Quantum Mechanics of Fundamental Systems 3

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Contributors

Daniel P. Arovas, Department of Physics, University of California at San Diego, La Jolla, California 92093

Tom Banks, Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855-0849

Laurent Baulieu, Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, 75005 Paris, France

Lars Brink, Chalmers University of Technology, Institute of Theoretical Physics, S-412 96 Göteborg, Sweden

Michael R. Douglas, Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855-0849

W. Fischler, Department of Physics, University of Texas, Austin, Texas 78712

G. F. Giudice, Department of Physics, University of Texas, Austin, Texas 78712

Marc Henneaux, Faculté des Sciences, Université Libre de Bruxelles, B-1050 Bruxelles, Belgium and Centro de Estudios Científicos de Santiago, Santiago 9, Chile

C. Itzykson, Service de Physique Théorique de Saclay, Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique, F-91191 Gif-sur-Yvette Cedex, France

R. G. Leigh, Department of Physics, University of Texas, Austin, Texas 78712

E. C. Marino, Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, 22452 Rio de Janeiro RJ, Brazil

Holger Bech Nielsen, Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

S. Paban, Department of Physics, University of Texas, Austin, Texas 78712

R. Rajaraman, Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012, India

Fidel A. Schaposnik, Departamento de Física, Universidad Nacional de La Plata, 1900 La Plata, Argentina

Nathan Seiberg, Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855-0849

Stephen H. Shenker, Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855-0849

J. B. Zuber, Service de Physique Théorique de Saclay, Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique, F-91191 Gif-sur-Yvette Cedex, France

Preface

From January 8-13, 1990, distinguished physicists from many nations came to Chile to share with each other and with Latin American students exciting recent developments. The occasion was the third of a series of meetings on *Quantum Mechanics of Fundamental Systems* which are held every two years at the Centro de Estudios Científicos de Santiago. This volume grew out from that gathering.

The meeting was possible thanks to the generous support of the Tinker Foundation, the John D. and Catherine T. MacArthur Foundation, the International Centre for Theoretical Physics, the Ministère des Affaires Etrangères et Service Culturel et de Coopération Scientifique et Technique de France, the Third World Academy of Sciences, and FONDECYT-Chile. The happy winds blowing over Chile at the time enhanced the joy provided by the beauty of the physics discussed.

Claudio Teitelboim
Jorge Zanelli

Santiago, Chile

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1. INTRODUCTION

One usually does not think of quantum statistics in terms of a continuous parameter, such as a coupling constant. We are accustomed to the notion that many-particle wave functions are either symmetric or antisymmetric:

$$\Psi(\cdots j \cdots i \cdots) = e^{i\theta}\Psi(\cdots i \cdots j \cdots), \tag{1}$$

where $\theta = 2n\pi$ for bosons and $\theta = (2n+1)\pi$ for fermions. Interpolating in θ, e.g., $\theta = \pi/2$, seems to make no sense because iterating Eq. (1) twice gives

$$\Psi(\cdots j \cdots i \cdots) = e^{2i\theta}\Psi(\cdots j \cdots i \cdots) \tag{2}$$

Daniel P. Arovas • Department of Physics, University of California at San Diego, La Jolla, California 92093.

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and the single-valuedness of Ψ demands that $e^{2i\theta} = 1$, so one concludes that Bose and Fermi statistics exhaust all the allowed values of θ.

What happens, though, if we relax the single-valuedness constraint and consider the wave function $\Psi(q)$ to be a *multivalued* function of its argument? One example of a multivalued function is the complex function $f(z) = z^\nu$, which changes by a factor $e^{2\pi i\nu}$ when z moves counterclockwise around a circle enclosing the origin. A path which winds around the origin n times accumulates a phase factor of $e^{2\pi i n\nu}$. If ν is not an integer, then $f(z)$ does not return to its original value. Although it may seem strange to consider multivalued wave functions, nothing prevents us from doing so! The Schrödinger equation is a differential equation and thus requires only that Ψ be *locally* well defined. In addition, physical quantities, such as probability densities, always depend on $|\Psi|^2$ and are appropriately single-valued. (The multivaluedness we are considering always occurs in the *phase* of the wave function.)

In the example $f(z) = z^\nu$, z takes its values in the complex plane. In the case of many-particle quantum mechanics, the argument q of $\Psi(q)$ exists in a more complicated space, called *configuration space.* It is the space of all N-tuples of coordinates $q = \{\mathbf{r}_1, \ldots, \mathbf{r}_N\}$ together with the equivalence relation $q \equiv \sigma q$, where σ is any element of the permutation group $\mathcal{S}_N$, and $\sigma q = \{\mathbf{r}_{\sigma(1)}, \ldots, \mathbf{r}_{\sigma(N)}\}$. The equivalence of q and σq means that the particles are indistinguishable. For technical purposes, it is necessary to impose the restriction that no two particles ever occupy the same position—this condition is necessary for the multivaluedness to be meaningful. This is analogous to the situation in our simple example $f(z) = z^\nu$, above, in which paths that intersect the orgin cannot be assigned a definite winding number. Physically, the restriction that no two particles occupy the same position can be accomplished by imposing an infinitely repulsive hard-core potential of vanishingly small range; this restriction has no effect on any physical properties.

We now ask what sorts of multivalued functions can be defined on this configuration space.* Recall that in the case of the simple example $f(z) = z^\nu$ paths could be classified by an integer winding number n; paths that have the same winding number are equivalent in the sense that they can be smoothly deformed into one another without crossing the origin. Associated to each path of winding number n is a phase $e^{2\pi i n\nu}$. If we append one path of winding number n' to a path of winding number n, the resultant path has winding number $n + n'$. Thus, we can think of the space of paths as a mathematical group, and in this simple case, group addition of two paths

* It should be remarked that conventional wave functions satisfying Fermi statistics *are* multivalued in configuration space, for the sign changes depending on the parity of the permutation associated with a given path.

of winding numbers n and n' produces a third path of winding number $n + n'$. Mathematically, this result is succinctly stated as

$$\pi_1(\mathbf{R}^2 - \{0\}) \cong \mathbf{Z}, \tag{3}$$

which means that the group of paths (under the operation of path addition) on the punctured plane (the plane minus the origin) is isomorphic to the group of integers (under the operation of addition). Mathematicians refer to the group of paths $\pi_1(M)$ as the "fundamental group" or "first homotopy group" of the manifold M. The fundamental group of the punctured plane is isomorphic to the integers.

The manifold associated with the N-particle configuration space is more complicated than the punctured plane. The difference is that rather than classifying paths by how they wind around the origin, we classify paths by how the particles wind around other particles. If the particles themselves exist in Euclidean space $\mathbf{R}^d$, then the configuration space is of dimension dN. Consider a closed path in this configuration space from a point q to an equivalent point σq. If d is three or larger, it is easy to see that any two paths from q to σq are deformable into one another. Just as loops in $\mathbf{R}^3$ cannot be classified by a winding number (each can be shrunk to a point without ever crossing the origin), any two configuration-space paths q to σq are "homotopically equivalent"—they can be deformed into one another. The paths are then classified by σ alone. Mathematicians would say that

$$\pi_1(N\text{-particle configuration space}) \cong \mathcal{S}_N \qquad \text{if } d > 2. \tag{4}$$

The phases associated with the paths form a unitary one-dimensional representation of π_1 (configuration space), and so for $d > 2$ we are left with unitary one-dimensional representations of $\mathcal{S}_N$, of which there are only two: the symmetric or Bose representation $e^{i\theta_\sigma} = +1$, and the antisymmetric or Fermi representation $e^{i\theta_\sigma} = \operatorname{sgn} \sigma$.

In two spatial dimensions, the notion of the relative winding of particles becomes well defined. As a consequence, the space of loops in the configuration space becomes more complicated. Indeed, a path in which a particle winds completely around another particle can no longer be deformed to a point without crossing that particle. The fundamental group of the configuration space is no longer $\mathcal{S}_N$, but rather is an infinite non-Abelian group known as the N-string *braid group*.[1] As its name suggests, the algebra of this group is associated with the weaving of "braids, which are world lines for our particles.* The phases associated with the paths in configuration space now form a unitary one-dimensional representation of the braid

* Here we are assuming that the particles exist in the plane $\mathbf{R}^2$. One could also consider the N-string braid group for particles on another two-dimensional surface, such as the torus or the sphere. Additional structure in the algebra of the braid group arises if the base space is compact or multiply connected.

group: to each pairwise exchange of particles one associates a factor $e^{i\theta}$. If we let $z_j = x_j + iy_j$ be the complex coordinate for particle j, the wave function takes the form

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i<j} (z_i - z_j)^{\theta/\pi} \Phi(\mathbf{r}_1, \ldots, \mathbf{r}_N), \tag{5}$$

where $\Phi(q)$ is a totally symmetric function. Note that $\theta = \pi$ leads to a function which satisfies Fermi statistics.

Fractional statistics was first clearly discussed in 1977 by Leinaas and Myrheim.[2] (For a pedagogical review, see MacKenzie and Wilczek.[3]) The above configuration-space analysis is due to Laidlaw and DeWitt,[4] who were mostly concerned with $d = 3$, and to Wu,[1,5] who discussed the case $d = 2$.

Paths in configuration space are central to the Feynman path integral description of the propagator,

$$K(q_1, t_1 | q_2, t_2) = \frac{1}{N!} \sum_{\sigma \in \mathcal{S}_N} \int_{\substack{q(t_1)=q_1 \\ q(t_2)=q_2}} \mathcal{D}q(t) \exp\left[\frac{i}{\hbar} \int_{t_1}^{t_2} dt \left(L(q, \dot{q}, t) + \hbar \frac{\theta}{\pi} \sum_{i<j} \dot{\varphi}_{ij} \right)\right]. \tag{6}$$

Here, $\varphi_{ij} = \tan^{-1}[(y_i - y_j)/(x_i - x_j)]$ is the relative angle between particles i and j. The $\dot{\varphi}_{ij}$ term in the Lagrangian keeps track of the relative winding of particles, associating a phase factor $e^{i\theta}$ to each interchange $\Delta\varphi_{ij} = \pi$. Thus, to shift the statistics by θ one must alter the many-particle Lagrangian

$$L \to L + \hbar \frac{\theta}{\pi} \sum_{i<j} \dot{\varphi}_{ij}. \tag{7}$$

Since the additional term is a total time derivative, the angle θ does not appear in the equations of motion. However, the quantity $\dot{\varphi}_{ij}\, dt$ cannot be regarded as an exact differential because it is not the differential of a *single-valued* function of the coordinates $\{\mathbf{r}_j\}$. Thus, the "statistical" part of the action leads to additional phase interference between paths of differing winding number. This is the essence of fractional statistics.

2. CHARGE–FLUX COMPOSITES

A particularly simple realization of fractional statistics was proposed by Wilczek,[6,7] who pointed out that a composite object consisting of a particle of charge e and a flux tube of strength $\phi = \theta\hbar c/e$ would possess fractional statistics. Recall that when a quantum-mechanical particle of charge q encircles a fixed solenoid of flux ϕ, its wavefunction accumulates a phase $e^{iq\phi/\hbar c}$; this is the celebrated Aharonov–Bohm effect. The same phase would result from a quantum-mechanical solenoid orbiting around a fixed charge. Now consider two of Wilczek's charge–flux composites and compute the phase they generate upon interchange (*half* a complete revolution). There are two contributions to the accumulated phase. A factor $e^{ie\phi/2\hbar c} = e^{i\theta/2}$ is generated from the *charge* of particle 1 moving in the field of the *flux* of particle 2, and an identical factor arises from the *flux* of particle 1 moving in the field of the *charge* of particle 2. The net accrued phase is thus $e^{i\theta}$.

A generic Lagrangian $L = \frac{1}{2}m\dot{q}^2 - V(q)$, altered to account for fractional statistics as in Eq. (7), results in the many-body Hamiltonian

$$H = \sum_i \frac{1}{2m}\left(\mathbf{p}_i - \hbar\frac{\theta}{\pi}\sum_{j(\neq i)}\frac{\hat{\mathbf{z}}\times(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\right)^2 + V(\mathbf{r}_1,\ldots,\mathbf{r}_N). \tag{8}$$

The θ-dependent term resembles a "statistical vector potential"

$$\begin{aligned}\mathbf{A}_i^{\text{stat}}(\mathbf{r}_i) &= \frac{\theta}{\pi}\frac{\phi_0}{2\pi}\sum_{j(\neq i)}\frac{\hat{\mathbf{z}}\times(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\\ &= \frac{\theta}{\pi}\frac{\hbar c}{e}\sum_{j(\neq i)}\nabla_i\varphi_{ij},\end{aligned} \tag{9}$$

where $\phi_0 = hc/e$ is the Dirac flux quantum. The form of the statistical vector potential is the same as the vector potential of a flux tube of strength $\phi = 2\theta\hbar c/e$, which is twice the flux of Wilczek's composite. The reason for this is that the statistical vector potential accounts for both the charge–flux and the flux–charge interactions.[8] Note that

$$\left(\mathbf{p}_i - \frac{e}{c}\mathbf{A}_i^{\text{stat}}(\mathbf{r}_i)\right) = \exp\left(+i\frac{\theta}{\pi}\sum_j{}'\varphi_{ij}\right)\mathbf{p}_i\exp\left(-i\frac{\theta}{\pi}\sum_j{}'\varphi_{ij}\right), \tag{10}$$

indicating that the statistical vector potential is a pure gauge, although a topologically nontrivial one, because the gauge factor is not single-valued. Application of this singular gauge transformation to a symmetric wave function yields a multivalued wave function of the kind in Eq. (5).

There are thus two equivalent ways to formulate the problem of fractional statistics. We can work with single-valued wave functions and include a statistical vector potential in our many-body Hamiltonian. This leads to long-ranged two-body and (from the $\mathbf{A}^2$ term) three-body interactions. Equivalently, we can employ a singular gauge transformation to "gauge away" the statistical vector potential at the cost of requiring multivalued wave functions, as in Eq. (5).

Wilczek named particles obeying fractional statistics *anyons*, presumably because they can have *any* statistics.

3. DILUTE ANYON GASES

The behavior of the anyon gas largely remains an open problem. Even for otherwise noninteracting anyons (e.g., $V = 0$), the statistical vector potential induces long-range interactions leading to divergences in Feynman diagrams, thus necessitating diagrammatic resummation techniques if perturbation theory is to be applied. Much progress has recently been made in this area; there is now, for example, good reason to believe that, for certain values of θ, the free anyon gas is a superconductor! We will review these developments later on.

Many-body effects are important in the study of dense gases. First, let us ask how a dilute anyon gas behaves. This issue was first discussed by Arovas, Schrieffer, Wilczek, and Zee in 1985.[9,10] At high temperatures or low densities n, when the mean particle spacing $n^{-1/2}$ is much larger than the thermal wavelength $\lambda_T = \sqrt{2\pi\hbar^2/mk_BT}$, deviations from the ideal gas law can be expanded in a series in $n\lambda_T^2$:

$$p = nk_BT\{1 + B_2 n + B_3 n^2 + \cdots\}. \tag{11}$$

The *virial coefficients* $B_j(\theta, T)$ are calculable from the j-body cluster integrals. Computation of B_2 is rendered simple by elimination of the center-of-mass degree of freedom, leaving only a one-body (relative coordinate) problem to solve. The second virial coefficient for the two-dimensional ideal Fermi gas is $B_2 = +\frac{1}{4}\lambda_T^2$; Pauli exclusion introduces an additional positive contribution to the pressure. For the Bose gas, one finds $B_2 = -\frac{1}{4}\lambda_T^2$; the negative sign of B_2 reflects the tendency of bosons to condense. What is $B_2(\theta, T)$ for anyons?

Let us complicate the problem slightly by introducing an external magnetic field of strength B. This will allow us to calculate the orbital ferromagnetic moment of the dilute anyon gas.[11] The two-body Hamiltonian is decomposed into center-of-mass and relative-coordinate contributions, i.e., $H_2 = H_{\text{CM}} + H_{\text{rel}}$, with

$$\begin{aligned} H_{\text{CM}} &= \frac{1}{2M}(\mathbf{P} - \tfrac{1}{2}M\omega_c\hat{\mathbf{z}} \times \mathbf{R})^2 \\ H_{\text{rel}} &= \frac{1}{2\mu}\left(\mathbf{p} - \tfrac{1}{2}\mu\omega_c\hat{\mathbf{z}} \times \mathbf{r} - \alpha\hbar\frac{\hat{\mathbf{z}} \times \mathbf{r}}{r^2}\right)^2, \end{aligned} \tag{12}$$

where $M = 2m$ is the total mass, $\mu = \frac{1}{2}m$ is the reduced mass, $\omega_c = eB/mc$ is the cyclotron frequency, and $\alpha = \theta/\pi$ (conforming with the notation of Ref. 12). The second virial coefficient is computed from the general expression

$$B_2 = \lim_{A\to\infty} A\left(\frac{1}{2} - \frac{Z_2}{Z_1^2}\right), \tag{13}$$

which can be expressed in terms of the unknown relative-coordinate partition function Z_{rel}, due to the relation

$$Z_{\text{CM}} = 2Z_1 = 2A\lambda_T^{-2} \frac{\frac{1}{2}\beta\hbar\omega_c}{\sinh \frac{1}{2}\beta\hbar\omega_c}. \tag{14}$$

The relative-coordinate partition function, from Eq. (6), is expressed as an integral over the imaginary time propagator

$$Z_{\text{rel}} = \tfrac{1}{2}\int d^2r\,[K(\mathbf{r},\mathbf{r};-i\beta\hbar) + K(\mathbf{r},-\mathbf{r};-i\beta\hbar)]. \tag{15}$$

We now need the relative coordinate propagator—the propagator for a charged particle in the presence of both a uniform magnetic field and a flux tube. Heroic efforts described in Refs. 13–15 and reviewed in Ref. 12 yield the result

$$\begin{aligned} K(\mathbf{r}',\mathbf{r}'';-i\beta\hbar) = {} & \frac{\mu\omega_c}{4\pi\hbar}\operatorname{csch}(\tfrac{1}{2}\beta\hbar\omega_c) \\ & \times \exp\left[\frac{-\mu\omega_c}{4\hbar}\operatorname{ctnh}(\tfrac{1}{2}\beta\hbar\omega_c)(r'^2 + r''^2)\right] e^{-\frac{1}{2}\beta\hbar\omega_c\alpha} \\ & \times \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}\beta\hbar\omega_c n}\, e^{in(\varphi'-\varphi'')}\, I_{|n+\alpha|}\left[\frac{\mu\omega_c}{2\hbar} r'r''\operatorname{csch}(\tfrac{1}{2}\beta\hbar\omega_c)\right], \end{aligned} \tag{16}$$

where $I_\tau(z)$ is the modified Bessel function of the first kind. Now Z_{rel} diverges as A, so that in evaluating B_2 one is presented with the delicate task of extracting a finite difference of two divergent expressions, namely $\frac{1}{2}A$ and $2AZ_{\text{rel}}/Z_1$. Let us employ a regularization procedure, substituting

$$d^2r \to \exp\left(-\varepsilon\frac{\mu\omega_c r^2}{2\hbar}\operatorname{csch}\tfrac{1}{2}\beta\hbar\omega_c\right) d^2r \tag{17}$$

for the integration measure, with $\varepsilon \to 0$ at the end of the day. For $\alpha = 2n + \nu$, with $|\nu| \le 1$, one obtains[12,16]

$$B_2 = -\frac{\lambda_T^2}{4\Delta}\tanh\Delta + \frac{\lambda_T^2}{2\Delta}\left[(1 - e^{-2\nu\Delta})\frac{e^{2\Delta\operatorname{sgn}\nu}}{\sinh 2\Delta} - \nu\right], \tag{18}$$

with $\Delta = \frac{1}{2}\beta\hbar\omega_c$. Note that the time reversal operation reverses both B and θ,

$$H^*(-B,-\theta) = h(B,\theta). \tag{19}$$

The thermodynamic potentials, being real functions, are therefore invariant under simultaneous reversal of both B and θ.

Even in zero field, the statistical vector potential breaks time reversal invariance (unless $\theta = k\pi$), and one should expect a nonzero orbital ferromagnetic moment, $M = -(\partial\Omega/\partial B)_{B=0}$, where Ω is the grand potential. In the high temperature-low density limit, the leading order contribution to the magnetic moment per particle is directly obtainable from the function B_2,

$$M/N = -\tfrac{1}{3}\mu_B n\lambda_T^2 \nu(|\nu|-1)(|\nu|-2), \tag{20}$$

where $\mu_B = e\hbar/2mc$ is the Bohr magneton. This result was first obtained by Johnson and Canright.[11]

M is neatly interpreted as a diamagnetic response to an effective field. The textbook weak-field moment per particle is $M/N = -\frac{1}{3}\mu_B^2 B/k_B T$, and comparing with Eq. (20), this suggests an effective field of strength

$$B_{\text{eff}} = n\phi_0 \cdot 2\nu(|\nu|-1)(|\nu|-2). \tag{21}$$

As proposed by Arovas *et al.*,[9] it is quite natural to think of each anyon as a particle moving in a net "statistical magnetic field" of magnitude $\bar{B} = (\theta/\pi)n\phi_0$. This is because an anyon moving along a path enclosing an area A will *on average* encircle nA other anyons and thus accrue a phase angle $\bar{B}A$. Indeed, from Eq. (9) we find that

$$\nabla \times \mathbf{A}_i^{\text{stat}}(\mathbf{r}) = \frac{\theta}{\pi}\phi_0 \sum_{j(\neq i)} \delta(\mathbf{r}-\mathbf{r}_j), \tag{22}$$

and hence $\langle \nabla \times \mathbf{A}_i^{\text{stat}}(\mathbf{r})\rangle = \bar{B}$ in the thermodynamic limit. (Since whole flux tubes are "invisible", the physics of the anyon gas is invariant under $\theta \to \theta + 2\pi$. Integer multiples of ϕ_0 can be gauged away, and the true mean field $\bar{B}$ is obtained by translating θ to the region $[-\pi, \pi]$.) This mean field theory (MFT) is the springboard to more sophisticated treatments of the anyon gas, which we shall soon discuss. For the moment, notice that for small θ, Eq. (21) gives $B_{\text{eff}} = 4\bar{B}$, which is correct in both its sign and its density dependence, but is larger than the true mean statistical field by a factor of 4. Johnson and Canright[11] have also discussed the extent to which low-density anyon physics can be interpreted in terms of conventional particles in an effective field.

4. FRACTIONAL STATISTICS IN THE QUANTIZED HALL EFFECT

While the fundamental particles of nature are all, to our present knowledge, either bosons or fermions, it is quite possible that localized quasiparticle excitations in two-dimensional condensed-matter systems might obey pecular statistics. How can we tell if this is the case? In what follows, we shall assume that quasiparticle dynamics takes place on time scales long compared with the characteristic time scales associated with the constituent particles. That is, we will treat the quasiparticle degrees of freedom in the adiabatic approximation.

Let the quasiparticles be described by their positions $\boldsymbol{\xi}_i$. We would like to know if there are long-range gauge interactions of the type in Eq. (6) in the effective quasiparticle Lagrangian. If an external field is present, we can also identify the quasiparticle charge e^* by examining how the quasiparticle current couples to the external vector potential. Both these issues can be assessed by deriving an effective quasiparticle propagator,

$$K_{\text{eff}}(\xi_1, t_1 | \xi_2, t_2) = \int \prod_{i=1}^{N_{\text{qp}}} \mathscr{D}\boldsymbol{\xi}(t) \exp \frac{i}{\hbar} \int_{t_1}^{t_2} dt \times \left[\sum_i \tfrac{1}{2} m^* \dot{\xi}_i^2 - V(\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{N_{\text{qp}}}) + \frac{e^*}{c} \sum_i \mathbf{A}_{\text{ext}}(\boldsymbol{\xi}_i) \cdot \dot{\boldsymbol{\xi}}_i + \hbar \frac{\theta}{\pi} \sum_{i<j} \dot{\varphi}_{ij} \right]. \tag{23}$$

We evaluate the effective Aharonov–Bohm phase by adiabatically dragging a quasiparticle around a loop of area A. In the presence of an external magnetic field, the action accumulates a phase

$$\gamma = \frac{e^*}{\hbar c} \oint \mathbf{A}_{\text{ext}} \cdot d\mathbf{l} = 2\pi \frac{e^*}{e} \frac{\phi}{\phi_0}, \tag{24}$$

where $\phi = BA$ is the magnetic flux enclosed by the loop. Berry[17] showed that the adiabatic phase accumulated over such a closed path includes a geometric quantity which is *independent* of how slowly the path is traversed. Let $\{H_\lambda\}$ be a family of Hamiltonians depending on a set of parameters $\lambda = \{\lambda_1, \lambda_2, \ldots\}$. If one varies the parameter λ sufficiently slowly, then according to the adiabatic theorem the solution to the full time-dependent Schrödinger equation,

$$i\hbar \frac{d}{dt} |\Phi(t)\rangle = H_{\lambda(t)} |\Phi(t)\rangle, \tag{25}$$

will differ by only a phase from the solution to the time-*in*dependent equation

$$H_\lambda |\psi(\lambda)\rangle = E(\lambda) |\psi(\lambda)\rangle, \tag{26}$$

i.e.,

$$|\Phi(t)\rangle \simeq \exp(i\gamma(t)) \exp\left(-\frac{i}{\hbar} \int^t dt' \, E(\lambda(t')) \right) |\psi(\lambda(t))\rangle. \tag{27}$$

The phase factor involving the integrated adiabatic energy depends linearly on the time for traversing the path. In the effective quasiparticle Lagrangian, this term corresponds to a potential energy.

The Berry phase $\gamma(t)$ is easily seen to satisfy the equation

$$\frac{d\gamma(t)}{dt} = i\langle\psi(\lambda(t))|\frac{d}{dt}|\psi(\lambda(t))\rangle. \tag{28}$$

If $\lambda(t)$ is a closed path, then the phase difference,

$$\gamma_c = i\oint d\lambda \cdot \langle\psi(\lambda)|\nabla_\lambda|\psi(\lambda)\rangle, \tag{29}$$

has a beautiful geometric interpretation, as noted by Simon[18] and by Wilczek and Zee.[19]

Let us now concentrate on a concrete example, that of the fractional quantized Hall effect (FQHE). Recall that Laughlin's[20] picture of the fractional quantized Hall effect is based on a discrete set of Jastrow-type trial wave functions of the form

$$\Psi_m(z_1, \ldots, z_N) = \mathcal{N}\prod_{j<k}^{N}(z_j - z_k)^m \prod_{s=1}^{N} e^{-\bar{z}_s z_s/4l^2}, \tag{30}$$

where M is an odd integer, $z_j = x_j + iy_j$ is the complex coordinate of particle j, $\mathcal{N}$ is a normalization constant, and $l = \sqrt{\hbar c/eB}$ is the magnetic length. Ψ_m describes an incompressible fluid state at filling fraction $\nu = 1/m$ (provided $m \lesssim 70$). The associated quasielectron and quasihole wave functions, respectively, are given by

$$\begin{aligned}\tilde{\Psi}_m[\boldsymbol{\eta}] &= \mathcal{N}(\boldsymbol{\eta})\prod_{s=1}^{N} e^{-\bar{z}_s z_s/4l^2}\prod_{i=1}^{N}\left(2l^2\frac{\partial}{\partial z_i} - \bar{\eta}\right)\prod_{j<k}^{N}(z_j - z_k)^m \\ \tilde{\Psi}_m[\boldsymbol{\xi}] &= \mathcal{N}(\boldsymbol{\xi})\prod_{s=1}^{N} e^{-\bar{z}_s z_s/4l^2}\prod_{i=1}^{N}(z_i - \xi)\prod_{j<k}^{N}(z_j - z_k)^m.\end{aligned} \tag{31}$$

The charge of these excitations was also discussed by Laughlin, who employed an argument analogous to that used in deducing the fractional charge of solitons in one-dimensional conductors. He concluded that for $\nu = 1/m$, the quasielectron and quasihole have charges $\mp e^* = \mp e/m$. These excitations are localized within a microscopic region whose size is dictated by the magnetic length and the filling fraction. The excitations described by Eq. (31) are centered at $\mathbf{r} = \boldsymbol{\eta}$ (quasielectron) and $\mathbf{r} = \boldsymbol{\xi}$ (quasihole), respectively. Roughly speaking, a quasihole in the incompressible fluid resembles a "bubble" of such a size that $1/m$ of an electron is absent. Here, I reproduce the calculations of Arovas, Schrieffer, and Wilczek,[21] which yield the quasiparticle charge and statistics from Berry's phase arguments. (In the following, I set the magnetic length l to unity.)

According to Eq. (31),

$$\frac{d}{dt}\tilde{\Psi}_m[\boldsymbol{\xi}] = \left[\frac{d}{dt}\ln \mathcal{N}[\boldsymbol{\xi}(t)] + \sum_{i=1}^{N}\frac{d}{dt}\ln[z_i - \xi(t)]\right]\tilde{\Psi}_m[\boldsymbol{\xi}], \tag{32}$$

so that

$$\frac{d}{dt}\gamma(t) = i\frac{d}{dt}\ln \mathcal{N}(\boldsymbol{\xi}) + i\langle\tilde{\Psi}_m[\boldsymbol{\xi}]|\frac{d}{dt}\sum_{i=1}^{N}\ln(z_i - \xi)|\tilde{\Psi}_m[\boldsymbol{\xi}]\rangle. \tag{33}$$

Using the single-particle density in the presence of the quasihole,

$$n\boldsymbol{\xi}(\mathbf{r}) = \langle\tilde{\Psi}_m[\boldsymbol{\xi}]|\sum_{i=1}^{N}\delta(\mathbf{r}-\mathbf{r}_i)|\tilde{\Psi}_m[\boldsymbol{\xi}]\rangle, \tag{34}$$

we obtain

$$\frac{d\gamma}{dt} = i\frac{d}{dt}\ln \mathcal{N}[\boldsymbol{\xi}(t)] + i\int d^2r\, n\boldsymbol{\xi}(\mathbf{r})\frac{d}{dt}\ln[z - \xi(t)]. \tag{35}$$

Since the normalization constant $\mathcal{N}(\boldsymbol{\xi})$ is a single-valued function of its argument, it will not contribute to the integral expression Eq. (29) for γ_c. We now write $n_{\boldsymbol{\xi}}(\mathbf{r}) = \bar{n} + \delta n_{\boldsymbol{\xi}}(\mathbf{r})$, where $\bar{n} = \nu/2\pi$ is the density in the Laughlin ground state Ψ_m and $\delta n_{\boldsymbol{\xi}}(\mathbf{r})$ is concentrated about the point $\mathbf{r} = \boldsymbol{\xi}$.* Concerning the $\bar{n}$ term, if ξ is integrated in a clockwise sense around a circle of radius R, one finds that

$$\oint_{|\xi|=R} dt\frac{d}{dt}\ln[z - \xi(t)] = \oint_{|\xi|=R} d\xi\frac{1}{\xi - z}$$
$$= 2\pi i\theta(R - |z|), \tag{36}$$

where $\theta(x)$ is a step function. Substituting this result into Eq. (35), we find that

$$\gamma_c = i\int^{R} d^2r\, 2\pi i\bar{n} = -2\pi\langle N\rangle_R = -2\pi\nu\phi/\phi_0, \tag{37}$$

where $\langle N\rangle_R$ denotes the mean number of electrons inside a circle of radius R. As Haldane[22] has argued, corrections to this result arising from the $\delta n_{\boldsymbol{\xi}}(\mathbf{r})$ term vanish due to the rotational symmetry of the quasihole.

A similar analysis shows that the charge of the quasielectron is $e^* = -\nu e$, although one must exercise some caution in dealing with the partial

* The uniform density $\bar{n} = \nu/2\pi$ of the trial ground state and the localized nature of the excitations are easily deduced from the plasma analogy of Laughlin.

derivative operators in Eq. (31). The adiabatic phase accumulated by a quasielectron is

$$\frac{d}{dt}\gamma(t) = i\frac{d}{dt}\ln \mathcal{N}(\boldsymbol{\eta}) + i\frac{d}{dt}\sum_{i=1}^{N}\langle\tilde{\Psi}_m[\boldsymbol{\eta}]| \times \exp\left(-\tfrac{1}{4}\sum_{r=1}^{N}|z_r|^2\right)\ln\left(2\frac{\partial}{\partial z_i}-\bar{\eta}\right)\exp\left(\tfrac{1}{4}\sum_{s=1}^{N}|z_s|^2\right)|\Psi_m[\boldsymbol{\eta}]\rangle. \tag{38}$$

The above matrix element is a many-particle generalization of the Bargmann–Fock space inner product,

$$\left\langle\langle g|\hat{\mathcal{O}}\left(2\frac{\partial}{\partial z}, z\right)|f\rangle\right\rangle \equiv \int\frac{dx\,dy}{2\pi}\,e^{-\frac{1}{2}\bar{z}z}\,\overline{g(z)}\hat{\mathcal{O}}\left(2\frac{\partial}{\partial z}, z\right)f(z). \tag{39}$$

It is easy to show[23] that if the operator $\hat{\mathcal{O}}$ is *normal ordered* such that all partial derivatives $\partial/\partial z$ appear to the *left* of all complex coordinates z, then the formal replacement $2\partial/\partial z \to \bar{z}$ is allowed, i.e.,

$$\left\langle\langle g| : \hat{\mathcal{O}}\left(2\frac{\partial}{\partial z}, z\right) : |f\rangle\right\rangle = \langle\langle g|\hat{\mathcal{O}}(\bar{z}, z)|f\rangle\rangle. \tag{40}$$

Making this substitution in Eq. (38), one recovers the result of Eq. (35), with $z \to \bar{z}$ and $\xi \to \bar{\eta}$. The charge of the quasielectron follows immediately. To determine the statistics of the excitations, we consider the state with quasiholes at $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$,

$$\tilde{\Psi}_m[\boldsymbol{\xi}, \boldsymbol{\xi}'] = \mathcal{N}(\boldsymbol{\xi}, \boldsymbol{\xi}')\prod_{i=1}^{N}[(z_i - \xi)(z_i - \xi')]\Psi_m. \tag{41}$$

As above, we adiabatically carry ξ around a closed loop of radius R. If ξ' lies outside the circle $|\xi| = R$ by a distance $d \gg a_0$, the above analysis is unchanged, i.e. $\gamma_c = -2\pi\nu\phi/\phi_0$. If, however, ξ' lies inside this loop, and $R - |\xi'| \ll a_0$, there is a deficit in $\langle N\rangle_R$ of $-\nu$, and the phase accrued is $\gamma_c' = \gamma_c + 2\pi\nu$. Therefore, when a quasihole adiabatically encircles another quasihole, an extra "statistical phase"

$$\Delta\gamma_c = 2\pi\nu \tag{42}$$

is accumulated. Again, an analogous result holds for a quasielectron. For the case of the filled Landau level, $\nu = 1$, $\Delta\gamma_c = 2\pi$, and the phase obtained upon interchanging quasiholes is $\Delta\gamma_c/2 = \pi$, corresponding to Fermi statistics. We identify the quantity $\Delta\gamma_c/2$ with the statistical angle θ; for non-integer ν, the excitations obey fractional statistics, in agreement with the conclusion of Halperin.[24] Clearly, when $e^{2i\theta} \neq 1$, the change of phase $\Delta\gamma_c$ accumulated when a *third* particle is in the vicinity will depend on the adiabatic path taken by the excitations as they are interchanged, a consequence of their fractional statistics. It should also be noted that the statistics of the quasiparticles is "long-distance physics" and becomes ill-defined in a dense quasiparticle gas.

4.1. Selection Rules for Anyon Production

The above calculations show that, for quasiparticle states of the form of Eq. (31), excitations with charge $e^* = \nu e$ have statistical angle $\theta = \pi\nu = \pi e^*/e$. When the condensate structure is more complicated, this relationship between the quasiparticle's charge and statistics is no longer valid.[22,25]

Let us consider a process in which p parent particles of charge e decay into q charge $e^* = (p/q)e$ quasiparticles with statistical angle $\theta\pi p/q$. Exchanging two groups of p parent particles produces a phase of $(\pm 1)^{p^2}$. On the other hand, viewed as an exchange of daughter quasiparticles, the statistical phase is $e^{i\theta q^2} = e^{i\pi pq}$. These phases must be equal, so we arrive at the selection rule

$$(\pm 1)^{p^2} = e^{i\pi pq}. \tag{43}$$

When p is even, this relation is always valid, independent of q. For p odd and q odd, one concludes that the parents must be fermions, whereas for p odd and q even, the parents are bosons. Tao[26] has argued for the FQHE odd-denominator rule based on these ideas. The recently discovered even-denominator FQHE states involve more complicated condensates in which both p and q may be even and in which spin plays a role.

4.2. Order Parameter and Landau–Ginzburg Theory of the FQHE

It was first suggested by Girvin and by Girvin and MacDonald[27] that Laughlin's wave function of Eq. (30) could be understood as a *condensate* of composite objects consisting of both charge and flux. Specifically, Girvin and MacDonald showed that if one were to adiabatically pierce each electron in the $\nu = 1/m$ state with a flux tube of strength $m\phi_0$, the resulting off-diagonal density matrix,

$$\rho(\mathbf{r}, \mathbf{r}') = \int \prod_{i=2}^{N} d^2 r_i \Psi^*(\mathbf{r}, \mathbf{r}_2, \ldots, \mathbf{r}_N) \exp\left(\frac{ie}{\hbar c} \int_{\mathbf{r}}^{\mathbf{r}'} \mathbf{A}_{\text{stat}} \cdot d\mathbf{l}\right) \Psi(\mathbf{r}', \mathbf{r}_2, \ldots, \mathbf{r}_N), \tag{44}$$

decays only *algebraically*, as $|\mathbf{r} - \mathbf{r}'|^{-m/2}$. The vector potential $\mathbf{A}_{\text{stat}}$ in Eq. (44) is of the form of Eq. (9), with $\theta = m\pi$. A Landau–Ginzburg theory including such a statistical gauge field was also suggested. The gauge field is *non-dynamical*, and instead satisfies the constraint of Eq. (22).

More recently, Read has constructed such a Landau-Ginzburg theory (see also the work of Zhang, Hansson, and Kivelson[28]) based on the composite order parameter

$$\phi(\mathbf{r}) = \psi^\dagger(\mathbf{r}) U^m(\mathbf{r}). \tag{45}$$

Here, $\psi^\dagger$ is an electron creation operator, and U is Laughlin's quasihole operator, which takes the form

$$U(\mathbf{r}) = \prod_{i=1}^{N} (z - z_i) \tag{46}$$

when acting on N-particle states. (I prefer to think of U as an "adiabatic ϕ_0 flux addition operator.") What makes Read's order parameter so lovely is that it is truly a boson and can condense. Let us calculate the statistical angle Θ which results when we interchange two of these composites. There are four contributions:

$$\begin{aligned} \Theta &= \pi + m\pi + m\pi + m^2\theta \\ &= (m+1)\pi \bmod 2\pi. \end{aligned} \tag{47}$$

The first of these contributions arises from exchange of the fermions created by the $\psi^\dagger$ operators. Next, a phase is accrued due to the fermions moving in the field of the added flux tubes. A single electron encircling m flux tubes accumulates a phase angle of $2\pi m$, so an exchange gives us half of this, $m\pi$. However, there are two composites, so, as discussed in section 2, there is an additional factor of 2 arising from the charge of composite 1 moving in the field of the flux of composite 2 and *vice versa.* Finally, there is the statistical angle due to the quasiparticles themselves (which are created by U). Each composite consists of m^2 quasiparticles, and therefore this contribution is $m^2\theta = m\pi$, since $\theta = \pi/m$. The net statistical phase is thus $\Theta = (m+1)\pi$, which is bosonic when m is odd.

The long-wavelength Landau-Ginzburg action is[27-29]

$$S = \int d^3x \left[\phi^* i D_0 \phi - \tfrac{1}{2}\kappa |D_j\phi|^2 - V(|\phi|) - \frac{e^2}{4\pi m}\, \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \right], \tag{48}$$

where κ is a "stiffness constant," an explicit expression for which is given by Read.[29] The object D_μ is the covariant derivative,

$$D_\mu = \partial_\mu - iea_\mu - ieA_\mu^{\text{ext}}, \tag{49}$$

where a_μ is a statistical gauge field, A_μ^{ext} is an external gauge field satisfying $\nabla \times \mathbf{A}^{\text{ext}} = B\hat{\mathbf{z}}$, and $\partial_\mu = (\partial_t, -\nabla)$ with metric $(+,-,-,-)$. The "potential" energy density is

$$V(|\phi|) = \lambda(|\phi|^2 - \rho_0)^2, \tag{50}$$

where ρ_0 is the bulk density, although Zhang, Hansson, and Kivelson[28] and Haldane[22] have remarked that a more realistic action would include a term

$$S_V[\phi^*, \phi] = \tfrac{1}{2}\int dt\, d^2r\, d^2r'(|\phi(\mathbf{r}, t)|^2 - \rho_0)v(\mathbf{r} - \mathbf{r}')(|\phi(\mathbf{r}', t)|^2 - \rho_0), \quad (51)$$

where $v(\mathbf{r} - \mathbf{r}')$ is the Coulomb potential. The $\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda$ part of the Lagrangian is known as the Chern–Simons (CS) term and is responsible for effecting the constraint of Eq. (22). Let us see how this works.

We evaluate the action S in the stationary phase approximation. Varying with respect to the field a_0, one finds

$$\frac{\delta S}{\delta a_0} = 0 = -\frac{e^2}{2\pi m}\,\varepsilon^{0ij}\partial_i a_j - e|\phi|^2, \quad (52)$$

which (in $\hbar = c = 1$ units) is just Eq. (22) for the statistical field: $b = \varepsilon^{ij}\partial_i a_j = 2\pi m|\phi|^2/e$. Varying with respect to a_i,

$$\frac{\delta S}{\delta a_i} = 0 = \tfrac{1}{2}e\kappa(\phi^* D_i\phi + \phi D^i\phi^*) - \frac{e^2}{2\pi m}\,\varepsilon^{i\mu\nu}\partial_\mu a_\nu. \quad (53)$$

Finally,

$$\frac{\delta S}{\delta\phi^*} = 0 = D_0\phi - \tfrac{1}{2}\kappa D_i D^i\phi - \frac{\delta}{\delta\phi^*}S_V[\phi^*, \phi]. \quad (54)$$

The optimal static solution to the equations of motion yields

$$\begin{aligned} |\phi|^2 &= \rho_0 \\ a_0 = A_0 &= 0 \\ a_i + A_i &= 0. \end{aligned} \quad (55)$$

Thus, the statistical field b exactly cancels the applied field B ($b + B = 0$). A solution of the equations therefore exists only if the filling factor is $\nu = 1/m$.

In the microscopic theory of the FQHE,[20] deviations from $\nu = 1/m$ are described by the localized quasiparticle defects of Eq. (31). In the Ginzburg–Landau theory, these quasiparticles correspond to vortex solutions of the equations of motion, analogous to vortices in other condensates such as superconductors and superfluids. Following Zhang, Hansson, and Kivelson,[28] consider a static ($a_0 = 0$) vortex solution in a uniform external field. As $r \to \infty$, the fields ϕ and a behave as

$$\begin{aligned} \phi(r, \phi) &= \sqrt{\rho_0}\, e^{\pm i\phi} \\ a(r, \phi) &= \pm\hat{\varphi}/er, \end{aligned} \quad (56)$$

and from Eq. (52) the charge on the vortex is

$$q = e \int d^2x |\phi|^2 = \frac{e^2}{2\pi m} \oint_{r=\infty} \mathbf{a} \cdot d\mathbf{l} = \pm e/m, \tag{57}$$

which was predicted by Laughlin[20] using his "plasma analogy" and deduced in section 4 from a Berry phase calculation.* Because it is the covariant derivative D_μ which appears in the action, the vortex of Eq. (56) does not have an infrared-divergent energy, as do vortices in, e.g., the $O(2)$ model.

As we shall see later on, the coefficient $e^2/4\pi m \equiv e^2/4\vartheta$ alters the statistics of the field ϕ. The vortices in the ϕ condensate have statistical angle θ, where

$$\theta/\pi = \pi/\vartheta, \tag{58}$$

and can likewise be described by a similar Chern–Simons theory with CS coefficient $e^2/4\theta$. This "duality" has been discussed by Wen and Zee,[31] and is central to some beautiful recent work by Lee and Fisher[32] and Lee and Kane.[33]

The collective mode spectrum as calculated within the Ginzburg-Landau theory predicts a $k = 0$ gap,[28,29] in conformity with the microscopic magnetophonon-magnetoroton theory of Girvin, MacDonald, and Platzman.[27,34] This is the Anderson–Higgs mechanism—the amplitude fluctuations of ϕ are massive, but the phase fluctuations are "eaten" by the statistical vector field a_μ.

Finally, although Read's order parameter leads to perfect off-diagonal long-range order (ODLRO) in the Laughlin state $|L\rangle$,

$$\langle L|\phi^\dagger(\mathbf{r})\phi(\mathbf{r}')|L\rangle \sim \text{constant}, \tag{59}$$

one expects, based on the Landau-Ginzburg theory, that the generic form for the ϕ–ϕ correlation function is

$$\langle L|\phi^\dagger(\mathbf{r})\phi(\mathbf{r}')|L\rangle \sim |\mathbf{r} - \mathbf{r}'|^{-m/2}, \tag{60}$$

with a universal power law exponent $\vartheta/2\pi$. In this sense, the Laughlin state is somewhat nongeneric.[35,36]

5. MANY-BODY THEORY OF THE ANYON GAS

Much of the recent (>1986) efforts in anyon theory were inspired by the work of Laughlin and collaborators on anyon superconductivity.[37,38]

* We should really make a distinction between the local charge deduced from, e.g., Laughlin's plasma analogy and the "topological" charge deduced from Berry phase arguments. Usually, these are one and the same.[30]

The possible relevance of such exotic physics to high-temperature superconductivity in copper oxide layers is an interesting chapter in the theory of anyons,[39] but one which is quite speculative and somewhat beyond the scope of these lectures. Let us confine our discussion to ideal anyon gases, which themselves are highly nontrivial.

The analysis of section 3 suggests that "quasibosons" (anyons whose statistical angle $\theta = 2n\pi + \delta$ deviates only slightly from the Bose value) can be viewed as ordinary bosons interacting through a weak repulsive force. Conversely, "quasifermions" with $\theta = (2n + 1)\pi + \delta$ should behave as weakly attractive conventional fermions. In hindsight, it is natural to suggest that such a system might be a superfluid and, if charged, a superconductor.

Recall how the mean field treatment of the anyon gas treats anyons with $\theta = k\pi + \delta$ as conventional particles with $\theta = k\pi$ in an average "statistical" field of strength $b = (|\delta|/\pi)\rho\phi_0$, where ρ is the bulk number density. For various reasons,[40] it is convenient to consider the quasifermion gas with $\theta = \pi(1 - 1/n)$, i.e., $\delta = -\pi/n$. The statistical field is then $\mathbf{b} = -b\hat{\mathbf{z}}$, with

$$b = \rho\phi_0/n. \tag{61}$$

Thus, at the mean field level, we have noninteracting fermions in a uniform magnetic field. The single-particle energy levels are simply Landau levels, where cyclotron energy $\hbar\omega_c$ and magnetic length l are ρ-dependent, viz.,

$$\begin{aligned} \hbar\omega_c &= \frac{\hbar eb}{mc} = \frac{1}{n}\frac{2\pi\hbar^2}{m}\rho \\ l &= \sqrt{\frac{\hbar c}{eb}} = \sqrt{\frac{n}{2\pi\rho}}. \end{aligned} \tag{62}$$

The filling fraction is

$$\nu = 2\pi l^2\rho = n, \tag{63}$$

and for integer n the mean field ground state is a Slater determinant of lowest n Landau levels, with energy

$$\begin{aligned} E &= N_L \sum_{k=0}^{n-1} (k + \tfrac{1}{2})\hbar\omega_c \\ &= A\frac{\pi\hbar^2}{m}\rho^2, \end{aligned} \tag{64}$$

where A is the area of the system and $N_L = bA/\phi_0$ is the Landau level degeneracy.

At this point, several remarks are in order. The mean field theory smears statistical flux into an effective magnetic field. Does this idea, "to boldly put flux where no flux has gone before,"* make any sense? After all, we know that the statistical angle θ does not enter into the classical equations of motion, which in the absence of interactions are purely ballistic. However, at the mean field level, particles are moving in cyclotron orbits, not straight lines. What is happening is this: when we calculate the stationary phase equations $\delta S/\delta q(t) = 0$, which yield the classical equations of motion, we compute *continuous* variations with respect to a path $q(t)$. Now the winding number $(1/\pi)\sum_{i<j}\Delta\varphi_{ij}$ is a "piecewise constant functional" of $q(t)$ (assuming $\delta q(t_2) = \delta q(t_1) = 0$), changing by *discrete* multiples of π when $q(t)$ changes winding number sectors. Thus, while the action-extremizing equations *within each winding number sector* yield ballistic classical trajectories, we must sum over *all* winding number sectors to obtain the semiclassical propagator. This issue also crops up in the study of a single charged particle in a lattice of flux tubes.[41] When the flux per tube is small ($n \to \infty$ for the anyon gas), the winding number itself can effectively be considered a continuous variable, since the phase change associated with a unit increase of winding number is correspondingly small. This suggests a mean field approximation, so one expects to find dispersionless (quasi-) Landau levels in one's band structure, which is exactly what one sees. Put another way,[40] the average number of particles in an area of size $2\pi l^2$ is n. When n is large, fluctuations are relatively small, and one puts one's faith in mean field theory.

If you believe in the mean field theory at this point, then I should try to sell you the Brooklyn Bridge as well, because there is one glaringly obvious problem. The only excited states one can construct all involve excitations to higher Landau levels, and thus MFT predicts a *gap* in the excitation spectrum. This is rather worrisome because on physical grounds one expects some finite compressibility. What has happened? The finite energy inter-Landau level excitations are particle–hole pairs. It is good that such excitations possess a gap, for this is just what happens in a superconductor. What is missing in the spectrum is a collective density mode responsible for the compressibility. Such a mode would involve a slow variation in the density ρ and hence also in the local field b. Indeed, from Eq. (64) we obtain a ground-state energy per particle of $\varepsilon = \pi\hbar^2/mv$ ($v = 1/\rho$ is the specific volume), *identical* to the ground-state energy of a free Fermi gas of the same density. This leads to a finite bulk modulus $\mathscr{B}$ and velocity c of first (thermodynamic) sound:

$$\begin{aligned} \mathscr{B} &= v\frac{\partial^2\varepsilon}{\partial v^2} = \frac{2\pi\hbar^2}{m}\rho^2 \\ c &= \sqrt{\mathscr{B}/m\rho} = \frac{\hbar}{m}\sqrt{2\pi\rho}. \end{aligned} \tag{65}$$

* I paraphrase *Star Trek* without correcting the split infinitive.

These quantities are independent of n and identical to the corresponding free Fermi gas values. The compressional sound wave appears in a more sophisticated random phase approximation (RPA) treatment of the anyon gas, as first shown by Fetter, Hanna, and Laughlin.[38]

Now suppose that an external field B is also present. A simple calculation, due to Chen, Wilczek, Witten, and Halperin,[40] suggests that the $\theta = \pi(1 - 1/n)$ anyon gas, despite breaking time-reversal symmetry, nonetheless desires to expel flux. Consider the anyon gas in the presence of an external magnetic field of strength B. Appealing again to mean field theory, let us calculate the ground-state energy when $\mathbf{B}$ and $\mathbf{b}$ are aligned so that the effective field strength is $b + B$. As we increase B from zero, the degeneracy N_L of the Landau levels increases. Since the particle number remains constant, a fraction x of the nth Landau level will be empty; number conservation gives $(n - x)(b + B) = nb$, which determines x. The energy of the ground state is easily found to be[40]

$$E = A\frac{\pi\hbar^2}{m}\rho\left[1 + \frac{B}{nb} - \left(1 - \frac{1}{n}\right)\frac{B^2}{b^2}\right]. \tag{66}$$

When $\mathbf{B}$ and $\mathbf{b}$ are antialigned, the effective field strength is $b - B$, the Landau level degeneracy is smaller, a fraction y of the $(n + 1)$th Landau level is occupied with $(n + y)(b - B) = nb$, and the energy is

$$E = A\frac{\pi\hbar^2}{m}\rho\left[1 + \frac{B}{nb} - \left(1 + \frac{1}{n}\right)\frac{B^2}{b^2}\right]. \tag{67}$$

Thus, application of a field in *either* direction results in an increase of energy. Is this surprising? After all, imposition of a field increases the energy of an ideal Fermi or Bose gas just as well. What is special about the $\theta \neq k\pi$ anyon gas, though, is that it *explicitly* breaks time-reversal symmetry. *A priori* one might then expect that a field oriented in one direction (e.g., $\hat{\mathbf{z}}$) would increase the energy of the anyon gas, while a field oriented in the opposite direction would decrease its energy. For $\theta = \pi(1 - 1/n)$, the mean field ground-state energy increases in an external field *independent* of the field's orientation. This is indeed surprising.

To demonstrate a Meissner effect, one must perform substantially more refined calculations. The linear response of a many-body system to an external electromagnetic field is embodied in the current–current correlation functions:[42]

$$\begin{aligned} 4\pi J_\mu(k) &= -K_{\mu\nu}(k)A^\nu(k) \\ K_{\mu\nu}(k) &= 4\pi R_{\mu\nu}(k) + \frac{4\pi\rho e^2}{m}\,\delta_{\mu\nu}(1 - \delta_{\nu 0}), \end{aligned} \tag{68}$$

where $k = (k_0, \mathbf{k})$, and the (retarded) real-time current–current correlation function is given by

$$R_{\mu\nu}(\mathbf{k}, t) = -i\langle[j_\mu(\mathbf{k}, t), j_\nu(-\mathbf{k}, 0)]\rangle\Theta(t). \tag{69}$$

The Meissner effect follows from

$$K_{ij}(k) = K_1(k)\left(\delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\right) + K_2(k) i k_0 \varepsilon^{0ij}, \tag{70}$$

whenever $K_1(q \to 0) > 0$. Note that K_2 must vanish in a time-reversal invariant system, but need not vanish for the anyon gas. The RPA calculations first performed by Fetter, Hanna, and Laughlin[38] and subsequently by Chen, Wilczek, Witten, and Halperin[40] yield $K_1(0) = 4\pi\rho e^2/m$, predicting a Meissner effect in a three-dimensional sample, with a London penetration depth of $\lambda_L = \sqrt{md/4\pi\rho e^2}$, where d is the interplane distance. The RPA calculations also yield a pole in the density-density response function at $k_0 = c|\mathbf{k}|$ with $c = (\hbar/m)\sqrt{2\pi\rho}$. This is the sound wave predicted in Eq. (65) above! A more detailed account of the electrodynamics of anyon superconductors is given by Fradkin.[43]

In sum, the excitation spectrum of the $\theta = \pi(1 - 1/n)$ anyon gas is qualitatively different from the spectrum of, e.g., a Fermi liquid. While a Fermi liquid exhibits various gapless sound modes, it also possesses a particle-hole excitation continuum which extends to zero energy. These gapless particle-hole excitations are responsible for various dissipative processes. In the anyon gas, the particle-hole continuum starts at a finite energy $\hbar\omega_c = 2\pi\hbar^2\rho/nm$ (which properly tends to zero in the fermion limit $n \to \infty$), and the only gapless excitation is the density wave. In field-theoretic language, this density wave is a Goldstone mode, which in an ordinary uncharged superconductor would correspond to phase fluctuation of the order parameter. When minimally coupled to electromagnetism, this mode is "eaten" via the Anderson-Higgs mechanism, and the photon becomes massive (the Meissner effect).

6. CHERN–SIMONS FIELD THEORY AND FRACTIONAL STATISTICS

We have seen how fractional statistics arises in quantum mechanics when particles move in a fictitious statistical gauge field whose curl is itself proportional to the particle density [cf. Eq. (22)]. Here, we shall discuss fractional statistics in the setting of quantum field theory.

Given a field theory with a conserved current j_μ, one can impart fractional statistics to the matter field by coupling this current to a $U(1)$ gauge field and adding a Chern-Simons term[44] to the action,

$$S_{\text{matter}} \to S_{\text{matter}} + e\int d^3x\, j^\mu a_\mu - \frac{e^2}{4\theta}\int d^3x\, \varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda. \tag{71}$$

Although the bare a_μ field is present in the Chern–Simons term, it remains gauge-invariant because $\mathcal{J}^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda$ is a conserved current. Applications of Chern–Simons field theories to fractional statistics were first discussed in Refs. 9 and 45–47.

If, for example, the matter fields are those of ordinary nonrelativistic particles, we have

$$S_{\text{matter}} = \int d^3x \left\{ \sum_i \tfrac{1}{2} m \dot{x}_i^2 - V[x] \right\}$$
$$j^\mu(x) = \int d\tau \sum_i \delta^3(x - x_i(\tau)) \frac{dx_i^\mu}{d\tau}, \tag{72}$$

where τ is proper time. As another example, consider the $(2+1)$-dimensional $O(3)$ nonlinear sigma model. The action is

$$S_{\text{matter}} = \frac{1}{g} \int d^3x (\partial_\mu \Omega^a)(\partial^\mu \Omega^a), \tag{73}$$

where $\Omega^a\Omega^a = 1$. The conserved topological (skyrmion) current is

$$j^\mu = \frac{1}{8\pi} \varepsilon_{abc} \varepsilon^{\mu\nu\lambda} \Omega^a \partial_\nu \Omega^b \partial_\lambda \Omega^c. \tag{74}$$

Since the action of Eq. (71) is quadratic in the a_μ field, this field can be integrated out directly by solving the equations of motion

$$\frac{\delta S}{\delta a_\mu} = 0 = e j^\mu - \frac{e^2}{2\theta} \varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda, \tag{75}$$

which says that the flux $f_{\nu\lambda}$ is confined to the particle world lines:

$$j^0 = \frac{e}{2\theta} f_{12} = \frac{eb}{2\theta} \tag{76}$$

in $\hbar = c = 1$ units. Note the correspondence with Eq. (52). One can now integrate out the a_μ field in, e.g., the $\partial_\mu a^\mu$ gauge and obtain an action

$$S_{\text{eff}} = S_{\text{matter}} - \theta \int d^3x\, d^3x'\, j^\mu(x) \varepsilon_{\mu\nu\lambda} \frac{\partial^\nu}{\partial^2} j^\lambda(x')$$
$$= S_{\text{matter}} + 2\theta N_{\text{link}}, \tag{77}$$

where N_{link} is the *linking number* of the particle trajectories. For a complete revolution of one particle around another, $N_{\text{link}} = 1$, and thus we associate θ with the statistical angle for interchange.

An explicit calculation is instructive. Define the formal nonlocal operator

$$K^{\nu}(x-x')=\frac{\partial^{\nu}}{\partial^{2}}, \tag{78}$$

which satisfies $\partial_{\nu}K^{\nu}(x-x')=\delta^{3}(x-x')$. Gauge freedom allows us to take

$$\begin{aligned} K^{0} &= 0 \\ K^{i}(x-x') &= k^{i}(\mathbf{x}-\mathbf{x}')\delta(x^{0}-x'^{0}) \\ k^{i}(\mathbf{x}-\mathbf{x}') &= \frac{1}{2\pi}\frac{x^{i}-x'^{i}}{|\mathbf{x}-\mathbf{x}'|}. \end{aligned} \tag{79}$$

The quantity $k^{i}(\mathbf{x}-\mathbf{x}')$ is recognized as the vector potential of a unit strength flux tube. Now let us wind one particle (Y) around another which stays fixed at the origin. The particle currents are

$$\begin{aligned} j^{0}(x) &= \delta(\mathbf{x})+\delta(\mathbf{x}-\mathbf{Y}) \\ j^{i}(x) &= \delta(\mathbf{x}-\mathbf{Y})\,\dot{Y}^{i}. \end{aligned} \tag{80}$$

The peculiar expression in Eq. (77) thus reduces to

$$\begin{aligned} \int d^{3}x\,d^{3}x'\,j^{\mu}(x)\varepsilon_{\mu\nu\lambda}K^{\nu}(x-x')j^{\lambda}(x') &= 2\int d\tau\,\varepsilon_{ij}k^{i}(-\mathbf{Y})\,\dot{Y}^{j} \\ &= -2\oint \varepsilon_{ij}\frac{Y^{i}\,dY^{j}}{\mathbf{Y}^{2}}, \end{aligned} \tag{81}$$

which is $-2N_{\text{link}}$.

Another way to see it: The flux enclosed by Y as it winds around the origin is

$$\phi=\oint \mathbf{a}\cdot d\mathbf{l}=\int f_{12}\,dS=\frac{2\theta}{e}\int j^{0}\,dS=2\theta q, \tag{82}$$

where q is the charge in units of e. The Aharonov–Bohm phase is $e^{iq\phi}=e^{2i\theta q^{2}}$, which is just what we expect.

The field theory of the many-anyon problem is then described by the Lagrangian density[40,43,48]

$$\mathscr{L}=\bar{\psi}(iD_{0}-\mu)\psi-\frac{1}{2m}|D_{k}\psi|^{2}-\frac{e^{2}}{4\theta}\varepsilon^{\mu\nu\lambda}a_{\mu}\partial_{\nu}a_{\lambda}-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{83}$$

where $D_{\mu}=\partial_{\mu}-ie(a_{\mu}+A_{\mu})$, A_{μ} is the vector potential of the physical electromagnetic field and μ is the chemical potential for the fermions. Several authors have used Eq. (83) as a point of departure for studying the anyon gas. Assuming that the physical fields are small and average out to zero, one separates the statistical vector field $a_\mu = \bar{a}_\mu + \delta a_\mu$ into a mean field contribution satisfying Eq. (22), $\varepsilon^{ij}\partial_i \bar{a}_j = 2\theta\rho(\hbar = c = 1)$, and a fluctuating part δa_μ. One can then integrate out the fermion fields ψ and $\bar{\psi}$, generating an effective action in terms of A_μ and δa_μ. Finally, one can attempt to integrate out the δa_μ field, leaving an effective action in terms of the physical A_μ field alone, from which one can study the electrodynamics of anyon gases. These developments are clearly discussed in the article of Fradkin,[43] to which the interested reader is referred.

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APPENDIX: MANY ANYONS IN A MAGNETIC FIELD

The notation of single-particle eigenstates is of dubious utility in the anyon problem: many-anyon states are not simple determinants or permanents of one-body wavefunctions. Wu[5] has derived some explicit 3-particle eigenfunctions (i.e., not a complete set) for anyons in an external harmonic potential, but to date (1990) there has been little progress in exhibiting exact many-anyon eigenstates for reasonable Hamiltonians.* (A welcome exception is the work of Ref. 49.)

Anyons in a magnetic field can exhibit a quantized Hall effect, and once can write down explicit N-particle wave functions in some cases. The general form of a many-anyon wave function is given in Eq. (5). One such wave function is the Laughlin state,[20]

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i<j} (z_i - z_j)^{2p+1+\alpha} \exp\left(-\tfrac{1}{4}\sum_k |z_k|^2\right), \tag{A1}$$

* We exclude blatantly stupid Hamiltonians of the $|ket\rangle\langle bra|$ variety.

with p an integer. This wave function, first discussed by Halperin,[24] is an eigenstate for noninteracting charge $+e$ anyons in a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ of strength $B = (2p + 1 + \alpha)\rho\phi_0$, where ρ is the number density of anyons. (Here and henceforth I take the anyon charge to be e and measure distances in units of the magnetic length $l = \sqrt{\hbar c/eB}$.) The wave function Ψ of Eq. (A1) is indeed a many-anyon eigenfunction because the prefactor of the Gaussian term is analytic in the $\{z_i\}$ and is hence annihilated by the cyclotron lowering operators $a_i = \sqrt{2}(\bar{\partial}_i + z_i/4)$. Since the guiding center operators $b_i = \sqrt{2}(\nabla_i + \bar{z}_i/4)$ are cyclic in the free particle Hamiltonian

$$H = \hbar\omega_c \sum_i (a_i^\dagger a_i + \tfrac{1}{2}), \tag{A2}$$

short-range two-body interactions of the type discussed by Haldane[50] can render Ψ an exact nondegenerate ground state of a nontrivial many-anyon Hamiltonian.

Consider a system of charged anyons at fixed density n in the presence of a magnetic field of fixed strength $B = (2p + 1)\rho\phi_0$. What is the character of the ground state of the system as a function of the statistical parameter α? I wish to remark how the fractional quantum Hall effect (FQHE) wave functions of MacDonald, Aers, and Dharma-wardana,[51] which describe Haldane's hierarchical condensates[50] may be used to construct many-anyon wavefunctions which should have favorable energies if the anyons also interact via short-range repulsive potentials. The basic mathematical problem is this: We are looking for wave functions of the form of Eq. (5),

$$\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \prod_{i<j} (z_i - z_j)^\alpha P(z_1, \ldots, z_N) \exp\left(-\tfrac{1}{4}\sum_k |z_k|^2\right), \tag{A3}$$

which have analytic and homogeneous polynomial factors $P[z]$. Since the density $\rho = \nu/2\pi l^2$ is related to the homogeneous degree of the polynomial P through[51]

$$\nu^{-1} = \alpha + \frac{2 \deg P}{N(N-1)}, \tag{A4}$$

and remains *fixed*, only certain polynomials P need apply for the job. It is clear, for example, that when $\alpha = 2k$ is an integer multiple of 2, one possible solution is $P = P_v^{2(p-k)+1}$, where $P_v = \prod_{i<j}(z_i - z_j)$ is the Vandermonde determinant.[52] That this solution should somehow be favored has been argued by Laughlin and by Halperin; by placing all the zeros of the particle positions, the wave function does its darnedest to keep the particles away from each other, thereby muting the short-range repulsion.

Let us briefly recall the basic recipe of Ref. 51 for making hierarchical wavefunctions. Starting from a principal (Laughlin) state, one obtains a hierarchy of homogeneous polynomials P_i through repeated application of either of the two basic steps

$$P_i^{(h)} = P_{i-1}^c P_V^{2p_i} \tag{A5a}$$

$$P_i^{(p)} = P_V\{P_{i-1}^c/P_V\}^\dagger P_V^{2pi}, \tag{A5b}$$

where p_i is an integer, P^c denotes the particle-hole conjugate of P as described by Girvin,[53] and $P^\dagger$ is the adjoint of P, obtained by sending $z \to 2\partial_z$.[23] The degree of P_i is obtained as follows.* First, note that deg $P_{i-1} = \frac{1}{2}N^2/\nu_{i-1}$, which gives the degree of the conjugate polynomial deg $P_{i-1}^c = \frac{1}{2}N^2/(1-\nu_{i-1})$. From deg $P_V = \frac{1}{2}N^2$, one finds that

$$\nu_i^{-1} = \frac{2}{N^2}\deg P_i$$

$$= 2p_i + 1 \pm \frac{1}{\nu_{i-1}^{-1} - 1}, \tag{A6}$$

as derived by MacDonald, Aers, and Dharma-wardana.[51] The fractions obtained are identical to those proposed by Haldane[50] and by Halperin[24]:

$$\nu_i = \cfrac{1}{2p_i + 1 + \cfrac{\sigma_{i-1}}{2p_{i-1} + \cfrac{\sigma_{i-2}}{\ddots + \cfrac{\sigma_1}{2p_1 + \cfrac{\sigma_0}{2p_0}}}}}$$

$$\equiv [p_i, \sigma_{i-1}p_{i-1}, \ldots, \sigma_0 p_0], \tag{A7}$$

where $\sigma_i = \pm 1$ determines whether one descends down the hole (+1) or the particle (−1) branch at level i of the hierarchy.

We now see from Eq. (A4) that whenever the statistical parameter α satisfies the relation

$$\alpha = \nu_0^{-1} - \nu_i^{-1}, \tag{A8}$$

with ν_i a member of the FQHE hierarchy of filling fractions and ν_0 a principal FQHE fraction, a legitimate multivalued, fixed density, many-anyon wave function can be constructed according to Eqs. (A3), (A5a), and (A5b). Call the associated polynomial $P_i[z]$. Note that the wave function, $P_i[z]\exp(-\frac{1}{4}z^\dagger z)$, is precisely the hierarchical FQHE wavefunction one would construct in a field $B = (2p_0 + 1 - \alpha)\rho\phi_0\hat{\mathbf{z}}$, which results from

* We are interested in the thermodynamic limit, in which $N(N-1)$ can be approximated by N^2.

subtracting from the physical magnetic field strength the fictitious "statistical magnetic field" obtained by smearing the effective anyon flux over the plane. As $z_i - z_j \to 0$, $\Psi_{\nu_i}[z]$ vanishes as $(z_i - z_j)^{s_i}$, with

$$s_i = \alpha + 2p_i + 1 = 2p_0 + 1 \mp \frac{1}{\nu_{i-1}^{-1} - 1}. \tag{A9}$$

Knowledge of the short-distance behavior of $P_i[z]$ may be useful in estimating the trial-state energy. Additional properties of the anyon gas in a magnetic field, such as cyclotron resonance, have been investigated by Johnson and Canright.[16]

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Chapter 2

Microscopic and Macroscopic Loops in Nonperturbative Two-Dimensional Gravity

Tom Banks, Michael R. Douglas, Nathan Seiberg, and Stephen H. Shenker

Two-dimensional quantum gravity is relevant both for string theory and as a toy model of higher-dimensional quantum gravity. The definition of pure 2D quantum gravity and quantum gravity coupled to matter in terms of matrix models[1] is very explicit and rigorous. Matrix realizations of pure gravity and gravity coupled to certain minimal conformal field theories (and their massive deformations) can be solved by the application of large-N techniques. Recently, an exact expression for the specific heat of some of these models was found in the continuum limit.[2–6] In this chapter, we will show that the correlation functions of operators in these models can also be easily computed. We distinguish between two kinds of operators, microscopic and macroscopic loops. By microscopic loops we mean expressions like Tr M^p in the matrix models with p finite. They contain all the information about integrals over the surface of local operators.

