). Really, every theory has an underlying logic, which is usually assumed to be classical logic, but which must be made explicit in philosophical studies concerning ontology and the foundations of science. That means that a primitive version of individuality is being assumed in every case through the logical basis. As we mentioned, the logical basis is responsible for the introduction of the greatest part of the components of the ontological level of commitments, such as individuality and identity. Then, it is to the underlying logic that one should look when studying individuality and non-individuality in the quantum case.

From the fact that most issues concerning identity and individuality arise at the logical level we should not then be surprised that most discussions concerning those topics in quantum mechanics are prey to the metaphysical underdetermination phenomena. Really, if we only take into account quantum mechanical facts then we do not deal with every aspect of the problem, we do not take into account the ontological level, as we have called it. Furthermore, to assume classical logic is already to adopt some kind of commitment with some form of minimal individuality, even if we do not acknowledge it explicitly. On the other hand, once we recognize that we must endow the underlying logic with some of the categorical features of our ontology, we are free to employ the most convenient logic according to what our purposes are; we determine which logic should be used for our ontological purposes, not the other way around. We don't have to be troubled by metaphysical underdetermination, that won't paralyze us once we acknowledge that there are other means of inquiry and other questions that must be answered in such a way that our decisions on those cases matter for our theoretical purposes. That is, a scientific theory is a body of (theoretical and practical) information about the world, but it does not come without an even more abstract underlying part, which concerns its logic and, in general, its mathematics. When dealing with ontological matters some of the relevant information, the most general and abstract ones, will have to be shaped in this vocabulary, in this part of the theory.

We shall suggest later that it is this role of the underlying logic that may be employed to grant that some form of realism may be adopted which is not necessarily ontic structural realism. Really, both Ladyman[6] as well as French and Ladyman[5] have claimed that since the theoretical content of quantum mechanics does not provide enough information about the *metaphysical nature* of quantum entities, then, one cannot be justified in adopting entity realism in that case. They employ underdetermination here, and conclude that entity realism failed. Our claim is that one must not look only at what is furnished by quantum mechanics, but must also take into account the underlying logic and its implications. Once that is done, more than naturalistic reasoning must be employed, and the underdetermination does not appear to be so harmful.

Now, to take a look at the other side of the coin, to deal properly with non-individuals one must somehow restrict the reflexive law of identity. One possible way to do that concerns the use of non-reflexive logics, such as Schrödinger Logics and Quasi-set Theory $\mathcal{Q}$. In those systems, identity holds at most for some kinds of objects, but does not make sense for others, the ones representing the non-individuals. Also, since $\mathcal{Q}$ is a set theory, one may encode the idea of cardinality and also adopt a primitive relation for indistinguishability. Items representing non-individual in $\mathcal{Q}$ are such that they may be indistinguishable and non-identical (for the details see French and Krause [3, chaps. 7 and 8]). To grant cardinality without identity we must separate the notion of counting from identity, and that separation is the source of interesting philosophical and formal insights (see Arenhart[21]). Some authors have proposed that non-individuals should also violate the condition of numerical definiteness, so that in many situations not only self-identity is violated, but also the number of entities involved is not determined. Those situations appear in quantum field theory, where situations arise in which we may only determine the probabilities that there are n or m particles in a given situation, but no such number is attributed to the system before a measurement of particle number (with n and m natural numbers; see Domenech and Holik[22] for such discussions). Other kinds of formalizations of non-reflexive logics are possible, but notice that they always follow some kind of intuitive motivation concerning the metaphysical features we wish to capture in our logic. The revolution began with the rise of non-classical logics is also a step towards the liberalization in ontology.

What goes for quantum mechanics and the individuality versus non-individuality debate goes also for other kinds of metaphysical disputes. The fact that we must choose the convenient logic, and that this choice is a debatable matter does not mean that we have only taken the discussion a step back. Really, ontological disputes are not easily solvable, and at least we may rely on quantum mechanics to know which of them is *not* right. If we have not arrived at a consensus about which one *is* the one that gets it right, that is no issue to wonder at, since we are here dealing with controversial matters at their best. Also, most of the ontological categories presented nowadays are only developed up to a point, and are not clearly and rigorously connected with the ontological investigation at the quantificational level; they are not fully articulated with the workings of the most important scientific theories available. Maybe such kind of work may give us further reasons to believe that some of them are more promising and that others should be definitively rejected as incompatible with the theories. That kind of work will demand more from the ontologists, who will have to develop their proposals in relation with our actual theories.

4. Realism About What?

Those considerations prompt us to reconsider the role of a too strict naturalistic methodology in ontological investigations. We shall argue in this section that perhaps some metaphysical optimism is well grounded in the face of the distinction made above between the two kinds of ontological levels of inquiry. Those two levels are accompanied by two distinct but interrelated methodologies. At the quantificational level, as mentioned, if the theory in question is non-relativistic quantum mechanics one must take into account the demands of the theory along with an interpretation. Some interpretations may not be allowed (for instance, naive realistic interpretations), and the considerations for their rejection may be looked for both on experimental and on theoretical grounds. However, other kinds of interpretations are allowed, and they may be seen as populating the world with distinct kinds of entities: waves, pilot waves and particles, only particles, quantum events and so on. It is important to notice that some experimental work can in principle furnish positive evidence in favor of some of those interpretations against the others (see for instance Cirkovic[23] for an interesting discussion of many proposals to experimentally test distinct kinds of interpretations by using experiments that at this moment are only *in principle* possible). On the other hand, at the ontological level the information does not come only from the scientific theory, but derives also from other sources of theoretical knowledge tied to the theory, in particular from its underlying logic and mathematics. We have also made it explicit that even though the ontological level deals with more general features of things, this investigation is not completely independent from the theory under consideration. That happens because some of the most important traits of things imposed by the theory may refuse some sets of ontological categories, as the case of bundle theories with PII in quantum mechanics well illustrate.

Those distinctions are a good starting point for a defense of some form of realism distinct from *ontic structural* realism. That is, eliminativism concerning particular objects as preconized by the most revisionary form of OSR is not the only option for the realist, and is not mandatory given the presence of metaphysical underdetermination reigning in quantum mechanics. Let us examine the main reason advanced for the adoption of OSR in the face of metaphysical underdetermination. The first thing one must recall is that the dispute between realists and antirealists in philosophy of science concerns explaining the tension between the evident empirical success of science and the apparent ontological discontinuity through scientific revolutions, *i.e.* situations that appear when a new theory is developed whose ontology is (at least apparently) completely disjoint from the ontology of the superseded generation of empirically successful scientific theories.

As the realist puts the point, only realism is able to account for the success of science. Really, theories are empirically successful because they refer successfully to a mind independent reality, or else because they are approximately true (there is no consensual formulation of what scientific realism amounts to, so that this rather vague formulation is given here). In other words, theories work i) either because they somehow capture features of reality, a fact that is reflected by the idea that its theoretical terms successfully refer, or else ii) a theory work because it provides approximately true knowledge about the entities populating the world. The challenge then is to accommodate situations of scientific revolutions in which drastic revision in our ontological beliefs appear, raising suspicions that we will never be able to know the true nature of the furniture of the world. As is known, the challenge gets even more pressing once we notice the fact that we have been constantly revising our theories, and there are no prospects that it will not always be so in the future; there is little hope of finding the one true final theory not liable to further revision; so, in the face of it, there is a big challenge to the claim that we may confidently know the furniture of the world, period. The anti-realist cleverly uses that fact, and glances at the history of science as a museum of failed theories, none of which is able to give us a true account of the inhabitants of this world, even though they recognize that some of those theories are empirically adequate (and they provide an explanation for that fact independently of concepts such as "approximate truth" and "denoting terms"). In the face of such a picture, they ask, why hold on to the view that theories may be true? Why not forget about ontology and keep only empirical adequacy?

Structural realism proposes one answer to those anti-realists' challenges (see also Ladyman[7] for a statement of the problem). The first idea is that theories may suffer radical ontological revisions at the level of their putative objects, but some relations (mainly mathematical equations) are preserved through theory change. As an instance of relational continuity with ontological revision at the level of underlying objects, Ladyman [6, p. 415] offers Ehrenfest's theorem, $F(\langle r \rangle) = md^2\langle r \rangle/dt^2$, which is similar to the classical mechanics' equation $\mathbf{F} = m\mathbf{a}$. It is generally argued that Ehrenfest's theorem makes the connection between classical and quantum mechanics; both have structural similarities, but the objects they relate are distinct: expectation values of Hermitean observables in the quantum case, real variables in the classical case. So, following the general idea that may be learned from this example, why not consider that what is preserved in theory change is the structural similarity as captured by such relations and leave talk about the relata behind? In that case, objects may be granted a secondary place, and relations occupy the center of the stage; one then is asked to be realist about relations and the structures constituted by them, not about objects.

Of course, the fact that relations appear to be the candidates to play the role of persisting characters in theory change is not enough to grant the *elimination* of objects. We may conclude from the discussion above that objects are somehow hidden from us, never to be known over and above the relations they enter in. That is the case of *Epistemic Structural Realism* (ESR). Another option would be to claim that the old theory did get some of the properties of the objects right, while leaving room for mistaking others, and that made us abandon the old theory for the new one, whose empirical adequacy in situations where the old one failed attests that it is in the right way of carving nature even better (this is a rather crude account of Chakravartty's *Semirealism*; see Chakravartty[24]). To avoid those possible paths and get what she wants from the realist/anti-realist dispute, the friend of OSR needs quantum mechanics and underdetermination to motivate the more audacious move of eliminating objects. The decisive step is right at hand; according to the proponents of OSR, the realist accepting objects must be able to give an account of their *metaphysical* nature: are they individuals? Are they non-individuals? The realist is supposed to give us a principled reason for her choice. Unfortunately for the object-oriented realist, metaphysical underdetermination prevents her from performing such a justification, according to the friends of OSR. As we have mentioned before, quantum mechanics is compatible with both views, and there is no prospect of a well-grounded decision based on quantum theory. As Ladyman [6, p. 420] has put it, "It is an *ersatz* form of realism that recommends belief in the existence of entities that have such an ambiguous metaphysical status". What to do then? The answer, it is said, is to forget about objects and their natures and shift to an ontology comprising only structures and relations.

There are some difficulties with that line of reasoning, though. First of all, the link between underdetermination and the failure of object-oriented realism is not so clear. Why should the realist feel obliged to answer metaphysical questions concerning individuality or non-individuality? Second: there are some object-oriented options which may do the job without such a radical revision in our metaphysical beliefs. Third, the structuralist owes us an explanation of what a structure is, and how we should understand the idea of relations without relata. Those are very substantial problems, and here we shall touch on them only lightly.

Consider the first problem. According to the characterization of the realism/anti-realism debate, what the realist must explain is why science is successful, and that is done by explaining how our knowledge about the world may be close to the truth, not only empirically adequate, or else by explaining that the theoretical terms of a theory refer, depending on how one understands realism. As we have mentioned before, Ladyman[6] and French and Ladyman[5] argue that realists must also provide an answer to the problem of the metaphysical nature of the entities dealt with. But the reason why the realist must perform that task is not clear. Why should the realist, only because she is a realist, be concerned with *metaphysical* problems at the ontological level? Are answers to the question about the metaphysical nature of entities really necessary for an account of the success of a theory and for its claims to be close to the truth? Are claims about the metaphysical nature of entities really necessary for an account that some terms refer? Not really. As we mentioned, one may only adopt such a view once a strict naturalistic approach to metaphysics is chosen. Now, one may either be a realist and not bother with metaphysical questions, or be a realist and not be strictly naturalist, accepting our distinction between quantificational and ontological levels. Really, taking the second option to an extreme, a "dogmatic" metaphysician could hold that every entity is primitively individuated from the start (or any other possibility, non-individuals included, that are compatible with quantum mechanics), and the underdetermination problem would never appear. So, the motivation behind OSR gets a point only from those wishing to be strict naturalists about ontology, and we have already argued in the previous section that this kind of view is not the best way to approach the problem.

Now, for those accepting that there is a distinction between the quantificational and the ontological level of inquiry the problem posed by underdetermination is not really surprising. Underdetermination appears for those wishing to extract conclusions where tools for such conclusion are absent. Then, the claim that we must shift to an ontology of structures, in which the theory is re-conceptualized in terms of relations amounts to just another possibility in the metaphysical array of options. In the face of it, as any metaphysical option, OSR will have to be grounded by arguments that make it plausible and show the explanatory benefits of suppressing objects. The argument based on underdetermination does not do that job. French and Ladyman[5] claim that underdetermination is a problem for the realist, since our theoretical knowledge about the nature of the posited entities is provided only by the theory. Since the theory does not provide us with information to decide the issue between individuals and non-individuals, ontic structuralism is thought to be justified as the better option. We have argued before that the underlying logic is part of the theoretical knowledge provided by the theory, and that it is here that the categories dealing with issues such as individuality should be investigated. Having that in mind, we state our view again: underdetermination is a problem for the strict naturalist. There is more to the investigation of the individuality issue than only that which is offered by the quantum mechanical theoretical apparatus. To approach the issue in quantum mechanics one must have made his mind on the ontological level first, and then extract the feedback from the theory.

Once we realize that OSR is really an addition to the metaphysical options concerning the issue, in the absence of compelling arguments for it we may stick with other simpler options. That presents us with the second problem mentioned before. There are some options allowing us to be realists, to accommodate the problems the realist must deal with, and still need not such a revisionary view in metaphysics (even though we must grant that being revisionary is not an accusation that by itself should compel us to reject any metaphysical view). We have mentioned one such option, Chakravartty's semirealism. According to semirealism, we may separate objects' properties into *detection properties* and *auxiliary properties*. A detection property concerns the qualities of an entity by means of which we detect it. Auxiliary properties are those that help characterizing the entity but which may be abandoned in the course of theoretical change. In the view of alternatives like this the ontic structural realist will have to engage in metaphysical disputes concerning which of the available options is preferable, and it is reasonable that this should be that way; isn't the friend of OSR willing to engage in metaphysics anyway? That may be bad news for the naturalistic structuralist, but someone advancing an ontology of relations may well be seen as not allergic to metaphysical arguments and metaphysical reasoning.

The fact that some metaphysics will have to be allowed into the dispute makes clear also the third point. Since OSR proposes that everything may be re-conceptualized as structure, and a structure is characterized by relations without the relata, one may wonder how to make such claims more rigorous. In other words, we present the Socratic question to OSR: what is the metaphysical nature of structures? Recall that the trouble with object oriented realism, at least by the lights of the structuralist, was that naturalistic style realism could not provide an answer to that question. But now, conceded that we shift to an ontology of structures, can the naturalistic structuralist explain the metaphysical nature of structures and avoid the difficulties that plagued object oriented realism? Really, it seems that the idea of an ontology of relations must be explained is some sense, with relations being either universals, tropes, or something else. In that sense, since the objects are to be re-conceptualized as nodes in the web of relations forming the theory – they are secondary to the relations – it is not easy not to think that structuralism collapses in some form of bundle theory. Objects are dispensed with, only relations are accepted, and objects are bundles of relations. That, however, is not as revolutionary as the structuralists of the eliminativist kind may be thinking about, but is not as easily dismissible when there is no other clear option at offer. The problem with that suggestion is that it brings back problems typical from an ontology of objects, such as the possibility to reduce relations to properties, the validity of some relational form of PII and the role of relations in individuation. If that understanding of relations is the one the friend of OSR is willing to endorse, then it seems that she is really proposing only a more refined version of bundle theories, allowing only relations and no properties. That is, it is a one category ontology, comprising only universals of relations (or tropes of relations), and it also falls inside our general scheme presented before. As we mentioned, just as any other metaphysical option, OSR will have to present its metaphysical credentials, and it follows from our discussion above that it will not do to throw the weight of the argument on the shoulders of metaphysical underdetermination. The metaphysical nature of relations and structures remains to be explained.

Another related metaphysical problem for the structuralist is the challenge of explaining the idea that structure is what is preserved through theory change. The idea is that some relations are preserved, with the underlying objects going away as disposable items. How do we know that the same relations are preserved when a theory changes? Or are they distinct relations? Do we have identity conditions for relations without the relata? Notice that from the fact that $\mathbf{F} = m\mathbf{a}$ and $F(\langle r \rangle) = md^2\langle r \rangle / dt^2$ have some similarities we cannot conclude that there is some relation that is identical in both cases. As we mentioned, both of those relations relate distinct objects; doesn't that make them distinct? Then why don't we fall prey to the incommensurability problem? If something is preserved in those cases, one could suggest, it is the "form" of the relation, not the relation itself. However, that suggestion only makes matters worse. The form of a relation is then a property of relations, so that distinct relations may have the same form, and distinct theories have instances of the same relation-form. OSR then could be seen as claiming that relation-form is what is preserved through theory change. But then we have only shifted the problem to a higher-order level: we must explain why it is the same higher-order relation-form that is being instantiated in both cases, and why that accounts for the success of the theory. That is no simple task, and may be more close to traditional hard core speculative metaphysics than the friend of OSR would like to admit.

The reasonable conclusion is that if ontic structural realism is advanced as an alternative to object-oriented metaphysics and hoping to keep a naturalistic methodology, then the naturalistic structuralist will have to recognize that she must face some of the same dilemmas of her old fashioned colleges. On the other hand, if it is proposed as an alternative metaphysical theory, then the dispute with other metaphysical options will have to be conducted on the grounds mentioned before: mixing quantum mechanical arguments with *a priori* reasoning that are typical of metaphysics. That brings us to our next final topic.

5. Back to First Philosophy?

As we have put the question, there are two kinds of investigation to be discerned: the ones from the quantificational level, and the ones from the ontological level. The scientific realist, when she is concerned with the existence of certain entities, is concerned with the quantificational level. The metaphysician, on the other hand, is concerned with answers to the ontological level, as they are asked from the entities posited by the investigation in the quantificational level. Science itself is only indirectly concerned with ontological questions (*i.e.* questions from the ontological level as opposed from questions arising from the quantificational level). By "indirectly" here we mean that its methods of inquiry are not primarily developed to deal with those questions. In fact, that is why those kinds of problems concern philosophers, and not physicists or biologists; there is no problem in physics about the individuality of quantum entities, but there is a metaphysical problem of individuality.

It was not the main goal of this paper to make explicit the method of metaphysics and how metaphysicians should proceed to determine answers to their questions. Indeed, that in itself is a very controversial metaphysical problem. However, it was our aim to promote the clarification of two distinct kinds of enquiry that should be clearly separated, so that we do not conduce with restricted tools, for instance, the kind of investigation that concerns the problem of identity and individuality of quantum particles. We believe that it is not through purely quantum mechanical reasoning that we will arrive at the answer to problems such as the one concerning individuality. Philosophy has its duties too.

That prompts another question: are we back to first philosophy then? Should we determine the answers to those questions directly from the armchair? Not really.

