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# Identity and Indiscernibility in Quantum Mechanics

Tomasz Bigaj

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# Identity and Indiscernibility in Quantum Mechanics

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# Preface and Acknowledgments

One of the most important purposes that philosophy of science can serve is bridging the gap between scientific inquiries and philosophical speculations. Nowhere is the need for building such bridges more pressing than in the case of metaphysical analysis on the one hand and fundamental physical theories on the other hand. If modern metaphysics aspires to be more than a mere footnote to the Great Old Masters, it must receive a generous influx of new ideas and concepts from the rapidly developing field of fundamental physical sciences. However, such an approach to metaphysics places a heavy burden on the practitioners of this ancient art of theoretical reflection. It requires that they enter the maze of highly abstract mathematical concepts that abound in modern theoretical physics. Even more treacherous territory is the issue of a proper physical (and metaphysical) interpretation of the mathematical formalisms of physical theories. Here even seasoned mathematical physicists admit that the task of actually reading physics off the mathematical equations, let alone the challenge of deriving useful metaphysical lessons, is not at all trivial.

This book is an attempt to distill some metaphysical contents from the quantum theory of many particles. The metaphysical problems that we will try to sort out with the help of modern quantum theory are old and venerated. They are questions about the most fundamental ontological concepts of identity, individuality and discernibility. Philosophers of different stripes have proposed numerous conceptions of what it is to be an individual object, how we manage to carve reality into separate entities of various kinds, what numerical identity and distinctness is, and how it relates to possessing differentiating qualitative properties and relations. A particularly popular view insists that numerical diversity and qualitative discernibility are intimately connected in that the former guarantees (as a matter of metaphysical necessity) the latter. Yet there are arguments that quantum mechanics may cast serious doubts on the validity of this view. This has something to do with the way quantum mechanics describes systems of many particles that belong to the same category (so-called indistinguishable particles). Given some mathematical restrictions placed on the available states and measurable properties of such systems, it may be argued that quantum particles of the same type are totally indiscernible with respect to their physical attributes.

However, this conclusion is by no means unquestionable. It relies on a number of tacit interpretational presuppositions which are open to debate. In this book I will carefully scrutinize the mathematical formalism of standard, non-relativistic quantum mechanics, and I will show that there is actually substantial freedom in choosing the right interpretation of some parts of this formalism. Even more importantly, depending on which interpretation to follow, the consequences related to the above-mentioned metaphysical issues may vary dramatically. I will give a broad presentation of two alternative readings of the mathematical apparatus used in the quantum theory of many particles (I refer to these readings as "orthodoxy" and "heterodoxy") that give rise to two distinct metaphysical conclusions regarding the fundamental characteristics of quantum objects, their identities and individualities. In order to fulfill this task properly, some degree of technicality turns out to be necessary. Thus at places this book may read like a textbook in quantum mechanics, with some mathematical theorems and proofs (all rather elementary, to be sure). However, all these technical issues should not obscure the fact that we are ultimately interested in the general lessons that quantum mechanics can teach us regarding the nature of the fundamental building blocks of the universe.

Allow me to briefly retrace my personal journey leading to the completion of this book, during which I received invaluable help from many people and incurred numerous debts. I got seriously involved in the topic

of the identity and individuality of quantum objects back in 2009 when visiting the University of Bristol as a Marie Curie fellow, thanks to James Ladyman. Out of our discussions grew our joint paper on the Principle of the Identity of Indiscernibles in quantum mechanics in which we criticized the role of so-called weak discernibility in restoring the objecthood and individuality of quantum particles. At that time I was leaning towards the orthodoxy in the form developed, among others, by Michael Redhead, Paul Teller, Steven French and Decio Krause, with its insistence on the lack of discernibility and individuality of quantum particles. Then, around the year 2010, Marek Kuś from the Center of Theoretical Physics in Warsaw pointed me towards a series of publications by GianCarlo Ghirardi with collaborators on the notion of entanglement applied to "indistinguishable" particles. These papers were a revelation to me. I realized that there is a clear formal sense in which electrons, photons and so on can be said to literally possess distinct and differentiating properties. I presented my take on Ghirardi et al. at a workshop in Bristol organized by James Ladyman in June of 2011, and later that year at the 14th Congress of Logic, Methodology and Philosophy of Science in Nancy, France. In attendance of these events were Simon Saunders, Fred Muller and Adam Caulton, whose helpful comments prevented me from making numerous embarrassing mistakes. As a result I prepared a paper commenting on the role of symmetric projectors in individuating quantum objects, which came out in print in 2015 as part of the CLMPS proceedings issue of *Philosophia Scientiae*. (I should also mention here the related work on the notion of entanglement with James Ladyman and Øystein Linnebo.) In the meantime, Simon Saunders and Adam Caulton wrote a number of excellent articles which essentially pointed in the same direction, developing an approach which Caulton calls "heterodoxy". Especially one beautiful paper of his, available on arXive, became a bible for me. I cannot fathom why this comprehensive 50-page-long formal analysis hasn't been published in any of the leading journals in philosophy of physics. In a sense the current book may be seen as a long and slightly verbose commentary to Adam's phenomenal paper.

I spent two extremely productive years from 2013 to 2015 at the University of California, San Diego, where I benefitted enormously from discussions with the Philosophy Department members and visiting

guests, including Chris Wüthrich, Craig Callender, John Dougherty, Kerry McKenzie, Nick Huggett, Josh Norton and Holger Lyre. Monthly meetings of the Southern California Philosophy of Physics group at UC Irvine were yet another source of inspiration for me. After that, one more year in Bristol gave me the opportunity to talk further about my developing ideas with James Ladyman, Karim Thébault and other faculty and visitors. In 2018 I was kindly invited by Kian Salimkhani and Tina Wachter to participate in a Bonn workshop on quantum individuality and entanglement, where again I had constructive exchanges with Simon Saunders, Adam Caulton, James Ladyman, Fred Muller as well as Cord Friebe, Dennis Dieks, Andrea Lubberdink and Jeremy Butterfield. In 2019 I was a visiting fellow at the Center for Philosophy of Science in Pittsburgh, where I presented and discussed my work. I am particularly grateful to John Norton, Naftali Weinberger and Chungyoung Lee for their incisive comments. I should also thank audiences in Lausanne, Pasadena, Chicago, Helsinki, Leeds, Barcelona, Warsaw, Cracow, Lublin and Santiago de Chile, where I gave presentations on various topics related to this book. To Steven French I extend my thanks for encouraging me to submit my work to the series *New Directions in Philosophy of Science*. Last but not least I would like to thank Ewa Bigaj for her linguistic corrections to the manuscript.

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# 1

# Introduction

In a nutshell, this book is about whether fundamental objects described by quantum mechanics can be distinguished from one another with the help of their physical properties. But why should we be even remotely interested in answering such a trifle question, let alone devote an entire book to its anatomization? And isn't it clear as day that we can and do differentiate quantum particles using various physical attributes? Surely, quantum mechanics breaks away from many firm beliefs about reality that we hold as self-evident. In classical physics bodies are assumed to be impenetrable, and thus each particle possesses its unique trajectory that differentiates it from any other particle. Quantum mechanics disposes with the idea of a well-defined spatial location, replacing it with a "smeared" presence encoded in a spatially extended wave function. Consequently, two quantum objects may temporarily share the same extended location; they may pass through each other like ghosts, or seemingly disappear and then reappear at the end of their interaction. But it is still perfectly possible for these quantum ghosts to be separated by huge spatial intervals, with virtually no overlap of their respective wave functions. Doesn't this prove that two electrons, photons or Higgs bosons can be distinguished even in a world governed by the bizarre quantum laws?



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Well, not so fast. Quantum theory has a special way of characterizing states of ensembles consisting of many particles that belong to the same type. Given that electrons (photons, neutrinos, etc.) do not differ from each other with respect to their uniquely identifying features, such as mass or electric charge, the quantum theory of many particles imposes the requirement of *permutation invariance* on the states these particles can jointly occupy. That is, the joint state of same-type particles should remain unchanged under any arbitrary permutation of these particles (this is the essence of what is typically referred to as the symmetrization postulate). And, on the face of it, it looks like attributing distinct properties to separate particles cannot be reconciled with the permutation symmetry of their joint states. Thus enter the Indiscernibility Thesis which proclaims that particles of the same type are indistinguishable with respect to *all* their properties. This means specifically that if you take, for example, two electrons, then whatever observable characterizing one of them you wish to consider (whether it is position, momentum, energy or spin), the expectation value of this observable should be the same for both electrons. The venerated Leibnizian Principle of the Identity of Indiscernibles seems to be universally violated in the quantum world.

Philosophers get all fired up at the prospect of the complete indiscernibility of quantum particles. Discernibility seems to be a condition sine qua non for individuating objects, selecting and naming them, making reference to one and not the other. Indiscernible entities, lacking the important feature of individuality, are objects only in the thinnest, Quinean sense of the word. A group of indiscernibles forms a whole that is often referred to as an *aggregate* rather than a *collection*. Some philosophers insist that for groups of indiscernible entities it is only possible to count them, but not to order them. Aggregates have a mere *cardinality* but no *ordinality*. For that reason perhaps the term "electron" should properly function as a mass term, referring to the whole electron mass of the universe. Alternatively, the individuality of the indiscernible quantum particles may be rescued by introducing non-qualitative *prinicipia individuationis*, such as *haecceities* (properties of being a particular object). From these remarks we can see that the issue of quantum (in)discernibility can acquire a strong metaphysical flavor. Now it looks more plausible that the question of whether quantum objects can be differentiated by properties could merit thorough philosophical scrutiny.

Characteristically, the majority of working physicists remain rather unimpressed by the metaphysical ramifications of the symmetrization postulate.[1] True, they admit that the postulate is very important from an empirical and practical point of view, especially when we are interested in describing the behavior of large collections of particles, for the statistical predictions regarding indiscernible quantum particles differ significantly from analogous predictions concerning classical, discernible particles. Generally speaking, quantum particles are divided into two categories—bosons and fermions—depending on the mathematical transformation their joint states undergo under the permutations of objects. Statistical behavior of bosons differs from that of classical particles in that the probability of finding a group of bosons occupying same states is higher than in the classical case (as if bosons "attracted" each other slightly). On the other hand, same-type fermions never occupy the same state (they "repel" each other strongly).[2]

However, when dealing with small numbers of same-type fermions or bosons, physicists often ignore the symmetrization postulate and write their states in a non-permutation-invariant form, as if they characterized distinguishable particles. Here is an interesting quote on that issue from a well-known textbook on quantum mechanics (Cohen-Tannoudji et al. 1978, p. 1406):

> If application of the symmetrization postulate were always indispensable, it would be impossible to study the properties of a system containing a restricted number of particles, because it would be necessary to take into account all the particles in the universe which are identical to those in the system. We shall see […] that this is not the case. In fact, under certain special conditions, identical particles behave as if they were actually

[1] Of course, there are some notable exceptions—famously including Erwin Schrödinger and Henry Margenau—but these are physicists who are already influenced by a philosophical way of thinking. See, for example, Schrödinger (1952); Margenau (1944, 1950).

[2] A careful reader may notice a strange inconsistency between this general characteristic of fermionic behavior (whose more precise expression bears the name of Pauli's exclusion principle) and the Indiscernibility Thesis mentioned earlier. Much of the subsequent discussions in this book will try to explain away this inconsistency.

different, and it is not necessary to take the symmetrization postulate into account in order to obtain correct physical predictions.

The quoted fragment is baffling. How can indiscernible objects "behave" as if they were discernible, even under "certain special conditions"? If the symmetrization postulate is a universal, exceptionless law of nature, and if its validity implies that whatever measurable property is possessed by one component of a system of "identical" particles, it is also possessed by any other component, then it becomes utterly mysterious how such particles could ever be treated "as if they were actually different".

It is difficult to make sense of a situation in which entirely indistinguishable objects behave as if they were distinguishable, unless we make some crucial changes in the way we identify these objects. And it turns out that this may be the key to understanding the above-mentioned quote: perhaps what justifies the suspension of the symmetrization postulate is an alternative method of "carving up" the totality of the composite system into smaller components, so that these new components not only behave "as if" they were distinguishable, but really *are*. Following this lead, in this book I will argue that there are actually two rival methods of individuating quantum particles that compose larger systems. For lack of a better term, I will refer to one of these methods as "orthodoxy", and the other as "heterodoxy" (I shamelessly borrow this terminology from Adam Caulton). The "orthodox" approach to individuality treats certain parts of the mathematical formalism, namely indices that are attached to the factors in the tensor products of Hilbert spaces, as referring to the components of the composite system under consideration. For the unorthodox approach, on the other hand, the task of individuating the components of a composite system is performed not by unphysical labels used to identify identical copies of a single-particle Hilbert space, but rather by physically meaningful symmetric operators of a certain kind. When this new individuating procedure is executed, it turns out that fermions occupying antisymmetric states are *always* discernible from each other by some properties, while bosons are not guaranteed to be discernible in that way. However, it is definitely possible to put two (or more) bosons in a state in which they will differ from each other with respect to their possessed properties.

This book is devoted to a systematic analysis (both formal and philosophical) and comparison of the two competing methods of individuating particles in quantum mechanics. In Chap. 2 I will lay out the formal and conceptual fundamentals of the orthodox approach to this problem, together with the ensuing Indiscernibility Thesis regarding quantum particles of the same type. Chapter 3 delves deeper into the problem of the justification of the symmetrization postulate. It also contains a brief formal description of the types of symmetry other than bosonic and fermionic, known collectively as *parastatistics*. Chapter 4 makes a small detour in order to discuss the logic and metaphysics of distinguishability. We identify and formalize three types of discernibility using the standard model-theoretical framework, and we connect them with the issue of symmetry. This chapter also includes a critical analysis of the weak discernibility program, which hopes to provide some semblance of quantum objecthood in the light of the apparent breakage of absolute discernibility. In Chap. 5 we will make first steps towards a non-standard conception of how to individuate quantum particles. The starting point will be an analysis of the physical meaning of certain symmetric projection operators acting in an appropriate tensor-product Hilbert space. Arguments will be presented to the effect that particular symmetric combinations of projectors should be interpreted as representing situations in which the components of a composite quantum system are discernible by their measurable properties. As it turns out, there are some serious objections to this interpretation, involving the concept of quantum measurement. The way to repel these objections will be to properly introduce spatial degrees of freedom into our general description of measurement processes.

Chapter 6, which is probably the most technical of all chapters in the book, contains proofs of several facts regarding the absolute discernibility of fermions and bosons under the unorthodox approach, and regarding a new concept of entanglement applicable to the case of same-type particles. One section of this chapter also addresses the more general question of whether it is logically possible to formulate symmetric sentences stating facts of absolute discernibility. The main goal of Chap. 7 is to present various pros and cons with respect to the two approaches to quantum individuation developed earlier in the book. In particular, we will discuss a serious problem affecting the unorthodox approach which is caused by the non-uniqueness (ambiguity) of individuation by symmetric operators. Chapter 8 adds the issue of diachronic and counterfactual identity of quantum objects to the discussion. In the closing section of this chapter, we will stress the non-classical character of the metaphysics emerging from the unorthodox approach to individuation. Even though the heterodoxy restores the validity of the Principle of the Identity of Indiscernibles in the majority of cases involving same-type particles, this does not lead to the rehabilitation of the classical picture of the quantum world. Particles of the same type may be individuated *synchronically*, but they are not full-blown classical individuals, since they typically fail to keep their identities over time and across possible scenarios, and their synchronic individualities suffer from unavoidable ambiguity.

I should mention here some presuppositions and limitations of this book. First off, for the most part I restrict myself to standard, non-relativistic quantum mechanics. I make some inroads into quantum field theory in Chap. 7, where I briefly present the Fock space formalism which enables us to talk about variable numbers of particles. However, I do not venture to discuss the ontological problem of identity and individuality in full-blown interacting QFT. One obvious reason for that reservation is that in spite of the tremendous effort of numerous philosophical commentators, the jury is still out whether QFT should be based on the fundamental ontology of particles or fields.[3] Another important limitation of the book is that it tries to stay clear of the notorious measurement problem and the ensuing question of the proper interpretation of quantum mechanics. The idea is to present the problem of identity and individuality within the confines of so-called textbook quantum mechanics, which is hopefully neutral with respect to the fundamental differences between various interpretations of QM.[4] Where reference to measurements becomes unavoidable, such as at the end of Chap. 5, I try to use a "unitary" account of measurement interactions as an alternative to the standard but philosophically controversial "collapse" account. Finally, I should warn the readers that they won't find a complete

[3] For an overview of this problem, see Kuhlmann (2020).

[4] Of course I am perfectly aware that adopting some specific interpretive variants of QM, such as Bohm's theory, may change radically our perspective on the issue of the discernibility of quantum objects. See, for example, Brown et al. (1999).

metaphysical conception of quantum objects in the book. While I consider myself a philosopher/metaphysician, the aim of this book is primarily to gather together some rather elementary facts, provable in the quantum-mechanical formalism, which appear salient with respect to the problem of identity, and raise the question of their proper physical and philosophical interpretation. Thus it is clearly a preparatory work. I do hope that someone may find this work useful while developing some more comprehensive proposals of quantum ontology.

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# 2

# Indiscernibility of Quantum Particles: A Road to Orthodoxy

It is impossible to constructively engage in a metaphysical discussion inspired by quantum mechanics without possessing rudimentary knowledge of the mathematical formalism of this theory. The first chapter of this book is meant primarily as a brief overview of the basics of the quantum formalism necessary to understand the debates on the problem of identity and individuality (further definitions and proofs are provided in the Appendix, albeit they are limited to a bare minimum). Obviously, this hasty presentation cannot be treated as a substitute for a thorough, step-by-step introduction to quantum theory, for which I can only refer the reader to one of many excellent textbooks on the market.[1] In what follows we will merely touch upon the standard mathematical representation of states and properties of composite systems, that is, physical systems consisting of many smaller components, with its primary tool of the

[1] Cohen-Tannoudji et al. (1978), Peres (2003) and Sakurai and Napolitano (2011) are recommended comprehensive physical textbooks on quantum mechanics. An elegant introduction to the quantum-mechanical formalism with an eye on the foundational issues is (Griffiths 2002), while the extensive compendium (Greenberger et al. 2009) offers an encyclopedic collection of short articles covering all fundamental concepts and results of quantum theory. My personal favorite among mathematically rigorous yet accessible introductions to philosophical issues in quantum mechanics is Hughes (1989).



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tensor product of vector spaces. We will see that it is not at all trivial to extend the single-system formalism of states and measurable properties to represent states and properties of composite systems. The key question here is how to express a given property of a particular quantum object in a framework which treats this object as part of a broader system consisting of many entities. The standard quantum-mechanical way to do this turns out to contain certain loopholes which will later prove to be crucial in the controversy regarding the proper method of individuating quantum particles of the same type.

Using the formal notion of a permutation operator, in the next step we will articulate the Indistinguishability Postulate which imposes an important restriction on joint states and properties of so-called indistinguishable particles (particles belonging to the same type). From this assumption a metaphysical consequence in the form of the Indiscernibility Thesis is usually derived. We will analyze typical arguments in favor of this claim, noting the indispensable role of the tacit assumption referred to as Factorism in these derivations. Finally, we will briefly discuss the metaphysical role of discernibility by properties in clarifying the notion of individuality, as well as its relation to numerical diversity.

## 2.1 Composite Systems and Their States

Let us start with a sketch of how the standard quantum-mechanical formalism describes states of composite systems, that is, systems consisting of smaller components, each of which can possess its own states. The mathematical representations of the states of any quantum system form a structure known as a *vector space*. More specifically, vector spaces used in quantum mechanics are spaces over the field of complex numbers, which bear the name of Hilbert spaces (see Appendix for a complete definition). Suppose that we have two quantum systems (e.g. two particles—an electron and a proton), whose states are represented in their respective individual Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. In order to describe states of the composite system consisting of the two particles (in our example this complex system can be a hydrogen atom, which is composed of one electron and one proton), we have to build a new Hilbert space. The standard mathematical recipe for doing this employs the concept of a *tensor product* of Hilbert spaces. The tensor product of spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, symbolized as $\mathscr{H}_1 \otimes \mathscr{H}_2$, is a vector space consisting of all pairs of vectors from $\mathscr{H}_1$ and $\mathscr{H}_2$ and their linear combinations. More precisely, the definition of $\mathscr{H}_1 \otimes \mathscr{H}_2$ can be given as follows. Let the set of vectors $|e_i^1\rangle$ be an orthonormal basis for $\mathscr{H}_1$, and $\{|e_j^2\rangle\}$ be an orthonormal basis for $\mathscr{H}_2$ (the definition of a basis is given in the Appendix). The tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ is a vector space spanned by the combinations $|e_i^1\rangle \otimes |e_j^2\rangle$, that is, the space of all linear combinations of the form.

$$\sum_{ij} c_{ij} \, |e_i^1\rangle \otimes |e_j^2\rangle, \tag{2.1}$$

where $c_{ij}$ are complex numbers.[2]

The key difference between classical and quantum composite systems lies in the fact that the state spaces of quantum compositions are in a sense much larger than in the classical case. In addition to the factorizable combinations of states of the form $|\varphi\rangle \otimes |\psi\rangle$, where $|\varphi\rangle \in \mathscr{H}_1$, $|\psi\rangle \in \mathscr{H}_2$, space $\mathscr{H}_1 \otimes \mathscr{H}_2$ includes vectors that cannot be presented as a product of two vectors. These non-factorizable combinations are also known as *entangled*. An example of an entangled state can be the combination $|e_1^1\rangle \otimes |e_1^2\rangle + |e_2^1\rangle \otimes |e_2^2\rangle$, where $|e_i^1\rangle$, $|e_j^2\rangle$ are, as before, some basis vectors of spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. It is an elementary fact from linear algebra that the above vector cannot be represented in the factorized form $|\varphi\rangle \otimes |\psi\rangle$.[3] Thus if the composite system occupies such a non-factorizable, entangled state, its components cannot be assigned any individual states in the form of vectors in their respective Hilbert spaces (states given in the form of vectors are typically referred to as *pure*). This does not imply that the

[2] To avoid clutter, sometimes the symbol $\otimes$ representing the operation of tensor product is omitted. That is, instead of writing $|\varphi\rangle \otimes |\psi\rangle$ we can simply write $|\varphi\rangle|\psi\rangle$.

[3] Given that each vector $|\varphi\rangle$, $|\psi\rangle$ has a unique decomposition in respective bases $\{|e_i^1\rangle\}$ and $\{|e_i^2\rangle\}$ (see Appendix for details), it follows that if these decompositions contain vectors $|e_1^1\rangle$, $|e_2^1\rangle$ and $|e_1^2\rangle$, $|e_2^2\rangle$, the product $|\varphi\rangle \otimes |\psi\rangle$ has to contain "cross terms" $|e_1^1\rangle \otimes |e_2^2\rangle$ and $|e_2^1\rangle \otimes |e_1^2\rangle$ in its decomposition and hence cannot be written as $|e_1^1\rangle \otimes |e_1^2\rangle + |e_2^1\rangle \otimes |e_2^2\rangle$.

components of the system cannot be characterized by *any* quantum-mechanical states whatsoever, but in the case of entanglement, these states will belong to a different category of *mixed states*, represented not by vectors but by density operators (see Appendix for further explanations). Regardless of these technical details, it is important to know that entangled states cannot be fully reduced to the "conjunction" of the (mixed) states of the individual components. That is, while knowing that one particle is in a pure state $|\varphi\rangle$ while the other particle occupies pure state $|\psi\rangle$ is sufficient to deduce that the joint state will be the product $|\varphi\rangle \otimes |\psi\rangle$, in the case of entangled states, determining the (mixed) states of the components does not uniquely determine the state of the entire system.

How do the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ relate to each other? It is an elementary mathematical fact that all Hilbert spaces over complex fields of the same dimensionality are isomorphic. The dimensionality of a state space in quantum mechanics is determined by the number of distinct values that can be possessed by a quantity (or quantities) taken to define states of systems. Thus spaces of states given in terms of position or momentum will be typically infinitely dimensional. On the other hand, spaces of the components of angular momentum (e.g. spin), which are discrete quantities, may have a finite number of dimensions. If we limit ourselves to either position/momentum spaces (spaces of wave functions), or to the cases of particles with the same total spin (e.g. spin $\frac{1}{2}$), we may assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ have the same number of dimensions and therefore are isomorphic. Consequently, we may stipulate that both $\mathcal{H}_1$ and $\mathcal{H}_2$ include "the same" vectors, and thus it is possible to drop the superscripts in the basis vectors written above. Henceforth, we will assume that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two copies of the same Hilbert space spanned by some orthonormal vectors $\{|e_i\rangle\}$. The only difference between them is that they are labeled with different numbers 1 and 2 (this labeling is often omitted for the sake of brevity, in which case the labels are assumed to be determined by the place a given vector occupies in the tensor product). However, this does not mean that the mere formal difference in labels does not reflect a deeper physical distinction. For instance, if we consider a system of two particles belonging to different kinds (such as a proton and an electron), the labels used to differentiate identical copies of one

Hilbert space are underlain by different state-independent properties defining appropriate kinds (e.g. rest mass).

Once we equate basis vectors spanning spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, it is possible to formally introduce the operation of permutation. The permutation $P_{12}$ is an operation that acts on the basis vectors of $\mathcal{H}_1 \otimes \mathcal{H}_2$ in the following way:

$$P_{12}(|e_i\rangle \otimes |e_j\rangle) = |e_j\rangle \otimes |e_i\rangle. \tag{2.2}$$

By linear extension, this operation applies to any vector in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Clearly, $P_{12}$ applied to any product state $|\varphi\rangle \otimes |\psi\rangle$ yields its "reverse" $|\psi\rangle \otimes |\varphi\rangle$. The most natural physical interpretation of the permutation operation is that it results in the situation in which particles swap their states, that is, particle 1 now occupies the state initially occupied by particle 2, and vice versa. However, we will have to look carefully at this interpretation later when we discuss the case of so-called indistinguishable particles (see Sect. 3.1 in Chap. 3).

## 2.2 Properties of Composite Systems

Measurable properties of a quantum system are represented by a particular type of linear operators, known as Hermitian (or self-adjoint), acting in the state space for this system. If the system is in a pure state $|\varphi\rangle$, the expectation value for the observable corresponding to a Hermitian operator $A$ is given by the inner product $\langle\varphi|A|\varphi\rangle$.[4] Suppose now that we know the state $|\psi(1,2)\rangle$ of an entire composite system consisting of two subsystems. State $|\psi(1,2)\rangle$ may or may not be factorizable into the product of the states of the components, so we can't assume that the pure states of these components taken separately are well defined. How, in that case, can we calculate the expectation value for an observable limited to one particle? The answer is given in terms of the tensor products of linear operators. Generally, if $A$ is a linear operator in space $\mathcal{H}_1$, and $B$ is a linear

[4] See a rationale behind this definition in Hughes (1989, pp. 70–71).

operator in $\mathscr{H}_2$, we can define a new operator $A \otimes B$ acting in the tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$ as follows:

$$A \otimes B(|e_i\rangle \otimes |e_j\rangle) = A|e_i\rangle \otimes B|e_j\rangle. \tag{2.3}$$

Any linear combination of the products of the form $A_k \otimes B_l$ is also a linear operator acting in $\mathscr{H}_1 \otimes \mathscr{H}_2$.

Consider now the product $A \otimes I$, where $I$ is the identity operator, that is, such that $I|\varphi\rangle = |\varphi\rangle$ for all vectors $|\varphi\rangle$. We can calculate its expectation value in the state $|e_i\rangle \otimes |e_j\rangle$ as follows[5]:

$$\langle e_i, e_j | A \otimes I | e_i, e_j\rangle = \langle e_i | A | e_i\rangle\langle e_j | e_j\rangle = \langle e_i | A | e_i\rangle, \text{ since } \langle e_j | e_j\rangle = 1. \tag{2.4}$$

Thus the expectation value of $A \otimes I$ in $|e_i\rangle \otimes |e_j\rangle$ turns out to be identical to the expectation value of $A$ in state $|e_i\rangle$. This gives us a reason to suspect that $A \otimes I$ in the product space $\mathscr{H}_1 \otimes \mathscr{H}_2$ represents the very same property as $A$ in $\mathscr{H}_1$. As it is sometimes put, $A \otimes I$ represents an observable of the composite system which corresponds to the measurement procedure consisting of measuring $A$ on the first component of the system and leaving the second component alone (this "leaving alone" is represented by the identity operator which does nothing to any state).

Another possible argument for the identification of the physical interpretations of operators $A \in \mathscr{H}_1$ and $A \otimes I \in \mathscr{H}_1 \otimes \mathscr{H}_2$ uses the standard notions of eigenstates and eigenvalues. The physical meaning of a given Hermitian operator is typically expressed by reference to mathematical eigenequations:

$$A|\lambda_a\rangle = a|\lambda_a\rangle, \tag{2.5}$$

where $a$ is a number. Number $a$, known as an eigenvalue of $A$ (which for Hermitian operators is always a real number—see Appendix), represents a particular value of observable $A$, while $|\lambda_a\rangle$ is a corresponding state, called an eigenstate. According to the interpretational rule called the

[5] The inner product $\langle\varphi\otimes\psi|\lambda\otimes\chi\rangle$ in the tensor product space $\mathscr{H}_1 \otimes \mathscr{H}_2$ is defined as $\langle\varphi|\lambda\rangle\langle\psi|\chi\rangle$ (see Appendix).

eigenstate-eigenvalue link (the e/e link),[6] a system objectively possesses a given value *a* of an observable *A*, iff the system is in an eigenstate $|\lambda_a\rangle$ corresponding to this value. Now, it can be easily shown that if a vector $|\lambda_a\rangle \in \mathscr{H}_1$ is an eigenstate for *A* with a value *a*, any vector of the form $|\lambda_a\rangle \otimes |\varphi\rangle$, where $|\varphi\rangle \in \mathscr{H}_2$, is an eigenvector for $A \otimes I$ with the same value *a*. The implication holds in the opposite direction too, which proves the following equivalence:

(2.6) Particle 1 possesses an objective value *a* of the observable represented by operator *A* iff the system of particles 1 and 2 possesses an objective value *a* of the observable represented by operator $A \otimes I$.

However, we should not forget the fact that operator $A \otimes I$, since it acts in a broader Hilbert space, is applicable in some cases which are not immediately covered by the one-particle operator *A*. That is, we can calculate the expectation value for $A \otimes I$ in entangled two-particle states, for which no pure state of individual particles exists, and thus the standard formula $\langle\varphi|A|\varphi\rangle$ for the expectation value of *A* cannot be applied. This fact may be seen as undermining the perfect physical equivalence between mathematical operators $A \otimes I$ and *A*. True, it may be pointed out that the full equivalence is restored by introducing the concept of a mixed state represented by density operators mentioned earlier, together with a new recipe of how to calculate the expectation values of operators when systems are assigned mixed states (the relevant formula is $\mathrm{Tr}(A\rho)$, where Tr is the trace operation, and ρ – a density matrix, see Appendix). However, in response we may observe that the assignment of a reduced mixed state ρ to an individual component of an entangled system is motivated precisely by the desire to keep the identity of the expectation values of $A \otimes I$ and *A* (the reduced state of a component is defined as the state which attributes to *A* the same expectation value as the expectation value assigned by the total state to $A \otimes I$—see Hughes 1989, pp. 149–150). Hence we can

[6] The term "eigenstate-eigenvalue link" is admittedly misleading, as observed, for example, in Muller and Leegwater (2020, ft. 18), since it suggests a purely mathematical relation rather than an interpretative rule that may or may not be accepted. I noticed this terminological problem in my Bigaj (2006, p. 375 ft. 1). Nevertheless, I will continue using this nomenclature which has become standard in the literature.

hardly use the identity between the expectation value of $A \otimes I$ calculated for the state of the entire system and the expectation value of $A$ for the reduced mixed state, as an argument that $A \otimes I$ and $A$ are physically equivalent in all scenarios. The only argument we can rely on is an "inductive" one: since the expectation values of $A \otimes I$ and $A$ coincide in the special cases when the components possess their own pure states (alternatively, since $A \otimes I$ possesses a given definite value iff $A$ possesses the very same value), we stipulate that the expectation values should also coincide in the remaining cases, and therefore we assume that operators $A \otimes I$ and $A$ represent the same physical quantity. We will see later in the book (Chap. 5) that analogous inductive arguments may be questioned in some other contexts.

In the previous section we have considered an important type of permutation operation, which was defined as a map on the total Hilbert space, transforming vectors into vectors. It turns out that permutations can also be applied to operators acting on Hilbert spaces rather than to vectors. The general method of how to turn a transformation on a given space $\mathscr{H}$ into the corresponding transformation on the set of operators on $\mathscr{H}$ can be presented as follows. Let $P$ be any linear transformation on $\mathscr{H}$ which has an inverse (thus $P$ must be one-to-one). We will seek a transformation corresponding to $P$ that sends any operator $A$ into its counterpart $A'$ while satisfying the following requirement:

$$\text{For any vectors } |\varphi\rangle \text{ and } |\psi\rangle, A'|\varphi\rangle = |\psi\rangle \text{ iff } AP|\varphi\rangle = P|\psi\rangle. \tag{2.7}$$

Expressing this condition informally: the transformed operator $A'$ acts in the original space $\mathscr{H}$ the same way as the original operator $A$ acts in the transformed space $P[\mathscr{H}]$. From this we can easily derive the form of the operator $A'$ in terms of $A$ and $P$. By applying the inverse operation $P^{-1}$ to both sides of the eq. $AP|\varphi\rangle = P|\psi\rangle$, we get $P^{-1}AP|\varphi\rangle = |\psi\rangle$, from which it follows (given that $|\varphi\rangle$ and $|\psi\rangle$ are selected completely arbitrarily) that $A' = P^{-1}AP$ (see Fig. 2.1).

We can now return to the case of permutation transformation. What is the result of applying permutation $P_{12}$ to the operator $A \otimes I$? In order to calculate the outcome of the transformation $P_{12}{}^{-1}(A \otimes I)\, P_{12}$, we have

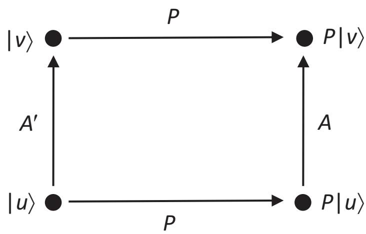


**Fig. 2.1** The action of transformation *P* on operator *A* yields operator *A*′. Note that the action of *A*′ on vector $|u\rangle$ is equivalent to the composition of transformations *P*, *A* and $P^{-1}$. Thus $A' = P^{-1}AP$ (operators are always put in reverse order—the last in sequence is the first to be applied)

to apply this transformed operator to an arbitrary basis vector $|e_i\rangle \otimes |e_j\rangle$, which yields the following:

$$P_{12}^{-1}(A \otimes I)P_{12}(|e_i\rangle \otimes |e_j\rangle) = P_{12}^{-1}(A \otimes I)\ (|e_j\rangle \otimes |e_i\rangle)$$
$$= P_{12}^{-1}(A|e_j\rangle \otimes |e_i\rangle) = |e_i\rangle \otimes A|e_j\rangle. \tag{2.8}$$

Thus, the transformed operator $P_{12}^{-1}(A \otimes I)\, P_{12}$ acts in the same way on the basis vectors as $I \otimes A$, and hence these operators are identical. It should not come as a surprise that applying permutation $P_{12}$ to the operator $A \otimes I$ yields the reversed-order variant $I \otimes A$ which, as we already know, represents the property *A* possessed by the second component of the composite system.

Of particular importance are operators acting on $\mathscr{H}_1 \otimes \mathscr{H}_2$ whose permutation doesn't change anything, that is, such that $P_{12}^{-1}\Omega P_{12} = \Omega$. Operators invariant under permutations in this sense are called "symmetric". A simple example of such an operator is the product $A \otimes A$, whose physical interpretation is obvious: it represents the same observable assigned to both components of the system. A slightly more complex example of a symmetric operator is provided by the following sum of appropriate operators: $A \otimes B + B \otimes A$. Here the issue of a proper physical interpretation is a bit trickier than in the previous example—we will return to it later in Chap. 5. For now, we will only mention that it would be inaccurate to interpret the operator $A \otimes B + B \otimes A$ as a representation of the disjunctive property "either $A$ for particle 1 and $B$ for particle 2 or $B$ for particle 1 and $A$ for particle 2".

## 2.3 Projection Operators

An important category of Hermitian operators are so-called *orthogonal projection operators* (or *projectors*, for short). While projectors can be interpreted as observables analogous to spin, position, momentum and so on, that is, as quantities capable of receiving different values from the admissible range, it is more typical to use them as representations of specific properties of quantum systems that may or may not be possessed in a given state. Formally, projection operators stand in one-to-one correspondence to subspaces of a given Hilbert space $\mathscr{H}$ (including $\mathscr{H}$ itself and the zero-subspace containing only the 0-vector). That is, to every subspace $S$ of $\mathscr{H}$, there corresponds a unique projector $E_S$, and each projector defines a subspace of $\mathscr{H}$ onto which it projects. Speaking loosely, the projector onto a subspace $S$ acts on an arbitrary vector $|\varphi\rangle$ in such a way that it decomposes $|\varphi\rangle$ into the component $|\varphi\rangle_S$ lying in $S$ and the component $|\varphi\rangle_{S'}$ perpendicular (orthogonal) to it, and then it selects $|\varphi\rangle_S$ as the outcome: $E_S|\varphi\rangle = |\varphi\rangle_S$ (see Fig. 2.2). From this loose characterization, it follows immediately that $E_S$ restricted to $S$ is the identity operator and that applying $E_S$ twice to any vector is equivalent to applying it only once (this property is called *idempotence*: $E_S^2 = E_S$).

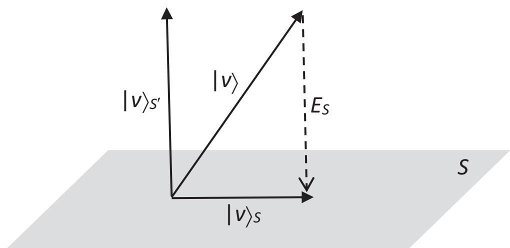


**Fig. 2.2** The orthogonal projection onto subspace *S*

The physical interpretation of a projector $E_S$ is rather straightforward: $E_S$ is an observable with two possible values 1 and 0 that “measures” whether the state of a given system lies within the corresponding subspace $S$.[7] If the state vector $|\varphi\rangle$ belongs to subspace $S$, the observable represented by $E_S$ assumes the eigenvalue 1 (since in that case $E_S|\varphi\rangle = |\varphi\rangle$). On the other hand, if the state vector is orthogonal to $S$, the value assumed by $E_S$ is 0 (projecting an orthogonal vector onto $S$ gives the zero vector). Any other vector (neither in $S$ nor perpendicular to it) is a non-eigenvector of $E_S$, and hence there is some probability that the value will be 1 and some probability that it will be 0. Projectors are quantum equivalents of the characteristic functions of certain sets of values and therefore can be interpreted as representations of the property of possessing one of the set of values associated with a given subspace. A special case of projection operators is one-dimensional projectors, whose corresponding subspace is a ray (one-dimensional subspace). If the ray signifies a state with a particular value of some measurable property (e.g. spin-up in a given spatial direction), the corresponding projector onto this ray can be assumed to represent this specific property (in the sense that the eigenvalue 1 of this projector corresponds to the system’s possession of this property, for instance, spin-up). A standard way to write the projector onto a ray containing a normalized vector $|\varphi\rangle$ is in the form of the so-called dyad $|\varphi\rangle\langle\varphi|$ (see Appendix for an explanation of this notation).

Projection operators acting in separate one-particle Hilbert spaces can be used to create new projectors acting in the tensor products of these spaces. Exactly as in the general case of Hermitian operators, we can consider projectors of the kind $E \otimes I$ and $I \otimes E$, where $E$ acts, respectively, in $\mathscr{H}_1$ or $\mathscr{H}_2$. If $E$ projects onto the subspace $S$ of, let’s say, $\mathscr{H}_1$, the subspace corresponding to $E \otimes I$ will be $S \otimes \mathscr{H}_2$. However, not all combinations of tensor products of one-particle projectors are themselves projectors. Consider, for instance, the following symmetric operator: $E \otimes F + F \otimes E$, where $E$ and $F$ are any projectors in $\mathscr{H}_1$ ($\mathscr{H}_2$). It turns out that this new operator will generally not be idempotent and therefore will not belong to the category of projectors in the tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$.

[7] Alternatively, projectors are interpreted as representing “yes” or “no” questions regarding whether the state of the system lies in a particular subspace.

Only when $E$ and $F$ are orthogonal to each other (i.e. they project onto orthogonal subspaces) is the above combination idempotent. This can be verified by direct calculation:

$$(E \otimes F + F \otimes E)(E \otimes F + F \otimes E) = E^2 \otimes F^2 + F^2 \otimes E^2 + EF \otimes FE + FE \otimes EF. \tag{2.9}$$

Thanks to the orthogonality assumption $EF = FE = 0$, and using the idempotence conditions $E^2 = E$ and $F^2 = F$, we arrive at the required outcome $E \otimes F + F \otimes E$. But if $E$ and $F$ are not orthogonal, this result is not guaranteed.

## 2.4 Systems of "Indistinguishable" Particles

Virtually all current discussions regarding the notions of the identity, individuality and discernibility of quantum particles center around the concept of "indistinguishable" particles. As is well known, fundamental particles of modern physics are categorized into kinds depending on their basic physical properties. The classification of particles used in particle physics is rather complex and the details need not concern us (see, e.g. Griffiths 2008 for a complete categorization). Suffice it to say that current physics distinguishes three broad types of truly elementary particles (i.e. particles with no proper components): leptons, quarks and mediators. Among leptons we classify electrons, muons and tau particles (plus their antiparticles) with three corresponding types of neutrinos. There are six types of quarks distinguishable by their flavors (up, down, strange, charm, bottom, top) and six corresponding antiquarks. Mediators (or mediating particles) carry forces: electromagnetic (photons), strong (gluons) and weak (particles $W^{\pm}$ and $Z^0$). In addition to genuinely elementary particles, there is a garden variety of particles composed of smaller elements (quarks), of which the best known and certainly most ubiquitous are protons and neutrons.

When physicists talk of *indistinguishable*, or *identical* particles, they usually mean particles belonging to the narrowest categories described above: electrons, muons, photons, strange quarks and so on. Their indistinguishability, or identity, does not involve all their properties, but rather a special kind of properties, the so-called *state-independent* ones. These are properties that do not and cannot change over time. For instance, every electron, no matter what state it occupies, is characterized by the same rest mass (0.511 MeV), the same electric charge ($-1.6 \times 10^{-19}$ C) and the same spin ($\frac{1}{2}\hbar$). No electron can lose these properties without ceasing to be an electron.[8] Thus identical, or indistinguishable, particles are those that share *all* of their state-independent properties. This of course does not exclude the possibility that, for instance, two electrons may differ with respect to their state-dependent features: position, momentum, energy, the spin component in a given direction and so on. For that reason, I will try to avoid using the potentially confusing terms "indistinguishable particles" and "identical particles" (if, for purely stylistic reasons, I occasionally revert to this terminology, I'll use scare quotes to indicate the metaphorical character of the terms), replacing them with the slightly more cumbersome phrase "particles of the same type", where type is meant as described above.[9]

Suppose that we are considering a system consisting of particles of the same type (e.g. a group of electrons). How does the fact that these particles share their state-independent properties bear on the way we should describe their joint state? The standard way of approaching this problem

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[8] However, we have to admit that the issue of whether properties such as rest mass are always state independent is a bit tricky. For instance, in the well-known phenomenon of neutrino oscillations, the state of a neutrino is a superposition of states with different rest masses, corresponding to different types of neutrinos (electron neutrino and muon neutrino). Hence the term "mass eigenstates" is introduced, which clearly suggests that mass becomes part of the state description that can change over time (for details, see Griffiths 2008, pp. 390–392). In response to that one may observe that mass is treated as a state-dependent property if we describe the process in terms of an "unspecific" neutrino that may manifest itself as an electron or muon type (in other words what we have here is a superposition of two types of particles). Once we limit ourselves to the states of a specific type of neutrino, mass can no longer vary over time.

[9] The fact that the terms "indistinguishability" and "identity" used in the above-mentioned contexts are most certainly misnomers has been noted by many authors (cf. van Fraassen 1991, p. 376; Butterfield 1993, p. 453). Another potential source of terminological confusion is the practice of referring to state-independent properties as "intrinsic", which is common in physical literature. This unfortunately interferes with the philosophical sense of the term, which roughly means "nonrelational" (see Chap. 4, Sect. 4.1, for a more precise characterization of intrinsic properties). For the rest of the book, I will use the term "intrinsic" in the philosophical sense only.

is through the concept of permutation invariance. Limiting ourselves to the simplest case of two particles of the same type, we may postulate that the states which differ only by the permutation of these particles should not be empirically distinguishable. Switching one electron for another should not create any observable, or measurable, difference in the total state of the system, since all electrons are "alike" (i.e. all electrons possess the same set of state-independent properties). This stands in contrast to the case of particles belonging to different types, such as an electron and a proton. If the electron initially occupies the state "spin-up" in a given direction, and the proton occupies the state "spin-down", then swapping them creates a new physical situation that may be experimentally distinguished from the previous one (we may, for instance, use a mass spectrometer to first select the electron, and then measure its spin, receiving different outcomes in two different scenarios before and after permutation).

The empirical indistinguishability of permuted states can be expressed in the form of the following principle, known as the Indistinguishability Postulate (IP)[10]:

(IP) Let $|\varphi\rangle$ be any available state of a system of $N$ particles of the same type, and $P$ – a permutation of the set of these particles. Then $\langle P\varphi|\Omega|P\varphi\rangle = \langle\varphi|\Omega|\varphi\rangle$ for any physically meaningful Hermitian operator $\Omega$.

Condition (IP) stipulates that the expectation values for physically meaningful operators be the same for all permuted states. This ensures that the permuted states will be indistinguishable by means of experimental procedures. There are two general ways to satisfy the equation in (IP). We can interpret it as a condition imposed on the states available to systems of particles of the same type, or as a condition on the set of admissible

[10] This terminology is used, for example, in Saunders (2009), while Bas van Fraassen calls IP "Permutation Invariance" (van Fraassen 1991, p. 382). On the other hand, Nick Huggett and Thomas Imbo in Huggett and Imbo (2009) use the term "Indistinguishability Postulate" slightly ambiguously—once as equivalent to our IP and once as the postulate limiting the set of observables to the symmetric ones. It may be true that IP, as defined above, is equivalent to the assumption of the symmetry of observables, but still conceptually these are two distinct postulates.

observables (cf. Messiah and Greenberg 1964). The second method of making (IP) true follows from a simple transformation of the formal condition of the permutation-invariance of expectation values (the transformation is based on the fact that permutation operators are *unitary*—see Appendix for an explanation):

$$\langle P\varphi \mid \Omega \mid P\varphi \rangle = \langle \varphi \mid P^{-1}\Omega P \mid \varphi \rangle = \langle \varphi \mid \Omega \mid \varphi \rangle. \tag{2.10}$$

The last two terms are guaranteed to be identical for all states $|\varphi\rangle$, if only the identity $P^{-1}\Omega P = \Omega$ holds. We immediately recognize this equality as the condition that observables be symmetric. However, it is much more common to interpret IP as applying not to observables (at least not directly) but to states.

When $N = 2$, there are two simple ways to make IP true by limiting the set of available states of two same-type particles. One way is to assume that permutation $P_{12}$ does not change the state of the system, that is, for all $|\varphi\rangle$, $P_{12}|\varphi\rangle = |\varphi\rangle$. An alternative option is to change the sign of the permuted state: $P_{12}|\varphi\rangle = -|\varphi\rangle$. In both cases the expectation value $\langle P_{12}\varphi|\Omega|P_{12}\varphi\rangle$ is guaranteed to be identical to $\langle\varphi|\Omega|\varphi\rangle$ *for all* operators $\Omega$. Vectors that remain the same under the permutation of two particles are called *symmetric*, while vectors that change their sign are referred to as *antisymmetric*. The definitions of symmetric and antisymmetric states can be easily extended for the cases when $N > 2$. Symmetric states of $N$ particles are such that any permutation of the set of particles leaves them unchanged. On the other hand, the case of antisymmetric states is a bit more complicated. We start with the assumption that an antisymmetric state will change its sign under the permutation of any two particles (the permutation swapping two objects is also known as a *transposition*). However, when we apply an even number of transpositions to a particular antisymmetric state, the result will be the same state. Thus, for antisymmetric states, odd permutations (i.e. permutations decomposable into an odd number of transpositions) change the sign of the state, while even permutations (consisting of an even number of transpositions) do not alter the initial state.

## 2.5 The Symmetrization Postulate

Now we are ready to formulate the thesis which has become the cornerstone of the modern debates on the metaphysics of quantum objects: the Symmetrization Postulate (SP).

(SP) For any system of particles of the same type, its states are either exclusively symmetric, or exclusively antisymmetric.

The Symmetrization Postulate effectively divides up all particles into two categories: those that form groups jointly described by symmetric states and those whose systems are described by antisymmetric states (and for that reason it is referred to as Dichotomy by van Fraassen, 1991, p. 383). Famously, particles of the first category are known as *bosons*, while the "antisymmetric" particles are referred to as *fermions*. While other types of symmetry are mathematically possible (and we will discuss them briefly in Chap. 3), so far there is no compelling evidence that particles other than bosons or fermions exist in nature.

From a formal point of view, SP amounts to the restriction of the initial $N$-fold tensor product of one-particle Hilbert spaces to appropriate subspaces (sections) containing only symmetric, or only antisymmetric, vectors. In the case when $N = 2$, the entire space $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be decomposed into two disjoint subspaces: the subspace $\mathcal{A}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of antisymmetric states and the subspace $\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of symmetric states. Subspace $\mathcal{A}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is spanned by antisymmetric vectors of the form $\frac{1}{\sqrt{2}}(|e_i\rangle|e_j\rangle - |e_j\rangle|e_i\rangle)$, where $i \neq j$, whereas the basis for the symmetric subspace $\mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ can be given in the form of vectors $\frac{1}{\sqrt{2}}(|e_i\rangle|e_j\rangle + |e_j\rangle|e_i\rangle)$ plus symmetric products $|e_i\rangle|e_i\rangle$. Note that all these vectors are orthogonal to each other due to the orthogonality relations $\langle e_i|e_j\rangle = \delta_{ij}$, where $\delta_{ij}$ – Kronecker's delta.[11]

[11] It can be easily verified that if the dimensionality of the one-particle Hilbert space $\mathcal{H}$ equals $n$ (and thus the tensor product $\mathcal{H} \otimes \mathcal{H}$ has $n^2$ dimensions), then the dimensionality of the antisymmetric subspace $\mathcal{A}(\mathcal{H} \otimes \mathcal{H})$ will be $\sum_{i=1}^{n-1} i$, while the dimensionality of $\mathcal{S}(\mathcal{H} \otimes \mathcal{H})$ equals $\sum_{i=1}^{n} i$. The

Adopting SP ensures that the Indistinguishability Postulate will be true regardless of any restrictions on the admissible observables. On the other hand, limiting the set of observables to those represented by symmetric Hermitian operators has the same desired consequence even without accepting SP. Thus it seems that there are two independent ways to satisfy IP. However, these ways are in fact not entirely independent. It turns out that the requirement of symmetry for observables follows from the Symmetrization Postulate. This is so because the symmetric and antisymmetric subspaces of the tensor product are not invariant under the action of non-symmetric operators. In other words, it is possible to transform a symmetric/antisymmetric vector into one that is neither by acting upon it with a non-symmetric operator. This can be proven as follows (as before, we limit ourselves to the case of $N = 2$). Let $|\varphi_S\rangle$ be an arbitrary symmetric vector (i.e. such that $P_{12}|\varphi_S\rangle = |\varphi_S\rangle$), and let $\Omega$ be a non-symmetric operator in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Given the assumption of the symmetry of state $|\varphi_S\rangle$, we have that $\Omega|\varphi_S\rangle = \Omega P_{12}|\varphi_S\rangle$. If vector $\Omega P_{12}|\varphi_S\rangle$ was guaranteed to be symmetric, this would imply that $P_{12}\Omega P_{12}|\varphi_S\rangle = \Omega P_{12}|\varphi_S\rangle$, which entails that $\Omega$ is a symmetric operator when limited to the symmetric subspace (taking into account that $P_{12}^{-1} = P_{12}$). An analogous argument can be produced for the case of antisymmetric states, which shows, given that in the case when $N = 2$ the entire tensor product space is spanned by the symmetric and antisymmetric sections, that $\Omega$ is a symmetric operator on the whole space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Since this contradicts our assumption, $\Omega$ has to take us outside either the symmetric or antisymmetric subspaces.

Why is the formal requirement that physically meaningful operators should not take us outside the space of available states so important? According to the spectral decomposition theorem, every Hermitian operator in a finitely dimensional vector space can be presented as a linear combination of mutually orthogonal projectors $E_i$ (see Hughes 1989, p. 50):

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sum of these two expressions is $n^2$. In particular, if $\mathcal{H}$ is two-dimensional, the antisymmetric subspace will be one-dimensional, while the symmetric subspace will have three dimensions. It is important to keep in mind that in the case of three or more particles ($N > 2$), the antisymmetric and symmetric subspaces do not span the entire tensor product space (see Sect. 3.3 for more details on that).

$$A = \sum_{i=1}^{k} a_i E_i. \tag{2.11}$$

The standard interpretation of this formula is that $a_i$ represents a possible value of observable $A$, while $E_i$ projects onto the corresponding eigenspace, that is, the space consisting of states for which observable $A$ is well defined and possesses value $a_i$. If an operator $A$, when applied to a vector $|\varphi\rangle$ from a subspace $V$, produces a vector $A|\varphi\rangle$ lying outside $V$, this means that there must be a projector in $A$'s spectral decomposition that projects onto a space which is neither a subspace of $V$ nor orthogonal to $V$. But this, in turn, means that some eigenstates of $A$ (states with well-defined values of $A$) are neither in $V$ nor orthogonal to $V$. Thus if a particular system occupies a state described by a vector lying in $V$, there is a non-zero probability that a measurement of $A$ will put the system in a state outside of $V$ (by the standard projection postulate). But SP precludes the possibility that a group of same-type fermions (or bosons) could ever occupy a state that is not antisymmetric (or symmetric). Hence no non-symmetric operators should be allowed to represent physically meaningful observables.

We have a curious situation now. Typically, the Symmetrization Postulate is argued for by reference to the Indistinguishability Postulate: the argument is that SP makes IP true, and this gives us a reason to adopt SP as a way to ensure the permutation-invariance of expectation values.[12] But now we know that SP necessitates the symmetry of admissible observables, and we also know that the condition that observables be symmetric is by itself sufficient to make IP true, regardless of whether we impose any additional restrictions on the available states of same-type particles. So, what additional reasons can we have for adopting SP? Surely, it would be much more cost-effective in terms of the number of extra assumptions to simply accept the symmetry postulate with respect to observables and

[12] There are some arguments for SP in the literature that seem to be independent from IP and yet under closer scrutiny turn out to be based on some unwarranted premises. For instance, van Fraassen in his van Fraassen (1991, pp. 389–392) analyzes a simple proof of SP due to Blokhintsev (1964, p. 399ff), which is based on the assumption that all operators on the tensor product are admissible. It is no surprise that if we do not place any restriction on available observables, the only way to satisfy IP is via the superselection rule in the form of SP applied to the available states.

forgo a similar postulate with respect to states. That is, unless we can give independent reasons for holding on to SP. We will return to the problem of independent justification for SP in Chap. 3. For now, following the standard approach, we will continue to accept SP as an extra rule governing the behavior of systems of same-type particles.[13]

## 2.6 The Indiscernibility Thesis

As we have pointed out, the "indistinguishability" of particles of the same type is limited to their state-independent properties. That is, two electrons possess the same rest mass and electric charge, but in principle may differ wildly with respect to their state-dependent properties, such as energy, position, spin components and so forth. However, this last statement has been challenged in what is known as the Indiscernibility Thesis. It has become part of orthodoxy in the philosophical foundations of quantum mechanics to argue that the Symmetrization Postulate implies that quantum particles of the same type possess the exact same physical properties and therefore cannot be discerned by *any* physical means. In this section we will discuss typical arguments in favor of this claim, and we will find them wanting.

The modern standards for an approach to the problem of the indiscernibility of same-type quantum particles have been set by Steven French and Michael Redhead (French and Redhead 1988). They start their discussion with formulating the Indistinguishability Postulate and then observing, as we did, that there are two ways of satisfying IP. French and Redhead's main goal is to argue that given IP, particles of the same type

[13] It should be pointed out that the symmetrization of admissible observables (operators) actually yields a principle very similar to SP, at least in the case of systems of two particles. If we limit observables (including Hamiltonians) to the symmetric ones, then it follows that no physical process (whether a Schrödinger evolution governed by an appropriate Hamiltonian or a measurement-induced collapse) can get us from a symmetric (antisymmetric) state to a non-symmetric (non-antisymmetric) one. This is a simple consequence of the fact that the symmetric/antisymmetric sectors are invariant under the action of symmetric observables. However, this does not mean that a certain type of particles *must* occupy a given type of state; only that once they start out in a state of a given symmetry type, they can never leave the particular section of states of this type. But restricting ourselves to symmetric operators does not exclude the possibility that a group of fermions could from the outset occupy a state that is not antisymmetric. This may be summarized by saying that SP limits the *availability* of the states of same-type particles, while the symmetrization of observables merely limits their *accessibility* (see French and Redhead 1988, p. 239).

must possess the same state-dependent quantum properties. In spite of the fact that IP can be interpreted as placing a restriction on observables only, they nevertheless use in their argument the assumption that fermionic and bosonic states must be antisymmetric/symmetric. The key premise of their argument is the assumption that monadic (i.e. non-relational, or intrinsic) properties of a particular component of a system of same-type particles are exhausted in statements regarding the probabilities of obtaining particular outcomes of measurements for each particle. The way they formalize these probabilistic properties is with the help of the tensor products of observables of the form $A \otimes B$. They first calculate, using the standard Born rule, the probability that a joint measurement of observable $A$ on particle 1 and observable $B$ on particle 2 will yield particular outcomes ($a$, $b$). The appropriate formula for this probability is the square of the inner product $|\langle\lambda_a \otimes \chi_b|\varphi\rangle|^2$, where $|\lambda_a\rangle$ and $|\chi_b\rangle$ are eigenvectors of $A$ and $B$, respectively, corresponding to values $a$ and $b$, and $|\varphi\rangle$ is the state of the two-particle system. Using the assumption that $|\varphi\rangle$ is antisymmetric or symmetric, they calculate the probabilities of revealing a given value on particle 1 and on particle 2 given that $A = B$, and they find these probabilities identical, which supports the claim that two fermions (or bosons) of the same type are never discerned by their properties.

French and Redhead's calculations are slightly complicated due to the necessity of going through the procedure of summing the probabilities of one outcome over all possible outcomes on the other particle. However, this can be significantly simplified by resorting to expectation values rather than probabilities (see Huggett 2003; Dieks and Versteegh 2008).[14] Let $A_1 = A \otimes I$ and $A_2 = I \otimes A$ represent observables associated, respectively, with the first and the second particles. Then, given the assumption that either $P_{12}|\varphi\rangle = |\varphi\rangle$ (for bosons), or $P_{12}|\varphi\rangle = -|\varphi\rangle$ (for fermions), we arrive at the following sequence of equations:

[14] In spite of appearances, there is no loss of generality in moving from probabilities to expectation values. This is so, because the probability of obtaining any outcome of any observable can be recovered as the expectation value of the projector onto the subspace corresponding to this outcome (cf. Hughes 1989, p. 71). Thus, fixing the expectation values of all projectors in a given Hilbert space automatically fixes all the probabilities of outcomes for any observable. This proves once again how flexible a tool projection operators are.

$$\langle \varphi | A_1 | \varphi \rangle = \langle \varphi | A \otimes I | \varphi \rangle = \langle P_{12}\varphi | A \otimes I | P_{12}\varphi \rangle = \langle \varphi | P_{12}^{-1}(A \otimes I)P_{12} | \varphi \rangle$$
$$= \langle \varphi | I \otimes A | \varphi \rangle = \langle \varphi | A_2 | \varphi \rangle. \tag{2.12}$$

Thus the expectation values for all observables pertaining to either particle are identical, and this is what the Indiscernibility Thesis amounts to.

The crucial premise in the above argument is the assumption that the operators $A \otimes I$ and $I \otimes A$ are indeed formally accurate representations of observable $A$ pertaining, respectively, to the first and second particles. But here we encounter an immediate stumbling block. Operators $A \otimes I$ and $I \otimes A$ are clearly not symmetric, so they should be disallowed on the basis of our earlier considerations. Since SP implies that only symmetric operators can have physical meaning when applied to systems of same-type bosons or fermions, it seems that we should not use the non-symmetric products in formalizing the argument for the Indiscernibility Thesis. French and Redhead are aware of that difficulty, but they are strangely dismissive about it. First, they interpret the symmetry postulate with respect to observables not as delimiting *physically meaningful* operators but operators that *can be observed.*[15] Having done this, they announce that "from the point of view of discussing PII [the Principle of the Identity of Indiscernibles – TB] it seems clear that we should not restrict the discussion to attributes which can actually be observed" (ibid. p. 239).

French and Redhead's response to the problem raises several questions. Firstly, interpreting the requirement of symmetry as applying to operators that can be observed misses the point of the permutation-invariance problem. Suppose that we have a non-symmetric operator $\Omega$ which, even though it cannot be observed, is still admissible as a representation of a particular objective property of the system. This means that an "actualization" of a given value of this operator (French and Redhead speak about

[15] Cf. (French and Redhead 1988, p. 239). We hasten to explain to logical purists that the phrase "operators that can be observed" is not supposed to be taken literally (operators, being mathematical objects, are *never* observable), but is a mere shorthand for the longer "operators representing observable properties". Another minor linguistic issue that may or may not be necessary to clarify here is that while the standard physical counterparts of mathematical operators are usually called *observables*, in the current context this terminology is not particularly felicitous (vide the term "unobservable observables").

actualizations rather than obtaining measurement outcomes, since we are dealing here with unobservable properties) can put the system into a state that is neither symmetric nor antisymmetric, thus violating the Symmetrization Postulate. The claim that we can't observe this transition does not nullify the fact that SP is made false by its existence. Secondly, and more importantly, the contention that the operator representing a *measurable* and *observable* property of one particle gets classified as unobservable when this particle is taken as part of a broader system together with another particle should be viewed with high suspicion. One possible reply in defense of French and Redhead could be that the reason for the unobservability of operators of the form $A \otimes I$ and $I \otimes A$ has nothing to do with the $A$-property per se, but rather comes from the fact that we don't have any empirical means to distinguish particle 1 from particle 2, so we don't know on which particle we are supposed to perform an appropriate measurement. This seems right, but the moral from this example should be to reevaluate the way we can make reference to individual particles, rather than blindly accept that the attributes of these particles represented by non-symmetric operators "can never be observed" (for more on that see Chap. 5).

Is there any other way to represent properties of individual particles without infringing upon the symmetrization requirement with respect to observables? Nick Huggett in (Huggett 2003) has suggested how to approach this problem in a more general fashion. His proposal is to formulate a set of minimal conditions that should be met by any operators in the tensor product space that can lay claim to representing attributes of individual particles. Let $\{O_1, O_2, \ldots O_N\}$ be a set of operators acting in the $N$-fold tensor product of one-particle Hilbert spaces $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$. Each operator $O_i$ is supposed to represent a particular observable $O$ attributed to the $i$-th particle. In order to be able to do that, the operators should satisfy two postulates, which Huggett dubs the conjugation condition (CC) and independence condition (IC). These conditions are as follows:

(CC) $P_{ij}^{-1} O_i P_{ij} = O_j$

(IC) $P_{ij}^{-1} O_k P_{ij} = O_k$, when $k \neq i$ and $k \neq j$.

The meaning of these requirements should be clear: the conjugation condition ensures that the permutation of two particles will swap their properties, whereas the independence condition guarantees that permuting two particles will not affect the properties of a third one distinct from the two.

Huggett then shows that conditions (CC) and (IC) are sufficient to obtain French and Redhead's indiscernibility result. In the case of monadic properties, the proof is immediate and requires only the conjugation condition (the case of relational properties will be evaluated in the next section).

$$\begin{aligned}\langle \varphi | O_i | \varphi \rangle &= \langle P_{ij}\varphi | O_i | P_{ij}\varphi \rangle && (\text{by the symmetry / antisymmetry of } |\varphi\rangle)\\ &= \langle \varphi | P_{ij}^{-1} O_i P_{ij} | \varphi \rangle && (\text{formal transformation})\\ &= \langle \varphi | O_j | \varphi \rangle && (\text{CC}).\end{aligned} \tag{2.13}$$

Conditions (CC) and (IC) are obviously satisfied by operators of the form $O_i = I \otimes \ldots \otimes A \otimes \ldots \otimes I$, where $A$ occupies the $i$-th place in the product. However, other operators can also be shown to conform to (CC) and (IC). Is it possible to find operators that would satisfy (CC) and (IC) and at the same time be symmetric? Generally, the answer is "yes", but the success turns out to be somewhat limited in scope. If the operators $O_i$ were to be symmetric, this would mean that $P_{ij}^{-1} O_i P_{ij} = O_i$ which, together with the conjugation condition $P_{ij}^{-1} O_i P_{ij} = O_j$, implies that $O_i = O_j$ for all $i$, $j$. Thus the only symmetric operators that could possibly represent properties of individual particles would be identical with each other. This obviously trivializes the question of whether particles of the same type can be discerned by their properties. It is hardly an exciting result proving that the expectation values of operators that are identical turn out to be identical too (we don't even need to rely on the Symmetrization Postulate with respect to states to prove that).

