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Josef Janyška
Marco Modugno

An Introduction to Covariant Quantum Mechanics

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Josef Janyška · Marco Modugno

An Introduction to Covariant Quantum Mechanics

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Josef Janyška
Department of Mathematics and Statistics
Masaryk University
Brno, Czech Republic

Marco Modugno
Department of Mathematics
and Informatics "U. Dini"
University of Florence
Florence, Italy

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La filosofia è scritta in questo grandissimo libro che continuamente ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non si impara a intender la lingua, e conoscer i caratteri, ne quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi ed altre figure geometriche, senza i quali mezzi è impossibile intenderne umanamente parola; senza questi è un aggirarsi vanamente per un oscuro labirinto.

G. Galilei, VI, 232, Il Saggiatore, 1623.

Preface

Quantum Mechanics and General Relativity are very successful fundamental theories of physics, but, unfortunately, they are not consistent together, as they stand. The deep reason of such an inconsistency is strictly related to the essential difference between the galilean framework of Quantum Mechanics and the lorentzian framework of General Relativity.

Several successful Relativistic Quantum Theories have been developed, the most notable example being perhaps the Standard Model of particle physics, which provides a unified description of electromagnetic, weak and strong interactions. However, most of these theories are formulated within the Quantum Field Theory approach, and they stand quite far from the core of standard Quantum Mechanics. Nevertheless, this is such an important theory that it deserves further consideration.

We observe that General Relativity involves two features which could be detached from each other: the lorentzian metric, on the one hand, and general covariance along with the geometric interpretation of gravity on the other hand. So, our aim is to show that we can reformulate standard Quantum Mechanics, still in a galilean framework, by including several features of General Relativity.

The present book is devoted to a review of a "covariant" approach to Quantum Mechanics named "*Covariant Quantum Mechanics*": we organise several previous scattered results into a consistent and systematic theory, along with several improvements, comments and many new results as well. In the Introduction, we explain what we mean by "covariant".

This research started some years ago with the aim at achieving an approach to standard Quantum Mechanics, in the framework of a flat galilean spacetime, through a manifestly equivariant formulation with respect to non-inertial observers. Then, as far as this goal had been achieved, an extension of this model to a curved galilean background was rather natural and easy. Actually, our curved framework is mostly chosen as a stimulus to find new methods and theoretical understanding of quantum objects.

Thus, in order to stand close to standard Quantum Mechanics as far as possible, we deal with an approach to Quantum Mechanics whose classical background is a curved "galilean" spacetime equipped with a fibring over absolute time, a given riemannian spacelike metric and a given "galilean" spacetime connection (interpreted as gravitational field).

So, the present approach involves several ideas of einsteinian General Relativity, such as the covariance and the geometric interpretation of the gravitational field. Nevertheless, we deal with the typical galilean aspects of standard Quantum Mechanics such as time fibring of spacetime and spacelike riemannian metric, which keep our approach far from the lorentzian framework.

We stress that we do not propose an alternative theory with respect to standard Quantum Mechanics, but discuss features which are complementary with respect to the usual approach.

We keep as our touchstone the main core of standard Quantum Mechanics in the flat case, including the probabilistic interpretation, the standard quantum operators and the Schrödinger operator. In fact, in the flat case, we recover these basic objects as particular cases. Moreover, in the case, we obtain a formulation referred to general observers.

That notwithstanding, the requirement of manifest covariance forces us to achieve standard objects via highly non-standard methods. Thus, covariance is the main guide of our approach.

Summing up, we emphasise the following original features of our approach:

– *minimal axioms* both in the classical and quantum theories;
– *manifest covariant formulation* of most classical and quantum objects, along with their expression with reference to *general observers*;
– systematic use of the *mathematical language of "scales"* and their essential role in some delicate features of covariance;
– *spacetime fibred over time*, but not split into the product of space and time;
– *essential role of time*, which does not play the role of a pure parameter;
– *odd dimensional classical phase space*, which replaces a more standard even dimensional phase space;
– *spacelike metric and timelike metric*; the first one plays a fundamental role in the classical and quantum theories, while the second one is just used for the definition of a few covariant forms;
– *fibrewise spacelike riemannian connection* and *fibrewise spacelike symplectic structure* induced by the spacelike metric play a secondary "statical role";
– fundamental role of the *cosymplectic structure* of classical phase space, which replaces the more standard symplectic structure;
– the *cosymplectic 2-form* of the classical phase space encodes the three fundamental fields of spacetime and plays a leading role in the classical and quantum theories; conversely, the *Poisson 2–vector* has a pure spacelike character and plays a secondary role;
– the *Hamiltonian formalism* in the cosymplectic framework deals with several unusual features with respect to the standard hamiltonian formalism in a more standard symplectic framework; such differences are related to the role of time and arise both in the classical and quantum theories;

- three fundamental fields of the classical theory: *spacelike metric field*, *gravitational field* and *electromagnetic field*;
- formal consistency with the galilean character of Quantum Mechanics forces us to deal with a *galilean version of electromagnetic field*; in fact, the standard Maxwell theory is essentially lorentzian, hence it is not consistent with the essentially galilean Quantum Mechanics (such an inconsistency is not perceived as far as one deals with a fixed inertial observer in a flat spacetime);
- systematic and fundamental role of different kinds of *connections* in the classical and quantum theories; in particular, the leading role played by the *galilean spacetime connection* in the classical theory and by the *galilean upper quantum connection* in the quantum theory;
- the *galilean spacetime connection* is not a standard Levi–Civita connection, hence it requires a more sophisticated approach based on the theory of general connections and the "Frölicher–Nijenhuis bracket";
- *covariant joining* of gravitational and electromagnetic fields starting from the galilean spacetime connection; the subsequent splitting of the joined objects yields automatically the expected classical and quantum laws for a charged particle and fluid;
- a fundamental role of the *Lie algebra of "special phase functions"* for the classification of classical and quantum symmetries, and, in particular, for the achievement of quantum operators, quantum currents and quantum expectation forms; a distinguished feature of special phase functions is that they admit a *spacetime lift*, hence a *holonomic lift* besides the more standard hamiltonian lift (holding for all phase functions); these two distinguished lifts play a role both in the classical and quantum theories;
- systematic analysis of *classical and quantum symmetries* and their comparison; a distinguished subalgebra of the Lie algebra of special phase functions generates in a covariant way both the classical and quantum symmetries;
- *quantum bundle defined over spacetime* by means of a minimal axiom; the *upper quantum bundle over the classical phase space* is achieved later via pullback; this is a strategic choice of our approach to the quantum theory;
- *intrinsic splitting* of the "proper quantum bundle" into the real "probability bundle" and "phase bundle";
- *intrinsic real features* of the quantum bundle and of the "proper quantum bundle" and their role in the quantum theory, emphasising the *two intrinsic real degrees of freedom of the scalar quantum particle*;
- *scaled vector valued hermitian metric* of the quantum bundle;
- *Galilean upper quantum connection* as fundamental objects of the quantum theory; a link of the curvature of this connection with the classical cosymplectic 2-form (involving the Planck constant) is the basic postulate of the quantum theory;
- the galilean upper quantum connection is equivalent to a *system of observed quantum connections* fulfilling a certain transition law;
- the galilean upper quantum connection induces *distinguished connections on the two real components of the proper quantum bundle*; these components play a role

in the intrinsic splitting of the Schrödinger equation and in the hydrodynamical picture;
- derivation of *quantum velocity*, *kinetic quantum tensor*, *quantum momentum*, *quantum probability current*, *Schrödinger operator*, *quantum lagrangian* and other fundamental quantum objects just from a requirement of covariance, via a "*criterion of projectability*" (without any reference to Fourier transform, hamiltonian formalism and other standard methods);
- "*game*" of potentials in the classical and quantum theories: "*observed potential form*" derived from the galilean spacetime connection, "*upper potential form*" derived from the cosymplectic 2-form and "*upper potential form*" derived from the galilean upper quantum connection; their relations and transformation rules;
- observer independent and gauge independent *distinguished timelike "potential seen by the quantum particle"*, which plays a role analogous to the proper mass in einsteinian General Relativity;
- *intrinsic polar splitting* of the Schrödinger equation, involving the distinguished potential seen by the quantum particle;
- *purely covariant derivation* of the Schrödinger operator via a procedure of "natural geometry" (including the covariance with respect to scales);
- covariant analysis of the *hydrodynamical picture of Quantum Mechanics*; this picture is derived from the intrinsic velocity of the quantum particle, and the quantum pressure is derived from the distinguished potential "seen" by the quantum particle;
- *classification of hermitian quantum vector fields* via a distinguished Lie algebra isomorphism between the Lie algebras of special phase functions and hermitian quantum vector fields;
- *quantum operators* achieved as a by-product of the classification of hermitian quantum vector fields;
- relation between *quantum currents* and *quantum expectation forms*;
- the covariance of the quantum theory requires an *infinite dimensional Hilbert quantum bundle over time* and prevents any distinguished splitting of this bundle; this fact emphasises that the standard Hilbert space of standard Quantum Mechanics is essentially observer dependent;
- interpretation of the *Feynman amplitudes* in term of the "upper quantum connection" and the "dynamical phase connection".

Our approach has some analogies with the well-known Geometric Quantisation, but several important differences arise as well. Indeed, Geometric Quantisation deals with a general quantisation programme starting from a symplectic structure, while Covariant Quantum Mechanics treats a specific model involved with a cosymplectic structure.

We are indebted to Geometric Quantisation for the ideas of quantum bundle and quantum connection. However, covariance is an essential requirement of Covariant Quantum Mechanics; this fact is reflected on the role of time, hence on several further consequences. In particular, for instance, our phase space is odd dimensional, cosymplectic techniques replace symplectic ones, and the special phase Lie algebra of special phase functions replaces the Poisson Lie algebra of all phase functions.

Typical no–go theorems of Geometric Quantisation do not apply to Covariant Quantum Mechanics, because of the difference of the two frameworks.

The first part of the book deals with a covariant formulation of galilean Classical Mechanics, which stands as a suitable background for Covariant Quantum Mechanics.

The second part deals with an introduction to Covariant Quantum Mechanics.

Further, in order to show how our covariant approach works in the framework of standard Classical Mechanics and standard Quantum Mechanics, the third part provides a detailed analysis of the standard flat galilean spacetime, along with three dynamical classical and quantum examples.

In the present book, we mainly deal with a scalar quantum particle effected by the gravitational and electromagnetic fields. However, our approach can be extended to a spin particle in a curved galilean framework. Moreover, several geometric techniques that we have developed for Covariant Classical and Quantum Mechanics in the galilean framework can be rephrased in the lorentzian general relativistic context. In the fourth part dealing with the Conclusions of the book, we shortly sketch these subjects with references to the related original papers.

Eventually, for the convenience of the reader, we conclude with an Appendix devoted to the exposition of some geometric methods largely used throughout the book.

As far as Quantum Mechanics is concerned, we mainly focus our attention on its geometric differential aspects, including the quantum bundle, the upper quantum connection, the quantum momentum, the Schrödinger equation, the probability current, the hydrodynamical picture, the quantum operators acting on sections of the quantum bundle and the quantum symmetries. Other topics dealing with the "sectional quantum bundle" are briefly sketched, leaving to the reader the task of further developments.

Thus, the book does not cover all aspects of Quantum Mechanics; nevertheless, we hope to have touched in a systematic way a large number of items that may yield new methods and hints for further developments.

The reader will likely realise at a glance that the style of the present book is quite unusual with respect to the dominant physical literature. We are aware of the fact that such a style might appear exotic and heavy for some readers; however, there are reasons for our choice.

First of all, the book is written in a quite formal mathematical way, by using the language of modern Differential Geometry.

All objects are systematically discussed in three ways. In fact, they are usually introduced in an intrinsic form; then we provide their "observed" expressions, with reference to "general observers", and, eventually, we add their coordinate expressions in curvilinear coordinates.

The intrinsic geometric language is part of sound heritage: we would like to show that such a language is quite natural and suitable for the manifest covariance of the theory, much more than the pure language of coordinates. In fact, the intrinsic language ensures covariance a priori and is theoretically very clean and heuristically efficient. Conversely, the coordinate language requires to check covariance at any step.

The systematic presentation in terms of general observers stands between the intrinsic and the coordinate languages and fosters the physical meanings of formal objects.

Further, let us comment the arrangement of the material. Most sentences are organised by well delimited statements (Definitions, Lemmas, Propositions, Corollaries, Theorems, Remarks), and there is no numbering of formulas, but only the statements are numbered.

In our book, we pay much attention to a thread of reasonings more than to a series of calculations; moreover, for each formula, we emphasise its specific background hypotheses. So, every formula holds in a certain framework that needs to be specified by a well delimited statement. This is the reason why we organise the book by means of statements. Accordingly, when we quote a formula, we refer to the related statement (which declares the relative full framework) rather than to the formula itself.

Additionally, in order to help the reader to follow the thread of the tangled developments of our exposition, we use very frequent internal cross references.

Eventually, another unusual feature of our language is that we adopt a formal mathematical approach to units of measurement, which allows us to explicitly emphasise the "scale" factor of every "scaled" physical object. Such a language is required by the broad covariance of our approach, which involves not only equivariance with respect to general observers but also with respect to units of a measurement.

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Contents

Chapter 1
Introduction

1.1 Historical Background

Formulations of Quantum Mechanics in a curved spacetime with absolute time and with reference to non inertial observers have been proposed by several authors with different aims and perspectives (see, for instance, [18, 20, 33, 46, 59, 60, 93, 103, 104, 108, 109, 112, 113, 118, 128, 134, 139, 161–163, 231, 259–263, 270, 272, 294–296, 298, 335, 364, 380–382, 391, 392, 395, 403, 404, 420, 421] and citations therein).

In particular, the present "*Covariant Quantum Mechanics*" is a geometric approach to Quantum Mechanics on a curved spacetime equipped with a *fibring over absolute time*, a *spacelike riemannian metric* and a *galilean spacetime connection* (see, for instance, [193, 194, 196, 214, 219–221, 223, 224, 226–228, 312, 358, 359, 386, 410] and citations therein).

This approach is aimed at implementing, in the context of quantum theory, the principle of general relativity, the covariant description of all fundamental objects with reference to general observers and the interpretation of gravity as a spacetime connection in a spacelike riemannian framework.

Thus, the set of our postulates, methods and results reflects closely fundamental aspects of both standard Quantum Mechanics and General Relativity, within the same language of modern Differential Geometry.

Actually, our choice to deal with a curved spacetime is mostly intended as a stimulus to develop new methods and viewpoints, that are able to provide a contribution also in the case of a flat spacetime, hence in the framework of standard Quantum Mechanics. In particular, even in the context of standard Quantum Mechanics, our approach can be used to get a systematic formulation with reference to non inertial observers and to find very new procedures to achieve standard objects such as, for instance, quantum momentum, quantum probability current, quantum operators and Schrödinger equation.

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Since the very beginning, Quantum Mechanics has been approached in several ways and investigated in many respects; the related literature is extremely huge. Indeed, the Schrödinger equation is one of the greatest successes of physics, hence it should be taken as a touchstone for any possible variation of the theory.

The most usual geometric approaches are based on hamiltonian techniques related to the quantisation of the classical hamiltonian function and the standard formalism is basically analytical [84, 190, 267, 302].

Among the numerous publications on Classical and Quantum Mechanics, which have inspired our approach, or which present analogies with it, or which deserve a comparison with it, we would like to make a specific mention on the works by:

D. Alba and L. Lusanna [2, 3], who deal with Quantum Mechanics in non-inertial frames,

V. Bargmann [18], who classifies the cocycles which yield the projective representations of Galilei group,

E. Cartan [46], who presents a geometric approach to Classical Mechanics,

M. Le Bellac and J.-M. Levy-Leblond [270], who study the galilean version of electromagnetism,

H. R. Brown and P. R. Holland [33], who deal with a four potential in Quantum Mechanics,

A. Camacho [38, 39], who discusses consistency problems of Quantum Mechanics in the framework of General Relativity and transformation rules in the framework of Haronov–Bohm effect,

S. Carrol [45], who deals with Quantum Theory in einsteinian curved spacetime,

J. Christian [52], who discusses the Newton–Cartan theory,

F. M. Ciaglia, A. Ibort, G. Marmo [53], who deal with a geometric picture of Quantum Mechanics,

G. Dautcourt [59, 60], who discusses the relation between galilean and einsteinian relativistic theories,

B. S. DeWitt [80], who studies Quantum Mechanics in a curved spacetime,

H. D. Dombrowski and Klaus Horneffer [93], who studies the galilean principle of relativity,

C. Duval et al. [102–104, 106], who study the Newton–Cartan theory and its relation with the Bargmann group,

J. Ehlers [112], who studies the newtonian limits of relativistic spacetimes,

J. R. Fanchi [118], who provides a review of invariant time formulations of Quantum Mechanics,

D. D. Ferrante [123], who discusses a bundle description of Quantum Mechanics,

M. Friedman [129], who provides a broad discussion on covariance and relativity in the framework of galilean and einsteinian theories,

T. Fülöp and S. D. Katz [134], who discuss a frame and gauge free formulation of Quantum Mechanics,

P. Havas [171], who discusses the covariant formulation of newtonian mechanics,

J. M. Jauch [231], who discusses a four potential in Quantum Mechanics and its relation with galilean invariance,

K. Kuchař [259], who approaches the Schrödinger operator via the Dirac constraint method in a coordinate independent way,

H. P. Künzle and C. Duval [109, 260–264], who compare the galilean and lorentzian structures, study a galilean version of spin and discuss a covariant version of Schrödinger equation,

A. Kyprianidis [265], who deals with covariant Schrödinger equation,

C. Lämmerzhal [266], who discusses the equivalence principle in Quantum Mechanics,

J.-M. Levy–Leblond [272–274], who discusses the galilean covariance group in Quantum Mechanics,

E. Massa and E. Pagani [293], who deal with a geometric approach to Classical Mechanics in the language of jets,

W. Pauli [335], who discusses the transition rule of wave functions under galilean transformations,

E. Prugovečki [344, 345], who deals with an approach to a quantum theory in the framework of General Relativity,

E. Schmutzer and J. Plebanski [364], who discuss the transition laws in Quantum Mechanics with respect to the change of frame of reference and coordinates,

S. Takagi [382], who deals with non inertial frames in Quantum Mechanics,

A. Trautman [391, 392], who discusses a comparison between newtonian and einsteinian theories,

W. M. Tulczjew [395], who proposes a geometric approach to Quantum Mechanics in a 5 dimensional spacetime.

A comparison with Geometric Quantisation deserves a more detailed attention: it will be discussed in Sect. 1.5.18.

1.2 General Relativity and Covariance

1.2.1 General Relativity and Covariant Quantum Mechanics

Preliminarily, it is worth discussing the terms like "relativistic", "covariant", "natural", "coordinate free", and so on, and state our conventions.

We stress that the einsteinian Special and General Relativity are based on two fundamental features: the principle of special and general relativity and the lorentzian metric. Actually, in the standard literature, these two features have been strongly linked together, for clear historical reasons, which go back to the fundamental work of A. Einstein (see [44, 172, 308, 318, 349]). So, a "relativistic quantum theory" is likely understood as a quantum theory formulated in a lorentzian framework.

However, the concepts of relativity and the lorentzian metric can be detached from each other. In a sense, the principle of relativity was known even to Galilei. We mention also the work of E. Cartan aimed at formulating a relativistic theory in a galilean framework (see, for instance, [46]). After this pioneering work, several other authors have investigated the "galilean relativity" (see, for instance, [18, 23, 59, 60, 66, 78, 79, 103, 104, 112, 113, 129, 161, 171, 196, 214, 259–264, 270, 272, 281, 292, 294, 296, 307, 324, 329, 330, 350, 391, 392, 407, 410]).

We have no doubts that, from a fundamental viewpoint, the lorentzian formulation of spacetime is much deeper and physically valuable than the galilean formulation; the latter can be just regarded as an approximation of the first one, which is very useful for practical purposes dealing with low velocities.

There exist well known and successful relativistic formulations of quantum theories in a lorentzian framework.

In particular, we start by mentioning the theory based on the Klein–Gordon equation (see [14, 148, 240]), which can be considered as a special relativistic version of the Schrödinger equation; actually, this theory can be easily extended to a general relativistic framework.

A special mention deserves the fundamental special relativistic Dirac theory (see [81, 82]); also this pioneering and successful theory can be easily extended to a general relativistic framework.

Along the thread of the Dirac theory, the Quantum Electrodynamics (see [55]) has been a very successful special relativistic quantum theory.

Further, several very successful versions of Quantum Field Theory have been developed in the framework of Special Relativity (see [28, 55, 339, 366, 367, 422, 430]). An extension of these theories to a general relativistic framework has also been proposed (see, for instance, [25, 41, 288, 417]).

Moreover, fully general relativistic quantum theories have been developed as well. In this respect, we mention the String Theory and the Quantum Gravity, in particular the Loop Quantum Gravity (see, for instance, [12, 77, 116, 289, 352–355]).

However, these theories stand rather far from standard Quantum Mechanics. Indeed, standard Quantum Mechanics is still such a fundamental physical theory, that it deserves to be better understood, beyond its relativistic developments in a lorentzian framework. Actually, standard Quantum Mechanics is strongly involved with absolute time and a spacelike riemannian metric; other quantum theories involving lorentzian metric move quite far from the original core of Quantum Mechanics.

In our present research, we have in mind at least two reasons for a further understanding of standard Quantum Mechanics. First of all, the above relativistic quantum theories can be regarded as successful theoretical developments of the standard quantum theory, but cannot replace it fully in practice. So, standard Quantum Mechanics still needs to be regarded as a fundamental physical theory. Moreover, standard Quantum Mechanics proposes several principles and methods that are more or less inherited by all further quantum developments above. So, any partial revisions of viewpoints and methods of standard Quantum Mechanics might be a hint also for further quantum theories.

In our approach, in order to be as close as possible to standard Quantum Mechanics and to link in a unified language the principle of General Relativity and standard Quantum Mechanics, we have been forced to skip the lorentzian metric. Clearly, in this way, our approach pays a price. In fact, we cannot avail of the Maxwell equations and of the Einstein equation in their full extent; in particular, we cannot speak of radiation and speed of light.

We stress that such feeble features hold formally also for standard Quantum Mechanics; for instance, the full usage of the Maxwell equations in the framework of standard Quantum Mechanics is formally inconsistent. Actually, the literature does not pay attention to the problem of such a formal inconsistency, as far as it deals with a given electromagnetic field and a fixed inertial observer in a flat spacetime.

For a general discussion on the compatibility between quantum theories and General Relativity, the reader could refer, for instance, to [122, 126, 169].

1.2.2 Lorentzian and Galilean Spacetimes

For the sake of clarity, let us emphasise the meaning of *lorentzian spacetime* and *galilean spacetime* in intrinsic geometric terms (see [22, 104, 129, 260, 309, 310]).

A *lorentzian spacetime* is defined to be a 4-dimensional affine space (special relativistic case), or a 4-dimensional manifold (general relativistic case), equipped with a *lorentzian metric* with signature $(- + ++)$.

The signature of the lorentzian metric just selects the timelike and spacelike directions, but does not yield any preferred splitting of the lorentzian spacetime into space and time. Such a possible splitting requires (locally) the arbitrary choice of an observer.

A *galilean spacetime* is defined to be a 4-dimensional affine space (special relativistic case), or a 4-dimensional manifold (general relativistic case), equipped with a *projection over absolute time* and a *galilean metric* (spacelike euclidean metric, or spacelike riemannian metric) with signature $(0 + ++)$ (see Postulates C.1 and C.2).

The projection over absolute time selects the spacelike vector fields, but does not yield any preferred splitting of the galilean spacetime into space and time. Such a possible splitting requires (locally) the arbitrary choice of an observer. In order to get a preferred splitting into space and time, we would need an additional preferred projection over space.

Thus, an essential comparison between the galilean spacetime and the einsteinian spacetime can be summarised as follows: in the 1st case we have a time fibring and a spacelike riemannian metric, in the 2nd case the time fibring is missing and the spacelike riemannian metric is replaced by a spacetime lorentzian metric.

1.2.3 Principle of Relativity

In the standard physical literature, for clear historical reasons, the words "covariance", "covariant", "relativity" and "relativistic" are largely used in strict connection with einsteinian Special and General Relativity. However, the above standard usage of these words might be quite misleading in the context of the present book. So, here we establish, without any pretension of completeness and full rigour, linguistic conventions which are suitable for our discussion.

Going back to the original Einstein's work, we might say, in a few words, that a *relativistic theory* is defined to be a physical theory whose fundamental laws can be expressed in an observer equivariant way. Such a condition requires to state which are the admissible observers of the theory we are dealing with. So, in Special and General Relativity the fundamental physical laws are, respectively, equivariant with respect to inertial and general observers.

Actually, in the Einstein theory, spacetime is a lorentzian affine space (Minkowski space of Special Relativity) or a lorentzian manifold (spacetime of General Relativity). Accordingly, the selection of distinguished observers (inertial or general observers) depends on the background lorentzian structure of spacetime. Therefore, in the Einstein theory, there is an essential interplay of the lorentzian structure of spacetime and the principle of relativity.

With reference to a generic physical theory, the principle of relativity, understood as equivariance of fundamental physical laws with respect to observers, can be detached from the possible lorentzian structure of spacetime.

For instance, we may formulate a theory of flat galilean spacetime in an equivariant way with respect to inertial observers. Indeed, such a formulation can also be extended to a curved galilean spacetime and to general observers. By keeping the above general meaning of relativistic theory, we might say that such galilean theories are relativistic.

Actually, in the standard literature, a relativistic quantum theory usually means a quantum theory based on a lorentzian spacetime.

So, in order to avoid confusion of language, in the present book, we shall reserve the name *relativistic* for a theory formulated in a lorentzian framework and fulfilling a principle of relativity, in the framework of the original Relativity Theory due to Einstein.

1.2.4 Principle of Covariance

There is another word which is strictly interrelated with relativity: *covariance*. In a few words, we can define this notion in the following way.

Let us consider a physical theory which is formalised by a well defined fundamental geometric framework. Then, we shall define the (local) *covariance group* to be the (local) group of automorphisms of this geometric framework. Accordingly, we say that the theory is *covariant* if its fundamental laws turn out to be equivariant with respect to the action of the covariance (local) group.

For instance, in the einsteinian Special Relativity, we deal with a lorentzian affine space (Minkowski space); hence, the covariance group is the group of affine isometries (Lorentz transformations). Analogously, in the einsteinian General Relativity, we deal with a lorentzian manifold; hence the covariance group is the group of isometric diffeomorphisms.

Our classical galilean theory is achieved by postulating a geometric structure of spacetime in three steps. Accordingly, the group of automorphisms and the induced covariance can be expressed in three steps.

(1) We start by postulating a spacetime manifold fibred over absolute time. So, *at this step*, the covariance group turns out to be the group of fibred automorphisms of spacetime over affine automorphisms of the base space. Accordingly, *at this step*, the physical laws are covariant if they are equivariant with respect to this group of automorphisms.
Indeed, the above fibred geometric structure can be fully represented by a suitable atlas of adapted charts. Hence, *at this step*, the covariance of the theory can be read, in coordinates, as *coordinate free expression* of physical laws.
(2) Then, we postulate a riemannian metric on the fibres of the spacetime fibred space. So, *at this step*, the covariance group turns out to be the group of fibred automorphisms of spacetime as in step 1, which further yield isometric diffeomorphisms of the fibres. Accordingly, *at this step*, the physical laws are covariant if they are equivariant with respect to this group of automorphisms.
In order to read the covariance of physical laws in coordinates, it would not be sufficient to refer to charts adapted to the fibring, but it would be necessary to refer also to suitable adapted frames.
(3) Eventually, we postulate a spacetime connection, which fulfills certain compatibility conditions with respect to the spacetime fibring and the spacelike metric. So, *at this step*, the covariance group turns out to be the group of fibred automorphisms of spacetime as in step 2, which further preserve the spacetime connection. Accordingly, *at this step*, the physical laws are covariant if they are equivariant with respect to this group of automorphisms.
Again, in order to read the covariance of physical laws in coordinates, it would not be sufficient to refer to charts adapted to the fibring, but it would be necessary to refer also to suitable adapted frames.
Our quantum galilean theory over the above galilean classical spacetime is achieved by postulating a further geometric structure in two steps.
(4) We start by postulating a hermitian quantum bundle over spacetime. So, *at this step*, the covariance group turns out to be the group of hermitian fibred automorphisms of the quantum bundle over the automorphisms of the spacetime base space as in step 3. Accordingly, *at this step*, the physical laws are covariant if they are equivariant with respect to this group of automorphisms.
Indeed, the above geometric structure of the quantum bundle can be fully represented by a suitable atlas of adapted linear charts. Hence, the covariance of physical laws can be read, in coordinates, as *coordinate free expression* of the physical laws.
(5) Then, we postulate a *galilean upper connection* on the upper quantum bundle, which fulfills a condition related to the geometric structure of the spacetime base. So, *at this step*, the covariance group turns out to be the group of fibred automorphisms of the quantum bundle which preserve the galilean upper quantum connection. Accordingly, *at this step*, the physical laws are covariant if they

are equivariant with respect to this group of automorphisms.
Again, in order to read the covariance of physical laws in coordinates, it would not be sufficient to refer to suitable charts of the quantum bundle, but it would be necessary to refer also to suitable adapted frames.

1.2.5 Naturality

Nowadays, the physical concept of "*covariance*" is mathematically covered by the modern concept of "*naturality*", which goes back to A. Nijenhuis and is largely discussed in the fundamental comprehensive book by I. Kolař, P. W. Michor, J. Slovak (see [246, 327] and see also Appendix J).

1.2.6 Intrinsic, Observed and Coordinate Languages

It is worth discussing three kinds of possible languages used in the formulation of a physical theory: the *intrinsic language*, the *language of coordinates* and the *language based on observers*.

In Geometry, some "original basic" concepts are *unavoidably* defined, through coordinates, by means of an explicit equivariance property with respect to a certain transition rule of charts. This is the case, for instance, of the concepts of manifolds and jet spaces. But, once these basic objects have been introduced in coordinates, one can proceed by means of formal "intrinsic methods", which do not require, at each step, the explicit mention of coordinates and their equivariance properties. In fact, such an equivariance is ensured a priory by those intrinsic methods. This is the case, just as an example, of the concepts of exterior differential and Lie derivatives of forms on manifolds.

So, there are at least two ways to deal with the covariance of a physical theory. Namely, if the mathematical language of the theory is systematically expressed in coordinates, then the covariance of the theory needs to be explicitly checked at any step. Conversely, if the physical concepts and laws of the theory are expressed in terms of an intrinsic geometric language, then the covariance is ensured a priori.

Most physical theories in standard literature are usually formulated in coordinates.

In the present book, in general, we first present the basic concepts and laws by means of an intrinsic geometric language. However, we systematically add a further description in coordinates, as well. Indeed, both languages turn out to be useful: the first one is basically convenient for its concise character, the second one is useful for emphasising further very useful features.

But, besides the intrinsic and coordinate formulations of a physical theory, it is also worth considering an intermediate approach which stands in between the intrinsic and the coordinate languages. Namely, this approach deals with *observers*.

Indeed, the equivariance of fundamental physical laws with respect to observers can be related to the original meaning of the word "relativity" used by Einstein in his original works.

For this purpose, let us have in mind the fact that, in einsteinian and galilean theories, each spacetime chart determines an observer, while many spacetime charts are adapted to a given observer (see, for instance, Definition 2.7.2 and Remark 2.7.4). So, often the coordinate expression of a physical law can be "factorised" through an "observed expression" by taking the quotient with respect to the equivalence relation of charts adapted to the same observer.

Indeed, we think that expressing physical laws in terms of observers, via such a quotient procedure, might be illuminating from physical viewpoint. In fact, often the "observed formulation" emphasises the essential features of physical phenomena, detaching them from the secondary numerical aspects.

By the way, we stress that our use of the word "observed" has nothing to do with the common use of this word in Quantum Mechanics with relation to quantum observables.

Actually, in the present book, we shall systematically pay attention to the language of observers.

After having discussed the electromagnetic field, we shall illustrate our language by an example in Sect. 1.4.5.

1.3 General Features of the Present Approach

The main general features of our approach and differences with respect to other approaches can be summarised as follows.

1.3.1 Covariance

Covariance is our main guideline; even more, we deal with *manifest covariance*, via intrinsic formulation of the basic objects. Indeed, covariance is not only a fundamental requirement of our approach, but it turns out to be also an effective heuristic tool.

In our formulation of the classical and quantum theories, we introduce all main objects by a manifestly covariant, intrinsic expression. Then, we systematically provide observed and coordinate expressions, as well. We pay attention to the fact whether every object be global, observer independent and gauge independent, or not.

1.3.2 Minimal Axioms

We essentially deal with *fundamental classical and quantum fields* and physical laws, which are introduced by a *few minimal axioms*.

By the way, we conventionally make a distinction between:

- the axioms which define the basic geometric structures of the classical and quantum models—which are conventionally called "*postulates*" (see, Postulates C.1–C.6, Q.1, and Q.2),
- the axioms which state physical laws within the above already established models—which are conventionally called "*assumptions*" (see Assumptions C.1–C.3 and Q.1–Q.4).

1.3.3 Limits Between Different Theories

Quite often in the literature we can find a usual comparison between einsteinian Classical Mechanics and galilean Classical Mechanics by taking the limit $c \to \infty$ and between Quantum Mechanics and Classical Mechanics by taking the limit $\hbar \to 0$.