The position defended in this paper was that a productive relation may obtain between metaphysics – and in particular ontology understood as a category ontology – and science. Both will have to contribute in finding answers to the problems. The metaphysician will have to look for the most apt set of ontological categories to give a reasonable account of the metaphysical nature of the entities existing according to scientific theories. Scientific theories, the well-established ones, will provide some sort of feedback. As we have exemplified before, the idea is that many possible sets of categories are available, and some of them are implausible on metaphysical grounds, others on scientific grounds (as is the case of some forms of bundle theories in quantum mechanics). However, many sets of categories remain that are compatible with our best scientific theories, and no possible experiment can at this moment determine which of them is the right one (if we believe there is a right one, as metaphysicians are almost always inclined to do). This position, allowing for the interference of science in metaphysics and metaphysics in science, is what we call *mitigated naturalism*, in opposition to what we labeled throughout this paper as *strict naturalism*. Recall that strict naturalism somehow proposed that we derive facts concerning the categories from the theory itself, and that those facts should work to the effect of giving positive support to a particular metaphysical view. Mitigated naturalism demands that the theory show us which categories are not right, and leave it for further investigation on both grounds (metaphysical and scientific) to delimit even more the range of plausible candidates (for further discussion, see Arenhart[25]).

Logic will have the role of dealing with the distinct options on the categorical ontology. In logic we will have to represent the distinct ontological categories that are being assumed by us. We do not say that logic does play that role directly, some interpretation is required. However, it is in the underlying logic that those notions appear. Here it is open to us to decide – based on arguments – whether there is only one logic or a plurality of them that will perform the job. One may be pluralist in ontology and logic. It is generally argued that the main purpose of a category ontology is to unify the distinct aspects of the world and the distinct theories in distinct branches of science with the intuitive idea that there is only one world, and one truth about it. Those are issues that will not be dealt with here, but we believe that the association made with logic may help us in determining more precisely concepts such as reduction of levels of reality and emergence. That idea will also involve an epistemology of logic which we have not touched on this paper (as we mentioned, it is our metaphysics which helps us to decide about which logic will be more appropriate for us in attaining our ontological goals). Now, it seems, logic will be related with the most general aspects of reality, and since we have distinct possible logics, they may be understood as describing distinct possibilities of reality. Some of those possibilities will show themselves more satisfactory than others, but it is not up logic alone to decide that issue.

The structuralist wanted to motivate her position with underdetermination and ground the choice of her option in the demands of science. Now, once we make the above distinction and hold to the view that strict naturalism may be abandoned, we must recognize the fact: given that the ontic structuralist's option is seen just as another metaphysical option, she finds herself in the same boat as everyone else. Either she holds tight to her strict naturalistic credos and admit that the proposed motivation coming from realism is not enough to justify structuralism, or else she admits that she will also have to cooperate in the metaphysical search for the nature of reality, and furnish the corresponding logic.

References

1. S. French, *Australasian Journal of Philosophy* **67**, 433 (1989).
2. E. J. Lowe, Individuation, in *The Oxford Handbook of Metaphysics*, eds. M. J. Loux and D. W. Zimmerman (Oxford Un. Press, Oxford, 2003) pp. 75–95.
3. S. French and D. Krause, *Identity in Physics. A historical, philosophical and formal analysis* (Oxford Un. Press, Oxford, 2006).
4. S. French, *Synthese* **180** (2011).
5. S. French and J. Ladyman, *Synthese* **136** (2003).
6. J. Ladyman, *Stud. Hist. Phil. Sci.* **29**, 409 (1998).

7. J. Ladyman, Structural realism (2009), `http://plato.stanford.edu/archives/sum2009/entries/structural-realism/`.
8. E. J. Lowe, *The Possibility of Metaphysics: Substance, Identity and Time* (Oxford University Press, Oxford, 1998).
9. S. Saunders, *Analysis* **66** (2006).
10. F. A. Muller and S. Saunders, *British Journal for the Philosophy of Science* **59** (2008).
11. F. A. Muller and M. P. Seevinck, *Philosophy of Science* **76** (2009).
12. F. A. Muller, *Synthese* **180** (2011).
13. J. R. B. Arenhart, Weak discernibility in quantum mechanics: does it save PII?, Forthcoming in *Axiomathes*, DOI 10.1007/s10516-012-9188-x, (2012).
14. E. J. Lowe, *The Four-Category Ontology. A metaphysical foundation for natural science* (Oxford Un. Press, Oxford, 2006).
15. G. Rodriguez-Pereyra, *Analysis* **64** (2004).
16. C. McGinn, *Logical Properties: Identity, Existence, Predication, Necessity, Truth* (Clarendon Press, Oxford, 2000).
17. N. C. A. da Costa, *Principia* **6**, 279 (2002).
18. N. C. A. da Costa and D. Krause, The logic of complementarity, in *The Age of Alternative Logics: Assessing Philosophy of Logic and Mathematics Today*, eds. H. G. R. M. van Benthem, J. and H. Visser (Springer, Dordrecht, 2006) pp. 103–120.
19. M. Dorato and M. Morganti, *Phil. Stud.* **163**, 591 (2013).
20. D. Krause, J. R. B. Arenhart and F. T. F. Moraes, *Foundations of Science* **16** (2011).
21. J. R. B. Arenhart, *Studia Logica* **100**, 437 (2012).
22. G. Domenech and F. Holik, *Found. Phys.* **37**, 855 (2007).
23. M. M. Cirkovic, *Found. Phys.* **35**, 817 (2005).
24. A. Chakravartty, *Stud. Hist. Phil. Sci.* **29**, 391 (1998).
25. J. R. B. Arenhart, Wither away individuals, Forthcoming in *Synthese*, DOI 10.1007/s11229-012-0204-x, (2013).

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Since its origins, quantum mechanics has presented problems with the concept of individuality. It is argued that quantum particles do not have individuality, and so, one can speak about "entities without identity". On the contrary, we claim that the problem of quantum non individuality goes deeper, and that one of its most important features is the fact that there are quantum systems for which particle number is not well defined. In this work, we continue this discussion in relation to the problem about the one and the many.

Keywords: Quasisets; particle number; quasicardinality; quantum indistinguishability.

1. Introduction

The concept of individuality in quantum mechanics clashes radically with its classical counterpart. In classical physics, particles can be considered as individuals without giving rise to consistency problems but, in quantum mechanics this is not the case. Problems arise if one intends to individuate elementary particles. The responses to this problem range from the claim that there are no elementary particles at all to the assertion that there are particles but they are intrinsically indistinguishable (i.e., indistinguishable in an ontological sense). Some authors talk about "entities without identity".[1] In this work, we claim that the problem of quantum non individuality is even worse: quantum non individuality clashes with the concept of number. In this work we discuss the significance of the superpositions of states with different particle number in the Fock-space formalism. We continue to interpret them as systems with an ontologically undefined particle number. And we conclude that quantum systems not only suffer the loss of transcendental identity; they also loss the property of having a definite number. A quantum system is not a *one*, nor a *many*. But in spite of it, it *is*.

In relation to the problem of quantum indistinguishability, Michael Readhead and Paul Teller claim[2] that:

> "Interpreters of quantum mechanics largely agree that classical concepts do not apply without alteration or restriction to quantum objects. In Bohr's formulation this means that one cannot simultaneously apply complementary concepts, such as position and momentum, without restriction. In particular, this means that one cannot attribute classical, well defined trajectories to quantum systems. But in a more fundamental respect it would seem that physicists, including Bohr, continue to think of quantum objects classically as individual things, capable, at least conceptually, of bearing labels. It is this presumption and its implications which we need to understand and critically examine." M. Readhead and P. Teller (Ref. 2, p.202)

When individuality of quanta is studied exhaustively, most investigations seem to point in the direction that quanta have no individuality at all (see for example Ref. 1 for a detailed analysis). To put it in Schrödinger's words:

> "I mean this: that the elementary particle is not an individual; it cannot be identified, it lacks 'sameness'. The fact is known to every physicist, but is rarely given any prominence in surveys readable by nonspecialists. In technical language it is covered by saying that the particles 'obey' a newfangled statistics, either Einstein-Bose or Fermi-Dirac statistics. [...] The implication, far from obvious, is that the unsuspected epithet 'this' is not quite properly applicable to, say, an electron, except with caution, in a restricted sense, and sometimes not at all." E. Schrödinger (Ref. 3, p.197)

This work should be considered as a tentative to enlarge the problem posed in the quotation of Readhead and Teller cited above. We claim that it is usually supposed that quantum systems can be considered as singular unities, (a quantum system as a "one"), or collections of them (a quantum system as a "many"). In this work we want to "understand and critically examine" this assumption, thus continuing the developments in Refs. 4 and 5.

In Section 2, we discuss the meaning of superpositions of particle number eigenstates in Fock-space. In Section 3 we discuss the assumptions and limitations of the notion of particles aggregate. Then, in Section 4, we review the approach of quasiset theory to quantum indistinguishability. In Section 5, we review the modifications to Quasiset-theory made in Ref. 4 and in Section 6 we discuss an enlargement of the Manin's problem.

2. Quantity in Quantum Mechanics

As is well known, performing a single measurement in a quantum system does not allow to attribute the result of this measurement to a property which the system possesses before the measurement is performed without giving rise to serious problems.[6] What is the relationship between this fact and the quantity of particles in a quantum system? Take for example the electromagnetic field (with a single frequency for simplicity) in the following state:

$$\mid \psi\rangle = \alpha \mid 1\rangle + \beta \mid 2\rangle \tag{1}$$

where $\mid 1\rangle$ and $\mid 2\rangle$ are eigenvectors of the particle number operator with eigenvalues 1 and 2 respectively, and α and β are complex numbers which satisfy $\mid \alpha \mid^2 + \mid \beta \mid^2 = 1$. If a measurement of the number of particles of the system is performed, one or two particles will be detected, with probabilities $\mid \alpha \mid^2$ and $\mid \beta \mid^2$ respectively. Any other possibility is excluded. Suppose that in a single measurement two particles are detected. What allows us to conclude that the system had two particles before the measurement was performed? The assertion that the number of particles is varying in time because particles are being constantly created and destroyed is also problematic, because it assumes that at each instant the number of particles is well defined. Only in case that it is known with certainty that the system is in an eigenstate of the particle number operator we can say that the system has a well defined cardinal. There would be no problem too if it is known with certainty that the system is prepared in an statistical mixture. In this case, the corresponding density operator would be:

$$\rho_m = \mid \alpha \mid^2 \mid 1\rangle\langle 1 \mid + \mid \beta \mid^2 \mid 2\rangle\langle 2 \mid \tag{2}$$

where the subindex "m" stands for statistical mixture. But the density operator corresponding to (1) is:

$$\rho = (\alpha \mid 1\rangle + \beta \mid 2\rangle)(\alpha^* \langle 1 \mid + \beta^* \langle 2 \mid) \tag{3}$$

which is the same as:

$$\rho = \mid \alpha \mid^2 \mid 1\rangle\langle 1 \mid + \mid \beta \mid^2 \mid 2\rangle\langle 2 \mid + \alpha\beta^* \mid 1\rangle\langle 2 \mid + \alpha^*\beta \mid 2\rangle\langle 1 \mid \tag{4}$$

The presence of interference terms in the last equation implies that difficulties will appear in stating that, after a single measurement, the system has the quantity of particles obtained as the result of the measurement. In this case, the incapability of knowing the particle number would not come from our ignorance about the system, but from the fact that in this state, the particle number is not even well defined. We see thus, that if superpositions of particle number are allowed, we are faced with an *indeterminancy* in the particle number. An indeterminancy of the same kind of that which appears in position or spin. But this time, the "property" affected is very special; *it is just the property linked to the dichotomy of being one or many.*

3. How Many?

Let us discuss the origin of the concept of "particle number". We start posing the questions: In which sense do we talk about quantum systems composed, for example, of a single photon? How do we decide if the field is in a single photon state or not? What do we mean when we use the words "single photon"? We could search for a clue to answer these questions in our laboratory experience, i.e., making measurements. In experiments, we often use a picture which allows us to speak about the photon as a particle (and so, as an individual). In a similar way, and always in relation to experiments, we talk about the other particles (electrons, protons, etc.). But in a deeper analysis we find that this supposed "particle behavior" of quantum systems is hardly compatible with its classical counterpart, and though experiments seem to suggest an idea of individuality, it is well established that this does not enable us to consider particles as individuals. Individuality is not compatible with the formal structure of quantum mechanics[5] and so elementary particles cannot be considered as individuals, as E. Schrödinger pointed out in the early days of the theory.[3] In spite of these difficulties, we continue speaking about photons, electrons, etc., using a jargon which has a lot of points in common with classical physics, source of conceptual confusion.

But it is just this interpretation, which presupposes the concept of 'particle' (or quantum object, to put it in more general terms) which lies at the heart of the notion of 'particles aggregate' (and so, a *defined* particle number, or a *defined* number of objects). There are definite experimental arrangements which force the appearance of particle characteristics as a final result of a single process. Experiments are designed to find out which is the particle number, but as we have already mentioned in Section 2, this does not imply that the resulting number is a property that pertains to the system under study. On the contrary, it refers to the definite *process* which takes place in each measurement. We are not allowed to consider the system as an aggregate of individuals as if they were classical objects, simply because the notion of "object" is incompatible with the formal structure of the theory.[5] The fall of the notion of "object", causes the fall of the notion of "objects aggregate".

We claim that superpositions in particle number discussed in Section 2 are a direct expression of the fall of the concept of "objects aggregate". We know that it is possible to assign to some quantum systems an associated number, take for example the electrons of a Litium atom, or single photon states. But particle number superpositions show that in general, it is not true that a definite number can be always assigned in a consistent manner. We are accustomed in our classical experience to assign always a definite number to the set of things that we are studying. But when we are faced to a quantum system, we are forced to abandon that habit.

4. Quasi-Set Theory and Indistinguishability

In the standard approach to quantum indistinguishability, particles are labeled as if they were individuals, and then indistinguishability is recovered via symmetrization postulates.[2] This is a variant of what in Ref. 7 was called the Weyl's strategy. Many authors pointed out the importance of developping alternative ways to describe quantum indistinguishability, reproducing the results obtained by standard techniques, but assumming in every step of the deduction that elementary particles of the same class are intrinsically indistinguishable from the beginning (see, for example, Refs. 1, 7 and 8), without making appeal to Weyl's strategy variants. Another claim is that quantum mechanics does not possess its own language, but it uses a portion of functional analysis which is itself based on set theory, and thus finally related to classical experience. This statement was posed by Y. Manin,[9] the Russian mathematician who suggested that standard set theories (as Zermelo-Fraenkel, ZF) are influenced by every day experience, and so it would be interesting to search for set theories which inspire its concepts in the quantum domain. This is known as Manin's problem.[10] In this spirit, and looking for a solution to Manin's problem a quasiset theory ($\mathcal{Q}$ in the following) was developed[7,11] (see also Refs. 12 and 13).

Quasiset theory seems to be adequate to represent as "sets" of some kind (quasisets) the collections of truly indistinguishable entities. This aim is reached in $\mathcal{Q}$ because equality is not a primitive concept, and there exist certain kinds of *urelemente* (m-atoms) for which only an indistinguishability relationship applies. So, in $\mathcal{Q}$, non individuality is incorporated by proposing the existence of entities for which it has no sense to assert that they are identical to themselves or different from others of the same class.

$\mathcal{Q}$ contains a copy of Zermelo-Fraenkel set theory plus Urelemente (ZFU). These Urelemente are called M-atoms. This feature divides the theory in two parts. One region involves only the elements of ZFU, and the other one contains quasisets whose elements can be truly indistinguishable entities. Quasisets containing only indistinguishable elements are called "pure quasisets". The ZFU copy of quasisets is called "the classical part of the theory " in Ref. 11. Indistinguishability is modeled in this theory using a primitive binary relation $\equiv$ (indistinguishability) and a new class of atoms, called m-atoms, which express the existence of quanta in the theory.[11] So, in the frame of $\mathcal{Q}$, when we speak of m-atoms of the same class, the only thing that we can assert about them is that they are indistinguishable, and nothing else makes sense, for expressions like $x = y$ are not well formed formulas. This is to say that we cannot make assertions about their identity, i.e., it has no sense to say that an m-atom is equal or different of other m-atom of the same class. It is important to remark that in $\mathcal{Q}$, indistinguishability does not imply identity, and so it is possible that even being indistinguishable, two m-atoms belong to different quasisets, thus avoiding the collapse of indistinguishability in classical identity.[11]

$\mathcal{Q}$ is constructed in such a way that allows the existence of collections of truly indistinguishable objects, and thereof it is impossible to label the elements of pure quasisets. For this reason, the construction used to assign cardinals to sets of standard ZFU theories cannot be applied any more. But even if electrons are indistinguishable (in an ontological sense), every physicist knows that it makes sense to assert that, for example, a Litium atom has three electrons. It is for that reason that $\mathcal{Q}$ should allow quasisets to have some kind of associated cardinal. In $\mathcal{Q}$ this is solved postulating that a cardinal number is assigned to every quasiset (remember that there is a copy of ZFU in $\mathcal{Q}$). Some other properties of the standard cardinal are postulated too. This rule for the assignment of cardinals uses a unary symbol $qc()$ as a primitive concept. So in $\mathcal{Q}$, the quasicardinal is a primitive concept alike ZF, in which the property that to every set corresponds a single cardinal number can be derived from the axioms.[14]

An important theorem of $\mathcal{Q}$ is related to the unobservability of permutations:

[Unobservability of Permutations] Let x be a finite quasi-set such that x does not contain all indistinguishable from z, where z is an m-atom such that $z \in x$. If $w \equiv z$ and $w \notin x$, then there exists w' such that

$$(x - z') \cup w' \equiv x$$

It is the assertion in the language of $\mathcal{Q}$ that permutation of indistinguishable quanta cannot imply any observable effect, or put in words of Penrose:

> "[a]ccording to quantum mechanics, any two electrons must necessarily be completely identical [in the physicist's jargon, that is, indistinguishable], and the same holds for any two protons and for any two particles whatever, of any particular kind. This is not merely to say that there is no way of telling the particles apart; the statement is considerably stronger than that. If an electron in a person's brain were to be exchanged with an electron in a brick, then the state of the system would be *exactly the same state* as it was before, not merely indistinguishable from it! The same holds for protons and for any other kind of particle, and for the whole atoms, molecules, etc. If the entire material content of a person were to be exchanged with the corresponding particles in the bricks of his house then, in a strong sense, nothing would be happened whatsoever. What distinguishes the person from his house is the *pattern* of how his constituents are arranged, not the individuality of the constituents themselves" (Ref. 15, Page 32).

In the next section, we go back to the problem of an undefined particle number and we relate it with $\mathcal{Q}$.

5. Modifications to the Theory of Quasi-Sets

The way in which the quasicardinal is introduced in $\mathcal{Q}$ implies that every quasiset has an associated cardinal, i.e., every quasiset has a well defined number of elements. But the idea that an aggregate of entities must necessarily have an associated number which represents the number of entities is based in our every day experience. As we have mentioned in Section 2, there are quantum systems to which it is not allowed to assign a number of particles in a consistent manner.

Taking into account these considerations, it is worth asking: is it possible to represent a system prepared in the state (4) in the frame of quasiset theory? Which place would correspond to a system like (4) in that theory? If such system could be represented as a quasiset, then it should have an associated quasicardinal, for every quasiset has it. But this does not seem to be proper, considering what we have disscussed in Section 2. It follows that it does not appear reasonable to assign a quasicardinal to every quasiset if quasiset theory has to include all quantum systems (in all their possible many particle states). Therefore, a system in the state (4) cannot be included in $\mathcal{Q}$ as a quasiset. Yet, it would be interesting to study the possibility of including systems in those states (such as (4)) in the formalism.