French and Redhead's proposal of how to formally represent properties of individual particles, as well as the general approach advocated by Huggett, both rely on the same implicit assumption, which is so basic that up to a certain point in the history of the debate no one even bothered to make it explicit. And yet this assumption, which is constitutive of the approach to the individuation of particles that may be called "orthodoxy", deserves to be seen in broad daylight. This claim, which some refer to as "Factorism",[16] may be spelled out as follows:

(F) In the $N$-fold tensor product of Hilbert spaces $\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_N$ that is meant to represent states and properties of systems of $N$ particles of the same type, and whose symmetric and antisymmetric sectors are assumed to contain all the admissible states of $N$ bosons and $N$ fermions respectively, each Hilbert space $\mathcal{H}_i$ represents states and properties of one individual particle.

Factorism seems to be presupposed by the way we defined tensor products of Hilbert spaces as representations of states of composite systems consisting of a number of component systems. Thus it may be claimed that Factorism is an essential part of the tensor product formalism[17] and as such cannot be called into question without abandoning the entire formalism. And yet on a certain level of abstraction this conclusion can be resisted. It is at least conceivable that we could treat the mathematical structure of the $N$-fold tensor product of Hilbert spaces purely formally, without attaching any physical interpretation to the individual factors in the product. Then, after imposing certain additional restrictions on the admissible states in this product space (e.g. in the form of the

[16] I follow the interpretation of Factorism as spelled out in Caulton (2014 p. 11). On the other hand, F.A. Muller and Gijs Leegwater in Muller and Leegwater (2020) introduce a broader reading of Factorism. They consider the general problem of how to factorize a given Hilbert space $\mathcal{H}$ into a tensor product of $N$ spaces, and they observe that typically there is more than one way to achieve such a factorization. Consequently they distinguish two general versions of Factorism: ∀-Factorism, stating that for all available factorizations the labels associated with the factors refer to the components of the system, and ∃-Factorism, asserting that *some* such factorizations play the referential role. Their main point is that ∃-Factorism may be preserved even for "indistinguishable" particles (see Sect. 4.3 for more on that). However, this conclusion does not invalidate the fact that Factorism as stated above is open to refutation. The variant of Factorism defined above involves one *specific* factorization—namely the factorization with respect to which we impose the requirement of permutation invariance, as explained in Sect. 2.4.

[17] Redhead and Teller (1991, 1992) use a longer term Labeled Tensor Product Hilbert Space Formalism.

symmetrization or antisymmetrization requirement), we may ask the question of how to identify in our formalism parts that could represent states and properties of individual components of the considered system. And it is by no means a foregone conclusion that the only way to do that is by interpreting the factors in the tensor product $\mathscr{H}_1 \otimes \ldots \otimes \mathscr{H}_N$ as corresponding to the individual components of the system. We will return to this problem in later parts of the book.

## 2.7 Relational Properties of Individual Particles

For the time being we will remain within the framework adopted by French and Redhead and extended by Huggett, with its commitment to Factorism. So far the Indiscernibility Thesis has been proven with respect to monadic properties interpreted with the help of the expectation values of individual operators (or, equivalently, the probabilities of obtaining particular outcomes of measurements). However, it is a well-known fact that monadic properties do not exhaust all the attributes that can differentiate between objects. Another important category is relational properties that characterize objects by reference to other entities. To illustrate this with a standard example: if we had two iron spheres identical in every respect (same size, mass, chemical makeup, etc.), it would still be possible to differentiate between them if we placed a third object (e.g. a pointer) in a way that breaks the overall symmetry. In that case we could say that one sphere differs from the other in that it is located closer to the pointer. Being at a certain distance from a pointer is a relational property of one sphere that differentiates it from the other sphere.

It is not immediately clear how to interpret relational properties within the standard quantum-mechanical formalism. French and Redhead propose to use conditional probabilities of obtaining particular outcomes as a representation of relations between quantum particles. They prove that the probability of obtaining outcome *a* when measuring observable $A_1 = A \otimes I$ conditional on obtaining outcome *b* of observable $A_2 = I \otimes A$ is equal to the "switched" probability of outcome *a* of observable $A_2$

conditional on obtaining outcome $b$ on $A_1$. Jeremy Butterfield extended this result in several directions (Butterfield 1993). First off, he considers the case when $N$ can be any number. Moreover, in the above result he replaces the second occurrence of observable $A$ with a different observable $B$. And, thirdly, he introduces a new case in which he conditionalizes the probability of a given outcome obtained on a third, arbitrarily selected particle, on probabilities of some outcome obtained on one of the two particles occupying the joint symmetric/antisymmetric state. As this case is probably the most interesting, we will have a closer look at it in the following paragraph.

Among the $N$ considered particles, we select three particles labeled $i$, $j$ and $k$. Interestingly, we don't have to assume that all three particles are of the same type, only that two of them (the $i$-th and $j$-th) are. Consequently, we assume that the joint state $|\varphi\rangle$ of the $N$ particles has the required symmetry property only with respect to the permutation $P_{ij}$, and not $P_{ik}$ or $P_{jk}$. Now, consider the probability that a particular observable $A_k$ has some value $a$, conditional on the fact that observable $B_i$ assumes value $b$. This probability can be calculated as follows:

$$\Pr\left(A_k = a \mid B_i = b\right) = \frac{\Pr\left(A_k = a \& B_i = b\right)}{\Pr\left(B_i = b\right)} \tag{2.14}$$

In order to calculate the probabilities in the numerator and denominator of the above expression, we can resort again to projectors. Let $A_k^a$ and $B_i^b$ be appropriate projectors corresponding to outcomes $a$ and $b$. Then $\Pr\left(A_k = a \,\&\, B_i = b\right) = \left\langle \varphi \middle| A_k^a B_i^b \middle| \varphi \right\rangle$ and $\Pr\left(B_i = b\right) = \left\langle \varphi \middle| B_i^b \middle| \varphi \right\rangle$. With respect to the second probability, it can be easily argued that $\Pr(B_i = b) = \Pr(B_j = b)$, essentially by repeating the derivation from the previous section. Regarding the first probability, we can proceed as follows:

$$\begin{aligned}\langle\varphi \mid A_k^a B_i^b \mid \varphi\rangle &= \langle P_{ij}\varphi \mid A_k^a B_i^b \mid P_{ij}\varphi\rangle && \text{(by the symmetry / antisymmetry of } |\varphi\rangle) \\ &= \langle\varphi \mid P_{ij} A_k^a P_{ij} P_{ij} B_i^b P_{ij} \mid \varphi\rangle && \text{(using the identity } P_{ij}^{-1} = P_{ij}) \\ &= \langle\varphi \mid A_k^a B_j^b \mid \varphi\rangle && (P_{ij} A_k^a P_{ij} = A_k^a \text{ from IC and } P_{ij} B_i^b P_{ij} = B_j^b \text{ from CC})\end{aligned} \tag{2.15}$$

This proves that the probabilistic relation that the *i*-th particle stands in to the *k*-th particle is exactly the same as the probabilistic relation that the same-type *j*-th particle stands to the *k*-th particle. Given the fact that all three particles and their properties have been selected completely arbitrarily, we may conclude that quantum particles of the same type cannot differ with respect to their relational properties. Whatever relation holds between particle number *i* and the rest of the world should hold between the same-type particle number *j* and the world.

The final conclusion from the last paragraph can be argued for in a much simpler and perhaps more elegant way, following Dieks and Versteegh (2008, p. 933). We can again consider two particles of the same type labeled by numbers *i* and *j* which are parts of a broader system of same-type particles, and in addition to that we consider a third particle labeled *k* which may or may not be of the same type as particles *i* and *j*. Let $A(i, k)$ be any Hermitian operator representing a measurable property of the system of two particles *i* and *k* (any Hermitian operator acting in the tensor product $\mathscr{H}_i \otimes \mathscr{H}_k$). Operators of the form $A(i, k)$ provide us with the most general way of expressing all physical relations between particles *i* and *k*, since the expectation values of such operators contain all the information necessary to calculate the probabilities of outcomes, conditional and unconditional. Formally, we can always extend the operator $A(i, k)$ to any tensor product of an arbitrary number of additional Hilbert spaces by adding a required number of the identity operators acting in all spaces except $\mathscr{H}_i$ and $\mathscr{H}_k$. For the sake of simplicity, we will refer to this extended operator using the same symbol $A(i, k)$. We can now stipulate, as Huggett did with respect to the operators representing monadic properties of individual particles, that $A(i, k)$ should satisfy the following joint conjugation and independence condition:

$$P_{ij} A(i,k) P_{ij} = A(j,k), \text{where } k \neq i, j. \tag{2.16}$$

Actually, given that all operators in the tensor product $\mathscr{H}_i \otimes \mathscr{H}_k$ can be presented as linear combinations of operators acting, respectively, in $\mathscr{H}_i$ and $\mathscr{H}_k$, our new condition follows from Huggett's original conjugation and independence conditions. Now, if $|\varphi\rangle$ represents the joint state of a system containing particles *i*, *j* and *k*, and particle number *i* belongs to the same type as particle number *j*, we can derive the following:

$$\langle\varphi|A(i,k)|\varphi\rangle = \langle P_{ij}\varphi|A(i,k)|P_{ij}\varphi\rangle = \langle\varphi|P_{ij}A(i,k)P_{ij}|\varphi\rangle = \langle\varphi|A(j,k)|\varphi\rangle. \tag{2.17}$$

This confirms that relations which connect each of the two same-type particles with a third one are identical and cannot discern these particles.

## 2.8 Indiscernibility and Individuality

The above formal results point unambiguously towards the rejection of the Leibnizian Principle of the Identity of Indiscernibles (PII) when applied to particles of the same type.[18] In its most basic form, PII asserts that if an object *a* possesses the exact same properties as an object *b*, *a* and *b* are numerically identical. When expressed contrapositively, the principle states that for every two distinct objects there is a property possessed by one and not the other. Of course, the exact meaning and validity of this principle depends on how broadly we understand the term "property". It is easy to observe that PII can be made trivially (i.e. logically) true if, for each object, we include among its attributes the property of being *this very object*. However, it is standard practice to exclude from the range of PII's quantifier so-called *impure* properties, that is, properties that somehow involve reference to individual objects.[19] It is also common to

[18] For a recent historical analysis of the role of the Principle of the Identity of Indiscernibles in Leibniz's philosophy, see Rodriguez-Pereyra (2014).

[19] More precisely, we should exclude properties that involve individual objects *and* the relation of numerical identity. Only those properties make PII trivially true. It turns out that properties whose

distinguish two versions of PII: the stronger PII that limits properties to intrinsic (non-relational, monadic) ones and the weaker PII that includes extrinsic (relational, polyadic) attributes as well. We have already seen that the indiscernibility arguments from the permutation-invariance of the states of same-type particles involve both intrinsic and extrinsic properties, and thus the quantum case seems to invalidate PII in its weaker form as well.

Why is PII such an important principle that its apparent violation in quantum mechanics stirs up so much controversy among philosophers of science? One reason may be the connection with the question of the metaphysical status of quantum objects, and in particular whether they deserve to be categorized as *individuals*. The notion of an individual is yet another buzzword in the contemporary metaphysics of science (see, e.g. the recent collection Guay and Pradeau 2016). Individuals are supposed to form a special subcategory of objects, sharply distinguished from non-individuals. One way of characterizing individuals is related precisely to the concept of discernibility: an individual is said to be an object that can be discerned from the rest of the universe (that possesses some unique combinations of properties which no other entity in the universe possesses). A minor problem with this definition is that it makes the property of being an individual extrinsic, as being discernible from other objects is a relational property. Thus, a perfect duplicate of an individual in our world may not be an individual in another possible world, if only in this alternative world there are two indiscernible copies of the original object (see Cortes 1976, p. 492; French 2019).

Setting this problem aside, we may conclude that PII is important because its truth guarantees the existence of a *principium individuationis* for every object in the universe. If PII fails, as described in previous sections, quantum particles of the same type are relegated to the shadowy category of non-individual entities. Alternatively, we may want to look for their *principia individuationis* in something other than physical, observable properties. This puts us on the treacherous path to the non-empirical notions of *primitive thisness*, *haecceity*, *bare substratum* and so

canonical linguistic representations contain proper names but exclude the identity symbol do not automatically make all distinct objects discernible. We will return to this issue in Chap. 4.

on, broadly described as *transcendental individualities*.[20] However, it seems that this “solution” of the problem of individuality in light of the possible demise of PII is just the old ploy to make PII trivially true in a new disguise. We can keep PII true by expanding the range of admissible properties to include haecceities and the like, or we can formally give up PII and still use these dubious attributes as a basis of the “transcendental individuality” of quantum particles. There is no substantial difference between the two strategies apart from the difference in terminology. So if we balk at the above-mentioned method of trivializing PII, we should similarly react with aversion to the idea of transcendental individuation, which involves precisely the same metaphysically suspicious “properties”.

So far, we have considered the possibility of individuating objects even without PII, thus questioning the assumption that the satisfaction of PII is necessary for objects to be individuals. But the complementary question can also be posed, whether satisfying PII is sufficient for objects to achieve the status of individuals. And I believe that the answer to this question may be negative. In order to discuss this problem further, we may use the standard example in the context of which the issue of individuality in quantum mechanics has arisen in the first place, that is, the case of quantum statistics. In the simplest possible case involving two particles labeled 1 and 2, and two possible states S and T, the problem is how probable various distributions of the particles among the available states are, given the assumption that the assignment is done purely randomly.[21] Generally speaking, there are four logical possibilities of how to distribute the particles among the available states: S(1)T(2), S(2)T(1), S(1)S(2) and T(1)T(2) (see Fig. 2.3). Under the assumption of the equiprobability of the basic arrangements (on the basis of what van Fraassen calls the Principle of Indifference), we arrive at the classical, Maxwell-Boltzmann distribution:

[20] The term “transcendental individuality” entered wide circulation thanks to Redhead and Teller (1991, 1992), who credit Heinz Post with introducing it in the context of quantum theory.

[21] This example, which originated from Reichenbach (1971, pp. 233–235), has been discussed in virtually all publications concerning the notion of identity and individuality in quantum mechanics. See, for example, French and Redhead (1988, pp. 236–238; van Fraassen 1991, p. 378; Redhead and Teller 1992, p. 204; French and Krause 2006, pp. 144–145, French 2019, Sect. 2).

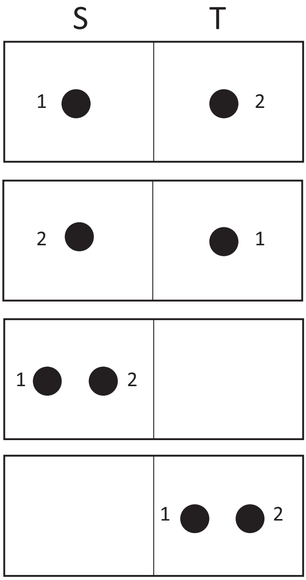


**Fig. 2.3** Four possible arrangements of two particles 1 and 2 among two states S and T

$$\Pr\big(S(1),T(2)\big)=\Pr\big(S(2),T(1)\big)=\Pr\big(S(1),S(2)\big)=\Pr\big(T(1),T(2)\big)=\frac{1}{4}, \quad (2.18)$$

from which it follows that the probability that the particles will occupy different states S and T (without saying which particle is in which state), given by:

$$\Pr\big(ST(1,2)\big)=\Pr\big(S(1),T(2)\big)+\Pr\big(S(2),T(1)\big) \quad (2.19)$$

equals $\frac{1}{2}$. But quantum particles do not obey this statistics. In the case of bosons, the probability Pr(ST(1,2)) equals 1/3, whereas for fermions this number goes up to 1 (these are, respectively, Bose-Einstein and Fermi-Dirac statistics). This discrepancy is often explained by noting that bosons and fermions are indistinguishable particles, and therefore cases S(1)T(2) and S(2)T(1) should be treated as one possibility ST(1, 2). Given the additional assumption that for fermions cases S(1)S(2) and T(1)T(2) are forbidden due to the Pauli exclusion principle, we arrive at the required probabilistic distribution:[22]

$$\begin{aligned}&\Pr\big(ST(1,2)\big)=\Pr\big(S(1),S(2)\big)=\Pr\big(T(1),T(2)\big)=1/3 \text{ for bosons, and}\\&\Pr\big(S(1),S(2)\big)=\Pr\big(T(1),T(2)\big)=0,\ \Pr\big(ST(1,2)\big)=1 \text{ for fermions.}\end{aligned} \tag{2.20}$$

The identification of the distributions S(1)T(2) and S(2)T(1) is sometimes interpreted as a sign of the loss of individuality for particles 1 and 2. But note that this has little to do with whether PII is valid for these particles or not. Suppose that particles 1 and 2 are photons (and thus they obey Bose-Einstein statistics). There is nothing inconsistent in the assumption that photons occupying the joint state ST are discernible thanks to the fact that one of them is in state S while the other is in T. We could even introduce labels that would reflect this discernibility (labeling the S-photon as 1 and the T-photon as 2). The only thing we can't do is to insist that photon labeled 1 could actually occupy state T (while photon 2 would assume state S) and that this would create a new situation, distinct from the original one. The loss of individuality is seen in the fact that we can't make an unambiguous counterfactual identification of a particular object, and not in the fact that this object could not be differentiated from the other entities in the actual scenario.[23]

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[22] One may ask the basic question: what does the statistical behavior described above have to do with the way we defined bosons and fermions as occupying, respectively, symmetric and antisymmetric states? In response we can observe that Bose-Einstein and Fermi-Dirac statistics can be recovered when we redescribe the entire situation properly in terms of symmetric or antisymmetric states. The four available distributions are represented by one antisymmetric state $|S\rangle|T\rangle - |T\rangle|S\rangle$ and three symmetric states $|S\rangle|T\rangle + |T\rangle|S\rangle$, $|S\rangle|S\rangle$ and $|T\rangle|T\rangle$, from which it follows that bosons can occupy three states with equal probabilities 1/3, while fermions have only one option available, and therefore its probability must be 1. Observe that in this analysis the states $|S\rangle|T\rangle - |T\rangle|S\rangle$ and $|S\rangle|T\rangle + |T\rangle|S\rangle$ correspond to situations that we would informally describe as cases in which one particle is in state S while the other in T, but we don't know which one is where.

[23] There is one more aspect of this commonly used example that merits closer inspection, as it is typically missing from standard presentations. We assumed, without much justification, that in the case of indistinguishable quantum particles the reduced number of distinct arrangements will nevertheless continue to be equiprobable. But it is instructive to observe that this assumption is most certainly violated for *classical* indistinguishable bodies. Suppose, for the sake of simplicity, that the precise state of each of two classical particles is represented by a continuous parameter in the form of a real number ("position" in a one-dimensional space). States S and T can be interpreted with the help of equal intervals of the position parameter—for instance, S can correspond to any value of position in the interval [–1, 0] and T in the interval [0, 1]. This reflects the fact that, classically

Thus it may be suggested that we should tie the concept of individuality to the possibility of the unambiguous identification of objects across different temporary instances and across different possible scenarios (different possible worlds). An individual is an object for which it is settled whether or not it is identical with any object at any moments in the actual world, and it is settled whether or not it is identical with any object in possible worlds.[24] One way of ensuring that some objects are individuals is to assume that every one of them is equipped with its unique *essence*, that is, the set of qualitative properties that unambiguously identify this object at any time and in any possible scenario. Another option, as always, is to insist that objects possess their unique non-qualitative haecceities, which unfortunately may not be empirically accessible to us. But the mere fact that an object is momentarily discernible from the rest of the universe by some of its *accidental* properties does not immediately

speaking, particles can occupy different precise locations within macroscopic "boxes" S and T. In the case of distinguishable particles, arrangements S(1)T(2), S(2)T(1), S(1)S(2) and T(1)T(2) will correspond to regions of equal volume in the phase space (a two-dimensional space of "positions" for both particles). These regions are, respectively, Cartesian products [−1, 0] × [0, 1], [0, 1] × [−1, 0], [−1, 0] × [−1, 0] and [0, 1] × [0, 1], which are equal squares. When we assume that particles 1 and 2 are indistinguishable, arrangements S(1)T(2), S(2)T(1) are to be identified. However, the corresponding region in phase space will be twice as big as the regions representing arrangements S(1)S(2) and T(1)T(2)! In the phase space for indistinguishable particles, we have to identify "permuted" points $(x, y)$ and $(y, x)$, as they represent one and the same state. This can be achieved by considering only half of the original space containing points for which $x \geq y$ (points lying below the diagonal $x = y$). In that case the region corresponding to the "amalgam" arrangement ST(1, 2) will be the square [0, 1] × [−1, 0], while the remaining two arrangements give rise to triangular regions half the volume of the square. (For instance, the region in phase space corresponding to arrangement S(1)S(2) will be the triangle defined by straight lines $x = 0$, $y = -1$ and $x = y$.) Consequently, the probability of state ST(1, 2) should be twice as big as the probability of state S(1)S(2), exactly as in the Maxwell-Boltzmann distribution. We may argue for the quantum statistics only if we assume that states S and T do not consist of more precise states. See Huggett (1999) and Albert (2000, pp. 45–47) for more on that.

[24] There is yet another notion of individuality that should be mentioned here, which I prefer to call "esoteric", since its meaning is rather difficult to grasp for me. Under this approach to individuality, an individual is an object identical with itself; hence by negation non-individuals are entities such that the notion of self-identity does not apply to them. Of course the esoteric conception of individuality requires that we modify classical logic, since classically it is a logical truth that every object in the universe is identical with itself. A lifelong promoter of this esoteric conception of individuality, who also insists that the quantum theory of many particles implies that quantum objects are non-individuals in this sense, is Decio Krause (see, e.g. Krause 1992, 2010; Krause and Arenhart 2016; Arenhart 2017; for some criticism, see Bueno 2014, pp. 329–330). Needless to say, one of the main upshots of this book will be that in fact quantum mechanics does not force us to adopt such a radical view.

guarantee that this object will be an individual. And I believe that the case of particles of the same type falls precisely into the category of non-individual objects that can in principle obey PII. As any two electrons possess the same set of state-independent properties (rest mass, electric charge, etc.), it is very implausible that they could have distinct essences. Thus it does not make sense to permute a group of electrons and expect that the resulting situation will be ontologically distinct from the original one. But still electrons could in principle differ from one another by their state-dependent properties; however, these properties cannot be used to identify them in different, alternative situations, precisely because state-dependent properties are accidental only.

Given that the link between PII and the individuality of objects is questionable, how else can we motivate the need for such a principle? There are two possible uses that PII can be put to, both rather critical from the perspective of an empiricist.[25] First of all, the validity of PII in one of its forms presented above guarantees the possibility of making reference to, naming, or generally selecting a particular object. Here we can quote a famous passage in the classic text on PII by Max Black, in which one of the disputants in a debate regarding two indistinguishable spheres in an otherwise empty universe starts off with saying "consider one of the spheres, *a*…", to which his opponent immediately retorts "How can I, since there is no way of telling them apart?" (Black 1952, p. 156). This highlights the fact that without qualitative differences between objects, making reference to one of them seems virtually impossible. Every act of referring to an object involves either using a description that uniquely identifies this object, or pointing at it (or making any other gesture unambiguously associated with the considered object), or—most typically—both. But in the case of genuinely indiscernible objects

[25] Jonas Arenhart distinguishes three possible reasons for accepting PII (Arenhart 2013). One is upholding the bundle theory of particulars, the second is enabling the possibility of counting objects, and the third is accepting a broad empirical stance which excludes the existence of facts that cannot be in principle empirically verified. I ignore the first reason, since I don't think we should be unconditionally committed to the bundle theory. As for the remaining two reasons, it seems to me that they both can be subsumed under my requirement to ground numerical diversity in qualitative facts. Counting obviously presupposes numerical diversity, while the postulate of empiricism is in line with the expectation that the non-empirical bare identity/diversity of objects should be grounded in an in-principle empirically accessible state of affairs.

(whether by intrinsic or extrinsic properties), none of this seems to be doable. As we remember, quantum particles of the same type possess the exact same intrinsic (i.e. non-relational) properties, so no description involving these properties can do the job. Introducing a third object (e.g. my finger) won't help either, since we have proven that the joint state of any other object and one particle of the group of same-type particles is exactly the same as for any other particle in the group. So when I point at any "single" electron, I am actually pointing at all the electrons in the universe!

The second use of PII has to do with the relations of numerical identity and its opposite, numerical distinctness. Numerical identity is one of the most fundamental notions without which we could not properly describe even the simplest empirical situations. Suffice it to say that without numerical identity we couldn't talk about the exact number of groups of things, for instance, pencils on my desk. And yet "bare" numerical identity/distinctness seems to be a non-empirical, non-qualitative notion. How, then, can we gain knowledge about which objects are numerically distinct from which objects? Here PII supplies one possible answer: we know that object *a* is distinct from object *b*, because we have found out that *a* possesses a qualitative, empirically accessible property that *b* does not possess. Now, to avoid possible misunderstandings, we have to stress that the inference from the existence of differentiating qualities to the numerical distinctness of the objects does not require PII; it is based entirely on the logical law known as the Leibniz law. However, PII gives us an assurance that such derivations could be made in *every instance* of numerical distinctness. To put it differently, if PII fails, there are cases in which the numerical distinctness of some objects could not be inferred from the facts about what properties are exemplified by these objects. Thus some facts of numerical distinctness/identity would seem to be in principle not accessible to us (vide the famous example of five identical hands used by Max Black in his classic article Black 1952). This of course raises further questions regarding systems of quantum particles: how do we know that a particular system of electrons consists of five, ten or $10^{12}$ electrons? How can we count them, if we cannot distinguish them in any way?

The problem of empirical justifications for statements regarding numerical identity/distinctness can be alternatively presented in a methodological or metaphysical guise. On the methodological level, we can ask whether the identity relation can be defined in an appropriate qualitative language. It is well known that identity can be easily defined in a second-order language, but such a definition has no practical use, since it is impossible (and perhaps also circular) to identify all the sets that an object belongs to. On the other hand, possible first-order definitions of identity require for their material adequacy that an appropriate form of PII is satisfied. The metaphysical counterpart of definability could be the notion of *grounding* (see Bliss and Trogdon 2016). Thus we can ask whether facts of numerical identity/distinctness are *grounded* in qualitative facts that do not involve the identity relation. Again, a positive answer to this question requires some form of PII; otherwise numerical identity cannot be fully reduced to qualitative first-order facts.

From what has been said, it should be clear that given the apparent plight of PII in the realm of quantum objects, some alternatives may be sought that could achieve goals similar to the two mentioned above: (1) to secure the possibility of making reference to individual objects and (2) to ground the relation of numerical identity in qualitative facts not involving identity. As it turns out, one way to do that is through extending the notion of *discernibility* beyond discernibility by intrinsic and extrinsic properties, towards relation-based types of discernibility. In order to see clearly how this might work, some preparatory work on terminology and logic needs to be done. This is what Chap. 4 will be about. But before that we will look closer at the Symmetrization Postulate and its possible alternative justifications.

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# 3

# The Source of the Symmetrization Postulate

We have noted in the previous chapter that the main source of the apparent breakage of the classical notions of individuality and identity in quantum mechanics is the requirement of permutation invariance imposed on the states and (or) properties of systems of same-type particles.[1] The key assumption leading to these restrictions is the Indistinguishability Postulate (IP), which demands that the expectation values of all meaningful (or at least empirically accessible) observables be invariant under permutations of the indices of particles. The logical options that the truth of IP leaves us with are the following. First, we may opt for an unconditional restriction on admissible observables in the form of the requirement of the symmetry of corresponding operators, without imposing any restrictions on the available states. One consequence of this approach is that if the system already occupies a state in a sector with a particular type of symmetry (i.e. symmetric, antisymmetric or higher-order paraparticle symmetries, of which we'll talk later in the chapter), it will never be able

[1] For a comprehensive formal and philosophical analysis of the notion of permutation symmetry in QM, see French and Rickles (2003).



T. Bigaj, *Identity and Indiscernibility in Quantum Mechanics*, New Directions in the Philosophy of Science, https://doi.org/10.1007/978-3-030-74870-8_3

to leave this sector by means of physical interactions. Admittedly, it is still logically possible that particles could occupy states that do not possess any symmetry properties (e.g. states that are products of distinct one-particle vectors). But once a physical process pushes them into a given permutation-invariant sector, there is no way back for such a group of particles.[2]

The second broad option is to impose the condition of symmetry on the available states of particles of the same type, without paying attention to their properties. Two main methods of realizing this strategy to make IP true are: to limit available states to the symmetric subspace and to limit them to the antisymmetric subspace of the tensor product. These options must be applied exclusively, since admitting the possibility of particles that could occupy both symmetric and antisymmetric states implies that states with no symmetry properties will also be available, thanks to the principle of superposition. A third broad option is to consider possible particles with states of a different type of symmetry—so-called paraparticles. We will discuss this option later in the chapter, for now limiting ourselves to the standard symmetric/antisymmetric dichotomy. As we have already observed in Chap. 2, the adoption of the Symmetrization Postulate with respect to states forces us to restrict admissible operators to the symmetric ones too, since the action of non-symmetric operators does not preserve the symmetry/antisymmetry of vectors. Thus the second option leads to the same thesis on which the first option relied—that only symmetric (i.e. permutation-invariant) operators are admissible.

The permutation-invariance restriction on operators seems an inevitable consequence of the Indistinguishability Postulate, since it follows

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[2] If among the physical processes we include measurements under the collapse interpretation, then it is straightforward to give a physical process that can transform a product state into a symmetric/antisymmetric one. Suppose we start with a product state $|\varphi\rangle|\psi\rangle$ where $|\varphi\rangle$ and $|\psi\rangle$ are orthogonal. Let us define the following vectors: $|\chi\rangle = \frac{1}{\sqrt{2}}\left(|\varphi\rangle|\psi\rangle + |\psi\rangle|\varphi\rangle\right)$ and $|\eta\rangle = \frac{1}{\sqrt{2}}\left(|\varphi\rangle|\psi\rangle - |\psi\rangle|\varphi\rangle\right)$.

Then it can be easily verified that the symmetric projectors $E_\chi \otimes E_\eta + E_\eta \otimes E_\chi$ and $E_\chi \otimes E_\chi$ (where $E_\chi = |\chi\rangle\langle\chi|$ and $E_\eta = |\eta\rangle\langle\eta|$) project the original state $|\varphi\rangle|\psi\rangle$ onto, respectively, the antisymmetric and symmetric sectors of the total Hilbert space. And since these projectors represent possible outcomes of measurements for admissible observables, it is physically possible to put the system into an antisymmetric/symmetric state by means of measurement procedures.

from both strategies described above; however, its exact implementation remains open to interpretation. Under the reading that Michael Redhead and Paul Teller (Redhead and Teller 1992, p. 208) call weak, the requirement of symmetry applies to observable properties, which means that non-symmetric operators may be meaningful but not observable. The strong interpretation, on the other hand, eliminates all non-symmetric operators as devoid of meaning. Choosing between the two interpretations is rather important, as this choice affects the way we can represent properties of individual components of larger collections of particles. If we insist that non-symmetric operators are meaningless, this blocks the standard representation of individual properties in the form of $N$-argument tensor products of one Hermitian operator and $N$-1 identity operators: $I \otimes I \otimes \ldots \otimes A \otimes \ldots \otimes I$, since such products are clearly not permutation-invariant. The only way to retain this representation is to insist that the non-symmetric operators are meaningful but not empirical.

One question that we posed in Chap. 2 is why we should insist on introducing the restrictions on available states in the form of the Symmetrization Postulate, given that the Indistinguishability Postulate can be satisfied by imposing restrictions on admissible operators. In this chapter we will look closer at some alternative justifications that SP receives in the literature. We will also dig deeper into the underside of the Indistinguishability Postulate itself, paying particular attention to the notion of the permutation of particles and its various possible interpretations. As it turns out, these issues cannot be properly approached without some philosophical insight into the problems of reference, labeling, modality *de re* and the like.

## 3.1 Indistinguishability Postulate and Permutations

What could be simpler and more straightforward than the mathematical concept of permutation? Formally, a permutation of an $n$-element set $X$ is a bijection σ mapping this set onto itself $\sigma: X \to X$. In the formalism of quantum theory, the set to which we apply permutations typically consists not of physical objects, but of indices, or labels. Thus, if we have a formula $\Psi(1,2, \ldots n)$ containing $n$ indices, the result of applying a permutation σ to this formula will be a new formula Ψ′ defined as follows: $\Psi' = \Psi(\sigma(1), \sigma(2), \ldots, \sigma(n))$. Now, depending on the mathematical interpretation of formula Ψ (on what mathematical objects Ψ is supposed to represent), the transformation $\Psi \rightarrow \Psi'$ may turn out to be the identity that changes nothing. To give a simple example: if formula Ψ is defined as $\Psi(1, 2) = a(1) + a(2) + b(1) + b(2)$, where $a(1)$, $a(2)$, $b(1)$ and $b(2)$ are stipulated to be real numbers, and symbol "+" represents ordinary addition, then the "permuted" formula $\Psi'(2, 1) = a(2) + a(1) + b(2) + b(1)$, even though distinct from Ψ when interpreted as a string of symbols, represents one and the same mathematical object due to the commutativity of addition. On the other hand, formula $\Phi(1, 2) = a(1) - a(2) + b(1) + b(2)$ does not possess the property of being permutation-invariant, unless $a(1) = a(2)$.

However, things get a bit complicated when we move from mathematical formulas and mathematical objects to their physical interpretations. We have to know how to physically interpret the permuted formula Ψ′, given that we already possess an interpretation for Ψ. In the quantum-mechanical context, indices, or labels, serve primarily as names for individual particles (as spelled out in the assumption of Factorism, Sect. 2.6); hence a natural counterpart of a permutation of indices is the physical process of "swapping" particles corresponding to these indices against the background of the situation described by the whole formula. But we must be a bit more explicit about how to interpret this "swapping" procedure. What does it mean, precisely, to exchange two objects (two electrons, chairs or galaxies)? The most natural way to think about this procedure is in terms of spatiotemporal location. If I asked you to swap this chair with that chair over there, you would most probably move one chair to the place where the other one stood while simultaneously bringing the other chair to the location previously occupied by the first one. All physical properties of one chair, except its location, would be taken with it, as it were. However, in physics in general, and in quantum mechanics in particular, the notion of a permutation of objects is introduced in the context of the joint state of these objects, and the spatiotemporal location of individual objects at best constitutes only part of this state. In classical physics, the other component characterizing the state of objects consists of their velocities (or momenta). Thus swapping two classical bodies amounts not only to changing their spatiotemporal positions but also velocities. In the quantum case, the state of a system is encompassed in a vector, a ray or, more generally, a density operator that represents the maximal information about all the available values of relevant observables. Characteristically, properties such as rest mass, charge, total spin, belong to the category of state-independent properties (which physicists call "intrinsic properties"), and thus do not get swapped during the permutation of objects. Thus, for instance, if the state of two particles is given as $|\varphi\rangle_1|\psi\rangle_2$, then the permuted state $|\psi\rangle_1|\varphi\rangle_2$ will represent a situation in which particle 1, with all its state-independent properties intact, occupies state $|\psi\rangle$ rather than $|\varphi\rangle$, and particle 2 takes up state $|\varphi\rangle$ instead of $|\psi\rangle$, retaining all its state-independent properties. Hence the permutation $|\varphi\rangle_1|\psi\rangle_2 \rightarrow |\psi\rangle_1|\varphi\rangle_2$ does not make particles 1 and 2 swap their state-independent properties.

This observation is important in the case of particles of different types, since we have to know that when we perform the formal exchange operation $|\varphi\rangle_1|\psi\rangle_2 \rightarrow |\psi\rangle_1|\varphi\rangle_2$, the particle referred to as 1 (or 2) after the permutation will retain its uniquely identifying state-independent properties, thus creating a situation empirically distinct from the original one. If we decided to permute objects with respect to *all their properties*, the result would be different—the final state would be empirically indistinguishable from the original one. For instance, if the initial situation involved a proton labeled 1 in state $|\varphi\rangle$ and an electron labeled 2 in state $|\psi\rangle$, the final, permuted state would be described as follows: particle 1 would become an electron in state $|\psi\rangle$, and particle 2 a proton in state $|\varphi\rangle$. This change would amount to nothing more than a simple relabeling of the original scenario. So it is paramount to exclude this understanding of permutation, if we want to insist that permuting particles of different types gives rise to empirically distinguishable situations.

On the other hand, when it comes to same-type particles, the distinction introduced above collapses. "Dragging" the state-dependent properties alongside the state-independent ones does not make any difference, since both particles possess the exact same state-independent properties. Still, we should keep in mind that the intention is to tie a given label with the actually possessed state-independent properties that define a given type of particle. Introducing some philosophical terminology, we may say that state-independent properties constitute the *essence* of a given particle, in that they identify this particle in a counterfactual scenario after the permutation (more on that will come in Chap. 8, Sect. 8.5). More precisely, the essence of an object *a* is a set of its properties $\wp$ such that possessing properties from $\wp$ is a necessary condition for any object to be identical with *a* (see Mackie 2006, pp. 4–6 for some formalizations of the notion of essential properties). Putting this in terms of possible worlds, for every possible world *w* in which *a* exists (or, in which a *counterpart* of *a* exists, if we follow David Lewis's conception of possible worlds and modality *de re*—see Lewis 1986; Beebee and MacBride 2015 for a recent critical analysis), *a* (or its counterpart) possesses $\wp$ in *w*. If by essentialism we understand the claim that all fundamental objects of a given theory (such as elementary particles) possess a set of essential properties, then essentialism seems to be an indispensable part of our interpretation of permutations in quantum mechanics.

Another possible view regarding modality *de re* (i.e. identification of objects in counterfactual scenarios) is known as *haecceitism*. It is basically the view that an object can acquire or lose any qualitative property while still remaining the same object, so the identity of an object is not tied to any of its qualitative characteristic. Metaphysicians sympathetic to this position (they are few and far between, to be sure) often add that the element responsible for retaining the identity of an object is its non-qualitative *haecceity*, that is, the property of being this very object (sometimes also called "primitive identity" or "primitive thisness", since it is not reducible to any more fundamental attributes).[3] Needless to say, haecceitism is at odds with empiricism, since haecceities are not empirically accessible to us. In addition to that problem, we should observe that haecceitism leads to an incorrect interpretation of permutations with respect to particles of different types, as explained above. Thus it seems that the quantum-mechanical formalism is more closely tied to the position of essentialism than haecceitism.

[3] The classical defense of the concepts of primitive thisness and primitive identity is Adams (1979). Among the proponents of haecceitism, one can count Gary Rosenkrantz—see his comprehensive book (Rosenkrantz 1993).

As we remember, permutations of labels in the quantum-mechanical formalism can be applied to states (vectors), or to operators. The latter application is less problematic from a philosophical point of view, since permuted operators do not refer to counterfactual, merely possible states of affairs, but instead represent different properties of the very same, actual objects. To give an example: if we apply permutation $P_{12}$ to operator $A \otimes I$, which represents a particular observable (measurable property) of particle number 1, the result will be $P_{12}^{-1} (A \otimes I) P_{12} = I \otimes A$, which is a mathematical representation of the same observable attributed to particle number 2. There is no need to decide how to identify permuted objects in a counterfactual scenario, since we do not create one—we just consider abstract properties attributed to different actual objects. To make things entirely clear—when we talk about attribution, we do not mean that the particle actually possesses a certain value of the considered parameter, but rather we say "whose" property a given operator represents. Operators $A \otimes I$ or $I \otimes A$ do not describe particular states of affairs of which only one can be realized; instead, they represent abstract measurable properties (spin, position, momentum, etc.) that are attributable to particle 1 and particle 2, respectively. In contrast to that, the permuted states of the form $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ are supposed to represent certain states of affairs that cannot occur "simultaneously". If the first vector correctly describes the actual state of particles 1 and 2, the permuted vector can be interpreted only as a representation of another, possible state of affairs that does not take place in actuality. And for that reason we have to be careful to specify what possible and not actual objects the labels in the second vector refer to.

## 3.2 The Argument from Exchange Degeneracy[4]

In this section we will analyze yet another argument in favor of the Symmetrization Postulate (distinct from the argument using the Indistinguishability Postulate) that appears in textbooks on quantum

[4] This is a shorter version of a more detailed analysis presented in Bigaj (2020).

mechanics (see, e.g. Cohen-Tannoudji et al. 1978, pp. 1375–1377). The argument from exchange degeneracy has the well-known form of a *reductio ad absurdum*. Suppose that indeed it is possible for a two-element system of same-type particles to occupy states that are neither symmetric nor antisymmetric, in particular states that are products of two non-identical (orthogonal) vectors. Let us select one such state of the form $|\varphi\rangle_1|\psi\rangle_2$. As we already know, the Hilbert space $\mathscr{H} \otimes \mathscr{H}$ also contains a permuted vector $|\psi\rangle_1|\varphi\rangle_2$ which, by assumption, represents a state that is empirically indistinguishable from $|\varphi\rangle_1|\psi\rangle_2$. Thus we have a case of what is known as *representational redundancy*: our mathematical framework contains distinct representations of the same physical, or empirical, situation.

Representational redundancy is a common occurrence in mathematical physics, thanks in part to the richness and flexibility of mathematical formalism (which implies in the majority of cases the existence of so-called *surplus structures*, in Redhead's terminology, see Redhead 2002). A well-known case of redundancy present in quantum mechanics is caused by the fact that two vectors that differ by a phase represent the same physical state. This redundancy can be eliminated by changing the domain of mathematical objects from vectors to rays, but the resulting mathematical structure loses certain nice formal features (e.g. the set of rays is no longer a vector space). In classical mechanics we encounter a case of representational redundancy related to permutation invariance very similar to the one considered above. When we consider the phase space for two indistinguishable classical particles, any physical state of these particles that is not singular can be represented by any of the two distinct points: $\langle q_1, p_1; q_2, p_2\rangle$ and $\langle q_2, p_2; q_1, p_1\rangle$, where $q_i$ represents the position of the $i$-th particle and $p_i$ its momentum.[5] The standard solution for this type of degeneracy is to identify the permuted points (to consider the reduced phase space with the permutation group "quotiented out"). Again, there is a mathematical price to be paid for this elimination of redundancy in the form of admitting a phase space that possesses some non-standard topological features (see Leinaas and Myrheim 1977).

[5] The same applies in the case when the particles are distinguishable, but their identifying properties are dynamically irrelevant. See Saunders (2015, p. 176ff) for a discussion of the classical case of permutation invariance.

The quantum case of permutation-based redundancy is different from the classical one in several respects. First of all, what makes the quantum case more difficult is that not only are the kets $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ distinct; they are also orthogonal, which means—according to the standard interpretational rule known as the Born rule—that the probability of finding the system in one state given that it occupies the other one should be zero. So it seems that the inclusion of both kets in our representational framework leads to a logical contradiction.[6] Moreover, in the quantum case, as opposed to the classical one, distinct states can be *superposed* to create new states, typically distinguishable from the original ones. The permuted kets $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ span an entire two-dimensional subspace $\mathscr{V}_{\varphi\psi}$ that contains all linear combinations of the form $a|\varphi\rangle_1|\psi\rangle_2 + b|\psi\rangle_1|\varphi\rangle_2$. The Born rule dictates that if a physical system is in a superposition of two orthogonal states, the probability of finding the system in any of the component states equals the squared modulus of the coefficient of this component. Thus if the system's state is the above linear combination, the probability that the system will be found in the state described by the first ket $|\varphi\rangle_1|\psi\rangle_2$ equals $|a|^2$, while the probability of the other option represented by $|\psi\rangle_1|\varphi\rangle_2$ equals $|b|^2$. But both kets are supposed to represent the same empirical situation; hence the probability of obtaining this situation should equal $|a|^2 + |b|^2 = 1$ (given the normalization of the linear combination above). It seems that every linear (normalized) combination $a|\varphi\rangle_1|\psi\rangle_2 + b|\psi\rangle_1|\varphi\rangle_2$ should represent precisely the same empirical state as kets $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$.

Now the stage is set for the argument from exchange degeneracy, which is indeed very simple. Let us consider a Hermitian operator $A$ acting in the one-particle Hilbert space $\mathscr{H}$, whose eigenvectors are orthogonal kets $\frac{1}{\sqrt{2}}\left(|\varphi\rangle+|\psi\rangle\right)$ and $\frac{1}{\sqrt{2}}\left(|\varphi\rangle-|\psi\rangle\right)$. It is easy to calculate that when we

[6] This contradiction can be avoided by limiting admissible observables to the symmetric ones, though, since in that case there will be no experimental procedure that could differentiate between permuted states. Thus the conditional statement regarding the zero probability of finding the system in state $|\psi\rangle_1|\varphi\rangle_2$ given that it occupies state $|\varphi\rangle_1|\psi\rangle_2$ would be vacuously true, since the antecedent ("If we performed such-and-such measurement, then…") would be necessarily false. Still, as a matter of formal elegance, it is better not to have both permuted states $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ in one representational framework.

perform a measurement of the same observable $A$ on each of the two particles (thus we are measuring the observable represented by the symmetric product $A \otimes A$) whose joint state is the superposition $a|\varphi\rangle_1|\psi\rangle_2 + b|\psi\rangle_1|\varphi\rangle_2$, the probability of finding both particles in state $\frac{1}{\sqrt{2}}(|\varphi\rangle + |\psi\rangle)$ will equal $\left|\frac{1}{2}(a+b)\right|^2$.[7] Hence this probability will be generally different for different values of $a$ and $b$. It looks like some elements of the subspace $\mathscr{V}_{\varphi\psi}$ can be differentiated by means of physically meaningful experiments, and therefore they can't possibly represent the same physical state.

The argument from exchange degeneracy shows that even if the Indistinguishability Postulate is satisfied, that is, there is no way to empirically discern permuted states, this does not exclude the possibility that there may be other unacceptable cases of apparent empirical differences where there shouldn't be any. The satisfaction of IP is ensured by the fact that the operator $A \otimes A$ used in the argument is symmetric, hence does not differentiate between any state $|\Phi(1,2)\rangle$ and its permuted variant $|\Phi(2,1)\rangle$, since $\langle\Phi(1,2)|A \otimes A|\Phi(1,2)\rangle = \langle\Phi(2,1)|A \otimes A|\Phi(2,1)\rangle$. Still, operator $A \otimes A$ differentiates between various combinations of the form $a\, |\varphi\rangle_1|\psi\rangle_2 + b\, |\psi\rangle_1|\varphi\rangle_2$, and this seems to be as bad as differentiating between permuted states. The key premise on which the argument relies, which I will refer to as Ontic Conservativeness of Superpositions (OCS), can be spelled out as follows:

(OCS) If vectors $|\eta\rangle$ and $|\chi\rangle$ represent the same physical state, then any linear combination $a|\eta\rangle + b|\chi\rangle$ represents the very same state.

[7] This probability can be equivalently presented as the expectation value for the projector onto the ray spanned by the ket $\frac{1}{\sqrt{2}}(|\varphi\rangle+|\psi\rangle)\otimes\frac{1}{\sqrt{2}}(|\varphi\rangle+|\psi\rangle)$.

As we have indicated earlier, OCS can be supported by reference to the Born rule, even though OCS does not seem to follow from it logically.[8] OCS, together with the assumption that vectors $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ represent the same physical state, leads to the absurd conclusion that one and the same physical state produces different expectation values of an admissible operator, merely because this state is mathematically represented by distinct vectors.

The proponents of a haecceitistic interpretation of permutations will want a premise slightly different from OCS, since they disagree with the assumption that kets $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ represent the same ontological state of affairs. They can only concede that $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ refer to physical facts which are *empirically indistinguishable* even though ontologically distinct; hence they need the following premise (called, for obvious reasons, Epistemic Conservativeness of Superpositions):

(ECS) If vectors $|\eta\rangle$ and $|\chi\rangle$ represent physical states that are *empirically indistinguishable*, then any linear combination $a|\eta\rangle + b|\chi\rangle$ represents a state that is also *empirically indistinguishable* from them.

Either way, the conservativeness of superpositions ensures that all superpositions of the form $a|\varphi\rangle_1|\psi\rangle_2 + b\,|\psi\rangle_1|\varphi\rangle_2$ should be empirically indistinguishable, which contradicts the result obtained above. Consequently, the initial assumption that both states $|\varphi\rangle_1|\psi\rangle_2$ and $|\psi\rangle_1|\varphi\rangle_2$ are available to the particles turns out to be unsustainable. However, this result by itself does not prove that the only available states in this case are symmetric or antisymmetric. Actually, there is an alternative way to deal with the challenge created by the argument from exchange degeneracy which seems to be prima facie acceptable. The idea is simply to eliminate the permutation-based redundancy by "brute force", as it were, that is, by

[8] The main obstacle on a path leading from the Born rule to OCS is the notorious gap between probability one and pre-measurement reality. The Born rule guarantees that it is certain that upon measurement the system will find itself in a particular physical state described as either $|\varphi\rangle$ or $|\psi\rangle$, but from this it does not logically follow that the system already occupies this state *before* measurement. We have to be wary of this type of reasoning, lest we are forced to accept the logical transition from the fact that it is guaranteed (with probability one) that a certain definite value of an observable will be revealed to the conclusion that a certain definite value of this observable is possessed before measurement.

stipulating that of any pair of permuted vectors $|\varphi(1, 2)\rangle$ and $|\varphi(2, 1)\rangle$, only one is admissible. A systematic way to do that is to select a subspace $\mathcal{R}(\mathcal{H} \otimes \mathcal{H})$ of the entire tensor product $\mathcal{H} \otimes \mathcal{H}$ spanned by the following selection of basis vectors:

$$\left\{|e_i\rangle|e_j\rangle : i \leq j\right\},$$

where $|e_i\rangle$ are basis vectors for $\mathcal{H}$. Thus out of $n^2$ dimensions of space $\mathcal{H} \otimes \mathcal{H}$, we consider only $k$ dimensions, where:[9]

$$k = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

It is easy to observe that for any vector $|\Phi(1, 2)\rangle \in \mathcal{R}(\mathcal{H} \otimes \mathcal{H})$, $|\Phi(2, 1)\rangle \in \mathcal{R}(\mathcal{H} \otimes \mathcal{H})$ iff $|\Phi(1, 2)\rangle = |\Phi(2, 1)\rangle$. This follows from the fact that each vector $|\Phi(1, 2)\rangle$ in $\mathcal{R}(\mathcal{H} \otimes \mathcal{H})$ has a unique decomposition in terms of the basis vectors: $|\Phi(1, 2)\rangle = \sum_{i \leq j} c_{ij}|e_i\rangle|e_j\rangle$, and the permuted vector $|\Phi(2, 1)\rangle = \sum_{i \leq j} c_{ij}|e_j\rangle|e_i\rangle$ belongs to $\mathcal{R}(\mathcal{H} \otimes \mathcal{H})$ only when $c_{ij} = 0$ for all $i \neq j$.

The restriction of the tensor product space to subspace $\mathcal{R}(\mathcal{H} \otimes \mathcal{H})$, coupled together with the limitation of admissible operators to the symmetric ones, necessary in order to guarantee the satisfaction of the Indistinguishability Postulates, seems to do the job. And yet there is one serious problem with this solution. As can be quickly verified, symmetric projectors of the form $E_\varphi \otimes E_\psi + E_\psi \otimes E_\varphi$ can take us outside subspace $\mathcal{R}(\mathcal{H} \otimes \mathcal{H})$ when applied to certain vectors. This happens, for instance, when:

$$|\varphi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle) \text{ and } |\psi\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle - |e_2\rangle),$$

[9] This method generalizes to the case of $N$ particles as follows: for the subspace of $\mathcal{H}^N$, we select the basis vectors from the set $\{|i_1, i_2, \ldots, i_N\rangle : i_1 \leq i_2 \leq \ldots \leq N$ and $i_k = 1, 2, \ldots, n\}$. Thus we take all non-descending $N$-element sequences of numbers from 1 to $n$ to form the required basis.

in which case:

$$\left(E_{\varphi} \otimes E_{\psi} + E_{\psi} \otimes E_{\varphi}\right) |e_1\rangle\, |e_2\rangle = \frac{1}{2}\left(|e_1\rangle|e_2\rangle - |e_2\rangle|e_1\rangle\right).$$

This is both formally and physically unacceptable. Formally, the set of operators on any given subspace $V$ must be limited to those operators $A$ acting in the entire space for which $A[V] \subseteq V$. Physically, the projector $E_{\varphi} \otimes E_{\psi} + E_{\psi} \otimes E_{\varphi}$ represents a certain possible outcome of an admissible measurement that, if obtained, will inevitably leave the system in a state that no longer belongs to $\mathscr{R}(\mathscr{H} \otimes \mathscr{H})$, which violates the assumption that only vectors in $\mathscr{R}(\mathscr{H} \otimes \mathscr{H})$ are admissible as representations of physically possible states of affairs. Thus we cannot simply eliminate the permutation-based redundancy by hand in the way described above, and the only other solution is to adopt the Symmetrization Postulate, that is, to limit available states to either the symmetric ones or the antisymmetric ones.

It is elementary to observe that with the states limited to either symmetric or antisymmetric, we avoid the problem with exchange degeneracy, since the combinations of the form $a\,|\varphi\rangle_1|\psi\rangle_2 + b\,|\varphi\rangle_1|\psi\rangle_2$, where $a \neq b$, are no longer admissible. Thus we have managed to form an argument in favor of SP that is quite independent from the need to keep the Indistinguishability Postulate true, even though it is still closely related to it.

## 3.3 Parastatistics

The discussion given in the previous section has been done under the simplifying assumption that the number of same-type particles equals two. Typically, generalizations for higher numbers of particles present problems that are little more than technical. Not this time, though. It turns out that there is a substantial difference between bipartite systems and systems of more than two particles in that only with respect to the former can we actually prove a dichotomy of two exclusive types of symmetry-preserving particles: bosons (symmetric) and fermions (antisymmetric). When the number of particles is larger than two, new types of symmetries become available. In this section we will show how to identify permutation-invariant subspaces containing vectors that display symmetries of types different than bosonic and fermionic in the case of three-particle systems (hypothetical particles obeying these new symmetries are called *paraparticles*). We will also see that there are some fundamental theoretical and conceptual difficulties with these new types of symmetries, which make the existence of paraparticles rather improbable, quite independently of the lack of strong empirical evidence supporting the paraparticle hypothesis.

To keep the level of formal complexity as low as possible, we will assume that each one-particle Hilbert space is three-dimensional and hence is spanned by three orthonormal vectors $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle$. Thus the entire product $\mathcal{H}^3 = \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$ will be 27-dimensional, spanned by the triples $|ijk\rangle$, where each index $i$, $j$, $k$ can assume any value $\alpha$, $\beta$ or $\gamma$. For every basis vector $|ijk\rangle$, we can define its antisymmetrization and symmetrization as follows:

$$\mathrm{Sym}|ijk\rangle = \frac{1}{\sqrt{6}}\sum_{\sigma\in S_3}|\sigma(i),\sigma(j),\sigma(k)\rangle, \tag{3.1}$$

$$\mathrm{Anti}|ijk\rangle = \frac{1}{\sqrt{6}}\sum_{\sigma\in S_3}\mathrm{sgn}\,\sigma|\sigma(i),\sigma(j),\sigma(k)\rangle, \tag{3.2}$$

where $S_3$ is the permutation group of a three-element set, and sgn$\sigma$ is the sign of permutation $\sigma$ (1 for permutations consisting of an even number of transpositions, and −1 for odd permutations). The operations of symmetrization and antisymmetrization are not one-to-one, that is, we can get the same result out of different initial vectors.[10] Thus the number of distinct (and orthonormal) symmetric and antisymmetric vectors will be less than 27.

Starting with the case of antisymmetric vectors, we can notice that there is only one possible non-zero result of the operation Anti $|ijk\rangle$. When any of the indices $i, j, k$ are identical, the result will be zero, since

[10] Formally, operations Sym and Anti can be treated as projection operators, projecting onto symmetric and antisymmetric subspaces of $\mathcal{H}^3$.

the odd permutation swapping two identical elements cannot possibly produce the minus sign, as required by (3.2). On the other hand, any vector $|ijk\rangle$, where $i, j$ and $k$ are non-identical, will produce the same (modulo the sign) antisymmetric vector, whose full form is as follows:

$$\frac{1}{\sqrt{6}}\left(|\alpha\beta\gamma\rangle+|\beta\gamma\alpha\rangle+|\gamma\alpha\beta\rangle-|\beta\alpha\gamma\rangle-|\gamma\beta\alpha\rangle-|\alpha\gamma\beta\rangle\right).$$

Thus the antisymmetric subspace is just one-dimensional.[11] However, the symmetric subspace is much larger. It consists, first, of a symmetric counterpart of the above-written antisymmetric vector:

$$\mathrm{Sym}|\alpha\beta\gamma\rangle=\frac{1}{\sqrt{6}}\left(|\alpha\beta\gamma\rangle+|\beta\gamma\alpha\rangle+|\gamma\alpha\beta\rangle+|\beta\alpha\gamma\rangle+|\gamma\beta\alpha\rangle+|\alpha\gamma\beta\rangle\right).$$

Then we have six vectors arising as a result of the symmetrization of triples with two repeated elements, as follows:

$$\mathrm{Sym}\,|\,\alpha\alpha\beta\rangle,\ \mathrm{Sym}\,|\,\alpha\alpha\gamma\rangle,\ \mathrm{Sym}\,|\,\beta\beta\alpha\rangle,\ \mathrm{Sym}\,|\,\beta\beta\gamma\rangle,\ \mathrm{Sym}\,|\,\gamma\gamma\alpha\rangle,\ \mathrm{Sym}\,|\,\gamma\gamma\beta\rangle.$$

For instance, the first vector on the list will have the following full form: $\frac{1}{\sqrt{3}}(|\,\alpha\alpha\beta\rangle+|\,\alpha\beta\alpha\rangle+|\,\beta\alpha\alpha\rangle)$. Note that the exact same vector will be obtained by symmetrizing the triples $|\alpha\beta\alpha\rangle$ and $|\beta\alpha\alpha\rangle$, which shows that the six symmetric vectors written above correspond to 18 basis vectors. Finally, we have three symmetric states which are the result of symmetrizing already symmetric triples:

$$\mathrm{Sym}|\alpha\alpha\alpha\rangle=|\alpha\alpha\alpha\rangle$$

[11] This is just a coincidence caused by the fact that the number of dimensions of $\mathscr{H}$ and the number of factors in the product of Hilbert spaces happen to be identical. In the general case when the number of particles is $N$ and the dimensionality of each Hilbert space is $K$ ($K > N$), there will be $\binom{K}{N}$ orthogonal antisymmetric vectors (the number of $N$-element combinations out of $K$ distinct elements).

$$\mathrm{Sym}|\beta\beta\beta\rangle = |\beta\beta\beta\rangle$$
$$\mathrm{Sym}|\gamma\gamma\gamma\rangle = |\gamma\gamma\gamma\rangle.$$

All in all, the symmetric subspace will have 10 dimensions, which leaves 16 dimensions of the product space $\mathscr{H}^3$ that are neither symmetric nor antisymmetric (27 = 10 + 1 + 16). Instead of working with this rather large 16-dimensional space, we will limit ourselves to the subspace of $\mathscr{H}^3$ spanned by six permutations of the triple $|\alpha\beta\gamma\rangle$. In that case both symmetric and antisymmetric subspaces are one-dimensional, which leaves us with four dimensions unaccounted for. Our task now will be to find within this four-dimensional space some subspaces that remain invariant under permutations and thus can possibly house states of particles with different types of symmetries (paraparticles).[12] Let us try to identify four vectors orthogonal to $\mathrm{Sym}|\alpha\beta\gamma\rangle$ and $\mathrm{Anti}|\alpha\beta\gamma\rangle$, that could span such permutation-invariant subspaces. Of course there is a substantial degree of freedom in selecting vectors orthogonal to the two above-mentioned, so we can start with some arbitrary choices. One vector orthogonal to $\mathrm{Sym}|\alpha\beta\gamma\rangle$ and $\mathrm{Anti}|\alpha\beta\gamma\rangle$ may be stipulated to have the following form:

$$\frac{1}{\sqrt{3}}\big(a|\alpha\beta\gamma\rangle + b|\beta\gamma\alpha\rangle + c|\gamma\alpha\beta\rangle\big). \tag{3.3}$$

As can be clearly seen, vector (3.3) consists of elements which are cyclic permutations of the initial triple $|\alpha\beta\gamma\rangle$. Given that (3.3) is supposed to be normalized, we may achieve that by putting $|a|^2 = |b|^2 = |c|^2 = 1$ (remember that $a$, $b$ and $c$ are complex numbers, so we take the squared modulus here). Moreover, we do not lose generality by laying $a = 1$, since multiplying (3.3) by any number whose norm equals one is irrelevant. Given the assumption that $|b|^2 = |c|^2 = 1$, the numbers we are searching for have the form $b = e^{i\theta_1}$ and $c = e^{i\theta_2}$. Next, thanks to the assumption that vector (3.3) must be orthogonal to $\mathrm{Sym}|\alpha\beta\gamma\rangle$ and $\mathrm{Anti}|\alpha\beta\gamma\rangle$, it follows that $1 + b + c = 0$.[13] From this, the following must hold: $\cos\theta_1 + \cos\theta_2 = -1$, $\sin\theta_1$

[12] The approach used below is closely modeled on Peres (2002, p. 131ff).

[13] To see that, take the inner product of (3.3) and $\mathrm{Sym}|\alpha\beta\gamma\rangle$ ($\mathrm{Anti}|\alpha\beta\gamma\rangle$).

= $-\sin\theta_2$ (using the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$). The second equation implies that $\theta_1 = 2\pi - \theta_2$, and taking into account that $\cos(2\pi - \theta) = \cos\theta$, we get from the first equation $\cos\theta_1 = \cos\theta_2 = -½$. Hence the solution is $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{4\pi}{3}$, which gives us the following form of the required vector:

$$\frac{1}{\sqrt{3}}\left(\left|\alpha\beta\gamma\right\rangle + e^{2\pi i/3}\left|\beta\gamma\alpha\right\rangle + e^{4\pi i/3}\left|\gamma\alpha\beta\right\rangle\right). \tag{3.4}$$

Now we can ask how the vector given in (3.4) will behave under permutations. It is easy to observe that cyclic permutations will produce multiples of (3.4), thanks to simple algebraic relations between the three roots of 1: 1, $e^{2\pi i/3}$ and $e^{4\pi i/3}$. As a matter of fact, these roots, which can be conveniently symbolized as 1, ω, and $\omega^2$, where $\omega = e^{2\pi i/3}$, form a multiplicative group which is precisely the cyclic group $Z_3$, that is, the group of cyclic permutations of a three-element set (cf. Penrose 2005, pp. 98–99). Thus any cyclic permutation of particles (labels) applied to (3.4) will exactly correspond to a multiplication of the coefficients by a constant.[14] It can be verified by direct calculation that, for instance, applying the permutation 123 → 231 will yield the following multiple of (3.4):

$$\frac{e^{4\pi i/3}}{\sqrt{3}}\left(\left|\alpha\beta\gamma\right\rangle + e^{2\pi i/3}\left|\beta\gamma\alpha\right\rangle + e^{4\pi i/3}\left|\gamma\alpha\beta\right\rangle\right). \tag{3.5}$$

However, applying transpositions to (3.4) will yield new vectors, orthogonal to (3.4). For instance, the transposition 123 → 213 will have the following effect:

[14] The fact that the $n$th roots of unity form a cyclic group $Z_n$ may be used as an explanation of why there are no one-dimensional permutation-invariant subspaces other than Sym$|\alpha\beta\gamma\rangle$ and Anti$|\alpha\beta\gamma\rangle$. Suppose that we consider an analogue of vector (3.4) containing all permuted triples $|ijk\rangle$ and thus potentially permutation-invariant. In order to make this new vector orthogonal to Sym$|\alpha\beta\gamma\rangle$ and Anti$|\alpha\beta\gamma\rangle$, the coefficients of its six components have to form the sixth roots of unity: 1, ω, $\omega^2$, $\omega^3$, $\omega^4$, $\omega^5$, where $\omega = e^{\pi i/3}$. But this set forms the cyclic group $Z_6$ which is not isomorphic with the permutation group $S_3$; hence permutations applied to the new vector will generally not produce the same vector.

$$\frac{1}{\sqrt{3}}\left(\left|\beta\alpha\gamma\right\rangle+e^{2\pi i/3}\left|\gamma\beta\alpha\right\rangle+e^{4\pi i/3}\left|\alpha\gamma\beta\right\rangle\right). \tag{3.6}$$

The remaining two transpositions will produce multiples of (3.6). Thus we have found a two-dimensional space spanned by vectors (3.4) and (3.6) which is orthogonal to symmetric and antisymmetric subspaces and remains invariant under permutations. We can write them down in an abbreviated form as follows:

$$\begin{aligned}\Psi_1 &= \frac{1}{\sqrt{3}}\left(\left|\alpha\beta\gamma\right\rangle+\omega\left|\beta\gamma\alpha\right\rangle+\omega^2\left|\gamma\alpha\beta\right\rangle\right)\\ \Psi_2 &= \frac{1}{\sqrt{3}}\left(\left|\beta\alpha\gamma\right\rangle+\omega\left|\gamma\beta\alpha\right\rangle+\omega^2\left|\alpha\gamma\beta\right\rangle\right)\end{aligned} \tag{3.7}$$

Now we only need to find two more orthogonal vectors with similar symmetry properties. Without going through tedious derivations on the basis of the orthogonality assumptions, we'll simply present the final result:

$$\begin{aligned}\Phi_1 &= \frac{1}{\sqrt{3}}\left(\left|\beta\alpha\gamma\right\rangle+\omega^2\left|\gamma\beta\alpha\right\rangle+\omega\left|\alpha\gamma\beta\right\rangle\right)\\ \Phi_2 &= \frac{1}{\sqrt{3}}\left(\left|\alpha\beta\gamma\right\rangle+\omega^2\left|\beta\gamma\alpha\right\rangle+\omega\left|\gamma\alpha\beta\right\rangle\right)\end{aligned} \tag{3.8}$$

The orthogonality relation between $\Psi_1$ and $\Phi_2$ and between $\Psi_2$ and $\Phi_1$ can be established using the following algebraic equalities (asterisk $^*$ indicates the operation of complex conjugate): $1 + (\omega^2)^*\omega + \omega^*\omega^2 = 1 + \omega\omega + \omega^2\omega^2 = 1 + \omega^2 + \omega$, and we have already proven earlier that this last formula equals zero. Vectors $\Psi_1$ and $\Phi_1$, as well as $\Psi_2$ and $\Phi_2$, are trivially orthogonal. Vectors $\Phi_1$ and $\Phi_2$ span yet another two-dimensional subspace that does not change under the action of permutations of particles.