Indeed, such limits of scaled quantities are heuristically useful, but cannot be taken too seriously, as their true physical meaning is much more questionable than it might appear at a first insight.

In fact, the basic mathematical settings of the above theories are rather *"rigid"*, as there is no true continuous transformation which maps one into another one. For instance, there is no observer independent continuous transformation which maps a metric with signature $(- + ++)$ into a metric with signature $(0 + ++)$. Indeed, there is a jump between these two metrics.

Moreover, just as an example, let us consider an equation, in a lorentzian framework, which involves the electric field E, the magnetic field B and the speed of light c. From a pure mathematical viewpoint, it might be possible to parametrise such an equation by substituting the fixed value c with a parametrised value $\lambda\, c$ and compute the limit of the equation for $\lambda \to \infty$. But, while we change the value of λ, the physical meaning of E and B pursues to be achieved in the lorentzian framework. So, at the limit $\lambda \to \infty$, we cannot say that the electric field E and the magnetic field B are the corresponding classical fields; in fact, in the galilean framework they are physically defined in a rather different way.

So, in the present book, we do not pay great attention to the limits $c \to \infty$ and $\hbar \to 0$, but we give more credit to a comparison of the structural differences of the different frameworks.

1.3.4 General Connections

The most elementary approach to connections is achieved in terms of the covariant differential and deals with the concepts of linear connection on a manifold and Levi–Civita connection on a pseudo-riemannian manifold (see, for instance, [27, 51, 92, 94–96, 138, 165, 241, 242, 251, 341]). This approach is largely used in physics.

Another important approach to connections deals with principal connections on a principal bundle and induced connections on associated bundles (see, for instance, [27, 241, 242, 341]). This approach provides a convenient machinery for bundles equipped with a symmetry group and is largely used in modern differential geometry and in gauge theories. Indeed, the 1st approach can be regarded as a particular case of the 2nd one.

A further more general approach to connections deals with connections on a fibred manifold (possibly without any symmetry group) and is based on tangent valued forms and their Frölicher–Nijenhuis bracket (see, for instance, [246, 284, 311]). Indeed, the 1st and 2nd approaches above can be regarded as particular cases of the 3rd one.

Actually, the tangent valued form representing the horizontal lift associated with a connection can be naturally regarded as a section of the 1st jet space of the fibred manifold. Thus, this turns out to be another useful general viewpoint to approach connections. In particular, this approach can be generalised to the notion of higher order connection on a fibred manifold.

In the present book we are dealing with several kinds of fibred manifolds and bundles, including the following ones:

- the *spacetime* $t : \boldsymbol{E} \to \boldsymbol{T}$, which is a fibred manifold over time (see Postulate C.1);
- the *tangent bundle of spacetime* $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$, which is a vector bundle over spacetime (see Definition 2.2.1);
- the *phase space* $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$, which is an affine bundle over spacetime (see Proposition 2.5.1);
- the *phase space* $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ which is a fibred manifold over time (see Proposition 2.5.1);
- the *tangent bundle of phase space* $\tau_{J_1\boldsymbol{E}} : TJ_1\boldsymbol{E} \to J_1\boldsymbol{E}$, which is a vector bundle over phase space (see Note 2.5.2);
- the *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, which is a complex vector bundle over spacetime (see Postulate Q.1);
- the *upper quantum bundle* $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ over the phase space, which is a complex vector bundle over phase space (see Definition 14.11.1);
- the *upper quantum fibred space* $t^1 \circ \pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \boldsymbol{T}$, which is a fibred space over time (see Theorem 15.3.1);
- along with several further derived fibred spaces and bundles.

Moreover, we are dealing with several kinds of connections, including the following ones:

- the *observers* $o : \boldsymbol{E} \to J_1\boldsymbol{E}$, which are general connections on the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Proposition 2.7.3),
- the *galilean spacetime connection* $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, which is a linear, torsion free, time preserving, metric preserving connection (with an additional property) on the spacetime vector bundle $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$ (see Definition 4.3.1),
- the phase connection $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$, which is an affine connection on the affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ (see Definition 9.1.1),
- the *dynamical phase connection* $\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, which is a general connection on the fibred manifold $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ and can also be regarded as a 2nd order connection on the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Definition 9.1.2),
- the *quantum connection* $\text{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$, which is a hermitian connection on the complex vector quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ (see Definition 15.1.1),
- the *upper quantum connection* $\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$, which is a hermitian connection on the complex vector upper quantum bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ (see Definition 15.1.5),
- the *upper quantum connection* $\underline{\text{Ч}}^{\uparrow} := \gamma \lrcorner \text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^* \otimes T\boldsymbol{Q}^{\uparrow}$, which is a connection on the fibred manifold $\pi^{\uparrow} \circ t^1 : \boldsymbol{Q}^{\uparrow} \to \boldsymbol{T}$ (see Theorem 15.3.1).

With relation to the above connections, we stress the following facts:

- in the generic case, the observer o are non linear and non principal connections;
- the spacetime connection K cannot be treated as a Levi–Civita connection, because the riemannian metric is only spacelike; hence, the standard methods are not sufficient to determine the connection and to achieve the Bianchi identities for the curvature;
- the phase connection Γ is an affine connection on an affine bundle, hence it is quite non standard; in particular, its torsion is a non standard object;
- the dynamical phase connection γ is a non standard object on an affine bundle;
- the upper quantum connection $\underline{\text{Ч}}^{\uparrow}$ is a non standard connection on a fibred manifold.

Therefore, by taking into account that we are dealing with several connections, which cannot be always treated by standard methods, we prefer to discuss all connections of the present book by means of the general unifying approach based on the language of tangent valued forms and their Frölicher–Nijenhuis bracket.

In forthcoming sections of the present Introduction we shall analyse further details of the galilean connections in the classical theory and on the upper connection on the quantum theory (see Sects. 1.4.3 and 1.5.6).

In the Appendix, we sketch a synthesis of this method, which is sufficient for our purposes (see Appendix F; for further details, the reader can refer to the quoted literature).

1.3.5 Scales

A characteristic feature of the mathematical language of the present book (and of further related literature on Covariant Quantum Mechanics, as well) is the systematic explicit use of "*scale spaces*" representing the units of measurement.

We stress that the world "scales" used in the present book has a conventional meaning, which should not be confused with other meanings used in the standard engineering literature.

Indeed, we stress that the literature dealing with units of measurements, under different perspectives, is very huge; here, we would like to quote, for instance [32, 175, 232, 239, 296, 389, 398].

So, let us explain what we mean.

In standard literature, one usually represents many physical objects as tensors. For instance, just to fix the ideas, the metric and electromagnetic fields, are usually represented by tensors of the type $g : \boldsymbol{M} \to T^*\boldsymbol{M} \otimes T^*\boldsymbol{M}$ and $F : \boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M}$.

However, to be more precise, such representations depend on the choice of units of measurement. In fact, if we change the units of measurement, then the above tensors change by a numerical factor determined by the ratio of those units of measurement. For instance, the scalar product of two vectors should be regarded as a number multiplied by the square of the unit of measurement of lengths.

So, it would be more appropriate to introduce the above tensors as a "*scaled tensors*" of the type $g : \boldsymbol{M} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{M} \otimes T^*\boldsymbol{M})$ and $F : \boldsymbol{M} \to \mathbb{F} \otimes \Lambda^2 T^*\boldsymbol{M}$, where $\mathbb{L}$ is suitable "*scale space*" representing the space of lengths and $\mathbb{F}$ a suitable "*scale space*" associated with the electromagnetic field.

We stress that such kind of considerations apply to many other classical objects, such as volumes, velocities, accelerations, forces, and so on (see Proposition 3.2.4, Definitions 2.4.1, 7.1.3, and 5.7.1, and so on). And also to quantum objects, such as the hermitian quantum metric. In fact, the quantum metric is not properly valued in $\mathbb{C}$, because it yields objects which should be integrated, hence should have the scale dimension of a volume (see Proposition 14.3.1).

There are also objects which have no scale dimension: this is the case, for instance, of connections and their curvature tensors. However, the Ricci tensor and the scalar curvature have scale dimension, because they involve the metric tensor.

So, it would be interesting to incorporate *explicitly and systematically* the units of measurement into the mathematical representation of physical objects, in a way which be formally rigorous and practically easy, at the same time. Such a kind of language might be useful in any text of physics, but it is even more desirable in our context, where we are dealing with a broad principle of covariance, which includes not only equivariance with respect to observers and gauges, but also the explicit equivariance with respect to units of measurement (in particular, see Remark 17.7.13).

Then, a question arises: how can we achieve such a goal? In the present book we refer to the formalism established in the paper [228], which is aimed at the above purpose (see also Appendix K, where we provide a concise summary of the above paper).

So, according to the paper [228], it seems natural to describe the set of each type of units of measurement as a semi-line $\mathbb{U}$. The elements of such a semi-line should be thought as points, that can be identified with real numbers of $\mathbb{R}^+$, *only after* having chosen a certain unit of measurement, i.e. a certain element of $\mathbb{U}$, which plays the role of a "basis". Actually, the ratio of units of measurement can be naturally thought as a number.

Then, we need mathematically well-defined algebraic operations concerning the *dual* of each scale space, *tensor products* of scale spaces, *rational powers* of each scale space and *tensor product* of a scale space with a standard tensor. Moreover, it is desirable that, under differential operations, the elements of these semi-lines behave as constant numbers.

After taking into account that scales are not properly numbers, how can we achieve this goal in a consistent way? Well, the paper [228] addresses this problem by means of abstract (hence rather sophisticated) definitions and proofs. But, luckily, at the end, if the reader trusts that paper, then he can proceed in practice by following a very intuitive and usual way, without taking in mind the abstract approach at any step. In practice, one has to take in mind a very few simple and intuitive practical rules. For instance, a simple useful rule is that the dual of a unit of measurement can be treated as the inverse element.

In order to deal with a simpler setting, one might think to represent such scale spaces as lines, instead as semi-lines; but such an approach would not be suitable for the definition of rational powers of scale spaces.

Actually, in the present book, we consider the following *basic spaces of scales* (see [228]):

(a) the positive space $\mathbb{T}$ of *time intervals*,
(b) the positive space $\mathbb{L}$ of *lengths*,
(c) the positive space $\mathbb{M}$ of *masses*.

Then, other *spaces of scales* are obtained by tensor products of "rational powers" of the above basic spaces $\mathbb{U} := \mathbb{T}^{q_1} \otimes \mathbb{L}^{q_2} \otimes \mathbb{M}^{q_3}$.

Indeed, the choice of fundamental scales is rather arbitrary; our choice is convenient as it reflects a standard usage.

We can naturally define the tensor product of spaces of scales with standard tensors; product tensors of this type will called "*scaled*", while standard tensors will be called "*unscaled*".

We denote a *time unit of measurement* and its dual, respectively, by $u_0 \in \mathbb{T}$ and $u^0 \simeq 1/u_0 \in \mathbb{T}^* \simeq \mathbb{T}^{-1}$.

We deal also with the following coupling scales, as "*universal scales*", the *Planck constant*, the *gravitational constant*, the *light speed*

$$\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}, \qquad \mathrm{r} \in \mathbb{T}^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^{-1}, \qquad c \in \mathbb{T}^{-1} \otimes \mathbb{L}$$

and with the *mass* and the *charge* of a given particle

$$m \in \mathbb{M} \quad \text{and} \quad q \in \mathbb{Q} := \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R}.$$

In principle, every physical field, whose definition depends on the choice of units of measurement, can be defined as a tensor carrying its own "scale space".

If we take into account physical equations which describe interaction between different fields, then we can reduce the family of independent scale spaces.

Let us explain such a procedure through the particular example of electromagnetic field in our galilean framework (in General Relativity we can proceed in the same way).

Thus, let us consider the Newton law of motion of a charged particle and the 2nd Maxwell equation (see Definitions 7.1.3, 5.7.1, Assumption C.2, Postulate C.5)

$$m\, g^{\flat}(a^{\natural}) = -q\, v \lrcorner F \quad \text{and} \quad \delta F = \rho\, dt.$$

We have

$$\begin{gathered} m \in \mathbb{M}, \qquad q \in \mathbb{Q}, \\ v : \boldsymbol{T} \to \mathbb{T}^{-1} \otimes T\boldsymbol{M}, \qquad a : \boldsymbol{T} \to \mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes T\boldsymbol{M}, \\ F : \boldsymbol{M} \to \mathbb{F} \otimes \Lambda^2 T^*\boldsymbol{M}, \qquad \delta F : \boldsymbol{M} \to (\mathbb{L}^{-2} \otimes \mathbb{F}) \otimes T^*\boldsymbol{M}, \\ \rho\, dt : \boldsymbol{M} \to (\mathbb{Q} \otimes \mathbb{T} \otimes \mathbb{L}^{-3}) \otimes T\boldsymbol{M}, \end{gathered}$$

where $\mathbb{Q}$ and $\mathbb{F}$ are suitable scale spaces, which should be expressed through tensor products of rational powers of $\mathbb{T}$, $\mathbb{L}$ and $\mathbb{M}$.

The two equations above imply the system

$$\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M} = \mathbb{T}^{-1} \otimes \mathbb{Q} \otimes \mathbb{F} \quad \text{and} \quad \mathbb{L}^{-2} \otimes \mathbb{F} = \mathbb{T} \otimes \mathbb{L}^{-3} \otimes \mathbb{Q},$$

which uniquely yields the equalities

$$\mathbb{Q} = \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R} \quad \text{and} \quad \mathbb{F} = \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}.$$

Having such a kind of procedure in mind, we expect to describe all possible scales and units of measurement by means of the scale spaces ($\mathbb{T}$, $\mathbb{L}$, $\mathbb{M}$), with reference to suitable equations describing physical interactions.

Throughout the book, we use frequently suitable coupling constants in order to eliminate the spaces $\mathbb{L}$ and $\mathbb{M}$, and use essentially the scale spaces $\mathbb{T}$ and overall $\mathbb{T}^*$. The reason for this procedure is related to the time fibring $t : \boldsymbol{E} \to \boldsymbol{T}$. In fact, the time fibring yields, by pullback, a natural inclusion $\boldsymbol{E} \times (\mathbb{T}^* \otimes \mathbb{R}) \subset T^*\boldsymbol{E}$. In this way, we can naturally identify a dual time scale $u^0 \in \mathbb{T}^*$ with a spacetime 1-form $u^0 : \boldsymbol{E} \to T^*\boldsymbol{E}$. Such a useful opportunity does not hold for other scales.

For further details, the reader can refer to Appendix K, which is devoted to a sketch of the mathematical approach to scales, following the ideas proposed in [228].

1.4 Features of Classical Theory

1.4.1 The Role of Time

In standard Classical Mechanics and standard Quantum Mechanics one usually deals with a flat spacetime $\boldsymbol{E}$ and a *given* inertial observer o, which yields a global splitting $\boldsymbol{E} = \boldsymbol{T} \times \boldsymbol{P}[o]$ of spacetime into the observer independent 1-dimensional time $\boldsymbol{T}$ and the observer dependent 3-dimensional space $\boldsymbol{P}[o]$ (see, for instance, Note 2.7.5). So, one can focus the attention to the *given* spacelike affine space $\boldsymbol{P}[o]$ and consider time as a parameter.

In our covariant approach we are dealing with all possible general observers o and, even more, we have chosen a manifestly covariant approach. So, according to a general principle of relativity, we do not deal with a distinguished observer o, a distinguished space $\boldsymbol{P}[o]$ and a distinguished splitting $\boldsymbol{E} = \boldsymbol{T} \times \boldsymbol{P}[o]$. However, according to a galilean scheme, we still have a distinguished projection on the observer independent time $\boldsymbol{T}$.

Indeed, the fact that a distinguished splitting of spacetime is missing prevents the possibility to regard time as a pure parameter. Actually, time plays an essential role at every step of the classical and quantum theories. In fact, such a fundamental role of time is reflected in several non standard features of our formulation, as we shall see in the forthcoming sections.

1.4.2 Galilean Metric

In einsteinian General Relativity, one deals with a scaled *spacetime lorentzian metric*

$$g : \boldsymbol{M} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{M} \otimes T^*\boldsymbol{M}), \qquad \text{with signature } (- + + +),$$

which generates the Levi–Civita connection $K^\natural$ and plays a further fundamental role via the "musical isomorphism" $g^\flat : T\boldsymbol{M} \to \mathbb{L}^2 \otimes T^*\boldsymbol{M}$ (for the scaling induced by $\mathbb{L}$, see Sect. 1.3.5).

Conversely, in our galilean framework, we deal with the scaled *spacelike galilean metric*

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}), \qquad \text{with signature } (0 + + +).$$

Indeed, this signature makes a great difference with respect to einsteinian General Relativity. In fact, in our contest, we *cannot avail* of a "spacetime musical isomorphism" $g^\flat : T\boldsymbol{E} \to \mathbb{L}^2 \otimes T^*\boldsymbol{E}$, but only of a "spacelike musical isomorphism" $g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}$. This fact turns out to be reflected in many physical laws that can be conceived in the present galilean framework.

Further, in the present galilean framework, we deal also with a scaled *spacetime timelike metric*

$$\mathbf{g} := dt \otimes dt : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}), \qquad \text{with signature } (+\,0\,0\,0),$$

which is naturally generated by the time fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Definition 3.1.1). This timelike metric yields the "spacetime musical morphism" $\mathbf{g}^\flat : T\boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E}$, whose rank is 1. Indeed, the timelike metric $\mathbf{g}$ plays a minor role. It is essentially used, via $\mathbf{g}^\flat$, for the definition of the timelike charge current and the timelike energy tensor (see Definition 8.1.1 and Proposition 8.2.1).

Unfortunately, in the present galilean framework, there is no covariant way to combine the two metrics $\mathbf{g}$ and g in order to obtain a non degenerate spacetime metric; in fact, such a possible combination would turn out to be observer dependent.

With reference to a given charged particle, of mass m, it is useful to define the rescaled covariant and contravariant metrics

$$G := \tfrac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}) \quad \text{and} \quad \bar{G} := \tfrac{\hbar}{m}\, \bar{g} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}),$$

where the scale $\mathbb{L}^2$ has been replaced by a convenient time scale $\mathbb{T}$.

In the main part of the present book, we deal with a *"given" spacelike galilean metric* g. Moreover, for the sake of completeness, we discuss also the Galilei–Einstein equation, which is a reduced galilean version of the true Einstein equation, linking the gravitational field $K^\natural$ with its matter source (see Proposition 8.2.1).

In our general theory, we postulate a generic scaled spacelike metric g. But, as far as we consider specific examples (see Definitions 24.1.1 and 28.1.1), the most physically reasonable energy tensor conceivable in our contest is only of timelike type (see Postulate C.6). Accordingly, we obtain a Ricci tensor $r^\natural$, whose spacelike restriction $\check{r}^\natural$ vanishes. As a consequence, the gravitational connection $\overset{\vee}{K}{}^\natural$ induced on the fibres of spacetime turns out to be flat. But, in virtue of the postulate $\nabla g = 0$, this fibrewise connection turns out to be fully determined by the metric g (see Theorem 4.2.13). Hence, under the above hypothesis, we deduce that the metric g is fibrewisely flat.

This fact has a consequence with respect to the Schrödinger equation. In fact, it has been proved that the most general covariant Schrödinger equation includes a possible term proportional to the scalar curvature, up to an undetermined scalar factor (see [219]). But, under the above hypothesis, the scalar curvature vanishes; hence, we can drop the above undetermined term in the Schrödinger equation.

1.4.3 Galilean Gravitational Field

In einsteinian General Relativity one deals with a lorentzian spacetime metric g and the associated Levi–Civita connection $K[g]$. Actually, the roles of the metric

g and of the Levi–Civita connection $K[g]$ are very different from an experimental viewpoint; in fact, the gravitational phenomena are described more directly by $K[g]$, rather than by g. The Levi–Civita connection $K[g]$ is fully determined by the lorentzian metric g, which plays, in a certain sense, the role of a potential. So, one is usually led to regard the metric g as the gravitational field. We think that, in spite of this circumstance occurring in einsteinian General Relativity, it would be theoretically more appropriate to regard $K[g]$ as the gravitational field and g as a metric field, suitable to describe different experiments and phenomena.

Actually, in the galilean framework, the galilean metric g has signature $(0+++)$, hence it is unable to determine a spacetime connection via the Levi–Civita procedure. So, we cannot identify the gravitational field with g. Accordingly, in the galilean framework, we describe the gravitational field by means of an appropriate *galilean connection* $K^{\natural}$.

We define a "*galilean spacetime connection*" to be a torsion free linear connection $K : \boldsymbol{TE} \to T^*\boldsymbol{E} \otimes \boldsymbol{TTE}$ of the vector bundle $\boldsymbol{TE} \to \boldsymbol{E}$ (see Definitions 4.1.1, 4.2.3 and 4.3.1) which is *time preserving* and *metric preserving*, i.e. which fulfills the conditions $\nabla dt = 0$ and $\nabla g = 0$, along with the *additional condition* $A\underline{R}[g, K] = 0$, with coordinate expression $R_{i\mu j\nu} = R_{j\nu i\mu}$.

Let us explain this additional condition. The curvature tensor of (pseudo-)riemannian connections fulfills certain well-known symmetry properties, which are direct consequence of the condition $\nabla g = 0$. But, in the galilean case, the above symmetry property of the connection should be considered as an additional condition, because the spacelike metric is unable to yield it.

We stress that this symmetry condition is essential in our context because, later, it turns out to be equivalent to the closure of the cosymplectic phase 2-form Ω and this fact turns out to be equivalent to the Bianchi identity for the upper quantum connection $\text{Ч}^{\uparrow}$ (see Theorem 9.2.15 and Remark 15.2.3). Actually, the above condition concerning a symmetry property of the curvature tensor has been considered by several authors (see, for instance, [259, 260, 263]).

In the present book, we analyse the galilean spacetime connections by considering successive hypotheses, step by step.

(1) We start by defining the "*time preserving*" torsion free linear connections K by means of the condition $\nabla dt = 0$. They are characterised in coordinates by the equality $K_{\lambda}{}^{0}{}_{\mu} = 0$ (see Definition 4.1.5 and Proposition 4.1.6).

(2) Then, with reference to an observer o and the rescaled metric G, we define the "*observed spacetime 2-form*" $\Phi[o] \equiv \Phi[G, K, o] := 2\,\mathrm{Ant}\big(\theta[o] \overset{2}{\lrcorner} \underline{\nabla}\text{д}[o]\big) : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$, with coordinate expression $\Phi[o] = -2\, G^0_{jh}\, (K_0{}^h{}_0\, d^0 \wedge d^j + K_i{}^h{}_0\, d^i \wedge d^j)$, (see Definition 4.2.11).

Indeed, this observer dependent 2-form plays an important role throughout the book.

(3) Further, we define the "*metric preserving*" time preserving torsion free spacetime connections K by means of the condition $\nabla g = 0$.

They are characterised by an observed splitting of the type (see Definition 4.2.3 and Theorem 4.2.13) $K = \mathsf{K}[G, o] - \frac{1}{2}\,(dt \otimes \widehat{\Phi}\,[G, K, o] + \widehat{\Phi}\,[G, K, o] \otimes dt)$, where $\mathsf{K}[G, o]$ is a certain metric preserving spacetime connection determined by the chosen observer o and the rescaled metric G. Unfortunately, an observer independent splitting as above does not exist. This formula, in coordinates, reads as

$$\begin{aligned} K_0{}^i{}_0 &= -G_0^{ij}\,\Phi_{0j},\\ K_0{}^i{}_h &= K_h{}^i{}_0 = -\tfrac{1}{2}\,G_0^{ij}\,(\partial_0 G^0_{hj} + \Phi_{hj}),\\ K_k{}^i{}_h &= K_h{}^i{}_k = -\tfrac{1}{2}\,G_0^{ij}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}). \end{aligned}$$

Even more, given an observer o, the above formula yields a bijection $K \mapsto \Phi$ between metric preserving time preserving torsion free linear spacetime connections K and spacetime 2-forms Φ.

(4) Eventually, we define the "*galilean spacetime connections*" K by adding the condition $A\underline{R}[g, K] = 0$, i.e., in coordinates, the condition $R_{i\mu j\nu} = R_{j\nu i\mu}$. Then, we prove that this additional condition is equivalent to the condition $d\Phi[K, G, o] = 0$ (see Definition 4.3.1 and Theorem 4.3.3).

Thus, in the case of a galilean spacetime connection, the observed spacetime 2-form $\Phi[o]$ admits locally a gauge dependent potential $A[\mathfrak{b}, o] : \boldsymbol{E} \to T^*\boldsymbol{E}$.

Accordingly, the coordinate expression of a galilean spacetime connection becomes

$$\begin{aligned} K_0{}^i{}_0 &= -G_0^{ij}\,(\partial_0 A_j - \partial_j A_0),\\ K_0{}^i{}_h &= K_h{}^i{}_0 = -\tfrac{1}{2}\,G_0^{ij}\,(\partial_0 G^0_{hj} + \partial_h A_j - \partial_j A_h),\\ K_k{}^i{}_h &= K_h{}^i{}_k = -\tfrac{1}{2}\,G_0^{ij}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}). \end{aligned}$$

This is the first case where we meet the potential; in fact, we shall find the potential in several other occurrences and frameworks, with mutual interplay.

So, after having introduced the above concepts, we postulate the gravitational field as a given galilean spacetime connection $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Postulate C.3).

Actually, in the main part of the present book, we deal with a given gravitational field. However, just for the sake of completeness, later we discuss the "*Galilei–Einstein*" equation, which is a reduced version of the true Einstein equation, hence it is able to relate, in the galilean framework, the gravitational field to its matter source (see Postulate C.6 and Proposition 8.2.1).

1.4.4 Galilean Electromagnetic Field

Joining standard Quantum Mechanics with the Maxwell theory of electromagnetism is a delicate problem.

The Maxwell theory of electromagnetism is one of the most successful achievements of classical physics (see, for instance, [164, 166, 189, 322, 333, 343, 346, 347, 418, 419]). It is very satisfactory from a phenomenological viewpoint, in complete agreement with einsteinian General Relativity and very elegant mathematically.

In standard literature, Quantum Mechanics and Maxwell theory are usually combined without specific care. However, these two theories are formally inconsistent. In fact, standard Quantum Mechanics is essentially a galilean theory, while the Maxwell theory is essentially a lorentzian theory. Actually, the above formal inconsistency does not manifestly appear when one deals with a fixed inertial observer: this fact explains why no serious troubles arise in the standard joining of the above two theories and why one does not pay too much attention to their formal inconsistency.

Now, as we are searching for a covariant formulation of Quantum Mechanics, with reference to any observer and possibly in a curved galilean spacetime, we have to overcome the above inconsistency. There are two ways: (1) to pass to a lorentzian formulation of Quantum Mechanics, (2) to search for a reduced galilean version of electromagnetism. Actually, as we have already pointed out, well-known successful lorentzian quantum theories have already been achieved, but they stand quite far from the original core of standard Quantum Mechanics and do not cover it fully. But the goal of the present book deals with a further understanding of Quantum Mechanics, within its original galilean framework. For this reason, despite ourselves, we are forced to deal with a reduced galilean version of electromagnetism, in order to get a formal consistency between the quantum theory and the electromagnetic theory.

Clearly, so doing we pay a price because we deal with a feeble version of electromagnetism. However, the price is not so high as one might expect at the first insight for the following reason. The 1st Maxwell equation $dF = 0$ is essentially the same in the true Maxwell theory and in its galilean reduced version. A relevant difference arises for the 2nd Maxwell equation $\delta F = J$, which deals with the source of the electromagnetic field. But, in our approach we basically deal with a given electromagnetic field; so we do not use the 2nd Maxwell approach in a relevant way.

Actually, such a joined scheme of quantum theory and electromagnetic theory surprisingly has sufficient interesting features, which deserve to be investigated. Thus, in the present book, we are forced to look for a theory of electromagnetism, which agrees with the galilean framework, is formally self-consistent and recovers, as far as possible, the scheme and the results of the pattern Maxwell theory.

In the literature the reader can find a thorough discussion on the galilean version of electromagnetism, which appears not as an alternative theory but as a low velocity version of the original lorentzian one (see, for instance, [78, 270, 350]). In the present book, we formulate, such a galilean theory of electromagnetism by means of our geometric language, in a way that is suitable for our purposes.

Now, let us explain, by our geometric language, how to possibly overcome the inconsistency of the true Maxwell theory with respect to the galilean framework, loosing as little as possible the good features of the original pattern theory.

Let us have in mind that the lorentzian spacetime is essentially a manifold equipped with a lorentzian metric and the galilean spacetime is essentially a manifold equipped with a fibring over absolute time and a spacelike riemannian metric.

In both frameworks, we can represent the electromagnetic field as an antisymmetric 2 tensor F, without any difference (see Definition 5.1.1). However, while in the lorentzian framework, thanks to the lorentzian metric, we can equivalently deal with a covariant or contravariant tensor F, in the galilean framework we cannot avail of such an opportunity. So, in the galilean theory, one has two physically inequivalent possibilities: dealing with a covariant F, or a contravariant F. We choose the 1st one, because it is more suitable for our purposes.

The definition of the electric and magnetic fields can be achieved in both theories via a splitting of the electromagnetic field F, with reference to an observer o. So, the observed electric field $E[o]$ can be defined in both frameworks by taking a contraction between the chosen observer o and the electromagnetic field F (see Definition 5.1.1). The observed magnetic field $B[o]$ can be defined in the lorentzian framework by taking the component of F orthogonal to the observer. But, in the galilean framework, the spacelike metric does not yield such an orthogonal component and the fibring over time yields an observer independent spacelike component of F. In this way, in the galilean framework, we obtain an observer independent magnetic field $\vec{B}$ (see Definition 5.2.1).

The 1st Maxwell equation $dF = 0$ (see Theorem 5.8.6) can be formulated in both frameworks on the same footing, as the exterior differential d involves only the differentiable structure of both manifolds. However, in the two frameworks, the splitting of this equation in terms of the magnetic and electric fields presents minor differences, which are due to the difference of the definition of these fields.

Greater differences between the two frameworks arise with respect to the 2nd Maxwell equation $\delta F = J$.

First of all, let us discuss the electric current. In both frameworks, we can define the contravariant electric current $\bar{J} = \rho\, v$, where ρ is the charge density and v the 4-velocity of the charged fluid (see Definition 7.3.2). We can also write the associated continuity equation, which accounts for the charge conservation (see Assumption C.3). However, in the lorentzian framework, the lorentzian metric allows us to achieve the covariant current form J. Unfortunately, in the galilean framework, this result cannot be achieved. The only reasonable galilean version of the covariant charge current seems to be the timelike form $\underline{\mathscr{J}} := \mathbf{g}^\flat(\bar{J}) = \rho\, dt$ (see Definition 8.1.1).

Further, the "full" divergence δF can be defined in the lorentzian framework, but not in the galilean framework, due to the signature of the metric. So, in the galilean framework, the reasonable divergence of the electromagnetic field is $\operatorname{div} F := C^1_1(\bar{g} \otimes \nabla F)$ (see Theorem 5.9.6). Accordingly, in the galilean framework, the reasonable analogue of the true 2nd Maxwell equation is $\operatorname{div} F = \rho\, dt$ (see Postulate C.5 and Proposition 8.1.2).

Actually, this equation accounts for the Coulomb law, but not for the electromagnetic effect of the movement of charges. This turns out to be the first important feeble feature of the galilean version of the Maxwell theory. However in the present book, we do not make an essential use of the 2nd "Galilei–Maxwell equation"; it is mainly discussed here for a sake of completeness.

Another, fundamental feeble feature of the galilean version of the Maxwell theory is that the "Galilei–Maxwell" equations describe a rather static theory and do not yield an electromagnetic radiation. This fact is consistent with the spacelike metric of the galilean framework.

We stress that in the definition of the magnetic field $\vec{B}$ we use the speed of light $c \in \mathbb{T}^* \otimes \mathbb{L}$ just as a convenient normalising factor, suitable for a comparison with the true Maxwell theory in a lorentzian framework (see Definition 5.2.1). However, this normalising factor does not produce any true physical effect in our galilean framework.

In conclusion, in standard Quantum Mechanics one usually couple the quantum theory with the true electromagnetic theory, so paying a theoretical prise for the formal inconsistency of the two frameworks; but, in practice, this problem does not appear manifestly as far as one deals with a fixed inertial observer. Actually, if one wants to keep the true electromagnetic theory, then standard Quantum Mechanics should be abandoned and replaced by an einsteinian relativistic quantum theory.

Our choice to keep Quantum Mechanics in a curved galilean framework and deal with a galilean version of the true electromagnetic theory is formally consistent and fits known phenomena in some respects. Moreover, our formulation is able to arise several new aspects of standard Quantum Mechanics.

In the present book, our formulation of the galilean electromagnetism is discussed in Chaps. 5 and 8.

1.4.5 Example of Intrinsic, Observed and Coordinate Languages

The electromagnetic theory gives us the opportunity to illustrate our discussion concerning the intrinsic, observed and coordinate languages (see the above Sect. 1.2.6). Most topics discussed in the present book are presented in an analogous way.

(1a) The *electromagnetic field* can be defined, in *intrinsic way*, as a scaled spacetime 2-form $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ (see Definition 5.1.1).

Actually, this definition holds both in the galilean and einsteinian frameworks, because it involves only the spacetime manifold $\boldsymbol{E}$, without reference to any further additional geometric structure.