A possible way out would be the introduction of a Fock-space, but constructed using the non classical part of $\mathcal{Q}$. This option has been considered in Ref. 16. In that paper, we reformulate Fock-space quantum mechanics using $\mathcal{Q}$. Due to the theorem of unobservability of permutations (see last section of this work), we avoid the use of the *labeled-tensor-product-Hilbert-space formalism* (LTPHSF), as called by Redhead and Teller.[2,17] So we can express states such as (4) as superpositions of different particle number in this novel space.

Another possibility is to reformulate $\mathcal{Q}$ in such a way that the quasicardinal is not to be taken as a primitive concept, but as a derived one, turning it into a property that some quasisets have and some others do not (in analogy with the property "being a prime number" of the integers). Those quasisets for which the property of having a quasicardinal is not satisfied, would be suitable to represent quantum systems with particle number not defined (such as equation (4)). This property would also fit well with the position that asserts that particle interpretation is not adequate in, for example, quantum electrodynamics. With such a modification of $\mathcal{Q}$, for example, a field (in any state) could always be represented as a quasiset, avoiding the necessity of regarding the field as a collection of classical "things". On the contrary, the field would be described by a quasiset which has a defined quasicardinal only in special cases, but not in general. Thus a field would be represented as something which genuinely is nor a one nor a many. And for that reason this quasiset could not be interpreted so simply as a collection of particles (because it seems reasonable to assume that a collection of particles, indistinguishable or not, must always have a well defined particle number).

We have followed the idea of modifying $\mathcal{Q}$ in Ref. 4. In that work, we have searched for a $\mathcal{Q}$-like theory in which the quasicardinal is not taken as a primitive concept alike $\mathcal{Q}$. We showed that it is possible to develop a theory about collections of indistinguishable entities in which quasicardinal is not a primitive concept. We define a singleton which allows us to extract "just one element" form a given quasiset X, and obtain a subquasiset X^-

such that:

$$X \supset X^{-}$$

It is important to remark that it has no sense to ask which is the extracted element, because this query is not defined in a theory without identity. It could happen that $X^{-} = \emptyset$ or not. If $X^{-} \neq_E \emptyset$ it follows that we can make the same operation again, and obtain X^{--}. Then we have:

$$X \supset X^{-} \supset X^{--}$$

Going ahead with this process, it could be the case that this chain of inclusions stopped (in case the last quasiset so obtained be the empty quasiset), or that it has no end. So we could conceive two qualitatively different situations:
Situation 1:

$$X \supset X^{-} \supset X^{--} \supset X^{---} \supset \cdots\cdots$$

(the inclusions chain continues indefinitely)
Situation 2:

$$X \supset X^{-} \supset X^{--} \supset X^{---} \supset \cdots\cdots \supset \emptyset$$

(the inclusion chain ends in the empty quasiset).
We call *descendent chains* to the chains of inclusions which appear in situations 1 and 2. In order to grant that one of these situations holds for every quasiset we postulate the existence of descendent chains. This is done by first translating the concept of descendent chain to first order language:

Definition 5.1.

$$CD_X(\gamma) \longleftrightarrow$$

$$(\gamma \in \wp(\wp(X)) \wedge X \in \gamma \wedge \forall z \forall y (z \in \gamma \wedge y \in \gamma \wedge z \neq_E y \longrightarrow (z \supset y \vee y \supset z))$$

$$\wedge \forall z (z \in \gamma \wedge z \neq_E \emptyset \longrightarrow \exists y (y \in \gamma \wedge DD_z(y) \wedge \forall w (w \in \gamma \wedge DD_z(w) \longrightarrow w =_E y))))$$

$CD_X(\gamma)$ is read as: γ is a descendant chain of X.

In the construction shown in Ref. 4 we have reobtained that every (finite) quasiset has a well defined quasicardinal. But in that construction, we can assert that a quasiset has a definite quasicardinal only if it satisfies the definition of finiteness, and nothing can be said about the quasisets which do not satisfy the definition. Though not necesarilly useful for the problem of the undefined particle number, the axiomatic variant exposed in

Ref. 4 shows explicitly that in a theory about collections of indistinguishable entities, the quasicardinal needs not be necessarily taken as a primitive concept. This result encourages the research of more complex axiomatic formulations, able to incorporate the quantum systems with undefined particle number as sets of some kind, thus enriching Manin's problem.

6. A New Turn on Manin's Problem

In this article, we have disscussed that quantum systems not only seem to lack individuality, but moreover, they seem to be undefinite in their cardinality. What is the meaning of this assertion? Perhaps, we have to abandon the supposition that we are dealing with something like "entities without identity", in the sense that we cannot consider a quantum system as a "one" or a "collection of ones". Perhaps, besides the ontological presupposition of individuality, we have to abandon the presupposition of number: so, we would have *neither name, nor number*. In this sense, the suggestions of Manin:

> "We should consider the possibilities of developing a totally new language to speak about infinity.[a] Classical critics of Cantor (Brouwer *et al.*) argued that, say, the general choice axiom is an illicit extrapolation of the finite case.
> I would like to point out that this is rather an extrapolation of common-place physics, where we can distinguish things, count them, put them in some order, etc. New quantum physics has shown us models of entities with quite different behavior. Even 'sets' of photons in a looking-glass box, or of electrons in a nickel piece are much less Cantorian than the 'set' of grains of sand. In general, a highly probabilistic 'physical infinity' looks considerably more complicated and interesting than a plain infinity of 'things'."

should be enlarged in order to include the question of undefined number. It would be very interesting to search for a "set theory" inspired on quantum phenomena, which could take into account the problem of undefined cardinality (perhaps, the term "set" would not be appropriate any more in a theory like this). It is quite possible that the development of a theory like this one, would yield a new way to approach the problem of quantum separability. Perhaps, when we speak of a quantum system composed of two subsystems we are thinking of it *as a collection of things*, and this could be

[a]Set theory is also known as the theory of the 'infinite'.

linked to the problems which emerge in quantum separability, because we have a wrong way to speak about it.

$\mathcal{Q}$ gives an answer to Manin's problem, in the sense that it is a "set theory" concerning collections of truly indistinguishable objects. But in order to solve the extended version of Manin's problem, further formal developments needs to be achieved. The axiomatic variant presented in Ref. 4 shows that in a theory like $\mathcal{Q}$, the quantity of elements needs not to be a primitive concept. This result encourages the search for a theory in which it is impossible to assign a quasicardinal to certain quasisets in a consistent manner, thus allowing to describe what it seems to happen with some quantum systems, in which non individuality expresses itself in the fact that particle number is not defined, besides ontological indistinguishability.

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References

1. French, S. and Krause, D., *Identity in Physics: A historical, Philosophical, and Formal Analysis*, Oxford University Press (2006).
2. Redhead, M. and Teller, P., Particle labels and the theory of indistinguishable particles in quantum mechanics, *British Journal for the Philosophy of Science* **43**, pp. 201-218, pp. 14-22 (1992).
3. Schrödinger, E., What is an elementary particle?, reprinted in Castellani, E. (ed.), *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*, Princeton University Press, pp. 197-210 (1998).
4. Domenech, G. and Holik, F., A discussion on particle number and quantum indistinguishability, *Foundations of Physics*, **37**, pp. 855-878 (2007).
5. Domenech, G., Holik, F. and de Ronde, C., Entities, identity and the formal structure of quantum mechanics, in press (2008).
6. Mittelstaedt, P., *The Interpretation of Quantum Mechanics and the Measurement Process*, Cambridge University Press, Cambridge (1998).
7. Krause, D., Why quasi-sets?, *Boletim da Sociedade Paranaense de Matematica* **20**, 73-92 (2003).
8. Post, H., Individuality and physics, *The Listener* **70**, 534-537; reprinted in *Vedanta for East and West* **32**, pp. 14-22 (1963), cited in Ref. 1.
9. Manin, Yu. I., A course in mathematical logic, Springer-Verlag, p. 84 (1977), cited in Ref. 1.
10. Manin, Yu. I., Mathematical Problems I: Foundations, in Browder, F. E. (ed.) (1976), p. 36, cited in Ref. 1.

11. Santorelli, A., Krause, D. and Sant'Anna, A., A critical study on the concept of identity in Zermelo-Fraenkel like axioms and its relationship with quantum statistics, *Logique & Analyse* **189-192**, pp. 231-260 (2005).
12. Dalla Chiara, M. L. and Toraldo di Francia, G., Identity questions from quantum theory, in Gavroglu, K. *et. al.*, (eds.), *Physics, Philosophy and the Scientific Community*, Dordrecht, Kluwer Academic Publishers, pp. 39-46 (1995).
13. Dalla Chiara, M. L., Giuntini, R. and Krause, D., Quasiset theories for microobjects: A comparision, in Castellani, E. (ed.), *Interpreting Bodies: Classical and Quantum Objects in Modern Physics*, Princeton University Press, Princeton, pp. 142-152 (1998).
14. Halmos P., *Naive Set Theory*, D. Van Nostrand Company (1963).
15. Penrose, R., *The Emperors New Mind*, Oxford University Press (1989).
16. Domenech, G., Holik, F. and Krause, D., Q-spaces and the foundations of quantum mechanics, submited to *Foundations of Physics* (2008).
17. Redhead, M. and Teller, P., Particles, particle labels, and quanta: The toll of unacknowledged metaphysics, *Foundations of Physics* **21**, pp. 43-62 (1991).

EPR CORRELATIONS, BELL INEQUALITIES AND COMMON CAUSE SYSTEMS

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Standard common causal explanations of the EPR situation assume a so-called joint common cause system that is a common cause for all correlations. However, the assumption of a joint common cause system together with some other physically motivated assumptions concerning locality and no-conspiracy results in various Bell inequalities. Since Bell inequalities are violated for appropriate measurement settings, a local, non-conspiratorial joint common causal explanation of the EPR situation is ruled out. But why do we assume that a common causal explanation of a set of correlation consists in finding a *joint* common cause system for *all* correlations and not just in finding *separate* common cause systems for the different correlations? What are the perspectives of a local, non-conspiratorial *separate* common causal explanation for the EPR scenario? And finally, how do Bell inequalities relate to the weaker assumption of *separate* common cause systems?

Keywords: EPR correlations; common cause; Bell inequality.

1. Introduction

In the history of probabilistic causation Reichenbach's definition[1] was the first formal grasp of the notion of common cause. The conceptual novelty of the Reichenbachian definition has attracted immense interest among philosophers of science from the very beginning.[2,3] From the physical side, the need for a common causal explanation of the EPR situation called attention to the definition of the common cause, even though in standard hidden variable strategies a slightly different common causal concept than the Reichenbachian has been applied.[4–6] An important step in the conceptual clarification of the common cause in the EPR-Bell situation was the paper of Belnap and Szabó[7] in which the difference between the so-called *joint* and *separate* common cause had been first recognized. Belnap and Szabó pointed out that in standard common causal explanations of the EPR correlations common cause is actually meant as a *joint* common cause accounting for *all* correlations.

Concerning the algebraic-probabilistic features of the Reichenbachian common cause Hofer-Szabó, Rédei and Szabó[8] proved the following proposition. Classical (and also non-classical) correlations can be given a probabilistic common causal explanation in the sense that any classical probability measure space with correlating pairs of events can be extended such that the extension contains a Reichenbachian *separate* common cause for each correlation. (For the precise definitions see below.) Then in Ref. 9 it was proven that this proposition does not apply if Reichenbachian *separate* common causes are replaced with Reichenbachian *joint* common causes. In other words, classical probability measure spaces containing correlating pairs of events generally cannot be extended such that the extension contains a Reichenbachian *joint* common cause for *all* correlations. Thus, being a joint common cause of a set of correlations turned out to be a much stronger demand than being a common cause of a single correlation.

The first to apply the concept of separate common cause to the EPR situation was Szabó.[10] Since factorizability, locality and no-conspiracy together entail various types of Bell inequalities, EPR correlations cannot be given a local, non-conspiratorial, joint common causal model. Now, Szabó's idea was to replace the joint common causes with separate common causes and thus to give a separate common causal model for the EPR scenario. He constructed a number of separate common causal models which were local and non-conspiratorial in the usual sense that the measurement settings were statistically independent of the different common causes. However, the models were conspiratorial on a deeper level. The measurement settings statistically correlated with various algebraic combinations of the separate common causes. This fact called attention to the subtle but important difference between the so-called *weak* no-conspiracy where statistically independence is required only from the measure settings and the common causes *themselves*, and *strong* no-conspiracy where statistically independence is required between *any Boolean combination* of the measure settings and *any Boolean combination* of the common causes. After numerous computer simulations aiming to remove the unwanted conspiracies Szabó concluded with the conjecture that EPR cannot be given a local, *strongly* non-conspiratorial, separate common causal model.

The conjecture of Szabó has been first proven by Grasshoff, Portmann and Wüthrich.[11] The proof consisted in deriving some Bell inequality from the same assumptions that Szabó intended to apply in his separate common causal models for the EPR correlations. A crucial premise of this derivation was that the (anti)correlation between some events be *perfect*. Assuming perfect anticorrelations, however, turned the *separate* common causal explanations into a *joint* common causal explanation. This fact has been shown in Ref. 12. In the same paper Hofer-Szabó eliminated the assumption of perfect anticorrelations and presented a separate common causal derivation of some Bell-like inequalities (Bell(δ) inequalities). At the same time Portmann and Wüthrich[13] presented a very similar result for the Clauser-Horne inequality replacing separate common causes with the more general notion of the so-called separate common cause systems (see below). Finally, in Ref. 14 and 15 a general recipe has been given how to derive any type of Bell(δ) inequality—that is an inequality differing from the corresponding Bell inequality in a term of order δ—provided that the original Bell inequality can be derived from a set of perfect anticorrelations.

Although due to the above proofs the separate common causal explanation of the EPR scenario has been excluded, there is a sense in which Szabó's conjecture is still not decided. Szabó's original conjecture referred to the so-called Clauser–Horne set that is a set of four correlations violating the Clauser–Horne inequality. His question was as to whether the Clauser–Horne set can be given a local, strongly non-conspiratorial, separate common causal model. Interestingly enough—in the face of the above results—this question is still open.

In Section 2 we make explicit the concepts and propositions introduced informally in the Introduction. In Section 3 the standard *joint* common causal explanation of EPR correlations will be recalled. In Section 4 and 5 we explicate what has been and what has not been proven in the local, non-conspiratorial, *separate* common causal explanation of the EPR scenario. We conclude the paper in Section 6.

2. Joint and Separate Common Cause Systems

Let us start the common causal explanation with Reichenbach's[1] definition of the common cause. Let (Σ, p) be a classical probability measure space and let $A, B \in \Sigma$ be two positively correlating events, i.e.

$$p(A \cap B) > p(A)p(B) \tag{1}$$

Reichenbach then defines the common cause of the correlation as follows:

Definition 2.1. An event $C \in \Sigma$ is said to be the *Reichenbachian common cause* of the correlation between A and B, if the events A, B and C satisfy the following relations:

$$p(A \cap B|C) = p(A|C)p(B|C) \tag{2}$$

$$p(A \cap B|\overline{C}) = p(A|\overline{C})p(B|\overline{C}) \tag{3}$$

$$p(A|C) > p(A|\overline{C}) \tag{4}$$

$$p(B|C) > p(B|\overline{C}) \tag{5}$$

where $\overline{C}$ denotes the complement of C and the conditional probability is defined in the usual way. Equations (2)-(3) are referred to as "screening-off" properties and inequalities (4)-(5) as "positive statistical relevance" conditions. (Here we do not discuss the problem as to whether conditions (2)-(5) are necessary or sufficient conditions for an event C to be a common cause and simply take them to be the *definition* of the common cause.)

Physicists use the notion of 'common cause' in a different meaning. We obtain this meaning if (i) we drop the positive statistical relevance conditions (4)-(5) from the definition, and (ii) we do not restrict the screening-off properties (2)-(3) to the partition $\{C, \overline{C}\}$ of Σ:

Definition 2.2. Let (Σ, p) be a classical probability measure space and let (A, B) be a correlating pair of events in Σ. A partition $\{C_k\}$ $(k \in K)$ of Σ is said to be the *common cause system* of the pair (A, B) if for all $k \in K$ the following conditions are satisfied:

$$p(A \cap B|C_k) = p(A|C_k)p(B|C_k) \tag{6}$$

The cardinality $|K|$ (the number of events in the partition) is called the *size* of the common cause system. We will refer to a common cause system of size 2 (that is of the form $\{C, \overline{C}\}$) as a *common cause.* (Sometimes we will also refer to C as a common cause.)

Now, let (Σ, p) be a classical probability measure space as before and let (A_1, B_1) and (A_2, B_2), respectively be two positively correlating pairs of events in Σ, i.e. for $i = 1, 2$

$$p(A_i \cap B_i) \neq p(A_i)p(B_i) \tag{7}$$

In order to provide a common causal explanation for *both* correlating pairs we have two options. Either we assume that the two correlations arise from the same causal source or we attribute different causal sources to the correlations. In the first case we explain the correlation by a so-called *joint* common cause system, in the second case we employ two *separate* common cause systems. The definition of joint and separate common cause systems, respectively are the following:

Definition 2.3. A partition $\{C_k\}$ $(k \in K)$ of Σ is said to be the *joint common cause system* of correlations (A_i, B_i) $(i = 1, 2)$, respectively if for $i = 1, 2$ and $k \in K$ the following relations are satisfied:

$$p(A_i \cap B_i | C_k) = p(A_i | C_k) p(B_i | C_k) \tag{8}$$

Definition 2.4. Two different partitions $\{C^i_k\}$ $(i = 1, 2;\ k(i) \in K(i))$ of Σ are said to be *separate common cause systems* of the correlations (A_i, B_i) $(i = 1, 2)$, respectively if for $i = 1, 2$ and $k(i) \in K(i)$ the following relations hold:

$$p(A_i \cap B_i | C^i_k) = p(A_i | C^i_k) p(B_i | C^i_k) \tag{9}$$

Having defined different common causal structures let us turn to the procedure of causal explanation. A common causal explanation of a given correlation is realized mathematically by the extension of the probabilistic measure space in such a way that for the original correlation there exists a common cause system in the extended probabilistic measure space. In the case of two (or more) correlations we can extend the algebra in two different ways according to our causal intuition. In order to model a joint common causal source of the correlations we extend the algebra such that in the extended algebra all correlations have a *joint* common cause system. On the other hand to account for separate causal mechanisms we extend the algebra such that in the extended algebra different correlations have *separate* common cause systems.