Tom Banks, Michael R. Douglas, Nathan Seiberg, and Stephen H. Shenker • Department of Physics and Astronomy, Rutgers University, Piscataway, New Jersey 08855-0849.

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Macroscopic loops are also given by Tr M^p, but p is taken to infinity in the continuum limit in such a way that they correspond to extended boundaries on the surface.

We start by deriving a free fermion representation for the correlation functions of arbitrary loops in theories of two-dimensional geometry based on a single large-N matrix integral. The methods we present generalize immediately to arbitrary one-dimensional chains of matrices of the type studied in Ref. 7. In the limit, when the matrix chain becomes infinite and continuous, our method reduces to the fermionic description of large-N matrix quantum mechanics discovered by Brézin, Itzykson, Parisi and Zuber.[8] We will indicate the form of this generalization only briefly here, reserving the details for a lengthier publication.[9] We will study correlation functions in the matrix model of the form

$$\frac{\int [dM]\, e^{-\mathrm{Tr}\, \nu(M)}(\mathrm{Tr}\, M^{p_1} \cdots \mathrm{Tr}\, M^{p_n})}{\int [dM]\, e^{-\mathrm{Tr}\, \nu(M)}}. \tag{1}$$

Geometrically these represent sums over random surfaces of arbitrary genus with boundaries of lengths $p_1 \cdots p_n$.

Introducing the decomposition of the Hermitian matrices M into unitary and diagonal matrices

$$M = U^\dagger \Lambda U,$$

this integral can be written

$$\frac{\int d^N\lambda \Delta^2(\lambda)\, e^{-\sum \nu(\lambda_i)} \sum \lambda_i^{p_1} \cdots \sum \lambda_i^{p_n}}{\int d^N\lambda \Delta^2(\lambda)\, e^{-\sum \nu(\lambda_i)}}, \tag{2}$$

where $\Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$ is a Vandermonde determinant. This expression may be viewed as the expectation value of a product of one-body operators in a Slater determinant[10] constructed from the first N members of the complete orthonormal set of one-body wave functions

$$\psi_n = P_n(\lambda)\, e^{-\frac{1}{2}\nu(\lambda)} \tag{3}$$

$$\int \psi_n \psi_m \, d\lambda = \delta_{nm}. \tag{4}$$

These functions are determined completely by Eq. (4) and by the requirement that P_n be a polynomial of order n. A set of recurrence relations determining them explicitly was given by Bessis, Itzykson, and Zuber.[11]

For large N we are dealing with a many-fermion system, and it is convenient to introduce second quantized notation. We define the fermion field

$$\Psi(\lambda) = \sum_{n=0}^{\infty} a_n \psi_n(\lambda), \tag{5}$$

where the a_n are annihilation operators. The correlation function, Eq. (1), can then be written

$$\langle F|\Psi^\dagger J^{p_1}\Psi \cdots \Psi^\dagger J^{p_n}\Psi|F\rangle, \tag{6}$$

where $|F\rangle$ is the filled Fermi sea for N fermions and J is the one-body operator of multiplication by λ. In order to take the continuum limit of this equation, it is best to work in the orthonormal polynomial basis,[11] where J is an infinite matrix given by

$$J_{mn} = \sqrt{R_m}\,\delta_{m,n+1} + \sqrt{R_n}\,\delta_{m+1,n}. \tag{7}$$

It is now easy to write down formulas for the correlation functions in terms of one-body operators by considering how the creation and annihilation operators act on the Fermi sea. For example, the connected one- and two-point functions are

$$\langle F|\Psi^\dagger J^p\Psi|F\rangle = \mathrm{Tr}\,\mathscr{S}J^p = \sum_{n=0}^{N-1} (J^p)_{nn} \tag{8}$$

$$\langle F|\Psi^\dagger J^{p_1}\Psi\Psi^\dagger J^{p_2}\Psi|F\rangle_c = \mathrm{Tr}\,\mathscr{S}J^{p_1}(1-\mathscr{S})J^{p_2}, \tag{9}$$

where $\mathscr{S}$ is the projection operator[10] on the subspace of one-body wave functions with $n \leq N-1$. Below we will show how to take the continuum limits of these formulas.

To facilitate computation of higher-order connected Green's functions, we can use Wick's theorem. In order to do this it is necessary to write our correlation functions in terms of time-ordered products. This is done by inventing a one-body Hamiltonian whose eigenstates are ψ_n and whose eigenvalues are monotonic functions of n. The details of the Hamiltonian are irrelevant because all of the correlation functions of interest are of operators at almost equal times. With respect to any such one-body Hamiltonian, our Slater determinant is the normalized N-fermion ground state. The expectation value of ordinary operator products that we want to compute may be written as time-ordered products by assigning the kth operator from the left a time $k\delta$. At the end of the calculation, δ is taken to zero. We can now use Wick's theorem with the fermion propagator for $s \approx t$:

$$\langle F|T\Psi(\lambda,t)\Psi^\dagger(\lambda',s)|F\rangle = \sum_{nm} \psi_n(\lambda)[S(s-t)(1-\mathscr{S}) - S(t-s)\mathscr{S}]_{nm}\psi_m^*(\lambda') \tag{10}$$

where $S(t)$ is an ordinary step function with support on the interval from zero to infinity. Note that we did not have to know anything about the spectrum of the fictitious fermion Hamiltonian because all times are taken to zero at the end of the calculation. We must, however, be cautious about one point: The fermions inside each bilinear operator are in ordinary rather than time-ordered products. This difficulty is easily remedied by writing

$$\Psi^\dagger O\Psi = \tfrac{1}{2}([\Psi^\dagger, O\Psi] + \{\Psi^\dagger, O\Psi\}) = \tfrac{1}{2}[\Psi^\dagger, O\Psi] + \tfrac{1}{2}\operatorname{Tr} O \qquad (11)$$

for any one-body operator O. The connected Green's functions of the time-ordered operators [the commutator part of Eq. (11)] are given by the usual ringlike Feynman diagrams for free-fermion bilinears, with the propagator given above. However, the time-ordered and ordinary products of two fermion operators differ by a c number. This subtraction thus affects only the connected one-point function. Thus higher point connected Green's functions of the operators that interest us are given correctly by applying Wick's theorem and ignoring the subtraction.

We now want to note an important property of the expressions we have derived for correlation functions. Consider the connected two-point function of Eq. (9) when the length of the loops is a small number of lattice spacings. As can be seen from Eq. (7), the operator J is "local" in fermion level space. It is a finite difference operator that connects only the $n \pm 1$ levels. Low powers of it have short range in fermion level space. Note that in the two-point function J^l and J^k are sandwiched between orthogonal projectors that project out the states above or below the Fermi surface. Only states in the neighborhood of the Fermi surface contribute. For example,

$$\langle \operatorname{Tr} M^2 \operatorname{Tr} M^2 \rangle_c = R_{N+1}R_N + R_N R_{N-1} \qquad (12)$$

$$\begin{aligned}\langle \operatorname{Tr} M^2 \operatorname{Tr} M^2 \operatorname{Tr} M^2 \rangle_c &= R_{N+1}R_N(R_{N+2} + R_{N+1} - R_N - R_{N-1}) \\ &\quad + R_N R_{N-1}(R_{N+1} + R_N - R_{N-1} - R_{N-2}). \qquad (13)\end{aligned}$$

This is important since the universal physics is located in the immediate neighborhoods of the Fermi surface. We see that the continuum theory is sensitive only to the states right near the Fermi surface. The space of levels near the Fermi surface becomes continuous, and operators like J, which are local in Fermi level space, become finite-order differential operators.

We have shown that general correlation functions in string models based on single large-N matrix integrals can be written as expectations values of products of fermion bilinears in a free fermion lattice field theory. These considerations are easily extended to matrix chain models.[7] The new element there is a transfer matrix along the chain. The correlations in these models can be written as expectation values of products of fermion bilinears and transfer matrices in the state $|F\rangle$ described above. Since this is not an eigenstate of the transfer matrix, an overlap integral must be computed. However, for infinitely long chains, the divergent part of the free energy is independent of the overlap and depends only on the ground state of the transfer matrix. This is a discrete version of the large-N quantum mechanics of Ref. 8, and our fermion formalism converges nicely to theirs in this limit. We note, however, that the matrix chain contains variables that cannot be described in terms of fermions. These are the unitary parts, or angular variables, of the matrices. They disappear from certain correlation functions because of a global $U(N)$ symmetry under which both the transfer matrix and the state $|F\rangle$ are invariant.[10] Other correlators involve the angular variables in an essential way, and are more difficult to study. We will give a detailed description of our results for matrix chain models in Ref. 9.

We now discuss the continuum limit of the correlation functions, using the scaling limit introduced in Refs. 2-4. For simplicity we present the formulas for $c = 0$, $m = 2$. We follow the notation of Ref. 3. We introduce a lattice spacing a, a renormalized cosmological constant $\mu = (\mu_0 - \mu_c/a^2)$, a renormalized string coupling $\lambda = a^{-5/2}\varepsilon(\varepsilon = 1/N)$, finite in the continuum limit, and a variable z that describes the universal infinitesimal region near the Fermi surface ($x = n/N \sim 1$), $e^{-a^2\mu}x = 1 - a^2 z$. The constants R_n are replaced by a function $r(x)$ whose universal part is $P(z)$, defined by $r - \rho = aP(z)$, where ρ is a nonuniversal constant. As shown in Refs. 2-4, $P(z)$ satisfies the Painlevé equation of the first kind.

We now consider macroscopic loops. These are operators of the form $\mathrm{Tr}\, M^p$ with $p \to \infty$ in the continuum limit so that $l = pa$, the physical length of the loop, is held fixed. It is convenient to write the Jacobi operator as

$$J = [\rho + aP(z)]^{1/2}\, e^{\varepsilon\partial_x} + e^{-\varepsilon\partial_x}[\rho + aP(z)]^{1/2}. \tag{14}$$

Introducing the above quantities and expanding to first order in a, we find that

$$J = 2\rho^{1/2} + a[\rho^{1/2}\lambda^2\,\partial_z^2 + \rho^{-1/2}P(z)] \tag{15}$$

and

$$\mathrm{Tr}\, J^p = \mathrm{Tr}\, J^{l/a} = e^{l/a\,\log(2\rho^{1/2})}\, e^{l/2(\lambda^2\partial_z^2 + \rho^{-1}P)/2}. \tag{16}$$

The first factor is a nonuniversal boundary energy which we absorb by a multiplicative renormalization. The remainder is universal. We see that the loop of length l is described by the heat kernel e^{-Hl} of the Schrödinger operator*

$$H = -\tfrac{1}{2}\lambda^2\partial_x^2 + V(z), \qquad V(z) = -\tfrac{1}{2}P(z), \tag{17}$$

where we have scaled ρ to one and $P(z)$ obeys the $c = 0$, $m = 2$ string equation

$$P^2 + \tfrac{1}{3}\lambda^2 P'' = z. \tag{18}$$

* This Schrödinger operator was derived independently by Gross and Migdal,[4] who showed that the multicritical string equations can be determined from its Seeley coefficients, and by Douglas and Shenker,[12] who showed that its heat kernel describes the macroscopic loop.

It is often useful to measure lengths in units of μ^{-1}. After rescaling z and P to accomplish this, Eqs. (17) and (18) remain unchanged except that λ is replaced by the dimensionless handle-counting parameter $\kappa = \lambda/\mu^{5/4}$.

The changes in scaling necessary to account for the negative-dimension operator to which the "cosmological constant" very probably couples in the general multicritical model are discussed in Ref. 5. The general string equation is discussed in Refs. 2–4.

Using the fermion formalism and the heat kernel H, we can now write the master formula for the expectation value of k macroscopic loops with lengths $l_1, l_2, \ldots, l_k$

$$\langle W_{l_1} W_{l_2} \cdots W_{l_k} \rangle_c = \left\langle \prod_{i=1}^{k} (\Psi^\dagger e^{-l_i H} \Psi) \right\rangle_c. \tag{19}$$

As an example, we calculate the expectation for one loop

$$\langle W_l \rangle = \int_{-\infty}^{\infty} dz\, S(z) \langle z | e^{-Hl} | z \rangle \tag{20}$$

and for two loops

$$\langle W_{l_1} W_{l_2} \rangle_c = \int_{-\infty}^{\infty} dz\, dw\, S(z) \langle z | e^{-Hl_1} | w \rangle (1 - S(w)) \langle w | e^{-Hl_2} | z \rangle. \tag{21}$$

The step functions $S(z)$ represent the existence of the Fermi sea, as explained above, and have support for $z > \mu$.

We now turn to a discussion of microscopic loops. The natural operators to examine are the scaling operators[12] O_k that couple to sources T_k in the tree level equation[2]* for the specific heat as

$$\mu = c_2 T_2 f^2 + c_3 T_3 f^3 + c_4 T_4 f^4 + \cdots + c_k T_k f^k + \cdots, \tag{22}$$

where c_i are normalization constants. Such a potential describes the general massive model interpolating between the multicritical fixed points.[13]

One way of isolating such scaling operators is to observe that as we take the lattice spacing to zero, lattice correlators like Eqs. (8) and (9) have expansions in powers of the lattice spacing whose coefficients are matrix elements of continuum scaling operators. In particular, these scaling operators can be found by examining the behavior of correlators for boundaries only a few lattice spacings long. The locality of small powers of J implies that the matrix elements of the scaling operators are given by polynomials of the Painlevé function and its derivatives. For example, at the scaling limit Eqs. (12) and (13) become

$$\langle \mathrm{Tr}\, M^2\, \mathrm{Tr}\, M^2 \rangle_c = 2\rho^2 + 4\rho a P(\mu) + O(a^2) \tag{23}$$

$$\langle \mathrm{Tr}\, M^2\, \mathrm{Tr}\, M^2\, \mathrm{Tr}\, M^2 \rangle_c = -8\lambda \rho^2 a^{3/2} P'(\mu) + O(a^2). \tag{24}$$

* Gross and Migdal[4] give a general formula for the correlation functions of scaling operators at tree level.

The additive constant $2\rho^2$ in Eq. (23) is not universal. It is not present in higher-n point functions such as Eq. (24). Such a constant exists also in the one-point function. It appears there because the expectation value depends on the entire Fermi sea and not just on its universal surface. A similar nonuniversal additive constant could appear in the calculation of the one-point function Eq. (20) where we integrate over the entire Fermi sea. However, as is clear from Eq. (16), because of the limit $p = l/a \to \infty$, the additive nonuniversal constant exponentiates and turns into a multiplicative constant. The contribution of the bottom of the Fermi sea ($z \to \infty$) is exponentially suppressed in Eq. (20) and therefore does not shift the answer.

In interpreting the correlation functions one should be careful not to forget the additive constants. These can usually be removed by differentiating a large enough number of times with respect to the cosmological constant μ. This has the effect of removing all the analytic dependence on μ. An equivalent way to understand this phenomenon is to study the theory at large (relative to the cutoff) fixed area A. Then the correlation functions on the sphere have the form A^r for some constant r. For $r \leq -1$, the integral over A diverges for small A. This is the origin of the additive constant. For $r < -1$ and not an integer, the universal term is proportional to μ^{-r-1}. For r an integer smaller than or equal to -1, the situation is more complicated. In this case, differentiating the answer $-r-1$ times with respect to μ, we expect to find $\log \mu$ in the answer. For example, for the one-point function of the energy operator in the Ising model, $r = -3$. Differentiating twice with respect to the cosmological constant, we expect $r = -1$, and the universal term is proportional to $\log \mu$. Since the exact result of the lattice calculation is a constant independent of μ, we conclude that the one-point function of the energy operator in the Ising model vanishes.* We would like to note, however, that such constants may very well have universal physical meaning.

A more elegant approach to constructing the correlation functions is to use the singular potentials introduced by Gross and Migdal[4] to pick out the pure scaling operators. They show that the matrix potential corresponding to a variation of the field T_k is $\mathrm{Tr}(2-M)^{k+1/2}$, which in the continuum limit is $H^{k+1/2}$. In the fermion formalism it is represented by the one-body operator $\Psi^\dagger H^{k+1/2}\Psi$. Thus we can write a general formula for the n-point correlation function of scaling operators (up to a normalization):

$$\langle O_{k_1} O_{k_2} \cdots O_{k_n} \rangle = \oint \frac{dl_1}{2\pi i} \cdots \oint \frac{dl_n}{2\pi i}\, l_1^{-(k_1+3/2)} \cdots \times l_n^{-(k_n+3/2)} \left\langle \prod_{i=1}^{k} (\Psi^\dagger e^{-l_i H} \Psi) \right\rangle. \tag{25}$$

* This is in contrast to the statement in Ref. 5.

For example, the one-point function of the operator conjugate to T_k is given (formally), up to a normalization, by

$$\langle O_k \rangle = \int_{\mu}^{\infty} dz \, \langle z | H^{k+1/2} | z \rangle. \tag{26}$$

For higher-point functions care must be taken with the distributions implicit in Eq. (25).

Equation (26) has important consequences. In order to explain them we temporarily shift notation to conform to Ref. 14, whose results we use extensively in what follows. Replace z by x, set $\lambda^2 = 2$, and let $u = V$. The Schrödinger operator becomes $H = -\partial_x^2 + u(x)$. Introduce the diagonal of the resolvent to define fractional powers of H,

$$\langle x | (H + \zeta)^{-1} | x \rangle = \sum_{l=0}^{\infty} \frac{R_l[u(x)]}{\zeta^{l+1/2}}. \tag{27}$$

The coefficients $R_l[u]$ are polynomials in u and its derivatives and are the generalized *KdV* potentials. Gross and Migdal[4] showed that the multicritical string equations are determined by these quantities. The diagonal of $H^{k+1/2}$ is determined up to a normalization by R_{k+1}. The string equation for the general massive model interpolating between multicritical points[3,4] is

$$x = \sum_{k=0}^{\infty} (k + \tfrac{1}{2}) T_k R_k[u] \tag{28}$$

(where we have now fixed the normalizations) or, using the identity $(\delta/\delta u) R_{k+1} = -(k + \frac{1}{2}) R_k$,

$$x = -\sum_{k=0}^{\infty} T_k \frac{\delta}{\delta u} R_{k+1}[u]. \tag{29}$$

We list the first few R_l:

$$\begin{aligned} R_0 &= \tfrac{1}{2} \\ R_1 &= -\tfrac{1}{4} u \\ R_2 &= \tfrac{1}{16}(3u^2 - u'') \\ R_3 &= -\tfrac{1}{64}(10u^3 - 10uu'' - 5(u')^2 + u''''). \end{aligned} \tag{30}$$

Noting that the specific heat $u \sim \partial_\mu^2 F$ and that $\langle O_k \rangle = (\partial / \partial T_k) F$ where F is the free energy, we see that Eq. (26) for $u(T_1, T_2 \ldots ; x)$ can be written, after differentiating twice,

$$\frac{\partial}{\partial T_k} u = \frac{\partial}{\partial x} R_{k+1}[u] \tag{31}$$

for every k. These are just the (generalized) KdV equations.* We see that the specific heat u as a function of the scaling fields and x is just a solution of the KdV hierarchy. This observation raises an important question: if we start at a given multicritical model, and then flow up to a higher one using Eq. (31), which special solution of the higher string equation (if any) do we come to?†

We can express Eq. (31) more compactly by introducing the vector fields which generate the KdV flows:

$$\xi_l = \sum_{i=0}^{\infty} R_l^{(i+1)} \frac{\delta}{\delta u^{(i)}}, \tag{32}$$

where the superscript refers to differentiation with respect to x and the $u^{(i)}$ are considered independent. Integrability of the KdV hierarchy depends crucially on the fact that these vector fields commute:

$$[\xi_l, \xi_m] = 0. \tag{33}$$

We can then write Eq. (31) as

$$\frac{\partial}{\partial T_k} u = \xi_{k+1} \cdot u. \tag{34}$$

Correlation functions of the general massive multicritical model are then given by the simple formula

$$\partial_\mu^2 \langle O_{k_1} O_{k_2} \cdots O_{k_n} \rangle_c = \xi_{k_n+1} \cdots \xi_{k_2+1} \xi_{k_1+1} \cdot u. \tag{35}$$

The ordering is unimportant because of Eq. (33). This expression is a polynomial in u and its derivatives, as the matrix expressions imply. It is straightforward to show, using identities that follow from Eq. (33), that the differential equation for the correlation function[4] derived from varying Eq. (28) with respect to T_k is satisfied by Eqs. (33) and (34).

We now discuss some of the physics of Eqs. (20) and (21). Since we are dealing with a free fermion theory, all the correlators can be written simply in terms of the heat kernel of H. Let us examine the properties of H. Figure 1 is a rough sketch of V for $m = 2$, when the string equation is Painlevé. Recall that the detailed shape depends on the nonperturbative free parameter. There is an infinite sequence of double poles as $z \to -\infty$ that asymptotically become periodic. We will call the location of the pth double pole z_p. Since the potential approaches $+\infty$ at these points as $(z - z_p)^{-2}$, the wave function must vanish there faster than $z - z_p$. With

* The connection to the free fermion formalism discussed above is quite likely to be made through the Grassmannian and its associated τ function.[15]

† Issues related to this have been considered by Witten[16] in his topological field-theory derivation of low-genus correlation functions.

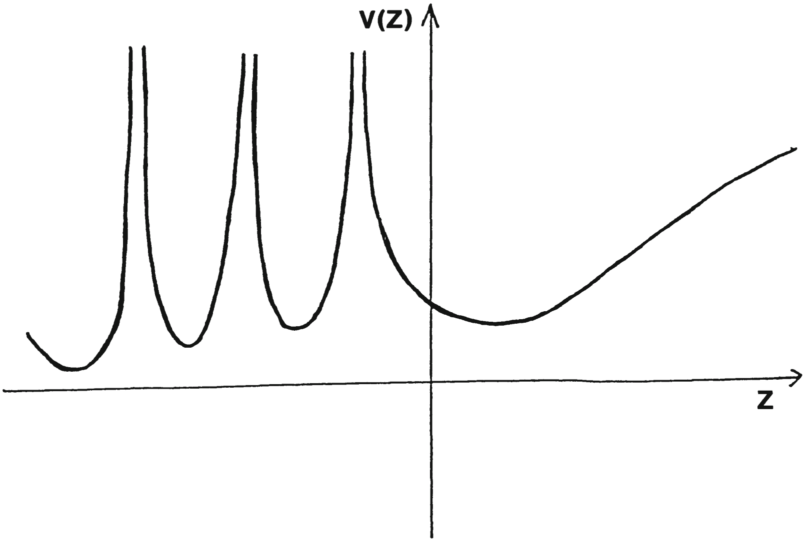

Figure 1. Approximation of V when the string equation is Painléve ($m = 2$).

such a behavior, the region between each pair of poles is disconnected from the others; i.e., the Hamiltonian H is self-adjoint once restricted to a single region. The region that joins onto perturbation theory is $z_1 < z < \infty$.

In the perturbative region $z \to \infty$, the potential $V \to \sqrt{z}$, and the wave function must decay. So there is a well-posed eigenvalue problem with discrete spectrum in this region. Referring to Eq. (20), we see that loops decay with an infinite number of distinct exponentials of their length due to this discrete spectrum. This is a dramatic and nonperturbative phenomenon in 2D quantum gravity. Note that the discreteness of the spectrum is a consequence of the first double pole, even if it is not visible in the free energy for "physical" $\mu > 0$. This suggests a physical role for these singularities.* We might wonder what happens if the free parameter is adjusted so that the Fermi level μ is in between two poles. We then couple to the part of the spectrum supported entirely between them. Perhaps this is a new strong-coupling phase of 2D quantum gravity? The spectrum of H would be gapped in each new "phase."

In the weak-coupling regime we might expect that geometrical intuition about sums over surfaces of different topologies would give a qualitatively correct picture of the physics of two-dimensional quantum gravity. At tree level David[17] has argued that the behavior of $\langle W_l \rangle$ is qualitatively similar to that of a large loop spanned by a surface of constant negative curvature

* That the double poles might have physical consequences was suggested by Brézin and Kazakov.[2]

$-\sqrt{\mu}$. The expectation value of the area

$$\partial_\mu \log\langle W_l\rangle \sim \begin{cases} l^2, & \text{for } l\sqrt{\mu} \ll 1; \\ l/\sqrt{\mu}, & \text{for } l\sqrt{\mu} \gg 1. \end{cases} \tag{36}$$

So at large l the area is large, and for small enough κ^2 we might expect to approximate the sum over surfaces by a dilute gas of handles. This approximation is the basis of wormhole physics.

In fact, at asymptotically large l, for any finite κ, the behavior of expectation values in two-dimensional gravity does not coincide with the dilute wormhole picture. The spectrum of the Hamiltonian is discrete, and at asymptotically large l, Eq. (20) is dominated by the ground-state energy. We find that

$$W_l = \int_\mu^\infty dz\, \phi_0^2(z)\, e^{-E_0 l}. \tag{37}$$

This looks more like what might be expected from a simple renormalization of the cosmological constant: the web of higher-genus surfaces seems to behave at large l like a genus-zero surface with an effective cosmological constant. Note that the value of this effective constant is not zero.

It should come as no surprise that the dilute wormhole gas is not a valid approximation for large-volume universes. There is no cluster expansion for wormholes as there is for ordinary instantons. The contribution of wormhole interactions (nonquadratic terms in the action for fluctuating couplings) to the logarithm of the partition function contains cubic and higher powers of the volume, while the dilute gas contribution is quadratic. Even if there is a small parameter (κ in the present context) controlling the wormhole density, the interaction terms dominate at large volumes. Our exact solution of the two-dimensional problem allows us to see the correct asymptotic behavior.

The above discussion was valid for asymptotically large volumes. As we let l become smaller, we see the possibility of a regime in which wormholes give a correct picture of the physics. The gap in the Schrödinger spectrum is of order κ, so if $l\sqrt{\mu}\kappa \ll 1$, we can no longer approximate the macroscopic loop by the contribution of the ground state alone. Thus for small κ and $1 \ll l\sqrt{\mu} \ll 1/\kappa$, we can expect to approximate the Schrödinger spectrum by a continuum and the result for the loop in this regime can be written as an integral over fluctuating values of the cosmological constant of the tree level result.[18]

This is a rather weak probe of the validity of wormhole ideas. To be more precise we can investigate the behavior of the loop expansion order by order. The diagonal matrix element of the heat kernel that appears in W_l can be written as a path integral with action

$$S = \int_0^l \frac{\dot{x}^2}{2\kappa^2} + \frac{\sqrt{x}}{2} + \sum_{p=1}^{\infty} \kappa^{2p} a_p z^{-(5p-1/2)}. \tag{38}$$

It can be expanded in powers of κ by writing $x(t) = z + \kappa\Delta(t)$. This generates, in each order in κ, an action which is polynomial in Δ. The path integral can then be written in terms of Feynman graphs whose l-dependence is determined by simple dimensional analysis.

When this analysis is performed in order κ^2 (genus one), we find a result consistent with wormhole ideas: a term proportional to l and a term proportional to l^2. The average area is again proportional to l, so the first of these resembles a renormalization of the cosmological constant while the second can be interpreted as a single wormhole contribution. However, at genus two and higher there appear to be contributions which do not fit into a wormhole picture, even when wormhole interactions are included. In particular, at genus g the leading large-l behavior appears to be l^{3g-1} rather than the l^{2g} one would expect from wormholes. This may be an indication that even in this perturbation regime, the contribution of "fat" surfaces to the path integral at large l dominates over that of wormhole configurations. Indeed, the arguments (such as they are) that have been adduced to justify restricting attention to wormhole configurations in the path integral over four-geometries are not obviously applicable in the present context.[19] We caution, however, that our understanding of this issue is not complete, and that a wormhole picture of the sum over two-geometries is not completely ruled out in the perturbative regime. What is clear is that, for asymptotically large l, all such arguments fail, and the behavior of the loop is controlled by aspects of the problem that are invisible in the genus expansion. We do not at present have an intuitive geometrical picture of the origin of these nonperturbative effects.

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Chapter 3

Supersymmetry and Gauge Invariance in Stochastic Quantization

Laurent Baulieu

1. INTRODUCTION

Stochastic quantization is an alternative to the Feynman path integral for quantizing a theory. In Ref. 1, Parisi and Wu have suggested applying stochastic quantization to gauge theories. Numerous works have followed.[2] One of the motivations of Parisi and Wu was that no gauge fixing is necessary to compute gauge-invariant quantities in stochastic quantization, since the stochastic evolution can be consistently defined from a drift force equal to minus the gradient of the classical action with respect to the gauge field, with no reference to the ghosts that occur in the ordinary path integral formalism. However, it has been realized that it is useful to introduce a kind of gauge fixing in stochastic quantization: a drift force can be defined along gauge orbits.[3] This permits a consistent renormalizability of the stochastically quantized gauge theory. Moreover, with a particular choice

LAURENT BAULIEU • Laboratoire de Physique Théorique et Hautes Energies, Université Pierre et Marie Curie, 75005 Paris, France.

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of this drift force, it seems that the gauge field is confined within the first Gribov horizon, and so one naturally escapes the Gribov problem.[3,4] The freedom in the Langevin equation of a gauge theory, which permits the introduction of the gauge-dependent drift force, follows in fact from the simple geometrical principle that stochastic evolution be compatible with the gauge symmetry.[5]

There is a supersymmetry that is inherent to any stochastically quantized theory, whether or not it has a gauge invariance.[2] This is a very general result, linked to the possibility of interpreting the Langevin equation as a constraint between the noises and the fields: this constraint can be exponentiated in the Boltzman weight involving the noise, provided a relevant Jacobian multiplies the measure; in turn, this Jacobian can be exponentiated, and one ends up with a supersymmetric action. It has been observed that this supersymmetry has a deep relationship with the notion of a topological gauge symmetry.[6,7] The stochastic supersymmetry is technically useful, since it implies Ward identities for correlation functions. The latter permit one to control and consistently renormalize the divergences which generally occur in stochastic quantization.[8,9]

If one considers a theory with a gauge invariance, the stochastic supersymmetry must be supplemented by some other symmetry acting as a reminder of the original gauge invariance. Previous attempts to express the gauge invariance in stochastic quantization to control the divergences of a stochastically quantized Yang–Mills theory can be found in Ref. 8. In Ref. 9, two separate Ward identities were written, one for the stochastic supersymmetry and one for the gauge symmetry. A basic tool in Ref. 10 is the definition of the ghost through its own Langevin equation, first introduced in Ref. 5.

Here we shall show that there is in fact a single symmetry that combines both stochastic supersymmetry and gauge invariance. We will work out in detail the case of theories with a Yang–Mills invariance and briefly sketch the case of the invariance under changes of coordinates, for which interesting phenomena seem to occur in two dimensions. We will find that the underlying invariance is of the topological type and has thus a geometrical meaning.

We shall mainly consider the case where the supersymmetric path integral representation is based on the parabolic differential operator $\partial/\partial t - \Delta$.

The other case where the representation of the supersymmetric path integral is based on elliptic or hyperbolic operators $\partial^2/\partial t^2 \pm \Delta$ necessitates another interpretation of the drift force along gauge orbits. This case has interesting applications for quantizating first-order systems. We will briefly display the method in the case of the 2D Chern–Simons action $\int_{M_2} \mathrm{Tr}\, F\phi$, and show how it provides a theory in three dimensions (the third dimension is stochastic time) related to the topological action for the second Chern class $\int_{M_4} \mathrm{Tr}\, FF$. (The quantization of the three-dimensional Chern–Simon theory is presented elsewhere.)[10]

2. LANGEVIN EQUATIONS FOR THEORIES WITH A YANG–MILLS SYMMETRY

Following earlier results,[5] we first construct the Langevin equations for the quantum theory by postulating that the ordinary Yang–Mills BRST transformations (which include as a subset the ordinary infinitesimal gauge transformations) must commute with the evolution along stochastic time. We start from the expression of the generator s of the ordinary Yang–Mills BRST symmetry: s is a graded differential operator, defined by its action on the G-valued Yang–Mills field 1-form $A = A_i dx^i$, with $1 \leq i \leq n$ and n the dimension of the physical space of the theory, and on the G-valued anticommuting 0-form ghost c:

$$sA = -Dc \qquad sc = -\tfrac{1}{2}[c, c]. \tag{1a}$$

Also, d is the ordinary exterior derivative $d = dx^i\, \partial_i$ and $D = d + [A,\quad]$ is the covariant derivative. We define the covariant BRST differential operator $S = s + [c,\quad]$. By definition, s and d anticommute. One has $s^2 = 0$, $S^2 = 0$, and $SD + DS = 0$. The BRST equations (1a) can be rewritten as

$$(d + s)(A + c) + \tfrac{1}{2}[A + c, A + c] = dA + [A, A]. \tag{1b}$$

In what follows, $\partial_0 = \partial/\partial t$ denotes the derivative with respect to the stochastic time g, and all fields are assumed to depend on x^i and t. Our postulate is that s and ∂_0 commute:

$$s\partial_0 = \partial_0 s. \tag{2}$$

One has also $s\partial_i = \partial_i s$. Applying ∂_0 to both sides of (1b) and using (2), we obtain

$$(D + S)(\partial_0(A + c)) = D(\partial_0 A). \tag{3}$$

This equation, once expanded in ghost number, gives:

$$S(\partial_0 c) = 0 \tag{4a}$$

$$S(\partial_0 A) + D(\partial_0 c) = 0. \tag{4b}$$

Equations (4) can be easily solved, simply by using power counting (the dimensions and ghost numbers of A and c are as usual) and the properties $S^2 = 0$ and $SD + DS = 0$. One obtains

$$\partial_0 c = Sv - \gamma = sv + [c, v] - \gamma \tag{5a}$$

$$\partial_0 A_i = D_i v + \frac{\delta I_{cl}[A]}{\delta A_i} + b_i, \tag{5b}$$

where γ and b_i, which can be identified as noises for c and A_i, respectively, are submitted to the BRST constraints

$$S\gamma = 0 \qquad Sb_i = D_i\gamma. \tag{6}$$

I_{cl} can be any given functional of the A_i, provided it is s-invariant, so that the equation of motion is s-covariant, $S\delta I_{cl}[A]/\delta A_i = 0$. This means that I_{cl} is a gauge-invariant classical action.

v is an arbitrary G-valued function, left undetermined when solving Eq. (3). The freedom in the choice of v, and thus in part of the drift force in the Langevin Eqs. (5), can be understood as the manifestation of gauge invariance.

Translating the Langevin equation, Eq. (5b) into a Fokker–Planck equation, it is possible to prove, in the limit $t \to \infty$, the independence of the choice of v of correlation functions of gauge-independent functionals of the A_i, computed from the Langevin equation, Eq. (5b).

In the next section we shall take v as a functional of the A_i, $v = v[A_i]$. In this case, Eq. (5b) is the modified Langevin equation introduced in Ref. [3] to induce a drift force along the gauge orbits without affecting the values of gauge-invariant quantities (a natural choice is $v = \partial_i A^i$). The stochastic ghost equation, Eq. (5a), means that now $\partial_0 c = (\delta v/\delta A_i)D_i c + [c, v] - \gamma$. The convergence toward the usual Faddeev distribution of the Fokker–Planck equation associated to the Langevin equation, Eq. (5b), has been demonstrated in Ref. 11, by using a certain functional $v[A]$.

If we compare Eqs. (5a) and (5b), we see that the evolution of A is not correlated to that of c, while that of c is correlated to that of A. (After introducing antighosts, one could imagine a more general situation where v is ghost- and antighost-dependent, implying a spurious ghost dependence in the evolution of A.)

The Langevin equation, Eq. (5b), for the ghost has been introduced in Ref. 5. It can be generalized for any given gauge theory. It was rederived by other means in Ref. 9, with $\gamma = 0$ and the choice $v = \partial_i A^i$, and used to investigate the Ward identities and the renormalization of the stochastically quantized Yang–Mills theory. (Choosing $\gamma = 0$ and thus $sb_i = -[c, b_i]$ reproduces the usual convention that the noise of the gauge field transforms covariantly and not as a gauge field.) Here, we will consider the general situation $\gamma \neq 0$ in order to construct a symmetry operator which unifies the stochastic supersymmetry and the ordinary gauge symmetry.

3. THE PARTITION FUNCTION

Up to the obvious requirement that the stochastic process induced by the Langevin equation, Eq. (5b), is meaningful, the correlation functions of gauge-independent functionals of the A_i do not depend on the choice of v. Let $J_i(x, t)$ be the sources of the A_is. The stochastic partition function is

$$Z[J] = \int [Db_i] \exp \int dt\, dx\, \mathrm{Tr}(-\tfrac{1}{2}b_i^2 + J_i(x, t)A_i(x, t)), \tag{7}$$

where the $b(x, t)$ are submitted to the constraints of Eq. (5b).

We wish to construct a supersymmetric functional representation of the Langevin equation that involves the G-valued Faddeev–Popov ghost c. We insert in the generating functional $Z[J]$ the identity $1 = \int [d\bar{c}][d\gamma] \exp - \int dt\, dx\, \bar{c}\gamma$ ($\bar{c}$ and γ are G-valued anticommuting fields). In this way, we have

$$Z[J] = \int [Db_i][d\bar{c}][d\gamma] \exp \int dt\, dx\, \mathrm{Tr}(-\tfrac{1}{2}b_i^2 - \bar{c}\gamma + J_i(x, t)A_i(x, t)). \tag{8}$$

It is convenient to define $F_{0i} = \partial_0 A_i - D_i v$ and $D_0 = \partial_0 + [v, \quad]$. Assuming that the super-Jacobian of the transformation $(A_i, c) \to (F_{0i} - \delta I_{cl}/\delta A_i,\ D_0 c - \delta v/\delta A_i D_i c)$ is not singular, we can insert in Eq. (8) the formal identity

$$1 = \int [DA_i][Dc]\delta\left(b_i - F_{0i} + \frac{\delta I_{cl}}{\delta A_i}\right)\delta\left(\gamma - D_0 c + \frac{\delta v}{\delta A_i} D_i c\right) \det \frac{\delta(b_i, \gamma)}{\delta(A_i, c)}. \tag{9}$$

The integration over b_i and γ is trivial. We get

$$Z[J] = \int [DA_i][D\bar{c}][Dc] \det \frac{\delta(b_i, \gamma)}{\delta(A_i, c)} \exp \int dt\, dx\, \mathrm{Tr}\left(-\frac{1}{2}\left(F_{0i} - \frac{\delta I_{cl}}{\delta A_i}\right)^2 - \bar{c}\left(D_0 c - D_i c \frac{\delta v}{\delta A_i}\right) + J_i(x, t)A_i(x, t)\right). \tag{10}$$

To exponentiate the superdeterminant in Eq. (10), we introduce G-valued anticommuting ghosts Ψ_i and $\bar{\Psi}_i$ and G-valued commuting ghosts for ghosts Φ and $\bar{\Phi}$. One has

$$\det \frac{\delta(b_i, \gamma)}{\delta(A_i, c)} = \int [D\Psi_i][D\bar{\Psi}_i][D\Phi][D\bar{\Phi}] \exp \int dt\, dx\, \mathrm{Tr}\,(\bar{\Psi}_i, \bar{\Phi}) \times \begin{pmatrix} D_0 \delta_{ij} - \dfrac{\delta^2 I_{cl}}{\delta A_i \delta A_j} - D_i \dfrac{\delta v}{\delta A_j} & 0 \\ -\dfrac{\delta^2 v}{\delta A_j \delta A_k} D_k c^a + \left[\dfrac{\delta v}{\delta A_j}, c\right] & D_0 - \dfrac{\delta v}{\delta A_k} D_k \end{pmatrix} \begin{pmatrix} \Psi_j \\ \Phi \end{pmatrix}. \tag{11}$$

We can therefore express the partition function as follows:

$$Z[J] = \int [DA_i][D\Psi_i][D\bar{\Psi}_i][Db_i][Dc][D\bar{c}][D\Phi][D\bar{\Phi}] \times \exp\left(I_{GF} + \int dt\, dx \operatorname{Tr} J_i A_i \right),$$

with

$$\begin{aligned} I_{GF} = \int dt\, dx \operatorname{Tr} \bigg(& \tfrac{1}{2} b_i^2 + b_i \left(F_{0i} - \frac{\delta I_{cl}}{\delta A_i} \right) - \bar{c} \left(D_0 c - D_i c \frac{\delta v}{\delta A_i} \right) \\ & - \bar{\Psi}_i \left(D_0 \delta_{ij} - D_i \frac{\delta v}{\delta A_j} - \frac{\delta^2 I_{cl}}{\delta A_i \delta A_j} \right) \Psi_j - \bar{\Phi} \left(D_0 - \frac{\delta v}{\delta A_k} D_k \right) \Phi \\ & - \bar{\Phi} \frac{\delta^2 v}{\delta A_j \delta A_k} D_k c \Psi_j + \bar{\Phi} \left[\left[\frac{\delta v}{\delta A_j}, c \right], \Psi_j \right] \bigg). \end{aligned} \tag{12}$$

[We have reintroduced Lagrange multiplier fields b in Eq. (12).] A nontrivial feature of our action, Eq. (12), is the trilinearity of the last term in the ghosts c, Ψ, $\bar{\Phi}$.

We can verify that I_{GF} is invariant under the action of the graded differential operator s_{top} defined as follows:

$$\begin{aligned} s_{\mathrm{top}} A_i &= \Psi_i \\ s_{\mathrm{top}} c &= \Phi \\ s_{\mathrm{top}} \Psi_i &= 0 \\ s_{\mathrm{top}} \Phi &= 0 \end{aligned} \tag{13a}$$

$$\begin{aligned} s_{\mathrm{top}} \bar{\Psi}_i &= b_i \qquad s_{\mathrm{top}} b_i = 0 \\ s_{\mathrm{top}} \bar{\Phi} &= \bar{c} \qquad s_{\mathrm{top}} \bar{c} = 0. \end{aligned} \tag{13b}$$

(All gradings are summarized by attributing ghost numbers to all fields: 0 for A_i and b_i; 1 for c and Ψ_i; -1 for $\bar{c}$ and $\bar{\Psi}_i$; 2 for Φ; and -2 for $\bar{\Phi}$. Moreover, the gradings of all forms and operators are defined as the sum modulo 2 of the ghost numbers and form degrees.)

One has $s_{\mathrm{top}}^2 = 0$. The s_{top}-invariance of the action is obvious, since one can write I_{GF} as an s_{top}-exact term:

$$I_{\mathrm{GF}} = \int dt\, dx\, s_{\mathrm{top}} \left(\bar{\Psi}_i \left(F_{0i} - \frac{\delta I_{cl}}{\delta A_i} + \tfrac{1}{2} b_i \right) - \bar{\Phi} \left(D_0 c - \frac{\delta v}{\delta A_i} D_i c \right) \right). \tag{14}$$

The last equation shows that $F_{0i} - \delta I_{cl}/\delta A_i$ and $D_0 c - D_i c \delta v / \delta A_i$ can be interpreted as gauge functions: functional integration is concentrated around the domains where these functions vanish.

If one eliminates the fields b_i by their algebraic equation of motion, the action of s_{top} on $\bar{\Psi}$, $s_{top}\bar{\Psi}_i = b_i$, is changed into $s_{top}\bar{\Psi}_i = F_{0i} - \delta I_{cl}/\delta A_i$. If one further Legendre-transforms I_{GF} into a Hamiltonian, one gets a supersymmetric Hamiltonian $H_{GF} = \frac{1}{2}[Q, \bar{Q}]$, where Q is the charge associated with s_{top}, with $Q^2 = 0$, and $\bar{Q}$ is the adjoint of Q.

One can identify s_{top} with the topological BRST Yang-Mills operator constructed in Ref. 12: if one performs the change of field-variables $\Psi_i \to \Psi_i + D_i c$, $\Phi \to \Phi - \frac{1}{2}[c, c]$, one gets, indeed,

$$\begin{aligned} s_{top}A_i &= \Psi_i + D_i c \\ s_{top}c &= \Phi - \tfrac{1}{2}[c, c] \\ s_{top}\Psi_i &= D_i\Phi - [c, \Psi_i] \\ s_{top}\Phi &= -[c, \Phi]. \end{aligned} \tag{15}$$

Under the form of Eq. (15), one sees that the stochastic supersymmetry is combined with the ordinary BRST symmetry. This phenomenon is made possible by the existence of the ghost of ghost Φ, a 0-form with ghost number 2. The geometrical interpretation follows from the possibility of expressing Eq. (15) as a generalization of Eq. (1b):[12]

$$(d + s)(A + c) + \tfrac{1}{2}[A + c, A + c] = F + \Psi_i\, dx^i + \Phi. \tag{16}$$

If we consider the ordinary Yang-Mills action, $I_{cl} = \int dx F_{ij}^2$, the case of interest is $v = \partial_i A^i$ (Zwanziger gauge). The action becomes

$$\begin{aligned} I_{GF} = \int dt\, dx\, \mathrm{Tr}\, (&\tfrac{1}{2}(\partial_0 A_i - D_j\partial^j A_i - [F_{ij}, A_j])^2 - \bar{c}(\partial_0 c - D_i\partial_i c) \\ &- \bar{\Psi}^i((\partial_0 - D_j D^j)\Psi_i - 2[F_{ij}, \Psi_j] + D_i[A^j, \Psi_j] \\ &- \bar{\Phi}((\partial_0 + D^j\partial_j)\Phi - [\Psi^i, \partial_i c]). \end{aligned} \tag{17}$$

We see that the theory is based on a parabolic differential operator of the type $\partial/\partial t - (\partial/\partial x^i)^2$. For a space dimension smaller than or equal to four, one has renormalizability by power counting, and the stability of the theory is due to the symmetry of Eq. (17), $s_{top}I_{GF} = 0$, with

$$\begin{aligned} s_{top}A_i &= \Psi_i \\ s_{top}c &= \Phi \\ s_{top}\Psi_i &= 0 \\ s_{top}\Phi &= 0 \end{aligned} \tag{18a}$$

$$\begin{aligned} s_{top}\bar{\Psi}_i &= \partial_0 A_i - D_j\partial^j A_i - [F_{ij}, A^j] \\ s_{top}\bar{\Phi} &= \bar{c} \qquad s_{top}\bar{c} = 0. \end{aligned} \tag{18b}$$

4. THE CASE OF DIFFEOMORPHISM INVARIANCE

The approach followed to obtain the Langevin equations, Eqs. (5), can be applied to other gauge invariances.[5] If one considers, for instance, diffeomorphism invariance, Eqs. (15) are replaced by the following ones:

$$\begin{aligned} s_{\rm top} g_{ij} &= \Psi_{ij} + g_{ik}\partial_j \xi^k + g_{jk}\partial_i \xi^k + \xi^k \partial_k g_{ij} \\ s_{\rm top}\xi^i &= \Phi^i + \xi^j \partial_j \xi^i \\ s_{\rm top}\Psi_{ij} &= g_{ik}\partial_j \Phi^k + g_{jk}\partial_i \Phi^k + \Phi^k \partial_k g_{ij} + \Psi_{ik}\partial_j \xi^k + \Psi_{jk}\partial_i \xi^k + \xi^k \partial_k \Psi_{ij} \\ s_{\rm top}\Phi &= \Phi^j \partial_j \xi^i - \xi^j \partial_j \Phi^i. \end{aligned} \tag{19}$$

In this case the drift force v is a vector field v^i. Again, there is an interpretation in terms of a topological gauge symmetry. If we consider the case of 2D gravity, one should take as conformally invariant variables the Beltrami differentials, so that $g_{ij}\, dx^i\, dx^j = \exp(\phi)(dz + \mu_{\bar z}^{z}\, d\bar z)(d\bar z + \mu_{z}^{\bar z}\, dz)$. The stochastic equations are separated into holomorphic and antiholomorphic sectors. The holomorphic sector is

$$\begin{aligned} s_{\rm top}\mu_{\bar z}^{z} &= \Psi_{\bar z}^{z} + \partial_{\bar z} c^z + c^z \partial_z \mu_{\bar z}^{z} - \mu_{\bar z}^{z}\partial_z c^z \\ s_{\rm top} c^z &= \Phi^z + c^z \partial_z c^z \\ s_{\rm top}\Psi_{\bar z}^{z} &= \partial_{\bar z}\Phi^z + \Phi^z \partial_z \mu_{\bar z}^{z} - \mu_{\bar z}^{z}\partial_z \Phi^z + c^z \partial_z \Psi_{\bar z}^{z} - \Psi_{\bar z}^{z}\partial_z c^z \\ s_{\rm top}\Phi^z &= \Phi^z \partial_z c^z - c^z \partial_z \Phi^z. \end{aligned} \tag{20}$$

It is convenient to rename the analog of v as μ_0^z. The Langevin equations, Eqs. (5), become

$$\begin{aligned} \partial_0 \mu_{\bar z}^{z} &= T^{zz} + \partial_{\bar z}\mu_0^z + \mu_0^z \partial_z \mu_{\bar z}^{z} - \mu_{\bar z}^{z}\partial_z \mu_0^z + b_{0\bar z}^{z} \\ \partial_0 c^z &= s\mu_0^z + c^z \partial_z \mu_0^z - \mu_0^z \partial_z c^z - \Psi_0^z, \end{aligned} \tag{21}$$

where T is the energy momentum tensor. A most interesting possibility is that of expressing all relevant equations for the stochastic quantization of a worldsheet as follows:

$$\begin{aligned} (d + s_{\rm top})\tilde A + \tfrac12[\tilde A, \tilde A] &= \tilde B \\ (d + s_{\rm top})\tilde B + [\tilde A, \tilde B] &= 0. \end{aligned} \tag{22}$$

We have defined

$$\begin{aligned} d &= dt\, \partial_0 + dz\, \partial_z + d\bar z\, \partial_{\bar z} \\ \tilde A &= (dz + d\bar z\, \mu_{\bar z}^{z} + dt\, \mu_0^z + c^z)\partial_z \\ \tilde B &= (dt\, d\bar z (T^{zz} + b_{0\bar z}^{z}) + d\bar z\, \Psi_{\bar z}^{z} + dt\, \Psi_0^z + \Phi^z)\partial_z. \end{aligned} \tag{23}$$

The analogy between Eqs. (15) and (20–23) is quite striking and will be used in a separate publication, where the link between 2D gravity and 3D Chern–Simons theory for the groups $SU(2) \times SU(2)$, $ISO(3)$, or $SL(2, C)$, depending on whether the worldsheet genus is 0, 1, or greater than 1, is examined.

5. THE CASE OF FIRST-ORDER SYSTEMS

First-order actions, i.e., actions only linear in the velocities, have vanishing Hamiltonians. Their quadratic approximations are not definitely positive, and therefore their quantization through ordinary Feynman path integral formalism is conceptually difficult to understand. In Ref. 10, in the case of the three-dimensional pure Chern–Simons action, stochastic quantization has been shown to get around this difficulty by giving a four-dimensional supersymmetric action whose bosonic part is of the ordinary Yang–Mills type, and thus second-order.

To show the generality of this transmutation of a first-order action into a second-order one, we present here another case, that of the two-dimensional Chern–Simon action

$$I_{cl} = \int_{M_2} \mathrm{Tr}\,(F\phi). \tag{24}$$

This action has a physical interest since, for particular choices of the gauge group, there are arguments for its relation to 2D gravity. $F = dA + AA$ is the curvature of a connection $A = A_z\, dz + A_{\bar{z}}\, d\bar{z}$. ϕ is a scalar field, valued in the same fundamental representation as A.

I_{cl} is first order, and the Hamiltonian vanishes modulo the classical constraint on A. The equations of motion are

$$\begin{aligned} \frac{\delta I_{cl}}{\delta A_i} &= D_i\phi = 0 \\ \frac{\delta I_{cl}}{\delta \phi} &= \varepsilon^{ij}F_{ij} = 0, \text{ where} \end{aligned} \tag{25}$$

i, j refer to the indices in M_2. The Langevin equations which describe stochastic quantization for the action [Eq. (24)] are therefore

$$\begin{aligned} F_{0i} &= D_i\phi + b_i \\ D_0\phi &= \varepsilon^{ij}F_{ij} + b. \end{aligned} \tag{26}$$

As in the previous sections, the index numbers zero refer to stochastic time, and we have defined $F_{0i} = \partial_0 A_i - \partial_i A_0 + [A_0, A_i]$ and $D_0 = \partial_0 + [A_0, \quad]$. (We have renamed the arbitrary function v as A_0). Since gauge-invariant quantities do not depend on the choice of $v = A_0$, we can functionally integrate over all possible choices of A_0, provided we define a stochastic evolution for A_0, for instance

$$\partial_0 A_0 = \partial_i A_i + b_0, \tag{27}$$

where b_0 is a Gaussian noise for A_0.

If we write the stochastic partition $\int [db] \exp - \int_{M_3} d^2z\, dt(\frac{1}{2}b_i^2 + \frac{1}{2}b_0^2)$ by expressing the noises in functions of the fields as in the previous sections (see Ref. 10 for the details in the case of the three-dimensional Chern–Simon action), we end up with a supersymmetric functional integral representation of the Langevin equations, Eqs. (26–27), defined from the following action:

$$\begin{aligned} I_{\mathrm{GF}} &= \int_{M_3} d^2z\, dt((F_{0i} - D_i\phi)^2 + (D_0\phi - \varepsilon^{ij}F_{ij})^2 + (\partial_0 A_0 - \partial_i A_i)^2 \\ &\quad + \text{supersymmetric terms}) \\ &= \int_{M_3} d^2z\, dt((F_{\alpha\beta}^2 + (D_\alpha\phi)^2 - 2\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}D_\gamma\phi + (\partial_\alpha A^\alpha)^2 \\ &\quad + \text{supersymmetric terms}). \end{aligned} \tag{28}$$

The greek indices α, β, γ, ... stand for three-dimensional indices for $M_2 \times S$, where S is the one-dimensional manifold in which the stochastic time runs. (The term $\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}D_\gamma\phi$ is a pure derivative and can be omitted.)

The right-hand side of Eq. (28) shows that the field A_0 can be truly interpreted as a gauge-field component along the stochastic direction. This interpretation of A_0 was already quite clear from Eq. (25).*

Equation (28) shows also that stochastic quantization provides us with an action that is second order: its quadratic field approximation is based on the elliptic operator $\sum_{i=1}^{3} (\partial/\partial x^i)^2$. The method which has yielded the second-order action [Eq. (28)] is quite general. Presumably, it can be used for any theory with a classical first-order action. Notice that gauge invariance (under a BRST form) has been maintained for the supersymmetric stochastic action by summing over all possibilities on the freedom of the Langevin equation.

As far as the specific example above is concerned, it is interesting to observe that the action [Eq. (28)] is the same as the one constructed in Ref. 15 for defining a quantum field theory from the magnetic monopole topological charge $\int_{M_3} \mathrm{Tr}\, FD\phi$. Moreover, by a trivial dimensional reduction, the latter is itself linked to the quantum field theory associated with the four-dimensional topological invariant $\int_{M_4} \mathrm{Tr}\, FF$ (the scalar field ϕ can be seen

* The idea of interpreting v as an additional gauge-field component first appeared in the work of Chan and Halpern,[14] in a different context.

as the fourth component of a connection over M_4). We have thus an example of a bidimensional quantum theory that has a deep relationship with a theory in four dimensions. What achieves stochastic quantization is a jump of one dimension, which could be called a generalization of Stokes' theorem at the quantum level.

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Chapter 4

Covariant Superstrings

Lars Brink

1. INTRODUCTION

Theoreticians have long striven to realize supersymmetry covariantly. They usually do so by introducing the concept of a superspace, a space with both bosonic and fermionic coordinates. The concept of a superspace led fairly quickly to new insights into supersymmetric field theories, such as proofs of nonrenormalization theorems.[1] There are, however, limits to what can be done. For theories with higher supersymmetry, or equivalently for theories in dimensions higher than six, no useful superspace formulation could be found—useful in the sense that it could be used for the quantum theory. One way out here was to relax the covariance by going to a light-cone frame gauge. This method proved very useful in proving the perturbative finiteness of the $N = 4$ Yang-Mills theory.[2] The method is, however, somewhat limited in that the Poincaré invariance is nonlinearly realized and the

LARS BRINK • Chalmers University of Technology, Institute of Theoretical Physics, S-412 96 Göteborg, Sweden.

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theories treated are in a specific gauge. The program of covariant supersymmetry in point-particle field theory is still unfinished.

It was also realized very early on in string theory that the introduction of a superspace is useful. In the early days we concentrated on the supersymmetry in the world sheet on which the Ramond-Neveu-Schwarz model was based.[3] The super-world sheet[4] proved to be a useful concept, which led to superconformal theories, supermoduli spaces, and the like. We might, however, also wonder if the space-time supersymmetry that the superstring carries can be realized covariantly. This has turned out to be a more complicated problem, the problem I will address in this talk.

Before going into the details, let me pause and ask whether or not this problem is important. Honestly, I do not know! We might argue that the real physics in string theory resides in the world sheet and the interpretation of certain fields over the world sheet as coordinates of a (super-) space-time should be dynamic. However, progress in physics often comes by investigating different routes. For the free superstring there is a covariantly supersymmetric action. The problem with it has been to quantize it covariantly. The hope here is that a solution of the problem will lead to new insight.

2. THE MINIMAL COVARIANT ACTION

To describe a string theory with space-time supersymmetry, the natural coordinates are $x^\mu(\sigma, \tau)$ and $\theta^\alpha(\sigma, \tau)$, where x^μ and θ^α are vectors and spinors, respectively, under $SO(1, d-1)$, with d being the dimension of space-time, which we take to be ten. The momentum density

$$\pi^\mu_\alpha = \partial_\alpha x^\mu - i\bar{\theta}\gamma^\mu\partial_\alpha\theta \qquad (\alpha = (\tau, \sigma)) \tag{2.1}$$

is the natural supersymmetrically invariant extension of the momentum density used for bosonic strings. To construct an action, the natural thing is to insert π^μ_α instead of $\partial_\alpha x^\mu$ in the action for the bosonic string. This is, however, not enough. To obtain a local fermionic symmetry that can eliminate unphysical fermionic degrees of freedom, Green and Schwarz added an extra (invariant) term, a Wess-Zumino-Novikov-Witten term in the language of σ-models and suggested the action[5]

$$S = -\tfrac{1}{2}\int d\tau\, d\sigma[\sqrt{-g}g^{\alpha\beta}\pi_\alpha \cdot \pi_\beta - 2i\varepsilon^{\alpha\beta}\partial_\alpha x^\mu\bar{\theta}\gamma_\mu\partial_\beta\theta]. \tag{2.2}$$

To understand the problems of this action it is easier and equally informative to study the point-particle limit[6]

$$S_p = -\tfrac{1}{2}\int d\tau\, e^{-1}\pi^\mu\pi_\mu, \tag{2.3}$$

where

$$\pi^\mu = \dot{x}^\mu - i\bar{\theta}\gamma^\mu\dot{\theta} \tag{2.4}$$

and e is the Einbein $\sim \sqrt{g_{\tau\tau}}$.

In a Hamiltonian formalism one gets the following primary constraints:

$$p_e = 0 \tag{2.5}$$

$$\bar{\chi} = \bar{p}_\theta + i\bar{\theta}\not{p} = 0, \tag{2.6}$$

where

$$p^\mu = -\frac{1}{e}\pi^\mu.$$

The secondary constraint is

$$p^2 = 0. \tag{2.7}$$

In order to check whether the constraints correspond to gauge symmetries of the action, we must check whether the constraint algebra, obtained by using the canonical Poisson brackets, closes. The critical bracket in the algebra turns out to be

$$\{\bar{\chi}_\alpha, \bar{\chi}_\beta\} = 2i(\gamma_0\not{p})_{\alpha\beta}, \tag{2.8}$$

where the RHS clearly is not a constraint. The rank of this 16×16 matrix is 8 because of the constraint, Eq. (2.7), and this fact shows that eight of the 16 constraints $\bar{\chi}$ are not gauge constraints, i.e., they are second class constraints, in Dirac's terminology, and should be eliminated.[7] However, there is no covariant way of dividing χ into two eight-component spinors. It is true that $\not{p}\chi$ is effectively an eight-component spinor because of Eq. (2.7), but there is no other vector satisfying, Eq. (2.7) in the theory that can be used to project out the other eight-component spinor. This is the root of the problem. If we allow ourselves to break covariance, there is no problem, and we can easily quantize the system in the light-cone gauge where only $SO(8)$ covariance is maintained, since $16 = 8_s + 8_c$ under the decomposition $SO(1.9) \to SO(8)$. (The representations 8_s and 8_c are the two eight-dimensional spinor representations of $SO(8)$.)

Various methods have been developed to treat systems with second-class constraints in the BRST treatment. In the case above one can start by constructing a BRST charge Q_B including all 16 constraints, hence introducing 16 ghost coordinates of bosonic type. This is an overcounting and must be compensated for by a new set of 16 ghosts for ghosts, which in turn must be compensated for by 16 ghosts for ghosts for ghosts, and this procedure goes on *ad infinitum*. There is, in fact, a problem even with the set of constraints $p^2 = 0$ and $\not{p}\chi = 0$, since they are not independent of each other. A proper BRST formulation needs an infinity of ghosts. This fact shows up in a Lagrangian formulation by the effect that the gauge symmetry related to constraint Eq. (2.6) (the κ-symmetry) does not close off-shell. Here one can also show that this fact leads to an infinity of ghosts.[8]

We could, in fact, have been suspicious from the beginning. From supersymmetry we conclude that P^0 is a positive operator in the quantum case, i.e., there are no negative energy states. However, a superstring clearly contains spinning states that for covariant descriptions need negative energy states. To rigorously deduce the statement that $P^0 > 0$ requires a proper time-gauge quantization in a positive definite Hilbert space, and this has not been done. However, we can certainly trust the result above and use the above reasoning as a reference for attempts to quantize covariantly.[9] It should be mentioned that the above obstructions do not apply in the light-cone gauge, since all states have positive energy in light-cone variables.

Perhaps most important here is the theorem of Jordan and Mukunda,[10] which states that no covariant commuting position operators for spinning particles can be defined on a Hilbert space spanned only by positive energy states. In quantizations of systems with only first-class constraints, the position operators certainly commute and the theorem above signals second-class constraints. In fact, by quantizing (2.3) in the light-cone gauge using the Dirac procedure, we arrive at

$$[x^\mu, x^\nu] = \frac{-p_\rho}{2p^2}\,\bar{\theta}\gamma^{\mu\nu\rho}\theta. \tag{2.9}$$

Since covariance is already broken, we can define a new position operator[6]

$$q^\mu = x^\mu + \frac{ip_\nu}{2p^+}\,\bar{\theta}\gamma^{\mu\nu+}\theta, \tag{2.10}$$

which does commute with itself and is canonically conjugate to p^μ. If we are to use fields which are functions of positions, we certainly need commuting position operators, and the reasoning above must be kept in mind.

Various methods have been devised to overcome the problems described above. In one attempt another set of constraints, all first class, is suggested.[11] The problem here is that the constraints are not independent of each other and will need an infinite set of ghosts in a BRST approach. In other attempts,[12] new coordinates are introduced, making it possible to eliminate the second-class constraints. The problem here is that these methods, although covariant, are in a one-to-one correspondence with the light-cone gauge approach. I will not describe these methods here but instead turn to a rather different approach, which I recently devised, to the covariant quantization of the superstring.

3. COVARIANT SUPERSTRINGS FROM THE LIGHT-CONE GAUGE

In this talk I will address the problem in a somewhat novel way. I will start from the light-cone gauge, which we know works, and work backwards to a covariant conformal gauge. This approach[13] has been triggered by some real progress achieved recently when a remarkable new formulation of the quantum action (bilinear in coordinates and ghosts) for a covariant superstring in a conformal gauge was derived.[8] The method uses a generalized BRST formulation of Batalin and Vilkovisky,[14] and the problems described in section 2 are seemingly avoided. The reason for this is unclear for the moment, and it is advantageous to reach this result from some other method. This is the main purpose of my approach. The procedure here hinges to a great extent on the results of Ref. 8 but has the virtue of showing fairly simply some of the miracles which have been found in the formalism used so far.

To formulate my method we start with the functional integral describing the free bosonic string

$$Z = \int Dg_{\alpha\beta}(\sigma) Dx^{\mu}(\sigma) \exp\left[\frac{1}{2\pi}\int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu\right]. \qquad (3.1)$$

We work in an Euclidean space to better define the integral but will use Minkowski notation in the world sheet. (We here follow the conventions of Ref. 15.) We also put the tension $T = 1/\pi$ for convenience.

Equation (3.1) can be regarded as the vacuum-to-vacuum amplitude. As such it is a pure number. An alternative meaning of Eq. (3.1) occurs if we put boundary conditions on the integral. Then Eq. (3.1) can be thought of as the propagator or the wave functional. However, introducing boundary conditions will break the gauge invariance and, for our arguments it will be necessary to consider Eq. (3.1) as the vacuum-to-vacuum amplitude.

It is now standard to perform a gauge fixing $g_{\alpha\beta} = \eta_{\alpha\beta}e^{\phi}$ and the ensuing Faddeev-Popov procedure of Eq. (3.1) to arrive at the expression

$$Z = \int D\phi(\sigma) Dx^\mu(\sigma) Db(\sigma) Dc(\sigma) D\bar{b}(\sigma) D\bar{c}(\sigma)$$

$$\times \exp\left[\frac{1}{2\pi}\int d^2\sigma(\eta^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x_\mu - 2c\partial_+ b - 2\bar{c}\partial_-\bar{b})\right]. \qquad (3.2)$$

This expression is conformally invariant in $d = 26$, which means that we can drop the integration over ϕ. The resulting expression is the vacuum-to-vacuum functional in the conformal gauge.

The introduction of the ghost coordinates is just a way to represent the $\det(\partial_+\partial_-)$ arising in the Faddeev-Popov procedure. The same determinant raised to the power -1 will arise if we perform an integration over two x-coordinates. This means that we can compensate the integral over the ghost coordinates by integrating out two x-coordinates. Such a procedure would in general lead to a remaining action which is not Lorentz invariant, and hence it will not be possible to use in order to introduce sources. However, there is one exception. If we integrate out x^0 and x^{d-1}, or equivalently x^+ and x^- in light-cone gauge terminology, we arrive at an action which in fact is still Lorentz invariant if we interpret the evolution parameter on the world sheet τ to be $\tau = x^+/p^+$.

Hence we have found that the vacuum-to-vacuum functional

$$Z = \int Dx^i(\sigma) \exp\left[\frac{1}{2\pi}\int d^2\sigma\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i\right], \tag{3.3}$$

with $\tau = x^+/p^+$, which has been obtained from the one in the conformal gauge by integrating out two bosonic and two fermionic coordinates, is as good a starting point as the one in the conformal gauge. This result has, in fact, been proven in the general case with sources and arbitrary topology by D'Hoker and Giddings.[16]

This argument, which can be seen as a proof of the light-cone gauge expression, can also be turned around. Suppose we are given the light-cone gauge functional; then we can try to covariantize it by introducing more functional integrals, which amounts to multiplication by the factor 1. The resulting expression should then be Lorentz invariant. Furthermore, we must demand that it be conformally invariant. If it is, one can undo the gauge fixing and return to a geometric action.