The transformation properties of vectors in (3.7) can be synthetically presented as follows (the formulas for vectors (3.8) will be identical except $\Phi$ replaces $\Psi$):

$$\begin{aligned} \sigma_{231}\Psi_1 &= \omega^2\Psi_1 \\ \sigma_{312}\Psi_1 &= \omega\Psi_1 \\ \sigma_{213}\Psi_1 &= \Psi_2 \\ \sigma_{132}\Psi_1 &= \omega\Psi_2 \\ \sigma_{321}\Psi_1 &= \omega^2\Psi_2 \end{aligned} \tag{3.9}$$

$$\begin{aligned} \sigma_{231}\Psi_2 &= \omega\Psi_2 \\ \sigma_{312}\Psi_2 &= \omega^2\Psi_2 \\ \sigma_{213}\Psi_2 &= \Psi_1 \\ \sigma_{132}\Psi_2 &= \omega\Psi_1 \\ \sigma_{321}\Psi_2 &= \omega^2\Psi_1 \end{aligned} \tag{3.10}$$

The above transformations can also be presented in matrix form, assuming that each permutation operator acts on a column vector with components $\Psi_1$ and $\Psi_2$ ($\Phi_1$ and $\Phi_2$). That way we can represent any permutation as a 2×2 matrix as follows:

$$\begin{aligned} \sigma_{231} &= \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix} \\ \sigma_{312} &= \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix} \\ \sigma_{213} &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_{132} &= \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix} \\ \sigma_{321} &= \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix} \end{aligned} \tag{3.11}$$

Incidentally, we have arrived at what is known as an *irreducible representation* of the group $S_3$ of permutations of a three-element set. The representation given above is two-dimensional, as opposed to one-dimensional representations associated with boson and fermions.

It may be interesting to observe that the choice of subspaces spanned by $\Psi_1$ and $\Psi_2$ and by $\Phi_1$ and $\Phi_2$ is by no means unique. An alternative decomposition (one of an infinite number) of the four-dimensional space into two two-dimensional permutation-invariant spaces can be given, for instance, by the following combinations[15]:

$$\Lambda_1 = \frac{1}{2}\Psi_1 - \frac{\sqrt{3}}{2}\Phi_1$$
$$\Lambda_2 = \frac{1}{2}\Psi_2 - \frac{\sqrt{3}}{2}\Phi_2 \tag{3.12}$$

$$\Xi_1 = \frac{\sqrt{3}}{2}\Psi_1 + \frac{1}{2}\Phi_1$$
$$\Xi_2 = \frac{\sqrt{3}}{2}\Psi_2 + \frac{1}{2}\Phi_2 \tag{3.13}$$

It can be easily verified that all vectors in (3.12) and (3.13) are mutually orthogonal. Also, given the transformation properties of vectors $\Psi_1$ and $\Psi_2$ presented in (3.9) and (3.10), as well as the mirror-image properties of $\Phi_1$ and $\Phi_2$, we can quickly check that spaces spanned by $\Lambda_1$ and $\Lambda_2$, and by $\Xi_1$ and $\Xi_2$, are permutation-invariant. In other words, permutations of particles turn vectors $\Lambda_1$ and $\Lambda_2$ into each other's multiples, and the same for $\Xi_1$ and $\Xi_2$.

It is rather curious that the choice of permutation-invariant subspaces of the remaining four-dimensional space turns out to be so heavily underdetermined. Since these subspaces are supposed to correspond to new types of particles whose states are assumed to reside exclusively within a particular subspace, this means that potentially we have an infinite number of types of such hypothetical particles, and it is difficult to see which ones would be selected in nature. However, ignoring this issue, let us

[15] A more general formula covering cases (5.12) and (5.13) can be found in Peres (2002, p. 134), however with an error (switched symbols $\Phi_\pm$ and $\Psi_\pm$ in (5.59)).

focus on the subspaces spanned by vectors $\Psi_1$ and $\Psi_2$ and by $\Phi_1$ and $\Phi_2$, and let us ask the question of what the properties of hypothetical particles occupying these spaces would be. In particular, we are interested in the problem of the indiscernibility of such particles. And it seems that because $\Psi_1$, $\Psi_2$, $\Phi_1$, $\Phi_2$ are themselves not permutation-invariant, it should be in principle possible to discern, by means of experimental procedures, particles occupying spaces spanned by these vectors. Take $\Psi_1$ and $\Psi_2$, for instance. Since $\Psi_1$ and $\Psi_2$ are orthogonal, we can find a Hermitian operator $A$ whose eigenvectors are precisely $\Psi_1$ and $\Psi_2$, and whose eigenvalues corresponding to these eigenvectors are distinct, therefore enabling us to differentiate between states that belong to the same permutation-invariant subspace of states for a given category of paraparticles.[16]

This argument can be countered by pointing out that we should limit ourselves to symmetric operators, since particles of the same type should not be discernible under permutations, no matter whether they are bosons, fermions or paraparticles. As we have already seen in Sect. 2.4, restricting operators to those that commute with all permutations has the effect of automatically guaranteeing the validity of the Indistinguishability Postulate (the assumption that the expectation values for all admissible observables in permuted states should be identical) without any restrictions on the available states. Seen in this perspective, the division of the entire product space into subspaces of paraparticle states (plus two subspaces for bosons and fermions) has the sole purpose of delineating spaces that remain invariant under the action of admissible, that is, symmetric operators. Here, again, we encounter the distinction between *available* and *accessible* states (see ft. 13 in Chap. 2): with only symmetric operators representing physically meaningful observables, states in any paraparticle subspace are the only ones physically *accessible* for particles that *already occupy this subspace*, but this does not mean that other states are not available for them, if only they could start their existence outside of the

[16] For Huggett the existence of non-permutation-invariant observables acting in spaces of states for a given type of paraparticles is proof that paraparticles *can* be discerned by properties. He insists that operators acting on allowed spaces (i.e. such that they do not take us outside a given allowed space) are perfectly admissible as representations of physical properties, even when they are not symmetric (Huggett 2003, p. 245ff). However, Peres disagrees with that, maintaining that all operators pertaining to "indistinguishable" particles, whether bosons, fermions or paraparticles, should be invariant under relabeling of particles (Peres 2002, p. 135).

considered subspace. But in that case paraparticles would still differ in one respect from bosons and fermions: while the latter occupy states such that they guarantee the satisfaction of IP without the need of the symmetrization postulate with respect to observables, paraparticles need the restriction on operators in order to be truly indiscernible. Needless to say, the restriction which limits operators to symmetric ones makes all particles "trivially" indiscernible (at least according to orthodoxy), regardless of what type they belong to and what states they occupy.

We will close this section with an observation, made in Peres (2002, pp. 136–137), which reveals another type of peculiarity affecting paraparticle states. Peres refers to this peculiarity as "cluster inseparability". In the case of bosons and fermions, it can be proven that they satisfy the condition of *cluster separability*, which ensures that smaller components of a bigger whole can be treated as a separate system provided these components are sufficiently remote from the rest of the system (Peres 2002, pp. 128–129). For instance, if we consider a system of three fermions or bosons whose states are initially (i.e. before the symmetrization/antisymmetrization) represented by $|\varphi\rangle$, $|\psi\rangle$ and $|\eta\rangle$, and if we take any operator $A$ representing a local measurement of the subsystem of the first two components and such that $A|\eta\rangle$ is vanishingly small (the condition of remoteness), then the expectation value of $A$ calculated for the state of the three particles will be the same as the expectation value for the state of the first two particles with the third particle ignored. However, it turns out that this result does not generalize to paraparticles. As we know, the state of two particles can display only two types of symmetry: bosonic or fermionic. Consequently, when considering a system of three paraparticles, the expectation values of observables limited to two of them will depend on whether we take the third one into account. It is as if "observable properties of the particles in our laboratory were affected by the possible existence on the Moon of another particle of the same species" (Peres 2002, p. 137).[17] Peres concludes that this consequence makes the

[17] Observe, incidentally, that Peres, like virtually all physicists, continues talking about distinguishable properties of particles of the same type (such as the location on the Moon versus the location in a lab), in spite of the symmetry of the joint state of these particles and the ensuing Indiscernibility Thesis. We will propose a firm theoretical foundation for this practice in Chap. 5 and subsequent chapters.

existence of paraparticles extremely unlikely. While we may have doubts regarding arguments that are ultimately based on some pretheoretical, pre-quantum intuitions (such as the intuition of separability), for the reminder of the book we will ignore the theoretical possibility of paraparticles.

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# 4

# Logic and Metaphysics of Discernibility

Chapter 2 has ended with a brief venture into the metaphysics of discernibility. In particular, we have posed several questions regarding the connection between the validity of the Principle of the Identity of Indiscernibles (PII) and the status of objects to which PII is applicable. Is the validity of PII a necessary condition for objects to achieve the metaphysical status of individuals? Is it sufficient? What other metaphysical uses can PII be put to? Can PII serve as an assurance that facts of numerical identity and distinctness will be reducible to (or grounded in) qualitative facts involving empirically accessible properties and/or relations? All these questions presuppose that we have a good grasp of the fundamental concept of discernibility. Yet it turns out that the notion itself admits various and inequivalent interpretations. In this chapter we will offer a formal analysis of three most popular variants of discernibility—absolute, relative and weak—together with some basic facts regarding their mutual logical connections. This analysis will be done in the framework of standard, first-order logic, which is a useful tool for that sort of considerations. We will point out that the answer to the question of whether some objects can be discerned may depend on the expressive power of the language in which this discernibility is to be formulated (in particular, on


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whether this language admits the predicate of numerical identity, or the individual names for all objects in the domain).

The requirement of permutation invariance, which plays the central role in the quantum theory of many particles, can be easily formalized in first-order logic. Given this formalization, it can be quickly shown that permutation-invariant languages do not admit expressions with the help of which we could discern objects in the domain either absolutely or relatively. The only type of discernibility available in permutation-invariant languages is weak discernibility. This logical fact underlies an attempt to build a new metaphysics of quantum particles based entirely on the relations of weak discernibility. We will critically evaluate this approach, pointing at the serious limitations of weakly discerning relations in their intended role as a metaphysical foundation for identity and individuality. Mere weak discernibility does not provide us with sufficient means to distinguish objects in a way that makes it possible to make reference to individual entities. Even the potentially viable goal of providing a qualitative ground for non-qualitative facts of numerical distinctness encounters serious obstacles, since it may be argued that the weakly discerning relations which can be realistically used in quantum mechanics implicitly involve the relation of numerical identity/distinctness.

## 4.1 From Absolute to Weak Discernibility

As we have already seen in Chap. 2, the most natural way (as some would even say, the only way) to interpret the concept of qualitative discernibility is in terms of properties and their possession. We qualitatively discern object *a* from object *b* if we can identify a property *P* such that *a* possesses *P* while *b* lacks it. Metaphysicians further inquire what type of property *P* has to be in order to speak about genuine qualitative discernibility and not some mere "formal" discernibility (e.g. discernibility with the help of *haecceities*, or the properties of being this or that particular object). However, we will not attempt here to make an ontological distinction between genuine qualitative properties and "impure" properties. Instead, we will simply assume that we have at our disposal a first-order language $\mathscr{L}$ whose primitive non-logical predicates denote admissible physical properties and relations. The assumption of the admissibility of corresponding properties and relations is part of a scientific theory formulated in language $\mathscr{L}$ and therefore cannot be argued for on the basis of some high-level metaphysical considerations. On top of that, we will not include among the non-logical expressions of language $\mathscr{L}$ the identity symbol =, nor will we admit any individual constants corresponding to the objects of the domain. The reason for these restrictions, which are commonly adopted in the literature, is that both the identity predicate and individual constants are usually considered non-qualitative, and therefore potentially inappropriate as representations of properties (relations) used to discern objects. However, later we will address the question of how discernibility in a language is affected by adding these expressions to its vocabulary.

A semantic interpretation of a first-order language $\mathscr{L}$ is typically given in the form of a relational structure $\mathfrak{R} = \langle D, R_1, \ldots, R_n \rangle$, where $D$ is a non-empty set (domain) and $R_1, \ldots, R_n$ are relations of various arities on $D$, corresponding to the primitive predicates of $\mathscr{L}$. Any well-formed formula $\Phi$ of $\mathscr{L}$ can receive an interpretation in $\mathfrak{R}$ using the standard procedure. In particular, formulas with $k$ free variables will correspond to $k$-argument relations in $\mathfrak{R}$ definable in terms of relations $R_1, \ldots, R_n$ and set-theoretical operations. Once we introduce the notion of *satisfaction* of formulas in model $\mathfrak{R}$, we can give a formal definition of what it means for two objects to be discerned by properties. Following standard practice, we will call this type of discernibility *absolute*, and we will relativize it to language $\mathscr{L}$ (with its intended interpretation in the form of structure $\mathfrak{R}$):[1]

(4.1) Objects $a, b \in D$ are *absolutely discernible* in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ iff there is an open formula $\Phi(x)$ in one variable in $\mathscr{L}$ such that $\mathfrak{R} \models \Phi(a)$ and $\mathfrak{R} \not\models \Phi(b)$.

[1] In my presentation of the logical analysis of discernibility, I follow closely Ladyman et al. (2012). This comprehensive overview of the logic of discernibility draws heavily on earlier works, including Ketland (2006, 2011), Caulton and Butterfield (2012). A formal introduction to the logical notions I am using in my exposition can be found in any textbook on model theory, for example, Hodges (1997).

Symbol $\vDash$ represents the relation of satisfaction in a model, whereas the metalogical expression $\mathfrak{R} \vDash \Phi(a)$ is an abbreviation of the more common but a bit cumbersome notation $\mathfrak{R} \vDash \Phi(x)[a]$. The latter formula highlights the fact that symbol $a$ does not belong to language $\mathscr{L}$ but to its metalanguage. The relation of satisfaction connects formula $\Phi(x)$, which belongs to $\mathscr{L}$, with a particular object $a$ in the domain of $\mathfrak{R}$.

We may want to further distinguish two subtypes of absolute discernibility: *intrinsic* and *extrinsic*. As we remember from Sects. 2.6 and 2.7, the Indiscernibility Thesis with respect to quantum particles is often split into two parts: one concerning non-relational properties and the other the relational ones. We will now make precise what counts as a relational property ("extrinsic" in our terminology). Speaking loosely, a relational (extrinsic) property of an object $a$ is a property that involves objects other than $a$. In our language $\mathscr{L}$, which does not contain individual constants in its vocabulary, the only way to "involve" other objects in a one-variable formula $\Phi(x)$ satisfied by $a$ is to use quantifiers to bind variables other than $x$. Thus, for instance, formula $\exists y\, \Psi(x, y)$ contains hidden reference to objects that can be distinct from $a$, since it states that any object satisfying it must stand in the relation denoted by $\Psi$ to some entity.[2] Following this intuition, we will define the concept of *absolute intrinsic discernibility*:

(4.2) Objects $a, b \in D$ are *absolutely intrinsically discernible* in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ iff there is an open formula $\Phi(x)$ in one variable in $\mathscr{L}$ such that $\Phi(x)$ does not contain any quantifiers, and $\mathfrak{R} \vDash \Phi(a)$ and $\mathfrak{R} \nvDash \Phi(b)$.

We will call two objects merely *extrinsically discernible*, if they are absolutely discernible but not absolutely intrinsically discernible (but see the caveat in ft. 2).

[2] Strictly speaking, the satisfaction of formula $\exists y\, \Psi(x, y)$ by element $a$ does not necessitate the fact that $a$ stands in the corresponding relation to a *distinct* object (it may be the case that $a$ stands in this relation to itself). In order to ensure that distinct objects are involved, we would have to use the identity predicate, which by assumption is not available in language $\mathscr{L}$. I will ignore this complication, noting only that because of it the definition of *intrinsic absolute discernibility* given in the main text may be too restrictive. This, however, does not have any detrimental consequences for our future discussions.

Even though this may be seen as unnecessarily repetitive, we will nevertheless specify what it means for a *particular formula* to discern two objects in the domain:

(4.3) A one-variable formula $\Phi(x)$ in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ *absolutely discerns* objects $a$ and $b$ iff either $\mathfrak{R} \vDash \Phi(a)$ and $\mathfrak{R} \nvDash \Phi(b)$ or $\mathfrak{R} \vDash \Phi(b)$ and $\mathfrak{R} \nvDash \Phi(a)$.

From definition (4.3) it follows immediately that if formula $\Phi(x)$ absolutely discerns $a$ from $b$, so does its negation $\neg\Phi(x)$. Also, the relation of absolute discernibility by a formula is symmetric, that is, if a formula discerns $a$ from $b$, it also discerns $b$ from $a$.

Once we have defined the logical concept of absolute discernibility, we may note two simple facts which can illustrate the metaphysical use this concept can be put to. First, if we assume that all objects in a given finite domain are absolutely discernible from each other, this immediately implies that for any object $a$ there is a formula that is *uniquely* satisfied by $a$. This formula is simply the conjunction of all one-variable formulas that absolutely discern $a$ from any other object.[3] Such a uniquely identifying formula can serve as a basis for *naming* the corresponding object, or making unambiguous reference to it. Second, it is an elementary logical fact that if objects $a$ and $b$ are absolutely discernible, they must be numerically distinct ($a \neq b$). Clearly, if $a = b$, then the existence of an absolutely discerning formula $\Phi(x)$ would be contradictory: $\Phi(a) \wedge \neg\Phi(a)$. Thus it may be claimed that in a domain in which all objects are absolutely discernible, all facts regarding their numerical distinctness can be inferred from (or grounded in) some qualitative facts regarding possession of particular properties.

As it turns out, absolute discernibility is not the only concept of discernibility available. W.v.O. Quine distinguished two further notions of

[3] If we drop the assumption of the finiteness of the domain, there may be no *formula* in language $\mathscr{L}$ that uniquely picks out a given object (since $\mathscr{L}$ admits only finite conjunctions), but we may still consider a *property* that corresponds to an infinite set of formulas from $\mathscr{L}$. Possession of such a property will uniquely differentiate a given object from any other object in the universe.

discernibility which are today known as *relative* and *weak* (Quine 1976).[4] Relative discernibility can be defined as follows:

(4.4) Objects $a, b \in D$ are *relatively discernible* in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ iff there is an open formula $\Phi(x, y)$ in two variables in $\mathscr{L}$ such that $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \nvDash \Phi(b, a)$.

Thus relative discernibility means that there is a formula which is satisfied by two objects in one order but is not satisfied in the opposite order. An example of such a situation is provided by numbers ordered by the "greater than" relation: $x > y$. Accordingly, we can talk about a formula relatively discerning two objects:

(4.5) A two-variable formula $\Phi(x, y)$ in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ *relatively discerns* objects $a$ and $b$ iff either $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \nvDash \Phi(b, a)$, or $\mathfrak{R} \vDash \Phi(b, a)$ and $\mathfrak{R} \nvDash \Phi(a, b)$.

Again, relative discernibility by a given formula is symmetric in the sense that if a formula discerns $a$ from $b$, it likewise discerns $b$ from $a$. Moreover, if a formula $\Phi(x, y)$ relatively discerns $a$ from $b$, then its converse $\Phi(y, x)$ discerns them too.

The appropriate definitions in the case of weak discernibility will be as follows:

(4.6) Objects $a, b \in D$ are *weakly discernible* in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ iff there is an open formula $\Phi(x, y)$ in two variables in $\mathscr{L}$ such that $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \nvDash \Phi(a, a)$.

[4] Quine in his book (1960) uses the terms "absolute discernibility" and "relative discernibility", but in the later article (Quine 1976) that we are referring to in the main text, these terms are replaced by "strong discriminability" and "moderate discriminability", with weak discriminability being a third concept.

(4.7) A two-variable formula $\Phi(x, y)$ in language $\mathscr{L}$ with its intended interpretation $\mathfrak{R}$ *weakly discerns* objects $a$ and $b$ iff $\mathfrak{R} \vDash \Phi(a, b)$, $\mathfrak{R} \vDash \Phi(b, a)$, $\mathfrak{R} \nvDash \Phi(a, a)$ and $\mathfrak{R} \nvDash \Phi(b, b)$.[5]

It is often said informally that two objects are weakly discerned by a relation that is symmetric and irreflexive on the set of objects that it discerns. This does not seem to be guaranteed by def. 4.6, but as a matter of fact it is implied by it. If a formula $\Phi(x, y)$ satisfies def. 4.6, we can define a new one as follows:

$$\Psi(x,y) =_{\text{df}} \left[\Phi(x,y) \vee \Phi(y,x)\right] \wedge \left[\neg\Phi(x,x) \vee \neg\Phi(y,y)\right].$$

It can be quickly verified that $\mathfrak{R} \vDash \Psi(a, b)$, $\mathfrak{R} \vDash \Psi(b, a)$, $\mathfrak{R} \nvDash \Psi(a, a)$ and $\mathfrak{R} \nvDash \Psi(b, b)$.[6] Thus we can accept the definition of weak discernibility based on a symmetric and irreflexive relation as an equivalent of (4.6).

## 4.2 The Meaning of Weak Discernibility

The notion of relative discernibility plays a minimal role in the discussion on the identity and individuality of quantum particles, so we mention it only for the sake of completeness.[7] However, weak discernibility has received an incomparably larger amount of attention from both logicians

[5] Logically, it is possible to introduce further subdivisions into intrinsic ("monadic") and extrinsic ("relational") relative discernibility, and into intrinsic and extrinsic weak discernibility, analogously to the case of absolute discernibility. However, such distinctions are absent from the literature. The reason for that is most probably that relative and weak discernibilities are already based on relations; hence an introduction of the intrinsic variants of these grades of discernibility lacks a clear ontological motivation.

[6] Note that $\Psi(x, y)$ by definition denotes a symmetric and irreflexive relation, since $\Psi(x, y)$ is logically equivalent to $\Psi(y, x)$, and $\Psi(x, x)$ is equivalent to the logically contradictory formula $\Phi(x, x) \wedge \neg\Phi(x, x)$.

[7] The relative unimportance of the relation of relative discernibility may have something to do with the fact, noted in Ladyman et al. (2012, p. 183) that—in contrast to the two remaining types of discernibility—its complement is not an equivalence relation. It can be proven that the relation of not being relatively discernible is not transitive, which is rather odd, since indiscernibility is some sort of identity (identity with respect to some qualitative facts).

and metaphysicians interested in the foundations of physical theories. We will discuss its possible applications to the case of quantum particles later, but for now let us look a bit closer at some logical features of weak discernibility. First of all, it may be asked what this rather odd-looking relation has got to do with the general idea of discerning, or distinguishing objects. Why does a symmetric and irreflexive relation holding between two entities enable us to say that we have somehow distinguished them qualitatively? In order to better see that weak discernibility indeed expresses a legitimate intuition associated with the notion of distinguishability, let us first introduce the concept of *utter indiscernibility* as follows:[8]

(4.8) Two elements $a$ and $b$ of the domain $D$ are *utterly indiscernible* in structure $\mathfrak{R}$ corresponding to language $\mathscr{L}$ iff for every formula $\Phi$ with $n + 1$ free variables in $\mathscr{L}$ (where $n$ can be any natural number, including 0), $\mathfrak{R} \vDash \forall x_1 \ldots \forall x_n\, [\Phi(a, x_1, \ldots, x_n) \leftrightarrow \Phi(b, x_1, \ldots, x_n)]$.

Utter indiscernibility means essentially that objects $a$ and $b$ are exchangeable *salva veritate* (preserving the truth) in all sentential contexts. Alternatively, we may express this concept by saying that $a$ and $b$ stand in the same relations to the same objects in structure $\mathfrak{R}$. It is worth emphasizing that objects to which both $a$ and $b$ stand in those relations may include, among others, $a$ and $b$ themselves. Thus if $a$ happens to stand in some relation $R$ to itself, and if $b$ is utterly indiscernible from $a$, then $b$ has to stand in the same relation $R$ to $a$.

Now, it can be easily proven that the condition of utter indiscernibility, as specified in def. 4.8, is equivalent to the negation of weak discernibility:

(4.9) Two objects $a$ and $b$ are utterly indiscernible iff $a$ and $b$ are not weakly discernible.

Proof (from left to right). Suppose that $a$ and $b$ are weakly discernible, that is, there is a formula $\Phi(x, y)$ such that $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \nvDash \Phi(a, a)$.

[8] Definition (4.8) is just a variant of Quine's definition of utter indiscriminability as given in Quine (1976, p. 114). It is also equivalent to the negation of what is sometimes called "Hilbert-Bernays discernibility" (Ladyman et al. 2012, pp. 175–176).

This straightforwardly implies that formula $\forall z[\Phi(z, x) \leftrightarrow \Phi(z, y)]$ cannot be satisfied by $(a, b)$, which shows that $a$ and $b$ are not utterly indiscernible.

Right to left. Suppose that $a$ and $b$ are not utterly indiscernible. This means that there is a formula $\Phi$ such that $\mathfrak{R} \vDash \neg\forall x_1\ldots\forall x_n\,[\Phi(a, x_1, \ldots, x_n) \leftrightarrow \Phi(b, x_1, \ldots, x_n)]$, and thus $\mathfrak{R} \vDash \exists x_1\ldots\exists x_n \neg\,[\Phi(a, x_1, \ldots, x_n) \leftrightarrow \Phi(b, x_1, \ldots, x_n)]$. Hence, formula $\exists x_1\ldots\exists x_n \neg\,[\Phi(x, x_1, \ldots, x_n) \leftrightarrow \Phi(y, x_1, \ldots, x_n)]$ weakly discerns $a$ from $b$, since $\exists x_1\ldots\exists x_n \neg\,[\Phi(a, x_1, \ldots, x_n) \leftrightarrow \Phi(a, x_1, \ldots, x_n)]$ is logically contradictory.

We can conclude that the weak discernibility of objects $a$ and $b$ ensures that there are some objects to which $a$ stands but $b$ does not stand in some relation $R$, and this justifies the assessment that $a$ and $b$ are somehow differentiated from each other. However, it would be too quick to jump to the conclusion that the weak discernibility of objects enables us to uniquely characterize each one of them in a way that would make it possible to refer to them individually. Suppose we wanted to make reference to object $a$ and not to a numerically distinct object $b$ merely on the basis of the fact that $a$ stands in relation $R$ to some object $c$, while $b$ does not stand in $R$ to $c$. It should be clear that this procedure may be executed successfully only on the condition that object $c$ has been already singled out and uniquely referred to. However, in the case when $a$ and $b$ are weakly discernible, the only guaranteed way to differentiate $a$ from $b$ is by saying that $a$ stands in some relation to $b$, while $b$ does not stand in this relation to $b$. But this is of no help, unless we have already distinguished $b$ from $a$. There is clearly some sort of circularity here that prevents us from making a definite separation between objects on the basis of their weak discernibility.

In order to make this argument more precise, Ladyman and Bigaj (2010) introduced the notion of *witness discernibility*,[9] by which they understand discernibility based on the fact that object $a$ stands in some relation to object $c$ (which may be called a witness), while $b$ does not stand in this relation to $c$, with an additional requirement that all objects absolutely indiscernible from $c$ should remain standing in the same relations to $a$ and $b$. This additional requirement, which I consider intuitive,

[9] This term was not originally used in Ladyman and Bigaj (2010) but was coined in a later critical analysis (Linnebo and Muller 2013). See also a response to Linnebo and Muller in (Bigaj 2015a).

excludes the possibility that there could be two absolutely indiscernible witnesses *c* and *d* that would stand in "opposite" relations to *a* and *b* (i.e. *c* stands in *R* to *a* and not to *b*, while *d* stands in *R* to *b* and not to *a*). In the case when *a* and *b* are weakly but not absolutely discernible, the above requirement is not satisfied, since a potential witness in the form of object *b* has an absolutely indistinguishable "counterpart" in object *a*, and its relations to *a* and *b* are reversed in the sense explained above. Øystein Linnebo and F.A. Muller in their (2013) critique have proven that witness discernibility is extensionally equivalent to absolute discernibility in finite domains. This result in my opinion shows two things. One minor thing is that witness discernibility and absolute discernibility are not *intensionally* equivalent, since they diverge in infinite domains, as Linnebo and Muller show. The other, more important consequence is that only absolute discernibility can guarantee that reference can be made to one and not the other object.

Given the above conclusion, the only metaphysical use that weak discernibility can be put to is to secure that non-qualitative facts of numerical distinctness will be grounded in qualitative relational facts. Indeed, if a weakly discerning relation connects objects *a* and *b*, we can logically infer from this that *a* and *b* are numerically distinct. However, we have to keep in mind that the relation used to weakly discern objects cannot involve numerical identity, since in that case the grounding could not occur. We will see in Sect. 4.6 that this assumption may not be satisfied in the most interesting case of relations that weakly discern quantum particles of the same type.

## 4.3 Grades of Discernibility in Extended Languages

The three grades of discernibility introduced above are connected by simple logical relations. It is straightforward to observe that absolute discernibility implies relative discernibility, which in turn implies weak discernibility. We will abbreviate these implications as follows:

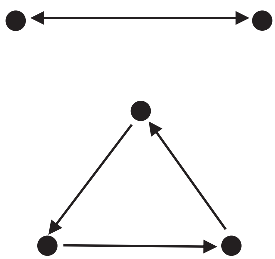

**Fig. 4.1** Structures whose elements are weakly but not relatively discernible (upper diagram) and relatively but not absolutely discernible (lower diagram)

(4.10) $\mathrm{Abs}_{\mathscr{L}}(a, b) \Rightarrow \mathrm{Rel}_{\mathscr{L}}(a, b) \Rightarrow \mathrm{Weak}_{\mathscr{L}}(a, b)$.

Proof. Let $a$ and $b$ be absolutely discerned by a formula $\Phi(x)$, i.e. $\mathfrak{R} \vDash \Phi(a)$ and $\mathfrak{R} \vDash \neg\Phi(b)$. In that case the two-argument formula $\Phi(x) \wedge \neg\Phi(y)$ relatively discerns $a$ and $b$. Next, suppose that $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \vDash \neg\Phi(b, a)$ for some formula $\Phi(x, y)$. From this it follows that $\Phi(x, y) \wedge \neg\Phi(y, x)$ is satisfied by $(a, b)$ and not by $(a, a)$, which is sufficient to prove that $a$ and $b$ are weakly discernible.

It is worth pointing out that implications in the opposite directions don't hold. That is, there are cases of objects that are weakly but not relatively discernible (and therefore not absolutely discernible), and cases of objects relatively but not absolutely discernible. Examples of these cases are usually depicted in the form of graphs, where vertices (nodes) represent objects, and connecting arrows represent binary relations (see Fig. 4.1).

In order to gain a better understanding of the introduced concepts of discernibility and their dependence on the expressive power of the language, it may be useful to briefly discuss languages whose vocabulary was expanded by adding either the symbol of identity, or individual constants for all elements of the domain (which, for that purpose, will be considered to be finite). Let $\mathscr{L}^{=}$ symbolize the extension of language $\mathscr{L}$ obtained by adding the identity symbol to its vocabulary, and let $\mathscr{L}^{*}$ stand for the language obtained from $\mathscr{L}$ by introducing individual constants for all objects in the domain. Obviously, all three grades of discernibility in languages $\mathscr{L}^{=}$ and $\mathscr{L}^{*}$ will be weaker than the corresponding grades in $\mathscr{L}$ (if two objects are discerned in $\mathscr{L}$, they'll be discerned in a language that has a larger vocabulary). What is more interesting is the logical relations between appropriate grades of discernibility in language $\mathscr{L}^=$ and in $\mathscr{L}^*$. As it turns out, absolute, relative and weak discernibility in $\mathscr{L}^=$ retain their mutual logical dependence known from language $\mathscr{L}$:

(4.11) $\text{Abs}_{\mathscr{L}^=}(a, b) \Rightarrow \text{Rel}_{\mathscr{L}^=}(a, b) \Rightarrow \text{Weak}_{\mathscr{L}^=}(a, b) \Leftrightarrow \neq$,

with the same absence of opposite implications as before. However, one important novelty is that weak discernibility in $\mathscr{L}^=$ collapses into numerical distinctness $\neq$, since obviously $\neq$ is a relation that is symmetric and irreflexive. Thus weak discernibility is not an interesting concept in a language equipped with the symbol of identity.

As for language $\mathscr{L}^*$, here all three grades of discernibility become logically equivalent, while weak discernibility in $\mathscr{L}^*$ turns out to be equivalent to weak discernibility in $\mathscr{L}$:

(4.12) $\text{Abs}_{\mathscr{L}^*}(a, b) \Leftrightarrow \text{Rel}_{\mathscr{L}^*}(a, b) \Leftrightarrow \text{Weak}_{\mathscr{L}^*}(a, b) \Leftrightarrow \text{Weak}_{\mathscr{L}}(a, b)$.

Here is a quick proof. Implications $\text{Abs}_{\mathscr{L}^*}(a, b) \Rightarrow \text{Rel}_{\mathscr{L}^*}(a, b) \Rightarrow \text{Weak}_{\mathscr{L}^*}(a, b)$ are proven analogously to the proof of (4.10) above. Implication $\text{Weak}_{\mathscr{L}^*}(a, b) \Rightarrow \text{Weak}_{\mathscr{L}}(a, b)$: let formula $\Phi(x, y, \underline{c}_1, \ldots, \underline{c}_k)$ weakly discerns $a$ from $b$, where $\underline{c}_1, \ldots, \underline{c}_k$ are all individual constants in the formula. In that case formula $\exists z_1 \ldots \exists z_k\, [\Phi(x, y, z_1, \ldots, z_k) \wedge \neg\Phi(x, x, z_1, \ldots, z_k)]$, obtained by replacing all constants with variables bound by existential quantifiers, weakly discerns $a$ and $b$. And obviously this formula belongs to language $\mathscr{L}$.

Implication $\text{Weak}_{\mathscr{L}}(a, b) \Rightarrow \text{Abs}_{\mathscr{L}^*}(a, b)$ can be proven as follows: let $\Phi(x, y)$ be such that $\mathfrak{R} \models \Phi(a, b)$ and $\mathfrak{R} \not\models \Phi(a, a)$. In that case the one-variable formula $\Phi(\underline{a}, x)$, where $\underline{a}$ is the constant denoting object $a$, absolutely discerns $a$ from $b$. This concludes proof of the equivalences in (4.12).

Let us note an interesting fact about discernibility in language $\mathscr{L}^*$. While it may seem that adding names for all elements of the domain should enable us to differentiate between any two distinct objects using their unique names, as a matter of fact this is technically incorrect. The reason for that is rather simple: in order to attribute a proper name to an object, we need the identity symbol. In language $\mathscr{L}^{=*}$ equipped both with individual constants and identity, for each object *a* in the domain, we can produce the expression of the form $x = \underline{a}$ which is satisfied uniquely by *a* (this expression can be said to represent *a*'s *haecceity*). Thus all grades of discernibility in $\mathscr{L}^{=*}$ collapse into numerical distinctness (which makes PII trivially true, exactly as observed earlier in Sect. 2.8). However, language $\mathscr{L}^{*}$ does not have the means to express *haecceities* for all elements of the domain. On the other hand, the presence of individual constants denoting all objects in the domain is sufficient to turn the negation of utter indiscernibility (4.8) into a "legitimate" absolute discernibility by monadic formulas. This is so, because now the fact that an object *a* stands in relation *R* to some *c* while *b* does not stand in *R* to *c* can be expressed using formula $\Phi_R(x, \underline{c})$, where $\Phi_R$ denotes relation *R*, and $\underline{c}$ is the name for *c*.

Similarly, the mere presence of the identity symbol in $\mathscr{L}^{=}$ does not "spoil" absolute and relative discernibility in the sense of making them trivially satisfied by all distinct objects.[10] Adding identity to our vocabulary makes it possible to perform more fine-grained distinctions between objects, taking, for instance, into account the number of related entities. In language $\mathscr{L}^{=}$ it is possible to discern *a* from *b* entirely on the basis of the fact that *a* stands in some relation *R* to a greater number of objects than *b*. I will not debate here whether counting the number of objects

[10] Claims to the contrary crop up in the literature even after the analysis in Ladyman et al. (2012). For instance, Muller writes "if identity were permitted in the sufficient condition of PII, then […] the truth of PII would become as trivial as any tautology" (Muller 2015, p. 8). Unfortunately, he is not careful enough to specify which variant of PII he has in mind. This is however crucial, because—as can be seen from (4.11)—while weak discernibility with identity reduces trivially to numerical distinctness, neither absolute nor relative discernibility suffers a similar fate. Thus PII based on absolute discernibility is not tautologous, even if we include identity in absolutely discerning formulas. Note that later in his article Muller repeats an ill-conceived restriction on the admissible predicates that can be used in discerning formulas, when he says that predicates containing "=" are forbidden to discern because they express "trivializing properties" (Muller 2015, p. 19). A similar error was made by Bas van Fraassen and Isabelle Peschard (van Fraassen and Peschard 2008, pp. 23–24). They claim that it is possible to absolutely discern all numerically distinct objects using a predicate built only out of the identity symbol (Argument 3). However, their way of presenting this alleged predicate with the help of the expression "$x = \ldots$" (with the intention that when this "predicate" is applied to any variable *y*, the ellipsis … is replaced by *y*) is rather non-standard. Again, we stress that including the identity predicate in absolutely discerning formulas does not trivialize PII, unless we also have proper names for all objects!

presupposes their absolute discernibility or not (most probably it doesn't). However, let us note that if we wanted to use the validity of PII as the basis for grounding facts of numerical distinctness in qualitative facts, then absolute (or relative) discernibility in $\mathscr{L}^{=}$ does not seem appropriate for the job, since the relation of numerical identity may be already involved in appropriate discerning formulas. In conclusion, while for some purposes discernibility in extended languages $\mathscr{L}^{=}$ and $\mathscr{L}^{*}$ seems legitimate, in the context of the problem of identity and individuation in quantum mechanics, it is better to limit ourselves to purely "qualitative" language $\mathscr{L}$.

## 4.4 Discernibility and Symmetry

The next important logical step is an analysis of the relation between various grades of discernibility in language $\mathscr{L}$ and the condition of permutation invariance, or symmetry. Let us start with formulating a precise definition of what it means for a language (with its specific interpretation $\mathfrak{R}$) to be symmetric:

(4.13) $\mathscr{L}$ is *symmetric* (resp. *permutation-invariant*) with respect to its intended interpretation $\mathfrak{R}$ iff for every open formula $\Phi(x_1, x_2, \ldots x_n)$ in $\mathscr{L}$ and any permutation $\sigma: D \to D$, $\mathfrak{R} \vDash \Phi(a_1, a_2, \ldots a_n)$ iff $\mathfrak{R} \vDash \Phi(\sigma(a_1), \sigma(a_2), \ldots \sigma(a_n))$.

It is relatively easy to show that condition (4.13) is equivalent to an analogous requirement imposed on structure $\mathfrak{R}$:

(4.14) Relational structure $\mathfrak{R}$ is *symmetric* iff for any $k$-element relation $R$ in $\mathfrak{R}$, any elements $a_1, \ldots, a_k \in D$ and any permutation $\sigma: D \to D$, $R(a_1 \ldots a_k)$ iff $R(\sigma(a_1) \ldots \sigma(a_k))$.

The equivalence of (4.13) and (4.14) can be established by using induction over the complexity of formulas in $\mathscr{L}$.

Now we should explore the connection between the condition of symmetry imposed on language $\mathscr{L}$ (or structure $\mathfrak{R}$) and the availability of discerning formulas in $\mathscr{L}$. As a matter of fact, it is easy to observe that both absolutely and relatively discerning formulas are excluded from symmetric languages, as stated in the following theorem:

(4.15) If language $\mathscr{L}$ is symmetric, then no two objects in $D$ are absolutely or relatively discerned in $\mathscr{L}$.

Proof. Suppose that $a$ and $b$ are relatively discernible in $\mathscr{L}$, that is, for some $\Phi(x, y)$ in $\mathscr{L}$, $\mathfrak{R} \vDash \Phi(a, b)$ and $\mathfrak{R} \nvDash \Phi(b, a)$. Taking any permutation σ such that $\sigma(a) = b$ and $\sigma(b) = a$, we immediately see that the condition of symmetry in (4.13) is violated. And given (4.10) this also means that there can be no formula in $\mathscr{L}$ that absolutely discerns $a$ and $b$. (This fact can be also proven directly, by pointing out that the symmetry of $\mathscr{L}$ requires that if a one-variable formula $\Phi(x)$ is satisfied by $a$, it must be satisfied by any other object in the domain.)

In contrast to absolute and relative discernibility, it can be shown that weak discernibility is not excluded in symmetric languages. The case depicted in Fig. 4.1 (upper diagram) shows a symmetric structure whose two elements are nevertheless weakly discerned by the relation represented by the double arrow. Indeed, if it is the case that $\mathfrak{R} \vDash \Phi(a, b)$ but $\mathfrak{R} \nvDash \Phi(a, a)$, as required when $a$ and $b$ are weakly discernible, this fact by itself does not violate the symmetry condition, since no permutation can map objects $a$ and $b$ onto the same element $a$.

The main conclusion from our elementary logical analysis is that weak discernibility appears to be the only type of discernibility that can be achieved in languages which satisfy the condition of permutation invariance. This result is of crucial importance to the metaphysical problem of identity and individuality in quantum mechanics. Given that the language in which we describe states and properties of systems of same-type particles should obey the condition of symmetry, as explained in Chaps. 2 and 3, we can conclude that in this language it is impossible to build a formula that would discern particles either absolutely or relatively. This consequence seems to be perfectly in line with the Indiscernibility Thesis as presented and discussed in Sects. 2.6 and 2.7. If the PII based on the absolute grade of discernibility fails, we may either give up entirely on the idea of quantum particles as individuals, or try to use weak discernibility for the purpose of restoring some elements of quantum individuality. The second option is precisely the strategy that we will analyze in detail in the next sections.

However, in anticipation of an unexpected turnaround that is coming up in later parts of the book, I would like to point out that the argument against discernibility in permutation-invariant languages contains an intriguing loophole. While it is unquestionable that such languages cannot host direct linguistic representations of absolutely (or relatively) discerning properties (or relations), this does not automatically exclude the possibility that such properties (relations) may objectively exist and that *the existence of such properties may be expressible even in symmetric sentences*. A more detailed analysis of this possibility will have to wait until Sect. 6.1. For now our attention will turn to the concept of weak discernibility.

## 4.5 Weak Discernibility in Quantum Mechanics

The notion of weak discernibility was fished out of the vast archives of logic and dusted off by Simon Saunders, who first noticed its potential to resolve (or at least alleviate) the metaphysical problem of quantum indiscernible particles. Saunders, in his article (2003), reported the equivalence between weak discernibility and the Hilbert-Bernays discernibility (see ft. 8). He also noted that while PII based on weak discernibility is itself a pretty weak principle, it is by no means so weak as to be logically trivial, since there are possible situations in which distinct objects are not even weakly discernible. While Saunders admits, as we did earlier, that the weak discernibility of objects does not guarantee that individual reference can be made to each of them, still its potential role in grounding the "bare" numerical distinctness of objects in empirical facts should not be ignored.

The success of the new strategy based on weak discernibility depends of course on whether quantum particles of the same type are indeed guaranteed to be distinguishable with the help of physically admissible relations that are symmetric and irreflexive. Saunders, in his articles (2003, 2006), makes the claim that all same-type fermions can be weakly discerned, citing as an example the relation of having opposite spins that connects two fermions in the singlet spin state. While this example of a weakly discerning relation cannot be straightforwardly generalized to all fermionic states, it may nevertheless be instructive to see in greater detail how this special case works, before we move on to a more universal argument. Let us, then, consider a system of two spin-half fermions (e.g. two electrons), focusing on its spin state. Each individual spin-state space is a two-dimensional space $\mathscr{H}$ spanned by vectors $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, representing values "up" and "down" of spin in an arbitrary direction $z$. Thus the state space of the two-particle system will be the tensor product $\mathscr{H} \otimes \mathscr{H}$ of two two-dimensional spaces, which is a space spanned by four vectors: $|\uparrow_z\rangle|\uparrow_z\rangle$, $|\uparrow_z\rangle|\downarrow_z\rangle$, $|\downarrow_z\rangle|\uparrow_z\rangle$ and $|\downarrow_z\rangle|\downarrow_z\rangle$.

The spin component of an individual particle is represented by a Hermitian operator $s_z$ whose action on the basis vectors is defined as follows:

$$s_z|\uparrow_z\rangle = \tfrac{1}{2}|\uparrow_z\rangle \text{ and } s_z|\downarrow_z\rangle = -\tfrac{1}{2}|\downarrow_z\rangle.$$

Thus vectors $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ are eigenvectors of $s_z$ with the corresponding eigenvalues ½ and –½. Now, it can be easily checked that the operator $s_z \otimes I + I \otimes s_z$ acting in the tensor product $\mathscr{H} \otimes \mathscr{H}$ represents the sum of the spin components for both particles. We can verify by direct calculation that vectors $|\uparrow_z\rangle|\uparrow_z\rangle$ and $|\downarrow_z\rangle|\downarrow_z\rangle$ are eigenvectors for $s_z \otimes I + I \otimes s_z$ with the corresponding eigenvalues equal, respectively, 1 and −1, while both $|\uparrow_z\rangle|\downarrow_z\rangle$ and $|\downarrow_z\rangle|\uparrow_z\rangle$ are eigenvectors corresponding to the same eigenvalue 0. This confirms that $s_z \otimes I + I \otimes s_z$ indeed represents the sum of the $z$-components of spin for both particles. Note that we have a case of degeneracy here, which is precisely the familiar exchange degeneracy discussed in Chap. 3.

The situation when two particles have opposite spins obviously corresponds to the eigenvalue 0 of the operator $s_z \otimes I + I \otimes s_z$. However, in order to argue that the particles are connected by a weakly discerning relation, we have to prove formally that the opposite-spin relation does not connect a particle with itself. What is the quantum-mechanical representation of the self-contradictory situation of having one's spin opposite to itself? The most natural way to present this impossible scenario is again with the help of the sum of spin $s_z$ and itself. Because we are considering a system of two particles, we have to use operators defined on the product $\mathscr{H} \otimes \mathscr{H}$, which leaves us with the choice of $s_z \otimes I + s_z \otimes I = 2(s_z \otimes I)$ for the first particle and analogously $2(I \otimes s_z)$ for the second particle. If the system were in an eigenstate for any of these operators with the eigenvalue 0 (which is impossible), we would say that the spin of one particle is opposite to itself.

All this can be concisely presented as follows. Let $s_z^{(x)}$, where variable $x$ ranges over the set of numbers {1, 2}, denote the tensor product of the single-particle operator $s_z$ and the identity operator, where $s_z$ occupies the $x$-th slot in the product. Then we can define the following binary relation $R$:

$$R(x,y) \text{ iff } \left(s_z^{(x)} + s_z^{(y)}\right)|\psi\rangle = \mathbf{0}, \tag{4.16}$$

where $|\psi\rangle$ is the state of both particles. Now, when the state $|\psi\rangle$ is the singlet state:

$$\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle|\downarrow_z - |\downarrow_z\rangle|\uparrow_z\rangle\right)$$

it can be quickly verified that $R(1, 2)$, as well as $R(2, 1)$, is true, but $R(1, 1)$ and $R(2, 2)$ are false. Thus formula $R(x, y)$ weakly discerns particles 1 and 2, as long as they occupy the singlet state.

The task of finding a more universal weakly discerning formula applicable to all fermionic states has been taken on by Muller and Saunders in 2008. Here is a brief summary of their proposal. They start with selecting a complete set of orthogonal one-dimensional projectors $P_i$ in a single-particle Hilbert space $\mathscr{H}$, that is, such that $\sum_{i=1}^{d} P_i = I$, where $d$ is the dimension of $\mathscr{H}$. Then they define $P_{ij} = P_i - P_j$, and subsequently build the following operators (not projectors!), acting in the tensor product $\mathscr{H} \otimes \mathscr{H}$:

$$P_{ij}^{(1)} = P_{ij} \otimes I, P_{ij}^{(2)} = I \otimes P_{ij}.$$

Finally, they pick the following Hermitian operators in $\mathscr{H} \otimes \mathscr{H}$:

$$\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(1)} = \sum_{i,j=1}^{d} P_{ij}^{2} \otimes I,$$
$$\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(2)} = \sum_{i,j=1}^{d} P_{ij} \otimes P_{ij}.$$

Muller and Saunders then prove that the following eigenequations hold for any antisymmetric state $|\psi_A\rangle$:[11]

$$\begin{aligned} \sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(1)} |\psi_A\rangle &= 2(d-1)|\psi_A\rangle \\ \sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(2)} |\psi_A\rangle &= -2|\psi_A\rangle \end{aligned} \tag{4.17}$$

Now, the key element of their strategy is the definition of a binary relation $R_{-2}$ that can be argued to weakly discern particles 1 and 2 (again, variables $x$ and $y$ range over the set of values {1, 2}):

$$R_{-2}(x, y) \text{ iff } \sum_{i,j=1}^{d} P_{ij}^{(x)} P_{ij}^{(y)} |\psi\rangle = -2|\psi\rangle. \tag{4.18}$$

On the basis of Eq. (4.17), we can conclude that if particles 1 and 2 occupy any antisymmetric state, relation $R_{-2}$ connects 1 and 2 but does not connect 1 with 1. Hence, formula $R_{-2}$ $(x, y)$ weakly discerns fermions in any state. Moreover, as Muller and Saunders are keen to emphasize, the weakly discerning relation $R_{-2}$ involves facts they call "categorical", that is, facts expressible in terms of the possession of definite values by certain observables. Notice that thanks to the eigenequations (4.17) both

[11] As a matter of fact, the first equation in (4.17) holds for any state whatsoever.

relational propositions symbolized as $R_{-2}(1, 2)$ and $\neg R_{-2}(1, 1)$ are made true by the possession of a definite value (either −2 or $2(d-1)$) by some observable ($\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(2)}$ or $\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(1)}$). This is important, because thanks to that feature the statement of weak discernibility does not depend on possible alternative interpretations of quantum mechanics that question the full eigenstate-eigenvalue link. As is well known, the fact that the quantum-mechanical state is *not* an eigenstate for a particular operator, under certain hidden variables hypotheses, does not imply that the observable does not objectively possess a definite value. On the other hand, virtually all interpretations accept the implication from being in an eigenstate to possessing the corresponding eigenvalue.

Thus, we have proven that fermions of the same type can be weakly discerned in any state. How about bosons? Here we have a problem. The method proposed above won't work in general, since the second equation in (4.17) is guaranteed to hold only in fermionic states. It may be instructive to see more closely where the problem lies. In order to do that, let us transform a bit the operator used in this equation:

$$\begin{aligned}\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(2)} &= \sum_{i,j=1}^{d} \left(P_i - P_j\right) \otimes \left(P_i - P_j\right) \\ &= \sum_{i,j=1}^{d} \left[P_i \otimes P_i + P_j \otimes P_j - P_i \otimes P_j - P_j \otimes P_i\right] \\ &= \sum_{i,j=1}^{d} \left(P_i \otimes P_i + P_j \otimes P_j\right) - \sum_{i,j=1}^{d} \left(P_i \otimes P_j + P_j \otimes P_i\right).\end{aligned}$$

It can be easily checked that the following algebraic equality holds: $\sum_{i,j=1}^{d} \left(A_i + B_j\right) = d\left(\sum_{i=1}^{d} A_i + \sum_{i=1}^{d} B_i\right)$, from which it follows that the above expression can be further expanded into the following formula:

$$2d\sum_{i=1}^{d} P_i \otimes P_i - \sum_{i,j=1}^{d} \left(P_i \otimes P_j + P_j \otimes P_i\right),$$

while the second summand can be rewritten as below:

$$\sum_{i,j=1}^{d}\left(P_i \otimes P_j + P_j \otimes P_i\right) = \sum_{i,j=1}^{d} P_i \otimes P_j + \sum_{i,j=1}^{d} P_j \otimes P_i$$
$$= \sum_{i=1}^{d} P_i \otimes \sum_{j=1}^{d} P_j + \sum_{j=1}^{d} P_j \otimes \sum_{i=1}^{d} P_i = I \otimes I + I \otimes I = 2I \otimes I,$$

given that $\sum_{i=1}^{d} P_i = \sum_{j=1}^{d} P_j = I$. As a result, the original operator has the following form:

$$\sum_{i,j=1}^{d} P_{ij}^{(1)} P_{ij}^{(2)} = 2d \sum_{i=1}^{d} P_i \otimes P_i - 2I \otimes I. \tag{4.19}$$

Observe that operator $2d \sum_{i=1}^{d} P_i \otimes P_i$ in (4.19) gives zero on all antisymmetric states (in order to see that, we have to write the selected antisymmetric state $|\psi_A\rangle$ in the orthogonal basis created by the eigenvectors $|\varphi_i\rangle$ of projectors $P_i$). Thus we have proven the second equation in (4.17). But we can also see that (4.17) is not guaranteed to hold for all bosonic states, since these states written in the basis $|\varphi_i\rangle$ may possess "diagonal" elements of the form $|\varphi_i\rangle \otimes |\varphi_i\rangle$. Only symmetric states that are obtained from the antisymmetric ones by replacing minus signs with pluses will satisfy the second equation in (4.17). But consider the symmetric state $|\varphi_i\rangle \otimes |\varphi_i\rangle$. When we apply to it the operator from (4.19), we can immediately see that the result will be $2(d-1)|\varphi_i\rangle \otimes |\varphi_i\rangle$. This means that for the bosons occupying this state both $R_{-2}(1, 2)$ and $R_{-2}(1, 1)$ hold, hence they are not weakly discerned by relation $R_{-2}$.

Drawing this conclusion, Muller and Saunders then proceed to argue that even in totally symmetric product states, it is still possible to weakly discern individual particles, albeit only probabilistically. This means that the expectation value of an operator representing a given relation holding between particles 1 and 2 may be different from the expectation value of an operator representing the same relation holding between one particle and itself. A simple example can illustrate this situation: let $A$ be any single-particle Hermitian operator. Let us stipulate that the expression $A^{(x)}A^{(y)}$ denote the product $A \otimes A$ when $x = 1$ and $y = 2$ (or $x = 2$ and $y = 1$), while it equals $A^2 \otimes I$ when $x = y = 1$ (and analogously for $x = y = 2$). Now we can define the following, probabilistic relation $R_t$:

(4.20) $R_t(x, y)$ iff $\langle\psi| A^{(x)}A^{(y)}|\psi\rangle = t$, where $|\psi\rangle$ is the state of both particles.

If $|\psi\rangle = |\varphi\rangle \otimes |\varphi\rangle$, then from the above it follows that $R_t(1, 2)$ iff $\langle\varphi|A|\varphi\rangle^2 = t$, and $R_t(1, 1)$ iff $\langle\varphi|A^2|\varphi\rangle = t$. Now, the difference $\langle\varphi|A|\varphi\rangle^2 - \langle\varphi|A^2|\varphi\rangle$ is the standard measure of what is known as the *dispersion* (or *variance*) of variable $A$, and it is well known that if $|\varphi\rangle$ is not an eigenvector of $A$, its dispersion is non-zero. Hence we can always find a meaningful observable $A$ such that for a certain number $t$, $R_t(1, 2)$ but not $R_t(1, 1)$. The bosons occupying the product state $|\varphi\rangle \otimes |\varphi\rangle$ are weakly probabilistically discernible.

In a surprising twist, Muller and Seevinck (2009) strengthen this claim by arguing that even categorical weak discernibility (i.e. discernibility with the help of possessing specific eigenvalues) is available in bosonic states. Their example is indeed very simple and is applicable to any state written in the position space. Let $P$ be the momentum operator for a single particle and $Q$ – position. Then we can introduce the usual notation $P^{(1)} = P \otimes I$, $P^{(2)} = I \otimes P$, $Q^{(1)} = Q \otimes I$, $Q^{(2)} = I \otimes Q$, and we can consider the familiar notion of the *commutator*: $[A, B] =_{\text{df}} AB - BA$. As can be easily established, the commutator of the momentum of one particle and the position of the other equals zero: $[P^{(1)}, Q^{(2)}] = [P \otimes I, I \otimes Q] = 0$. However, it is a well-known fact that the position and momentum of one particle do not commute. As a matter of fact, their commutator equals $[P^{(1)}, Q^{(1)}] = -i\hbar$. This leads to the definition of the following, weakly discerning relation:

$$C(x,y) \text{ iff } \left[P^{(x)}, Q^{(y)}\right] = 0. \tag{4.21}$$

Given the equality $[P^{(1)}, Q^{(1)}] = -i\hbar$, it may be argued that particles 1 and 2 are categorically weakly discerned in any state $|\psi\rangle$, since $C(1, 2)$ is equivalent to saying that $|\psi\rangle$ is an eigenstate for the commutator $[P^{(1)},$

$Q^{(2)}$] with the eigenvalue equal 0, while $\neg C(1, 1)$ is made true by the fact that $|\psi\rangle$ is an eigenstate of the commutator $[P^{(1)}, Q^{(1)}]$ with the different value $-i\hbar$.

Alternative weakly discerning relations have been proposed by other authors as well (Caulton 2013; Huggett and Norton 2014; for an overview see Bigaj 2015b). However, we will not discuss them here. Instead, we will proceed directly to the question of what the existence of weakly discerning relations tells us about the metaphysics of quantum particles. As a result of analyzing this problem, it will be argued that the usefulness of these particular relations for restoring some of the metaphysical uses of PII may be legitimately called into question.

## 4.6 Quantum Weak Discernibility and Identity

The weak discernibility program in quantum mechanics, as it may be called, has the ambition of reversing the dominating trend in the metaphysics of quantum objects, which centers around the Indiscernibility Thesis and its negative consequences for the individuality of these entities. Saunders in his article (2006) stresses that in the light of the weak discernibility of quantum particles, we may safely say that these particles are *objects* (presumably as opposed to less object-like entities such as dollars in a bank account). However, he stops short of categorizing them as *individuals*, noting only that the notion of an individual itself is in need of further clarifications. Muller and Saunders subsequently supply such a clarification, when they stipulate that "we call objects that are absolutely discernible from all other objects *individuals*; those that are only relationally discernible from all other objects we call *relationals*" (Muller and Saunders 2008, p. 504). In that way they introduce a new category of entities called *relationals*, in which they include all objects that are not absolutely but merely weakly discernible from each other.

In spite of later efforts by Muller to promote the introduction of the new category of relationals as a breakthrough in the metaphysics of science (in Muller 2011, 2015), to my best knowledge this concept has never entered the mainstream of scientifically informed metaphysics. Regardless of whether we agree with the metaphysical importance of the new distinction between individuals and relationals, still the fact that quantum objects turn out to be (merely) weakly discernible from each other deserves to be closely evaluated from the metaphysical perspective. Muller touts the discovery of the weak discernibility of quanta as a great success of "naturalistic" metaphysics, based on science rather than armchair speculations (Muller 2015, p. 24). He stresses that PII, instead of being undermined by the development of modern physics, receives a strong boost and therefore is vindicated as "a metaphysical crown on our most fundamental knowledge of the universe". Be that as it may, one can still ask what is so precious about this piece of the most fundamental knowledge to deserve such glowing praise. Again, we should recall the two fundamental uses of different variants of PII: one in supporting the possibility of making reference to separate objects in the domain, and the other in securing the grounding of numerical identity/distinctness in qualitative facts. Since the first use is excluded in the case of weak discernibility, as noted in Sect. 4.2, we have to turn to the second one.

A successful grounding of numerical identity in qualitative facts about objects is definitely something to strive for. It directly responds to the worry of empiricists, from John Locke to Max Black, that "bare" identity is not directly accessible to us. Here is how this is supposed to work in the case of weak discernibility. If one asks, as Black did in his seminal article (Black 1952), how do we know that there are two distinct objects and not one, even though they possess the exact same properties, one answer may be "because they are connected by a weakly discerning relation, for instance being spatially separated by one meter". Since weakly discerning relations can never connect an object with itself, we have an assurance that the number of objects is indeed two and not one. However, this explanation relies on the assumption that facts of numerical diversity are not in some form "smuggled" into the weakly discerning relations. And some authors claim that this is precisely the case.

The *circularity charge* against weak discernibility can be found, for example, in French and Krause (2006; Hawley 2006, 2009; Wüthrich 2009). French and Krause (pp. 169–171), followed by Christian Wüthrich (p. 1048), insist that the numerical diversity of objects is *presupposed* by the relation that weakly discern them. However, it is not clear what sense of presupposition they have in mind. If presuppositions are simply identified with logical consequences, then they are right that weak discernibility "presupposes" numerical diversity, since it logically entails it. However, in the same manner numerical diversity is implied by absolute discernibility (discernibility by monadic properties), and yet to my knowledge nobody charges absolute discernibility with circularity. French and Krause probably have an epistemic rather than logical notion of presupposition in mind when they write in (2006, p. 175) that "to know that a relation is irreflexive presupposes that one knows that the relata are diverse". If presupposition implies here a temporal order (i.e. in order to know that an irreflexive relation connects objects *a* and *b*, I have to know *beforehand* that *a* is distinct from *b*), then it seems to me that this statement is plainly wrong. I can clearly learn that *a* is distinct from *b* on the basis of the information that an irreflexive relation connects *a* and *b*, not vice versa (I discuss a straightforward example of such a situation later in this section).

Katherine Hawley offers a slightly different take on the circularity problem. She claims that if objects *a* and *b* are connected by a weakly discerning relation *R*, then *a* and *b* differ with respect to possessing properties "standing in *R* to *a*" and "standing in *R* to *b*" (Hawley 2009, p. 109). But then Hawley observes that the distinctness of these two properties is grounded in the fact that *a* is numerically distinct from *b*. Hence the distinction between *a* and *b* based on weak discernibility is grounded in the fact that $a \neq b$ and therefore cannot ground it. This argument is incorrect on several counts. First of all, we don't need to use the properties "standing in *R* to *a*" and "standing in *R* to *b*" to make a qualitative distinction between *a* and *b*—to establish their absolute discernibility it is perfectly sufficient to show that one object exemplifies the property "standing in *R* to *a*", while the second does not exemplify it. But even more fundamentally, we don't need (as a matter of fact, we are not even allowed) to invoke discernibility by dubious "impure" properties involving direct reference to *a* and *b*. Weak discernibility by relation *R* is a different type of discernibility that does not rely on possessing distinct properties. We discern objects *a* and *b* weakly by verifying that they stand in an irreflexive relation to each other, not by attributing distinct and incompatible monadic properties to each of them. So Hawley's argument falls short of proving that the fact that *a* and *b* are connected by an irreflexive relation *R* must be grounded in the numerical distinctness of *a* and *b*.

I don't know of any strong arguments showing that *all* weakly discerning relations necessarily suffer from the circularity problem. However, it is rather clear that circularity may be present if the weakly discerning relation used in a particular case contains an *indispensable reference* to the fact of the numerical distinctness of discerned objects. For in that case we cannot justify the claim that there are two numerically distinct objects by relying on their weak discernibility, since this discernibility in turn relies on their being distinct objects.