The extendability of the probabilistic measure spaces by joint respectively separate common causal structures crucially depends on the size of the common cause system. In the case of a common cause system of size 2 that is in the case of a common cause there is a great difference between joint and separate common cause extensions as it is shown in the following two propositions:

Proposition 2.1 (Hofer-Szabó, Rédei, Szabó, 1999). *Let (Σ, p) be a classical probability measure space and let (A_1, B_1) and (A_2, B_2), respectively be two correlating pairs of events in Σ. Then there always exists a (Σ', p') extension of (Σ, p) such that for the correlation (A_1, B_1) there exists a common cause C^1 and for the correlation (A_2, B_2) there exists a common cause C^2 in (Σ', p').*

Proposition 2.2 (Hofer-Szabó, Rédei, Szabó, 2002). *There exists a* (Σ, p) *classical probability measure space and two correlating pairs* (A_1, B_1) *and* (A_2, B_2)*, respectively in* Σ *such that there is no* (Σ', p') *extension of* (Σ, p) *which contains a joint common cause* C *in* (Σ', p') *for both correlations.*

Proposition 2.1 claims that for two correlating pairs a separate common causal explanation is always possible by extending the probability measure space in an appropriate way. (Moreover, if Σ contains $n \in \mathbb{N}$ correlating pairs, *each* correlation can be given a *separate* common causal explanation.) However, according to Proposition 2.2 this strategy does not work generally if we are going to obtain the same common cause for the two (or more) correlating pairs. Thus, being a joint common cause imposes much stronger demand on C than simply being a separate common cause.

However, strangely enough this difference between the common and separate common causal extendability of a probability measure space disappears if the size of the common cause system is *not* specified. In other words, to find a joint common cause system *of arbitrary size* for a set of correlations is *not* a stronger demand than to find separate common cause systems for the same set. To see this, let (A_1, B_1) and (A_2, B_2) be two arbitrary correlating pairs in Σ. Then the partition

$$\{A_1 \cap B_1, A_1 \cap B_2, A_2 \cap B_1, A_2 \cap B_2, \}$$

is always a joint common cause system in Σ for both correlations. Obviously, this partition can be regarded only as a *trivial* joint common cause system of the correlations. This fact makes it clear that without further specification a joint common causal explanation is not more compelling than a separate common causal explanation. In the following sections we will see how these two types of explanations diverge due to extra requirements.

3. No Local, Non-Conspiratorial Joint Common Cause System for the EPR

Consider the standard EPR-Bohm experimental setup with a source emitting pairs of spin-$\frac{1}{2}$ particles prepared in the singlet state $|\Psi_s\rangle$. Let $p(a_i)$ denote the probability that the spin measurement apparatus is set to measure the spin in direction $\vec{a}_i$ $(i \in I)$ in the left wing and let $p(b_j)$ denote the same for direction $\vec{b}_j$ $(j \in J)$ in the right wing. Furthermore, let $p(A_i)$ stand for the probability that the spin measurement in direction $\vec{a}_i$ in the left wing yields the result $+1$ ('up') and let $p(\overline{A_i})$ denote the probability of the result -1 ('down'). Let $p(B_j)$ and $p(\overline{B}_j)$ be defined in a similar way in the right wing for direction $\vec{b}_j$. (See Fig. 1.) Quantum mechanics then

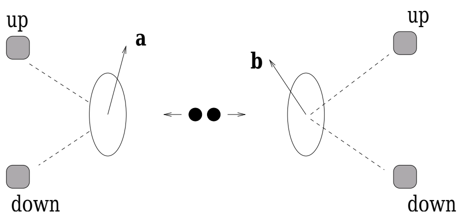

Fig. 1. EPR–Bohm setup for spin-$\frac{1}{2}$ particles

yields the following conditional probabilities for the events in question:

$$p(A_i \cap B_j | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(P_{A_i} \otimes P_{B_j})) = \frac{1}{2}\sin^2\left(\frac{\theta_{a_i b_j}}{2}\right) \tag{10}$$

$$p(A_i | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(P_{A_i} \otimes I)) = \frac{1}{2} \tag{11}$$

$$p(B_j | a_i \cap b_j) = Tr(W_{|\Psi_s\rangle}\,(I \otimes P_{B_j})) = \frac{1}{2} \tag{12}$$

where $W_{|\Psi_s\rangle}$ is the density operator pertaining to the pure state $|\Psi_s\rangle$; P_{A_i} and P_{B_j} denote projections on the eigensubspaces with eigenvalue $+1$ of the spin operators associated with directions $\vec{a}_i$ and $\vec{b}_j$, respectively; and $\theta_{a_i b_j}$ denotes the angle between directions $\vec{a}_i$ and $\vec{b}_j$.

Thus, for non-perpendicular directions $\vec{a}_i$ and $\vec{b}_j$ there is a conditional correlation

$$p(A_i \cap B_j | a_i \cap b_j) \neq p(A_i | a_i \cap b_j) p(B_j | a_i \cap b_j) \tag{13}$$

and for parallel directions there is a perfect anticorrelation between the outcomes:

$$p(A_i \cap B_j | a_i \cap b_j) = 0 \tag{14}$$

Now, consider a set $\{(A_i, B_j)\}_{(i,j) \in I \times J}$ of EPR correlations in the sense of Eq. (13). A full-fledged common causal explanation of the set $\{(A_i, B_j)\}_{(i,j) \in I \times J}$ must comply with three demands on the statistical level. Firstly, all the correlations must be screened-off by a joint common cause system. Secondly, statistical relations among the measurement outcomes and the measurement settings must reflect the spacetime location of these events in the sense that spatially separated events have to be statistically independent. Thirdly, the measurement settings and the common cause should not influence each other, they have to be statistically independent. We refer to these requirements in turn as 'joint common cause system', 'locality' and 'no-conspiracy'. In the case of 'no-conspiracy' we will distinguish two types: the 'weak' and the 'strong no-conspiracy'. The precise probabilistic formulation of these demands is the following:

(1) *Joint common cause system:* There exists a partition $\{C_k\}$ of Σ such that for every A_i, B_j, a_i and b_j in Σ $(i \in I, j \in J)$ and for any $k \in K$ the following factorization holds:

$$p(A_i \cap B_j | a_i \cap b_j \cap C_k) = p(A_i | a_i \cap b_j \cap C_k) p(B_j | a_i \cap b_j \cap C_k) \quad (15)$$

(2) *Locality:* For every A_i, B_j, a_i, b_j and C_k in Σ $(i \in I, j \in J, k \in K)$ the following screening-off relations hold:

$$p(A_i | a_i \cap b_j \cap C_k) = p(A_i | a_i \cap C_k) \quad (16)$$

$$p(B_j | a_i \cap b_j \cap C_k) = p(B_j | b_j \cap C_k) \quad (17)$$

(3) *a. Weak no-conspiracy:* For every a_i, b_j and C_k in Σ $(i \in I, j \in J, k \in K)$ the following independence holds:

$$p(a_i \cap b_j \cap C_k) = p(a_i \cap b_j) p(C_k) \quad (18)$$

b. Strong no-conspiracy: Consider two Boolean subalgebras $\mathfrak{A}$ and $\mathfrak{C}$ of Σ such that $\mathfrak{A}$ is generated by the partition of the different measurement choices $\{a_i b_j\}$ $(i \in I, j \in J)$ on the opposite wings, and $\mathfrak{C}$ is generated by the partition of the common cause system $\{C_k\}$ $(k \in K)$. Then for any element $E \in \mathfrak{A}$ and $F \in \mathfrak{C}$ the following independence holds:

$$p(E \cap F) = p(E) p(F) \quad (19)$$

It is straightforward to see that in the case of *joint* common cause systems Eqs. (18) and (19) are equivalent, the probabilistic independence of the *Boolean combinations* of common causes and the measurement settings does not demand more than simply the probabilistic independence of the common causes and the measurement settings *themselves*. Thus, in the case of the joint common cause system type explanations Eq. (18) will suffice as a no-conspiracy requirement.

However, as it is well-known Eqs. (15)-(18) result in various Bell inequalities which are violated for special measurement settings in the EPR experiment. For the simplest set of correlations, namely for the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j) \in CH}$ where $CH = I \times J$ with $I = \{1, 2\}$ and $J = \{3, 4\}$ the Bell theorem is the following:

Proposition 3.1 (Clauser, Horne, 1974). *For some measurement directions $\vec{a}_1, \vec{a}_2$ and $\vec{b}_3, \vec{b}_4$ there cannot exist extension of the probability space (Σ, p) such that the extension contains local, (weakly or strongly) non-conspiratorial joint common cause systems for all EPR correlations of $\{(A_i, B_j)\}_{(i,j)\in CH}$.*

Consequently, EPR correlations fall short of a local, non-conspiratorial, joint common cause system type explanation. One premise has to be given up.

4. Local, Weakly Non-Conspiratorial Separate Common Cause Systems Do Exist for the EPR

Strategies aiming to avoid Bell inequalities and to provide a common causal explanation for the EPR correlations can be grouped according the abandoned premise. The first group consists of approaches abandoning locality and preserving the joint common causal background and no-conspiracy. Bohmian mechanics is an eminent representative of this group. The second group consists of less attractive models in which no-conspiracy is given up. Examples of this approach are Brans' and Szabó's models.[10,18] In these models the authors relinquished no-conspiracy and provided a local, deterministic but conspiratorial joint common cause system type explanation for the EPR. (For the problem of no-conspiracy and free will see Ref. 19). In this paper, however, we will follow a third strategy which gives up the hypothesis of a joint common cause system. The key idea here is to replace the concept of joint common cause system with that of separate common cause systems and to provide a local, non-conspiratorial, *separate* common cause system type explanation for the EPR. A *separate* common cause system type explanation for a set $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ consists in finding for every $(i,j) \in I \times J$ index pair a *separate* partition $\{C_k^{ij}\}$ $(k(ij) \in K(ij))$ such that screening-off, locality, and (weak or strong) no-conspiracies holds in the following sense:

(1) *Separate common cause systems:* For every A_i, B_j, a_i and b_j in Σ $(i \in I, j \in J)$ there exists a separate partition $\left\{C_k^{ij}\right\}$ of Σ such that for any $k(ij) \in K(ij)$ the following factorization holds:

$$p(A_i \cap B_j|a_i \cap b_j \cap C_k^{ij}) = p(A_i|a_i \cap b_j \cap C_k^{ij})p(B_j|a_i \cap b_j \cap C_k^{ij}) \quad (20)$$

(2) *Locality:* For every $i \in I, j \in J$ and $k(ij) \in K(ij)$ the following screening-off relations hold:

$$p(A_i|a_i \cap b_j \cap C_k^{ij}) = p(A_i|a_i \cap C_k^{ij}) \tag{21}$$

$$p(B_j|a_i \cap b_j \cap C_k^{ij}) = p(B_j|b_j \cap C_k^{ij}) \tag{22}$$

(3) *a. Weak no-conspiracy:* For every a_i, b_j and $C_k^{i'j'}$ in Σ ($i, i' \in I$; $j, j' \in J$; $k(i'j') \in K(i'j')$) the following independence holds:

$$p(a_i \cap b_j \cap C_k^{i'j'}) = p(a_i \cap b_j)p(C_k^{i'j'}) \tag{23}$$

b. Strong no-conspiracy: Consider again two Boolean subalgebras $\mathfrak{A}$ and $\mathfrak{C}$ of Σ such that $\mathfrak{A}$ is generated by the partition of the different measurement choices $\{a_i b_j\}$ ($i \in I, j \in J$) and $\mathfrak{C}$ is generated by the partition of *all the different common cause systems* $\left\{\cap_{ij} C_k^{ij}\right\}$ ($k \in K$). Then for any element $E \in \mathfrak{A}$ and $F \in \mathfrak{C}$ the following independence holds:

$$p(E \cap F) = p(E)p(F) \tag{24}$$

Here, requirement (23) does *not* entail Eq. (24), that is the independence of the separate common cause systems of the choice of the measurement settings does *not* assure that *any Boolean combination* of the common causes will also be independent of *any Boolean combination* of the measurement settings. Thus, in the case of separate common cause system type explanations one has to take into consideration two different versions of no-conspiracy.

The idea to replace the concept of a joint common cause system with that of separate common cause systems and to provide a local, non-conspiratorial separate common causal explanation for the EPR was first raised by Szabó.[10] Actually, Szabó replaced the joint common cause system with separate common cause systems of size 2 that is with separate common causes. Szabó provided a number of separate common causal models for the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j)\in CH}$ such that the models were local and non-conspiratorial in the *weak* sense of Eq. (24). In a precise form, Szabó's proposition was the following:

Proposition 4.1 (Szabó, 2000). *Let $\{(A_i, B_j)\}_{(i,j)\in CH}$ be the Clauser–Horne set of correlations in (Σ, p). Then for any measurement directions $\vec{a}_1, \vec{a}_2$ and $\vec{b}_3, \vec{b}_4$ there exists an extension of the probability space (Σ, p) such that the extension contains local, weakly non-conspiratorial separate common causes for the correlations of $\{(A_i, B_j)\}_{(i,j)\in CH}$.*

The common causal models provided by Szabó, however, were all conspiratorial in the *strong* sense of Eq. (24). After numerous computer simulations aiming to remove the unwanted conspiracies Szabó finally concluded with the conjecture that EPR *cannot* be given any local, separate common causal model free from *all* type of conspiracies.

5. Local, Strongly Non-Conspiratorial Separate Common Cause Systems for the EPR?

Szabó's conjecture is then the following:

Conjecture 5.1 (Szabó, 2000). *For some measurement directions* $\vec{a}_1, \vec{a}_2$ *and* $\vec{b}_3, \vec{b}_4$ *there cannot exist extension of the probability space* (Σ, p) *such that the extension contains local, strongly non-conspiratorial separate common cause systems for the correlations of* $\{(A_i, B_j)\}_{(i,j)\in CH}$.

Although a lot has happened since 2000 in understanding the status of the separate common causal explanation of the EPR scenario, Szabó's conjecture in its original form is *still an open question*. What has actually been excluded, is not a local, strongly non-conspiratorial separate common causal explanation of the the Clauser–Horne set $\{(A_i, B_j)\}_{(i,j)\in CH}$, but that of *another set*. Let $I = J = \{1, 2, 3, 4\}$ and let PA be the following subset of $I \times J$:

$$PA = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

Then one can prove the following proposition:

Proposition 5.1. *For some measurement directions* $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4\}$ *and* $\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4\}$ *there cannot exist extension of the probability space* (Σ, p) *such that the extension contains local, strongly non-conspiratorial separate common cause systems for all EPR correlations of* $\{(A_i, B_j)\}_{(i,j)\in PA}$.

The above proposition was first proved by Grasshoff, Portmann and Wüthrich.[11] They have shown that no local, *strongly* non-conspiratorial separate common cause systems are possible for all correlations of $\{(A_i, B_j)\}_{(i,j)\in PA}$, *if* for any index pair $(i, j) \in PA$ there is a *perfect anticorrelation* (hence the denotation 'PA') in the sense of Eq. (14).

The assumption of perfect anticorrelations, however, was unsatisfactory in two respects. The first problem concerns experimental testability. Since perfect anticorrelations cannot be tested experimentally with absolute precision, the proof of Grasshoff, Portmann and Wüthrich did not provide an experimentally verifiable refutation of a separate common causal explanation of the EPR.

The second problem was more conceptual. Standard derivations of the Bell inequalities assume a joint common cause system. The chief virtue of the proof of Grasshoff, Portmann and Wüthrich was that it avoided this strong concept of a *joint* common cause system and used the weaker concept of *separate* common cause systems instead. However, in the perfect anticorrelation case the assumptions of separate common cause systems turned out to be reducible to the assumptions of the standard joint common cause system as it was shown in the following proposition:[12]

Proposition 5.2 (Hofer-Szabó, 2008). *Let* $\left\{C_k^{ij}\right\}_{(i,j)\in PA}$ *be local, strongly non-conspiratorial separate common cause systems for the correlations of* $\{(A_i, B_j)\}_{(i,j)\in PA}$*. Then the partition* $\{D_l\} := \left\{\cap_{ij} C_k^{ij}\right\}$ *generated by the intersections of the different separate common cause systems is a local, non-conspiratorial joint common cause system of the same correlations of* $\{(A_i, B_j)\}_{(i,j)\in PA}$*.*

The assumption of perfect anticorrelations, however, turned out not to be indispensable in the proof of Proposition 5.1. Portmann and Wüthrich[13] and Hofer-Szabó[12] have shown that Proposition 5.1 also holds if one only assumes that the correlations to be explained form an *almost perfect anticorrelation* set, $\{(A_i, B_j)\}_{(i,j)\in PA(\delta)}$, in the sense that there exists a δ of some small but not zero value such that

$$p(A_i \cap B_j | a_i \cap b_j) \leqslant \delta \tag{25}$$

for any index pair $(i, j) \in PA(\delta)$.

Finally, Hofer-Szabó[14,15] generalized this proof by deriving arbitrary Bell(δ) inequality—that is to say, an inequality differing from the corresponding Bell inequality in a term of order δ. The recipe of this derivation is roughly the following. Consider a Bell inequality resulting from the local, non-conspiratorial *joint* common causal explanation of a given set of correlations $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ (not necessarily $\{(A_i, B_j)\}_{CH}$). Now, define the set PA for $\{(A_i, B_j)\}_{(i,j)\in I\times J}$ as follows: let PA contain all the index pairs (k, k) in $(I \cup J) \times (I \cup J)$ that is all indices appearing either on the left or on the right hand side of the correlations in $\{(A_i, B_j)\}_{(i,j)\in I\times J}$.

Now consider the set $\{(A_i, B_j)\}_{PA(\delta)}$ of almost perfect anticorrelations and suppose that it has a local, strongly non-conspiratorial separate common causal explanation. This assumption results in a Bell(δ) inequality differing from the original Bell inequality in a term of order of δ where the exact magnitude of this term is the function of the approximation. Choose the setting which violates the Bell inequality maximally. If the δ term is smaller than the violation of the original Bell inequality, then the Bell(δ) inequality will also be violated, excluding a local, strongly non-conspiratorial separate common causal explanation of the set $\{(A_i, B_j)\}_{PA(\delta)}$.

6. Conclusions

In the paper, first, different common causal concepts ranging from Reichenbach's definition to the most general concept of the common cause system have been listed. Then the role of the different causal notions in the common causal explanation of the EPR scenario has been exposed. It was said that a completely satisfactory common causal explanations of the EPR would consist in finding a *joint* common causal source for all correlations which is local and non-conspiratorial. Since these assumptions together entail various Bell inequalities one assumption has to be abandoned. The ambition of the separate common cause system type approach of the EPR was to preserve the latter two physically motivated assumptions of locality and no-conspiracy at the expense of replacing the strong concept of the joint common cause system with the weaker concept of separate common cause systems. It has been shown, however, that the weakening of the common causal concept does not provide a solution to this problem since the weakened assumptions still entail some Bell or Bell(δ) inequalities. Consequently, there exists no local, (weakly or strongly) non-conspiratorial *separate* common causal explanation of the EPR.

A weakness of all the above no-go theorems, however, is that they are all based on either *perfect* or *almost perfect* EPR correlations. As it was made clear in Proposition 5.2 the separate common causal explanation of such correlations is always parasitic on some joint common causal explanation. Therefore it would be highly desirable to derive some Bell inequality form a local, strongly non-conspiratorial separate common causal explanation of a set of *genuine* (not almost perfect) EPR correlations. For example it would be widely wanted to prove or falsify Szabó's original conjecture (Conjecture 5.1)—that is for the set $\{(A_i, B_j)\}_{(i,j)\in CH}$ violating the Clauser–Horne inequality

(i) either to derive the Clauser–Horne inequality (or some other constraint) from the assumption that $\{(A_i, B_j)\}_{(i,j)\in CH}$ has a local, strongly non-conspiratorial separate common causal explanation;
(ii) or to show up local, strongly non-conspiratorial separate common cause systems for the set $\{(A_i, B_j)\}_{(i,j)\in CH}$.