Before applying this technique to the superstring, let us consider the spinning string. In the light-cone gauge the vacuum-to-vacuum functional is

$$Z_s = \int Dx^i(\sigma) D\lambda^i_A(\sigma) \exp[-S], \tag{3.4}$$

with

$$S = -\frac{1}{2\pi}\int d^2\sigma[\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i + i\lambda^i_1\partial_+\lambda^i_1 + i\lambda^i_2\partial_-\lambda^i_2]. \tag{3.5}$$

We know how to covariantize the bosonic part from the example with the bosonic string. Similarly, we can covariantize the fermionic parts and compensate the new functional integrals with functional integrals over bosonic ghosts. We then end up with the following expression

$$Z_s = \int Dx^\mu(\sigma) D\lambda^\mu_A(\sigma) Db(\sigma) Dc(\sigma) D\bar{b}(\sigma) D\bar{c}(\sigma) \times D\beta(\sigma) D\gamma(\sigma) D\bar{\beta}(\sigma) D\bar{\gamma}(\sigma) \exp[-S], \tag{3.6}$$

with

$$S = -\frac{1}{2\pi}\int d^2\sigma[\eta^{\alpha\beta}\partial_\alpha x^\eta \partial_\beta x_\mu + i\lambda_1^\mu \partial_+ \lambda_{1\mu} + i\lambda_2^\mu \partial_- \lambda_{2\mu} - 2c\partial_+ b$$
$$- 2\bar{c}\partial_-\bar{b} - 2\gamma\partial_+\beta - 2\bar{\gamma}\partial_+\bar{\beta}]. \tag{3.7}$$

The final step now is to show that the action is conformally invariant. To check, we need only to see that the c-number is zero. Since we know the c-numbers from x, λ, and (b, c), we need

$$\tfrac{1}{12}(10 + 5 - 26 + c_{\beta\gamma}) = 0$$
$$c_{\beta\gamma} = 11. \tag{3.8}$$

For a pair like (β, γ), the c-number is

$$c_{\beta\gamma} = -1 + 3(2J - 1)^2, \tag{3.9}$$

where J is the conformal weight of either of the ghosts. (The expression is symmetric in J and $1 - J$.) Combining Eqs. (3.8) and (3.9) we find that

$$J = 3/2, \quad \text{or} \quad -1/2, \tag{3.10}$$

which, of course, is the result from covariant methods.

If the covariant geometric methods were not known, one could have used the result on the conformal weights to argue for what kind of local symmetry the geometric action should have. The c and γ ghosts correspond to local symmetries with parameters with conformal weight -1 and $-1/2$, respectively. It would be natural to guess at a reparametrization and a supersymmetry invariance.

These two examples are quite straightforward and not very illuminating. If, however, we turn to superstrings, the problem is more complex. Again we start with the light-cone gauge functional (for type IIb):

$$Z_{ss} = \int Dx^i(\sigma) DS_A^a(\sigma)$$
$$\times \exp\left[\frac{1}{2\pi}\int d^2\sigma(\eta^{\alpha\beta}\partial_\alpha x^i \partial_\beta x^i + iS_1^a\partial_+ S_1^a + iS_2^a \partial_- S_2^a)\right], \tag{3.11}$$

where the index $a = 1, \ldots, 8$ and denotes an $SO(8)$ spinor. As in the other cases, we want to multiply this integral by another integral which equals one, but which adds terms to the action such that the new action is Lorentz and conformally invariant.

The covariantization of the bosonic part is managed as in the previous examples by introducing the reparametrization ghosts. For the fermionic part new ideas are, however, necessary. There is no way that we can add another eight-component spinor to S_1 to make a Lorentz scalar bilinear expression in a 16-component spinor. This is because of the Weyl property of θ. In fact, the only scalar expression bilinear in spinors we can write is of the form

$$\bar{\pi}\partial_+\theta, \tag{3.12}$$

where we also have had to introduce a spinor π conjugate to θ.

Hence we find that in this case we cannot just covariantize by adding terms. Instead, we have to change variables in the functional integral. This, however, is simplified by the fact that we can compute the integrals over the fermionic coordinates. In fact,

$$\int DS_1^a(\sigma)\exp\left[\frac{1}{2\pi}\int d^2\sigma\, S_1^a\partial_+S_1^a\right] = (\det\partial_+)^4. \tag{3.13}$$

We must now look for a functional integral over a covariant action which leads to this result. If we use a term of the form above, we obtain

$$\int D\pi(\sigma)D\theta(\sigma)\exp\left[\frac{1}{\pi}\int d^2\sigma\bar{\pi}\partial_+\theta\right] = (\det\partial_+)^{\pm 16}, \tag{3.14}$$

with the plus sign if π, θ are fermionic and the minus sign if they are bosonic.

It is clear that a finite number of such integrals can give the determinant only to the power of a multiple of 16. The only way out is to use an infinite number. By organizing an infinite sum of fermionic and bosonic integrals such that the exponent of the determinant becomes

$$16(1-2+3-4+\cdots) = -\lim_{q\to 1}\sum_{n=1}^{\infty}16\frac{d}{dq}(-q)^n = \tfrac{1}{4}\cdot 16 = 4, \tag{3.15}$$

we can find an expression in terms of a covariant action which equals the integral in Eq. (3.13). Accepting the regularization in Eq. (3.15), we have found that

$$\int DS^a(\sigma)\exp\left[\frac{1}{2\pi}\int d^2\sigma\, S_1^a\partial_+S_1^a\right]$$
$$= \int\prod_{n=0}^{\infty}\prod_{i=0}^{n}D\pi_{ni}(\sigma)D\theta_{ni}(\sigma)\exp\left[\frac{i}{\pi}\int d^2\sigma\sum_{n=0}^{\infty}\sum_{i=0}^{n}\bar{\pi}_{ni}\partial_+\theta_{ni}\right], \tag{3.16}$$

where the Grassmann property is alternating in the sum over n. We have then ended up with a covariant action

$$Z_{ss} = \int Dx^\mu(\sigma)Db(\sigma)Dc(\sigma)D\bar{b}(\sigma)D\bar{c}(\sigma)\prod_{n=0}^{\infty}\prod_{i=0}^{n}D\pi_{Ani}(\sigma)D\theta_{Ani}(\sigma)$$
$$\times\exp\left[\frac{1}{2\pi}\int d^2\sigma\left\{\eta^{\alpha\beta}\partial_\alpha x^\mu\partial_\beta x_\mu - 2c\partial_+b - 2\bar{c}\partial_-\bar{b}\right.\right.$$
$$\left.\left.+2i\sum_{n=0}^{\infty}\sum_{i=0}^{n}(\bar{\pi}_{1ni}\partial_+\theta_{1ni}+\bar{\pi}_{2ni}\partial_-\theta_{2ni})\right\}\right]. \tag{3.17}$$

It remains to show that the conformal weights of the new spinorial coordinates can be chosen so that the action in Eq. (3.17) is indeed conformally invariant. From our point of view this choice must be guesswork. The weights of the θ_{00}, which should be the θ-coordinate of the Green–Schwarz action, and θ_{11}, which should be the ghost related to the κ-symmetry of that action, are known. The remaining ones we simply choose such that the c-number c_θ adds to the c-numbers from the bosonic part such that the sum is zero:

$$\begin{aligned} \tfrac{1}{12}(10 - 26 + c_\theta) &= 0, \\ c_\theta &= 16. \end{aligned} \tag{3.18}$$

If we choose the conformal weight for θ_{ni} to be $-i$, we can use Eq. (3.9) for the c-number from a bilinear expression $\bar{\pi}\partial_+\theta$. Then

$$\begin{aligned} c_\theta &= \lim_{q\to 1} 16 \sum_{n=0}^{\infty} \sum_{i=-n}^{0} (-1)^{n+1} q^n 2[6i^2 - 6i + 1] \\ &= \lim_{q\to 1} \sum_{n=0}^{\infty} (-1)^{n+1} 32(2n^3 + 6n^2 + 5n + 1) q^n \\ &= \lim_{q\to 1} 32\left[\frac{-12}{(1+q)^4} + \frac{12}{(1+q)^3} - \frac{1}{(1+q)^2}\right] = 16. \end{aligned} \tag{3.19}$$

We have hence found that the action [Eq. (3.17)] is indeed conformally invariant given the choice of conformal weights above. Admittedly, the choice would be difficult to make if the results of Ref. 8 had not been known, but not impossible. We must also keep in mind that there could be other choices which also solve Eq. (3.18).

The arguments above have been given for a trivial world-sheet topology. It is not clear from the arguments that one can reach a similar simplicity for nontrivial topologies.

The functional in Eq. (3.17) is the one derived in Ref. 8. We have now shown that, given that we accept the various regularizations performed and given that we have extended the concept of functional integrals to those with an infinity of functional integrations, the light-cone functional gives the same result as the covariant functional of Ref. 8. This is a useful result. In the covariant approaches one is not explicitly eliminating the second-class constraints. This should lead to a resulting spectrum with negative-norm states. However, the light-cone theory is explicitly built from just positive-norm states, so the results from the covariant method must somehow solve the problem with second-class constraints yet indeed contain only positive-norm states. It remains, though, to understand this problem by covariant methods. In the procedure above we have concentrated on covariantizing the Poincaré symmetry. We have not argued at all about the supersymmetry.

In fact, we have to accept whatever we get here. In Ref. 17 it has been shown that the concept of Poincaré invariance is indeed generalized to an $OSp(9, 1/4)$ symmetry, since we have an infinity of coordinates. This fact, which could lead to a deeper insight, is not very transparent in our approach. Note that there is a difference between the procedure for the spinning string and the superstring. In the former case new terms are just added to the action and the symmetries are not changed while in the latter case a change of variables is performed, possibly changing the symmetries. Let us also note here that our formalism bears resemblance to Siegel and Zwiebach's construction of covariant expressions in the Hamiltonian form, where they also add new coordinates to covariantize the light-cone expressions for the generators of the Poincaré group.[18]

The results in this paper show rather clearly the possibility of acquiring a covariant action. One should, of course, look for other solutions within the given framework.

We should, finally, address the question of why we are searching for covariant quantization methods. One reason is that it can be advantageous to have an alternative to the Ramond-Neveu-Schwarz formalism, which is covariantly supersymmetric. It could help, for example, in computing multiloop amplitudes. Possibly deeper is the question of how to find coordinates that can be used, for example, in the description of the very early universe. We know that the concept of space-time breaks down at Planck energies and also that there is a possible phase transition there. A proper understanding of the problem above can shed light on this fascinating problem.

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Chapter 5

Constraints on the Baryogenesis Scale from Neutrino Masses

W. Fischler, G. F. Giudice, R. G. Leigh, and S. Paban

1. INTRODUCTION

Over the past few years it has been realized[1] that at temperatures above $\sim m_W/\alpha_W$, transitions that violate baryon (B) and lepton (L) number occur rapidly since, due to the electroweak anomaly, these quantum numbers are not conserved. Indeed,

$$\partial_\mu j^B_\mu = \frac{g_2^2}{32\pi^2} n_f (\tfrac{1}{3} \times 3) F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a$$

$$\partial_\mu j^L_\mu = \frac{g_2^2}{32\pi^2} n_f F^a_{\mu\nu} \tilde{F}^{\mu\nu}_a$$

$$\partial_\mu j^{B-L}_\mu = 0.$$

One can understand how the baryon number and lepton number can be changed at high temperature by looking at the structure of the weak gauge

W. Fischler, G. F. Giudice, R. G. Leigh, and S. Paban • Department of Physics, University of Texas, Austin, Texas 78712.

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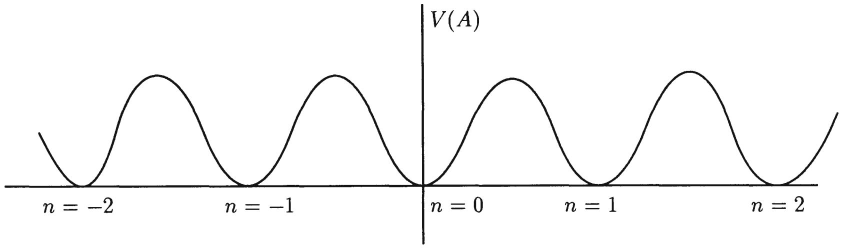

Figure 1. The vacua of the weak gauge group.

group vacuum (Fig. 1). There are an infinite number of vacua for $SU_L(2)$ labelled by an integer n:

$$A_i = g_n^{-1}\partial_i g_n,$$

where $g_n = e^{in\pi\bar{\tau}\hat{x}f(r)}$, $f(0) = 0$ and $f(\infty) = 1$, and

$$n = \int d^3x\, K_0 = \int d^3x\, \varepsilon_{ijk}(A_i^a F_{jk}^a + \tfrac{2}{3} f_{abc} A_i^a A_j^b A_k^c).$$

The presence of fermion doublets lifts the degeneracy of these vacua (Fig. 2). As the temperature is raised above the W mass, field configurations responsible for transitions among the various vacua are generated in thermal equilibrium. The bias in the potential due to the presence of fermions guarantees that all over the universe the change in baryon and lepton numbers has the same sign. These transitions are potentially important because they may erase any cosmic B-asymmetry, which, according to conventional ideas, is generated at some GUT energy scale. Within the standard model there is one linear combination, $B - L$, which is non-anomalous, and thus in order that a pre-existing B-asymmetry survive, a $B - L$ excess must have existed at very early times. It is in fact not difficult to generate such an excess in most GUTs. However, in this type of scenario, it is crucial that the $B - L$ asymmetry not be eliminated through some other mechanism, which might act in concert with the anomalous electroweak processes. Effectively, we require that no such interaction come into equilibrium after the $B - L$ asymmetry is produced.

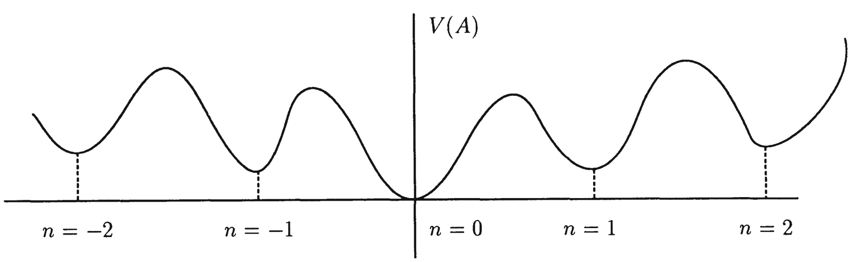

Figure 2. Vacua in Fig. 1 altered by fermion doublets.

The lowest-dimension operator that violates B or L in the standard model is the following $\Delta L = 2$ dimension-five operator[2]:

$$\frac{m_\nu}{v^2}(l_L H)^2. \tag{1}$$

Below the electroweak symmetry-breaking scale, $v \simeq 246\,\text{GeV} \equiv \sqrt{2}\langle H\rangle$; this operator generates a neutrino Majorana mass m_ν. The interaction seen in Eq. (1) also mediates such scattering processes as $l_L l_L \leftrightarrow H^* H^*$, $\bar{l}_L \bar{l}_L \leftrightarrow HH$, and $l_L H \leftrightarrow \bar{l}_L H^*$, all of which violate L.

Recently, two groups, Harvey and Turner and Barr and Nelson,[3] have made the interesting observation that the $B - L$ violation due to these scattering processes, together with the electroweak B-violating interactions, would have the effect of erasing any B and/or L pre-existing asymmetries* should they come into equilibrium. In particular, they find that the presently observed B-asymmetry imposes a bound on the neutrino mass:

$$m_\nu \lesssim \frac{4\,\text{eV}}{(T_{B-L}/10^{10}\,\text{GeV})^{1/2}}, \tag{2}$$

where T_{B-L} is the temperature at which the $B - L$ asymmetry is generated.

In this chapter we would like to elaborate further on the argument of Ref. 3. We consider explicit realizations of $B - L$ violation in various extensions of the standard model and derive the conditions under which a B-asymmetry generated at the GUT scale can survive today. In particular, in any given model, there are interactions involving heavy particles; below their mass scale, the operator in Eq. (1) is produced. At energies above their mass scale, one should consider interactions involving these particles, such as decays, which may lead to stronger bounds. Any experimental evidence for $B - L$ violation above the limits presented here should be regarded as a strong indication for some new mechanism of baryogenesis at low temperatures (see Refs. 5, 6, 7, and 8 for specific examples).

2. HEAVY NEUTRINO DECAYS

The most familiar way to generate the interaction in Eq. (1) is to introduce a gauge-singlet right-handed neutrino N with a Majorana mass M and a Yukawa interaction

$$\lambda \bar{N}(H l_L), \tag{3}$$

* Fukugita and Yanagida[4] have previously made a similar observation, but considered only the case of B-violating interactions just below the electroweak phase transition.

where λ is the coupling constant. This yields a neutrino mass

$$m_\nu = \lambda^2 \frac{v^2}{2M}. \tag{4}$$

Let us assume that $M < T_{B-L}$, the scale where the $B-L$ asymmetry is generated. Because of its Majorana nature, the particle N can decay into a lepton and a Higgs boson, thus violating the L-number and potentially erasing the pre-existing $B-L$ asymmetry. The relevant quantity is of course the ratio of the decay rate to the Hubble expansion at temperature T,

$$\frac{\Gamma_D}{H} \simeq \left(\frac{\lambda^2 M M_P}{16\pi g_*^{1/2}}\right) \frac{M}{T^2(M^2+T^2)^{1/2}}. \tag{5}$$

Here, g_* counts the effective number of degrees of freedom ($g_* \simeq 100$). In order that $B-L$ not be washed out, the right-handed N should start decaying at temperatures below their mass (where the back-reaction is suppressed and equilibrium cannot be established) or, equivalently, that $\Gamma_D \ll H$ for all $T > M$. Combining this with Eq. (4), we find a bound on the neutrino mass that is a function of only the Newton and Fermi constants:

$$m_\nu \lesssim 8\pi g_*^{1/2} \frac{G_N^{1/2}}{\sqrt{2}G_F} \approx 10^{-3}\,\text{eV} \equiv m_*. \tag{6}$$

As noticed in Ref. [3], the bound in Eq. (6) holds as long as there is mixing among generations. Thus there is a simple loophole to the limit: if some global $U(1)$ is exactly conserved up to an electroweak anomaly, then there are two linearly independent conserved nonanomalous currents which protect the baryon number as long as there is an excess of at least one linear combination of charges. A simple example is the situation in which one of the lepton numbers is separately conserved.

The condition on the decay rate in Eq. (5) in fact automatically implies that the scattering processes $\bar{l}_L \bar{l}_L \leftrightarrow HH$, etc., are out of equilibrium. The scattering rate in units of the Hubble constant is

$$\frac{\Gamma_S}{H} \simeq \frac{4\lambda^2}{\pi^2}\left(\frac{\lambda^2 M_P}{16\pi g_*^{1/2} M}\right)\frac{TM^3}{(T^2+M^2)^2}. \tag{7}$$

Using Eq. (5), we obtain $\Gamma_S/H \ll 1$, assuming a perturbative coupling constant λ. Therefore the bound on the neutrino mass from the decay process Eq. (6), is more stringent than the analog derived from the scattering rate Eq. (2).

Now, if a neutrino mass of $m_\nu \gtrsim m_*$ were discovered, we would conclude that the particle N was in thermal equilibrium when it became nonrelativistic. A cosmic asymmetry *could not* be produced by the N decay, and the conclusion that $m_\nu \gtrsim m_*$ erases $B-L$ is still valid. Our conclusion in this event, then, is that the present baryon asymmetry cannot be explained in terms of physics at some scale larger than M. That is,

$$m_\nu \gtrsim m_* \quad \Rightarrow \quad T_{B-L} \lesssim M. \tag{8}$$

We can write M in terms of other parameters:

$$M = \frac{m_D^2}{m_\nu} = \lambda^2 \frac{v^2}{2m_\nu} = \frac{\lambda^2}{2}\left(\frac{m_*}{m_\nu}\right) 10^{17}\,\text{GeV}. \tag{9}$$

The bound in Eq. (2) obtained in Ref. 3 can be rewritten in the form

$$\frac{T_{B-L}}{10^{17}\,\text{GeV}} \lesssim \left(\frac{m_*}{m_\nu}\right)^2, \tag{10}$$

and thus the new bound, $T_{B-L} \lesssim M$, is stronger for $(\lambda^2/2)(m_v/m_*) \lesssim 1$. The important point here is that for reasonable values of the coupling constant, the $B-L$ generation must take place at scales significantly below the GUT scale. For example, if we take a Dirac mass for the neutrino of $m_D = \lambda v/\sqrt{2} \sim m_{\text{charm}}$ and a neutrino mass of 1 eV, we find $T_{B-L} \lesssim 10^9$ GeV. As a second example, consider a Dirac mass for the neutrino $m_D \sim m_e$; then $T_{B-L} \lesssim 300$ GeV. If baryogenesis at the GUT scale is ruled out, then we must rely on some other mechanism, perhaps through the (first order) phase transition at the weak scale.[5]

3. CONCLUSIONS

In conclusion, any violation of $B-L$ through processes below the GUT scale is potentially a powerful constraint for the mechanism of baryogenesis. Observation of $B-L$ violation, most notably of a neutrino mass, would provide a strong indication that baryogenesis occurred at some low scale.

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Chapter 6

The Antifield-BRST Formalism for Gauge Theories

Marc Henneaux

1. INTRODUCTION

It has been recognized for some time now that the BRST method provides one of the most powerful tools for quantizing theories endowed with a local gauge freedom. This method is extremely useful not only in the path-integral approach, but also in the operator formalism.

A striking development in the last few years has been the emergence of many gauge-theoretical models for which the BRST method appears to be the only satisfactory (covariant) method of quantization. These models are characterized by the fact that the gauge transformations close only on-shell: if one computes the commutator of two infinitesimal gauge transformations, denoted by $\delta_\varepsilon \phi^i$ and $\delta_\eta \phi^i$, one finds a transformation of the same type, denoted by $\delta_{[\varepsilon,\eta]}\phi^i$, but modulo the equations of motion

$$[\delta_\varepsilon, \delta_\eta]\phi^i = \delta_{[\varepsilon,\eta]}\phi^i + \text{field equations.}$$

MARC HENNEAUX • Faculté des Sciences, Université Libre de Bruxelles, B-1050 Bruxelles, Belgium and Centro de Estudios Científicos de Santiago, Santiago 9, Chile.

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Models with "open gauge algebras" include supergravity theories,[1] the Green–Schwarz superstring,[2] and the superparticle,[3] among others.

If one determines the gauge-fixed action in covariant gauges by means of the standard Faddeev–Popov method,[4] designed for true gauge groups, one gets an incorrect (nonunitary and non-gauge independent) answer in the open algebra case. Hence, it is not enough to add to the gauge-invariant action the standard gauge-fixing and Faddeev–Popov determinant terms.

To determine the correct path integral, two approaches can be used, and both are characterized by the fact that they strongly rely on the BRST symmetry. The first is based on the Hamiltonian formalism.[5] The second starts from the Lagrangian formulation.[6–8] This second approach is slightly less general and appears to be formally less precise in what concerns the local measure in the path integral.* However, it has the definite advantage of preserving manifest covariance throughout and on this ground deserves to be studied.

The Hamiltonian construction of the BRST symmetry has been reviewed elsewhere.[9] We will therefore analyze here only the Lagrangian antifield formalism of Batalin and Vilkovisky.[6]

The main goals of these lectures are, first, to derive the correct gauge-fixed Lagrangians containing all the necessary ghost vertices and, second, to show explicitly how the derivation incorporates gauge invariance throughout. (We will in particular indicate why BRST invariance can be used as a substitute for gauge invariance.) Both purposes lead to interesting algebraic and geometric features, which we believe provide the key to the rationale behind the antifield formalism.

The gauge-fixed Lagrangians obtained by BRST methods generate the correct set of Feynman diagrams. As such, they provide the appropriate starting point for studying the perturbative quantum properties of the theory (renormalization, anomalies). There also, the BRST symmetry proves to be a crucial tool in the analysis. These issues, however, will not be addressed in the lectures.

2. STRUCTURE OF THE GAUGE SYMMETRIES

The structure of the gauge symmetries may appear to be somewhat puzzling in the "open algebra" case, as it may wrongly be felt that the group structure is completely lost. Our first task, therefore, is to clarify the structure of the gauge symmetries in the general case.

* It is not inconceivable that this shortcoming could be overcome some day by pure Lagrangian means, without having to resort to the Hamiltonian. Also, note that the validity of the Lagrangian path integral appears to restrict the form of the Lagrangian (see comments in section 8.6).

2.1. The Action Principle and the Equations of Motion

Our starting point is the action $S_0[\phi^i]$, which we assume to be a local functional of the fields:

$$S_0[\phi^i] = \int d^D x \, \mathscr{L}_0(\phi^i, \partial_\mu \phi^i, \partial_{\mu_1 \mu_2} \phi^i, \ldots, \partial_{\mu_1 \mu_2 \ldots \mu_k} \phi^i). \tag{1}$$

As will be seen, the subsequent structure is completely encoded in the action and is not an independent input.

The field equations are

$$\frac{\delta S_0}{\delta \phi^i(x)} \equiv \frac{\delta \mathscr{L}_0}{\delta \phi^i}(x) = 0, \tag{2a}$$

where the "variational derivatives" $\delta \mathscr{L}_0 / \delta \phi^i$ of the Lagrangian density $\mathscr{L}_0$ are defined by

$$\frac{\delta \mathscr{L}_0}{\delta \phi^i} \equiv \frac{\partial \mathscr{L}_0}{\partial \phi^i} - \partial_\mu \frac{\partial \mathscr{L}_0}{\partial(\partial_\mu \phi^i)} + \partial_\mu \partial_\nu \frac{\partial \mathscr{L}_0}{\partial(\partial_{\mu\nu} \phi^i)} - \cdots + (-1)^k \frac{\partial \mathscr{L}_0}{\partial(\partial_{\mu_1 \ldots \mu_k} \phi^i)}. \tag{2b}$$

The derivatives $\delta S_0 / \delta \phi^i(x)$ are referred to as the "functional derivatives" of S_0.

For notational simplicity, we take the fields to be commuting. Our considerations can be extended straightforwardly to theories with fermions provided the appropriate phases are included.

2.2. Gauge Transformations

A gauge transformation is a transformation that can be prescribed independently at each space-time point and that leaves the action invariant up to a surface term. Thus gauge transformations are parametrized by arbitrary space-time functions (as opposed to rigid symmetry transformations) and typically take the form

$$\delta_\varepsilon \phi^i = \bar{R}^i_\alpha \varepsilon^\alpha + \bar{R}^{i\mu}_\alpha \partial_\mu \varepsilon^\alpha + \cdots + \bar{R}^{i\mu_1 \ldots \mu_s} \partial_{\mu_1 \ldots \mu_s} \varepsilon^\alpha. \tag{3}$$

Here, the coefficients $\bar{R}^i$, $\bar{R}^{i\mu}, \ldots, \bar{R}^{i\mu_1 \ldots \mu_s}$ depend on the ϕ^i and their derivatives up to a finite order, and $\varepsilon^\alpha(x)$ are arbitrary gauge parameters. Invariance of the action under Eq. (3) means that for any choice of $\varepsilon^\alpha(x)$, one has

$$\delta_\varepsilon \mathscr{L}_0 = \partial_\mu K^\mu_\varepsilon \tag{4}$$

for some local functions $K^\mu_\varepsilon(\phi^i, \partial_\nu \phi^i, \partial_{\nu_1 \ldots \nu_t} \phi^i, \varepsilon^\alpha, \ldots, \partial_{\nu_1 \ldots \nu_m} \varepsilon^\alpha)$.

2.3. Noether Identities

It is at this point convenient to adopt De Witt's condensed notation, where the indices i, α also include x (i.e., $i \leftrightarrow (i, x)$, $\alpha \leftrightarrow (\alpha, x)$) and a summation over i, α implies an integration over x.[10] In this notation, Eq. (3) becomes

$$\delta_\varepsilon \phi^i = R^i_{\ \alpha} \varepsilon^\alpha \quad \left[\Leftrightarrow \quad \delta_\varepsilon \phi^i(x) = \int d^D y R^i_\alpha(x, y) \varepsilon^\alpha(y)\right], \tag{5a}$$

with

$$R^i_\alpha(x, y) = \bar{R}^i_\alpha(x)\delta(x-y) + \bar{R}^{i\mu}_\alpha(x)\delta_{,\mu}(x-y) + \cdots. \tag{5b}$$

The Noether identities on the field equations are derived by starting from the invariance of the action:

$$\delta S_0 = \frac{\delta S_0}{\delta \phi^i} \delta_\varepsilon \phi^i = \frac{\delta S_0}{\delta \phi^i} R^i_{\ \alpha} \varepsilon^\alpha = 0. \tag{6a}$$

Since Eq. (6a) holds for any function $\varepsilon^\alpha(y)$, one gets the local identities

$$\frac{\delta S_0}{\delta \phi^i} R^i_\alpha = 0 \quad \Leftrightarrow \quad \int \frac{\delta S_0}{\delta \phi^i(x)} R^i_\alpha(x, y)\, d^D x = 0$$

$$\Leftrightarrow \quad \left(\frac{\delta \mathcal{L}_0}{\delta \phi^i} \bar{R}^i_\alpha - \left(\frac{\delta \mathcal{L}_0}{\delta \phi^i} \bar{R}^{i\mu}_\alpha\right)_{,\mu} + \cdots\right)(y) = 0. \tag{6b}$$

[Strictly speaking, there could be a surface term in Eq. (6a). This term vanishes if the gauge parameters are zero outside some finite domain, as one can assume. The local identities at y inferred from Eq. (6a) are thus certainly valid under this assumption. However, as these identities are local, they do not depend on the behavior of ε^α away from y (actually, in this case, they do not depend on ε^α at all), so they are clearly valid without the restriction that ε^α should vanish outside some finite domain. This is a useful line of reasoning that is frequently followed in deriving local identities.]

One consequence of the Noether identities [Eq. (6b)] is that the field equations are not independent. This is of course all right, as the existence of a gauge symmetry implies the presence of arbitrary functions in the general solution of the equations of motion which must then underdetermine $\phi^i(x)$.

2.4. Gauge Group

For a given action functional, there are a certain number of gauge transformations. What is the structure of the set containing all the gauge transformations?

One thing that can be said without having to make any calculations is that the infinitesimal gauge transformations form a Lie algebra.* There is

* Accordingly, the finite gauge transformations formally form a Lie group (formally because the gauge Lie algebras are infinite-dimensional continuous).

no escape from that result because (invertible) transformations leaving something (here the action) invariant *always obey the group axioms.* The group of all gauge transformations is denoted by $\mathscr{G}$ in the sequel.

The unconvinced reader may easily check that if $\delta_\eta \phi^i$ such that

$$\delta_\eta \phi^i = S^i_{A} \eta^A$$

is another gauge transformation ($\delta_\eta S_0 = 0$), then both $\lambda \delta_\varepsilon \phi^i + \mu \delta_\eta \phi^i$ (λ, $\mu \in R$) and $[\delta_\varepsilon, \delta_\eta]\phi^i = \delta_\varepsilon(\delta_\eta \phi^i) - \delta_\eta(\delta_\varepsilon \phi^i)$ obey Eq. (6a). Furthermore, $[\lambda\delta_\varepsilon, \delta_\eta] = \lambda[\delta_\varepsilon, \delta_\eta]$, $\lambda \in R$. So one clearly has a Lie algebra.

What is then meant by "open gauge algebras"? In order to answer this question precisely, it is necessary to introduce some new concepts. The guiding principle of the following developments is to determine the minimum number of independent Noether identities.

2.5. Trivial Gauge Transformations

Consider the transformations

$$\delta_\mu \phi^i = \mu^{ij} \frac{\delta S_0}{\delta \phi^j} \tag{7a}$$

where μ^{ij} is an arbitrary antisymmetric function

$$\mu^{ij} = -\mu^{ji} \tag{7b}$$

Exercise. Write explicitly Eqs. (7a) and (7b) for

$$\mu^{ij}(x, y) = k_1^{ij}(x)\delta(x, y) + k_2^{ij\mu}(x)\delta_{,\mu}(x, y) + \cdots + k_s^{ij\mu_1\cdots\mu_{s-1}}(x)\delta_{,\mu_1\cdots\mu_{s-1}}(x, y).$$

The arbitrary functions $k_1, \ldots, k_s$ may involve the fields and their derivatives up to some finite order.

It is easy to see that Eq. (7) leaves the action S_0 invariant:

$$\delta S_0 = \frac{\delta S_0}{\delta \phi^j} \frac{\delta S_0}{\delta \phi^i} \mu^{ij} = 0. \tag{8}$$

This is because the product $(\delta S_0/\delta\phi^i)(\delta S_0/\delta\phi^j)$ is symmetric in i, j while μ^{ij} is antisymmetric. Thus Eq. (7) defines a gauge transformation.*

The commutator of any gauge transformation of the type of Eq. (7a) with an arbitrary gauge transformation is a transformation of the type of Eq. (7a). Indeed, if $\delta S_0 = 0$ for $\delta_t \phi^i = t^i$ with

$$\frac{\delta S_0}{\delta \phi^i} t^i = 0, \tag{9}$$

* Let us insist that one does not need to use the equations of motion to prove Eq. (8).

then one finds, using Eq. (9),

$$[\delta_\mu, \delta_t]\phi^i = \left(\frac{\delta t^i}{\delta\phi^k}\mu^{kj} - \frac{\delta t^j}{\delta\phi^k}\mu^{ki} - t^k\frac{\delta\mu^{ij}}{\delta\phi^k}\right)\frac{\delta S_0}{\delta\phi^j}, \tag{10}$$

which is of the form of Eq. (7).

We can thus conclude that the set of all gauge transformations shown in Eq. (7) form a normal (i.e., invariant) subgroup $\mathcal{N}$ of the full gauge group $\bar{\mathcal{G}}$.

How significant are the transformations of Eq. (7)? Are they really new symmetry transformations? It is easy to convince oneself that these transformations are of no physical significance because

1. They exist independently of what the action is; in other words, they do not restrict at all the form of the Lagrangian and, indeed, no nontrivial Noether identities are associated with them.
2. They thus imply no degeneracy of the action, and in the Hamiltonian formalism there is no corresponding constraint. Actually, the conserved charges associated with Eq. (7), when rewritten as phase space functions by using the equations of motion, if necessary, vanish identically.
3. The transformations in Eq. (7) vanish on-shell, i.e., do not map solutions of the equations of motion onto new, different solutions.
4. There is accordingly no need for a "gauge-fixing" of Eq. (7). This is fortunate, as it is impossible to gauge-fix Eq. (7), which exists for any action!

On these grounds, it is legitimate to disregard the transformations of Eq. (7). The relevant invariance group of the action is thus given by the factor group $\mathcal{G} \equiv \bar{\mathcal{G}}/\mathcal{N}$ of all gauge transformations modulo the transformations of Eq. (7), a concept that is mathematically well defined as the transformations of Eq. (7) form a normal subgroup.

For this reason, the transformations of Eq. (7) are usually not even mentioned in standard textbooks on mechanics or field theory. These transformations have never been a source of concern for theories without gauge invariance, even though they are already present there.

Before closing this section, we mention the following useful theorem:

Theorem. Under suitable regularity assumptions on the functions $\delta S_0/\delta\phi^i$, to be made precise below, any gauge transformation that vanishes on-shell can be written as in Eq. (7);

$$\delta\phi^i \approx 0 \quad \text{and} \quad \delta\phi^i\frac{\delta S_0}{\delta\phi^i} = 0 \qquad \Rightarrow \qquad \delta\phi^i = \varepsilon^{ij}\frac{\delta S_0}{\delta\phi^j}, \tag{11}$$

for some $\varepsilon^{ij} = -\varepsilon^{ji}$.

The theorem will be proved below (p. 95).

Exercises. 1. Consider the action $S_0[A_\mu] = \int d^3x\, \varepsilon^{\mu\nu e} F_{\mu\nu} A_e$ for pure Chern–Simons (Abelian) Yang–Mills theory in three dimensions.[11] This action is invariant under ordinary gauge transformations $\delta A_\mu = \partial_\mu \Lambda$ and diffeomorphisms, $\delta A_\mu = \xi^\rho\, \partial_\rho A_\mu + \partial_\mu \xi^\rho A_\rho$. Show that the diffeomorphisms differ from the ordinary gauge transformations by a trivial gauge transformation.

2. Consider the action $S[q, p] = \int (p\dot{q} - H)\, dt$. Show that it is invariant under $\delta q = \varepsilon(\dot{q} - \partial H/\partial p)$, $\delta p = \varepsilon(\dot{p} + \partial H/\partial q)$. Check that the algebra of these trivial gauge transformations is isomorphic with the algebra of diffeomorphisms in one dimension. Observe that when $H = 0$, these transformations actually reduce to standard diffeomorphisms along the world line.

2.6. Factorization of $\mathcal{N}$

If the gauge group $\bar{\mathcal{G}}$ is the semidirect product of $\mathcal{N}$ by $\mathcal{G}$, $\bar{\mathcal{G}} = \mathcal{N} x_\sigma \mathcal{G}$, i.e., if the quotient group $\mathcal{G} \equiv \bar{\mathcal{G}}/\mathcal{N}$ can be realized as a subgroup of $\bar{\mathcal{G}}$ complementary to $\mathcal{N}$, then it is easy to disregard the transformations of $\mathcal{G}$. One simply works with the gauge transformations of $\mathcal{G}$ (viewed as a subgroup of $\bar{\mathcal{G}}$) and forgets about $\mathcal{N}$. This is permissible, as the commutator of two gauge transformations of $\mathcal{G}$ is again in $\mathcal{G}$ and does not generate a trivial transformation.

However, it may turn out that $\bar{\mathcal{G}} \neq \mathcal{N} x_\sigma \mathcal{G}$. In other words, the gauge algebra may symbolically read

$$[\text{trivial, trivial}] = \text{trivial} \tag{12a}$$

$$[\text{trivial, nontrivial}] = \text{trivial} \tag{12b}$$

$$[\text{nontrivial, nontrivial}] = \text{nontrivial} + \text{trivial}, \tag{12c}$$

where the trivial part in Eq. (12c) does not identically vanish and cannot be removed by redefinition of the nontrivial transformations (compatible with locality, covariance etc. . . .).

In that case, one cannot forget about the trivial transformations as they are generated through the commutators [Eq. (12c)]. One must work with all the gauge transformations and build the formalism so that the addition of trivial transformations to any transformation is ultimately irrelevant.

2.7. Independent Noether Identities

The factorization of the trivial gauge transformations was motivated by the fact that they imply no Noether identity and hence lead by themselves to no independent degeneracy of the equations of motion and do not need any gauge-fixing condition.

This is not the end of the story, however. Indeed, the remaining gauge transformations do not all lead to independent Noether identities. This can be seen as follows:

Let $\delta_\varepsilon \phi^i$ be gauge transformations

$$\delta_\varepsilon \phi^i = R^i_{\ \alpha} \varepsilon^\alpha, \tag{13a}$$

leading to the Noether identities

$$\frac{\delta S_0}{\delta \phi^i} R^i_{\ \alpha} = 0. \tag{13b}$$

Consider next the transformations

$$\delta_\eta \phi^i = (R^i_{\ \beta} M^\beta_A) \eta^A, \tag{14a}$$

where M^β_A is some matrix *allowed to depend on the fields*, $M^\beta_A(\phi^i)$. These transformations also leave the action invariant. From the point of view of Lie algebra theory, the transformation $\delta_\eta \phi^i$ are linearly independent of the transformation $R^i_{\ \alpha}$ as one cannot write $\delta_\eta \phi^i$ as a combination of $R^i_{\ \alpha}$ with coefficients that belong to the ground field, i.e., that are real (or complex) numbers.

However, the Noether identities that follow from $\delta_\eta S_0 = 0$,

$$\frac{\delta S_0}{\delta \phi^i} R^i_{\ \beta} M^\beta_A = 0 \tag{14b}$$

are clearly not independent of the Noether identities in Eq. (13b), as Eq. (14b) is a consequence of Eq. (13b). Hence, there is no new information in $\delta_\eta \phi^i$.

To take the example of electromagnetism, the gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda, \tag{15}$$

with $\Lambda(x, A_\nu)$ a functional of A_ν, are independent from the Lie-algebraic point of view of the transformations $\delta A_\mu = \partial_\mu \Lambda$ with $\Lambda = \Lambda(x)$. However, if one completely freezes the gauge freedom associated with the second set of transformations (e.g., $\nabla \cdot \mathbf{A} = 0$, $A_0 = 0$), one automatically freezes the gauge freedom associated with the first set. So, there is nothing new in Eq. (15).

2.8. Generating Sets

This leads one to the concept of generating set. A set G of gauge transformations

$$\delta_\varepsilon \phi^i = R^i_{\ \alpha} \varepsilon^\alpha \tag{16a}$$

is a *generating set* if it contains all the information about the Noether identities. More precisely, G is a generating set if any gauge transformation can be written in terms of the elements of G as

$$\delta\phi^i \frac{\delta S_0}{\delta\phi^i} = 0 \quad \Rightarrow \quad \delta\phi^i = \lambda^\alpha R^i_\alpha + M^{ij}\frac{\delta S_0}{\delta\phi^j}, \qquad M^{(ij)} = 0, \tag{16b}$$

with coefficients λ^α and M^{ij} *that may involve the fields.* Given a generating set, the Lie algebra of all the gauge transformations is spanned by Eq. (16b). Note that a generating set is in general *not* a basis in the Lie-algebraic sense.

As the bracket of two elements of the generating set is a gauge transformation, it must be expressible as in Eq. (16b). So one has

$$R^j_\alpha \frac{\delta R^i_\beta}{\delta\phi^j} - R^j_\beta \frac{\delta R^i_\alpha}{\delta\phi^j} = C^\gamma_{\alpha\beta}(\phi) R^i_\gamma + M^{ij}_{\alpha\beta}(\phi)\frac{\delta S_0}{\delta\phi^j}. \tag{17}$$

From the physical point of view, it is enough to consider only generating sets. This is because generating sets contain all the information about the Noether identities, about the degeneracy of the action principle, and about the number of required gauge conditions.

The situation is analogous to the following finite-dimensional geometrical setting: Consider a manifold A in $\mathbf{R}^n$. The vector fields tangent to A form an infinite-dimensional Lie algebra. However, for describing functions that are constant along A, the number of relevant vector fields is really finite and equal to the dimension n_a of A. If $\mathbf{X}_a$ $(a = 1, \ldots, n_a)$ provides at each point of A a basis of tangent vectors ($\mathbf{Y}$ tangent $\Rightarrow \mathbf{Y} = y^a(x)\mathbf{X}_a$), then the n_a equations $\mathbf{X}_a f = 0$ imply the infinite number of equations $\mathbf{Y}f = 0$ for all vectors tangent to A.

The vector fields $\mathbf{X}_a$ obey

$$[\mathbf{X}_a, \mathbf{X}_b] = C^c_{ab}(x)\mathbf{X}_c \tag{18}$$

and may not form a Lie subalgebra.

To select a point on A, it is enough to impose n^a coordinate conditions. One does not need an infinite number of them.

2.9. Open Algebras

It is now clear what the terminology "open algebra" means. It really applies to the generating sets and not to the gauge groups $\mathcal{G}$ or $\bar{\mathcal{G}}$.

Thus one says that a given generating set is *open* if $M^{ij}_{\alpha\beta}(\phi)$ in Eq. (17) is different from zero. It is *closed* if $M^{ij}_{\alpha\beta}(\phi) = 0$. It defines a Lie algebra if, in addition, $C^\alpha_{\beta\gamma}$ does not depend on the fields.

This last case includes the usual gauge theories (Yang–Mills, gravity), but not all the interesting ones. Gauge theories with a generating set G that forms a Lie algebra are very special in that one can think of the transformations of G "abstractly," i.e., independently of what the dynamics or the field content are. Furthermore, the BRST construction is then much simpler. However, this is a very lucky instance, which misses some of the important ingredients of the general case.

2.10. Reducible Generating Sets

Although generating sets should be complete, they can contain some redundancy. This occurs when there are some relations among the generators, i.e., when there exist some nontrivial λ^α such that the following identities hold:

$$\lambda^\alpha R^i_\alpha = N^{ij} \frac{\delta S_0}{\delta \phi^j} \tag{19}$$

The coefficients N^{ij} are antisymmetric because the right-hand side of Eq. (19) should be a gauge transformation. One says that such a generating set is *reducible*. A generating set is *irreducible* otherwise.*

The consideration of reducible generating sets is permissible within the formalism. However, the ghost spectrum associated with a reducible set is more complicated: besides the usual ghosts, one needs ghosts for ghosts.

An example of a reducible theory is given by p-form gauge fields. For a 2-form $A_{\mu\nu}$ with field strength $F_{\mu\nu\rho} = \partial_{[\mu}A_{\nu\rho]}$, the gauge transformations read

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{20a}$$

$$\Rightarrow \quad R^i_\alpha \sim R^\lambda_{\mu\nu}(x, y) = -\delta^\lambda_\mu \delta_{,\nu}(x-y) + \delta^\lambda_\nu \delta_{,\mu}(x-y). \tag{20b}$$

Then one finds that

$$\lambda^\alpha R^i_\alpha = 0, \tag{20c}$$

with

$$\lambda_\lambda(x, y) = \delta_{,\lambda}(x, y). \tag{20d}$$

2.11. Relation between Different Generating Sets

Although the gauge groups $\bar{\mathscr{G}}$, $\mathscr{N}$ and $\mathscr{G}$ are entirely determined by the action S_0 itself, there is clearly an enormous freedom in the choice of the

* Thus an irreducible theory is such that the only solution of Eq. (19) reads $\lambda^\alpha = M^{\alpha i}(\delta S_0/\delta\phi^i)$, i.e., vanishes on-shell. [One then finds $\lambda^\alpha R^i_\alpha = M^{\alpha j}(\delta S_0/\delta\phi^j)R^i_\alpha = (M^{\alpha j}R^i_\alpha - M^{\alpha i}R^j_\alpha) \times (\delta S_0/\delta\phi^j)$ since $R^j_\alpha(\delta S_0/\delta\phi^j) = 0$. Accordingly, $\lambda^\alpha R^i_\alpha$ indeed defines a trivial gauge transformation.] In the irreducible-group case, one says that the group has a "free action."

generating sets. *Two generating sets* R^i_α and R^i_A ($\alpha = 1, \ldots, m$; $A = 1, \ldots, M \geq m$) are related as

$$R^i_\alpha = t^A_\alpha R^i_A + M^{ij}_\alpha \frac{\delta S_0}{\delta \phi^j}, \qquad M^{ij}_\alpha = -M^{ji}_\alpha \tag{21a}$$

$$R^i_A = \bar{t}^\alpha_A R^i_\alpha + M^{ij}_A \frac{\delta S_0}{\delta \phi^j}, \qquad M^{ij}_A = -M^{ji}_A, \tag{21b}$$

where t^A_α and $\bar{t}^\alpha_A$ are of maximum rank m.

The requirement of covariance and locality in spacetime does, however, narrow down the choice of available generating sets. (One could otherwise always find one that is Abelian[12,13]—but usually not covariant, not local in spacetime, or not globally defined. A geometrical proof of Abelianization is given in the appendix.)

It turns out that the BRST methods incorporate not only gauge invariance, but also formal independence of the choice of generating set. On this ground, all the generating sets—which, of course, describe the same gauge symmetry—are equivalent.

2.12. Generating Sets and Gauge Orbits

On the stationary surface where the equations of motion $\delta S_0/\delta\phi^i = 0$ hold, the transformations generated by the elements of any generating set are integrable, i.e., obey Frobenius integrability condition (the Lie bracket $[X_i, X_j]$ is proportional to X_k). Therefore, these transformations generate well-defined surfaces, the "gauge orbits." The gauge orbits do not depend on the choice of generating set, on account of Eq. (21).

The number of elements in an irreducible generating set is equal to the dimension of the gauge orbits on the stationary surface. This gives a geometrical explanation of why generating sets are so relevant. By contrast, the dimension of the Lie algebra $\mathscr{G}$ containing all the gauge transformations is much greater: $\mathscr{G}$ is far from having a free action on the gauge orbits.

The above observation yields a criterion more practical than Eq. (16b) for deciding whether a set of gauge transformations is complete, i.e., generating. The set $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha$ is complete if and only if it accounts for all the degeneracies in the general solution of the equations of motion. More precisely, any two solutions ϕ^i and $\bar{\phi}^i$ fulfilling the same initial conditions must be related by iteration of $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha$.

This will be the case if all the null eigenvectors of the matrix $\delta^2 S_0/\delta\phi^i \delta\phi^j$ are spanned on-shell by R^i_α, i.e.,

$$\frac{\delta^2 S_0}{\delta\phi^j \delta\phi^i} \xi^j \approx 0 \;\Rightarrow\; \xi^j \approx \lambda^\alpha R^j_\alpha. \tag{22}$$

Here, $\approx$ means "equal modulo field equations." Indeed, if $\delta\phi^i$ is a gauge transformation, one finds by differentiation of $(\delta S_0/\delta\phi^i)\delta\phi^i = 0$ and upon use of Eq. (22) that $\delta\phi^i$ is equal to $\lambda^\alpha R^i_\alpha$ on the stationary surface. Hence, $\delta\phi^i - \lambda^\alpha R^i_\alpha$ is a gauge transformation that vanishes on-shell, and thus it is a trivial gauge transformation by the theorem of section 2.5.

A constructive method for getting a complete set of gauge transformations from any given action S_0 is given in Ref. 14. This method is based on the Hamiltonian formalism of Dirac.[15]

2.13. Why Ghosts Are Ghosts Fields

While the dimension of the Lie algebra of the gauge transformations is huge [it is given by the number of functionals $\lambda^\alpha(\phi)$ and $M^{ij}(\phi)$ in Eq. (16b)], the number of elements in a standard generating set is smaller and parametrized by spacetime fields: the index α in Eq. (16a) ranges over both R^n (it contains x) and a discrete set.

This is very important because, as we shall see, the number of ghosts in the BRST formalism is determined by the number of elements in the chosen generating set. So the ghosts are *ghost fields*, and one can apply the usual methods of local field theory for analyzing the gauge-fixed action. It is therefore of crucial importance that the BRST construction is based on generating sets and not on the full group $\bar{\mathcal{G}}$ containing all the gauge transformations.

From now on, therefore, we will deal exclusively with generating sets. These may have to be open and reducible in order to define transformations that are local in spacetime or covariant. The group structure of the set $\bar{\mathcal{G}}$ of all the gauge transformations is only of marginal interest in the BRST context.

3. GAUGE INVARIANCE AND BRST INVARIANCE—BASIC REQUIREMENTS

The derivation of the gauge-fixed Lagrangian is performed in two steps. First one replaces the original local gauge invariance by an equivalent global symmetry, the BRST symmetry. The replacement is carried out in such a manner that BRST invariance can be used as a substitute for gauge invariance. This first step is completely intrinsic and does not require any gauge-fixing condition. Second, one chooses appropriate gauge-fixing conditions and works out the corresponding gauge-fixed action in a way that incorporates BRST invariance.

The key requirements for constructing the BRST symmetry are the following:

1. The BRST symmetry acts as a graded odd derivation on the original fields ϕ^i and on some extra fields to be determined, i.e., for any A, B with B of definite Grassmann parity ε_B, one finds*

$$s(AB) = A(sB) + (-)^{\varepsilon_{\hat{B}}}(sA)B \qquad \text{(Leibnitz rule)} \tag{23a}$$

and

$$s^2 = 0 \qquad \text{(nilpotency)}. \tag{23b}$$

The grading of s is called the ghost number, and one has

$$gh(sA) = gh(A) + 1 \tag{23c}$$

$$\varepsilon(sA) = \varepsilon_A + 1 \qquad (\text{mod } 2) \tag{23d}$$

2. The zeroth cohomological group $H^0(s) \equiv (\text{Ker } s/\text{Im } s)^0$ is isomorphic with the set of gauge-invariant functions ("observables")

$$H^0(s) = \{\text{gauge-invariant functions}\}. \tag{24}$$

In other words, if one identifies two BRST-invariant functions that differ by a BRST-exact one,

$$sA = 0, \qquad sA' = 0, \qquad A \sim A' \;\Leftrightarrow\; A - A' = sB \tag{25}$$

one just finds, at ghost number zero, the gauge-invariant functions.

3. The BRST symmetry is a canonical transformation in an appropriate bracket structure (,) to be defined below. Hence

$$sA = (A, S), \tag{26}$$

where S is the canonical generator of s.

These three requirements completely determine S up to a canonical transformation, at least in the so-called "minimal sector" (see below). Accordingly, they completely capture the BRST symmetry.

4. RELATIVISTIC DESCRIPTION OF GAUGE-INVARIANT FUNCTIONS

In order to construct a nilpotent symmetry obeying Eq. (24), it is necessary to recall first how gauge-invariant functions ("observables") are described. As we want to develop a manifestly covariant formalism, we need a manifestly relativistic description.

* We choose an action from the right for s. Also, when we say "fields ϕ^i," we really mean "field histories $\phi^i(x)$" (condensed notation and terminology).

4.1. Covariant Phase Space in the Absence of Gauge Invariance

Let us first assume for a moment that there is no gauge invariance. The observables are then usually realized as the phase-space functions $F(q, p)$. This is, however, not fully satisfactory, as a phase-space point refers to the state of the system at a given instant of time.

As (q, p) at $t = t_0$ completely determines $(q(t), p(t))$ through the Hamiltonian equations, one can alternatively view phase space as the space of all solutions of the equations of motion. One can then drop reference to the momenta and consider the solutions $q(t)$ of the equations of motion for q obtained by eliminating p from the Hamiltonian equations. These equations for q usually take a manifestly covariant form.

The space of all solutions of the equations of motion is known as the *covariant phase space*. Its consideration goes back to the work of Peierls, who showed how to determine the Poisson bracket structure directly in the covariant phase space.[16,10] More recent work includes Refs. 17–19. The same idea applies, of course, to field theory, where observables can be viewed as functions* $f(\phi^i)$ of the solutions ϕ^i of the equations of motion $\delta S_0/\delta\phi^i = 0$.

As the explicit description of the solutions of the equations of motion may be involved, it is convenient to push the reformulation of the concept of observables one step further. This is done as follows:

Denote by I the (infinite-dimensional functional) space of all possible field histories. Therefore a point of I is an arbitrary entire history that may not solve $\delta S_0/\delta\phi^i = 0$. In I, the equations of motion $\delta S_0/\delta\phi^i = 0$ determine a submanifold Σ, which we call the stationary surface. This submanifold is just the covariant phase space (in the absence of gauge invariance).

The observables are the functions defined on Σ, i.e., the elements of $C^\infty(\Sigma)$ ("smooth" functions on Σ). Now, any function f on Σ can be extended off Σ to a function $F(\phi^i)$ defined on I, i.e., to an element of $C^\infty(I)$ ("smooth" functions on I). Two different extensions F and F' differ by a function that vanishes on Σ. These functions form an ideal $\mathcal{N}$ as FG vanishes on Σ whenever F (or G) does. The algebra $C^\infty(\Sigma)$ of the smooth functions on Σ is thus the quotient algebra $C^\infty(I)/\mathcal{N}$ of the smooth functions on I by the functions that vanish on Σ.

It should be stressed that our considerations based on the use of the equations of motion can be extended to cover quantum mechanics. This is because the observables can still be identified with the (operator-valued) functions of $\hat{q}$ and $\hat{p}$ at a given instant of time. These functions are again in bijective correspondence with, and hence can be realized as, the functions

* The words *functions* and *functionals* are used interchangeably in the sequel. The suggestive terminology and notations of finite-dimensional manifold theory will also be adopted without any analysis of the (complicated) functional aspects.

on the space of solutions $\hat{q}(t)$, $\hat{p}(t)$ of the equations of motion. The absence of conflict with the principles of quantum mechanics is particularly obvious in the manifestly covariant Heisenberg picture, where the field operators obey the appropriately ordered equations of motion $\delta S_0/\delta\phi^i = 0$.

4.2. Boundary Conditions

In order to contain all the solutions of the equations of motion and not just the one corresponding to a definite set of initial data, the space I of histories should not be restricted by boundary conditions at the initial and final times t_i and t_f. The stationary surface Σ contains, then, all the possible dynamical states of the system.

For this reason, the space I is not the space $I_{i\to f}$ over which one integrates in the path-integral representation of a definite quantum-mechanical amplitude between given *in* and *out* states (I is too large). The space I is actually the union over all possible pairs of *in* and *out* states of the spaces $I_{i\to f}$.

Furthermore, as one does not vary the boundary data at t_1 and t_2 in the action principle, the functional derivatives $\delta S_0/\delta\phi^i$ in the field equations $\delta S_0/\delta\phi^i = 0$ do not refer to the derivatives of S_0 with respect to the boundary data. More precisely, if we write

$$\phi^i(\mathbf{x}, t) = \bar{\phi}^i(\mathbf{x}, t) + f^i[\alpha_1, \alpha_2](\mathbf{x}, t),$$

where, first, $f^i[\alpha_1, \alpha_2](\mathbf{x}, t)$ is, for given α_1, α_2, a fixed history such that $f^i[\alpha_1, \alpha_2](\mathbf{x}, t_1) = \alpha_1^i(\mathbf{x})$, $f^i[\alpha_1, \alpha_2](\mathbf{x}, t_2) = \alpha_2^i(\mathbf{x})$; and, second, $\bar{\phi}^i(\mathbf{x}, t_1) = \bar{\phi}^i(\mathbf{x}, t_2) = 0$. Then the field equations are $\delta S_0/\delta\bar{\phi}^i = 0$. This will always be implicitly understood in the sequel, even though the above decomposition will never be used.

4.3. Covariant Phase Space in the Presence of a Gauge Freedom

If there is a gauge freedom, the observables should, in addition, be gauge invariant.

We have pointed out that the gauge transformations $\delta_\varepsilon\phi^i = R^i_{\ \alpha}\varepsilon^\alpha$ are integrable when the equations of motion hold. Accordingly, they generate well-defined orbits on Σ, the dimension of which is equal to the number of independent $R^i_{\ \alpha}$. The gauge invariant functions are constant along the gauge orbits and hence induce definite functions on the quotient space Σ/G of the stationary surface by the gauge orbits. Formally, one can thus write the space of observables as $C^\infty(\Sigma/G)$, i.e., as the space of smooth functions on Σ/G. (In general, Σ/G is not a smooth manifold, but we will nevertheless use this suggestive notation.) The gauge orbits are obtained by integrating $\delta_\varepsilon\phi^i = R^i_{\ \alpha}\varepsilon^\alpha$ on Σ, as shown in Fig. 1.

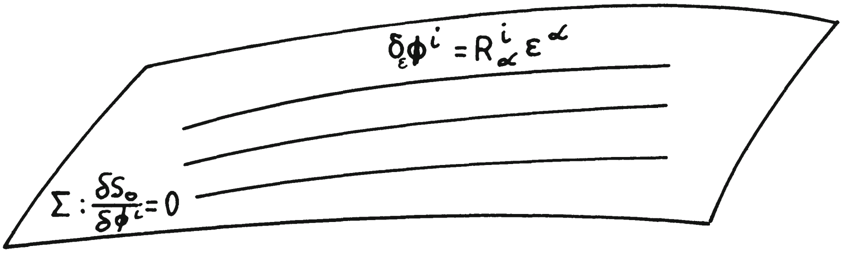


Figure 1. Integration of $\delta_\varepsilon \phi^i$ on Σ, giving rise to gauge orbits.

The gauge-invariant observables are thus reached in two steps. First, one goes from I to Σ; then, from Σ to Σ/G. To solve Eq. (24), one must find a nilpotent operator s that implements these two steps through its cohomology:

$$H^0(s) = C^\infty(\Sigma/G) \tag{27}$$

It is not surprising that the searched-for nilpotent s actually contains two nilpotent operators. Each of these two differentials implements one of the steps.

The first differential δ provides what is known as a "Koszul–Tate resolution" of $C^\infty(\Sigma)$, i.e., is such that $H_0(\delta) \equiv (\mathrm{Ker}\, \delta/\mathrm{Im}\, \delta)_0 = C^\infty(\Sigma)$. It implements the first step, from I to Σ. The second differential d is the vertical exterior derivative along the gauge orbits and implements the second step, from Σ to Σ/G; $H^0(d) = C^\infty(\Sigma/G)$. The BRST derivative s is formally the sum of δ and d.

4.4. Regularity Conditions and a Useful Theorem

In order to develop the formalism, it is necessary to make some regularity assumptions on the derivatives $\delta S_0/\delta\phi^i$. One assumes that one can split on Σ the derivatives $\delta S_0/\delta\phi^i$ into independent functions y^a and dependent ones z^α, $(\delta S^0/\delta\phi^i) = (y^a, z^\alpha)$, in such a way that the equations $\delta S_0/\delta\phi^i = 0$ are completely equivalent to $y^a = 0$, i.e., $z^\alpha = 0$ is a consequence of $y^a = 0$; and the exterior form $\bigwedge_a dy^a$ does not vanish on Σ. This means that zero is a regular value of the map defined by y^a and that one can locally take the y^a as first coordinates of a new regular coordinate system on I, $(\phi^i) \leftrightarrow (y^a, \tau^\alpha)$.

To illustrate these conditions in the simpler, finite-dimensional situation, let us consider a two-dimensional space with coordinates (u, v) instead of I. If the action S_0 is $\frac{1}{2}(u-v)^2$, one finds $\partial S_0/\partial u = (u - v)$, $\partial S_0/\partial v = -(u - v)$. One can take the first equation as independent; the second is clearly a consequence of the first. Hence, $y \equiv \partial S_0/\partial u$, $z \equiv \partial S_0/\partial v$, and

$y = 0 \Rightarrow z = 0$. The gradient $dy = d(u - v)$ does not vanish on Σ. The regularity conditions are fulfilled, and $y \equiv u - v$ can be taken as first coordinate of a new regular coordinate system (y, τ), e.g., with $\tau = v$.

If S_0 were to be replaced by $\bar{S}_0 \equiv \frac{1}{6}(u - v)^3$, the regularity conditions would not be fulfilled because both $d(\partial \bar{S}_0/\partial u) = u - v$ and $d(\partial \bar{S}_0/\partial v) = v - u$ vanish on Σ. The function $y \equiv \partial \bar{S}_0/\partial u = \frac{1}{2}(u - v)^2 = \partial \bar{S}_0/\partial v$ cannot be taken as first coordinate of a new regular coordinate system in the vicinity of Σ, as the inverse transformation $u - v = \sqrt{2y}$ is not smooth on Σ.

When the regularity conditions hold, the following theorem is immediate:

Theorem. Any smooth function $F(\phi^i)$ vanishing on Σ can be written as

$$F(\phi^i) = \lambda^i(\phi) \frac{\delta S_0}{\delta \phi^i} \tag{28}$$

with smooth coefficients $\lambda^i(\phi)$.

Proof. The proof is standard. In the (y, τ) coordinate system, one finds that

$$F(\phi^i) = F(y, \tau) = F(y = 0, \tau) + \int_0^1 d\mu \frac{dF}{d\mu}(\mu y, \tau)$$
$$= y^a \int_0^1 d\mu \frac{\partial F}{\partial y^a}(\mu y, \tau) = \lambda^i(\phi) \frac{\delta S_0}{\delta \phi^i}$$

as $F(y = 0, \tau)$ vanishes by assumption and as y^a are some of the field equations.

This proof is local in field space because the coordinates (y, τ) are usually defined only locally. However, it is easy to see, using for instance partitions of unity, that it can be extended to cover the whole of I.

In algebraic terms, the theorem expresses that the ideal $\mathcal{N}$ of the functions that vanish on Σ is the same as the ideal $\mathcal{N}'$ of the functions that are combinations $\lambda^i(\phi)(\delta S_0 | \delta \phi^i)$ of the field equations.

We have been a bit cavalier with the functional aspects of the theorem and have proceeded as if the space I of all histories were finite-dimensional. So our discussion is rather formal. However, things are not as bad as one may think at first sight, because space-time locality comes as a help. Indeed, only local functionals occur below in the construction of the solution of the master equation and of the gauge-fixed action, i.e., functionals that take the form

$$F = \int k(\phi^i, \phi^i_{,m}, \ldots, \phi^i_{,\mu_1 \ldots \mu_s})\, d^D x, \tag{29}$$

where k involves a finite number of derivatives. One can then reformulate the question in terms of k $(\phi^i, \phi^i_{,\mu}, \ldots, \phi^i_{,\mu_1\ldots\mu_s})$ as a finite-dimensional problem. This enables one to prove as a bonus that λ^i in Eq. (29) is also local in space-time.[20] But it should be added immediately that in spite of the space-time locality of the gauge-fixed action and of the generator of the BRST transformation, gauge-invariant functionals that are not local in space-time can be of great interest. So nonlocal functionals should also be considered. For these, the above formal derivations must be supplemented by appropriate functional-analysis arguments which will not be given here.

The regularity conditions are usually fulfilled by all models of physical interest. The only exception that I know of is given by the Siegel model for chiral bosons[21] where, as pointed out in Ref. 22, some of the relevant field equations vanish only quadratically on the stationary surface. This seems to prevent a consistent, physically meaningful Lagrangian BRST formulation of the model, already at the classical level. This difficulty does not appear to have been fully appreciated in the literature. (By contrast, the Hamiltonian formulation is straightforward. For more information, see Ref. 23 and works cited therein.)

5. THE KOSZUL–TATE RESOLUTION

5.1. The Problem

The first step in the BRST construction is to implement the restriction from I to Σ. Therefore one needs to define a differential δ that acts as a (nilpotent) graded derivative on polynomials in some generators (to be specified) with coefficients that are functions on I (just as d in the standard exterior calculus acts on polynomials in dx, dy, $dz, \ldots$ with coefficients that are functions on the manifold); and that computes $C^\infty(\Sigma)$ through its homology.

The grading of δ is called the antighost number. As δ decreases the antighost number by one unit, it behaves like a boundary operator. The requirement that δ compute $C^\infty(\Sigma)$ through its homology reads

$$H_0(\delta) \equiv \left(\frac{\operatorname{Ker}\delta}{\operatorname{Im}\delta}\right)_0 = C^\infty(\Sigma) = \frac{C^\infty(I)}{\mathcal{N}}. \tag{30}$$

We will actually ask more: namely, that Eq. (30) contain *all* the homology of δ. In other words, we require

$$H_k(\delta) = 0 \qquad k \neq 0. \tag{31}$$

This requirement turns out to be essential not only for guaranteeing that the BRST cohomology at ghost number zero is given by the gauge-invariant functions, but also for being able to prove the existence, in general, of the BRST symmetry itself.

A differential complex with the properties shown in Eqs. (30) and (31) is said to provide a *resolution* of the quotient algebra $C^\infty(I)/\mathcal{N}$. In the present context, the relevant resolution is due to Koszul,[24] Borel,[25] and Tate.[26]

One can describe δ in an intrinsic manner, without having to work with a specific representation of the surface Σ. However, our ultimate goal is to derive the gauge-fixed action with a definite set of fields. For this reason, we will not strive for intrinsicality.

5.2. Actions without Gauge Invariance

In the absence of gauge invariance, the construction of δ is very simple. Because $(\text{Ker}\,\delta/\text{Im}\,\delta)_0$ should be equal to $C^\infty(I)/\mathcal{N}$, we simply define δ so that

$$(\text{Ker}\,\delta)_0 = C^\infty(I) \tag{32a}$$

and

$$(\text{Im}\,\delta)_0 = \mathcal{N}. \tag{32b}$$

Consequently, we set

$$\delta\phi^i = 0. \tag{33a}$$

Using the Leibnitz rule, this implies that $\delta F(\phi^i) = 0$ for any functions on I and hence, $(\text{Ker}\,\delta)_0 = C^\infty(I)$.

To implement $(\text{Im}\,\delta)_0 = \mathcal{N}$, we observe that due to our regularity assumptions, the elements of $\mathcal{N}$ are given by the combinations of the field equations,

$$G(\phi^i) \in \mathcal{N} \quad \Leftrightarrow \quad G(\phi^i) = \lambda^i(\phi^j)\frac{\delta S_0}{\delta\phi^i}.$$

Therefore, we introduce as many new generators ϕ_i^* as there are field equations and simply set

$$\delta\phi_i^* = -\frac{\delta S_0}{\delta\phi^i}. \tag{33b}$$

The minus sign is inserted for later convenience. This implies that $G = \delta(-\lambda^i\phi_i^*)$ and $(\text{Im}\,\delta)_0 = \mathcal{N}$. With this definition, our first goal is achieved: Eq. (30) holds.

The generators ϕ_i^* are known as the antifields associated with the original fields ϕ^i. They are equal in number to the ϕ^i, since the number of field equations is equal to the number of fields (the field equations set the gradient of S_0 equal to zero).

To preserve the grading properties of δ, one must impose

$$\varepsilon(\phi_i^*) = 1 \tag{34a}$$

(as we assume the fields to be bosonic) and

$$\mathit{antigh}\, \phi_i^* = 1 \tag{34b}$$

(of course, $\mathit{antigh}\, \phi^i = 0$). The action of δ on a general polynomial in ϕ^i, ϕ_i^* is obtained by using the Leibnitz rule, and one easily checks nilpotency, $\delta^2 = 0$.