Let us delve deeper into what should count as an *indispensable* use of identity, and what use can be seen as dispensable. The starting point is that, unless a particular weakly discerning formula is considered to be primitive, it should be reducible to, or definable in terms of more fundamental predicates. An obvious suggestion seems to be that in a proper definition no use of the identity symbol should be allowed. However, there are some uses of identity that are clearly harmless, for instance, when we apply the following equivalence:

$$R(x,y) \text{ iff } [x \neq y \wedge R(x,y)] \vee [x = y \wedge R(x,x)]. \tag{4.22}$$

Apart from the fact that the above statement can hardly be seen as a proper definition of formula *R*(*x*, *y*), since symbol *R* is present on both sides of the equivalence, the rhs contains a totally harmless occurrence of the identity symbol. This can be seen in the fact that both sides are logically equivalent, so the use of symbol "=" can be eliminated without any loss of meaning. However, let us consider the following characterization of *R*:

$$R(x,y) \text{ iff } [x \neq y \wedge T] \vee [x = y \wedge F], \tag{4.23}$$

where *T* is any true sentence, while *F* – a false sentence. Clearly, *R* is a weakly discerning relation, since it is both symmetric and irreflexive (it follows from (4.23) that $R$ holds between all and only distinct objects). However, it should be obvious that the use of the identity symbol is essential here. As a matter of fact, the presence of sentences $T$ and $F$ is in a sense a red herring, since the main job of weak discernibility is done by formulas $x \neq y$ and $x = y$. In order to decide whether particular objects $a$ and $b$ satisfy $R(x, y)$, we have to verify first whether $a \neq b$ or $a = b$, and this defeats the purpose for which weak discernibility was introduced in the first place.

Incidentally, the weakly discerning formula introduced by Huggett and Norton (2014) as a corrected variant of Muller and Saunders's original proposal, with an intention of avoiding some of the formal shortcomings of the latter,[12] falls precisely under the category (4.23). Thus in this case the cure turns out to be worse than the disease (for more details on that, see Bigaj 2015b, pp. 48–49). However, this does not mean that Muller and Saunders's relation is free from an analogous problem. In order to better see a potential issue with their definition (4.18), let us consider yet another possible form a definition of a weakly discerning formula might take:

$$R(x,y) \text{ iff } \left[x \neq y \wedge T(x,y)\right] \vee \left[x = y \wedge F(x,x)\right], \qquad (4.24)$$

where it is assumed that $T(x, y)$ is satisfied by any pair of distinct objects, whereas $F(x, y)$ is irreflexive (i.e. $\mathfrak{R} \nvDash F(a, a)$ for any $a$). Again, under these assumptions $R$ is guaranteed to be symmetric and reflexive in the entire domain, and thus it weakly discerns all its elements. But, as in the previous example, the use of the identity symbol is essential and indispensable, since without it there is no possibility to connect formulas $T(x, y)$ and $F(x, x)$ in a way which ensures that formula $R(x, y)$ will be weakly discerning. The key difference between cases (4.22) and (4.24) lies in the fact that formulas $T(x, y)$ and $F(x, y)$ denote distinct relations which are juxtaposed in order to create a new, "gerrymandered" relation

[12] These shortcomings, in a nutshell, are that operators used in (4.17) are not symmetric when the number of particles is greater than 2. Huggett and Norton propose to use the totally symmetrized variants of these operators, but in that way they lose one crucial element of the original definition, namely the presence of variables $x$ and $y$, and this is why their definition of a relation weakly discerning same-type fermions has to take the form of (4.23).

$R$, while in (4.22) we have the same relation $R$ that is artificially split into two cases: one when its arguments are identical, and the other when they are distinct. Let us stress that the problem with formula (4.24) is not simply that relation $R$ is a combination of two other relations $T$ and $F$, but rather that in order to create $R$ out of $T$ and $F$, we have to resort to the notion of numerical identity, in order to apply $T$ when two objects are distinct, and $F$ when there is one and the same object.

Let us now present the weakly discerning relation proposed by Muller and Saunders in a similar fashion:

$$R_{-2}(x,y) \text{iff} \left( x \neq y \wedge \sum_{i,j=1}^{d} P_{ij}^{(x)} P_{ij}^{(y)} |\psi\rangle = -2|\psi\rangle \right) \vee \left( x = y \wedge \sum_{i,j=1}^{d} P_{ij}^{(x)} P_{ij}^{(x)} |\psi\rangle = -2|\psi\rangle \right). \tag{4.25}$$

At first sight it looks like formula (4.25) is of the type (4.22), and therefore the use of the identity symbol in the rhs is dispensable. However, this optimistic conclusion may be too hasty. The notation used by Muller and Saunders suggests that expression $\sum_{i,j=1}^{d} P_{ij}^{(x)} P_{ij}^{(y)} |\psi\rangle = -2|\psi\rangle$ denotes the "same" relation regardless of whether $x = y$ or $x \neq y$. But let us look closer at the components of the above formula in the form of operators $P_{ij}^{(x)} P_{ij}^{(y)}$. It should be clear that the meaning of the operation of taking the product of two projectors $P_{ij}^{(x)}$ and $P_{ij}^{(y)}$ strongly depends on whether variables $x$ and $y$ denote two different objects or one object. When $x \neq y$, operator $P_{ij}^{(x)} P_{ij}^{(y)}$ has the form of $(P_{ij} \otimes I)(I \otimes P_{ij}) = P_{ij} \otimes P_{ij}$ (given that the number of particles is 2). On the other hand, in the case of $x = y$, the product operation yields either $P_{ij}P_{ij} \otimes I$ or $I \otimes P_{ij}P_{ij}$.

It is worth stressing that the problem we are identifying is not merely that the outcome of a given procedure varies depending on whether we are dealing with the same object or two different objects. This is to be expected from a weakly discerning relation, which is supposed to differentiate between cases when $a = b$ and cases when $a \neq b$. The problem is though that we don't know how to execute the procedure necessary to produce the definite outcome, unless we know which one is the case, since ultimately we have *two* procedures and not one. To repeat: in order to verify whether the relation expressed in formula $\sum_{i,j=1}^{d} P_{ij}^{(x)} P_{ij}^{(y)} | \psi\rangle = -2 | \psi\rangle$ holds for some objects $a$ and $b$, we have to be able to multiply operators $P_{ij}^{(a)}$ and $P_{ij}^{(b)}$. But this can't be done until we verify whether $a$ is the same as $b$, or different.

Thus I submit that formula (4.25) is closer in spirit to the general case described in (4.24) than to the "innocuous" use of identity in (4.22). And the same problem affects all other relations that are supposed to weakly discern quantum particles, including relations (4.16), (4.20) and (4.21) discussed in Sect. 4.5. To consider just one more example: in order to check whether the categorical weakly discerning relation $C$ defined in (4.21) holds between two bosons $a$ and $b$, we have to compute the commutator of the operators we denote as $P^{(a)}$ and $Q^{(b)}$. But again, in order to multiply two operators $P^{(a)}$ and $Q^{(b)}$ we have to know their exact form (whether they have the single-particle operators $P$ and $Q$ in the same slot or in different slots of appropriate tensor products). And this in turn requires that we know in advance whether or not $a = b$.

It is instructive to observe that the problem we have just described does not necessarily affect all weakly discerning relations. Consider the standard case of the Euclidean spatial distance between points. Regardless of whether points $a$ and $b$ are distinct or identical, the procedure of weak discernment is the same: we select any Cartesian coordinate system, assign to $a$ and $b$ their particular coordinates $(x_a, y_a, z_a)$ and $(x_b, y_b, z_b)$ and calculate the value of the expression:

$$\sqrt{(x_a - x_b)^2 + (y_a - y_b)^2 + (z_a - z_b)^2}.$$

If the value is non-zero, we have weakly discerned points $a$ and $b$.[13] There is no reason to distinguish two independent relations (or two procedures): one applicable when $a \neq b$, and the other when $a = b$.

[13] It may be objected that once we have assigned the coordinates to points $a$ and $b$, the rest of the procedure is moot: we already know that if the coordinates are different, $a$ must be distinct from $b$.

It may be argued that the flaw in the quantum-mechanical weakly discerning relations that we have just uncovered is an artifact of the tensor-product formalism, which works on the condition of the distinctness of the particles whose joint state is represented in the *N*-fold tensor product of Hilbert spaces.[14] It is true that vectors representing joint states, as well as operators representing properties of composite systems, are defined under the assumption that we know the exact number of the components in a given system, and thus that we have already numerically differentiated between these components. No wonder, then, that when we try to define in such a formalism a weakly discerning relation that is supposed to ground numerical distinctness, we run into the circularity problem. Perhaps it is possible to come up with an alternative formalism that could treat systems with different numbers of particles in a uniform way without prejudging their identity/diversity. Actually, such a formalism may be already available in the form of the so-called Fock spaces. We will talk more about this in Sect. 7.2. But the lesson from the currently discussed cases seems to be that if we take the standard formalism at face value, there is a serious problem with all relations that are supposed to weakly discern individual particles. As these relations, when defined within the formalism, presuppose an indispensable use of numerical identity, we cannot use them to ground numerical distinctness.

As we remember from Chap. 2, taking the tensor-product formalism at face value is an essential part of the orthodoxy with its thesis of the complete indiscernibility of quantum particles of the same type. Now, we have seen that the same literal interpretation of the formalism prevents us from using weak discernibility as a way to restore some meaningful form of PII. In the light of this fact we may choose to slip back into the revisionary metaphysics of non-individuals, but we may also reconsider our commitment to the literal interpretation of the formalism. Perhaps we

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That may be true, but again the assignment of coordinates to any points does not presuppose that we should know in advance whether these points are identical or not. And, besides, possessing any coordinates by a point is not its genuine property, since it depends on the choice of the framework of reference, whereas the Euclidean distance between points is invariant. An alternative weakly discerning relation whose application does not require making a choice of particular coordinates may be given in the form of the length of the shortest continuous curve connecting *a* and *b*.

[14] This has been noted in Arenhart (2013, p. 476).

were too hasty in associating each factor in the product of Hilbert spaces with an individual component (Factorism). In the next chapters, we will see how this suggestion may be turned into a working alternative to orthodoxy.

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# 5

# Qualitative Individuation of Same-Type Particles: Beyond Orthodoxy

One of the pillars of the orthodox approach to quantum individuation is the Indiscernibility Thesis that we have formulated and discussed in Chap. 2 (Sect. 2.6). Yet, as we remember, proofs of this thesis rely on one dubious assumption: that properties of individual particles composing a larger physical system are represented by non-symmetric operators (products of a number of identity operators and one non-identity Hermitian operator representing a given property). Operators that are not permutation-invariant should be disallowed on the basis of the Indistinguishability Postulate, as we have observed at the beginning of Chap. 3, and yet they seem indispensable as representations of "individualized" properties. In the following sections, we will try to question this apparent indispensability of non-symmetric operators by considering an alternative way to represent such individuating properties with the help of totally symmetric operators. In that way we will challenge the claim, put forward, for example, by French and Redhead (1988, p. 239) that "from the point of discussing PII we should not restrict the discussion […] to symmetric combinations such as $Q_1 + Q_2$" (where $Q_1 = Q \otimes I$ and $Q_2 = I \otimes Q$). Actually, it is quite possible to fruitfully discuss the validity of PII using only symmetric operators, but the conclusions from such a




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discussion will markedly depart from orthodoxy. It will turn out that there is a way to individuate particles of the same type, in the sense of attributing to them distinct properties, using only symmetric operators applied to the whole system. But before we can show how to do that, we have to ask what is the correct physical interpretation of a certain kind of symmetric operators.

## 5.1 Symmetric Operators and Their Meaning

We will begin this section with identifying and correcting a surprisingly persistent confusion regarding the meaning of certain symmetrized Hermitian operators acting in tensor products of Hilbert spaces vis-à-vis their components acting in the individual spaces (factors). Suppose that we have a one-particle Hermitian operator $A$ defined on $\mathscr{H}$ (for simplicity we'll assume that $A$ is non-degenerate). Let $|\lambda_a\rangle$ be an eigenstate of $A$ corresponding to some eigenvalue $a$. Now, we already know that the tensor product $A \otimes I$ acting in $\mathscr{H} \otimes \mathscr{H}$ is interpreted as representing the observable corresponding to $A$ attributed to particle number one, while particle number two is left to its own devices. But we also know that $A \otimes I$ does not satisfy the requirement of permutation invariance and therefore is inappropriate as a mathematical representation of observables for systems of same-type particles. Is there any way to symmetrize this product operator in order to avoid the aforementioned difficulty while retaining its intended meaning (which is that it should represent observable $A$ attributed to one component of the system)? One possible answer to this question, tacitly presupposed by many authors without much explanation, is that the correct symmetrized variant of $A$ should be $A \otimes I + I \otimes A$, or—as some authors insist—its "normalized" version $\frac{1}{2}(A \otimes I + I \otimes A)$.[1] This solution certainly "looks" right, given its close

[1] For instance, Peres in his well-known textbook writes "The operator **A** [equal $A \otimes I$ – T.B.], which was used to refer to the 'first' system, must now be replaced by a new operator, namely $A \otimes I + I \otimes A$" (Peres 2003, p. 129). Huggett and Norton follow this lead, except that they maintain that the correct symmetrized version of $A \otimes I$ should be normalized (see formula 17 in Huggett and Norton 2014, p. 49). F.A. Muller and Gijs Leegwater in their recent paper present a general argument in favor of this interpretation, which is nevertheless questionable (Muller and Leegwater

similarity to the standard way of symmetrizing/antisymmetrizing states of same-type particles. And yet a simple calculation can immediately convince us that the above symmetrized operators cannot possibly represent observable $A$ attributed to one of the two particles (without saying which particle it is).

If the correct interpretation of the operator $A \otimes I + I \otimes A$ was as it is commonly suggested, the product vectors of the form $|\lambda_a\rangle|\varphi\rangle$ and $|\varphi\rangle|\lambda_a\rangle$, where $|\varphi\rangle$ is any vector in $\mathscr{H}$, would have to be its eigenvectors with the corresponding value $a$, since these vectors clearly reflect situations in which one particle assumes value $a$ for observable $A$. But the algebra does not confirm this supposition. Applying the operator to the first vector yields this:

$$\left(A \otimes I + I \otimes A\right)|\lambda_a\rangle|\varphi\rangle = a|\lambda_a\rangle|\varphi\rangle + |\lambda_a\rangle A|\varphi\rangle.$$

Because vector $|\varphi\rangle$ has been selected completely arbitrarily, the result of the operation $A|\varphi\rangle$ can be any vector in $\mathscr{H}$, which shows that $|\lambda_a\rangle|\varphi\rangle$ is generally not an eigenvector of $A \otimes I + I \otimes A$ with the corresponding value equal $a$.

It may be quickly verified that products of the form $|\lambda_a\rangle|\lambda_{a'}\rangle$, where $|\lambda_a\rangle$ and $|\lambda_{a'}\rangle$ are eigenvectors of $A$ corresponding to eigenvalues $a$ and $a'$, *are* eigenvectors for $A \otimes I + I \otimes A$:

$$\left(A \otimes I + I \otimes A\right)|\lambda_a\rangle|\lambda_{a'}\rangle = a|\lambda_a\rangle|\lambda_{a'}\rangle + a'|\lambda_a\rangle|\lambda_{a'}\rangle = \left(a + a'\right)|\lambda_a\rangle|\lambda_{a'}\rangle.$$

2020, p. 5). They observe that the expectation value of any single-particle operator $A_i$ in the reduced state represented by density operator $\rho_i$ is equal to the expectation value of the symmetric operator $\mathrm{A} = \frac{1}{2}(A_1 \otimes I + I \otimes A_2)$ in the joint state $\rho$ of the entire system (formula 5). However, the equality $\mathrm{Tr}(A_i\rho_i) = \mathrm{Tr}(\mathrm{A}\rho)$ holds only for symmetric/antisymmetric states $\rho$. According to the definition of the reduced state (see def. A.16 in Appendix), $\mathrm{Tr}(A_1\rho_1) = \mathrm{Tr}[(A_1 \otimes I)\rho]$ and $\mathrm{Tr}(A_2\rho_2) = \mathrm{Tr}[(I \otimes A_2)\rho]$. But because for symmetric/antisymmetric states all reduced single-particle states are identical ($\rho_1 = \rho_2$), it follows that $\mathrm{Tr}(A_1\rho_1) = \mathrm{Tr}(A_2\rho_2) = \mathrm{Tr}(\mathrm{A}\rho)$. Yet this equality fails for other joint states (e.g. products of orthogonal states); hence it cannot be used as a general argument in favor of interpreting the symmetric normalized operator $A = \frac{1}{2}(A_1 \otimes I + I \otimes A_2)$ as representing property $A$ associated with one of the components. As Caulton (2013, p. 61) correctly observes, operator A represents the statistical mean of $A$ taken over the two particles, and not $A$ attributed to one of them.

From this it follows that the considered symmetrized operator actually corresponds to the following experimental procedure: we perform measurements of $A$ on *both* particles and add the obtained results. In other words, the corresponding observable is in fact *the sum* of the values of observable $A$ on both particles (or the average value, in the case of the normalized operator $\frac{1}{2}(A \otimes I + I \otimes A)$). So, if, for instance, $A$ is spin $s_z$ in a given direction $z$, the operator $s_z \otimes I + I \otimes s_z$ is just the total spin in this direction for the entire system. But it definitely does not represent spin $s_z$ attributed to one of the two particles (as we have already noted in Sect. 4.5).

It may be interesting to observe that in spite of the fact that $A$ is assumed to be non-degenerate, operator $A \otimes I + I \otimes A$ suffers from an inevitable degeneracy related to the permutation-induced freedom of choice. For instance, both vector $|\lambda_a\rangle|\lambda_{a'}\rangle$ and its permuted variant $|\lambda_{a'}\rangle|\lambda_a\rangle$ represent the same value $a + a'$. The permutation-induced degeneracy disappears though when we limit ourselves to the symmetric or antisymmetric sectors of the product $\mathscr{H} \otimes \mathscr{H}$ (as we have already observed in Chap. 3, the elimination of this degeneracy is one of the main reasons why we introduce the Symmetrization Postulate with respect to states). For instance, the antisymmetric subspace $\mathscr{A}(\mathscr{H} \otimes \mathscr{H})$ is spanned by the mutually orthogonal vectors of the form $\frac{1}{\sqrt{2}}(|\lambda_\alpha\rangle|\lambda_{\alpha'}\rangle - |\lambda_{\alpha'}\rangle|\lambda_\alpha\rangle)$, and in principle all eigenvalues $a + a'$ associated with these vectors could be distinct (it is definitely possible to find a set of numbers such that every two of them add up to a different number—for example, the numbers 1, 2, 3). On the other hand, it is also possible that some degeneracy will remain, since the sum can be the same while its summands differ. An example of such a scenario is the case with four eigenvalues 1, 2, 3, 4, in which case the eigenvectors $\frac{1}{\sqrt{2}}(|\lambda_1\rangle|\lambda_4\rangle - |\lambda_4\rangle|\lambda_1\rangle)$ and $\frac{1}{\sqrt{2}}(|\lambda_2\rangle|\lambda_3\rangle - |\lambda_3\rangle|\lambda_2\rangle)$ will correspond to the same eigenvalue 5. Thus the operator $A \otimes I + I \otimes A$ is "unable" to differentiate the case in which one measurement reveals outcome 1 and the other 4 from the case in which one measurement reveals 2 and the other 3. This further confirms the already established fact that

$A \otimes I + I \otimes A$ has nothing to do with measuring $A$ separately on one or the other particle.

Generalizing a bit further, we may observe that by analogy it would be incorrect to interpret the symmetric operator $A \otimes B + B \otimes A$ as encoding the "disjunctive" property of $A$ applied to one particle and $B$ applied to the other. For starters, if $A$ and $B$ are incompatible, that is, they do not share all their eigenvectors, this creates an immediate difficulty in that the products of the form $|\lambda_a\rangle|\chi_b\rangle$ will generally not be eigenvectors for the entire operator, even though $|\lambda_a\rangle$ and $|\chi_b\rangle$ are eigenvectors, respectively, for $A$ and $B$. But even if $A$ and $B$ are compatible, the action of the operator on the vector $|\lambda_a\rangle|\chi_b\rangle$ will yield a not-so-straightforward algebraic combination of values of $A$ and $B$ as the corresponding eigenvalue: $ad' + bb'$, where $d'$ is the value $A$ assumes when a particle is in state $|\chi_b\rangle$ (which, by assumption, is an eigenvector for $A$), and $b'$ is the value attributed to $B$ when the state is $|\lambda_a\rangle$.

Does this mean that the quantum-mechanical formalism has no means to express the intuitively acceptable disjunctive properties of many-particle systems? The answer is that we should not accept defeat just yet. We haven't exhausted all the weaponry in the arsenal of quantum theory, including projection operators. And it turns out that projectors may be just what we need to solve the problem at hand. Suppose, then, that $E_a$ is the projector onto a one-dimensional space spanned by vector $|\lambda_a\rangle$ (i.e. $E_a$ is the dyad $|\lambda_a\rangle\langle\lambda_a|$). In other words, $E_a$ represents the specific property of $A = a$ possessed by a single particle. Now, what would be the correct interpretation of the sum $E_a \otimes I + I \otimes E_a$? Again, as before, it does not represent what we want it to, that is, the property of either particle one or particle two possessing the value $a$ of observable $A$ (one reason for that is that it is not even a projection operator anymore, since it is not idempotent, as can be quickly checked by multiplying it by itself). But now let us consider a small correction to the above formula in the form of the following operator:

$$\Omega(E_a) = E_a \otimes I + I \otimes E_a - E_a \otimes E_a. \tag{5.1}$$

Its action on vectors of the form $|\lambda_a\rangle|\varphi\rangle$ and $|\varphi\rangle|\lambda_a\rangle$ reveals that we may be onto something:

$$\Omega(E_a)|\lambda_a\rangle|\varphi\rangle = |\lambda_a\rangle|\varphi\rangle + |\lambda_a\rangle E_a|\varphi\rangle - |\lambda_a\rangle E_a|\varphi\rangle = |\lambda_a\rangle|\varphi\rangle,$$

and similarly:

$$\Omega(E_a)|\varphi\rangle|\lambda_a\rangle = E_a|\varphi\rangle|\lambda_a\rangle + |\varphi\rangle|\lambda_a\rangle - E_a|\varphi\rangle|\lambda_a\rangle = |\varphi\rangle|\lambda_a\rangle.$$

It is not difficult to verify that indeed $\Omega(E_a)$ is a projector in $\mathscr{H} \otimes \mathscr{H}$ whose image is spanned by vectors $|\lambda_a\rangle|\varphi\rangle$ and $|\varphi\rangle|\lambda_a\rangle$ (formally: $\Omega(E_a)[\mathscr{H} \otimes \mathscr{H}] = \{E_a[\mathscr{H}] \otimes \mathscr{H}\} \oplus \{\mathscr{H} \otimes E_a[\mathscr{H}]\}$. Thus it can be claimed that $\Omega(E_a)$ represents the required property of one particle (we don't know which) possessing the value $A = a$.[2]

It turns out, then, that there may be a way to express in the quantum-mechanical formalism facts regarding possession of measurable properties by unspecified components of composite systems. The proper method to do that, as it seems, is to use symmetrized projection operators of the form $\Omega(E)$ given in (5.1). In the next section we will indicate some possible objections to that method of encoding appropriate properties of composite systems, but for now let us briefly address one more technical question of whether (and why) it is really the case that the proposed method of symmetrizing operators in $\mathscr{H} \otimes \mathscr{H}$ works only for projectors, while it fails in the general case of any Hermitian operators. In particular, one may ask why we can't rely on the spectral decomposition theorem and build an appropriate symmetric counterpart of $A$ acting in $\mathscr{H} \otimes \mathscr{H}$ using projectors of the form $\Omega(E_a)$. As we recall (Sect. 2.5), the spectral decomposition theorem guarantees that operator $A$ can be presented in the form of the sum $\sum_{a\in\Delta} aE_a$, where $\Delta$ is the range of values of $A$. It seems natural, then, that the sum $\sum_{a\in\Delta} a\Omega(E_a)$ acting in $\mathscr{H} \otimes \mathscr{H}$ should

[2] It is interesting to observe that an analogous trick won't work for general Hermitian operators, that is, it is not the case that the operator $A \otimes I + I \otimes A - A \otimes A$ represents observable $A$ associated with one of the two particles (the reader is invited to do the required calculations confirming this negative result). The key fact responsible for this difference between projectors and any Hermitian operators is that projectors have only the numbers 0 and 1 as their eigenvalues.

represent the required observable $A$ applied to any of the two components of the system.

Yet this method won't work, for the simple reason that projectors $\Omega(E_a)$ are not mutually orthogonal (or even disjoint, for that matter). In the spectral decomposition of a given Hermitian operator, all projection operators corresponding to different eigenvalues must be orthogonal and therefore also disjoint (in the sense that the only common subspace for corresponding eigenspaces is the zero space). However, in the case of projectors $\Omega(E_a)$ and $\Omega(E_b)$, where $a \neq b$, their corresponding subspaces have non-zero vectors in common. The product vector $|\lambda_a\rangle|\lambda_b\rangle$ lies both in the subspace projected onto by $\Omega(E_a)$ and in the subspace corresponding to $\Omega(E_b)$. Even if we restrict ourselves to symmetric/antisymmetric sectors of the product $\mathscr{H} \otimes \mathscr{H}$, the problem remains, since vector $\frac{1}{\sqrt{2}}(|\lambda_a\rangle|\lambda_b\rangle \pm |\lambda_b\rangle|\lambda_b\rangle)$ still belongs to both subspaces. This is to be expected: after all, while one particle cannot simultaneously possess two unequal values of a given operator, it is perfectly possible that one of the two particles possesses value $a$ of $A$ and one of the particles (not the same!) possesses a different value $b$ of the same operator $A$. But this means that we can't hope to build a new Hermitian operator out of projectors corresponding to such non-exclusive properties of the entire system.[3]

## 5.2 Symmetric Projectors and Disjunctive Properties

It might seem that we have stumbled on the projectors of the form $\Omega(E_a)$ presented in (5.1) entirely by accident. Actually, this accidental discovery can be put on a firmer footing by approaching the problem more generally. The question we will consider now is what formal requirements have to be imposed on operators acting in $\mathscr{H} \otimes \mathscr{H}$ in order for them to be able to represent some symmetric combinations of one-particle

[3] However, as we will see in Sect. 5.3, there is a way to find a symmetric "counterpart" to non-symmetric product operators $A \otimes I$ and $I \otimes A$ which will recover the appropriate expectation values in some special cases (formula 5.11).

properties. Let us start generally with any projection operator $E$ acting in $\mathscr{H}$ and therefore representing some property $\Pi_E$ of a single particle. Given that we would like to consider a system of two such particles, what are the formal restrictions to be placed on any linear operator $\Omega(E)$ in $\mathscr{H} \otimes \mathscr{H}$ that could possibly represent the very same property $\Pi_E$ applied to one component of the composite system? It is relatively straightforward to notice that such an operator $\Omega(E)$ should satisfy the following desiderata:[4]

(5.2) (i) $\Omega(E)$ should be Hermitian,
(ii) $\Omega(E)$ should be symmetric,
(iii) $\Omega(E)$ should be a projector (and therefore idempotent),
(iv) $\Omega(E)$ should be the sum of tensor products of one-particle operators involving only $E$ and $I$ (the identity).

Requirements (5.2) are self-explanatory and don't need any extensive comments. From conditions (ii) and (iv), it follows that the most general form $\Omega(E)$ can have is the following:

$$\Omega(E) = aE \otimes I + aI \otimes E + bE \otimes E.$$

Given that $\Omega(E)$ is assumed to be Hermitian, coefficients $a$ and $b$ have to be real. Now we can apply requirement (iii):

$$\Omega(E)^2 = \Omega(E).$$

Let us calculate the square of $\Omega(E)$ (using the fact that $E^2 = E$):

$$\Omega(E)^2 = a^2 E \otimes I + a^2 I \otimes E + \left(2a^2 + 4ab + b^2\right) E \otimes E.$$

Comparing formulas for $\Omega(E)$ and $\Omega(E)^2$, we can first derive $a^2 = a$. This equation obviously has two solutions in real numbers (0 and 1), but we can discard the value 0, as the operator $E \otimes E$ clearly represents the situation in which both particles have the same property $\Pi_E$. If we put $a = 1$, we can easily solve the quadratic equation in $b$ which arises as the

[4] The following analysis is based on my Bigaj (2015).

result of equating the coefficients of the component $E \otimes E$ in the expansions of $\Omega(E)$ and $\Omega(E)^2$: $b = 2 + 4b + b^2$. This equation has two solutions $b = -1, -2$, from which it follows that there are two operators satisfying (5.2):

$$\Omega_1(E) = E \otimes I + I \otimes E - E \otimes E$$
$$\Omega_2(E) = E \otimes I + I \otimes E - 2E \otimes E. \quad (5.3)$$

We can immediately recognize that $\Omega_1(E)$ is the same as the operator (5.1) discussed in the previous section, which, as we have agreed, represents the situation in which one particle has property $\Pi_E$, while the other is in any state whatsoever (including the possibility that it can also be in a state with property $\Pi_E$, which justifies the proposal to interpret $\Omega_1(E)$ as stating that *at least* one particle possesses property $\Pi_E$). In order to interpret the second operator $\Omega_2(E)$, we may first present it in an equivalent form:

$$\Omega_2(E) = E \otimes (I - E) + (I - E) \otimes E.$$

It can be easily verified that $\bar{E} = I - E$ is a projector that projects onto the subspace orthogonal to the subspace $E[\mathscr{H}]$ and complementing it to the entire Hilbert space (i.e. each vector in $\mathscr{H}$ can be presented as a sum of vectors from $E[\mathscr{H}]$ and $\bar{E}[\mathscr{H}]$). This follows from the fact that for any vector $|\varphi\rangle$ orthogonal to $E[\mathscr{H}]$,

$$(I - E)|\varphi\rangle = |\varphi\rangle - E|\varphi\rangle = |\varphi\rangle \left(\text{since } E|\varphi\rangle = 0\right),$$

plus the observation that if $|\psi\rangle$ is not orthogonal to $E[\mathscr{H}]$, $E|\psi\rangle \neq 0$. Thus it is to be expected that $\Omega_2(E)$ represents the situation in which one particle has property $\Pi_E$, while the other particle *definitely* does not have property $\Pi_E$ (in the sense that it possesses a property *inconsistent* with $\Pi_E$—for instance, a different value of the same observable). This can be confirmed by calculating the action of $\Omega_2(E)$ on vectors $|\lambda_E\rangle|\lambda_{\bar{E}}\rangle$ and $|\lambda_{\bar{E}}\rangle|\lambda_E\rangle$, where $E|\lambda_E\rangle = |\lambda_E\rangle$ and $\bar{E}|\lambda_{\bar{E}}\rangle = |\lambda_{\bar{E}}\rangle$. Clearly, these vectors are eigenvectors of $\Omega_2(E)$ with 1 as the eigenvalue; hence it may be inferred that $\Omega_2(E)$ indeed has the above-stated meaning, sometimes expressed in the statement that *exactly one* particle possesses property $\Pi_E$.[5]

However, an argument can be put forward questioning the proposed interpretation of operators $\Omega_1(E)$ and $\Omega_2(E)$. In order to better see which elements of this interpretation are objectionable, let us again consider the case of a non-degenerate one-particle observable $A$ and a one-dimensional projector $E_a$ corresponding to the value $a$ of $A$. While it is true that product vectors $|\lambda_a\rangle|\varphi\rangle$ and $|\varphi\rangle|\lambda_a\rangle$ are eigenvectors of $\Omega_1(E_a)$ with the eigenvalue 1, so is their superposition $|\lambda_a\rangle|\varphi\rangle \pm |\varphi\rangle|\lambda_a\rangle$. And the standard interpretation of states like this one is that individual particles occupying it do not possess definite states $|\lambda_a\rangle$ or $|\varphi\rangle$. The state is entangled, and the components are characterized by identical mixed states. Consequently, the argument goes, projector $\Omega_1(E_a)$ cannot represent the situation in which one particle *definitely* possesses property $A = a$, since some of the states in the subspace onto which $\Omega_1(E_a)$ projects clearly do not describe such a situation, but rather are entangled states that are not reducible to the products of pure states. An analogous argument can be presented in the case of the second operator $\Omega_2(E_a)$, except now the eigenvectors will be of the form $|\lambda_a\rangle|\lambda_b\rangle$, $|\lambda_b\rangle|\lambda_a\rangle$ and, most importantly, $|\lambda_a\rangle|\lambda_b\rangle \pm |\lambda_b\rangle|\lambda_a\rangle$. Again, if the system is in the latter state, no definite value $a$ or $b$ of observable $A$ should be attributed to any component of the system. This argument seems especially damaging in the case of particles of the same type, since the antisymmetric and symmetric subspaces contain primarily entangled states (with an exception of bosonic product states consisting of identical component states). Thus the asymmetric product states $|\lambda_a\rangle|\varphi\rangle$ and $|\lambda_a\rangle|\lambda_b\rangle$, which we used to argue for a particular interpretation of the operators $\Omega_1(E)$ and $\Omega_2(E)$, are not even admissible in the case of same-type particles.

[5] Observe that the arguments in support of the proposed readings of operators $\Omega_1(E)$ and $\Omega_2(E)$ are analogous to the "inductive" argument in favor of the empirical equivalence between an observable $A$ acting in a single-particle Hilbert space $\mathcal{H}$ and the tensor-product operator $A \otimes I$ acting in $\mathcal{H} \otimes \mathcal{H}$ that we considered in Sect. 2.2. In both cases we derive the required conclusions from analyzing how appropriate operators act on selected, special case vectors while ignoring the remaining cases.

This is a serious interpretive issue, and we will have to give it full attention (for a comprehensive treatment of this problem, we will have to wait until Sects. 5.4 and 5.5). The issue of the physical interpretation of the projection operators $\Omega_1(E)$ and $\Omega_2(E)$ is the crossroads at which two competing approaches to the problem of the individuation of quantum particles start to part ways. The orthodox interpretation insists that projectors $\Omega_1(E)$ and $\Omega_2(E)$ cannot possibly be taken as representing the properties discussed above, while the emerging heterodox approach will question that. Observe, further, that if we follow the orthodox approach, which rejects the proposed interpretations of the discussed projectors, then immediately two questions arise. One question is: what is the *correct* interpretation of operators $\Omega_1(E)$ and $\Omega_2(E)$ as opposed to the rejected one? The second question complements the first one: how can we, within the formalism of quantum mechanics, alternatively represent the physical propositions incorrectly associated with $\Omega_1(E)$ and $\Omega_2(E)$? We will take up these questions in turn, starting with the second one.

To remind ourselves: the heterodox approach associates with the projectors $\Omega_1(E_a)$ and $\Omega_2(E_a)$ the following propositions:[6]

(5.4) "At least one of the two particles possesses property $A = a$",

(5.5) "Exactly one of the two particles possesses property $A = a$",

respectively (with the proviso that the phrase "exactly one" is to be interpreted as stating that the other particle possesses the property $A = b$ for some $b \neq a$). Now, given that the orthodoxy questions that association, what other formal representations for (5.4) and (5.5), which are clearly intelligible, well-defined propositions, can we find? As seen in the formal argument presented earlier, there is no other projector on $\mathcal{H} \otimes \mathcal{H}$ that could do the job. The only option left is to split these propositions into disjunctions of more specific statements, as follows:

[6] We may note that this interpretation is assumed without hesitation (and without even noting that there may be something unorthodox about it) in the comprehensive paper (Ghirardi et al. 2002) as well as follow-up papers (Ghirardi and Marinatto 2003, 2004).

(5.6) “Either particle 1 possesses property $A = a$, or particle 2 possesses property $A = a$”

(5.7) “For some $b$ such that $a \neq b$, either particle 1 possesses property $A = a$ and particle 2 possesses property $A = b$, or particle 1 possesses property $A = b$ and particle 2 possesses property $A = a$”.

Each disjunct in the above propositions can be given its representation in the form of an appropriate projection operator as follows: $E_a \otimes I$ for the first disjunct in (5.6) and $I \otimes E_a$ for the second disjunct in (5.6); $E_a \otimes E_b$ for the first disjunct in (5.7) and $E_b \otimes E_a$ for the second disjunct in (5.7). Thus the entire propositions would be translated into the following, classical disjunctions:

(5.8) “The state of the system lies either in subspace $E_a[\mathcal{H}] \otimes \mathcal{H}$ or in subspace $\mathcal{H} \otimes E_a[\mathcal{H}]$”

(5.9) “For some $b$ such that $a \neq b$, the state of the system lies either in subspace $E_a[\mathcal{H}] \otimes E_b[\mathcal{H}]$ or in subspace $E_b[\mathcal{H}] \otimes E_a[\mathcal{H}]$”.

But observe now that none of these statements are literally available in the case of particles of the same type, since joint states of these particles can never occupy subspaces outside of the antisymmetric/symmetric sectors. One possible reaction to that may be that this simply confirms the Indiscernibility Thesis, but I think there is more to it than that. Statements (5.8) and (5.9) are not just false—they are devoid of physical meaning, since they use parts of the formalism (such as non-symmetric projectors given above) that are not allowed to be used in descriptions of the states of same-type particles. I admit that the distinction between falsity and meaninglessness is a controversial issue, but it is at least possible to argue that no fully adequate representations of the propositions (5.4) and (5.5) exist under the orthodox approach to individuation.

The other question posed above was how to properly interpret operators $\Omega_1(E)$ and $\Omega_2(E)$, which are formally admissible and therefore should possess well-defined empirical meaning. The orthodox approach questions their connection with propositions (5.4) and (5.5), but the only other option seems to be to treat the eigenstates of $\Omega_1(E_a)$ and $\Omega_2(E_a)$ as characterizing the system's dispositions to reveal particular values of an observable under measurement. Thus if the system's state resides in the subspace corresponding to $\Omega_1(E_a)$, it is guaranteed that when simultaneous measurements of observable $A$ are performed on both particles, at least one measurement will show value $a$. If the state of the system is, for instance, represented by the ket $|\lambda_a\rangle|\varphi\rangle$, in addition it is guaranteed that it will be particle 1 and not particle 2 that will reveal value $a$. However, the superposition $|\lambda_a\rangle\ |\varphi\rangle \pm |\varphi\rangle\ |\lambda_a\rangle$ only ensures that one particle will produce outcome $a$, not that any specific particle will. Similarly, in the case of projector $\Omega_2(E_a)$, it is certain that the $A$-measurements will reveal two *distinct* values, exactly one of which must be $a$.

The main problem with this "conditional" (or "dispositional") interpretation of $\Omega_1(E)$ and $\Omega_2(E)$ is that it presupposes that the $A$-measurements must reveal determinate outcomes on particles 1 and 2. But given the ordinary projection postulate (or, alternatively, given the eigenstate-eigenvalue link), this means that after measurement the system must find itself in one of the states $|\lambda_a\rangle|\varphi\rangle$ or $|\varphi\rangle|\lambda_a\rangle$. However, this is impossible for systems of same-type particles! No matter what type of interaction a system of such particles participates in—whether measurement-induced or not—both before and after interaction the system's state must reside in one of the two sectors: symmetric or antisymmetric, according to the Symmetrization Postulate. Thus it is inappropriate to insist that the system characterized by the projection operators $\Omega_1(E_a)$ or $\Omega_2(E_a)$ has the dispositions to reveal particular outcomes under measurement, since literally these dispositions can never be actualized. Moreover, we are facing another, closely related problem: how can we perform measurements on particle 1 and particle 2, given that there is no way to tell them apart? The dispositional account of the meaning of operators $\Omega_1(E_a)$ and $\Omega_2(E_a)$ thus stumbles over two problems: the purported dispositions can never be manifested, because such a manifestation would apparently violate SP, and because the triggering event (separate measurements on particles 1 and 2) can never be made to happen.

The last statement may be countered by the proponents of orthodoxy in the following way: while it is impossible to select for measurement the particle that bears label 1, we can still select one particle of the two-particle system without knowing whether it is particle 1 or 2. So perhaps the double measurement of observable $A$ on the system of two same-type particles is possible after all; only we can't identify the labels of the particles undergoing individual measurements.[7] But this solution is short-lived. The problem returns when we consider possible outcomes of such experiments. Suppose that the initial state of the system was the antisymmetric combination:

$$\frac{1}{\sqrt{2}}\left(|\lambda_a\rangle|\lambda_b\rangle - |\lambda_b\rangle|\lambda_b\rangle\right),$$

in which case it is guaranteed that one outcome of the joint measurement will be $a$ and the other $b$. But after measurement we have two particles associated with two mutually exclusive outcomes $a$ and $b$, of which we know that they can't possibly be particles 1 and 2, since this would mean that the system is in one of the forbidden states $|\lambda_a\rangle|\lambda_b\rangle$ or $|\lambda_b\rangle|\lambda_b\rangle$. So what are they? The supporters of orthodoxy are in a bind here. They have to assume either that the outcomes are not associated with any particles (free-floating outcomes?), or that we have managed to discover and select new particles that can't be identified with any of the particles 1 or 2. There is also a third option possible, which unfortunately is not available to the followers of orthodoxy, because it would imply abandoning one of the pillars on which the orthodoxy rests. Namely, we could simply say that labels 1 and 2 *never* referred to any individual particles to begin with, but served merely as parts of the formalism with no independent meaning (in philosophical parlance we often call such elements of language "syncategorematic"). In order to make reference to individual elements of a composite system, we have to rely on parts of the formalism other than the labels—parts that carry clear physical meanings, reflected in legitimate experimental procedures. This is precisely what the heterodox conception attempts to do, and the key to doing it are the symmetric operators of the type described above as $\Omega_1(E)$ and $\Omega_2(E)$.

[7] This solution has definitely a haecceitistic feel about it, as it implicitly acknowledges that particles do possess different non-qualitative features corresponding to the distinct labels, only we cannot know them. John Locke would be very unhappy about it.

## 5.3 Qualitative Individuation via Symmetric Projectors

Let us suppose again that we have two particles of the same type, whose states are represented by vectors in the symmetric or antisymmetric sector of the tensor product $\mathscr{H} \otimes \mathscr{H}$. The orthodox approach to individuation in this case prescribes that the first and second factors in the product correspond to individual particles, in accordance with the assumption we have dubbed Factorism (see Sect. 2.6). However, let us adopt a different approach—instead of stipulating upfront which parts of the formalism represent individual components of the system, let us try to figure this out for ourselves. Thus the starting point will be that the two-particle system occupies a joint state $|\psi(1,2)\rangle$, which possesses the required symmetry properties with respect to the permutation of *formal indices* (not particles!) 1 and 2, but we don't know yet how to make reference to individual components occupying this state. On the basis of the analysis of the symmetric projectors $\Omega_1(E)$ and $\Omega_2(E)$ given in the previous section, we may suggest that the right way to individuate the particles comprising the entire system is through the possession of the joint property corresponding to one of these projectors. Since successful reference arguably requires the uniqueness of the intended referent (we don't want to make reference to an object using a property that may be possessed by more than one entity), it should be clear that the best option is to use operator $\Omega_2(E)$ with its assumed interpretation (5.5). Thus the proposed criterion of reference (individuation) would be such that if the system occupies a state which is an eigenstate for $\Omega_2(E)$ with the corresponding eigenvalue equal 1, then we can make reference to one particle using the property corresponding to projector $E$.

Analogously, we can make reference to the other component of the two-particle system any time we can find a projector $F$ which is orthogonal to $E$ and such that the state $|\psi(1,2)\rangle$ is also an eigenvector for $\Omega_2(F)$. The condition of orthogonality is necessary in order to exclude the possibility that one and the same particle could have a non-zero probability of possessing $E$ and of possessing $F$. Projector $F$ may be the orthogonal complement of $E$: $F = I - E$, but it may also be a more specific projector corresponding to any subspace of $\mathscr{H}$ orthogonal to $E[\mathscr{H}]$. Instead of two conditions of reference formulated separately for each of the two orthogonal one-particle projectors, we may actually use one condition as follows:

(5.10) System *s* consists of two particles such that one particle possesses property associated with projector $E_a$, while the other particle possesses property associated with projector $E_b$, where $E_a$ is orthogonal to $E_b$, iff the state of *s* is an eigenstate of the projector $E_a \otimes E_b + E_b \otimes E_a$ with the corresponding eigenvalue equal 1.

It can be easily verified that the above condition is equivalent to the requirement that the state of *s* be a common value-one eigenvector for projectors $\Omega_2(E_a)$ and $\Omega_2(E_b)$—this follows from the fact that $\Omega_2(E_a)\Omega_2(E_b) = \Omega_2(E_b)\,\Omega_2(E_a) = E_a \otimes E_b + E_b \otimes E_a$. The criterion of individuation given in (5.10) can be generalized to *N* particles by introducing the set of *N* mutually orthogonal projectors $E_{a_i}$ $i = 1, \ldots, N$, and by stipulating that the state of the system must be an eigenstate of the symmetric projector $\sum_{\sigma \in S_N} E_{\sigma(a_1)} \otimes \cdots \otimes E_{\sigma(a_N)}$.[8]

In the proposed approach, reference to the individual particles is made not with the help of the factors in the tensor product $\mathscr{H} \otimes \mathscr{H}$, but qualitatively, using selected properties. However, it is possible to redescribe the symmetric/antisymmetric states satisfying (5.10) in such a way that individual particles referred to by projectors $E_a$ and $E_b$ will actually correspond to factors in a *new* tensor product of appropriately selected subspaces of $\mathscr{H}$. Caulton (2014a) has shown how to do that, using the notion of *unitary equivalence*. Following his approach, let us first identify appropriate subspaces of $\mathscr{H} \otimes \mathscr{H}$ that can "host" states of same-type particles which satisfy condition (5.10). Let $\mathscr{A}$ be the antisymmetric

[8] In the case of *N* particles where $N > 2$, it is also possible to make less specific individuations that pick out not single particles but their assemblies. For instance, we can individuate *k* particles out of *N* when we apply the projector that is the permutation-invariant sum of all tensor products containing *k* identical single-particle projectors *E* and $N - k$ complementary projectors $I - E$. See Caulton (2016) and Bigaj (2016) for more on that.

sector of $\mathscr{H} \otimes \mathscr{H}$, $\mathscr{S}$, the symmetric sector of $\mathscr{H} \otimes \mathscr{H}$, and let $\mathcal{E} = E_a \otimes E_b + E_b \otimes E_a$. Then, the considered subspaces for fermions and bosons, respectively, are $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$. Now, it can be shown that states and properties of systems occupying these subspaces can be equivalently represented in the tensor product $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$.[9] The mapping that secures the equivalence between $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$ on the one hand and $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ on the other is given by the operator:

$$U_{ab} = \sqrt{2}E_a \otimes E_b.$$

It is elementary to observe that $U_{ab}$ applied to any vector from either $\mathcal{E}[\mathscr{A}]$ or $\mathcal{E}[\mathscr{S}]$ will yield a vector in $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$. Moreover, operator $U_{ab}$ limited to subspaces $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$ is *unitary*, that is, it preserves the length of the transformed vectors. Here is a proof of this fact. Let $|v\rangle$ be any vector from $\mathcal{E}[\mathscr{A}]$ or $\mathcal{E}[\mathscr{S}]$. We want to prove that the transformed vector $U_{ab}|v\rangle$ will have the same length as $|v\rangle$, which can be shown as follows:

| | | | |
|---|---|---|---|
| $\lvert U_{ab}\lvert v\rangle\rvert^2$ | = | $\langle v\lvert U_{ab}^{\dagger} U_{ab}\rvert v\rangle$ | |
| | = | $2\langle v\lvert(E_a \otimes E_b)(E_a \otimes E_b)\rvert v\rangle$ | (the hermiticity of $E_a$ and $E_b$) |
| | = | $2\langle v\lvert E_a \otimes E_b\rvert v\rangle$ | (the idempotence of $E_a$ and $E_b$) |
| | = | $\langle v\lvert E_a \otimes E_b\rvert v\rangle + \langle v\lvert E_b \otimes E_a\rvert v\rangle$ | (the (anti-)symmetry of $\lvert v\rangle$) |
| | = | $\langle v\lvert\mathcal{E}\rvert v\rangle$ | (definition of $\mathcal{E}$) |
| | = | $\langle v\vert v\rangle$ | (because $\lvert v\rangle \in \mathcal{E}[\mathscr{A}]$ or $\mathcal{E}[\mathscr{S}]$) |

The existence of unitary transformation $U_{ab}$ connecting $\mathcal{E}[\mathscr{A}]$ and $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$, as well as $\mathcal{E}[\mathscr{S}]$ and $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$, shows that the algebras of operators defined on these spaces are unitarily equivalent, that is, they may be considered two different representations of the same physical observables. That is, if we have a Hermitian operator $A$ acting in the product space $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$, it can be interpreted as representing the same observable as the operator $U_{ab}^{-1}AU_{ab}$ acting in $\mathcal{E}(\mathscr{A})$ and in $\mathcal{E}(\mathscr{S})$. Even though operator $U_{ab} = \sqrt{2}E_a \otimes E_b$ does not have an inverse in the whole range $\mathscr{H} \otimes \mathscr{H}$, it does have inverses when restricted to

[9] Caulton calls the product $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ an "individuation block".

subspaces $\mathcal{E}(\mathscr{A})$ and $\mathcal{E}(\mathscr{S})$, and these inverses are antisymmetrization/symmetrization operators $\frac{1}{\sqrt{2}}(I - P_{12})$ and $\frac{1}{\sqrt{2}}(I + P_{12})$, respectively.[10] Thus, operators with the same physical meaning as $A$ that belong to the algebras of operators in $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$ will have the following forms:

$$(I + P_{12})\mathrm{A}(E_a \otimes E_b) \text{ in } \varepsilon[\mathcal{S}]$$
$$(I - P_{12})\mathrm{A}(E_a \otimes E_b) \text{ in } \varepsilon[\mathcal{A}].$$

Let us now apply the above transformation rules in order to determine the operators that can represent properties of particles individuated by projectors $E_a$ and $E_b$. In the representation given by the algebra of the operators acting in the tensor product $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$, the Hermitian operators $A \otimes I$ and $I \otimes A$ should naturally represent measurable properties of the $E_a$-individuated particle and the $E_b$-individuated particle, respectively. But we can transform these operators back to the "original" representation in the symmetric and antisymmetric sectors of the total Hilbert space, using the above expressions. The result of these transformations will be the following symmetric operators:

$$\mathrm{A}_a = AE_a \otimes E_b \pm P_{12}(AE_a \otimes E_b)$$
$$\mathrm{A}_b = E_a \otimes AE_b \pm P_{12}(E_a \otimes AE_b).$$

Given that for all vectors $|v\rangle$ in $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$ the following holds—$P_{12}|v\rangle = \pm|v\rangle$, we may simplify the above formulas:

$$\mathrm{A}_a = AE_a \otimes E_b + E_b \otimes AE_a$$
$$\mathrm{A}_b = E_a \otimes AE_b + AE_b \otimes E_a. \tag{5.11}$$

[10] This can be easily verified as follows. Let $|v\rangle \in \mathcal{E}(\mathscr{A})$ or $\mathcal{E}(\mathscr{S})$. Then, $\frac{1}{\sqrt{2}}(I \pm P_{12})\sqrt{2}E_a \otimes E_b \,|v$ $= (E_a \otimes E_b)\,|\,v\rangle \pm P_{12}(E_a \otimes E_b) \pm P_{12}\,|\,v\rangle = (E_a \otimes E_b + E_b \otimes E_a)\,|\,v\rangle = |v\rangle$.

Thus, if we have a system of two same-type fermions or bosons occupying a joint state $|v\rangle$ in which they are individuated by properties $E_a$ and $E_b$ in the above-defined sense (5.10), and if we are asked to calculate the expectation value of any single-particle Hermitian operator $A$ attributed to the $E_a$ (or $E_b$) particle, the answer will be given in terms of the usual formula for the expectation value of operators $A_a$ and $A_b$ in state $|v\rangle$ as follows:

(5.12)
$$\langle v|A_a|v\rangle = \langle v|AE_a \otimes E_b|v\rangle + \langle v|E_b \otimes AE_a|v\rangle$$
$$\langle v|A_b|v\rangle = \langle v|E_a \otimes AE_b|v\rangle + \langle v|AE_b \otimes E_a|v\rangle.$$ [11]

Let us illustrate these considerations with a simple example. Suppose that the joint state of a system of two fermions (bosons) is the well-known combination:

$$|\varphi_{ab}\rangle = \frac{1}{\sqrt{2}}\left(|\lambda_a\rangle|\lambda_b\rangle \pm |\lambda_b\rangle|\lambda_a\rangle\right),$$

where $|\lambda_a\rangle$ and $|\lambda_b\rangle$ are orthogonal. In that case the particles are individuated by one-dimensional projectors $E_a = |\lambda_a\rangle\langle\lambda_a|$ and $E_b = |\lambda_b\rangle\langle\lambda_b|$. That is, the state $|\varphi_{ab}\rangle$ lies in the subspace projected onto by the symmetric operator $E_a \otimes E_b + E_b \otimes E_a$. An alternative representation of state $|\varphi_{ab}\rangle$ in the tensor product $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ will be the result of the following transformation:

[11] Caulton (2014a) develops a slightly different approach to the problem of how to calculate the expectation values of single-particle Hermitian operators in connection with the individuation by projectors $E_a$ and $E_b$. First of all, he does not assume that the projector $E_a$ individuates exactly one particle, leaving the possibility open that both particles may be individuated by the same projector $E_a$. Further, he proposes to use the symmetric operator $E_aAE_a \otimes I + I \otimes E_aAE_a$ in order to calculate the expectation value of the single-particle observable $A$ attributed to one of the $E_a$-individuated particles. In spite of the superficial difference between Caulton's operator and my suggested operator given in (5.11), it actually turns out that they both produce the same expectation values, as long as the state $|v\rangle$ of the system belongs to the subspace projected onto by $E_a \otimes E_b + E_b \otimes E_a$. This can be proven as follows: $\langle v|AE_a \otimes E_b + E_b \otimes AE_a|v\rangle = \langle v|(E_a \otimes E_b + E_b \otimes E_a)^\dagger(AE_a \otimes E_b + E_b \otimes AE_a)|v\rangle = \langle v|\, E_aAE_a \otimes E_b + E_b \otimes E_aA\, E_a|v\rangle = \langle v|(E_aAE_a \otimes I + I \otimes E_aAE_a)\,(E_a \otimes E_b + E_b \otimes E_a)|v\rangle = \langle v|(E_aAE_a \otimes I + I \otimes E_a\, AE_a)|v\rangle$. In the derivation we have used, as always, the following facts: $E_a^\dagger = E_a$, $E_b^\dagger = E_b$, $E_aE_b = E_bE_a = 0$, $E_a^2 = E_a$, $E_b^2 = E_b$.

$$\sqrt{2}\left(E_a \otimes E_b\right)|\varphi_{ab}\rangle = |\lambda_a\rangle|\lambda_b\rangle,$$

whereas the symmetrizator $\frac{1}{\sqrt{2}}\left(I + P_{12}\right)$ or antisymmetrizator $\frac{1}{\sqrt{2}}\left(I - P_{12}\right)$ will recover back the original state $|\varphi_{ab}\rangle$. If we are interested in calculating the expectation value for any single-particle Hermitian operator $A$ attributed to the particle individuated by $E_a$, all we have to do is apply the first formula in (5.12) to the state $|\varphi_{ab}\rangle$, which in our case will produce the result $\langle\lambda_a|A|\lambda_a\rangle$ which coincides with the standard way of calculating expectation values for systems in state $|\lambda_a\rangle$. In the special case when $A = E_a$, the expectation value equals 1, which confirms that the $E_a$-individuated particle indeed possesses the property associated with $E_a$. It should be clear, then, that the proposed method of individuation leads to the conclusion that systems possessing states of the form $|\varphi_{ab}\rangle$ behave as if consisting of two particles, each of which occupies its own pure state ($|\lambda_a\rangle$ or $|\lambda_b\rangle$).[12]

We have to remember, however, that the new method of qualitative individuation described above has its limitations. A pair of particles can be individuated using certain projectors $E_a$ and $E_b$ as long as the assembly occupies a joint state that lies in the subspace projected onto by $E_a \otimes E_b + E_b \otimes E_a$. If the evolution of the system takes it outside this subspace, as may very well happen, the individuation done previously no longer applies, and we have to use different sets of projectors to individuate the particles. Thus the representation of states and properties in particular individuating blocks cannot be guaranteed to be available throughout the entire evolution of the system, and it may be even surmised that its availability is usually very short-lived (momentary). In

[12] Muller and Leegwater argue in their paper (Muller and Leegwater 2020) that the possibility of representing symmetric/antisymmetric states in the tensor product $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ amounts to the rehabilitation of the thesis of Factorism, since the factors in this product host states of individual particles. However, as I explained in Chap. 2 ft. 16, Factorism presupposed by orthodoxy (and required for the Indiscernibility Thesis) contains the additional assumption that the factor Hilbert spaces corresponding to individual particles must figure in the original Symmetrization Postulate restricting the available states to symmetric/antisymmetric sectors of the whole product. Thus the rewriting of the states of same-type particles in the tensor product of individuating blocks $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ does not reinstate *this* interpretation of Factorism.

consequence, while we may be able to distinguish and individuate particles synchronically, their diachronic identification across different temporal instants poses a separate challenge that needs to be addressed independently (see Chap. 8).

Another problem that should be looked into is whether qualitative individuation done with the help of symmetric projection operators $E_a \otimes E_b + E_b \otimes E_a$ is guaranteed to be attainable in all available states of fermionic and bosonic systems. As it turns out, the answer in the case of fermions is positive, that is, for every antisymmetric state of two fermions, there are non-trivial orthogonal projectors $E_a$ and $E_b$ (possibly more than one-dimensional, though) such that the state lies in the subspace projected onto by $E_a \otimes E_b + E_b \otimes E_a$. However, this result does not carry over to the case of bosons. This can be clearly seen when we consider totally symmetric product states of the form $|\varphi\rangle|\varphi\rangle$, which can never be eigenstates of any non-trivial projector of the form $E_a \otimes E_b + E_b \otimes E_a$.[13] Thus bosons are not guaranteed to be always discerned in the way described above. We will return to this issue in Chap. 6, where all the relevant facts will be stated and proven.

## 5.4 Singlet-Spin State and Qualitative Individuation

As it stands, the concept of qualitative individuation introduced in the previous section is open to a serious objection that we have already hinted at in Sect. 5.2. The proposed method of individuation based on symmetric projectors may be accused of being at odds with well-known and widely discussed facts regarding entangled systems and the measurable properties of their components. To see where the problem may lie, let us consider one of the most famous examples of quantum states that has entered wide circulation in philosophical literature: the singlet state of two spin-half particles (e.g. two electrons). Let $|\uparrow_z\rangle$ denote the state of

[13] Here is a quick proof of this fact. The result of the action of the projector $E_a \otimes E_b + E_b \otimes E_a$ on the state $|\varphi\rangle\,|\varphi\rangle$ is either zero (when $|\varphi\rangle$ is orthogonal to one of the subspaces associated with $E_a$ or $E_b$), or has the form of $|\mu\rangle|\chi\rangle + |\chi\rangle|\mu\rangle$ where $\langle\mu|\chi\rangle = 0$. Thus it can never produce back the symmetric product vector $|\varphi\rangle|\varphi\rangle$.

one particle possessing the definite value "up" of its $z$-spin, and $|\downarrow_z\rangle$ the state corresponding to the value "down". Then, the singlet state is the state of two particles which has the following form:

$$\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle\otimes|\downarrow_z\rangle-|\downarrow_z\rangle\otimes|\uparrow_z\rangle\right). \tag{5.13}$$

The singlet state is a standard example of an entangled state, that is, a state that cannot be written as a product of two pure states. It features prominently in the foundational debates on quantum mechanics related to the EPR argument, Bell's inequalities, non-locality, holism and the like. The key property of state (5.13), responsible for all the non-classical phenomena mentioned above, is the fact that individual particles occupying it do not possess well-defined spins of their own, and yet the total spin of the system is determinate and equals zero. Thus it is guaranteed that simultaneous measurements performed on both particles will reveal that spins are anticorrelated, even though no measurement taken separately has its outcome predetermined. The standard interpretation of state (5.13) implies that individual particles occupying it possess the exact same reduced state which prescribes that the probability of obtaining any value of spin in any direction equals 1/2. Needless to say, these predictions are unambiguously confirmed in experiments.

However, when we apply the method of individuation introduced in the previous section, the results seem to contradict the above-mentioned analysis. Clearly, state (5.13) is an eigenstate for the operator $E_{\uparrow z}\otimes E_{\downarrow z}+E_{\downarrow z}\otimes E_{\uparrow z}$, where $E_{\uparrow z}=|\uparrow_z\rangle\langle\uparrow_z|$ and $E_{\downarrow z}=|\downarrow_z\rangle\langle\downarrow_z|$. Consequently, following the suggestion of Sect. 5.3, we should conclude that within the system characterized by state (5.13), there is one particle possessing spin "up" and one particle possessing spin "down". An alternative, unitarily equivalent representation of state (5.13) in the individuation block $E_{\uparrow z}[\mathscr{H}]\otimes E_{\downarrow z}[\mathscr{H}]$ discussed earlier is just the separable product state $|\uparrow_z\rangle\otimes|\downarrow_z\rangle$; hence no real entanglement is present here. This also implies that there shouldn't be any non-local correlations present, nor should there be any violations of Bell's inequality. It seems that the heterodox conception of qualitative individuation leads to unacceptable consequences that are in disagreement with the empirically well-confirmed facts regarding particles occupying states of the form (5.13).[14]

Before we discuss possible responses to the above-revealed challenge, let us notice that the orthodox account of the singlet state is not free from conceptual difficulties either. The main question, which relates to the problem considered at the end of Sect. 5.2, is whether the actualization of particular outcomes of spin-measurements can be reconciled with the Symmetrization Postulate. Under the assumption that the singlet state (5.13) characterizes a system of two "indistinguishable" fermions, we have to make sure that the state of these fermions is properly antisymmetrized, before and after measurement. The singlet state itself is obviously antisymmetric; however, the measurement of spin on both particles presumably leaves the system in one of the two following product states: $|\uparrow_z\rangle \otimes |\downarrow_z\rangle$ or $|\downarrow_z\rangle \otimes |\uparrow_z\rangle$. But these states are patently not permutation-invariant, and when we apply the necessary antisymmetrization, we end up with the same singlet state that characterized the system before the measurement. In consequence, the orthodox approach leads to the following dilemma: either the post-measurement state of two fermions violates the Symmetrization Postulate, or the measurement does not affect the state of the system. The second horn of the dilemma is unacceptable for the proponent of orthodoxy, because under the standard collapse interpretation, the only case in which a measurement of a given quantity $A$ does not change the initial state of the system is when observable $A$ already possesses a well-defined value in this state. But, as we pointed out, it is part of orthodoxy that particles occupying the singlet state do not have their spins well defined.

There is some confusion surrounding the proper interpretation of the singlet-spin state, and it may be worthwhile to try clearing it up before we move any further. This state is antisymmetric, but its antisymmetry is not a consequence of the Symmetrization Postulate applied to fermions of the same type. In fact, the singlet state can be occupied by pairs of "distinguishable" particles, such as an electron and a proton, as long as both particles possess the same total spin number (1/2 in the case of electrons

[14] Caulton (2014b, p. 7) notes this problem as well. His response to it is essentially the same as the one developed in this section.

and protons). And yet even if we attribute the singlet state to a proton and an electron, it does not lose its antisymmetric property. The singlet state is usually introduced in the context of the well-known problem of how to add two spins (or, more generally, two angular momenta).[15] The spin state of any individual particle can be completely characterized by determining the values of two commuting operators: the square of the total spin $\mathbf{S}^2$ and the component of $\mathbf{S}$ in an arbitrary direction $z$: $S_z$. Given that the total spin of a specific particle is state-independent (does not change in time), the operator $\mathbf{S}^2$ is just a multiple of the identity operator, whereas $S_z$ can assume different values (e.g. –½ and ½ for spin-half particles).

When we consider two particles with their respective spins $\mathbf{S}_1$ and $\mathbf{S}_2$, their joint spin state may be characterized again by determining the values of $\mathbf{S}^2 = (\mathbf{S}_1 + \mathbf{S}_2)^2$ and $S_z = S_{1z} + S_{2z}$. In the case of two spin-half particles, the four-dimensional spin-state space can be proven to be spanned by the common eigenvectors for $\mathbf{S}^2$ and $S_z$. As it turns out, the singlet state (5.13) is one such common eigenvector which corresponds to the value 0 of the total spin and the value 0 for spin in the $z$-direction. The three remaining vectors represent the situation when the total spin equals 1, which splits into three possibilities regarding the $z$ component $S_z$: −1, 0, 1. These three states are known as *triplet states*, and their exact forms in the basis created by vectors $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ are as follows:

$$\begin{aligned}&|\uparrow_z\rangle \otimes |\uparrow_z\rangle\\&|\downarrow_z\rangle \otimes |\downarrow_z\rangle\\&\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle \otimes |\downarrow_z\rangle + |\uparrow_z\rangle \otimes |\downarrow_z\rangle\right)\end{aligned} \tag{5.14}$$

All triplet vectors are symmetric, and yet they are supposed to represent spin states of two fermions. This clearly shows that the symmetry/antisymmetry of the common eigenvectors for operators $\mathbf{S}^2$ and $S_z$ has little if anything to do with the Symmetrization Postulate and is an

[15] See, for example, Cohen-Tannoudji et al. (1978, 1003–1008) for a standard treatment of this problem.

accidental feature possessed by the singlet and triplet states regardless of whether they are applied to "distinguishable" or "indistinguishable" particles. However, the question remains how to properly antisymmetrize the available spin states (5.13) and (5.14) of two fermions of the same type. Naturally, with respect to the triplet states, this can't be done directly, since the antisymmetrization of a symmetric state yields the zero vector. In order to achieve the proper antisymmetrization, we have to rewrite states (5.13) and (5.14) taking into account not only internal but also spatial degrees of freedom.