Neither option seems to be a trivial mathematical task.

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References

1. H. Reichenbach, *The Direction of Time* (University of California Press, Berkeley, 1956).
2. W. C. Salmon, Theoretical explanation, in Stephan Körner (ed.), *Explanation*, Oxford, pp. 118-143 (1975).
3. B. C. Van Fraassen, Rational belief and common cause principle, in R. McLaughlin (ed.), *What? Where? When? Why?*, Reidel, pp. 193-209 (1982).
4. J. S. Bell, Introduction to the hidden variable question, in B. d'Espagnat (ed.), *Foundation of Quantum Mechanics*, New York, pp. 171-181 (1971).
5. J. P. Jarrett, On the physical significance of the locality conditions in the Bell inequality, *Noûs* **18**, 569-589 (1984).
6. B. C. Van Fraassen, The charybdis of realism: Epistomological implications of Bell's inequality, in J. T. Cushing and E. McMullin (eds.), *Philosophical Consequences of Quantum Theory*, University of Notre Dame Press, Ind., pp. 97-113 (1989).
7. N. Belnap and L. E. Szabó, Branching space-time analysis of the GHZ theorem, *Foundations of Physics*, **26**, 982-1002 (1996).
8. G. Hofer-Szabó, M. Rédei and L. E. Szabó, On Reichenbach's common cause principle and on Reichenbach's notion of common cause, *The British Journal for the Philosophy of Science*, **50**, 377-399 (1999).
9. G. Hofer-Szabó, M. Rédei and L. E. Szabó, Common causes are not common common causes, *Philosophy of Science*, **69**, 623-633 (2002).
10. L. E. Szabó, On an attempt to resolve the EPR–Bell paradox via Reichenbachian concept of common cause, *International Journal of Theoretical Physics*, **39**, 911 (2000).
11. G. Grasshoff, S. Portmann and A. Wüthrich, Minimal assumption derivation of a Bell-type inequality, *The British Journal for the Philosophy of Science*, **56**, 663-680 (2005).
12. G. Hofer-Szabó, Separate- versus *common*-common-cause-type derivations of the Bell inequalities, *Synthese*, **163/2**, 199-215 (2008).
13. S. Portmann and A. Wüthrich, Minimal assumption derivation of a weak Clauser–Horne inequality, *Studies in History and Philosophy of Modern Physics*, **38/4**, 844-862 (2007).
14. G. Hofer-Szabó, Bell(δ) inequalities derived from separate common causal explanation of almost perfect EPR anticorrelations, *Foundations of Physics*, **41**, 1398-1413 (2011).
15. G. Hofer-Szabó, Separate common causal explanation and the Bell inequalities, *International Journal of Theoretical Physics*, **51**, 110-123 (2012).

16. J. F. Clauser and M. A. Horne, Experimental consequences of objective local theories, *Phys. Rev. D*, **10**, 526-535 (1974).
17. G. Hofer-Szabó, M. Rédei and L. E. Szabó, *The Principle of the Common Cause* (Cambridge: Cambridge University Press, 2013, in press).
18. C. H. Brans, Bell's theorem does not eliminate fully causal hidden variables, *International Journal of Theoretical Physics*, **27**, 219 (1988).
19. I. SanPedro, Free will and no-conspiracy, in A. Wüthrich and T. Sauer (eds.), *Current Interpretational Problems of Quantum Theory* (2013, forthcoming).

A LOGIC-ALGEBRAIC FRAMEWORK FOR CONTEXTUALITY AND MODALITY IN QUANTUM SYSTEMS

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In this work we develop a modal structure for the simultaneous treatment of actual and possible properties of quantum systems. A logical system based on orthomodular lattices enriched with a modal operator is given, obtaining algebraic completeness and completeness with respect to a Kripke-style semantic. We show that, in spite of the fact that, the language is enriched with the addition of a modal operator, contextuality remains a central feature of quantum systems.

Keywords: Contextuality; modal orthomodular logic.

0. Introduction

In their 1936 seminal paper 1, Birkhoff and von Neumann made the proposal of a non-classical logic for quantum mechanics founded on the basic lattice-order properties of all closed subspaces of a Hilbert space. This lattice-order properties are captured in the orthomodular lattice structure. More precisely, in the usual terms of quantum logic,[1,21] a property of a system is related to a subspace of the Hilbert space $\mathcal{H}$ of its (pure) states or, analogously, to the projector operator onto that subspace. A physical magnitude $\mathcal{M}$ is represented by an operator $\mathbf{M}$ acting over the state space. For bounded self-adjoint operators, conditions for the existence of the spectral decomposition $\mathbf{M} = \sum_i a_i \mathbf{P}_i = \sum_i a_i |a_i >< a_i|$ are satisfied. The real numbers a_i are related to the outcomes of measurements of the magnitude $\mathcal{M}$ and projectors $|a_i >< a_i|$ to the mentioned properties. Thus, the physical properties of the system are organized in the lattice of closed subspaces $\mathcal{L}(\mathcal{H})$.

Contextuality is one of the main features of the discourse about quantum systems and has been studied from different approaches. We are interested here in algebraic versions related to partial valuations of the orthomodular lattice of closed subspaces of Hilbert space. This proposal allows to identify the constraints imposed by the structure to the relation between actuality and possibility and the discourse that includes both type of propositions.

The consideration of *possibility* is of course always present in quantum theories. We have propose an algebraic consideration of this type of sentences and their articulation with those about actual properties. Several attempts to obtain modal extensions of the orthomodular systems are found in Refs. 4, 5, 9, 17, 18. One possibility developed in Ref. 9 allows to embed orthomodular propositional systems in modal systems. Another extension is provided by adding quantifiers to the orthomodular structure,[22,23] so generalizing the monadic extension of the Boolean algebras.[19] In our case, we enrich the orthomodular structure with a modal operator thus obtaining an algebraic variety such that each orthomodular lattice can be represented by an algebra of this variety. On the other hand, this operator acts as a quantifier in the sense of Refs. 22, 23. The physical motivation for this construction is the purpose to link consistently the propositions about actual and possible properties of the system in a single structure.

The paper is organized as follows: Section 1 contains generalities on universal algebra, orthomodular lattices and Baer *-semigroups. In Section 2 we discuss the contextual character of quantum systems from an algebraic perspective. Section 3 introduces the desiderata of modal interpretations of quantum mechanics involved in our treatment. We devote Section 4 to expose the algebraic structure which represents the orthomodular lattice enriched with modal operators. More precicely, we introduce the class of Boolean saturated orthomodular lattices $\mathcal{OML}^{\Box}$. In Section 5 we show how the discourse about properties is genuinely enlarged giving an adequate framework to represent the Born rule. Moreover we use Kochen-Specker theorem to show that the contextual character of quantum mechanics is maintained even when the discourse is enriched with modalities. In Section 6, a Hilbert-style calculus is explained obtaining a strong completeness theorem for the variety $\mathcal{OML}^{\Box}$. In Section 7, we give a representation theorem by means of a sub-class of Baer *-semigroups for $\mathcal{OML}^{\Box}$. This allows to develop a Kripke-style semantic for the calculus of the precedent section following the approach given in Ref. 28. A strong completeness theorem for these Kripke-style models is also obtained. Finally, we outline our conclusions.

1. Basic Notions

We freely use all basic notions of universal algebra that can be found in Ref. 3. If K is a class of algebras of the same type then we denote by $\mathcal{V}(K)$ the variety generated by K. Let $\mathcal{A}$ be a variety of algebras of type σ. We denote by $Term_{\mathcal{A}}$ the *absolutely free algebra* of type σ built from the set of variables $V = \{x_1, x_2, ...\}$. Each element of $Term_{\mathcal{A}}$ is referred to as a *term*. We denote by $Comp(t)$ the complexity of the term t. Let $A \in \mathcal{A}$. If $t \in Term_{\mathcal{A}}$ and $a_1, \ldots, a_n \in A$, by $t^A(a_1, \ldots, a_n)$ we denote the result of the application of the term operation t^A to the elements $a_1, \ldots, a_n$. A *valuation* in A is a map $v : V \to A$. Of course, any valuation v in A can be uniquely extended to an $\mathcal{A}$-homomorphism $v : Term_{\mathcal{A}} \to A$ in the usual way, i.e., if $t_1, \ldots, t_n \in Term_{\mathcal{A}}$ then $v(t(t_1, \ldots, t_n)) = t^A(v(t_1), \ldots, v(t_n))$. Thus, valuations are identified with $\mathcal{A}$-homomorphisms from the absolutely free algebra. If $t, s \in Term_{\mathcal{A}}$, $\models_A t = s$ means that for each valuation v in A, $v(t) = v(s)$ and $\models_{\mathcal{A}} t = s$ means that for each $A \in \mathcal{A}$, $\models_A t = s$. We denote by $Con(A)$ the lattice of congruences of A. A *discriminator term* for A is a term $t(x, y, z)$ such that:

$$t^A(x,y,z) = \begin{cases} x, & \text{if } x \neq y; \\ z, & \text{if } x = y. \end{cases}$$

The variety $\mathcal{A}$ is a *discriminator variety* iff there exists a class of algebras K with a common discriminator term $t(x, y, z)$ such that $\mathcal{A} = \mathcal{V}(K)$.

Now we recall from Refs. 13, 25, 27 some notions about orthomodular lattices. A *lattice with involution*[24] is an algebra $\langle L, \vee, \wedge, \neg\rangle$ such that $\langle L, \vee, \wedge\rangle$ is a lattice and $\neg$ is a unary operation on L that fulfills the following conditions: $\neg\neg x = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. Let $L = \langle L, \vee, \wedge, 0, 1\rangle$ be a bounded lattice. Given a, b, c in L, we write: $(a, b, c)D$ iff $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$; $(a, b, c)D^*$ iff $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$ and $(a, b, c)T$ iff $(a, b, c)D$, $(a, b, c)D^*$ hold for all permutations of a, b, c. An element z of a lattice L is called *central* iff for all elements $a, b \in L$ we have $(a, b, z)T$. We denote by $Z(L)$ the set of all central elements of L and it is called the *center* of L. $Z(L)$ is a Boolean sublattice of L [27, Theorem 4.15]. An *orthomodular lattice* is an algebra $\langle L, \wedge, \vee, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ that satisfies the following conditions:

(1) $\langle L, \wedge, \vee, \neg, 0, 1\rangle$ is a bounded lattice with involution,
(2) $x \wedge \neg x = 0$,
(3) $x \vee (\neg x \wedge (x \vee y)) = x \vee y$.

We denote by $\mathcal{OML}$ the variety of orthomodular lattices. An important characterization of the equations in $\mathcal{OML}$ is given by:

$$\models_{\mathcal{OML}} t = s \quad \text{iff} \quad \models_{\mathcal{OML}} (t \wedge s) \vee (\neg t \wedge \neg s) = 1 \tag{1}$$

Therefore we can safely assume that all $\mathcal{OML}$-equations are of the form $t = 1$, where $t \in Term_{\mathcal{OML}}$.

Remark 1.1. It is clear that, this characterization is maintained for each variety $\mathcal{A}$ such that there are terms of the language of $\mathcal{A}$ defining on each $A \in \mathcal{A}$ operations $\vee$, $\wedge$, $\neg$, $0, 1$ such that $L(A) = \langle A, \vee, \wedge, \neg, 0, 1\rangle$ is an orthomodular lattice.

Proposition 1.1. *Let L be an orthomodular lattice then we have that:*

(1) $z \in Z(L)$ if and only if $a = (a \wedge z) \vee (a \wedge \neg z)$ for each $a \in L$.
(2) If L is complete then, $Z(L)$ is a complete lattice and for each family (z_i) in $Z(L)$ and $a \in L$, $a \wedge \bigvee z_i = \bigvee (a \wedge z_i)$.

Proof. See [27, Lemma 29.9 and Lemma 29.16]. □

Now we recall from Refs. 14, 27, 28 some notions about Baer *-semigroups. A **-semigroup* is an algebra $\langle G, \cdot, \star, 0\rangle$ of type $\langle 2, 1, 0\rangle$ that satisfies the following equations:

(1) $\langle G, \cdot\rangle$ is a semigroup,
(2) $0 \cdot x = x \cdot 0 = 0$,
(3) $(x \cdot y)^\star = y^\star \cdot x^\star$,
(4) $x^{\star\star} = x$.

Let G be a *-semigroup. An element $e \in G$ is a *projection* iff $e = e^\star = e\cdot e$. The set of all projections of G is denoted by $P(G)$. Let M be a non empty subset of G. If $x \in G$ we define $x \cdot M = \{x \cdot m \in G : m \in M\}$ and $M \cdot x = \{m \cdot x \in G : m \in M\}$. Moreover x is said to be a *left annihilator* of M iff $x \cdot M = \{0\}$ and it is said to be a *right annihilator* of M iff $M \cdot x = \{0\}$. We denote by M^l the set of left annihilators of M and by M^r the set of right annihilators of M. A *-semigroup is called *Baer *-semigroup* iff for each $x \in G$ there exists $e \in P(G)$ such that

$$\{x\}^r = \{y \in G : x \cdot y = 0\} = e \cdot G$$

We do not assume in general that any $e \in P(G)$ can be represented as $\{x\}^r = e\cdot G$ for some $x \in G$. Thus we say that $e \in P(G)$ is a *closed projection* iff there exists $x \in G$ such that $\{x\}^r = e\cdot G$. The set of all closed projections is denoted by $P_c(G)$. Let G be a Baer*-semigroup. From [28, Corollary 3.5], for each $x \in G$ there exists a unique projection $e_x \in P(G)$ such that $\{x\}^r = e \cdot G$. We denote this e_x by x'. Moreover $0'$ is denoted as 1. We can define a partial order $\langle P(G), \leq \rangle$ as follows:

$$e \leq f \Longleftrightarrow e \cdot f = e$$

Proposition 1.2. *Let G be a Baer *-semigroup. For any $e_1, e_2 \in P_c(G)$, we have that:*

(1) $e \leq f$ *iff* $e \cdot G \subseteq f \cdot G$.
(2) $x \cdot 1 = 1 \cdot x = x$.

Proof. See [27, Theorem 37.2 and Theorem 37.4]. □

Theorem 1.1. *Let G be a Baer *-semigroup. For any $e_1, e_2 \in P_c(G)$, we define the following operation:*

(1) $e_1 \wedge e_2 = e_1 \cdot (e_2' \cdot e_1)'$,
(2) $e_1 \vee e_2 = (e_1' \wedge e_2')'$.

Then $\langle P_c(G), \wedge, \vee, ', 0, 1 \rangle$ is an orthomodular lattice with respect to the order $\langle P(G), \leq \rangle$.

Proof. See [27, Theorem 37.8]. □

On the other hand we can build a Baer *-semigroup from a partial ordered set. Let $\langle A, \leq, 0 \rangle$ be a partial ordered set with smallest element 0. If $\varphi : A \to A$ is an order homomorphism then, a *residual map* for φ is an order homomorphism $\varphi^\natural : A \to A$ such that $(\varphi \circ \varphi^\natural)(x) \leq x \leq (\varphi^\natural \circ \varphi)(x)$ where $\circ$ is the composition of order-homomorphisms. We denote by $G(A)$ the set of order-homomorphisms in A admitting residual maps. If we consider the constant order-homomorphism θ, given by $\theta(x) = 0$, then $\theta \in G(A)$ and $\langle G(A), \circ, \theta \rangle$ is a semigroup.

Theorem 1.2. *Let A be an orthomodular lattice and we consider the semigroup $\langle G(A), \circ, \theta \rangle$. If for each $\varphi \in G(A)$ we define $\varphi^\star$ as $\varphi^\star(x) = \neg\varphi^\natural(\neg x)$ then we have that:*

(1) $\langle G(A), \circ, \star, \theta \rangle$ *is a Baer *-semigroup,*
(2) if we define $\mu_a(x) = (x \vee \neg a) \wedge a$ for each $a \in A$ then, $P_c(G(A)) = \{\mu_a : a \in A\}$,
(3) $f : A \to P_c(G(A))$ such that $f(a) = \mu_a$ is a $\mathcal{OML}$-homomorphism.

Proof. See [14, Theorem 8]. □

2. Contextuality in Quantum Systems

In Ref. 10, we dealt with the problem of the limits imposed by the orthomodular structure of projection operators to the possibility of thinking of properties possessed by an isolated quantum system. This is an important point in the discussion about quantum systems because almost every problem in the relation between the mathematical formalism and what may be called our experience about the behavior of physical objects can be encoded in the question about the possible meaning of the proposition *the physical magnitude $\mathcal{A}$ has a value and the value is this or that real number*. Already from the first formalizations, this point was recognized. For example, P. A. M. Dirac stated in his famous book: "The expression that an observable 'has a particular value' for a particular state is permissible in quantum mechanics in the special case when a measurement of the observable is certain to lead to the particular value, so that the state is an eigenstate of the observable. It may easily be verified from the algebra that, with this restricted meaning for an observable 'having a value', if two observables have values for a particular state, then for this state the sum of the two observables (if the sum is an observable) has a value equal to the sum of the values of the two observables separately and the product of the two observables (if this product is an observable) has a value equal to the product of the values of the two observables separately".[8] This last point is the requirement of the functional compatibility condition (FUNC), to which we will return later. As long as we limit ourselves to speaking about measuring results and avoid being concerned with what happens to nature when she is not measured, quantum mechanics carries out predictions with great accuracy. But if we naively try to interpret eigenvalues as actual values of the physical properties of a system, we are faced with all kind of no-go theorems that preclude this possibility. Most remarkable is the Kochen-Specker (KS) theorem that rules out the non-contextual assignment of values to physical magnitudes.[26] In Ref. 10 we gave algebraic, topological and categorial versions of the KS theorem. So we recall the main features of our discussion there.

Let $\mathcal{H}$ be the Hilbert space associated with the physical system and $L(\mathcal{H})$ be the set of closed subspaces on $\mathcal{H}$. It is well known that each self-adjoint operator $\mathbf{A}$ representing a physical magnitude $\mathcal{A}$ has associated a Boolean sublattice W_A of $L(\mathcal{H})$. More precisely, W_A is the Boolean algebra of projectors $\mathbf{P}_i$ of the spectral decomposition $\mathbf{A} = \sum_i a_i \mathbf{P}_i$. We will refer to W_A as the spectral algebra of the operator $\mathbf{A}$. Assigning values to a physical quantity $\mathcal{A}$ is equivalent to establishing a Boolean homomorphism $v : W_A \to \mathbf{2}$,[20] being $\mathbf{2}$ the two elements Boolean algebra. So it is natural to consider the following definition:

Definition 2.1. Let $(W_i)_{i \in I}$ be the family of Boolean sublattices of $L(\mathcal{H})$. A global valuation over $L(\mathcal{H})$ is a family $(v_i : W_i \to \mathbf{2})_{i \in I}$ of Boolean homomorphisms such that, $v_i \mid W_i \cap W_j = v_j \mid W_i \cap W_j$ for each $i, j \in I$.