To see whether δ provides a resolution of $C^\infty(I)/\mathcal{N}$, it remains to compute $H_k(\delta)$. It is here that the assumed absence of gauge invariance plays a key role. Indeed, the equations of motion are then independent, so that the number of new objects ϕ_i^* in degree one is exactly equal to the number of independent equations of motion. Using this property, one easily proves,[24,25,9] that

$$H_k(\delta) = 0 \qquad k \neq 0. \tag{35}$$

5.3. Actions with a Gauge Freedom

5.3.1. Irreducible Case

The above definition (33), (34) of δ can still be used if there is a gauge freedom, and one still finds $H_0(\delta) = C^\infty(\Sigma)$. However, it is no longer true that $H_k(\delta) \neq 0$. Because of the Noether identity,

$$\frac{\delta S_0}{\delta \phi^i} R^i_{\ \alpha} = 0, \tag{36}$$

one actually finds nontrivial δ-closed polynomials in degree one. These are given by

$$R^i_{\ \alpha} \phi_i^* . \tag{37a}$$

Indeed, one checks that the $R^i_{\ \alpha}\phi_i^*$ are δ-closed

$$\delta(R^i_{\ \alpha}\phi_i^*) = -R^i_{\ \alpha}\frac{\delta S_0}{\delta \phi^i} = 0 \tag{37b}$$

and exhausts all nontrivial δ-closed polynomials of degree one:

$$\delta(\lambda^i \phi_i^*) = 0 \quad \Rightarrow \quad \lambda^i \phi_i^* = \mu^\alpha R^i_{\ \alpha} \phi_i^* + \delta(\tfrac{1}{2}\varepsilon^{ij} \phi_i^* \phi_j^*). \tag{37c}$$

Furthermore, the $R^i \phi_i^*$ are nonexact. So, $H_1(\delta) \neq 0$.

To understand how this problem can be remedied, let us first assume that the gauge transformations are independent, so that all the nontrivial cycles in Eq. (37a) are independent.

One can then recover $H_l(\delta) = 0$ and at the same time $H_k(\delta) = 0$ for all $k \neq 0$ in an extremely elegant way devised by Tate.[26] One simply adds one new generator ϕ_α^* for each cycle in Eq. (37a) and defines

$$\delta \phi_\alpha^* = R^i_{\ \alpha} \phi_i^*. \tag{38a}$$

Because $\delta(R^i_{\ \alpha} \phi_i^*) = 0$, one has $\delta^2 \phi_\alpha^* = 0$. Furthermore, by taking

$$antigh(\phi_\alpha^*) = 2, \qquad \varepsilon(\phi_\alpha^*) = 0 \tag{38b}$$

(recall that $\varepsilon(\phi_i^*) = 1$) and extending δ as a graded derivation to any polynomial in ϕ^i, ϕ_i^*, and ϕ_α^* one maintains $\delta^2 = 0$.

With the introduction of the antifields ϕ_α^*, the cycles $R^i_{\ \alpha} \phi_i^*$ that were nonexact become exact. Thus $H_1(\delta)$ is now zero. Furthermore, using the assumed irreducibility of the gauge transformations, one easily shows that $H_k(\delta) = 0$ for all $k > 0$.[22,26,27]

5.3.2. Reducible Case

The construction of δ in the reducible case proceeds along the same lines as in the irreducible one. First, one observes that with Eqs. (33) and (38), the homology group $H_2(\delta)$ is nonzero even though $H_1(\delta) = 0$. This is because the polynomials

$$Z^\alpha_A \phi_\alpha^* + \tfrac{1}{2} C^{ij}_A \phi_i^* \phi_j^* \tag{39}$$

are closed but not exact. Here, the Z terms form a complete set of reducibility functions; i.e., they are such that

$$\mu^\alpha R^i_{\ \alpha} = N^{ij} \frac{\delta S_0}{\delta \phi^j} \quad \Rightarrow \quad \mu^\alpha = \nu^A Z^\alpha_A + M^{\alpha i} \frac{\delta S_0}{\delta \phi^i}. \tag{40a}$$

One has for the Z terms

$$Z^\alpha_A R^i_{\ \alpha} = C^{ij}_A \frac{\delta S_0}{\delta \phi^j}, \qquad C^{ij}_A = -C^{ji}_A, \tag{40b}$$

and this property guarantees that the polynomials in Eq. (39) are closed. These polynomials are not exact, because the Z^α_A cannot all vanish on Σ when the gauge transformations are truly reducible.

One therefore introduces new generators (antifields) ϕ^*_A at antighost number three and sets

$$\delta\phi^*_A = -Z^\alpha_A \phi^*_\alpha - \tfrac{1}{2} C^{ij}_A \phi^*_i \phi^*_j \tag{41a}$$

$$\textit{antigh}\, \phi^*_A = 3, \qquad \varepsilon(\phi^*_A) = 1. \tag{41b}$$

This kills the homology at degree two; i.e., $H_2(\delta)$ is now zero. The minus signs in Eq. (41a) are again inserted for later convenience.

If the reducibility equations, Eq. (40b), are all independent, this is the end of the story. Not only are $H_1(\delta)$ and $H_2(\delta)$ zero, but also all the higher homology groups $H_k(\delta)$, $k = 3, 4, \ldots$.[22,26,27]

But if the reducibility equations [Eq. (40b)] are not independent, the analysis is not finished and one has to keep going. This is because $H_3(\delta)$ is then non-zero when δ is defined by Eqs. (33), (38), (41). For each nontrivial relation

$$Z^A_{(i)} Z^\alpha_A = C^{\alpha j}_{(i)} \frac{\delta S_0}{\delta \phi^j} \tag{42}$$

on the reducibility functions, one must therefore introduce one antifield at antighost number four. This kills $H_3(\delta)$.

One can, if desired, introduce more antifields; i.e., one can consider an overcomplete set of reducibility relations [Eq. (42)]. But one must then compensate at the next order by adding antifields of antighost number five that take into account the relations on the $Z^A_{(i)}$. The general idea of passing from order k to order $k+1$ is always the same.

We refer the reader to Ref. 27, where the construction of δ is more explicitly analyzed and the complete proofs of its properties are given. These proofs are explicitly worked out within the Hamiltonian formalism. But, as briefly indicated below, the algebraic features of that approach to the BRST symmetry are identical. For this reason, we list here only the salient facts:

1. δ can be constructed recursively, antighost level by antighost level, and the spectrum of antifields can be chosen at each step so that $\delta^2 = 0$, $H_k(\delta) = 0$, $k > 0$ [and of course, $H_0(\delta) = C^\infty(\Sigma)$].

2. The explicit expression for δ becomes awkward when the coefficients C^{ij}_A in Eq. (40b), $C^{\alpha j}_{(i)}$ in Eq. (42), etc. are nonzero (on-shell reducibility), but this affects neither $\delta^2 = 0$ nor $H_k(\delta) = 0$, $k > 0$ (it complicates the proofs only technically).

3. The requirement of acyclicity of δ at antighost number $k > 0$ turns out to be equivalent to the "proper solution" requirement of Batalin and Vilkovisky,[6] as both demands lead to the same spectrum of fields and antifields.

4. Let λ^i be a gauge transformation that vanishes on-shell,

$$\lambda^i \frac{\delta S_0}{\delta \phi^i} = 0, \qquad \lambda^i = \lambda^{ij} \frac{\delta S_0}{\delta \phi^j}$$

A priori, λ^{ij} may not be antisymmetric in (i, j). However, λ^{ij} is not uniquely defined by λ^i, as one can replace λ^{ij} by $\lambda^{ij} + \mu^{ijk}\delta S_0/\delta\phi^k$ with $\mu^{ijk} = -\mu^{ikj}$. One can use this freedom to set $\lambda^{ij} = -\lambda^{ji}$. Indeed, one has $\delta(\lambda^i \phi_i^*) = 0$, and hence $\lambda^i \phi_i^* = \delta(\lambda^\alpha \phi_\alpha^* + \nu^{ij}\phi_i^*\phi_j^*)$ with $\nu^{ij} = -\nu^{ji}$. Since $\lambda^i \approx 0$, the λ^α must be such that $\lambda^\alpha R^i_\alpha \approx 0$, i.e., $\lambda^\alpha = Z^\alpha_A t^A + \sigma^{\alpha i}\delta S_0/\delta\phi^i$. This implies that $\lambda^\alpha \phi^*_\alpha = Z^\alpha_A t^A \phi^*_\alpha + \sigma^{\alpha i}\phi^*_\alpha \delta S_0/\delta\phi^i = \delta(t^A\phi^*_A + \sigma^{\alpha i}\phi^*_\alpha\phi^*_i) + \varepsilon^{ij}\phi^*_i\phi^*_j$ for some $\varepsilon^{ij} = -\varepsilon^{ji}$. Accordingly, $\lambda^i\phi^*_i = \delta[(\nu^{ij} + \varepsilon^{ij})\phi^*_i\phi^*_j]$; i.e., $\lambda^i = \lambda^{ij}(\delta S_0/\delta\phi^j)$ with $\lambda^{ij} = -\lambda^{ji}(=\nu^{ij} + \varepsilon^{ij})$, as required.

5. For definiteness, we will explicitly develop the subsequent formalism in the case of reducible gauge theories of the first order, i.e., of gauge theories with reducibility functions Z^α_A that are independent. The antifield spectrum is then given by ϕ^*_i, ϕ^*_α, and ϕ^*_A. The general case is treated along the lines of these lectures in Refs. 22 and 27.

6. THE EXTERIOR DERIVATIVE ALONG THE GAUGE ORBITS

6.1. Definition

As the orbits generated by the gauge transformations $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha$ are integrable on the stationary surface Σ, one can define, on Σ, an exterior derivative operator d that takes antisymmetrized derivatives along the gauge orbits. This operator acts on p-forms along the gauge orbits.

Borrowing the terminology of fiber bundle theory where the fibers (here, the orbits) are drawn vertically, one can call a vector tangent to the orbits a "vertical vector." A p-form along the gauge orbits is then named a "vertical p-form" and d is the "vertical exterior derivative."

So, if F is a function on Σ, dF is the vertical 1-form defined by

$$dF(X) = \partial_X F \tag{43a}$$

for all vertical vectors X. dF vanishes iff F is invariant along the orbits. The exterior derivative of a vertical 1-form α is given by

$$(d\alpha)(X, Y) = -\mathscr{L}_y \alpha(X) + \mathscr{L}_x \alpha(Y) + \alpha([X, Y]), \tag{43b}$$

where X and Y are vertical vector fields. The Lie bracket $[X, Y]$ of X and Y in Eq. (43b) is also a vertical vector field as the gauge transformations are integrable. Similar formulas hold for higher-rank p-forms, and one finds $d^2 = 0$ (on the stationary surface Σ, where d is defined). We follow the exterior calculus conventions of Ref. 27.

Because d only takes derivatives along the gauge orbits, it is clear that $H^0(d) \equiv (\text{Ker } d/\text{Im } d)^0$ is isomorphic with the algebra of gauge-invariant functions. Note that the higher-order cohomology groups $H^k(d)$, $k > 0$, may be nontrivial.

6.2. Representation

6.2.1. Irreducible Case

In the irreducible case, the dimension of the gauge orbits is equal to the number of gauge transformations $\delta_\varepsilon \phi^i = R^i_{\ \alpha} \varepsilon^\alpha$. A basis at each point X_α of vertical vectors is thus given by

$$X_\alpha F \equiv \frac{\delta F}{\delta \phi^i} R^i_{\ \alpha}, \tag{44a}$$

and one has

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta}(\phi) X_\gamma \qquad (\text{on } \Sigma). \tag{44b}$$

If $\{C^\alpha\}$ stands for the basis dual to $\{X_\alpha\}$, one can represent the vertical exterior derivative along the gauge orbits as

$$dF = (X_\alpha F) C^\alpha \tag{45a}$$

$$dC^\alpha = \tfrac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma. \tag{45b}$$

The dual one-forms C^α are anticommuting and will be interpreted as the ghosts below. The form degree will therefore already now be called the "pure ghost number." The operator α increases the pure ghost number by one unit.

6.2.2. Reducible Case

In the reducible case, the vertical vectors X_α associated with the gauge transformations no longer form a basis. Rather, they form an overcomplete set, subject to the following reducibility equations:

$$Z^\alpha_A X_\alpha = 0. \tag{46a}$$

Accordingly, the components $\alpha_{\alpha_1 \ldots \alpha_p} \equiv \alpha(X_{\alpha_1}, \ldots, X_{\alpha_p})$ of a vertical p-form in the overcomplete set $\{X_\alpha\}$ are also subject to the same conditions

$$Z^{\alpha_i}_A \alpha_{\alpha_1 \ldots \alpha_i \ldots \alpha_p} = 0. \tag{46b}$$

We will assume for simplicity that there is no relation on the Z^α_As; i.e., that these functions are all independent.

Although there is no dual basis to $\{X_\alpha\}$, one can introduce formal objects C^α. By saturating the indices of the components $\alpha_{\alpha_1 \ldots \alpha_p}$ with $C^{\alpha_1} \ldots C^{\alpha_p}$, one can identify vertical p-forms with polynomials of order p

in C^α whose coefficients obey the algebraic condition in Eq. (46b). One can then compute d using the same formulas [Eq. (45)] as in the irreducible case.

It is, however, more convenient to relax the algebraic condition, Eq. (46b), as follows: Introduce as many additional objects C^A as there are conditions on X_α and modify dC^α as

$$dC^\alpha = \tfrac{1}{2}C^\alpha_{\beta\gamma}C^\beta C^\gamma + Z^\alpha_A C^A. \tag{47}$$

The new term in the right-hand side of Eq. (47) does not affect the vertical derivative of a vertical p-form, because of Eq. (46b). To preserve the grading properties of d, the generators C^A should be even and of pure ghost number two. These generators will be identified later with the ghosts of ghosts.

It is possible to define dC^A so that $d^2 = 0$ on arbitrary polynomials in C^α, C^A (on Σ). The proof will not be given here (see Ref. 27). The mathematical structure defined by d, ϕ^i, C^α, and C^A is known as a *free differential algebra.*

What is the effect of the new term added to dC^α? Its effect is to enforce the algebraic condition [Eq. (46b)] through the closedness relation. So, for instance, a one-form $\alpha \equiv \alpha_\alpha C^\alpha$ is closed iff

$$d\alpha = 0 \quad \Leftrightarrow \quad d^{\text{old}}\alpha + \alpha_\alpha Z^\alpha_A C^A = 0. \tag{48}$$

This implies both the algebraic equation $\alpha_\alpha Z^\alpha_A = 0$ and the requirement $d^{\text{old}}\alpha = 0$.

Thus, one can represent the vertical exterior derivative d in the space of arbitrary polynomials in C^α and C^A. The algebraic condition [Eq. (46b)] is automatically enforced when passing to the cohomology. This is made possible through the introduction of the ghosts of ghosts.

We refer again the reader to Ref. 27 for the details. This reference also analyzes the case when the reducibility equations are not independent, which requires further ghosts of ghosts of ghosts.

7. BRST SYMMETRY—MASTER EQUATION

7.1. The Problem

With the Koszul–Tate operator and the vertical exterior derivative at hand, all the building blocks of the BRST symmetry have been constructed. What is required now is to put these ingredients together in a manner that preserves the crucial nilpotency.

To that end, we tentatively first define

$$s\phi^i \overset{?}{=} d\phi^i, \qquad sC^\alpha \overset{?}{=} dC^\alpha, \qquad sC^A \overset{?}{=} dC^A, \tag{49a}$$

$$s\phi^*_i \overset{?}{=} \delta\phi^*_i, \qquad s\phi^*_\alpha \overset{?}{=} \delta\phi^*_\alpha, \qquad s\phi^*_A \overset{?}{=} \delta\phi^*_A, \tag{49b}$$

where d is extended off Σ by using the same formulas, Eqs. (45a) and (45b) or (47), as before. This makes sense because the gauge transformations and the reducibility functions are well defined for all histories, not just for histories that obey the equations of motion. So, in Eq. (49), the BRST transformation s reduces to d in the sector containing the fields ϕ^i and the ghosts C^α (and the ghosts of ghosts C^A if any). It reduces to δ in the antifield sector.

The grading of s is called the *ghost number* and is given by the difference between the pure ghost number and the antighost number: s increases the ghost number by one unit.

The problem with the simple definition in Eq. (49) is that it does not yield a nilpotent operator. This is because $\delta d + d\delta \neq 0$ off Σ, except when the gauge transformations form an Abelian group, and because $d^2 \neq 0$ off Σ in the open-algebra case. The gauge transformations $\delta_\varepsilon \phi^i = R^i_{\ \alpha}\varepsilon^\alpha$ are then nonintegrable off Σ.

In order to remedy this situation, one needs to improve s by terms of a higher antighost number,

$$s = \delta + d + \text{more}. \tag{50a}$$

More precisely, the appropriate definitions read

$$s\phi^i = d\phi^i + S^i \tag{50b}$$

$$sC^\alpha = dC^\alpha + S^\alpha \tag{50c}$$

$$sC^A = dC^A + S^A \tag{50d}$$

$$s\phi^*_i = \delta\phi^*_i + T_i \tag{50e}$$

$$s\phi^*_\alpha = \delta\phi^*_\alpha + T_\alpha \tag{50f}$$

$$s\phi^*_A = \delta\phi^*_A + T_A, \tag{50g}$$

with

$$antigh(S^i) \geq antigh(d\phi^i) + 1 = 1 \tag{50h}$$

$$antigh(S^\alpha) \geq antigh(dC^\alpha) + 1 = 1 \tag{50i}$$

$$antigh(S^A) \geq antigh(dC^A) + 1 = 1 \tag{50j}$$

$$antigh(T_i) \geq antigh(\delta\phi^*_i) + 1 = 1 \tag{50k}$$

$$antigh(T_\alpha) \geq antigh(\delta\phi^*_\alpha) + 1 = 2 \tag{50l}$$

$$antigh(T_A) \geq antigh(\delta\phi^*_A) + 1 = 3. \tag{50m}$$

The improvement terms are determined by requiring that $s^2 = 0$. [Note: the extra terms in Eq. (50) are in general not of definite antighost numbers. The inequalities in Eqs. (50h–m) should therefore be understood as inequalities on the components of S^i, S^α, S^A, T_i, T_α, and T_A of minimum antighost number].

7.2. Homological Perturbation Theory

It turns out that it is always possible to find S^i, S^α, S_A, T_i, T_α, and T_A with the above properties such that $s^2 = 0$. This is a general theorem of "homological perturbation theory,"[28] a branch of algebraic topology.

Furthermore, the cohomology of s is given by the cohomology of d on Σ. As anticipated, the role of the antifields is to enforce $\delta S_0/\delta\phi^i = 0$ when one passes to the cohomology. The operator s then reduces to d, and one finds $H^0(s) = H^0(d) = \{\text{gauge invariant functions}\}$. More generally, one gets

$$H^g(s) = \begin{cases} 0 & g < 0 \quad (51a) \\ H^g(d) & g \geq 0 \quad (51b) \end{cases}$$

Here, the cohomologies of s and d can be either the local cohomologies, or the cohomologies in the space of all local and nonlocal functionals. In this latter case, the arguments leading to Eq. (51) are more formal. Note that the local cohomology may be nontrivial for $g > 0$ even if the corresponding cohomology in the space of all functionals is trivial.

The proof of Eq. (51), which is immediate in the Abelian case for which Eq. (49) is correct, is given in Refs. 20, 27, 29, and 30 and will not be reproduced here. Rather, we will prove only the existence of s.

Because of Eq. (51), one can conclude that *the BRST symmetry completely incorporates gauge invariance at ghost number zero.* This is a general feature valid even if the elements R^i_α of the generating set under consideration do not form a group. The group structure is actually never used, and one can thus say that the *properties of the BRST symmetry rely on a more primitive structure.* It is the author's opinion that works on the BRST symmetry that overemphasize the group structure are sometimes misleading.

The same primitive structure is also encountered in the Hamiltonian formalism, where again the description of the gauge-invariant observables involves two steps: first, the restriction to the "constraint" surface; second, the passage to the quotient of the constraint surface by the gauge orbits. These same ingredients lead to the same algebraic construction of the BRST symmetry; see Refs. 9, 27, and 30.* The techniques of homological perturbation theory were actually first rediscovered by physicists in the Hamiltonian context.

* The major difference between the Lagrangian and Hamiltonian constructions lies in the bracket structure that is naturally defined. While the Lagrangian bracket ("antibracket") to be defined below does not appear to be realized quantum-mechanically, the Hamiltonian bracket ("Poisson bracket") becomes the physical, graded commutator (times $(i\hbar)^{-1}$) in the quantum theory.

In order to prove the existence of the extra higher-order terms S^i, S^α, S^A, T_i, T_α, and T_A needed to secure nilpotency, we will take advantage of one extra property of the BRST symmetry. This extra property is that the BRST transformation preserves a natural bracket structure in the space of fields, ghosts, and antifields. Accordingly, rather than trying to work out S^i, S^α, S^A, T_i, T_α, and T_A individually, it is more economical to construct directly the BRST generator for the BRST symmetry, which is a single function(al).

7.3. Antibracket

What is the natural bracket structure in the Lagrangian formalism?

It is well known that the Hamiltonian Poisson bracket does not induce a Poisson bracket in the Lagrangian space I of all field histories: a physically meaningful symplectic structure is defined only on the stationary surface Σ, modulo G.

However, because the stationary surface has the property of being obtained by equating the gradient $\delta S_0/\delta\phi^i$ of a single function S_0 to zero, it turns out that one can nevertheless define a (rather odd) bracket structure among the variables of the Lagrangian BRST complex. This bracket structure possesses strange features and is named the *antibracket.* It is very useful when developing the formalism but disappears when one fixes the gauge and does not seem to have a direct quantum-mechanical analog.

The definition of the antibracket suggests itself once it is realized that there is a remarkable symmetry between the fields and the ghosts on the one hand and the antifields on the other hand. This symmetry, in turn, is a consequence of the fact that it is the same functional S_0 which determines both the ghost spectrum (through the gauge symmetries) and the antifield spectrum (through the Noether identities).

Taking again, for definiteness, a reducible gauge theory without a reducibility equation on the coefficients Z^α_A, one finds that

$$\begin{array}{cccccccc} -3 & -2 & -1 & 0 & 1 & 2 & & \\ \hline \phi^*_A & \phi^*_\alpha & \phi^*_i & \phi^i & C^\alpha & C^A & \text{ghost number} & \longrightarrow \end{array} \quad . \tag{52}$$

So it is natural to declare that the pairs ϕ^i, ϕ^*_i; C^α, ϕ^*_α; and C^A, ϕ^*_A, are conjugate:

$$(\phi^i, \phi^*_j) = \delta^i_j \tag{53a}$$

$$(C^\alpha, \phi^*_\beta) = \delta^\alpha_\beta \tag{53b}$$

$$(C^A, \phi^*_B) = \delta^A_B. \tag{53c}$$

[Recall that the indices i, α, A stand for both a discrete index and a continuous one. Explicitly, $(\phi^i(x), \phi^*_j(y)) = \delta^i_j\delta^D(x-y)$, etc. The expressions in Eq. (53) are manifestly covariant in spacetime.] The Lagrangian antibracket (,) is extended to arbitrary functionals A, B of the fields, the ghosts, and the antifields as follows:

$$(A, B) = \frac{\delta^r A}{\delta\phi^i}\frac{\delta^l B}{\delta\phi_i^*} - \frac{\delta^r A}{\delta\phi_i^*}\frac{\delta^l B}{\delta\phi^i} + \frac{\delta^r A}{\delta C^\alpha}\frac{\delta^l B}{\delta\phi_\alpha^*} - \frac{\delta^r A}{\delta\phi_\alpha^*}\frac{\delta^l B}{\delta C^\alpha} + \frac{\delta^r A}{\delta C^A}\frac{\delta^l B}{\delta\phi_A^*} - \frac{\delta^r A}{\delta\phi_A^*}\frac{\delta^l B}{\delta C^A}. \tag{53d}$$

The striking features of the antibracket are the following:

(i) the antibracket carries ghost number +1, i.e.,

$$gh((A, B)) = ghA + ghB + 1; \tag{54a}$$

(ii) it is odd, i.e.,

$$\varepsilon((A, B)) = \varepsilon_A + \varepsilon_B + 1; \tag{54b}$$

and

(iii) it obeys symmetry properties that are opposite to the usual ones,

$$(A, B) = -(-1)^{(\varepsilon_A+1)(\varepsilon_B+1)}(B, A). \tag{54c}$$

So, in particular,

$$(\text{Boson}_1, \text{Boson}_2) = (\text{Boson}_2, \text{Boson}_1) \tag{54d}$$

$$(\text{Fermion}, \text{Boston}) = -(\text{Boson}, \text{Fermion}) \tag{54e}$$

$$(\text{Fermion}_1, \text{Fermion}_2) = -(\text{Fermion}_2, \text{Fermion}_1). \tag{54f}$$

A further important property of the antibracket, an immediate consequence of its definition, is the Jacobi identity

$$(-)^{(\varepsilon_A+1)(\varepsilon_C+1)}(A, (B, C)) + \text{“cyclic”} = 0. \tag{55}$$

Also, the antibracket acts as a derivation

$$(AB, C) = A(B, C) + (-1)^{\varepsilon_B(\varepsilon_C+1)}(A, C)B \tag{56a}$$

$$(A, BC) = (A, B)C + (-1)^{\varepsilon_B(\varepsilon_A+1)}B(A, C). \tag{56b}$$

Because of Eq. (54d), it is in general not true that an arbitrary bosonic functional A has a vanishing antibracket with itself. If $\varepsilon_A = 0$, one may have $(A, A) \neq 0$. However, by the Jacobi identity, $((A, A), A) = 0$.

The antibracket is easily defined along exactly the same lines for more general reducible theories requiring ghosts of ghosts of ghosts. We leave this problem as an exercise to the reader. One finds the same features as in the case investigated explicitly here. In particular, the conjugate to a variable A of parity ε_A and ghost number g_A has itself parity $\varepsilon_A + 1$ and ghost number $-g_A - 1$.

7.4. The Master Equation

As we mentioned above, the BRST symmetry is a canonical transformation in the antibracket. So the BRST variation sA of an arbitrary functional A is given by

$$sA = (A, S). \tag{57a}$$

On account of the parity and ghost number properties of the antibracket, the BRST generator S should be even and have ghost number zero

$$\varepsilon(S) = 0 \qquad gh(S) = 0. \tag{57b}$$

Furthermore, the nilpotency of s is equivalent to

$$(S, S) = 0, \tag{57c}$$

as it follows from the Jacobi identity and the fact that there is no c-number of ghost number one.

The first few terms in S should generate δ and d. This means that, in the expansion of S according to antighost number,

$$S = \sum_{m \geq 0} \overset{(m)}{S} \tag{58a}$$

$$antigh \overset{(m)}{S} = m, \tag{58b}$$

one should have

(i)

$$\overset{(0)}{S} = S_0, \tag{58c}$$

so that $(\phi_i^*, S) = \delta\phi_i^* + \text{more}$;

(ii)

$$\overset{(1)}{S} = \phi_i^* R^i_\alpha C^\alpha, \tag{58d}$$

so that $(\phi^*_\alpha, S) = \delta\phi^*_\alpha$ + more, and $(\phi^i, S) = d\phi^i$ + more $= R^i_\alpha C^\alpha$ + more; and

(iii)

$$\overset{(2)}{S} = \phi^*_\alpha Z^\alpha_A C^A + \text{terms not containing } \phi^*_\alpha C^A, \tag{58e}$$

so that the first terms in $\delta\phi^*_A$ and dc^α are appropriately reproduced.

The problem of finding the BRST symmetry s—i.e., of finding the extra terms S^i, S^α, S^A, T_i, T_α, T_A in Eq. (50)—can thus be reformulated as the problem of finding the solution S of Eq. (57c), with the boundary conditions in Eq. (58). It is remarkable that the first piece in the BRST generator S is just the gauge-invariant action S_0.

Equation (57c) is named the *master equation.* The boundary conditions in Eq. (58) define what is known as a *proper solution.*[6] These boundary conditions follow in our presentation from the form of δ (and d). The structure of the Koszul–Tate differential δ was in turn determined by the requirement $H_k(\delta) = 0$ for $k > 0$ (acyclicity of δ).

7.5. Solution of the Master Equation

The solution of the master equation is derived as follows. As $\overset{(0)}{S}$ and $\overset{(1)}{S}$ are completely fixed by the boundary condition, the first question is to find $\overset{(2)}{S}$, given only incompletely by Eq. (58c). One has

$$\overset{(2)}{S} = \phi^*_\alpha(Z^\alpha_A C^A + k^\alpha_{\beta\gamma} C^\beta C^\gamma) + \phi^*_i\phi^*_j(f^{ij}_A C^A + f^{ij}_{\alpha\beta} C^\alpha C^\beta), \tag{59}$$

where $k^\alpha_{\beta\gamma}$, f^{ij}_A and $f^{ij}_{\alpha\beta}$ are unknown.

The condition $(S, S) = 0$ leads, at antighost number one, to the equation

$$2\delta \overset{(2)}{S} + \overset{(2)}{D} = 0, \tag{60a}$$

where $\overset{(2)}{D}$ is given by

$$\begin{aligned} \overset{(2)}{D} &= (\overset{(1)}{S}, \overset{(1)}{S}) \\ &= -\phi^*_i[R_\alpha, R_\beta]^i C^\alpha C^\beta \\ &= -\phi^*_i C^\gamma_{\alpha\beta} R^i_\gamma C^\alpha C^\beta - \phi^*_i M^{ij}_{\alpha\beta} \frac{\delta S_0}{\delta\phi^j} C^\alpha C^\beta. \end{aligned} \tag{60b}$$

We have used here Eq. (17) and $(\overset{(0)}{S}, \overset{(1)}{S}) = 0$.

To prove the existence of a solution of Eq. (60a), one must check that $\overset{(2)}{D}$ is δ-closed, $\delta\overset{(2)}{D} = 0$. This is easy to do and is left to the reader. The acyclicity of δ (i.e., $H_k(\delta) = 0$ for $k > 0$) implies then that $\overset{(2)}{D}$ is also δ-exact, $\overset{(2)}{D} = -2\delta\bar{S}^{(2)}$. This equation defines $\bar{S}^{(2)}$ only up to a δ-exact term. One can use part of this ambiguity to set the coefficient of $\phi^*_\alpha C^A$ in $\bar{S}^{(2)}$ equal to Z^α_A. With this adjustment, $\bar{S}^{(2)} = S^{(2)}$, and $S^{(2)}$ indeed exists. Actually, the solution $S^{(2)}$ reads explicitly, in terms of its components $k^\alpha_{\beta\gamma}, f^{ij}_A$, and $f^{ij}_{\alpha\beta}$,

$$k^\alpha_{\beta\gamma} = \tfrac{1}{2}C^\alpha_{\beta\gamma} \tag{60c}$$

$$f^{ij}_A = \tfrac{1}{2}C^{ij}_A \tag{60d}$$

$$f^{ij}_{\alpha\beta} = -\tfrac{1}{4}M^{ij}_{\alpha\beta}, \tag{60e}$$

where the structure functions C^{ij}_A are those that appear in Eq. (40b).

The terms from Eqs. (60d) and (60c) complete $\delta\phi^*_A$ and dC^α, i.e., are such that $(\phi^*_A, S) = \delta\phi^*_A +$ "higher order" and $(C^\alpha, S) = dC^\alpha +$ "higher order." This is as it should be, and we could have included Eqs. (60c, d) as part of the boundary condition, Eq. (58e). What our analysis shows is that this is not necessary as Eqs. (60c, d) are in fact forced by $(S, S) = 0$ (which contains $\delta^2 = 0$, $d^2 = 0$).

Once $\overset{(2)}{S}$ is constructed, the analysis of the remaining terms in the master equation proceeds recursively along similar lines. Assume that S has been constructed up to order $n - 1$ ($n \geq 3$), and let

$$\overset{(n-1)}{R} = \sum_{i \leq n-1} \overset{(i)}{S}. \tag{61}$$

It is easy to check that for any A of antighost number k_a, the component of antighost number $k_a - 1$ in $(A, R^{(n-1)})$ reads

$$(A, \overset{(n-1)}{R}) = \delta A + \text{higher orders } (n \geq 3) \tag{62a}$$

$$antigh\, A = k_a \tag{62b}$$

$$antigh(\text{higher orders}) > k_a - 1. \tag{62c}$$

This is because only the pieces at most linear in the ghosts in $\overset{(n-1)}{R}$ contribute to the antighost number $k_a - 1$ component of $(A, \overset{(n-1)}{R})$. This property selects $\overset{(0)}{S}$, $\overset{(1)}{S}$ and $\overset{(2)}{S}$ because $\overset{(k)}{S}$ for $k \geq 3$ is at least quadratic in the ghosts. One verifies that $\overset{(0)}{S}$, $\overset{(1)}{S}$ and the linear piece of $\overset{(2)}{S}$ indeed yields δA.

The equation $(S, S) = 0$ then reads, at antighost number $n - 1$,

$$2\delta\overset{(n)}{S} + \overset{(n-1)}{D} = 0. \tag{63}$$

Here $\overset{(n-1)}{D}$ is the component of antighost number $n-1$ of $(\overset{(n-1)}{R}, \overset{(n-1)}{R})$ and depends only on the functions $\overset{(k)}{S}$, $k \leq n-1$.*

Now, for Eq. (63) to possess a solution $\overset{(n)}{S}$ given $\overset{(k)}{S}$ with $k \leq n-1$, it is necessary (by the nilpotency of δ) and sufficient (by the acyclicity of δ) that $\delta \overset{(n-1)}{D} = 0$. But this simply follows from the Jacobi identity $0 = ((\overset{(n-1)}{R}, \overset{(n-1)}{R}), \overset{(n-1)}{R})$, which implies $\delta \overset{(n-1)}{D} = 0$ at antighost number $n-2$.

We can thus conclude that the solution S of the master equation exists. The solution is not unique because, at each stage, one can add a δ-exact term to $\overset{(n)}{S}$. However, because of Eq. (62), this only modifies S by a canonical transformation (in the antibracket). Hence the solution of the master equation with the boundary conditions of Eq. (58) exists and is unique up to canonical transformations.[12] As a result, the BRST symmetry $sA = (A, S)$ also exists.

It is an easy exercise to check that canonical transformations also enable one to pass from one generating set R^i_α to any other generating set $\bar{R}^i_\alpha$ of the same dimension. The enlargement of the generating sets by adding trivial gauge transformations and increasing the ghost spectrum requires a further concept, that of a "nonminimal solution," and will be discussed below (section 7.8).

It should be pointed out that, in general, the components $\overset{(3)}{S}, \overset{(4)}{S}, \ldots$ are different from zero, so that S contains multighost vertices. In the irreducible group case, only $\overset{(0)}{S}$, $\overset{(1)}{S}$, and $\overset{(2)}{S}$ are different from zero, but this is an accident not representative of the general situation. It should also be stressed that nowhere was it necessary to fix the gauge so far, and that the existence of S is global in field space, because the acyclicity of δ is a global statement. Global obstructions may be relevant when discussing gauge-fixing conditions (Gribov problem) but do not afflict the gauge-independent BRST symmetry in the space of the fields, the ghosts, and the antifields.

7.6. Space-time Locality of S

In order to apply the usual methods of quantum field theory, it is necessary that S be a local functional in spacetime,

$$S = \int \mathscr{L} d^D x \tag{64a}$$

* One has $(\overset{(n-1)}{R}, \overset{(n-1)}{R}) = \overset{(n-1)}{D}$ + higher orders. The lower antighost number components of $(\overset{(n-1)}{R}, \overset{(n-1)}{R})$ vanish as the functions $\overset{(0)}{S}, \ldots, \overset{(n-1)}{S}$ obey $(S, S) = 0$ up to order $n-2$.

where $\mathscr{L}$ is a function of the fields, the ghosts, the antifields, and their derivatives up to some finite order ("local function"). This is equivalent to

$$\overset{(n)}{S} = \int \overset{(n)}{\mathscr{L}} d^D x, \tag{64b}$$

where the $\overset{(n)}{\mathscr{L}}$ are also local functions.

Is it guaranteed that Eqs. (64a, b) hold? To investigate this question, let us assume again that the $\overset{(k)}{S}$ have been constructed up to order $n-1$ and are local functionals. [$\overset{(0)}{S}$ is clearly a local functional, as well as $\overset{(1)}{S}$ and the given piece of $\overset{(2)}{S}$ if the gauge transformations and the reducibility equations are local]. Eq. (63) reads, in terms of the searched-for local function $\overset{(n)}{\mathscr{L}}$,

$$2\delta \overset{(n)}{\mathscr{L}} + \overset{(n-1)}{d} = \partial_\mu \overset{(n-1)\mu}{k}, \tag{65a}$$

where $\overset{(n-1)}{D} = \int \overset{(n-1)}{d} d^D x$ and where k^μ is some local current, which yields a surface term when one integrates both sides of Eq. (65). As $\overset{(n-1)}{R}$ is a local functional, $\overset{(n-1)}{D} = (\overset{(n-1)}{R}, \overset{(n-1)}{R})$ is also a local functional, so that $\overset{(n-1)}{d}$ in $\overset{(n-1)}{D} = \int \overset{(n-1)}{d} d^D x$ is indeed a local function. Furthermore, because $\delta \overset{(n-1)}{D} = 0$, $\overset{(n-1)}{d}$ is subject to

$$\delta \overset{(n-1)}{d} = \partial_\mu j^\mu \tag{65b}$$

The known function in Eq. (65a) is $\overset{(n-1)}{d}$, which is really determined from $\overset{(n-1)}{D}$ only up to a local divergence $\partial_\mu \beta^\mu$, but we assume that some definite choice has been made. The unknown functions are $\overset{(n)}{\mathscr{L}}$ and $\overset{(n-1)\mu}{k}$. Once these are found, S obeys $(S, S) = \int \partial_\mu k^\mu$, with $k^\mu = \sum \overset{(i)\mu}{k}$. The boundary conditions must be such that the surface term is zero.

So, the question of space-time locality of S can be reformulated as the problem of the local homology of δ: given a local function f such that $\delta f = \partial_\mu j^\mu$ with j^μ local, is it guaranteed that $f = \delta g + \partial_\mu k^\mu$ where both g and k^μ are local? The function f is also known to be of strictly positive pure ghost number ($\overset{(n-1)}{D}$ involves the ghosts).

It turns out that the answer to this question is positive, provided the gauge transformations $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha$ obey the following local completeness condition: any local identity on the field equations $\delta S_0/\delta\phi^i$ can be derived from the Noether identity $(\delta S_0/\delta\phi^i)R^i_\alpha = 0$ by local means, i.e., by differentiation and algebraic manipulations (but no integration). This assumption is very mild as it appears to be always fulfilled by appropriate redefinitions of the gauge transformations if necessary.

To give an example of gauge transformations that do not obey the local completeness condition, consider electromagnetism, invariant under

$$\delta A_\mu = \partial_\mu(\Delta\varepsilon). \tag{66a}$$

With appropriate boundary conditions at spatial infinity, this parametrization of the gauge transformations is equivalent to the standard one, $\delta A_\mu = \partial_\mu\Lambda$, $\Lambda = \Delta\varepsilon$, $\varepsilon = \Delta^{-1}\Lambda$. The Noether identities that follow from Eq. (66) are

$$\Delta\partial_\mu \frac{\delta S_0}{\delta A_\mu} = 0. \tag{66b}$$

The identities $\partial_\mu(\delta S_0/\delta A_\mu) = 0$ cannot be derived from Eq. (66b) by local means, as one needs to invert the Laplacian. So Eq. (66a) does not obey the local completeness condition. However, the change of gauge parameters $\Delta\varepsilon = \Lambda$ yields a form of the gauge transformations that obey the local completeness assumption.

One can then prove the following:

Theorem. If the local function f of antighost number $k > 0$ obeys $\delta f = \partial_\mu j^\mu$ (with j^μ local) and is of strictly positive pure ghost number, then

$$f = \delta g + \partial_\mu k^\mu,$$

where g and k^μ are local functions. In other words, the local homology of δ modulo the space-time exterior differential d is trivial.

The proof of this theorem is given in Ref. 20. Let us simply indicate that the restriction on the pure ghost number is important. Consider for example the free particle, $S_0 = \frac{1}{2}\int dt\, \dot{q}^2$. One has one antifield q^* with $\delta q^* = \ddot{q}$ and no ghost. The function $f \equiv q^*$ obeys $\delta f = (d/dt)(\dot{q})$ but cannot be written as $\delta g + dk/dt$ with local and regular g, k. This does not contradict the theorem because the pure ghost number of f is zero.

If the gauge transformations are reducible, a similar local completeness assumption must be made on the reducibility functions.

The theorem guarantees the space-time locality of the solution S of the master equation—at least if the rank of the theory, i.e., the highest n for which $\overset{(n)}{S}$ is nonvanishing, is finite.

7.7. Antibracket and Equivalence Classes of BRST-invariant Observables

Given two BRST-invariant functionals A and B, the antibracket (A, B) depends only on the cohomological classes of A and B: if $A' = A + (K, S)$, $B' = B + (L, S)$, then (A, B) and (A', B') are in the same cohomological class. So there is a well-defined antibracket structure in the space of cohomological classes of BRST-invariant functions.

Because the antibracket (A, B) of two BRST-invariant functions of ghost number zero possesses ghost number one, it is clear that the induced antibracket has no direct connection with the Poisson bracket that can be defined among gauge-invariant observables.[16–19] Furthermore, although we have no complete proof of this property, there is some evidence that (A, B) is actually cohomologically trivial, i.e., (A, B) is BRST-exact, $(A, B) = (K, S)$. The induced structure appears thus to be completely trivial.

These are the first indications that the antibracket has no obvious physical meaning in spite of its usefulness in the construction of S.

7.8. Nonminimal Solutions

The requirement $H^0(s) = \{\text{gauge-invariant functions}\}$ does not completely fix s. Indeed, it is always possible to add to a given solution further variables that are cohomologically trivial and hence do not modify $H^k(s)$. The uniqueness theorem given above was derived with the specific set of fields ϕ^i, C^α, C^A and antifields ϕ_i^*, ϕ_α^*, ϕ_A^* and would not apply if this set had been enlarged.

Cohomologically trivial variables can be assumed, with appropriate redefinitions, to fulfill

$$s\bar{c} = \pi, \qquad s\pi = 0, \qquad gh(\bar{c}) = gh(\pi) - 1. \tag{67a}$$

The condition $sF = 0$ eliminates $\bar{C}$. The further passage to the quotient by BRST-exact functions eliminates π. So $\bar{C}$ and π do not contribute to $H^k(s)$.

If one requires a canonical action for the BRST symmetry, one must introduce antifields $\bar{C}^*$ and π^*, respectively, conjugate to $\bar{C}$ and π,

$$(\bar{C}, \bar{C}^*) = 1, \qquad (\pi, \pi^*) = 1 \tag{67b}$$

$$gh\bar{C}^* = -ghC - 1, \qquad gh\pi^* = -gh\pi - 1 \tag{67c}$$

The term that generates Eq. (67a) through the antibracket reads

$$\bar{C}^*\pi. \tag{67d}$$

One has $s\pi^* = \bar{C}^*$, $s\bar{C}^* = 0$, so the pair π^*, $\bar{C}^*$ is also cohomologically trivial.

The general solution of the master equation is given by

$$\bar{S}(\phi^i, C^\alpha, C^A, \phi_i^*, \phi_\alpha^*, \phi_A^*; \pi, \bar{C}, \pi^*, \bar{C}^*) = S(\phi^i, c^\alpha, C^A, \phi_i^*, \phi_\alpha^*, \phi_A^*) + \Sigma\bar{C}^*\pi, \tag{68}$$

where S is the minimal solution described above, depending on the minimal set of fields ϕ^i, C^α, C^A, ϕ_i^*, ϕ_α^*, ϕ_A^*, and where π, $\bar{C}$, π^*, $\bar{C}^*$ stand for all the trivial variables that are added. The solution $\bar{S}$ containing extra variables is known as a *nonminimal* solution. Nonminimal solutions are unique modulo canonical transformations *and* addition of cohomologically trivial pairs.

Whether or not extra pairs are required depends on the type of gauge-fixing condition desired. This point will be illustrated in the examples below. Let us simply mention now that the usual antighosts are part of the non-minimal sector.

The relation between the reducible and irreducible descriptions of the same gauge symmetry also becomes clear: the corresponding S are related by a canonical transformation and by the addition of cohomologically trivial pairs.

7.9. Abelian Form of S

As we indicated in section 2.11, it is always possible to Abelianize the gauge transformations. Furthermore, one can redefine the field variables $\phi^i \to \chi^i = \chi^i(\phi^j)$, $\chi^i = (\chi^{\bar{a}}, \chi^{\bar{\alpha}})$, in such a way that the first variables $\chi^{\bar{a}}$ are gauge-invariant and the gauge transformations are just shifts in the last variables $\chi^{\bar{\alpha}}$. This change of variables is generically nonlocal and full of functional subtleties, which we will not address here.

The action S_0 depends only on $\chi^{\bar{a}}$ as it is gauge invariant. Together with the boundary conditions, the equations $\delta S_0/\delta\chi^{\bar{a}} = 0$ completely determine $\chi^{\bar{a}}$. The gauge components $\chi^{\bar{\alpha}}$ are completely arbitrary.

The fields $\chi^{\bar{a}}$ may not be all propagating (the equations $\delta S_0/\delta\chi^{\bar{a}} = 0$ may imply $\chi^{\bar{a}} = 0$ for some $\bar{a}$), so that the number of true degrees of freedom is in general smaller than the number of $\chi^{\bar{a}}$.

A complete set of gauge transformations is given by

$$\delta\chi^{\bar{a}} = 0, \qquad \delta\chi^{\bar{\alpha}} = \varepsilon^{\bar{\alpha}}. \tag{69a}$$

The remaining (reducible) gauge transformations can be taken to be

$$\delta\chi^i = O \cdot \varepsilon^A \tag{69b}$$

($\alpha = (\bar{\alpha}, A)$). So one has $R^i_A = 0$ and the reducibility equations read $Z^A_B R^i_A = 0$, with $Z^A_B = \delta^A_B$.

The solution of the master equation is given by

$$S = S_0(\chi^{\bar{a}}) + \chi^*_{\bar{\alpha}} C^{\bar{\alpha}} + C^*_A C^A, \tag{70}$$

where C^*_A are the antifields of ghost number 2 conjugate to the ordinary ghosts associated with the ineffective gauge transformations in Eq. (69b)—and not the antifields conjugate to the ghosts of ghosts C^A. The noticeable feature of S is that it differs from the gauge-invariant action $S_0(\chi_{\bar{a}})$ by manifestly cohomologically trivial terms that possess exactly the same structure as the nonminimal terms in Eq. (68).

8. PATH INTEGRAL

8.1. Gauge Invariance of the Master Equation

We now turn to the problem of writing down the path integral. If we were working within the Hamiltonian formalism, there would not be much to say because all the work has been done once the BRST symmetry in Hamiltonian form is constructed. The path integral is simply $\int D$ (Hamiltonian variables) $\exp i \int$ [Hamiltonian kinetic term $- H$] dt, where

1. *all* the variables of the Hamiltonian formalism occur in the path integral;
2. the Hamiltonian kinetic term is the one that yields the Hamiltonian Poisson bracket among the canonical variables (momenta times time derivative of coordinates in canonical coordinates); and
3. the Hamiltonian H is one representative in the BRST cohomological class associated with the original, gauge-invariant Hamiltonian H_0.

Different choices of representatives amount to different choices of gauge.[5,9] The path integral is well defined because the action that appears in the integrand is not degenerate: the Hamiltonian equations of motion that follow from it are in normal form and hence possess a unique solution for given initial data.

The same approach cannot be applied to the Lagrangian case, and the straightforward attempt

$$\text{Path integral} = \int D(\text{fields})\, D(\text{antifields})\, D(\text{ghosts}) \exp \frac{i}{\hbar} S, \tag{71}$$

where S is the solution of the master equation, does not work. This is because S is gauge-invariant and thus Eq. (71) as it stands is meaningless.

The gauge invariance of the solution S of the master equation is easy to work out. Let us denote collectively the original fields ϕ^i, the ghosts C^α, and all the necessary ghosts of ghosts by ϕ^A $(A = 1, \ldots, N)$. These also include the antighosts and the auxiliary fields of the nonminimal sector, if any. We will refer to ϕ^A as the "fields". All the remaining variables will be denoted by ϕ^*_A: i.e., the antifields ϕ^*_i, $\phi^*_\alpha, \ldots, \bar{C}^*$, π^*, etc.* Finally, let

* So ϕ^*_A stands from now on for all the antifields and not just for the antifields of antighost number three associated with the reducibility equations.

us set

$$z^a = (\phi^A; \phi^*_A), \qquad a = 1, \ldots, 2N \tag{72a}$$

and

$$\varepsilon(z^a) = \varepsilon_a. \tag{72b}$$

The antibracket can then be written as

$$(A, B) = \frac{\delta^r A}{\delta z^a} \zeta^{ab} \frac{\delta^l B}{\delta z^b}, \tag{73a}$$

where the (inverse of the) "symplectic form" ζ^{ab} reads

$$\zeta^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}, \qquad \zeta^{ab} = -\zeta^{ba}. \tag{73b}$$

In these notations, the master equation becomes

$$(S, S) = \frac{\delta^r S}{\delta z^a} \zeta^{ab} \frac{\delta^l S}{\delta z^b} = 0, \tag{74a}$$

from which one easily derives, upon differentiation with respect to z^c, that

$$\frac{\delta^r S}{\delta z^a} \mathscr{R}^a_c = 0. \tag{74b}$$

Here, we have set

$$\mathscr{R}^a_c = \zeta^{ab} \frac{\delta^l \delta^r S}{\delta z^b \delta z^c}. \tag{74c}$$

These equations, which express the fact that $s^2 z^c$ is zero, indicate that the functional S is gauge-invariant under

$$\delta z^a = \mathscr{R}^a_c \varepsilon^c \quad \left(\Leftrightarrow \quad \delta z^a(x) = \int \mathscr{R}^a_c(x, y) \varepsilon^c(y)\, dy\right), \tag{75}$$

where $\varepsilon^c(y)$ are arbitrary space-time functions.

How many gauge invariances does S possess? Superficially, $2N$, which is the total number of fields and antifields. It actually turns out that the action S has fewer independent gauge invariances, because the matrix $\mathscr{R}^a_c$ defining the gauge transformations is nilpotent on-shell,

$$\mathscr{R}^a_b \mathscr{R}^b_c = 0 \quad \left(\text{when equations of motion } \frac{\delta S}{\delta z^a} = 0 \text{ hold}\right). \tag{76}$$

This can be seen by a further differentiation of Eq. (74b). The gauge transformations [Eq. (75)] are thus not all independent; there is "on-shell reducibility."

Because $\mathscr{R}^2 \approx 0$, the number of independent gauge transformations in Eq. (75) is at most equal to N. It is actually precisely equal to N because the general solution of $\mathcal{R}v \approx 0$ is given by $v \approx \mathcal{R}t$, so that the nilpotent matrix $\mathcal{R}$ contains only (rank one) two-dimensional Jordan blocks $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, making its total rank equal to $2N/2 = N$.

Furthermore, one can also show that Eq. (75) exhausts all the gauge symmetries of S. So the solution of the master equation possesses exactly N independent gauge transformations.

The proofs of these statements are most conveniently derived by making the canonical change of variables $z^a \to \bar{z}^a$ such that $S(\bar{z}^a)$ takes the simple form [Eq. (70)]. This is permissible, as the matrix $\mathcal{R}^a_b$ transforms on-shell as a $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ tensor under canonical transformations, so its rank properties are unchanged in $z^a \to \bar{z}^a$. One can easily check that the gauge transformations of $S(\bar{z}^a)$ are just the arbitrary shifts in the N variables (say z_2^A) that do not occur in $S(z^a) \equiv S(z_1^A)$, and that these transformations can be written as in Eq. (75) because the matrix $\delta^2 S/\delta z_1^A \delta z_1^B$ is invertible.

It is remarkable that the solution S of the master equation contains all its gauge symmetries in the sense that these are just obtained by differentiation of S. Furthermore, for each field–antifield pair, there is one gauge symmetry. This property was the motivation of Ref. 6.

8.2. BRST Invariance as the Guiding Principle for Deriving the Gauge-fixed Action

Because the solution of the master equation is still gauge invariant, with a gauge-algebraic structure that presents no obvious simplification over the original one, it might be felt that nothing has been gained in the construction and that one is exactly back to the original difficulty of writing the correct gauge-fixed action, without new insight. Something has been gained, however, and this is that we now have the BRST symmetry at our disposal. Because BRST invariance can be used as a substitute for gauge invariance, one can completely forget about the gauge symmetries and simply focus on the BRST symmetry. If one can write down a gauge-fixed action that incorporates BRST invariance, then one has also automatically incorporated into the path integral the gauge symmetry of the original action.

This is the point of view developed in the sequel. This means that no attempt will be made to devise an appropriate gauge fixing of the gauge symmetries of S along conventional lines. These gauge symmetries will cease to be of any concern from now on, and were mentioned only to point out that S is not a propagating action. By a "propagating action," we mean one without gauge invariance. Our only concern will be to extract a propagating action S_ψ from S in a manner that incorporates all the properties of the BRST formalism. Because the BRST symmetry is a global invariance rather than a local gauge symmetry, one can find BRST-invariant actions that are nondegenerate.

8.3. Gauge-fixed Action

One possibility for getting a nondegenerate (=gauge-fixed) action is simply to eliminate N of the $2N$ fields/antifields z^a by means of N equations $\Omega^A(z^a) = 0$ (N "gauge conditions" for the N independent gauge invariances of S).

It turns out that the properties of the BRST formalism are preserved if one takes the functions Ω^A to be in involution, i.e.,

$$(\Omega^A, \Omega^B) = 0. \tag{76a}$$

One motivation for Eq. (76a) is that these conditions are invariant under canonical transformations. This is important, as canonical transformations account for the ambiguity in S, which should be ultimately irrelevant. Another motivation is that the equations $\Omega^A = 0$ describing the gauge fixing actually involve, with Eq. (76a), a single arbitrary function, as in the Hamiltonian formalism.[5,9] This fact will be crucial in proving the independence of physical amplitudes on the choice of Ω^A.

To see that the Ω^A involve a single function, let us solve $\Omega^A = 0$ for the antifields. (We assume that this can be done.) Then, the equations $\phi^*_A - \omega_A(\phi) = 0$ are in involution iff

$$\Omega^A = 0 \quad \Leftrightarrow \quad \phi^*_A = \frac{\delta\psi}{\delta\phi^A} \tag{76b}$$

for some function $\psi(\phi^A)$ of ghost number -1 and Grassmann parity 1,

$$\psi(\phi^A), \qquad gh\psi = -1, \qquad \varepsilon(\psi) = 1. \tag{76c}$$

The functional $\psi(\phi^A)$ must be local in space-time, $\psi = \int \rho \, d^Dx$, so that the antifields ϕ^*_A are local functions of the fields and their derivatives.

If one inserts Eq. (76b) inside the solution $S(\phi, \phi^*)$ of the master equation, one gets the gauge-fixed action S_ψ:

$$S_\psi = S\left(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}\right). \tag{77}$$

The remaining part of these lectures will be devoted to showing that the action S_ψ correctly governs the path integral. In particular, the path integral will be proved not to depend on ψ.

Before carrying on the analysis, it is necessary to make some comments on Eq. (77):

1. Different choices of ψ effectively correspond not only to different choices of gauge conditions but also to different ways to enforce them in the path integral (delta functions, Gaussian average). This will be illustrated below in the case of electromagnetism.

2. The function(al) $\psi(\phi)$ is required to be such that S_ψ is propagating. That is, S_ψ should have no gauge invariance. [This excludes $\psi = 0$ in the case of Eq. (76), where it is the antifields that are eliminated, since then S_ψ reduces to $S_0(\phi^i)$. In the path integral, the integration over the gauge directions would yield infinity, while the integration over the ghosts would yield zero. This appears to be a generic feature of bad choices of ψ. The resulting path integral is intrinsically ill-defined—rather than giving a well-defined but incorrect answer.]

3. If one makes the canonical "phase" transformation

$$\phi^A, \phi^*_A \rightarrow \bar{\phi}^A = \phi^A, \qquad \bar{\phi}^*_A = \phi^*_A - (\delta^r\psi/\delta\phi^A),$$

the "gauge conditions" in Eq. (76b) can be written as $\bar{\phi}^*_A = 0$. The functional form of S is not invariant under the canonical transformation if ψ defines a propagating action.

4. It will be seen that in the path integral, one does not sum over the antifields as these no longer appear in S_ψ. The integration variables ϕ^A obey $(\phi^A, \phi^B) = 0$, so reference to the antibracket are completely lost.

5. One could in principle eliminate some of the fields in favor of the corresponding antifields, i.e., solve $\Omega^A = 0$ for some of the fields. This will be illustrated below. The integration variables that are left over in that more general case are obtained by picking out, from each conjugate pair, either the field or the antifield. These integration variables have, again, vanishing brackets. Reference to the antibracket is thus again lost. The symmetry between fields and antifields is not complete, however, as the requirement that S_ψ is propagating forces one to keep the gauge invariant fields among the integration variables.

6. If some $\overset{(n)}{S}$, $n \geq 3$ are different from zero, S_ψ will contain quartic, sixtic, ... ghost interactions. These ghost interactions are crucial and follow from BRST invariance. They would be missed by a naive application of the Faddeev–Popov determinant method.

8.4. Gauge-fixed BRST Symmetry—Gauge-fixed BRST cohomology

The gauge-fixed form $\bar{s}$ of the BRST symmetry is defined by

$$\bar{s}\phi^A = (s\phi^A)\left(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}\right) \equiv \frac{\delta^l S}{\delta\phi^{*A}}\left(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}\right). \qquad (78)$$

If $s\phi^A$ depends on the antifields, i.e., if S is more than linear in the antifields, the gauge-fixed BRST symmetry (in Lagrangian form) *depends on the gauge-fixing fermion* ψ.

We leave it to the reader to check the following straightforward assertions:

1. The gauge-fixed action is BRST invariant under Eq. (78):

$$\bar{s}S_\psi = 0. \tag{79}$$

Hence, one can define a conserved Noether charge Ω_ψ that depends in general on ψ.

2. The BRST variation $\bar{s}\phi^*_A$ of the antifields viewed as functions of the fields differs from $(s\phi^*_A)(\phi, \phi^* = (\delta\psi/\delta\phi))$ by equation-of-motion terms:

$$\bar{s}\phi^*_A = s\phi^*_A + \frac{\delta^l S_\psi}{\delta\phi^A}. \tag{80a}$$

Hence

$$\bar{s}B = sB + \frac{\delta^r B}{\delta\phi^*_A}\frac{\delta^l S_\psi}{\delta\phi^A}. \tag{80b}$$

3. Because of this, the gauge-fixed BRST symmetry (in Lagrangian form) is in general only on-shell nilpotent,

$$\bar{s}^2\phi^A = \text{field equations}, \tag{81}$$

where the field equations in Eq. (81) are those of the *gauge-fixed action.* The right-hand side of Eq. (81) identically vanishes if and only if $\delta^2 S/\delta\phi^A\delta\phi^B$ is zero. For open algebras, $\delta^2 S/\delta\phi^A\delta\phi^B \neq 0$ and $\bar{s}^2 \neq 0$.

4. Because of the invariance of the action S_ψ, the surface $\delta S_\psi/\delta\phi^A = 0$ is left invariant under Eq. (78). One can thus define the gauge-fixed BRST cohomology as the space of equivalence classes of weakly BRST invariant functions $A(\phi)$ modulo the weakly BRST exact ones,

$$\bar{s}A = \lambda^A \frac{\delta S_\psi}{\delta\phi^A} \tag{82a}$$

$$A \sim B \quad \text{iff} \quad A = B + \bar{s}C + \mu^A \frac{\delta S_\psi}{\delta\phi^A}. \tag{82b}$$

Here, "weakly" means "on the surface $\delta S_\psi/\delta\phi^A = 0$."

5. One can define a Koszul resolution $\bar{\delta}$ for the surface $\delta S_\psi/\delta\phi^A = 0$. The antifields $\bar{\phi}^*_A \equiv \phi^*_A - \delta\psi/\delta\phi^A$ can be viewed as the generators of this resolution. No further generator is needed as the equations $\delta S_\psi/\delta\phi^A = 0$ are independent. The relationship between $(\bar{\delta}, \bar{s})$ and (δ, s) is algebraically identical to the relationship between (δ, d) and (δ, s). The same algebraic techniques imply, therefore, $H^k(s) = H^k(\bar{s})$, where $\bar{s}$ is understood to act on $\delta S_\psi/\delta\phi^A = 0$. Hence, the gauge-fixed BRST cohomology at ghost number zero is again given by the gauge-invariant functions.

Given an element $A(\phi, \phi^*)$ in $H^k(s)$, the corresponding element in $H^k(\bar{s})$, can be taken to be

$$A_\psi(\phi) = A\left(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}\right). \tag{83a}$$

Indeed, the equation $(A, S) = sA = 0$ and Eq. (80) imply

$$\bar{s}A_\psi = \lambda^A \frac{\delta^l S_\psi}{\delta\phi^A}, \qquad \lambda^A = \frac{\delta^r A}{\delta\phi^*_A}. \tag{83b}$$

Conversely, given a solution $\bar{A}(\phi)$ of $\bar{s}\bar{A} = (\delta S_\psi/\delta\phi^A)\lambda^A$, one can recursively construct a solution $A(\phi, \phi^*) = \bar{A}(\phi) - (\phi^*_A - \delta\psi/\delta\phi^A)\lambda^A + 0((\phi^*_A - (\delta\psi/\delta\phi^A))^2)$ of $(A, S) = 0$ using the acyclicity of $\bar{\delta}$ and the Jacobi identity for the antibracket.

6. When the solution of the master equation is linear in the antifields, the gauge-fixed action S_ψ can be written as $S_\psi = S_0 + s\psi$. One recovers the familiar formulas of Ref. 31, applicable to closed algebras.

7. As we have just seen, the action S_ψ is not linear in the gauge-fixing fermion ψ when the solution of the master equation is not linear in the antifields. Furthermore, the BRST variation $s\phi^A$ of the fields involves the antifields. Let $\tilde{s}\phi^A$ be the ϕ^*-independent component of $s\phi^A$ [i.e., $\tilde{s}\phi^A = \tilde{s}\phi^A(\phi, \phi^* = 0)$], and let $\tilde{S}_\psi = S_0 + \tilde{s}\psi$. One finds that $\tilde{S}_\psi$ is not invariant under $\tilde{s}$. However, $\tilde{s}\tilde{S}_\psi = 0(\psi)$, and the nonvanishing terms in $\tilde{s}\tilde{S}_\psi$ are proportional to the functional derivatives of S_0, $\tilde{s}\tilde{S}_\psi \sim \delta S_0/\delta\phi^i$. Thus, by modifying $\tilde{s}\phi^i$, one can remove these terms. Following Noether lines, one then constructs recursively both $\bar{s} = \tilde{s} + 0(\psi)$ and $S_\psi = \tilde{S}_\psi + 0(\psi^2)$ so that $\bar{s}S_\psi = 0$. These $\bar{s}$ and S_ψ just coincide with the ones obtained by the above methods. The Noether approach was followed in the original work.[7,8] As our remark indicates, this approach has close connections with the methods of homological perturbation theory.

8.5. Hamiltonian Formulation

The gauge-fixed action S_ψ is a local functional and possesses no gauge invariance. Hence, it can be rewritten in Hamiltonian form without difficulty. If there were problems in going to the Hamiltonian formalism, this would mean, by definition, that the gauge-fixing procedure has not been correctly performed, and we assume that this is not so.*

We will also assume that the original gauge-invariant Lagrangian is nonpathological. By this we mean that the Lagrangian does not provide a counterexample to the Dirac conjecture,[14,15] i.e., that it exhibits all the relevant gauge symmetries. Under these conditions, the Lagrangian and Hamiltonian gauge transformations are equivalent. For more information, see Ref. 14. The usual Lagrangian of physical interest fall into this class.

* The Hamiltonian formalism can be developed even if the Lagrangian contains higher-order time derivatives. One simply needs more conjugate pairs. Also, there could be some second-class constraints in the Hamiltonian formalism. But these can be eliminated by means of the Dirac bracket method, and we assume that this has been done. The Hamiltonian formulation is then free of constraints, and all the equations of motion are dynamical.

For such Lagrangians, the Lagrangian and Hamiltonian concepts of gauge-invariant functions are equivalent, and there is a single bracket structure defined among them. The dynamics of the gauge-invariant functions is, of course, also the same in either description.

Now, to any local-in-time functional A of the fields and their time derivatives up to some finite order, one can associate, by using the equations of motion if necessary, a well-defined phase-space function. In particular, if one expresses the Noether charge Ω_ψ in terms of the canonical variables, one gets a phase-space function with the following features: (a) Ω_ψ is off-shell nilpotent because the (Dirac) bracket $[\Omega_\psi, \Omega_\psi]$, which should be zero on-shell, does not contain the time derivatives, i.e., cannot involve the equations of motion. Thus, it must identically vanish, $[\Omega_\psi, \Omega_\psi] = 0$. (b) The canonical transformation generated by Ω_ψ starts like a gauge transformation because $\bar{s}\phi^i = R^i_{\ \alpha} C^\alpha +$ "more."

The properties (a) and (b) are just the defining properties of the Hamiltonian BRST charge. From the general theorems on the existence and uniqueness of the BRST charge in the Hamiltonian formalism, one can thus infer that Ω_ψ differs from the gauge-independent BRST charge Ω constructed along Hamiltonian lines[9] at most by a canonical change of variables (in the Dirac bracket) and the possible addition of cohomologically trivial pairs. The canonical transformation relating Ω_ψ to Ω may have a complicated, ψ-dependent structure, but it is nevertheless *canonical.*

Similarly, in each cohomological class of the gauge-fixed BRST cohomology, one can find one function $A_\psi(t)$ that involves only the fields and their independent time derivatives at time t (initial data at t). This is because one can add equation-of-motion terms in Eq. (82b). If one rewrites $A_\psi(t)$ in terms of the canonical variables, one gets a phase-space function such that $[A_\psi, \Omega_\psi] = 0$. This implies that the Hamiltonian BRST cohomology and the gauge-fixed cohomology are also isomorphic. Therefore, the Hamiltonian BRST cohomology at ghost number zero is given by the gauge invariant observables, a result derived differently, along purely Hamiltonian lines, in Refs. 9, 27, 29.

We can thus conclude that the Lagrangian and Hamiltonian BRST formalisms are equivalent for standard Lagrangians. The equivalence is revealed upon making the Legendre transformation on S_ψ—if that action is not already in first-order form. Further discussion on the comparison between the Lagrangian and Hamiltonian formulations of the BRST symmetry may be found in Refs. 32–34.

We can, at this point, develop the path integral formalism along two different lines.

1. One possibility is to base the whole discussion on the Hamiltonian formalism. The path integral is then clearly related to definite expectation

values of operators and yields manifestly unitary answers. The Hilbert-space apparatus can be used to define what is meant by the path-integral expressions. This approach has the advantage of being self-contained—at least formally, i.e., if the operator formalism indeed exists. Furthermore, with the introduction of the conjugate momenta, which are quantum-mechanically realized as operators, off-shell nilpotency is achieved even in the open-algebra case. This greatly simplifies the discussion and is a key element of the operator formulation of the quantum theory.

2. Another possibility is to write down directly the Lagrangian path integral in such a manner that it fulfills the following important requirement: in the Abelian representation, it should reduce to a path integral over the gauge-invariant degrees of freedom only. The gauge and ghost modes should decouple and drop out of the theory, which becomes manifestly equivalent to the theory in which only the gauge-invariant degrees of freedom are present. This non-Hamiltonian approach possesses a high degree of inner consistency, but is less precise than the Hamiltonian approach. For instance, as we shall see, it fails to yield the complete expression for the integration measure. This is because the measure for the gauge-invariant degrees of freedom is not determined by the above requirement. The ambiguity, however, affects only terms that are of formal higher order in $\hbar$ ("quantum corrections"), but which nevertheless may play an important role. It is not inconceivable that this shortcoming could be overcome some day by non-Hamiltonian means.

We will follow the Lagrangian lines here. The Hamiltonian results are mentioned only for the purpose of providing some insight into the Lagrangian derivation.