(1b) In any spacetime chart, the *coordinate expression* of the electromagnetic field is $F = 2\, F_{0j}\, d^0 \wedge d^j + F_{ij}\, d^i \wedge d^j$.

With reference to suitable spacetime charts, this expression holds both in the galilean and einsteinian frameworks.

(1c) In the galilean framework, an *observer* is defined to be a normalised scaled spacetime vector field $д[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$, with coordinate expression $д[o] = u^0 \otimes (\partial_0 + o_0^i\, \partial_i)$, (see Definition 2.7.1 and Proposition 2.7.3).

We observe that, in the einsteinian framework, the observers can be defined in a rather analogous way. But differences between the galilean and the einsteinian cases arise due to the fact that in the two cases, respectively, the time fibring and the lorentzian metric are involved.

Then, in the galilean framework, the *magnetic vector field* and the *observed magnetic 1-form* are defined as the scaled spacetime sections (see Definition 5.2.1)

$$\vec{B} \;:= \tfrac{c}{2}\, i_{\check{F}}\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E},$$
$$B[o] \;:= \theta[o] \lrcorner \vec{B} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}.$$

Moreover, in the galilean framework, the *observed electric 1-form* and the *observed electric vector field* are defined as the scaled spacetime sections (see Definition 5.3.1)

$$E[o] \;:= -\, д[o] \lrcorner F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E},$$
$$\vec{E}[o] \;:= g^{\sharp}(\check{E}[o]) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}.$$

Their *coordinate expressions* are

$$\vec{B} \;= (\tfrac{c}{2}\, \tfrac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, F_{hk})\, \partial_i, \qquad B[o] = \tfrac{c}{2}\, \sqrt{|g|}\, \epsilon_{hki}\, F^{hk}\, (d^i - o^i_0\, d^0),$$
$$\vec{E}[o] = -g^{hj}\, (F_{0j} + F_{ij}\, o^i_0)\, u^0 \otimes \partial_h, \quad E[o] = -(F_{0j} + F_{ij}\, o^i_0)\, u^0 \otimes d^j.$$

Then, in the galilean case, the observed expression of the electromagnetic field, in terms of the observed electric and magnetic fields turns out to be (see Proposition 5.4.1)

$$F = -2\, dt \wedge E[o] + 2\, \tfrac{1}{c}\, i_{\vec{B}}\, \eta[o].$$

Thus, the observed expression of the electromagnetic field turns out to be intermediate between the intrinsic expression and the coordinate expression.

Actually, in the einsteinian case, the observed splitting of the electromagnetic field can be written in a rather analogous way. We stress that in the galilean case we can define an observer independent magnetic vector field, by using the time fibring of spacetime. Conversely, in the einsteinian framework, there is no observer independent version of the magnetic field.

(2a) The 1st Maxwell equation can be written in *intrinsic form* as $dF = 0$ (see Definition 5.8.3).

Indeed, this equation, in the above intrinsic form, turns out to be essentially the same in einsteinian Special Relativity, in einsteinian General Relativity and in our flat and curved galilean frameworks. In fact, this equation involves only a generic spacetime manifold (flat or curved, without any reference to a possible lorentzian metric or time fibring), a 2-form F and the exterior differential d. Indeed, the covariance properties of these concepts can be taken as already granted by their original geometric definitions. Thus, this equation turns out to be automatically covariant with reference to the diffeomorphisms group of spacetime.

(2b) The 1st Maxwell equation can be written *in generic coordinates*, both in a einsteinian and galilean frameworks, as $\partial_\lambda F_{\mu\nu}\, d^\lambda \wedge d^\mu \wedge d^\nu = 0$.

Clearly, this coordinate formulation turns out to be automatically equivariant with respect to the change of spacetime charts.

Further, in the einsteinian and galilean frameworks we might refer to spacetime charts adapted to the lorentzian metric or to the time fibring of spacetime. In such a case, the above coordinate expression of the 1st Maxwell equation would acquire different specifications in the einsteinian and galilean frameworks.

For instance, in the galilean framework, we can write, in adapted coordinates,

$$(\partial_0 F_{ij} - 2\, \partial_i F_{0j})\, d^0 \wedge d^i \wedge d^j + \partial_h F_{ij}\, d^i \wedge d^j \wedge d^h = 0.$$

(2c) Further, in the galilean case, the 1st Maxwell equation can also be expressed, with reference to a general observer o, as (see Theorem 5.8.6)

$$\operatorname{curl} \vec{E}[o] + \tfrac{1}{c}\, L_{д[o]}\, \vec{B} + \tfrac{1}{c}\, (\operatorname{div}_\eta д[o])\, \vec{B} = 0 \quad \text{and} \quad \operatorname{div}_\eta \vec{B} = 0,$$

in terms of the magnetic vector field $\vec{B}$ and the observed electric vector field $\vec{E}[o]$ (without explicit mention of the coordinate components of the observed fields).

Indeed, in this way we make a clear distinction between the fields $\vec{B}$ and $\vec{E}[o]$ (which depend on the laboratory which makes the observation) and the unessential mention of the less important components B^i_0 and E^i_0 (which depend on the chosen adapted chart).

Additionally, in the galilean case, the coordinate expression of the above observed equation is, in any spacetime chart adapted to o,

$$\frac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, \partial_h (g_{kr}\, E^r_0) + \tfrac{1}{c_0}\, \partial_0 B^i_0 + \tfrac{1}{c_0}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, B^i_0 = 0$$

and

$$\frac{\partial_i (B^i_0 \sqrt{|g|})}{\sqrt{|g|}} = 0.$$

We conclude the present discussion by showing how the Maxwell equation $dF = 0$ provides an effective opportunity to illustrate the heuristic value of our intrinsic approach.

Let us start by mentioning the following standard Maxwell equations in a flat spacetime, referred to an inertial observer and to a cartesian chart,

$$\epsilon^{hki}\, \partial_h E_{r\,0} + \tfrac{1}{c_0}\, \partial_0 B^i_0 = 0 \quad \text{and} \quad \partial_i B^i_0 = 0.$$

Then, suppose that we realise that the pair (B^i_0, E^i_0) can be synthetically formulated, in an observer independent way, by means of the electromagnetic 2-form F and further that the above Maxwell equations can be synthetically formulated, in an observer independent way, by the equation $dF = 0$.

Then, the above equation $dF = 0$ is the natural candidate to be generalised to the special relativistic and general relativistic contexts, in a straightforward way. Further, finding the observed and coordinate expressions of this generalised equation turns out to be a simple unambiguous computation. Indeed, it would be a much more complicated affair to guess the generalised observed and coordinate expressions directly from their standard coordinate formulation above. In our opinion, this is a heuristic way to fully exploit the spirit of einsteinian General Relativity.

Well, we systematically follow the above heuristic criterion of intrinsic approach to all classical and quantum objects discussed in the present book. Indeed, this is the main reason why our language appears rather unusual in comparison with the standard language of physical literature.

1.4.6 Joined Spacetime Connection

We "join", by means of a *covariant minimal coupling*, the *gravitational field* and the *electromagnetic field* $K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ and $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ into a "*joined spacetime connection*" (see Theorem 6.3.1)

$$K \equiv K^\natural + K^\mathfrak{e} := K^\natural - \tfrac{1}{2}\,\mathrm{k}\,(dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

with coordinate expression

$$\begin{aligned} K_0{}^i{}_0 &= K^\natural{}_0{}^i{}_0 - \mathrm{k}_0\, F_0{}^i, \\ K_0{}^i{}_j &= K_0{}^i{}_j = K^\natural{}_0{}^i{}_j - \tfrac{1}{2}\,\mathrm{k}_0\, F_j{}^i = K^\natural{}_j{}^i{}_0 - \tfrac{1}{2}\,\mathrm{k}_0\, F_j{}^i, \\ K_h{}^i{}_k &= K^\natural{}_h{}^i{}_k. \end{aligned}$$

In the present book, we deal with two specifications of the above coupling constant $\mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}$.

The most relevant case is provided by the "*electromagnetic joining scale*"

$$\mathrm{k} := \tfrac{q}{m} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R},$$

which is systematically used for a concise formulation of Classical and Quantum Mechanics, with reference to a particle of mass m and change q.

A further, minor case is provided by the "*gravitational joining scale*"

$$\mathrm{k} := \sqrt{\mathrm{r}} \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2},$$

which is used for the formulation of "Galilei–Einstein equation".

Indeed, in the classical theory, the joined spacetime connection K yields, in a covariant way, several other classical objects, including the observer dependent joined spacetime 2-form $\Phi[o]$, the joined cosymplectic phase 2-form Ω, the gauge dependent horizontal upper potential $A^\uparrow[\mathrm{b}]$, the observer dependent and gauge dependent joined spacetime potential $A[\mathrm{b}, o]$.

Moreover, in the quantum theory, the curvature tensor of the "upper quantum connection" $Ч^\uparrow$, which turns out to be the main source of our approach to Covariant Quantum Mechanics, is proportional to the classical joined cosymplectic phase 2-form Ω.

So, our joining of the gravitational field $K^\natural$ with the electromagnetic field F introduced by the joined spacetime connection K is reflected at any further developments of the classical and quantum theories.

Actually, an analogous joining is usually present in the standard literature with reference to the geometric structure of the phase space (see, for instance, the so called "canonical momentum"), but we obtain a result of this type at a deeper and unifying level, starting from the connection structure of spacetime.

We stress that in einsteinian General Relativity it is not possible to achieve, in a covariant way, a similar joined spacetime connection K on the whole tangent space $\boldsymbol{TE}$ of spacetime, but it can be achieved on the timelike cone $T^{-}\boldsymbol{E} \subset \boldsymbol{TE}$ and on the classical phase space (see [213, 214]).

The above joining procedure suggests a hint for a possible extension of our scheme to a larger physical context.

Usually, in Quantum Field Theory, all fields are regarded on the same footing and classified in terms of their symmetry group. Conversely, our approach concerning the joined spacetime connection suggests a rather hierarchic viewpoint.

In fact, in our basic setting, we deal with three fundamental fields: the gravitational field, the metric field and the electromagnetic field.

First of all, we stress that the gravitational field is not a linear field; also the metric field is not a linear field, due to the signature constraint. Only the electromagnetic field is a linear field. The gravitational field and the metric field are "*seen*" by all particles, while the electromagnetic field is "*seen*" only by charged particles. In a sense, the gravitational field appears to be the most fundamental field and the metric field plays the role of joining the electromagnetic field with the gravitational field. Each particle with a certain ratio $\frac{q}{m}$ "*sees*" a joined spacetime connection field, which encodes a full information on the acting fields. Different charged particles with different ratios $\frac{q}{m}$ see different joined spacetime connections.

1.4.7 Connection Formalism in Classical Mechanics

There are several methods to formulate the dynamics of Classical Mechanics, including the hamiltonian formalism, the lagrangian formalism and the connection formalism. Historically, the original formulation of the Newton law of motion is formulated in terms that nowadays are geometrically regarded as a connection. Later, the lagrangian and hamiltonian approaches have been developed with specific aims. Actually, all three approaches above play an important role and have specific features and goals.

In our covariant approach to Classical Mechanics we start with a formulation of Classical Dynamics in terms of a connection. Actually, after having postulated the gravitational connection $K^{\natural}$ and the electromagnetic field F we join them, by a minimal coupling, into a "*joined spacetime connection*" K and use it for the formulation of Classical Dynamics in a compact way as $\nabla_{ds} ds = 0$ (see Proposition 7.2.2).

1.4.8 Classical Phase Space

In standard Classical Mechanics, with reference to a *given* inertial observer o, one often chooses as classical *phase space* the manifolds $\boldsymbol{T} \times T^{*}\boldsymbol{P}[o]$, or $\boldsymbol{T} \times T\boldsymbol{P}[o]$, where $\boldsymbol{P}[o]$ is the observed spacelike configuration space. Clearly, these possible choices do not fulfill a manifest covariance requirement, as these configuration spaces depend on the observer.

Looking for a more intrinsic, i.e. observer independent, choices of the classical phase space, one might choose $V^*\boldsymbol{E}$, or $V\boldsymbol{E}$. However, if we try to formulate the laws of classical kinematics and dynamics starting from these phase spaces, we soon realise that such spacelike frameworks are insufficient and require the choice of an observer. In fact, in a sense, these possible phase spaces have a rather static feature, than a dynamical suitability.

One might think to bypass the above problem by choosing as phase space the bundles $T^*\boldsymbol{E}$, or $T\boldsymbol{E}$. However, two problems arise immediately.

First of all, we represent time as a 1-dimensional affine space $\boldsymbol{T}$, not just as $\mathbb{R}$, and every motion is described by a section $s : \boldsymbol{T} \to \boldsymbol{E}$ (see Postulate C.1 and Definition 2.4.1). Accordingly, the velocity of s turns out to be a scaled section $ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

Therefore, it would be more advisable to choose $\mathbb{T} \otimes T^*\boldsymbol{E}$, or $\mathbb{T}^* \otimes T\boldsymbol{E}$, as classical phase space.

The second problem is that $\mathbb{T} \otimes T^*\boldsymbol{E}$ and $\mathbb{T}^* \otimes T\boldsymbol{E}$ have a reduntant dimension. In fact, being s a section, its velocity fulfills the constraint $dt \lrcorner ds = 1$ (see Definition 2.4.1). Therefore, it is advisable to choose as classical phase space the 7-dimensional subbundles of $\mathbb{T} \otimes T^*\boldsymbol{E}$, or $\mathbb{T}^* \otimes T\boldsymbol{E}$, which fulfill the above constraint. In fact, in our opinion it is advisable to skip fields with explicit constraints; actually, this is a criterion of our approach.

By the way, we notice that, in all above possible choices, we have considered a certain cotangent space or the dual tangent space. We stress that, in general, the cotangent choice is more suitable for the hamiltonian formalism; for instance, on the cotangent space of a manifold we have the natural Liouville form λ and the associated natural symplectic form $\omega = d\lambda$. The tangent choice is suitable not only for the hamiltonian formalism, but also for the lagrangian formalism and for the formulation of classical dynamics via a connection.

Therefore, our best minimal choice is to take, as classical phase space, the above mentioned constrained 7-dimensional subbundle $J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ (see, also, [56, 144, 177, 293]). Indeed, this is nothing but the *1st jet space* of motions (see Proposition 2.5.1).

Thus, our choice of $J_1\boldsymbol{E}$ as classical phase space reflects the fundamental role of time, fulfills the requirement of manifest covariance and follows a criterion of "minimality", allowing us to skip constraints.

Actually, we stress the fact that phase space is odd dimensional yields several important differences with respect to more usual geometric approaches to Classical and Quantum Mechanics. We examine these features in the forthcoming sections.

1.4.8.1 Cosymplectic Structure of Phase Space

Symplectic Geometry and Poisson Geometry are very popular mathematical theories (see, for instance, [1, 57, 61, 146, 199, 275, 425]). Further, a description of classical and quantum physical theories in terms of Symplectic Geometry and Poisson Geometry is also well established and plays an important role in a very huge literature (see, for instance, [1, 4, 21, 143, 145, 146, 149–155, 159, 167, 168, 222, 223, 238, 250, 297, 348, 372, 373, 395, 396, 399–402, 425]).

Symplectic Geometry and Poisson Geometry are very successful in the mathematical formalisation of classical and quantum theories described by even dimensional manifolds, such as theories referred to a given observer, where manifest covariance is not required. However, for physical theories where time does not play the simple role of a parameter, Symplectic and Poisson Geometry are not sufficient.

Sometimes, a better role is played by Contact Geometry. This is, for instance, the case of Classical Mechanics in einsteinian general relativistic framework where Contact Geometry is well suited (see, for instance, [213]).

However, Symplectic Geometry, Poisson Geometry and Contact Geometry are not well suited for our manifest covariant approach to galilean Classical Mechanics and Quantum Mechanics. Indeed, in our classical framework, we do have a natural fibrewise symplectic structure $\omega := d\lambda$ in $V^*\boldsymbol{E}$ induced by the smooth structure of spacetime and a natural fibrewise Poisson structure $\Lambda[g]$ in $V\boldsymbol{E}$ induced by the spacelike metric g (see [146] and Proposition 3.2.14). But these structures have a rather "static" role and are not suitable for a manifestly covariant formulation of Classical Dynamics and Quantum Dynamics (see Proposition 3.2.14).

Actually, there is another analogous geometric theory, namely Cosymplectic Geometry, which deals with odd dimensional manifolds and is suitable for our purposes. Cosymplectic Geometry is much less popular than Symplectic Geometry and the related literature is not so wide (see, for instance, [67, 68, 70–76] and also see Appendix: Definition I.1.10).

As we have already noticed, our best choice for modelling our classical phase space is the odd dimensional manifold $J_1\boldsymbol{E}$. Accordingly, the geometric structure well suited for our manifestly covariant approach to Classical Mechanics is a "*cosymplectic structure*" (dt, Ω), consisting of the forms $dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}$ and $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ (see Theorem 10.1.1 and Definition 15.2.1).

We stress that the replacement of symplectic structure with a cosymplectic structure yields deep changes in the hamiltonian formalism, which are reflected in many steps throughout the development of our classical and quantum theories.

Indeed, the cosymplectic structure plays a fundamental role both in our approach to Classical and Quantum Mechanics. We stress that the phase 2-form Ω accounts for all classical fields postulated in the classical theory: the metric field g, the gravitational field $K^\natural$ and the electromagnetic field F.

Thus, again, our choice based on cosymplectic structure fulfills the requirement of covariance and emphasises the role of time.

1.4.8.2 Hamiltonian Formalism

In standard hamiltonian models applied to physics one usually starts with a given even dimensional manifold $\boldsymbol{M}$ equipped with a *given* symplectic 2-form $\omega : \boldsymbol{M} \to$

$\Lambda^2 T^* \boldsymbol{M}$ and a *given* hamiltonian function $\mathcal{H} : \boldsymbol{M} \to \mathbb{R}$. Then, the dynamics is described by a hamiltonian vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$, which is uniquely characterised by the condition $i_X \omega = -d\mathcal{H}$.

In our approach to Classical Mechanics, as we have already noticed, the even dimensional manifold $\boldsymbol{M}$ is replaced by the odd dimensional phase space $J_1\boldsymbol{E}$, whose choice reflects our requirement of manifest covariance, the geometric role of time (which does not play the simple role of a parameter) and a minimality criterion.

Accordingly, we start with a cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, which encodes the given metric, gravitational and electromagnetic fields.

We stress that, in this context, we *do not* postulate any *given* hamiltonian function.

Actually, the role of the usual hamiltonian vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$ is played by a 2nd order connection $\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$, which is uniquely characterised by the condition $i_\gamma \Omega = 0$ (see Theorem 9.1.8).

So, in a sense, in our cosymplectic context, the role of the usual hamiltonian function $\mathcal{H} : \boldsymbol{M} \to \mathbb{R}$ is played by the zero function. Indeed, the above 2nd order connection γ plays the role of the Reeb vector field in classical Contact and Cosymplectic Geometry.

In our framework, the cosymplectic phase 2-form Ω admits locally a *horizontal* gauge dependent and observer independent potential $A^\uparrow[\mathrm{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, such that $\Omega = dA^\uparrow[\mathrm{b}]$.

Then, chosen an observer o, we define the gauge dependent and observer dependent hamiltonian function $\mathcal{H}[\mathrm{b}, o] := -\, д[o] \lrcorner A^\uparrow[\mathrm{b}]$ (see Theorem 10.1.8).

Thus, in our scheme, the hamiltonian function $\mathcal{H}[\mathrm{b}, o]$ is not postulated, but it turns out to be encoded in the cosymplectic phase 2-form Ω, via the choice of a gauge b and of an observer o.

In Symplectic Geometry, the symplectic form ω provides the linear isomorphism $\omega^\flat : T\boldsymbol{M} \to T^*\boldsymbol{M}$, whose inverse is given by the Poisson 2-vector $\Lambda : \boldsymbol{M} \to \Lambda^2 T\boldsymbol{M}$.

So, in Symplectic Geometry, for each function $f : \boldsymbol{M} \to \mathbb{R}$, one deals with the linear hamiltonian lift $X[f] := i_{df}\Lambda : \boldsymbol{M} \to T\boldsymbol{M}$.

In our cosymplectic context, the linear map $\Omega^\flat : TJ_1\boldsymbol{E} \to T^*J_1\boldsymbol{E} : X^\uparrow \mapsto i_{X^\uparrow}\Omega$ is not an isomorphism (see Proposition 11.2.2).

Moreover, in our cosymplectic framework, the induced 2-vector $\Lambda : J_1\boldsymbol{E} \to \Lambda^2 TJ_1\boldsymbol{E}$ carries less information than Ω.

So, in our framework, for each phase function $f : J_1\boldsymbol{E} \to \mathbb{R}$, we deal with the affine hamiltonian lift $X^\uparrow[f] := f'' \lrcorner \gamma + i_{df}\Lambda$, where $f'' : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$ is a certain scale factor "extracted" in a covariant way from f, (see Definition 11.3.6).

Indeed, the "horizontal term" $f'' \lrcorner \gamma$ yields relevant changes with respect to standard procedures of Symplectic Geometry.

In particular, for each pair of phase functions $f, \acute{f} : J_1\boldsymbol{E} \to \mathbb{R}$, we can define in a covariant way the Poisson bracket $\{f, \acute{f}\} := \Lambda^\sharp(df \wedge d\acute{f})$, but this bracket plays a less relevant role than in Symplectic Geometry, due to the fact that the phase 2-vector Λ does not carry a full information on the geometric structure of the phase space (see Definition 11.4.1). Actually, this Poisson bracket has essentially a spacelike character and is not sufficient for a covariant formulation of dynamics. In fact, for this purpose we need to use the cosymplectic phase 2-form Ω or, equivalently, the phase pair (γ, Λ).

In several respects, the role played in Symplectic Geometry by the Poisson bracket for all phase functions will be played by the "*special phase bracket*" for special phase functions (see Definition 12.5.1).

1.4.8.3 Lagrangian Formalism

In standard 1st order lagrangian models applied to physics one often starts with a given bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, with $\dim \boldsymbol{B} = m$, equipped with a *given* lagrangian density $\mathcal{L} : J_1\boldsymbol{F} \to \Lambda^m T^*\boldsymbol{B}$ and postulates, as dynamical equation, the *Euler–Lagrange equation* $\mathcal{E}[\mathcal{L}] \circ j_2 s = 0$, where $\mathcal{E}[\mathcal{L}] : J_2\boldsymbol{F} \to V^*\boldsymbol{F} \otimes \Lambda^m T^*\boldsymbol{B}$ is the *Euler–Lagrange operator*.

In our approach to Classical Mechanics, we *do not* postulate any *given* lagrangian function. As we have already noticed, we start with a cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, which encodes the given metric, gravitational and electromagnetic fields. Moreover, Ω admits locally a *horizontal* gauge dependent and observer independent potential $A^{\uparrow}[\mathrm{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$. Then, we define the gauge dependent and observer independent lagrangian function $\mathcal{L}[\mathrm{b}] := д \lrcorner A^{\uparrow}[\mathrm{b}]$, where $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ is the contact map (see Theorem 10.1.8). The associated Euler–Lagrange equation $\mathcal{E}[\mathcal{L}] \circ j_2 s = 0$ turns out to be equivalent to the observer independent and gauge independent *Newton law of motion* $\nabla_{ds} ds = 0$.

Thus, in our scheme, the lagrangian function $\mathcal{L}[\mathrm{b}]$ is not postulated, but it turns out to be encoded in the cosymplectic phase 2-form Ω, via the choice of a gauge b.

1.4.9 Lie Algebra of Special Phase Functions

1.4.9.1 Special Phase Functions

In standard Classical Mechanics, where time plays the role of a parameter, the Poisson Lie algebra of all phase functions turns out to be a very important tool (see, for instance, [1, 146, 275, 399–402]). However, in our covariant framework, where time cannot be regarded as a pure parameter, the Poisson Lie bracket of all phase functions appears to have a rather spacelike (hence "static") character, due the spacelike character of the Poisson phase 2-vector Λ (see Definitions 11.4.1 and 9.1.4). So, the Poisson Lie algebra of all phase functions is unable to fully account for Classical Mechanics in a manifestly observer independent formulation.

Conversely, in our approach, a fundamental role is played by the "*Lie algebra of special phase functions*". In principle, the Lie algebra of special phase functions is an object of the classical cosymplectic framework, but it can be better understood in the framework of Covariant Quantum Mechanics, due to a striking interplay of the transition laws of special phase functions and of quantum potential.

Indeed, the Lie algebra of special phase functions plays a fundamental role in our developments of Covariant Classical Mechanics and Covariant Quantum Mechanics for the classification of infinitesimal symmetries. Even more, in Covariant Quantum Mechanics, the Lie algebra of special phase functions turns out to be the source of quantum operators, quantum currents and quantum expectation forms. We stress that, in the developments of Covariant Quantum Mechanics, the special phase Lie algebra allows us to treat spacetime functions, momentum and energy on the same footing.

Thus, the subsheaf of *special phase functions* $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$ consists of the phase functions f, which are characterised, in a covariant way, by the condition $D^2 f = f'' \otimes G$, with $f'' \in \operatorname{map}(\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R})$, i.e., whose coordinate expression is of the type $f = f^0 \frac{1}{2} G^0_{ij} x^i_0 x^j_0 + f^i G^0_{ij} x^j_0 + \breve{f}$, with $f^0, f^i, \breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ (see Definition 12.1.1).

We stress that, in general, the product of two special phase functions is *not* a special phase function as well.

It is worth mentioning the following distinguished "*algebraic subsheaves*" of the sheaf $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ (see Definition 12.1.3)

the *subsheaf of projectable s.p.f.*	$:= \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$	$:=$	$\{f \mid \partial_j f^0 = 0\}$.
the *subsheaf of time preserving s.p.f.*	$:= \operatorname{timspe}(J_1\boldsymbol{E}, \mathbb{R})$	$:=$	$\{f \mid \partial_\lambda f^0 = 0\}$,
the *subsheaf of affine s.p.f.*	$:= \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$	$:=$	$\{f \mid f^0 = 0\}$,
the *subsheaf of spacetime s.p.f.*	$:= \operatorname{map}(\boldsymbol{E}, \mathbb{R})$	$:=$	$\{f \mid f^\lambda = 0\}$.

So, special phase functions turn out to be fundamental functions of the classical phase space, including the spacetime coordinates (x^λ), the components of the classical momentum $(\mathcal{P}_j)$, the classical hamiltonian $(\mathcal{H}_0)$ the classical lagrangian $\mathcal{L}_0$ and the square of the classical momentum $\mathcal{P}^2_0$.

At a first insight, the definitions of special phase functions and of their special phase Lie bracket might appear quite arbitrary, but several facts strongly support them. In fact, for instance, the special phase functions naturally appear in the classification of hermitian quantum vector fields and of classical and quantum infinitesimal symmetries (see Theorems 19.1.7, 13.1.3, 13.2.6, 19.2.2 and 19.3.2).

It is also remarkable the already mentioned striking interplay of the transition laws of special phase functions and of quantum potential (see Corollary 12.2.10 and Theorem 15.2.26).

Further, we emphasise that the special phase functions naturally arise in the condition of projectability of the hamiltonian lift of phase functions (see Theorem 12.4.4).

1.4.9.2 Tangent Lift of Special Phase Functions

In standard Classical Mechanics (and in our approach as well) all phase functions admit a hamiltonian phase lift (see, for instance, [1] and also Sect. 1.4.8.2).

It is worth noticing that special phase functions f admit, in a covariant way, the unusual "*tangent lift*" $X[f] = f'' \lrcorner д - G^{\sharp}(Df) \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, with coordinate expression $X[f] = f^0\, \partial_0 - f^i\, \partial_i$. (see Theorem 12.2.1).

Indeed, this tangent lift turns out to be a relevant feature of special phase functions, as it is the source of several developments.

First of all, the tangent lift yields two distinguished splittings of each special phase function (see Proposition 12.2.9):

(1) the *observed splittings*

$$\begin{aligned} f &= \bar{f}[o] + \check{f}[o] = -X[f] \lrcorner \mathcal{C}[o] + \check{f}[o] \\ &= f'' \lrcorner \mathcal{K}[o] + f'[o] \lrcorner \mathcal{Q}[o] + \check{f}[o] = (f^0\, \mathcal{K}_0 + f^i\, \mathcal{Q}_i) + \check{f}, \end{aligned}$$

(2) the *gauge splittings*

$$\begin{aligned} f &= \tilde{f}[\mathfrak{b}] + \hat{f}[\mathfrak{b}] = -X[f] \lrcorner A^{\uparrow}[\mathfrak{b}] + \hat{f}[\mathfrak{b}] \\ &= f'' \lrcorner \mathcal{H}[\mathfrak{b}, o] + f'[o] \lrcorner \mathcal{P}[\mathfrak{b}, o] + \hat{f}[\mathfrak{b}] = (f^0\, \mathcal{H}_0 + f^i\, \mathcal{P}_i) + \hat{f}. \end{aligned}$$

Then, we mention the "*divergence*" $\operatorname{div}_\eta f := \operatorname{div}_\eta X[f] \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ of projectable special phase functions, with coordinate expression (see Definition 12.2.7)

$$\operatorname{div}_\eta f = f^0\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i\, \sqrt{|g|})}{\sqrt{|g|}}.$$

1.4.9.3 Phase Lifts of Special Phase Functions

Moreover, we deal with the following distinguished phase lifts of special phase functions:

- the unusual "*holonomic phase lift*" $X^{\uparrow}{}_{\mathrm{hol}}[f] := r^1 \circ J_1 X[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, with coordinate expression (see Definition 12.3.2),

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = f^0\, \partial_0 - f^i\, \partial_i - (\partial_0 f^i + \partial_j f^i\, x^j_0 + \partial_0 f^0\, x^i_0 + \partial_j f^0\, x^j_0\, x^i_0)\, \partial^0_i,$$

- the "*hamiltonian phase lift*" $X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$, with coordinate expression (see Definition 12.4.1)

$$\begin{aligned} &X^{\uparrow}{}_{\mathrm{ham}}[f] \\ &\quad = f^0\, \partial_0 - f^i\, \partial_i + G^{ij}_0\, (-f^0\, (\partial_0 \mathcal{P}_j - \partial_j A_0) \\ &\qquad + f^h\, (\partial_h \mathcal{P}_j - \partial_j A_h) + \partial_j f^0\, \mathcal{K}_0 + \partial_j f^h\, \mathcal{Q}_h + \partial_j \check{f})\, \partial^0_i. \end{aligned}$$

1.4.9.4 Special Phase Lie Bracket

The sheaf of special phase functions is naturally equipped with the "*special phase Lie bracket*" $[\![f, \acute{f}\,]\!] := \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f$, (see Definition 12.5.1 and Theorem 12.5.3), with coordinate expression

$$\begin{aligned}
[\![f, \acute{f}\,]\!]^{\lambda} &= X[f]^{\mu}\, \partial_{\mu} X[\acute{f}\,]^{\lambda} - X[\acute{f}\,]^{\mu}\, \partial_{\mu} X[f]^{\lambda}, \\
\breve{[\![f, \acute{f}\,]\!]} &= X[f]^{\mu}\, \partial_{\mu} \breve{\acute{f}} - X[\acute{f}\,]^{\mu}\, \partial_{\mu} \breve{f} + X[f]^{\lambda}\, X[\acute{f}\,]^{\mu}\, (\partial_{\lambda} A_{\mu} - \partial_{\mu} A_{\lambda}), \\
\hat{[\![f, \acute{f}\,]\!]} &= X[f]^{\mu}\, \partial_{\mu} \hat{\acute{f}} - X[\acute{f}\,]^{\mu}\, \partial_{\mu} \hat{f},
\end{aligned}$$

and further equivalent expressions

$$\begin{aligned}
[\![f, \acute{f}\,]\!] &= -\big[X[f], X[\acute{f}\,]\big] \lrcorner\, \mathcal{C}[o] + X[f].\breve{\acute{f}} - X[\acute{f}\,].\breve{f} + \Phi[o]\big(X[f], X[\acute{f}\,]\big), \\
[\![f, \acute{f}\,]\!] &= -\big[X[f], X[\acute{f}\,]\big] \lrcorner\, A^{\uparrow}[\mathrm{b}] + X[f].\hat{\acute{f}} - X[\acute{f}\,].\hat{f}, \\
[\![f, \acute{f}\,]\!] &= X^{\uparrow}[f].\acute{f} - X^{\uparrow}[\acute{f}\,].f + 2\,\Omega\big(X^{\uparrow}[f], X^{\uparrow}[\acute{f}\,]\big).
\end{aligned}$$

We stress that the special phase Lie bracket reduces to the Poisson Lie bracket in the particular case of affine phase functions (see Definitions 12.1.3 and 12.5.1).

So, in our approach, the special phase Lie bracket plays an alternative role with respect to the more standard Poisson Lie bracket essentially for "quadratic" phase functions, including kinetic energy function $\mathcal{K}_0[o]$, hamiltonian function $\mathcal{H}_0[\mathrm{b}, o]$, lagrangian function $\mathcal{L}_0[\mathrm{b}]$ and square of the momentum function $\mathcal{P}_0^2[\mathrm{b}, o]$.