Were it possible, this global valuation would give the values of all magnitudes at the same time maintaining a *compatibility condition* in the sense that whenever two magnitudes share one or more projectors, the values assigned to those projectors are the same from every context. But the KS theorem assures that we cannot assign real numbers pertaining to their spectra to operators $\mathbf{A}$. In the algebraic terms of the previous definition, the KS theorem reads:

Theorem 2.1. *If $\mathcal{H}$ is a Hilbert space such that dim$(\mathcal{H}) > 2$, then a global valuation over $L(\mathcal{H})$ is not possible.*

Of course contextual valuations allow us to refer to different sets of actual properties of the system which define its state in each case. Algebraically, a *contextual valuation* is a Boolean valuation over one chosen spectral algebra. In classical particle mechanics it is possible to define a Boolean valuation of all propositions, that is to say, it is possible to give a value to all the properties in such a way of satisfying FUNC. This possibility is lost in the quantum case.

Now we intend to show how this discussion is able to include *modalities*, i.e. to consider also possibility and necessity of propositions about the properties of physical systems.

3. Modal Interpretations

Modal interpretations of quantum mechanics[6,7,31,32] face the problem of finding an objective reading of the accepted mathematical formalism of the theory, a reading "in terms of properties possessed by physical systems, independently of consciousness and measurements (in the sense of human interventions)".[7] These interpretations intend to consistently include the possible properties of the system in the discourse and so find a new link between the state of the system and the probabilistic character of its properties, namely, sustaining that the interpretation of the quantum state must contain a modal aspect. The name *modal interpretation* was for the first time used by B. van Fraassen[32] following *modal logic*, precisely the logic that deals with possibility and necessity. The fundamental point is the purpose of interpreting "the formalism as providing information about properties of physical systems".[7] In this context, a physical property of a system is "a definite value of a physical quantity belonging to this system; i.e., a feature of physical reality".[7] As usual, definite values of physical magnitudes correspond to yes/no propositions represented by orthogonal projection operators acting on vectors belonging to the Hilbert space of the (pure) states of the system.

Modal interpretations may be thought to be a study of the constraints under which one is able to talk a consistent classical discourse without contradiction with the quantum formalism. To study this issue and in order to avoid inconsistencies, we face the problem of modalities in the frame of algebraic logic. To do so, we build a variety that is an expansion of the orthomodular lattices by adding an operator, *the possibility operator*, to these structures. It will represent the possibility of occurrence of a property, measurable in terms of the Born rule. The analysis of the changes introduced by allowing modalities will be performed, as in Ref. 10, for the case of pure states. In spite of the restrictions this imposes to the comparison with the general case of modal interpretations, we think it contributes to enlighten the discussion all the same. In a following step, we will extend the treatment to the factorized space of subsystems.

4. Orthomodularuty and Modality

The impossibility to assign values to the properties at the same time satisfying compatibility conditions is a weighty obstacle for the interpretation of the formalism. B. van Fraassen was the first one to formally include the reasoning of modal logic to circumvent these difficulties presenting a modal interpretation of quantum logic in terms of its semantical analysis.[33] In our case, the modal component was introduced with different purposes: to provide a rigorous framework for the Born rule and mainly, to discuss the restrictions posed by the KS theorem to modalities.[11]

To do so we enriched the orthomodular structure with a modal operator taking into account the following considerations:

1) Propositions about the properties of the physical system are interpreted in the orthomodular lattice of closed subspaces of the Hilbert space of the (pure) states of the system. Thus we will retain this structure in our extension.

2) Given a proposition about the system, it is possible to define a context from which one can predicate with certainty about it together with a set of propositions that are compatible with it and, at the same time, predicate probabilities about the other ones (Born rule). In other words, one may predicate truth or falsity of all possibilities at the same time, i.e. possibilities allow an interpretation in a Boolean algebra. In rigorous terms, for each proposition P, if we refer with $\Diamond P$ to the possibility of P, then $\Diamond P$ will be a central element of a orthomodular structure.

3) If P is a proposition about the system and P occurs, then it is trivially possible that P occurs. This is expressed as $P \leq \Diamond P$.

4) Assuming an actual property and a complete set of properties that are compatible with it determines a context in which the classical discourse holds. Classical consequences that are compatible with it, for example probability assignments to the actuality of other propositions, shear the classical frame. These consequences are the same ones as those which would be obtained by considering the original actual property as a possible one. This is interpreted in the following way: if P is a property of the system, $\Diamond P$ is the smallest central element greater than P.

From consideration 1) it follows that the original orthomodular structure is maintained. The other considerations are satisfied if we consider a modal operator $\Diamond$ over an orthomodular lattice L defined as $\Diamond a = Min\{z \in Z(L) : a \leq z\}$ with $Z(L)$ the center of L under the assumption that this minimum exists for every $a \in L$. In the following section we explicitly show our construction. For technical reasons this algebraic study will be performed using the necessity operator $\Box$ instead of the possibility operator $\Diamond$. As usual, it will be then possible to define the possibility operator from the necessity operator.

Now, we describe an expanded class orthomodular structure by adding a modal operator, mainly following Ref. 11.

Definition 4.1. Let A be an orthomodular lattice. We say that A is *Boolean saturated* if and only if for all $a \in A$ the set $\{z \in Z(A) : z \leq a\}$ has a maximum. In this case such maximum is denoted by $\Box a$.

In view of Proposition 1.1, orthomodular complete lattices considering $e(a) = \bigvee\{z \in Z(L) : z \leq a\}$ as operator, are examples of boolean saturated orthomodular lattices. Note that any Hilbert-space orthomodular lattice is irreducible (i.e. its center consists of two elements). Hence, in this case, the necessity operator behaves as follows:

$$\Box a = \begin{cases} 1, & \text{if } a = 1; \\ 0, & \text{otherwise} . \end{cases}$$

As a consequence, $\Box$ represents a kind of "absolute necessity" that behaves in the same way for all pure states.

Proposition 4.1. *Let A be an orthomodular lattice. Then A is boolean saturated iff there exists an unary operator $\Box$ satisfying*

S1. $\Box x \leq x$,
S2. $\Box 1 = 1$,
S3. $\Box\Box x = \Box x$,
S4. $\Box(x \wedge y) = \Box(x) \wedge \Box(y)$,
S5. $y = (y \wedge \Box x) \vee (y \wedge \neg\Box x)$,
S6. $\Box(x \vee \Box y) = \Box x \vee \Box y$,
S7. $\Box(\neg x \vee (y \wedge x)) \leq \neg\Box x \vee \Box y$.

Proof. Suppose that A is is Boolean saturated. S1), S2) and S3) are trivial. S4) Since $x \wedge y \leq x$ and $x \wedge y \leq y$ then $\Box(x \wedge y) \leq \Box(x) \wedge \Box(y)$. For the converse, $\Box(x) \leq x$ and $\Box(y) \leq y$, thus $\Box(x) \wedge \Box(y) \leq \Box(x \wedge y)$. S5) Follows from Proposition 1.1 since $\Box(x) \in Z(A)$. S6) For simplicity, let $z = \Box y$. From the precedent item and taking into account that $z \in Z(L)$ we have that $\Box(z \vee x) \wedge \Box(\neg z \vee x) = \Box((z \vee x) \wedge (\neg z \vee x)) = \Box(x)$. Since $\neg z \leq \Box(\neg z \vee x)$ then we have that $1 = z \vee \neg z \leq z \vee \Box(\neg z \vee x)$. Also we have $z \leq \Box(z \vee x)$. Finally $z \vee \Box(x) = (z \vee \Box(z \vee x)) \wedge (z \vee \Box(\neg z \vee x)) = (z \vee \Box(z \vee x)) \wedge 1 = \Box(z \vee x)$ i.e. $\Box(x \vee \Box y) = \Box x \vee \Box y$. S7) Since $\Box(x) \leq x$ then $\neg x \leq \neg\Box x$, we have that $\neg x \vee (y \wedge x) \leq \neg\Box x \vee y$. Using the precedent item $\Box(\neg x \vee (y \wedge x)) \leq \Box(\neg\Box x \vee y) = \neg\Box x \vee \Box y$ since $\neg\Box x \in Z(A)$.

For the converse, let $a \in A$ and $\{z \in Z(A) : z \leq a\}$. By $S1$ and $S5$ it is clear that $\Box a \in \{z \in Z(A) : z \leq a\}$. We see that $\Box a$ is the upper bound of the set. Let $z \in Z(A)$ such that $z \leq a$ then $1 = \neg z \vee (a \wedge z)$. Using $S2$ and $S7$ we have $1 = \Box 1 = \Box(\neg z \vee (a \wedge z)) \leq \neg\Box z \vee a = \neg z \vee a$. Therefore $z = z \wedge (\neg z \vee \Box a)$ and since z is central $z = z \wedge \Box a$ resulting $z \leq \Box a$. Finally $\Box a = Max\{z \in Z(A) : z \leq a\}$. $\Box$

Note that the operator $\Box$ is an example of quantifier in the sense of Janowitz.[22]

The following theorem is an immediate consequence of Proposition 4.1.

Theorem 4.1. *The class of Boolean saturated orthomodular lattices constitutes a variety which is axiomatized by:*

(1) Axioms of $\mathcal{OML}$,
(2) $S1, ..., S7$.

Boolean saturated orthomodular lattices are algebras $\langle A, \wedge, \vee, \neg, \Box, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$. The variety of this algebras is noted as $\mathcal{OML}^{\Box}$. By simplicity, the set $Term_{\mathcal{OML}^{\Box}}$ will be denoted by $Term^{\Box}$. Since $\mathcal{OML}$ is a reduct of $\mathcal{OML}^{\Box}$ we can also suume that all $\mathcal{OML}^{\Box}$-equations are of the form $t = 1$. It is well known that $\mathcal{OML}$ is congruence distributive and congruence permutable. Therefore if $A \in \mathcal{OML}^{\Box}$ and we consider the OML-reduct of A it is clear that $Con_{\mathcal{OML}^{\Box}}(A) \subseteq Con_{\mathcal{OML}}(A)$ resulting A congruence distributive and congruence permutable in the sence of $\mathcal{OML}^{\Box}$-congruences. Hence the variety $\mathcal{OML}^{\Box}$ is congruence distributive and congruence permutable. The following lemma gives basic properties that will be used later:

Lemma 4.1. *Let* $A \in \mathcal{OML}^{\Box}$ *and* $a, b \in A$ *and* $z_1, z_2 \in Z(A)$. *Then we have:*

(1) $\neg\Box a \vee a = 1$,
(2) $\neg(a \vee \neg b) \vee (a \vee \neg\Box b) = 1$,
(3) $\neg(\neg z_1 \vee z_2) \vee ((\neg(z_1 \vee a) \vee (z_2 \vee a) = 1$,
(4) $\Box a \vee \Box b \leq \Box(a \vee b)$,
(5) $(\neg\Box a \wedge \neg\Box b) \vee \Box(a \vee b) = 1$,
(6) if $x \leq y$ *then* $\Box x \leq \Box y$.

Proof. 1) Since $\Box a \leq a$ then $\neg a \leq \neg\Box a$ and $1 = a \vee \neg a \leq a \vee \neg\Box a$. 2) Since $\neg\Box b \in Z(A)$ and by item 1 we have that $\neg(a \vee \neg b) \vee (a \vee \neg\Box b) = (\neg a \wedge b) \vee (a \vee \neg\Box b) = ((\neg a \vee \neg\Box b) \wedge (b \vee \neg\Box b)) \vee a = ((\neg a \vee \neg\Box b) \wedge 1) \vee a = 1$. 3) $\neg(\neg z_1 \vee z_2) \vee ((\neg(z_1 \vee a) \vee (z_2 \vee a) = \neg((\neg z_1 \vee z_2) \wedge (z_1 \vee a)) \vee (z_2 \vee a) = \neg((\neg z_1 \wedge a) \vee (z_1 \wedge z_2) \vee (z_2 \wedge a)) \vee (z_2 \vee a) = ((z_1 \vee \neg a) \wedge (\neg z_1 \vee \neg z_2) \wedge (\neg z_2 \vee \neg a)) \vee (z_2 \vee a) = ((z_1 \vee \neg a \vee z_2) \wedge (\neg z_2 \vee \neg x \vee z_2) \wedge (\neg z_2 \vee \neg a \vee z_2)) \vee a = z_1 \vee \neg a \vee z_2 \vee a = 1$. 4) $\Box a \leq a$ and $\Box b \leq b$, $\Box a \vee \Box b \leq a \vee b$. Since $\Box a \vee \Box b \in Z(A)$ it is clear that $\Box a \vee \Box b \leq \Box(a \vee b)$. 5) Immediately from item 4. 6) Suppose that $x \leq y$ then $x = x \wedge y$. By Axiom $S4$ we have that $\Box x = \Box(x \wedge y) = \Box x \wedge \Box y)$, hence $\Box x \leq \Box y$. □

Lemma 4.2. *Let* $A \in \mathcal{OML}^{\Box}$ *and* $z \in Z(A)$. *Then the binary relation* Θ_z *on* A *defined by* $a\Theta_z b$ *iff* $a \wedge z = b \wedge z$ *is a congruence on* A, *such that* $A \cong A/\Theta_z \times A/\Theta_{\neg z}$.

Proof. It is well known that Θ_z is a $\mathcal{OML}$-congruence and A is $\mathcal{OML}$-isomorphic to $A/\Theta_z \times A/\Theta_{\neg z}$. Therefore we only need to see the $\Box$-compatibility. In fact: suppose that $a\Theta_z b$ then $a \wedge z = b \wedge z$. Therefore $\Box(a) \wedge z = \Box(a) \wedge \Box(z) = \Box(a \wedge z) = \Box(b \wedge z) = \Box(b) \wedge \Box(z) = \Box(b) \wedge z$. Hence $\Box(a)\Theta_z\Box(b)$. $\square$

Proposition 4.2. *Let $A \in \mathcal{OML}^{\Box}$ then we have that:*

(1) The map $z \to \Theta_z$ is a lattice isomorphism between $Z(L)$ and the Boolean subalgebra of $Con(L)$ of factor congruences.
(2) A is directly indecomposable iff $Z(A) = \{0, 1\}$.

Proof. 1) Follows from Lemma 4.2 using the same arguments that prove the analog result for orthomodular lattices [2, Proposition 5.2]. 2) Follows form item 1. $\square$

Proposition 4.3. *Let A be a directly indecomposable $\mathcal{OML}^{\Box}$-algebra. Then:*

$$t(x, y, z) = (x \wedge \neg\Box((x \wedge y) \vee (\neg x \wedge \neg y))) \vee (z \wedge \Box((x \wedge y) \vee (\neg x \wedge \neg y)))$$

is a discriminator term for A.

Proof. By Proposition 4.2, $Z(A) = \{0, 1\}$. Therefore for each $a \in A - \{1\}$, $\Box a = 0$. Let $a, b, c \in A$. Suppose that $a \neq b$. By the characterization of the equations in $\mathcal{OML}^{\Box}$ we have that $(a \wedge b) \vee (\neg a \wedge \neg b) \neq 1$ and then $t(a, b, c) = a$. If we suppose that $a = b$ then it is clear that $t(a, b, c) = c$. Hence $t(x, y, z)$ is a discriminator term for A. $\square$

Proposition 4.4. *If $\mathcal{A}$ is a subvariety of $\mathcal{OML}^{\Box}$ then $\mathcal{A}$ is a discriminator variety.*

Proof. Let $SI_{\mathcal{A}}$ be the class of subdirectly irreducible algebras of $\mathcal{A}$. Each algebra of $SI_{\mathcal{A}}$ is directly indecomposable. Therefore by Proposition 4.3 $SI_{\mathcal{A}}$ admit a common discriminator term. Since $\mathcal{A} = \mathcal{V}(SI_{\mathcal{A}})$ we have that $\mathcal{A}$ is a discriminator variety. $\square$

5. Modalities and Contextuality

It is clear that the addition of modalities gives by itself greater expressive power to the language of propositions about the system. But what we want to emphasize is that it gives an adequate framework to represent, for example, the Born rule for the probability of actualization of a property, something that has no place in the orthomodular lattice alone. In order to develop these ideas, we need to prove which conditions on elements of a subset A are necessary to make $\langle A \rangle_L$, the sublattice generated by A, a Boolean sublattice.

Definition 5.1. Let L be an orthomodular lattice and $a, b \in L$. Then a commutes with b if and only if $a = (a \wedge b) \vee (a \wedge \neg b)$. A non-empty subset A is called a Greechie set iff for any three different elements of A, at least one commutes with the other two.

Proposition 5.1. *Let L be an orthomodular lattice. If A is a Greechie set in L such that for each $a \in A, \neg a \in A$ then, $\langle A \rangle_L$ is Boolean sublattice.*

Proof. It is well known from Ref. 16 that $\langle A \rangle_L$ is a distributive sublattice of L. Since distributive orthomodular lattices are Boolean algebras, we only need to see that $\langle A \rangle_L$ is closed by $\neg$. To do that we use induction on the complexity of terms of the sub universe generated by A. For $comp(a) = 0$, it follows from the fact that A is closed by negation. Assume validity for terms of the complexity less than n. Let τ be a term such that $comp(\tau) = n$. If $\tau = \neg \tau_1$ then $\neg \tau \in \langle A \rangle_L$ since $\neg \tau = \neg\neg \tau_1 = \tau_1$ and $\tau_1 \in \langle A \rangle_L$. If $\tau = \tau_1 \wedge \tau_2$, $\neg \tau = \neg \tau_1 \vee \neg \tau_2$. Since $comp(\tau_i) < n$, $\neg \tau_i \in \langle A \rangle_L$ for $i = 1, 2$ resulting $\neg \tau \in \langle A \rangle_L$. We use the same argument in the case $\tau = \tau_1 \vee \tau_2$. Finally $\langle A \rangle_L$ is a Boolean sublattice. □

Definition 5.2. Let L be an orthomodular lattice and $L^\Box \in \mathcal{OML}^\Box$ be a modal extension of L. We define the *possibility space* of L in $L^\Box$ as

$$\Diamond L = \langle \{ \Diamond p : p \in L \} \rangle_{L^\Box}$$

The *possibility space* represents the modal content added to the discourse about properties of the system.

Proposition 5.2. *Let L be an orthomodular lattice, W a Boolean sublattice of L and $L^\Box \in \mathcal{OML}^\Box$ a modal extension of L. Then $\langle W \cup \Diamond L \rangle_{L^\Box}$ is a Boolean sublattice of $L^\Box$. In particular $\Diamond L$ is a Boolean sublattice of $Z(L^\Box)$.*

Proof. Follows from Proposition 5.1 since $W \cup \Diamond L$ is a Greechie set closed by $\neg$. □

We know that, in the orthomodular lattice of the properties of the system, it is always possible to choose a context in which any possible property pertaining to this context can be considered as an actual property. We formalize this fact in the following definition and then we prove that this is always possible in our modal structure.

Definition 5.3. Let L be an orthomodular lattice, W a Boolean sublattice of L, $p \in W$ and $L^{\Box}$ be a modal extension of L. If $f : \Diamond L \to \mathbf{2}$ is a Boolean homomorphism such that $f(\Diamond p) = 1$ then an actualization of p compatible with f is a Boolean homomorphism $f_p : W \to \mathbf{2}$ such that:

(1) $f_p(p) = 1$.
(2) There exists a Boolean homomorphism $g : \langle W \cup \Diamond L \rangle_{L^{\Box}} \to \mathbf{2}$ such that $g \mid W = f_p$. and $g \mid \Diamond L = f$.