8.6. The Integration Measure: the Problem

The gauge-fixed action enables one to compute transition amplitudes as Lagrangian path integrals,

$$Z_\psi = \int [D\mu]\, e^{(i/\hbar)S_\psi}. \tag{84a}$$

Here, $[D\mu]$ is the integration measure

$$[D\mu] = [D\phi^A]\mu \tag{84b}$$

$$\mu = \mu_0(1 + \hbar\mu_1 + \hbar^2\mu_2 + \cdots). \tag{84c}$$

If desired, one can incorporate the measure into the action by exponentiating it:

$$Z_\psi = \int [D\phi^A]\, e^{(i/\hbar)(S_\psi + (\hbar/i)\ln \mu_0 + O(\hbar^2))}. \tag{84d}$$

A correct way to determine the integration measure is to start from the Hamiltonian path integral, for which the measure is known to be the product over time of the Liouville measure $d\phi^A d\pi_A$. Here, the π_A are the momenta canonically conjugate to ϕ^A. So one has

$$Z_\psi = \int [D\phi^A D\pi_A] \, e^{(i/\hbar)S_\psi^H}, \tag{85a}$$

with

$$S_\psi^H = \int dt(\pi\dot{\phi} - H_\psi). \tag{85b}$$

If it is permissible to evaluate the integral over the momenta by the stationary phase method—and we assume this to be the case, for otherwise it would not seem that the path integral can be expressed in Lagrangian form with a local measure—one gets Eq. (84) with a definite expression for the integration measure. This measure is local *in time* because of the structure of the Hamiltonian action [Eq. (85b)]. Accordingly, the measure terms in Eq. (84d) are generically singular and formally contain $\delta(0)$.

Similarly, the quantum average of phase space observables

$$\langle A \rangle = \int [D\phi^A D\pi_A] \, e^{(i/\hbar)S_\psi^H} A(\phi, \pi) \tag{86a}$$

can be rewritten in Lagrangian form as

$$\langle A \rangle = \int [D\mu] \, e^{(i/\hbar)S_\psi}(A + \hbar\alpha_1 + \hbar^2\alpha_2 + \cdots). \tag{86b}$$

The corrections $\alpha_1, \alpha_2, \ldots$ to the value $A(\phi) = A(\phi, \pi = \pi(\phi))$ of A at the extremum for π are just the higher-order terms in the stationary phase method and take again a definite form. For operators that are local in time, these corrections are singular ($\sim\delta(0)$).

Contrary to the Hamiltonian expressions, Eqs. (85a) or (86a), the Lagrangian measure and the Lagrangian corrections to A are not universal and depend on the dynamics. These corrections arise because the integration over the conjugate momenta π_A may not simply amount to replacing the momenta by their classical value in the Hamiltonian path integral. Nevertheless, something can be said about $[D\mu]$ and $\alpha_1, \alpha_2, \ldots$ on general grounds, without using the Hamiltonian formalism.

8.7. Dimensional Regularization

The simplest approach consists in using a regularization method which sets to zero the singular terms proportional to $\delta(0)$ in the local measure

$[D\mu]$. Such a method exists, based on dimensional regularization. The local measure is then irrelevant,[35] and Eq. (84a) becomes

$$Z_\psi = \int [D\phi^A]\, e^{(i/\hbar)S_\psi}. \tag{87}$$

Similarly, the simplest regularization of the singular terms $\alpha_1, \alpha_2, \ldots$ proportional to $\delta(0)$ in Eq. (86b) is again to set them equal to zero. Thus, one replaces Eq. (86b) by

$$\langle A \rangle = \int [D\phi^A]\, e^{(i/\hbar)S_\psi} A. \tag{88}$$

Equation (88) is usually singular since one has dropped a singular term from $\langle A \rangle$. This singularity appears because A contains products of operators evaluated at coincident times. One regularizes these terms by splitting the times (e.g., $\dot{q}^2(t) \to \dot{q}(t+\varepsilon)\dot{q}(t)$) and taking the limit as the times coincide.[36] This regularization is, as a rule, compatible with setting $\alpha_1 = \alpha_2 = \cdots = 0$ in Eq. (86b).

With these drastic regularization prescriptions, the Lagrangian path integrals in Eqs. (87) or (88) are completely determined. The Lagrangian methods are entirely self-contained.

It is then easy to check that Eq. (87) is the correct path integral. First of all, the measure is BRST-invariant because its variation is proportional to $\delta(0)$,

$$\frac{\delta(\bar{s}\phi^A)}{\delta\phi^A} \sim \int \delta(0) = 0.$$

Second, the change of variables

$$\phi^A \to \delta^A - (\bar{s}\phi^A)\mu, \tag{89a}$$

where μ is not a constant parameter but a functional of the fields given by

$$\mu = \frac{i}{\hbar}(\psi' - \psi), \tag{89b}$$

shows that

$$Z_\psi = Z_{\psi'}. \tag{89c}$$

The transformation of the measure is proportional to $\delta\mu/\delta\phi$ and accounts for the change $\psi \to \psi'$ in the gauge-fixed action.

Third, the gauge-fixed BRST cohomology is also seen to be incorporated in the path integral. Indeed, one finds that the quantum average of any BRST invariant operator does not depend on ψ:

$$\langle A_\psi \rangle_{S_\psi} = \langle A_{\psi'} \rangle_{S_{\psi'}}. \tag{90a}$$

Here, we have defined

$$\langle A_\psi \rangle_{S_\psi} = \int [D\phi^A]\, e^{(i/\hbar)S_\psi} A\left(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}\right). \tag{90b}$$

Furthermore,

$$\left\langle \lambda^A \frac{\delta S_\psi}{\delta\phi^A} + \bar{s}B \right\rangle_{S_\psi} = 0. \tag{90c}$$

In Eq. (90), A and λ^A are assumed to be local functions. The path integral associates therefore a well-defined quantum average to any cohomological class of BRST-invariant operators, i.e., to any gauge-invariant operator.

An important tool in the proof of Eq. (90) is the Schwinger–Dyson equation

$$\left\langle F \frac{\delta^l S_\psi}{\delta\phi^A} \right\rangle = -\frac{\hbar}{i} \left\langle \frac{\delta^l F}{\delta\phi^A} \right\rangle (-1)^{\varepsilon_F}. \tag{91}$$

This equation is obtained by making a shift of integration variables in the path integral, $\phi^A \to \phi^A + \varepsilon^A$ (see Ref. 36). An alternative, very interesting, derivation of the Schwinger–Dyson equation based on the BRST symmetry has recently been given in Ref. 37.

From Eq. (91), it follows that

$$\left\langle \lambda^A \frac{\delta S_\psi}{\delta\phi^A} \right\rangle = 0$$

if λ^A is a local function, since then $\delta\lambda^A/\delta\phi^A$ is singular ($\sim\delta(0)$) and, hence, regularized to zero. Similarly, the use of Eqs. (91) and (83b) and $\delta A/\delta\phi^A\delta\phi^*_A \sim \delta(0) = 0$, combined with the change of variables in Eq. (89), leads to Eq. (90a). Finally, the change of variables

$$\phi^A \to \phi^A + (\bar{s}\phi^A)\varepsilon, \qquad \varepsilon = \text{const.}$$

in

$$\int [D\phi^A] B(\phi)\, e^{(i/\hbar)S_\psi}$$

yields

$$\langle \bar{s}B \rangle_{S_\psi} = \int [D\phi^A](\bar{s}B)\, e^{(i/\hbar)S_\psi} = 0.$$

8.8. More Careful Incorporation of the Measure

Although dimensional regularization provides a consistent and self-contained formalism, it is not always justified: $\delta(0)$ is not always equal to zero and the local measure may be important. So one needs a formalism that handles more carefully the singular terms without ascribing any definite value to them.

Such a formalism exists. Because of lack of space, we will not explain it here, but will only report the results. We refer to Refs. 6 and 13 for the proofs of the main statements. The proofs of the properties not given in Refs. 6 and 13 are left as exercises.

The requirement of invariance of the formalism under antibracket canonical transformations enables one to describe the Lagrangian measure and the $\hbar$-corrections to BRST-invariant operators in terms of the antibracket structure. One finds that the path integral is given by

$$\langle A_\psi \rangle_{S_\psi} = \int [D\phi^A]\alpha\left(\phi, \phi^* = \frac{\delta W}{\delta\phi}\right) \exp\frac{i}{\hbar} W\left(\phi, \phi^* = \frac{\delta W}{\delta\phi}\right), \quad (92a)$$

where

$$W(\phi, \phi^*) = S(\phi, \phi^*) + \hbar M_1(\phi, \phi^*) + \hbar^2 M_2(\phi, \phi^*) + \cdots \quad (92b)$$

$$\alpha(\phi, \phi^*) = A(\phi, \phi^*) + \hbar\alpha_1(\phi, \phi^*) + \hbar^2\alpha_2(\phi, \phi^*) + \cdots \quad (92c)$$

obey the equations

$$\tfrac{1}{2}(W, W) = i\hbar\Delta W \quad (92d)$$

$$(\alpha, W) = i\hbar\Delta\alpha. \quad (92e)$$

Here, Δ is defined by

$$\Delta\alpha = \frac{\delta^r}{\delta\phi^A}\frac{\delta^r\alpha}{\delta\phi^*_A}(-)^{\varepsilon_A+1}, \qquad \varepsilon(\Delta) = 1, \quad (92f)$$

and one has

$$\begin{aligned} \Delta^2 = 0; \qquad \Delta(\alpha, \beta) &= (\alpha, \Delta\beta) - (-)^{\varepsilon_\beta}(\Delta\alpha, \beta); \\ \Delta(\alpha\beta) &= \alpha\Delta\beta + (-)^{\varepsilon_\beta}(\Delta\alpha)\beta + (-)^{\varepsilon_\beta}(\alpha, \beta). \end{aligned} \quad (92g)$$

To zeroth order in $\hbar$, the Eqs. (92d–e) reduce to $(S, S) = 0$ and $(A, S) = 0$. The terms $M_1, M_2, \ldots$ describe the Lagrangian integration measure, while the terms $\alpha_1, \alpha_2, \ldots$ describe the "quantum corrections" to A.

Equation (92e) can rewritten in terms of a nilpotent operator σ:

$$\sigma\alpha \equiv (\alpha, W) - i\hbar\Delta\alpha, \qquad (92e) \Leftrightarrow \sigma\alpha = 0 \quad (93a)$$

$$\sigma^2 = 0, \quad (93b)$$

which coincides with s at zeroth order in $\hbar$,

$$\sigma = s + O(\hbar). \quad (93c)$$

This operator can be thought of as a quantum deformation of s that takes into account the quantum fluctuations in the integration over the momenta. The deformation preserves nilpotency, but not the Leibnitz rule: in general, σ does not act as a derivation:

$$\sigma(\alpha\beta) \neq \alpha(\sigma\beta) + (-1)^{\varepsilon_\beta}(\sigma\alpha)\beta \tag{93d}$$

[see Eq. (92g)]. The fact that σ does not act as a derivation is not surprising, as the integration over the momenta does not preserve the product structure: the expectation value of a product is, in general, different from the product of the expectation values.

Provided $H^1(d) = 0$, as we will assume, one can easily show that the cohomology of σ at ghost number zero is isomorphic with the set of $\hbar$-dependent elements in $H^0(s)$. Hence, $H^0(\sigma)$ is also isomorphic with the set of $\hbar$-dependent, gauge-invariant functions. However, the correspondence between $H^0(\sigma)$ and $H^0(s)$ is not universal and depends on the dynamics. Given A obeying $(A, S) = 0$, there is no natural element in $H^0(\sigma)$ associated with it. Equation (92e) alone, which just expresses "quantum BRST invariance," allows for the possibility of adding an independent gauge-invariant operator at each order in $\hbar$.

Similarly, given S, the Eq. (92d) for W leaves the same freedom of adding a new, independent, gauge-invariant term at each order in $\hbar$. *The principle of BRST symmetry alone determines the Lagrangian integration measure only up to BRST invariant terms.* This appears to be the best one can do if one does not want to analyze the detailed structure of S_ψ and A_ψ.

By making the same change of variables as in the previous section and using the Schwinger–Dyson equation, one can formally prove that

$$\langle A_\psi \rangle = \langle A_{\psi'} \rangle \tag{94a}$$

and

$$\langle \sigma\beta \rangle = 0. \tag{94b}$$

In particular, for $A = 1$, one gets again

$$Z_\psi = Z_{\psi'}. \tag{94c}$$

This time, however, it is not necessary to eliminate singular terms by hand to reach Eq. (94). So the Lagrangian path integral incorporates the quantum BRST cohomology and does not depend on the choice of gauge-fixing fermion.

Finally, it should be stressed that the full Lagrangian integration measure is in general not invariant under the original BRST symmetry s (or $\bar{s}$), even though the action S_ψ and the Hamiltonian integration measure are. The effect of the integration over the momenta amounts in general to more than just replacing the momenta by their on-shell value. There are quantum fluctuations, which force one to replace, in the Lagrangian path integral, s by σ:

$$s\phi^A \rightarrow \sigma\phi^A = (\phi^A, W) \\ = s\phi^A + O(\hbar). \tag{95}$$

It should be kept in mind, however, that the considerations of this section are *very formal* since the correction terms are, as a rule, divergent.

It should also be observed that the possibility of adding equation-of-motion terms to the classical observables,

$$A(\phi) \rightarrow A(\phi) + \lambda^A(\phi)\frac{\delta S_\psi}{\delta\phi^A} \tag{A}$$

is replaced, in the quantum theory, by the possibility of adding Schwinger-Dyson-equation terms

$$\alpha(\phi) \rightarrow \alpha(\phi) + \lambda^A(\phi)\frac{\delta S_\psi}{\delta\phi^A} + \frac{\hbar}{i}\frac{\delta\lambda^A}{\delta\phi^A}. \tag{B}$$

While the first freedom does not modify the classical expectation values, the second freedom does not modify the quantum ones.

That (A) and (B) are indeed incorporated in the formalism is particularly clear in the case of systems without gauge freedom, for which one finds that

$$s(F(\phi)\phi_i^*) = F(\phi)\frac{\delta S_0}{\delta\phi^i}$$

while

$$\sigma(F(\phi)\phi_i^*) = F(\phi)\frac{\delta S_0}{\delta\phi^i} + \frac{\hbar}{i}\frac{\delta F}{\delta\phi^i},$$

so that $\langle\sigma(F\phi_i^*)\rangle = 0$ is just the Schwinger-Dyson equation.

Because the last term in the right-hand side of (B) contains $\hbar$, one can formally think of it as a quantum correction to (A). The freedom (A, B) has been implicitly fixed in the previous discussion by assuming that the observables $A(\phi)$ were local in time and depended only on the independent initial data (and not on their time derivatives). Once this is done, the only unknown in α, given A, is related to the integration over the momenta as analyzed in section 8.6.

8.9. Invariance under Canonical Transformation in the Antibracket

The canonical covariance of the formalism is straightforward and follows from the fact that the gauge conditions $\Omega^A = 0$ used to eliminate the antifields are in involution, $(\Omega^A, \Omega^B) = 0$. This is a statement invariant under canonical transformations. So the conditions $\phi^*_A = \delta\psi/\delta\phi^A$ in one canonical coordinate system are equivalent to the conditions $\bar{\phi}^*_A = \delta\bar{\psi}/\delta\bar{\phi}^A$ in any other canonical coordinate system, with, in general, a different $\bar{\psi}$.

If one rewrites the path integral in terms of the bare variables, one finds the same expression, with the only exceptions being that

(i) ψ is replaced by $\bar{\psi}$, but the physical amplitudes do not depend on ψ.
(ii) there are some Jacobian factors that modify the integration measure.

If the canonical transformation is local in space-time, the Jacobian factors differ from unity by terms proportional to $\delta(0)$. Within the framework of dimensional regularization, these terms vanish. Therefore, the physical amplitudes take exactly the same form [Eqs. (87) and (88)] in any canonical basis. This shows in particular that all the representations of the gauge symmetry that are local in space-time and that can be obtained from one another by local transformations are equivalent.[12] In space-time local bases, the gauge-fixed action is local, the measure is set equal to one by dimensional regularization, and one can use the usual methods of quantum field theory.

Nontrivial measure factors appear when one makes nonlocal changes of variables. To handle these, one needs to use the more careful formalism of section 8.8. The effect of the Jacobian is to modify the functional W. One can show that Eq. (92d, e), with the new W, are form-invariant under canonical transformations.[13] From this property, it easily follows that the quantum averages are also invariant under canonical transformations. The same conclusions are thus reached as in the case of local transformations.

An interesting application of the invariance of the path integral under canonical transformations is obtained by going to the Abelian representation [Eq. (70)]. It is easy to check that the solution [Eq. (70)] of the master equation obeys $\Delta S = 0$. Accordingly, W can be taken to differ from S by a function of the gauge-invariant variables $\chi^{\bar{a}}$ only. Similarly, α can also be assumed to depend on $\chi^{\bar{a}}$ only and then obeys $(\alpha, W) = 0$, $\Delta\alpha = 0$.

To evaluate Eq. (92a) in the representation [Eq. (70)], one takes a gauge-fixing fermion that does not depend on the gauge-invariant variables $\chi^{\bar{a}}$. With that choice, the gauge-fixed action takes the form

$$S_\psi = S_0(\chi^{\bar{a}}) + \bar{S}_\psi, \tag{96a}$$

where $\bar{S}_\psi$ involves only the gauge degrees of freedom and the ghosts. There is complete decoupling between the gauge-invariant sector and the gauge-ghost sector. The integration over these latter variables yields a factor independent of $\chi^{\bar{a}}$, and so the path integral takes the manifestly gauge-invariant and correct form

$$\langle A \rangle = \int [D\chi^{\bar{a}}]\mu(\chi^{\bar{a}})\alpha(\chi^{\bar{a}}) \exp \frac{i}{\hbar} S_0(\chi^{\bar{a}}) \tag{96b}$$

for some measure $\mu(\chi^{\bar{a}})$. This gives a justification of the formalism which is not based on the comparison with the Hamiltonian. [The integration over the gauge and ghost modes may be more tricky than our discussion indicates, but we assume here that there is no subtlety.] Similar arguments show that cohomologically trivial pairs decouple with appropriate choices of ψ and hence do not modify the path integral.

8.10. Zinn–Justin Equation

The antifield formalism and the master equation find their roots in developments due to Zinn-Justin[38] in the context of the renormalization of Yang–Mills fields.

Let us introduce sources j_A and K_A for the fields and their BRST variations, and let us define

$$\begin{aligned} Z[j_A, K_A] &= \int [D\phi^A] \exp \frac{i}{\hbar}\left[S\left(\phi, K + \frac{\delta\psi}{\delta\phi}\right) + j_A\phi_A \right] \\ &= \int [D\phi^A] \exp \frac{i}{\hbar} [S_\psi(\phi) + K_A(\bar{s}\phi^A) + O(K^2) + j_A\phi^A]. \end{aligned} \tag{97}$$

The sources j_A occur linearly, but the dependence on K_A is more complicated unless the gauge algebra is closed. If one constructs the effective action $\Gamma[\langle\phi\rangle, K]$ as the Legendre transform of $(\hbar/i)\ln Z$ with respect to the sources j^A, one finds that

$$(\Gamma, \Gamma) = 0 \tag{98}$$

as a result of the master equation (with $(\langle\phi^A\rangle, K_B) = \delta^A_B$). This form of the Ward identity was written for the first time by Zinn-Justin in the case of the Yang–Mills theory.[38] It is useful in the analysis of the renormalization of the theory, where the antibracket turns out to play again an important role.[39]

8.11. Conclusions

We have shown that the path integral incorporates the BRST cohomology, and hence gauge invariance, in a satisfactory manner. This result (i) holds even in the open-algebra case, where the gauge-fixed BRST symmetry $\bar{s}$ is only nilpotent modulo the field equations of the gauge-fixed action and (ii) indicates that the operator BRST cohomology at ghost number zero is isomorphic with the set of "transverse," i.e., gauge-invariant, operators.

Our conclusions should, of course, be taken with a grain of salt. Formal path-integral manipulations may miss important aspects of the operator formalism (operator ordering) which have not been addressed at all here. A more careful analysis of these subtleties may reveal departures from the above straightforward derivations.

Lastly, we emphasize that only infinitesimal transformations are covered by the BRST formalism developed here. So, in the case of a group, BRST invariance is equivalent to invariance under the gauge transformations in the connected component of the identity, but does not imply invariance under "large" gauge transformations. In spite of this, it should be stressed, as some confusion has arisen, that the BRST transformation is *globally defined*, i.e., it is well defined everywhere in I. This is because the vector fields R^i_α representing the infinitesimal gauge transformations are also well defined everywhere.

9. EXAMPLES

9.1. Electromagnetism

The action S_0 is

$$S_0 = -\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^Dx \tag{99a}$$

and is invariant under

$$\delta_\varepsilon A_\mu = \partial_\mu \varepsilon. \tag{99b}$$

The gauge transformations are irreducible.

The minimal solution of the master equation reads

$$S = -\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^Dx + \int A^{*\mu}\, \partial_\mu C\, d^Dx. \tag{100}$$

To implement the covariant Lorentz gauge, one needs to add to Eq. (100) the nonminimal term

$$\int d^Dx\, \bar{C}^* b, \tag{101}$$

where $\bar{C}$ is the antighost of ghost number minus one, b is the Takanishi-Lautrup auxiliary field, and $\bar{C}^*$, b^* are the corresponding antifields.

If one takes as the gauge fermion

$$\psi = -\int \bar{C}\partial^\mu A_\mu\, d^Dx \tag{102a}$$

and eliminates all the antifields, the path integral becomes

$$\int [DA_\mu][DC][D\bar{C}]\Pi\delta(\partial_\mu A^\mu)$$

$$\times \exp\frac{i}{\hbar}\left[-\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,d^Dx + \int \partial^\mu\bar{C}\partial_\mu C\,d^Dx\right]. \qquad (102b)$$

It involves a δ-function of the gauge condition.

If, on the other hand, one adopts

$$\psi = \int \bar{C}\left(\frac{1}{2\beta}b - \partial^\mu A_\mu\right) d^Dx, \qquad (103a)$$

one finds, after integration over b, the "Gaussian average" representation

$$\int [DA_\mu][DC][D\bar{C}]\exp\frac{i}{\hbar}\int d^Dx\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\beta}{2}(\partial^\mu A_\mu)^2 + \partial^\mu\bar{C}\,\partial_\mu C\right]. \qquad (103b)$$

Finally, the temporal gauge is reached by sticking to the minimal solution and eliminating C^* and A_0 by means of $\psi = 0$, which is here permissible. One gets

$$S_\psi = -\tfrac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,d^Dx + \int A^{*0}\partial_0 C\,d^Dx, \qquad (104)$$

where $F_{\mu\nu}$, A_0 is set equal to zero. The antifield A^{*0} plays the role of the usual antighost. Note that $\psi = 0$ is permissible precisely because one keeps the antifield A^{*0}. If one had eliminated all the antifields in favor of the fields, one would have obtained $S_\psi = S_0$, which leads to an ill-defined path integral.

9.2. Abelian 2-form Gauge Field

The action S_0 is

$$S_0 = -\tfrac{1}{12}\int F_{\mu\nu\rho}F^{\mu\nu\rho}\,d^Dx, \qquad (105a)$$

with

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu}. \qquad (105b)$$

It is invariant under

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu. \tag{105c}$$

The gauge transformations are now reducible: if Λ_ν is equal to $\partial_\nu \varepsilon$, Eq. (105c) reduces to $\delta A_{\mu\nu} = 0$. One needs the following minimal spectrum of fields and antifields:

$$\begin{array}{cccccc} -3 & -2 & -1 & 0 & 1 & 2 \\ \hline C^* & C^{*\mu} & A^{*\mu\nu} & A_{\mu\nu} & C_\mu & C \end{array} \longrightarrow . \tag{106a}$$

The minimal solution reads

$$S = \int [-\tfrac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + A^{*\mu\nu}(\partial_\mu C_\nu - \partial_\nu C_\mu) + C^{*\mu} \partial_\mu C] \, d^D x. \tag{106b}$$

To mimic the electromagnetic case, one first tentatively introduces antighosts $\bar{C}^\mu$ (for the gauge fixing of $A_{\mu\nu}$) and $\bar{C}$ (for the gauge fixing of C_μ, $C_\mu \to C_\mu + \partial_\mu \varepsilon$), and considers the nonminimal solution

$$S^{\text{nonmin}} = S + \int (\bar{C}^*_\mu b^\mu + \bar{C}^* b) \, d^D x. \tag{107a}$$

Here, b_μ and b are auxiliary fields. The gauge fixing fermion

$$\psi = \int [\bar{C}^\mu (\partial^\nu A_{\nu\mu}) + \bar{C} \partial^\nu C_\nu] \, d^D x \tag{107b}$$

leads to

$$\begin{aligned} S_\psi = \int [& -\tfrac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + \tfrac{1}{2}(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\mu C_\nu - \partial_\nu C_\mu) \\ & + \partial^\nu A_{\nu\mu} b^\mu + \partial^\nu C_\nu b - \partial^\mu \bar{C} \partial_\mu C] \, d^D x. \end{aligned} \tag{107c}$$

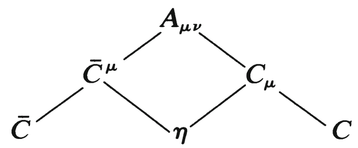
This cannot be the final answer, however, because

(i) the integration over b_μ yields $\delta(\partial^\nu A_{\nu\mu})$ in the path integral. This product of delta functions contains $\delta(0)$ because the arguments $\partial^\nu A_{\nu\mu}$ are not independent, $\partial^\mu \partial^\nu A_{\mu\nu} \equiv 0$.
(ii) the action [Eq. (107c)] is gauge-invariant under $\bar{C}^\mu \to \bar{C}^\mu + \partial^\mu \Lambda$. This formally yields a "compensating" zero in the path integral.

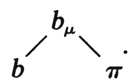
To remedy these problems, one extends the nonminimal sector by adding the term

$$\int \eta^* \pi \, d^D x,$$

with $gh(\eta^*) = -1$, $gh\pi = 1$, $gh\pi^* = -2$, and $gh\eta = 0$. The ghost–antighost spectrum is given by



while the auxiliary field spectrum reads



An appropriate gauge-fixing fermion is given by

$$\psi = \int [\bar{C}^{\mu}(\partial^{\nu} A_{\nu\mu}) + \bar{C}\partial^{\nu} C_{\nu} + \bar{C}^{\nu}\partial_{\nu}\eta] d^D x. \tag{108a}$$

The gauge-fixed action is then

$$S_{\psi} = \int [-\tfrac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + \tfrac{1}{2}(\partial^{\mu}\bar{C}^{\nu} - \partial^{\nu}\bar{C}^{\mu})(\partial_{\mu} C_{\nu} - \partial_{\nu} C_{\mu})$$
$$+ (\partial^{\nu} A_{\nu\mu} + \partial_{\mu}\eta) b^{\mu} + (\partial^{\nu} C_{\nu}) b + (\partial_{\nu}\bar{C}^{\nu})\pi] d^D x. \tag{108b}$$

The gauge freedom of the antighost $\bar{C}^{\mu}$ is now fixed. Furthermore, the integration over b_{μ} yields $\delta(\partial^{\nu} A_{\nu\mu} + \partial_{\mu}\eta)$, which is sensible [$\partial^{\mu}(\partial^{\nu} A_{\nu\mu} + \partial_{\mu}\eta) = \Box\eta$ no longer vanishes identically. The delta functions enforce $\Box\eta = 0$, i.e., $\eta = 0$, and hence also $\partial^{\nu} A_{\nu\mu} = 0$. The arguments of the delta functions become independent with the introduction of η]. Equation (108b) has been derived by various authors along various lines.[6,40]

To reach a Gaussian average representation, one adds to Eq. (108a) the term

$$\int (\alpha \bar{C}^{\mu} b_{\mu} + \beta \bar{C} b + \gamma \eta \pi) \, d^D x, \tag{109}$$

which is linear in the auxiliary fields.

9.3. Remark on the Gribov Ambiguity

As the previous examples indicate, the path integral contains a delta function of the gauge conditions when the gauge-fixing fermion ψ does not depend on the auxiliary fields b. The gauge conditions are just the coefficients of the antighosts in ψ.

The class of available ψ is much larger, however. For more complicated ψ, the path integral does not reduce to an integral in a definite gauge. For instance, gauge-fixing fermions that are linear in the auxiliary fields lead to a Gaussian average over different gauges.* One virtue of the BRST formalism is that it incorporates these more general ψ from the very beginning since there is no *a priori* restriction on the choice of ψ except that ψ should define an action without gauge invariance through $S_\psi = S(\phi, \phi^* = \delta\psi/\delta\phi)$.

The important cohomological and invariance features of the BRST formalism do not depend on the existence of global sections transverse to the gauge orbits. We believe that this is a definite advantage for theories afflicted by the Gribov ambiguity, for which no such section exists. As the BRST construction nevertheless goes through in that case, the actions S_ψ appear to be still the correct objects to be path-integrated. The only requirement on ψ is that S_ψ be propagating. This may force some nontrivial dependence of ψ on the auxiliary fields. It would be of interest to completely settle this issue.

The global significance of the BRST symmetry for systems with Gribov horizons has also been pointed out along different lines in Ref. 41. That the Gribov ambiguity does not signal a true physical pathology is well known and has been observed earlier. Attempts to overcome the Gribov problem in the path integrals may be found in Ref. 42. These attempts are consistent with the BRST approach.

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APPENDIX: ABELIANIZATION OF THE GAUGE TRANSFORMATIONS

We will prove the Abelianization theorem in the finite-dimensional case where functional difficulties are absent. Let $S(q^i)$ be a function of

* In spite of this, we still call ψ the gauge-fixing fermion, as its purpose is to yield a propagating action without gauge invariance.

$q^i \in R^n$. Assume that the equations $\partial S/\partial q^i = 0$ are degenerate. Then $\partial S/\partial q^i = 0$ defines a manifold Σ of dimension m, with $0 < m \leq n$. The gauge transformations are

$$\delta_\varepsilon q^i = R^i_\alpha \varepsilon^\alpha, \qquad \delta_\varepsilon S = 0, \qquad \alpha = 1, \ldots, m, \tag{A.1}$$

where the matrix $R^i_\alpha(q)$ is of rank m.

Without loss of generality, we can assume that the coordinates $q^i = (q^a, q^\alpha)$ are locally such that R^α_β is invertible. In that case, $q^\alpha = \overset{0}{q}{}^\alpha$ are good gauge conditions and the equations $\partial S/\partial q^\alpha = 0$ are consequences of the equations $\partial S/\partial q^a = 0$,

$$\frac{\partial S}{\partial q^a} R^a_\alpha + \frac{\partial S}{\partial q^\alpha} R^\alpha_\beta = 0 \quad \Rightarrow \quad \frac{\partial S}{\partial q^\alpha} = t^a_\alpha \frac{\partial S}{\partial q^a}. \tag{A.2}$$

By the regularity condition put on the action S, the functions $\partial S/\partial q^a$ can be used as first coordinates in the vicinity of $\partial S/\partial q^i = 0$. This means that the matrix $T_{ia} \equiv \partial/\partial q^i(\partial S/\partial q^a)$ at the stationary point is of rank $n - m$:

$$T_{ia}\mu^a = 0 \quad \Rightarrow \quad \mu^a = 0.$$

But this condition implies in turn that T_{ab} is invertible, because $T_{\alpha a}$ can be expressed in terms of T_{ab} by means of Eq. (A.2) at $\partial S/\partial q^i = 0$.

If one fixes the gauge variables q^α, the stationary problem $\partial S/\partial q^a = 0$ determines q^a uniquely as a function of q^α, $q^a = Q^a(q^\alpha)$. By the above remark, the critical point $Q^a(q^\alpha)$ is furthermore nondegenerate. Thus, using Morse's lemma, one can make a q^α-dependent, invertible, smooth change of coordinates

$$q^a \to x^a = x^a(q^b, q^\alpha),$$

such that S takes the canonical form

$$S = \eta_{ab} x^a c^b, \qquad \eta_{ab} = \mathrm{diag}(\pm 1)$$

in the vicinity of the critical point $x^a = 0$.

The change of variables $q^a \leftrightarrow x^a$ can be extended to $q^i \leftrightarrow x^a, q^\alpha$. In the new coordinates, S does not depend on q^α. It is thus invariant under the Abelian shifts $q^\alpha \to q^\alpha + \varepsilon^\alpha$. This exhausts the gauge freedom, as $\partial S/\partial q^i = 0$ completely determines x^a. The Abelianization theorem is thereby proven.

The theorem is easily extended to the case of an action $S(q^i, \alpha^A)$ that depends not only on the dynamical variables q^i, but also on unvaried extra variables α^A.

An alternative proof of the Abelianization theorem is given in Refs. 12 and 13.

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Chapter 7

Combinatorics of Mapping Class Groups and Matrix Integration

C. Itzykson and J. B. Zuber

1. INTRODUCTION

In the mid-seventies 't Hooft introduced a clever device to keep track of $SU(N)$-group theoretic factors in the perturbative expansion of gauge field models.[1] The goal was to find an approximation in the large N (planar) limit. A similar approximation for vector-valued fields singles out one-loop graphs, easily handled, and provides an interesting model in a number of problems—for instance, critical phenomena. Alas, in the case of gauge theories apart from phenomenological applications, the scheme was not very successful, since the leading terms still involve the computation of infinitely many perturbative terms. The story is recorded in a 1979 report by S. Coleman[2] entitled "$1/N$."

C. Itzykson and J. B. Zuber • Service de Physique Théorique de Saclay, Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique, F-91191 Gif-sur-Yvette Cedex, France.

Quantum Mechanics of Fundamental Systems 3, edited by Claudio Teitelboim and Jorge Zanelli. Plenum Press, New York, 1992.

What 't Hooft had shown was that after insertion of an appropriate factor of N in front of a path-integral action, vertices carry a factor N, propagators a factor $1/N$. To follow matrix indices (of gauge fields) he found it convenient to represent propagators as oppositely oriented double lines, indicating the flow of indices and connected at vertices to cyclically ordered hooks (representing traces of powers of the field), in such a way that they close on F-index loops, each one responsible for a factor N by summation over a dummy index. For a vacuum-connected graph with V vertices, L propagators (or links), and F index loops, the total power of N is therefore

$$N^{V-L+F}.$$

On the other hand, we can think of this collection of V vertices, L links, and F loops with obvious incidence relations as a two-dimensional complex that inherits a consistent orientation from the one on loops and is connected by definition. This leads to an identification of the combination $V - L + F$ with the Euler characteristic χ of an orientable compact surface of genus g such that $\chi = 2 - 2g$. Because χ takes its maximal value 2 for the spherical topology (or the planar, with an added point at infinity), the leading term was called the "planar" approximation, meaning that the corresponding "fat graphs" can be drawn on a sheet of paper without crossings.

Koplik, Neveu, and Nussinov[3] than suggested a drastic reduction to a zero-dimensional toy model (simple integrals) in order to understand the purely combinatorial aspects, using techniques from graph theory. By 1978, in a collaboration which included first E. Brézin and G. Parisi then D. Bessis[5] we investigated the zero- and one-dimensional problems at leading order, using saddle-point methods, then the various subleading corrections with the help of orthogonal polynomials as suggested by Bessis. One must admit that at the time the physical motivation was slim, except that we found the leading approximation quite accurate in simple quantum-mechanical problems where it relates to the semiclassical approximation.

Interest was revived in these questions in the mid-eighties when it was realized by David, Kazakov, and Fröhlich,[6] among others, that the above techniques are very effective in studying two-dimensional field (or statistical) models coupled to a random geometry in the context of a discretized regularized version, then looking at fixed points where a continuous geometry is restored. This is then called "quantum gravity" and generalizes to coupled analogues. At first the study was performed at fixed genus; then it was recently extended to a resummation over all genera with surprising and exciting new results by Brézin and Kazakov,[7] Gross and Migdal,[8] and Douglas and Shenker.[9] This resummation has relied up to now on the use of orthogonal polynomial methods and a nontrivial scaling limit. This subject, now widely studied by various groups, will not be pursued here. It is noticeable that in a different mathematical context Penner[10] used matrix integration to illuminate a computation performed by Harer and Zagier[11] pertaining to the topological properties of the mapping class group of Riemann surfaces.

Here we would like to survey some of the combinatorics involved in these calculations (they are dual to each other in a sense that will be made precise as we proceed) and point out a number of connections with the representation theory of linear and permutation groups. These might suggest further developments.

We should not give the reader the false impression that we are conversant with algebraic topology. For these aspects we rely on the original articles of Harer and Zagier,[11] Penner,[10] and Ivanov[12] quoted in the references.

Briefly stated, the mapping class group is an infinite discrete group describing the classes of (continuous or differentiable) one-to-one maps of a manifold onto itself up to equivalence under those homotopic (continuously deformable) to the identity. A familiar example is the modular group for two-dimensional genus-one tori. These groups are essential in defining fundamental domains of integration over moduli spaces (of complex structures) in perturbative string theories. Their explicit form is still poorly understood in general so that even the topology of these fundamental domains is not easy to describe. This justifies an interest in the most global aspects, an example being the virtual Euler characteristic. The reason for the terminology will be commented on below.

Returning to matrix integration, it will be shown that one can keep track of perturbative diagrams by coding them using pairs of permutations up to overall conjugacy. This idea may have antecedents in the literature on graphs. We learnt it from J. M. Drouffe, who presented it in an appendix in our joint paper with D. Bessis (which can be consulted for general background).[5] This technique enables one to exhibit in a neat way the Poincaré duality. On the other hand the (Frobenius) duality between the linear and the permutation groups is at the heart of the evaluation of some integrals. It would certainly be interesting to develop q-analogs of the integration scheme. We mention some work of Andrews and Onofri in this direction.[13]

Hermitian matrices, with which we deal exclusively, can be considered as spanning the Lie algebra of the unitary group. With some adaptation, the calculations presented below could be extended to any Lie algebra of a compact Lie group. For orthogonal and symplectic groups, one can presumably develop a topological interpretation (involving possibly non-orientable surfaces).

2. MATRIX INTEGRATION REVISITED

We will deal with $N \times N$ Hermitian matrices denoted generically M and depending therefore on N^2 real parameters. We set

$$dM = \prod_i dM_{ii} \prod_{i<j} d\,\mathrm{Re}\,M_{ij}\, d\,\mathrm{Im}\,M_{ij}$$

for the Lebesgue measure on the (linear) manifold of these matrices and define the normalized Gaussian measure

$$DM = \frac{dM \exp -\frac{1}{2}\mathrm{Tr}\,M^2}{\int dM \exp -\frac{1}{2}\mathrm{Tr}\,M^2}.$$

Both measures are invariant under the adjoint action of the unitary group, $M \to UMU^{-1}$, $UU^+ = \mathbb{1}$. As a result we are mostly interested in averages over functions of M invariant under this action, i.e., which are symmetric functions of the eigenvalues of M.

For $\underline{\nu}$ a finite sequence $\nu_1, \nu_2, \ldots,$ of nonnegative integers, we define

$$t_{\underline{\nu}}(M) = (\mathrm{Tr}\,M)^{\nu_1}(\mathrm{Tr}\,M^2)^{\nu_2}\ldots.$$

The sequence $\underline{\nu}$ can be identified with a partition of the integer $\sum k\nu_k$, the overall degree of homogeneity in M of $t_{\underline{\nu}}(M)$. Let us first compute the Gaussian average

$$\langle t_{\underline{\nu}}\rangle = \int D(M)t_{\underline{\nu}}(M). \tag{2.1}$$

This vanishes unless $\sum k\nu_k$ is an even integer denoted by $2n$. Using Wick's theorem, or generating function,

$$\int D(M)\, e^{\mathrm{TR}\,JM} = e^{\frac{1}{2}\mathrm{Tr}\,J^2},$$

with J an $N \times N$ Hermitian matrix, we can compute Eq. (2.1) as a sum of contributions pertaining to Feynman graphs with double propagators and vertices, as in Fig. 1. Each factor $\mathrm{Tr}\,M^k$ corresponds to a valence k vertex.

Figure 1. Double-line representation of a propagator and of a vertex $\mathrm{Tr}\,M^3$.

Propagators arise from the contraction

$$\overline{MM} \equiv \langle M_{ij} M_{kl} \rangle = \delta_{il}\delta_{jk}.$$

Each graph factors into connected components, labelled by an index a. The $V = \sum V_a = \sum \nu_k$ vertices and $L = \sum L_a = \frac{1}{2}\sum k\nu_k = n$ links thus correspond to a set of orientable surfaces with $F = \sum F_a$ faces (or index loops) and yield a term $N^F = \prod_a N^{F_a}$ in the computation of $\langle t_{\underline{\nu}} \rangle$. Each component has genus g_a given by $2 - 2g_a = V_a - L_a + F_a$. To sum all these contributions, we code the graphs as follows: Cut the n propagators in $2n$ hooks attached to the vertices. Label them arbitrarily from 1 to $2n$. Define two permutations σ and τ on $2n$ letters as follows: Take the hook labelled j. One of its double lines directed to a vertex re-emerges in the hook labelled $\sigma(j)$. This defines a permutation σ whose conjugacy class codes the vertices and is $1^{\nu_1}2^{\nu_2}\ldots k^{\nu_k}\ldots$, hence independent of the arbitrary labelling of the vertices. We identify the conjugacy class $[\sigma]$ with the partition $\underline{\nu}$, where brackets around a permutation denote its class. Thus if S_p is the permutation group on p letters, $[S_p]$ its set of conjugacy classes, we have

$$\sigma \in S_{2n} \qquad [\sigma] = \underline{\nu}.$$

On the other hand the contractions joining two hooks (attached to distinct or identical vertices) define a second permutation τ. Clearly $\tau^2 = 1$, and τ belongs to the class 2^n (n cycles of length two). The number of cycles in $\sigma\tau$ is the number of index loops or faces, by the very definition of σ and τ. We write

$$[\sigma\tau] = \underline{\mu}.$$

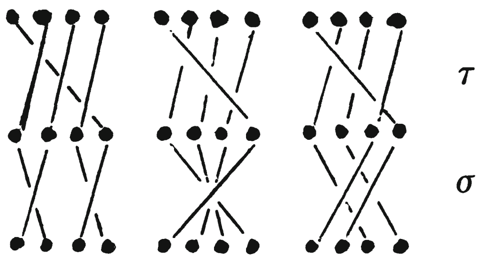
Thus if σ is a fixed representative of the class $\underline{\nu}$, we find

$$\begin{aligned}\langle t_{\underline{\nu}} \rangle &= \sum_{\underline{\mu} \in [S_{2n}]} \sum_{\substack{\tau \in [2^n] \\ [\sigma\tau] = \underline{\mu}}} N^{\sum \mu_k} \\ &= \sum_{\underline{\mu} \in [S_{2n}]} N^{\sum \mu_k} \sum_{\tau \in S_{2n}} \delta_{[\tau],[2^n]} \delta_{[\sigma\tau],\underline{\mu}},\end{aligned}$$

where in the first expression the sum over τ represents the set of all contractions and is split into contributions according to the corresponding power of N. In other words it simply states the Feynman rules in an abstract but efficient way. As an example, consider the average of $\mathrm{Tr}\, M^4$ which is coded as $\nu = \{\underline{\nu}_k = \delta_{4,k}\}$. This is

$$\langle \mathrm{Tr}\, M^4 \rangle = \mathrm{Tr}\, \overline{MM}\,\overline{MM} + \mathrm{Tr}\, \overline{M\overline{MM}M} + \mathrm{Tr}\, MMMM = N^3 + N^3 + N.$$

On the other hand, we take σ as the cyclic permutation on four letters. We find three possible permutations of the type τ, as shown in the schematic, and from the above formula we get the expected result $2N^3 + N$.



$[\sigma\tau]$ $\quad (1)^2(2) \quad (1)^2(2) \quad (4)$

$\quad N^3 \quad N^3 \quad N$

We can now make use of the characters of the permutation group S_{2n} indexed by Young tableaux (denoted Y) also in correspondence with partitions (of $2n$). Let $\|[\sigma]\|$ be the number of elements in the class $[\sigma]$, which we also identify with ν, then

$$\|[\sigma]\| = |\underline{\nu}| = \frac{(2n)!}{\prod\limits_{k\geq 1} k^{\nu_k}\nu_k!}.$$

We write $\chi^Y([\sigma])$ for the value of the (real, integral) character pertaining to the representation Y evaluated on the class $[\sigma]$. The orthogonality and completeness relations on characters read

$$\sum_{\tau\in S_{2n}} \chi^Y([\tau])\chi^{Y'}([\sigma\tau]) = (2n)!\delta^{Y,Y'}\frac{\chi^Y([\sigma])}{\chi^Y([1^{2n}])}$$

$$\sum_Y \chi^Y([\sigma])\chi^Y([\sigma']) = \frac{(2n)!}{\|[\sigma]\|}\delta_{[\sigma],[\sigma']},$$

where $\chi^Y([1^{2n}])$ is the dimension of the representation Y. Thus, using the second of these relations twice,

$$\langle t_{\underline{\nu}}\rangle = \sum_{\underline{\mu}\in[S_{2n}]} N^{\sum\mu_k} \sum_{Y,Y',\tau\in S_{2n}} \frac{\|[2^n]\|\,|\underline{\mu}|}{(2n)!2}\chi^Y([\tau])\chi^Y([2^n])\chi^{Y'}([\sigma\tau])\chi^Y(\underline{\mu}).$$

We can now perform the free sum over τ, and obtain

$$\langle t_{\underline{\nu}}\rangle = \sum_{\underline{\mu}\in[S_{2n}]} \langle t_{\underline{\nu}}\rangle_{\underline{\mu}}, \tag{2.2a}$$

$$\frac{\langle t_{\underline{\nu}}\rangle_{\underline{\mu}}}{B^{\sum\mu_k}|\underline{\mu}|} = \frac{\|[2^n]\|}{(2n)!}\sum_Y \frac{\chi^Y([2^n])\chi^Y(\underline{\mu})\chi^Y(\underline{\nu})}{\chi^Y([1^{2n}])} \tag{2.2b}$$

Thus $\langle t_{\underline{\nu}}\rangle$ can be obtained as a sum of contributions over classes $\underline{\mu}$, $N^{\sum\mu_k}$ may be interpreted as $t_{\underline{\mu}}(\mathbb{1})$, and $\|[2^n]\|/(2n)!$ is $1/2^n n!$. Equation (2.2b) exhibits a symmetry in the interchange $\underline{\mu} \leftrightarrow \underline{\nu}$ reflecting the (Poincaré) duality between vertices and faces of the cell decompositions of surfaces. Note that $\sum \mu_k$ (number of faces) somehow plays the rôle of area, and $\ln N$ plays the rôle of its conjugate variable. As a check of Eq. (2.2) setting $\underline{\nu} = [1^{2n}]$, we get the obvious result

$$\langle (\mathrm{Tr}\, M)^{2n} \rangle = N^n (2n-1)!!,$$

obtained from all possible pairings of valence-1 vertices. On the other hand, the highest possible power N^{2n} only occurs, with coefficient unity, in $\langle (\mathrm{Tr}\, M^2)^n \rangle$ corresponding to the contractions of the various M' under each trace. Apart from these two extreme cases, the above sum remains difficult to write in closed form, although perhaps one could develop a generating function. As a final remark we observe that in spite of the appearance of denominators $\langle t_{\underline{\nu}} \rangle_{\underline{\mu}}$ is, from its very definition an integer for each N; thus the coefficient of $N^{\sum \mu_k}$ is an integer (a positive one). *A fortiori* $\langle t_{\underline{\nu}} \rangle$ is an integer (for each integer N).

Let now the characters of the linear group $GL(N)$ associated with the same set of Young tableaux (with $2n$ boxes) be denoted ch_Y. As they are polynomials in the matrix elements, they can be naturally extended to any matrix M, so that it makes sense to consider $\mathrm{ch}_Y(M)$. We have then the beautiful Frobenius reciprocity relation

$$t_{\underline{\nu}}(M) = \sum_Y \mathrm{ch}_Y(M) \chi^Y(\underline{\nu}). \tag{2.3a}$$

Or, by inverting this relation,

$$\mathrm{ch}_Y(M) = \frac{1}{(2n)!} \sum_{\underline{\nu} \in [S_{2n}]} |\underline{\nu}| \chi^Y(\underline{\nu}) t_{\underline{\nu}}(M). \tag{2.3b}$$

Thus with the help of (2.2) we can obtain the average of a character trough

$$\langle \mathrm{ch}_Y \rangle = \frac{1}{(2n)!^2} \sum_{\underline{\nu}, \underline{\mu} \in [S_{2n}]} |\underline{\nu}| |\underline{\mu}| N^{\sum \mu_k} |[2^n]| \chi^Y(\underline{\nu})$$
$$\times \sum_{Y'} \frac{\chi^{Y'}([2^n]) \chi^{Y'}(\underline{\mu}) \chi^{Y'}(\underline{\nu})}{\chi^{Y'}([1^{2n}])}.$$

Summing over $\underline{\nu}$ and using the orthogonality of characters of S_{2n}, this is

$$\langle \mathrm{ch}_Y \rangle = |[2^n]| \frac{\chi^Y([2^n])}{\chi^Y([1^{2n}])} \sum_{\underline{\mu} \in [S_{2n}]} \frac{\chi^Y(\underline{\mu}) N^{\sum \mu_k} |\underline{\mu}|}{(2n)!}. \tag{2.4}$$

Since, as was noted above, $t_{\underline{\nu}}(\mathbb{1}) = N^{\sum \mu_k}$, the last sum is identified from Eq. (2.3b) as $\mathrm{ch}_Y(\mathbb{1})$, the dimension of the Y-representation of the linear group.

Using $|[2^n]| = (2n-1)!!$, we conclude that

$$\langle \mathrm{ch}_Y \rangle = (2n-1)!!\chi^Y([2^n]) \cdot \frac{\mathrm{ch}_Y(\mathbb{1})}{\chi^Y([1^{2n}])}, \tag{2.5a}$$

which admits the easy generalization

$$\langle \mathrm{ch}_Y(MA) \rangle_M = (2n-1)!!\chi^Y([2^n]) \frac{\mathrm{ch}_Y(A)}{\chi^Y([1^{2n}])}. \tag{2.5b}$$

Each factor on the right-hand side of Eq. (2.5a) is clearly identified: $(2n-1)!!$ is the number of contractions in a homogeneous polynomial in M of degree $|Y| = 2n$; next, $\chi^Y([2^n])$ appears as the signature of Gaussian averages; finally the last term is the ratio of the dimensions of the representations of the linear and permutation groups corresponding to Y. As far as we can tell, Eq. (2.5) is new. It should admit generalizations for other compact Lie groups. Table I gives $\langle \mathrm{ch}_Y(M) \rangle$ for $2n = 2$ and 4.

The reader will note that for positive integral N these values are always integers (sometimes negative). This is true in general, and we digress to give an argument communicated by G. Segal.[14]

The set of characters $\mathrm{ch}_Y(M)$ attached to Young tableaux with a given number $|Y| = 2n$ of boxes forms a basis of symmetric functions of degree $2n$ of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$. Another basis, attached to partition $\underline{\nu} = 1^{\nu_1}2^{\nu_2} \cdots 2n^{\nu_{2n}}$ of $2n$, is provided by

$$m_{\underline{\nu}} = \sum_{\substack{1 \le i_1 < i_2 < \cdots < i_{\nu_1} \le N \\ 1 \le j_1 < j_2 < \cdots < j_{\nu_2} \le N \\ \vdots}} \lambda_{i_1}^1 \lambda_{i_2}^1 \cdots \lambda_{i_{\nu_1}}^1 \lambda_{j_1}^2 \cdots \lambda_{j_{\nu_2}}^2 \cdots .$$

Table I. Values of $\langle \mathrm{ch}_Y(M) \rangle$ for $2n = 2$ and 4

$2n$	Y	$\langle \mathrm{ch}_Y \rangle$
2	(2)	$\frac{N(N+1)}{2}$
	(1^2)	$\frac{N(N-1)}{2}$
4	(4)	$\frac{N(N+1)(N+2)(N+3)}{8}$
	(3)(1)	$\frac{-N(N^2-1)(N+2)}{8}$
	$(2)^2$	$\frac{N^2(N^2-1)}{4}$
	$(2)(1)^2$	$\frac{-N(N^2-1)(N-2)}{8}$
	$(1)^4$	$\frac{N(N-1)(N-2)(N-3)}{8}$

It is known that the matrix that connect these two bases and its inverse both have integer entries. The integrality of the $\langle \mathrm{ch}_Y \rangle$ is thus equivalent to the integrality of the $\langle m_{\underline{\nu}} \rangle$.

A generating function for the $m_{\underline{\nu}}(\lambda)$ (for $|\underline{\nu}|$ bounded by some r) is provided by

$$\det \sum_0^r t_k M^k = \prod_{i=1}^N (1 + t_1\lambda_i + t_2\lambda_i^2 + \cdots + t_t\lambda_i^r)$$

$$= \sum_{\underline{\nu}} t_{\underline{\nu}} m_{\underline{\nu}}(\lambda),$$

where $t_{\underline{\nu}} = r_1^{\nu_1} t_2^{\nu_2} \cdots$. Using Hermite monic orthogonal polynomials (see section 3), it is easy to see that

$$\left\langle \det \sum_0^r t_k M^k \right\rangle = \det\left(\sum_k t_k J^k \right)$$

in terms of the $N \times N$ principal minor of the (infinite) Jacobi matrix.

$$\lambda P_n(\lambda) = P_{n+1}(\lambda) + nP_{n-1}(\lambda) = \sum_m J_{nm} P_m(\lambda).$$

This is a polynomial in t' with integer coefficients, and the result follows.

Some topological application of matrix integration, to be discussed below, requires the computation of

$$\langle t_{[2n]} \rangle \equiv \mathrm{Tr}\, M^{2n} \rangle = \int DM\, \mathrm{Tr}\, M^{2n}.$$

In terms of a graphical expansion, this average involves only (connected) graphs with a single vertex which we call marguerite graphs. According to the previous analysis these have 1 vertex, n links, and a variable number of faces appearing in the power $N^F = N^{\sum \mu_i}$. Correspondingly, the surface will have genus g given by $2 - 2g = 1 + F - n$. Thus we can regard the factor N^F to be N^{n+1-2g}, with $g \leq n/2$, since the power of N is at least 1. Let $\varepsilon_g(n)$ be the number of these graphs for given g, so that

$$\langle t_{[2n]} \rangle = \sum_{2g \leq n} N^{n+1-2g} \varepsilon_g(n) = \sum_Y \chi^Y([2n]) \langle \mathrm{ch}_Y \rangle.$$

According to Eq. (2.5a), this is

$$\langle t_{[2n]} \rangle = (2n-1)!! \sum_Y \frac{\chi^Y([2n]) \chi^Y([2^n])}{\chi^Y([1^{2n}])} \mathrm{ch}_Y(\mathbb{1}).$$

A drastic simplification occurs here, since for $\chi^Y([2n])$ to be nonzero, the corresponding Young tableau, which we denote $Y_{p,q}$, must belong to a subclass with at most one row of length larger than one, equal to $1 + q \geq 1$;

thus $Y \equiv (1+q)(1)^p$, with $1+q+p=2n$. Furthermore,

$$\chi^{Y_{p,q}}([2n]) = (-1)^p$$

$$\chi^{Y_{p,q}}([2^n]) = \begin{cases} p \text{ even} & (-1)^{(p/2)}\binom{n-1}{p/2} \\ p \text{ odd} & (-1)^{(p+1)/2}\binom{n-1}{(p-1)/2} \end{cases}$$

$$\frac{\mathrm{ch}_{Y_{p,q}}(\mathbb{1})}{\chi^{Y_{p,q}}(1^{2n})} = \binom{N+q}{2n}, \qquad p+q+1=2n.$$

The constraint $p+1 \le N$ is automatically taken into account by the vanishing of the combinatorial factor. Thus, splitting the sum over p into even and odd parts,

$$\begin{aligned}
\frac{\langle t_{[2n]}\rangle}{(2n-1)!!} &= \sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} \\
&\quad \times \left[\binom{N+2n-2p-1}{2n} + \binom{N-2n-2p-2}{2n}\right] \\
&= \sum_{p=0}^{n-1} (-1)^p \binom{n-1}{p} \\
&\quad \times \oint \frac{dx}{2i\pi}\left[\frac{(1+x)^{N+2n-2p-1}}{x^{N-2p}} + \frac{(1+x)^{N+2n-2p-2}}{x^{N-2p-1}}\right] \\
&= \oint \frac{dx}{2i\pi}\,\frac{(1+x)^{N+2n-1}}{x^N}\left[1+\frac{x}{1+x}\right]\sum_{p=0}^{n-1}\left(\frac{x}{1+x}\right)^{2p}(-1)^p\binom{n-1}{p} \\
&= \oint \frac{dx}{2i\pi}\,\frac{1+2x}{1+x}\,\frac{(1+x)^{N+2n-1}}{x^N}\left[1-\frac{x^2}{(1+x)^2}\right]^{n-1} \\
&= \oint \frac{dx}{2i\pi}\,\frac{(1+2x)^n(1+x)^N}{x^N} \\
&= \frac{1}{2}\int \frac{dy}{2i\pi}\,\frac{1}{y^{n+2}}\left(\frac{1+y}{1-y}\right)^N,
\end{aligned}$$

where in the last step we changed variable, setting $y = 1/(1+2x)$. One concludes therefore that

$$\begin{aligned}
\langle t_{[2n]}\rangle &= \sum_{1\le g\le n/2} N^{n+1-2g}\varepsilon_g(n) \\
&= (2n-1)!!\frac{1}{2}\oint \frac{dy}{2i\pi}\,\frac{1}{y^{n+2}}\left(\frac{1+y}{1-y}\right)^N.
\end{aligned} \tag{2.6}$$

The right-hand side, which is $\frac{1}{2}(2n-1)!!$ times the coefficient of y^{n+1} in the expansion of $(1+y)/(1-y)^N$ at the origin is a polynomial in N of degree $n-1$ with integral coefficients, where N can now be considered as an arbitrary parameter.

3. HARMONIC OSCILLATOR AND FERMIONS

Let us give an alternative derivation of Eq. (2.6). Consider the generating function

$$T(x) = 1 + 2\sum_{n=0}^{\infty} x^{n+1}\frac{\langle t_{[2n]}\rangle}{(2n-1)!!}.$$

Setting $1/(2n-1)!! = 2^n n!/(2n)!$, we use Euler's representation of $n!$ to write for x positive,

$$\begin{aligned} T(x) &= 1 + 2\int_0^{\infty} y\,dy\,e^{-y^2/2x}\sum_{n=0}^{\infty} y^{2n}\frac{\langle t_{[2n]}\rangle}{(2n)!} \\ &= 1 + 2\int_0^{\infty} y\,dy\,e^{-(-y^2/2x)}\langle \mathrm{Tr}\, e^{yM}\rangle, \end{aligned}$$

where in the last step we noticed that odd powers of M have zero average. The Gaussian average of a class function, i.e., such that $f(M) = f(UMU^{-1})$ for U unitary, can be expressed as an average over the eigenvalues $\lambda_0, \ldots, \lambda_{N-1}$ of M arranged in a diagonal matrix Λ, in the form

$$\langle f(M)\rangle = \frac{\displaystyle\int \prod_{k=0}^{N-1} d\lambda_k\, e^{-\frac{1}{2}\lambda_k^2}\Delta^2(\Lambda) f(\Lambda)}{\displaystyle\int \prod_{k=0}^{N-1} d\lambda_k\, e^{-\frac{1}{2}\lambda_k^2}\Delta^2(\Lambda)}, \tag{3.1}$$

where

$$\Delta(x) = \prod_{N-1\geq k>l\geq 0} (\lambda_k - \lambda_l) = \det(\lambda_k)^l|_{0\leq k,l\leq N-1}$$

is a Vandermonde determinant. This brings us to a second theme, free fermions, and, in the present case, the harmonic oscillator eigenfunctions. Indeed, let $P_l(\lambda)$ stand for a monic polynomial of degree l, $P_l(\lambda) = \lambda^l + \cdots$. Then

$$\Delta(\lambda) = \det P_l(\lambda_k)|_{0\leq k,l\leq N-1}.$$

Choose the P_l orthogonal with respect to the measure $d\lambda\, e^{-\lambda^2/2}$, i.e.,

$$\int d\lambda\, e^{-\lambda^2/2} P_l(\lambda) P_k(\lambda) = 0 \qquad \text{if } l \neq k.$$

Then, by expanding determinants,

$$\begin{aligned} \langle \mathrm{Tr}\, e^{yM}\rangle = \mathcal{N}^{-1}\int \prod_{k=0}^{N-1} d\lambda_k\, e^{-\lambda_k^2/2} P_0(\lambda_0)\cdots P_{N-1}(\lambda_{N-1})\left(\sum_{s=0}^{N-1} e^{y\lambda_s}\right) \\ \times \sum_{\mathcal{P}\in S_N} (-1)^{\mathcal{P}} P_0(\lambda_{\mathcal{P}_0})\cdots P_{N-1}(\lambda_{\mathcal{P}_{N-1}}), \end{aligned}$$

where the normalization factor is

$$\mathcal{N} = \prod_{k=0}^{N-1} \int d\lambda\, e^{-\lambda^2/2} P_k(\lambda)^2.$$

For each term in the sum $\sum_{s=0}^{N-1} e^{y\lambda_s}$, the orthogonality property forces the permutation $\mathcal{P}$ to be the identity, and therefore

$$\langle \mathrm{Tr}\, e^{yM} \rangle = \sum_{s=0}^{N-1} \frac{\int d\lambda\, e^{-\lambda^2/2+y\lambda} P_s^2(\lambda)}{\int d\lambda\, e^{-\lambda^2/2} P_s^2(\lambda)}.$$

The reader will recognize a standard expression for the average of a $\mathbb{1}$-body operator in the theory of an N-fermion system. Should one wish to do so, one could generalize the following discussion to various many-body operators, therefore obtaining generating functions for other averages $\langle t_\nu \rangle$. We shall refrain from doing so here.

Returning to $T(x)$, we see that the calculation is reduced to a single harmonic oscillator,

$$T(x) = 1 + 2\int_0^\infty y\, dy\, e^{-y^2/2x} \sum_{s=0}^{N-1} \langle s|e^{y\lambda}|s\rangle,$$

where we have used the *bra-ket* notation of quantum mechanics

$$\langle \lambda | s \rangle = \frac{e^{-\lambda^2/2} P_s(\lambda)}{[\int d\lambda\, e^{-\lambda^2/2} P_s^2(\lambda)]^{1/2}}.$$

We also introduce creation and annihilation operators

$$a = \frac{\lambda}{2} + \frac{\partial}{\partial\lambda} \qquad a^\dagger = \frac{\lambda}{2} - \frac{\partial}{\partial\lambda} \qquad [a, a^\dagger] = \mathbb{1} \qquad \lambda = a + a^\dagger.$$

The vacuum state $|s = 0\rangle$ is such that $a|0\rangle = 0$ and

$$|s\rangle = \frac{(a^\dagger)^s}{\sqrt{s!}}|0\rangle.$$

Let us once more compute a generating function for an arbitrary complex v

$$G(y, v) = \sum_{s=0}^\infty \frac{(\bar{v}v)^s}{s!} \langle s|e^{y\lambda}|s\rangle$$

$$= \sum_{s=0}^\infty \frac{1}{(s!)^2} \langle 0|(\bar{v}a)^s\, e^{y(a+a^\dagger)} (va^\dagger)^s|0\rangle.$$

Setting $z = e^{i\theta}v$, this is

$$G(y, v) = \int_0^{2\pi} \frac{d\theta}{2\pi} \langle 0|e^{\bar{z}a}\, e^{y(a+a^\dagger)}\, e^{za^\dagger}|0\rangle,$$

and from the commutation rules

$$G(y, v) = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{y^2/2+\bar{z}z+y(z+\bar{z})}$$
$$= e^{y^2/2+\bar{v}v} \sum_{n=0}^{\infty} \frac{y^{2n}}{(2n)!} \int_0^{2\pi} \frac{d\theta}{2\pi} (v e^{i\theta} + \bar{v} e^{-i\theta})^{2n}$$
$$= e^{y^2/2+\bar{v}v} \sum_{n=0}^{\infty} y^{2n} \frac{(v\bar{v})^n}{n!^2}.$$

We therefore have

$$\sum_{s=0}^{\infty} \frac{(v\bar{v})^s}{s!} 2 \int_0^{\infty} y\, dy\, {}^{-y^2/2x} \langle s|e^{\Lambda y}|s\rangle$$
$$= \int_0^{\infty} dy^2\, e^{-y^2/2x+\frac{1}{2}y^2+v\bar{v}} \sum_{n=0}^{\infty} \frac{y^{2n}}{n!^2} (v\bar{v})^n$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} (v\bar{v})^n \left(\frac{2x}{1-x}\right)^n e^{v\bar{v}} = \frac{2x}{1-x} e^{v\bar{v}(1+x)/(1-x)}$$
$$= \frac{2x}{1-x} \sum_{s=0}^{\infty} \frac{(v\bar{v})^s}{s!} \left(\frac{1+x}{1-x}\right)^s.$$

We conclude that

$$2 \int_0^{\infty} y\, dy\, e^{-y^2/2x} \langle s|e^{\Lambda y}|s\rangle = \frac{2x}{1-x} \left(\frac{1+x}{1-x}\right)^s$$

and

$$T(x) = 1 + 2 \sum_{n=0}^{\infty} x^{n+1} \frac{\langle t_{[2n]}\rangle}{(2n-1)!!}$$
$$= 1 + \sum_{s=0}^{N-1} \frac{2x}{1-x} \left(\frac{1+x}{1-x}\right)^s = \left(\frac{1+x}{1-x}\right)^N, \tag{3.2}$$

which agrees with our previous result, Eq. (2.6).

4. THE VIRTUAL EULER CHARACTERISTIC OF THE MAPPING CLASS GROUP

Given a closed orientable connected surface of genus g, one considers the smooth (i.e., continuous—possibly with continuous derivatives; this is not what matters here) one-to-one orientation-preserving maps. These form a group which possesses an invariant subgroup of these maps homotopic to the identity, with the discrete mapping class group as the factor group. The one-to-one maps act on the homologies, in particular on the group H_1, the only nontrivial one. Given a set of generators of H_1, the mapping class group transforms them linearly preserving the intersection matrix, i.e., as $\mathrm{Sp}(2g, \mathbb{Z})$ transformations, and it is asserted that this homomorphism is surjective. In the case $g = 1$, with $\mathrm{Sp}(2, \mathbb{Z}) \sim SL(2, \mathbb{Z})$ this homomorphism is an isomorphism so that the mapping class group is a double covering of the standard modular group $PSL(2, \mathbb{Z})$. The modular group acts on the ratio τ of two independent periods of elliptic curves, considered as a complex variable with positive imaginary part. The quotient of the upper half plane by $PSL(2, \mathbb{Z})$ is depicted by a fundamental region which is a quadrangle $\mathrm{Im}\, \tau > 0, |\tau| \geq 1, |\mathrm{Re}\, \tau \leq 1$ with the identification of sides through $\tau \to \tau + 1$ and $\tau \to -\tau^{-1}$. This identification leads to an orbifold, topologically a sphere with a point (at infinity) deleted and Euler characteristic 1 (an oriented surface obtained from the sphere by gluing g handles and deleting s disks or s points has Euler characteristic $2 - 2g - s$).

Although one can introduce the modular invariant function $j(\tau)$ mapping the upper half τ-plane mod $PSL(2, \mathbb{Z})$ one-to-one on the complex plane to depict the situation, the smooth differentiable structure in the j-plane is not equivalent to the one in the τ-plane at the preimages of $j = 1728$ and $j = 0$ ($\tau = \sqrt{-1}$, $\sqrt[3]{-1}$ mod $PSL(2, \mathbb{Z})$, respectively). This is due to modular transformations with fixed points ($\tau' = -\tau^{-1}$, $\tau' = -(\tau + 1)^{-1}$). However there exist subgroups of finite index acting without fixed points in the upper half plane. Such is the case of the modular subgroup of level 2, with index 6. If the elliptic curve is represented as $y^2 = P_4(x)$, where the right-hand side is a polynomial of degree four in x, this modular subgroup leaves invariant the cross ratio of the four roots of this polynomial. This cross ratio is defined only up to permutation of the roots under which it assumes six values distinct from 0, 1, and ∞ (which would correspond to a coincidence of two roots and hence depict degeneracies of the topological torus). At these exceptional values, j is infinite. Thus we have a sixfold smooth covering of the modular space as a sphere with three punctures. This suggests that we define the virtual (or orbifold) Euler characteristic of the modular group of genus 1 as $\frac{1}{6}\chi(S_2 - \{0, 1, \infty\}) = -\frac{1}{6}$. However, this is not quite what is required, since $PSL(2, \mathbb{Z})$ is only a factor group of $SL(2, \mathbb{Z})$, being covered twice. This is responsible for an extra division by a factor 2, so with χ the virtual Euler characteristic, one has

$$\chi(SL(2, \mathbb{Z})) = -\tfrac{1}{12}. \tag{4.1}$$

This slightly paradoxical result is what is generalizable to arbitrary genus and arbitrary number of punctures. One looks for a contractible space where a subgroup of finite index of the corresponding mapping class group acts without fixed points, computes the ordinary Euler characteristics of the corresponding factor space, and divides by the index of the subgroup. One shows that the construction is independent of the various arbitrary choices.