1.4.9.5 Differential Lie Subalgebras

The special phase Lie bracket well interplays with the holonomic and hamiltonian lifts of special phase functions, according to the following results:

- for each $f, \acute{f} \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have $\big[X^{\uparrow}{}_{\mathrm{hol}}[f],\ X^{\uparrow}{}_{\mathrm{hol}}[\acute{f}\,]\big] = X^{\uparrow}{}_{\mathrm{hol}}\big[\,[\![f, \acute{f}\,]\!]\,\big]$ (see Proposition 12.6.5)
- for each $f, \acute{f} \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have $\big[X^{\uparrow}{}_{\mathrm{ham}}[f],\ X^{\uparrow}{}_{\mathrm{ham}}[\acute{f}\,]\big] = X^{\uparrow}{}_{\mathrm{ham}}\big[\,[\![f, \acute{f}\,]\!]\,\big]$ (see Proposition 12.6.6).

1.4.9.6 Differential Subalgebras

A relevant role is played by the following "*differential subalgebras*" of the Lie algebra of special phase functions:

- the subalgebra of "*holonomic special phase functions*" (see Definition 12.6.8)

$$\mathrm{holspe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

characterised by the condition $X^{\uparrow}{}_{\mathrm{ham}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f]$,

- the subalgebra of "*conserved special phase functions*" (see Definition 12.6.10)

$$\mathrm{cns}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}),$$

characterised by the condition $\gamma.f = 0$.

Indeed, we obtain $\mathrm{cns}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{holspe}(J_1\boldsymbol{E}, \mathrm{I\!R})$ (see Theorem 12.6.17).

1.4.10 Classical Symmetries

We observe that the cosymplectic pair (dt, Ω) fully encodes the geometric structure of spacetime and its fields. Accordingly, we define the *infinitesimal symmetries of classical structure* to be the phase vector fields $X^{\uparrow} \in \mathrm{pro}_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, which fulfill the conditions $L_{X^{\uparrow}} dt = 0$ and $L_{X^{\uparrow}} \Omega = 0$ (see Definition 13.1.1).

Actually, the infinitesimal symmetries of the classical structure turn out to be of the type $X^{\uparrow} = X^{\uparrow}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$, with $f \in \mathrm{cns}\,\mathrm{timspe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, i.e., in coordinates of the type $X^{\uparrow} = X^{\uparrow}[f] = f^0\,\partial_0 - f^i\,\partial_i + X^i_0\,\partial^0_i$, with $f^0 \in \mathrm{I\!R}, f^i, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$, and $X^i_0 = G^{ij}_0\big(-f^0\,(\partial_0\mathcal{P}_j - \partial_j A_0) + f^h\,(\partial_h\mathcal{P}_j - \partial_j A_h) + \partial_j f^h\,\mathcal{Q}_h + \partial_j\breve{f}\big)$.

Hence, the Lie algebra of infinitesimal symmetries of classical structure is generated, in a covariant way, by the Lie algebra of conserved time preserving special phase functions.

Moreover, we observe that the pair $(dt, \mathcal{L}[\mathfrak{b}])$ fully encodes the geometric structure of classical dynamics. Accordingly, we define the *infinitesimal symmetries of classical dynamics* to be the spacetime vector fields $X \in \mathrm{pro}\,\mathrm{sec}(\boldsymbol{E}, \boldsymbol{TE})$, which fulfill the conditions $L_{X^1} dt = 0$ and $L_{X^1}\mathcal{L}[\mathfrak{b}] = 0$, where $X^1 \in \mathrm{pro}_{\boldsymbol{E},\boldsymbol{T}}\,\mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ is the 1-jet holonomic prolongation of X, (see Definition 13.2.4 and Theorem 13.2.6).

Actually, the infinitesimal symmetries of the classical structure turn out to be of the type $X^1 = X^{\uparrow}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$, with (see Definition 12.6.2) $f \in \mathrm{cns}\,\mathrm{timspe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, fulfilling the condition $d\breve{f}[o] = -d(i_{X[f]}A[\mathfrak{b}, o])$, which can be expressed in coordinates as $f = f^0\,\mathcal{H}_0[\mathfrak{b}, o] + f^i\,\mathcal{P}_i[\mathfrak{b}, o]$, with $f^\lambda \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})\,;\,\hat{f} \in \mathrm{I\!R}$.

Hence, the Lie algebra of infinitesimal symmetries of classical dynamics is generated, in a covariant way, by a certain Lie subalgebra of conserved time preserving special phase functions.

Therefore, the Lie algebra of infinitesimal symmetries of classical structure is smaller than Lie algebra of infinitesimal symmetries of classical dynamics; the reason of this discrepancy is due to the fact that the pair (dt, Ω) is gauge independent, while the pair $(dt, \mathcal{L}[\mathfrak{b}])$ is gauge dependent.

1.5 Features of Quantum Theory

1.5.1 Standard Quantum Mechanics as Touchstone

We keep *standard Quantum Mechanics* in a flat spacetime, formulated with reference to an inertial observer (including the wave function, the probabilistic interpretation, the Schrödinger equation and the quantum operators), as *our touchstone* (see, for instance, [15, 18, 31, 34, 54, 83–85, 115, 119, 127, 135–137, 170, 190, 234, 267, 278, 279, 301, 302, 319, 321, 323, 325, 328, 334, 337, 356, 357, 361–363, 365, 369, 377, 383, 384, 390, 406, 423]). So, in the flat case, we recover the usual Schrödinger operator, probability current and quantum operators of standard Quantum Mechanics.

Indeed, the requirements of manifest covariance and equivariance with respect to general observers force us to replace several standard approaches by means of other non standard geometric procedures.

1.5.2 Quantum Bundle Based on Spacetime

The "*quantum bundle*" is our basic arena of our approach to Quantum Mechanics (see, for instance, [368]).

Indeed, our "*quantum bundle*" $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ is based on the spacetime $\boldsymbol{E}$, instead of the classical phase space $J_1\boldsymbol{E}$, (see Postulate Q.1 and Note 14.4.5). This choice fulfills a requirement of minimality and turns out to be strategic for further developments.

In view of the subsequent dynamical postulate on the "upper quantum connection" $Ч^{\uparrow}$, we introduce the "*upper quantum bundle*" $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$, based on the classical phase space $J_1\boldsymbol{E}$, (see Definition 14.11.1). But, this enlargement of the base space is achieved via a pullback, so preserving the criterion of minimality. In this context, the enlarged base space accounts for all classical possible observers.

We would like to mention that a similar minimality choice can be found in the work of H. P. Künzle (see, for instance, [263]).

1.5.3 Real and Complex Quantum Bundle

Since the very beginning, Quantum Mechanics has been formulated in a complex language. Apparently, this fact conflicts with the real language of Classical Mechanics: such a discrepancy might look quite strange. So, one might ask why Quantum Mechanics should be a complex theory and whether there is a mysterious deep reason for this.

Actually, this strange anomaly disappears if we realise that the notion of a hermitian 1-dimensional complex bundle is fully equivalent to the notion of a euclidean oriented 2-dimensional real vector bundle (see, for instance [242, Vol. II]).

Indeed, we give due weight to this equivalence. So, besides the usual approach to the quantum bundle in terms of a *hermitian 1-dimensional complex bundle*, we discuss also the approach in terms of an *oriented euclidean 2-dimensional real bundle* (see Postulate Q.1 and Note 14.4.5).

These two approaches are equivalent from a formal geometric viewpoint but they emphasise different physical meanings and provide different practical advantages.

We stress that, from a physical viewpoint, the real language emphasises 2 (internal) real degrees of freedom of the scalar quantum particle; actually, this fact can be easily understood in comparison with Classical Mechanics. Moreover, the real language is very convenient for the development of geometric aspects of Quantum Mechanics which are based on real geometric methods, such as lagrangian theory, symmetries and jet spaces. Actually, in standard literature these aspects are usually treated in complex terms. But we stress that a rigorous translation in complex terms of geometric theories which are deeply real is much more subtle and delicate than it might appear at a first insight.

However, the complex language provides a quick and compact expression of several developments and formulas. So, in our opinion, this is the very reason why Quantum Mechanics needs the complex language.

Actually, in the book we use both languages: we use the real language just for the topics which are essentially real, but then translate the results in the usual complex language.

1.5.4 Proper Quantum Bundle and Its Polar Real Splitting

There is also another interesting way to compare the complex and real languages for Quantum Mechanics and to discuss the 2 real (internal) degrees of freedom or the 1 complex (internal) degree of freedom of the scalar quantum particle.

Let us start by considering the "proper complex plane" $\mathbb{C}_{/0} \subset \mathbb{C}$, obtained by dropping the zero element. Analogously, by considering the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ as a vector bundle with type fibre $\mathbb{C}$, we define the "*proper quantum bundle*" to be the subbundle $\pi_{/0} : \boldsymbol{Q}_{/0} \subset \boldsymbol{Q} \to \boldsymbol{E}$ obtained by dropping the zero section (see Definition 14.6.1).

We can regard the proper quantum bundle as the restriction of the quantum bundle to the domain where the quantum particle has non vanishing probability to be detected.

The "polar splitting" yields a bijection $\mathbb{C}_{/0} \to \mathbb{R}^+ \times \mathbb{R}/2\pi : (c_1, c_2) \mapsto (\varrho, \varphi)$. Analogously, with reference to a quantum basis, the proper quantum bundle splits locally as $\boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\Vert}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{\varphi}_{/0} : \Psi_e \mapsto \big(\|\Psi\|_e, \varphi(\Psi)_e\big)$, where $\pi_{\Vert} : \boldsymbol{Q}^{\Vert}_{/0} := (\boldsymbol{E} \times \mathbb{R}^+) \to \boldsymbol{E}$ is a trivial bundle with type fibre $\mathbb{R}^+$ and $\pi_{\varphi} : \boldsymbol{Q}^{\varphi}_{/0} \to \boldsymbol{E}$ is a bundle with type fibre $U(1) = \mathbb{R}/2\pi$. This is the *usual* "polar splitting of the quantum bundle", which is largely used in the literature, for instance in view of the splitting of Schrödinger equation and of the hydrodynamical picture of Quantum Mechanics. We stress that the above phase φ is, by definition, gauge dependent, i.e. it depends on the choice of a base of the quantum bundle. Hence, we cannot fully regard this phase φ, as it stands, as representing a true real (internal) degree of freedom of the quantum particle, in an intrinsic way.

However, there is another more intrinsic way to address the above splitting. In fact, without reference to any gauge or quantum basis, we can define the global splitting $\boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{(\!)}_{/0} : \Psi_e \mapsto \big(\|\Psi\|_e, (\!(\Psi)\!)_e\big)$, where $\pi^{(\!)} : \boldsymbol{Q}^{(\!)}_{/0} \to \boldsymbol{E}$ is a bundle with type fibre $\boldsymbol{S}_1$ (see Proposition 14.7.1).

Indeed, $\pi^{(\!)} : \boldsymbol{Q}^{(\!)}_{/0} \to \boldsymbol{E}$ turns out to be a principal bundle associated with the trivial bundle $\boldsymbol{E} \times U(1)$. Thus, the replacement of the gauge dependent fibre $U(1)$ with the gauge independent fibre $\boldsymbol{S}_1$ allows us to regard the phase $\phi \in \boldsymbol{S}_1$ as a true real (internal) degree of freedom of the quantum particle.

In this respect, we stress that, while the standard phase φ is gauge dependent, the difference of two standard phases $\acute{\varphi} - \varphi$ is gauge independent. It also worth mentioning that the hermitian product can be read in a very intuitive way through the above polar splitting; in fact, in real terms, it turns out to be split into the product of the norms and the difference of the intrinsic phases (see Proposition 14.7.5).

Thus, we adopt the above *intrinsic polar splitting* of the proper quantum bundle (see Proposition 14.7.2) and largely use it in many steps of the quantum theory, for instance, in the contest of the quantum kinetic tensor, of the Schrödinger equation and of the hydrodynamical picture of Quantum Mechanics (see Corollaries 17.3.3, 17.3.5, and 17.6.18, Proposition 17.6.17, and Theorem 18.2.2).

Thus, the fact that we deal with an intrinsic phase gives more emphasis to the physical interpretation of several results.

1.5.5 η-Hermitian Quantum Metric

We deal with an "*η-hermitian quantum metric*" $\mathrm{h}_\eta := \mathrm{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^*\boldsymbol{E} \otimes \mathbb{C}$ valued in the space of complexified spacelike volume forms of spacetime (see Definition 14.5.1). This unusual choice seems to be quite natural. Indeed, it turns out to be convenient for integration on the fibres of spacetime and allows us to skip semi-forms.

1.5.6 Galilean Upper Quantum Connection

After the quantum bundle based on spacetime and the η-hermitian quantum metric, our main postulate deals with the galilean upper quantum connection. Thus, we postulate a "*galilean upper quantum connection*" $\mathrm{Ч}^{\uparrow} : J_1\boldsymbol{E} \to T^*J_1\boldsymbol{E} \otimes \boldsymbol{Q}^{\uparrow}$, which is defined to be a connection of the upper quantum bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ (see Definitions 15.1.5 and 15.2.1 and Postulate Q.2),

(1) which is "*hermitian*" (see Definition 15.1.12),
(2) which is "*reducible to a system of observed quantum connections*" of the quantum bundle $\{\mathrm{Ч}[o]\}$, with $\mathrm{Ч}[o] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ (see Definition 15.1.6),
(3) whose *curvature tensor* $R[\mathrm{Ч}^{\uparrow}] : \boldsymbol{Q}^{\uparrow} \to \Lambda^2 T^*J_1\boldsymbol{E} \otimes \boldsymbol{Q}^{\uparrow}$ is proportional (through the mass of the quantum particle and the Planck constant) to the classical joined cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^*J_1\boldsymbol{E}$ of the classical phase space $J_1\boldsymbol{E}$ (see Theorem 10.1.1).

By the way, we stress that the Planck constant $\hbar$ has been incorporated into Ω, via its scale normalisation.

Thus, the choice of such a galilean upper quantum connection $\mathrm{Ч}^{\uparrow}$ is equivalent to the choice of a *system of observed hermitian quantum connections* $\{\mathrm{Ч}[o]\}$, which fulfill the transition rule $\mathrm{Ч}[\acute{o}] = \mathrm{Ч}[o] + \mathfrak{i}\,\big(\theta[o] \lrcorner G^{\flat}(\vec{v}) - \frac{1}{2}\,G(\vec{v},\vec{v})\big) \otimes \mathbb{I}$ (see Theorem 15.2.7).

1.5.7 The "Game" of Potentials and Distinguished Observer

In our approach to Covariant Classical Mechanics and Covariant Quantum Mechanics we meet the potential in several forms and in several contexts (see [224]).

(1) First of all, we mention the joined observed spacetime 2-form $\Phi[o] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$, which is associated with the joined spacetime connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Corollary 6.3.3).

This 2-form is closed, hence admits a, gauge dependent, local *joined observed spacetime potential* $A[\mathfrak{b}, o] : \boldsymbol{E} \to T^*\boldsymbol{E}$, according to $\Phi[o] = 2\,dA[\mathfrak{b}, o]$.

Here, in the classical context, the symbol $\mathfrak{b}$ just denotes the choice of a gauge.

(2) Then, we consider the *joined cosymplectic phase 2-form* $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, which is closed, hence admits a joined gauge dependent and observer independent, potential $A^{\uparrow}[\mathfrak{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$, which can be chosen to be horizontal according to $\Omega = dA^{\uparrow}[\mathfrak{b}]$ (see Theorem 10.1.4). Here, again, in the classical context, the symbol $\mathfrak{b}$ just denotes the choice of a gauge.

(3) There is a natural way to link the above classical potentials and to compare the choices of the corresponding gauges $\mathfrak{b}$. In fact, for each observer o, we obtain the equality $\Phi[o] = 2\,o^*\Omega$, which yields $A[\mathfrak{b}, o] = o^*A^{\uparrow}[\mathfrak{b}]$.

Hence, we have the observed and coordinate expressions (see Theorem 10.1.8)

$$\begin{aligned} A^{\uparrow}[\mathfrak{b}] &= -\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o] \\ &= -(\mathcal{K}[o] - д[o] \lrcorner A[\mathfrak{b}, o]) + (\mathcal{Q}[o] + \theta[o] \lrcorner A[\mathfrak{b}, o]) \\ &= -(\tfrac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 - A_0)\,d^0 + (G^0_{ij}\,x^j_0 + A_j)\,d^i . \end{aligned}$$

(4) Indeed, the upper potential $A^{\uparrow}[\mathrm{b}]$ turns out to be the source of the classical lagrangian, hamiltonian and momentum (see Theorem 10.1.8)

$$\begin{aligned}
\mathcal{L}[\mathrm{b}] &:= \quad д \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}),\\
\mathcal{H}[\mathrm{b}, o] &:= -\, д[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}),\\
\mathcal{P}[\mathrm{b}, o] &:= \quad \theta[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\mathcal{L}[\mathrm{b}] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0,\\
\mathcal{H}[\mathrm{b}, o] &= (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0,\\
\mathcal{P}[\mathrm{b}, o] &= (G^0_{ij}\, x^j_0 + A_i)\, d^i.
\end{aligned}$$

Moreover, $A^{\uparrow}[\mathrm{b}]$ turns out to be the Poincaré–Cartan form associated with $\mathcal{L}[\mathrm{b}]$.

(5) Further, in the quantum contest, the upper potential $A^{\uparrow}[\mathrm{b}]$ appears in the gauge dependent expression of the upper quantum connection $Ч^{\uparrow}$, where it plays the role of an *upper quantum potential*, according to the equality (see Theorem 15.2.4)

$$\begin{aligned}
Ч^{\uparrow} &= \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\, A^{\uparrow}[\mathrm{b}] \otimes \mathbb{I}^{\uparrow}\\
&= \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\big(-\mathcal{K}[o] + \mathcal{Q}[o] + A[\mathrm{b}, o]\big) \otimes \mathbb{I}^{\uparrow}\\
&= \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\big(-\mathcal{H}[\mathrm{b}, o] + \mathcal{P}[\mathrm{b}, o]\big) \otimes \mathbb{I}^{\uparrow},
\end{aligned}$$

i.e., in coordinates,

$$Ч^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + \mathrm{i}\big(-(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i\big) \otimes \mathbb{I}^{\uparrow}.$$

We stress that, in the classical theory, the symbol b has no true geometric meaning; conversely, in the quantum theory, every gauge b can be interpreted as a basis of the quantum bundle (see Remark 10.1.6). The link between the two cases can be easily obtained by observing the relation between the two gauge independent and observer independent objects Ω and $Ч^{\uparrow}$ and the fact that the same upper potential $A^{\uparrow}[\mathrm{b}]$ yields both Ω and $Ч^{\uparrow}$.

(6) The choice of the upper quantum connection $Ч^{\uparrow}$ yields, the following transition rules, with reference to the quantum bases b and $\acute{\mathrm{b}} = \mathrm{b}\, \exp(\mathrm{i}\, \vartheta)$ and to the observers o and $\acute{o} = o + \vec{v}$, (see Theorem 15.2.26)

$$\begin{aligned}
A^{\uparrow}[\acute{\mathrm{b}}] &= A^{\uparrow}[\acute{\mathrm{b}}] + d\vartheta,\\
A[\acute{\mathrm{b}}, \acute{o}] &= A[\mathrm{b}, o] - d\vartheta + \theta[o] \lrcorner G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v}),
\end{aligned}$$

i.e., in coordinates,

$$A^{\uparrow}[\acute{\mathrm{b}}] = A^{\uparrow}{}_{\lambda}\, d^{\lambda} + \partial_{\lambda}\vartheta\, d^{\lambda},$$
$$A[\acute{\mathrm{b}}, \acute{o}] = A_{\lambda}\, d^{\lambda} - \partial_{\lambda}\vartheta\, d^{\lambda} + G^{0}_{ij}\, v^{i}_{0}\, d^{j} - \tfrac{1}{2}\, G^{0}_{ij}\, v^{i}_{0}\, v^{j}_{0}\, d^{0}.$$

(7) As a consequence, we find the following gauge dependent *observer equivariant objects* (see Corollary 15.2.28)

$$\nu[\mathrm{b}] := д[o] - G^{\sharp}(\check{A}[\mathrm{b}, o]) \in \sec(\boldsymbol{E},\ \mathbb{T}^{*} \otimes T\boldsymbol{E})$$
$$\alpha[\mathrm{b}] := д[o] \lrcorner A[\mathrm{b}, o] - \tfrac{1}{2}\, \bar{G}\big(\check{A}[\mathrm{b}, o],\ \check{A}[\mathrm{b}, o]\big) \in \sec(\boldsymbol{E}, H^{*}\boldsymbol{E}),$$

with coordinate expressions

$$\nu[\mathrm{b}] = u^{0} \otimes (\partial_{0} - A^{i}_{0}\, \partial_{i}),$$
$$\alpha[\mathrm{b}] = (A_{0} - \tfrac{1}{2}\, A_{i}\, A^{i}_{0})\, u^{0}.$$

Indeed, these objects appear several times throughout the book.

(8) Moreover, every proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, yields a *distinguished observer* o_{Ψ}, such that the associated observed potential $A[\Psi] := A\big[\mathrm{b}_{\Psi}, o_{\Psi}\big] \in \sec(\boldsymbol{E}, H^{*}\boldsymbol{E})$ be timelike (see Theorem 15.2.31).

With reference to any quantum basis b and any observer o, we have the coordinates expressions

$$o^{i}_{0}[\Psi] = G^{ij}_{0}\, (\partial_{j}\varphi - A_{j}),$$
$$A[\Psi] = -\Big(\big(\partial_{0}\varphi - A_{0}\big) + \tfrac{1}{2}\, G^{ij}_{0}\, \big(\partial_{i}\varphi - A_{i}\big)\, \big(\partial_{j}\varphi - A_{j}\big)\Big)\, d^{0}.$$

Moreover, with reference to the quantum basis b_{Ψ} and any observer o, we obtain the equalities $o[\Psi] = o - G^{\sharp}\big(A\big[\mathrm{b}_{\Psi}, o\big]\big)$ and $A[\Psi] = \alpha[\mathrm{b}_{\Psi}]$.

Thus, we can say that every proper quantum section Ψ "*sees*" the distinguished timelike observed potential $A[\Psi]$, which is associated with the distinguished quantum basis b_{Ψ} and the distinguished observer o_{Ψ}, which are uniquely determined by Ψ.

In a sense, there is a certain analogy between the distinguished potential $A[\Psi]$ in our formulation of Quantum Mechanics and the rest mass m in einsteinian General Relativity.

With relation to the above observation, we emphasise the following equalities (see Theorem 15.2.31 and Note 14.6.3, and also Propositions 16.1.17, 16.2.7 and 16.4.5)

$$\nabla^{⦾}[o_{\Psi}](\!(\Psi)\!) = -A[\Psi],$$
$$\nabla^{⦾ 2}[o_{\Psi}](\!(\Psi)\!) = -\nabla[K]A[\Psi],$$
$$\nabla^{\uparrow ⦾}(\!(\Psi)\!) = -A^{\uparrow}[\mathrm{b}_{\Psi}],$$

which exhibit a remarkable link between the distinguished observed quantum potential $A[\Psi]$ and the distinguished upper quantum potential $A^{\uparrow}[\mathrm{b}_\Psi]$ "seen" by the proper quantum section Ψ with the phase quantum covariant differentials and the phase upper quantum covariant differential.

Indeed, the distinguished observer o_Ψ turns out to be also the *rest observer* of the hydrodynamical picture of a proper quantum section Ψ (see Theorem 18.1.1).

(9) Further, the distinguished timelike potential $A[\Psi]$ turns out to be equal (up to sign) to the hamiltonian of every classical particle of the flow $\mathscr{C}[\Psi]$ in the hydrodynamical picture of Quantum Mechanics (see Corollary 18.1.10 and Remark 18.1.11).

(10) With reference to any proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the distinguished observer o_Ψ and the distinguished observed timelike potential $A[\Psi]$ yield remarkable and unusual simple expressions of several quantum objects, which deserve a physical interpretation. Indeed, such expressions hold also in the flat case, but can be hardly achieved in standard Quantum Mechanics, because they involve the possibly highly non inertial observer o_Ψ.

For instance, we emphasise the following cases:

(a) The *quantum velocity* can be expressed as (see Theorem 17.2.2)

$$\mathrm{V}[\Psi] = д[o_\Psi].$$

(b) The *kinetic quantum tensor* can be expressed as (see Corollary 17.3.3)

$$\mathrm{Q}[\Psi] = \big(д[o_\Psi] - \mathrm{i}\,\vec{d}\,\log\|\Psi\|\big) \otimes \Psi.$$

(c) The *kinetic quantum vector field* can be expressed as (see Corollary 17.3.5)

$$\mathrm{Q}[\Psi]/\Psi = д[o_\Psi] + \mathrm{i}\,\vec{d}\,\|\Psi\|.$$

(d) The *probability quantum current* can be expressed as (see Theorem 17.4.2)

$$\mathrm{J}[\Psi] = \|\Psi\|^2\, д[o_\Psi].$$

(e) The *quantum lagrangian* can be expressed as (see Corollary 17.5.3)

$$\mathrm{L}[\Psi] = (\tfrac{1}{2}\,\bar{G}\,(d\,\|\Psi\|,\, d\,\|\Psi\|) + A[\Psi]\,\|\Psi\|^2) \otimes \upsilon.$$

(f) The *Schrödiger operator* can be expressed as (see Corollary 17.6.9)

$$\begin{aligned}&\mathrm{S}[\Psi]\\ &= \Big(д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\ \mathrm{div}_\eta\, д[o_\Psi] - \mathrm{i}\,\big(\tfrac{1}{2}\,\Delta[G]\|\Psi\| + A[\Psi]\,\|\Psi\|\big)\Big) \otimes \mathrm{b}_\Psi,\end{aligned}$$

hence, the Schrödinger equation admits the polar splitting (see Corollary 17.6.18)

$$0 = д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\ \mathrm{div}_\eta\, д[o_\Psi],$$
$$0 = \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi].$$

We stress that, in the above equations, the explicit mention of the phase polar degree of freedom of the quantum particle $(\!(\Psi)\!)$ disappears; however, it is implicitly encoded in o_Ψ and $A[\Psi]$.

(g) The law of motion of the fluid associated with every proper quantum section Ψ through its hydrodynamical picture can be expressed as (see Theorem 18.2.1)

$$\mu[\Psi]\,\mathscr{A}^\natural[\Psi] = -\rho[\Psi]\,g^\sharp(д[o_\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\boldsymbol{p}[\Psi],$$

where the quantum pressure is given by $\boldsymbol{p}[\Psi] = -\frac{\hbar}{m}\,A[\Psi]$.

1.5.8 Criterion of Projectability

Our quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ lives on the spacetime $\boldsymbol{E}$, but the cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ lives on the phase space $J_1\boldsymbol{E}$. Therefore, in order to define a galilean upper quantum connection $Ч^\uparrow : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ suitable to be linked to Ω, we have been forced to introduce, *by pullback*, the upper quantum bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$, which lives on the phase space $J_1\boldsymbol{E}$ (see Definition 9.1.3, 14.11.1, 15.1.6 and 15.1.5, Corollary 9.2.4, Theorems 9.2.8, 10.1.1, Postulate Q.1, Q.2, and Lemma 15.1.3).

In this way, we deal with an upper quantum bundle $\boldsymbol{Q}^\uparrow$, whose base space $J_1\boldsymbol{E}$ involves all observers o. Accordingly, also the upper quantum connection $Ч^\uparrow$, involves all observers o implicitly, and, equivalently, the associated system of observed quantum connections $\{Ч[o]\}$, involves all observers o explicitly.

Indeed, our quantum theory is aimed at deriving all dynamical quantum objects from the upper quantum connection $Ч^\uparrow$, getting rid of observers, in order to fulfill a covariance requirement. Accordingly, our procedure is based on the search of quantum objects which, on the one hand, are *derived*, in a natural way, from the upper quantum connection $Ч^\uparrow$, but, on the other hand, are *projectable* on the quantum bundle $\boldsymbol{Q}$. We call this type of procedure *"criterion of projectability"*: actually it turns out to be a way to implement the covariance of the quantum theory (see Note 17.1.1).

1.5.9 Dynamical Quantum Objects

According to the above covariant procedure based on the criterion of projectability, we exhibit the following *dynamical quantum objects*.

Actually, all objects below are achieved in two steps:

(1) we show two distinguished quantum objects, which live on the same source space $J_1\boldsymbol{E}$ and have the same target space,

(2) for each of the above pairs, we show that their difference projects on the spacetime $\boldsymbol{E}$, so getting rid of the observers which are encoded in the phase space.

Indeed, this distinguished difference turns out to be our natural candidate for the searched dynamical quantum object.

Some of the following dynamical quantum objects are rather unusual and cannot be found in the literature, at least in our way of presentation.

We stress that all dynamical quantum objects above are *global, observer independent and gauge independent*. Actually, this property is related to the fact that they deal with the gravitational and electromagnetic fields effecting the quantum particle and do not involve other possible phenomenological fields. Of course, other additional phenomenological fields might be added by hand; but, so doing, we would possibly brake the covariance of the theory.

Moreover, we stress that the above procedure based on the criterion of projectability yields the dynamical quantum objects above without any relation to hamiltonian methods.

Indeed, these objects can be achieved by other differential geometric approaches related to the above criterion of projectability. In particular, one can prove that S and L are determined by the only requirement of covariance (see [219]).

1.5.9.1 Quantum Velocity

For each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we obtain the gauge independent and observer independent "*quantum velocity*" (see Theorem 17.2.2)

$$\mathsf{V}[\Psi] := д + \vec{\nabla}^{\uparrow \circledcirc}(\!(\Psi)\!) \in \sec(\boldsymbol{E}, \ \mathbb{T}^* \otimes T\boldsymbol{E}),$$

with observed and coordinate expressions

$$\begin{aligned} \mathsf{V}[\Psi] &= д[o] + \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!) \\ &= u^0 \otimes \Big(\partial_0 + \big(G^{ij}_0 \, \partial_j \varphi - A^i_0\big) \, \partial_i\Big). \end{aligned}$$

In particular, with reference to the distinguished observer o_Ψ associated with the proper quantum section Ψ, we obtain the equality $\mathsf{V}[\Psi] = д[o_\Psi]$.

The quantum velocity has a close relation with the kinetic quantum tensor, the kinetic quantum vector field and the quantum probability current (see Corollary 17.3.3 and Theorem 17.4.2). Moreover, the quantum velocity plays a key role in the context of the hydrodynamical picture of Quantum Mechanics (see Theorem 18.1.1).

The quantum velocity is a rather usual object of standard Quantum Mechanics. But, our 4-dimensional intrinsic presentation, the link with the "rest observer" and the related physical interpretation can be hardly achieved in standard Quantum Mechanics.

1.5.9.2 Kinetic Quantum Tensor

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the gauge independent and observer independent "*kinetic quantum tensor*" (see Theorem 17.3.2)

$$\mathrm{Q}[\Psi] := д \otimes \Psi - \mathrm{i}\, \vec{\nabla}^{\uparrow} \Psi \in \sec\big(\boldsymbol{E},\ \mathbb{T}^* \otimes (\boldsymbol{TE} \otimes \boldsymbol{Q})\big),$$

with observed and coordinate expressions

$$\begin{aligned} \mathrm{Q}[\Psi] &= д[o] \otimes \Psi - \mathrm{i}\, \vec{\nabla}[o]\Psi \\ &= \big(\psi\, \partial_0 - \mathrm{i}\, G_0^{ij}\, (\partial_j \psi - \mathrm{i} A_j\, \psi)\, \partial_i\big) \otimes u^0 \otimes \mathrm{b}. \end{aligned}$$

In particular, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we have the splitting

$$\mathrm{Q}[\Psi] = \big(\mathrm{V}[\Psi] - \mathrm{i}\, \vec{d} \log \|\Psi\|\big) \otimes \Psi.$$

The kinetic quantum tensor is a rather unusual object, with respect to standard Quantum Mechanics but this object plays a relevant role in our approach, as it is the source of the Schrödinger operator via the projectability criterion (see Theorem 17.6.5).

Indeed, the kinetic quantum tensor has a close relation with the quantum velocity (see Corollary 17.3.3). Furthermore, the kinetic quantum tensor has a close relation with the quantum momentum operator (see Example 20.1.12).

In this respect, we stress that the standard approach to Quantum Mechanics, the notion of quantum momentum is usually strictly linked with the Fourier formalism. However, in a curved spacetime, the Fourier methods can be hardly proposed in a covariant way. So, in our general curved framework, we are forced to follow a completely different geometric way, which, in the flat case, reproduces known objects and results.

1.5.9.3 Kinetic Quantum Vector Field

For each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we obtain the gauge independent and observer independent "*kinetic quantum vector field*" (see Corollary 17.3.5)

$$\mathrm{Q}[\Psi]/\Psi \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \boldsymbol{TE} \otimes \mathbb{C})\big),$$

with observed and coordinate expression

$$\begin{aligned} \mathrm{Q}[\Psi]/\Psi &= \big(д[o] + \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!) + \mathrm{i}\, \vec{d}\, \|\Psi\|\big) \\ &= \big(\partial_0 - (\mathrm{i}\, G_0^{ij}\, \partial_j \psi / \psi + A_0^i)\, \partial_i\big) \otimes u^0. \end{aligned}$$

In particular, with reference to the distinguished "rest observer" o_Ψ associated with the proper quantum section Ψ, we have the splitting (see Corollary 17.3.5)

$$\mathrm{Q}[\Psi]/\Psi = \mathrm{V}[\Psi] - \mathrm{i}\, \vec{d} \log \|\Psi\|$$

into real and imaginary components, which are related, respectively, to the real polar degrees of freedom of the quantum particle (see sect. 1.5.4 and Proposition 14.7.2).