Theorem 5.1. *Let L be an orthomodular lattice, W a Boolean sublattice of L, $p \in W$ and $L^{\Box}$ be a modal extension of L. If $f : \Diamond L \to \mathbf{2}$ is a Boolean homomorphism such that $f(\Diamond p) = 1$ then there exists an actualization of p compatible with f.*

Proof. Let F be the filter associated with the Boolean homomorphism f. We consider the $\langle W \cup \Diamond L \rangle_{L^{\Box}}$-filter F_p generated by $F \cup \{p\}$. We want to see that F_p is a proper filter. If F_p is not proper, then there exists $a \in F$ such that $a \wedge p \leq 0$. Thus $p \leq \neg a$ being $\neg a$ a central element. But $\Diamond p$ is the smallest Boolean element greater than p, then $\Diamond p \leq \neg a$ or equivalently $\Diamond p \wedge a = 0$ And this is a contradiction since $\Diamond p, a \in F$ being F a proper filter. Thus we may extend F_p to be a maximal filter F_M in $\langle W \cup \Diamond L \rangle_{L^{\Box}}$, resulting the natural projection $\langle W \cup \Diamond L \rangle_{L^{\Box}} \to \langle W \cup \Diamond L \rangle_{L^{\Box}} / F_M \approx \mathbf{2}$ an actualization of p compatible with f. $\Box$

The next theorem allows an algebraic representation of the Born rule which quantifies possibilities from a chosen spectral algebra.

Theorem 5.2. *Let L be an orthomodular lattice, W be a Boolean sublattice of L and $f : W \to \mathbf{2}$ be a Boolean homomorphism. If we consider a modal extension $L^{\Box}$ of L then there exists a Boolean homomorphism $f^* : \langle W \cup \Diamond L \rangle_{L^{\Box}} \to \mathbf{2}$ such that $f^* \mid W = f$.*

Proof. Let $i : W \to \langle W \cup \Diamond L \rangle_{L^{\Box}}$ be the Boolean canonical embedding. If we consider the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{f} & \mathbf{2} \\ {\scriptstyle i}\downarrow & & \\ \langle W \cup \Diamond L \rangle_{L^\Box} & & \end{array}$$

we see that there exists a Boolean homomorphism $f^* : \langle W \cup \Diamond L \rangle_{L^\Box} \to \mathbf{2}$ such that $f^* \mid W_A = f$ because $\mathbf{2}$ is injective in the variety of Boolean algebras.[29] $\Box$

We note that this reading of the Born rule is a kind of converse of the possibility of actualizing properties given by Theorem 5.1.

The addition of modalities to the discourse about the properties of a quantum system enlarges its expressive power. At first sight it may be thought that this could help to circumvent contextuality, allowing to refer to physical properties belonging to the system in an objective way that resembles the classical picture. To prove it here, we introduce an algebraic representation of the notion of global actualization:

Definition 5.4. Let L be an orthomodular lattice, $(W_i)_{i \in I}$ the family of Boolean sublattices of L and $L^\Box$ a modal extension of L. If $f : \Diamond L \to \mathbf{2}$ is a Boolean homomorphism, an *actualization compatible with* f is a global valuation $(v_i : W_i \to \mathbf{2})_{i \in I}$ such that $v_i \mid W_i \cap \Diamond L = f \mid W_i \cap \Diamond L$ for each $i \in I$.

Compatible actualizations represent the passage from possibility to actuality.

Theorem 5.3. *Let L be an orthomodular lattice. Then L admits a global valuation iff for each possibility space there exists a Boolean homomorphism $f : \Diamond L \to \mathbf{2}$ that admits a compatible actualization.*

Proof. Suppose that L admits a global valuation $(v_i : W_i \to \mathbf{2})_{i \in I}$. Let $L^\Box$ be a modal extension of L and consider $A_i = W_i \cap \Diamond L$. Let $f_0 = \bigcup_i A_i \to \mathbf{2}$ such that $f_0(x) = v_i(x)$ if $x \in W_i$. f_0 is well defined since $(v_i)_i$ is a global valuation. If we consider $\langle \bigcup_i A_i \rangle_{L^\Box}$, the subalgebra of $L^\Box$ generated by the join of the family (A_i), it may be proved that it is a Boolean subalgebra of the possibility space $\Diamond L$. We can extended f_0 to a Boolean homomorphism $f_0^* : \langle \bigcup_i A_i \rangle_{L^\Box} \to \mathbf{2}$. Since $\mathbf{2}$ is injective in the variety of Boolean algebras,[29] there exits a Boolean homomorphism $f : \Diamond L \to \mathbf{2}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \langle \bigcup_i A_i \rangle_{L^\Box} & \xrightarrow{f_0^*} & \mathbf{2} \\ i \big\downarrow & \equiv & \nearrow f \\ \Diamond L & & \end{array}$$

Thus $f : \Diamond L \rightarrow \mathbf{2}$ admits a compatible actualization. The converse is immediate. $\Box$

Since the possibility space is a Boolean algebra, there exists a Boolean valuation of the possible properties. But in view of the last theorem, an actualization that would correspond to a family of compatible global valuations is prohibited. The theorem states that the contextual character of quantum mechanics is maintained even when the discourse is enriched with modalities. Thus, moving to the realm of logical possibility, things do not improve due to the fact that possible properties do not admit compatible actualizations.

6. $\mathcal{OML}^\Box$-Logic

We have elsewhere that it is possible to build a logic to deal with actual properties pertaining to different contexts.[10] It is also possible to build a modal logic to take into account possible properties.[12] Indeed this is a deductive system to formally treat quantum properties taking into account our type of modality.

6.1. *Hilbert-style calculus for* $\mathcal{OML}^\Box$

In this subsection we build a Hilbert-style calculus $\langle Term^\Box, \vdash \rangle$ for $\mathcal{OML}^\Box$. We first introduce some notation. $\alpha \in Term^\Box$ is a *tautology* iff $\models_{\mathcal{OML}^\Box} \alpha = 1$. Each subset T of $Term^\Box$ is referred as *theory*. If v is a valuation, $v(T) = 1$ means that $v(\gamma) = 1$ for each $\gamma \in T$. We use $T \models_{\mathcal{OML}^\Box} \alpha$ (read α is *semantic consequence* of T) in the case in which when $v(T) = 1$ then $v(\alpha) = 1$ for each valuation v. The consequence relation that is here chosen represents a kind of weak consequence that does not correspond to the lattice partial order. In fact, unlike other semantic characterizations of QL, here the relation $\alpha \models \beta$ iff for any valuation v, $v(\alpha) \leq v(\beta)$ does not hold.

Lemma 6.1. *Let* γ *and* $\alpha \in Term^\Box$*. Then we have*

(1) If v *is a valuation then* $v(\alpha) = 1$ *iff* $v(\Box\alpha) = 1$*.*

(2) $\gamma \models_{\mathcal{OML}^{\Box}} \alpha$ *iff* $\gamma \models_{\mathcal{OML}^{\Box}} \Box\alpha$ *iff* $\Box\gamma \models_{\mathcal{OML}^{\Box}} \alpha$ *iff* $\Box\gamma \models_{\mathcal{OML}^{\Box}} \Box\alpha$.

Proof. 1) If $v(\alpha) = 1$ then $1 = \Box 1 = \Box(v(\alpha)) = v(\Box\alpha)$. The converse follows from the fact $1 = v(\Box\alpha) = \Box(v(\alpha)) \leq v(\alpha)$. 2) Immediate from the item 1. □

Definition 6.1. Consider by definition the following binary connective

$$\alpha R\beta \text{ for } (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$$

The calculus $\langle Term^{\Box}, \vdash \rangle$ is given by the following axioms:

A0. $1R(\alpha \vee \neg\alpha)$ and $0R(\alpha \wedge \neg\alpha)$,
A1. $\alpha R\alpha$,
A2. $\neg(\alpha R\beta) \vee (\neg(\beta R\gamma) \vee (\alpha R\gamma))$,
A3. $\neg(\alpha R\beta) \vee (\neg\alpha R\neg\beta)$,
A4. $\neg(\alpha R\beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma))$,
A5. $(\alpha \wedge \beta)R(\beta \wedge \alpha)$,
A6. $(\alpha \wedge (\beta \wedge \gamma))R((\alpha \wedge \beta) \wedge \gamma)$,
A7. $(\alpha \wedge (\alpha \vee \beta))R\alpha$,
A8. $(\neg\alpha \wedge \alpha)R((\neg\alpha \wedge \alpha) \wedge \beta)$,
A9. $\alpha R\neg\neg\alpha$,
A10. $\neg(\alpha \vee \beta)R(\neg\alpha \wedge \neg\beta)$,
A11. $(\alpha \vee (\neg\alpha \wedge (\alpha \vee \beta))R(\alpha \vee \beta)$,
A12. $(\alpha R\beta)R(\beta R\alpha)$,
A13. $\neg(\alpha R\beta) \vee (\neg\alpha \vee \beta)$,
A14. $(\Box\alpha \vee \alpha)R\alpha$,
A15. $\Box(\alpha \vee \neg\alpha)R(\alpha \vee \neg\alpha)$,
A16. $\Box\Box\alpha R\Box\alpha$,
A17. $\Box(\alpha \wedge \beta)R(\Box\alpha \wedge \Box\beta)$,
A18. $((\alpha \wedge \Box\beta) \vee (\alpha \wedge \neg\Box\beta))R\alpha$,
A19. $\Box(\alpha \vee \neg\Box\beta)R(\Box\alpha \vee \neg\Box\beta)$,
A20. $\Box(\alpha \vee \Box\beta)R(\Box\alpha \vee \Box\beta)$,
A21. $(\Box(\neg\alpha \vee (\beta \wedge \alpha)) \vee (\neg\Box\alpha \vee \Box\beta))R(\neg\Box\alpha \vee \Box\beta)$,
A22. $\neg(\alpha \vee \neg\beta) \vee (\alpha \vee \neg\Box\beta)$,
A23. $\neg(\gamma \vee \neg\beta) \vee (\neg(\beta \vee \alpha) \vee (\gamma \vee \alpha))$,
A24. $\Box(\alpha \vee \beta) \vee (\neg\Box\alpha \wedge \neg\Box\beta)$.

and the following inference rules:

$$\frac{\alpha, \neg\alpha \vee \beta}{\beta}, \qquad \textit{disjunctive syllogism (DS)}$$

$$\frac{\alpha}{\Box\alpha}, \qquad \textit{necessitation (N)}$$

Let T be a theory. A *proof* from T is a sequence $\alpha_1, ..., \alpha_n$ in $Term^{\Box}$ such that each member is either an axiom or a member of T or follows from some preceding member of the sequence using *DS* or *N*. $T \vdash \alpha$ means that α is provable in T, that is, α is the last element of a proof from T. If $T = \emptyset$, we use the notation $\vdash \alpha$ and in this case we will say that α is a theorem of $\langle Term^{\Box}, \vdash\rangle$. T is *inconsistent* if and only if $T \vdash \alpha$ for each $\alpha \in Term^{\Box}$; otherwise it is *consistent.*

Proposition 6.1. *Let T be a theory and $\alpha, \beta, \gamma \in Term^{\Box}$. Then:*

(1) $T \vdash \alpha R\beta \Longrightarrow T \vdash \beta R\alpha$,
(2) $T \vdash \alpha R\beta$ *and* $T \vdash \beta R\gamma \;\Longrightarrow\; T \vdash \alpha R\gamma$,
(3) $T \vdash \alpha R\beta \;\Longrightarrow\; T \vdash \neg\alpha R\neg\beta$,
(4) $T \vdash \alpha R\beta$ *and* $T \vdash \alpha \wedge \gamma \;\Longrightarrow\; T \vdash \beta \wedge \gamma$,
(5) $T \vdash \alpha R\beta$ *and* $T \vdash \alpha \vee \gamma \;\Longrightarrow\; T \vdash \beta \vee \gamma$,
(6) $T \vdash \alpha R\beta \;\Longrightarrow\; T \vdash \Box\alpha R\Box\beta$,
(7) $\vdash \alpha \vee \neg\alpha$,
(8) $T \vdash \alpha \;\Longrightarrow\; T \vdash \alpha \vee \beta$.

Proof. (1)

1. $T \vdash \alpha R\beta$
2. $T \vdash (\alpha R\beta)R(\beta R\alpha)$ — by A12
3. $T \vdash \neg((\alpha R\beta)R(\beta R\alpha)) \vee (\neg(\alpha R\beta) \vee (\beta R\alpha))$ — by A13
4. $T \vdash (\neg(\alpha R\beta) \vee (\beta R\alpha))$ — by *DS* 2,3
5. $T \vdash \beta R\alpha$ — by *DS* 1,4

(2) Is easily from A2 and two application of the *DS*.
(3) Follows from A3.
(4)

1. $T \vdash \alpha R\beta$
2. $T \vdash \alpha \wedge \gamma$
3. $T \vdash \neg(\alpha R\beta) \vee ((\alpha \wedge \gamma)R(\beta \wedge \gamma))$ — by A4
4. $T \vdash (\alpha \wedge \gamma)R(\beta \wedge \gamma)$ — by *DS* 1
5. $\neg(\alpha \wedge \gamma)R(\beta \wedge \gamma) \vee (\neg(\alpha \wedge \gamma) \vee (\beta \wedge \gamma))$ — by A4
6. $T \vdash \beta \wedge \gamma$ — by *DS* 5,4,2

(5) Follows by item 4, A9 and A10.

(6)

1. $T \vdash \alpha R\beta$
2. $T \vdash (\alpha \wedge \beta) \vee (\neg\alpha \wedge \neg\beta)$ equiv. 1
3. $T \vdash (\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)$ by item 5 and A10
4. $\vdash \neg((\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)) \vee ((\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta))$ by A22
5. $T \vdash (\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta)$ by *DS* 4,3
6. $T \vdash \Box((\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta))$ by *DS* 4,3
7. $T \vdash \Box(\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta)$ by A13, A19
8. $T \vdash (\Box\alpha \wedge \Box\beta) \vee \neg\Box(\alpha \vee \beta)$ by item 5 and A17
9. $\neg((\Box\alpha \wedge \Box\beta) \vee \neg\Box(\alpha \vee \beta)) \vee (\neg(\Box(\alpha \vee \beta) \vee (\neg\Box\alpha \wedge \neg\Box\beta)) \vee ((\Box\alpha \wedge \Box\beta) \vee (\neg\Box\alpha \wedge \neg\Box\beta))$ by A23
10. $\Box(\alpha \vee \beta) \vee (\neg\Box\alpha \wedge \neg\Box\beta)$ by A24
11. $(\Box\alpha \wedge \Box\beta) \vee (\neg\Box\alpha \vee \neg\Box\beta))$ by *SD* 8,9,10
12. $T \vdash \Box\alpha R\Box\beta$ equiv. 1

7)Follows from A1 and A13.

8)

1. $T \vdash \alpha$
2. $T \vdash (\alpha \vee \neg\alpha)R((\alpha \vee \neg\alpha) \vee \beta)$ by A3, A8, A10
3. $T \vdash (\alpha \vee \neg\alpha) \vee \beta$ by item 7 and A13
4. $\vdash \neg((\alpha \wedge \beta) \vee \neg(\alpha \vee \beta)) \vee ((\alpha \wedge \beta) \vee \neg\Box(\alpha \vee \beta))$ by A22
5. $\vdash \neg\alpha \vee (\alpha \vee \beta)$ by 4, A5, A3, A10
6. $\vdash \alpha \vee \beta$ by *DS* 1,5 □

Proposition 6.2. *Axioms of the* $\langle Term^{\Box}, \vdash\rangle$ *are tautologies.*

Proof. For A0... A13 see [25, Chapter 4.15]. A22... A24 follow from Proposition 4.1. □

Theorem 6.1. *Let* T *be a theory. If for each* $\alpha \in Term^{\Box}$ *we consider the set* $[\alpha] = \{\beta : T \vdash \alpha R\beta\}$ *then* $L_T = \{[\alpha] : \alpha \in Term^{\Box}\}$ *determines a partition in equivalence classes of* $Term^{\Box}$*. Defining the following operation in* L_T*:*

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta], \qquad \neg[\alpha] = [\neg\alpha], \qquad 0 = [0]$$
$$[\alpha] \vee [\beta] = [\alpha \vee \beta], \qquad \Box[\alpha] = [\Box\alpha], \qquad 1 = [1]$$

we have that

(1) $T \vdash \alpha$ *if and only if* $[\alpha] = 1$.

(2) $\langle L_T, \vee, \wedge, \neg, \Box, 0, 1\rangle$ *is a Boolean saturated orthomodular lattice.*

Proof. By A1 and Proposition 6.1 (item 1 and 2) $L_T = \{[\alpha] : \alpha \in Term^{\Box}\}$ is a partition in equivalence classes of $Term^{\Box}$.

1) Assume that $T \vdash \alpha$, then we have that:

(1) $T \vdash \alpha$	
(2) $T \vdash \alpha R(\alpha \wedge (\alpha \vee \neg\alpha))$	by A7
(3) $T \vdash \alpha \wedge (\alpha \vee \neg\alpha)$	by 1 and A13
(4) $T \vdash (\alpha \wedge (\alpha \vee \neg\alpha)) \vee (\neg\alpha \wedge \neg(\alpha \vee \neg\alpha))$	by 3 and Prop. 6.1 (8)
(5) $T \vdash \alpha R(\alpha \vee \neg\alpha)$	equiv. in 4

resulting $[\alpha] = 1$. On the other hand, if $[\alpha] = 1$, we have that $T \vdash \alpha R(\alpha \vee \neg\alpha)$. Using Proposition 6.1 (7) and A13, it results $T \vdash \alpha$.

2) By Proposition 6.1 (item 3 and 6) $\vee, \wedge, \neg, \Box$ are well defined in L_T. By A0...A13 and ([25, Proposition 4.15. 1]) L_T is an orthomodular lattice. By A14...A21 and Proposition 6.2, L_T is boolean saturated. □

The following theorem establishes the strong completeness for $\langle Term^{\Box}, \vdash\rangle$ with respect to the variety $\mathcal{OML}^{\Box}$.

Theorem 6.2. *Let* $\alpha \in Term^{\Box}$ *and* T *be a theory. Then we have that:*

$$T \vdash \alpha \Longleftrightarrow T \models_{\mathcal{OML}^{\Box}} \alpha$$

Proof. If T is inconsistent, this result is trivial. Assume that T is consistent. $\Longrightarrow$) Immediate. $\Longleftarrow$) Suppose that T does not prove α. Then, by Proposition 6.1, $[\alpha] \neq 1$. Then the projection $p : Term^{\Box} \to L_T$ with $p(\varphi) = [\varphi]$ is a valuation such that $p(\varphi) = 1$ for each $\varphi \in T$ and $p(\alpha) \neq 1$. Finally we have that not $T \models_{\mathcal{OML}^{\Box}} \alpha$. □

Corollary 6.1 (Compactness). *Let* $\alpha \in Term^{\Box}$ *and* T *be a theory. Then we have that,* $T \models_{\mathcal{OML}^{\Box}} \alpha$ *iff there exists a finite subset* $T_0 \subseteq T$ *such that* $T_0 \models_{\mathcal{OML}^{\Box}} \alpha$.