A complete state of a spin-half particle has to be written in the tensor product of two Hilbert spaces: the spin state space $\mathscr{H}_s = \mathbb{C}^2$ and the position state space $\mathscr{H}_r = \mathscr{L}^2(\mathrm{R}^3)$ (consisting of square-integrable functions on $\mathrm{R}^3$). Let $|L\rangle$ and $|R\rangle$ indicate states in $\mathscr{H}_r$ (wave functions), whose support is limited to two well-localized and non-overlapping regions in space $L$ and $R$ (thus $|L\rangle$ and $|R\rangle$ are orthogonal to each other). Then vectors $|\uparrow_z\rangle|L\rangle$, $|\downarrow_z\rangle|R\rangle$ and so on will represent states of a particle that possesses an appropriate value of $z$-spin and is located in a particular region of space. Now, let us consider the following antisymmetric combination of vectors (labels 1 and 2 are added for greater clarity):[16]

$$\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_1|L\rangle_1 \otimes |\downarrow_z\rangle_2|R\rangle_2 - |\downarrow_z\rangle_1|R\rangle_1 \otimes |\uparrow_z\rangle_2|L\rangle_2\right) \tag{5.15}$$

In spite of its superficial similarity with state (5.13), vector (5.15) is not the singlet-spin state. For instance, (5.15) lacks an important feature of the singlet state which is known as spherical symmetry. Vector (5.13) can be rewritten in a basis consisting of the states representing spin components in any arbitrary direction—instead of values "up" and "down" in a given direction $z$, we could choose values of spin in any direction $n$, and

[16] As Muller and Leegwater correctly observe (Muller and Leegwater 2020, Sect. 4), the tensor product in this context plays two different roles. The product may connect the Hilbert spaces associated with distinct particles or with distinct degrees of freedom for one particle. In order to keep these two roles separate, for the rest of this chapter I will use the symbol ⊗ to indicate the product of states and operators which refer to distinct particles, while the products of states and operators associated with distinct degrees of freedom of the same particle will be simply symbolized by concatenation.

the resulting vector would be mathematically identical with (5.13). However, inserting the eigenstates of spin along a different spatial direction into formula (5.15) will yield a different vector. Consequently, state (5.15) does not guarantee that spins measured in various directions will always be anticorrelated—the anticorrelation is only secured for the direction chosen to write particular states $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$ used in (5.15). One—perhaps slightly biased—way of reading the state given in (5.15) is such that it describes a superposition of two situations that differ from one another only with respect to their assignment of conventional labels: one component of state (5.15) ascribes label 1 to the left particle with spin up and label 2 to the right particle with spin down, while the other component switches the labels. The proper version of the singlet state with spatial degrees of freedom has to contain different combinations of spins and positions, as in the following vector:

$$\frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle_1|L\rangle_1\otimes|\downarrow_z\rangle_2|R\rangle_2-|\downarrow_z\rangle_1|L\rangle_1\otimes|\uparrow_z\rangle_2|R\rangle_2\right). \quad (5.16)$$

In state (5.16) we have a superposition of the situation in which the "left" particle has spin up and the "right" particle has spin down with the *physically distinct* situation where the left particle has spin down and the right particle spin up. However, state (5.16) is not antisymmetric, as can be easily verified by permuting indices 1 and 2. In order to use (5.16) as a representation of the state of two indistinguishable fermions, we have to antisymmetrize it, which produces the following, rather complicated vector:

$$\frac{1}{2}\begin{pmatrix}|\uparrow_z\rangle_1|L\rangle_1\otimes|\downarrow_z\rangle_2|R\rangle_2-|\downarrow_z\rangle_1|L\rangle_1\otimes|\uparrow_z\rangle_2|R\rangle_2\\-|\downarrow_z\rangle_1|R\rangle_1\otimes|\uparrow_z\rangle_2|L\rangle_2+|\uparrow_z\rangle_1|R\rangle_1\otimes|\downarrow_z\rangle_2|L\rangle_2\end{pmatrix}, \quad (5.17)$$

which can be simplified as follows:

$$\frac{1}{2}\left(|\uparrow_z\rangle_1\otimes|\downarrow_z\rangle_2-|\downarrow_z\rangle_1\otimes|\uparrow_z\rangle_2\right)\left(|L\rangle_1\otimes|R\rangle_2+|R\rangle_1\otimes|L\rangle_2\right). \quad (5.18)$$

The form of vector (5.18) immediately reveals that this state is spherically symmetric with respect to spin (since the first component of the tensor product is just the standard singlet-spin state with no positions). Hence this is the right variant of the singlet-spin state (5.13) in the case when the spatial degrees of freedom and the symmetrization postulate are taken into account.

## 5.5 Measurements for "Indistinguishable" Particles

Now we can tackle the problem of how to account for measurements and their outcomes under the inevitable symmetrization postulate. In the first part of this section, we will assume the validity of the Projection Postulate, which prescribes that the act of measurement of an observable $A$ on a system prepared in state $|\varphi\rangle$ "collapses" the initial state $|\varphi\rangle$ onto one of the eigenstates $|\lambda_a\rangle$ of $A$ corresponding to the revealed value $a$ (see Goldstein 2009). In order to formally calculate the final, reduced state $|\lambda_a\rangle$, we may apply the projection operator $E_a$ (such that $E_a[\mathcal{H}]$ is the eigenspace of $A$ with the corresponding value $a$) to the initial state:

$$|\lambda_a\rangle = \frac{1}{\sqrt{\langle\varphi|E_a|\varphi\rangle}} E_a|\varphi\rangle,$$

where $\frac{1}{\sqrt{\langle\varphi|E_a|\varphi\rangle}}$ is the normalization constant. In the orthodox approach to quantum individuation based on Factorism, measurements on individual components of the composite system are associated with labels identifying separate factors in the tensor product of Hilbert spaces. That is, we could talk about measuring the spin of the particle labeled 1, and the projector corresponding to a particular outcome $a$ of this measurement would have the form of:

$$E_a^{(1)} I_r^{(1)} \otimes I_s^{(2)} I_r^{(2)}, \tag{5.19}$$

where $I_r^{(1)}$, $I_s^{(2)}$ and $I_r^{(2)}$ are the identity operators acting, respectively, in the position space $\mathcal{H}_r^{(1)}$ of the first particle, the spin space $\mathcal{H}_s^{(2)}$ of the second particle and the position space $\mathcal{H}_r^{(2)}$ of the second particle.

However, in the case of indistinguishable particles this representation seems inappropriate for the following reasons. First of all, projectors of the form (5.19) are not symmetric, so when applied to antisymmetric states of fermions, they will not produce states of the required type of symmetry (we have already witnessed this feature in Sect. 5.4 when we have discussed measurements done on singlet-spin states without the spatial degrees of freedom). Moreover, when we consider particles of the same type, individual labels that are supposed to identify particular components of the composite system lose their clear physical meaning, since they cannot be associated with any physically differentiating properties. Electron number one does not differ in any empirically verifiable way from electron number two, so it is unclear how we can select one and not the other for measurement.

An alternative proposal is to use again qualitative properties in order to pick out one particle for measurement. It is rather uncontroversial to observe that in the majority of cases the role of such a selection property is played by spatial location. Measurements are localized affairs due to the use of macroscopic instruments whose position in space is well defined. Thus if we are dealing with a system consisting of "indistinguishable" particles, then instead of asking what the spin of the particle number 1 (or 2) is, it is better to ask about the spin of the particle that we locate in region $L$ ($R$), since we can simply place our spin-measuring device in an appropriate location. And the aforementioned experimental question has to be formulated in a symmetric way, in order to preserve the permutation invariance of the description of the system. Thus the projector that corresponds to the spin-outcome "up" obtained on the particle located in region $L$ will look as follows:

$$E_{\uparrow z}^{(1)} E_L^{(1)} \otimes I_s^{(2)} I_r^{(2)} + I_s^{(1)} I_r^{(1)} \otimes E_{\uparrow z}^{(2)} E_L^{(2)} - E_{\uparrow z}^{(1)} E_L^{(1)} \otimes E_{\uparrow z}^{(2)} E_L^{(2)} \tag{5.20}$$

Note that the operator in (5.20) has the general form of projector $\Omega_1$ from formula (5.3). This projector, according to the heterodoxy, represents the property of the system expressed in the sentence "At least one component (either 1 or 2 or both) possesses spin "up" and is located in $L$". Thus when we apply projector (5.20) to the initial state of the system, we should obtain the state in which the system will be found after undergoing spin-measurement in location $L$ which revealed value "up".

It can be easily confirmed that when we act with the projector (5.20) on states (5.15) or (5.17), the result will be the same in both cases, namely the state (5.15). We can use this fact to show how the unorthodox approach to quantum individuation avoids the problem with the singlet state described in Sect. 5.4. The "pseudo" singlet state (5.15) does not change under measurements whose outcome is spin "up" in location $L$, which is consistent with the contention that this state corresponds to a situation in which one particle is already located in $L$ and possesses spin "up" (while the other particle is located in $R$ and has spin "down"). Indeed, the fact that state (5.15) is an eigenstate of the projector (5.20) shows that under the heterodox approach to individuation laid out in Sect. 5.3, the system contains a particle in location $L$ and possessing spin "up".[17] On the other hand, the genuine singlet state (5.17) changes under an appropriate measurement. This shows that before measurement the system occupying state (5.17) does not contain particles with well-defined positions and spins. The intuition that the singlet state does not determine spins of separate components of the system is preserved under the proposed approach, if only we write the singlet state in the correct form (5.17).

The discussion done so far in this section may be accused of relying too heavily on the controversial account of quantum measurements in terms of the projection postulate (collapse of the wave function). As is well known, the issue of a proper interpretation of measurements is one of the most hotly debated topics in the foundational analysis of quantum mechanics. Apart from the collapse interpretations (whether in the

[17] Of course, when we consider a measurement of spin in any arbitrary direction $n$, as a result the original state (5.15) will change. However, observe that in this case the components of state (5.15) corresponding to the $R$-particle will remain unchanged, as the total state will now be either $\frac{1}{\sqrt{2}}\left(|\uparrow_n\rangle_1 |L\rangle_1 \otimes |\downarrow_z\rangle_2 |R\rangle_2 - |\downarrow_z\rangle_1 |R\rangle_1 \otimes |\uparrow_n\rangle_2 |L\rangle_2\right)$ or $\frac{1}{\sqrt{2}}\left(|\downarrow_n\rangle_1 |L\rangle_1 \otimes |\downarrow_z\rangle_2 |R\rangle_2 - |\downarrow_z\rangle_1 |R\rangle_1 \otimes |\downarrow_n\rangle_2 |L\rangle_2\right)$.

Hence the system behaves as if it consisted of two independent subsystems, $L$ and $R$. See Muller and Leegwater (2020, Sect. 5) for a discussion.

standard, much-vilified Copenhagen version, or in the newest, spontaneous localization or GRW form), there are famous alternatives, such as the many-worlds interpretation, that dispense with the suspicious non-unitary reductions of the wave function upon measurements. I do not wish to commit myself to any specific proposal of how to deal with the measurement problem. Fortunately, it is possible to restate virtually all the above results in a neutral framework that is arguably presupposed by all interpretations of the quantum theory. This neutral framework is provided by a well-known unitary description of the measurement process understood as a physical interaction between the quantum system and the measuring device, subject to certain natural constraints (see Albert 1992, pp. 73–79, Barrett 1999, p. 28ff).

We will assume that the measuring device in the case of spin-half is any physical system that can be in one of the three states: the neutral state "ready" $|r\rangle_m$, the state associated with the recorded outcome "up" $|u\rangle_m$ and the state associated with the outcome "down" $|d\rangle_m$. Given the assumption that the measuring device functions properly (faithfully represents the states of the measured system), we stipulate that the unitary evolution of the entire composite system consisting of a measuring device $m$ and a measured system $s$ should obey the following constraints:

$$\begin{aligned} |\uparrow\rangle_s \otimes |r\rangle_m &\to |\uparrow\rangle_s \otimes |u\rangle_m \\ |\downarrow\rangle_s \otimes |r\rangle_m &\to |\downarrow\rangle_s \otimes |d\rangle_m. \end{aligned} \tag{5.21}$$

That is, if the measured system already possesses a determinate value of spin, the interaction between this system and the measurement apparatus should put the latter in the state corresponding to this possessed value. The transformation rules (5.21) can be extended to any states of system $s$ by linearity, from which it follows that if the initial state of $s$ is a superposition of states with well-defined values of the measured observable, the entire composite system after measurement will occupy an entangled state, and therefore only mixed states can be attributed to the individual components.

The difficulty caused by the orthodox analysis of measurements on the singlet state (5.13) reappears in this new approach. Using the rules (5.21) we can predict the final state of the measuring device together with the two-particle system, assuming that the measurement is done on particle number 1:

$$\begin{aligned}&\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\right) \otimes |r\rangle_m \rightarrow \\ &\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 \otimes | \downarrow_2 \otimes |u\rangle_m - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2 \otimes |d\rangle_m\right).\end{aligned} \tag{5.22}$$

As can be easily verified, the final state in (5.22) is not invariant with respect to the permutation of "indistinguishable" particles 1 and 2. Thus the general unitary approach to measurement cannot be reconciled with the universal validity of the Symmetrization Postulate, exactly as in the case of measurements with collapse. But now let us specify the constraints on the unitary evolution in the case of spin-measurements associated not with particles' labels but with specific locations. Suppose, as before, that the measurement of spin is performed in location $L$. The conditions on the evolution that ensure the faithfulness of measurement can be written as follows:

$$\begin{aligned}&|x\rangle_1\,|L\rangle_1 \otimes |y\rangle_2\,|R\rangle_2 \otimes |r\rangle_m \rightarrow |x\rangle_1\,|L\rangle_1 \otimes |y\rangle_2\,|R\rangle_2 \otimes |x\rangle_m \\ &|x\rangle_1\,|R\rangle_1 \otimes |y\rangle_2\,|L\rangle_2 \otimes |r\rangle_m \rightarrow |x\rangle_1\,|R\rangle_1 \otimes |y\rangle_2\,|L\rangle_2 \otimes |y\rangle_m,\end{aligned} \tag{5.23}$$

where $x = \uparrow$ or $\downarrow$ and $y = \uparrow$ or $\downarrow$. We don't have to specify the evolution of the system $1 + 2 + m$ in the case when both particles occupy the same region (both in $R$ or both in $L$), since such states are absent from the considered superpositions (5.15) and (5.17). However, for completeness we may stipulate that in this case the measuring device will remain in the neutral state "ready".

Applying the rules of evolution (5.23) to states (5.15) and (5.17), we will get the following final states:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |L\rangle_1 \otimes |\downarrow\rangle_2 |R\rangle_2 - |\downarrow\rangle_1 |R\rangle_1 \otimes |\uparrow\rangle_2 |L\rangle_2\right) \otimes |u\rangle_m \tag{5.24}$$

$$\frac{1}{2}\Big[\big(|\uparrow\rangle_1|L\rangle_1\otimes|\downarrow\rangle_2|R\rangle_2-|\downarrow\rangle_1|R\rangle_1\otimes|\uparrow\rangle_2|L\rangle_2\big)\otimes|u\rangle_m$$
$$+\big(|\uparrow\rangle_1|R\rangle_1\otimes|\downarrow\rangle_2|L\rangle_2-|\downarrow\rangle_1|L\rangle_1\otimes|\uparrow\rangle_2|R\rangle_2\big)\otimes|d\rangle_m\Big]. \tag{5.25}$$

Both states (5.24) and (5.25) are antisymmetric with respect to the permutation of labels 1 and 2. The state in (5.24) is a product of the pseudo singlet state (5.17) and the state of the measuring device indicating the outcome "up". Thus in this case the measured system 1 + 2 does not get entangled with the measuring device and retains its original state, as expected. On the other hand, (5.25) is a genuinely entangled state of the system 1 + 2 + *m*. Seen from the perspective of the many-worlds interpretation, (5.25) describes two branches ("worlds") that the world splits into as a result of the interaction between systems *m* and 1 + 2. The branch corresponding to the outcome "up" associates with the two-particle system the state which is an antisymmetric representation of the situation described as "particle in *L* has spin "up" while particle in *R* has spin "down"", while the other branch contains the alternative scenario corresponding to the outcome "down" in region *L*. Taken together, the state (5.25) attributes to the measured system a mixed state which is a symmetric combination of the projectors projecting onto the rays associated with the separate branches "up" and "down". What is important from our perspective is that the unitary, non-collapse interpretation of measurement produces essentially the same consequences regarding singlet-spin states with spatial degrees of freedom as the collapse interpretation.

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# 6

# The Heterodox Approach to Absolute Discernibility and Entanglement

We have already witnessed that it is possible to introduce a new, unorthodox method of individuating same-type particles which relies on qualitative properties rather than on unphysical labels. Qualitative individuation of that kind is expressed in terms of symmetric combinations of orthogonal single-particle projection operators. We have argued that if a system of same-type bosons or fermions occupies a state which lies in the range of an appropriate symmetric operator Ω, we may individuate the components of this system by attributing to them properties represented by the single-particle projectors composing Ω. Given that these single-particle projectors are assumed to be orthogonal, we can already draw the conclusion that the particles individuated in such a way possess distinct and mutually exclusive properties and thus are absolutely discernible by their intrinsic (i.e. non-relational) properties.

In this chapter we will probe deeper the issue of the absolute discernibility of same-type particles under the considered unorthodox approach to individuation. In particular, we will address the question of whether the absolute discernibility claim regarding two same-type particles is guaranteed to hold true in all available states. Given that a particular state occupied by these particles indeed ensures that they possess discerning




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properties, we may also ask whether they are maximally specific properties, such as possessing precise values of certain observables, or less-specific properties defined by broader ranges of admissible values. Related to these issues are questions regarding the notion of entanglement and its connection with the possession of maximally specific properties by the individual components of a composite system. As it turns out, the new approach to individuation strongly suggests abandoning the standard definition of entanglement in terms of non-factorizability. It may be argued that we should use a different notion of entanglement applicable to the states of same-type particles. This concept, which we refer to as GMW-entanglement, meshes well with the adopted conception of individuation and measurement presented in the previous chapter. In particular, it may be shown that only GMW-entangled systems display well-known features such as non-local correlations between experimental outcomes and violation of Bell's inequalities. But before we get to these problems, we will begin with a brief logical prelude that can cast new light on the thesis, argued for in Chap. 4, that absolute discernment is unattainable in symmetric languages.

## 6.1 Absolute Discernibility in Symmetric Languages

In Chap. 4 Sect. 4.4, we have reported a simple logical fact regarding languages whose intended interpretations (in the form of relational structures) are invariant with respect to permutations of objects. As we recall, in such languages (we call them symmetric, for short), it is impossible to construct formulas that would discern objects absolutely or relatively. Obviously, any formula $\Phi(x)$ in one variable such that $\Phi(x)$ is satisfied in a structure $\mathfrak{R}$ by some object $a$ but not by an object $b$ violates the symmetry of the language $\mathscr{L}$ under the interpretation $\mathfrak{R}$. This is so, because for $\mathscr{L}$ to be symmetric, all its predicates must receive set-theoretical interpretations in $\mathfrak{R}$ which remain invariant under permutations of objects, and in the case of one-argument predicates (or formulas), this means that their interpretation has to be either the empty set or the full set.

It may seem that this result proves once and for all that symmetry (i.e. permutation invariance) and absolute discernibility are irreconcilable, as suggested in Chap. 4. However, a more thorough analysis of this problem reveals that this pessimistic conclusion (pessimistic at least from the perspective of a proponent of the absolute discernibility of quanta) may be resisted. First of all, we have to distinguish two related but distinct questions. One question is whether in a particular language $\mathscr{L}$ there is a formula $\Phi$ that absolutely distinguishes some objects in the domain of $\mathscr{L}$, while another question is whether in $\mathscr{L}$ it is possible to formulate a *sentence* which states that some (or all) objects in the domain are discernible by their properties. As it turns out, these two questions are not identical, which means that a negative answer to one of them does not necessarily imply an analogous response to the other one. To be sure, if $\mathscr{L}$ contains an absolutely discerning formula $\Phi$, it is to be expected that a sentence stating the fact of absolute discernibility (i.e. a sentence true if and only if some objects are indeed absolutely discernible) will be expressible in $\mathscr{L}$. But the opposite implication is by no means a foregone conclusion. In what follows we will present a formal argument showing that indeed we may build a fully symmetric sentence which states that all objects in the domain are discerned from each other by their properties.

The starting point will be the following theorem due to Simon Saunders (Saunders 2006, 2013).

(6.1) Let $\mathscr{L}^=$ be a first-order language without proper names but with the identity symbol. Then for every sentence $T$ in $\mathscr{L}^=$ and every natural number $N$, there is a sentence $S$ in $\mathscr{L}^=$ of the form $S = \exists x_1 \ldots \exists x_N\, G(x_1, \ldots, x_N)$ such that open formula $G$ is symmetric, and $S$ is equivalent to $T$ in all models of cardinality $N$.

The sketch of a proof for this theorem is as follows (for details see Saunders 2006, pp. 209–210). Every sentence $T$ can be presented in the standard prenex form as $Q_1, \ldots, Q_n\, F(x_1, \ldots, x_n)$, where $Q_i$ is either $\exists x_i$ or $\forall x_i$. In order to construct the corresponding symmetric sentence $S$, we eliminate every quantifier $Q_i$ step by step, starting with $Q_n$, while simultaneously replacing formula $F(x_1, \ldots, x_n)$ with either a conjunction (when $Q_i$ is the universal quantifier) or a disjunction (when $Q_i$ is the existential quantifier) of formulas $F(\ldots a_1\ldots)$, …, $F(\ldots a_N\ldots)$, where all occurrences of the variable $x_i$ are replaced with unique names $a_1$, …, $a_N$ of all elements in the domain. For instance, the first step in the procedure in the case when $Q_n$ is universal will give us formula:

$$Q_1 \ldots Q_{n-1}\left[F\left(x_1, \ldots, x_{n-1}, a_1\right) \wedge F\left(x_1, \ldots, x_{n-1}, a_2\right) \wedge \ldots \wedge F\left(x_1, \ldots, x_{n-1}, a_N\right)\right].$$

After finishing this procedure, we end up with a formula with no quantifiers but instead containing constants $a_1$, …, $a_N$. Finally, we replace every occurrence of $a_i$ with a variable $x_i$ bound by an existential quantifier, and we add to the entire sentence an expression stating that all $x_i$'s are distinct and that they exhaust the entire domain (every object in the domain is identical with some $x_i$). The sentence $S$ obtained during this procedure is symmetric by design, and it is also not difficult to observe that it must be equivalent to $T$ in all models of cardinality $N$.

One consequence of theorem (6.1) is that if we have a language $\mathscr{L}^{=}$ in which all objects in its finite domain are absolutely discernible by monadic formulas, then the sentence in $\mathscr{L}^{=}$ expressing this fact can be equivalently formulated in a language built out of symmetric combinations of predicates in $\mathscr{L}^{=}$. Let us show explicitly how this can be done. Suppose, then, that the domain of $\mathscr{L}^{=}$ consists of $N$ elements $a_1$, …, $a_N$. In addition, we assume that for all $a_i$ and $a_j$, where $i \neq j$, there is a formula $\Phi_{ij}$ in $\mathscr{L}^{=}$ that absolutely discerns $a_i$ and $a_j$, that is, it is the case that:

$$\mathfrak{R} \vDash \left[\Phi_{ij}\left(a_i\right) \wedge \neg\Phi_{ij}\left(a_j\right)\right] \vee \left[\Phi_{ij}\left(a_j\right) \wedge \neg\Phi_{ij}\left(a_i\right)\right].$$

If we take the disjunction of all of the above formulas over all possible $i \neq j$, it is clear that the resulting expression must be satisfied by every pair of distinct objects, and that the truth of the universal generalization of this expression is equivalent to the statement that all distinct objects are absolutely discernible:

$$\forall x \forall y\left\{x \neq y \rightarrow \vee_{i \neq j}^{N}\left[\Phi_{ij}(x) \wedge \neg\Phi_{ij}(y)\right] \vee \left[\Phi_{ij}(y) \wedge \neg\Phi_{ij}(x)\right]\right\}. \quad (6.2)$$

Now, theorem (6.1) guarantees that there must exist a sentence which is equivalent to (6.2) in all domains of cardinality $N$, and which is built out of a totally symmetric combination of predicates from $\mathscr{L}^=$. Applying the procedure described above in the sketch of proof for (6.1), we arrive at the following symmetric equivalent of (6.2):

$$\exists x_1 \ldots \exists x_N \left\{ \begin{array}{l} \rho\left(x_1,\ldots,x_N\right) \wedge \wedge_{k \neq l}^N \vee_{i \neq j}^N \left[ \Phi_{ij}\left(x_k\right) \wedge \neg \Phi_{ij}\left(x_l\right) \right] \\ \vee \left[ \Phi_{ij}\left(x_l\right) \wedge \neg \Phi_{ij}\left(x_k\right) \right] \end{array} \right\}, \tag{6.3}$$

where $\rho(x_1,\ldots,x_N)$ abbreviates the following: $\wedge_{i \neq j}^N x_i \neq x_j \wedge \forall x \vee_{k=1}^N x = x_k$, that is, the formula ensuring that there are exactly $N$ objects in the domain. The formula:

$$\begin{array}{l} \Psi\left(x_1,\ldots,x_N\right) \equiv \wedge_{k \neq l}^N \vee_{i \neq j}^N \left[ \Phi_{ij}\left(x_k\right) \wedge \neg \Phi_{ij}\left(x_l\right) \right] \\ \vee \left[ \Phi_{ij}\left(x_l\right) \wedge \neg \Phi_{ij}\left(x_k\right) \right] \end{array} \tag{6.4}$$

is totally symmetric in that if $\Psi(x_1, \ldots, x_N)$ is satisfied by the sequence $(a_1, \ldots, a_N)$, it is satisfied by any permuted sequence $(\sigma(a_1), \ldots, \sigma(a_N))$.[1] (Note also that $\Psi(x_1, \ldots, x_N)$ can never be satisfied by a sequence containing repeated elements, since in that case one of the conjuncts in (6.4) is guaranteed to be logically false.) Now, it is true that the expression in (6.4) contains open formulas $\Phi_{ij}(x_k)$ that are not symmetric; hence it is still part of a non-symmetric language. But we can formally consider a new language $\mathscr{L}^{\text{sym}}$ in which $\Psi(x_1, \ldots, x_N)$ is a primitive predicate with the same intended interpretation as the original expression (6.4). In such a symmetric language, there is no formula that would absolutely discern

[1] There is a certain subtlety that we are glossing over here. A formula can *happen* to be symmetric under a particular semantic interpretation, or it can be *necessarily* symmetric, that is, symmetric under any possible interpretation, in virtue of its logical form. Formula $\Psi$ defined in (6.4) belongs to the second category, that is, its symmetry (in models of cardinality $N$, to be precise) is guaranteed by its logical form. On the other hand, the two-variable formula in the consequent of the implication in (6.2) is symmetric under the assumption that all objects in the domain are absolutely discernible by formulas $\Phi_{ij}$. Saunders's theorem ensures that the discernibility statements can be expressed with the help of logically symmetric formulas, and not only formulas that happen to have permutation-invariant interpretations.

objects in the domain; however we can still express the statement that all objects in the domain are absolutely discernible.[2]

In the simplest case when $N = 2$, the symmetric formula $\Psi(x_1, x_2)$ has the following form:

$$\Psi(x_1, x_2) \equiv [\Phi_{12}(x_1) \wedge \neg\Phi_{12}(x_2)] \vee [\Phi_{12}(x_2) \wedge \neg\Phi_{12}(x_1)], \qquad (6.5)$$

where $\Phi_{12}$ absolutely discerns elements $a_1$ and $a_2$ of the domain. We may observe that there is a striking formal similarity between the logical structure of formula $\Psi(x_1, x_2)$ written in terms of $\Phi_{12}$ and the way the compound symmetric projector $\Omega_2(E)$ introduced in Sect. 5.3 (formula 5.3) is built with the help of a single-particle projector $E$. As we recall, $\Omega_2(E)$ can be written as follows:

$$\Omega_2(E) = E \otimes \bar{E} + \bar{E} \otimes E, \qquad (6.6)$$

where $\bar{E} = I - E$ is the orthogonal complement of $E$. Given that $E$ represents a particular property $\Pi_E$ (i.e. when a system's state is an eigenstate of $E$ corresponding to the eigenvalue 1, the system possesses property $\Pi_E$), the standard interpretation of $\bar{E}$ is that it represents the negation of property $\Pi_E$. Similarly, the sum of two orthogonal projectors $E + F$ is typically associated with the disjunctive property "either $\Pi_E$ or $\Pi_F$". This interpretation may be argued for as follows: if a system's state lies in the subspace $(E + F)[\mathcal{H}] = E[\mathcal{H}] \oplus F[\mathcal{H}]$, this implies that upon a measurement aiming at revealing properties associated with $E$ (or $F$), the system will be found occupying a state lying either in $E[\mathcal{H}]$ or in $F[\mathcal{H}]$.[3] Finally, the system's state being an eigenvector of the tensor product of two

[2] It is unclear to me though whether we have to go to such lengths with respect to the permutation-invariance requirement in quantum mechanics. This requirement imposes the symmetry restriction on physically meaningful formulas representing states and properties of systems, but it does not demand that we eliminate any expressions from our language that do not satisfy the permutation-invariance requirement. Thus the symmetry requirement admits as meaningful operators of the form $A \otimes I + I \otimes A$, even though their constitutive elements $A \otimes I$ and $I \otimes A$ are patently non-symmetric. I am grateful to Joanna Luc and Tomasz Placek for a recent discussion which helped me rethink this issue.

[3] This argument belongs to the category of "inductive" arguments that I have already mentioned on two occasions: once in Chap. 2 when explaining why the product $A \otimes I$ represents property $A$

projectors $E \otimes F$ clearly represents the situation in which the first particle has property $\Pi_E$ and the second particle has property $\Pi_F$.[4]

Given the formal analogy between (6.5) and (6.6), we may argue that the symmetric sentence (6.4) expressing the fact of the absolute discernibility of objects can be translated (limiting ourselves to the case when $N = 2$) into the following quantum-mechanical statement:

$$\left(E \otimes \bar{E} + \bar{E} \otimes E\right) | \psi(1,2)\rangle = | \psi(1,2)\rangle, \tag{6.7}$$

where $|\psi(1,2)\rangle$ is the state occupied by the system of two particles, and $E$ is the projector representing the same property as the monadic formula $\Phi_{12}$. Thus, reasoning from analogy, we have come to the conclusion that the quantum-mechanical sentence (6.7) states the fact that the two particles are discerned by the property $\Pi_E$ corresponding to the projector $E$. This further confirms the unorthodox interpretation of symmetric projectors of the form $\Omega_2(E)$ and the ensuing heterodox approach to individuation and discernibility. At the very least, we have cleared an important logical obstacle on the path to absolute discernibility in languages obeying the postulate of permutation invariance.

However, we have to keep in mind that the argument given above may be criticized on the grounds of weak analogy. The correspondence between the classical logical connectives of negation and disjunction on the one hand, and their quantum counterparts of the orthogonal complement and the sum of projectors on the other, no matter how striking, may be questioned by the proponents of orthodoxy. Suppose, for instance,

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associated with particle 1, and later in Chap. 5 (ft. 5), discussing the meaning of operators $\Omega_1(E)$ and $\Omega_2(E)$. As such, the argument is open to similar objections as the previous ones.

[4] The general case when $N > 2$ is a bit more complicated and requires a slightly different approach in order to draw the required analogy with the quantum case. Instead of pairwise discerning formulas $\Phi_{ij}$, it is now more useful to introduce $N$ formulas $\Phi_i$ such that each formula uniquely characterizes one object $a_i$, that is, it is the case that $\mathcal{A} \vDash \Phi_i(a_i) \wedge \neg\Phi_i(a_j)$ for all $i \neq j$. Then it can be argued that the symmetric sentence stating the fact that all $N$ objects in the domain are mutually discernible can be presented in the following form: $\exists x_1 \ldots \exists x_N[\rho(x_1,\ldots,x_N) \wedge \vee_{\sigma \in S_N} \wedge_{i=1}^{N} \Phi_i(x_{\sigma(i)})]$. The operator corresponding to the formula $\vee_{\sigma \in S_N} \wedge_{i=1}^{N} \Phi_i(x_{\sigma(i)})$ will have the easily recognizable form of the symmetrized tensor product of $N$ mutually orthogonal projectors $E_i$: $\sum_{\sigma \in S_N} E_{\sigma(1)} \otimes E_{\sigma(2)} \otimes \cdots \otimes E_{\sigma(N)}$.

that the state of a two-particle system is a superposition of an eigenstate for the operator $E \otimes \bar{E}$ and an eigenstate for its permuted variant $\bar{E} \otimes E$:

$$\frac{1}{\sqrt{2}}\left(\left|\lambda_E\right\rangle\left|\lambda_{\bar{E}}\right\rangle + \left|\lambda_{\bar{E}}\right\rangle\left|\lambda_E\right\rangle\right), \tag{6.8}$$

where $|\lambda_E\rangle \in E(\mathscr{H})$ and $|\lambda_{\bar{E}}\rangle \in \bar{E}(\mathscr{H})$. Vector (6.8) clearly satisfies condition (6.7), but this will hardly convince the follower of the orthodoxy that in this state it is true that either particle 1 has property $\Pi_E$ while particle 2 doesn't have $\Pi_E$, or vice versa. This disjunction, taken classically, is true only if the system is in one of the product states: $|\lambda_E\rangle|\lambda_{\bar{E}}\rangle$ or $|\lambda_{\bar{E}}\rangle|\lambda_E\rangle$, not when the state is a superposition of both. Thus while it is indeed possible to speak about absolute discernibility in symmetric languages using sentences of the form (6.3), this does not prove that the same strategy is applicable in the case of quantum particles of the same type.

Indeed, there is some sleight of hand in the transition from the general logical formula (6.5) to the quantum-mechanical analogue (6.7). Observe first that the non-symmetric absolutely discerning formulas $\Phi_{ij}$ and the symmetric predicates of the form (6.4) are defined on the same set of objects constituting the fixed domain of the language. Arguably, this is not so in the quantum case. When we move from the non-permutation-invariant language describing a system of two "distinguishable" particles to the symmetric language of a theory of "indistinguishable" particles, we apparently change the interpretation of the referring parts of the formalism, at least according to the heterodox approach. If a system of two particles admits the product vector $|\lambda_E\rangle|\lambda_{\bar{E}}\rangle$ as representing one of its available states, we agree that it is the particle bearing label "1" and not "2" that possesses property $\Pi_E$ associated with projector $E$. But if we consider a system of two "indistinguishable" particles occupying the symmetric state (6.8), it is no longer the case that the domain contains two particles, one bearing label "1", the other label "2". If we followed such an approach to individuation, we would have to conclude, in line with the orthodoxy, that both particles are entirely indiscernible with respect to their properties, and thus sentence (6.4) would come out false of them.

The heterodox approach changes the way the permutation-invariant formalism refers to individual objects in the domain in comparison to the non-symmetric formalism, so it is unclear if we can still rely on the method of "symmetrizing" absolute discernibility statements with the help of the general theorem (6.1).[5] We will conclude this section with a remark that the logical result expressed in theorem (6.1) can at best serve as a motivation based on analogy for accepting the heterodox approach to individuation, not as a watertight argument in favor of it.

## 6.2 The Scope of Absolute Discernibility for Fermions and Bosons

As we have already pointed out, the new approach to quantum individuation implies that particles of the same type *can* differ from each other with respect to their possessed properties. However, the question remains whether absolute discernibility is guaranteed to hold in all available states of such particles, or only in some of them. This question is crucial to the philosophical analysis of the universal validity of the Principle of the Identity of Indiscernibles that we have discussed in Chaps. 2 and 4. If it could be proven that quantum particles of the same type are discernible by their properties in *all* of their available states, this would vindicate PII in its entirety. On the other hand, a weaker result showing that only *some* states enable us to differentiate particles by their properties while some other states do not does not rescue PII as an exceptionless metaphysical principle. Still, even such a weaker result would go against the orthodox Indiscernibility Thesis which claims that the violation of PII is

[5] Indeed, it may be claimed that the argument from Chap. 4 against the possibility of absolute discernment in symmetric languages does not even apply to the heterodox approach to individuation, since in this approach permutation invariance involving *labels* in the formalism no longer reflects permutation invariance involving *objects*. If we adopt this stance, then the entire exercise involving theorem (6.1) turns out to be unnecessary. Nevertheless, the logical possibility of absolute discernment in symmetric languages is an interesting formal result worth mentioning quite independently from its applicability to the quantum case. For a more detailed analysis of this logical result, see Bigaj (2020).

necessitated by all symmetric and antisymmetric states.[6] Logically, there are three options here: either PII fails in all available states for systems of same-type particles, or is true in all available states, or else it fails in some states and remains valid in some other states. We will have to make our choice on the basis of the proposed approach to individuation.

We will start our analysis with the case of same-type fermions, limiting ourselves to systems of two particles. At the beginning we will quote a technical result from matrix theory which is directly relevant to the problem at hand. It can be proven that all antisymmetric $n \times n$ matrices (i.e. matrices such that for each of their elements $c_{ij}$, $c_{ij} = - c_{ji}$) can be unitarily transformed into the following, block-diagonal form:

$$Z = \operatorname{diag}\left[Z_1, \ldots, Z_m, Z_0\right],$$

where:

$$Z_i = \begin{pmatrix} 0 & z_i \\ -z_i & 0 \end{pmatrix} \text{for } i > 0,$$

and $Z_0$ is a $(n - 2m) \times (n - 2m)$ matrix consisting entirely of zeros. The operation diag results in placing all $m$ matrices $Z_i$ diagonally one after another with all the remaining entries filled by zeros.[7]

[6] I don't have space here to acknowledge numerous commentaries to the apparent violation of PII in quantum mechanics that have been made in the literature. Let me only mention one particular response by Michela Massimi in Massimi (2001). She attempts to defend PII in its stronger version (involving monadic properties) by pointing out that PII is not applicable to the case of same-type fermions, since the components of fermionic systems do not possess individual pure states. Consequently, no well-defined values of measurable quantities (observables) can be attributed to the components. My objection to this argument is twofold. First, I don't see a reason why monadic properties should be limited to possessing well-defined values of observables, since probabilistic attributions seem equally legitimate. Second, even if we agreed to this restriction on admissible properties, still PII would be violated if we presented it in the form of the implication "if *a* is distinct from *b*, then there is a property possessed by *a* and not by *b*", since the consequent would be guaranteed to be false.

[7] See Schliemann et al. (2001, p. 022303–2; Eckert et al. 2002, p. 94).

This result is relevant to the issue of how to conveniently represent antisymmetric states of two particles, since all antisymmetric vectors in the product $\mathcal{H} \otimes \mathcal{H}$, where $\dim\mathcal{H} = n$, have the form of:

$$\sum_{i,j=1}^{n} c_{ij} |\varphi_i\rangle |\varphi_j\rangle,$$

where $|\varphi_i\rangle$ $|\varphi_i\rangle$ are orthonormal vectors (basis vectors) in $\mathcal{H}$, and $c_{ij} = -c_{ji}$. Given that a unitary transformation represents a change of basis, it may be concluded from the above result regarding antisymmetric matrices that every antisymmetric vector can be presented in the following form (we will call this representation the Schmidt-Slater decomposition)[8]:

$$\sum_{i=1}^{m} z_i \left( |\eta_{2i-1}\rangle |\eta_{2i}\rangle - |\eta_{2i}\rangle |\eta_{2i-1}\rangle \right), \tag{6.9}$$

where $2m \leq n$ and $|\eta_i\rangle$ are some orthonormal vectors in $\mathcal{H}$. Now, it can be easily proven that vector (6.9) lies in the range of the symmetric projector $E \otimes F + F \otimes E$, where $E$ projects on the subspace of $\mathcal{H}$ spanned by "odd" vectors $|\eta_{2i-1}\rangle$, and $F$ projects on the orthogonal subspace spanned by "even" vectors $|\eta_{2i}\rangle$.[9] Thus, according to the heterodox interpretation, the composite system occupying state (6.9) can be individuated into two components, one possessing the property associated with projector $E$ and the other equipped with the property corresponding to projector $F$. In consequence, it is guaranteed that two fermions of the same type will always be discerned by their mutually exclusive properties $\Pi_E$ and $\Pi_F$. However, their discerning properties will usually not be maximally specific, that is, will not be represented by one-dimensional projectors. The maximally specific discernibility only happens if the number of non-zero coefficients $z_i$ in the Schmidt-Slater decomposition (6.9) equals one.

[8] We have to note that the Schmidt-Slater decomposition of antisymmetric states is not unique, which will become rather important later. However, what is unique is the number of non-zero coefficients $z_i$ in the decomposition (6.9).

[9] We should observe that projectors $E$ and $F$ defined above are by no means the only choice available. We could have taken any projector $E'$ whose range is spanned by any set of vectors $S$ such that out of the pair $|\eta_{2i-1}\rangle$ and $|\eta_{2i}\rangle$ exactly one belongs to $S$, and the result would be the same. The resulting arbitrariness of the choice of discerning properties will be the subject of intensive scrutiny in the next chapter.

In conclusion, we have proven that fermions occupying antisymmetric states are *always* discerned by some properties; hence they always obey PII. However, an analogous result does not obtain in the case of bosons. In order to deal with this case, we will formulate a general criterion of discernibility applicable to symmetric states, in the form of the following equivalence[10]:

(6.10) For every normalized symmetric vector $|\psi\rangle$ in $\mathscr{H} \otimes \mathscr{H}$, there are orthogonal projectors $E$ and $F$ such that $(E \otimes F + F \otimes E)|\psi\rangle = |\psi\rangle$, iff $|\psi\rangle = \sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} (|\varphi_i\rangle |\varphi_j\rangle + |\varphi_j\rangle |\varphi_i\rangle)$, for some orthogonal $|\varphi_i\rangle$.

Proof. From right to left. Let $E$ project on the subspace of $\mathscr{H}$ spanned by vectors $\{|\varphi_i\rangle\}_{i=1}^{k}$, and let $F$ project on the subspace spanned by vectors $\{|\varphi_j\rangle\}_{j=k+1}^{n}$ (obviously these subspaces are orthogonal). It is easy to verify that $(E \otimes F + F \otimes E) \left(\sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} \left(|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle\right)\right) = \sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} \left(|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle\right)$.

From left to right. This time we select any orthogonal vectors $\{|\varphi_i\rangle\}_{i=1}^{k}$ spanning the subspace $E[\mathscr{H}]$ and any orthogonal vectors $\{|\varphi_j\rangle\}_{j=k+1}^{n}$ spanning $F[\mathscr{H}]$. Every symmetric vector $|\psi\rangle$ can be written as the sum $\sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} \, |\varphi_i\rangle |\varphi_j\rangle$, where $c_{ij} = c_{ji}$ and $d$ – the dimensionality of $\mathscr{H}$. When we act upon this vector with the projector $E \otimes F + F \otimes E$, the result will be:

$$\sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} |\varphi_i\rangle|\varphi_j\rangle + \sum_{i=k+1}^{n} \sum_{j=1}^{k} c_{ij} |\varphi_i\rangle|\varphi_j\rangle =$$

[10] An analogous equivalence holds for fermionic antisymmetric states, with the plus sign replaced by the minus sign in the rhs of the equivalence. Since we know that all fermionic states can be written in the form of the minus-sign combination in the rhs of (6.10), this proves immediately that fermions are always discerned by some properties.

$$\sum_{i=1}^{k}\sum_{j=k+1}^{n}\left(c_{ij}|\varphi_i\rangle|\varphi_j\rangle + c_{ji}|\varphi_j\rangle|\varphi_i\rangle\right) =$$
$$\sum_{i=1}^{k}\sum_{j=k+1}^{n}c_{ij}\left(|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle\right).$$

Given this fact, we infer that the eigenequation $(E \otimes F + F \otimes E)|\psi\rangle = |\psi\rangle$ holds true only if $|\psi\rangle = \sum_{i=1}^{k}\sum_{j=k+1}^{n}c_{ij}(|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle)$; hence we have proven the rhs of the equivalence in (6.10).

The equivalence in (6.10) gives us a convenient positive criterion of absolute discernibility for same-type bosons. It immediately identifies symmetric states whose occupation by a composite system guarantees that its components will be absolutely discerned by appropriate properties. The "discerning" states are those that can be divided into symmetric blocks $|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle$ such that vectors $|\varphi_i\rangle$ and $|\varphi_j\rangle$ are taken from two orthogonal spaces $U$ and $V$, same for each block. This already provides us with an easy way to identify projectors $E$ and $F$ that discern the particles: they are, namely, projectors whose ranges are orthogonal spaces $U$ and $V$. However, (6.10) is not a particularly useful criterion for the *indiscernibility* of same-type bosons. This is so, because it is typically more difficult to prove that a given vector *cannot* be written in a particular mathematical form, than to prove that it *can* be written in this form.

One straightforward example of a bosonic state in which no absolute discernibility is possible is the symmetric product of two identical vectors: $|\varphi\rangle|\varphi\rangle$. It is rather obvious to observe that such a vector can never be presented in the form of the combination $\sum_{i=1}^{k}\sum_{j=k+1}^{n}c_{ij}(|\varphi_i\rangle|\varphi_j\rangle + |\varphi_j\rangle|\varphi_i\rangle)$. No matter what basis vectors we decide to write vector $|\varphi\rangle|\varphi\rangle$ in, symmetric combinations of the form $|\varphi_i\rangle|\varphi_i\rangle$ will always be present in the expansion. However, we have to be careful not to jump to conclusions in more complex cases. For instance, it may feel intuitive to expect that linear combinations of symmetric products $|\varphi_i\rangle|\varphi_i\rangle$ should also prevent absolute discernibility. After all, it seems plausible that superpositions of entirely indiscernible states should themselves be indiscernible. But this contention is completely wrong. As a matter of fact, it can be proven that *all* symmetric states are expressible in the form of the sum $\sum_{i=1}^{k} c_i \,|\varphi_i\rangle\,|\varphi_i\rangle$ for some orthogonal vectors $|\varphi_i\rangle$. Among these vectors there must be states that enable absolute discernibility. Thus we should not be deceived by a particular form a given vector takes in some basis, since this form can be radically altered when we select an alternative basis.

To illustrate this further, let us consider any symmetric binary combination of the form:

$$|\Psi\rangle = c_1|\varphi_1\rangle|\varphi_1\rangle + c_2|\varphi_2\rangle|\varphi_2\rangle \tag{6.11}$$

and let us ask the question of when (if at all) such a state enables us to discern the components of the composite system occupying it. Given the criterion (6.10), this question is equivalent to asking whether there are two orthogonal vectors $|\chi_1\rangle$ and $|\chi_2\rangle$ such that the state $|\Psi\rangle$ can be presented as a combination:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\chi_1\rangle|\chi_2\rangle + |\chi_2\rangle|\chi_1\rangle\right). \tag{6.12}$$

Obviously, vectors $|\chi_1\rangle$ and $|\chi_2\rangle$ must belong to the two-dimensional space spanned by $|\varphi_1\rangle$ and $|\varphi_2\rangle$; hence their most general form is:

$$\begin{aligned} |\chi_1\rangle &= a|\varphi_1\rangle + b|\varphi_2\rangle \\ |\chi_2\rangle &= b^*|\varphi_1\rangle - a^*|\varphi_2\rangle \end{aligned} \tag{6.13}$$

Inserting these into formula (6.12) and working out the appropriate products, we can immediately see that the resulting formula will have the form (6.11) only if the following holds:

$$|a|^2 = |b|^2,$$

which, given the normalization $|a|^2 + |b|^2 = 1$, leads to the following determination of the coefficients $a$ and $b$:

$$a = \frac{1}{\sqrt{2}} e^{i\alpha}$$

$$b = \frac{1}{\sqrt{2}} e^{i\beta},$$

for any $\alpha$, $\beta \in [0, 2\pi]$. Again, inserting this into (6.13) and (6.12), we obtain the following expression for the state $|\Psi\rangle$, under the assumption that particles are absolutely discerned when occupying it (where $\gamma = \alpha - \beta$ and $\delta = \beta - \alpha$):

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(e^{i\gamma}|\varphi_1\rangle|\varphi_1\rangle + e^{i\delta}|\varphi_2\rangle|\varphi_2\rangle\right). \tag{6.14}$$

Formula (6.14) covers all the cases in which absolute discernibility is attainable, given that the state is as in (6.11). All the other combinations of type (6.11) with the coefficients of unequal moduli will describe a situation in which no property can discern one particle from the other.

To conclude this section: the question of the universality of absolute discernibility and the validity of the Principle of the Identity of Indiscernibles depends on whether we consider fermions or bosons. Fermions occupying antisymmetric states are always discerned by some properties. These properties may be maximally specific (represented by one-dimensional projectors) or less than maximally specific (represented by many-dimensional projectors, whose dimensionality is however no larger than half of the dimensionality of the single-particle space in the case of two particles). The dimensionality of the discerning projectors is determined by the number of non-zero coefficient in the Schmidt-Slater decomposition (6.9) of the state vector. Regardless of these fine details, the philosophical moral is that under the heterodox approach to quantum individuation, all fermions satisfy the PII.

On the other hand, bosonic states may be divided into those that attribute mutually exclusive properties to individual components and those that do not. In some admissible states of same-type bosons, no property is definitely possessed by one particle while definitely not being possessed by the other. Apart from the general if somewhat abstract criterion (6.10) of the (in-)discernibility of bosons that we have proposed, we have identified two specific types of symmetric states in which the components of the system are not discernible by their properties. These are symmetric products of individual states, and superpositions consisting of two such products with coefficients of uneven moduli. Thus bosons generally do not obey the Principle of the Identity of Indiscernibles, even though the cases of its violation are not as ubiquitous as one might expect on the basis of the orthodox approach.

Our analysis provided us with an answer to the question posed at the end of Sect. 5.3 regarding the scope of the proposed method of individuation of same-type particles. The unorthodox method of individuating "indistinguishable" quantum particles is guaranteed to work in the case of fermions; however, not all bosonic states admit such an individuation.

## 6.3 Entanglement and Properties

If we were to choose just one concept unique to quantum mechanics that epitomizes its non-classical, intuition-defying character, it would most probably be the concept of entanglement.[11] Entangled systems of many particles display astonishing features such as non-local correlations that do not depend on the spatial separation of the components, and the irreducibility of the properties of a whole to the (intrinsic) properties of its components. Because of the intimate connection between entanglement and the possession of properties by the components of a composite, entangled system, we should expect that a new proposal of how to individuate same-type particles will have an impact on our understanding of the notion of entanglement applied to these particles. And indeed this is what all the proponents of the heterodox approach to quantum individuation agree upon: we have to modify the standard concept of entanglement in order to apply it to the case of "indistinguishable" particles.

[11] See the recent special issue "The metaphysics of entanglement" of *Synthese*, Vol. 197 No 10, 2020 for a multifaceted philosophical analysis of the concept of entanglement.

The standard definition of entangled states, which we have already encountered in Sect. 2.1, is purely formal. An entangled state of $N$ particles is a state represented by a vector in the $N$-fold tensor product of Hilbert spaces which cannot be written as a product of $N$ vectors, each from one single-particle Hilbert space. Given this well-known definition, an immediate consequence regarding systems of same-type particles is that the vast majority of their admissible states will have to be classified as entangled. More precisely, all fermionic states are formally entangled, since no antisymmetric state can be written in the form of a product of vectors. As far as bosons are concerned, the only states that would be classified as non-entangled would be symmetric products of identical vectors: $|\varphi\rangle \otimes |\varphi\rangle \otimes \ldots \otimes |\varphi\rangle$. However, this apparent prevalence of entanglement has no support in experimental facts. For instance, we do not observe non-local correlations connecting all electrons in the universe. Thus there is a need to come up with a new concept of entanglement that would be better suited to the task of describing systems of same-type particles.

Such a new concept of entangled states has been proposed and thoroughly discussed in an extensive paper by Giancarlo Ghirardi, Luca Marinatto and Tullio Weber (Ghirardi et al. 2002), followed by (Ghirardi and Marinatto 2003, 2004). In order to distinguish it from the old, standard notion of entanglement, I will use the term "GMW-entanglement", which has already gained currency in the literature. The authors of the mentioned paper take their cue from one of the features of "ordinary" entanglement, which is the fact that the components of entangled systems are *not* characterized by precise values of any observables (since no pure state can be attributed to the components of entangled systems). By negation, we may define a *non-entangled* state of two particles as such in which *both* components are characterized by precise values of a complete set of commuting observables (or, equivalently, of a maximal, non-degenerate observable).

So far, this definition can be taken as being in perfect agreement with the standard characterization of non-entangled states as product states, as long as we accept the formal representation of observables attributed to individual components in terms of direct, non-symmetric tensor products of operators, containing identity operators (to remind ourselves: in the case of two particles, an observable $A$ attributed to particle number one is represented by the product of operators $A \otimes I$). It can be formally proven that the state of a two-particle system $s(1, 2)$ can be presented in the form of a product of two vectors if and only if there are single-particle non-degenerate observables $A$ and $B$ such that the system $s(1, 2)$ is in an eigenstate of the operator $A \otimes I$ and $I \otimes B$.[12] But now we can try to substitute a new, unorthodox understanding of what it means for both components to be characterized by a precise value of an observable, for the old one. As we remember, in the case of two same-type particles, we have argued that the only way to represent their individual properties while respecting the symmetry restriction is with the help of operators $\Omega_1(E)$ and $\Omega_2(E)$ (formula 5.3 in Sect. 5.2). To that we will add a third, straightforward operator $\Omega_3(E)$ representing the case in which both particles possess the same precise values of commuting observables, which we excluded in Sect. 5.2, since now we are less interested in the issue of discernibility and more in the problem of entanglement:

$$\Omega_1(E) = E \otimes I + I \otimes E - E \otimes E$$

$$\Omega_2(E) = E \otimes I + I \otimes E - 2E \otimes E = E \otimes \bar{E} + \bar{E} \otimes E$$

$$\Omega_3(E) = E \otimes E$$

The idea is to use some of these operators to define a new concept of GMW-entanglement. However, it is not immediately obvious to decide which ones are the right choice (and in what combination). The least controversial case is when the system's state is an eigenstate of operator $\Omega_3(E)$ for some one-dimensional projector $E$, since in that case clearly both particles possess the corresponding maximally specific property $\Pi_E$. Operator $\Omega_2(E)$ in turn represents a situation in which one particle

[12] Proof of this equivalence is indeed very simple. If the state of a two-particle system is given as $|\varphi\rangle|\psi\rangle$, this state is an eigenstate of the operators $|\varphi\rangle\langle\varphi| \otimes I$ and $I \otimes |\psi\rangle\langle\psi|$; hence the required non-degenerate observables $A$ and $B$ are based on the projectors $|\varphi\rangle\langle\varphi|$ and $|\psi\rangle\langle\psi|$. On the other hand, let us assume that the state of the system is an eigenstate for some $A \otimes I$. Selecting eigenvectors $|\lambda_i\rangle$ for $A$ as a basis, we can write the state of the system generally as $\Sigma_{ij}c_{ij}|\lambda_i\rangle|\lambda_j\rangle$. Given that there is no degeneracy, vector $\Sigma_{ij}c_{ij}|\lambda_i\rangle|\lambda_j\rangle$ can be an eigenvector for $A \otimes I$ only if it has the form $\Sigma_j c_{ij}|\lambda_i\rangle|\lambda_j\rangle$ for some $i$, which can be rewritten as the product $|\lambda_i\rangle \otimes \Sigma_j c_{ij}|\lambda_j\rangle$.

definitely possesses the property associated with $E$, while the other particle definitely does not possess $E$. Hence if the system's state lies in the subspace projected onto by $\Omega_2(E)$, where $E$ is one-dimensional, this implies that both particles possess precise values of some maximal, non-degenerate observable, and thus the state should be classified as non-entangled. In order to make sure that this is indeed the case, let us formulate and prove the following theorem.

(6.15) If the state $|\psi(1,2)\rangle$ of two same-type particles is an eigenstate of projector $\Omega_2(E)$ where $E$ is one-dimensional, then there is a one-dimensional projector $E_\perp$ orthogonal to $E$ such that $|\psi(1,2)\rangle$ is an eigenstate of $\Omega_2(E_\perp)$.

Theorem (6.15) effectively says that if *exactly* one component of a system of same-type particles possesses a precise value for a given non-degenerate observable, the other component must also possess a precise value for some non-degenerate observable. In order to prove this, let us present generally the state $|\psi(1,2)\rangle$ as the combination $\sum_{ij=0}^{n} c_{ij} \, |\varphi_i\rangle \, |\varphi_j\rangle$, where $|\varphi_0\rangle$ is selected as a vector lying in the ray projected onto by $E$ (i.e. $E = |\varphi_0\rangle\langle\varphi_0|$). In that case we have:

$$\left(E \otimes \bar{E} + \bar{E} \otimes E\right) \sum_{ij=0}^{n} c_{ij} \, |\varphi_i\rangle \otimes |\varphi_j\rangle = \sum_{i=1}^{n} c_{0i} \left(|\varphi_0\rangle \otimes |\varphi_i\rangle\right) + \sum_{i=1}^{n} c_{i0} \left(|\varphi_i\rangle \otimes |\varphi_0\rangle\right)$$
$$= \left|\varphi_0\right\rangle \otimes \sum_{i=1}^{n} c_{0i} \left|\varphi_i\right\rangle + \sum_{i=1}^{n} c_{i0} \, |\varphi_i\rangle \otimes |\varphi_0\rangle.$$

Hence the state $|\psi(1,2)\rangle$ must have the form $|\varphi_0\rangle \otimes \sum_{i=1}^{n} c_{0i} \, |\varphi_i\rangle \pm \sum_{i=1}^{n} c_{0i} \, |\varphi_i\rangle \otimes \varphi_0\rangle$ (given that $c_{ij} = \pm c_{ji}$). From this it follows that $|\psi(1,2)\rangle$ is an eigenvector of $\Omega_2(E_\perp)$, where $E_\perp$ is a one-dimensional projector whose range is the ray spanned by vector $\sum_{i=1}^{n} c_{0i} \, |\varphi_i\rangle$. Since $\sum_{i=1}^{n} c_{0i} \, |\varphi_i\rangle$ is orthogonal to $|\varphi_0\rangle$, projectors $E$ and $E_\perp$ must be orthogonal too.

In conclusion, we have established that both operators $\Omega_2(E)$ and $\Omega_3(E)$ can serve as a criterion of non-entanglement in the case of particles of the same type. However, it may be argued that there is a third case of non-entanglement not covered by these operators. In the case of "distinguishable" particles, the product state $|\varphi\rangle|\chi\rangle$, where $|\varphi\rangle$ is neither orthogonal to $|\chi\rangle$ nor parallel to it, is still considered non-entangled, since particle number one possesses the definite value 1 of the projector $|\varphi\rangle\langle\varphi|$, and particle two possesses the value 1 of the projector $|\chi\rangle\langle\chi|$. Yet these projectors are neither identical nor orthogonal; hence their possession cannot be expressed by $\Omega_2(E)$ or $\Omega_3(E)$ in the case of "indistinguishable" particles. It may seem, then, that we will need a third case covered by the projector $\Omega_1(E)$. The suggested criterion could be as follows: if there are two one-dimensional projectors $E$ and $F$ (not necessarily orthogonal!) such that $\Omega_1(E)|\psi\rangle = |\psi\rangle$ and $\Omega_1(F)|\psi\rangle = |\psi\rangle$, then the state $|\psi\rangle$ is classified as non-entangled.

Surprisingly, GMW explicitly reject this third criterion of non-entanglement (Ghirardi et al. 2002, p. 78). Their argument against it is that when a state $|\psi\rangle$ is a joint eigenstate for $\Omega_1(E)$ and $\Omega_1(F)$ with $E$ not being orthogonal to $F$, there is still a non-zero probability that both particles will be found in the same state (e.g. in the state corresponding to $E$).[13] But this is baffling. Take, for example, an unquestionably non-entangled product state $|\varphi\rangle|\chi\rangle$. We can observe that when $|\varphi\rangle$ is not orthogonal to $|\chi\rangle$, there is also a non-vanishing probability that measurement of the observable represented by $|\varphi\rangle\langle\varphi|$ will find both particles in state $|\varphi\rangle$, since by assumption $|\langle\chi|\varphi\rangle|^2 > 0$. But this does not invalidate the fact that *before* measurement both particles possessed definite values of the projectors $|\varphi\rangle\langle\varphi|$ and $|\chi\rangle\langle\chi|$, and it similarly should not affect the assessment that $|\varphi\rangle|\chi\rangle$ is non-entangled. The only serious argument against the considered criterion of non-entanglement could be that somehow being in an eigenstate of both $\Omega_1(E)$ and $\Omega_1(F)$ does not exclude the possibility that one and the same particle will have both properties represented by $E$ and by $F$. After all, the condition that $\Omega_1(E)|\psi\rangle = |\psi\rangle$ and $\Omega_1(F)|\psi\rangle = |\psi\rangle$ only guarantees that (at least) one particle has property $\Pi_E$, and (at least) one particle has property $\Pi_F$, without saying which particle possesses which property. Thus it is at least logically possible that both properties will be attributed to one and the same particle.

[13] This argument is also repeated in Caulton (2014a).

But is this a real quantum-mechanical possibility? The only way to express the statement that one of the two particles possesses a certain property is again with the help of the operator $\Omega_1(E)$ applied to the single-particle projector $E$ representing this property. But there is no single-particle projector representing the joint possession of two incompatible properties! The operator $EF$ is not a projector, since $E$ and $F$ do not commute. So, if the quantum-mechanical formalism cannot even represent the situation in which one particle possesses both property $\Pi_E$ and $\Pi_F$, shouldn't we conclude that such a possibility is excluded by our best theory? And if we agree with this conclusion, then the only option is to admit that if the condition $\Omega_1(E)|\psi\rangle = |\psi\rangle$ and $\Omega_1(F)|\psi\rangle = |\psi\rangle$ is satisfied, it must be the case that one particle possesses a definite value of $E$ while *the other* possesses a definite value of $F$.

## 6.4 Fermionic Entanglement Versus Bosonic Entanglement

The above-mentioned controversy does not affect the case of fermions, since the symmetric operator $E \otimes E$, where $E$ is a one-dimensional projector, is equal to zero on the space of antisymmetric vectors.[14] Thus in this case operators $\Omega_1(E)$ and $\Omega_2(E)$ become identical, while $\Omega_3(E)$ turns out to be zero. Consequently, if there are two non-orthogonal one-dimensional projectors $E$ and $F$ such that $\Omega_1(E)|\psi\rangle = |\psi\rangle$ and $\Omega_1(F)|\psi\rangle = |\psi\rangle$, it is also guaranteed that there are two *orthogonal* one-dimensional projectors $E$ and $E_\perp$ such that $\Omega_2(E)|\psi\rangle = |\psi\rangle$ and $\Omega_2(E_\perp)|\psi\rangle = |\psi\rangle$, and the state $|\psi\rangle$ turns out to be non-entangled on the basis of the uncontroversial criterion involving projectors of the type $\Omega_2$. As a result, in the case of fermions of the same type, we can formulate the following definition of GMW-non-entanglement:

[14] Here is a quick proof of this fact. We can decompose any antisymmetric vector in an orthogonal basis $|\varphi_i\rangle$ $i = 0, \ldots, n$, such that $E = |\varphi_0\rangle\langle\varphi_0|$. Given that the decomposition has the form $\Sigma_{i \neq j} c_{ij} \mid \varphi_i\rangle \mid \varphi_j\rangle$, we can immediately see that the action of $E \otimes E$ on this vector produces zero, since every component of the sum contains at least one vector $|\varphi_i\rangle$ where $i \neq 0$.

(6.16) The normalized state $|\psi(1,2)\rangle$ of two same-type fermions is GMW-non-entangled iff $(E \otimes \bar{E} + \bar{E} \otimes E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ for some one-dimensional projector $E$.

There is a particularly elegant and intuitive alternative way to define the notion of fermionic GMW-(non-)entanglement which relies directly on the form of vector $|\psi(1,2)\rangle$, as stated in the following theorem.

(6.17) The normalized state $|\psi(1,2)\rangle$ of two same-type fermions is GMW-non-entangled iff $|\psi(1,2)\rangle$ has the form of $\frac{1}{\sqrt{2}}(|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle)$ for some orthogonal vectors $|\varphi\rangle$ and $|\chi\rangle$.