We stress that an analogous splitting would have no meaning for $\mathrm{Q}[\Psi]$.

1.5.9.4 Quantum Probability Current

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the gauge independent and observer independent *quantum probability current* (see Theorem 17.4.2)

$$\mathrm{J}[\Psi] := \text{д} \otimes \|\Psi\|^2 - \mathrm{re}\, \mathrm{h}(\Psi,\ \mathrm{i}\, \vec{\nabla}^{\uparrow} \Psi) \in \sec\big(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big),$$

with observed and coordinate expressions

$$\begin{aligned}\mathrm{J}[\Psi] &= \|\Psi\|^2\, \text{д}[o] - \mathrm{re}\, \mathrm{h}(\Psi, \mathrm{i}\, \vec{\nabla}[o]\Psi) \\ &= \big(|\psi|^2\, \partial_0 + (\mathrm{i}\, \tfrac{1}{2}\, G^{ij}_0\, (\psi\, \partial_j \bar{\psi} - \bar{\psi}\, \partial_j \psi) - A^i_0\, |\psi|^2)\, \partial_i\big) \otimes u^0 .\end{aligned}$$

In particular, for each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we have the expression $\mathrm{J}[\Psi] = \|\Psi\|^2\, \mathrm{V}[\Psi]$, which emphasises the role of the two real polar degrees of freedom of the quantum particle.

In view of the discussion of quantum currents, it is convenient to introduce also the *quantum probability form* $\underline{\mathrm{J}}[\Psi] := i_{\mathrm{J}[\Psi]}\, \upsilon \in \sec(\boldsymbol{E}, \Lambda^3\, T\boldsymbol{E})$, with coordinate expression $\underline{\mathrm{J}}[\Psi] = \Big(|\psi|^2\, \upsilon^0_0 + \big(\mathrm{i}\, \tfrac{1}{2}\, G^{ij}_0\, (\psi\, \partial_j \bar{\psi} - \bar{\psi}\, \partial_j \psi) - A^i_0\, |\psi|^2\big)\, \upsilon^0_i\Big)$ (see Proposition 3.2.4).

1.5.9.5 Quantum Lagrangian

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the gauge independent and observer independent *quantum lagrangian* (see Theorem 17.5.2)

$$\mathrm{L}[\Psi] := -\, dt \wedge \Big(\mathrm{im}\, \mathrm{h}_\eta(\Psi,\ \text{д} \lrcorner \nabla^{\uparrow}\Psi) + \tfrac{1}{2}\, (\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\ \check{\nabla}^{\uparrow}\Psi)\Big) : \boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E},$$

with observed and coordinate expressions

$$\begin{aligned}\mathrm{L}[\Psi] &= -dt \wedge \big(\mathrm{im}\, \mathrm{h}_\eta(\Psi,\ \nabla_{\text{д}[o]}[o]\Psi) + \tfrac{1}{2}\, (\bar{G} \otimes \mathrm{h}_\eta)\, (\nabla[o]\Psi,\ \nabla[o]\Psi)\big), \\ \mathrm{L}[\Psi] &= \tfrac{1}{2}\, \big(-\, G^{ij}_0\, \partial_i \bar{\psi}\, \partial_j \psi + \mathrm{i}\, (\bar{\psi}\, \partial_0 \psi - \psi\, \partial_0 \bar{\psi}) - \mathrm{i} A^j_0\, (\bar{\psi}\, \partial_j \psi - \psi\, \partial_j \bar{\psi}) + 2\, \alpha_0\, \bar{\psi}\, \psi\big)\, \upsilon^0 ;\end{aligned}$$

With reference to the distinguished quantum basis b_Ψ, the distinguished observer o_Ψ, and the potential $A[\Psi]$ "seen by" Ψ, the above expression can be written in the following remarkable way (see Theorem 15.2.31 and Corollary 17.5.3)

$$\mathsf{L}[\Psi] = (\tfrac{1}{2}\,\bar{G}\,(d\,\|\Psi\|,\; d\,\|\Psi\|) + A[\Psi]\,\|\Psi\|^2) \otimes \upsilon.$$

We stress that, here again, the explicit mention of the phase polar degree of freedom of the quantum particle $(\!(\Psi)\!)$ disappears; however, it is implicitly encoded in $A[\Psi]$.

According to the standard lagrangian formalism, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the "*quantum momentum form*" (see Proposition 17.5.7)

$$\mathsf{P} := \vartheta_{\boldsymbol{Q}} \,\bar{\wedge}\, V_{\boldsymbol{Q}} \mathsf{L} : J_1 \boldsymbol{Q} \to \Lambda^4 T^* \boldsymbol{Q},$$

with coordinate expression

$$\begin{aligned}\mathsf{P} = \tfrac{1}{2}\,\mathfrak{i}\,(\bar{z}\,dz - z\,d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2}\,\big(G^{ij}_0\,(\bar{z}_i\,dz + z_i\,d\bar{z}) + \mathfrak{i}\,A^j_0\,(z\,d\bar{z} - \bar{z}\,dz)\big) \wedge \upsilon^0_j \\ + \Big(-\tfrac{1}{2}\,\mathfrak{i}\,(\bar{z}\,z_0 - z\,\bar{z}_0) + \tfrac{1}{2}\,\big(G^{ij}_0\,(\bar{z}_i\,z_j + z_i\,\bar{z}_j) + \mathfrak{i}\,A^j_0\,(z\,\bar{z}_j - \bar{z}\,z_j)\big)\Big)\,\upsilon^0.\end{aligned}$$

It is remarkable the following link between the kinetic quantum tensor and the quantum lagrangian (see Proposition 17.5.9)

$$\mathsf{Q}[\Psi] = -\mathfrak{i}\,(\mathrm{re}\,\mathrm{h})^\sharp (i_{\bar{\upsilon}} V_{\boldsymbol{Q}} \mathsf{L})[\Psi].$$

Further, the quantum lagrangian yields the gauge independent and observer independent "*quantum Poincaré–Cartan form*" (see Theorem 17.5.10)

$$\mathsf{C} := \mathsf{L} + \mathsf{P} : J_1 \boldsymbol{Q} \to \Lambda^4 T^* \boldsymbol{Q},$$

with coordinate expression

$$\begin{aligned}\mathsf{C} = \tfrac{1}{2}\,\mathfrak{i}\,(\bar{z}\,dz - z\,d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2}\,\big(G^{ij}_0\,(\bar{z}_i\,dz + z_i\,d\bar{z}) + \mathfrak{i}\,A^j_0\,(\bar{z}\,dz - z\,d\bar{z})\big) \wedge \upsilon^0_j \\ + \big(\tfrac{1}{2}\,G^{ij}_0\,\bar{z}_i\,z_j + \alpha_0\,\bar{z}\,z\big)\,\upsilon^0.\end{aligned}$$

Later, this form will play a role in the discussion of quantum symmetries.

1.5.9.6 Schrödinger Operator

For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the gauge independent and observer independent *Schrödinger operator* (see Theorem 17.6.5 and [219])

$$\mathsf{S}[\Psi] := \tfrac{1}{2}\,\big(\text{д} \lrcorner \nabla^\uparrow \Psi + \delta^\uparrow(\mathsf{Q}[\Psi])\big) \in \sec(\boldsymbol{E},\; \mathbb{T}^* \otimes \boldsymbol{Q}),$$

with observed and coordinate expression

$$\mathrm{S}[\Psi] = \nabla[o]_{\text{д}[o]}\Psi + \tfrac{1}{2}\,\mathrm{div}_\eta\,\text{д}[o]\,\Psi - \mathrm{i}\,\tfrac{1}{2}\,\Delta[G,o]\,\Psi,$$

$$\mathrm{S}[\Psi] = \Big(\partial_0\psi - \tfrac{1}{2}\,\mathrm{i}\,G_0^{ij}\,\partial_{ij}\psi - \Big(A_0^j + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\Big)\,\partial_j\psi$$
$$+ \tfrac{1}{2}\,\Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\;2\,\alpha_0\Big)\;\psi\Big)\,u^0 \otimes \mathrm{b}.$$

Several authors say that the Schrödinger equation is observer dependent; this fact happens if we consider an arbitrary phenomenological potential. But, if we deal with the joined gravitational and electromagnetic potential, then the Schrödinger equation turns out to be observer equivariant. In fact, the above joined potential fulfills a distinguished transition law (determined by the upper quantum connection), which turns out to be responsible for the observer equivariance of the Schrödinger equation. Of course, a possible additional phenomenological potential might be added by hand to our Schrödinger equation, but so doing we would break the covariance of the equation.

With reference to the distinguished quantum basis b_Ψ, the distinguished observer o_Ψ, and the potential $A[\Psi]$ "seen by" Ψ, the above expression can be written in the following remarkable way (see Corollary 17.6.9)

$$\mathrm{S}[\Psi] = \big(\text{д}[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\;\mathrm{div}_\eta\,\text{д}[o_\Psi] - \mathrm{i}\,(\tfrac{1}{2}\,\Delta[G]\|\Psi\| + A[\Psi])\,\|\Psi\|\big) \otimes \mathrm{b}_\Psi.$$

We stress that, once more, the explicit mention of the phase polar degree of freedom of the quantum particle $(\!(\Psi)\!)$ disappears in the above equation; however, it is implicitly encoded in o_Ψ and $A[\Psi]$.

Later, in the context of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ (see Proposition 22.6.2), we show that the Schrödinger operator S can be regarded as a connection of this infinite dimensional bundle. Actually, the role of the infinite dimensional symbols of this connection is played by the spacelike differential operator appearing in the expression of S.

1.5.10 Lagrangian Formalism in Quantum Mechanics

The procedures of standard Quantum Mechanics are largely inspired by the hamiltonian formalism, while the lagrangian formalism usually plays a secondary role, with a few exceptions in standard literature (see, for instance, [125, 267]).

In our covariant approach to Quantum Mechanics, we deal with the lagrangian formalism in several contexts.

We are not explicitly involved with the usual hamiltonian methods. The classical cosymplectic 2-form Ω, which appears in our formulation of Quantum Mechanics as the curvature of the upper quantum connection $\text{Ч}^\uparrow$, turns out to be the source of several geometric procedures, which, in a sense, might be regarded as hamiltonian type methods.

1.5.11 Hydrodynamical Picture of Quantum Mechanics

The *hydrodynamical picture* of standard Quantum Mechanics is a well known representation, which arose from the very beginning of the Quantum Theory (see, for instance, [176, 230, 280]). Nowadays, it is involved in the bohmian approach to Quantum Mechanics and is largely used in the physics of Bose–Einstein condensates (see, for instance, [8, 62, 179, 181] and [58, 340]).

In the present book, we provide a covariant version, in our curved galilean framework, of the standard hydrodynamical picture.

Thus, we show that each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ yields, in a covariant way, an observer independent and gauge independent 3-plet (see Theorem 18.1.1)

$$\mathfrak{F}[\Psi] := \big(\mathscr{C}[\Psi], \mu[\Psi], \rho[\Psi]]\big),$$

where (see Theorem 17.2.2)

$$\begin{aligned} \mathscr{C}[\Psi] &:= \partial \mathrm{V}[\Psi] \in \mathrm{map}\,\big((\mathbb{T} \times \mathbb{R}) \times \boldsymbol{E},\ \boldsymbol{E}\big), \\ \mu[\Psi] &:= m\, \|\Psi\|^2 \in \mathrm{map}(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes \mathbb{M}), \\ \rho[\Psi] &:= q\, \|\Psi\|^2 \in \mathrm{map}\,\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big). \end{aligned}$$

This, 3-plet can be regarded as a classical charged fluid moving in the classical spacetime $\boldsymbol{E}$ and effected by the gravitational field $K^{\natural}$ and the electromagnetic field F.

We show the clear intrinsic relations of the mass density $\mu[\Psi]$ and the velocity $\mathrm{V}[\Psi]$ with the two intrinsic real polar degrees of freedom $\|\Psi\|$ and $(\!(\Psi)\!)$ (see Theorem 18.1.1)

$$\mu[\Psi] = m\, \|\Psi\|^2 \quad \text{and} \quad \mathrm{V}[\Psi] = д + \vec{\nabla}^{\uparrow 0} (\!(\Psi)\!).$$

We discuss the correspondence between quantum sections and their hydrodynamical representation, showing that it is not a full bijection (see Remark 18.1.9).

We have also the equality $\mathrm{V}[\Psi] = д[o_{\psi}]$.

Hence, in the present context, we can reinterpret the distinguished observer o_{Ψ}, which has previously been introduced by Theorem 15.2.31, as the *"rest observer"* of the fluid (see Definition 18.1.3).

Moreover, we define the gauge independent and observer independent, *mass density current* and *charge density current* (see Definition 18.1.14)

$$\begin{aligned} \mathscr{P} &\equiv \mathscr{P}[\Psi] := \mu[\Psi]\, \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes \boldsymbol{TE}\big), \\ \mathscr{J} &\equiv \mathscr{J}[\Psi] := \rho[\Psi]\, \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \boldsymbol{TE}\big). \end{aligned}$$

Further, regardless of the Schrödinger equation, we prove the observer independent and gauge independent equality $\mathscr{A}[\Psi] = -\vec{d} A[\Psi]$, with coordinate expression $\mathscr{A}[\Psi] = -G^{ij}_0\, \partial_j A_0[\Psi]\, u^0 \otimes u^0 \otimes \partial_i$, which expresses the acceleration $\mathscr{A}[\Psi]$ of the fluid through the gradient of the distinguished gauge independent and observed independent timelike observed joined potential $A[\Psi]$ "seen" by the quantum particle (see Theorems 18.1.17 and 15.2.31).

Then, we achieve the dynamical law of motion in two steps.

First of all, regardless of the Schrödinger equation, we prove the following gauge independent and observer independent law of motion (see Theorem 18.2.1)

$$\mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] = -\rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\boldsymbol{p}[\Psi],$$

where the "*quantum pressure*" $\boldsymbol{p}[\Psi]$ is given, up to an arbitrary time function, by the equality

$$\boldsymbol{p}[\Psi] = -\tfrac{\hbar}{m}\,A[\Psi] \in \operatorname{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{2}) \otimes \mathbb{R})\big).$$

Thus, we interpret, up to a scale factor, the quantum pressure in terms of the distinguished timelike potential "seen" by the quantum particle.

Then, by considering the Schrödinger equation and its intrinsic polar splitting, we obtain the system (see Theorem 18.2.2)

$$\begin{aligned} \mathscr{V}[\Psi].\mu[\Psi] + \mu[\Psi]\ \operatorname{div}_{\eta} \mathscr{V}[\Psi] &= 0, \\ \mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] + \rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) - \mu[\Psi]\,\vec{d}\,\mathfrak{p}[\Psi] &= 0, \end{aligned}$$

where the "quantum pressure" $\mathfrak{p}[\Psi]$ is given, in terms of the mass density $\mu[\Psi]$, by the following "*constitutive equation*"

$$\mathfrak{p}[\Psi] = \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\,.$$

In comparison with the standard approach to the hydrodynamical picture of Quantum Mechanics, besides the generalisation to a curved spacetime and to observed expressions referred to any observer, we emphasise the physical interpretation of the quantum pressure in terms of the distinguished timelike potential "seen" by the quantum particle and the intrinsic polar splitting of the law of motion via the two real polar degrees of freedom.

1.5.12 Quantum Symmetries

1.5.12.1 Symmetries of the η-Quantum Metric

We start by discussing the *"η-hermitian quantum vector fields"*, i.e. the *infinitesimal symmetries* of the η-hermitian quantum metric $\mathfrak{h}_{\eta}$.

They are defined to be the projectable quantum vector fields $Y_{\eta} \in \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\boldsymbol{Q})$, such that $L_{Y_{\eta}} \mathfrak{h}_{\eta} = 0$.

We find that they are generated by projectable special phase functions according to the equalities (see Theorem 19.1.7)

$$Y_\eta = Y_\eta[f] = X[f] \lrcorner \chi[\mathrm{b}] + \big(\mathfrak{i}\hat{f}[\mathrm{b}] - \tfrac{1}{2}\,\mathrm{div}_\eta f\big)\,\mathbb{I}$$
$$= X[f] \lrcorner \text{Ч}[o] + \big(\mathfrak{i}\check{f}[o] - \tfrac{1}{2}\,\mathrm{div}_\eta f\big)\,\mathbb{I}, \quad \text{with } f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

i.e., in coordinates,

$$Y_\eta = Y_\eta[f] = f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,(\check{f} + A_0 f^0 - A_i f^i) - \tfrac{1}{2}\,\mathrm{div}_\eta f\big)\,\mathbb{I}$$
$$= f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\hat{f} - \tfrac{1}{2}\,\mathrm{div}_\eta f\big)\,\mathbb{I}.$$

Moreover, we prove that the map $Y_\eta : \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y_\eta[f]$ turns out to be an $\mathbb{R}$-Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.

1.5.12.2 Symmetries of the Quantum Structure

Next, we observe that the quantum triplet $(dt, \mathrm{h}^\uparrow_\eta, \text{Ч}^\uparrow)$ fully encodes the geometric structure of quantum theory. Accordingly, we define the *infinitesimal symmetries of quantum structure* to be the phase vector fields $Y^\uparrow{}_\eta \in \mathrm{lin}_{\mathbb{R}}\,\mathrm{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T},\boldsymbol{Q}}(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow)$, which fulfill the condition (see Definition 19.2.1)

$$L_{Y^\uparrow{}_\eta} dt = 0, \qquad L_{Y^\uparrow{}_\eta} \mathrm{h}^\uparrow_\eta = 0, \qquad L_{Y^\uparrow{}_\eta} \text{Ч}^\uparrow = 0.$$

Actually, the infinitesimal symmetries of quantum structure are generated by conserved time preserving special phase functions according to the equality

$$Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] = \text{Ч}^\uparrow\big(X^\uparrow[f]\big) + \mathfrak{i} f\,\mathbb{I}^\uparrow,$$

where $f \in \mathrm{cns\,timspe}(J_1\boldsymbol{E},\ \mathbb{R})$ and $X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f]$ (see Theorem 19.2.2).

1.5.12.3 Symmetries of the Quantum Dynamics

Moreover, we observe that the pair (dt, L) fully encodes the geometric structure of quantum dynamics. Accordingly, we define the *infinitesimal symmetries of quantum dynamics* to be the quantum vector fields $Y_\eta \in \mathrm{lin}_{\mathbb{R}}\,\mathrm{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$, which fulfill the condition (see Definition 19.3.1)

$$L_{Y^1_\eta} dt = 0 \quad \text{and} \quad L_{Y^1_\eta} \mathrm{L} = 0,$$

where $Y^1_\eta \in \sec(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q})$ is the 1-jet holonomic prolongation of Y_η.

Actually, the infinitesimal symmetries of quantum dynamics are generated by conserved time preserving special phase functions according to the equality (see Theorem 19.3.2)

$$Y_\eta = Y_\eta[f], \quad \text{with } f \in \text{cns timspe}(J_1\boldsymbol{E}, \mathbb{R}),$$

i.e., in coordinates,

$$Y_\eta = Y_\eta[f] = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\breve{f} + A_0 f^0 - A_i f^i)\,\mathbb{I},$$

where the functions $f^0, f^i, \breve{f} \in \text{map}(\boldsymbol{E}, \mathbb{R})$ fulfill the conditions stated in Corollary 12.6.14.

1.5.13 Quantum Differential Operators

We do not start with a postulate on a "*correspondence principle*" between classical and quantum theories. Our classical structure plays just the role of implicit background of the quantum structure. Namely, the primitive relation between classical and quantum structures is essentially played by the classical spacetime $\boldsymbol{E}$, as base space of the quantum bundle $\boldsymbol{Q}$, and by the classical cosymplectic 2-form Ω, as source of the curvature tensor of the "upper quantum connection" $\text{Ч}^\uparrow$ (see Postulates C.1, Q.1, and Q.2 and Definition 15.2.1).

Actually, afterwards, we do obtain a "*correspondence rule*" between classical and quantum objects (special phase functions on one side and quantum operators, quantum currents and quantum expectation forms on the other side) (see [218]).

However, this correspondence arises as a natural byproduct of a "*classification theorem*" which states the natural isomorphism between the "*Lie algebra of η-hermitian quantum vector fields*" and the "*Lie algebra of projectable special phase functions*". In fact, the η-hermitian quantum vector fields, generated by "*projectable special phase functions*", turn out to be the covariant source of *quantum operators*, *quantum currents* and *quantum expectation forms* (see Theorems 19.1.7, 20.1.9, 19.2.2, 19.3.2, 21.1.4, 21.1.9, 21.2.4 and 21.3.7). We achieve the *quantum differential operators* associated with time preserving special phase functions by means of the following covariant procedure.

We start by showing that the projectable quantum vector fields $Y \in \text{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ naturally act on quantum sections Ψ as 1st order differential operators (see Lemma 20.1.3), via the Lie derivative $Y(\Psi) \equiv L_Y\Psi \in \text{sec}(\boldsymbol{E}, \boldsymbol{Q})$, whose real coordinate expression is $Y(\Psi) = (X^\mu\,\partial_\mu\Psi^{\mathrm{a}} - \Psi^{\mathrm{b}}\,\partial_{\mathrm{b}} Y^{\mathrm{a}})\,\mathfrak{b}_{\mathrm{a}}$.

In particular, we obtain the following distinguished quantum differential operators (see Example 20.1.6)

$$\begin{aligned}
Y_\eta[x^\lambda](\Psi) &= -\mathrm{i}\, x^\lambda\, \psi\, \mathfrak{b}, \\
Y_\eta[\mathcal{Q}_j](\Psi) &= -\Big(\nabla_j \psi - \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathfrak{b}, \\
Y_\eta[\mathcal{P}_j](\Psi) &= -\Big(\partial_j \psi - \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathfrak{b}, \\
Y_\eta[\mathcal{K}_0](\Psi) &= \Big(\nabla_0 \psi + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} \psi\Big)\, \mathfrak{b}, \\
Y_\eta[\mathcal{H}_0](\Psi) &= \Big(\partial_0 \psi - \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathfrak{b}.
\end{aligned}$$

Now, the quantum differential operators $Y_\eta[f]$ associated with affine special phase functions turn out to be spacelike differential operators, while the quantum differential operator $Y_\eta[\mathcal{H}_0]$ involves the time derivative. Actually, a criterion of standard Quantum Mechanics requires that a measurement performed at a certain time $\mathrm{t} \in \boldsymbol{T}$ should depend on the restriction $\Psi_{\mathrm{t}} \in \sec(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}})$ of the quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$. So, for each affine special phase function f, we might regard $Y_\eta[f]$ as the corresponding quantum operator, but $Y_\eta[\mathcal{H}_0]$, as it stands, does not fulfill the above criterion.

We overcome this obstruction by taking into account the Schrödinger operator and postulating (see Theorem 20.1.9), for each time preserving special phase function f, the *quantum differential operator* $\mathrm{O}[f] = \mathrm{i}\,\big(Y_\eta[f] - f'' \lrcorner\, \mathrm{S}\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$, with coordinate expression $\mathrm{O}[f](\Psi) = \Big(\big(\breve{f} - A_i f^i - \mathrm{i}\,(f^i\, \partial_i + \tfrac{1}{2} \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}}) - \tfrac{1}{2} f^0\, \Delta_0\big)\, \psi\Big)\, \mathfrak{b}$.

In particular, we have the following distinguished cases (see Example 20.1.12)

$$\begin{aligned}
\mathrm{O}[x^\lambda](\Psi) &= x^\lambda\, \psi\, \mathfrak{b}, \\
\mathrm{O}[\mathcal{Q}_j](\Psi) &= -\mathrm{i}\,\Big(\nabla_j \psi + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathfrak{b}, \\
\mathrm{O}[\mathcal{P}_j](\Psi) &= -\mathrm{i}\,\Big(\partial_j \psi + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathfrak{b}, \\
\mathrm{O}[\mathcal{K}_0](\Psi) &= -(\tfrac{1}{2}\, \Delta_0\, \psi)\, \mathfrak{b}, \\
\mathrm{O}[\mathcal{H}_0](\Psi) &= -\big(\tfrac{1}{2}\, \Delta_0\, \psi + A_0\, \psi\big)\, \mathfrak{b}.
\end{aligned}$$

We stress that the above procedure, originated by the classification of η-hermitian quantum vector fields and the Schrödinger operator, deals with spacetime coordinates, components of momentum and energy on the same footing.

It is worth mentioning that, for each affine special phase function $f \in \operatorname{aff\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ and each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the real and imaginary components of the spacetime complex function $\mathrm{O}[f](\Psi)/\Psi \in \sec(\boldsymbol{E}, \mathbb{C})$ factorise through the norm and phase components of Ψ, respectively (see Corollary 20.1.19).

For instance, we have the following equalities (see Example 20.1.20)

$$\operatorname{re} \frac{\mathrm{O}[x^\lambda](\Psi)}{\Psi} = x^\lambda, \qquad \operatorname{im} \frac{\mathrm{O}[x^\lambda](\Psi)}{\Psi} = 0,$$

$$\operatorname{re} \frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} = \partial_j \varphi, \qquad \operatorname{im} \frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} = -\partial_j \log|\psi| - \tfrac{1}{2} \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}.$$

Further, we define the commutator

$$\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] := -\,\mathrm{i}\,\big((\mathrm{O}[\acute{f}] \circ \mathrm{O}[f]) - (\mathrm{O}[f] \circ \mathrm{O}[\acute{f}])\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$$

and show the equality (see Theorem 20.1.24)

$$\begin{aligned}\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] = \mathrm{i}\, Y_\eta\big[[\![\acute{f}, f]\!] \big] &+ \mathrm{i}\,\big(Y_\eta[f] \circ \mathrm{S}[\acute{f}] - \mathrm{S}[\acute{f}] \circ Y_\eta[f]\big) \\ &+ \mathrm{i}\,\big(\mathrm{S}[f] \circ Y_\eta[\acute{f}] - Y_\eta[\acute{f}] \circ \mathrm{S}[f]\big).\end{aligned}$$

In particular, we obtain the following distinguished cases (see Example 20.1.28)

$$\begin{aligned}
&(0) \quad \big[\mathrm{O}[x^0], \mathrm{O}[f]\big](\Psi) = \mathrm{O}\big[[\![x^0, f]\!] \big](\Psi) = 0,\\
&(1) \quad \big[\mathrm{O}[x^\lambda], \mathrm{O}[x^\mu]\big](\Psi) = \mathrm{O}\big[[\![x^\lambda, x^\mu]\!] \big](\Psi) = 0,\\
&(2) \quad \big[\mathrm{O}[x^\lambda], \mathrm{O}[\mathcal{P}_j]\big](\Psi) = \mathrm{O}\big[[\![x^\lambda, \mathcal{P}_j]\!] \big](\Psi) = \delta^\lambda_j\, \Psi,\\
&(3) \quad \big[\mathrm{O}[\mathcal{P}_i], \mathrm{O}[\mathcal{P}_j]\big](\Psi) = \mathrm{O}\big[[\![\mathcal{P}_i, \mathcal{P}_j]\!] \big](\Psi) = 0,\\
&(4) \quad \big[\mathrm{O}[x^0], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = 0,\\
&(5) \quad \big[\mathrm{O}[x^i], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = \mathrm{i}\, Y_\eta[x^i_0](\Psi),\\
&(6) \quad \big[\mathrm{O}[\mathcal{P}_i], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = \mathrm{i}\,\Big(\mathrm{S}_0\big(Y_\eta[\mathcal{P}_j](\Psi)\big) - Y_\eta[\mathcal{P}_j]\,\big(\mathrm{S}_0(\Psi)\big)\Big).
\end{aligned}$$

In order to compare the above commutators with the corresponding commutators of standard Quantum Mechanics, we recall that our special phase functions $\mathcal{P}_i$ and $\mathcal{H}_0$ are normalised through the Planck constant $\hbar$ via the rescaled metric $G := \frac{m}{\hbar}\, g$.

1.5.14 Quantum Currents

We propose a systematic lagrangian method to achieve a quantum current for every projectable special phase function.

Actually, a general notion of quantum current (besides the probability current) and its systematic link with quantum operators can be hardly found in the literature concerning standard Quantum Mechanics. In fact, our systematic theory on quantum currents is strictly related to the unusual Lie algebra of special phase functions and to the lagrangian formulation of Quantum Mechanics.

Thus, we define the *quantum current*, associated with a projectable special phase function $f \in \text{pro}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})$, to be the local gauge dependent and observer independent the horizontal 3-form $\mathrm{j}_\eta[f] := -\, i_{Y^{\uparrow}_\eta[f]}\, \mathrm{C} = -i_{Y_\eta[f]}\, \mathrm{C} \in \sec(J_1\boldsymbol{Q}, \Lambda^3\, T^*\boldsymbol{Q})$ (see Definition 21.1.3).

Then, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the *quantum current form* (see Proposition 21.2.2) $\mathrm{j}_\eta[f](\Psi) := (j_1\Psi)^*\mathrm{j}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3 T^*\boldsymbol{E})$, with coordinate expression

$$\begin{aligned}
\mathrm{j}_\eta[f](\Psi) = \tfrac{1}{2}\,\big(&G^{hk}_0\, \partial_h\bar\psi\, \partial_k\psi + \mathrm{i}\, A^h_0\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi) \\
&- 2\,(A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, |\psi|^2\big)\, \big(f^0\, \upsilon^0_0 - f^j\, \upsilon^0_j\big) \\
&- f^0\, \tfrac{1}{2}\, \big(G^{hj}_0\, (\partial_h\bar\psi\, \partial_0\psi + \partial_h\psi\, \partial_0\bar\psi) + \mathrm{i}\, A^j_0\, (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\big)\, \upsilon^0_j \\
&+ f^i\, \tfrac{1}{2}\, \big(G^{jh}_0\, (\partial_i\bar\psi\, \partial_h\psi + \partial_i\psi\, \partial_h\bar\psi) + \mathrm{i}\, A^j_0\, (\bar\psi\, \partial_i\psi - \psi\, \partial_i\bar\psi)\big)\, \upsilon^0_j \\
&- f^j\, \tfrac{1}{2}\, \mathrm{i}\, \big((\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \upsilon^0_0 - (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\, \upsilon^0_j\big) \\
&- \hat f\, \big(\tfrac{1}{2}\, \mathrm{i}\, G^{hj}_0\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi)\, \upsilon^0_j - |\psi|^2\, (\upsilon^0_0 - A^j_0\, \upsilon^0_j)\big) \\
&- \tfrac{1}{4}\, \mathrm{div}_\eta f\; G^{hj}_0\, \partial_h|\psi|^2\, \upsilon^0_j .
\end{aligned}$$

In particular, we have the following distinguished examples (see Example 21.2.3)

$$\begin{aligned}
&\mathrm{j}_\eta[x^\lambda](\Psi) \\
&= x^\lambda\, \big(-\tfrac{1}{2}\, \mathrm{i}\, G^{hk}_0\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi)\, \upsilon^0_k + |\psi|^2\, (\upsilon^0_0 - A^h_0\, \upsilon^0_h)\big), \\
&\mathrm{j}_\eta[\mathcal{P}_j](\Psi) \\
&= -\tfrac{1}{2}\, \big(G^{hk}_0\, \partial_h\bar\psi\, \partial_k\psi + \mathrm{i} A^h_0\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi) - 2\,(A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, |\psi|^2\big)\, \upsilon^0_j \\
&\quad + \tfrac{1}{2}\, \big(G^{hk}_0\, (\partial_j\bar\psi\, \partial_h\psi + \partial_j\psi\, \partial_h\bar\psi) + \mathrm{i}\, A^k_0\, (\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\big)\, \upsilon^0_k \\
&\quad - \tfrac{1}{2}\, \mathrm{i}\, \big((\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \upsilon^0_0 - (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\, \upsilon^0_j\big) \\
&\quad + \tfrac{1}{4}\, \frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\, G^{hk}_0\, \partial_h|\psi|^2\, \upsilon^0_k, \\
&\mathrm{j}_\eta[\mathcal{H}_0](\Psi) \\
&= \tfrac{1}{2}\, \big(G^{ij}_0\, \partial_i\bar\psi\, \partial_j\psi + \mathrm{i} A^i_0\, (\bar\psi\, \partial_i\psi - \psi\, \partial_i\bar\psi) - 2\,(A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, |\psi|^2\big)\, \upsilon^0_0 \\
&\quad - \tfrac{1}{2}\, \big(G^{ij}_0\, (\partial_i\bar\psi\, \partial_0\psi + \partial_i\psi\, \partial_0\bar\psi) + \mathrm{i}\, A^j_0\, (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\big)\, \upsilon^0_j \\
&\quad - \tfrac{1}{4}\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, G^{hj}_0\, \partial_h|\psi|^2\, \upsilon^0_j .
\end{aligned}$$

Indeed, the quantum current associated with the distinguished projectable special phase function $f = 1$ turns out to be just the Hodge star of the *quantum probability current* $\mathrm{j}_\eta[1] = *_\upsilon(\mathrm{J})$ (see Theorem 17.4.2).

So, we can regard the well known quantum probability current as a particular result of our general procedure by which we associate, in a covariant way, a quantum current with each projectable special phase function.