However, before proceeding to the general case, we would like, pedestrian physicists as some of us are, to rederive the $-\frac{1}{6}$ for $PSL(2, \mathbb{Z})$ in another, more direct (but equivalent) way, exposing perhaps some "naiveté" on our part. As is well known, the upper half τ-plane can be endowed naturally with an $SL(2, \mathbb{R})$ invariant metric of constant negative curvature, say -1. A geodesic triangle (meaning a triangle with arcs of geodesics as sides) has area $(\pi - \alpha - \beta - \gamma)$ if α, β, γ denote the (interior) angles at the vertices. The modular fundamental domain consists of two such triangles, each one with angles 0, $\pi/2$, and $\pi/3$. On the other hand, for curvature -1 the Gauss–Bonnet formula reads

$$\chi = \frac{1}{2\pi} \sum_{\text{triangles}} (\alpha + \beta + \gamma - \pi)$$

for a decomposition into geodesic triangles (the formula as it stands also holds for the sphere of curvature $+1$, where it correctly yields $\chi = 2$). In the present case, a blind application yields

$$\chi(PSL(2, \mathbb{Z})) = 2 \cdot \frac{1}{2\pi}\left(0 + \frac{\pi}{2} + \frac{\pi}{3} - \pi\right) = -\frac{1}{6}.$$

Of course the use of the formula does not yield the true Euler characteristic $(+1)$, but instead the virtual one because of the orbifold conical points where the differential structure is not smooth. Should one, however, similarly dissect the sixfold covering discussed above into twelve similar triangles, one would get the correct integral result -1. So we see on this "trivial" example why and how it is much easier to compute virtual characteristics. As emphasized by Penner[10] and as implicitly recognized by Harer and Zagier[11] this is naturally done in the context of matrix integration.

In the sequel, one denotes Γ_g^1 the mapping class group for genus g and 1 puncture (Γ_g^s for s punctures, $\Gamma_g \equiv \Gamma_g^0$). It acts on the space of conformal equivalence classes of such surfaces but also on cellular complexes defined by arc decompositions of such a surface. A clever choice and some topological analysis lead Harer and Zagier[11] to a specific construction, where the number of $(6g - 3 - n)$-dimensional cells weighted by the inverse of the order of their isotropy group is $\lambda_g(n)/2n$ and the virtual characteristic is obtained as the finite sum

$$\chi(\Gamma_g^1) = \sum_{2g \le n \le 6g-3} (-1)^{n+1} \frac{\lambda_g(n)}{2n}. \tag{4.2}$$

To define $\lambda_g(n)$, a combinatorial factor, one proceeds by intermediate steps as follows: Consider a $2n$-gon, label the sides, and identify them pairwise to get an orientable surface of genus g. The number of distinct ways to do so is called $\varepsilon_g(n)$. Note that after such an identification one gets a connected graph of a matrix theory with V vertices (of varying valence), n

links (or propagators), and 1 face (or index loop) such that $V - n + 1 = 2 - 2g$; hence $n + 1 - 2g > 0$. So $\varepsilon_g(n)$ is the number of such graphs. Each of these graphs may contain an arbitrary number of vertices of valence 1 (or "tadpoles"). Let $\mu_g(n)$ the number of graphs defined as before but without tadpoles. Since tadpoles may be attached to any of the two "lips" of a propagator, one clearly has

$$\varepsilon_g(n) = \sum_m \binom{2n}{m} \mu_g(n-m). \tag{4.3}$$

Furthermore graphs without tadpoles may still contain vertices of valence 2 of "self-energy" type. Let $\lambda_g(n)$ denote the number of graphs with n-propagators of genus g and without valence 1 or 2 vertices. One has

$$\mu_g(n) \sum_m \binom{n}{m} \lambda_g(n-m). \tag{4.4}$$

In $\lambda_g(n)$ we count only graphs with vertices of valence k larger than or equal to 3. For a given graph, let V_k denote their number. We have

$$V = \sum_{k\geq 3} V_k = n + 1 - 2g \qquad \sum_{k\geq 3} kV_k = 2n.$$

Therefore $2n - 3V$ is a nonnegative integer, showing that $\lambda_g(n)$ is non-vanishing only if n satisfies the inequalities

$$2g \leq n \leq 6g - 3$$

as claimed in Eq. (4.2).

When g is equal to 1, the base point is irrelevant; $\Gamma_1^1 \sim \Gamma_1$. In the computation of $\lambda_1(n)$, n can take only the values 2 and 3, and one readily finds a unique graph in each case: $\lambda_1(2) = 1$, $\lambda_1(3) = 1$. Therefore, again,

$$\chi(SL(2,\mathbb{Z})) = \chi(\Gamma_1) = -\tfrac{1}{4} + \tfrac{1}{6} = -\tfrac{1}{12}.$$

As g increases, the simple enumeration needed to apply (4.2) becomes an impossible task and one requires more effective tools. Two techniques have been developed. In the first one computes $\varepsilon_g(n)$ and then uses Eqs. (4.3) and (4.4) to get $\lambda_g(n)$ and then χ. This is the method followed by Harer and Zagier,[11] which we shall reproduce in this section. The second procedure, due to Penner,[10] leads at once to $\chi(\Gamma_g^s)$. It will be explained in the following section.

To obtain $\varepsilon_g(n)$, one observes that instead of enumerating the graphs obtained by the above gluing procedure, one can exchange the roles of faces and vertices using duality. For this purpose one chooses a point inside the $2n$-gon and draws a (flat) $2n$-vertex with $2n$ hooks. The latter, extended to the sides of the polygon, generate for each pairwise identification of these sides a marguerite graph as exemplified in Fig. 2.

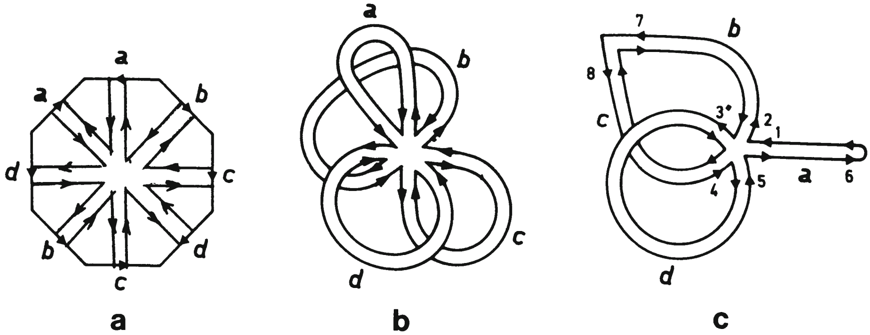


Figure 2. (a) Pairwise identification of a $2n$-gon, (b) associated marguerite graph, (c) initial graph obtained by gluing the sides. Graphs (b) and (c) are dual.

The marguerite graph has a 1 vertex, n links, and a number of faces $F = n + 1 - 2g$. Moreover, each such graph arises from the Gaussian average of Tr M^{2n}. Using Eq. (2.6), we have for fixed n

$$\begin{aligned}\langle t_{[2n]}\rangle &= \sum_{2g\leq n} N^{n+1-2g}\varepsilon_g(n)\\ &= \frac{(2n-1)!!}{2}\oint \frac{dy}{2i\pi}\frac{1}{y^{n+2}}\left(\frac{1+y}{1-y}\right)^N.\end{aligned}$$

Thus

$$\begin{aligned}\langle t_{[2]}\rangle &= N^2\\ \langle t_{[4]}\rangle &= 2N^3 + N\\ \langle t_{[6]}\rangle &= 5N^4 + 10N^2\\ \langle t_{[8]}\rangle &= 14N^5 + 70N^3 + 21N\\ \langle t_{[10]}\rangle &= 42N^6 + 420N^4 + 483N^2 \cdots.\end{aligned}$$

As a side remark we note the interpretation of $\langle t_{[2n]}\rangle$ as $(2n-1)!!$ times the sum of the dimensions of the irreducible representations of the linear groups in N variables with Young tableaux of $n + 1$ boxes and at most one line of length larger than one, so that

$$\begin{aligned}N = 1 &\qquad \langle t_{[2n]}\rangle = (2n-1)!!\\ N = 2 &\qquad \langle t_{[2n]}\rangle = (2n-1)!!2(n+1)\\ N = 3 &\qquad \langle t_{[2n]}\rangle = (2n-1)!!(2n^2 + 4n + 3) \cdots.\end{aligned}$$

One can, moreover, obtain $\varepsilon_g(n)$ as follows: In the above integral expression for $\langle t_{[2n]}\rangle$, one sets

$$y = \tanh\frac{t}{2} \qquad \frac{1+y}{1-y} = e^t,$$

and integrates by parts with the result

$$\sum_{0\leq g\leq n/2} \varepsilon_g(n)N^{n-2g} = \frac{(2n)!}{(n+1)!2^{n+1}} \oint \frac{dt}{2i\pi} \frac{e^{Nt}}{(\tanh t/2)^{n+1}}$$

$$= \frac{(2n)!}{(n+1)!} \oint \frac{dt}{2i\pi t^{n+1}} \left(\frac{t/2}{\tanh t/2}\right)^{n+1} e^{Nt}$$

Upon expanding e^{Nt} up to order t^n, one finds, for the right-hand side:

$$\frac{(2n)!}{(n+1)!} \sum_{p=0}^{n} \frac{N^{n-p}}{(n-p)!} \oint \frac{dt}{2i\pi t^{p+1}} \left(\frac{t/2}{\tanh t/2}\right)^{n+1}$$

Noticing that $(t/2)/\tanh t/2$ is even in t, this yields

$$\varepsilon_g(n) = \frac{(2n)!}{(n+1)!(n-2g)!} \oint \frac{dt}{2i\pi t^{2g+1}} \left(\frac{t/2}{\tanh t/2}\right)^{n+1} \tag{4.5}$$

The Bernouilli numbers are defined through

$$\frac{e^t}{e^t-1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}. \tag{4.6a}$$

The odd ones all vanish except B_1, and

$$B_0 = 1, \qquad B_1 = -\tfrac{1}{2}, \qquad B_2 = \tfrac{1}{6}, \qquad B_4 = -\tfrac{1}{30} \cdots,$$
$$B_{2n} = (-1)^{n+1}|B_{2n}|. \tag{4.6b}$$

Therefore

$$\left(\frac{t/2}{\tanh t/2}\right) = \sum_{n=0}^{\infty} B_{2n} \frac{t^{2n}}{(2n)!}, \tag{4.7}$$

and

$$\varepsilon_g(n) = \frac{(2n)!}{(n+1)!(n-2g)!} \text{coefficient of } t^{2g} \text{ in} \left(\frac{t/2}{\tanh t/2}\right)^{n+1}. \tag{4.8}$$

In order to contrast it with the direct method of Penner,[10] we now copy the remaining calculation of Harer and Zagier.[11] Define the three generating functions in n with reference to Eqs. (4.3) and (4.4):

$$E_g(x) = \sum_{n\geq 0} \varepsilon_g(n)x^n$$
$$M_g(x) = \sum_{n\geq 0} \mu_g(n)x^n \tag{4.9}$$
$$L_g(x) = \sum_{n\geq 0} \lambda_g(n)x^n.$$

One is in fact interested in $L_g(x)$, which is a polynomial in x. We find successively, for $|x|$ small enough,

$$E_g(x) = \sum_{m\geq 0} \mu_g(m)x^m \sum_{n\geq 0} x^n \binom{2n+2m}{n}$$

$$= \sum_{m\geq 0} \mu_g(m)x^m \oint \frac{du}{2i\pi} \frac{(1+u)^{2m}}{u - x(1+u)^2}$$

$$= \frac{1}{\sqrt{1-4x}} M_g\left(\frac{1-2x-\sqrt{1-4x}}{2x}\right)$$

Similarly,

$$M_g(x) = \frac{1}{1-x} L_g\left(\frac{x}{1-x}\right).$$

So, altogether,

$$L_g(x) = \frac{1}{(1+x)(1+2x)} E_g\left(\frac{x(1+x)}{(1+2x)^2}\right). \tag{4.10}$$

Instead of $E_g(x)$, it is more convenient to define an equivalent generating function which codes the quantities $\varepsilon_g(n)$. For that purpose notice that in Eq. (4.8) the coefficient of t^{2g} in $[(t/2)/(\tanh t/2)]^{n+1}$ is a polynomial in n of degree g, which vanishes when $n = -1$, while the prefactor, which can be rewritten

$$\frac{(2n)!}{(n+1)!(n-2g)!} = \binom{2n}{n+1} \frac{(n-1)!}{(n-2g)!},$$

involves the ratio $(n-1)!/(n-2g)!$, which is also a polynomial in n of degree $2g-1$, so altogether we find as a factor of $\binom{2n}{n+1}$ a polynomial in n of degree $d = 3g - 1$, with $n + 1$ as a factor. We can expand it not in powers of n but equivalently as $(n+1)$ times a combination of the d $(=3g-1)$ quantities

$$1,\ n-1,\ (n-1)(n-2), \ldots, (n-1)(n-2)\cdots(n-d+1),$$

which all vanish except for the first when $n = 1$, and are of the form

$$\frac{(n-1)!}{(n-1)!}, \frac{(n-1)!}{(n-2)!}, \ldots, \frac{(n-1)!}{(n-d)!}.$$

So, altogether, inserting a factor $r!/(2r)!$ for convenience

$$e_g(n) = \frac{(2n)!}{n!} \sum_{r=1}^{d} \frac{r!}{(2r)!} \frac{k(r)}{(n-r)!}$$

for some coefficients $k(r)$ which depend on g. This leads one to introduce the polynomial

$$K_g(x) = \sum_{r=1}^{d} k(r)x^r, \tag{4.11}$$

in terms of which

$$\begin{aligned} E_g(x) &= \sum_{n\geq 0} \varepsilon_g(n)x^n = \sum_{n\geq 0} x^n \frac{(2n)!}{n!} \sum_{r=1}^{d} \frac{r!}{(2r)!} \frac{k(r)}{(n-r)!} \\ &= \sum_{r=1}^{d} k(r)\frac{r!}{(2r)!} \sum_{n\geq r} \frac{x^n(2n)!}{(n-r)!n!} \\ &= \sum_{r=1}^{d} k(r)\frac{r!}{(2r)!} x^r \left(\frac{d}{dx}\right)^r \sum_{n\geq 0} x^n \binom{2n}{n} \\ &= \sum_{r=1}^{d} k(r)\frac{r!}{(2r)!} x^r \left(\frac{d}{dx}\right)^r \frac{1}{\sqrt{1-4x}} \\ &= \frac{1}{\sqrt{1-4x}} \sum_{r=1}^{d} k(r)\left(\frac{x}{1-4x}\right)^r. \end{aligned}$$

Consequently,

$$E_g(x) = \frac{1}{\sqrt{1-4x}} K_g\left(\frac{x}{1-4x}\right), \tag{4.12}$$

and from Eq. (4.10) the two polynomials $L_g(x)$ and $K_g(x)$ are related through

$$L_g(x) = \frac{K_g(x(1+x))}{(1+x)}. \tag{4.13}$$

Since K is a polynomial of degree $d = 3g - 1$ without constant term, the right-hand side of this expression is indeed a polynomial in x of degree $6g - 3$. From the definitions in Eqs. (4.2) and (4.9), it follows that

$$\begin{aligned} \chi(\Gamma_g^1) &= -\frac{1}{2}\int_0^1 \frac{dx}{x} L_g(-x) \\ &= -\frac{1}{2}\int_0^1 \frac{dx}{x(1-x)} K_g(-x(1-x)) \\ &= \frac{1}{2}\int_0^1 dx \sum_{r=1}^{d} (-1)^{r-1} k(r) x^{r-1}(1-x)^{r-1} \\ &= \sum_{r=1}^{d} (-1)^{r-1} k(r) \frac{r!(r-1)!}{(2r)!}. \end{aligned}$$

One recalls that

$$\varepsilon_g(n) = \binom{2n}{n+1}(n+1)\sum_{r=1}^{d} k(r)\frac{r!}{(2r)!}\frac{(n-1)!}{(n-r)!}$$

$$= \binom{2n}{n+1}(n-1)(n-2)\cdots(n-2g+1)$$

$$\times \text{ coeff. of } t^{2g} \text{ in } \left(\frac{t/2}{\tanh t/2}\right)^{n+1}$$

Comparing with the above, we see that $\chi(\Gamma_g^1)$ is obtained by setting $n = 0$ in the factor multiplying $\binom{2n}{n+1}$ on the right-hand side of these expressions. Hence

$$\chi(\Gamma_g^1) = -(2g-1)! \times \text{ coeff. of } t^{2g} \text{ in } \frac{t/2}{\tanh t/2}$$

$$= -(2g-1)!\frac{B_{2g}}{(2g)!}.$$

One concludes that

$$\chi(\Gamma_g^1) = -\frac{B_{2g}}{2g} = \zeta(1-2g) \qquad g \geq 1, \tag{4.14}$$

an elegant result in terms of Riemann's ζ-function at odd negative integers, where one recalls the functional equation (for integer g)

$$\zeta(1-2g) = \frac{(-1)^g}{(2\pi)^{2g}}\frac{(2g)!}{g}\zeta(2g).$$

It can be shown that the virtual characteristic of moduli space without any puncture is given by

$$\chi(\Gamma_g) = \frac{\chi(\Gamma_g^1)}{2-2g} = \frac{\zeta(1-2g)}{2-2g} \qquad g > 1, \tag{4.15}$$

while for genus 1 one has $\chi(\Gamma_1) = \chi(\Gamma_1^1)$. This is recorded in the Table II together with the true Euler characteristics $e(\Gamma_g^1)$ and $e(\Gamma_g)$, also obtained by Harer and Zagier.[11] Although the trend cannot yet be seen in this table, these authors also prove that in both cases of Γ_g and Γ_g^1 the ratio e/χ tends to 1 as $g \to \infty$.

Table II. Virtual (χ) and True (e) Euler Characteristics for $1 \le g \le 10$

g	$\chi(\Gamma_g^1)$	$e(\Gamma_g^1)$	$\chi(\Gamma_g)$	$e(g)$
1	$-\frac{1}{12}$	1	$-\frac{1}{12}$	1
2	$\frac{1}{120}$	2	$-\frac{1}{240}$	1
3	$-\frac{1}{252}$	6	$\frac{1}{1008}$	3
4	$\frac{1}{240}$	2	$-\frac{1}{1440}$	2
5	$-\frac{1}{132}$	6	$\frac{1}{1056}$	3
6	$\frac{691}{32760}$	8	$-\frac{691}{327600}$	4
7	$-\frac{1}{12}$	8	$\frac{1}{144}$	1
8	$\frac{3617}{8160}$	-34	$-\frac{3617}{114240}$	-6
9	$-\frac{43867}{14364}$	164	$\frac{43867}{229824}$	45
10	$\frac{174611}{6600}$	-350	$-\frac{174611}{118800}$	-86

5. DIRECT METHOD

Interchanging the roles of the marguerite graphs and their duals obtained by gluing the sides of a $2n$-gon amounts to using the Poincaré duality discussed in section 2. According to Eq. (2.2), we can write

$$\varepsilon_g(n) = \frac{1}{N} \sum_{\substack{\mu \\ \sum \mu_k = n+1-g}} \langle t_\mu \rangle_{[2n]} \frac{2n}{\prod_k k^{\mu_k} \mu_k!}. \tag{5.1}$$

To obtain $\lambda_g(n)$, defined in the previous section, all that is required is to modify the summation on the right-hand side by insisting that $\mu_1 = \mu_2 = 0$, since these give the numbers of vertices of valence 1 and 2, respectively. Therefore the virtual characteristic is

$$\begin{aligned} N\chi(\Gamma_g^1) &= N \sum_n (-1)^{n+1} \frac{\lambda_g(n)}{2n} \\ &= \sum_n (-1)^{n+1} \sum_{\substack{\mu, \mu_1 = \mu_2 = 0 \\ \sum \mu_k = n+1-2g}} \langle t_\mu \rangle_{[2n]} \frac{1}{\prod_{k \ge 3} k^{\mu_k} \mu_k!}. \end{aligned} \tag{5.2}$$

This may now interpreted, as noted by Penner,[10] as the perturbative contribution of order N to the logarithm of a partition function with interaction Lagrangian

$$\sum_{k\geq 3} \frac{x^{k-2}}{k} \operatorname{Tr} M^k$$

if we extract the coefficient of

$$x^{\sum_{k\geq s}(k-2)\mu_k} = x^{2n-2(n+1-2g)} = x^{4g-2}.$$

Thus finally (as an asymptotic expansion in $x \to +0$),

$$\sum_g x^{4g-2}\chi(\Gamma_g^1) = \text{coeffic. of } N \text{ in } \ln Z(x, N) \tag{5.3a}$$

$$Z(x, N) = \frac{\displaystyle\int dM \exp -\frac{1}{x^2} \sum_{k\geq 2} \operatorname{Tr} \frac{M^k x^k}{k}}{\displaystyle\int dM \exp -\tfrac{1}{2} \operatorname{Tr} M^2}. \tag{5.3b}$$

In fact, taking the logarithm, to restrict oneself to connected graphs, does not affect the coefficient linear in N, since the graphs having only one index loop are necessarily connected.

There is only one little problem. As it stands, $Z(x, N)$ is meaningless (except in a term-by-term perturbative expansion), since the integrand in the numerator

$$\det(\mathbb{1} - xM)^{1/x^2} \exp\left(\frac{1}{x} \operatorname{Tr} M\right)$$

is undefined unless $\mathbb{1} - xM > 0$, and moreover divergent (the inequality stands for each eigenvalue of M, or equivalently for the corresponding sesquilinear form). Henceforth we assume $x > 0$. The above uneasiness is familiar to anyone who has tried to derive Sterling's asymptotic formula for Euler's Γ-function.

Recall that

$$\Gamma(s+1) = \int_0^\infty du \exp(-u + s \ln u).$$

Assuming s real, positive, and large, the integrand has a maximum for $u = s$, so that changing variable from u to m through

$$u = s - m\sqrt{s},$$

we find

$$s\Gamma(s) = \left(\frac{s}{e}\right)^s \sqrt{s} \int_{-\infty}^{\sqrt{s}} dm \exp\left[-s \sum_{k=2}^{\infty} \frac{1}{k}\left(\frac{m}{\sqrt{s}}\right)^k\right]$$

or writing

$$s = \frac{1}{x^2}, \qquad x > 0,$$

$$\frac{(ex^2)^{\frac{1}{x^2}}}{\sqrt{2\pi x^2}}\Gamma\left(\frac{1}{x^2}\right) = \frac{\int_{-\infty}^{1/x} dm \exp -\frac{1}{x^2}\sum_{k=2}^{\infty}\frac{m^k x^k}{k}}{\int_{-\infty}^{+\infty} dm \exp -\frac{m^2}{2}}. \tag{5.4}$$

The analogy with the previous matrix integral is clear, as is the method for obtaining Stirling's formula. Asymptotically in the perturbative expansion in powers of x, we can drop exponentially small terms by extending the range of integration to $-\infty < m < +\infty$. This shows, by the way, that the expansion, although asymptotic, is necessarily divergent. On the other hand, it yields immediately the cure to a sensible $Z(x, N)$ without affecting its asymptotic expansion at the origin. All that is needed is the insertion in the integrand of the numerator a factor $\theta(\mathbb{1} - xM)$ with $\theta(y)$ being the step function, equal to zero for $y < 0$, to 1 for $y > 0$.

To complete the calculation is now straightforward. We factor the integral over M in terms of an integral over the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_{N-1}$, times an integral over the (diagonalizing) unitary matrices, as in section 3, introducing as a Jacobian the square of a Vandermonde determinant $\Delta(\Lambda)^2 = \prod_{k\neq l}(\lambda_k - \lambda_l)$.

The amended $Z(x, N)$ takes, therefore, the form

$$Z(x, N) = \frac{1}{(2\pi)^{N/2}\prod_1^N p!}\int \Delta^2(\Lambda) \prod_{0\le k\le N-1} d\lambda_k\, \theta(1 - x\lambda_k)$$
$$\times (1 - x\lambda_k)^{x^{-2}} \exp(\lambda_k/x). \tag{5.5}$$

Writing $\Delta(\Lambda)$ as a determinant, using the symmetry of the integral under permutations of the arguments λ_k, and finally changing the variables of integration from λ_k to $y_k = (1 - x\lambda_k)/x^2$, one obtains, from Eq. (5.4),

$$Z(x, N) = \left[\frac{(ex^2)^{\frac{1}{x^2}}}{\sqrt{2\pi}}\right]^N \frac{x^{N^2}}{\prod_1^{N-1} p!}\det \Gamma\left(\frac{1}{x^2} + r + s + 1\right)\Bigg|_{0\le r,s\le N-1}. \tag{5.6}$$

From

$$\Gamma\left(\frac{1}{x^2} + n + 1\right) = \frac{1}{(x^2)^{n+1}}\Gamma\left(\frac{1}{x^2}\right)\prod_0^n (1 + px^2),$$

it follows that

$$Z(x, N) = \left[\frac{(ex^2)^{\frac{1}{x^2}}}{\sqrt{2\pi}}\Gamma\left(\frac{1}{x^2}\right)\right]^N \frac{x^{-N^2}}{\prod_1^{N-1} p!} \det \prod_{p=0}^{r+s} (1 + px^2)|_{0\le r,s\le N-1}. \quad (5.7)$$

The last determinant is evaluated as

$$\det \prod_{p=0}^{r+s} (1 + px^2)|_{0\le r,s\le N-1} = x^{N(N-1)} \prod_{\Lambda}^{N-1} p! \prod_{p=1}^{N-1} (1 + px^2)^{N-p},$$

so that

$$Z(x, N) = \left[\frac{(ex^2)^{\frac{1}{x^2}}}{\sqrt{2\pi x^2}}\Gamma\left(\frac{1}{x^2}\right)\right]^N \prod_{p=1}^{N-1} (1 + px^2)^{N-p}. \quad (5.8)$$

To extract the required quantities, one uses the asymptotic (divergent) expansion

$$\ln\left[\frac{(ex^2)^{\frac{1}{x^2}}}{\sqrt{2\pi x^2}}\Gamma\left(\frac{1}{x^2}\right)\right] \underset{x\to+0}{\sim} \sum_{n\ge1} \frac{B_{2n}}{2n(2n-1)} x^{4n-2}, \quad (5.9)$$

as well as the Bernoulli polynomials defined by [compare with Eq. (4.6)]

$$\frac{te^{tx}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \quad (5.10a)$$

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r}, \quad (5.10b)$$

such that for $k \ge 1$

$$\sum_{p=1}^{N-1} p^k = \frac{B_{k+1}(N) - B_{k+1}}{k+1}. \quad (5.10c)$$

Therefore, as $x \to +0$,

$\ln Z(x, N)$

$$\sim N \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)} x^{4n-2} + \sum_{k=1}^{\infty} (-1)^{k-1}\frac{x^{2k}}{k}$$
$$\times \left[\frac{1}{k+1}\sum_{r=1}^{k+1} N^{r+1}\binom{k+1}{r} B_{k+1-r} - \frac{1}{k+2}\sum_{r=1}^{k+2} N^r\binom{k+2}{r} B_{k+2-r}\right]. \quad (5.11)$$

The coefficient linear in N is the asymptotic series

$$\sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g-1)} x^{4g-2} + \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k} B_{k+1}.$$

For $k \geq 1$, B_{k+1} vanishes unless $k+1$ is even; therefore,

$$\sum_{g\geq 1} \chi(\Gamma_g^1) x^{4g-2} \sim -\sum_{g\geq 1} \frac{B_{2g}}{2g} x^{4g-2}. \tag{5.12}$$

Thus we recover the previous result, valid for $g > 0$:

$$\chi(\Gamma_g^1) = -\frac{B_{2g}}{2g} = \zeta(1-2g). \tag{5.13}$$

In general, the coefficient of $N^s (s \geq 1)$ in $\ln Z(x, N)$, involving connected diagrams with s faces, is for $g > 0$

$$\sum_{2g > \sup(0,2-s)} x^{4g-4+2s} \frac{B_{2g}}{2g(2g+s-2)} \binom{2g+s-2}{s}. \tag{5.14}$$

According to Penner,[10] the coefficient of $x^{4g-4+2s}$ is the virtual Euler characteristic $\chi(\Gamma_g^s)$ of the mapping class group of surfaces of genus g with s punctures (allowing permutations of the punctures) so that, for $s > 0$ and $2g - 2 + s > 0$, one has

$$\chi(\Gamma_g^s) = (-1)^{s-1} \frac{\zeta(1-2g)}{2g+s-2} \binom{2g+s-2}{s}. \tag{5.15}$$

We see that Penner's method is in fact very effective.

6. CONCLUDING REMARKS

Various extensions and generalizations of the above calculations look natural and worth investigating. The first observation, looking at Eq. (5.8), is that $Z(x, N)$ as a function of x for fixed (large) N has poles of order p at each value $x^2 = -1/p$, for every positive p (responsible for the essential singularity at $x = 0$). It would perhaps be interesting to interpret the data pertaining to these singularities in topological terms or perhaps to relate them to some recent results in 2D quantum gravity. Also, one could try to compute some mean values of observables in this theory and find a suitable interpretation.

Matrix integration is not limited to Hermitian matrices. One could think of using real symmetric or antisymmetric matrices, quaternionic matrices, or matrices involving Grassmannian variables to describe moduli spaces of nonorientable or supersymmetric surfaces.

Also, instead of integrating over noncompact vector spaces, one could consider integrals over compact groups with various weights, the more so since we saw in section 2 how matrix integration is intimately related to group theory. Work by Andrews and Onofri[13] using the heat kernel over the Cartan torus of unitary groups, seems very suggestive in this direction of relations with "quantum groups." Moreover, further extensions are possible to integrals over coupled matrices or even matrix quantum mechanics. Finally, one can wonder about the status of matrix integration in relation to the topological properties of moduli spaces. Is this a simple combinatorial trick, or is there a deeper, natural explanation for its occurrence?

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Chapter 8

Field-theoretical Description of High-T_c Superconductors: Topological Excitations, Generalized Statistics, and Doping

E. C. Marino

1. INTRODUCTION

In the last few years, an intense theoretical activity has been employed in the study of quantum field theories (QFT) formulated in three-dimensional space-time. The interest in this kind of theory was, to a large degree, generated by the recent discovery of substances which become superconductors at a temperature quite a bit higher than the old BCS-type materials.[1] The reason for this unexpected connection between areas of physics which at first sight are completely unrelated begins to be understood with the observation that a common feature in all high-T_c superconductors is the occurrence of a layered structure consisting of CuO_2 planes in which Cu^{++} ions are assembled on the nodes and O^{--} ions on the links of a square lattice. There is experimental evidence that the superconducting properties are highly anisotropic and the planar nature of the system, it has been argued, plays a key role in the mechanism of superconductivity.

E. C. MARINO • Departamento de Física, Pontifícia Universidade Católica do Rio de Janeiro, 22452 Rio de Janeiro RJ, Brazil.

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The standard model for the new superconductors is a two-dimensional theory describing the interaction of the active electron of the Cu^{++} ions on the 2D square lattice. In the continuum limit, the dynamics of the system is described by a $(2+1)$D quantum field theory, and the abovementioned connection is therefore established.

A lot of extremely interesting ideas arose, using the peculiarities of $(2+1)$D QFT in the effort to explain the new mechanism of superconductivity.

In this work, we describe (in section 2) how the continuum field-theoretic description of the new superconductors is obtained, starting from the standard theory for the original condensed-matter system.

In section 3, we describe a method for the quantization of the topological excitations (skyrmions) which occur in the continuum model supposed to describe the new superconductors: the CP^1/nonlinear σ-model in $2+1$ dimensions. A general expression for the skyrmion correlation functions is obtained, and it is shown that when the topological angle is equal to zero, the skyrmions condense.

In section 4, we study the basic properties of spin statistics in two spatial dimensions and their connection with topology. We comment on the nontrivial spin statistics of skyrmions and finally point out the interesting consequences these peculiar properties of the $(2+1)$D system may have in the theoretical description of the new kind of superconductivity.

In section 5, we try to describe a possible way to include the dopants in the continuum QFT description.

2. THE CONTINUUM MODEL FOR HIGH-T_c SUPERCONDUCTORS

We show here how one arrives at the CP^1/nonlinear σ-model in $(2+1)$D, starting from the discrete version of the standard model for the high-T_c superconductors.

Let us take the typical compound La_2CuO_4, which becomes a superconductor upon doping, say with barium: $La_{2-\delta}Ba_\delta CuO_4$. There is evidence that superconductivity occurs in the CuO_2 planes, which contain C^{++} and O^{--} atoms in the configurations $3d^9$ and $2p^6$, respectively. Cu^{++} has spin $S = 1/2$ and one active electron and orbital. O^{--} has a perfect gas configuration, having therefore $S = 0$ and no active electron. The Cu^{++} are on the nodes and the O^{--} on the links of a square lattice. The standard model for the system is the two-dimensional Hubbard model, described by

$$H = -t \sum_{\langle ij\rangle,\sigma} (\psi^+_{i\sigma}\psi_{j\sigma} + \text{H.C.}) + U \sum_i (\psi^+_{i\uparrow}\psi_{i\uparrow})(\psi^+_{i\downarrow}\psi_{i\downarrow}). \tag{2.1}$$

In this expression $\psi^+_{i\sigma}$ is the creation operator for the active electron of Cu^{++}, with spin $\alpha = \pm 1/2$. Of course, there is one electron per site, corresponding to a half-filled band. U is positive, corresponding to in-site repulsion.

We may rewrite Eq. (2.1) in the form

$$H = -t \sum_{\langle ij\rangle,\sigma} (\psi^+_{i\sigma}\psi_{i\sigma} + \text{H.C.}) + U\left[\tfrac{1}{2}\sum_i \mathbf{S}_i \cdot \mathbf{S}_i + \sum_i \mathbf{S}_i \cdot (\psi^+_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{i\beta})\right], \quad (2.2)$$

where $\mathbf{S}_i$ is an auxiliary vector field. One can see that Eqs. (2.1) and (2.2) are equivalent by integrating over $\mathbf{S}_i$ and using the properties of the Pauli matrices $\boldsymbol{\sigma}_{\alpha\beta}$. The equation of motion of $\mathbf{S}_i$ is

$$\mathbf{S}_i = \psi^+_{i\alpha}\boldsymbol{\sigma}_{\alpha\beta}\psi_{i\beta}, \quad (2.3)$$

which shows that $\mathbf{S}_i$ is the spin operator of the active electron of the Cu^{++} ions.

Integrating over the fermions, one obtains, in the limit $U \gg t$,

$$H = J \sum_{\langle ij\rangle} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (2.4)$$

with $J = 4t^2/U$. Since $U > 0$ and $J > 0$, [Eq. (2.4)] corresponds to the antiferromagnetic Heisenberg model.

The continuum limit of the model described by Eq. (2.4) has been studied in terms of the plaquette staggered spin

$$\vec{S}_{\text{PL}} = \frac{\vec{S}_a - \vec{S}_b + \vec{S}_c - \vec{S}_d}{4\sqrt{S(S+1)}}, \quad (2.5)$$

where a, b, c, d are the vertices of a given lattice plaquette. In the continuum limit $\mathbf{S}_{PL} \to \mathbf{n}(\mathbf{x}, t)$, with $|\mathbf{n}|^2 = 1$. It may be shown, then, that the continuum limit of Eq. (2.4) gives a field theory in $2 + 1$ dimensions whose action is[2]

$$S = \int d^3x \, \tfrac{1}{2}\, \partial_\mu n^a \, \partial^\mu n^a, \qquad |\mathbf{n}|^2 = 1, \quad (2.6)$$

that is, the nonlinear σ-model.

We see that the nonlinear σ-field can be thought of as the continuum limit of the staggered spin of the Cu^{++} atoms of the superconducting (upon doping) material La_2CuO_4.

The nonlinear σ-model is well known to be equivalent to the CP^1 model.[3] To get the CP^1 representation, we write the $\mathbf{n}$ field as $\mathbf{n} = Z^+\boldsymbol{\sigma}Z$, where $Z = \binom{Z_1}{Z_1}$ is a doublet of complex scalar fields, such that $Z^+Z = |Z_1|^2 + |Z_2|^2 = 1$ and $\boldsymbol{\sigma}$ are the Pauli matrices. In terms of Z, the Euclidean action corresponding to Eq. (2.6) may be written as[3]

$$S = \int d^3x[|D_\mu Z_1|^2 + |D_\mu Z_2|^2]; \qquad Z^+Z = 1. \quad (2.7)$$

In this expression, $D_\mu = \partial_\mu + iA_\mu$ and $A_\mu = iZ^+\partial_\mu Z$.

The $(2+1)$D nonlinear σ-model possesses the identically conserved (topological) current

$$J^\mu = \frac{1}{8\pi} \varepsilon^{\mu\alpha\beta} \varepsilon_{abc} n^a \partial_\alpha n^b \partial_\beta n^c, \tag{2.8}$$

whose topological charge $Q = \int d^2x J^0$ measures the class of the mappings between $\mathbf{x} \in \mathbb{R}^2 \sim S^2$ and $n^a \in S^2$, induced by a static configuration $n^a(\mathbf{x}, 0)$. The nontriviality of this mapping makes possible the existence of solitons in the theory. We will study the quantization of these excitations in the next section.

There is also a nontrivial topology in the mappings between S^3 and S^2, discovered by Hopf. The inverse image of a point in S^2 is a closed curve in S^3. The Hopf invariant, which measures the class of the mappings between S^3 and S^2, expresses the number of times the curves belonging to S^3 link one another. It is therefore called the linking number. It may be written in terms of J^μ in the following way: Defining W_μ such that

$$J^\mu \equiv \varepsilon^{\mu\alpha\beta} \partial_\alpha W_\beta, \tag{2.9}$$

the Hopf invariant or linking number is given by

$$S_H = \int d^3x\, J^\mu W_\mu = \int d^3x\, \varepsilon^{\mu\alpha\beta} W_\mu \partial_\alpha W_\beta. \tag{2.10}$$

S_H is also known as the Chern–Simons action.

It is an amazing discovery,[4] the fact that, due to the nontrivial topology measured by S_H, the particles of the theory may change their statistics or suffer a so-called statistics transmutation. We will explain this phenomenon in section 4.

Taking into account the Hopf term, the nonlinear σ-model action may be generalized to

$$S = S_{NL\sigma M} + \theta S_H, \tag{2.11}$$

where $S_{NL\sigma M}$ is given by Eq. (2.6) and S_H by (2.10). θ is the topological angle.

In CP^1 language, the topological current J^μ, Eq. (2.8), is given by $J^\mu = \varepsilon^{\mu\alpha\beta} \partial_\alpha A_\beta$. We see that in this description, $W_\mu \equiv A_\mu$ and the Hopf extension of the CP^1 action is

$$S = S_{CP^1} + \theta S_{CS}, \tag{2.12}$$

where S_{CP^1} is given by Eq. (2.7) and S_{CS} is the Chern–Simons action for A_μ.

Later on we will comment on the surprising consequences the existence of a nontrivial topology in the mapping $S^3 \to S^2$ may have on the original condensed-matter system we started with.

3. QUANTIZATION OF TOPOLOGICAL EXCITATIONS IN THE CONTINUUM MODELS FOR THE NEW SUPERCONDUCTORS

As we saw in the last section, the existence of a nontrivial topology in the mapping $S^2 \to S^2$ allows the existence of solitons. We will expose here a method for the full description of the quantum field theory of these solitons.

The nonlinear σ-model possesses static soliton solutions with $Q = \int J^0\, d^2x = 1$, as was shown by Belavin and Polyakov.[5] The solution is given by

$$\mathbf{n}_S(\mathbf{x}, t) = (\sin f(r)\hat{\mathbf{x}}, \cos f(r)); \qquad \mathbf{x} = r\hat{\mathbf{x}}. \tag{3.1}$$

In the CP^1 version, the solution reads

$$Z_S = \begin{pmatrix} \cos\dfrac{f(r)}{2} & e^{-(i/2)\arg(\mathbf{x})} \\ \sin\dfrac{f(r)}{2} & e^{(i/2)\arg(\mathbf{x})} \end{pmatrix} \tag{3.2}$$

$$A_i^S(\mathbf{x}, t) = \tfrac{1}{2}\cos f(r)\partial_i \arg(\mathbf{x}); \qquad A_0^S = 0. \tag{3.3}$$

For $\theta = 0$, an exact solution may be found for $f(r)$, namely,

$$f(r) = 2\arctan\frac{\lambda}{r}; \qquad f(r) \to \begin{cases} 0 & r \to \infty \\ \pi & r \to \infty. \end{cases} \tag{3.4}$$

For $\theta \neq 0$, $f(r)$ has the same asymptotic behavior as above.

Making a gauge transformation $Z_i \to e^{-i\Lambda}Z_i$; $A_\mu \to A_\mu + \partial_\mu A$, with $\Lambda = -\frac{1}{2}\arg(\mathbf{x})$, we may write

$$Z_S = \begin{pmatrix} \cos f/2 & e^{-i}\arg(\mathbf{x}) \\ \sin f/2 & \end{pmatrix}; \qquad A_i^S = \tfrac{1}{2}[\cos f(r) + 1]\partial_i \arg(\mathbf{x}); \qquad A_0^S = 0. \tag{3.5}$$

We see that in the CP^1 version the soliton is vortex-like. These solitons were called skyrmions, in connection with the ones appearing in the Skyrme model.[6]

Let us now apply the dual method of soliton quantization to the skyrmions of the CP^1/nonlinear σ-model.[7]

In the order-disorder duality quantization scheme, the topological excitation creation operator is defined through the so-called dual algebra in which the asymptotic behavior of the classical topological solution is applied to the Lagrangian fields by commutation with the soliton operator.[8-10] In $2+1$ dimensions, the dual algebra involving the local Lagrangian fields Z and A_μ require a nonlocal soliton operator $\mu(\mathbf{x}, t; c)$ defined on a closed curve C. This was first observed in the case of vortices in the Abelian Higgs model[9] and also in globally symmetric theories in $2+1$ dimensions.[11]

In the case of skyrmions, the appropriate dual algebra which takes the above asymptotic behavior into account is

$$\mu(\mathbf{x}, t; c)A_i(\mathbf{y}, t) = \begin{cases} [A_i(\mathbf{y}, t) + (1/2)\partial_i \arg(\mathbf{y}-\mathbf{x})]\mu(\mathbf{x}, t; c) & \mathbf{y} \notin T_x(c) \\ [A_i(\mathbf{y}, t) - (1/2)\partial_i \arg(\mathbf{y}-\mathbf{x})]\mu(\mathbf{x}, t; c) & \mathbf{y} \in T_x(c) \end{cases} \tag{3.6a}$$

$$\mu(\mathbf{x}, t; c)Z_1(\mathbf{y}, t) = \begin{cases} e^{-i/2\arg(\mathbf{y}-\mathbf{x})}Z_1(\mathbf{y}, t)\mu(\mathbf{x}, t; c) & \mathbf{y} \notin T_x(c) \\ Z_1(\mathbf{y}, t)\mu(\mathbf{x}, t; c) & \mathbf{y} \in T_x(c) \end{cases} \tag{3.6b}$$

$$\mu(\mathbf{x}, t; c)Z_2(\mathbf{y}, t) = \begin{cases} Z_2(\mathbf{y}, t)\mu(\mathbf{x}, t; c) & \mathbf{y} \notin T_x(c) \\ e^{i/2\arg(\mathbf{y}-\mathbf{x})}Z_2(\mathbf{y}, t)\mu(\mathbf{x}, t; c) & \mathbf{y} \in T_x(c) \end{cases} \tag{3.6c}$$

In the above expressions, C is a closed plane curve contained in the $t =$ constant plane and $T_x(c)$ is the minimal surface bounded by it.

We are going to determine the operator $\mu(x; c)$ satisfying the above algebra. In order to do that, we must first determine the basic commutators among the Lagrangian variables Z_i and A_μ. The system contains the following set of second-class constraints:

$$\varphi_1 = Z_i^+ Z_i - 1 \approx 0; \ \varphi_2 = \pi_i Z_i + \pi_i^* Z_i^* \approx 0; \ \varphi_3 = P_1 - (\theta/\pi^2)A_2 \approx 0;$$

$$\varphi_4 = P_2 + (\theta/\pi^2)A_1 \approx 0.$$

In the above equations, $\pi_i = \partial\mathcal{L}/\partial\dot{Z}_i = (D_0 Z_i)^*$ and

$$P_i = (\partial\mathcal{L}/\partial(\dot{A}_i)) = (\theta/\pi^2)\varepsilon^{ij}A_j.$$

The first-class (gauge) constraints are implemented *à la* Gupta–Bleuler. Using the method of Dirac, we obtain the following nonzero commutators:

$$\begin{aligned} [Z_i, \pi_j] &= i[\delta_{ij} - \tfrac{1}{2}Z_i Z_j^*]\delta^2(\mathbf{x}-\mathbf{y}) \\ [Z_i^*, \pi_j] &= \frac{i}{2} Z_i^* Z_j^* \delta^2(\mathbf{x}-\mathbf{y}) \\ [\pi_i, \pi_j] &= \frac{i}{2}[\pi_i Z_j^* - Z_i^* \pi_j]\delta^2(\mathbf{x}-\mathbf{y}) \\ [\pi_i, \pi_j^*] &= \frac{i}{2}[\pi_i Z_j - Z_i^* \pi_j^*]\delta^2(\mathbf{x}-\mathbf{y}) \\ [A_i, A_j] &= \frac{i}{2\pi^2\theta}\varepsilon^{ij}\delta^2(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{3.7}$$

Using these commutation rules, we find the expression for $\mu(x; c)$ satisfying the dual algebra in Eq. (3.6):

$$\mu(\mathbf{x}, t; c) = \exp\left\{ -(1/2) \int_{\mathbb{R}^2 - Tx} d^2x' \arg(\mathbf{x}' - \mathbf{x})[Z_1^*(\mathbf{x}', t)\pi_1^*(\mathbf{x}', t) \right.$$
$$- \pi_1(\mathbf{x}', t)Z_1(\mathbf{x}', t)] + (1/2)\int_{Tx} d^2x' \arg(\mathbf{x}' - \mathbf{x})$$
$$\times [Z_2^*(\mathbf{x}', t)\pi_2^*(\mathbf{x}', t) - \pi_2(\mathbf{x}', t)Z_2(\mathbf{x}', t)] - (i\theta/\pi^2)$$
$$\left. \times \left[\int_{\mathbb{R}^2 - Tx} - \int_{Tx}\right] d^2x' \arg(\mathbf{x}' - \mathbf{x})\varepsilon^{0ij}\partial_i' A_j(\mathbf{x}', t)\right\}$$
$$\equiv e^{B(\mathbf{x}, t; c)}. \tag{3.8}$$

We may write the skyrmion operator μ in the compact form

$$\mu(\mathbf{x}, t; c) = \exp\left\{ i\int d^3z[j_\mu^{(1)}\tilde{A}^{(1)\mu} + j_\mu^{(2)}\tilde{A}^{(2)\mu} - (4\theta/\pi)J^\mu(\tilde{A}_\mu^{(1)} + \tilde{A}_\mu^{(2)})]\right\} \tag{3.9}$$

by using the currents $j_\mu^{(a)} = i[Z_a^+(D_\mu Z_a) - (D_\mu Z_a)^+ Z_a]$, $a = 1, 2$, the topological current J^μ, and defining the external fields

$$\tilde{A}_\mu^{(a)}(z, x; T_x(c)) = (-1)^{a+1}(1/2)\arg(\mathbf{z} - \mathbf{x})\int_{\Omega a} d^2\xi_\mu \delta^3(\xi - z), \qquad a = 1, 2, \tag{3.10}$$

where $\Omega_1 = \mathbb{R}^2 - T_x, \Omega_2 = T_x, z = (\mathbf{z}, z^0),\ x = (\mathbf{x}, t),\ d^2\xi_\mu = \delta_{\mu 0}d^2\xi$, and $\xi^0 = t$.

Observe that in the dual algebra in Eq. (3.6) as well as in our construction of μ, Eqs. (3.8)-(3.10), only the asymptotic behavior of the classical configuration was used. This classical configuration, however, contains in addition the smeared-out Heaviside functions $\cos f/2$, $\sin f/2$, and $\cos f$. Since we want to work with unsmeared fields, in the spirit of local field theory, we exchanged these smooth functions in Eqs. (3.8)-(3.10) by the corresponding true Heaviside functions centered on C. This immediately allows us to interprete R, the radius of C, as a measure of the skyrmion size. The classical analog of R would then be R_0 such that $f(R_0) = \pi/2$.

Let us consider now the correlation function $\langle\mu(\mathbf{x}, x^0; c_1)\mu^*(\mathbf{y}, y^0; c_2)\rangle$. This is most conveniently expressed in the functional integral framework. Of course we expect the occurrence of divergences associated with the time-localized infinite surfaces in Eqs. (3.9)-(3.10). These divergences also appear in the case of vortices and kinks and, as in those cases, they may be eliminated by the introduction of counterterms whose explicit form the requirement of the surface invariance of $\langle\mu\mu^*\rangle$ determines.[8-11] Inserting (3.9) into the Euclidean functional integral

$$\langle\mu\mu^*\rangle = Z^{-1}\int \prod_{a=1}^{2} DZ_a DZ_a^* DA_\mu\, e^{-[S[Z_a, A_\mu]+S_{\text{count}}]}\mu\mu^*\delta[|Z|^2-1], \quad (3.11)$$

and introducing the appropriate counterterms, we obtain[7]

$$\begin{aligned}\langle\mu(x;c)\mu^*(y;c)\rangle \\ = Z^{-1}\int DA_\mu DZ_a DZ_a^* \exp\Big\{-\int d^3z\Big[|[\partial_\mu + i(A_\mu + \tilde{A}_\mu^{(1)})]Z_1|^2 \\ + |[\partial_\mu + i(A_\mu + \tilde{A}_\mu^{(2)})]Z_2|^2 i\frac{2\theta}{\pi}J_\mu A^\mu \\ + \frac{i2\theta}{\pi}J^\mu[\tilde{A}_\mu^{(1)} + \tilde{A}_\mu^{(2)})] + \frac{i2\theta}{\pi}[\tilde{J}^\mu_{(1)} + \tilde{J}^\mu_{(2)})]A_\mu\Big]\Big\}\delta[|Z|^2-1]. \quad (3.12)\end{aligned}$$

In this expression, $\tilde{A}_\mu^{(a)} = \tilde{A}_\mu^{(a)}(z;x) - \tilde{A}_\mu^{(a)}(z;y);\ a = 1,2$.

One may easily see that surface invariance is a consequence of gauge invariance. The θ terms are surface invariant except for a term

$$I_\theta = -\int d^3z\left[i\frac{2\theta}{\pi}(\tilde{J}_\mu^{(1)} + \tilde{J}_\mu^{(2)})(\tilde{A}^{(1)\mu} + \tilde{A}^{(2)\mu})\right]. \quad (3.13)$$

It happens that

$$I_\theta = \begin{cases} 0 \\ i\theta \end{cases}, \quad (3.14)$$

depending on the surface chosen. This ambiguity is similar to the ones found previously in various systems. As we saw, it must reflect the commutation rule for the μ. Indeed, taking the operator $\mu(x;c)$ and using the basic commutators [Eq. (3.7)], we find (equal times)

$$\mu(x;c)\mu(y;c) = e^{i\theta}\mu(y;c)\mu(x;c). \quad (3.15)$$

For $\theta \neq 0,\ \pi$, the skyrmions obey generalized statistics in agreement with the semiclassical analysis of Wilczek and Zee.[4]

Using the relation $n^a = Z^+\sigma^a Z$, we may evaluate the commutation rules of the skyrmion field μ with the nonlinear σ-field:

$$\begin{aligned}\mu(x;c)n^\pm(y) &= e^{\pm i\arg(\mathbf{y}-\mathbf{x})}n^\pm(y)\mu(x;c) \\ [\mu(x;c), n^3(y)] &= 0,\end{aligned} \quad (3.16)$$

where $n^\pm = n^1 \pm in^2$. We see that μ is dual to $n^\pm$.

Let us show now an interesting property the skyrmions have at $\theta = 0$. For this value of θ, let us make the shift $A_\mu \to A'_\mu = A_\mu + \tilde{A}_\mu^{(1)}$ in the functional integral [Eq. (3.12)]. The effect of this (at $\theta = 0$) is that $\tilde{A}_\mu^{(1)}$ decouples from Z_1 and the new external field coupled to Z_2 becomes $\tilde{A}_\mu^{(2)} - \tilde{A}_\mu^{(1)}$. It happens that $\tilde{A}_\mu^{(2)} - \tilde{A}_\mu^{(1)}$ is a pure gauge:

$$\tilde{A}_\mu^{(2)} - \tilde{A}_\mu^{(1)} = \partial_\mu \Lambda$$

$$\Lambda = \tfrac{1}{2}[\arg(\mathbf{z} - \mathbf{x})\theta(z^3 - x^3) - \arg(\mathbf{z} - \mathbf{y})\theta(z^3 - y^3)]; \tag{3.17}$$

we may therefore completely eliminate the external field from Eq. (3.12) through a gauge transformation, obtaining the result

$$\langle \mu\mu^* \rangle_{\mu=0} = 1. \tag{3.18}$$

The above argument, of course, may be extended to show that for an arbitrary correlation function of the μ, we have

$$\langle \mu \cdots \mu^* \cdots \rangle_{\theta=0} = 1. \tag{3.19}$$

The skyrmions condense at $\theta = 0$. This fact, combined with the observation that the skyrmion is dual to the transversal components of $\mathbf{n}$, [Eq. (3.16)] leads us conclude that $\langle n^\pm \rangle_{\theta=0} = 0$. This is still compatible with ordering in the 3-direction.

4. GENERALIZED STATISTICS AND SUPERCONDUCTIVITY

A very interesting peculiar feature of systems of two (space) dimensions is the fact that the spin (or statistics) is not necessarily an integer or semi-integer as in three spatial dimensions.

Let us consider a one-particle system described by a wave function $\psi_i(\mathbf{x}, t)$, transforming under some irreducible representation of the rotation group.

Then, under a rotation of θ along a direction $\hat{\boldsymbol{\theta}}$, we have

$$\psi_i(\mathbf{x}, t) \xrightarrow[R(\boldsymbol{\theta})]{} (e^{-(i/\hbar)\boldsymbol{\theta}\cdot\mathbf{J}})_{ij}\psi_j(R\mathbf{x}, t), \tag{4.1}$$

where $\mathbf{J}$ is the angular momentum operator.

If $\boldsymbol{\theta} = 2\pi\hat{\boldsymbol{\theta}}$, then $R\mathbf{x} = \mathbf{x}$ and the representation of the rotation group is diagonal:

$$\psi_i(\mathbf{x}, t) \xrightarrow[R(2\pi\hat{\boldsymbol{\theta}})]{} e^{-i2\pi s}\psi_i(\mathbf{x}, t). \tag{4.2}$$

The number s is the spin of the system.

In an analogous way we may define the statistics in the following way: Let $\psi(\mathbf{x}_1, \mathbf{x}_2, t)$ be a wave function of two identical particles. Under a permutation, this wave function may differ from the original one by at most a phase (since the particles are identical):

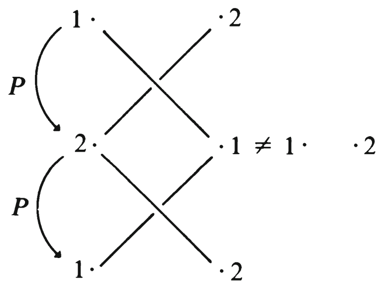
$$\psi(\mathbf{x}_1, \mathbf{x}_2, t) \xrightarrow[P]{} \psi(\mathbf{x}_2, \mathbf{x}_1, t) = e^{-i2\pi e}\psi(\mathbf{x}_1, \mathbf{x}_2, t). \tag{4.3}$$

The number e is the statistics. There is a remarkable theorem which states that $s = e$, called the spin-statistics theorem.[12]

In three spatial dimensions, the rotation group is non-Abelian $[SU(2)]$ and the algebra of angular momentum restricts the values of s to integers or semi-integers. In two spatial dimensions, on the other hand, the rotation group is Abelian $[U(1)]$ and the spin can be any real number. The same fact may be viewed from the point of view of statistics. In three spatial dimensions, two permutations are equivalent to the identity $P^2 = 1 \rightarrow P = \pm 1$, which implies, by Eq. (4.3), that

$$e^{-i2\pi e} = \pm 1 \Rightarrow e = \begin{cases} \text{integer} & \text{(bosons)} \\ \text{semi-integer} & \text{(fermions)} \end{cases} \tag{4.4}$$

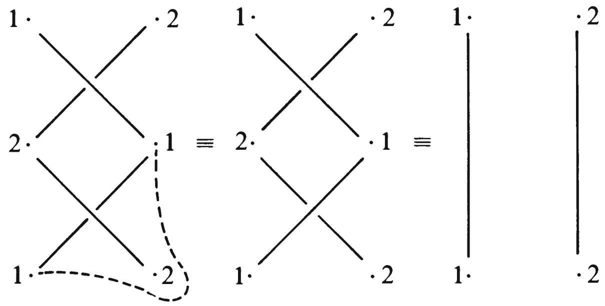
In three spatial dimensions, the wave functions for identical articles have definite parity under permutations and the Hilbert space yields a representation of the permutation group. In two spatial dimensions, it is no longer true that $P^2 = 1$. Imagine that the particles being exchanged are linked by elastic strings. One immediately recognizes that



$$P^2\psi(\mathbf{x}_1, \mathbf{x}_2) \neq \psi(\mathbf{x}_1, \mathbf{x}_2)$$

$D = 2$

In three spatial dimensions, however, we may undo the nontrivial braid



$D = 3$

In two spatial dimensions, the Hilbert space yields a representation of the braid group.

If $P^2 \neq 1$, e may be any real integer. For $s = e \neq$ integer or semi-integer, the spin-statistics is generalized, neither Fermi nor Bose. The particles are then called *anyons.* The wave function for anyons, of course, must be multivalued. The number of sheets is determined by $s = e$. It may be seen[13] that for a rational spin-statistics $s = e = 1/N$, the wave function contains N sheets, and for $s = e =$ irrational, it contains an infinite number of sheets. Representing by $\psi^{(n)}(\mathbf{x}_1, \mathbf{x}_2, t)$ the nth sheet of ψ, we have, under permutations,[13]

$$\psi^{(1)}(\mathbf{x}_1, \mathbf{x}_2) \xrightarrow[P]{} \psi^{(1)}(\mathbf{x}_2, \mathbf{x}_1) \xrightarrow[P]{} \psi^{(2)}(\mathbf{x}_1, \mathbf{x}_2) \xrightarrow[P]{} \psi^{(2)}(\mathbf{x}_2, \mathbf{x}_1) \xrightarrow[P]{} \cdots \psi^{(1)}(\mathbf{x}_1, \mathbf{x}_2). \tag{4.5}$$

A first example of a wave function possessing generalized statistics in two-dimensional space was given in Ref. 13, in the form

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = \exp\{E_{\text{INT}}(\mathbf{x}_1, \mathbf{x}_2)\}, \tag{4.6a}$$

where E_{INT} is the classical interaction energy of a system of two particles placed in $\mathbf{x}_1$ and $\mathbf{x}_2$ and each possessing a charge a and a magnetic flux b. The expression of $\psi(\mathbf{x}_1, \mathbf{x}_2)$ is[13]

$$\psi(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|^{-(a^2+b^2)/2\pi} \exp\left\{ i\frac{ab}{2\pi}[\arg(\mathbf{x}_1 - \mathbf{x}_2) + \arg(\mathbf{x}_2 - \mathbf{x}_1)] \right\}. \tag{4.6b}$$

Since $\arg(-\mathbf{r}) = \arg(\mathbf{r}) + \pi$, we see that, under a permutation, $\psi(\mathbf{x}_1, \mathbf{x}_2)$ acquires a factor e^{iab} and therefore $s = ab/2\pi$.

Given an arbitrary one-particle lagrangian $L(t)$, it is in general possible to change the statistics of this particle by the introduction of an additional interaction with a point magnetic flux on the particle. A point magnetic flux is associated with a magnetic field $B_s = \Phi/2\pi\, \delta^2(\mathbf{x})$. The vector potential associated to B is $\mathbf{A}_s = (\Phi/2\pi)\, \boldsymbol{\nabla}[\arg(\mathbf{x})]$. The interaction of the particle with $\mathbf{A}$ is given by the Lagrangian

$$L_s = q\dot{\mathbf{x}} \cdot \mathbf{A}_s = \frac{q\Phi}{2\pi}\frac{d}{dt}[\arg(\mathbf{x})]. \tag{4.7}$$

Observe that the charge q as well as the vector potential $\mathbf{A}_s$ are fictitious.

Since

$$\psi(\mathbf{x}, t) = \int d^2x_0\, G(\mathbf{x}, \mathbf{x}_0; t, t_0)\psi(\mathbf{x}_0, t_0) \tag{4.8a}$$

with

$$G(\mathbf{x}, \mathbf{x}_0; t, t_0) = \int_{\substack{\mathbf{x}(t)=\mathbf{x} \\ \mathbf{x}(t_0)=\mathbf{x}_0}} D\mathbf{x}\, e^{i\int_{t_0}^{t} L(\xi)\, d\xi}, \tag{4.8b}$$

it follows from Eq. (4.7) that for $L_{\text{ef}} = L + L_s$,

$$G_{\text{ef}} = e^{i(q\Phi/2\pi)[\arg(\mathbf{x}_0) - \arg(\mathbf{x})]} G. \tag{4.9}$$

Hence from Eq. (4.8a), we see that

$$\psi_{\text{ef}}(\mathbf{x}) \xrightarrow[R(2\pi)]{} e^{-i2\pi(q\Phi/2\pi)} \psi_{\text{ef}}(\mathbf{x}). \tag{4.10}$$

We see that, in the presence of L_s the spin-statistics of the particle suffers a transmutation. The new value of the spin is $s = q\Phi/2\pi$.

The fact that particles with generalized spin-statistics could be described in terms of charges and magnetic fluxes was first observed in[13] and later on generalized in.[14] The lagrangian L_s, Eq. (4.7) is called the "statistical interaction," for obvious reasons.

We may also introduce a statistical interaction in the framework of quantum field theory. Consider a general QFT in $(2+1)D$, described by a Lagrangian $\mathcal{L}$ possessing a conserved current j^μ. Let us introduce an additional Lagrangian $\mathcal{L}_s$ given by

$$\mathcal{L}_s = j^\mu A_\mu + \frac{1}{\theta} \varepsilon^{\mu\alpha\beta} \partial_\alpha A_\beta, \tag{4.11}$$

where A_μ is an arbitrary vector field. The field equation associated to $\mathcal{L}_s$ is

$$j^\mu = \frac{1}{2\theta} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta}, \tag{4.12}$$

where $F_{\alpha\beta}$ is the field-intensity tensor of A_μ. Observe that A_μ does not have independent dynamics, being completely determined by j_μ. For a j^μ corresponding to a static point particle,

$$j^\mu = \begin{cases} j^0 = \delta^2(\mathbf{x}) \\ j^k = 0 \end{cases} \Rightarrow F^{ij} = \theta \varepsilon^{ij} \delta^2(\mathbf{x}) \Rightarrow B(\mathbf{x}, t) = \theta \delta^2(\mathbf{x}). \tag{4.13}$$

We see that $\mathcal{L}_s$ induces the coupling of each particle of the theory with a statistical field B identical to the one found in the one-particle example. We conclude that in the presence of $\mathcal{L}_s$, the particles associated with j^μ suffer a transmutation in statistics determined by the value of θ, the topological angle.

The second term of Eq. (4.11) is the Chern–Simons Lagrangian whose action, as we saw, is the Hopf term, which measures the nontrivial topological classes of the mapping $S^3 \to S^2$. In this way, we clearly see the relation between topology and the transmutation in statistics in $(2+1)$ dimensional quantum field theories.

In view of the above analysis, we immediately conclude that in the presence of the Chern–Simons term, the Z particles of the CP^1 model change their statistics, since their current $j_\mu = iZ^+ \overleftrightarrow{\partial}_\mu Z$ is coupled to the

Chern–Simons field A_μ. This fact was first noticed in Ref. 4. We may also see that the topological excitations (skyrmions) of the CP^1 model will change their statistics in the presence of the Chern–Simons term, since in terms of the topological current $J^\mu = \varepsilon^{\mu\alpha\beta}\partial_\alpha A_\beta$, we have

$$S_{\text{Chern–Simons}} = \tfrac{1}{2}\int d^3x\, J^\mu A_\mu + \tfrac{1}{2}\int d^3x\, \varepsilon^{\mu\alpha\beta} A_\mu \partial_\alpha A_\beta, \qquad (4.14)$$

which displays a coupling between the skyrmion current J_μ and the Chern–Simons field.

The fact that skyrmions change their statistics in the presence of a Chern–Simons term was first observed by Wilczek and Zee.[4] We can see clearly that this is indeed true in our operator formulation for the skyrmion. Taking Eq. (3.8), using the equal time commutation relations [Eq. (3.7)] and the identity $\varepsilon^{ij}\partial_i\partial_j \arg(\mathbf{x}) = 2\pi\delta^2(\mathbf{x})$, which is a consequence of the Cauchy–Riemann equation for the function ln x, we find $[B(\mathbf{x}, t), B(\mathbf{y}, t)] = (i\theta/\pi)\,[\arg(\mathbf{x}-\mathbf{y}) - \arg(\mathbf{y}-\mathbf{x})]$. Using the Baker–Hausdorff formula and the fact that $\arg(x) - \arg(-x) = \pi$, we immediately see that $\mu(\mathbf{x}, t; c_1)\mu(\mathbf{y}, t; c_2) = \mu(\mathbf{y}, t; c_2)\mu(\mathbf{x}, t; c_1)e^{i\theta}$. It is clear that for $\theta = 0$ the skyrmion is a boson, whereas for $\theta = \pi$ it is a fermion. For other values of θ it obeys a generalized statistics $(0 \le \theta < 2\pi)$.

Let us explore now the possible physical consequences that the occurrence of generalized statistics may have on the description of the new type of superconductivity.

It was pointed out by Laughlin[15] that anyons are natural candidates for the formation of Cooper pairs leading to superconductivity. It is well known[16] that bosons (fermions) may be described as classical particles plus an attractive (repulsive) potential. Anyons, on the other hand, may be described as classical particles plus a repulsive potential weaker than that of fermions.[17] But this is equivalent to saying that anyons are fermions plus an attractive potential. As was shown by Cooper[18] a long time ago, fermions in the presence of an arbitrary attractive potential always form bound states, at least at zero temperature. Anyons, therefore, are very likely to form bound states and condense, leading to superconductivity if they are charged. The statistical interaction would provide the attractive interaction responsible for this new kind of superconductivity (see Ref. 19 for a review).

There is an experimental way to detect this kind of anyon superconductivity. Observe that the Lagrangian L_s, (4.7), leading to a transmutation in statistics violates $P(\mathbf{r} = (x, y) \to (x, -y))$ and $T(t \to -t)$, since $(d/dt) \times (\arg(\mathbf{x}))$ is odd under P and T separately. An anyonic superconducting state would display this violation, an effect that in principle could be measured.

Another interesting feature of the anyonic superconducting state is the fact that, since anyons bear a magnetic flux, a condensate of anyons would contain this magnetic fluxes on it. As a consequence of the Bohm–Aharonov effect, two translations T_1 and T_2 would no longer commute ($T_1 T_2 \neq T_2 T_1$), since they differ by a closed loop around the magnetic fluxes. This spontaneous breakdown of commutativity of translations is known as the spontaneous generation of quantum holonomy and leads us into the branch of mathematics called noncommutative geometry.

Several authors have shown that when taking the continuum limit starting from the undoped system—that is, the antiferromagnetic Heisenberg model at half filling—one arrives at the nonlinear σ-model with topological angle $\theta = 0$. Upon doping, however, θ is no longer zero, in spite of the fact that its dependence on the doping parameter δ is still unknown. We see that, assuming that the field-theoretic description of the superconducting system is correct, the process of doping changes the statistics of the particles in the continuum associate model! This fact, combined with the potentiality of anyons for superconductivity and the experimental fact that superconductivity is introduced upon doping the original system, makes obvious the reason for the great interest in the study of $(2+1)D$ QFT and its peculiar properties in statistics and topology, in connection with high-T_c superconductors.

We are currently investigating the field-theoretical description of the doped system: the introduction of dopants, their interation with topological excitations, and so on.

5. INTRODUCTION OF DOPANTS: A POSSIBLE MECHANISM OF SUPERCONDUCTIVITY

Let us describe in this section how the charge carriers (holes) introduced through doping could be included in a QFT description.

As we saw, the pure system is described by the CP^1/nonlinear σ-model with $\theta = 0$. It was stressed[2] that in the presence of doping—i.e., starting from the Hubbard model out of the half-filling regime—one arrives at a continuum QFT with a topological angle θ that is no longer zero but an unknown function of the doping parameter δ. One could expect, therefore, that the dynamics of the original spin system would be described, in the presence of doping, by

$$S_\delta = S_{CP^1} + \theta(\delta) S_{CS}, \tag{5.1}$$

where S_{CP^1} is the CP^1 action and S_{CS} is the Chern–Simons term.