Proof of this biconditional is straightforward. From right to left: we can select $E = |\varphi\rangle\langle\varphi|$, and it immediately follows that $(E \otimes \bar{E} + \bar{E} \otimes E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$. From left to right: we have already proven when establishing the truth of (6.15) that if $(E \otimes \bar{E} + \bar{E} \otimes E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$, then state $|\psi(1,2)\rangle$ can be represented as:

$$|\varphi_0\rangle \otimes \sum_{i=1}^{n} c_{0i} |\varphi_i\rangle \pm \sum_{i=1}^{n} c_{0i} |\varphi_i\rangle \otimes |\varphi_0\rangle,$$

which, after normalization, is exactly of the required form $\frac{1}{\sqrt{2}}(|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle)$.

As a matter of fact, the condition of orthogonality in (6.17) can be dropped without affecting the equivalence, thanks to a simple fact regarding antisymmetric states. That is, the following biconditional can be proven to hold:

(6.18) The (normalized) state $|\psi(1,2)\rangle$ of two same-type fermions is GMW-non-entangled iff $|\psi(1,2)\rangle$ has the form of $\frac{1}{\sqrt{2(1-|\langle\varphi|\chi\rangle|^2)}}(|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle)$ for some vectors $|\varphi\rangle$ and $|\chi\rangle$.

Proof from right to left (left to right follows obviously from (6.17)). Suppose that vectors $|\varphi\rangle$ and $|\chi\rangle$ are not orthogonal. In that case we can write $|\chi\rangle$ as the sum $a|\varphi\rangle + b|\varphi_\perp\rangle$, where $|\varphi_\perp\rangle$ is a normalized vector orthogonal to $|\varphi\rangle$, and reformulate vector $|\psi(1,2)\rangle$ as follows (taking into account that $\langle\varphi|\chi\rangle = a$ and $|a|^2 + |b|^2 = 1$):

$$\frac{1}{\sqrt{2\left(1-|a|^2\right)}}\left(|\varphi\rangle|\chi\rangle-|\chi\rangle|\varphi\rangle\right)=\frac{1}{\sqrt{2\left(1-|a|^2\right)}}$$
$$\left[a|\varphi\rangle|\varphi\rangle+b|\varphi\rangle|\varphi_\perp\rangle-\left(a|\varphi\rangle|\varphi\rangle+b|\varphi_\perp\rangle|\varphi\rangle\right)\right]=\frac{1}{\sqrt{2}}\left(|\varphi\rangle|\varphi_\perp\rangle-|\varphi_\perp\rangle|\varphi\rangle\right).$$

Given (6.17), we can now infer that the state $|\psi(1,2)\rangle$ is indeed GMW-non-entangled. Note that the above derivation does not work for bosons, since the plus sign makes it impossible for the symmetric components $|\varphi\rangle|\varphi\rangle$ to cancel each other out. Only antisymmetric states are guaranteed to be expressed with the help of the combinations of orthogonal vectors (vide the Schmidt-Slater decomposition discussed in Sect. 6.2).

Yet another compelling characteristic of GMW-(non-)entanglement can be given in terms of individuation blocks discussed in Sect. 5.3. To remind ourselves: if there are two orthogonal single-particle projectors $E_a$ and $E_b$ such that the state $|\psi(1,2)\rangle$ of a system of same-type particles lies in the subspace projected onto by $\mathcal{E} = E_a \otimes E_b + E_b \otimes E_a$, then $|\psi(1,2)\rangle$ can be equivalently represented as a vector in the individuation block $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$. Each factor in this tensor product is supposed to host states of one component of the total system, which is considered to consist of a particle possessing the property corresponding to $E_a$, and another particle characterized by the property associated with $E_b$. The "old" representations of states in the subspaces $\mathcal{E}[\mathscr{A}]$ and $\mathcal{E}[\mathscr{S}]$, where $\mathscr{A}$ and $\mathscr{S}$ are the antisymmetric and symmetric sectors of $\mathscr{H}$, are connected to the "new" representations in the individuation block $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ via the mapping given by the operator $U_{ab} = \sqrt{2}E_a \otimes E_b$ and its inverses: $\frac{1}{\sqrt{2}}(I - P_{12})$ onto $\mathcal{E}[\mathscr{A}]$ and $\frac{1}{\sqrt{2}}(I + P_{12})$ onto $\mathcal{E}[\mathscr{S}]$.

The criterion of entanglement for fermions individuated by orthogonal projectors $E_a$ and $E_b$ is indeed very simple:

(6.19) If a system of two fermions occupies a state $|\psi(1,2)\rangle$ such that $(E_a \otimes E_b + E_b \otimes E_a)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ for some orthogonal projectors $E_a$ and $E_b$, then $|\psi(1,2)\rangle$ is GMW-non-entangled iff the representation of $|\psi(1,2)\rangle$ in the individuation block $E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ is a product vector $|\varphi\rangle|\chi\rangle$, where $|\varphi\rangle$ is orthogonal to $|\chi\rangle$.

Proof from right to left. The unitarily equivalent counterpart of $|\varphi\rangle|\chi\rangle \in E_a[\mathscr{H}] \otimes E_b[\mathscr{H}]$ in $\mathcal{E}[\mathscr{A}]$ is given by $\frac{1}{\sqrt{2}}(I - P_{12})|\varphi\rangle|\chi\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle)$, which is a GMW-non-entangled state according to (6.17).

Left to right. On the basis of (6.17) we know that $|\psi(1,2)\rangle$ can be presented as $\frac{1}{\sqrt{2}}(|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle)$ for some orthogonal vectors $|\varphi\rangle$ and $|\chi\rangle$. Let us select a basis $|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle$ of $\mathscr{H}$, such that $|e_1\rangle, \ldots |e_k\rangle$ span the subspace $E_a[\mathscr{H}]$, and $|e_{k+1}\rangle, \ldots, |e_l\rangle$ span the subspace $E_b[\mathscr{H}]$. Let $|\varphi\rangle = \sum_{i=1}^{n} c_i |e_i\rangle$ and $|\chi\rangle = \sum_{i=1}^{n} d_i |e_i\rangle$. Given these assumptions, the result of the action of $E_a \otimes E_b + E_b \otimes E_a$ on $|\psi(1,2)\rangle$ can be written as follows:

$$\sum_{i=1}^{k} c_i |e_i\rangle \otimes \sum_{j=k+1}^{l} d_j |e_j\rangle - \sum_{j=k+1}^{l} d_j |e_j\rangle \otimes \sum_{i=1}^{k} c_i |e_i\rangle$$
$$+ \sum_{i=k+1}^{l} c_i |e_i\rangle \otimes \sum_{j=1}^{k} d_j |e_j\rangle - \sum_{j=1}^{k} d_j |e_j\rangle \otimes \sum_{i=k+1}^{l} c_i |e_i\rangle$$

This vector has to be identical to $|\psi(1,2)\rangle$, that is to:

$$\sum_{i=1}^{n} c_i |e_i\rangle \otimes \sum_{j=1}^{n} d_j |e_j\rangle - \sum_{j=1}^{n} d_j |e_j\rangle \otimes \sum_{i=1}^{n} c_i |e_i\rangle.$$

By inspection we can verify that the identity between these two expressions can hold only in two cases: (a) when for all $i > k$ $c_i = 0$, and therefore also for all $j \leq k$, $d_j = 0$, or (b) when for all $i \leq k$, $c_i = 0$, and therefore also for all $j > k$, $d_j = 0$ (plus clearly for all $i > l$, $c_i = 0$ and $d_i = 0$). This means that either $|\varphi\rangle = \sum_{i=1}^{k} c_i \,|\, e_i\rangle$ and $|\chi\rangle = \sum_{i=k+1}^{l} d_i \,|\, e_i\rangle$, or $|\varphi\rangle = \sum_{i=k+1}^{l} c_i \,|\, e_i\rangle$ and $|\chi\rangle = \sum_{i=1}^{k} d_i \,|\, e_i\rangle$. Consequently, the action of the operator $\sqrt{2}E_a \otimes E_b$ on $\frac{1}{\sqrt{2}}(|\,\varphi\rangle\,|\,\chi\rangle - |\,\chi\rangle\,|\,\varphi\rangle)$ will produce either $|\varphi\rangle|\chi\rangle$ or $-|\chi\rangle|\varphi\rangle$, which proves the required implication.

It is worth noting that criterion (6.19) implies that non-entangled product states in the representation based on the individuation blocks must consist of orthogonal states rather than any states. Indeed, there can be no product states consisting of non-orthogonal vectors in the individuation block $E_a[\mathcal{H}] \otimes E_b[\mathcal{H}]$ for the simple reason that $E_a$ and $E_b$ themselves are orthogonal to each other. Thus no vector of the form $|\varphi\rangle|\chi\rangle$, where $\langle\varphi|\chi\rangle \neq 0$, can ever represent a (non-entangled) state of two fermions of the same type.[15]

Moving on to the case of bosons, we will first follow the proposal put forward by GMW. They essentially repeat the definition of fermionic entanglement given in (6.16), adding to it a condition which covers the case of products of identical states. This leads to the following definition:

(6.20) The normalized state $|\psi(1,2)\rangle$ of two same-type bosons is GMW-non-entangled iff $\left(E \otimes \bar{E} + \bar{E} \otimes E\right)\left|\psi(1,2)\right\rangle = \left|\psi(1,2)\right\rangle$ or $E \otimes E\ |\psi(1,2)\rangle = |\psi(1,2)\rangle$ for some one-dimensional projector $E$.

It is easy to observe that (6.20) can be equivalently written in the form of an equivalence analogous to (6.17):

[15] Ladyman et al. (2013) give yet another analysis of non-entangled fermionic states in terms of the Grassman ("wedge") antisymmetric product $\wedge$, according to which a state is entangled iff it has the product form $|\varphi\rangle \wedge |\psi\rangle$ for some orthogonal states $|\varphi\rangle$and $|\psi\rangle$.

(6.21) The normalized state $|\psi(1,2)\rangle$ of two same-type bosons is GMW-non-entangled iff $|\psi(1,2)\rangle$ has the form of $\frac{1}{\sqrt{2}}(|\varphi\rangle|\chi\rangle+|\chi\rangle|\varphi\rangle)$ for some orthogonal vectors $|\varphi\rangle$ and $|\chi\rangle$, or $|\psi(1,2)\rangle$ has the form of $|\varphi\rangle|\varphi\rangle$.

However, there is no bosonic analogue of theorem (6.18). As a matter of fact, the following theorem can be proven:

(6.22) If the normalized state $|\psi(1,2)\rangle$ of two same-type bosons has the form of $\frac{1}{\sqrt{2(1-|\langle\varphi|\chi\rangle|^2)}}(|\varphi\rangle|\chi\rangle+|\chi\rangle|\varphi\rangle)$, where $|\langle\varphi|\chi\rangle|^2 > 0$, then $|\psi(1,2)\rangle$ is GMW-entangled.

We can prove (6.22) by showing that vector $\frac{1}{\sqrt{2(1-|\langle\varphi|\chi\rangle|^2)}}(|\varphi\rangle|\chi\rangle+|\chi\rangle|\varphi\rangle)$ cannot be alternatively written as a combination $\frac{1}{\sqrt{2}}(|\eta\rangle|\mu\rangle+|\mu\rangle|\eta\rangle)$, where $\langle\mu|\eta\rangle = 0$. Presenting generally vectors $|\eta\rangle$ and $|\mu\rangle$ as:

$$|\eta\rangle = a|\varphi\rangle + b|\chi\rangle$$
$$|\mu\rangle = c|\varphi\rangle + d|\chi\rangle$$

we can reformulate the expression $\frac{1}{\sqrt{2}}(|\eta\rangle|\mu\rangle+|\mu\rangle|\eta\rangle)$ as follows:

$$\frac{1}{\sqrt{2}}\left[2ac|\varphi\rangle|\varphi\rangle + 2bd|\chi\rangle|\chi\rangle + (ad+cb)\left(|\varphi\rangle|\chi\rangle + |\chi\rangle|\varphi\rangle\right)\right].$$

This expression is identical to $\frac{1}{\sqrt{2(1-|\langle\varphi|\chi\rangle|^2}}(|\varphi\rangle|\chi\rangle+|\chi\rangle|\varphi\rangle)$, only if either $a = 0$ or $c = 0$ and either $b = 0$ or $d = 0$. Either way, this implies that $\langle\mu|\eta\rangle = \langle\varphi|\chi\rangle$, which violates the assumption that $|\langle\varphi|\chi\rangle|^2 > 0$.

Regarding the bosonic counterpart of criterion (6.19), surprisingly it can be repeated with no amendments in the case of systems of two bosons. However, its meaning will be slightly different. In the case of fermionic states, the antecedent of (6.19) is always guaranteed to be satisfied (this is an immediate consequence of the existence of the Schmidt-Slater decomposition (6.9) and the ensuing absolute discernibility of fermions). But this is not so in the case of bosons. The condition (6.19) does not cover the situation in which bosons occupy product states of the form $|\varphi\rangle|\varphi\rangle$, since there are no individuation blocks that could provide us with an alternative representation of such states. Similarly, (6.19) cannot apply to the controversial case of vectors of the form $\frac{1}{\sqrt{2\left(1-|\langle\varphi|\chi\rangle|^2\right)}}\left(|\varphi\rangle|\chi\rangle+|\chi\rangle|\varphi\rangle\right)$, where $|\langle\varphi|\chi\rangle|^2 > 0$.

Thus we cannot obtain an independent justification of the claim that this vector represents an entangled state.

At the end of Sect. 6.3, we have voiced our doubts regarding the classification of "skewed" combinations of the form used in (6.22) as entangled. The argument given by GMW falls short of proving that particles occupying such states do not possess well-defined values of maximal, non-degenerate observables. Given that there are no additional plausibility arguments, for instance, in terms of some alternative criteria of entanglement, in favor of GMW's classification (and, as we will see later, there may be indirect arguments supporting the opposite classification), I will suggest an alternative approach. In order to distinguish it from GMW's original concept, I'll call the extended notion of entanglement, perhaps somewhat immodestly, GMWB-entanglement:

(6.23) A normalized state $|\psi(1,2)\rangle$ of two same-type bosons is GMWB-non-entangled iff $\Omega_1(E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ and $\Omega_1(F)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ for some one-dimensional distinct projectors $E$ and $F$, or $(E \otimes E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ for some one-dimensional projector $E$.

According to the definition (6.23), bosonic states of the form $\frac{1}{\sqrt{2\left(1-|\langle\varphi|\chi\rangle|^2\right)}}\left(|\varphi\rangle|\chi\rangle-|\chi\rangle|\varphi\rangle\right)$ are GMWB-non-entangled, since the above vector lies in the subspace projected onto by $\Omega_1(|\varphi\rangle\langle\varphi|)$ and in the subspace projected onto by $\Omega_1(|\chi\rangle\langle\chi|)$. In the absence of arguments to the contrary, we take it that in this case it makes sense to speak of the system as consisting of two particles such that one of them possesses the property represented by $|\varphi\rangle\langle\varphi|$, and *the other* possesses the property associated with $|\chi\rangle\langle\chi|$.

## 6.5 Entanglement and Non-local Correlations

A surefire method to confirm the presence of quantum entanglement is via detecting the occurrence of non-local correlations, that is, statistical correlations between outcomes of measurements obtained at arbitrary spatial separations from each other. The correlations are present if the probability of obtaining some joint outcomes of measurements does not factorize, that is, cannot be written as the product of two probabilities, each associated with a separate outcome. Another way to express this is in terms of expectation values: we will say that there are non-local correlations between subsystems 1 and 2, if there are some observables $O_1$ and $O_2$ pertaining to spatially separated systems 1 and 2, such that $\langle O_1O_2\rangle \neq \langle O_1\rangle\langle O_2\rangle$, where brackets $\langle\ldots\rangle$ denote expectation values (averages). A well-known example of non-local correlations in the case of "distinguishable" particles can be based on the singlet-spin state (see formula 5.13 in Chap. 5):

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow_z\rangle\otimes|\downarrow_z\rangle-|\downarrow_z\rangle\otimes|\uparrow_z\rangle\right).$$

Let $O_1 = |\uparrow_z\rangle\langle\uparrow_z| \otimes I$ and $O_2 = I \otimes |\uparrow_z\rangle\langle\uparrow_z|$. Thus $O_1$ represents some measurable property of the first particle (possessing a definite spin "up"), while $O_2$ represents the same property attributed to the second particle. The operator $O_{12} = |\uparrow_z\rangle\langle\uparrow_z| \otimes |\uparrow_z\rangle\langle\uparrow_z|$ on the other hand denotes the property "spin up" attributed to both particles. It can be quickly verified that:

$$\langle O_1\rangle_{\psi_s} = \langle \psi_s | O_1 | \psi_s \rangle = \frac{1}{2}$$
$$\langle O_2\rangle_{\psi_s} = \langle \psi_s | O_2 | \psi_s \rangle = \frac{1}{2}$$
$$\langle O_{12}\rangle_{\psi_s} = \langle \psi_s | O_{12} | \psi_s \rangle = 0,$$

and thus:

$$\langle O_{12}\rangle_{\psi_s} \neq \langle O_1\rangle_{\psi_s} \langle O_2\rangle_{\psi_s}.$$

This confirms that state $|\psi_s\rangle$ is indeed entangled. On the other hand, if we take a non-entangled product state $|\varphi\rangle|\chi\rangle$ and any observable $A \otimes B$, it immediately follows that:

$$\langle \varphi | \langle \chi | A \otimes B | \varphi \rangle | \chi \rangle = \langle \varphi | A | \varphi \rangle \langle \chi | B | \chi \rangle,$$

thus the expectation values factorize.

Now, let us apply a similar test to GMW-non-entangled states of "indiscernible" particles, to see that there are no detectable non-local correlations in these cases.[16] In order to do it properly, we have to rely on the account of measurements presented in Sect. 5.5. The central idea of this approach is to tie measurements done on separate particles to specific *locations* rather than to unphysical *labels*. In order to be able to do that formally, we have to consider both internal and spatial degrees of freedom. Thus, the single-particle state space has to be decomposable into

[16] Caulton (2014b) proves generally that antisymmetric GMW-non-entangled states do not violate any Bell inequality (see in particular his Theorem 4.1 and Corollary 4.4.). In his proof he generalizes a theorem from Gisin (1991) which deals with "distinguishable" particles. Caulton defines a symmetric variant of the "local" operations which uses the idea of individuation blocks given by orthogonal projectors $E_a$ and $E_b$, as explained in Sect. 4.3. Local operations in Caulton's sense cannot transform the initial state into a GMW-entangled state, and hence it can be shown that the appropriate expectation values of spin-like observables (represented by Pauli matrices) factorize.

the product of two Hilbert spaces: $\mathscr{H} = \mathscr{H}_s \otimes \mathscr{H}_r$, where $\mathscr{H}_s$ contains states describing internal degrees of freedom (e.g. spin), and $\mathscr{H}_r$ is the space of positions. The operator that represents the situation in which a particle located in a region $L$ possesses the property associated with some projector $E_s$ has the following form:

$$\Omega_1\left(E_s|L\rangle\langle L|\right) = \left[E_s|L\rangle\langle L|\right]_1 \otimes \left[I_s I_r\right]_2 + \left[I_s I_r\right]_1 \otimes \left[(E_s|L\rangle\langle L|\right]_2 \\ -\left[E_s|L\rangle\langle L|\right]_1 \otimes \left[(E_s|L\rangle\langle L|\right]_2,$$

and analogously for the operator representing a property $F_s$ of a particle located in a region $R$ non-overlapping with $L$:

$$\Omega_1\left(F_s|R\rangle\langle R|\right) = \left[F_s|R\rangle\langle R|\right]_1 \otimes \left[I_s I_r\right]_2 + \left[I_s I_r\right]_1 \otimes \left[F_s|R\rangle\langle R|\right]_2 \\ -\left[F_s|R\rangle\langle R|\right]_1 \otimes \left[(F_s|R\rangle\langle R|\right]_2.$$

Calculating the product $\Omega_1(E_s\,|L\rangle\langle L|)\Omega_1(F_s\,|R\rangle\langle R|)$, we will obtain the following operator (the operators $\Omega_1(E)$ and $\Omega_1(F)$ commute if $E$ and $F$ are orthogonal, which is the case, given that $|L\rangle$ is orthogonal to $|R\rangle$):

$$\left[E_s|L\rangle\langle L|\right]_1 \otimes \left[F_s|R\rangle\langle R|\right]_2 + \left[F_s|R\rangle\langle R|\right]_1 \otimes \left[E_s|L\rangle\langle L|\right]_2.$$

This projector represents the situation in which the "left" particle possesses property $E_s$ while the "right" particle possesses $F_s$. Now, let us assume that the state of the system is a GMW-non-entangled state of the form[17]:

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}}(|\varphi\rangle|L\rangle \otimes |\chi\rangle|R\rangle \pm |\chi\rangle|R\rangle \otimes |\varphi\rangle|L\rangle),$$

where $|\varphi\rangle$ and $|\chi\rangle$ are any normalized vectors in the internal degrees space $\mathscr{H}_s$, and $|L\rangle$ and $|R\rangle$ are normalized orthogonal vectors in $\mathscr{H}_r$ associated

[17] We call this state "pseudo singlet state" in the case when $|\varphi\rangle$ and $|\chi\rangle$ are orthogonal spin states (e.g. $|\uparrow\rangle$ and $|\downarrow\rangle$). See formula (5.15) and the subsequent discussion in Chap. 5.

with space regions $L$ and $R$. Computing the expectation value of the operator $\Omega_1(E_s \otimes |L\rangle\langle L|)\Omega_1(F_s \otimes |R\rangle\langle R|)$ in state $|\psi(1,2)\rangle$, we get the following:

$$\langle\psi(1,2)|\,[E_s|L\rangle\langle L|]_1 \otimes [(F_s|R\rangle\langle R|]_2 + [(F_s|R\rangle\langle R|]_1\rangle \\ \otimes [E_s|L\rangle\langle L|]_2|\psi(1,2)\rangle = \langle\varphi|E_s|\varphi\rangle\langle\chi|F_s|\chi\rangle. \tag{6.24}$$

This result proves that no statistical correlations between the outcomes of measurement of observables $E_s$ and $F_s$ are present, as long as measurements are performed in separate locations. Thus, systems occupying GMW-non-entangled states indeed behave as if consisting of separate, statistically independent components.[18]

Interestingly, we can apply an analogous procedure of checking the existence of non-local correlations to the controversial case of GMW-entangled but GMWB-non-entangled states (see definition 6.23). Suppose, then, that a system of two bosons occupies the following state:

$$|\Phi(1,2)\rangle = N(|\varphi\rangle|S\rangle \otimes |\chi\rangle|T\rangle + |\chi\rangle|T\rangle \otimes |\varphi\rangle|S\rangle),$$

where $|S\rangle$ and $|T\rangle$ are wave functions whose supports are limited, respectively, to spatial region $S$ and $T$ such that $S$ and $T$ overlap; $|\varphi\rangle$ and $|\chi\rangle$ are

[18] F.A. Muller and Gijs Leegwater give an example of a GMW-non-entangled antisymmetric state ("tangled" in their terminology) which they claim violates a Bell inequality and hence gives rise to non-local correlations (Muller and Leegwater 2020, Sect. 7). The proposed state has the following form: $\frac{1}{2}(|L\rangle_1 + |R\rangle_1) \otimes (|L\rangle_2 + |R\rangle_2)\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2)$. However, particles described by this state cannot be individuated by separate locations, as both have identical states in the spatial degrees of freedom: $\frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$. In order to perform joint measurements aimed at verifying Bell's inequalities, we have to first locate one particle in region $L$ and the other in $R$. This can be done by means of a pre-measurement of position in which we will discard all pairs of particles whose locations turned out to be identical (either $|L\rangle|L\rangle$ or $|R\rangle|R\rangle$), leaving only pairs with distinct locations $|L\rangle|R\rangle$ or $|R\rangle|L\rangle$ (note that this pre-measurement is not a local operation in Caulton's sense—see ft. 16). But the state of such an ensemble will no longer be given by the above formula, but will be a GMW-entangled combination of the form $\frac{1}{\sqrt{2}}(|L\rangle_1 \otimes |R\rangle_2 + |R\rangle_2 \otimes |L\rangle_1)\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_2 \otimes |\uparrow\rangle_1)$.

non-orthogonal vectors in the space of internal degrees of freedom; and $N = \frac{1}{\sqrt{2\left(1-\left|\langle\varphi|\chi\rangle\langle S|T\rangle\right|^2\right)}}$ is the normalization constant. Now, let us select the following projection operator:

$$\Xi = [E_s X_L]_1 \otimes [F_s X_R]_2 + [F_s X_R]_1 \otimes [E_s X_L]_2 .$$

Projectors $E_s$ and $F_s$ act in the space of internal degrees of freedom (they may or may not be orthogonal), while $X_L$ and $X_R$ are projectors in space $\mathcal{H}_r$ that project onto the subspaces containing all wave functions whose support is limited, respectively, to regions $L$ and $R$ that do not overlap.[19] Furthermore, we assume that region $L$ does not overlap with $T$ but $L$ is included in $S$, and $R$ does not overlap with $S$ but is included in $T$. The spatial relations between regions $L$, $R$, $S$ and $T$ are depicted on Fig. 6.1. The idea behind these stipulations should be clear: $L$ and $R$ are regions in which we will place our measuring devices, and we want to be sure that each device will interact with only one "branch" of the state $|\Phi(1,2)\rangle$. Formally, this means that $X_L|T\rangle = 0$ and $X_R|S\rangle = 0$.

Given these stipulations, we can calculate the expectation value of the projector $\Xi$ in state $|\Phi(1,2)\rangle$:

$$\begin{aligned}\langle\Phi(1,2)|\Xi|\Phi(1,2)\rangle &= \langle\Phi(1,2)|\big[E_s X_L\big]_1 \otimes \big[F_s X_R\big]_2 \\ &+ \big[F_s X_R\big]_1 \otimes \big[E_s X_L\big]_2 |\Phi(1,2)\rangle = 2N^2\langle\varphi|E_s|\varphi\rangle\langle S|X_L|S\rangle\langle\chi|F_s|\chi\rangle\langle T|X_R|T\rangle. \end{aligned} \tag{6.25}$$

As can be seen from the above result, the expectation value for the joint outcome of measurements of property $E_s$ in region $L$ and property $F_s$ in region $R$ can be presented in the factorizable form as the product of independent expectation values for $E_s \otimes X_L$ and for $F_s \otimes X_R$. The reason why expressions $\langle S|X_L|S\rangle$ and $\langle T|X_R|T\rangle$ were absent from the previous case of GMW-entangled states is that now it is not guaranteed that a particle will be found in region $L$ ($R$), so these expectation values are no longer equal

[19] The action of the projector $X_L$ ($X_R$) on a given wave function $\psi$ will produce a wave function $\psi'$ which is zero outside $L$ ($R$) and coincides with $\psi$ inside $L$ (R). See Griffiths (2002, p. 52).

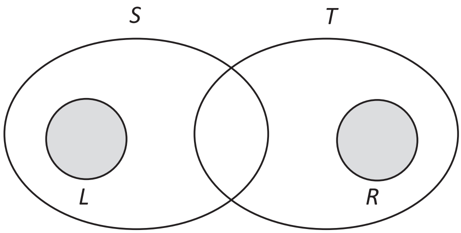


**Fig. 6.1** Spatial regions used in an example of a GMWB-non-entangled state. Regions *L* and *R* represent the locations of measuring devices

to 1. Another difference between formulas (6.24) and (6.25) is the presence of the factor $2N^2 = \dfrac{1}{1-|\langle\varphi|\chi\rangle\langle S|T\rangle|^2}$. This can be explained by pointing out that the formula $1 - |\langle\varphi|\chi\rangle\langle S|T\rangle|^2$ represents the probability that the system occupying state $|\varphi\rangle|S\rangle$ will *not* be found in the state $|\chi\rangle|T\rangle$, which was equal 1 in the case of orthogonal components. Anyway, the presence of this constant factor, which is independent of the selection of operators $E_s$, $F_s$, $X_L$ and $X_R$, amounts merely to a rescaling of probabilities, and does not affect the conclusion that the probabilities factorize, and hence there is no clear sign of statistical correlations between spatially separated outcomes that are typical for entangled systems.

The above-mentioned result points towards the suggested classification of bosonic states of the form written in (6.22) as non-entangled (GMWB-non-entangled). This is not to say that states which result from the symmetrization of the products of non-orthogonal states do not differ in any significant ways from the states resulting from the symmetrization of orthogonal states (e.g. in the latter particles are discernible by "orthogonal" properties, while in the former they are not), but there is no clear indication that it is the presence or absence of entanglement that is responsible for these differences.

## 6.6 Entanglement and Discernibility: Summary

At the end of this chapter, we will put together all its main results regarding discernibility and entanglement of systems of two particles of the same type (fermions and bosons). As we have found out, the relation between these two important concepts is not at all straightforward. In particular, it is not the case that entanglement prevents discernibility, but it is also not the case that non-entanglement guarantees it. The details of the relations between these concepts depend on whether we are dealing with fermions or bosons. As before, we will start with the case of fermions, which is less controversial. The table below contains characteristics of the types of fermionic states discussed earlier with respect to whether these states are entangled and whether they enable discernibility by properties.

The first two cases distinguished in Table 6.1 are actually identical, as we know from the proof of theorem (6.18), but we keep them separate in order to facilitate quick comparisons with the bosonic case. It can be clearly seen that fermions are always discernible by their properties regardless of whether they are entangled or not; however, non-entangled states ensure that their discernibility will be of the maximally specific type.

The case of two bosons is more complicated, not only because it includes a larger number of genuinely distinct possibilities but also

**Table 6.1** Comparison of entanglement and discernibility for fermions

| Fermionic states | GMW-entanglement | Discernibility |
|---|---|---|
| $\frac{1}{\sqrt{2}}(\lvert\varphi\rangle\lvert\chi\rangle - \lvert\chi\rangle\lvert\varphi\rangle)$, where $\langle\varphi\lvert\chi\rangle = 0$ | No | Yes, by maximally specific properties |
| $\frac{1}{\sqrt{2(1-\lvert\langle\varphi\lvert\chi\rangle\rvert^2)}}(\lvert\varphi\rangle\lvert\chi\rangle - \lvert\chi\rangle\lvert\varphi\rangle)$, where $\langle\varphi\lvert\chi\rangle \neq 0$ | No | Yes, by maximally specific properties |
| All the remaining states | Yes | Yes, by less than maximally specific properties |

because some cases turn out to be rather controversial, as the above-mentioned distinction between GMW-entanglement and GMWB-entanglement attests. Actually, the problems with the correct classifications of bosonic states are not limited to the issue of entanglement, since the very same case that led us to question the GMW classification of entanglement can also cast some doubts on the issue of discernibility. We have so far ignored the fact that there may be a second, weaker concept of absolute discernibility by properties that may be applicable to the case of bosonic states that arise by symmetrizing products of two non-orthogonal vectors. Thus before we can write a bosonic variant of Table 6.1, we should address this issue in greater detail.

The concept of discernibility used by us so far (as expressed, e.g. in Eq. 6.7) is based on the assumption that for two quantum objects to be discerned by state-dependent properties, one of them has to possess the property corresponding to a certain projector $E$, while the other possesses the property represented by the *orthogonal complement* of $E$, that is, $\overline{E} = I - E$. This looks like a natural way of interpreting quantum mechanically the logical concept of absolute discernibility, which requires—as we recall—that for any two absolutely discerned objects one has to definitely possess a property that the other one definitely does not possess. But such an interpretation may seem to be unnecessarily strict, as it excludes cases which arguably do not deserve to be classified as straightforward examples of indiscernibility. Consider a simple product state $|\varphi\rangle|\chi\rangle$ where $\langle\varphi|\chi\rangle \neq 0$. In that case there is clearly no projector $E$ such that $\langle\varphi|E|\varphi\rangle = 1$ and $\langle\chi|\overline{E}|\chi\rangle = 1$, and yet we have a strong intuition that particles 1 and 2 differ somehow with respect to their state-dependent properties. For instance, they are assigned different probabilities of obtaining the same outcomes of particular measurements. In order to include such cases in the broad concept of discernibility, we may want to introduce a distinction between two senses of discerning same-type particles by their properties. For the purpose of bookkeeping, the notion of discernibility used in earlier sections can be called "categorical":

(6.26) Two particles of the same type are *categorically* discerned by their properties in a state $|\psi(1,2)\rangle$ iff there is a non-trivial single-particle projector $E$ such that $\left(E \otimes \overline{E} + \overline{E} \otimes E\right)|\psi(1,2)\rangle = |\psi(1,2)\rangle$.

Categorical discernibility can be further divided into maximally specific discernibility (when $E$ is one-dimensional) or non-maximally specific discernibility (when $E$ is more than one-dimensional). On the other hand, a new, broader concept of discernibility can be introduced as follows:

(6.27) Two particles of the same type are *broadly* discerned by their properties in a state $|\psi(1,2)\rangle$ iff there are two distinct one-dimensional projectors $E$ and $F$ such that $\Omega_1(E)|\psi(1,2)\rangle = |\psi(1,2)\rangle$ and $\Omega_1(F)|\psi(1,2)\rangle = |\psi(1,2)\rangle$.

Broad discernibility defined in (6.27) may be seen as somewhat peculiar, since it includes maximally specific categorical discernibility as a special case (when $E$ and $F$ are orthogonal), but excludes non-maximally specific categorical discernibility. This looks like a downside, but I see no easy way to avoid this consequence. It won't do to stipulate that $E$ and $F$ not be orthogonal, for in the case of fermionic states condition (6.26) will be still satisfied for $E$ (and for $F$), so maximally specific categorical discernibility is guaranteed.[20] On the other hand, in the case of bosons, we may have states in which particles are broadly discerned without being categorically discerned.

In order to probe deeper the concept of broad discernibility applied to bosonic states, let us consider an arbitrary one-dimensional projector $E = |\varphi_0\rangle\langle\varphi_0|$. Selecting orthogonal vectors $|\varphi_i\rangle$, $i = 0, \ldots, n$ as a basis, we can expand any ket $|\psi(1,2)\rangle$ as follows:

$$|\psi(1,2)\rangle = \sum_{i,j=0}^{n} c_{ij} |\varphi_i\rangle |\varphi_j\rangle$$

and then we calculate the action of $\Omega_1(E)$ on $|\psi(1,2)\rangle$ (given that $c_{ij} = c_{ji}$):

[20] However, it has to be stressed that the maximally specific categorical discernibility in this case is not by $E$ and $F$ *jointly*. We may discern fermions that satisfy (6.27) either by $E$ and its complement $\bar{E}$, or by $F$ and its complement $\bar{F}$. The choice of a particular projector to individuate fermions is arbitrary—we will return to this problem in Chap. 7.

$$\Omega_1(E)|\psi(1,2)\rangle = \sum_{j=0}^{n} c_{0j}|\varphi_0\rangle|\varphi_j\rangle + \sum_{i=0}^{n} c_{i0}|\varphi_i\rangle|\varphi_0\rangle - c_{00}|\varphi_0\rangle|\varphi_0\rangle$$
$$= |\varphi_0\rangle \otimes \sum_{j=0}^{n} c_{0j}|\varphi_j\rangle + \sum_{j=0}^{n} c_{0j}|\varphi_j\rangle \otimes |\varphi_0\rangle - c_{00}|\varphi_0\rangle|\varphi_0\rangle \tag{6.28}$$

In order for $|\psi(1,2)\rangle$ to be in the range of $\Omega_1(E)$, it has to be of the form (6.28) written above (modulo normalization). Given this assumption, we can consider the following cases.

1. $c_{00} = 0$. In that case $|\psi(1,2)\rangle = |\varphi_0\rangle \otimes \sum_{j=1}^{n} c_{0j}|\varphi_j\rangle + \sum_{j=1}^{n} c_{0j}|\varphi_j\rangle \otimes |\varphi_0\rangle$, thus it is of the form $|\varphi\rangle|\chi\rangle + |\chi\rangle|\varphi\rangle$, where $\langle\varphi|\chi\rangle = 0$. Thus bosons occupying such a state are discerned categorically by maximally specific properties.
2. $c_{00} \neq 0$ and $c_{0i} = 0$ for all $i = 1,\ldots, n$. In that case $|\psi(1,2)\rangle = |\varphi_0\rangle|\varphi_0\rangle$ (after normalization), and thus the bosons are indiscernible.
3. $c_{00} \neq 0$ and $c_{0i} \neq 0$ for some $i = 1,\ldots, n$. Then we can rewrite the vector $|\psi(1,2)\rangle$ as follows:

$$|\psi(1,2)\rangle = |\varphi_0\rangle \otimes \left(\sum_{j=0}^{n} c_{0j}|\varphi_j\rangle - \frac{1}{2}c_{00}|\varphi_0\rangle\right) + \left(\sum_{j=0}^{n} c_{0j}|\varphi_j\rangle - \frac{1}{2}c_{00}|\varphi_0\rangle\right) \otimes |\varphi_0\rangle.$$

Since vector $|\chi\rangle = \sum_{j=0}^{n} c_{0j}|\varphi_j\rangle - \frac{1}{2}c_{00}|\varphi_0\rangle$ is neither orthogonal nor parallel to $|\varphi_0\rangle$, this is the case when $|\psi(1,2)\rangle$ is the result of the symmetrization of the product of two "skewed" vectors. Thus in that scenario the two bosons are *merely* broadly discernible by non-orthogonal projectors $E$ and $F = |\chi\rangle\langle\chi|$, but they are not categorically discernible.

The conclusion from the above is that the condition in (6.27) applied to bosons is satisfied if and only if the state is of the form $|\varphi\rangle|\chi\rangle + |\chi\rangle|\varphi\rangle$ for distinct kets $|\varphi\rangle$ and $|\chi\rangle$. Moreover, if $|\varphi\rangle$ and $|\chi\rangle$ are orthogonal, the discernibility is categorical, whereas when they are not orthogonal (but not parallel either), we have a case of mere broad discernibility. All facts that we have established regarding the discernibility and entanglement of bosons are put together in Table 6.2. The general form of symmetric

**Table 6.2** Comparison of entanglement and discernibility for bosons

| Bosonic states | GMW-entanglement | GMWB-entanglement | Discernibility |
|---|---|---|---|
| $\lvert\varphi\rangle\lvert\varphi\rangle$ | No | No | No |
| $\frac{1}{\sqrt{2}}\left(\lvert\varphi\rangle\lvert\chi\rangle+\lvert\chi\rangle\lvert\varphi\rangle\right)$, where $\langle\varphi\lvert\chi\rangle = 0$ | No | No | Categorical, by maximally specific properties |
| $\frac{1}{\sqrt{2\left(1-\lvert\langle\varphi\lvert\chi\rangle\rvert^2\right)}}\left(\lvert\varphi\rangle\lvert\chi\rangle-\lvert\chi\rangle\lvert\varphi\rangle\right)$, where $\langle\varphi\lvert\chi\rangle \neq 0$ | Yes | No | No categorical discernibility, but broad discernibility still applies |
| None of the above but of the general form $\sum_{i=1}^{k}\sum_{j=k+1}^{n}c_{ij}\left(\lvert\varphi_i\rangle\lvert\varphi_j\rangle+\lvert\varphi_j\rangle\lvert\varphi_i\rangle\right)$ | Yes | Yes | Categorical, by non-maximally specific properties |
| All the remaining states | Yes | Yes | No |

vectors used in the fourth row of the table is taken from theorem (6.10) which expressed the necessary and sufficient condition for categorical discernment of bosons.

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# 7

# Two Views on Quantum Individuation: A Comparison

Having established that there are two alternative and incongruous ways to identify and individuate quantum particles of the same type, we should now address the issue of which one ought to be preferred. As often happens with interpretive questions of that sort, there is no simple answer to be had. Each approach has its own advantages and disadvantages, and it is hard to find decisive arguments (empirical or theoretical) that could break this impasse. Still, some arguments can be seen as stronger, or more compelling, than the others. In this chapter we will task ourselves with discussing some of the most important problems that each of the two approaches to quantum individuation runs into. While I try to remain impartial and objective in my assessments, I will not hide the fact that in my view the orthodox approach faces more severe challenges than its rival, heterodox conception. My preferences lie squarely with the new conception and its insistence on the distinguishability of elementary particles in the majority of cases. It has to be noted though that the appeal of heterodoxy does not necessarily result from the mere fact that it rehabilitates the classical Principle of the Identity of Indiscernibles. By introducing the concept of qualitative individuation in terms of appropriate projectors, we uncover a new and unexpected feature of quantum reality:


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the possible relativization of the very *existence* of individual particles composing greater wholes to the experimental choices of what sets of parameters to measure. And while this fact may be used as an argument against heterodoxy, it may also be seen as a chance to develop a new and exciting metaphysics of objects in the quantum regime. We will talk more about this in the last section of this chapter, as well as in the next chapter. For now, let us focus on the fundamental shortcomings of the orthodox interpretation of quantum individuation.

## 7.1 The Troubles with Orthodoxy

As we remember from Chap. 2, the orthodox approach to individuation is rooted in the standard interpretation of the tensor-product formalism used to describe systems of multiple particles. Admittedly, this fact counts as a clear advantage of the orthodoxy—after all, the tensor-product formalism has been designed specifically to represent states of the components of a composite system within the factor Hilbert spaces. Any atypical way of reading the formalism may be accused of being a misinterpretation. However, when coupled with the Symmetrization Postulate, the standard reading of the formalism leads to rather troubling consequences, famously including the Indiscernibility Thesis. From the literal treatment of formalism, a revisionary metaphysics of quantum objects follows.

There is a growing body of works devoted to the formal analysis of the concept of non-individual objects—objects which not only can't be differentiated from each other, but to which it is even impossible to apply the notion of self-identity.[1] I will not attempt to give a survey of these logical considerations, which are interesting in their own right. Instead, I would like to point out that from the perspective of a working scientist the idea of the complete indistinguishability of fundamental objects of a certain kind is hard to accept. Any experimental procedures involving those fundamental objects are based on the implicit assumption of *separability*, which ensures that the objects undergoing scientific scrutiny can

[1] We have already mentioned some works that develop this conception in Chap. 2, ft. 24. See also French and Krause (2006, Chap. 7).

be somehow set apart from the rest of the universe. The most natural way of separating the objects of experimental inquiry from the rest of the world is in terms of their spatial location: electrons used in an experiment are *in the lab*, and not in the Andromeda Galaxy. And yet in the orthodox approach to individuation, this separation is impossible to achieve: all electrons in the universe have the precise same reduced state, including their spatial location. In fact, no *single* electron can ever occupy a particular location, because this would immediately differentiate it from the rest of the electrons, in violation of the Indiscernibility Thesis. Every spatial region seemingly occupied by an electron is occupied by *all of them*, albeit in a tiny proportion, so to speak.

I should stress that when I say *all of them*, I really mean *all* electrons in the universe with no exceptions. This is yet another example of the peculiarity of the metaphysical picture emerging from the orthodox approach: the state of a single electron appears to depend on what system of indistinguishable electrons it is considered a member of. This can be seen using a simple example. Suppose we pick out three same-type fermions initially described as occupying some orthogonal states $|\varphi\rangle$, $|\chi\rangle$ and $|\eta\rangle$. Taking into account only the first two fermions, we have to properly antisymmetrize their joint state, which produces the combination $|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle$ (the normalization constant omitted). As is well known, the (identical) reduced states for both particles in this case will be given by the density operator $\frac{1}{2}|\varphi\rangle\langle\varphi| + \frac{1}{2}|\chi\rangle\langle\chi|$ (see Appendix). But when we consider the antisymmetric state of the three particles, consisting of six permutations of the sequence $|\varphi\rangle|\chi\rangle|\eta\rangle$ with appropriate signs, the reduced state of each fermion will be $\frac{1}{3}|\varphi\rangle\langle\varphi| + \frac{1}{3}|\chi\rangle\langle\chi| + \frac{1}{3}|\eta\rangle\langle\eta|$, which is markedly different from the previous one. Consequently, if we want to write down the *actual* state of any electron, we have to take into account the joint state of all electrons in the universe and then reduce it. Considering any system of electrons smaller than the universal one as the starting point will produce a different result with respect to the states attributed to individual electrons, and this result has to be seen as strictly speaking incorrect.[2] This reveals a radically *holistic* and *non-local* nature of composite systems of same-type particles under the orthodox approach.

In contrast to that, the unorthodox conception of individuality does not imply any holism of that sort. The particles composing the binary system in state $|\varphi\rangle|\chi\rangle - |\chi\rangle|\varphi\rangle$ are individuated by the projectors $|\varphi\rangle\langle\varphi|$ and $|\chi\rangle\langle\chi|$, while the individuation of the three-particle system produces particles with properties represented by $|\varphi\rangle\langle\varphi|$, $|\chi\rangle\langle\chi|$ and $|\eta\rangle\langle\eta|$. Thus we simply have a system which can be seen as consisting of two particles occupying states $|\varphi\rangle$ and $|\chi\rangle$, with the third one being in state $|\eta\rangle$. The states of the first two particles do not depend on whether or not we take into account the third one. Consequently, we can limit ourselves to considering relatively small systems of particles without running the risk of attributing incorrect properties to single particles.

The fact that under the orthodox interpretation all particles of the same type possess identical reduced (mixed) states has other, difficult to accept consequences. One of the theoretical requirements regarding quantum mechanics is that in the classical limit, when we consider macroscopic objects consisting of a huge number of elementary particles, the theory should reproduce the observable phenomena that we all are familiar with (and that are predicted by classical mechanics which predates the quantum revolution). In particular, quantum theory should explain the fact that we perceive macroscopic bodies as possessing well-defined and distinguishing physical properties: spatiotemporal location, momentum, kinetic energy and so forth. For all we know, the Leibnizian Principle of the Identity of Indiscernibles works well when applied to the objects of our experience. And yet the orthodox interpretation of the quantum formalism spectacularly fails to achieve the goal of recovering the classical predictions when taken in the macroscopic limit. Any macroscopic body consists of an enormous number of fermions: mostly electrons, protons

[2] In fact, given that any system *s* of electrons that we may want to discuss in scientific practice is a subsystem of the universal one, attributing to *s* any pure state as we did above is always wrong, since a reduced state of any antisymmetric state is always mixed. So the situation in a sense is even worse – we can never know the exact state of *any* system of electrons unless we first find out the state of the total system. As a result, we can never properly do any physics that we routinely do in a lab, when we for instance calculate the energy levels of a single hydrogen atom, or verify experimentally the non-local correlations between two electrons in the EPR state.

and neutrons. If these particles occupy a mixed state comprising, for instance, a gigantic number of various positions (i.e. all locations in the universe in which a particle of a certain kind is present), then their macroscopic collection will "inherit" a similar mixed state. In consequence, no macroscopic object should have a well-defined position (even approximately). Instead, all bodies consisting of electrons (protons, neutrons) ought to appear "smeared" over all locations in the universe containing some electrons (protons, neutrons). This is as far from what we actually observe as it gets.[3]

In fact, even the application of the Indiscernibility Thesis to microscopic, unobservable entities can raise justified doubts as to its real "metaphysical" meaning. As Dieks and Lubberdink (2011) point out, the Indiscernibility Thesis regarding quantum particles can be reproduced in the case of classical particles obeying the laws of Newtonian mechanics. The states of $N$ classical particles are represented in a $6N$-dimensional phase space whose points are identified by three position coordinates and three momentum components for each particle. In the case when $N = 3$, a particular state can be written as a sequence of numbers $(x, p; y, q; z, r)$, where $(x, p)$, $(y, q)$ and $(z, r)$ represent the positions and momenta of, respectively, the first, second and third particles. However, if the particles do not differ with respect to their state-independent properties, the state of the entire system can be equivalently written using permuted sequences $(y, q; x, p; z, r)$, $(z, r; x, p; y, q)$ and so forth. Thus we have a case of representational redundancy, perfectly analogous to the quantum-mechanical redundancy. And it is possible to deal with this classical redundancy in an analogous way by introducing a "symmetrization postulate" which stipulates that the state of the three-particle system is represented by the set $\Lambda$ of all permuted sequences. Under this interpretation it is natural to define a "reduced" state of the first (second, third) particle as the set of all sequences $(x, p)$ that occur in the first (second, third) place in some sequence belonging to $\Lambda$. And it is easy to observe that in this case the reduced state for all three particles will be identical: $\{(x, p), (y, q), (z, r)\}$.

[3] This argument has been developed by Dennis Dieks and Andrea Lubberdink in Dieks and Lubberdink (2011, 2020). Adam Caulton (2014) reports the fact of the incorrect predictions in the classical limit as one of the most convincing arguments against orthodoxy.

Consequently, the classical particles become indiscernible with respect to their physical properties.

This consequence should alert us to the fact that we might have attached too much importance to the quantum version of the Indiscernibility Thesis. The proponents of the metaphysical conception of quantum objects as non-individuals want us to believe that the peculiar character of these objects is a unique feature of quantum theory that sets it apart from the classical worldview which assumes the individuation of bodies by their spatiotemporal trajectories. And yet, as we have seen above, the exact same indiscernibility claim that led to the development of the non-classical logic of non-individuals can be recovered in the case of Newtonian mechanics. Thus either the phenomenon of non-individuality is much more ubiquitous than we initially suspected, or there is something wrong with our assessment of the quantum version of indiscernibility. And there are good reasons to believe that the second option is closer to the truth. When we look more carefully at the classical indiscernibility argument given in the previous paragraph, we should notice that the objects to which we apply permutations and which as a result become totally indiscernible are not ordinary bodies equipped with properties as we experience them, but rather something akin to "bare substrata" with no qualitative characteristics attached to them (with an exception perhaps of the properties that classical particles of a particular type may share with each other, such as mass). There is nothing surprising in the claim that if we strip any objects of their state-dependent qualitative characteristics (such as position and momentum), then what will be left (if anything at all) may be qualitatively indistinguishable from each other. The only difference between them that remains is their bare numerical distinctness.

And it seems that something very much like that happens in the quantum case. Strictly speaking, the referents of labels in the tensor product formalism retain some rudimentary qualitative attributes in the form of state-independent properties (rest mass, charge, total spin). But because all particles of a given type possess the same state-independent properties, the indiscernibility claim follows immediately. However, it has to be stressed that this type of indiscernibility concerns not "ordinary" electrons (photons, neutrinos), but electrons stripped of all their properties except state-independent ones. Putting this in terms of the bundle theory of objects: while ordinary particles can be seen as bundles of all properties, both state-dependent and state-independent, co-instantiated at a given moment, the referents of labels in the orthodox approach are smaller bundles consisting only of the latter kind properties.[4] Thus, the orthodox conception of individuation can be accused of picking the wrong objects as the referents of labels. The unorthodox approach, on the other hand, offers us a method of making reference to full-blown quantum objects with all their applicable attributes, whether state-dependent or not.

Another powerful objection raised against orthodoxy considers a different sort of inter-theoretical correspondence relation of quantum theory (for details see Caulton 2014). It is well known that quantum field theory (QFT), which is a more universal theory of matter and its interactions, should turn into elementary quantum mechanics when the total number of particles is conserved. However, this creates an immediate difficulty, since the states of QFT-quanta, built out of the vacuum state by applying to it a number of creation operators, are pure states and not statistical mixtures. Thus the required correspondence between QFT and the quantum theory of many particles under the orthodox cannot be achieved, since according to the latter, particles of the same type always occupy mixed states. In the next section we will discuss in greater detail the formalism which constitutes the basis for QFT, known as the Fock space formalism, and its relations to the problem of how to individuate quantum particles.

## 7.2 The Fock Space Formalism

It is often claimed that the Fock space formalism, which marks the transition from non-relativistic quantum mechanics to quantum field theory, supports the "revisionary" ontology of quanta as indiscernible non-individuals that seemingly follows from the orthodox approach to quantum individuation. As Paul Teller writes:

[4] This statement is not to be construed as expressing my commitment to the bundle theory of quantum objects. I give some arguments against the bundle-based approach in Sect. 8.6.

> (…) Fock space realizes the idea of quanta understood as entities that can be (merely) aggregated, as opposed to particles, which can be labeled, counted and thought of as switched. (Teller 1995, p. 37)

> The Fock space formalism – or, as I will call it, the occupation number formalism – invites us to think about the subject matter of quantum mechanics as free of haecceities, free from things that support labels or admit of counterfactual switching. There are only amounts, or "heaps" of stuff, coming in discrete units, thought of in analogy to dollars in a bank account, with no this one or that one about ones with the same properties (ones in the same bank account), and no sense to a different case arising if they could somehow be switched. (Teller 1998, p. 128)

In this section we will have a closer look at the occupation number formalism, and we will argue that it actually gives more support to the heterodox approach to quantum individuation than to the orthodox one with its touted qualitative indiscernibility of quanta. This is not to say that Teller's observations are completely off the mark—in particular, he may be right about the issue of the counterfactual switching of quanta and the abandonment of haecceities. However, the impossibility of a substantial counterfactual switching of quanta is perfectly compatible with the approach to quantum individuation that allows for the absolute discernibility of quantum particles, in contrast to the "dollars in a bank account" picture.[5] We will return to this problem in the final chapter.

We can begin by introducing a seemingly innocuous alternative notation that can be used to describe possible states of a fixed number of particles. Let us first consider a single-particle Hilbert space $\mathscr{H}$, and let us select a complete set of orthogonal vectors $|\lambda_1\rangle, |\lambda_2\rangle, \ldots \in \mathscr{H}$, which can be thought of as the eigenvectors of some maximal, non-degenerate observable $A$ corresponding to distinct eigenvalues $k_1, k_2, \ldots$. Now, instead of using "single" kets $|\lambda_1\rangle, |\lambda_2\rangle, \ldots$, to represent possible eigenstates of $A$, we may suggest the following notation. Let the infinitely long sequence of zeros and one $|1, 0, 0, \ldots\rangle$ represent the situation in which the particle occupies state $|\lambda_1\rangle$, and let $|0, 1, 0, 0, \ldots\rangle$ represent state $|\lambda_2\rangle$, and so on. The underlying interpretational rule is that the $i$-th slot in a

[5] The analogy with money in a bank account was probably used for the first time in Teller (1983, p. 317).

sequence of numbers constituting a new ket corresponds to a particular state $|\lambda_i\rangle$, and the presence of the number 1 in this slot indicates that the particle occupies precisely this state. So far this is just a fancy way of writing the old single-particle states, as there is a one-to-one correspondence between the old vectors and the new ones, consisting of sequences of zeros with exactly one number 1 among them.

But now let us move to the case of multiple particles, where things get a bit more interesting. When $N = 2$, there is an obvious correspondence between the tensor product kets in $\mathscr{H} \otimes \mathscr{H}$ and the occupation number kets (the cases of $N > 2$ are easy generalizations of this case):

$$|\lambda_1\rangle \otimes |\lambda_1\rangle \rightarrow |2,0,0,\ldots\rangle,$$
$$|\lambda_2\rangle \otimes |\lambda_2\rangle \rightarrow |0,2,0,\ldots\rangle,$$
$$|\lambda_1\rangle \otimes |\lambda_2\rangle \rightarrow |1,1,0,\ldots\rangle,$$

and so on. However, this time the correspondence is not one-to-one. Clearly, permuted kets are represented by the same occupation number sequences and so are superpositions of permuted states:

$$|\lambda_2\rangle \otimes |\lambda_1\rangle \rightarrow |1,1,0,\ldots\rangle,$$
$$|\lambda_1\rangle \otimes |\lambda_2\rangle + |\lambda_2\rangle \otimes |\lambda_1\rangle \rightarrow |1,1,0,\ldots\rangle.$$

Thus the new occupation number notation does not seem to be particularly useful in the general case of multiple particle systems, as it identifies states that may not be identical after all. However, the situation changes when we consider particles of the same type. Suppose we limit ourselves to the symmetric sector ($\mathscr{H} \otimes \mathscr{H}$) of the tensor product $\mathscr{H} \otimes \mathscr{H}$. In that case the correspondence between the old and new notations is guaranteed to be one-to-one (up to the choice of the multiplicative constant):

$$|\lambda_1\rangle \otimes |\lambda_1\rangle \leftrightarrow |2,0,0,\ldots\rangle,$$
$$|\lambda_2\rangle \otimes |\lambda_2\rangle \leftrightarrow |0,2,0,\ldots\rangle,$$
$$|\lambda_1\rangle \otimes |\lambda_2\rangle + |\lambda_2\rangle \otimes |\lambda_1\rangle \leftrightarrow |1,1,0,\ldots\rangle, \text{and so on.}$$

Same applies to the antisymmetric sector ($\mathcal{H} \otimes \mathcal{H}$) housing fermionic states:

$$|\lambda_1\rangle \otimes |\lambda_2\rangle - |\lambda_2\rangle \otimes |\lambda_1\rangle \leftrightarrow |1,1,0,\ldots\rangle$$
$$|\lambda_1\rangle \otimes |\lambda_3\rangle - |\lambda_3\rangle \otimes |\lambda_1\rangle \leftrightarrow |1,0,1,0,\ldots\rangle, \text{and so on.}$$

Observe that the fermionic occupation number kets can never contain in their slots any numbers greater than 1, since it is impossible to have an antisymmetric state in which two or more particles will be assigned the same pure state. Now, one unquestionable advantage of the new notation is that it enables us to represent states with various numbers of particles in the same framework. For instance, we can write down the following bosonic states: $|2, 0, 3, 5, 0, \ldots\rangle$ (two particles in state $|\lambda_1\rangle$, three in $|\lambda_3\rangle$ and five in $|\lambda_4\rangle$) and $|1, 1, 0, 0, 4, 0, 0\ldots\rangle$ (one particle in state $|\lambda_1\rangle$, one in $|\lambda_2\rangle$ and four in $|\lambda_5\rangle$), which do not belong to the same tensor product of Hilbert spaces. We can even consider their superpositions, indicating states with no well-defined number of particles. A mathematically rigorous way to do all that is by introducing the concept of a Fock space $\mathcal{F}$ built out of a single-particle Hilbert space $\mathcal{H}$:

$$\mathcal{F} = \oplus_{n=0}^{\infty} \mathcal{H}^n,$$

where $\mathcal{H}^0 = \mathbb{C}$ (a zero-dimensional Hilbert space), $\mathcal{H}^1 = \mathcal{H}$, and $\mathcal{H}^n$ for $n > 1$ is the $n$-fold tensor product of spaces $\mathcal{H}$. Symbol $\oplus$ represents the direct sum of vector spaces (see Appendix for a formal definition). Of particular interest to us are symmetric and antisymmetric Fock spaces:

$$\mathcal{F}_+ = \oplus_{n=0}^{\infty} \mathcal{S}(\mathcal{H}^n)$$
$$\mathcal{F}_- = \oplus_{n=0}^{\infty} \mathcal{A}(\mathcal{H}^n),$$

where, as always, symbols $\mathcal{S}(\mathcal{H}^n)$ and $\mathcal{A}(\mathcal{H}^n)$ represent, respectively, symmetric and antisymmetric subspaces of appropriate tensor products of spaces. From the above definitions it follows immediately that $\mathcal{F}_+$ is spanned by kets of the form $|n(1), n(2), \ldots, n(k), \ldots\rangle$, where $n(i)$ are any natural numbers, whereas the basis for $\mathcal{F}_-$ can be constructed out of sequences $|p(1), p(2), \ldots, p(k), \ldots\rangle$, where $p(i) = 0$ or 1.

To complete this whirlwind exposition of the Fock space formalism, we should note that the basis vectors for $\mathcal{F}_+$ and $\mathcal{F}_-$ can be alternatively written using only the "vacuum" state with zero particles $\mathbf{0} = |0, 0, 0, \ldots\rangle$ and certain operators. In the case of the bosonic Fock space $\mathcal{F}_+$, these operators are defined as follows ($\hat{a}_i^\dagger$ is called a *creation* or *raising* operator, while $\hat{a}_i$ is an *annihilation* or *lowering* operator):

$$\hat{a}_i^\dagger \mid n(1),\ldots,n(i)\ldots\rangle = \sqrt{n(i)+1} \mid n(1),\ldots,n(i)+1\ldots\rangle,$$

$$\hat{a}_i \mid n(1),\ldots,n(i)\ldots\rangle = \sqrt{n(i)} \mid n(1),\ldots,n(i)-1\ldots\rangle, \text{when } n(i) > 0,$$

$$\hat{a}_i \mid n(1),\ldots,n(i)\ldots\rangle = 0, \text{when } n(i) = 0.$$

As can be quickly verified, the commutation relations of the above-defined operators are as follows:

$$\left[\hat{a}_i^\dagger,\hat{a}_j^\dagger\right] =_{\mathrm{df}} \hat{a}_i^\dagger\hat{a}_j^\dagger - \hat{a}_j^\dagger\hat{a}_i^\dagger = 0$$

$$\left[\hat{a}_i,\hat{a}_j\right] =_{\mathrm{df}} \hat{a}_i\hat{a}_j - \hat{a}_j\hat{a}_i = 0,$$

thus bosonic creation and annihilation operators commute. On the other hand, fermionic counterparts of the above-defined operators obey different commutation relations. First off, this is how they are usually defined:

$$\hat{c}_i^\dagger \mid p(1),\ldots,p(i)\ldots\rangle = \pm \mid p(1),\ldots,p(i)+1\ldots\rangle, \text{when } p(i) = 0,$$

$$\hat{c}_i^\dagger \mid p(1),\ldots,p(i)\ldots\rangle = 0, \text{when } p(i) = 1,$$

$$\hat{c}_i \mid p(1),\ldots,p(i)\ldots\rangle = \pm \mid p(1),\ldots,p(i)-1\ldots\rangle, \text{when } p(i) = 1,$$

$$\hat{c}_i \mid p(1),\ldots,p(i)\ldots\rangle = 0, \text{when } p(i) = 0.$$

The choice of sign + or – depends on the specific form of the vector acted upon by $\hat{c}_i^\dagger$ or $\hat{c}_i$. The sign convention has to be chosen properly, so that the following anticommutation relations will hold:

$$\left\{\hat{c}_i^\dagger, \hat{c}_j^\dagger\right\} =_{\text{df}} \hat{c}_i^\dagger \hat{c}_j^\dagger + \hat{c}_j^\dagger \hat{c}_i^\dagger = 0.$$

$$\left\{\hat{c}_i, \hat{c}_j\right\} =_{\text{df}} \hat{c}_i \hat{c}_j + \hat{c}_j \hat{c}_i = 0.$$

Using creation operators $\hat{a}_i^\dagger$ and $\hat{c}_i^\dagger$, we can construct any basis vector in $\mathscr{F}_+$ and $\mathscr{F}_-$ by acting with these operators an appropriate number of times on the vacuum state. Moreover, we can introduce new and rather useful operators, known as *number operators*, as follows:

$$\hat{N}_i = \hat{a}_i^\dagger \hat{a}_i \text{ for bosons}$$

$$\hat{N}_i = \hat{c}_i^\dagger \hat{c}_i \text{ for fermions.}$$

As can be easily verified, the result of an action of $\hat{N}_i$ on a given occupation number ket is the following:

$$\hat{N}_i \mid n(1),\ldots,n(i)\ldots\rangle = n(i) \mid n(1),\ldots,n(i)\ldots\rangle,$$

which means that $\hat{N}_i$ "measures" the number of particles occupying the *i*-th slot (possessing property $k_i$).

The main formal feature of the Fock space formalism which virtually all philosophical commentators jump on is the conspicuous absence of labels attached to individual particles. An occupation number ket contains the information of how many particles occupy various states, but does not refer to these particles by their individual labels. Supposedly, this fact makes it impossible to even formally introduce the concept of a permutation of particles, since permutations conceived as mathematical operations act on labels. But in fact the Fock space formalism does enable us to find a mathematical representation for the permutations of particles, if we build vectors in $\mathscr{F}_+$ and $\mathscr{F}_-$ by applying sequentially a number of creation operators to the vacuum state. In that case permutations of particles are represented by permutations of creation operators occurring in an appropriate sequence (see Sakurai & Napolitano 2011, p. 462). Given the (anti)commutation relations for bosonic and fermionic creation operators presented above, the result of such permutations is identity in the case of bosons, while for fermions the odd permutations result in the change of sign, as expected.

Still, for some philosophers the absence of labels in the occupation number formalism indicates a transition from the ontology of particles equipped with primitive thisnesses (or *haecceities*) to the ontology of *quanta*, which act not as individual objects but as units of something (as the analogy with dollars in a bank account suggest).[6] Teller (2001, p. 383) goes so far as to use yet another analogy with a continuous substance that comes in variable amounts, even though he admits that in contrast to this case, the quanta stuff comes only in discrete units, but he insists, somewhat mysteriously, that these units are nevertheless measures of amounts, not numbers. I believe that even the analogy with money in a bank account is not entirely accurate, since amounts of money can be measured in different units (cents, pounds, euros) which leads to using various non-integer numbers representing the same amount, while in the case of the "amounts" of quanta only specific natural numbers are employed in that role. However, I will not mount an extensive critique of Teller's suggested ontology of quanta, moving instead to my main point of this section.