Proof. In view of Theorem 6.2, if $T \models_{\mathcal{OML}^{\Box}} \alpha$ then $T \vdash \alpha$. If $\varphi_1, \cdots \varphi_m, \alpha$ is a proof of α from T, we can consider the finite set $T_0 = \{\varphi_i \in T : \varphi_i \in \{\alpha_1, \cdots \alpha_n\}\}$. Using again Theorem 6.2 we have $T_0 \models_{\mathcal{OML}^{\Box}} \alpha$. □

We can also establish a kind of deduction theorem.

Corollary 6.2. *Let* $\gamma, \alpha \in Term^{\Box}$ *and* T *be a theory. Then:*

$$T \cup \{\gamma\} \vdash \alpha \quad iff \quad T \vdash \neg\Box\gamma \vee \alpha$$

Proof. By Theorem 6.2 we will prove that $T \cup \{\gamma\} \models_{\mathcal{OML}^\Box} \alpha$ iff $T \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$. By Corollary 6.1 $T \cup \{\gamma\} \models_{\mathcal{OML}^\Box} \alpha$ iff there exists $\varphi_1 \ldots \varphi_n \in T$ such that $(\varphi_1 \wedge \ldots \wedge \varphi_n) \wedge \gamma \models_{\mathcal{OML}^\Box} \alpha$. Let $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$. Then $\varphi \wedge \gamma \models_{\mathcal{OML}^\Box} \alpha$ implies that $(\varphi \wedge \gamma) \vee \neg\Box(\gamma) \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$ and then $\varphi \vee \neg\Box\gamma \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$ since for each valuation v, $v(\Box\gamma)$ is a central element and $v(\gamma \vee \neg\Box\gamma) = 1$. Since $\varphi \models_{\mathcal{OML}^\Box} \varphi \vee \neg\Box\gamma$ we have that $\varphi \models_{\mathcal{OML}^\Box} \Box\gamma \vee \alpha$ thus $T \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$.

On the other hand, if $T \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$ we can consider again $\varphi = \varphi_1 \wedge \ldots \wedge \varphi_n$ such that $\varphi_1 \ldots \varphi_n \in T$ and $\varphi \models_{\mathcal{OML}^\Box} \neg\Box\gamma \vee \alpha$. Therefore $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\Box} \Box\gamma \wedge (\neg\Box\gamma \vee \alpha)$ and then $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\Box} \Box\gamma \wedge \alpha$ since for each valuation v, $v(\Box\gamma \wedge (\neg\Box\gamma \vee \alpha)) = v(\Box\gamma \wedge \alpha)$ taking into account that $v(\Box\alpha)$ is always a central element. Since $\Box\gamma \wedge \alpha \models_{\mathcal{OML}^\Box} \alpha$ we have that $\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\Box} \alpha$. Applying Lemma 6.1 we have that $\Box(\varphi \wedge \Box\gamma) \models_{\mathcal{OML}^\Box} \alpha$ hence $\Box\varphi \wedge \Box\gamma \models_{\mathcal{OML}^\Box} \alpha$ and $\varphi \wedge \gamma \models_{\mathcal{OML}^\Box} \alpha$ in view of Axiom $S4$ of $\mathcal{OML}^\Box$. Thus $T \cup \{\gamma\} \models_{\mathcal{OML}^\Box} \alpha$. $\Box$

6.2. *Modal orthomodular frames and Kripke-style semantics*

The orthomodular structure is characterized by a weak form of distributivity called *orthomodular law*. This "weak distributivity", which is the essential difference with the Boolean structure, makes it extremely intractable in certain aspects. In fact, a general representation theorem for a class of algebras, which has as particular instances the representation theorems as algebras of sets for Boolean algebras and distributive lattices, allows in many cases and in a uniform way the choice of a Kripke-style model and to establish a direct relationship with the algebraic model.[30] In this procedure the distributive law plays a very important role. In absence of distributivity this general technique is not applicable, consequently to obtain Kripke-style semantics may be complicated. Such is the case for the orthomodular logic. Indeed, in Ref. 15, Goldblatt gives a Kripke-style semantic for the orthomodular logic based on an imposed restriction on the Kripke-style semantic for the orthologic. This restriction is not first order expressible. Thus the obtained semantic is not very attractive. In Ref. 28, author introduced another approach to the Kripke-style semantic for the orthomodular logic based on the representation theorem by Baer semigroups given by Foulis in Ref. 14 for orthomodular lattices. In this way a Kripke-style model is obtained whose universe is given by semigroups with additional operations. In order to establish a Kripke-style semantics $\langle Term^\Box, \vdash \rangle$ we first introduce el concept of modal Baer semigroups which constitute a sub-class of Baer *-semigroups.

Definition 6.2. A *modal Baer semigroup* is a Baer *-semigroup G such that $\langle P_c(G), \wedge, \vee, ', 0, 1\rangle$ is a Boolean saturated orthomodular lattice. A *modal orthomodular frame* is a pair $\langle G, u\rangle$ such that G is a modal Baer semigroup and u is a valuation $u : Term^\Box \to P_c(G)$

We denote by $\mathcal{MOF}$ the class of all modal orthomodular frames. The following result is a representation theorem by modal Baer semigroups of Boolean saturated orthomodular lattices.

Theorem 6.3. *Let $A \in \mathcal{OML}^\Box$, then there exists a modal Baer semigroup $G(A)$ such that A is $\mathcal{OML}^\Box$-isomorphic to $P_c(G(A))$.*

Proof. Let $A \in \mathcal{OML}^\Box$. By Theorem 1.2 there exists a Baer *-semigroup G such that A is $\mathcal{OML}$-isomorphic to $P_c(G(A))$. Since $\mathcal{OML}$-isomorphisms preserve supremum of central elements we have that $P_c(G(A)) \in \mathcal{OML}^\Box$ and then, $G(A) \in \mathcal{MBS}$. $\Box$

Note that we can easily prove that $\models_{\mathcal{OML}^\Box} t = 1$ iff for all modal Baer semigroups G we have that $\models_{P_c(G)} t = 1$.

Proposition 6.3. *Let $\langle G, u\rangle$ be a modal orhomodular frame and $t, s \in Term^\Box$. Then we have that:*

(1) $u(t \wedge s) \cdot G = (u(t) \cdot G) \cap (v(t) \cdot G)$,
(2) $u(\neg t) \cdot G = \{x \in G : \forall y \in u(t) \cdot G, \;\; y^\star \cdot x = 0\}$
(3) $u(\Box t) \cdot G = \bigcup\{x \cdot G : x \in Z(P_c(G)) \;\; and \;\; x \leq v(t)\}$.

Proof. 1) Follows from an analogous argument used in [28, Theorem 3.13]. 2) See the proof of [28, Lemma 3.16-3]. 3)We first note that $u(t) \geq \Box u(t) = u(\Box t) \in Z(P_c(G))$. Thus $u(\Box t) \cdot G \in \{x \cdot G : x \in Z(P_c(G)) \;\; and \;\; x \leq u(t)\}$ and then $u(t) \cdot G \subseteq \bigcup\{x \cdot G : x \in Z(P_c(G)) \;\; and \;\; x \leq u(t)\}$. On the other hand, if $x \in Z(P_c(G))$ and $x \leq u(t)$ then $x \cdot G \subseteq u(\Box t) \cdot G$ since $x \leq u(\Box t)$. Hence $\bigcup\{x \cdot G : x \in Z(P_c(G)) \;\; and \;\; x \leq u(t)\} \subseteq u(t) \cdot G$. $\Box$

Definition 6.3. Let $\langle G, u\rangle$ be a modal orhomodular frame. Then we define inductively the *forcing relation* $\models^x_{\langle G,u\rangle} \subseteq G \times Term^\Box$ as follows:

(1) $\models^x_{\langle G,u\rangle} p$ iff $x \in u(p) \cdot G$, for each variable $p \in Term^\Box$,
(2) $\models^x_{\langle G,u\rangle} \alpha \wedge \beta$ iff $\models^x_{\langle G,u\rangle} \alpha$ and $\models^x_{\langle G,u\rangle} \beta$,
(3) $\models^x_{\langle G,u\rangle} \neg\alpha$ iff $\forall g \in G$, $\models^g_{\langle G,u\rangle} \alpha \Longrightarrow g' \cdot x = 0$,
(4) $\models^x_{\langle G,u\rangle} \Box\alpha$ iff $x = z \cdot g$ such that $z \in Z(P_c(G))$ and $\models^z_{\langle G,u\rangle} \alpha$.

The relation $\models^x_{\langle G,u\rangle} \alpha$ is read as α *is true at the point* x *in the modal orthomodular frame* $\langle G,u\rangle$ and by $\models_{\langle G,u\rangle} \alpha$ we understand that for each $x \in G$, $\models^x_{\langle G,u\rangle} \alpha$. Generalizing, if T is a theory, $\models_{\langle G,u\rangle} T$ means that, for each $\beta \in T$ we have that $\models_{\langle G,u\rangle} \beta$. With these elements we can establish a notion of *consequence* in the Kripke-style sense that will be noted by $T \models_{\mathcal{MOF}} \alpha$.

$$T \models_{\mathcal{MOF}} \alpha \quad \text{iff} \quad \forall \langle G,u\rangle \in \mathcal{MOF}, \ \models_{\langle G,u\rangle} T \Longrightarrow \models_{\langle G,u\rangle} \alpha$$

Let $\alpha \in Term^{\Box}$, T be a theory and $\langle G,u\rangle$ be an orthomodular frame. Then we consider the following sets:

$$\mid \alpha \mid_{\langle G,u\rangle} = \{x \in G : \models^x_{\langle G,u\rangle} \alpha\}$$

$$\mid T \mid_{\langle G,u\rangle} = \bigcap_{\beta \in T} \mid \beta \mid_{\langle G,u\rangle}$$

Proposition 6.4. *Let* $\alpha \in Term^{\Box}$, T *be a theory and* $\langle G,u\rangle$ *be a modal orthomodular frame. Then we have that:*

(1) $\mid \alpha \mid_{\langle G,u\rangle} = u(\alpha) \cdot G$,
(2) $\models_{\langle G,u\rangle} T$ *iff* $\mid T \mid_{\langle G,u\rangle} = G$.

Proof. 1) We use induction on the complexity of terms. If α is a variable the proposition results trivial. If α is $\beta \wedge \gamma$ or $\neg\beta$ we refer to [28, Lemma 3.16]. Suppose that α is $\Box\beta$. We prove that $u(\Box\beta) \cdot G \subseteq \mid \Box\beta \mid_{\langle G,u\rangle}$. By Proposition 1.2-1 and by inductive hypothesis, we have that $u(\Box\beta) \cdot G = \Box(u(\beta)) \cdot G \subseteq u(\beta) \cdot G = \mid \beta \mid_{\langle G,u\rangle}$. Then $\Box(u(\beta)) \cdot 1 = \Box(u(\beta)) \in \mid \beta \mid_{\langle G,u\rangle}$ i.e., $\models^{\Box(u(\beta))}_{\langle G,u\rangle} \beta$. Thus if $x \in u(\Box\beta) \cdot G$ then $x = \Box(u(\beta)) \cdot g$ and, taking into account that $\Box(u(\beta)) \in Z(P_c(G))$, it results that $x \in \mid \Box\beta \mid_{\langle G,u\rangle}$. On the other hand, if $x \in \mid \Box\beta \mid_{\langle G,u\rangle}$, then $x = z \cdot g$ such that $z \in Z(P_c(G))$ and $z \in \mid \beta \mid_{\langle G,u\rangle}$. Then by inductive hypothesis we have that $z = u(\beta) \cdot g_0 \leq u(\beta) \cdot 1 = u(\beta)$. By the lattice-order definition of $\Box(u(\beta))$ it is clear that $z \leq \Box(u(\beta)) = u(\Box(\beta))$. Therefore $z \cdot G \subseteq u(\Box(\beta)) \cdot G$. Hence $x \in u(\Box(\beta)) \cdot G$.

2) Using the above item, $\models_{\langle G,u\rangle} T$ iff $\forall \beta \in T$, $\models_{\langle G,u\rangle} \beta$ iff $\forall \beta \in T$, $\mid \beta \mid_{\langle G,u\rangle} = G$ iff $G = \bigcap_{\beta \in T} \mid \beta \mid_{\langle G,u\rangle} = \mid T \mid_{\langle G,u\rangle}$ □

Theorem 6.4 (Kripke style completeness). *Let* $\alpha \in Term^{\Box}$ *and* T *be a theory. Then:*

$$T \models_{\mathcal{OML}^{\Box}} \alpha \Longleftrightarrow T \models_{\mathcal{MOF}} \alpha$$

Proof. Suppose that $T \models_{\mathcal{OML}^{\Box}} \alpha$. By Corollary 6.1 there exists $\gamma_1, \ldots \gamma_n \in T$ such that if we consider γ as $\gamma_1 \wedge \ldots \wedge \gamma_n$ then $\gamma \models_{\mathcal{OML}^{\Box}} \alpha$. By Corollary 6.2 we have that $\models_{\mathcal{OML}^{\Box}} \neg\Box\gamma \vee \alpha$. Let $\langle G, u\rangle$ be a modal orthomodular frame such that $\models_{\langle G,u\rangle} T$. By Proposition 6.4 and Proposition 1.2-1 we have that $\mid \gamma \mid_{\langle G,u\rangle} = G$ and then $u(\gamma) = 1$. But $u(\gamma) = 1$ implies $u(\Box\gamma) = 1$ and $\neg u(\Box\gamma) = 0$. Therefore, necessarily $u(\alpha) = 1$, and $\mid \alpha \mid_{\langle G,u\rangle} = G$. Hence $\models_{\langle G,u\rangle} \alpha$.

On the other hand we assume that $T \models_{\mathcal{MOF}} \alpha$. Suppose that $T \not\models_{\mathcal{OML}^{\Box}} \alpha$. Then there exists $A \in \mathcal{OML}^{\Box}$ and a valuation $v : Term^{\Box} \to A$ such that $v(T) = 1$ but $v(\alpha) \neq 1$. By Theorem 6.3 there exists a modal Baer semigroup $G(A)$ such that A is $\mathcal{OML}^{\Box}$-isomorphic to $P_c(G(A))$ being $f : A \to P_c(G(A))$ such isomorphism. Consider the modal orthomodular frame $\langle G(A), fv\rangle$. Then for each $\beta \in T$ we have that $\mid \beta \mid_{\langle G(A),fv\rangle} = fv(\beta) \cdot G(A) = 1 \cdot G(A) = G(A)$. Therefore $\mid T \mid_{\langle G(A),fv\rangle} = G(A)$ and $\models_{\langle G(A),fv\rangle} T$ in view of Proposition 6.4. By Proposition 1.2-1 $\mid \alpha \mid_{\langle G(A),fv\rangle} = fv(\alpha) \cdot G(A) \neq G(A)$ again since $fv(\alpha) < 1$. Then $\not\models_{\langle G(A),fv\rangle} \alpha$ which is a contradiction. Hence $T \models_{\mathcal{OML}^{\Box}} \alpha$. □

Conclusions

We have given first an algebraic description of the contextual character of actual properties. We have shown that, when this structure is enriched with modal operators, the discourse about properties is genuinely enlarged. However, taking into account the algebraic characterization of contextuality, given by the non existence of compatible global valuations over the orthomodular structure, it is possible to prove that the contextual character of the complete language is maintained. Thus contextuality remains a main feature of quantum systems even when modalities are taken into account.

Moreover we have developed a logical system based on the orthomodular structure of propositions about quantum systems enriched with a modal operator. We have obtained algebraic completeness and completeness with respect to a Kripke-style semantic founded on Baer *-semigroups. The importance of this structure from a physical perspective deals with the interpretation of quantum mechanics in terms of modalities.

References

1. G. Birkhoff and J. von Neumann, *Ann. Math.* **27**, 823 (1936).
2. G. Bruns and J. Harding, in *Current Research in Operational Quantum Logic* (Kluwer Academic Publishers, Dordrecht, 2000).

3. S. Burris and H. P. Sankappanavar, *A Course in Universal Algebra* (Graduate Text in Mathematics, Springer-Verlag, New York, 1981).
4. M. Dalla Chiara, *J. Philosophical Logic* **6**, 391 (1977).
5. M. Dalla Chiara, R. Giuntini and R. Greechie, *Reasoning in Quantum Theory, Sharp and Unsharp Quantum Logics* (Kluwer Academic Publishers, Dordrecht, 2004).
6. D. Dieks, *Foundations of Physics* **19**, 1397 (1989).
7. D. Dieks, *Foundations of Physics* **35**, 399 (2005).
8. P. A. M. Dirac, *The Principle of Quantum Mechanics* (Oxford University Press, 1958).
9. H. Dishkant, *J. Symbolic Logic* **42**, 421 (1977).
10. G. Domenech and H. Freytes, *J. Math. Phys.* **46**, 012102-1 (2005).
11. G. Domenech, H. Freytes and C. de Ronde, *Ann. der Physik* **15**, 853 (2006).
12. G. Domenech, H. Freytes and C. de Ronde, *Math. Log. Quart.* **55**, 287 (2009).
13. A. Dvurečenskij and S. Pulmannova, *New Trends in Quantum Structures* (Kluwer Academic Publishers, 2000).
14. D. J. Foulis, *Proc. Amer. Math. Soc.* **11**, 648 (1960).
15. R. I. Goldblatt, *J. Symbolic Logic* **49**, 401 (1984).
16. R. Greechie, *Proc. Amer. Math. Soc.* **67**, 17 (1977).
17. G. Hardegree, *Notre Dame J. Formal Logic* **22**, 163 (1981).
18. L. Herman and R. Piziak, *J. Symbolic Logic* **39**, 478 (1974).
19. P. Halmos, *Compositio Math.* **12**, 217 (1955).
20. C. Isham and J. Butterfield, *Int. J. Theor. Phys.* **37**, 2669 (1998).
21. J. M. Jauch, *Foundations of Quantum Mechanics* (Addison Wesley Reading, 1968).
22. M. F. Janowitz, *Pacific J. Math* **13**, 1241 (1963).
23. M. F. Janowitz, *Ill. J. Math.* **9**, 660 (1965).
24. J. A. Kalman, *Trans. Amer. Math. Soc.* **87**, 485 (1958).
25. G. Kalmbach, *Ortomodular Lattices* (Academic Press, London, 1983).
26. S. Kochen and E. P. Specker, *J. Math. Mech.* **17**, 9 (1967).
27. F. Maeda, S. Maeda, *Theory of Symmetric Lattices* (Springer-Verlag, Berlin, 1970).
28. Y. Miyazaki, *Math. Log. Quart.* **47**, 341 (2001).
29. R. Sikorski, *Ann. Soc. Pol. Math.* **21**, 332 (1948).
30. V. Sofronie-Stokkermans, in *Theory and Applications of Multiple-Valued Logic* (Springer-Verlag Series Studies in Fuzziness and Soft Computing, 2003).
31. B. C. van Fraassen, *Quantum Mechanics: An Empiricist View* (Clarendon, Oxford, 1991).
32. B. C. van Fraassen, in *Current Issues in Quantum Logic* (Plenum Press, New York, 1981).
33. B. C. van Fraassen, in *Contemporary Research in the Foundations and Philosophy of Quantum Theory* (C. A. Hooker Ed., Reidel, Dordrecht, 1973).