Let us describe the charge carriers, which are experimentally known to be located on the CuO_2 planes, by an independent fermionic field ψ in the $(2+1)$-dimensional space-time we have been considering.

It was firmly established[20] that in a $(1+1)D$ QFT description, the dopant field ψ would interact with the spins through a minimal coupling with the CP^1 field A_μ. We will assume the same holds true here and, as a working hypothesis, take

$$S = \int d^3x \, [|D_\mu Z_1|^2 + |D_\mu Z_2|^2 + \theta(\delta)\varepsilon^{\mu\alpha\beta}A_\mu\partial_\alpha A_\beta + i\bar{\psi}\partial\!\!\!/\psi + e\bar{\psi}\gamma^\mu\psi A_\mu] \tag{5.2}$$

as the action for the complete system including doping.[21]

Integrating over the fermion field,[22] one obtains

$$S_{\text{ef}} = \int d^3x \left[|D_\mu Z_1|^2 + |D_\mu Z_2|^2 + \left[\theta(\delta) + \frac{e^2}{8\pi} \right] \varepsilon^{\mu\alpha\beta}A_\mu\partial_\alpha A_\beta \right] + O[(A_\mu)^4]. \tag{5.3}$$

Since $A_\mu = iZ^+\partial_\mu Z$, the higher-order terms in A_μ contain higher derivatives and can be neglected in the continuum limit description of the lattice system. We see that we arrive at a CP^1 model with an effective topological angle $\theta_{\text{ef}} = \theta(\delta) + (e^2/8\pi)$ depending on the doping parameter.

The statistics of topological excitations depends now on the doping parameter δ. As we showed, in the absence of a topological term, the skyrmions condense. We could expect that for a certain critical value $\delta = \delta_c$, $\theta_{\text{ef}}[\delta_c] = 0$, implying the condensation of skyrmions at this amount of doping.

We strongly suspect[21] that the skyrmions become charged upon doping, as the topological excitations of the associated discrete model do.[23] In this case, skyrmion condensation at $\theta_{\text{ef}} = 0$ $(\delta = \delta_c)$ would provide a mechanism of superconductivity. We are presently investigating this possibility.

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Chapter 9

Random Dynamics, Three Generations, and Skewness

Holger Bech Nielsen

1. RANDOM DYNAMICS

The two talks I shall present here will in a way be almost the same one, the second time given in reverse order. In the first one—called *random dynamics*—I shall present a project which I and others have worked on for several years.[1,2] We seek to derive the already known laws of nature as a result of almost any very complicated fundamental model. In the second talk we look at the Standard Model—especially the structure of the gauge group—and, inspired by that, seek to get information on the fundamental laws of nature at a deeper level. I, of course, want to be able to say that nature points in the direction of a random rather chaotic fundamental model, and thus the second talk becomes number one in reverse order. A major prediction[3,4]—which I want to stress we made before the LEP experiments confirmed it—is that there should be only three generations of quarks and leptons!

HOLGER BECH NIELSEN • Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark.

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The idea of random dynamics may be presented in a mild and in a strong form.

In the *mild form*, random dynamics reduces to a warning against a certain bad way of drawing conclusions from empirical observations. The bad way of concluding, which random dynamics in the mild form warns against, consists in extrapolating an empirical law in one regime of experiment to an underlying regime without first investigating whether it could be more generally derived. It could be that the validity of the observed law of regularity could be understood under much more general assumptions than the simple possibility that it should also be valid at the "deeper" level.

An example of such a bad way way of concluding is the following hypothetical one; Suppose we look at Hook's law of elasticity as an empirical fact, from which we want to get inspiration for making a theory in solid-state physics. Imagine we did not know much about the structure of matter, but wanted to speculate theoretically about it. Taking it very seriously, we might conclude—actually, with some right on our side—that the fundamental potential $V(\mathbf{x}, \mathbf{y})$ between the constituents of the material (atoms) must be of a quadratic form in the relative position of the atoms which interact. In other words, the force between atoms must be harmonic. The potential would have to be of the form $V(\mathbf{x} - \mathbf{y}) = (\mathbf{x} - \mathbf{y} + \mathbf{a})^2 \times$ constant. (Actually, the constant three-vector $\mathbf{a}$ should be $\mathbf{0}$ in order to have rotational invariance, and if so the—of course wrong—"theory" would have a collapse problem: why does matter not collapse to zero volume?) But of course we now know, as I suppose most sensible physicists have known all the time, that this is not a trustworthy way of arguing and that the conclusion is wrong, too. The truth is that forces between atoms are only very approximately harmonic near the equilibrium situation. Indeed, there would be very long-range interactions of a very strong type between pieces of matter if atoms really interacted with quadratic potentials.

In fact, I would like to say that a little bit of thinking should be enough to avoid the wrong conclusion in this case of deriving harmonic forces. One can easily see that Hook's law could be derived by a very general Taylor expansion method like the following: Consider, e.g., a piece of rubber under the pull of a force F. Denote the length of the rubber piece by l, or assume that in equilibrium it is a function just of the force F, $l(F)$. In physics we usually assume in practice that all relevant functions are analytical so as to have a Taylor expansion. Let us do so for $l(F)$. I.e., we assume that $l(F) = l(0) + F \times dl/dF|_{F=0} + (1/2!) \times F^2 \times d^2l/dl^2|_{F=0} + \cdots$. Now the extension of the length of the rubber is $\Delta l(f) \stackrel{\text{def}}{=} l(f) - l(0) = F \times dl/dF|_{F=0} + \cdots$. Restricting our attention to small forces F, we may ignore all but the linear term $dl/dF|_{F=0} \times F$ in F in the Taylor expansion, and in this way we could "derive" Hook's law under very general assumptions. Indeed, this "derivation" contains a lot of truth, since we know that after all Hook's law is no longer valid for strong forces F. In fact they can even lead to the breaking of the material: a situation which must signify that Hook's law fails. So there is really no reason to believe that the higher-order terms in the Taylor expansion should not be there even though Hook's law does not need them. It would rather be a very strange accident if they were not present! Just thinking this way should be sufficient to make it obvious that it would be unjustified to accept harmonic forces between atoms from the validity of Hook's law. One should rather say that, to a large extent, one can ignore Hook's law as a major source of inspiration in building up a theory of solids at a more fundamental level. Rather, one should learn only some general idea from it since there is some equilibrium of the material and some smoothness of the forces. It really cannot tell us much since it can be derived so easily!

A historically true example is the theory of Heisenberg,[5] which is an attempt to create a unified theory of physics. It is now known to be a violation of the warning, which I call the mild version of random dynamics. The Heisenberg theory is a field theory in which isospin symmetry is used as one of the guiding principles.[5] Now we know that isospin symmetry is not fundamental, but that it can relatively easily be understood—as explained by Steven Weinberg[6]—as a consequence of the QCD model for strong interactions together with the assumption that u and d quarks are light. It is indeed true that both the up quark and the down quark are very light compared to the scale of QCD, the Λ_{QCD} scale. Roughly, we have $\Lambda_{QCD} = 210$ MeV, $m_u = 4$ MeV, and $m_d = 7$ MeV, so $m_{u,d}/\Lambda_{QCD} \approx 0$. Notice that it is now supposed that the ratio of the up quark's mass to that of the down quark is on the order of $4/7 = 1/1.8$ and thus deviates from unity much more than the one or a few percent by which the isospin symmetry is broken phenomenologically! Weinberg's point is that on the scale of hadrons—which is given by Λ_{QCD}—both the up- and the down-quark masses are so small that to a very good approximation we can consider them 0. Then it does *not* matter that their ratio is far from unity. According to the warning of "mild random dynamics," Heisenberg should have investigated whether there would possibly exist some way of deriving the isospin symmetry, but of course they were very well excused by the fact that no one had dreamt about QCD yet at that time.

A third example of the type of argument we warn against would be to take the fact that the kinetic energy of a nonrelativistic particle, is quadratic in the momentum as a deep fundamental principle. The principle one might like to conclude to be valid from this fact might be that the kinetic energy has to be quadratic in the conjugate momentum. In fact we know that this is just because a nonrelativistic particle is one with small momentum and that thus a Taylor expansion—this time in the square of the momentum because of rotational invariance—can be chopped off: the energy $E(\mathbf{p}^2)$ of the particle is expanded $E(\mathbf{p}^2) = E(0) + \mathbf{p}^2 \times E'(0) + \cdots$, ignoring the potential energy part. Then normalizing the energy for the zero momentum situation to zero by defining the (genuine) kinetic energy to be $T(\mathbf{p}^2) = E(\mathbf{p}^2) - E(0)$, we find $T(\mathbf{p}^2) = \mathbf{p}^2 \times E'(0) +$ terms chopped off for small $\mathbf{p}$, so we can define $1/2m = E'(0)$ and obtain the usual nonrelativistic formula for the kinetic energy $T(\mathbf{p}^2) = \mathbf{p}^2/2m$. So the kinetic energy formula is really just a consequence of a Taylor expansion and can not tell us much about what is really behind it. Without help from other principles it would not have been easy to find the formula behind it from just starting at the nonrelativistic $T(\mathbf{p}^2) = \mathbf{p}^2/2m$. From special relativity, we now believe that $T(\mathbf{p}^2) = \sqrt{\mathbf{p}^2 + m^2} - m^2$ is the truth behind it. Certainly a straightaway postulation of the nonrelativistic formula being the fundamental truth—in spite of its relative simplicity—would have been a mistake from what we now know (after Einstein)!

One can rather easily find more examples like this and the warning (= the mild form of random dynamics) is well motivated. It might even seem a useful idea to speculate whether a *strong version* of random dynamics could be true, although that of course is much less convincing: could it be that all the empirical laws which scientists have found could find some explanation without any severe assumptions, only using mild assumptions like the existence of Taylor expansions which are chopped off so that the argument would work almost whatever the basic theory might be? If we indeed could find such explanations in all cases, we could understand all the empirical knowledge without assuming what the fundamental law behind it was. We would not really need any fundamental law, or we could take a random one. It would not matter. This is the strong version of random dynamics: some day we might derive all the empirically found laws of physics and thus it does not matter or almost does not matter what physics will turn out to be at the fundamental level (at very short distances, say the at the Planck scale). We could explain the accessible phenomena anyway.

If it happened to be so, the best theory of everything (TOE) would be to imagine that a random theory was used as the world mechanism. In other words, the true TOE would be randomly chosen from a large class of mostly very complicated models.

Because the basic idea that the fundamental physics does not matter could be realized by calculating with a randomly chosen fundamental dynamics (= TOE model), we call this idea or project "random dynamics."

We may distinguish two possibilities that might each represent a success of the random dynamics idea, but would be impossible to distinguish for a very long time to come:

1. The true theory is really a chaotic one at the bottom.

2. Some possibility for the fundamental true theory exists, other than a chaotic one, but it does not matter because all our known physics is derivable rather independently of the precise true model anyway.

In order to argue for the likelihood of such a scenario I would like to remind you of the quantum staircase, i.e., that physics falls into a series of subsciences that to a large extent study phenomena concerned with shorter and shorter distances, smaller and smaller objects.

The idea of reductionism is that one can reduce the science at one level of the quantum staircase to the science field one level above on this staircase. For instance one may hope to understand and reproduce chemistry from atomic physics. In a dream of reduction—which can hardly be realized in practice—chemistry should all be understood in terms of atomic physics. We express this sort of relationship in the quantum staircase by letting the step "chemistry" be the one just below "atomic physics." At least for the part of the quantum staircase involved with branches of microphysics and chemistry, it is so that the higher one goes on the staircase the higher the energy of the elementary particles used in studying the field. This means that the distances relevant for the science get smaller the higher one walks up the staircase.

Typically the laws valid in one field are to a large extent taken over from the ones a level above. So, for instance, the law of the conservation of energy is valid at all the known steps of the quantum staircase, while, for instance, the abovementioned nonrelativistic form $T(\mathbf{p}^2) = \mathbf{p}^2/2m$ of the kinetic energy for a particle is not valid in high-energy physics where particles have relativistic velocities, but is (approximately) valid in atomic physics where velocities of the relevant particles are small compared to that of light. The regularity of approximately stable chemical bonds is also valid only at the rather moderate collision energies used in chemistry and atomic physics but becomes completely unreliable for high-energy collisions.

To obtain an idea about how often it happens that new laws appear on a rather general ground as one proceeds down the quantum staircase, one compares the number of symmetries that are input into the Standard Model with the number of those that appear just as a consequence of this model. The symmetry principles clearly assumed in the Standard Model are

1. Poincaré invariance, which really consists of translational invariance and Lorentz invariance, which again represents the principle of relativity and rotational invariance.
2. Gauge invariance under the gauge Lie algebra $U(1) \times SU(2) \times SU(3)$ (personally I like to talk about the gauge *group*, $S(U(2) \times U(3))$; see below).

You may say that this is six symmetries: three under (1) and three under (2), but you may also say that it is only two or three. However, the number of symmetries appearing as a result of the structure of the Standard Model and/or some limit taken, ignoring the weak interactions for instance is quite large:

A. There are a series of symmetries following from the fact that some of the quarks have small masses compared to the QCD scale Λ_{QCD}: Chiral symmetry $U(3) \times U(3)$, say, and, as a subgroup, isospin symmetry. Actually the axial $U(1)$ symmetry contained in this chiral group is broken by an anomaly, but there is still some sign of it.
B. Without any assumption of small masses for quarks, there are some conservation laws result from neglecting weak interactions. These are the conservation laws for quantum numbers such as strangeness, charm, bottom, top, the third component of isospin, and baryon number, which is conserved to even better accuracy.
C. Still ignoring weak interactions, one finds in the Standard Model parity P, charge conjugation C, and time reversal symmetry T.
D. Even with weak interactions, you find conservation of the various lepton numbers: electron lepton number, muon lepton number, and τ lepton number.

So even more symmetries seem to come out than one puts into the Standard Model. But we may stress that the resulting symmetries are only approximate, while the symmetries put in are presumably, but not necessarily, exact, although you do not know for sure. Some similar phenomena are likely to occur for any effective theory at any level of the quantum staircase, and so we do indeed expect that as one proceeds down the quantum staircase, new symmetries will appear as result of looking only at the relatively low energies at that level. This means that only some of the symmetries are simply inherited from the level above, while others have appeared as a result of taking the limit of low energy (or some similar limit). If we now imagine a very long quantum staircase with many steps to be found by future research, it seems likely that the percentage of the symmetries that are simply inherited from the fundamental theory at the top, reaching down to some of the at-present accessible levels is very low. As we go down the staircase, we will in fact expect that the relative number of simply inherited symmetries will go down exponentially: at each level we expect a certain fraction of the symmetries to appear due to some limit of low energy, say. What we here suspect for symmetries is presumably also true for other laws of nature, such as we have already suggested for the formula for the kinetic energy of a nonrelativistic particle.

The hypothesis of random dynamics in the *strong form* is the guess that the laws and symmetries appearing due to taking the limits of low energy, *et cetera*, make up 100% of the laws and symmetries. If this is so, there is no need for any fundamental level at all, since no known law or symmetry has survived from there anyway. More correctly: there might or there might not be any highest and most fundamental level on the staircase. We could hardly know anyway, since no regularity would survive from that level to tell us anything about it. We could imagine living at that level with its chaotic "laws" (or, better, chaotic dynamics, laws should not be chaotic by definition, one might say). Anyway, the point would be that it does not matter.

In order to justify such a hypothesis—that fundamental physics does not matter—we should be able to derive rather generally all the known laws. We have indeed made attempts at almost all of them, but I must admit that most of these attempts are not very convincing.

All the physics known today except for gravity is summarized in the Standard Model if we include into this such general, empirically justified principles as the theory of relativity and the idea that it is a quantum field theory and thus includes the principles of quantum mechanics.

In Table I, we list the nine assumptions or features which can be considered the input needed to construct the Standard Model in this quite complete sense. The number of asterisks is supposed to symbolize the degree of success with which we want to claim that we can deduce the feature in question from so general arguments that there is hope of saying that it can be explained by random dynamics. There are no three-star items, and that symbolizes that there is really no feature, we must admit, which can be

Table I. Features of the Standard Model "Derived" from Random Dynamics

*	1	Quantum features of Standard Model
(why not *)	2	Field theory
**	3	Gauge symmetry
**	4	With Lie algebra $U(1) \times SU(2) \times SU(3)$
(**)	5	and left-handed fermions in representations $3(y = 1, \underline{1}, \underline{1})$, $3(y = 1/2, \underline{2}, \underline{1})$, $3(y = 1/6, \underline{2}, \underline{3})$, $3(y = 2/3, \underline{1}, \bar{\underline{3}})$, $3(y = -1/3, \underline{1}, \bar{\underline{3}})$ (simplest system without anomalies)
(why not *)	6	Breaking by the Higgs field (or whatever) of $(y = -1/2, \underline{2}, \underline{1})$-representation (hierarchy problem)
(why not *)	7	Translational and Lorentz invariance ≈ gauge symmetry for general relativity
*	8	In (3 + 1)-dimensional space-time

perfectly derived from a random model without any extra input. The only two features to which we have given two stars without doubt are gauge symmetry and the claim that the gauge group is $U(1) \times SU(2) \times SU(3)$ or, we should rather say, $S(U(2) \times U(3))$. As we shall see below, these two features can be speculatively calculated from an amorphous space-time quantum field theory called a "field theory glass."

For some of the items in the table we have written by the asterisk "why not" in order to say that there is presumably no real need for explaining *that* feature, but that it could almost be an accident once the other principles have been either derived or assumed. To say that we can explain Poincaré invariance or, better, general relativity diffeomorphism symmetry may be a bit exaggerated, because we basically just argue that diffeomorphism symmetry (i.e., reparametrization invariance) is very analogous to the gauge symmetry of Yang-Mills field theories, and then we refer to the fact that we can argue that such gauge symmetries can appear in a quantum field theory (on a lattice, say) without being put in exactly, i.e., without fine tuning of any parameters. So we may say that by analogy there is also hope that diffeomorphism symmetry can appear without fine tuning, and thus at least with less good luck needed than one at first might have thought. Actually M. Lehto, M. Ninomiya, and I wrote some articles seeking to support this hope.[7]

The star for Quantum at item (1) was given for the understanding of the linearity of, say, the Schrödinger equation as derived from a chopped-off Taylor expansion. This is the way one should usually derive any linearity law.

The resulting Schrödinger equation may then be thought of as an equation for the time development of a wave functional of a quantum field theory. Then what is needed for the derivation is smallness of the wave functional and analyticity of the time derivative as a function of the functional.

The star for $3 + 1$ space-time dimensions refers to the remark that the Weyl equation in $3 + 1$ dimensions has special stability in a model in which Lorentz invariance is not assumed *a priori*, compared to equations for other dimensions.[1] In fact, a Taylor expansion of a very general non-Lorentz invariant operator for a linearized equation of motion for a fermion field around a point in energy-momentum space at which two states are degenerate leads to a $(3 + 1)$-dimensional Weyl equation.

In this review, let me also mention that D. Bennett, I. Picek, and I[3,4] claim that using a fit to a model inspired by random dynamics predicted that there should be only three generations, and not for instance four as could easily have been allowed experimentally at that time. It is as much as to say that there should, for instance, be only the three already known types of neutrinos: ν_e, ν_μ, and ν_τ. This model, inspired by random dynamics, was based on the following basic assumptions:

1. An "anti-grand-unification scheme," which reminds one of grand unification GUT, but does not quite look like the usual GUT. The latter makes use of a simple group or one in which you have at least some discrete symmetry between the couplings if there are several groups, while our model does not even use a semisimple one. A very general group—a random one—say, is of the form

$$U(1) \times U(1) \times \cdots \times U(1) \times SU(2) \times SU(2) \times \cdots \times SU(2) \times SU(3) \times \cdots \times SU(3) \times SO(5) \times \cdots \times SO(5) \times \cdots.$$

   It then breaks partly down to a group that is a cross product of as many copies of the Standard Model group as there are generations, so that each of these copies can have its own generation. There may still be some other, but basically irrelevant crossproduct factors at this level:

$$SMG \times SMG \times \cdots \times SMG(N_{\text{gen}}; \text{factors}) \times ? \cdots ?, \tag{1}$$

   where

$$SMG = S(U(2) \times U(3)) \quad \text{or} \quad U(1) \times SU(2) \times SU(3) \tag{2}$$

   is the Standard Model group, or Lie algebra. Successively this group breaks down again to its diagonal subgroup, which then is the Standard Model group found empirically. The diagonal subgroup of this group means

$$SMG_{\text{diag}} = \{(h, h, \ldots, h) \mid h \in SMG\}. \tag{3}$$

2. The gauge groups in the intermediate step—the N_{gen} SMG—have coupling constants just strong enough that it is barely avoided that there would already be confinement at the Planck scale of energy. This is taken to mean that at the Planck scale, where the breakings are all supposed to take place, these groups have just critical couplings in the approximation called the mean field approximation. It is an approximation that can be made in a lattice formulation of a gauge theory.

The point was that we fitted the number of generations N_{gen} to the fine structure constants, i.e., to the electromagnetic fine structure constant, the Weinberg angle, and Λ_{QCD} in our model. That was how we got the number 3 out.

2. FIELD THEORY GLASS

It should not really matter very much exactly how one implemented the idea of random dynamics, if indeed it did work. In fact, the hypothesis was that almost any model would be good enough. This should especially be true if one considers various classes of models from which a random one is chosen. And so it should not matter so much exactly which class of model and which measure on it is chosen in order to realize in some way the idea of a random model. In order to give a reasonable example of how one might implement the idea of a random theory, let us define what we may call a field theory glass[8] (or perhaps better a fluid, if we let the structure of the system of the space-time points be functionally integrated over it). First, imagine some pregeometric random discrete point structure, much like what one would construct by seeking to implement quantum gravity—general relativity theory with quantum mechanics—as a "lattice" theory, in the sense that there is supposed to exist only some discrete points in space-time. Second, assume that for each little neighborhood of this "lattice" gravity there is a random quantum field theory in the following sense: for each site of the irregular "lattice" structure, there is a random system of degrees of freedom and a random local contribution to the action. This is a way to implement the idea of a random "fundamental" theory or dynamics: for each site—which really means each point in space-time—you assume an *a priori* different model field theory, and you imagine that you can take these "theories" or "dynamics" to be chosen—by God—so that we can at least effectively calculate as if they were randomly chosen.

In principle we might think of realizing our theory by means of a Monte Carlo computer program. First the program should by means of random numbers set up a random manifold for each site in the dynamical "lattice." This random manifold should be the manifold on which the field defined on that site should take its values. For another site the manifold would be another random manifold. In this way there would really be a huge number of models from which the actually realized one is thought to be chosen randomly.

The idea is that not only the target space (=value space) is chosen randomly, but also the local action. That is to say, the functional form of each local action contribution is also supposed to be chosen randomly among all the possible functional forms for such a bit of action.

Now one may try to estimate what such an essentially random model would show up like at the long-wave length limit.

Assume that we made the model so much a discretized quantum gravity that it has got diffeomorphism symmetry and effectively also Poincarè invariance. Then there will be particles with masses, but these are *a priori* expected to be of the order of magnitude of the "fundamental" mass scale of the random model. Supposing that the input scale is the Planck scale, these masses would be *a priori* of the order of the Planck mass. So at first you would expect that the resulting masses would be too high for any of the particles accessible with the experimental technique of today. If this were true, random dynamics would predict that there would be no particles as light as electrons, hadrons, or anything known today. Luckily enough for the hope that random dynamics may have a chance, the argument that all masses would be of the order of the assumed fundamental scale has an important exception! The two-star origin of gauge symmetry (item 3 in Table I) represents this exception. The exception is that symmetries can appear "accidentally" without fine tuning and cause some particles to be light or exactly massless! The mechanism that can produce massless particles due to gauge symmetries obtained without fine tuning is a rather remarkable effect that can be illustrated by a model study exemplified by a lattice gauge theory with an explicit breaking of the gauge symmetry introduced into the action.

To be concrete, you might consider an ordinary $U(1)$-lattice gauge (on a regular lattice, or on an irregular one; it does not matter), but—and this is important—with an explicit gauge symmetry-breaking term added to the action. For instance, one might add a latticed mass term for the photon. That could be the sum over all the links of the lattice of the real part of the link variable

$$\alpha \sum_{-} \mathrm{Re}(U(-)), \tag{4}$$

with some coefficient α proportional to the naive (bare) photon mass. The full action becomes thus (of the form)

$$S = \beta \sum_{\square} \mathrm{Re}(U(M)U(\;|)U(\overline{})U(|\;)) + \alpha \sum \mathrm{Re}(U(-)), \tag{5}$$

where the term proportional to β is the "latticised" $-1/4g^2 \int F_{\mu\nu}^2 \, d^4x$.

Now one might *a priori* think that adding such a term would give the photon a mass roughly proportional to the coefficient of this gauge-breaking term. The surprise, however, is that α has to reach a finite value before any photon mass really appears! It may seem unexpected at first, but it is practically beyond doubt that there is a whole phase in this model in which there is a photon particle with zero mass. It will seem, however, less surprising if one considers the following change of variables:

Replace the original link variables $U(-)$ by the somewhat larger set of variables consisting not only of an analogous set of link variables $U_h(-)$, but also of a set of site-defined variables $H(\cdot)$. The relation is defined to be

$$U(x - x + \delta_\mu) = H(x) U_h(x - x + \delta_\mu) H(x + \delta_\mu)^{-1}. \tag{6}$$

Since we got superfluously many new (or "human") variables, both $U_h(-)$ and $H(\cdot)$, it is not surprising that there is the possibility of making transformations on these "human" variables without changing the original variables $U(-)$ at all. In fact, we can transform

$$\begin{aligned} U_h(x - x + \delta_\mu) &\to \Lambda(x) U_h(x - x + \delta_\mu) \Lambda(x + \delta_\mu)^{-1} \\ H(\cdot) &\to H(\cdot)\Lambda(\cdot)^{-1} \end{aligned} \tag{7}$$

and it will leave the original variables $U(-)$ invariant. So any action formed from the original variables will of course be left invariant under this transformation.

In this way we see that it is quite easy to introduce a completely formal gauge symmetry, just by choosing an appropriate notation with too many degrees of freedom.

Using a block spin method to introduce yet more new variables (some that are averages over the fields within small blocks), it is not difficult to see that the model achieved is actually a latticed Higgs model! Indeed the $U_h(-)$ field now transforms as a gauge field on the lattice and the field $H(\cdot)$ may act as the Higgs field. Now, however, the crucial point is that there is a possibility that the model is not really in the Higgs phase, but that rather the effective mass of the would-be Higgs field has become an ordinary bradyonic mass (i.e., real mass, m^2 positive) rather than a tachyonic one as is needed to get the Higgs phenomenon. So the Higgs mechanism does not work for a whole region in the space of parameters α and β. Here β is the coefficient of the usual gauge-invariant lattice action, i.e., $\beta = 2/g^2$, where g is the gauge coupling constant. In this range it was thus possible for God to make an exactly zero mass photon without having to fine tune anything. It was needed only to have these parameters in the right phase to get no Higgs mechanism and just a massless photon and an ordinary scalar charged particle!

The range with the zero-mass gauge particle, a bit dependent upon how one would measure it, may really be rather small although of order unity, but the most remarkable is that it is not a set of measure zero [for $U(1)$ at least].

For this mechanism of getting a massless particle without fine tuning to work it was not really important whether the discretized space-time model was a regular lattice or a more amorphous structure or even a discretized structure set up to make a "latticification" of quantum gravity. We have not proven this generality rigorously, but the mechanism has a general nature. In any case, we would expect that if somehow the model gets close enough to a discretized gauge theory, so that it has an approximate gauge invariance and a sufficiently weak coupling, then it will turn out to be in a gauge theory phase and show an exactly massless photon, if it is Abelian. If it approximates instead a non-Abelian gauge theory, it can be argued that although you normally will not see truly and exactly massless gauge particles because of confinement, there can without any exact fine tuning easily be several orders of magnitude of energy scales over which one effectively will see massless gauge particles as approximately asymptotically free. However, after all, at some low energies, particles get confined or spoiled somehow by the complicated vacuum.

What we finally want to conclude is that a random and chaotic field theory glass or a quantum-gravity discretized model with a lot of degrees of freedom taken as random can very easily—by accident—develop some effective gauge theories on it. We imagine as a natural assumption that the fundamental mass scale in a model of this type valid in nature is on the order of magnitude of the Planck mass. Then *a priori* one would expect that resulting effective particles as far as they have masses at all will have masses on the order of the Planck mass. However, as just stressed, there is the exception that a gauge symmetry can be constructed artificially and the theory comes into a phase in which this symmetry causes some particles to be massless. If this is the only exception to the intuitive expectation of all particles getting masses on the order of the Planck mass, then we would expect to observe at presently accessible energy scales only particles that should be massless due to gauge symmetry. In fact that is what we see in nature. All the particles of the Standard Model would be massless because of gauge invariance if it were not for the mysterious Weinberg-Salam-Higgs field or whatever the mechanism might be, e.g., Technicolor.[9] We may consider this fact an agreement of phenomenology with random dynamics. Of course it tells us only that the Standard Model is a low-energy limit of some other theory and that it carries signals of being that. But it is promising for the success of random dynamics that the Standard Model contains just the types of particles that can appear as massless without fine tuning, gauge particles and chiral fermions (except for the never-seen Higgs particle(s)). In this way it is a sign of agreement with random dynamics that the empirically supported model is a gauge theory, but of course competing theories such as superstring theory would also produce gauge theories.

However, random dynamics has a prediction or at least suggestions for the gauge group expected at the low-energy limit, where we find the Standard Model. In fact it suggests that we shall find each cross-product factor such as the $U(1)$, the $SU(2)$, and the $SU(3)$ in the Standard Model Lie algebra $U(1) \times SU(2) \times SU(3)$ only once. Probably we shall find more simple groups in the cross-product (but again, each only once!), as we go up in energy. In fact, a major idea coming out of random dynamics is what we call the "confusion mechanism." Roughly the idea is that in the amorphous type of vacuum speculated in random dynamics, there is a difficulty in keeping track of which cross-product factor is which if the gauge group has several isomorphic cross-product factors. Indeed, it would need some convention to tell in each little region of space-time which degrees of freedom locally correspond to the first of a couple of isomorphic factors (call it Peter) and which corresponds to the second one (call it Paul). Well, you might attempt to start with an arbitrary convention for which fields should be called Paul and which Peter in one little region and then extend the convention to a neighboring region, and so on. This extension of the convention in small steps would mostly be determined by the condition that it is the Peter field in one region that interacts with the Peter field in the next region, while the Paul field interacts with the Paul field. Now, however, will that extension of convention turn out successful if it is extended along some series of small regions lying on a closed curve in space-time? Presumably such an extension in a random (amorphous, say) model will turn out to be inconsistent in many cases. That is to say that in the glassy model in most cases there will be no satisfactory way of giving names to the gauge fields, so that neighboring small regions have interaction only between fields of the same name. What shall we see in the long-wavelength limit in such a confused model, where no consistent naming can be performed? N. Brene and I[10] have estimated that one will only see one set of gauge particles for each type of simple group, even if it appeared *a priori* as a factor many times!

The other gauge particles (i.e., the ones corresponding to antisymmetric linear combinations of the original gauge particles) acquire nonzero masses. So they do not show up in the low-energy limit. There will occur a back-driving force if one attempts to have nonzero potential $A_\mu^{\text{anti}} = A_\mu^{\text{Peter}} - A_\mu^{\text{Paul}}$. This signals a mass term $m^2(A_\mu^{\text{anti}})^2$.

We talk about the confusion mechanism being active when a gauge Lie algebra in some gauge model breaks down effectively to a subalgebra, because there are problems in setting up an appropriate distinction, between, for instance, various invariant subalgebras. It should be stressed that the permutation symmetry of a couple of isomorphic invariant subalgebras is not the only possibility for confusion. It could also happen in an amorphous model after going around a series of small regions along a closed curve in space-time, that one would be forced to identify the fields in a gauge field theory with their images under some other isomorphism of the gauge Lie algebra. If this isomorphism is only an inner one, i.e., if it is of the form

$$h \to aha^{-1},$$

it can be shown that an inconsistency after going around a closed curve will just represent some gauge fields present in the vacuum, and thus it can be removed by setting up a compensating field. But if it is outer, it cannot be compensated away, and we expect only massless gauge particles for that subgroup which is left invariant under the automorphism.

We see that random dynamics assumes that the gauge group varies *a priori* from place to place, but that the *a priori* gauge group breaks down and is effectively replaced by a subgroup. The tendency in this breakdown is to break it each time there is an (outer) automorphism, and the breaking goes into the subgroup invariant under the automorphism in question. Especially a gauge group that is a crossproduct of isomorphic groups tends to break down to the diagonal subgroup so that there is only one factor of each type at the end. This is indeed very close to the Standard Model Lie algebra, which happens to consist of the three lowest-dimensional (reductive) simple Lie algebras, each used only once! Strictly speaking, even the Standard Model group—even taken to be the *group* $S(U(2) \times U(3))$, as the electric-charge quantization rule suggests, as we shall see in the next lecture—has an outer automorphism and thus, strictly speaking, should break down under the conditions of an amorphous gauge theory. However, this is very little compared to other groups, and we can claim that random dynamics makes an approximately correct prediction for the gauge group. Also this single outer automorphism corresponds to charge conjugation and is strongly broken by the many chiral (Weyl) matter fields.

One may imagine that the Weyl fermions could somehow play a role in keeping order in the gauge fields by preventing confusion in some cases. One may indeed think that it could be avoided that there is confusion on return along some closed curve between a gauge group and its image under a charge-conjugation symmetry by an effect of the Weyl particles. Charge conjugation corresponds to an automorphism shifting the sign of the hypercharge (Abelian) gauge field and complex conjugation for $SU(3)$. The idea should be that the Weyl particle fields could be used all along the closed curve to mark which sign is to be used for the hypercharge gauge field and thus to keep track of the gauge fields so that the convention needed could be based on the Weyl fields rather than on just extending the definition from one place to the next. I.e., the convention could be, e.g., specifying the hypercharge sign by telling the sign of the lefthanded Weyl particle.

Assume now that indeed the Weyl particles do help in preventing confusion.

Then we expect all the accidental occurrences of gauge groups at different places in the field theory glass to produce confusion related to their complex conjugation symmetry (if they have complex conjugation) unless they have some charge conjugation-breaking Weyl fermions associated with the group in question. We take this to mean that groups having complex conjugation [or for that matter other (outer) automorphisms] not broken by their fermion matter fields or in other ways will suffer a breakdown by "confusion." We then take this approximately into account by speculating that only those $SU(3)$ and $U(1)$ cross-product factors which have a "generation" (family) of Weyl particles associated with them survive confusion. After the breaking, we would have only those $SU(3)$ and $U(1)$ groups which have (at least) one generation attached. We have at first no similar argument for the idea that all $SU(2)$ groups but one per generation should break by confusion, but one could perhaps imagine that, for instance, a $U(1)$ and an $SU(2)$ group could have occurred rather as one $U(2)$ group and thus the complex conjugation of the latter could make us hope for a similar effect. The problem with $SU(2)$ is that its complex conjugation automorphism is an inner one. Speculating that the $SU(2)$ somehow falls in line, we would have an intermediate scale region in which we have just

N_{gen} cross-product factors isomorphic to $U(1)$, N_{gen} factors isomorphic to $SU(2)$, and N_{gen} factors isomorphic to $SU(3)$. From the Lie algebra point of view, this speculation means that we have really at this intermediate stage the (N_{gen})th cross-product power of the Standard Model group $SMG = S(U(2) \times U(3)) \equiv U(1) \times SU(2) \times SU(3)$:

$$SMG \times SMG \times \cdots \times SMG(N_{\text{gen}} \text{ factors}). \tag{8}$$

Successively this intermediate group or algebra will then by confusion break down to its diagonal subgroup *SMG*, and that is the part of it we see experimentally.

The important point is that we here guess that we can relate the number of, say, $SU(3)$-groups which have been confused to break down to the QCD-SU(3), gauge group to the number of generations of fermions and leptons. The $SU(3)$ found in QCD as the phenomenological model is thus the descendent of N_{gen} parent groups.

Now we remember that in the random dynamics model the gauge theories came about by accident because a gauge symmetry that was accidentally found with sufficient accuracy in the field theory glass would show up as a theory with truly massless gauge particles. This phenomenon—this phase—happens in a whole region of parameters, so that fine tuning is avoided, but all over this region the couplings must be so that there is no MFA-confinement (at the lattice scale). Since, supposedly, a non-Abelian gauge theory without too many types of matter fields will be in the confining phase, strictly speaking we cannot get the phenomenon of appearance of a gauge theory without fine tuning for the non-Abelian case. In an approximate sense there is, however, no problem in having this mechanism also working for non-Abelian Yang–Mills fields. It is needed only that the coupling be small enough that there is no "confinement at the lattice scale." That must, however, be avoided even without use of the no-fine tuning mechanism. So we have to have in random dynamics, as well as in almost any model, that the running coupling at the Planck scale—which is here taken to be the smallest distance scale at which physics make sense—must be weak enough not to cause immediate confinement at that scale.

In our calculations, we take the avoidance of confinement at the Planck scale to mean that the running coupling $g(\mu_{\text{Planck}})$ at that scale is weaker than the critical coupling $g_{\text{crit } MFA}$ calculated in the mean field approximation. Calculated in this approximation, it is this critical coupling which separates the Coulomb phase with a gauge particle massless without fine tuning from the confining phase in which there is no massless gauge boson.

Now the mean field approximation is used for gauge theories which are described as lattice gauge theories. In principle one can therefore obtain a MFA phase for each plaquette action one may choose. This means that one has, strictly speaking, infinitely many parameters corresponding to the gauge coupling constants. In fact, one can use as a plaquette action any superposition of the characters (traces) of the various representations of the gauge group, so there is an infinite ambiguity in the plaquette action S_{LATTICE}. However in practice one is accustomed to the fact that one may almost guess the form of the plaquette action just from smoothness. It would not contain many more parameters than the number of continuum theory couplings. In the continuum gauge theory, there is one gauge coupling constant for each minimal invariant subgroup, i.e., for the Standard Model there are three gauge coupling constants, corresponding to the three subgroups $U(1)$, $SU(2)$, and $SU(3)$. Usually, information contained in these coupling constants is presented in the fine structure constant α, the Weinberg angle θ_W, and the QCD scale parameter $\Lambda_{\overline{MS}}$, say. Our main point concerning coupling constant predictions is to require that in the mean field approximation we shall have for the actual values the Coulomb phase, so that in first approximation there are massless gauge particles. Now, however, remember that in random dynamics there was at first only approximate gauge invariance. In order to get the Coulomb phase, it is then required both that the would-be gauge coupling is sufficiently weak and that the breaking is small enough. If there are large fluctuations in the would-be physical directions, one may expect the gauge noninvariant terms to be easier to be averaged out. Large fluctuations—say in the Euclideanized lattice—seems to be the most likely. This is taken to mean large entropy. So random dynamics suggests that the actual couplings at the Planck scale are so as to correspond to a maximal entropy in the gauge field degrees of freedom under the important requirement that in MFA there be Coulomb-phase physics at the Planck scale (i.e., if the lattice constant is taken to be the Planck length).[11]

This maximal entropy will be found—we think—in a corner of the region of coupling parameter space corresponding to the Coulomb phase. This corner is—we think—squeezed in between phases in which part of the standard model gauge group is confining, while another part is in the Coulomb phase. For instance one can have a phase in the mean field approximation in which, say, $U(2)$ [i.e. $U(1)$ and $SU(2)$] is in the Coulomb phase, while the $SU(3)$-degrees of freedom confine even in the mean field approximation (and not only at very long distances as usual). Where various phases of this type meet with each other and the fully confining and the fully Coulomb phase (all made in the mean field approximation, where all these phases exist) there will be corners. So random dynamics implies that the corner with the largest entropy of this type to be the truth, i.e., to correspond to the actual coupling constants. They must then be corrected for the dependence of the coupling constants on the renormalization point μ. That is to say, one must take into account the variation of the coupling constant due to the renormalization group in order to calculate the values measured experimentally in terms of those at the Planck scale assumed to be the ones in the critical corner corrected by a factor $1/\sqrt{N_{gen}}$.

If we just extrapolated the critical corner couplings down to the experimental scales we would not get good agreement; we would predict couplings that are too strong. However, it goes much better if we include the following correction: We say that the gauge couplings which at the Planck scale take on the "corner"-critical values in the mean field approximation are not the same as the observed ones. Rather the observed gauge group is the diagonal subgroup $SMG = S(U(2) \times U(3))$ of a cross product of three with this Standard Model Group isomorphic groups, one per generation. In other words: at the Planck scale, our model has the gauge group SMG_{diag}, which contains the experimentally found degrees of freedom appearing as the diagonal subgroup of the cross product $SMG \times SMG \times SMG$. This cross-product group may then be embedded in a larger, more general group. This hypothesis of an intermediate group that is a cross product of one Standard Model Group for each generation is quite natural in the scheme of random dynamics, as was explained above. The importance of this intermediate cross-product hypothesis is that under the breakdown of a group of the form $G \times G \times G$ (call the groups G_{Peter}, G_{Paul}, G_{Maria}), the inverse squares of the coupling constants are additive in the cross-product factors:

$$\frac{1}{g^2_{i,\,diag}} = \frac{1}{g^2_{i,\,Peter}} + \frac{1}{g^2_{i,\,Paul}} + \frac{1}{g^2_{i,\,Maria}}. \tag{9}$$

For all the couplings of the Peter, Paul, and Maria groups being in the same corner in the phase diagram, this assumption of the observed group being the diagonal one leads to a correct factor of three, making the Planck scale prediction for the couplings weaker in g^2. In the coupling it means that we predict this way a factor $\sqrt{3}$ weaker couplings at the Planck energy scale. This turns out to make the agreement with experiment rather good. We have not yet calculated the smaller details of the prediction for the gauge couplings from the corner-critical couplings interpreted this way, but it is very close to the older calculations[3] in which we had a parameter (x) to fit, and we can at least say that the corner couplings corrected with the factor $\sqrt{3}$ and extrapolated down will not be far away from agreeing with experiment.[11]

3. SKEWNESS: THE GOLDEN PRINCIPLE

Let us now, as announced, roughly present the first lecture again in reverse order in the sense that we look at the physical laws known today—the Standard Model—and seek to find signs in the model which indicate some special property that can serve as a *guiding principle* in constructing models behind the Standard Model. Or, perhaps, the same golden principles themselves governing the laws of physics can be found simply by looking for a characteristic property of the Standard Model. What we want to claim is that one of the most characteristic features of the Standard Model is its lack of symmetry![12] We can say the Standard Model is very skew. In fact many of the questions of the type, "Why should it just be like that?" are answered by: "This makes the Standard Model most skew."

The inspiration for the idea that the Standard Model should be especially poor in symmetries could appear from noticing that the gauge Lie algebra is the direct sum of the three lowest-dimensional invariant subalgebras that could be used: the Abelian algebra $U(1)$, the algebra $SU(2)$ of weak isospin, and finally the QCD algebra $SU(3)$. These three algebras have dimensions, respectively, of 1, 3, and 8. There are no lower-dimensional simple or abelian Lie algebras, only some nonreductive Lie algebras that would not be useful for the construction of Yang–Mills theories. By counting Lie algebras composed from simple and Abelian ones with dimension of the same order of magnitude as the Standard Model, Niels Brene and I[13] found that the remarkable feature is not so much that the invariant subalgebras have low dimension, but that none of them are used more than once! In other words, the remarkable fact is that we do not find several $U(1)$ or several $SU(2)$ etc. One can naturally state that fact by saying that the Standard Model lacks some symmetries. A typical Lie algebra such as $U(1) \times U(1) \times \cdots U(1) \times U(1) \times SU(2) \times SU(2)$ would have symmetries corresponding to permutations of the various isomorphic direct product factors. They are lacking in the Standard Model, because there are no algebras to permute, since each of them occurs only once. For a cross product of, for example, two factors $G \times G = \{(a, b) \mid a, b \in G\}$, the transformation corresponding to the permutation of these two factors is the mapping $(a, b) \to (b, a)$.

We clearly cannot say that nature does not like the gauge symmetries, but we want to claim that it does attempt to avoid the symmetries involving transformations shifting the gauge fields around without being gauge transformations themselves. This means the "outer automorphisms" of the gauge group (or algebra). It seems that nature likes the symmetries not touching the gauge fields, since it might have avoided such symmetries by choosing fewer generations. It seems that indeed nature attempts to avoid, for example, scalings of the hypercharge (field) and charge conjugation, say of the QCD $SU(3)$ Lie algebra. It tends to avoid the outer automorphisms. What nature attempts to do with symmetries which do not transform the gauge fields is less clear. Superficially, at least it seems that nature is not afraid of piling one generation on top of the other, thereby opening the way for lots of flavor mixing symmetries.

Many features of the Standard Model are explained by this horror of the outer automorphisms, or—equivalently—by the preference for skewness:

1. Why are there just three colors in QCD, i.e., why $N_c = 3$? Our answer is: $N_c = 3$ is the smallest number of colors that would not bring nature into the problem of having to keep track of two isomorphic gauge groups, namely an $SU(2)_{\text{weak}}$ and an $SU(2)_{\text{strong}}$, say. (In fact the only sensible number of colors leading to any gluons at all lower than $N_c = 3$ is the possibility $N_c = 2$, and that would produce the confusion with the weak gauge group.)
2. Why should quarks have the 1/3-charges they are known to have? This produces an electric charge quantization rule

$$Q = -\frac{t}{3} \pmod 1 \tag{10}$$

linking charge conjugation for the $SU(3)$ to that for electric charge. If, e.g., quarks had zero charge, there would exist two separate charge conjugation symmetries C_1 and C_2 for the model. Here C_1 transforms the electrically charged particles into (anti) particles with the opposite electric charge, but it leaves the $SU(3)$ representation the same:

$$C_1 : Q \to -Q, \qquad SU(3) \text{ invariant}, \tag{11}$$

and the other charge conjugation C_2 leaves the electric charge, but lets the $SU(3)$ representation go into the one of the conjugate:

$$C_2 ; Q \to Q, \qquad SU(3) \text{ is complex conjugated}. \tag{12}$$

Even the charge quantization rule is not invariant under these two operations separately. In nontrivial cases, the representations obeying the rule will be even less invariant. We may say that nature made a rule linking the two charge conjugations C_1 and C_2 in order to stress as strongly as possible that they are not separately good symmetries. Nature really attempted to make the Standard Model as skew as possible.
3. Why should there be parity P and charge conjugation C violations? The reason is to have the charge conjugation violation to break the symmetry corresponding to the combined symmetry $C = C_2 \circ C_1$ that corresponds to an outer automorphism of the Gauge *group* $S(U(2) \times U(3))$ and is in fact charge conjugation symmetry.
4. Why should there be so relatively many generations as three? Well, we could even claim that nature is not satisfied with breaking this charge conjugation C once, but wants to break it several times!

(Why just three times by the three generations we do not know, but we connect it with the smallness of the fine-structure constants.)

5. C. Jarlskog[14] suggested that we have just three generations because we thereby get *CP* violation, whereas further generations would bring nothing new. If this is the reason, it might also look as if nature were keen to get one more symmetry broken. Again it is fighting to get asymmetry, skewness. (This point is less clear.)

We made our claim about nature choosing the most skew among many possibilities relatively precise by saying how it selects the most skew of all nonsemisimple *groups* $SMG = S(U(2) \times U(3))$ of dimension up to 18, i.e., with up to 18 species of gauge particles (see Table II). Here it is important to have in mind that—following Michel and O'Rafeartaigh[15]—one can assign a meaning to the gauge *group* and not only to the gauge Lie algebra. Various *groups* with the same Lie algebra allow only some of the representations of this Lie algebra. One may therefore look at the matter fields and how they transform under gauge transformations in order to see if they should all happen to be such that they would be allowed as representations for some special *group* with the Lie algebra $\mathbf{R} \times SU(2) \times SU(3)$ corresponding to the gauge fields found in nature. Indeed it turns out that the restriction on the matter field representations imposed by the requirement that the electric charge be quantized in the well-known way—colorless particles have integer charge and quarks have charges of $-1/3 +$ integer—can be interpreted as the requirement that the matter fields are representations of the *group* $S(U(2) \times U(3))$. This *group* has the same Lie algebra as $U(1) \times SU(2) \times SU(3)$ and is defined as the group of 5×5 matrices which are 2-3 block-diagonal, unitary, and of determinant 1:

$$S(U(2) \times U(3)) = \left\{ \left(\begin{array}{ccccc} \multicolumn{2}{c}{\mathbf{U}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & & \\ 0 & 0 & & \mathbf{U}_3 & \end{array} \right) \middle| \; \mathbf{U}_2 \in U(2) \wedge \mathbf{U}_3 \in U(3) \wedge \det = 1 \right\}$$

Table II. Numbers of Nonsemisimple Groups up to and Including Dimension 12, with Various Properties Meaning Successively Greater and Greater Degree of Skewness.

194	nonsemisimple groups up to $d = 12$
13	only the unavoidable outer automorphism
6	no $G/Z_2 \cong G$
1	neither $G/Z_3 \cong G$
1	neither $G/Z_4 \cong G$
0	neither $G/Z_5 \cong G$

Here det stands for the determinant of the whole 5×5 matrix; i.e., it is really the product of the determinants of $\mathbf{U_2}$ and $\mathbf{U_3}$. We can obtain the *group* $S(U(2) \times U(3))$ as the factor group (=quotient group) derived from the "covering group" $\mathbf{R} \times SU(2) \times SU(3)$ by identifying the elements generated by the element $(2\pi, -I^{2\times 2}, \exp(i2\pi/3)I^{3\times 3})$ with the identity

$$S(U(2) \times U(3)) = (\mathbf{R} \times SU(2) \times SU(3))/\{(2\pi, -I^{2\times 2}, \exp(i2\pi/3)I^{3\times 3})^n \mid n \in \mathbf{Z}\}.$$

The representations of the Lie algebra are the same as those of the covering group of that Lie algebra. The condition to be imposed on a representation of the gauge Lie algebra of the Standard Model, in order for it to be a representation of $S(U(2) \times U(3))$, is that it represent the elements generated by $(2\pi, -I^{2\times 2}, \exp(i2\pi/3)I^{3\times 3})$ trivially, i.e., by the identity operation. This condition is easily seen[15] to be equivalent to the restriction

$$y + d/2 + t/3 = 0 \qquad (\text{mod } 1) \tag{13}$$

on the weak hypercharge y. This is the same as the condition of electric charge (Q) quantization the usual way:

$$Q = -t/3 \qquad (\text{mod } 1). \tag{14}$$

Here d denotes the "duality" of the $SU(2)$ component in the representation (i.e., $d = 0$ when the weak isospin is integer and $d = 1$ when it is half integer), and t is the "triality" of the color representation (i.e., for example, $t = 1$ for quarks and $t = -1 = 2\ (\text{mod } 3)$ for antiquarks, while the gluons and the leptons have triality $t = 0$, all counted modulo 3).

The remarkable empirical fact that all the matter fields, all the (Weyl) fermion fields, belong to representations of the gauge Lie algebra obeying the usual electric charge quantization rule means that there is an empirical suggestion for "the gauge *group*" to be $S(U(2) \times U(3))$ rather than, say, just its covering group.

Our main result is formulated by considering the *groups* which could be used as gauge groups. But, and this is to some extent a problem for our work, we ignore the semisimple groups, i.e., those that do not have a center of at least dimension one. I.e., we ignore those with no $U(1)$, so to speak. The ones that cannot be used are the ones with a radical that is not the center. If they really were chosen as the gauge group there would be some components of the gauge field for which no kinetic term in the action could be found. We shall restrict ourselves to the groups with the radical being the center. They are called reductive groups.

Just to give some numbers, let me say that the number of reductive nonsemisimple groups of dimension less than or equal to 12 is 194. Twelve is the dimension of the Standard Model itself. In Table II we show these groups up to dimension 4. It can easily be shown that the groups of this type must always have at least one nontrivial outer automorphism, in fact a kind of "charge conjugation." It has the property of inverting the sign of the center part of the Lie algebra: inverting the hypercharge, we may say. It also complex conjugates under the other parts of the Lie algebra. Thirteen of the 194 groups have only this outer automorphism. They are therefore the 13 most skew among the 194 groups. As we also want to make some separation between groups that are more skew than others among these 13 groups, we shall take into account some outer automorphisms of the Lie algebra which are, however, only in an approximate way outer automorphisms of the *group* structure. We may call such automorphisms generalized or approximate (outer) automorphisms, and we define them to be isomorphisms between (different) factor groups of the *group* in question. We consider the factor groups the more important the smaller the invariant subgroups (of the center) which are divided out. Since a group can be considered a factor group of itself with the invariant subgroup consisting of the unit element alone, any automorphism is also an isomorphism between factor groups and thereby a generalized automorphism. After this most important type, the next most important generalized automorphisms should be the ones we got as isomorphisms of the group itself with a factor group obtained by dividing out an invariant subgroup isomorphic to the two-element group $\mathbf{Z}_2$. Out of the 13 groups having only the one nontrivial outer automorphism, there are seven which have such a generalized automorphism. That is to say that there are seven nonsemisimple groups up to dimension 12 which have an isomorphism so that

$$G/\mathbf{Z}_2 \cong G. \tag{15}$$

Here G denotes the group considered. For seven out of 13 groups with only the unavoidable outer automorphism, there is indeed such an isomorphism between the group itself and a factor group obtained by dividing an invariant subgroup with only two elements. These seven are thus the least skew among the 13. Out of the remaining $6 = 13 - 7$ groups, there are five which have an isomorphism corresponding to

$$G/\mathbf{Z}_3 \cong G; \tag{16}$$

i.e., there are 5 that are isomorphic to one of their own factor groups obtained by dividing out an invariant subgroup with three elements, $\mathbf{Z}_3$. In order to find an isomorphic factor group for the last one ($=6 - 5$) group it is not enough to divide out a four-element group; no, you need to divide out one with five elements, $\mathbf{Z}_5$. This group is thus not only the most skew among the 194 nonsemisimple reductive groups of up to 12 dimensions; it even seems to stand somewhat clearly ahead with respect to skewness.* Needless to say, this exceptionally skew group is precisely $S(U(2) \times U(3))$, the group selected by nature!

As one goes up in dimension, one finds more and more groups increasing exponentially, and we can rigorously[12] show that at dimension 19 you first find a group that is as skew as the Standard Model group $S(U(2) \times U(3))$. This competing group has a structure similar to that of the Standard Model, but with $SU(2)$ replaced by the covering group of $SO(5)$, $Spin(5)$. At dimension 36 you find a group that is even more skew than the Standard Model group, but it is essentially the Standard Model group extended by the addition of an $SU(5)$ factor. If you imagined that nature had chosen that group, there would presumably be confinement of the $SU(5)$ at some energy scale above present experimentally accessible energies. The variation of the coupling constant with scale would for similarly many fermions be faster for $SU(5)$ than for, say, $SU(3)$.

The result to be seen at present would then be just the Standard Model group anyway! So it would not make much difference. *Nature has—so it seems—chosen the group of the Standard Model among several hundred nonsemisimple reductive groups, at least* 194, *as the most skew.*

This is as remarkable as making a prediction of a number to an accuracy on the order of $\frac{1}{2}\%$ from some principle, so we have indeed some evidence for the truth of the idea that nature has selected the Standard Model *because it is utterly skew!*

As we mentioned above under point (3) about parity violation, you may interpret the relatively high number of generations as an attempt by nature to break the only nontrivial outer automorphism of the Standard Model group *SMG*. This is evidence for skewness in the Standard Model not counted into the selection of *SMG* among the at least 194 groups. So the evidence for skewness is even a bit greater.

The piling up of several generations with the same representations repeated again and again would make the symmetries under scaling of the weak hypercharge even more solidly broken, but would of course make some symmetry between the generations appear. It seems that nature cares mostly to avoid those potential symmetries that transform the gauge fields in a nontrivial way. However, it does not seem to mind so much the symmetries among flavors not transforming the gauge fields.

But why is nature so keen to avoid those symmetries, those automorphisms? This is a question which we are now encouraged to speculate about. We may expect that finding—guessing—the correct answer to this question could be very helpful in finding the correct theory behind the standard

* It is easy to think through that there are no groups that can be equally skew before dimension 19 simply by using the classification of Lie groups.[12]

model. Which models would favor giving a very skew, model in the low energy limit? A most natural explanation for why we empirically see an extremely skew (asymmetric) group could be the following:

1. At some (more) fundamental level—small distances, Planck level of energy and length scale, say—nature tries a bigger, not necessarily very skew group, but one more like what most groups are like:

$$G_{\text{nature}} = (U(1) \times U(1) \times \cdots \times U(1) \times SU(2) \times SU(2) \times \cdots \times SU(2) \times SU(3) \times SU(3) \times \cdots \times SU(3) \times Spin(5) \times \cdots)/H.$$

   Here H is some discrete subgroup of the center that is being divided out of the covering group. (All reductive connected Lie groups must be of this form.)
2. Then successively nature somehow cannot keep track of the various possibilities for transforming this group around by outer automorphisms and breaks it down to a subgroup invariant under the (outer) automorphisms. The idea is that if the original group G has an (outer) automorphism f: $G \to G$, then there is some mechanism that causes a *breakdown* of the group G into the subgroup $\{g \in G | f(g) = g\}$, which is left invariant under the automorphism. If such a breakdown mechanism is applied to several (or all) outer automorphisms, it would be highly expected that the resulting surviving subgroup would have very few outer automorphisms left, since the procedure would simply remove them successively. Such a mechanism could be an explanation for the very high degree of skewness of the gauge group found in nature to have survived.

A model of this character of cleaning away the nontrivial outer automorphisms by making them trivial after a breakdown of the group is rather reasonable as a proposal to explain such a negative property as skewness. The property skewness is negative in the sense that it is the lack of something: the outer automorphisms. To explain such a lack of something a mechanism which makes it disappear successively is very reasonable. Clearly a nontrivial outer automorphism represents the possibility of confusing the gauge fields with one another or themselves. In fact, the group structure can at least not make any distinction between the gauge fields $A_\mu^a(x)$ and their images under the automorphism f, i.e., $f(A_\mu^a)$. How can such a possibility for mixing up a field with its image now be imagined to cause a breakdown of the original group? Well, if they got locked in so that the field $A_\mu^a(x)$ and its image $f(A_\mu^a)$ had to take on the same value unless the energy of the field were appreciably increased, then it would cost energy to perform those field vibrations which correspond to gauge fields not left invariant by the automorphism f. This cost in energy can be shown to correspond to that at which masses appear for other gauge field vibrations than the invariant ones.

A locking in so that the field $A^a_\mu(x)$ has to be equal to $f(A^a_\mu(x))$ could be imagined in several ways:

1. Above we mentioned the field theory glass leading to a gauge glass which is so chaotic a model that one gets into contradictions if one seeks to keep track of the gauge fields so as to distinguish a gauge field from its image under the automorphism f. This meant that the interaction of the degrees of freedom describing the gauge fields were such as to make $A^a_\mu(x)$ and $f(A^a_\mu(x))$ indistinguishable in a well-defined manner, and thus they really get locked together, since they should then take the same values.
2. You may also think of an ordinary continuum field theory formulation in a general relativity space-time forming a lot of space-time foam due to quantum fluctuations. Provided the space-time foam is not simply connected, it is possible to modify the laws of physics in a way that represents a true change only in the global, but not locally. This is done by introducing what we call confusion surfaces. The latter means a surface in space-time (of co-dimension one, i.e., of dimension 1 less than that of space-time itself) across which the notation changes as under an automorphism f of the gauge group. Thus one has at the confusion surface instead of the usual continuity property of the fields a discontinuity rule:

$$\lim_{\text{from side 1}} f(A^a_\mu(x)) = \lim_{\text{from side 2}} A^a_\mu(x).$$

   This rule tells us that on passage of the confusion surface the fields go into their images under the automorphism. If in a simply connected space you introduce such confusion surfaces it just means that you use different notation in different regions of space-time, i.e., on the different sides of the confusion surface. That must of course be without any physical content. If space-time is, however, not simply connected, you can introduce surfaces with discontinuity rules just as if they were just surfaces of change of notation, but now in the global case there could indeed be a true physical change in the laws of nature postulated in the model. It is in fact possible to find in this case surfaces which do not divide space-time, but with the sides 1 and 2 in fact continuously connected. This fact may be revealed on return around some handle in the space-time foam.

   If we made the assumption that physical laws are of an entirely local nature one could argue that a modification of the physical

> field theory which could only be detected globally, but not inside any sufficiently small neighborhood, could not be prevented. There should therefore under this assumption be no possibility of preventing the existence of the proposed "confusion surfaces" and thus for having the confusion mechanism work unless the symmetry corresponding to the automorphism is at least in some way broken (for example, by the fermion fields not being invariant under it). So with space-time foam a confusion type of mechanism is indeed very likely to come into play if any automorphisms are not broken.

This relatively great ease with which a mechanism attacking symmetries could come about together with the strong phenomenological evidence for skewness of the Standard Model Gauge group suggests rather strongly that this skewness is not accidental, but that there is indeed some mechanism that has caused this skewness!

If we want to take seriously point (4) telling that the presence of relatively many generations is a skewness sign we must imagine that whatever the mechanism—confusion, say—that breaks the group to make the resulting one more skew, it is at least to some extent influenced by the existence of fermion fields breaking the symmetry. That is to say we must expect that the mechanism breaking the group down to the subgroup $\{h \mid h \in G$ and $f(h) = h\}$ is prevented by the existence of fermions to which the automorphism f cannot be extended as a good symmetry. If the mechanism that produces skewness preferentially attacks symmetries that are also symmetries for the matter fields (the fermions, say) then this mechanism would presumably first remove all factors in the gauge group that have no chiral matter fields associated with them but nevertheless have the charge conjugation automorphism—complex conjugation—as an outer automorphism. Successively isomorphic cross-product factors of the gauge group resulting from the first step would break down to their diagonal subgroup because of the attack from the "mechanism ensuring skewness," but this would happen only if the matter fields associated with these isomorphic factors can be considered in correspondence. If the different isomorphic cross product factors do not have the same representations of matter fields, the symmetries permuting the groups Peter, Paul, etc. would not be good symmetries of these matter fields and the attack from the "mechanism ensuring skewness" would not work. Only when the matter fields are in the same representations will the breakdown to the diagonal subgroup take place without any suppression, we must expect. If we at any stage of the breakdown of the group(s) avoid gauge anomalies and mixed anomalies, it is hard to see how matter fields for groups that are the standard model group or cross-product powers of it could avoid coming in full generations. We may define a generation as a set of chiral fermion fields having the anomalies adding up to zero, but without any subrepresentation with this property.

If the Standard Model gauge group as we find it experimentally—phenomenologically—originates as the diagonal of the cross product of say n isomorphic factors, each of these factors must have the same number of generations. Since the diagonal subgroup couples to a generation for one of the groups Peter, Paul, etc., in the same way as the one of these to which it belonged, the number of generations for the diagonal subgroup becomes the sum of the number of generations for the groups Peter, etc., and since each of them should have equally many generations, we get

$$N_{\text{gen}} = n \times N_{\text{gen, Peter}}, \tag{18}$$

where $N_{\text{gen, Peter}}$ is the number of generation associated with the group SMG_{Peter} or one of the other groups from which the experimentally accessible SMG descends. Clearly, then, the number of isomorphic factors n from which the observed group descends must be a factor in the experimentally observed number of generations N_{gen}, which is now known to be 3. Since the number 3 is a prime there are only the possibilities $n = 3$ or $n = 1$, the latter meaning that there was no diagonal subgroup breakdown in the last step at all. The possibility $n = 3$ is the one corresponding to the "anti-grand unification" model which we have used to predict that the number of generations should be 3.

A scheme in which the standard model group results as the diagonal subgroup by a breakdown from the cross product of as many factors isomorphic to the standard model group as there are generations N_{gen} has the advantage that horizontal quantum numbers separating the different flavors now have a rather natural way to correspond to gauge symmetries at a higher energy level. The various Peter, Paul, etc., gauge charges will act as horizontal charges partially suppressing quark and lepton masses—once the Weinberg-Salam-Higgs field is added to the picture.

Taking the skewness of the Standard Model as a guiding principle, we are indeed led very much in the direction of the anti-grand unification scheme which we also said was an effective ingredient as an intermediate step in the random dynamics model.

The good agreement of expectations for the relation between the fine-structure constants and the number of generations derived from the random dynamics model means that if we extrapolate the experimentally observed couplings up to the Planck energy scale and assume that here the standard model group is found as the diagonal subgroup of a cross product of N_{gen} with the Standard Model group isomorphic groups, then the couplings for the latter are just on the borderline of confinement in the lattice approximation that is called the "mean field approximation." If the gauge couplings in such a scheme were stronger than they are experimentally, we would have the strange situation that there should have been confinement already at the Planck scale. This would hardly be believable. E.g., the QCD coupling must be so weak at the Planck scale that the Peter, Paul, and Maria $SU(3)$ couplings avoid confinement at the Planck scale. We may claim mainly from ideas based on extrapolating the skewness idea to have argued that also, for instance, the fine-structure constant α had better not be stronger than it happens to be experimentally. So we have from the skewness-inspired scheme turned the usual mystery, "Why is the fine-structure constant so weak?" into the almost opposite one, "What is the reason that the fine-structure contants have just their strongest allowed values?" This remarkable saturation of the upper bound for the gauge couplings in our scheme has an explanation in the random dynamics model: Small gauge-symmetry breakings are more easily hidden and unimportant if the gauge couplings are strong, because the larger quantum fluctuations wash them out.

Although this saturation may be taken as evidence for random dynamics, this is relatively weak. The conclusion of the part of my talk concerned with the skewness principle should rather be the following:

The part of the random dynamics scheme that is relevant for the predictions can essentially be argued for by extracting a principle of low symmetry (skewness) of the Standard Model group and imagining a mechanism leading to the survival of a skew group only. Apart from an assumption of saturation—the gauge couplings are as strong as allowed—and minor "helping assumptions" not derived in random dynamics either, there is no need for random dynamics to derive the scheme of ours giving the gauge coupling constants related to the number of generations. So the experimental agreement which I claim that we have should be taken rather as an agreement of the scheme of anti-grand unification, and skewness as the golden principle, than as evidence for the random dynamics idea, which however, first brought us to these ideas.

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Chapter 10

Gauge Anomalies in Two Dimensions

R. Rajaraman

1. INTRODUCTION

In these lectures, I will try to explain how two-dimensional field theories in which gauge invariance is anomalously broken may be treated consistently and to outline the resultant interesting and unfamiliar features of such theories. As is well known, such anomalies are not special to two dimensions; they can occur in any number of dimensions where chiral fermions exist, and it was believed until a few years ago that in all such cases the gauge anomaly dealt a fatal blow to the theory. The anomaly was believed, variously, to lead to the failure of renormalizability, unitarity, Lorentz invariance, and even canonical consistency. This last was believed to fail because of the apparent conflict between the current-anomaly equation $D_\nu J^\nu = R(A_\mu)$, where $R(A_\mu)$ is the anomalous divergence, and the gauge field equation $D_\mu F^{\mu\nu} = J^\nu$, which implies $D_\nu J^\nu = 0$.

R. RAJARAMAN • Centre for Theoretical Studies, Indian Institute of Science, Bangalore 560012, India.

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While such anomalous gauge theories and their alleged problems could exist in any (even) number of dimensions, we choose to discuss them here in the two-dimensional context for the following reason. Except for the nonrenormalizability problem, which is diminished in two dimensions, concern about the other problems such as loss of Lorentz invariance, unitarity, or canonical consistency has been expressed as much here as in higher dimensions, and these problems need to be investigated critically. At the same time, two dimensionality offers simplifications, not only at the technical and algebraic level, but also through the availability of bosonization, which, as we will see, is a great asset in investigating anomalous theories. Taking advantage of all this, one can show that anomalous gauge theories in two dimensions are in fact free of the suspected pathologies. They turn out to be constrained but not gauge-invariant theories, i.e., they carry second-class constraints, as was first pointed out by Faddeev.[1] When treated according to standard constraint theory, they yield perfectly consistent structures. In fact, the Abelian case (the Chiral Schwinger Model) is exactly soluble, and the absence of the alleged pathologies can be explicitly verified.[2] The spectrum is relativistic and the theory unitary.