[6] We should mention here that it is debatable whether the occupation number formalism is indeed entirely free of the commitment to labels. Given that a symmetric/antisymmetric Fock space is defined as the direct sum of "ordinary" symmetric/antisymmetric subspaces of tensor products, it may be surmised that the occupation number formalism only "masks" the commitments of the more fundamental tensor-product formalism. French and Krause (2006, p. 390) argue, for instance, that in order to maintain the label-free interpretation of the occupation number formalism, we would have to be able to build an entire Fock space "from scratch", without relying on the interpretational rule connecting the occupation number sequences with symmetric or antisymmetric vectors in an appropriate tensor product of spaces. But, as they observe, it is mathematically impossible to impose the structure of a vector space on sequences of natural numbers, since the product of a vector by a scalar is not a sequence of natural numbers. In a similar vein van Fraassen (1991, p. 443) warns not to interpret occupation numbers too literally, since ultimately the interpretation of the Fock space kets relies on symmetric/antisymmetric vectors in tensor-product spaces. Thus, according to van Frasseen, $|1, 1, 0\ldots\rangle$ does not denote the state in which two fermions occupy distinct pure states, since it corresponds to the antisymmetric state of two fermions which attributes identical reduced mixed states to both fermions. Clearly, van Fraassen follows here the orthodox interpretation of quantum individuation.

What we have so far overlooked regarding the occupation number formalism is that its most natural reading supports the view that quantum particles are discernible by properties in many situations (see Bigaj 2018, pp. 152–153). Take any fermionic ket, for instance, |1, 1, 0, 1, 0…⟩. The standard reading of this ket is that it describes three fermions which possess distinct properties $A = k_1$, $A = k_2$ and $A = k_4$, and therefore are clearly discernible entities. In the case of bosons we may have numerically distinct particles occupying the same physical state, as in |2, 0, 0,…⟩; nevertheless qualitative discernibility of bosons is still very much possible. This, if anything, points towards the unorthodox conception of qualitative individuation. The lack of labels in the Fock formalism is no obstacle to the qualitative discernibility of particles. Actually, given the role of labels in the assumption of Factorism and in the proofs of the Indiscernibility Thesis presented in Chap. 2, we may come to the conclusion that the elimination of labels from the formalism only helps bring to the surface the fact that quantum particles often differ with respect to their properties.[7]

It has to be admitted that Teller acknowledges, albeit in a footnote, that "there is a sense in which fermions are individuated by their properties" (Teller 1998, p. 140, ft. 25). Yet he somehow ignores the fact that such individuation stands in contrast to his preferred ontological view of "quantum stuff". Clearly, it doesn't make much sense to try to individuate dollars in a bank account by their properties, since they do not possess any distinguishing features (these may be possessed by dollar bills, but not by dollars written in a bank ledger). My guess is that Teller's way of picturing the Fock space formalism is in terms of "boxes"—represented by distinct slots in an occupation number sequence—and qualitatively identical, "bare" entities that can be "sprinkled" over these boxes. Thus the fermionic state |1, 1, 0…⟩ does not represent two physically different electrons, but rather two physically identical electron-like entities

[7] Teller mentions in various places the possibility of interpreting labels used in the standard tensor-product formalism as mere "cogs" in the mathematical machinery with no direct physical meaning (Teller 1995, p. 27; 1998, p. 131). This is precisely an interpretation adopted in the heterodox approach to individuation.

occupying distinct boxes (or, better yet, the amount of electron-stuff whose measure equals two units, equally spread over two distinct boxes). And, in a sense, the occupation number formalism is conducive to that sort of mental imagery. But the main problem with such a picture is that it portrays state-dependent physical properties, such as spin components, energy, momentum and so on, as if they were "external" with respect to the objects possessing them (we have already encountered this problem in Sect. 7.1, where we have noted that the orthodox approach picks the "impoverished" bundles of properties as the bearers of quantum properties). Only state-independent properties truly belong to a given object, and for that reason we must treat all electrons as fundamentally indistinguishable. But I see no reason why the mass of an electron should be ontologically unlike its spin component or momentum. An electron can be distinguished from a proton by its mass in the same way as it can be distinguished from another electron by its momentum. The only genuine difference between these two cases is that the first distinction is permanent (as electrons and protons can't change their rest masses), while the other is temporary only.

## 7.3 The Ambiguity of Qualitative Individuation: Fermions and Bosons

Let us now switch gears and begin a critical analysis of the heterodox approach. In this and the following sections, we will discuss in detail what can be considered the greatest challenge to the unorthodox conception of quantum individuation by properties. The problem, in a nutshell, is that the criterion of individuation presented in Chap. 5 (formula 5.10) admits the possibility of the existence of alternative and even incompatible individuations with respect to a given multiparticle system. That is, criterion (5.10) may be satisfied in some states of a two-particle system by various pairs of projectors: $E_{a_1}$ and $E_{b_1}$, $E_{a_2}$ and $E_{b_2}$, and so on, where projectors $E_{a_i}$, $E_{b_i}$ are distinct from $E_{a_j}, E_{b_j}$ for $i \neq j$, and in some cases are even incompatible. This feature of qualitative individuation is variously referred to in the literature as "conventionality" (Caulton 2014, p. 47) or "arbitrariness" (Ghirardi et al. 2002, p. 84), but I prefer to use the slightly stronger term "ambiguity". I will distinguish two types of ambiguity that may affect qualitative individuation. One type occurs when we individuate particles with the help of many-dimensional projectors (individuation by less than maximally specific properties). This ambiguity affects both fermionic and bosonic states. In contrast to that, the second type of ambiguity afflicts only fermions and is applicable to the case when they are discerned by one-dimensional projectors. We will talk about this case in the next section.

Let us start by recalling the general form in which every fermionic state of two particles can be written (the Schmidt-Slater decomposition—see formula 6.9):

$$\sum_{i=1}^{m} z_i \left( |\varphi_{2i-1}\rangle |\varphi_{2i}\rangle - |\varphi_{2i}\rangle |\varphi_{2i-1}\rangle \right), \tag{7.1}$$

where all $|\varphi_i\rangle$ are mutually orthogonal. This essentially means that every bipartite fermionic state can be written as the sum of $m$ blocks, each of which consists of an antisymmetric combination of two orthogonal vectors, and in addition no vector occurs in more than one block. As we have observed in Sect. 6.2, if we select the following subspaces:

$$\mathcal{V}_E = \text{Span}\{|\varphi_{2i-1}\rangle\}_{i=1,\ldots,m}$$

$$\mathcal{V}_F = \text{Span}\{|\varphi_{2i}\rangle\}_{i=1,\ldots,m}$$

then the corresponding orthogonal projectors $E$ and $F$ qualitatively discern particles in the sense that state (7.1) lies in the subspace projected onto by $E \otimes F + F \otimes E$. However, we have also noted (see ft. 9 in Chap. 6) that the choice of the subspaces with the above-mentioned property is generally not unique. Basically, we can choose any two subspaces of the space spanned by all $2m$ vectors $|\varphi_i\rangle$ in such a way that for each antisymmetric block in (7.1) one vector from the block is picked to span one subspace and the other vector to span the other subspace (see Fig. 7.1).

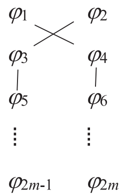


**Fig. 7.1** One particular partition of the set $\{|\varphi_i\rangle\}$. There are exactly $2^{m-1}$ such partitions

It can be quickly verified that there are exactly $2^{m-1}$ distinct pairs of single-particle projectors $E_i$ and $F_i$ such that state (7.1) is an eigenstate of $E_i \otimes F_i + F_i \otimes E_i$.[8] Thus, there are $2^{m-1}$ distinct methods of selecting and individuating the components of the system occupying state (7.1).

Notice that the same problem affects bosonic counterparts of states of the form (7.1), where all minus signs are replaced by pluses. A similar kind of ambiguity can be found in more general, discernibility-enabling bosonic states of the form given in criterion (6.10), which is as follows:

$$\sum_{i=1}^{k} \sum_{j=k+1}^{n} c_{ij} \left( |\varphi_i\rangle |\varphi_j\rangle + |\varphi_j\rangle |\varphi_i\rangle \right) \tag{7.2}$$

as long as some specific coefficients $c_{ij}$ vanish. For instance, if the considered bosonic state is of the form:

$$\begin{aligned} &c_{13} \left( |\varphi_1\rangle |\varphi_3\rangle + |\varphi_3\rangle |\varphi_1\rangle \right) + c_{14} \left( |\varphi_1\rangle |\varphi_4\rangle + |\varphi_4\rangle |\varphi_1\rangle \right) \\ &\quad + c_{25} \left( |\varphi_2\rangle |\varphi_5\rangle + |\varphi_5\rangle |\varphi_2\rangle \right), \end{aligned} \tag{7.3}$$

then there are two ways of differentiating between the components of the system; one given by the pair of subspaces Span$\{|\varphi_1\rangle, |\varphi_5\rangle\}$ and Span$\{|\varphi_2\rangle,$

[8] There are exactly $2^m$ ways of selecting $m$ vectors by taking exactly one vector out of each pair in the $i$-th antisymmetric block in (7.1), but selecting a particular set of vectors in that way and selecting its complement should count as the same individuation; hence the number of distinct individuations equals $2^m/2 = 2^{m-1}$. From this it follows that in the case when $m = 1$ the number of distinct individuations is 1, and therefore there is no ambiguity (at least not of the type we are currently considering).


**Fig. 7.2** Two ways of grouping vectors from the combination (7.3). Vectors in one column are taken from the same symmetric block

$|\varphi_3\rangle$, $|\varphi_4\rangle\}$, and the other by Span$\{|\varphi_1\rangle$, $|\varphi_2\rangle\}$, Span$\{|\varphi_3\rangle$, $|\varphi_4\rangle$, $|\varphi_5\rangle\}$. (Observe that vectors which span together a given subspace can never occur together in the same block.) See Fig. 7.2 for an explanation. On the other hand, if all coefficients $c_{ij}$ figuring in (7.2) are non-zero, there is only one individuation available (as identified in the proof of theorem 6.10 given in Sect. 6.2).

How bad is the aforementioned ambiguity? At first glance it looks like it doesn't have to be fatal to the unorthodox conception of individuation. In everyday life we often individuate macroscopic objects using various alternative properties—we may distinguish them using their colors, or their shapes, or sizes, and so on. However, the crucial thing is that all these qualitative individuations lead to the selection of the same individuals, thanks to the fact that one and the same entity can simultaneously possess various types of properties (a given color, shape and size). But the question is: does the quantum mechanical formalism allow us to attribute properties represented by different projectors to the same individual objects? Generally, the answer is "no", since the projectors corresponding to different properties may be incompatible, and therefore may not have any eigenvectors in common. But in the case currently under consideration, the alternative projectors *are* compatible with each other. A well-known criterion of compatibility for subspaces of a Hilbert space, and consequently for the projectors corresponding to these subspaces, is that compatible subspaces $\mathscr{V}_1$ and $\mathscr{V}_2$ should be presentable as $\mathscr{V}_1 = \mathscr{V}_a \oplus \mathscr{V}_c$ and $\mathscr{V}_1 = \mathscr{V}_b \oplus \mathscr{V}_c$, where $\mathscr{V}_a$, $\mathscr{V}_b$ and $\mathscr{V}_c$ are mutually orthogonal (and possibly zero) subspaces (cf. Hughes 1989, p. 103). Since in the above-considered cases all the subspaces involved are spanned by combinations of mutually orthogonal vectors, they clearly satisfy the criterion of compatibility.

Thus it is not formally prohibited that the alternative individuations by various distinct pairs of projectors could ultimately pick out unique objects. But does this solve the problem of ambiguity? Consider as an example the specific bosonic state written in (7.3). In order for the two above-given alternative individuations to pick out the same components of the system, we have to join together the properties used in these individuations, so that each component could possess one property from each alternative individuation. In our case this means that the individuating properties will be represented by either subspaces:

$$\mathrm{Span}\{|\varphi_1\rangle,|\varphi_2\rangle\} \cap \mathrm{Span}\{|\varphi_1\rangle,|\varphi_5\rangle\} \text{ and}$$
$$\mathrm{Span}\{|\varphi_3\rangle,|\varphi_4\rangle,|\varphi_5\rangle\} \cap \mathrm{Span}\{|\varphi_2\rangle,|\varphi_3\rangle,|\varphi_4\rangle\},$$

or:

$$\mathrm{Span}\{|\varphi_1\rangle,|\varphi_2\rangle\} \cap \mathrm{Span}\{|\varphi_2\rangle,|\varphi_3\rangle,|\varphi_4\rangle\} \text{ and}$$
$$\mathrm{Span}\{|\varphi_3\rangle,|\varphi_4\rangle,|\varphi_5\rangle\} \cap \mathrm{Span}\{|\varphi_1\rangle,|\varphi_5\rangle\}.$$

Consequently, we have two options: either one particle possesses the property associated with subspace Span$\{|\varphi_1\rangle\}$, while the other particle possesses the property represented by Span$\{|\varphi_3\rangle, |\varphi_4\rangle\}$, or the individuating properties are represented by Span$\{|\varphi_2\rangle\}$ and Span$\{|\varphi_5\rangle\}$, respectively. Clearly these options are exclusive, since one particle cannot simultaneously possess properties represented by vectors $|\varphi_1\rangle$ and $|\varphi_2\rangle$ (or $|\varphi_5\rangle$). Hence, in spite of our effort, the ambiguity of individuation is still present. This time the ambiguity comes from the fact that there are many ways to connect properties figuring in alternative individuations in the sense of attributing them to the same underlying object.

The problem of the ambiguity of attribution, as it may be called to distinguish it from the ambiguity of individuation, gets even worse in the case of fermionic states of the form (7.1). The way of grouping the properties taken from the $2^{m-1}$ alternative individuations in such a way that each group of properties could be attributed to one and the same object (i.e. that all the subspaces representing these properties have a non-zero subspace in common) is by selecting a particular vector $|\varphi_i\rangle$ and choosing all the subspaces containing this vector. It is not difficult to see that in this case all the selected subspaces from the alternative individuating pairs will only have the one-dimensional subspace spanned by $|\varphi_i\rangle$ in common, whereas for the remaining subspaces the common subspace will be spanned by the vector (let's symbolize it as $|\varphi_{i\pm1}\rangle$) which figures in the same antisymmetric block as $|\varphi_i\rangle$ in the Schmidt-Slater decomposition (7.1). Therefore, if we wanted to insist that all the alternative individuations by properties ultimately pick out the same objects, we would have to conclude that the components of the system are individuated by some pure states $|\varphi_i\rangle$ and $|\varphi_{i\pm1}\rangle$. But clearly the choice of the state $|\varphi_i\rangle$ was completely arbitrary—we might have chosen any other pure state $|\varphi_j\rangle$ and group together all the subspaces containing this vector. Thus, there are $m$ alternative individuations of objects composing the entire system.[9]

But an even more serious problem is that the conclusion that the particles composing the system in state (7.1) actually possess pure states $|\varphi_i\rangle$ and $|\varphi_{i\pm1}\rangle$ has no backing in the quantum-mechanical formalism whatsoever. As a matter of fact, the formalism unambiguously speaks against such an interpretation. Clearly, if the composite system occupies state (7.1), then for any pure state $|\varphi_j\rangle$ there is a non-vanishing probability that an appropriate measurement will find one particle precisely in this state. This is incompatible with the assumption that before measurements particles occupy specific pure states $|\varphi_i\rangle$ and $|\varphi_{i\pm1}\rangle$.[10] Consequently, we cannot claim that the alternative individuations of the components of the system in state (7.1) pick out the same particles. The ambiguity of individuation runs deeper than the level of properties—it affects our ontology of property-bearing objects as well. To each potential way of qualitatively individuating objects, there corresponds a distinct set of possible objects composing the system.

[9] The reason why the number of alternative individuations is $m$ and not 2 $m$ is that selecting a vector and selecting its "partner" occupying the same antisymmetric block in (7.1) leads to the same individuation.

[10] One rather desperate attempt to overcome this objection could be to insist that the actual state of the system is an appropriate mixture of pure states $|\varphi_{2i-1}\rangle|\varphi_{2i}\rangle$, interpreted along the lines of the ignorance interpretation. This solution could reproduce the correct probabilities of revealing particles in various states $|\varphi_i\rangle$. However, mixed states are never fully empirically equivalent to pure states; hence, there will always be some formally admissible measurements that could tell us that the system actually occupies state (7.1) and not the corresponding mixture.

In order to explain this further, let us use a specific physical example instead of the universal fermionic state as written in (7.1). We will use the toy model already discussed in Sect. 5.4, that is, two fermions that may be characterized by two spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ (in some specified direction) and two position states $|L\rangle$ and $|R\rangle$. Thus the total state space of an individual particle composing such a system is spanned by the four orthogonal vectors $|\uparrow\rangle|L\rangle$, $|\uparrow\rangle|R\rangle$, $|\downarrow\rangle|L\rangle$ and $|\downarrow\rangle|R\rangle$. Let us now consider the (genuine) singlet-spin state of the two fermions (see formula 5.17):

$$\frac{1}{2}\begin{pmatrix} |\uparrow\rangle_1 |L\rangle_1 \otimes |\downarrow\rangle_2 |R\rangle_2 - |\downarrow\rangle_1 |L\rangle_1 \otimes |\uparrow\rangle_2 |R\rangle_2 \\ - |\downarrow\rangle_1 |R\rangle_1 \otimes |\uparrow\rangle_2 |L\rangle_2 + |\uparrow\rangle_1 |R\rangle_1 \otimes |\downarrow\rangle_2 |L\rangle_2 \end{pmatrix}, \tag{7.4}$$

which can be rewritten in the form corresponding to the general structure of formula (7.1) as follows:

$$\begin{aligned} &\frac{1}{2}\left(|\uparrow\rangle_1 |L\rangle_1 \otimes |\downarrow\rangle_2 |R\rangle_2 - |\downarrow\rangle_1 |R\rangle_1 \otimes |\uparrow\rangle_2 |L\rangle_2\right) \\ &+\frac{1}{2}\left(|\uparrow\rangle_1 |R\rangle_1 \otimes |\downarrow\rangle_2 |L\rangle_2 - |\downarrow\rangle_1 |L\rangle_1 \otimes |\uparrow\rangle_2 |R\rangle_2\right). \end{aligned} \tag{7.5}$$

On the basis of our previous analysis, we can immediately distinguish two ways of individuating the components of the system (since $m = 2$ in this case, the number $2^{m-1}$ of distinct individuations equals 2 as well). One individuation can be done using subspaces Span$\{|\uparrow\rangle|L\rangle, |\uparrow\rangle|R\rangle\}$ and Span$\{|\downarrow\rangle|R\rangle, |\downarrow\rangle|L\rangle\}$, while the alternative individuation is based on subspaces Span$\{|\uparrow\rangle|L\rangle, |\downarrow\rangle|L\rangle\}$ and Span$\{|\uparrow\rangle|R\rangle, |\downarrow\rangle|R\rangle\}$. Now, these formal individuations have a clear physical meaning. The first individuation corresponds to selecting particles with well-defined spin whose position is characterized only as some unspecified superposition of two distinct locations *L* and *R* (a quantum-mechanical disjunction of being in *L* and being in *R*). If we followed this method of individuation, we would have to admit that the system consists of a particle having spin "up" in a given direction whose location is undetermined between *L* and *R*, and a particle with determined spin "down" and similarly located "partially" in *L* and "partially" in *R*, so to speak. An alternative individuation leaves us with one particle located in *L* but with no well-defined spin, and one particle located in *R* whose spin is not well-defined either. It should be clear now that in both approaches we talk about different possible components of the entire system. These may be either localized particles with no spin determined, or "un-localized" particles possessing definite spins.

Before we discuss possible ways of dealing with the above-discussed ambiguity of individuation, let us turn to the second type of ambiguity that affects only fermionic states.

## 7.4 Ambiguity for Fermions

The non-uniqueness of the type discussed so far applies exclusively to the cases in which individuation is done by projectors of dimensionality greater than one, hence is not maximally specific. If the Schmidt-Slater decomposition of a fermionic state consists of only one antisymmetric block, then, as we have explained in ft. 8, there is no ambiguity resulting from the alternative selections of vectors $|\varphi_i\rangle$ spanning multidimensional subspaces used to individuate particles (the same applies to the bosonic versions of such single-block states). However, an even more acute type of ambiguity is still present in the fermionic case. As it turns out, there are an infinite number of alternative and *mutually incompatible* ways of performing individuation in the case of fermionic states. In order to identify the source of this radical form of ambiguity, let us consider the following, GMW-non-entangled state of two fermions (see def. 6.17):

$$\frac{1}{\sqrt{2}}\left(|\varphi_1\rangle|\varphi_2\rangle - |\varphi_2\rangle|\varphi_1\rangle\right). \tag{7.6}$$

As we have stated numerous times, under the unorthodox interpretation of individuation, particles occupying this state can be said to possess the individuating properties corresponding to orthogonal projectors $E_{\varphi_1} = |\varphi_1\rangle\langle\varphi_1|$ and $E_{\varphi_2} = |\varphi_2\rangle\langle\varphi_2|$. However, it can be easily verified that vector (7.6) can be equivalently written using any pair of orthogonal vectors from the two-dimensional subspace spanned by $|\varphi_1\rangle$ and $|\varphi_2\rangle$. To see that, let us write down the general form of any two orthogonal vectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$ in subspace Span$\{|\varphi_1\rangle, |\varphi_2\rangle\}$:

$$|\lambda_1\rangle = a|\varphi_1\rangle + b|\varphi_2\rangle$$
$$|\lambda_2\rangle = -b^*|\varphi_1\rangle + a^*|\varphi_2\rangle,$$

where $|a|^2 + |b|^2 = 1$. It is elementary to calculate that the following vector:

$$\frac{1}{\sqrt{2}}\left(|\lambda_1\rangle|\lambda_2\rangle - |\lambda_2\rangle|\lambda_1\rangle\right) \tag{7.7}$$

is identical to (7.6). In conclusion, we have to admit that the components of the system occupying state (7.6) can be alternatively individuated using projectors $E_{\lambda_1} = |\lambda_1\rangle\langle\lambda_1|$ and $E_{\lambda_2} = |\lambda_2\rangle\langle\lambda_2|$.[11]

Since complex numbers $a$ and $b$ have been selected completely arbitrarily, there are an infinite number of alternative individuating projectors $E_{\lambda_1}$ and $E_{\lambda_2}$. Even more worrisome is the fact that the projectors corresponding to different choices of orthogonal vectors $|\lambda_1\rangle$, $|\lambda_2\rangle$ are mutually incompatible. This means that standard quantum mechanics does not permit to attribute the properties corresponding to these projectors to a single particle. This can be clearly seen when we use spin-half as an example. If vectors $|\varphi_1\rangle$ and $|\varphi_2\rangle$ used in (7.6) are identified as representing the values of spin in a certain direction, then particular linear

[11] A more general analysis of the fermionic type of ambiguity discussed here can be found in Caulton (2016). Caulton's analysis applies to the case of $N$ fermions occupying a GMW-non-entangled state and takes into account all possible individuations of the subsystems not limited to single particles. He proves a theorem that can be roughly summarized as follows. Let the state of the $N$-particle system be the result of the antisymmetrization of $N$ orthogonal vectors $|\varphi_1\rangle, \ldots, |\varphi_N\rangle$, and let $\mathcal{V}_N = \text{Span}\{|\varphi_1\rangle, \ldots, |\varphi_N\rangle\}$. Then there is a one-to-one correspondence between all subspaces of $\mathcal{V}_N$ and the subsystems of the system that can be individuated using these subspaces. Consequently, to each subspace $\mathcal{V}_k \subseteq \mathcal{V}_N$ of dimensionality $k$ there corresponds a possible separation of the $N$-particle system into a $k$-element subsystem and the complementing ($N$–$k$)-element subsystem, such that the $k$-element system consists of all particles possessing the property represented by $\mathcal{V}_k$. It is easy to observe that the ambiguity discussed above constitutes a special case of Caulton's theorem when $N = 2$, since Span$\{|\lambda_1\rangle\}$ and Span$\{|\lambda_2\rangle\}$are subspaces of Span$\{|\varphi_1\rangle, |\varphi_2\rangle\}$.

combinations $|\lambda_1\rangle$, $|\lambda_2\rangle$ will be eigenvectors of spin in a different spatial direction. But it is well known that according to standard quantum mechanics, no particle can possess determined values of spin in different directions (i.e. if spin is determined in a particular direction, all the other spatial components of spin are undetermined). And yet, the heterodox approach seems to suggest that when the system of two spin-half fermions is in state (7.6), we are justified in saying that there is one particle with spin "up" in the $z$ direction and one particle with spin "down" in the $z$ direction, but also that there is one particle with spin "up" in the $x$ direction and one particle with spin "down" in the $x$ direction, and analogously for any spatial direction $n$. In order not to infringe upon the principles of standard quantum mechanics, we would have to admit that particles possessing various spin components are actually numerically distinct, but this leads to an even more troublesome conclusion that the system that we have initially described as consisting of two particles actually includes an infinite number of components.[12]

The problem described above cannot be easily brushed aside as just another example of quantum-mechanical "weirdness". The consequences of the dramatic freedom of choice with respect to the individuating projectors are far-reaching and troubling, as they come dangerously close to internal inconsistency, or even logical contradiction.[13] Recall that in Sect. 5.3 we have introduced formulas (5.12) that enable us to calculate the

[12] Caulton (2016) observes that in addition to this inflation of the number of existing components of a given system, it can be proved that their structure violates some basic rules of mereology (the part-whole theory).

[13] Thus I do not share the optimism of the authors of Ghirardi et al. (2002) when they write that "there are at least two reasons for which one can ignore this, at first sight, puzzling situation, one of formal and physical nature, the second having more to do with the laboratory practice" (p. 85). The formal reason to dismiss the ambiguity problem they offer is that the quantum-mechanical formalism allows for the situations in which incompatible observables have some (but not all) eigenvectors in common. And indeed, in the case considered above projectors $E_{\varphi_1} \otimes E_{\varphi_2} + E_{\varphi_2} \otimes E_{\varphi_1}$ and $E_{\lambda_1} \otimes E_{\lambda_2} + E_{\lambda_2} \otimes E_{\lambda_1}$, in spite of being incompatible, possess common eigenvectors (vector 7.6 being an example). But the problem runs deeper than that, since according to the unorthodox approach, we are allowed to attribute to the individuated components of the system properties associated with selected single-particle projectors $E_{\lambda_i}$, and these projectors do not share any eigenvectors. Regarding the second reason mentioned in Ghirardi et al. (2002), we'll discuss it in ft. 14.

expectation values of any single-particle observable $A$ associated with the components individuated by specific projectors $E_a$ and $E_b$:

$$\begin{aligned} \langle v | A_a | v \rangle &= \langle v | AE_a \otimes E_b | v \rangle + \langle v | E_b \otimes AE_a | v \rangle \\ \langle v | A_b | v \rangle &= \langle v | E_a \otimes AE_b | v \rangle + \langle v | AE_b \otimes E_a | v \rangle \end{aligned} \tag{7.8}$$

It is straightforward to notice that different choices of individuating projectors $E_a$ and $E_b$ will lead to different expectation values for the same observables according to (7.8). This effectively thwarts any attempt to identify particles individuated by alternative projectors, since in that case one and the same particle would admit different numerical values of the expectation values for the same observables. But this is a conceptual impossibility. Thus we cannot claim that the particle individuated by, let's say, spin "up" in the $z$-direction is the same object as the particle individuated by spin "down" in the $x$ direction, since in that case the expectation values for any other components of spin would receive inconsistent values. But the alternative seems to be no less disconcerting: the existence of an infinity of distinct particles within the two-particle system is as close to logical contradiction as it gets.

One conceivable way to tackle the problem of the ambiguity of qualitative individuation is to try to eliminate all but one of the alternative individuations on the basis of their unphysicality. A natural strategy to do that could be to use the spatial degrees of freedom as the preferred method of individuating objects. Let us illustrate this method with some examples. We start with the specific antisymmetric state (5.15) from Chap. 5 which is of the general form written in (7.6):

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |L\rangle_1 \otimes |\downarrow\rangle_2 |R\rangle_2 - |\downarrow\rangle_1 |R\rangle_1 \otimes |\uparrow\rangle_2 |L\rangle_2\right). \tag{7.9}$$

One straightforward way to individuate particles occupying this state is with the help of the following projectors:

$$\begin{aligned} E_{uL} &= |\uparrow\rangle\langle\uparrow|\ |L\rangle\langle L| \\ E_{dR} &= |\downarrow\rangle\langle\downarrow|\ |R\rangle\langle R|, \end{aligned} \tag{7.10}$$

which means that we can distinguish one particle occupying region $L$ and possessing spin "up", and one particle located in $R$ with spin "down".

However, as we have already seen, we can introduce new single-particle vectors:

$$\begin{aligned} |\Lambda\rangle &= a|\uparrow\rangle|L\rangle + b|\downarrow\rangle|R\rangle \\ |\Gamma\rangle &= -b^*|\uparrow\rangle|L\rangle + a^*|\downarrow\rangle|R\rangle, \end{aligned} \tag{7.11}$$

and we can show that the following vector is identical with (7.9):

$$\frac{1}{\sqrt{2}}\left(|\Lambda\rangle|\Gamma\rangle - |\Gamma\rangle|\Lambda\rangle\right). \tag{7.12}$$

This means that alternative individuations are given by projectors corresponding to rays spanned by vectors $|\Lambda\rangle$ and $|\Gamma\rangle$. But observe that these vectors are non-trivial superpositions of states with different positions and spin components (given that coefficients $a$ and $b$ are non-zero). Thus if we introduced a rule which prescribes that only subspaces which factorize into an internal state subspace and a spatial state subspace are physically acceptable as a means to individuate particles, we could eliminate all the individuations except the one in terms of the projectors written in (7.10).[14]

[14] Ghirardi et al. (2002, p. 86) justify eliminating the individuations done with the help of vectors $|\Lambda\rangle$ and $|\Gamma\rangle$ by observing that "measurements involving states like [these] are extremely difficult to perform and of no practical interest". This is obviously a legitimate pragmatic reason for disregarding such individuations, but we should give a more substantial argument if we wanted to make the metaphysical claim that nothing in reality corresponds to the individuations excised on such pragmatic grounds.

## 7.5 Two Solutions of the Problem of Ambiguity

Such a rule, when appropriately formulated, will also hopefully help resolve some cases of the ambiguity of the first kind, presented in the previous section. Consider again state (7.4). We have noted that there are two alternative ways of individuating particles occupying this state: one with the help of the projectors onto subspaces Span$\{|\uparrow\rangle|L\rangle, |\uparrow\rangle|R\rangle\}$ and Span$\{|\downarrow\rangle|R\rangle, |\downarrow\rangle|L\rangle\}$, and the other using subspaces Span$\{|\uparrow\rangle|L\rangle, |\downarrow\rangle|L\rangle\}$ and Span$\{|\uparrow\rangle|R\rangle, |\downarrow\rangle|R\rangle\}$. These two pairs of subspaces can be alternatively presented as the following products:

$$\text{Span}\{|\uparrow\rangle\} \otimes \text{Span}\{|L\rangle, |R\rangle\}; \text{Span}\{|\downarrow\rangle\} \otimes \text{Span}\{|L\rangle, |R\rangle\}.$$

$$\text{Span}\{|\uparrow\rangle, |\downarrow\rangle\} \otimes \text{Span}\{|L\rangle\}; \text{Span}\{|\uparrow\rangle, |\downarrow\rangle\} \otimes \text{Span}\{|R\rangle\}.$$

Only the second method of individuation ascribes to the individuated particles definite position states; the first individuation implies merely that the position state of each single particle is a vector in the two-dimensional subspace Span$\{|L\rangle, |R\rangle\}$ without specifying which vector it is.[15] Hence we can propose the following rule that will eliminate the first individuation as unphysical:

(7.13) Projectors $E_a$ and $E_b$ used to individuate particles of the same type should be such that $E_a[\mathscr{H}] = \mathcal{V}_s^a \otimes \mathcal{V}_r^a$ and $E_b[\mathscr{H}] = \mathcal{V}_s^b \otimes \mathcal{V}_r^b$, where $\mathcal{V}_s^a$, $\mathcal{V}_s^b$ – any subspaces in the internal state space, and $\mathcal{V}_r^a$, $\mathcal{V}_r^b$ – *one-dimensional* subspaces in the position state space.

[15] Observe that the requirement that particles be individuated by states with well-defined positions also eliminates the ambiguity of the second kind present in the case of state (7.4), since all the alternative individuations will be in terms of subspaces that cannot be factorized into internal and spatial degrees of freedom.

Thus, according to the condition expressed in (7.13), the only acceptable method of individuation is when the individuating subspaces factorize into a product of the internal and position subspaces, with the additional requirement that the position subspace be one-dimensional. This essentially means that we should individuate the components of the system with the help of well-defined position states (specific wave functions) rather than multidimensional subspaces spanned by orthogonal wave functions.

While restriction (7.13) solves the problem of ambiguity in many cases, it is far from being a panacea. First off, the requirement of the one-dimensionality of the position subspaces may be argued to eliminate too many cases of non-maximally specific individuation. In order to see that, let us consider the following fermionic state:

$$\frac{1}{2}\left(|\uparrow\rangle_1|L\rangle_1\otimes|\downarrow\rangle_2|R\rangle_2-|\downarrow\rangle_1|R\rangle_1\otimes|\uparrow\rangle_2|L\rangle_2\right)$$
$$+\frac{1}{2}\left(|\uparrow\rangle_1|S\rangle_1\otimes|\downarrow\rangle_2|T\rangle_2-|\downarrow\rangle_1|T\rangle_1\otimes|\uparrow\rangle_2|S\rangle_2\right), \tag{7.14}$$

where $|L\rangle$, $|R\rangle$, $|S\rangle$, $|T\rangle$ are mutually orthogonal vectors in the position space (e.g. wave functions whose supports are non-overlapping). The state written in (7.14) makes it possible to individuate the two particles occupying it, albeit only with the help of the less then maximally specific properties represented by two-dimensional projectors/subspaces. Thus one individuation is given by the subspaces $\mathrm{Span}\{|\uparrow\rangle|L\rangle, |\uparrow\rangle|S\rangle\}$ and $\mathrm{Span}\{|\downarrow\rangle|R\rangle, |\downarrow\rangle|T\rangle\}$, and the other by $\mathrm{Span}\{|\uparrow\rangle|L\rangle, |\downarrow\rangle|T\rangle\}$ and $\mathrm{Span}\{|\downarrow\rangle|R\rangle, |\uparrow\rangle|S\rangle\}$.[16] None of these subspaces can be presented in the product form required by condition (7.13) with a well-defined spatial component; hence both individuations should be treated as non-physical. In other words, particles occupying (7.14) cannot be individuated in a physically meaningful way. Hence, the criterion of physicality presented

[16] I am ignoring the fact that each antisymmetric block in (7.14) can be alternatively rewritten in the way explained with respect to state (7.9). These additional alternative individuations only make the considered problem worse, since none of them involve well-defined position states as required by condition (7.13).

in (7.13) appears to be too strong, as it excludes all alternative individuations available in this case, rather than picking up exactly one of them.

On the other hand, criterion (7.13) can also be proved to be too weak. Some cases of ambiguity remain even after applying the condition formulated in (7.13). Consider the following, antisymmetric state of two electrons:

$$\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1|L\rangle_1 \otimes |\downarrow\rangle_2|L\rangle_2 - |\downarrow\rangle_1|L\rangle_1 \otimes |\uparrow\rangle_2|L\rangle_2\right)$$
$$= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 \otimes |\downarrow\rangle_2 - |\downarrow\rangle_1 \otimes |\uparrow\rangle_2\right)\left(|L\rangle_1 \otimes |L\rangle_2\right). \tag{7.15}$$

This is the state of two electrons that occupy the same spatial region $L$ (they have the exact same spinless wave functions), while their spins are anticorrelated (a standard example of such a situation is provided by two electrons in a helium atom which occupy the lowest energy level, and thus are characterized by the same spatial wave function, while at the same time the total spin of the electrons equals zero). In that case there are an infinite number of individuating pairs of projectors which differ only with respect to the spin state. That is, each pair of projectors $E_{\uparrow_n} = |\uparrow_n\rangle\langle\uparrow_n||L\rangle\langle L|$ and $E_{\downarrow_n} = |\downarrow_n\rangle\langle\downarrow_n||L\rangle\langle L|$, where $|\uparrow_n\rangle$ and $|\downarrow_n\rangle$ are states "up" and "down" for spin in an arbitrary direction $n$, individuates the two electrons. But it is straightforward to observe that all pairs of the form $E_{\uparrow_n}, E_{\downarrow_n}$ satisfy (7.13), and therefore the ambiguity is still present.

The above-described case of ambiguity can be dismissed on pragmatic grounds, since no experimental procedure can determine what the spin state of the individual electrons occupying (7.15) is, as long as they are not spatially separated. In order to experimentally verify the hypothesis that each electron possesses well-defined spin in multiple directions, we would have to separate them spatially in order to perform individual measurements. But this preparatory procedure would change the total state of the system which would no longer have the form given in (7.15), but instead would be described by formula (7.4). And, as we have seen, in this case criterion (7.13) does eliminate the existing ambiguity, since the only individuation admissible in the light of this criterion is the one defined by subspaces Span$\{|\uparrow\rangle, |\downarrow\rangle\} \otimes$ Span$\{|L\rangle\}$ and Span$\{|\uparrow\rangle, |\downarrow\rangle\} \otimes$ Span$\{|R\rangle\}$. However, this argument does not show that there is no ambiguity in the initial state (7.14), only that its existence should have no empirically verifiable consequences that would be inconsistent with standard quantum-mechanical predictions. The irreducible ambiguity that remains in this case comes from the fact that there is no answer to the question of how to "distribute" definite values of spin in various directions among the two particles, given that the only constraint on the "true" values of spin in a particular direction is that they have to be opposite for both particles.

A more fundamental objection can be raised against the proposed, position-based method of dealing with the ambiguity challenge to the hederodox approach. It may be asked what justifies the choice of position as the key parameter in the criterion that draws the line between legitimate and illegitimate individuations. What makes properties that involve intricate combinations of position and internal states less appropriate as a means of identifying the components of a system? As we have already stressed, there are good pragmatic reasons to choose the criteria of individuation that rely on relatively well-defined localizations, since measurements are localized states of affairs. But why should nature choose properties that are useful to us as objectively possessed by observer-independent objects?[17] It is true that there are interpretations of quantum mechanics which elevate the ontological status of position in comparison to other measurable parameters. The most famous of such interpretations is Bohmian mechanics, which assumes that all particles possess well-defined trajectories and that measurements of various observables ultimately reduce to measurements of the location of an object within its guiding pilot wave. Another interpretation which recognizes the privileged ontological status of position is the GRW theory (the spontaneous localization theory). This theory supplements the ordinary laws of quantum mechanics with a probabilistic rule that describes a fundamentally indeterministic transition ("jump") from an unlocalized state to a state with (almost) perfect localization. Given these two prominent interpretations, it may be less controversial to admit that spatial localization plays the key role in an objective selection of the components of a composite system.

[17] There is one important way of answering this question within the so-called Everettian interpretation of quantum mechanics. Namely, the physical process responsible for selecting a basis with (approximately) well-defined positions is via decoherence, arising as a result of an interaction of the quantum system with its environment. I am grateful to Simon Saunders for pointing this out to me.

Still, there are alternative interpretations of quantum mechanics which do not accord a special ontological status to position (at least not directly). And taking into account the serious limitations of the position-based strategy described earlier in this section, we may want to look for alternatives. A second option is simply to accept the irreducibly ambiguous character of individuations done with the help of qualitative properties. More specifically, we may introduce the concept of a *relative* individuation, where individuations are relativized to a particular complete set of orthonormal vectors in the single-particle Hilbert space. Alternatively, and equivalently, we may select any maximal set of compatible observables, and with respect to this set we may admit the existence of particular components with certain properties as specified by appropriate projectors/subspaces.

Let us then choose any orthogonal basis $\mathcal{B} = \{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ for the $n$-dimensional Hilbert space $\mathcal{H}$. Let $|\varphi(1,2)\rangle$ be the state of a two-particle system $S$ of "indistinguishable" fermions or bosons, and let $E_a$ and $E_b$ be some orthogonal projectors that individuate this system (i.e. $|\varphi(1,2)\rangle$ is assumed to lie in the range of projector $E_a \otimes E_b + E_b \otimes E_a$). Then we will say that:

(7.16) The components of system $S$ individuated by projectors $E_a$ and $E_b$ exist *with respect to* $\mathcal{B}$, if $E_a[\mathcal{H}] = \text{Span}\{|e_i\rangle\}_{i=1,\ldots,k}$ and $E_b[\mathcal{H}] = \text{Span}\{|e_j\rangle\}_{j=1,\ldots,l}$, where $|e_i\rangle, |e_j\rangle \in \mathcal{B}$ for all $i, j$.

The condition expressed in (7.16) can be stated equivalently by saying that the components of the system exist with respect to a maximal set of compatible observables, if $E_a$ and $E_b$ belong to that set. Thus we abandon the notion of the "absolute" existence of the components of a given system in favor of the notion that ties the existence of particularly individuated components to the preselection of a "perspective" in the form of a maximal set of observables whose precise values may be known simultaneously. In that way we defuse the danger posed by the existence of a multitude of alternative and incompatible individuations, since we no longer claim that these individuations pick out objects that exist "together", independently of anything else. On the contrary, alternative sets of components exist only conditionally on the choice of a particular orthogonal basis.

This proposal, which may be referred to as *perspectivalism*, is particularly suited to deal with the second kind of ambiguity, affecting fermionic states. Let us consider again state (7.6). The alternative individuations available in this state are all given in terms of orthogonal projectors $E_{\lambda_1} = |\lambda_1\rangle\langle\lambda_1|$ and $E_{\lambda_2} = |\lambda_2\rangle\langle\lambda_2|$, where $|\lambda_1\rangle$ and $|\lambda_2\rangle$ are any orthogonal vectors spanning the two-dimensional subspace Span$\{|\varphi_1\rangle, |\varphi_2\rangle\}$. Each pair of individuating projectors correspond to a different selection of an orthogonal basis, and consequently of a different maximal set of compatible observables. Thus we may say that for a given pair of projectors $E_{\lambda_1}$ and $E_{\lambda_2}$, the components of the system identified by these projectors exist only relative to the choice of vectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$ as elements of an orthogonal basis. If the initial vectors $|\varphi_1\rangle$, $|\varphi_2\rangle$ are identified, for instance, as eigenvectors of spin in some direction $z$, this means that we are no longer obliged to admit that the components of the bipartite systems possess various incompatible properties (well-defined spins in different directions). Rather, we should say that depending on which spin direction we select, the system is certain to contain particles with opposite values of this particular spin component *with respect to this selection*.

How are we supposed to understand the above-formulated relativization of existence, which in philosophy is typically considered an absolute, objective concept? And doesn't this relativization reduce an ontological notion to an epistemological one? The most natural way to explain the proposed relativization is of course in terms of future measurements and their outcomes. It is a well-known and perhaps even overstated feature of quantum mechanics that measurements can reveal precise values of only a handful of physical quantities. We have to decide in advance which observable we want to measure, and in that way we limit our ability to learn the exact values of properties to the set of observables compatible with the selected one. This experimental choice corresponds precisely to the selection of an orthogonal basis to which we relativize the existence of particular components of the composite system. Thus it may be said that the components exist relative to a future choice of observables to measure. Or, to put it differently and perhaps in a more philosophical language, the existence of particular components becomes a *dispositional property* of the total system. The system of two electrons with anticorrelated spins possesses the dispositional property that it will reveal well-defined and opposite spins in any direction given an appropriately selected spatial orientation of the measuring apparatus. We interpret these dispositional properties as expressing the *relative* existence of particles with well-defined spins even before measurement. Yet another way to express this idea may be in terms of *potential* existence, where again potentiality can be actualized only via a selection of a particular direction of spin to measure.[18]

Unfortunately, perspectivalism does not offer a solution to the first kind of ambiguity, due to the fact that all alternative individuations in this case are based on mutually compatible projectors. Since all single-particle vectors $|\varphi_i\rangle$ constituting fermionic states (7.1) or bosonic states (7.2) form (a part of) an orthogonal basis, and since vectors $|\varphi_i\rangle$ are directly used to build alternative individuating subspaces, all the alternatively individuated components of the system are relativized to the same orthogonal basis, and therefore must be assumed to exist "simultaneously". At this point I have no suggestion of how to further deal with this problem. More fine-grained relativization to some subsets of the set of vectors $|\varphi_i\rangle$ won't do, since this would effectively mean that any admissible individuation picks out existing components relative to itself, and this would obviously trivialize the perspectivalist solution. As a stopgap measure, we may revert in this case to using position as the privileged parameter, with all its limitations that we have discussed earlier.[19]

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[18] It may be worth reminding ourselves that the actualization mentioned here does not necessarily involve any change in the quantum-mechanical state of the entire system. As we have already stressed (see Sect. 5.4), due to the Symmetrization Postulate, the final state of the two fermions can never be a product of two states. Thus it may happen that a particular measurement does not "collapse" the initial state of the form (7.6), and yet this measurement may still be seen as actualizing the existence of particles with well-defined values of the measured observable.

[19] In my Bigaj (2016, Sect. 5) I opted for a solution of the ambiguity problem which I called "mixed view". This is a combination of the preferred basis approach (based on the privileged role of position) and the "incompatible properties" view, according to which each component of a fermionic systems may possess properties represented by incompatible observables (such as spin components in different directions). The incompatibility analysis is applicable when the components cannot be individuated by separate locations (as in the case of two electrons occupying the same energy level in an atom), and hence no conflict with the predictions of standard quantum mechanics can be revealed by means of spatially separated measurements. On the other hand, when the particles become separable by their spatial locations, the preferred basis approach dictates that only one of the infinite numbers of alternative and incompatible individuations represents reality. Since in my 2016 article I concentrated only on the ambiguity of the second kind (affecting fermions not bosons), I did not discuss the problems, indicated above, which this approach may encounter when considering less than maximally specific individuations (individuations by many-dimensional projectors). Regardless of the viability of the "mixed view" solution, I think that the perspectivalist approach deserves to be taken into account as a serious alternative due to its (mostly) uniform analysis of all cases of ambiguity, and its intriguing similarities to some interpretations of quantum mechanics, such as the Everettian relative state interpretation and Rovelli's relational quantum mechanics (cf. Barrett 2018; Laudisa and Rovelli 2019).

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# 8

# The Metaphysics of Quantum Objects: Transtemporal and Transworld Identities

The last two sections of the previous chapter strongly suggest that the heterodox approach to quantum individuation departs from the classical views on identity and individuality no less dramatically than the orthodoxy. In the final chapter of the book, we will further explore the nonclassical character of the emerging metaphysical conception of quantum objects, taking into account two so far relatively neglected aspects of their identities: one related to their identifications over time and the other present in counterfactual scenarios. We will argue that it is fundamentally impossible to uniquely identify a given quantum particle in an alternative, merely possible situation. Regarding the transtemporal identification of quantum objects, the situation appears to be more subtle. The heterodox approach, in contrast to the orthodox one, enables us in certain circumstances to make definite identifications of quantum objects across extended periods of time. However, in the majority of cases transtemporal identifications fail, and this means that the components of a complex system identified at different temporal moments will generally not be connected by the relation of genidentity, even though the system as a whole retains its identity. At the end of the chapter, we will try to sketch a future metaphysical theory of objects that could account for the revealed facts regarding various types of identity in the quantum regime.



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## 8.1 Synchronic Versus Diachronic Identity

Discussions about the validity of the PII and the relations between numerical distinctness and qualitative discernibility are typically conducted under the tacit assumption of synchrony. That is, we stipulate that at a given moment *t* there exist two numerically distinct objects, and we ask the question of whether these objects can be differentiated using properties and relations exemplified by them at moment *t*. In the context of quantum mechanics, this question is considered by taking into account the instantaneous state of the composite system at the very same moment *t*. Thus the perspective that is adopted here may be called synchronic. However, for objects that persist in time a different type of perspective may be taken: the diachronic one. Diachronic (transtemporal) considerations involve distinct moments $t_1$ and $t_2$ at which certain objects exist. Interestingly, with the change of perspective from synchronic to diachronic, the type of identity-related questions that are usually raised changes too. We are no longer interested in the issue of discernibility at different temporal points, since even one and the same object can possess distinct and mutually exclusive properties at various moments of its existence. Thus diachronic qualitative identity is not a necessary condition of numerical identity over time. Instead, the main question regarding diachronic identity is how to decide the truth of numerical identifications over time. How do we answer the question whether an object characterized at $t_1$ is the same as (or distinct from) an object characterized at $t_2$? And is such a question always guaranteed to have a unique and unambiguous answer? In other words, are facts of diachronic identity/distinctness objectively determined, regardless of the practical issue of how we can know them?

Let us start with some preliminary assumptions and terminological distinctions. The notion of diachronic identity can be conceptualized in many different ways. Hans Reichenbach introduced the relation of *gen-identity* which, strictly speaking, is not identity at all, since it connects events occurring on a particular object at different moments (Reichenbach 1971, p. 38). And clearly events taking place at different times are numerically distinct.[1] However, what underlies the relation of genidentity connecting distinct events is precisely the numerical identity of the object on which these events occur. Consequently, we will adopt the convention according to which diachronic identity is not a new type of identity, but rather numerical identity considered in the context of separate moments of time. More precisely, we will not think of diachronic and synchronic identity as genera of the relation of identity, but instead we will talk about *diachronic identity statements* and *synchronic identity statements*. A synchronic identity statement is a proposition that connects two descriptions taken at the same time with the symbol of identity: $\Phi$ at $t = \Psi$ at $t$ (e.g. "The blue object on my desk at $t$ is identical with the pen on my desk at $t$"). A diachronic identity statement, on the other hand, involves descriptions taken at different times: $\Phi$ at $t_1 = \Psi$ at $t_2$ (e.g. "The man I met yesterday at the park is the same as the man that stole my bicycle today"). However, in both sentences the symbol "=" denotes the same fundamental relation of the numerical identity between any object and itself.

Formulating appropriate criteria for statements of transtemporal identity is one of the core problems in the metaphysics of persistence. Common answers include spatiotemporal continuity (continuous trajectories) and transmission of certain identifying marks (see Reichenbach 1971, pp. 224–227, and for an overview see Gallois 2016). These criteria can be criticized on many grounds, and it may also be argued that they provide no solution to the most egregious cases, such as the celebrated example of the ship of Theseus. Yet we will not discuss cases involving macroscopic referents of terms from everyday language with their unavoidable vagueness and ambiguity. Instead, we will focus on fundamental entities, whether taken as classical particles or quantum-mechanical objects. In the case of classical bodies, they are guaranteed to have well-defined spatiotemporal trajectories throughout their existence, and this can lay ground for the facts of diachronic identity. Thus we can identify objects picked out at two different times if they are connected by

[1] Compare the standard analysis of events by Jaegwon Kim in Kim (1976), where events are defined as triples consisting of an object, a property and a temporal point at which the object possesses the property.

a continuous curve in spacetime whose points are occupied by successive stages of some material entity. Observe that this criterion relies on the assumption of *impenetrability*, that is, the claim that no two material bodies can occupy the exact same spatial location (i.e. trajectories of distinct bodies can never cross).

Still, there are possible scenarios when even in the case of classical objects the continuity criterion may be insufficient. Suppose that a particle splits in half at a certain point of its existence and that the products of the splitting fly off in opposite directions. According to the criterion of continuity, each half should be seen as identical with the original particle, and yet this seems impossible given that the halves are clearly numerically distinct.[2] In this case the criterion of mark transmission may help. It may be claimed that some identifying features of the original particle (e.g. its mass) are not preserved after the split, and thus the products are not identical with their source particle. However, given that many properties of a body change throughout its existence, we have to be careful to distinguish attributes that must be preserved from the remaining ones. The choice of properties that may be called essential to the continuation of existence depends on the presupposed conception of objects of interest (to use a well-worn example: a statue would have a different set of essential attributes than the lump of clay it is made of). Or, to put it differently, by selecting a particular set of essential properties, we clarify what type of objects we have in mind when we refer to a given entity. The specification of essences gives us an insight into the "conditions of survival" of a particular entity, and hence it provides us with a general characterization of the temporal boundaries of this entity.

According to a widely accepted view, the criterion of diachronic identity based on spatiotemporal continuity is not applicable to the case of quantum particles. Reichenbach puts this failure of continuity down to quantum indeterminacy and the related wave-particle duality (Reichenbach 1971, p. 228). Steven French and Decio Krause offer a

[2] A more sophisticated variant of this problem is known as the "amputation case" (van Inwagen 1981; Heller 1984). If the particle in question is genuinely elementary, that is, does not possess any parts smaller than itself, then the "splitting" is better thought of as a process of decay where new particles are created. Sill, from the classical perspective the condition of temporal continuity is satisfied.

similar evaluation, citing Erwin Schrödinger's observation that quantum particles cannot be attributed definite spatiotemporal trajectories (French and Krause 2006, p. 123). And yet Simon Saunders points out that the criterion of continuity does not have to be entirely abandoned (Saunders 2015, p. 169). In quantum mechanics, instead of spatiotemporal trajectories, we can talk about *orbits*, that is, continuous evolutions of an initial state vector under the action of a particular unitary evolution operator *U*.[3] Given the Schrödinger equation, the temporary evolution of a system whose initial state is $|\psi(t_1)\rangle$ can be presented as follows:

$$|\psi(t_2)\rangle = U(t_2 - t_1)\,|\psi(t_1)\rangle,$$

where the unitary time-dependent evolution operator $U(t)$ is a function of the Hamiltonian $H$ of the system:

$$U(t) = e^{-\frac{itH}{\hbar}}.$$

Consequently, it is possible to introduce the following continuity criterion of transtemporal identity applicable to the case of quantum particles. An object *a* characterized by its quantum-mechanical state $|\psi(t_1)\rangle$ at time $t_1$ is identical with an object *b* occupying state $|\psi(t_2)\rangle$ at $t_2$, if there is a continuous curve in the appropriate Hilbert space parametrized by a real-valued function $t$ which connects vectors $|\psi(t_1)\rangle$ and $|\psi(t_2)\rangle$, and such that for each $t \in [t_1, t_2]$, vector $|\psi(t)\rangle$ represents the physical state of some actual system. However, we may have justified doubts as to the practicality of this criterion, given the well-known fact that we cannot monitor quantum-mechanical states of any given system over an extended period of time. So it seems that in practice we must rely on some other

[3] As Nick Huggett and Tom Imbo (Huggett and Imbo 2009) point out, the non-existence of continuous spatiotemporal trajectories in quantum mechanics follows precisely from the fact that the Schrödinger evolution of quantum systems is continuous in Hilbert spaces. Since classical trajectories in space consist of points corresponding to eigenstates of the position operator, and since distinct eigenvectors are orthogonal to each other, it follows that the continuous evolution of a quantum system from one eigenvector to another must take the system through states with no well-defined position.

criteria of transtemporal identity, and only when we establish that we are dealing with the same physical system at two different moments of time, we can try to introduce an appropriate Hamiltonian which will enable us to theoretically calculate an orbit representing the evolution of the system. This epistemological issue raises the question of which fact is ontologically more fundamental: the continuity of a given orbit, or the diachronic identity connecting various stages of a particular system.

Is it feasible to use instead the criterion of mark transmission in the quantum regime? Generally, the answer is "yes", as long as we are dealing with "distinguishable" particles. Considering a system of a proton and an electron, we have no problems connecting appropriate temporal stages by genidentity thanks to the unique features of the particles involved (electric charge, rest mass). Thus state-independent properties can definitely be used as marks. However, this criterion is of no use when we consider systems of "indistinguishable" particles, such as a pair of electrons, since they share all their state-independent, identifying marks. But perhaps we could make transtemporal identifications on the basis of state-dependent properties. If a particular evolution of a system does not affect certain magnitudes, we may use the value of such a magnitude to trace back the history of this system, even though the quantity in question is not generally time-independent and hence may depend on the state of the system. However, that sort of identification presents us with a fundamental obstacle in the form of the synchronic Indiscernibility Thesis.

If we follow the orthodox conception of individuation, we have to accept that particles of the same type are never discernible synchronically by their physical properties, and this prevents us from making definite qualitative identifications across time. Clearly, if we have two objects *a* and *b* which at time $t_1$ share all their properties, and objects *c* and *d* that are similarly indiscernible with respect to their properties at $t_2$, then whatever argument based on properties we will come up with for the identification of, let's say, *c* at $t_2$ with *b* at $t_1$, will also apply to *c* at $t_2$ and *a* at $t_1$. Observe, further, that this situation does not change even if we assume that pairs *a*, *b* and *c*, *d* are weakly discernible at respective moments $t_1$ and $t_2$. No wonder, then, that the main proponent of the weak discernibility approach to orthodoxy, F.A. Muller, states authoritatively (Muller 2015, p. 23, emphasis original):

> [w]hen I have a physical system composed of *N* absolutely indiscernible particles at time *t*, then not a single particle can be re-identified at a later time $t' > t$ because it cannot be identified at all: they are not individuals but relationals. Elementary particles have no *genidentity*, as Reichenbach […] put it. The relations I have employed to discern particles weakly use the quantum-mechanical state *at a time* and therefore discern them *synchronically*. Relations to discern particles *diachronically* are not forthcoming in QM. There are no persistence conditions for single particles.

In what follows we will see that under the unorthodox approach to individuation the situation with respect to diachronic identification is not as dire as portrayed above.

## 8.2 Diachronic Identity of Same-Type Particles

On the basis of our current analysis, we can distinguish three general types of criteria of diachronic identity that can be potentially used for quantum objects. Diachronic identity statements regarding quantum objects can be based on the continuity of orbits, on the identity of state-independent properties, or on the identity of state-dependent properties (given the assumption that these properties are not affected by the interactions at the intervening moments). In the case of quantum particles of the same type, the second criterion is of no use, since such particles share all their state-independent properties. Hence the only hope for a working criterion of transtemporal identity is either continuity of orbits or preservation of selected state-dependent properties.

As we have seen above, both criteria fail under the orthodox approach to quantum individuation. The orthodox approach implies that the components of an ensemble of "indistinguishable" particles are assigned the exact same reduced (mixed) states, so the criterion of continuity obviously breaks here (each particle will have an identical temporal evolution). The same applies to the criterion based on state-dependent properties, since the Indiscernibility Thesis proves that all measurable properties will be identical among the components of the ensemble. The only possibility left for those who want to admit objective facts regarding transtemporal identifications is that these facts are primitive and thus not grounded in more fundamental qualitative facts. One could, for instance, insist that the labels used in the description of a joint state actually trace individual but qualitatively indistinguishable particles. The particle labeled 1 at time $t_1$ is *in fact* identical with the particle labeled 1 at time $t_2$, even though there is no physical way to separate this particle from the rest of the ensemble. Needless to say, this approach should not please empirically oriented philosophers and scientists.

The unorthodox conception, on the other hand, is more conducive to the idea of the preservation of numerical identity across temporal moments. Given that qualitative individuation with the help of measurable properties is available in the majority of states, we may hope for a successful application of one of the two admissible criteria of diachronic identity in some cases involving same-type particles. And indeed, Saunders observes that the criterion of the continuity of orbits can be applied to assemblies of "indistinguishable" particles which occupy GMW-non-entangled states (Saunders 2015, pp. 169–170). Here is how this might work (as usual, we will limit ourselves to the case of two particles). Suppose that a system of two same-type fermions occupy the following antisymmetric state at $t_1$:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{2}}(|\lambda_a\rangle \otimes |\lambda_b\rangle - |\lambda_b\rangle \otimes |\lambda_a\rangle), \tag{8.1}$$

where $|\lambda_a\rangle$ and $|\lambda_b\rangle$ are orthogonal. In that case, according to the heterodox approach, we can treat this assembly as consisting of two particles, one of which occupies state $|\lambda_a\rangle$ and the other state $|\lambda_b\rangle$. Now, let us suppose that the unitary evolution operator has the form $U(t) \otimes U(t)$, where $U(t)$ acts in the single-particle Hilbert space and $t = t_2 - t_1$. When applied to the initial state $|\psi(t_1)\rangle$, this operator produces the following:

$$\begin{aligned}|\psi(t_2)\rangle &= U(t) \otimes U(t))|\psi(t_1)\rangle \\ &= \frac{1}{\sqrt{2}}(U(t)|\lambda_a\rangle \otimes U(t)|\lambda_b\rangle - U(t)|\lambda_b\rangle \otimes U(t)|\lambda_a\rangle).\end{aligned} \tag{8.2}$$

Given that $U(t)$ is unitary, vectors $U(t)|\lambda_a\rangle$ and $U(t)|\lambda_b\rangle$ are orthogonal;[4] thus again we can individuate the particles occupying the evolved state $|\psi(t_2)\rangle$ using the properties associated with states $U(t)|\lambda_a\rangle$ and $U(t)|\lambda_b\rangle$. Consequently, we can connect state $|\lambda_a\rangle$ with $U(t)|\lambda_a\rangle$ and $|\lambda_b\rangle$ with $U(t)|\lambda_b\rangle$ by continuous orbits. Using the representation in terms of individuation blocks, as explained in Chap. 5, we can treat the evolution of the system as if it was described in terms of factorizable states, from $|\lambda_a\rangle \otimes |\lambda_b\rangle$ to $U(t)|\lambda_a\rangle \otimes U(t)|\lambda_a\rangle$, where each particle traces its own trajectory in a respective Hilbert space.

However, there are some serious limitations of the above-described method of transtemporal identification based on the continuity of orbits. First of all, we should not forget that in the case of fermionic states there are a multitude of alternative synchronic individuations. Thus at any moment $t$ of the continuous evolution of the system, we might have selected a different pair of orthogonal one-dimensional projectors to individuate the particles, and there is no way to connect thus individuated particles with the initial states $|\lambda_a\rangle$ and $|\lambda_b\rangle$ by continuous orbits. The method of transtemporal identification described above amounts to a selection of a privileged basis which consists of orthogonal vectors $|\lambda_a\rangle$ and $|\lambda_b\rangle$ evolved in time by the action of unitary operator $U(t)$. With respect to this continuously evolving basis, we can state the existence of the components of the entire system that indeed retain their identity over time (recall the concept of the relative existence of the components of a given fermionic system outlined in Sect. 7.5, formula 7.16). However, now the argument in support of the transtemporal identity of these particles looks dangerously close to being circular. Surely the selected components retain their identity thanks to the continuity of orbits, since we have selected these components using the continuously evolving basis vectors. But select some other bases, and the continuity goes out the window.

Another problem is more general, as it equally affects states of distinguishable particles. The continuity criterion obviously works with respect to unitary evolutions of systems, but when we consider non-unitary,

[4] Here is a quick proof: $\langle U\lambda_a|U\lambda_b\rangle = \langle\lambda_a|U^\dagger U|\lambda_b\rangle = \langle\lambda_a|\lambda_b\rangle = 0$, since $U^\dagger = U^{-1}$ for unitary operators.

discontinuous processes, such as measurements, this criterion is no longer applicable. If we want to talk about diachronic identity of particles undergoing measurement-like processes (including the Copenhagen-school collapses of the wave function, or spontaneous localizations propounded in the GRW theory), we have to rely on the preservation of selected state-dependent properties as the only available criterion. And in principle it seems that this method may work under the heterodox approach to individuation, since state-dependent properties are what we use in this approach to differentiate synchronically between same-type particles. If a particular (unitary or not) evolution of a system of same-type particles preserves certain quantities, we may use the values of these quantities to identify specific particles taken at different moments, even if the total state of the system is not preserved. Of course this method of diachronic identification presupposes that the particles differ from one another synchronically with respect to the value of the quantities in question, so that we can trace the evolution of each individual particle by following its uniquely identifying value at every moment. This presupposition has a chance of coming true only under the heterodox conception of individuation and discernibility.

In the next sections we will probe deeper the possibility of diachronic identifications by properties using a standard quantum-mechanical example of scattering experiments. These experiments involve a pair of interacting particles, whose identities pre- and post-interaction may be put into question. As it turns out, in some of these scattering interactions, the diachronic identities of participating particles are "lost", while in others they may be seen as preserved. We will see that quantum mechanics provides us with a simple empirical criterion that can distinguish between identity-preserving interactions and identity-erasing interactions. This criterion refers to the presence or absence of interference effects in the angular distribution of scattered particles. Subsequently we will have a closer look at this interference-based criterion.

## 8.3 Scattering Experiments

Scattering events are one of the most ubiquitous occurrences in particle physics. As David Griffiths points out, virtually all experimental information in particle physics comes from three sources: decays, bound states and scattering (Griffiths 2008, p. 2). The simplest type of scattering events involves two incoming particles that collide and then fly off in opposite directions (in the center of mass frame of reference).[5] The main parameter characterizing a given interaction is the *differential cross section*, which roughly represents the proportion of the incident particles registered at a selected angle after scattering. In the current context we are not interested in particular methods and techniques to calculate the cross sections for a given potential,[6] but instead we will focus on more general descriptions of scattering processes that may give us clues regarding how to identify particles pre- and post-interaction. Thus our approach will be broadly schematic, ignoring the fine details of concrete interactions between particles.