1.5.15 Quantum Expectation Forms

We define the *quantum expectation form* associated with a projectable special phase function to be the vertical 3-form $\epsilon_\eta[f](\Psi) := \mathrm{re}\, \mathrm{h}_\eta\big(\Psi,\ \mathrm{O}[f](\Psi)\big) \in \sec(\boldsymbol{E}, \Lambda^3 V^*\boldsymbol{E})$, with coordinate expression (see Definition 21.3.1)

$$\begin{aligned}&\epsilon_\eta[f](\Psi)\\ &= \big((\check{f} - A_i f^i)\,|\psi|^2 - \tfrac{1}{2}\,\mathrm{i} f^i\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi}) - \tfrac{1}{4} f^0\,(\bar{\psi}\,\Delta_0\psi + \psi\,\overline{\Delta_0\psi})\big)\,\eta.\end{aligned}$$

In particular, we have the following distinguished examples (see Example 21.3.3):

$$\begin{aligned}\epsilon_\eta[x^\lambda](\Psi) &= x^\lambda |\psi|^2\,\eta,\\ \epsilon_\eta[\mathcal{P}_j](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi)\,\eta,\\ \epsilon_\eta[\mathcal{H}_0](\Psi) &= -\big(A_0\,|\psi|^2 + \tfrac{1}{4}\,(\psi\,\overline{\Delta[G]_0\psi} + \bar{\psi}\,\Delta[G]_0\psi)\big)\,\eta.\end{aligned}$$

Now, the vertical restriction of quantum current forms and the quantum expectation forms have the same source and target, even if they have been achieved by very different procedures. So, it is natural to compare them. Thus, we consider the difference

$$\mathfrak{d}_\eta[f](\Psi) := \check{\mathfrak{j}}_\eta[f](\Psi) - \epsilon_\eta[f](\Psi) \in \sec(\boldsymbol{E}, \Lambda^3 V^*\boldsymbol{E}),$$

with coordinate expression $\mathfrak{d}_\eta[f](\Psi) = \frac{1}{4} f^0 \left(G_0^{ij}\,\partial_{ij}|\psi|^2 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi|^2\right)\eta$, and prove that its spacelike integral $\int \mathfrak{d}_\eta[f](\Psi)$ vanishes (see Theorem 21.3.7).

1.5.16 Hilbert Quantum Bundle

The main part of the present book deals with differentiable aspects of Quantum Mechanics discussed in terms of the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and its sections $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$.

We provide a preliminary introduction to the Hilbert stuff in a way which is suitable for our manifestly covariant approach, leaving further developments to the reader.

In standard Quantum Mechanics, one usually deals with a given inertial observer o, an affine space time $\boldsymbol{E}$, which splits as $\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}[o]$, and a globally trivialised quantum bundle $\boldsymbol{Q} \simeq \boldsymbol{E} \times \mathbb{C}$. Accordingly, the quantum sections Ψ are usually represented by wave functions $\psi : \boldsymbol{T} \times \boldsymbol{P}[o] \to \mathbb{C}$.

In this contest, one usually deals with a Hilbert space $\boldsymbol{H}[o]$ consisting of suitable maps $\boldsymbol{P}[o] \to \mathbb{C}$. So, usually one represents the evolution of a quantum state by a map of the type $\hat{\psi} : \boldsymbol{T} \to \boldsymbol{H}[o]$.

Indeed, such a framework is unsuitable for our manifestly covariant approach, which is aimed at an observer independent and gauge independent formulation of Quantum Mechanics.

So, we define a "*sectional quantum bundle*" $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$, whose fibres $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ consist of *smooth sections with compact support* $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ (see Definition 22.2.6).

Then, after a completion procedure, the η-hermitian metric h_{η} makes, for every time $\mathrm{t} \in \boldsymbol{T}$, the fibres $\widehat{\boldsymbol{Q}}_{\mathrm{t}} \subset \widehat{\boldsymbol{Q}}$ observer independent Hilbert spaces $\boldsymbol{H}_{\mathrm{t}}$. Accordingly, the sectional quantum bundle turns out to be a "*Hilbert bundle*" $\boldsymbol{H} \to \boldsymbol{T}$ over time (see Proposition 22.4.1).

We stress that there is no distinguished splitting of this bundle.

The fact that we deal with different Hilbert spaces for different times suggests the need of a possible extension of the postulates of standard Quantum Mechanics concerning quantum operators, in order to include in an appropriate way "quantum operators" acting between different Hilbert spaces at different times. Likely, this is, for instance, the case of "arrival time" (see, for instance, [26, 192, 235–237, 320] and literature therein).

We take these considerations just has a hint for possible developments, which are out of the scope of the present book.

In the present introductory presentation, in order to skip some hard problems of infinite dimensional differential geometry for handling the above material, we use the geometric language of F-smooth spaces, which is able to achieve some geometric results without preliminary strong hypotheses (see [225]). Of course, for further detailed developments, the techniques of infinite dimensional geometry could not be omitted.

In particular, we show that the quantum operators $\mathrm{O}[f]$ associated with time preserving projectable special phase functions are symmetric (hermitian) (see Theorem 22.5.5).

Moreover, as we have already mentioned, the Schrödinger operator $\mathrm{S} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}$ can be naturally regarded as the covariant differential associated with a connection $\text{Ш} : \widehat{\boldsymbol{Q}} \to \mathbb{T}^* \otimes T\widehat{\boldsymbol{Q}}$ of the sectional quantum bundle, where the 2nd order term of S plays the role of the *infinite dimensional symbols of this connection* (see Proposition 22.6.2).

1.5.17 Feynman Amplitudes

We provide an introductory covariant approach to the Feynman amplitudes for path integral (see Theorem 23.2.2).

In fact, we show that the composition of the upper quantum connection $\text{Ч}^{\uparrow}$ and the dynamical phase connection γ yields a connection $\underline{\text{Ч}}^{\uparrow}$ of the fibred manifold over time given by the composition $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E} \to \boldsymbol{T}$. It is remarkable that this connection naturally involves the classical lagrangian $\mathcal{L}$.

The above result allows us to compute the covariant differential of a quantum section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ along a motion $s : \boldsymbol{T} \to \boldsymbol{E}$ (see Proposition 23.1.1)

$$\nabla^{\uparrow}{}_{s}\Psi := ds \lrcorner (\nabla^{\uparrow}\Psi^{\uparrow}) \circ j_1 s : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{Q},$$

with coordinate expression

$$(x^0, x^i;\ u_0 \otimes z) \circ \nabla^{\uparrow}{}_s \Psi = \big(x^0, s^i;\ \partial_0(\psi \circ s) - \mathrm{i}\,(\mathcal{L}_0 \circ j_1 s)\,(\psi \circ s)\big).$$

Then, we obtain $\nabla^{\uparrow}{}_s \Psi = 0$ if and only if (see Lemma 23.2.1)

$$(\Psi \circ s)(\mathrm{t}) = (\Psi \circ s)(\mathrm{t}_0)\ \exp\Big(\mathrm{i} \int_{[\mathrm{t}_0, \mathrm{t}]} (\mathcal{L} \circ j_1 s)\Big), \quad \text{with } \mathrm{t}_0,\ \mathrm{t} \in \boldsymbol{I}.$$

1.5.18 Comparison with Geometric Quantisation

Among the geometric approaches to Quantum Mechanics, we must mention the well known *Geometric Quantisation* (see, for instance, [1, 9–11, 21, 24, 50, 51, 64, 65, 84, 102, 105, 109, 133, 142, 143, 149–151, 151–160, 167, 168, 233, 238, 250, 263, 313, 336, 338, 348, 369, 372, 373, 379, 387, 395, 396, 399–402, 425]).

Geometric Quantisation is aimed at an ambitious general programme of quantisation of classical systems via a suitable quantum representation of the classical Poisson Lie algebra.

This method has been successful in several respects, even if it met difficulties related to “polarisations” and “ordering”. Eventually, some interesting examples of quantum systems can be included in this scheme.

By comparing Geometric Quantisation with Covariant Quantum Mechanics, we can mention analogies and differences, as well.

First of all, we stress that we do not propose a general quantisation programme of a classical system, but we develop an approach to Quantum Mechanics based on a specific model of spacetime and guided by the principle of covariance.

By taking into account the requirement of covariance, the essential role of time and a criterion of minimality, we choose, as classical phase space, the odd dimensional 1st-jet space $J_1\boldsymbol{E} \to \boldsymbol{E}$ of the fibred spacetime manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Proposition 2.5.1). This odd dimensional classical phase space $J_1\boldsymbol{E}$ replaces the typical even dimensional phase space $\boldsymbol{M}$ of Geometric Quantisation.

As a consequence of our postulates on the gravitational and electromagnetic fields, the classical phase space $J_1\boldsymbol{E}$ turns out to be naturally equipped with a cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ (see Definition 9.1.3 and Theorem 9.2.8). So, the cosymplectic structure (dt, Ω) of the phase space $J_1\boldsymbol{E}$ replaces the typical symplectic structure (ω) of the even dimensional phase space $\boldsymbol{M}$ considered by Geometric Quantisation (see Theorem 10.1.1). Actually, a symplectic 2-form ω of the vertical bundle $V\boldsymbol{E}$ arises also in our model (see Proposition 3.2.14 and Remark 10.1.15). But, such a spacelike object, is not suitable for achieving the classical and quantum dynamics in a covariant way.

In the two theories, the difference between the odd and even dimension of the phase space, which arises from the difference of the role of time, yields great differences in the hamiltonian formalism (see Theorems 10.1.8, 11.2.6 and 11.2.11, Proposition 11.1.4, and Definition 11.3.6).

In our approach a fundamental role is played by the "*Lie algebra of special phase functions*", which replaces the role played by the Poisson Lie algebra of all phase functions in Geometric Quantisation (see Definitions 12.1.1, 12.5.1 and Theorem 20.1.9). Indeed, the Lie algebra of special phase functions deals with energy and momentum on the same footing.

The fact that, in our approach, the phase space $J_1\boldsymbol{E}$ is a bundle over the spacetime $\boldsymbol{E}$ and that the cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ admits locally a horizontal potential $A^\uparrow : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ allows us to follow the following procedure, according to a minimality requirement (see Proposition 2.5.1 and Theorem 10.1.4).

Our quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ is based on the spacetime $\boldsymbol{E}$, not on the phase space $J_1\boldsymbol{E}$ (see Postulate Q.1, Propositions 14.2.1 and 14.3.1). We introduce, by pullback, the upper quantum bundle $\pi^\uparrow : \boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$ based on the classical phase space $J_1\boldsymbol{E}$ (see Definition 14.11.1).

Analogously to Geometric Quantisation, we postulate an upper quantum connection $Ч^\uparrow : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ (see Definition 15.2.1 and Postulate Q.2). Analogously to Geometric Quantisation, we require hermitianity of $Ч^\uparrow$ and proportionality of the curvature $R[Ч^\uparrow]$ with the cosymplectic phase 2-form Ω. However, besides these hypotheses, we postulate the "*reducibility*" of the upper quantum connection $Ч^\uparrow$ (see Definition 15.1.6). Indeed, such an additional hypothesis, fulfills a criterion of minimality and is allowed by the construction of the upper quantum bundle $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$ by pullback and by the fact that Ω admits horizontal potentials.

The upper quantum bundle $\boldsymbol{Q}^\uparrow$, which has been defined by pullback, has already the appropriate minimal dimension; so, we do not need to "kill" some dimensions and search for polarisations. In a sense, we deal with a fixed distinguished polarisation, which is naturally induced by the fibring of phase space over spacetime. So, we can skip a difficult problem faced by Geometric Quantisation.

In our approach, the upper quantum connection $Ч^\uparrow$ is the source of all dynamical quantum objects. The base space of the upper quantum bundle $\boldsymbol{Q}^\uparrow$ is the classical phase space $J_1\boldsymbol{E}$, which encodes all classical observers. So, in principle, the objects derived from $Ч^\uparrow$ would be observer dependent explicitly. But, the covariance of the quantum theory requires to get rid of observers. In order to address this problem, we propose an original and effective "*criterion of projectability*", as an implementation of covariance (see Note 17.1.1). In a few words, we show that, for all main dynamical quantum objects, there exist two natural objects living on $J_1\boldsymbol{E}$ and a combination which project on the spacetime $\boldsymbol{E}$. Thus, such a projection turns out to be the searched observer independent dynamical quantum object (see Theorems 17.2.2, 17.3.2, Corollary 17.3.5, Theorems 17.4.2, 17.5.2, and 17.6.5). In a sense, this method replaces the quotient procedure with respect to polarisations, which is typical of Geometric Quantisation.

Typical no-go results of Geometric Quantisation (for instance the Van Hove Theorem [405]) cannot be applied to Covariant Quantum Mechanics because the contexts and the goals of the two theories are rather different.

We do not postulate a principle of correspondence; actually, our quantum operators are suggested by a classification theorem concerning hermitian quantum vector fields (see Theorems 19.1.7 and 20.1.9).

In the flat case and with reference to inertial observers, Covariant Quantum Mechanics reproduces well-known examples of standard Quantum Mechanics.

Thus, we stress that, in spite of the differences between the two theories, our approach to Covariant Quantum Mechanics has been inspired by some successful ideas of Geometric Quantisation. In particular, we have borrowed the idea of a quantum bundle and the link of the curvature of the upper quantum connection with a classical phase 2-form (see Definition 15.2.1 and Postulate Q.2).

1.5.19 Open Problem: Angular Momentum

The angular momentum is highly problematic for our covariant approach to Classical Mechanics and Quantum Mechanics, for the following reasons.

(a) The standard notion of angular momentum $\mathfrak{L}$ involves the "distance vector" $\vec{r}$ between the position of the test particle and a "reference centre position". Indeed, the definition of such a vector demands an affine background. By the way, we stress that the same problem arises also in einsteinian General Relativity.
So, the implementation of the standard notion of angular momentum in our curved spacetime requires a model of spacetime that has, at least, affine fibres. Actually, in our covariant approach, we consider the example of the standard flat newtonian spacetime and other models of spacetime with affine fibres; thus, in these models of spacetime, the standard definition of angular momentum can be achieved.

(b) In the standard theory of angular momentum, an important role is played by the Poisson identities fulfilled by its components (see, for instance, [111]). Indeed, such identities hold in cartesian coordinates, but need not to be true in curvilinear coordinates; in the book we provide counter-examples of this fact. So, this aspect of the theory of angular momentum fails in curvilinear coordinates, even in the standard flat newtonian spacetime! Thus, this turns out to be a feeble formal feature, with respect to covariance, not only of our approach, but also of standard Classical Mechanics and standard Quantum Mechanics.

(c) The square $\mathfrak{L}^2$ of angular momentum plays an important role in standard Classical Mechanics and standard Quantum Mechanics. Unfortunately, this function is not a special phase function. So, we cannot apply to $\mathfrak{L}^2$ our covariant approach to quantisation, as we do, for instance, for $\mathcal{H}$ and $\mathcal{P}$.
For the above reasons, the treatment of angular momentum in our covariant approach has feeble features. So far, this is an open problem, at our knowledge. Perhaps, the solution might pass through a very different covariant definition of the notion of angular momentum in a curved background, that yields the standard results in the standard flat case.

1.5.20 Examples

In the 3rd Part of the book (see Part III), we discuss the *standard flat model of classical spacetime* within our geometric language. This spacetime gives us the opportunity to systematically provide an explicit expression of the main classical and quantum objects in the arena of standard Quantum Mechanics.

We consider three cases: vanishing electromagnetic field, radial electric field, constant magnetic field. Moreover, we deal with three kinds of observers: inertial observer, uniformly accelerated observer, uniformly rotating observer. Accordingly, we refer to cartesian and curvilinear coordinates adapted to the above observers.

According to the introductory character of the present book, we leave to the reader the task to develop and discuss concrete physical examples, by using the above formalism.

By the way, we add a discussion of the *curved newtonian spacetime*, which gives us the opportunity to recover the Newton law of gravitation within our general covariant approach.

1.6 Algebraic and Geometric Language

1.6.1 Fibred Manifolds and Bundles

In the present book, all manifolds and all maps between manifolds are are supposed to be smooth (unless a different hypothesis is explicitly mentioned).

We recall that a manifold $\boldsymbol{F}$ is said to be "*fibred*" on a "*base manifold*" $\boldsymbol{B}$ if it is equipped with a surjective map $p : \boldsymbol{F} \to \boldsymbol{B}$, whose *rank* equals the dimension of $\boldsymbol{B}$ (see, for instance, [47, 341]).

Thus, by hypothesis, in virtue of the rank theorem, for each point $f \in \boldsymbol{F}$ there is a splitting $\Phi : \boldsymbol{V} \simeq \boldsymbol{U} \times \mathbb{F}$ of an open neighbourhood $\boldsymbol{V} \subset \boldsymbol{F}$ of f as the cartesian product of an open subset $\boldsymbol{U} \subset \boldsymbol{B}$ and *local type fibre* $\mathbb{F}$, which makes the following diagram commutative

$$\begin{array}{ccc} \boldsymbol{V} & \xrightarrow{\ \Phi\ } & \boldsymbol{U} \times \mathbb{F} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pro}_1} \\ \boldsymbol{U} & \xrightarrow{\ \mathrm{id}\ } & \boldsymbol{U} \end{array} \quad .$$

In particular, a "*(fibre) bundle*" can be defined to be a fibred manifold which, for each $f \in \boldsymbol{F}$, admits "*tubelike*" trivialisations over the base space $\Phi : p^{-1}(\boldsymbol{U}) \simeq \boldsymbol{U} \times \mathbb{F}$, where $\boldsymbol{U} \subset \boldsymbol{B}$ is an open neighbourhood of $p(f)$.

We stress that, if $p : \boldsymbol{F} \to \boldsymbol{B}$ is a bundle and we drop a point of $\boldsymbol{F}$, then it is no longer a bundle, but it is still a fibred manifold. So, in a sense, we can consider the concept of fibred manifold as a generalisation of the concept of bundle, suitable to account for singular points.

In the present book, in the context of sections, fibred morphisms and differential operators, we frequently use the concepts of *sheaf* and *sheaf morphism*.

In simple words, these concepts just indicate, in a quick synthetic way, a *family of local objects* and a *map between such families*, which behave in a natural way with respect to local restrictions of the domain and glueing of domains (see Appendix: Note A.1.3).

Part I
Covariant Classical Mechanics

The first part of the book is devoted to a covariant formulation of Classical Mechanics in a curved galilean framework, by regarding it as the classical background of Covariant Quantum Mechanics (see, for instance, [193, 196, 214, 223, 224, 226, 227, 358] and citations therein).

We introduce the *galilean curved spacetime manifold fibred over absolute time* and equipped with the *galilean metric*, *gravitational* and *electromagnetic fields*. In this context, we discuss the classical phase space equipped with a *cosymplectic structure*.

Moreover, we introduce the *Lie algebra of special phase functions*, which will play a fundamental role in our approach to Quantum Mechanics.

Eventually, we discuss the *Lie algebra of infinitesimal symmetries of galilean Classical Mechanics*.

Chapter 2
Spacetime

We introduce our model of *spacetime* regarded as a 4-dimensional manifold fibred over absolute time (Sect. 2.1).

Then, we analyse the *tangent* and *cotangent bundles* of spacetime, along with their vertical and horizontal subbundles (Sect. 2.2). We introduce the notions of *particle motions* and *continuum motions* (Sect. 2.4). The classical *phase space* is defined to be the 7-dimensional 1st jet bundle of spacetime fibred manifold; it is equipped with the natural *contact map* and *complementary contact map* (Sects. 2.5 and 2.6). Classical *observers* are defined as sections of the phase bundle (Sect. 2.7). Every observer yields a splitting of the tangent and cotangent bundles and locally a splitting of the spacetime fibred manifold.

We stress that our hypothesis that the fibres of spacetime be 3-dimensional reflects the standard model of classical spacetime. Actually, this hypothesis plays a crucial role only in a few developments of the theory. Throughout the present book, we maintain the hypothesis that the fibres of spacetime be 3-dimensional. However, one can use our approach also in different contexts, by regarding our spacetime as a "configuration space" of a different model (see, for instance, the model for the configuration space of a rigid body [314, 385, 408]). In such a case, it might be required to postulate a different dimension of the fibres of spacetime.

2.1 Spacetime Fibring

We start by introducing the *curved galilean spacetime* as a fibred manifold over time $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Appendix: Definition A.1.1), along with the *spacetime charts* (x^0, x^i) and the *spacetime covariance group* of fibred isomorphisms $f_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E}$.

Postulate C.1 *We postulate* time *to be an oriented 1-dimensional affine space* $\boldsymbol{T}$*, associated with the vector space* $\bar{\mathbb{T}} := \mathbb{T} \otimes I\!R$ *(see Sect. 1.3.5), and* spacetime *to be an oriented 4-dimensional manifold* $\boldsymbol{E}$ *equipped with a* time fibring*, i.e. a surjective projection of rank 1 (see Appendix: Definition A.1.1)*

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$$t : \boldsymbol{E} \to \boldsymbol{T} . \quad \square$$

In particular, we might assume the more restrictive hypothesis that the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ be a bundle (see Appendix: Definition A.2.1). Here, we have postulated, more generally, that $t : \boldsymbol{E} \to \boldsymbol{T}$ be a fibred manifold, in order to include possible singular events. Actually, in most steps of the book this generality does not yield any additional difficulty.

Remark 2.1.1 Let us recall a classical Theorem of Differential Geometry (see Appendix: Theorem A.2.6), which states that any bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ with a "contractible" base space $\boldsymbol{B}$ is globally trivialisable (see, for instance, [374, Corollary 1.6], [191, p. 488]).

Now, being $\boldsymbol{T}$ contractible, in virtue of the above Theorem, if the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ is a bundle, then it is globally trivialisable, hence we have a global bundle isomorphism of the type $\boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{S}$. But, we stress that, even in the case when spacetime is a bundle, we do not postulate any distinguished bundle splitting as above. $\square$

Note 2.1.2 We do not represent time as $\mathbb{R}$ (as it is often done), because we do not choose any distinguished origin $\mathrm{t} \in \boldsymbol{T}$ and any distinguished unit of measurement $u_0 \in \mathbb{T}$. This fact is reflected in many steps of the classical and quantum theory.

The vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$ represents the space of "*time intervals*".

The "affine translation" $\tau : \bar{\mathbb{T}} \times \boldsymbol{T} \to \boldsymbol{T}$ agrees with the standard postulate by which two time intervals with different origin can be compared; this is just the postulate of existence of "good clocks".

In the present galilean framework, the affine space $\boldsymbol{T}$ is "absolute", as it does not depend on any motion or observer.

Every fibre of the spacetime fibring $\boldsymbol{E}_{\mathrm{t}} \subset \boldsymbol{E}$, with $\mathrm{t} \in \boldsymbol{T}$, represents the equivalence class of simultaneous events at the time t. Indeed, these classes are "absolute", in the sense that they do not depend on any observer. $\square$

Remark 2.1.3 In einsteinian General Relativity, each timelike world line $s \in \boldsymbol{E}$ can be regarded (at least locally) as an affine space $\boldsymbol{T}_s$. If s and s' are different motions, there is no distinguished isomorphism between the affine spaces $\boldsymbol{T}_s$ and $\boldsymbol{T}_{s'}$. These affine spaces $\boldsymbol{T}_s$ and $\boldsymbol{T}_{s'}$ are associated (at least locally) with the *same* vector space $\bar{\mathbb{T}} := \mathbb{T} \otimes \mathbb{R}$. It means that we can compare *time intervals* along different timelike word lines, in spite of the fact that there is no distinguished simultaneity rule; again, this is just, also in einsteinian General Relativity, the postulate of existence of "good clocks". $\square$

In the general theory, it is convenient to refer to "unscaled" spacetime charts. In fact, coordinates with possible different scaling would require, case by case, different scale specifications of the single components of tensors (see Sect. 1.3.5).

Of course, this criterion, would not prevent the use of scaled coordinates, along with suitable scale information, in specific examples (see, for instance Definitions 24.2.4, 24.2.5, 24.4.2, 24.5.2 and 24.5.3).

Definition 2.1.4 We shall refer to (local) *spacetime charts*

$$(x^\lambda) \equiv (x^0, x^i) : \boldsymbol{E} \to \mathbb{R} \times \mathbb{R}^3,$$

defined as charts of the manifold $\boldsymbol{E}$, which are adapted to the time fibring, the affine structure of $\boldsymbol{T}$ and the orientation of $\boldsymbol{E}$ and $\boldsymbol{T}$. Thus, in particular, the time coordinate is of the type

$$x^0 : \boldsymbol{E} \to \mathbb{R} : e \mapsto u^0\,(t(e) - \mathrm{t}_0),$$

with reference to a time origin $\mathrm{t}_0 \in \boldsymbol{T}$ and a time unit of measurement $u^0 \in \mathbb{T}$.

Hence, the choice of a spacetime chart implies the choice of a time unit of measurement $u_0 := 1/u^0 \in \mathbb{T}^*$.

The associated coordinate curves are denoted by (see Fig. 2.1)

$$x_\lambda : \mathbb{R} \times \boldsymbol{E} \to \boldsymbol{E}.$$

The transition rule between two spacetime charts (x^λ) and $(\acute{x}^\lambda)$ is of the type

$$\begin{aligned} &\acute{x}^i \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}), && \text{with } \det(\partial_j \acute{x}^i) > 0, \\ &\acute{x}^0 = a^0_0\, x^0 + a^0, && \text{with } a^0_0 \in \mathbb{R}^+, a^0 \in \mathbb{R}. \quad \square \end{aligned}$$

Proposition 2.1.5 *The* covariance (local) group *(see, Introduction: Sect. 1.2.4) of the present galilean spacetime is constituted by the (local) orientation preserving fibred diffeomorphism $f_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E}$ over orientation preserving affine isomorphisms $f_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}$, according to the commutative diagram*

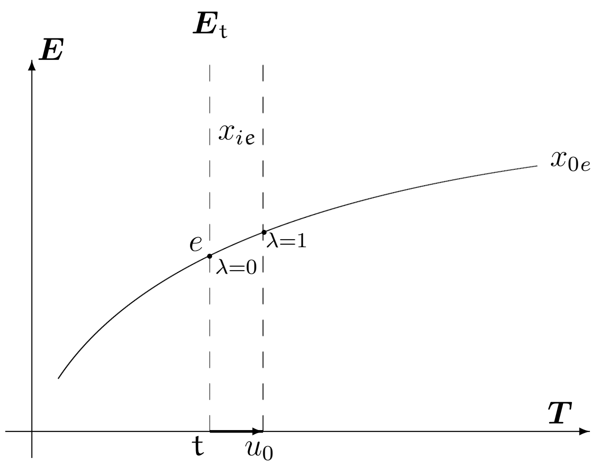

Fig. 2.1 Timelike coordinate curve

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{f_{\boldsymbol{E}}} & \boldsymbol{E} \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ \boldsymbol{T} & \xrightarrow{f_{\boldsymbol{T}}} & \boldsymbol{T} \end{array} .$$

Their coordinate expressions are of the type

$$f^i := x^i \circ f_{\boldsymbol{E}}, \in \text{map}(\boldsymbol{E}, I\!R), \quad \text{with } \det(\partial_j f^i) > 0,$$
$$f^0 := x^0 \circ f_{\boldsymbol{T}} = f_0^0\, x^0 + \check{f}^0, \quad \text{with } f_0^0 \in I\!R^+,\ \check{f}^0 \in I\!R. \quad \square$$

2.2 Tangent Space of Spacetime

Throughout the book, we shall be largely involved with the *tangent* and *cotangent bundles* of spacetime $T\boldsymbol{E} \to \boldsymbol{E}$ and $T^*\boldsymbol{E} \to \boldsymbol{E}$, along with their *natural vertical subbundle* and *natural horizontal subbundle* $V\boldsymbol{E} \subset T\boldsymbol{E}$ and $H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$.

We stress that we do not have observer independent complementary subbundles, hence we do not have observer independent splittings of the tangent and cotangent bundles.

The fibring of spacetime over time yields the distinguished subsheaf of *projectable spacetime vector fields*, which will play an important role throughout the present book.

For further details on the tangent bundle, the reader can refer, for instance, to [146] and to Appendix: Sect. B.

Definition 2.2.1 We denote the *tangent and cotangent bundles* of spacetime by

$$\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E} \qquad \text{and} \qquad \tau^{\boldsymbol{E}} : T^*\boldsymbol{E} \to \boldsymbol{E}.$$

Each spacetime chart (x^λ) induces the following fibred spacetime charts of $T\boldsymbol{E}$ and $T^*\boldsymbol{E}$ (see Appendix: Definition B.1.2)

$$(x^\lambda, \dot{x}^\lambda) \equiv (x^0, x^i,\ \dot{x}^0, \dot{x}^i) : T\boldsymbol{E} \to \mathrm{I\!R}^4 \times \mathrm{I\!R}^4,$$
$$(x^\lambda, \dot{x}_\lambda) \equiv (x^0, x^i,\ \dot{x}_0, \dot{x}_i) : T^*\boldsymbol{E} \to \mathrm{I\!R}^4 \times \mathrm{I\!R}^4.$$

The associated bases of vector fields and forms of $\boldsymbol{E}$ are denoted by

$$(\partial_\lambda) \equiv (\partial_0, \partial_i) \qquad \text{and} \qquad (d^\lambda) \equiv (d^0, d^i). \quad \square$$

Note 2.2.2 The time fibring yields the distinguished scaled *time form*

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}, \quad \text{with coordinate expression } dt = u_0 \otimes d^0.$$

We stress that the *time scale* $u^0 \in \mathbb{T}^*$ associated with the spacetime chart (x^λ) is uniquely defined by the equality

$$d^0 = dt(u^0). \quad \square$$

Remark 2.2.3 Some authors dealing with covariant Classical Mechanics and Quantum Mechanics, instead of assuming a fibring of spacetime over time and an affine structure of time, assume a closed spacetime 1-form τ, which replaces our time form dt.

Clearly, such a model of spacetime is more general than ours. However, we are not convinced of the physical reasons and advantages of such a generality. In particular, a model of this kind does not account for the existence of "good clocks", which seems to be a physically inalienable assumption.

Indeed, even in einsteinian General Relativity, such hypothesis is implicitly present. In fact, the proper time intervals do not depend on the observers and are measured via the pullback of the spacetime lorentzian metric on the world lines of particles. □

Proposition 2.2.4 *The tangent bundle $T\boldsymbol{E}$ and the cotangent bundle $T^*\boldsymbol{E}$ of spacetime have the following distinguished natural subbundles, along with their characterisations in coordinates:*

- *the* $(4+3)$*-dimensional* vertical subbundle

$$V\boldsymbol{E} \;\equiv\; V_{\boldsymbol{T}}\boldsymbol{E} := \ker(dt) \subset T\boldsymbol{E} \;\mid\; \dot{x}^0 = 0,$$

- *the* $(4+1)$*-dimensional* horizontal subbundle

$$H^*\boldsymbol{E} \;\equiv\; H^*_{\boldsymbol{T}}\boldsymbol{E} := \mathrm{im}\,(dt) \subset T^*\boldsymbol{E} \;\mid\; \dot{x}_i = 0.$$

Thus, the vertical subbundle is generated by the spacetime vectors tangent to the fibres of spacetime and the horizontal subbundle is generated by spacetime 1–forms which are the pullback of 1-forms of $\boldsymbol{T}$.

The vertical vectors are also called "spacelike" *and the horizontal forms are also called* "timelike"*.*

The transpose of the natural "vertical fibred inclusion" *over $\boldsymbol{E}$*

$$j_{\boldsymbol{E}} : V\boldsymbol{E} \subset T\boldsymbol{E}$$

is the natural surjective "vertical fibred projection" *over $\boldsymbol{E}$*

$$\vee : T^*\boldsymbol{E} \to V^*\boldsymbol{E} : \alpha \mapsto \check{\alpha},$$

which turns out to be the restriction of spacetime forms to spacelike forms.

Accordingly, $(\check{d}^i)$ turns out to be a basis of $V^*\boldsymbol{E}$. □

Actually, throughout the book, the vertical projection of forms will be denoted by the above "check symbol" ˇ.

Remark 2.2.5 We stress that we DO NOT have any distinguished complementary subbundle and any distinguished inclusion

$$H\boldsymbol{E} \subset T\boldsymbol{E} \qquad \text{and} \qquad V^*\boldsymbol{E} \subset T^*\boldsymbol{E}.$$

Moreover, we DO NOT have any distinguished projection

$$T\boldsymbol{E} \to V\boldsymbol{E} \qquad \text{and} \qquad T^*\boldsymbol{E} \to H^*\boldsymbol{E}.$$

Actually, the above missing objects can be achieved by means of the choice of an observer (see Proposition 2.7.3).

Indeed, the above facts play an important role with respect to the covariance of the present framework. □

Definition 2.2.6 A spacetime vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ is said to be *projectable*, or *time preserving*, if it is, respectively, projectable on $\boldsymbol{T}$, or with constant time component. These types of vector fields are characterised, in coordinates, respectively, by the conditions

$$\partial_j X^0 = 0, \qquad \text{or} \qquad \partial_\lambda X^0 = 0.$$

The subsheaves

$$\operatorname{pro}\sec(\boldsymbol{E}, T\boldsymbol{E}) \subset \sec(\boldsymbol{E}, T\boldsymbol{E}) \qquad \text{and} \qquad \operatorname{tim}\sec(\boldsymbol{E}, T\boldsymbol{E}) \subset \operatorname{pro}\sec(\boldsymbol{E}, T\boldsymbol{E})$$

of projectable and time preserving spacetime vector fields are closed with respect to the Lie bracket of vector fields. □

We shall often be involved with the following algebraic Lemma.