The non-Abelian generalization (chiral 2D QCD) cannot be solved exactly, but short of solving it explicitly one can show, using bosonization, that it has all the desirable healthy features of a quantum field theory, and its independent degrees of freedom can be identified.[3] With appropriate generalization, Chiral $(\mathrm{QCD})_2$ shares some of the properties of the Abelian case.[4]

2. THE CHIRAL SCHWINGER MODEL: THE EFFECTIVE ACTION

The possibility that some anomalous gauge theories may be quite sensible if treated properly can be very explicitly realized in the chiral Schwinger model (CSM). This is a two-dimensional model of a massless fermion ψ coupled to a vector meson A_μ. It is very similar to the famous Schwinger model, with the sole difference that A_μ couples not to the vector current but just to the right (or just to the left) chiral current. This is the simplest anomalous gauge theory and, as we will show below, it can be solved exactly in the charge-zero sector. When the theory is appropriately regularized, it yields a relativistic spectrum of particles with real masses, and it is unitary. The theory contains many new and interesting features which we will discuss. But first, let us derive the solution of the model.

The model is described by the action

$$S_F(\psi, \bar{\psi}, A_\mu) = \int d^2x \left[i\bar{\psi}\not{\partial}\psi - \frac{e}{2}\bar{\psi}\gamma^\mu(1-\gamma_5)\psi A_\mu - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \right]. \quad (2.1)$$

We are using the following representation and notation:

$$\gamma^0 = \sigma_1, \qquad \gamma^1 = -i\sigma_2, \qquad \gamma_5 \equiv \gamma^0\gamma^1 = \sigma_3. \tag{2.2}$$

These satisfy

$$\gamma^\mu\gamma_5 = -\varepsilon^{\mu\nu}\gamma_\nu \quad \text{with} \quad \varepsilon^{01} = +1. \tag{2.3}$$

The action (2.1) is gauge-invariant at its classical level under the Abelian chiral gauge transformations

$$\psi_R \equiv \tfrac{1}{2}(1-\gamma_5)\psi \to e^{-ie\Lambda(x)}\,\psi_R,$$
$$\psi_L \equiv \tfrac{1}{2}(1+\gamma_5)\psi \to \psi_L,$$

and

$$A_\mu \to A_\mu + \partial_\mu\Lambda. \tag{2.4}$$

Associated with this invariance, the chiral current is classically conserved:

$$\partial_\mu j^\mu = 0 \qquad \text{where } j^\mu = \frac{e}{2}\,\bar{\psi}\gamma^\mu(1-\gamma_5)\psi. \tag{2.5}$$

However, when the model is quantized, the chiral anomaly caused by fermionic fluctuations destroys both the current conservation and the gauge invariance. Let us first show these features.

The quantum theory is described by the functional integral

$$Z[I_\mu] = \int DA_\mu D\psi D\bar{\psi}\, \exp\left[iS_F(\psi\bar{\psi}A_\mu) + i\int d^2x\, A_\mu I^\mu\right], \tag{2.6}$$

where I_μ is the source of the gauge field A_μ. Derivatives of $Z[I_\mu]$ with respect to $I_\mu(x)$ yield, through familiar methods, all n-point functions of A_μ. We will be interested for our purposes in only the charge-zero sector; hence source terms for the Fermi field ψ have not been included.

One can rewrite $Z[I]$ in the form

$$Z[I_\mu] = \int DA_\mu\, \exp\left\{iS_{\text{eff}}(A_\mu) + i\int d^2x\, A^\mu I_\mu\right\}, \tag{2.7}$$

where the "effective action" $S_{\text{eff}}\,[A]$ of the vector field is given by

$$S_{\text{eff}}[A_\mu] = W[A_\mu] - \int \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\, d^2x, \tag{2.8}$$

with

$$e^{iW[A_\mu]} \equiv \int D\psi D\bar{\psi}\, \exp\left\{i\int d^2x\, (\bar{\psi}i\partial\!\!\!/\psi - j^\mu A_\mu)\right\}. \tag{2.9}$$

On differentiating both sides with respect to $A_\mu(x)$, we get

$$\frac{\partial W}{\partial A_\mu(x)} = e^{-iW}\int D\psi D\bar\psi\, e^{i\int d^2x'(\bar\psi i\partial\!\!\!/\psi - j^\mu A_\mu)}(-j^\mu(x))$$
$$= -\langle j^\mu(x)\rangle, \tag{2.10}$$

where $\langle\cdot\rangle$ denotes expectation value in the fermionic vacuum in the presence of its chiral coupling to an external field A_μ.

Our strategy for solving this theory in the A_μ sector is to first obtain $W[A_\mu]$. This, in turn, we will do by evaluating $\langle j^\mu\rangle$ and integrating Eq. (2.10) over A_μ. Note that the bilinear expression for $j^\mu(x)$ given in Eq. (2.5) becomes formally divergent in the quantum theory. It has to be regularized. We write

$$j^\mu(x) \approx \frac{e}{2}\bar\psi_i(x)(\gamma^\mu - \gamma^\mu\gamma_5)_{ij}\psi_j(x)$$
$$= \lim_{\substack{x\to y\\ y_0 > x_0}} \frac{e}{2}\left\{\bar\psi_i(y)\gamma^\mu_{ij}\, e^{-ie\alpha\int_x^y A_\sigma d\xi^\sigma}\,\psi_j(x)\right.$$
$$\left. - \bar\psi_i(y)(\gamma^\mu\gamma_5)_{ij}\, e^{-ie\beta\int_x^y A_\sigma d\xi^\sigma}\,\psi_j(x)\right\}, \tag{2.11}$$

where α and β are arbitrary real constants.

Classically, as $s^\mu \equiv x^\mu - y^\mu \to 0$, the elaborate expression in Eq. (211) reduces to just $(e/2)\bar\psi(x)\gamma^\mu\gamma_5\psi(x)$, and the exponentials involving A_μ have no effect. In the quantum theory, however, $\psi(x)\bar\psi(y)$ diverges as $1/s$, and the linear term in these exponentials leaves a residual contribution. It can be checked, incidentally, that Eq. (2.11) represents the most general regularization of j^μ in this theory.

Now the Feynman propagator $S(x-y)$ in the presence of A_μ is

$$[S(x-y)]_{ij} = -i\langle T\psi_j(x)\bar\psi_i(y)\rangle \tag{2.12}$$

and obeys

$$\left[i\gamma^\mu\frac{\partial}{\partial x^\mu} - \frac{e}{2}\gamma^\mu(1-\gamma_5)A_\mu(x)\right]S(x-y) = \delta^2(x-y). \tag{2.13}$$

In terms of S, the vacuum expectation value of j^μ given in Eq. (2.11) can be written as

$$\langle j^\mu(x)\rangle = \lim_{y\to x}\left(\frac{-ie}{2}\right)\{(1 + ie\alpha(x-y)^\sigma A_\sigma + \cdots)\,\mathrm{Tr}[\gamma^\mu S(x-y)]$$
$$- (1 + ie\beta(x-y)^\sigma A_\sigma + \cdots)\,\mathrm{Tr}[\gamma^\mu\gamma_5 S(x-y)]\}. \tag{2.14}$$

Our task reduces to calculating the propagator $S(x-y)$, which obeys Eq. (2.13). To do this, recall that in two dimensions, one can write any vector field $A_\mu(x)$ in the form

$$A_\mu(x) = (g^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\nu a(x) + (g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\nu b(x), \tag{2.15}$$

where a and b are two scalar fields.

Using Eqs. (2.15) and (2.3), we can write Eq. (2.13) as

$$\left[i\gamma^\mu \frac{\partial}{\partial x^\mu} - e\gamma^\mu(1-\gamma_5)\partial_\mu b(x)\right] S(x-y) = \delta^2(x-y). \tag{2.16}$$

This is solved by

$$S(x-y) = e^{-ie(1-\gamma_5)b(x)}\, S_0(x-y)\, e^{ie(1+\gamma_5)b(y)}, \tag{2.17}$$

where

$$iS_0(x-y) = \frac{1}{2\pi i}\frac{\gamma^\mu(x-y)_\mu}{(x-y)^2 - i\varepsilon} \tag{2.18}$$

is the free propagator satisfying

$$i\gamma^\mu \frac{\partial}{\partial x^\mu} S_0(x-y) = \delta^2(x-y). \tag{2.19}$$

Inserting Eqs. (2.17)-(2.18) into Eq. (2.14) and using

$$\operatorname{Tr} \gamma^\mu\gamma^\nu = 2g^{\mu\nu}, \qquad \operatorname{Tr} \gamma^\mu\gamma^\nu\gamma_5 = 2\varepsilon^{\mu\nu}, \quad \text{and} \quad s^\mu \equiv x^\mu - y^\mu,$$

$$\begin{aligned}
\langle j^\mu \rangle &= \lim_{s\to 0} \frac{ie}{4\pi} \operatorname{Tr}\Big[((1 + ie\alpha A_\sigma s^\sigma)\gamma^\mu - (1 + ie\beta A_\sigma s^\sigma)\gamma^\mu\gamma_5) \\
&\qquad \times (1 - ie(1-\gamma_5)\partial_\sigma b s^\sigma)\left(\frac{\gamma^\nu s_\nu}{s^2}\right)\Big] \\
&= \frac{ie}{2\pi}\left[(g^{\mu\nu} - \varepsilon^{\mu\nu})\frac{s_\nu}{s^2}\right]_{s\to 0} \\
&\quad + \lim_{s\to 0}\frac{-e^2}{2\pi}[(\alpha g^{\mu\nu} + \beta\varepsilon^{\mu\nu})A_\sigma - 2(g^{\mu\nu} + \varepsilon^{\mu\nu})\partial_\sigma b]\frac{s^\sigma s_\nu}{s^2}.
\end{aligned} \tag{2.20}$$

The first term is divergent and A_μ-independent. We subtract it out as part of our regularization procedure. This amounts to normal-ordering the free current. The remaining second term is finite, if we take the limit $s \to 0$ symmetrically ($\lim_{s\to 0}(s^\sigma s_\nu / s^2) = \frac{1}{2}g^\sigma_\nu$). The result is

$$\langle j^\mu \rangle_{\text{reg}} = \frac{-e^2}{4\pi}\left[\alpha A^\mu + \beta\varepsilon^{\mu\nu}A_\nu - (g^{\mu\nu} + \varepsilon^{\mu\nu})\frac{\partial_\nu\,\partial_\rho}{\Box}(g^{\rho\sigma} - \varepsilon^{\rho\sigma})A_\sigma\right], \tag{2.21}$$

where we have used

$$b = \frac{1}{2\Box}(g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu A_\nu, \tag{2.22}$$

which can be obtained by inverting Eq. (2.15).

Equation (2.21) gives us the vacuum expectation value of the most general point-split current operator [Eq. (2.11)], in terms of two parameters α and β. But the requirement of Eq. (2.10) forces $\beta = 0$, since the piece $\varepsilon^{\mu\nu}A_\nu$ in Eq. (2.21) is not integrable. However, α remains an arbitrary parameter. Inserting Eq. (2.21) into Eq. (2.10) and integrating over the field $A_\mu(x)$, we have

$$\begin{aligned} W[A_\mu] &= -\int D[A_\mu(x)]\langle j^\mu(x)\rangle \\ &= \frac{e^2}{i\pi}\int d^2x\left[\alpha A_\mu A^\mu - A_\mu\cdot(g^{\mu\nu}+\varepsilon^{\mu\nu})\frac{\partial_\nu\,\partial_\rho}{\Box}(g^{\rho\sigma}-\varepsilon^{\rho\sigma})A_\sigma\right]. \end{aligned} \tag{2.23}$$

In a corresponding analysis of the Schwinger model, the value of α is uniquely fixed by the requirement of gauge invariance. In the present chiral model, however, gauge invariance is necessarily lost because of the anomaly. Note that $W[A_\mu]$ is not invariant under $A_\mu \to A_\mu + \partial_\mu\Lambda$ for any value of α. The anomaly, obtained from Eq. (2.21), is (with $\beta = 0$)

$$\partial\mu\langle j^\mu\rangle = \frac{e^2}{4\pi}[(1-\alpha)\partial_\mu A^\mu - \varepsilon^{\mu\nu}\partial_\mu A_\nu]. \tag{2.24}$$

This anomaly, too, cannot be made to vanish identically for any value of α. At this stage of our analysis, there are no preferred values of α, and we will study the entire family of theories characterized by the real parameter α.

3. SOLUTIONS OF THE CSM

Since $W[A_\mu]$ in Eq. (2.23) is quadratic, so is the full effective action $S_{\text{eff}}(A_\mu)$ given in Eq. (2.8), which can be written in the form

$$S_{\text{eff}}(A_\mu) = \tfrac{1}{2}\int d^2x\, A^\mu M_{\mu\nu} A^\nu, \tag{3.1}$$

where $M_{\mu\nu}$ is a nonlocal differential operator. It is also a 2×2 matrix in the Lorentz indices μ, ν. The field equations obtained from $S_{\text{eff}}(A_\mu)$ are nonlocal, but linear. The system is therefore exactly soluble and will consist of some free fields. The solutions can be obtained in a variety of ways.

First, the exact propagator for A_μ can be obtained by inverting the $M_{\mu\nu}$ defined in Eq. (3.1). The expression for $M_{\mu\nu}$ can be read off from Eqs. (2.8) and (2.23), and it is straightforward to invert it in momentum space to yield the propagator $G_{\mu\nu}(k)$. The answer is

$$
\begin{aligned}
-iG_{\mu\nu}(k) &\equiv (M^{-1})_{\mu\nu} \\
&= \left(\frac{1}{k^2 - m^2}\right)\Bigg[-g_{\mu\nu} + \frac{1}{\alpha - 1} \\
&\quad \times \left\{k_\mu k_\nu\left(\frac{4\pi}{e^2} - \frac{2}{k^2}\right) - \varepsilon_{\mu\sigma}\frac{k^\sigma k_\nu}{k^2} - \varepsilon_{\nu\sigma}\frac{k^\sigma k_\mu}{k^2}\right\}\Bigg], \qquad (3.2)
\end{aligned}
$$

where

$$m^2 \equiv \frac{e^2\alpha^2}{4\pi(\alpha - 1)}. \qquad (3.3)$$

This propagator has poles at $k^2 = m^2$ and $k^2 = 0$. The residues at these poles will be 2×2 matrices that can be diagonalized. The resulting physical content depends on the value of α.

1. $\alpha > 1$. Then $m^2 > 0$, and it can also be checked that the residues of the poles at $k^2 = m^2$ and at $k^2 = 0$ are all positive. Thus, for $\alpha > 1$, the theory is unitary and has two species of relativistic free particles, of masses m and 0, respectively.
2. $\alpha < 1$. Here $m^2 < 0$, and further the residue at m^2 is not positive. The theory contains tachyons and is nonunitary.
3. $\alpha = 1$. This is a special value of the regularization parameter α. If we approach $\alpha \to 1$ from above, we see $m^2 \to \infty$ and can expect the massive particle to decouple, leaving behind only the massless excitations in the spectrum. An *ab initio* Hamiltonian analysis of the system with $\alpha = 1$ confirms this (see below).

In summary, the exact propagator for A_μ tells us that the theory is unitary for $\alpha \geq 1$, with a spectrum of free relativistic particles. For $\alpha < 1$, the theory is indeed nonsensical.

To get a fuller understanding of the content of this model, let us do a canonical Hamiltonian analysis. We cannot work directly with the effective action $S_{\text{eff}}(A_\mu)$ since, thanks to $W[A_\mu]$, it is nonlocal. However, it can be rendered local by introducing an auxiliary scalar field $\phi(x)$. It is easy to verify that

$$
\begin{aligned}
\int D\psi D\bar{\psi} DA_\mu \exp\{iS_F(\psi, \bar{\psi}, A_\mu)\} &= \int DA_\mu \exp\{iS_{\text{eff}}(A_\mu)\} \\
&= \int DA_\mu D_\phi \exp\{iS_B(\phi, A_\mu)\}, \qquad (3.4)
\end{aligned}
$$

where

$$S_B(\phi, A_\mu) = \int d^2x\, L(\phi, A_\mu)$$
$$= \int d^2x \Big[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi$$
$$+ \frac{e}{\sqrt{4\pi}}(g^{\mu\nu} - \varepsilon^{\mu\nu})\partial_\mu\phi A_\nu + \frac{\alpha e^2}{8\pi} A_\mu A^\mu \Big]. \tag{3.5}$$

Upon performing the Gaussian integral of $\exp(iS_B(\phi, A_\mu))$ over $\phi(x)$, one recovers $S_{\text{eff}}(A_\mu)$ as given in Eqs. (2.8) and (2.23). We have essentially derived just the "bosonized" action for the chiral Schwinger model, with α representing the sole regularization ambiguity in this bosonization.

A canonical Hamiltonian analysis starting from the bosonized action [Eq. (3.5)] reveals the altered constraint structure brought about by the anomaly. Note that the bosonized action (3.5), which is equivalent to the gauge field action $S_{\text{eff}}[A_\mu]$ obtained by integrating over the fermionic fluctuations, already contains the anomaly. Thus, a classical Hamiltonian analysis of the bosonized system already incorporates anomalous effects. Once this is done, quantization can be carried out by replacing classical (Dirac) brackets by quantum commutators.

Let us denote by π_0, E, and π the canonial momenta conjugate to A_0, A_1, and ϕ, respectively. Early steps in the canonical analysis proceed as in anomaly-free gauge theories.

We have

$$E = \frac{\partial L}{\partial \dot{A}_1} = \partial_0 A_1 - \partial_1 A_0,$$

$$\pi = \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} + \frac{e}{\sqrt{4\pi}}(A_0 - A_1),$$

and

$$\pi_0 = 0, \qquad \text{as a constraint.} \tag{3.6}$$

The Hamiltonian is

$$H = \int dx (E\dot{A}_1 + \pi\dot{\phi} - L + \pi_0 v_0)$$
$$= \int dx \Big[\tfrac{1}{2}(\pi^2 + (\partial_1\phi)^2 + E^2) - \partial_1 E A_0 - \frac{\alpha e^2}{8\pi}(A_0^2 - A_1^2)$$
$$+ \frac{e^2}{8\pi}(A_0 - A_1)^2 - \frac{e}{\sqrt{4\pi}}(A_0 - A_1)(\pi + \partial_1\phi) + \pi_0 v_0 \Big]. \tag{3.7}$$

The last term proportional to the constraint $\pi_0(x)$ is added on, as required in Dirac's theory of constraints, where $v_0(x)$ is an as yet undetermined velocity. Then the consistency requirement

$$\begin{aligned}0 &= \{\pi_0(x), H\}_{\text{P.B.}} \\ &= \partial_1 E + \frac{e}{\sqrt{4\pi}}(\pi + \partial_1\phi) + \frac{e^2}{4\pi}(A_1 + (\alpha - 1)A_0) \\ &\equiv G(x) \end{aligned} \tag{3.8}$$

yields a second constraint, which is the analogue here of Gauss's law. However, as distinct from an anomaly-free gauge theory, the constraints here are of the second class for all $\alpha \neq 1$. (We will return to $\alpha = 1$ later.)

$$\{\pi_0(x), G(y)\} = \frac{e^2}{4\pi}(1 - \alpha)\delta(x - y). \tag{3.9}$$

There are are no first-class constraints and there is no gauge freedom. The velocity field $v_0(x)$ in the Hamiltonian [Eq. (3.7)] is uniquely fixed by the consistency of the Gauss's Law constraint, i.e., by $\{G(x), H\} = 0$. This leads to

$$v_0 = \partial_1 A_1 + \frac{1}{(1-\alpha)} E.$$

Inserting this v_0 into Eq. (3.7), we can see that

$$\partial_0 A_0 = \{A_0, H\} = \partial_1 A_1 + \frac{1}{(1-\alpha)} E. \tag{3.10}$$

In other words, the anomaly [Eq. (2.24)] vanishes dynamically. Next we just follow Dirac's procedure for treating second-class constraints. Dirac brackets are constructed to replace Poisson brackets. This makes both constraints Eqs. (3.6) and (3.8) hold "strongly"; π_0 and A_0 can be eliminated using these constraints. The Hamiltonian [Eq. (3.7)] becomes

$$\begin{aligned} H = \tfrac{1}{2}\int dx\Bigg[\left(\pi + \frac{e}{\sqrt{4\pi}}A_1\right)^2 + \left(\partial_1\phi + \frac{e}{\sqrt{4\pi}}A_1\right)^2 + E^2 \\ + \frac{e^2}{4\pi}(\alpha - 1)(A_0^2 + A_1^2)\Bigg]. \end{aligned} \tag{3.11}$$

Clearly, this is real, unique, *and* positive for $\alpha > 1$. It is quadratic and easily diagonalized to yield the spectrum mentioned earlier. Upon replacing *Dirac* brackets by commutators, quantization is effected with no further complications, yielding a Hermitian, bounded-from-below Hamiltonian. The quantum theory is unitary and self-consistent. Its Lorentz invariance, although not manifest in the Hamiltonian formalism, was proved by Mitra and myself.[5]

Analysis of the $\alpha = 1$ case involves a few more steps, since now $\{\pi_0, G\} = 0$ forces two more constraints, $E = 0$ and $(A_0 - A_1) = 0$. Collectively, the four constraints are all of the second class. Upon using Dirac brackets, all gauge-field variables A_0, A_1, π_0, and E can be eliminated. It can be checked that the remaining variables ϕ and π (the matter field) are governed by a free, massless Hamiltonian:

$$H_{\alpha=1} = \tfrac{1}{2}\int dx[\pi^2 + (\partial_1\phi)^2]. \tag{3.12}$$

Notice that, unlike the gauge-invariant Schwinger model, where again the gauge field can be eliminated, here the matter field remains massless.

Lessons from CSM about Anomalous Theories in General

The chiral Schwinger model has offered a valuable prototype example for demonstrating the fact that anomalous gauge theories (AGT) need not be inconsistent, violate unitarity and Lorentz invariance, or in any other way be nonsensical. Being exactly soluble, it allows us to see explicitly how the different alleged problems of anomalous theories get resolved, except for the question of renormalizability, which is absent as a serious problem in this two-dimensional model.

From the study of the CSM, one can abstract some lessons which may be expected to hold in more complicated AGT as well, even though the latter are not exactly soluable. Let us list a few such lessons:

1. The pair of equations $D_\mu F^{\mu\nu} = j^\nu$ and $D_\nu j^\nu = \text{anomaly} \equiv R(A_\mu)$ are not necessarily mutually inconsistent, as was once believed. Of course these equations together imply that the anomaly $R(A_\mu)$ obeys

$$R(A_\mu) = D_\nu j^\nu = D_\mu D_\nu F^{\mu\nu} = 0.$$

This means only that the *anomaly must vanish dynamically*, by virtue of the operator field equations, and not identically for arbitrary $A_\mu(x, t)$. Any solution of any given AGT requires, for its consistency, that it satisfy $R(A_\mu) = 0$ for the Heisenberg field operator $A_\mu(x, t)$. In CSM we saw that the anomaly $(e^2/4\pi)[(1-\alpha)\partial_\mu A^\mu - \varepsilon_{\mu\nu}(F^{\mu\nu}/2)]$ does vanish by virtue of Hamilton's equation [Eq. (3.10)].

2. Despite the fact that they must satisfy the vanishing of the anomaly, the space of solutions of an AGT can be nontrivial. In CSM with $\alpha > 1$, we saw that any initial data for $A_1(x)$ and $\phi(x)$ leads to a solution. The spectrum consisted of one massive and one massless particle, whereas the anomaly-free Schwinger model contained only one massive particle. In general, other AGTs, if consistent, can be expected to have a larger space of solutions (more degrees of freedom) than the corresponding anomaly-free gauge theory.

3. Lorentz invariance need not be violated in an AGT. In the CSM, we explicitly found a relativistic spectrum corresponding to one massive and one massless species of particles. The Poincaré algebra has also been established for this model. One reason why some authors may have obtained non-Lorentz invariant results could be their use of the "Weyl" ($A_0 = 0$) gauge. In an AGT, gauge invariance is broken by the anomaly. Consequently, one does not have the freedom to fix any gauge condition. To impose an $A_0 = 0$ condition in the face of this is to explicitly break Lorentz invariance by hand. For instance in CSM, the field A_0 is fully determined by the constraint of Eq. (3.8). It does not vanish identically. The resulting Lorentz-invariant content of the theory would be destroyed if one required $A_0 = 0$. Of course, if one starts with some alternate gauge-invariant reformulation of an AGT (with some other action), then various gauges may be fixed, without violating Lorentz invariance.

4. NON-ABELIAN CHIRAL GAUGE THEORY

Next, consider the non-Abelian generalization of the two-dimensional model in Eq. (2.1), i.e., N massless Dirac fermions whose right chiral current is coupled to a $U(N)$ gauge field

$$(A_\mu)_{ij} = \lambda^a_{ij} A^a_\mu,$$

where λ^a are $U(N)$ group generators satisfying $\mathrm{Tr}\,\lambda^a\lambda^b = \delta^{ab}$ and

$$[\lambda^a, \lambda^b] = f_{abc}\lambda^c.$$

The action is

$$S_F(\psi^i, \bar{\psi}^i, A^a_\mu) = \bar{\psi}^i i\partial\!\!\!/\psi^i - \frac{e}{2}\bar{\psi}^i A\!\!\!/_{ij}(1-\gamma_5)\psi^j - \tfrac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}. \tag{4.1}$$

The bosonized action associated with this is[3,6]

$$\begin{aligned} S[U, A_\mu] &= \frac{1}{4\pi}\int d^2x\,\mathrm{Tr}\left[\tfrac{1}{2}\partial_\mu U \partial^\mu U^{-1} - ie(g^{\mu\nu} + \varepsilon^{\mu\nu})U^{-1}\partial_\nu U A_\mu + \frac{\alpha e^2}{2} A_\mu A^\mu\right] \\ &\quad - \tfrac{1}{4}\int d^2x\,\mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] \\ &\quad + \frac{1}{12\pi}\int_V d^3y\,\mathrm{Tr}[\varepsilon^{ijk}U^{-1}\partial_i U U^{-1}\partial_j U U^{-1}\partial_k U_i]. \end{aligned} \tag{4.2}$$

Here, U is a $U(N)$ group-valued field, A_μ is the matrix-valued gauge field, and the last term is the famous Wess–Zumino term, defined on a 3-surface

V whose boundary is compactified space-time. The right current, coupled to the gauge field, is

$$j_R^\mu = \frac{i}{4\pi}(g^{\mu\nu} + \varepsilon^{\mu\nu})U^{-1}\partial_\nu U. \tag{4.3}$$

The term $(\alpha e^2/8\pi)$ Tr $A_\mu A^\mu$ comes from regularizing, just as in the Abelian case. What is done next is a canonically constrained Hamiltonian analysis of the bosonized system of Eq. (4.2) in the same spirit as described earlier for the Abelian case. But technically, the problem is more complicated because one of the fields lies on a non-Abelian group manifold, and because of the presence of the *WZ* term. We will omit details of derivation. The Poisson brackets and the Hamiltonian can be obtained by using the canonical symplectic 2-form associated with the action [Eq. (4.2)]. Alternatively, one can work in terms of group coordinates ϕ^a. The results are most conveniently expressed in terms of matter-charge densities ρ_L^a and ρ_R^a of the left and right chiralities. The Hamiltonian is

$$H = \int dx \Big[\pi(\rho_R^a \rho_R^a + \rho_L^a \rho_L^a) + e\left(\rho_R^a + \frac{e}{8\pi}(A_0 - A_1)^a\right)(A_0 - A_1)^a$$
$$- \frac{\alpha e^2}{8\pi}(A_0^a A_0^a - A_1^a A_1^a) + \tfrac{1}{2}E^a E^a - (D_1 E)^a A_0^a + \pi_0^a v^a \Big], \tag{4.3}$$

with Poisson brackets

$$\{\rho_{L,R}^a(x), \rho_{L,R}^b(y)\} = -f^{abc}\rho_{L,R}^c(x)\delta(x-y) \mp \delta^{ab}\cdot\frac{1}{2\pi}\delta'(x-y)$$
$$\{A_0^a(x), \pi_0^b(y)\} = \{A_1^a(x), E^b(y)\} = \delta^{ab}\delta(x-y), \tag{4.4}$$

and all other brackets vanishing.

As before,

$$\pi_0^a(x) = 0 \tag{4.5}$$

is a constraint and $v^a(x)$ is the associated (at this stage undetermined) velocity. A second family of (Gauss's law) constraints comes from

$$0 = \{\pi_0^a(x), H\}$$
$$= (D_1 E)^a + \frac{e^2}{4\pi}((\alpha - 1)A_0^a + A_1^a) - e\rho_R^a$$
$$\equiv G^a(x). \tag{4.6}$$

These constraints in Eqs. (4.5) and (4.6) are again all of the second class, with brackets

$$\{\pi_0^a(x), G^b(y)\} = \frac{e^2}{4\pi}(1-\alpha)\delta^{ab}\delta(x-y)$$

$$\{G^a(x), G^b(y)\} = ef^{abc}\left[G^c(x) + \frac{e^2}{4\pi}(A_1 + (1-\alpha)A_0)^c\right]\delta(x-y). \quad (4.7)$$

The requirement that $\{G^a, H\} = 0$ can be satisfied when $\alpha \neq 1$ by adjusting the function $v^a(x)$. Dirac brackets can again be set up to render these constraints strong. The Hamiltonian reduces to

$$H = \int dx \left[\pi\left(\rho_R^a - \frac{eA_1^a}{2\pi}\right)^2 + \pi\rho_L^a\rho_L^a + \tfrac{1}{2}E^aE^a + \frac{e^2}{8\pi}(\alpha-1)(A_0^aA_0^a + A_1^aA_1^a)\right]. \quad (4.8)$$

Clearly, for $\alpha > 1$, the Hamiltonian is real and positive. When the Dirac brackets are turned into commutators, the resultant quantum theory is consistent and unitary. The equations of motion given by this Hamiltonian, along with the constraint equation [Eq. (4.6)], can be checked as being the same as the Lorentz covariant field equations obtained from the action [Eq. (4.2)]. Of course, the spectrum cannot be obtained here, unlike the situation in the Abelian case, since the system is nonlinear and not exactly soluble by present-day techniques. But unitarity and Lorentz invariance have been formally established.[5]

The $\alpha = 1$ regularization is again a special case. Here the Poisson bracket between π_0^b and G^a vanishes [see Eq. (4.7)]. The requirement $\partial_0 G^a = \{G^a, H\} = 0$ can no longer be met by adjusting the velocity $v^a(x)$, and that requirement leads, as in the Abelian case, to further constraints. John Lott and I found[4] that there are altogether 3 dim[G] + Rank[G] constraints in phase space (at each x) on the gauge field. Since there are 4 dim[G] gauge field variables in phase space to start with (π_0^a, A_0^a, E^a, A_1^a), there are left behind dim[G] − Rank[G] dynamical fields, in addition to matter fields, in phase space. Note that this number is always even for any compact [G]. Hence the number of dynamical fields in coordinate space is still an integer, but it is not proportional to dim[G]. Recall that in a gauge-invariant theory in two dimensions there are *no* dynamical gauge-field degrees of freedom left. The Hamiltonian of the $\alpha = 1$ case is again positive, and the theory unitary.

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Chapter 11

Some Topics in Topological Quantum Field Theories

Fidel A. Schaposnik

1. INTRODUCTION

I will describe in this talk some work on topological quantum field theories (TQFTs) that I have done in collaboration with Leticia Cugliandolo, Gustavo Lozano, Hugo Montani, and George Thompson. In the first part of the talk I will review some relevant features of topological field theory quantization. Then, in section 2, I will discuss the connection among various approaches to quantization by using a stochastic process description. In section 3 I will present some interesting two-dimensional TQFTs, while in section 4 I discuss topological invariants.

Antecedents of TQFTs can be found in the works of A. Schwartz[1]; S. Deser, R. Jackiw, and S. Templeton[2]; and C. R. Hagen.[3] Interest in these theories has been triggered by recent works of E. Witten on Yang-Mills,[4] sigma model,[5] Chern-Simons,[6] and gravitation[7] field theories.

FIDEL A. SCHAPOSNIK • Departamento de Física, Universidad Nacional de La Plata, 1900 La Plata, Argentina.

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Apart from their mathematical relevance—TQFTs provide a way of evaluating topological invariants for low-dimensional manifolds—these theories are of interest from the point of view of physics: they have a rich structure related to instantons, monopoles, vortices; they are connected to Nicolai maps in supersymmetric theories; in the case of three-dimensional Chern–Simons theory there is a relation with two-dimensional conformal field theories.[6,8–9]

In TQFTs the role of symmetries is in some sense more important than the role of Lagrangians. This is the reason meaningful TQFTs can be constructed starting from "trivial" actions (such as $S = 0$ modulo topological invariants).

Once one has chosen the fields that will describe the system, one looks for the largest local symmetry possible for these fields and then considers a classical action which is invariant under this topological large symmetry. All interesting features of topological field theories must arise upon quantization. The favorite quantization approach to deal with the large symmetry referred above is the BRST method.

Let us first describe how L. Baulieu and I. M. Singer[10] derived the TQFT proposed by Witten for Yang–Mills fields[4] by BRST quantization of a classical action, the Chern–Pontryagin invariant in four dimensions.

Consider the quantity

$$S_T = c \int_{M_4} \mathrm{Tr}\, F^{\mu\nu} {}^*F_{\mu\nu}\, dV, \tag{1}$$

which measures the first Pontryagin class of the vector bundle on which A_μ is the connection, $F_{\mu\nu}$ the curvature and

$${}^*F_{\mu\nu} = \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \tag{2}$$

The integral in Eq. (1) extends over some compact four-dimensional manifold M_4, and c is a constant chosen so that $S_T = n \varepsilon \mathbf{Z}$. Of course, S_T is invariant under ordinary gauge transformations:

$$\begin{aligned} \delta A_\mu &= -D_\mu \varepsilon \\ \delta F_{\mu\nu} &= [\varepsilon, F_{\mu\nu}] \qquad (3) \\ D_\mu &= \partial_\mu + e[A_\mu,\ \ ] \\ \delta S_T &= 0, \qquad (4) \end{aligned}$$

with $\varepsilon = \varepsilon(x) = \varepsilon^a t^a$, taking values in the Lie algebra of some group G with generators t^a.

Since S_T is a topological invariant, it has a symmetry larger than that of Eq. (3). Indeed, S_T is in essence invariant under arbitrary variations of the gauge field:

$$\delta A_\mu = \varepsilon_\mu(x), \tag{5}$$

where the one-form $\varepsilon_\mu(x)$ takes values in the Lie algebra of G. To see this, note that $F_{\mu\nu}$ changes under such a variation like a gauge field under ordinary gauge transformations:

$$\delta F_{\mu\nu} = D_\mu \varepsilon_\nu - D_\nu \varepsilon_\mu \equiv D_{[\mu,}\varepsilon_{\nu]}, \tag{6}$$

and δS_T can be written in the form

$$\delta S_T = -4c \int \varepsilon_\mu D_\nu {}^*F^{\mu\nu}\, dV, \tag{7}$$

which vanishes due to the Bianchi identity.

Of course, at the classical level, the action in Eq. (1) leads to no dynamics, since the Pontryagin density can be written as a total derivative:

$$S_T = \int \partial_\mu K^\mu \, dV \tag{8}$$

$$K^\mu = c\varepsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}(A_\nu \partial_\alpha A_\beta + \tfrac{2}{3} A_\nu A_\alpha A_\beta).$$

Since S_T is a topological invariant, it does not depend on the metric. This is a characteristic of an action taken as the starting point for a topological theory. The point is now to see if this property resists quantization.

In view of the large symmetry of a topological action such as S_T, one should proceed carefully when quantizing the model. One can use the BRST approach to fix the symmetry, and this may in principle introduce a metric dependence into the quantum action. We shall see, however, that the corresponding generating functional is still metric-independent.

Let us write the BRST variation of some functional F of the fields in the form

$$\delta_{\mathrm{BRST}} F = i\varepsilon\{Q, F\}, \tag{9}$$

where ε is the anticommuting global BRST parameter and $\{Q,\ \}$ is just a notation for the linear transformation δ_{BRST}. (In a Hamiltonian approach $\{\ ,\ \}$ would represent a graded commutator, and then Q would correspond to the BRST nilpotent charge.)

After the usual BRST procedure, one ends with the generating functional

$$Z_{\mathrm{Top}} = \int D\,\text{fields} \exp[-S_Q], \tag{10}$$

with

$$S_Q = S_T + \int \{Q, F\}\, dV. \tag{11}$$

The second term in Eq. (11) represents the gauge fixing and the ghosts terms which can be written as the BRST variation of some functional F of original fields, ghost fields, and auxiliary fields. Of course this term introduces a metric dependence. However, it is easy to see that the generating functional does not depend on the metric $g_{\mu\nu}$. Indeed,

$$\frac{1}{Z}\frac{\delta Z}{\delta g^{\mu\nu}} = -\frac{1}{Z}\int \exp[-S_Q] \times \frac{1}{\sqrt{g}}\left\{Q, \frac{\delta F}{\delta g^{\mu\nu}}\right\} dV\, D\,\text{fields}, \tag{12}$$

or

$$\frac{1}{Z}\frac{\delta Z}{\delta g^{\mu\nu}} = \left\langle \left\{Q, \frac{1}{\sqrt{g}}\frac{\delta F}{\delta g^{\mu\nu}}\right\} dV \right\rangle = 0, \tag{13}$$

since the vacuum expectation value of any BRST variation is zero. (In a Hamiltonian approach this results from the definition of physical states $|\text{phys}\rangle$ such that $Q|\text{phys}\rangle = 0$. In the present approach, Eq. (13) follows from BRST invariance of the path-integral measure and of S_Q.)

We have then arrived at a very important point: in a TQFT, the energy-momentum tensor $T_{\mu\nu}$, which is nothing but the variation of S_Q under a change of the metric, is the BRST variation of some functional, and hence its vacuum expectation value is zero. We can use this property to give a definition of a TQFT in a broad sense:

A topological quantum field theory is a quantum field theory with a metric-independent partition function. That is, Z depends on the smooth structure of the manifold but not on the metric.

In many cases (as for example in the Yang–Mills case we have taken as our example) not only is the variation of S_Q with respect to the metric a BRST variation, but S_Q can already be written in the form

$$S_Q = \{Q, V\}. \tag{14}$$

In view of these features, one should naively conclude that all observables in TQFTs are trivial. Although this is true locally, local properties cannot be simply extended into global ones, and hence global invariants are not trivial.[11]

There are other TQFTs, like that of Chern-Simons, for which S_Q cannot be written in the form of Eq. (14). However, Z is still metric-independent due to the arguments leading to Eq. (13).

Baulieu and Singer[10] explicitly constructed the quantum action S_Q [Eq. (11)] associated with the classical action of Eq. (1). They showed that an appropriate gauge fixing of the large symmetry [Eq. (5)], with the instanton defining equation

$$F_{\mu\nu} = {}^*F_{\mu\nu} \tag{15}$$

as one of the gauge conditions, leads to the TQFT proposed by Witten in Ref. 4. In their approach, the fermions introduced by Witten are ghost fields that arise in the BRST procedure. There are also ghosts of ghosts due to the existence of a second-generation gauge invariance, which we shall discuss below.

An alternative approach to TQFT was proposed by Labastida and Pernici[12] and also developed by Birmingham, Rakowski, and Thompson.[13-14] In this approach, instead of starting from a topological classical action like Eq. (1), one starts from a different kind of "trivial" action: a Gaussian action S_G where, apart from the original fields (A_μ fields for the Yang-Mills case), auxiliary fields $G_{\mu\nu}$ are introduced:

$$S_G = \int [G_{\mu\nu} - \xi_{\mu\nu}[A_\nu]]^2\, dV. \tag{16}$$

Here $\xi_{\mu\nu}[A_\mu]$ is an adequately chosen self-dual functional. Again, the topological generating functional is

$$Z = \int D\,\text{fields}\, \exp\left[-S_{G^+} \int \{Q, F\}\, dV\right], \tag{17}$$

where D fields includes measures of gauge fields, ghost fields, and auxiliary fields. The BRST term $\{Q, F\}$ arises after gauge fixing of the Gaussian action, which has the following large local symmetry:

$$\delta A_\mu = \varepsilon_\mu \tag{18a}$$

$$\delta G_{\mu\nu} = \frac{\delta \xi_{\mu\nu\delta}}{\delta A_\alpha}\, \varepsilon_\alpha, \tag{18b}$$

where the $G_{\mu\nu}$ variation is chosen so as to make the action invariant under the largest local symmetry.

It is obvious that the action in Eq. (16) becomes trivial if one makes the shift $G_{\mu\nu} \to G_{\mu\nu} + \xi_{\mu\nu}$. All interesting effects should arise, again, after quantization.

In order to obtain Witten's TQFT for the Yang-Mills case, one has to choose

$$\xi_{\mu\nu} = \tfrac{1}{2}(F_{\mu\nu} + {}^*F_{\mu\nu}) \equiv {}^+F_{\mu\nu}. \tag{19}$$

We shall rewrite the transformation laws [Eq. (18)] in the form

$$\delta A_\mu = \varepsilon_\mu - D_\mu \varepsilon \tag{20a}$$

$$\delta G_{\mu\nu} = D_\mu \varepsilon_\nu - D_\nu \varepsilon_\mu + \varepsilon_{\mu\nu\alpha\beta} D_\alpha \varepsilon_\beta - [\varepsilon, G_{\mu\nu}]. \tag{20b}$$

In Eq. (20a) we have distinguished from the general family of transformations [Eq. (18a)] those that correspond to ordinary gauge transformations. With this, the field strength changes in the form

$$\delta F_{\mu\nu} = D_\mu \varepsilon_\nu - D_\nu \varepsilon_\mu - [\varepsilon, F_{\mu\nu}]. \tag{21}$$

Written in this form, not all four components of ε_μ are effective regarding gauge invariance. Namely, if we choose $\varepsilon_\mu = D_\mu \Lambda$ and $\varepsilon = \Lambda$, one has

$$\delta^\Lambda A_\mu = 0, \qquad \delta^\Lambda F_{\mu\nu} = 0. \tag{22}$$

There is, however, a change for $G_{\mu\nu}$:

$$\delta^\Lambda G_{\mu\nu} = [\Lambda, {}^+F_{\mu\nu} - G_{\mu\nu}]. \tag{23}$$

Now, if one uses the equations of motion arising from the action of Eq. (16),

$${}^+F_{\mu\nu} = G_{\mu\nu}, \tag{24}$$

the variation in Eq. (23) vanishes:

$$\delta^\Lambda G_{\mu\nu}|_{\text{on shell}} = 0. \tag{25}$$

Equations (2) and (25) mean that there is a second-generation gauge invariance, and then the naive Faddeev–Popov method cannot be applied. One can instead use Batalin–Vilkovisky approach[15] specialized to the case of a "first-stage reducible theory" (which corresponds in their classification to the presence of this observed second-generation invariance). I will not describe in detail the calculations (see Refs. 12–13). The resulting quantum action is

$$\begin{aligned} S_Q = \operatorname{Tr} \int_{M_4} & dV[\tfrac{1}{2}({}^+F_{\mu\nu}{}^+F^{\mu\nu} - G_{\mu\nu}G^{\mu\nu}) + iD_\alpha\psi\beta\chi^{\alpha\beta} - i\eta D^\alpha\psi_\alpha \\ & + \tfrac{1}{2}\lambda D_\mu D^\mu\phi + \tfrac{1}{2}G_{\mu\nu}d^{\mu\nu} - \tfrac{1}{2}i\lambda\{\psi_\mu, \psi^\mu\} \\ & - \tfrac{1}{8}i\phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} + d\partial_\mu A^\mu - ib\partial_\mu\psi^\mu + b\partial_\mu D^\mu c]. \end{aligned} \tag{26}$$

Apart from the fields A_μ and $G_{\mu\nu}$, which are Grassmann even, Batalin–Vilkovisky construction makes appear a first generation of ghosts c and ψ_μ (with ghost number 1, Grassmann odd) related to symmetries with parameters ε and ε_μ, respectively; a second generation ghost ϕ (ghost number 2, Grassmann even) associated with the second-generation gauge invariance; antighosts fields $\chi_{\alpha\beta}$, b, and λ which are, respectively, self-dual Grassmann odd, odd, and even and have ghost numbers $(-1, -1, -2)$. Fields $d_{\mu\nu}$, d, and η (even, even, and odd), with ghost numbers $(0, 0, -1)$, respectively, are Lagrange multipliers.

The gauge conditions that fix invariance [Eqs. (20)–(25)] are

$$G_{\mu\nu} = 0; \qquad \partial^\mu A_\mu = 0; \qquad D_\mu \Psi^\mu = 0. \tag{27}$$

Once these conditions are used and the Lagrange multipliers eliminated, the action of Eq. (26) coincides with the one proposed by Witten.[11] Moreover, using the BRST transformations for the fields, one can show that Eq. (26) can be rewritten in the form

$$S_Q = \int \{Q, F\}\, dV, \tag{28}$$

with

$$F = \mathrm{Tr}[\tfrac{1}{4}\chi^{\mu\nu}({}^{+}F_{\mu\nu} + G_{\mu\nu}) - \tfrac{1}{2}\lambda D_\mu \psi^\mu], \tag{29}$$

which trivially shows that

$$\langle T_{\mu\nu}\rangle = 0. \tag{30}$$

We have been able to construct a Yang-Mills TQFT by starting from a Gaussian action in which the instanton equation plays a central role. An analogous construction can be envisaged by starting from the Prasad-Sommerfield monopole equations (Bogomol'nyi equations)[16-17] in three dimensions, obtaining a TQFT associated to $SO(3)$ monopoles. Also, two-dimensional TQFTs can be built starting from the Bogomol'nyi equations for Abelian and non-Abelian vortices.[18-19]

In the Baulieu-Singer approach[10] to these theories, an alternative procedure is followed: one starts from topological actions which are the monopole magnetic charge or the vortex flux in three and two dimensions, respectively, and then one fixes the topological symmetry *à la* BRST, using Bogomol'nyi equations as gauge conditions.

To see the central role played by instanton defining (Bogomol'nyi) equations in the resulting TQFT, let us note that one can easily show that the generating functional is not only independent under a change in the metric, but also under a change of the coupling constant. For example, in the Yang-Mills case, if one redefines fields so as to factor out the squared coupling constant from S_Q, one has

$$\begin{aligned} Z &= \int \exp\left[-\frac{1}{e^2} S_Q\right] D\,\text{fields} \\ \frac{\delta Z}{\delta e^2} &= e^{-4}\{Q, F\} = 0. \end{aligned} \tag{31}$$

One can then evaluate Z by taking the limit of the very small coupling constant where the action is dominated by classical minima. Now, the gauge field terms in S_Q are such that classical minima correspond to the equation

$$F_{\mu\nu} = -{}^{*}F_{\mu\nu}.$$

Then evaluation of Z depends on expansion around instantons! The same is true for monopoles and vortices.

2. TQFTs AND STOCHASTIC PROCESSES

I discuss in this section the relation between Baulieu-Singer[10] and Labastida-Pernici[12] approaches by deriving a connection between a Langevin equation (in real time) for a certain stochastic process and the Bogomol'nyi equation used in the construction of a TQFT.

Schematically, Bogomol'nyi equations can be obtained whenever the action defining a model can be written in the form

$$S = \frac{1}{2} \int_M \sum_i [a^i \pm \tilde{a}^i]^2 \, dV \mp \int_M \sum_i a^i \tilde{a}^i \, dV, \tag{32}$$

with a^i and $\tilde{a}^i$ two functionals of the fields describing the system on some manifold M and the second term in Eq. (32) related to some topological invariant Q_T:

$$Q_T = \alpha \int_M \sum_i a^i \tilde{a}^i \, dV, \tag{33}$$

with α a normalization constant.

From Eq. (31) one has

$$S \geq \mp Q_T, \tag{34}$$

and the bound is saturated when Bogomol'nyi equations hold:

$$\xi^i = a^i \pm \tilde{a}^i = 0. \tag{35}$$

As a first example, consider the Yang-Mills theory in four dimensions. For $a^1 \equiv F_{\mu\nu}/2$ and $\tilde{a}^1 \equiv {}^*F_{\mu\nu}/2$, Q_T is the Chern-Pontryagin invariant and Eq. (35) becomes the instanton equation [Eq. (15)]. For a $U(1)$ gauge field A and a complex scalar ϕ, in two dimensions, if one takes $a^1 \equiv F/\sqrt{2}$ (F, the curvature), $a^2 \equiv \varepsilon_{ab} D\phi_a/\sqrt{2}$ (D, the covariant derivative and $a, b = 1, 2$) Eq. (35) becomes the vortex Bogomol'nyi equations,[17] provided one chooses $\tilde{a}^1 \equiv {}^*(\phi^2 - \eta^2)e/\sqrt{2}$ and $\tilde{a}^2 \equiv {}^*D\phi_a$. (Here $*$ is the Hodge star operator, η the Higgs field vacuum expectation value and e the gauge-coupling constant.) The same can be done for non-Abelian vortices,[20] Prasad-Sommerfield monopoles, and CP^n instantons.

In Refs. 13–14, a connection between the Gaussian action [Eq. (16)] and a Langevin equation with the auxiliary field G acting as a random field was pointed out. I shall now discuss this connection, trying to clarify its origin and relating it with the Baulieu-Singer[10] approach to TQFTs. I shall closely follow Ref. 21.

Consider the Langevin equation for some generic field Φ:

$$\frac{\partial \Phi}{\partial t} - \frac{\partial W}{\delta \Phi} = G, \tag{36}$$

with t the real time, G a random field, and W the potential providing a drift to the stochastic process described by Eq. (36). Now, the point is that one can make a choice of W such that this Langevin equation becomes a Bogomol'nyi equation. Moreover, this identification allows one to establish a connection between the two different approaches to the construction of TQFTs.

Let us again take as example the case of Yang-Mills fields in four dimensions. We have seen that the topological charge [Eq. (33)] can be written in the form of Eq. (8), namely:

$$Q_T = \int \partial_\mu K^\mu \, dV, \tag{37}$$

with K_μ given by Eq. (8). Now, one can easily prove that in the $A_0 = 0$ gauge, Q_T can be written in the form

$$Q_T = \lim_{t\to\infty} W(t), \tag{38}$$

with

$$W(t) = \frac{1}{\alpha} \int K_0 \, d^3x. \tag{39}$$

(We have chosen initial conditions $A_i(-\infty) = 0$ so that $W(-\infty) = 0$.) Now, if one chooses $W(t)$ as the potential W in the Langevin equation [Eq. (36)] for a Yang-Mills field $A_i = \Phi$,

$$\frac{\partial A_i}{\partial t} - \frac{\delta W(t)}{\delta A_i} - G_{0i} = 0, \tag{40}$$

one discovers that this equation is nothing but the instanton-defining equation, in the presence of a noise, (in the $A_0 = 0$ gauge).

In view of the behavior in Eq. (38), one can conclude that the stochastic process described by Eq. (40) evolves toward an equilibrium state governed by $\exp(-Q_T)$.

We are now ready to see the connection between Baulieu-Singer[10] and Labastida-Pernici[12] TQFT construction. Indeed, it has been known since the work of Parisi and Sourlas[22] that one can arrive at SUSY QFTs by starting from a real-time Langevin equation of the type of Eq. (40) for the bosonic sector. The corresponding stochastic generating functional Z_{stoch} (defined as a path integral over bosonic and random fields with an action which is the squared Langevin equation) has a large symmetry to be fixed: an arbitrary change in the bosonic fields can always be compensated for by an appropriate change in the random fields. In the gauge-fixing procedure, ghosts come into play and the resulting quantum theory is supersymmetric. For the Yang-Mills example we are discussing, the Parisi-Sourlas approach corresponds to taking the square of the Langevin equation [Eq. (40)] as starting (Gaussian) action and then proceed to a BRST quantization. Z_{stoch} then reads

$$Z_{\text{stoch}} = \int D \text{ fields} \exp\left[-\int \left(G_{0i} - \frac{\partial A_i}{\partial t} - \frac{\partial W}{\partial A_i}\right)^2 dV + \text{BRST terms}\right]. \tag{41}$$

Now, in view of what we have said, this is nothing but the generating functional [Eq. (17)] in the $A_0 = 0$ gauge, used in the Labastida-Pernici approach.[12] From the stochastic point of view, one can interpret Eq. (41) as arising from a stochastic process evolving toward an equilibrium governed by $W(t = \infty) = Q_T$. But precisely Q_T was the starting action considered by Baulieu and Singer in their approach to TQFTs.[10] This is the announced connection between the two approaches.

3. TWO-DIMENSIONAL MODELS

The link described above can be established for other theories admitting instanton-like solutions.

Indeed, apart from the monopole case referred to above, two-dimensional vortices can be used to generate interesting TQFTs. Consider first a $U(1)$ gauge field A_μ and a charged scalar ϕ^a $(a = 1, 2)$ in two space-time dimensions. A topological invariant (which can be interpreted as a magnetic vortex flux) can be defined in different ways. A useful one is

$$Q_T = \int d^2x \left[\frac{e}{2} F_{\mu\nu}(\eta^2 - \phi^2) - D_\mu\phi^a D_\nu\phi^b \varepsilon_{ab}\right]\varepsilon_{\mu\nu}, \tag{42}$$

with $\alpha = \frac{1}{2}\pi\eta^2$. Again,

$$Q^T = \int d^2x \partial_\mu K^\mu, \tag{43}$$

with

$$K_\mu = \frac{\alpha}{2}[e\eta^2 A_\mu + \varepsilon_{ab}\phi^a D_\mu \phi^b]. \tag{44}$$

Taking $W(t)$ in the form

$$W(t) = \frac{1}{\alpha}\int K_0 \, dx, \tag{45}$$

one has

$$\lim_{t\to\infty} W(t) = Q_T. \tag{46}$$

The corresponding Langevin equations read:

$$E^a \equiv \frac{\partial \phi^a}{\partial t} - \varepsilon^{ab}(D_1\phi)^b = G_0^a \tag{47}$$

$$E \equiv \frac{\partial A_1}{\partial t} - \frac{e}{2}(\eta^2 - \phi^2) = G. \tag{48}$$

In the $A_0 = 0$ gauge, these are the Bogomol'nyi equations for Nielsen–Olesen vortices,[17] valid when the following relation between coupling constants holds:

$$\lambda = e^2. \tag{49}$$

In Ref. 18 we have constructed the TQFT associated with two-dimensional vortices, thus providing the first example of a topological field theory with (explicit) symmetry breaking. (In the monopole case studied in Refs. 23–25, the symmetry breaking was realized *à la* Prasad-Sommerfield,[16] which corresponds to the $\lambda = 0$ limit.)

Working in the Landau gauge, one starts from the generating functional

$$Z = \int DG_a DG D\phi^a DA_\mu \exp(-S_G), \tag{50}$$

with

$$S_G = \tfrac{1}{2}(E - G)^2 + \tfrac{1}{2}(E_\mu^a + G_\mu^a)^2. \tag{51}$$

The Gaussian action S_G has the following large symmetry:

$$\begin{aligned} \delta A_\mu &= \varepsilon_\mu - \partial_\mu \varepsilon \\ \delta G &= \varepsilon^{\mu\nu}\partial_\mu \varepsilon_\nu - 2\lambda^a \phi^a \\ \delta \phi^a &= \lambda^a - i\varepsilon_b^a \varepsilon \phi^b \end{aligned} \tag{52}$$

$$\delta G_\mu^a = \tfrac{1}{2}[D_\mu \lambda^a + \varepsilon_\mu^\nu \varepsilon_b^a D_\nu \lambda^b] - \frac{i}{2}[\varepsilon_b^a \varepsilon_\mu \phi^b + \varepsilon_\mu^\nu \varepsilon_\nu \phi^a] - i\varepsilon_b^a G_\mu^b,$$

where $\varepsilon(x)$ is the parameter associated with ordinary gauge transformations and $\varepsilon_\mu(x)$ and $\lambda^a(x)$ the parameters associated with the large topological symmetry. All those parameters are not effective: there is again a second-generation invariance which becomes apparent by writing

$$\begin{aligned} \varepsilon &= \Lambda \\ \varepsilon_\mu &= D_\mu \Lambda \\ \lambda^a &= i\varepsilon_b^a \Lambda \phi^b. \end{aligned} \tag{53}$$

All variations [Eq. (52)] vanish for the particular choice [Eq. (53)] except that for G^a_μ, which reduces to

$$\delta^\Lambda G^a_\mu = -i\varepsilon^a_b \Lambda[G^b_\mu - \tfrac{1}{2}(D_\mu \phi^b + \varepsilon^\nu_\mu \varepsilon^a_b D_\nu \phi^b)]. \tag{54}$$

Now, the variation in Eq. (54) vanishes when equations of motion are taken into account:

$$\delta^\Lambda G^a_\mu = 0|_{\text{on shell}}. \tag{55}$$

Following Batalin-Vilkovisky procedure[15] we assign a ghost field to each of the symmetries associated with Eq. (52):

$$\begin{aligned} \varepsilon &\to c \\ \varepsilon_\mu &\to \psi_\mu \\ \lambda^a &\to \rho^a \\ \Lambda &\to \eta, \end{aligned} \tag{56}$$

so that the minimal set of fields is $(A_\mu, \phi^a, G, G^a_\mu, \psi_\mu, c, \rho^a, \eta)$ with ghost numbers (0, 0, 0, 0, 1, 1, 1, 2). We choose as gauge conditions

$$G = G^a_\mu = \partial_\mu A^\mu = \partial_\mu \psi^\mu = 0. \tag{57}$$

The final form of the TQFT Lagrangian is

$$\begin{aligned} L = {} & [\tfrac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu} + (1-\phi)^2]\Omega + \tfrac{1}{2}D_\mu\phi^a d^\mu_a + d^\mu_a d^a_\mu + 2\Omega^2 + b(-\partial^2 c + \partial^\mu\psi_\mu) \\ & + \omega(\varepsilon^{\mu\nu}\partial_\mu\psi^\nu - 2\rho^a\phi_a) + \chi^\mu_a D_\mu\rho^a - i\chi^\mu_a\varepsilon^a_b\psi_\mu\phi^b + \lambda\partial^2\eta + d\partial_\mu A^\mu \\ & - \sigma\partial^\mu\psi_\mu + \chi^\mu_a\chi^\nu_b\varepsilon^{ab}\delta_{\mu\nu}\eta. \end{aligned} \tag{58}$$

The BRST variations are

$$\begin{aligned} &\delta A_\mu = \varepsilon(\partial_\mu c - \psi_\mu), \qquad \delta c = \varepsilon\eta, \qquad \delta\omega = \varepsilon\Omega, \\ &\delta\phi^a = -\varepsilon(\rho^a - ic\varepsilon^a_b\phi^b), \qquad \delta\eta = \delta\sigma = \delta c = \delta d = \delta\Omega = 0, \\ &\delta\psi_\mu = \varepsilon\partial_\mu\eta, \qquad \delta\chi^a_\mu = \varepsilon(d^a_\mu + i\varepsilon^a_b c\chi^b_\mu), \qquad \delta\lambda = \varepsilon\sigma, \\ &\delta\rho^a = \varepsilon(i\varepsilon^a_b\phi^b\eta + ic\varepsilon^a_b\rho^b), \qquad \delta b = \varepsilon d, \qquad \delta d^a_\mu = i\varepsilon^a_b c d^b_\mu. \end{aligned} \tag{59}$$

Using Eq. (59), the Lagrangian of Eq. (58) can be written in the form

$$L = \{Q, V\} \tag{60}$$

with

$$V = [\omega(\tfrac{1}{2}\varepsilon^{\mu\nu}F_{\mu\nu} + 1 - \phi^2 + 2\Omega) + b\partial_\mu A^\mu + \lambda\partial^\mu\psi_\mu + \chi^\mu_a(d^a_\mu + D_\mu\phi)]. \tag{61}$$

Due to peculiarities of two-dimensional space-time, one can manage to rewrite the Lagrangian (58) using the fermion–boson connection, as a supersymmetric Lagrangian. In terms of new fermionic fields ψ, $\bar{\psi}$, χ, and $\bar{\chi}$ and scalars A and B adequately defined in terms of the original fields, the Lagrangian in Eq. (58) reads

$$L = L_{\text{Gaussian}} + \bar{\chi}\not{D}\chi + \bar{\psi}\not{\partial}\psi + \sqrt{2}\,e(\bar{\chi}\psi\phi^{-} + \bar{\psi}\chi\phi^{+})$$
$$+ 2\sqrt{2}\,ie\bar{\chi}(A + \gamma_5 B)\chi + e^2|\phi|^2(A^2 + B^2) + \tfrac{1}{2}A\partial^2 A + \tfrac{1}{2}B\partial^2 B, \qquad (62)$$

which has a striking resemblance to the Lagrangian for a $U(1)$ SUSY Abelian Higgs model introduced by Salam and Strathdee.[26] It is interesting to note that these authors, as well as Fayet[27] also found the restriction $\lambda = e^2$ in order to have a parity-conserved theory, with all fields having the same mass. The only difference (apart from the fact we are here dealing with a two-dimensional theory) is that we have $N = 2$ SUSY rather than the $N = 1$ one.

An analogous treatment of non-Abelian vortices can be envisaged, starting from the corresponding Bogomol'nyi equations.[20] For example, in the $SU(2)$ case, where one needs two Higgs fields in order to have complete symmetry breaking, one has the following Bogomol'nyi equations:

$$E \equiv \tfrac{1}{2}\varepsilon_{\mu\nu}F^{\mu\nu} - \psi\,\mathrm{Tr}(\phi^2 - \phi_0^2) = 0$$
$$E_\mu \equiv D^+_\mu\phi - i[\psi, D^+_\mu\phi] = 0 \qquad (63)$$
$$F_\mu \equiv D^+_\mu\psi = 0,$$

where the gauge field A_μ takes values in the Lie algebra of $SU(2)$, ϕ and ψ are in the adjoint representation, D_μ is the covariant derivative, and

$$D^+_\mu = \tfrac{1}{2}(D_\mu + i\varepsilon_{\mu\nu}D^\nu). \qquad (64)$$

The topological charge associated with the Z_2 vortex flux is given, modulo 2, by the expression:

$$Q_T = \frac{1}{2\pi}\,\mathrm{Tr}\int \psi\varepsilon^{\mu\nu}_{\mu\nu}\,d^2x. \qquad (65)$$

Starting from the Gaussian action

$$S_G = \int d^2x(|G - E|^2 + |G_\mu - E_\mu|^2 + |H_\mu - F_\mu|^2), \qquad (66)$$

where G, G_μ, and H_μ are random fields taking values in the Lie algebra of $SU(2)$ (with G_μ and H_μ self-dual fields, i.e., $G_\mu = i\varepsilon_{\mu\nu}G^\nu$, and the same for F_μ) and gauge fixing the large topological invariance under variations

$$\begin{aligned} \delta A_\mu &= \varepsilon_\mu(x) - D_\mu \omega(x) \\ \delta\phi &= \lambda(x) - [\phi, \omega] \\ \delta\psi &= \eta(x) - [\psi, \omega], \end{aligned} \tag{67}$$

(with infinitesimal local parameters ε_μ, ω, λ, and η), supplemented with the appropriate transformation laws for G, G_μ, and H_μ, one can construct a TQFT by using the BRST procedure.

Again there is an on-shell second-generation gauge invariance which can be discovered through the following relations between parameters:

$$\begin{aligned} \varepsilon_\mu &= D_\mu \Lambda \\ \omega &= \Lambda \\ \lambda &= -[\Lambda, \phi] \\ \eta &= -[\Lambda, \psi]. \end{aligned} \tag{68}$$

After introducing ghosts fields associated with each one of the parameters in Eq. (67):

$$\begin{aligned} \varepsilon_\mu &\to \psi_\mu \\ \omega &\to c \\ \lambda &\to \rho \\ \eta &\to \xi \\ \Lambda &\to \sigma, \end{aligned} \tag{69}$$

one ends with a topological action which again can be written in the form

$$S = \{Q, V\}, \tag{70}$$

with V in the form:

$$\begin{aligned} V = \int d^2x \{ \tfrac{1}{4} X d - \tfrac{1}{4}\varepsilon_{\mu\nu} F^{\mu\nu} - \psi \operatorname{Tr}(\phi^2 - \phi_0^2) + \tfrac{1}{4} X_\mu (d^\mu - D_\mu^+ \phi \\ + i[\psi, D_\mu^+ \phi]) + \tau_\mu (\tfrac{1}{4} e_\mu - D_\mu^+ \psi) - \bar{c}\partial_\mu A^\mu - \bar{\sigma} D_\mu \psi^\mu \}, \end{aligned} \tag{71}$$

where $\bar{c}, X, X_\mu, \tau_\mu, \bar{\sigma}$ are antighosts with ghost number $(-1, -1, -1, -1, -2)$, and d, d_μ, and e_μ are Lagrange multipliers. Also in this case $\delta Z/\delta e^2 = 0$, and hence one can evaluate Z by taking the $e^2 \to 0$ limit where the path integral is dominated by the classical limit, which in this case corresponds to the solution to the Bogomol'nyi equations, Eq. (63). Since these solutions are known to exist,[28] one can hope to have a complete understanding of the quantum theory by performing an instanton expansion. A topological quantum field theory for vortices has also been studied by Grossman and Chapline[29] starting from a topological invariant action and quantizing it *à la* Baulieu-Singer.[10] In this way a link between conformal field theory and Donaldson theory of 4-manifolds was set up.

4. TOPOLOGICAL INVARIANTS

Witten has shown[4] that one can compute relevant topological invariants using TQFTs by evaluating vacuum expectation values of operators O, adequately chosen:

$$\langle O\rangle = \frac{1}{Z}\int d \text{ fields} \exp(-S)O. \tag{72}$$

The conditions that these operators should satisfy are

$$1 - \{Q, O\} = 0 \text{ modulo those of the form } O = \{Q, V\} \tag{73}$$

$$\frac{\delta O}{\delta g}\mu\nu = \{Q, V\} \quad \text{(eventually } V = 0). \tag{74}$$

These conditions guarantee that

$$\frac{\delta\langle O\rangle}{\delta g^{\mu\nu}} = 0. \tag{75}$$

I will explain how one finds these invariants using the two-dimensional models discussed in the previous section.

By inspection of the BRST transformations [Eq. (59)] for the Abelian model, a candidate satisfying conditions of Eqs. (1)–(2) is the η ghost of a ghost. Let us call it $W_o[P]$, where P denotes a point in two-dimensional space. One can easily show that $\langle W_o[P]\rangle$ is independent of P. Indeed,

$$\frac{\partial W_o}{\partial x_\mu} = \partial^\mu \eta = \{Q, \psi^\mu\}, \tag{76a}$$

and then

$$\langle W_o[P]\rangle - \langle W_o[P']\rangle = \left\langle \int_{P'}^{P} \frac{\partial W_o}{\partial x_\mu} dx_\mu \right\rangle = \left\langle \left\{ Q, \int_{P'}^{P} \psi^\mu \, dx_\mu \right\} \right\rangle = 0. \tag{76b}$$

From Eq. (76), recursively, other topological invariants can be constructed. Writing Eq. (76) in the form

$$dW_o = \{Q, W_1\} \tag{77}$$

with $\{Q, W_o\} = 0$ and

$$W_1 = \psi = \psi_\mu \, dx^\mu, \tag{78}$$

one finds that

$$dW_1 = \{Q, W_2\}, \tag{79}$$

with

$$W_2 = F = \tfrac{1}{2}\varepsilon_{\mu\nu}F^{\mu\nu} \tag{80}$$

and

$$dW^2 = 0. \tag{81}$$

From W_2, one constructs the obvious topological invariant for the two-dimensional Abelian TQFT:

$$Q_T = \int F. \tag{82}$$

One can find interesting operators from the intermediate W_i (in this case, W_1). Define

$$I[C] = \int_C W_1, \tag{83}$$

where C is a closed curve. Then,

$$I[C] = \int_{S_C} dW_1 = \int_{S_C} \{Q, W_2\} \tag{84}$$

(S_C is some surface with border C) is a generalization of Eq. (76b) satisfying the condition $\{Q, I[C]\} = 0$, since

$$\{Q, I\} = \int_C \{Q, W_1\} = \int_C dW_0 = 0. \tag{85}$$

From products of integrals of the type of Eq. (83), one can obtain relevant topological invariants (for example, in the Yang-Mills TQFT[4]).

Concerning the non-Abelian example we discussed in section 3, an analogous sequence can be constructed:

$$\begin{aligned} \{Q, W_o\} &= 0, \qquad & W_o &= \tfrac{1}{2}\mathrm{Tr}\,\sigma^2 \\ dW_o &= \{Q, W_1\}, & W_1 &= \mathrm{Tr}\,\sigma\psi \\ dW_1 &= \{Q, W_2\}, & W_2 &= \mathrm{Tr}\,\sigma F \\ dW_2 &= 0. \end{aligned} \tag{86}$$

As in the Abelian case, W_2 is associated with the Z_2 topological charge of vortex configurations. In fact, note that W_2 has an expression analogous to Eq. (65), except that in the former the ghost of ghost σ replaces the Higgs field appearing in the topological charge definition.

The description of many interesting models and properties of TQFTs have been omitted in this talk. They can be found in Refs. 30–40.

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