At the beginning we will discuss the case of elastic scattering involving two "distinguishable" particles (e.g. a proton and an electron).[7] The reason for this choice is that later on we will be able to compare appropriate formulas for so-called transition amplitudes derived in the case of "indistinguishable" particles with the currently obtained formulas. This comparison will enable us to discern cases in which diachronic identities of "indistinguishable" particles are preserved from cases in which these identities are lost. So let us assume that the incoming particles are characterized by respective wave functions $|\psi_L\rangle$ ("coming from the left") and $|\psi_R\rangle$ ("coming from the right"). Thus the state of the entire system of particles before the interaction will be the product of two wave functions:

$$|\psi(t_1)\rangle = |\psi^a{}_L\rangle \otimes |\psi^b{}_R\rangle \tag{8.3}$$

[5] More general processes, such as rearrangement collisions, lead to the creation of new particles and thus belong to the category of *reactions* (see Cohen-Tannoudji et al. 1978, pp. 903–904).

[6] For more details on that, see, for example, Cohen-Tannoudji et al. (1978, p. 903ff).

[7] The subsequent analysis is based on Cohen-Tannoudji et al. (1978, pp. 1403–1408). See also Bigaj (2020).

where particles characterized by wave packets $|\psi^a{}_L\rangle$ and $|\psi^b{}_R\rangle$ are identifiable and therefore discernible by their state-independent properties (hence the distinct superscripts *a* and *b*). The interaction between the particles can be most generally presented with the help of a unitary evolution operator $U(t)$, where $t = t_2 - t_1$. Acting with this operator on the initial state, we can calculate the state of the entire system at any time $t_2 > t_1$ as follows:

$$|\psi(t_2)\rangle = U^{ab}(t)(|\psi^a{}_L\rangle \otimes |\psi^b{}_R\rangle) \tag{8.4}$$

The total wave function $|\psi(t_2)\rangle$ after the interaction (i.e. at such a time $t_2$ when the part of the Hamiltonian describing the interaction between the particles becomes negligible) will typically take the form of spherical waves spreading in all directions. In order to calculate the probability that the particles will be detected at a selected angle, we have to take the inner product of the final state $|\psi(t_2)\rangle$ and the state describing the particles after the detection. Suppose that the detectors are located at places *A* and *B*, as depicted on Fig. 8.1. The final state after the detection should be the product of the wave function $|\psi_A\rangle$ detected at *A* and the wave function $|\psi_B\rangle$ detected at *B*. However, we have to keep in mind the physical difference between the two particles. Therefore, there are actually two ways of realizing the final scenario of detection, described by the following product states:

$$\begin{gathered} |\psi_A^a\rangle \otimes |\psi_B^b\rangle \\ |\psi_B^a\rangle \otimes |\psi_A^b\rangle. \end{gathered} \tag{8.5}$$

The first state obviously corresponds to the situation in which particle *a* was detected as coming in the direction of detector *A* and particle *b* in the direction of *B*, while the second state describes the "reversed" distribution (particle *a* in *B* and particle *b* in *A*). The probabilities of finding the system in the appropriate states above are given as follows:

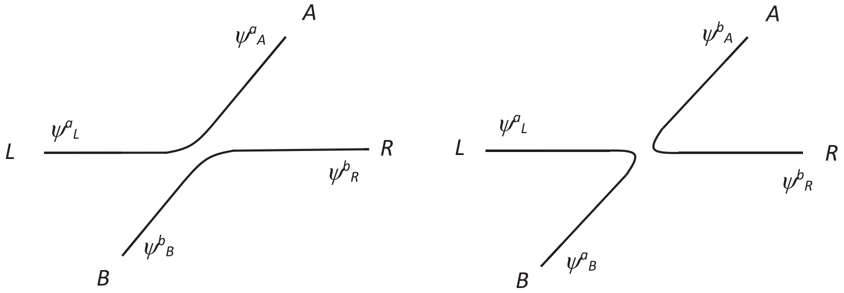


**Fig. 8.1** Two scenarios involving scatterings of distinguishable particles *a* and *b*

$$\begin{aligned} &|\langle\psi_A^a|\otimes\langle\psi_B^b|U^{ab}(t)|\psi_L^a\rangle\otimes|\psi_R^b\rangle|^2 \\ &|\langle\psi_B^a|\otimes\langle\psi_A^b|U^{ab}(t)|\psi_L^a\rangle\otimes|\psi_R^b\rangle|^2 . \end{aligned} \tag{8.6}$$

Supposing that the unitary evolution operator $U^{ab}(t)$ factorizes into the product of the components governing the evolution of particle *a* and particle *b* separately[8] (i.e. that $U^{ab}(t) = U^a(t) \otimes U^b(t)$), we can rewrite these formulas in the following way:

$$\begin{aligned} &|\langle\psi_A^a|U^a(t)|\psi_L^a\rangle\langle\psi_B^b|(U^b(t)|\psi_R^b\rangle|^2 \\ &|\langle\psi_B^a|U^a(t)|\psi_L^a\rangle\langle\psi_A^b|U^b(t)|\psi_R^b\rangle|^2 \end{aligned} \tag{8.7}$$

Numbers $\langle\psi^a{}_A|U^a(t)|\psi^a{}_L\rangle$ $\langle\psi^b{}_B|U^b(t)|\psi^b{}_R\rangle$ and $\langle\psi^a{}_B|U^a(t)|\psi^a{}_L\rangle$ $\langle\psi^b{}_A|U^b(t)|\psi^b{}_R\rangle$ are called "probability amplitudes" or "transition amplitudes". The first amplitude obviously corresponds to the transition of particle *a* from *L* to *A* and the simultaneous transition of particle *b* from *R* to *B*, while the second amplitude describes the transitions from *L* to *B* and from *R* to *A*. If we are not interested in identifying particles *a* and *b* detected in appropriate locations *A* and *B*, then in order to calculate the probability that *one* particle will be detected in *A* and the other in *B*,

[8] This assumption is not necessary for the subsequent discussion, but it simplifies appropriate formulas.

without telling which particles we have detected, we should add the probabilities in an entirely classical fashion:

$$\Pr(LR \to AB) = |\langle \psi_A^a | U^a(t) | \psi_L^a \rangle \langle \psi_B^b | U^b(t) | \psi_R^b \rangle|^2 + |\langle \psi_B^a | (U^a(t) | \psi_L^a \rangle \langle \psi_A^b | U^b(t) | \psi_R^b \rangle|^2 \quad (8.8)$$

Suppose now that the interacting particles belong to the same type—they are, for instance, two electrons. In that case the initial and the final states must be properly antisymmetrized, since these are states of two "indistinguishable" fermions. Thus the appropriate formulas will be as follows:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{2}}(|\psi_L\rangle_1 \otimes |\psi_R\rangle_2 - |\psi_R\rangle_1 \otimes |\psi_L\rangle_2)$$

$$|\psi(t_2)\rangle = U_{12}(t)\frac{1}{\sqrt{2}}(|\psi_L\rangle_1 \otimes |\psi_R\rangle_2 - |\psi_R\rangle_1 \otimes |\psi_L\rangle_2). \quad (8.9)$$

For the state after detecting particles in $A$ and $B$, the correct formula is:

$$\frac{1}{\sqrt{2}}\left(|\psi_A\rangle_1 \otimes |\psi_B\rangle_2 - |\psi_B\rangle_1 \otimes |\psi_A\rangle_2\right). \quad (8.10)$$

Assuming, as before, that the unitary evolution operator factorizes ($U_{12}(t) = U(t) \otimes U(t)$; note that the operator has to be symmetric; thus we omit superscripts), we can calculate the transition amplitude from the initial state at $t_1$ to the final state after the detection to be the following:

$$\langle \psi_A | U(t) | \psi_L \rangle \langle \psi_B | U(t) | \psi_R \rangle - \langle \psi_B | U(t) | \psi_L \rangle \langle \psi_A | U(t) | \psi_R \rangle. \quad (8.11)$$

Note that this time, in contrast to the case of distinguishable particles, there is no separation into two physical processes with two distinct amplitudes. This possibility of distinguishing the scattering when the $L$-particles ends up deflected in the $A$ direction from the scattering in which the $L$-particle is detected at $B$ is excluded from the outset by the antisymmetrization of the initial and final states. In particular we can't distinguish two possible final states after detection $|\psi_A\rangle \otimes |\psi_B\rangle$ and $|\psi_B\rangle \otimes |\psi_A\rangle$, since these kets are not permutation-invariant. Taking this into account, we can write directly the probability of detecting *any* particles in $A$ and $B$ as the square of the modulus of the probability amplitude in (8.11):

$$\begin{aligned}\Pr(LR \to AB) = |\langle\psi_A|U(t)|\psi_L\rangle\langle\psi_B|U(t)|\psi_R\rangle \\ -\langle\psi_B|U(t)|\psi_L\rangle\langle\psi_A|U(t)|\psi_R\rangle|^2\end{aligned} \tag{8.12}$$

This formula is markedly different from expression (8.8) derived in the case of distinguishable particles. Instead of adding the squared moduli of appropriate amplitudes, we add (or, in the case of fermions, subtract) the amplitudes and then square them. As a result, expression (8.12) contains the so-called interference term (the third term in the sum below), absent from the previous formula:

$$\begin{aligned}\Pr\left(LR \to AB\right) = |\langle\psi_A|U(t)|\psi_L\rangle\langle\psi_B|U(t)|\psi_R\rangle|^2 \\ +|\langle\psi_B|U(t)|\psi_L\rangle\langle\psi_A|U(t)|\psi_R\rangle|^2 \\ -2\,\mathrm{Re}\langle\psi_A|U(t)|\psi_L\rangle\langle\psi_B|U(t)|\psi_R\rangle \\ \langle\psi_B|U(t)|\psi_L\rangle^*\langle\psi_A|U(t)|\psi_R\rangle^*.\end{aligned} \tag{8.13}$$

The interference term, as the name suggests, is responsible for the interference effects, that is, the variation in the angular distribution of the probabilities for detecting the scattered particles (at some angles the probabilities are higher than in the "classical" case, and at some other angles the probabilities are lower). The presence of interference effects is a clear indication that we are dealing with same-type particles rather than particles which can in principle be discerned.[9]

Finally, we will consider the case of scattering that again involves same-type particles (electrons), but this time we will take into account their

[9] See Liu et al. (1998) for a detailed description of a real collision experiment involving electrons that gives rise to interference effects.

internal, that is, spin degrees of freedom. Moreover, one crucial assumption will be that the interaction does not affect their spins. Suppose, then, that the initial state of two incoming electrons is as follows:

$$|\psi(t_1)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |\uparrow\rangle_1 |\psi_L\rangle_1 \otimes |\downarrow\rangle_2 |\psi_R\rangle_2 \\ -|\downarrow\rangle_1 |\psi_R\rangle_1 \otimes |\uparrow\rangle_2 |\psi_L\rangle_2 \end{pmatrix}, \tag{8.14}$$

where, as always, $|\uparrow\rangle$ and $|\downarrow\rangle$ are spin states corresponding to values "up" and "down" in a particular direction. This state describes a situation in which one electron coming from the left has spin "up" in a given direction (perpendicular to the direction of motion), while the other electron coming from the right has the opposite spin "down" in the same direction. Provided that the magnetic forces during the interaction are not strong enough to "flip" the spins of the particles,[10] we assume that the evolution operator in the spin space of both electrons is the identity *I*. This means that after the interaction the state of the pair will be the following:

$$|\psi(t_2)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} |\uparrow\rangle_1 U(t) |\psi_L\rangle_1 \otimes |\downarrow\rangle_2 U(t) |\psi_R\rangle_2 \\ -|\downarrow\rangle_1 U(t) |\psi_R\rangle_1 \otimes |\uparrow\rangle_2 U(t) |\psi_L\rangle_2 \end{pmatrix}. \tag{8.15}$$

Since now there are two experimentally distinguishable ways of detecting scattered particles (one with spin "up" at location *A* and spin "down" at *B* and the other with the spins switched), we have to write down two final states and calculate two probabilities:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\psi_A\rangle_1 \otimes |\downarrow\rangle_2 |\psi_B\rangle_2 - |\downarrow\rangle_1 |\psi_B\rangle_1 \otimes |\uparrow\rangle_2 |\psi_\mathrm{A}\rangle_2),$$
$$\frac{1}{\sqrt{2}}(|\downarrow\rangle_1 |\psi_A))_1 \otimes |\uparrow\rangle_2 |\psi_B\rangle_2 - |\uparrow\rangle_1 |\psi_B\rangle_1 \otimes |\downarrow\rangle_2) |\psi_\mathrm{A}\rangle_2), \tag{8.16}$$

[10] See the discussion of this assumption in Feynman et al. (1965, chap. 3). As I point out in Bigaj (2020, p. 14), this assumption does not have to be realistic for the argument to go through.

Given the orthonormality relations between spin states $|\uparrow\rangle$ and $|\downarrow\rangle$, the corresponding transition amplitudes between the initial state and each of the above states will turn out to be as follows:

$$\langle\psi_A|U(t)|\psi_L\rangle\,\langle\psi_B|U(t)|\psi_R\rangle,$$
$$\langle\psi_B|U(t)|\psi_L\rangle\,\langle\psi_A|U(t)|\psi_R\rangle, \tag{8.17}$$

from which we can calculate the total probability of registering any particles coming in the $A$ and $B$ directions while disregarding their spins:

$$\Pr(LR \rightarrow AB) = |\langle\psi_A|U(t)|\psi_L\rangle\langle\psi_B|U(t)(\psi_R\rangle|^2 + |\langle\psi_B|U(t)|\psi_L\rangle\langle\psi_A|U(t)|\psi_R\rangle|^2\,. \tag{8.18}$$

Observe that this formula is identical to the probability formula (8.8) derived in the case of "distinguishable" particles (barring the insignificant absence of superscripts referring to individual particles). That is, there is no interference term in this expression, in contrast to formula (8.13). Consequently, no interference effects can be experimentally observed in this case. It seems that the presence of spins which in principle enable us to identify the direction from which a detected particle has arrived has the effect of erasing the interference pattern that would otherwise characterize the scattering of two same-type particles.

## 8.4 Diachronic Identity in Scatterings: Orthodoxy Versus Heterodoxy

Let us now analyze the mathematical results laid out in the previous section, taking into account the two competing approaches to quantum individuation. We will focus our discussion on the possibility of making definite diachronic identifications between particles pre- and post-interaction in each of the above-analyzed scenarios. Recall that we have formally described three cases of scattering interactions: one involving "distinguishable" particles and two concerning particles of the same type. The derived formulas for transition amplitudes, and consequently for the probabilities of detecting particles scattered at a certain angle, show that in the case of distinguishable particles there are no interference effects, and the total probability equals the sum of the probabilities describing alternative "trajectories". On the other hand, scatterings involving same-type particles with no identifying features clearly show the presence of interference effects, since the total probability is not equal the sum of the squared moduli of appropriate amplitudes. Finally, the formal analysis of the case of same-type particles equipped with identifying properties (distinguishable properties that are preserved during the particles' interaction) implies that the interference effects disappear, and the resulting probability formula is structurally identical to the formula derived in the case of distinguishable particles.

The case of distinguishable particles does not present us with any major difficulty regarding the in-principle possibility of identifying particles before and after interaction, due to the existence of state-independent properties that uniquely characterize the interacting particles. The problem arises when we consider same-type particles. Let us first assume the orthodox approach to quantum individuation. In that case the indices 1 and 2 occurring in formulas (8.9), (8.10), (8.14), (8.15) and (8.16) play the referential role, picking out totally indiscernible elements of the composite system of interacting particles. Thus the question of diachronic identity can be formulated as follows: "Is particle 1 (or 2), which constitutes an element of the system after interaction and detection, identical with particle 1 (or 2) before interaction?". And it should be clear that because of the symmetry of the initial and final states, any conceivable argument in favor of one particular identification can be turned into an argument for an alternative identification. We have already seen at the end of Sect. 8.1 that orthodoxy relinquishes diachronic identifications *en bloc*, regardless of the details of the temporal evolution of the system. Thus for the proponent of orthodoxy cases of "indistinguishable" particles with and without identifying features are treated equally as instances where transtemporal identities fail.[11]

The unorthodox approach, on the other hand, differentiates between the two considered cases of same-type particles' scattering. In the first discussed scenario, the initial state of two electrons given in (8.9) enables us to individuate synchronically the particles thanks to the fact that this state is an eigenstate for the projector $E_L \otimes E_R + E_R \otimes E_L$, where $E_L$ projects onto the ray spanned by $|\psi_L\rangle$ and $E_R$ onto the ray spanned by $|\psi_R\rangle$. To put it simply, we are allowed to distinguish here one particle that is coming from the left (associated with the wave packet $|\psi_L\rangle$), and one coming from the right (described by $|\psi_R\rangle$). Similarly, the final state (8.10) after detection describes two particles individuated by their respective wave packets $|\psi_A\rangle$ and $|\psi_B\rangle$ ("coming towards $A$" and "coming towards $B$"). But now the question arises whether we have sufficient grounds to connect particles individuated qualitatively after detection with particles from before interaction. And it seems clear that no such grounds could be given. It is equally plausible (or implausible) to insist that the particle associated with the wave packet $|\psi_A\rangle$ came from the left, as to claim that it came from the right.

More specifically, we can notice that no criterion of transtemporal identity listed at the beginning of Sect. 8.2 can be applied in the current scenario. No state-dependent property can connect particles described by wave packets $|\psi_A\rangle$ and $|\psi_B\rangle$ with particles whose initial states are $|\psi_L\rangle$ and $|\psi_R\rangle$. As for the criterion based on the continuity of orbits, it can be employed in the current case as long as we are limiting ourselves to the unitary evolution of the system. That is, we can easily link the particles occupying state $|\psi(t_1)\rangle$ with those jointly characterized by state $|\psi(t_2)\rangle$ as given in (8.9): the particle individuated at $t_2$ as occupying state $U(t)|\psi_L\rangle$ will be diachronically identified with the particle associated with $|\psi_L\rangle$ at $t_1$, and analogously for the second particle whose orbit connects $|\psi_R\rangle$ and $U(t)|\psi_R\rangle$. However, after detection, the wave functions $U(t)|\psi_L\rangle$ and $U(t)|\psi_R\rangle$ (which typically will have the form of spherical waves spreading

[11] This argument against the possibility of diachronic identifications for objects that are synchronically indistinguishable is similar to the argument against the possibility of history-based individuation of objects that share their momentary properties, as presented in Cortes (1976, pp. 503–504).

in all directions from the point of collision) will collapse onto wave packets $|\psi_A\rangle$ and $|\psi_B\rangle$ describing particles propagating in specific directions where the detectors have been placed. From this moment on the possibility of identifying the detected particles with the original ones is lost—no continuity is preserved. It is fundamentally impossible to find out whether the particle associated with the wave packet $\psi_A$ after detection came from the particle individuated before measurement as occupying $U(t)|\psi_L\rangle$ or $U(t)|\psi_R\rangle$.

The situation is different with respect to the second scenario involving "indistinguishable" particles. Here we can resort to the criterion of trans-temporal identity that is based on the preservation of some state-dependent properties. Recall, first, that there are two possibilities with respect to the final state of the scattered particles after they have been detected at locations *A* and *B*, as written in formulas (8.16). According to the heterodox approach, these two possibilities can be described as follows. After measurement we can have either one particle with spin "up" scattered in the direction of *A* and one particle with spin "down" moving towards *B*, or one particle with spin "down" at *A* and one with spin "up" at *B*. Given that the interaction does not affect spins, it is most natural to identify particles according to their spin. That is, in the first case the particle detected at *A* (with spin up) should be identified with the one coming from the left, while the *B*-particle (with spin down) can be traced back to the one coming from the right. Similarly in the second case, except now the identifications will be switched. Note that the appropriate diachronic identities can be assumed to hold even if for any reasons we are not able to verify which scenario has been actualized. If we do not measure the spin of the outgoing particles, we lose the ability to tell whether the measured particles came from the left or the right, but this is purely an epistemic issue, not an ontological one.

The key experimentally verifiable difference between the two cases of same-type particles' scattering is the presence or absence of interference patterns. The heterodox approach to individuation offers a simple and convincing explanation of this difference: interference occurs only when particles lose their diachronic identity, and consequently the total transition amplitude (8.11) is the sum of transition amplitudes associated with separate channels. On the other hand, the absence of interference effects in scatterings with identifying features is accounted for by assuming that particles retain their identities throughout the entire process, including detection, and that consequently there are two ways the scatterings can develop, described by independent probabilities which must be added to obtain the total probability formula (8.18). No explanation of this kind is available to the proponent of orthodoxy, though. In both cases, whether with or without identifying features, the particles lack diachronic identities, so the presence or absence of interference effects must be treated as brute facts, following from the formalism but not connected with some deeper ontological features of the involved processes. I believe that this inability to account for the clearly visible patterns in scattering processes gives an additional argument against orthodoxy and in favor of the heterodox approach to individuality. Moreover, orthodoxy has nothing to say regarding the conspicuous formal correspondence between formula (8.8) derived in the case of "distinguishable" particles and formula (8.18) describing same-type particles with identifying features. The heterodox approach, on the other hand, takes this close formal analogy as a clear indication that diachronic identities are preserved in both scenarios.

## 8.5 Identity Across Possible Worlds

We have already considered two types of identity statements: synchronic and diachronic. With respect to these statements, the heterodox approach to quantum individuation departs from the orthodoxy in that in some cases it enables us to distinguish qualitatively objects that are synchronically distinct and to make definite diachronic identifications. However, there is a third category of identity statements that we have not yet discussed, namely counterfactual identity statements. Counterfactual identity connects objects described in alternative scenarios, where those scenarios often bear the name "possible worlds" (hence other optional terms for counterfactual identity are "transworld identity" or "identity across possible worlds"). For an individual object it generally should make sense to formulate statements that ascribe to it possible but not actually possessed properties. For instance, we may wonder whether it was possible for Niels Bohr to be a pianist rather than a physicist. The intelligibility of such a question presupposes that there are possible worlds containing someone who stands in the relation of counterfactual identity to the actual Bohr.

The exact nature of the relation of transworld identity is hotly debated by philosophers. Some, following Saul Kripke, insist that transworld identity is just numerical identity (Kripke 1980). Others, like David Lewis, question the very idea of objects that occupy more than one world. To emphasize that counterfactual identity is not numerical identity, Lewis uses the neutral term "counterpart" (Lewis 1968, 1986). What is important in Lewis's analysis is that the counterpart relation need not have the formal features of numerical identity. In particular, an actual object can have more than one counterpart in a given possible world, and several actual objects can share their otherworldly counterparts.

The question we would like to consider now is whether the two competing conceptions of individuation make it possible to speak intelligibly about quantum objects remaining the same in counterfactual scenarios. Since we have two broad philosophical conceptions of counterfactual identity, generally there are four possibilities regarding the conceptual framework in which we can ask the aforementioned question (below we will add to that two more options). Suppose, first, that we follow Kripke and his conception based on stipulating which actual objects exist in the counterfactual scenarios under consideration. For Kripke it is perfectly admissible to characterize possible worlds not merely qualitatively, but individualistically as well. That is, we can take any actual object *o* possessing some property *P* and consider an alternative scenario described as containing the very same object *o* possessing some other property *Q* rather than *P*. The only limitation of such a procedure adopted by Kripke is that the selected object cannot "lose" any of its essential properties. For instance, an alternative scenario in which Niels Bohr is a tree is a metaphysical (or conceptual) impossibility, since plausibly one of his essential properties is being a human. In the case of quantum particles, we will assume, as seems natural, that their essences are constituted by state-independent properties characterizing a particular type of particles (rest mass, electric charge, total spin, etc.). Thus it is conceptually impossible to consider, for example, a counterfactual scenario in which an electron has a positive charge.

Now, let us consider an example involving two fermions (e.g. electrons) occupying jointly some antisymmetric state, such as the well-known state:

$$\frac{1}{\sqrt{2}}\left(|\lambda_a\rangle_1|\lambda_b\rangle_2 - |\lambda_b\rangle_1|\lambda_a\rangle_2\right). \tag{8.19}$$

According to the orthodoxy, labels "1" and "2" refer to two otherwise qualitatively indiscernible electrons. Our testing ground for the possibility of definitive counterfactual identifications will be the thought experiment of switching the electrons (so-called counterfactual switching in Paul Teller's terminology—see Teller 2001). Is it intelligible to consider the case in which the two electrons, bearing their respective labels, have been swapped? Does that lead to the creation of a new scenario distinct from the original one?[12] In Kripke's approach, we are generally allowed to consider an alternative scenario (a possible world) whose description contains individual names (rigid designators) "1" and "2". However, the physical description of the electrons occupying state (8.19) after the supposed switching will be identical with the original one, both qualitatively and individualistically. Since before the switching both particles 1 and 2 occupy the same reduced state, the switching does not change this a bit. We are still in the same possible world, so to speak. A real alternative may be, for instance, the situation in which both electrons occupy a different antisymmetric state, but the difference will be "global" (with respect to the entire joint state) and not specifically between particle number 1 and particle number 2. Thus in the orthodox approach the counterfactual identification of separate particles becomes an impossibility. The particles are truly non-individuals.

In the heterodox approach the outcome of the counterfactual switching is different. Here in the actual world the particles can be differentiated not by their labels but by their qualitative properties represented by orthogonal projectors $E_a$ and $E_b$. In other words, we can individuate the

[12] Recall that we have already encountered a similar question in Sect. 2.8, when we discussed quantum statistics and the problem of how to count distinct distributions of particles over available states. See also Sect. 3.1 and the analysis of the notion of permutation therein.

particles using qualitative descriptions. Each description can subsequently serve as a rigid designator, picking out in any possible world the object that in the actual world satisfies the description. For the purpose of keeping track of the actually discerned particles, we can introduce new labels "*a*" and "*b*" such that each label refers to the same particle across all possible worlds. In the actual world the following statements are true: "Particle *a* possesses the property corresponding to $E_a$" and "Particle *b* possesses the property corresponding to $E_b$". But in a "switched" world, the opposite holds: it is particle *b* that possesses property $E_a$ and particle *a* that is characterized by $E_b$. Thus the switching creates a new possible world, distinct from the original one. However, the difference is purely individualistic, not qualitative. Both the actual world and the switched world are qualitatively indistinguishable, meaning that all true statements that do not contain labels "*a*" and "*b*" are identical in both worlds. Consequently, in Kripke's approach to modality de re, we have to admit so-called haecceitistic differences, not grounded in any qualitative facts. This is a high price to be paid for the possibility of speaking intelligibly about the counterfactual switching of quantum particles.

On the other hand, Lewis famously renounces haecceitistic differences between possible worlds. In his approach possible worlds are to be characterized purely qualitatively, and hence the relation of transworld identity must be determined qualitatively as well. A counterpart of a given actual object is any object that is "sufficiently similar" to it (Lewis 1968). This vague specification is not particularly helpful in the current context; however, we can make it more precise. Again, we can resort to the notion of essential properties and our adopted interpretation of essences in terms of state-independent quantities. Hence a counterpart of a given actual electron in a possible world is any particle that possesses all its state-independent properties—in short, any electron. Note that under this interpretation the counterpart relation becomes many-to-many, rather than one-to-one, as is the case with numerical identity. All actual electrons in the universe share all their counterparts in the form of all electrons in any possible universe. Any attempt to avoid this consequence by expanding the set of essential properties for individual electrons will inevitably lead to unintuitive consequences. If we insisted that some state-dependent properties (such as spin components in particular directions) should be counted among the essential properties of a given electron, this would imply that this electron cannot change its current state without losing its identity.

Given the above-mentioned assumptions of Lewis's original counterpart theory, we can now evaluate the counterfactual switching experiment under both approaches to quantum individuation. The orthodoxy clearly implies the impossibility of substantial counterfactual switching (in the sense of creating a new, numerically distinct scenario), since the two electrons occupying an antisymmetric state are totally indiscernible with respect to all their properties.[13] How about the heterodox conception of individuation? While electrons occupying the initial state (8.19) are qualitatively discernible, the properties enabling us to discern them in the actual world (represented by projectors $E_a$ and $E_b$) are not sufficient to identify them in alternative scenarios, since these properties are state-dependent. With respect to their essences, both electrons are identical. Thus the switched electrons scenario is qualitatively indiscernible from the original one, and since Lewis does not admit haecceististic (individualistic) differences between possible worlds, the switching does not produce a new possibility.

It has to be added, though, that Lewis in his later variant of the counterpart theory (known as "cheap haecceitism") makes room for admitting possibilities that do not give rise to any qualitative difference (Lewis 1986, pp. 230–235). The way to do that is through the assumption that objects can have counterparts other than themselves in the actual world. Thus, for any given electron, all other electrons in our universe are its counterparts. The motivation behind this stipulation is that actual counterparts can represent the possibility of counterfactual switching even if this switching does not produce any qualitative differences. According to cheap haecceitism, there are two ways to represent alternative, possible scenarios. One is the standard interpretation in the form of possible worlds, which have to be qualitatively distinct from the actual world. But another option is to represent the possibility of the counterfactual

[13] Matteo Morganti argues for the impossibility of counterfactual switching on the basis of quantum holism. See Morganti (2009, 2013, pp. 55–56).

switching of two actual objects *a* and *b* in the form of the pair (*a*, *b*), as long as *a* is a counterpart of *b*.

I will not discuss here the merits and demerits of cheap haecceitism (but see Fara 2009 for some criticism). This view strikes me though as slightly disingenuous, since it admits distinct *possibilities* that do not differ qualitatively while refusing to do the same with respect to *possible worlds*. Moreover, it seems that in this approach the distinctness of possibilities is ultimately grounded in the numerical distinctness of counterparts *a* and *b*, and this violates the idea that the fundamental facts about physical reality should be qualitative. It is true that for Lewis the counterpart relation is qualitative, but this should not obscure the fact that according to cheap haecceitism, non-qualitative numerical diversity plays part in distinguishing various possibilities.

Another reason for being skeptical regarding the assumption of cheap haecceitism in the context of the counterfactual switching of elementary particles is that this approach turns out to be even more problematic from an empirical point of view than Kripke's stipulation-based essentialism. For Kripke the counterfactual switching of two completely indiscernible objects is an impossibility, since the result will be both qualitatively and individualistically indistinguishable from the original one.[14] But Lewisian cheap haecceitism admits the possibility of counterfactual switching even in the case of completely indiscernible objects, since this possibility is grounded in the fact of numerical distinctness of these objects. Consequently, cheap haecceitism turns out to be in a sense even more extremely haecceitistic than Kripke's variant of haecceitistic essentialism.

The options discussed above are concisely summarized in Table 8.1. In conclusion, we can say that while the heterodox approach to individuality makes it possible to admit that counterfactual switching creates a new, distinct possibility, this requires an introduction of decidedly unempirical elements (haecceities, bare numerical diversity). In my opinion the most attractive option for a scientifically and naturalistically oriented metaphysician is to accept essentialism in the version associated with

[14] That is, any sentence expressed with the help of individual constants (names) true in the original world will remain true in the "switched" world.

**Table 8.1** The possibility of counterfactual switching in various approaches to quantum individuation and modality de re

| | | Lewis's counterpart theory | |
|---|---|---|---|
| | Kripke's theory | Original | Cheap haecceitism |
| **Orthodoxy** | No counterfactual switching | No counterfactual switching | Counterfactual switching possible |
| **Heterodoxy** | Counterfactual switching possible | No counterfactual switching | Counterfactual switching possible |

Lewis's original counterpart theory. Once we adopt this metaphysical stance, the inevitable consequence is that counterfactual identity is not well defined (i.e. unique) for quantum particles. There is no scientifically justified way to pick out any electron in an alternative scenario as *the* electron that we have singled out in the actual world using a legitimate experimental procedure. Even though according to the heterodoxy all electrons are synchronically discernible by some of their actually and contingently possessed properties, electrons are *modally* indiscernible in the sense that everything that is possible about one electron is possible about any other electron. Electrons are not individuals in a deeper, modal sense.

## 8.6 Towards the Metaphysics of Quantum Objects

Where does all this leave us? Let us take stock first. Limiting ourselves to fermions, we have established the following facts, given the heterodox approach to individuation. For any system of many fermions of the same type, it is always possible to decompose it at any moment into separate particles with distinguishing measurable properties represented by orthogonal projectors. However, as we have noted in the previous chapter, this decomposition is relativized to a choice of an orthogonal basis (or, equivalently, a maximal set of compatible observables). In the majority of cases, some of these choices can be seen as "better" or more "natural" than others, but it is unclear whether this pragmatic preference (typically based on the privileged status of position) is underlain by a genuine ontological difference.

Setting the relativization problem aside, the next question is whether the components of a fermionic system individuated synchronically can retain their unique identities throughout time and in counterfactual scenarios. With respect to transtemporal (diachronic) identity, the situation is not entirely clear-cut. Generally, particles individuated at different times lose their specific identities in that it is fundamentally impossible to connect them via an empirically accessible relation of continuation. However, in some special cases such identifications across temporal moments seem possible—these cases involve interactions of particles which preserve some of their state-dependent properties. Moreover, there is an empirical criterion which enables us to distinguish between cases with and without transtemporal identity. This criterion relies on the interference effects, which disappear when particles retain their diachronic identities. As for the notion of counterfactual identity, under the most natural philosophical analysis of modality de re, quantum particles of the same type do not appear to possess their unique identities which could single them out in possible scenarios. Consequently, the possibility of a substantial counterfactual switching is excluded, regardless of the issue of contingent synchronic discernibility in the actual world.

Is it possible to develop a metaphysics of quantum objects which would account for the above-mentioned features of quantum particles? But what do we mean by a metaphysics of objects (whether quantum or not)? One possible idea is to give a characterization of a particular category of objects using standard concepts and tools from the toolbox of analytic metaphysics, so that the resulting metaphysical theory of objects will have the required consequences. An example of such a theory can be provided by the conception of quantum objects proposed and developed by Olimpia Lombardi, Newton da Costa and others (see Lombardi and Castagnino 2008; da Costa et al. 2013; da Costa and Lombardi 2014; Lombardi and Dieks 2016). This conception takes its cue from the bundle theory of objects which reduces objects to bundles of properties. It has to be noted that this proposal is developed within the orthodox approach to quantum individuation with its central thesis of absolute indiscernibility and the ensuing collapse of the notion of an individual.

Given that, it may come as a slight surprise that Lombardi et al. decided to turn to the metaphysical conception of objects as bundles of properties, since one of the widely discussed objections to this conception is that it makes the Principle of the Identity of Indiscernibles trivially true.[15] Clearly, two sets containing the same elements (in this case properties) are identical thanks to the principle of extensionality. This seems to exclude the possibility of identifying quantum particles with the sets of appropriate properties, even in the heterodox approach (let alone the orthodox one).

In fact, the proposed ontology is not exactly the bundle theory in its original version. Rather, it is a property-based ontology in which the fundamental objects are properties (divided into type-properties and case-properties), and the fundamental facts about these properties involve their instantiations. The cases of two or more particles occupying symmetric or antisymmetric states are described as cases of multiple instantiations of the same properties (this situation is also alternatively described as an "aggregation" or "merging" of bundles). Since Lombardi et al. insist that bundles are not individuals (and therefore the PII does not apply to them, instead of being simply false), it seems to me that ultimately the proposed ontology is not so much the ontology of objects as bundles of properties but of properties themselves. The fundamental objects are just properties, and their "bundling" together plays a secondary role. Consequently, the issue of the reidentification of particles over periods of time does not arise in this approach, since properties are not temporal objects, and their instantiations at particular moments belong to the category of "occurrents", that is, momentary objects which do not persist.[16]

My proposed approach to the construction of an appropriate metaphysics of quantum objects is more conservative. I suggest keeping the concept of a quantum object central in the theory, whether considered

[15] See, for example, van Cleve (1985). However, see an argument that the bundle theory can be reconciled with the falsity of PII in Rodriguez-Pereyra (2004).

[16] Another proposal for an ontological theory of quantum particles as bundles of properties is outlined in Friebe (2014). It is notable that Friebe formulates his proposal in the context of the GMW conception of entanglement (as presented in Chap. 6), and thus broadly in terms of what we call the heterodox approach (even though he does not make the distinction between two alternative conceptions of individuation in the quantum theory of many particles). A detailed critical analysis of Friebe's approach has to be left for another occasion.

primitive or reducible to some constructs of other entities (properties, etc.). That is, I do not seek an eliminative conception which gets rid altogether of objects and interprets the ordinary object-based discourse in an alternative language with a different underlying ontology.[17] The postulate of "conservatism within limits" is one of the desiderata that I propose to impose on any quantum metaphysics. Other desiderata follow from the established facts regarding various types of identity (synchronic, diachronic, counterfactual) and (in)discernibility applied to quantum objects. Thus, we need a metaphysical theory of objects that makes it possible to speak about distinct entities possessing differentiating properties at the same moment, but also leaves room for cases in which distinct particles (i.e. bosons) become entirely indiscernible. The sought-after theory should also account for the loss of counterfactual identity, that is, it should explain why it is so that counterfactual switching is not a genuine possibility. The final, and probably the hardest to satisfy, desideratum concerns diachronic identity. The metaphysical conception of objects we are after should be capable of admitting that in some cases transtemporal identity of objects is preserved, while in other cases it is lost. Moreover, as we have pointed out, the loss of diachronic identity is often associated with a peculiar merger (or confluence) of the identities of objects involved in mutual interactions. Thus we need to be able to explicate in basic ontological terms a scenario in which we have a composite system consisting of a number of individual components, such that the system as a whole retains its identity over a period of time, whereas each individual element of the system at a later moment is somehow genetically connected but not fully and uniquely identical with the components at an earlier time.

I have to admit that at this point I am unable to offer a fully developed metaphysical conception that would satisfy the desiderata listed above. My impression is that no standard approach known from the philosophical literature can do the required job. In particular, straightforward variants of the bundle theory of objects seem inadequate, since from the

[17] Thus I am not a proponent of one of many eliminative ontologies of the physical world that dispense with the concept of a physical object as a fundamental entity, replacing it, for instance, with more fundamental abstract structures. See French (1998), French and Ladyman (2003, 2011) and Rickles and Bloom (2016).

outset they struggle with the issue of persistence in the light of the qualitative changes that temporal objects undergo. We need a new and fresh approach that can perhaps begin in the form of a set of axioms (formal characteristics). It may turn out to be necessary to expand the list of primitive predicates to include, for instance, a relation of "partial genidentity" that can connect different stages of the components of the same composite system. Contrary to what we have stipulated in Sect. 8.1, diachronic identity may turn out not to be numerical identity after all. Instead, it may be advisable to resort to the ontology of momentary objects, or more specifically momentary decompositions of a persisting composite system, which can be connected by various transtemporal relations similar but not fully equivalent to genidentity.

Let us try to take a few tentative steps in the direction suggested above. Let variables *a*, *b*, *c*… range over the set of objects persisting in time, and let variables *x*, *y*, *z*, … range over momentary objects (objects existing at precise moments). The set of primitive binary predicates $S_t(x, a)$ indexed by temporal moments *t* will denote the relation of being a temporary slice of an object (*x* is a temporal slice of *a* at *t*). Another primitive predicate will be $P_t(x, a)$, which denotes the relation of being a part of *a* at a given moment *t* (being a component of *a* at *t*). The binary predicate $GI(x, y)$ will refer to the relation of "full" genidentity between two momentary entities. A straightforward postulate connects predicates *GI* and $S_t$:

$$GI(x,y) \rightarrow \exists a\left[S_{t1}(x,a) \wedge S_{t2}(y,a)\right], \tag{8.20}$$

which means that if momentary objects are linked by the relation of genidentity, they are temporal slices of the same persisting entity.[18] Another important predicate represents the uniquely quantum relation of "partial" genidentity that we can symbolize as $PI(x, y)$. Partial genidentity becomes full genidentity if the following uniqueness condition is satisfied:

[18] It is quite possible that we could use equivalence here instead of implication, which would mean that the relation denoted by *GI* is actually definable in terms of $S_t$.

$$\left\{PI(x,y)\wedge E_t(y)\wedge\forall z\left(E_t(z)\wedge PI(x,z)\right)\rightarrow z=y\right\}\rightarrow GI(x,y) \qquad (8.21)$$

where $E_t(x)$ means that $x$ exists at time $t$. In other words, if an object $x$ is partially genidentical with exactly one object $y$, they are genidentical *simpliciter*.

The case of two-particle interactions with a loss of identity presented in Sect. 8.3 can be formally described in our language as follows:

$$\begin{aligned}
&P_{t_1}(x_1,a)\wedge P_{t_1}(y_1,a)\wedge x_1\neq y_1\wedge\forall z\left[P_{t_1}(z,a)\rightarrow z=x_1\vee z=y_1\right],\\
&P_{t_2}(x_2,a)\wedge P_{t_1}(y_2,a)\wedge x_2\neq y_2\wedge\forall z\left[P_{t_2}(z,a)\rightarrow z=x_2\vee z=y_2\right],\\
&PI(x_1,x_2)\wedge PI(x_1,y_2)\wedge PI(y_1,x_2)\wedge PI(y_1,y_2).\\
&\neg\exists b\left[S_{t_1}(x_1,b)\wedge S_{t_2}(x_2,b)\right]\wedge\neg\exists b\left[S_{t_1}(x_1,b)\wedge S_{t_2}(y_2,b)\right]\wedge\\
&\neg\exists b\left[S_{t_1}(y_1,b)\wedge S_{t_2}(x_2,b)\right]\wedge\neg\exists b\left[S_{t_1}(y_1,b)\wedge S_{t_2}(y_2,b)\right].
\end{aligned} \qquad (8.22)$$

The first two formulas in (8.22) indicate that a persisting object $a$ (intuitively, the composite system of same-type particles) consists of exactly two "momentary" objects at two times,[19] while the third formula expresses the fact that each particle at one moment can be "partially" traced back to either particle at an earlier moment. In the final formula, we stress that there are no persistent components of the entire system that retain their identities in the period between $t_1$ and $t_2$.

Much more needs to be done in order to come up with a complete metaphysical theory of objects implied by quantum mechanics. In particular, the issue of counterfactual identity, presumably cast in terms of essential properties, has to be introduced into the formalism.[20] This task

[19] In a fully developed metaphysical theory of quantum particles, we would have to find a way to indicate that the existence of momentary objects $x_i$ and $y_i$ composing system $a$ at a time $t_i$ is relativized to the choice of an orthogonal basis.

[20] Apart from that we have to solve the following conceptual conundrum: how is it possible that object $a$ (the system of two particles) retains its identity while its components do not? The whole temporal slices of $a$ at times $t_1$ and $t_2$ should be connected by the relation of full genidentity $GI$, and yet the spatial parts of these slices are linked by mere partial genidentities $PI$. Clearly, some intuitions regarding the relations between components and their diachronic identities must be abandoned. It is advisable to spell out these intuitions and show precisely why they fail.

must be left for another occasion. However, I would like to stress that even though the metaphysical conception that emerges from the above preliminary formalizations looks decisively non-classical (due to the introduction of the relation of partial genidentity), the whole theory is cast in perfectly classical logic with no need to abandon the fundamental logical laws such as the law of self-identity $\forall x\, x = x$. It seems to me that this is the right way to do things in the context of quantum mechanics. While quantum particles are definitely not individuals in the classical sense, their non-individuality can hopefully be expressed without sacrificing classical logic that has served us well for two millennia.

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# Appendix: Basic Concepts of the Quantum-Mechanical Formalism

The fundamental notion used in the mathematical formalism of quantum mechanics is that of a vector space.

(A.1) $\mathscr{V}$ is a vector space over a field $\mathscr{F}$ (a set of scalars) if $\mathscr{V} = \langle V, +, \cdot, \mathbf{0} \rangle$, where $V$ – a non-empty set, $+: V \times V \to V$, $\cdot: \mathscr{F} \times V \to V$, $\mathbf{0} \in V$ such that for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathscr{F}$,

$\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$

$\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$

$\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$

$(ab) \cdot \boldsymbol{u} = a \cdot (b \cdot \boldsymbol{u})$

$a \cdot (\boldsymbol{u} + \boldsymbol{v}) = a \cdot \boldsymbol{u} + a \cdot \boldsymbol{v}$

$(a + b) \cdot \boldsymbol{u} = a \cdot \boldsymbol{u} + b \cdot \boldsymbol{u}$

$0 \cdot \boldsymbol{u} = \mathbf{0}$

$1 \cdot \boldsymbol{u} = \boldsymbol{u}$

In quantum-mechanical applications, $\mathscr{F}$ is assumed to be the field of complex numbers C. On space $\mathscr{V}$ we can define an important operation of the *inner product* of vectors. The result of the inner product of two vectors $\boldsymbol{u}$ and $\boldsymbol{w}$, written as $\langle \boldsymbol{u} | \boldsymbol{v} \rangle$, is a complex number: $\langle \boldsymbol{u} | \boldsymbol{v} \rangle \in$ C. The inner product satisfies the following conditions:


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$$\langle \boldsymbol{v}|\boldsymbol{v}\rangle \geq 0 \text{ (positivity)} \quad (A.2)$$

$$\langle \boldsymbol{v}|\boldsymbol{v}\rangle = 0 \text{ iff } \boldsymbol{v} = \mathbf{0} \quad (A.3)$$

$$\langle \boldsymbol{u}|a\boldsymbol{v}\rangle = a\langle \boldsymbol{u}|\boldsymbol{v}\rangle \text{(linearity in second argument)} \quad (A.4)$$

$$\langle \boldsymbol{u}|\boldsymbol{v}+\boldsymbol{w}\rangle = \langle \boldsymbol{u}|\boldsymbol{v}\rangle + \langle \boldsymbol{u}|\boldsymbol{w}\rangle \text{(linearity in second argument.)} \quad (A.5)$$

$$\langle \boldsymbol{u}|\boldsymbol{w}\rangle = \langle \boldsymbol{w}|\boldsymbol{u}\rangle^{*} \quad (A.6)$$

The asterisk indicates the operation of taking the complex conjugate, which transforms any complex number $a + ib$ into $(a + ib)^{*} = a - ib$. From conditions (A.4), (A.5) and (A.6), it follows that the inner product is *antilinear* in the first argument, meaning that $\langle \boldsymbol{u} + \boldsymbol{v}|\boldsymbol{w}\rangle = \langle \boldsymbol{u}|\boldsymbol{w}\rangle + \langle \boldsymbol{v}|\boldsymbol{w}\rangle$ and that $\langle a\boldsymbol{u}|\boldsymbol{v}\rangle = a^{*}\langle \boldsymbol{u}|\boldsymbol{v}\rangle$. Vector spaces equipped with the above-defined inner product and satisfying the condition of completeness (any converging set of vectors has a limit) are called Hilbert spaces.

Vector space $\mathcal{V}$ is said to be spanned by vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, if every vector $\boldsymbol{u}$ in $\mathcal{V}$ can be presented as a linear combination $\boldsymbol{u} = c_1\boldsymbol{e}_1 + \ldots + c_n\boldsymbol{e}_n$. The set $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ spanning $\mathcal{V}$ forms an *orthonormal basis for* $\mathcal{V}$ if $\langle \boldsymbol{e}_i|\boldsymbol{e}_j\rangle = 0$ when $i \neq j$ and $\langle \boldsymbol{e}_i|\boldsymbol{e}_j\rangle = 1$ when $i = j$. In that case every vector $\boldsymbol{u}$ has a unique decomposition in the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of the form $\langle \boldsymbol{e}_1|\boldsymbol{u}\rangle\, \boldsymbol{e}_1 + \ldots + \langle \boldsymbol{e}_n|\boldsymbol{u}\rangle\, \boldsymbol{e}_n$. Number $n$ is called the dimensionality of $\mathcal{V}$.

Let us take two vector spaces: $\mathcal{V}$ with an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and $\mathcal{W}$ with an orthonormal basis $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_m$. The direct (tensor) product of $\mathcal{V}$ and $\mathcal{W}$ (written $\mathcal{V} \otimes \mathcal{W}$) is formally defined as a $n \times m$ dimensional vector space spanned by ordered pairs $(\boldsymbol{e}_i, \boldsymbol{f}_j)$. In addition, it is stipulated that the pair $(\boldsymbol{u}, \boldsymbol{v})$, where $\boldsymbol{u} = \sum_i c_i\boldsymbol{e}_i$ and $\boldsymbol{v} = \sum_j d_j\boldsymbol{f}_j$, represents the "multiplicative" combination $\sum_{ij} c_i d_j(\boldsymbol{e}_i,\boldsymbol{f}_j)$. The inner product of two vectors $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{w}, \boldsymbol{z})$ in $\mathcal{V} \otimes \mathcal{W}$ equals $\langle \boldsymbol{u}|\boldsymbol{w}\rangle\langle \boldsymbol{v}|\boldsymbol{z}\rangle$. The direct sum of $\mathcal{V}$ and $\mathcal{W}$ (written $\mathcal{V} \oplus \mathcal{W}$) is a $n + m$ dimensional vector space spanned by pairs $(\boldsymbol{e}_i, 0)$ and $(0, \boldsymbol{f}_j)$. The pair $(\boldsymbol{u}, \boldsymbol{v})$ represents the "additive" combination $\sum_i c_i(\boldsymbol{e}_i, 0) + \sum_j d_j(0,\boldsymbol{f}_j)$. The inner product of $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{w}, \boldsymbol{z})$ in $\mathcal{V} \oplus \mathcal{W}$ is defined as $\langle \boldsymbol{u}|\boldsymbol{w}\rangle + \langle \boldsymbol{v}|\boldsymbol{z}\rangle$.

In quantum mechanics it is standard to adopt Dirac's notation, where vectors are written in the form of *kets*: $|\varphi\rangle$, $|\psi\rangle$ and so on. The inner product of kets $|\varphi\rangle$ and $|\psi\rangle$ will continue to be symbolized as $\langle\varphi|\psi\rangle$. The flexibility of Dirac's notation allows us to "split" the bracket symbol $\langle\varphi|\psi\rangle$ and introduce a new kind of objects: *bras* of the form $\langle\varphi|$. Formally, a bra is a *linear functional*, that is, a function from $\mathscr{V}$ to C whose action on vectors (kets) is defined as follows: $\langle\varphi|(|\psi\rangle) := \langle\varphi|\psi\rangle$. It can be easily proven that the set of all bras satisfies all the conditions of a vector space; moreover the resulting vector space $\mathscr{V}^*$ (which is called a space dual to $\mathscr{V}$) is provably isomorphic to $\mathscr{V}$, with the natural isomorphism given by the map $|\varphi\rangle \rightarrow \langle\varphi|$ combined with the complex conjugate transformation $a \rightarrow a^*$ on the field of scalars.

An important category of objects comprises linear operators acting on $\mathscr{V}$. $A$: $\mathscr{V} \rightarrow \mathscr{V}$ is a linear operator, if $A(a|\varphi\rangle + b|\psi\rangle) = aA|\varphi\rangle + bA|\psi\rangle$ for any $a, b \in$ C and $|\varphi\rangle, |\psi\rangle \in \mathscr{V}$. An operator $A$ is called self-adjoined (or Hermitian), if $\langle\varphi|A|\psi\rangle = \langle A\varphi|\psi\rangle$ for all $|\varphi\rangle, |\psi\rangle \in \mathscr{V}$.[1] This definition (together with condition (A.6) satisfied by the inner product) implies that if $A$ is self-adjoined, $\langle\varphi|A|\psi\rangle = \langle\psi|A|\varphi\rangle^*$ for all $|\varphi\rangle, |\psi\rangle$. Defining the operation of *adjoint* ("dagger") as follows:

$$A^\dagger \text{ is the operator satisfying the equality}$$
$$\langle\varphi \mid A^\dagger \mid \psi\rangle = \langle\psi \mid A \mid \varphi\rangle^* \text{for all } |\varphi\rangle, |\psi\rangle, \qquad \text{(A.7)}$$

we can succinctly categorize self-adjoint operators as those for which $A^\dagger = A$.Using the dagger operation, we can "move" an operator acting on the first argument of the inner product to the second argument as follows:

$$\langle A\varphi \mid \psi\rangle = \langle\psi \mid A\varphi\rangle^* = \langle\varphi \mid A^\dagger \mid \psi\rangle. \qquad \text{(A.8)}$$

This formal transformation is particularly useful when we consider *unitary* operators, that is, operators for which the adjoint is their inverse: $U^\dagger = U^{-1}$. Unitary operators preserve the inner product of vectors: $\langle U\varphi|U\psi\rangle = \langle\varphi|U^\dagger U|\psi\rangle = \langle\varphi|U^{-1}U|\psi\rangle = \langle\varphi|\psi\rangle$. From this it follows that both the length of vectors and the "angles" between vectors remain

[1] The so-called matrix element $\langle\varphi|A|\psi\rangle$ is defined as the inner product of $|\varphi\rangle$ and $A|\psi\rangle$. Analogously, $\langle A\varphi|\psi\rangle$ is the product of $A|\varphi\rangle$ times $|\psi\rangle$.

unchanged under unitary transformations. Thus a unitary transformation of a given orthonormal basis produces an alternative orthonormal basis. Two structures consisting of a vector space, a selected basis and a set of operators are called "unitarily equivalent" if there is a unitary transformation connecting both. It is typically assumed that two unitarily equivalent structures may represent the same physical reality. In particular, the expectation values of operators (which encompass the complete physical information about the probabilities of outcomes) are preserved under unitary transformations, as can be seen from the following:

$$\langle U\varphi \,|\, A \,|\, U\psi\rangle = \langle\varphi \,|\, U^{\dagger}AU \,|\, \psi\rangle = \langle\varphi \,|\, U^{-1}AU \,|\, \psi\rangle. \tag{A.9}$$

An arbitrary permutation operator acting on the *n*-fold tensor product of identical vector spaces is unitary. A permutation operator $P$ acts on any basis vector $|e^{a_1}\rangle|e^{a_2}\rangle\ldots|e^{a_n}\rangle$ by "shuffling" the vectors as follows: $|e^{a_{\sigma(1)}}\rangle|e^{a_{\sigma(2)}}\rangle\ldots|e^{a_{\sigma(n)}}\rangle$, where σ is a permutation of $n$ numbers. By definition, $\langle\varphi\,|\,P^{\dagger}|\psi\rangle = \langle P\varphi\,|\,\psi\rangle$. If $|\varphi\rangle = |e^{a_1}\rangle|e^{a_2}\rangle\ldots|e^{a_n}\rangle$ and $|\psi\rangle = |e^{b_1}\rangle|e^{b_2}\rangle\ldots|e^{b_n}\rangle$, then the result of the last product will be: $\langle e^{a_{\sigma(1)}}\,|\,e^{b_1}\rangle\langle e^{a_{\sigma(2)}}\,|\,e^{b_2}\rangle\ldots\langle e^{a_{\sigma(n)}}\,|\,e^{b_n}\rangle$. A quick reflection reveals that this product can be equivalently written as $\langle e^{a_1}\,|\,e^{b_{\sigma^{-1}(1)}}\rangle\langle e^{a_2}\,|\,e^{b_{\sigma^{-1}(2)}}\rangle\ldots\langle e^{a_n}\,|\,e^{b_{\sigma^{-1}(n)}}\rangle$, which shows that $\langle P\varphi\,|\,\psi\rangle = \langle\varphi\,|\,P^{-1}\psi\rangle$, and ultimately $P^{\dagger} = P^{-1}$.

Next we define the notions of an *eigenvector* and *eigenvalues* for a given linear operator. A vector $|\lambda_a\rangle$ is an eigenvector for operator $A$ iff:

$$A\,|\,\lambda_a\rangle = a\,|\,\lambda_a\rangle, \tag{A.10}$$

where number $a$ is the corresponding eigenvalue. It can be easily verified that all eigenvalues for Hermitian operators must be real numbers (since $\langle\lambda_a|A|\lambda_a\rangle = \langle\lambda_a|A|\lambda_a\rangle^*$ for Hermitian operators, it follows that $a^* = a$). This justifies interpreting eigenvalues as the results of measurements for the observable represented by a given Hermitian operator.

An *orthogonal projection operator* (*projector*) $E$ is defined as a Hermitian operator which is idempotent, meaning that $E^2 = E$. From this definition it follows that projectors have only two eigenvalues: 0 and 1. Let us write down an eigenequation of the form (A.10) for a particular projector $E$: $E|\lambda_a\rangle = a|\lambda_a\rangle$. From idempotence we have $E^2|\lambda_a\rangle = E|\lambda_a\rangle$, and since $E^2|\lambda_a\rangle = a^2|\lambda_a\rangle$, we get the equation $a = a^2$ which has only two solutions in real numbers: 1 and 0. The set of vectors $E[\mathscr{V}]$ (containing all vectors of the form $E|\psi\rangle$) form a subspace $S_E$ of $\mathscr{V}$ onto which $E$ projects. It immediately follows that all vectors in $S_E$ are eigenvectors of $E$ with the eigenvalue equal 1. Eigenvectors of $E$ corresponding to the zero eigenvalue form a space orthogonal to $S_E$.

For a given normalized vector $|\varphi\rangle$ (i.e. a vector whose length is 1), we can define an operator (a *dyad*) $|\varphi\rangle\langle\varphi|$ whose action on any vector $|\psi\rangle$ gives $\langle\varphi|\psi\rangle\,|\varphi\rangle$. A dyad is obviously a projector, since when applied twice to $|\psi\rangle$ it gives the same vector $\langle\varphi|\psi\rangle\,|\varphi\rangle$ (provided that $\langle\varphi|\varphi\rangle = 1$). It is also elementary to prove that dyads are self-adjoint. The dyad $|\varphi\rangle\langle\varphi|$ projects onto the one-dimensional space (called a ray) spanned by vector $|\varphi\rangle$.

Vectors in a Hilbert space represent so-called pure states of physical systems. In addition to that, there are states which cannot be represented by vectors, but instead are encoded in more general mathematical objects, called *density operators* (alternatively, *statistical operators*). In order to introduce density operators, we have to first define the concept of a *trace*. Let $A$ be a linear operator, and let $|e_i\rangle$ be any orthonormal basis of $\mathscr{V}$. Then the trace of $A$ is defined as follows:

$$\mathrm{Tr}(A) = \sum_i \langle e_i|A|e_i\rangle. \tag{A.11}$$

It can be proven that the value of $\mathrm{Tr}(A)$ does not depend on the choice of the basis. Given that, we define density operators as positive operators (i.e. such that $\langle\varphi|A|\varphi\rangle \geq 0$ for all $|\varphi\rangle$) whose trace equals 1. All one-dimensional projectors are density operators, but not all density operators are projectors. However, it is the case that every density operator can be presented as a *convex sum* of one-dimensional orthogonal projectors $E_i$, that is, the sum $\sum_i p_i E_i$, where $p_i \geq 0$ and $\sum_i p_i = 1$.

One of the main interpretational rules of QM prescribes that if the system is in a pure state $|\varphi\rangle$, the expectation value of a given observable $A$ is calculated using formula $\langle\varphi|A|\varphi\rangle$. The generalization of this rule with respect to mixed states represented by density operators is as follows: the expectation value for $A$ is given by $\mathrm{Tr}(A\rho)$, where $\rho$ is the density operator representing the state of the system. Given that $\rho$ can be expressed as the convex sum $\sum_i p_i E_i$, it can be easily proven that $\mathrm{Tr}(A\rho) = \sum_i p_i\langle e_i|A|e_i\rangle$, where $|e_i\rangle$ is a normalized vector spanning the ray onto which $E_i$ projects. The proof of this fact is as follows. Let us select as a basis the set $\{|e_i\rangle\}$ of normalized vectors corresponding to the projectors $E_i$ in the decomposition of $\rho$ (i.e. $E_i = |e_i\rangle\langle e_i|$). Then the following holds:

$$\mathrm{Tr}(A\rho) = \sum_i \langle e_i|A\rho|e_i\rangle = \sum_i\sum_j p_j\langle e_i|A|e_j\rangle\langle e_j|e_i\rangle = \sum_i p_i\langle e_i|A|e_i\rangle. \quad \text{(A.12)}$$

Thus the expectation value for $A$ is a weighted sum of the expectation values of $A$ in states $|e_i\rangle$. This fact justifies an interpretation of the operator $\rho = \sum_i p_i E_i$ as representing a statistical mixture of pure states $|e_i\rangle$. In addition, observe that when $\rho = E_i$, formula $\mathrm{Tr}(A\rho)$ reduces to $\langle e_i|A|e_i\rangle$, which is the standard expectation value of $A$ in pure state $|e_i\rangle$. Thus it may be claimed that using density operators as representations of physical states is more general than using vectors. Pure states are special cases of states that are represented by density operators which are also projectors.

Suppose now that we have a system of two particles, and that their state is most generally represented by a density operator $\rho_{12}$ acting in the tensor product of two Hilbert spaces. Then the expectation value of any operator $\Omega$ in state $\rho_{12}$ is calculated as follows:

$$\mathrm{Tr}(\Omega\rho_{12}) = \sum_{ij}\langle e_i|\otimes\langle f_j|\Omega\rho_{12}|f_j\rangle\otimes|e_i\rangle, \quad \text{(A.13)}$$

where $|e_i\rangle$ are basis vectors for the first Hilbert space, and $|f_j\rangle$ for the second. Now let us assume that operator $\Omega$ has the form $A \otimes I$, and let us rewrite appropriately the expression above:

$$\begin{aligned}\mathrm{Tr}\left[(A\otimes I)\rho_{12}\right] &= \sum_{ij}\langle e_i|\otimes\langle f_j|(A\otimes I)\rho_{12}|f_j\rangle\otimes|e_i\rangle \\ &= \sum_i\langle e_i|A\sum_j\langle f_j|\rho_{12}|f_j\rangle|e_i\rangle. \end{aligned} \quad \text{(A.14)}$$

The formula:

$$\mathrm{Tr}_2(\rho_{12}) = \sum_j \langle f_j | \rho_{12} | f_j \rangle \tag{A.15}$$

is called a *partial trace* over the state space for system 2. Observe that the object denoted by $\mathrm{Tr}_2(\rho_{12})$ is not a number but an operator acting in the first Hilbert space. Moreover, obviously this operator is a density matrix, since its trace must be 1. Let us symbolize it by $\rho_1 = \mathrm{Tr}_2(\rho_{12})$. Going back to formula (A.14), we can now rewrite the last expression as follows:

$$\mathrm{Tr}[(A \otimes I)\rho_{12}] = \mathrm{Tr}_1[A\mathrm{Tr}_2(\rho_{12})] = \mathrm{Tr}_1(A\rho_1). \tag{A.16}$$

The last expression in (A.16) is identical to the general formula for calculating the expectation value of operator $A$ in state $\rho_1$. This justifies calling $\rho_1$ the reduced state of the first system. However, observe that the claim that $\rho_1$ can represent the state of system 1 "in isolation" from the second system is based on the assumption that $A \otimes I$ should represent the property $A$ associated with particle number 1. Thus we cannot use the fact that the expectation value of $A$ when system 1 is in state $\rho_1$ is equal to $\mathrm{Tr}[(A \otimes I)\, \rho_{12}]$ as an argument that $A \otimes I$ is the correct representation of $A$ attributed to first particle. We simply assume that this is the case, and on this basis we define the reduced states of individual components.

Let us illustrate the concept of a reduced state using as an example the singlet spin of two fermions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2). \tag{A.17}$$

This state can be equivalently presented with the help of a projector (and thus a density operator):

$$\rho_{12} = \frac{1}{2}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)({}_1\langle\uparrow|{}_2\langle\downarrow| - {}_1\langle\downarrow|{}_2\langle\uparrow|). \tag{A.18}$$

The reduced state $\rho_1$ is obtained by calculating the partial trace of (A.18) over the space of the second particle. Selecting a basis that consists of vectors $|\uparrow\rangle$ and $|\downarrow\rangle$, we can obtain the following (using the orthonormality relations $\langle\uparrow|\uparrow\rangle = \langle\downarrow|\downarrow\rangle = 1$, $\langle\uparrow|\downarrow\rangle = \langle\downarrow|\uparrow\rangle = 0$):

$$\begin{aligned}
&Tr_2(\rho_{12}) = {}_2\langle\uparrow|\,\rho_{12}\,|\uparrow\rangle_2 + {}_2\langle\downarrow|\,\rho_{12}\,|\downarrow\rangle_2 = \\
&= \frac{1}{2}\left(|\uparrow\rangle_1\langle\uparrow|\downarrow\rangle_2 - |\downarrow\rangle_1\langle\uparrow|\uparrow\rangle_2\right)\left({}_1\langle\uparrow|\,\langle\downarrow|\uparrow\rangle_2 - {}_1\langle\downarrow|\,\langle\uparrow|\uparrow\rangle_2\right) + \\
&\frac{1}{2}\left(|\uparrow\rangle_1\langle\downarrow|\downarrow\rangle_2 - |\downarrow\rangle_1\langle\downarrow|\uparrow\rangle_2\right)\left({}_1\langle\uparrow|\,\langle\downarrow|\downarrow\rangle_2 - {}_1\langle\downarrow|\,\langle\uparrow|\downarrow\rangle_2\right) = \\
&\frac{1}{2}|\downarrow\rangle\langle\downarrow| + \frac{1}{2}|\uparrow\rangle\langle\uparrow|
\end{aligned} \tag{A.19}$$

Thus the reduced state of particle 1 (and, analogously, of particle 2) is an equal mixture of states spin down and spin up.

# Index[1]



[1] Note: Page numbers followed by 'n' refer to notes.


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