Lemma 2.2.7 *For each section* $\alpha : \boldsymbol{E} \to \Lambda^r V^*\boldsymbol{E}$, *with* $1 \le r \le 3$, *the wedge product*

$$dt \wedge \alpha : \boldsymbol{E} \to \mathbb{T} \otimes \Lambda^{r+1} T^*\boldsymbol{E}$$

is well defined, *in spite of the fact that* dt *is a scaled* spacetime *1-form,* α *is a* spacelike r*-form and that there is no natural inclusion* $\Lambda^r V^*E \subset \Lambda^r T^*\boldsymbol{E}$.

We have the coordinate expression

$$dt \wedge \alpha = r!\, \alpha_{1\ldots r}\, u_0 \otimes d^0 \wedge d^1 \wedge \ldots \wedge d^r.$$

Proof. Let us consider any (local) spacetime extension $\tilde{\alpha} : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}$ of the spacelike form α, with coordinate expression

$$\tilde{\alpha} = \alpha_{i_1\ldots i_r}\, (d^{i_1} - c_0^{i_1}\, d^0) \wedge \ldots \wedge (d^{i_r} - c_0^{i_r}\, d^0), \qquad \text{with} \qquad c_0^i \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}).$$

Then, the wedge product $dt \wedge \tilde{\alpha} : \boldsymbol{E} \to \mathbb{T} \otimes \Lambda^4 T^*\boldsymbol{E}$, is well defined and has coordinate expression

$$dt \wedge \tilde{\alpha} = \alpha_{i_1 \dots i_3} u_0 \otimes d^0 \wedge d^{i_1} \wedge \dots \wedge d^{i_r}.$$

In fact, this wedge product turns out to be independent of the choice of the extension $\widetilde{\alpha}$, because the wedge product with the time form dt kills the terms $c_0^i d^0$. □

We shall often use the following Lemma, which allows us to extend the concept of Lie derivative to the case of a vertical valued covariant tensor and a projectable spacetime vector field.

Lemma 2.2.8 *If* $\alpha \in \sec(\boldsymbol{E}, \otimes^r V^*\boldsymbol{E})$ *and* $X \in \operatorname{pro}\sec(\boldsymbol{E}, T\boldsymbol{E})$, *then the vertical restriction*

$$L_X \alpha := (L_X \widetilde{\alpha})^{\vee} \in \sec(\boldsymbol{E}, \otimes^r V^*\boldsymbol{E})$$

of the standard Lie derivative of any extension $\widetilde{\alpha} \in \sec(\boldsymbol{E}, \otimes^r T^*\boldsymbol{E})$ *of* α *does not depend on the extension* $\widetilde{\alpha}$.

Actually, we have the following equality, for each $Y_1, \dots, Y_r \in \sec(\boldsymbol{E}, V\boldsymbol{E})$,

$$L_X \alpha (Y_1, \dots, Y_r) = X.(\alpha(Y_1, \dots, Y_r)) - \sum_i \alpha (Y_1, \dots, [X, Y_i], \dots, Y_r)$$

and the coordinate expression

$$\begin{aligned} &L_X \alpha \\ &= (X^0 \partial_0 \alpha_{i_1 \dots i_r} + X^j \partial_j \alpha_{i_1 \dots i_r} + \sum_{1 \le k \le r} \alpha_{i_1 \dots i_{k-1} j i_{k+1} \dots i_r} \partial_{i_k} X^j) \check{d}^{i_1} \otimes \dots \otimes \check{d}^{i_r}. \end{aligned}$$ □

We shall say that $L_X \alpha$ is the *Lie derivative* of α.

Remark 2.2.9 We stress that, in the above coordinate expression, the terms appearing in the standard Lie derivative and containing the derivatives $\partial_i X^0$ are missing. □

2.3 Iterated Tangent Space of Spacetime

Throughout the book, we shall also be involved with the *iterated tangent bundle of spacetime* $\tau_{T\boldsymbol{E}} : TT\boldsymbol{E} \to T\boldsymbol{E}$, along with its fibred charts $(x^\lambda, \dot{x}^\lambda; \overset{\shortparallel}{x}{}^\lambda, \ddot{x}^\lambda)$ and natural distinguished subbundles.

For further details on the iterated tangent bundle, the reader can refer, for instance, to [146] and to Appendix B.

Definition 2.3.1 We denote the *iterated tangent bundle* of spacetime by (see Appendix: Note B.4.1)

$$\tau_{T\boldsymbol{E}} : TT\boldsymbol{E} \to T\boldsymbol{E}.$$

Each spacetime chart (x^λ) induces the following fibred spacetime chart of $TT\boldsymbol{E}$

$$\begin{aligned}&(x^\lambda, \dot{x}^\lambda;\; \overset{\shortparallel}{x}{}^\lambda, \ddot{x}^\lambda)\\ &\qquad \equiv (x^0, x^i,\; \dot{x}^0, \dot{x}^i;\;\; \overset{\shortparallel}{x}{}^0, \overset{\shortparallel}{x}{}^i,\; \ddot{x}^0, \ddot{x}^i) : TT\boldsymbol{E} \to (\mathbb{R}^4 \times \mathbb{R}^4) \times (\mathbb{R}^4 \times \mathbb{R}^4).\end{aligned}$$

The associated bases of vector fields and forms of $T\boldsymbol{E}$ are denoted by

$$(\partial_\lambda; \dot{\partial}_\lambda) \equiv (\partial_0, \partial_i;\; \dot{\partial}_0, \dot{\partial}_i) \qquad \text{and} \qquad (d^\lambda; \dot{d}^\lambda) \equiv (d^0, d^i;\; \dot{d}^0, \dot{d}^i). \quad \square$$

Proposition 2.3.2 *The iterated tangent bundle of spacetime $TT\boldsymbol{E}$ has the following distinguished natural subbundles, along with their characterisations in coordinates (see Appendix: Note B.4.1):*

(1) the $(8+7)$*-dimensional* $\boldsymbol{T}$-vertical subbundle *of* $TT\boldsymbol{E}$

$$V_{\boldsymbol{T}} T\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \overset{\shortparallel}{x}{}^0 = 0,$$

(2) the $(8+4)$*-dimensional* $\boldsymbol{E}$-vertical subbundle *of* $TT\boldsymbol{E}$

$$V_{\boldsymbol{E}} T\boldsymbol{E} \simeq T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \overset{\shortparallel}{x}{}^0 = \overset{\shortparallel}{x}{}^i = 0,$$

(3) the $(7+7)$*-dimensional* tangent bundle *of* $V\boldsymbol{E}$

$$TV\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \dot{x}^0 = \ddot{x}^0 = 0,$$

(4) the $(7+6)$*-dimensional* $\boldsymbol{T}$-vertical subbundle *of* $TV\boldsymbol{E}$

$$V_{\boldsymbol{T}} V\boldsymbol{E} \subset TV\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \dot{x}^0 = \overset{\shortparallel}{x}{}^0 = \ddot{x}^0 = 0,$$

(5) the $(7+3)$*-dimensional* $\boldsymbol{E}$-vertical subbundle *of* $TV\boldsymbol{E}$

$$V_{\boldsymbol{E}} V\boldsymbol{E} \simeq V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} \subset TV\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \overset{\shortparallel}{x}{}^i = \dot{x}^0 = \overset{\shortparallel}{x}{}^0 = \ddot{x}^0 = 0,$$

(6) the $(8+4)$*-dimensional* symmetric subbundle *of* $TT\boldsymbol{E}$

$$ST\boldsymbol{E} \subset TT\boldsymbol{E} \;\mid\; \overset{\shortparallel}{x}{}^\lambda = \dot{x}^\lambda,$$

which turns out to be an affine bundle over $T\boldsymbol{E}$ *associated with the vector bundle* $T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}$ *(see Appendix: Proposition B.4.5).* $\square$

2.4 Particle and Continuum Motions

We define the (observer independent) classical *particle motions* $s : \boldsymbol{T} \to \boldsymbol{E}$ and *continuum motions* $\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$, along with their *velocity* $ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and $\partial\mathscr{C} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

Definition 2.4.1 We define a *(particle) motion* to be a (local) section (see Fig. 2.2)

$$s : \boldsymbol{T} \to \boldsymbol{E},$$

with coordinate expression

$$(x^0, x^i) \circ s = (x^0, s^i), \quad \text{where } s^i \in \text{map}(\boldsymbol{T}, \mathbb{R}).$$

The *velocity* of a motion s is defined to be the scaled section (see Fig. 2.3)

$$ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expression $ds = u^0 \otimes ((\partial_0 \circ s) + \partial_0 s^i \, (\partial_i \circ s))$.

Indeed, the condition $t \circ s = \text{id}_{\boldsymbol{T}}$ implies the identity

$$dt \lrcorner ds = 1. \quad \square$$

Definition 2.4.2 We define a *continuum motion* (see Fig. 2.4) to be a (local) 1-parameter group of fibred diffeomorphisms, over the time translation $\tau : \bar{\mathbb{T}} \times \boldsymbol{T} \to \boldsymbol{T}$,

$$\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}.$$

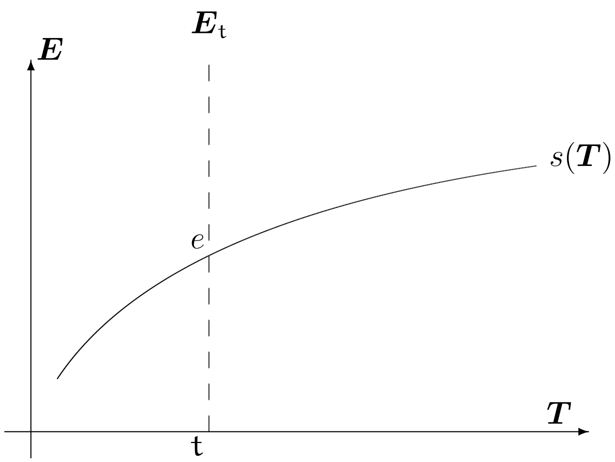

Fig. 2.2 Motion

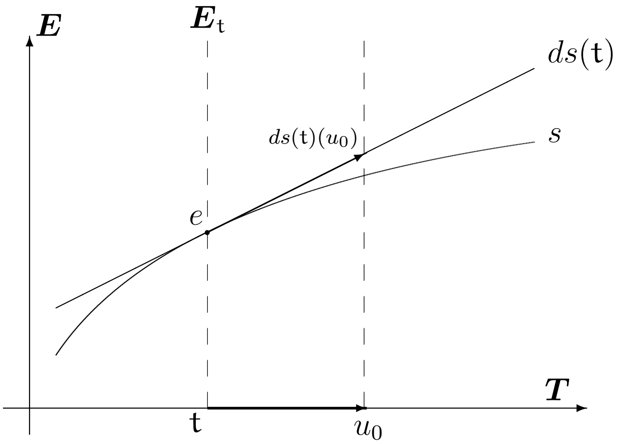

Fig. 2.3 Velocity of motion

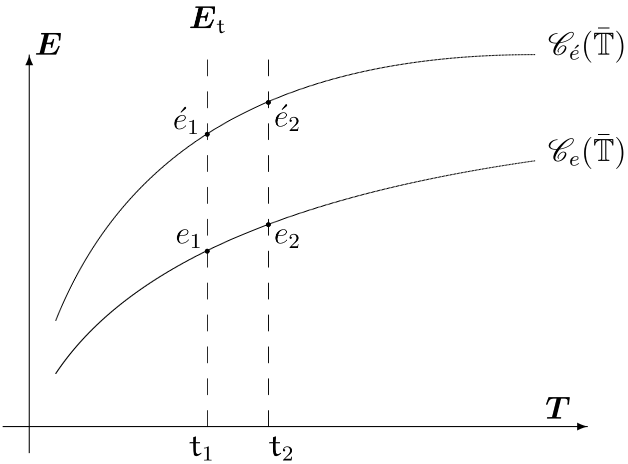

Fig. 2.4 Continuum motion

Thus, by definition, a continuum motion $\mathscr{C}$ makes the following diagram commutative

$$\begin{array}{ccc} \bar{\mathbb{T}} \times \boldsymbol{E} & \xrightarrow{\mathscr{C}} & \boldsymbol{E} \\ {\scriptstyle \mathrm{id}_{\bar{\mathbb{T}}} \times t}\downarrow & & \downarrow{\scriptstyle t} \\ \bar{\mathbb{T}} \times \boldsymbol{T} & \xrightarrow{\tau} & \boldsymbol{T} \end{array} \quad .$$

The *velocity* of a continuum motion $\mathscr{C}$ is defined to be the scaled section

$$\partial\mathscr{C} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expression

$$\partial\mathscr{C} = u^0 \otimes (\partial_0 + \partial_0\mathscr{C}^i\, \partial_i), \quad \text{where } \partial_0\mathscr{C}^i \in \text{map}(\boldsymbol{E}, \mathbb{R}),$$

where ∂ denotes partial derivative with respect to the parameter, evaluated at the zero value of the parameter.

For each $e \in \boldsymbol{E}$, the map $s : \boldsymbol{T} \to \boldsymbol{E} : \mathrm{t} \mapsto \mathscr{C}\big(\mathrm{t} - t(e), e\big)$ turns out to be a particle motion and, for each $\mathrm{t} \in \boldsymbol{T}$, we have $ds(\mathrm{t}) = \partial\mathscr{C}\big(s(\mathrm{t})\big)$.

Indeed, for every $e \in \boldsymbol{E}$, we can easily identify the above motion $s : \boldsymbol{T} \to \boldsymbol{E}$ with the map $\mathscr{C}_e : \bar{\mathbb{T}} \to \boldsymbol{E}$. □

2.5 Classical Phase Space

As *classical phase space* we consider the odd dimensional 1st jet space $J_1\boldsymbol{E}$ of classical motions.

Here, we just briefly introduce the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, along with its affine structure associated with the vector bundle $\mathbb{T}^* \otimes V\boldsymbol{E}$ and its natural inclusion into the scaled tangent space of spacetime $\mathbb{T}^* \otimes T\boldsymbol{E}$, which is expressed by the contact map $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

Indeed, the above structures will play an essential role throughout the book.

Later, we shall see that the classical fundamental fields of spacetime (the metric field g, the gravitational field $K^\natural$ and the electromagnetic field F) naturally yield a rich structure on the space $J_1\boldsymbol{E}$, namely a cosymplectic structure (dt, Ω) and also a coPoisson structure (γ, Λ) (see Chap. 10).

Indeed, the present choice of classical phase space $J_1\boldsymbol{E}$ turns out to be strategical for our approach: in fact, it reflects the fundamental role of time, hence the covariance of the theory, and a criterion of minimality.

Our phase space $J_1\boldsymbol{E}$ replaces the more usual even dimensional symplectic phase space. In particular, the possible alternative phase spaces $V\boldsymbol{E}$ and $V^*\boldsymbol{E}$ would be more suitable for a theory where time is just a parameter; hence, they would be unable to properly account for covariance. Moreover, the possible alternative spaces $\mathbb{T}^* \otimes T\boldsymbol{E}$ and $\mathbb{T} \otimes T^*\boldsymbol{E}$ would not take into proper account the fact that the galilean metric is spacelike and would not fulfill a principle of minimality.

For further details on jet spaces, the reader can refer, for instance, to [246, 360, 427] and to Appendix G.

In the introduction we have discussed our reasons for the present choice of phase space (see Introduction: Sect. 1.4.8).

An essential comparison between the phase space in the galilean and einsteinian frameworks can be summarised as follows: in the 1st case, phase space is the affine

1st jet bundle of sections of spacetime, in the 2nd case, phase space is the Grassmannian 1st jet bundle of timelike 1-dimensional submanifolds of spacetime (see [222]).

For the jet description of phase space, see also, for instance, [144, 293].

Proposition 2.5.1 *We say that two motions* $s, \acute{s} \in \sec(\boldsymbol{T}, \boldsymbol{E})$ *are* equivalent, at 1st order, *in* $\mathrm{t} \in \boldsymbol{T}$, *if they fulfill the (coordinate equivariant) conditions (see, for instance, [360])*

$$s^i(\mathrm{t}) = \acute{s}^i(\mathrm{t}) \qquad \text{and} \qquad \partial_0 s^i(\mathrm{t}) = \partial_0 \acute{s}^i(\mathrm{t}).$$

We denote the equivalence class, at $\mathrm{t} \in \boldsymbol{T}$, *of a motion* s *and the set of such equivalence classes, respectively, by*

$$j_1 s(\mathrm{t}) \qquad \text{and} \qquad J_{1\mathrm{t}}\boldsymbol{E} := \{j_1 s(\mathrm{t})\}.$$

Then, we define the 1st jet space *of the spacetime fibring* $t : \boldsymbol{E} \to \boldsymbol{T}$ *and the* 1st jet prolongation *of a motion* s *to be, respectively, the set and the section*

$$J_1\boldsymbol{E} := \sqcup_{\mathrm{t}\in \boldsymbol{T}} J_{1\mathrm{t}}\boldsymbol{E} \qquad \text{and} \qquad j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E} : \mathrm{t} \mapsto j_1 s(\mathrm{t}).$$

A spacetime chart (x^λ) *naturally yields the fibred charts* (x^λ, x^i_0) *of the space* $J_1\boldsymbol{E}$, *according to the following equality, for each motion* s,

$$x^0\big(j_1 s(\mathrm{t})\big) := x^0(\mathrm{t}), \qquad x^i\big(j_1 s(\mathrm{t})\big) := s^i(\mathrm{t}), \qquad x^i_0\big(j_1 s(\mathrm{t})\big) = \partial_0 s^i(\mathrm{t}).$$

The transition rule between the fibred charts (x^λ, x^i_0) *and* $(\acute{x}^\lambda, \acute{x}^i_0)$ *is given by (see Definition 2.1.4)*

$$\acute{x}^i_0 = \acute{\partial}_0 x^0\,(\partial_j \acute{x}^i\, x^j_0 + \partial_0 \acute{x}^i). \quad \square$$

We can easily prove that the above charts make the 1st jet space $J_1\boldsymbol{E}$ *a 7-dimensional manifold. Moreover, we naturally obtain the fibred manifold and the bundle*

$$t^1 : J_1\boldsymbol{E} \to \boldsymbol{T} : j_1 s(\mathrm{t}) \mapsto \mathrm{t} \qquad \text{and} \qquad t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E} : j_1 s(\mathrm{t}) \mapsto s(\mathrm{t}). \quad \square$$

Indeed, the phase bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ *turns out to be an affine bundle associated with the vector bundle* $\mathbb{T}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$.

Proof. The proof of the affine structure can be easily obtained from the transition rule of fibred coordinates of $J_1\boldsymbol{E}$. □

Note 2.5.2 In the following we shall also be involved with the *tangent bundle* of phase space

$$\tau_{J_1\boldsymbol{E}} : TJ_1\boldsymbol{E} \to J_1\boldsymbol{E}$$

and its fibred charts

$$(x^0, x^i, x^i_0;\ \dot{x}^0, \dot{x}^i, \dot{x}^i_0) : TJ_1\boldsymbol{E} \to (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \times (\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3). \quad \square$$

Assumption C.1 As *classical phase space* we choose the 7 dimensional 1st jet space of motions

$$t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}. \quad \square$$

2.6 Contact Map

The phase space $J_1\boldsymbol{E}$ is equipped with the "*contact map*" $\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and the "*complementary contact map*" $\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$, which will be largely used throughout the book.

For further details on the contact structure, the reader can refer, for instance, to [360] and to Appendix G.3.

Proposition 2.6.1 *The phase space $J_1\boldsymbol{E}$ is naturally equipped with the injective fibred morphism over $\boldsymbol{E}$, called* 1st order contact map*,*

$$\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

which is characterised by the following condition, for each motion $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$,

$$\text{д} \circ j_1 s = ds,$$

according to the commutative diagram

$$\begin{array}{ccc} J_1\boldsymbol{E} & \xrightarrow{\ \text{д}\ } & \mathbb{T}^* \otimes T\boldsymbol{E} \\ {\scriptstyle j_1 s}\uparrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \boldsymbol{T} & \xrightarrow{\ ds\ } & \mathbb{T}^* \otimes T\boldsymbol{E} \end{array} .$$

The coordinate expression of the contact map is

$$\text{д} = u^0 \otimes (\partial_0 + x^i_0\, \partial_i).$$

Thus, the injective contact map д yields an affine fibred isomorphism over $\boldsymbol{E}$ between the affine bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ and the affine subbundle $\boldsymbol{U} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ which projects on $\mathbf{1}_{\boldsymbol{T}}$, according to the following commutative diagram

$$\begin{array}{ccccc} J_1\boldsymbol{E} & \xrightarrow{\text{д}} & \boldsymbol{U} & \xrightarrow{\subset} & \mathbb{T}^* \otimes T\boldsymbol{E} \\ {\scriptstyle t^1}\downarrow & & & & \downarrow{\scriptstyle \mathrm{id} \otimes Tt} \\ \boldsymbol{T} & & \xrightarrow{\quad \boldsymbol{1}_{\boldsymbol{T}} \quad} & & \mathbb{T}^* \otimes T\boldsymbol{T} \end{array} \quad .$$

In other words, $J_1\boldsymbol{E} \to \boldsymbol{E}$ *turns out to be the affine subbundle of* $\mathbb{T}^* \otimes T\boldsymbol{E} \to \boldsymbol{E}$, *which is characterised, in coordinates, by the normalising constraint*

$$\dot{x}_0^0 = u_0 \otimes \dot{x}^0 = 1.$$

For this reason, the affine subbundle

$$J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$$

reflects a criterion of minimality. □

Corollary 2.6.2 *The complementary map of the contact map* д *is the surjective* complementary contact map

$$\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}, \text{ whose coordinate expression is } \theta = (d^i - x_0^i\, d^0) \otimes \partial_i. \quad □$$

Corollary 2.6.3 *The contact map* д *and the complementary contact map* θ *yield a splitting of spacetime vector fields into their horizontal and vertical components "over the phase space"* $J_1\boldsymbol{E}$ *induced by the following* vertical projection *and* vertical inclusion

$$J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to V\boldsymbol{E} \qquad \text{and} \qquad J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V^*\boldsymbol{E} \to T^*\boldsymbol{E},$$

with coordinate expressions

$$X^\lambda\, \partial_\lambda \mapsto (X^i - X^0\, x_0^i)\, \partial_i \qquad \text{and} \qquad \alpha_i\, \check{d}^i \mapsto \alpha_i\, (d^i - x_0^i\, d^0). \quad □$$

2.7 Observers

Our definition of observer can be regarded as a mathematical formalisation of a laboratory which moves in spacetime and observes physical phenomena.

Indeed, in our approach the concept of observer plays an important role as an intermediate tool between the intrinsic and the coordinate expressions of physical objects.

In the Introduction we have discussed our approach to observers (see Sect. 1.4.5).

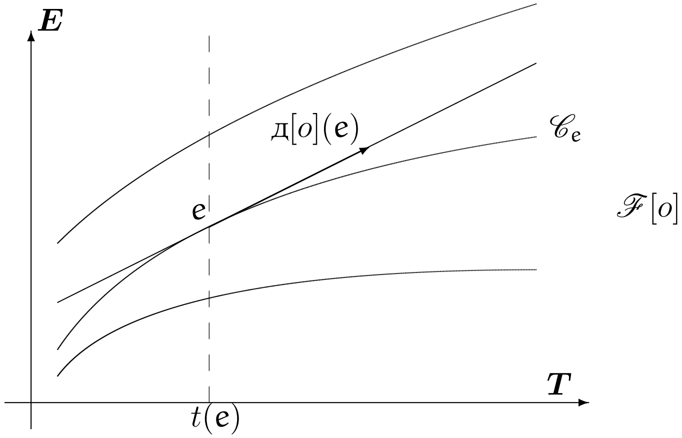

Fig. 2.5 Flow of an observer

Definition 2.7.1 A classical *observer* is defined to be a (local) section

$$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}.$$

Thus, an observer o can be regarded as the velocity field of the motion of a classical continuum, whose flow is described by a local group of fibred isomorphisms over the affine time translation $\bar{\mathbb{T}} \times \boldsymbol{T} \to \boldsymbol{T}$ (see Fig. 2.5)

$$\mathscr{F}[o] : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}. \quad \square$$

Definition 2.7.2 A spacetime chart (x^λ) is said to be *adapted* to an observer o if

$$o_0^i := x_0^i \circ o = 0,$$

i.e. if the spacelike coordinates x^i are constant along the particle motions of the flow $\mathscr{F}[o]$. □

Proposition 2.7.3 *The choice of an observer yields several* "observed" *objects, including a splitting of the tangent and cotangent bundles of spacetime (see Fig. 2.6)*

$$T\boldsymbol{E} = T_o\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} V\boldsymbol{E} \qquad \text{and} \qquad T^*\boldsymbol{E} = H^*\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} V_o^*\boldsymbol{E},$$

with coordinate expressions in an adapted chart

$$X = X^0\,\partial_0 + X^i\,\partial_i \qquad \text{and} \qquad \alpha = \alpha_0\,d^0 + \alpha_i\,d^i.$$

An observer o can be regarded as a connection of the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$.

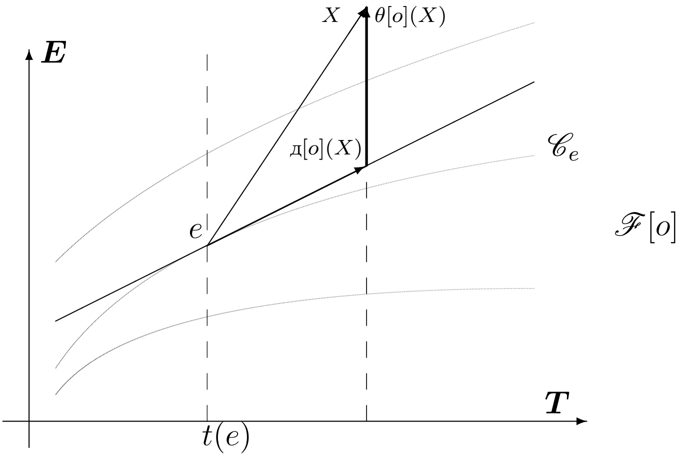

Fig. 2.6 Observed splitting of the tangent space of spacetime

Accordingly, it yields the associated observed covariant differential

$$\nabla[o] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}, \ \textit{with coordinate expression} \ \nabla[o] = (x^i_0 - o^i_0)\, d^0 \otimes \partial_i .$$

An observer o is characterised by the "observed" contact map *and* complementary contact map *(see Fig. 2.6)*

$$\text{д}[o] := \text{д} \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \qquad \text{and} \qquad \theta[o] := \theta \circ o : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

whose coordinate expressions are

$$\text{д}[o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i) \qquad \text{and} \qquad \theta[o] = (d^i - o^i_0\, d^0) \otimes \partial_i . \quad \square$$

Remark 2.7.4 We stress that, given an observer o, there are many adapted spacetime charts. In fact, let (x^λ) be adapted; then $(\acute{x}^\lambda)$ is adapted if and only if $\partial_0 \acute{x}^i = 0$.

However, a spacetime chart (x^λ) is adapted to the unique observer o such that $\text{д}[o] = u^0 \otimes \partial_0$. $\square$

Note 2.7.5 Every observer o yields locally the equivalence relation in $\boldsymbol{E}$

$$e \simeq \acute{e} \qquad \Leftrightarrow \qquad \acute{e} = \mathscr{F}[o](t(\acute{e}) - t(e),\ e).$$

In other words, two events $e, \acute{e} \in \boldsymbol{E}$ are equivalent if they are touched by the same particle motion of the flow $\mathscr{F}[o]$.

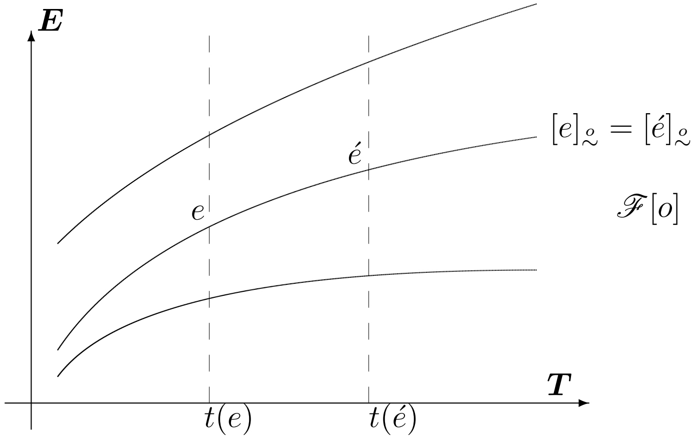

Fig. 2.7 Observed space

The above local equivalence relation yields locally the quotient manifold, called *observed space*, along with the *observed projection* (see Fig. 2.7)

$$\boldsymbol{P}[o] := \boldsymbol{E}/\simeq \qquad \text{and} \qquad p[o] : \boldsymbol{E} \to \boldsymbol{P}[o].$$

Then, we obtain the local observed splitting

$$\boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}[o].$$

Each spacetime chart (x^λ) adapted to o yields a chart on $\boldsymbol{P}[o]$, which will still be denoted by (x^i). □

Note 2.7.6 Given two observers o and $\acute{o}$, we can uniquely write

$$\acute{o} = o + \vec{v}, \qquad \text{where} \qquad \vec{v} \in \operatorname{sec}(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E}).$$

Accordingly, we have the following *transition rules*

$$\nabla[\acute{o}] = \nabla[o] - \vec{v}, \qquad д[\acute{o}] = д[o] + \vec{v}, \qquad \theta[\acute{o}] = \theta[o] - dt \otimes \vec{v}.$$

Hence, the equivariance of physical objects with respect to observers is ruled by the abelian (local) group of scaled vector fields $\operatorname{sec}(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$. □

Definition 2.7.7 For each motion $s \in \operatorname{sec}(\boldsymbol{T}, \boldsymbol{E})$, we define the *observed velocity* of s, with respect to an observer o, to be the scaled spacelike vector field (see Definition 2.4.1 and Proposition 2.7.3)

$$\nabla[o]s \equiv d[o]s := \theta[o] \lrcorner ds \in \operatorname{sec}(\boldsymbol{T}, \mathbb{T}^* \otimes V\boldsymbol{E}),$$

with coordinate expression

$$\nabla[o]s = (\partial_0 s^i - o^i_0 \circ s))\,(\partial_i \circ s).$$

Thus, we have the observed splitting of the velocity

$$ds = д[o] \circ s + \nabla[o]s \;\equiv\; д[o] \circ s + d[o]s.$$

Hence, with reference to two observers o and $\acute{o} = o + \vec{v}$, we obtain the usual transition rule

$$\nabla[\acute{o}]s = \nabla[o]s - \vec{v} \circ s. \quad \square$$

Definition 2.7.8 For each continuum motion $\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$, we define the *observed velocity* of $\mathscr{C}$, with respect to the observer o, to be the scaled spacelike vector field (see Definition 2.4.2 and Proposition 2.7.3)

$$\partial[o]\mathscr{C} := \theta[o] \lrcorner\, \partial\mathscr{C} : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E},$$

with coordinate expression

$$\partial[o]\mathscr{C} = (\delta_0 \mathscr{C}^i - o^i_0)\,\partial_i.$$

Thus, we have the observed splitting of the velocity

$$\partial\mathscr{C} = д[o] + \partial[o]\mathscr{C}.$$

Hence, with reference to two observers o and $\acute{o} = o + \vec{v}$, we obtain the usual transition rule

$$\partial[\acute{o}]\mathscr{C} = \partial[o]\mathscr{C} - \vec{v}. \quad \square$$

Remark 2.7.9 Clearly, the velocity of a continuum motion $\mathscr{C}$ coincides with the velocity of its particles. In other words, for each $e \in \boldsymbol{E}$, we have

$$\partial C(e) = d(\mathscr{C}_e). \quad \square$$

Chapter 3
Galilean Metric Field

The fibring of spacetime over time naturally yields a natural scaled *timelike metric* with signature $(+\,0\,0\,0)$, which will be used in a few occasions (Sect. 3.1).

Besides this, we postulate a scaled *spacelike galilean metric* g with signature $(0\,+\,+\,+)$, which plays a fundamental role throughout the book (Sect. 3.2).

With reference to a given particle it turns out to be very convenient to consider also a *rescaled spacelike metric* G.

The galilean metric naturally yields several geometric objects: such as the spacelike and spacetime *volume forms* (Sect. 3.2.2), the observed *kinetic energy*, *kinetic momentum* and *Poincar–Cartan form* (Sect. 3.2.4), a spacelike *riemannian connection* (Sect. 3.2.6) and a spacelike *symplectic form* (Sect. 3.2.7), a spacelike *gradient operator*, a spacelike *divergence operator*, a spacelike *curl operator* and a spacelike *laplacian operator* on the fibres of spacetime (Sect. 3.2.8).

3.1 Timelike Galilean Metric

The time fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ yields naturally a scaled *timelike metric* of spacetime $\mathbf{g} := dt \otimes dt$ with signature $(+\,0\,0\,0)$. Later, we shall use the associated *metric musical morphism* $\mathbf{g}^\flat : T\boldsymbol{E} \to \mathbb{T}^2 \otimes H^*\boldsymbol{E}$, which turns out to be degenerate, but is defined on the whole tangent space $T\boldsymbol{E}$, while the metric musical morphism $g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}$ will be defined only on the vertical subspace (see Definitions 7.3.3, 8.1.1, 3.2.2 and also Proposition 8.2.1).

Definition 3.1.1 We define the *timelike galilean metric* to be the scaled spacetime metric

$$\mathbf{g} := dt \otimes dt : \boldsymbol{E} \to \mathbb{T}^2 \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}) \subset \mathbb{T}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with signature $\operatorname{sign}(\mathbf{g}) = (+\,0\,0\,0)$ and coordinate expression

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*,
Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1_3

$$\mathbf{g} = (u_0 \otimes u_0) \otimes d^0 \otimes d^0 .$$

We define the *metric timelike musical morphism* to be the linear fibred morphism over $\boldsymbol{E}$

$$\mathbf{g}^\flat : T\boldsymbol{E} \to \mathbb{T}^2 \otimes H^*\boldsymbol{E} : X \mapsto X \lrcorner \mathbf{g},$$

with coordinate expression

$$\mathbf{g}^\flat(X) = X^0 \, (u_0 \otimes u_0) \otimes d^0 .$$

The image and the kernel of $\mathbf{g}^\flat$ are the subbundles (see Proposition 2.2.4)

$$\mathrm{im}\,(\mathbf{g}^\flat) = \mathbb{T}^2 \otimes H^*\boldsymbol{E} \subset \mathbb{T}^2 \otimes T^*\boldsymbol{E} \qquad \text{and} \qquad \ker\,(\mathbf{g}^\flat) = V\boldsymbol{E} \subset T\boldsymbol{E}. \quad \square$$

Note 3.1.2 By taking into account the speed of light $c \in \mathbb{T}^* \otimes \mathbb{L}$, we can rescale the timelike galilean metric $\mathbf{g}$ with a scale dimension which will turn out to be convenient later (see Postulate C.6)

$$\mathfrak{g} := c^2\,\mathbf{g} = c^2\, dt \otimes dt : \boldsymbol{E} \to \mathbb{L}^2 \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}) \subset \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}). \quad \square$$

3.2 Spacelike Galilean Metric

The *spacelike galilean metric* $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$ turns out to be one of our basic objects in the classical and quantum theory.

Indeed, it naturally yields several objects: the *rescaled galilean metric*, the *musical spacelike isomorphisms*, the *spacelike volume form*, the *spacetime volume form*, the *observed kinetic energy*, the *observed kinetic momentum*, the *observed kinetic Poincaré–Cartan form*, the *fibrewise riemannian connection* the *fibrewise symplectic 2-form* the *spacelike gradient*, the *spacelike divergence*, the *spacelike curl*, the *spacelike laplacian*.

3.2.1 Definition of Spacelike Galilean Metric

We postulate the *spacelike galilean metric* $g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$ as a scaled fibrewise riemannian metric of spacetime.

Further, with reference to a particle of mass m, we define the *rescaled spacelike galilean metric* $G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$, which carries a convenient scale factor.

The galilean metric and the rescaled galilean metric yield the *metric musical spacelike isomorphisms* $g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}$ and $G^\flat : V\boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E}$.

Postulate C.2 *We postulate the* galilean metric *to be a scaled spacelike riemannian metric*

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}),$$

with coordinate expression

$$g = g_{ij}\, \check{d}^i \otimes \check{d}^j, \quad \text{with } g_{ij} = g_{ji} \in \text{map}(\boldsymbol{E}, \mathbb{L}^2 \otimes I\!R),\ \det(g_{ij}) > 0. \quad \square$$

Definition 3.2.1 With reference to a particle of mass $m \in \mathbb{M}$, it is convenient to introduce (by using, also in the classical context, the Planck constant $\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$) the *rescaled spacelike galilean metric*

$$G := \tfrac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}),$$

which carries the convenient time scale $\mathbb{T}$.

We have the coordinate expression

$$G = G^0_{ij}\, u_0 \otimes \check{d}^i \otimes \check{d}^j, \quad \text{with } G^0_{ij} \in \text{map}(\boldsymbol{E}, I\!R). \quad \square$$

Thus, all classical and quantum objects derived from the rescaled galilean metric G will incorporate the mass m and the Planck constant $\hbar$.

Definition 3.2.2 We denote the *contravariant galilean metric* and the *contravariant rescaled galilean metric* by

$$\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}) \quad \text{and} \quad \bar{G} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}).$$

Their coordinate expressions are

$$\bar{g} = g^{ij}\, \partial_i \otimes \partial_j \quad \text{and} \quad \bar{G} = G^{ij}_0\, u^0 \otimes \partial_i \otimes \partial_j,$$

with $g^{ij} \in \text{map}(\boldsymbol{E}, \mathbb{L}^{-2} \otimes I\!R)$ and $G^{ij}_0 \in \text{map}(\boldsymbol{E}, I\!R)$.

Moreover, the galilean metric g and the rescaled metric G yield the *metric musical spacelike isomorphisms*

$$\begin{aligned} g^\flat &: V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}, \qquad g^\sharp : V^*\boldsymbol{E} \to \mathbb{L}^{-2} \otimes V\boldsymbol{E}, \\ G^\flat &: V\boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E}, \qquad G^\sharp : V^*\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}, \end{aligned}$$

with coordinate expressions

$$\begin{aligned} g^\flat(X) &= g_{ij}\, X^i\, \check{d}^j, \qquad g^\sharp(\alpha) = g^{ij}\, \alpha_i\, \partial_j, \\ G^\flat(X) &= G^0_{ij}\, X^i\, u_0 \otimes \check{d}^j, \qquad G^\sharp(\alpha) = G^{ij}_0\, \alpha_i\, u^0 \otimes \partial_j. \end{aligned}$$

Remark 3.2.3 We recall that the time form dt yields the natural fibred isomorphism over $\boldsymbol{E}$

$$dt : \boldsymbol{E} \times \bar{\mathbb{T}}^* \to H^*\boldsymbol{E} \subset T^*\boldsymbol{E}.$$

In other words, we transform $\boldsymbol{E} \times \bar{\mathbb{T}}^*$ into $T^*\boldsymbol{E}$ by pullback over $\boldsymbol{E}$.

Accordingly, the scale factor $\mathbb{T}^*$, which is involved in the rescaled galilean metric $\bar{G} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$, yields a convenient interplay with the cotangent space $T^*\boldsymbol{E}$ of spacetime. Actually, a similar useful property does not hold for the scale $\mathbb{L}$. □

3.2.2 Volumes

The galilean metric g, along with the orientation of the fibres of spacetime, yields the distinguished scaled *spacelike volume form* $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}$ and *spacelike volume vector* $\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E}$.

Moreover, the spacelike volume form η and volume vector $\bar{\eta}$, along with the time fibring, yield the distinguished scaled *spacetime volume form* $\upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E}$ and *spacetime volume vector* $\bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{-3}) \otimes \Lambda^4 T^*\boldsymbol{E}$.

Proposition 3.2.4 *The spacelike metric g and the spacetime orientation yield the scaled* spacelike volume form

$$\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3\, V^*\boldsymbol{E},$$

with coordinate expression

$$\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3.$$

Then, according to Lemma 2.2.7, we obtain the scaled spacetime volume form

$$\upsilon := dt \wedge \eta : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E},$$

with coordinate expression

$$\upsilon = \sqrt{|g|}\, u_0 \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3.$$

We denote the dual spacelike volume 3-vector and spacetime volume 4-vector by

$$\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E} \qquad \text{and} \qquad \bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{-3}) \otimes \Lambda^4 T\boldsymbol{E},$$

with coordinate expressions

$$\bar{\eta} = \frac{1}{\sqrt{|g|}}\, \partial_1 \wedge \partial_2 \wedge \partial_3 \qquad \text{and} \qquad \bar{\upsilon} = \frac{1}{\sqrt{|g|}}\, u^0 \otimes (\partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3).$$

Thus, we have

$$i_{\bar{\eta}}\eta = 1 \qquad \text{and} \qquad i_{\bar{\upsilon}}\upsilon = 1. \quad \square$$

Notation 3.2.5 We set

$$\begin{aligned} \mathrm{d}^0_\lambda &:= i_{\partial_\lambda}(d^0 \wedge d^1 \wedge d^2 \wedge d^3), \\ \upsilon^0 &:= i_{u^0}\upsilon = \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3, \\ \upsilon^0_\lambda &:= i_{\partial_\lambda}\upsilon^0 = \sqrt{|g|}\, i_{\partial_\lambda}(d^0 \wedge d^1 \wedge d^2 \wedge d^3) = \sqrt{|g|}\, \mathrm{d}^0_\lambda \end{aligned}$$

and

$$\begin{aligned} &\mathrm{d}^0_0 = d^1 \wedge d^2 \wedge d^3, \qquad \mathrm{d}^0_i = (-1)^i\, d^0 \wedge \widehat{d^i} \wedge d^3, \\ &\upsilon^0_0 = \sqrt{|g|}\, d^1 \wedge d^2 \wedge d^3 = \sqrt{|g|}\, \mathrm{d}^0_0, \quad \upsilon^0_i = (-1)^i \sqrt{|g|}\, d^0 \wedge \widehat{d^i} \wedge d^3 = \sqrt{|g|}\, \mathrm{d}^0_i. \quad \square \end{aligned}$$

The spacelike volume form is valued in the cotangent vertical bundle $V^*\boldsymbol{E}$.

We stress that we do not have a natural inclusion $V^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ (see Remark 2.2.5). So, if we want to regard η as valued in the cotangent bundle $T^*\boldsymbol{E}$, we need to choose an observer o and define observed volume form $\eta[o]$.

Definition 3.2.6 We reference to an observer o, we define the *observed spacelike volume form* to be the scaled section

$$\eta[o] := \theta^*[o]\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 T^*\boldsymbol{E},$$

with coordinate expression

$$\eta[o] = \sqrt{|g|}\, (d^1 - o^1_0\, d^0) \wedge (d^2 - o^2_0\, d^0) \wedge (d^3 - o^3_0\, d^0). \quad \square$$

3.2.3 Hodge Star and Cross Product

Further, the spacelike metric isomorphism $g^\flat$ and the spacelike volume form η yield the *Hodge isomorphism* $*_\eta : \Lambda^r V^*\boldsymbol{E} \to \mathbb{L}^{3-2r} \otimes \Lambda^{3-r} V^*\boldsymbol{E}$ and the scaled *cross product* $X \times Y \in \sec(\boldsymbol{E}, \mathbb{L} \otimes V\boldsymbol{E})$.

Corollary 3.2.7 *The metric* g *and the spacelike volume yield the* spacelike Hodge isomorphism, *for* $0 \le r \le 3$,

$$*_\eta : \Lambda^r V^*\boldsymbol{E} \to \mathbb{L}^{3-2r} \otimes \Lambda^{3-r} V^*\boldsymbol{E} : \phi \mapsto i_{g^\sharp(\phi)}\eta,$$

with coordinate expression

$$*_\eta : \check{d}^{i_1} \wedge \ldots \wedge \check{d}^{i_r} \mapsto \sqrt{|g|}\, g^{i_1 j_1} \ldots g^{i_r j_r}\, i_{\partial_{j_1}} \ldots i_{\partial_{j_r}}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3. \quad \square$$

Corollary 3.2.8 *We define the spacelike* cross product *of two spacelike vector fields* $X, Y \in \sec(\boldsymbol{E}, V\boldsymbol{E})$ *by the equality*

$$X \times Y := i_{g^\flat(X) \wedge g^\flat(Y)}\, \bar{\eta} \in \sec(\boldsymbol{E},\, \mathbb{L} \otimes V\boldsymbol{E}),$$

with coordinate expression

$$X \times Y = \frac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, X_h\, Y_k\, \partial_i.$$

For each $X, Y, Z \in \sec(\boldsymbol{E}, V\boldsymbol{E})$, *we have the following identities*

$$\begin{aligned} &X \times Y = -Y \times X, \\ &(X \times Y) \times Z + (Z \times X) \times Y + (Y \times Z) \times X = 0. \end{aligned}$$

In an analogous way, we define the spacelike cross product *of two spacelike 1-forms* $\alpha, \beta \in \sec(\boldsymbol{E}, V^*\boldsymbol{E})$ *by the equality*

$$\alpha \times \beta := i_{g^\sharp(\alpha) \wedge g^\sharp(\beta)}\, \eta \in \sec(\boldsymbol{E},\, \mathbb{L}^{-1} \otimes V^*\boldsymbol{E}),$$

with coordinate expression

$$\alpha \times \beta = \sqrt{|g|}\, \epsilon_{hki}\, \alpha^h\, \beta^k\, \check{d}^i.$$

For each $\alpha, \beta, \gamma \in \sec(\boldsymbol{E}, V^*\boldsymbol{E})$, *we have the following identities*

$$\begin{aligned} &\alpha \times \beta = -\beta \times \alpha, \\ &(\alpha \times \beta) \times \gamma + (\gamma \times \alpha) \times \beta + (\beta \times \gamma) \times \alpha = 0. \quad \square \end{aligned}$$

3.2.4 Observed Kinetic Objects

With reference to a particle of mass $m \in \mathbb{M}$, the galilean metric G, along with an observer o, yields the *"observed" kinetic energy* $\mathcal{K}[o] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$, the *"observed" kinetic momentum* $\mathcal{Q}[o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$, and the *"observed" kinetic Poincaré–Cartan form* $\mathcal{C}[o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$.

Later, we shall discuss the dynamical extensions of the above objects (see Theorem 10.1.8).

Definition 3.2.9 With reference to a particle of mass $m \in \mathbb{M}$ and to an observer o, we define the following *observed* objects:

$$\begin{aligned}
&\textit{kinetic energy } \mathcal{K}[o] && \equiv \mathcal{K}[G, o] := \tfrac{1}{2}\, G\,(\nabla[o], \nabla[o]) \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}),\\
&\textit{kinetic momentum } \mathcal{Q}[o] && \equiv \mathcal{Q}[G, o] := \theta[o] \lrcorner (G^\flat\, \nabla[o]) \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),\\
&\textit{kinetic Poincaré–Cartan form } \mathcal{C}[o] && \equiv \mathcal{C}[G, o] := -\,\mathcal{K}[o] + \mathcal{Q}[o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),
\end{aligned}$$

with coordinate expressions,

$$\begin{aligned}
\mathcal{K}[o] &= \tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0,\\
\mathcal{Q}[o] &= G^0_{ij}\, (x^j_0 - o^j_0)\, (d^i - o^i_0\, d^0),\\
\mathcal{C}[o] &= -\tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 + G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - o^j_0\, d^0).
\end{aligned}$$

i.e., in a spacetime chart adapted to o,

$$\begin{aligned}
\mathcal{K}[o] &= \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, d^0,\\
\mathcal{Q}[o] &= G^0_{ij}\, x^j_0\, d^i,\\
\mathcal{C}[o] &= -\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0\, d^0 + G^0_{ij}\, x^j_0\, d^i. \quad \square
\end{aligned}$$

The above notion of kinetic Poincaré–Cartan form can be clarified by taking into account the following result (for the concept of momentum and Poincaré–Cartan form associated with a lagrangian, see, for instance, [117, 147, 243, 283, 411]).

Remark 3.2.10 Let us consider the following *observed spacelike lagrangian*

$$\mathcal{L}^\curlyvee[o] := \mathcal{K}[o] : J_1\boldsymbol{E} \to T^*\boldsymbol{T},$$

with coordinate expression

$$\mathcal{L}^\curlyvee[o] = \tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0.$$

Then, the *momentum* associated with the above observed spacelike lagrangian $\mathcal{L}^\curlyvee$ turns out to be the spacetime 1-form

$$\mathcal{N}^\curlyvee[o] = \theta \lrcorner (V_{\boldsymbol{E}}\mathcal{L}^\curlyvee[o]) : J_1\boldsymbol{E} \to T^*\boldsymbol{E},$$

with coordinate expression

$$\mathcal{N}^\curlyvee[o] = G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - x^j_0\, d^0).$$

Hence, the *Poincaré–Cartan form* associated with the above observed spacelike lagrangian $\mathcal{L}^\curlyvee$ turns out to be the spacetime 1-form (see, for instance, [117, 147, 243, 283, 411])

$$\mathcal{C}^{\Upsilon}[o] = \mathcal{L}^{\Upsilon}[o] + \mathcal{N}^{\Upsilon}[o],$$

with coordinate expression

$$\mathcal{C}^{\Upsilon}[o] = \tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 + G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - x^j_0\, d^0).$$

Indeed, we have the equalities

$$\mathcal{N}^{\Upsilon}[o] = \theta \lrcorner (V_{\boldsymbol{E}} \mathcal{L}^{\Upsilon}[o]), \quad \mathcal{Q}[o] = \theta[o] \lrcorner (V_{\boldsymbol{E}} \mathcal{L}^{\Upsilon}[o]),$$
$$\check{\mathcal{N}}^{\Upsilon}[o] = \check{\mathcal{Q}}[o]$$

and

$$\mathcal{C}^{\Upsilon}[o] = \mathcal{C}[o] = -\mathcal{K}[o] + \mathcal{Q}[o].$$

Proof. We have

$$\begin{aligned}
\mathcal{C}^{\Upsilon}[o] &= \tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 + G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - x^j_0\, d^0) \\
&= \tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 + G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - o^j_0\, d^0) \\
&\quad - G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 \\
&= -\tfrac{1}{2}\, G^0_{ij}\, (x^i_0 - o^i_0)\, (x^j_0 - o^j_0)\, d^0 + G^0_{ij}\, (x^i_0 - o^i_0)\, (d^j - o^j_0\, d^0) \\
&= -\mathcal{K}[o] + \mathcal{Q}[o] = \mathcal{C}[o]. \quad \square
\end{aligned}$$

Proposition 3.2.11 *Given two observers o and $\acute{o} = o + \vec{v}$, we obtain the following* transition rules *(see Note 2.7.6)*

$$\begin{aligned}
\mathcal{K}[\acute{o}] &= \mathcal{K}[o] - G(\vec{v}, \nabla[o]) + \tfrac{1}{2}\, G(\vec{v}, \vec{v}), \\
\mathcal{Q}[\acute{o}] &= \mathcal{Q}[o] - G(\vec{v}, \nabla[o]) - G(\vec{v}, \theta[o]) + G(\vec{v}, \vec{v}), \\
\mathcal{C}[\acute{o}] &= \mathcal{C}[o] - G(\vec{v}, \theta[o]) + \tfrac{1}{2}\, G(\vec{v}, \vec{v}). \quad \square
\end{aligned}$$

3.2.5 Observed Angular Momentum

The observed kinetic energy, kinetic momentum and kinetic Poincaré–Cartan form discussed in the above Section can be defined in any curved galilean spacetime.

Conversely, the standard definition of *angular momentum* (see, for instance, [302]) does not work in a generic curved galilean spacetime. In fact, we cannot define a vector of the type $p - q$, for a pair of points (p, q) belonging to a fibre $\boldsymbol{E}_t$ of a generic curved galilean spacetime, unless the fibre is an affine space.

So, *in the present section* (Sect. 3.2.5), we suppose that the fibres of spacetime be affine and introduce the concept of *observed angular momentum*.

This topic has been discussed in Introduction: Sect. 1.5.19 and will be further discussed later, after having introduced models of spacetime with affine fibres (see Definition 24.1.1 Propositions 25.1.2 and 28.1.4).

Thus, in Sect. 3.2.5, we suppose that the fibres of spacetime be affine spaces.

Definition 3.2.12 With reference to a motion $s : \boldsymbol{T} \to \boldsymbol{E}$ and an observer o, we define the rescaled *observed kinetic angular momentum* to be the scaled spacelike form (see Proposition 2.7.3, Corollary 3.2.8 and Definition 3.2.9)

$$\mathfrak{L} \equiv \mathfrak{L}[s, o] := \underline{r} \times \check{\mathfrak{Q}}[o] = g^{\flat}\big(i_{\underline{r} \wedge \check{\mathfrak{Q}}[o]} \bar{\eta}\big) = \tfrac{m}{\hbar}\, \big(i_{\vec{r} \wedge \nabla[o]} \eta\big) : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E},$$

where we have set

$$\vec{r} : \boldsymbol{E} \to V\boldsymbol{E} : e \mapsto \big(e - s\big(t(e)\big)\big) \qquad \text{and} \qquad \underline{r} := g^{\flat}(\vec{r}) : \boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}.$$

Moreover, we define the phase function

$$\mathfrak{L}^2[s, o] := \bar{g}(\mathfrak{L}, \mathfrak{L}) : J_1\boldsymbol{E} \to \mathbb{R}. \quad \square$$

For the coordinate expression of the observed angular momentum, see Proposition 25.1.2, Remarks 25.1.6 and 25.1.8.

3.2.6 Fibrewise Riemannian Structure

The galilean metric g and, equivalently, the rescaled galilean metric G, yield, in a covariant way, a *riemannian connection* $\varkappa[g] = \varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{T}}V\boldsymbol{E}$ on the fibres of spacetime. Indeed, this fibrewise riemannian structure has a rather static nature and is insufficient for classical and quantum dynamics.

However, later, we shall deal with a spacetime connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Proposition 4.1.2, Postulate C.3 and Theorem 6.3.1). We shall see that the present fibrewise riemannian structure can be derived from the spacetime connection K via a suitable spacelike restriction (see Corollary 4.2.14). We shall also see that the scalar curvatures of the fibrewise riemannian connection $\varkappa$ and of the galilean spacetime connection K coincide (see Corollary 4.2.22).

Proposition 3.2.13 *The galilean metric g (or, equivalently, the rescaled galilean metric G) naturally yield the following fibre-wise objects:*

- *the fibre-wise* riemannian connection

$$\varkappa[g] = \varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{T}}V\boldsymbol{E},$$

- *the fibre-wise* curvature

$$R[g] = R[G] : \boldsymbol{E} \to \Lambda^2 V^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

- *the fibre-wise* Ricci tensor

$$r[g] = r[G] := C^1_2 R[g] = C^1_2 R[G] : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

- *the fibre-wise scaled* scalar curvatures

$$\begin{aligned} C[g] &:= \bar{g} \lrcorner r : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes I\!R, \\ C[G] &:= \bar{G} \lrcorner r : \boldsymbol{E} \to \mathbb{T}^* \otimes I\!R. \end{aligned}$$

Their coordinate expressions are

$$\begin{aligned} \varkappa[g] = \varkappa[G] &= \check{d}^i \otimes (\partial_i + \varkappa_i{}^h{}_k \, \dot{x}^k \, \dot{\partial}_j), \\ R[g] = R[G] &= -2\,(\partial_h \kappa_k{}^i{}_j + \varkappa_h{}^r{}_j \, \varkappa_k{}^i{}_r)\, \check{d}^h \wedge \check{d}^k \otimes \partial_i \otimes \check{d}^j, \\ r[g] = r[G] &= (\partial_h \varkappa_i{}^h{}_j - \partial_i \varkappa_h{}^h{}_j + \varkappa_h{}^k{}_j \, \varkappa_i{}^h{}_k - \varkappa_i{}^k{}_j \, \varkappa_h{}^h{}_k)\, \check{d}^i \otimes \check{d}^j, \\ C[g] &= g^{ij}\, (\partial_h \varkappa_i{}^h{}_j - \partial_i \varkappa_h{}^h{}_j + \varkappa_h{}^k{}_j \, \varkappa_i{}^h{}_k - \varkappa_i{}^k{}_j \, \varkappa_h{}^h{}_k), \\ C[G] &= G^{ij}_0\, (\partial_h \varkappa_i{}^h{}_j - \partial_i \varkappa_h{}^h{}_j + \varkappa_h{}^k{}_j \, \varkappa_i{}^h{}_k - \varkappa_i{}^k{}_j \, \varkappa_h{}^h{}_k)\, u^0, \end{aligned}$$

where

$$\varkappa_h{}^i{}_k = -\tfrac{1}{2}\, g^{ij}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) = -\tfrac{1}{2}\, G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}),$$

hence

$$\begin{aligned} C[g] = &- g^{kl}\, g^{ip}\, (\partial_{il} g_{pk} - \partial_{ip} g_{kl}) + \tfrac{1}{4}\, g^{kl}\, \partial_k g^{ip}\, \partial_l g_{pi} \\ &- g^{kl}\, (\tfrac{1}{2}\, \partial_i g^{ip} + \tfrac{1}{4}\, g^{rp}\, g^{iq}\, \partial_r g_{qi})\, (2\, \partial_k g_{pl} - \partial_p g_{kl}) \\ &+ \tfrac{1}{4}\, g^{kl}\, g^{rp}\, g^{iq}\, (\partial_r g_{qk} - \partial_q g_{kr})\, (\partial_i g_{pl} - \partial_p g_{il}), \\ C[G] = &- G^{kl}_0\, G^{ip}_0\, (\partial_{il} G^0_{pk} - \partial_{ip} G^0_{kl}) + \tfrac{1}{4}\, G^{kl}_0\, \partial_k G^{ip}_0\, \partial_l G^0_{pi} \\ &- G^{kl}_0\, (\tfrac{1}{2}\, \partial_i G^{ip}_0 + \tfrac{1}{4}\, G^{rp}_0\, G^{iq}_0\, \partial_r G^0_{qi})\, (2\, \partial_k G^0_{pl} - \partial_p G^0_{kl}) \\ &+ \tfrac{1}{4}\, G^{kl}_0\, G^{rp}_0\, G^{iq}_0\, (\partial_r G^0_{qk} - \partial_q G^0_{kr})\, (\partial_i G^0_{pl} - \partial_p G^0_{il}). \quad \square \end{aligned}$$

We stress that our Christoffel symbols $\varkappa_h{}^i{}_k$ are the negative of the standard ones, because they fit the definition of a connection via the horizontal prolongation instead of the vertical projection (see Appendix: Definition F.2.1).

3.2.7 Fibrewise Symplectic Structure

The galilean metric g yields, in a covariant way, a scaled *symplectic form* $\omega[g] := \check{d}_v g^{\square} : V\boldsymbol{E} \to \mathbb{L}^2 \otimes \Lambda^2 V^*_{\boldsymbol{E}} V\boldsymbol{E}$ on the fibres of spacetime (see, for instance, [146].

The above spacelike symplectic form ω can be also obtained in a covariant way from the spacelike riemannian connection $\varkappa$ (see, for instance, [198]).

Indeed, this symplectic structure ω has a rather kinematical nature and is insufficient for classical and quantum dynamics (see, Introduction, Sect. 10.1). However, later, we shall find a cosymplectic structure $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ of the phase space, which does our job (see Theorem 10.1.1). We shall see that the present fibrewise symplectic structure can be derived from the cosymplectic structure Ω via a suitable spacelike restriction (see Remark 10.1.15).

Proposition 3.2.14 *The galilean metric g naturally yields the following fibre-wise objects (see, for instance, [146]):*

- *the scaled spacelike* metric quadratic function *(see, for instance, [146])*

$$g^{\square} : V\boldsymbol{E} \to \mathbb{L}^2 \otimes I\!R : X \mapsto \tfrac{1}{2}\, g(X, X),$$

- *the scaled spacelike* Liouville 1-form *(see Note B.4.7 and [146, pag. 163])*

$$\lambda[g] := d_v g^{\square} : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*_{\boldsymbol{E}} V\boldsymbol{E},$$

- *the scaled spacelike* symplectic 2-form

$$\omega[g] = \check{d}\lambda[g] : V\boldsymbol{E} \to \mathbb{L}^2 \otimes \Lambda^2 V^*_{\boldsymbol{E}} V\boldsymbol{E}.$$

Their coordinate expressions are

$$\begin{aligned} g^{\square} &= \tfrac{1}{2}\, g_{ij}\, \dot{x}^i\, \dot{x}^j, \\ \lambda[g] &= g_{ij}\, \dot{x}^j\, \check{d}^i, \\ \omega[g] &= g_{ij}\, \check{\dot{d}}^i \wedge \check{d}^j + \partial_i g_{jh}\, \dot{x}^h\, \check{d}^i \wedge \check{d}^j. \quad \square \end{aligned}$$

The above spacelike symplectic form $\omega[g]$ can also be derived from the spacelike riemannian connection $\varkappa[g]$ as follows.

Proposition 3.2.15 *Let us consider the projection $\nu[\varkappa]$ associated with the spacelike riemannian connection $\varkappa[g]$ and the natural spacelike vertical valued 1-form ν*

$$\nu[\varkappa] : V_{\boldsymbol{T}} V\boldsymbol{E} \to V\boldsymbol{E} \qquad \text{and} \qquad \nu : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

with coordinate expressions

$$\nu[\varkappa] = (\check{\dot{d}}^i - \varkappa_h{}^i{}_k \, \dot{x}^k \, \check{d}^h) \otimes \partial_i \qquad \text{and} \qquad \nu = \check{d}^j \otimes \partial_j .$$

Then, we have the equality

$$\omega[g] = g \lrcorner \big(\nu[\varkappa] \wedge \nu\big).$$

Proof. We have

$$\begin{aligned} g \lrcorner \big(\nu[\varkappa] \wedge \nu\big) &= g_{ij} \, \big(\check{\dot{d}}^i - \varkappa_h{}^i{}_k \, \dot{x}^k \, \check{d}^h\big) \wedge \check{d}^j \\ &= g_{ij} \, \check{\dot{d}}^i \wedge \check{d}^j + \tfrac{1}{2} \, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) \, \dot{x}^k \, \check{d}^h \wedge \check{d}^j \\ &= g_{ij} \, \check{\dot{d}}^i \wedge \check{d}^j + \partial_i g_{jh} \, \dot{x}^h \, \check{d}^i \wedge \check{d}^j . \quad \square \end{aligned}$$

3.2.8 Metric Differential Operators

We discuss the *gradient* and *rescaled gradient* of spacetime functions, the *spacetime divergence* of spacetime vector fields, the *spacelike divergence* of projectable spacetime vector fields, the *metric laplacian* of spacetime functions, the *rescaled metric laplacian* of spacetime functions and the *curl* of spacelike vector fields.

Indeed, these subjects are rather standard. However, a special attention is required by some specifications.

Definition 3.2.16 We define the *gradient* and *rescaled gradient operators* to be the sheaf morphisms

$$\vec{d} : \text{map}(\boldsymbol{E}, \mathrm{I\!R}) \to \text{map}(\boldsymbol{E}, \, \mathbb{L}^{-2} \otimes V\boldsymbol{E}) : f \mapsto g^\sharp(df),$$
$$\overrightarrow{d} : \text{map}(\boldsymbol{E}, \mathrm{I\!R}) \to \text{map}(\boldsymbol{E}, \, \mathbb{T}^* \otimes V\boldsymbol{E}) : f \mapsto G^\sharp(df),$$

with coordinate expressions

$$\vec{d} f = g^{ij} \, \partial_j f \, \partial_i \qquad \text{and} \qquad \overrightarrow{d} f = G_0^{ij} \, \partial_j f \, u^0 \otimes \partial_i . \quad \square$$

Definition 3.2.17 We define the *spacetime divergence* and the *spacelike divergence* of spacetime vector fields, respectively, as follows (see Definition 2.2.6)

$$\begin{aligned} &\text{div}_\upsilon \, X := i_{\bar{\upsilon}}(L_X \upsilon) \in \text{map}(\boldsymbol{E}, \, \mathrm{I\!R}), \qquad \text{for each} \qquad X \in \text{sec}(\boldsymbol{E}, T\boldsymbol{E}), \\ &\text{div}_\eta \, X := i_{\bar{\eta}}(L_X \eta) \in \text{map}(\boldsymbol{E}, \, \mathrm{I\!R}), \qquad \text{for each} \qquad X \in \text{pro}(\boldsymbol{E}, T\boldsymbol{E}), \end{aligned}$$

We have the coordinate expressions

$$\operatorname{div}_\upsilon X = \frac{\partial_0(X^0\sqrt{|g|})}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}},$$
$$\operatorname{div}_\eta X = X^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(X^i\sqrt{|g|})}{\sqrt{|g|}}.$$

We have the following identities

$$\operatorname{div}_\upsilon[X, \acute{X}] = X.\operatorname{div}_\upsilon \acute{X} - \acute{X}.\operatorname{div}_\upsilon X,$$
$$\operatorname{div}_\eta[X, \acute{X}] = X.\operatorname{div}_\eta \acute{X} - \acute{X}.\operatorname{div}_\eta X. \quad \square$$

Example 3.2.18 We have the following divergences of the observer o

$$\operatorname{div}_\upsilon \text{д}[o] \in \operatorname{map}(\boldsymbol{E},\ \mathbb{T}^* \otimes \mathbb{R}),$$
$$\operatorname{div}_\eta \text{д}[o] \in \operatorname{map}(\boldsymbol{E},\ \mathbb{T}^* \otimes \mathbb{R}),$$

with coordinate expressions

$$\operatorname{div}_\upsilon \text{д}[o] = \left(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(o_0^i\sqrt{|g|})}{\sqrt{|g|}}\right) u^0,$$
$$\operatorname{div}_\eta \text{д}[o] = \left(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_i(o_0^i\sqrt{|g|})}{\sqrt{|g|}}\right) u^0. \quad \square$$

Definition 3.2.19 We define the *curl* of a spacelike vector field $X \in \sec(\boldsymbol{E}, V\boldsymbol{E})$ to be the scaled spacelike vector field

$$\operatorname{curl} X := i_{\check{d}(g^\flat(X))}\,\bar{\eta} \in \sec\big(\boldsymbol{E},\ \mathbb{L}^{-1} \otimes V\boldsymbol{E}),$$

with coordinate expression

$$\operatorname{curl} X = \frac{1}{\sqrt{|g|}}\,\epsilon^{ijh}\,\partial_i X_j\,\partial_h.$$

For each $X, Y \in \sec(\boldsymbol{E}, V\boldsymbol{E})$, we have the equality

$$\operatorname{curl}(X \times Y) = X\ \operatorname{div}_\eta Y - Y\ \operatorname{div}_\eta X - [X, Y]. \quad \square$$

Definition 3.2.20 We define the spacelike *metric laplacian* and the spacelike *rescaled metric laplacian* to be the sheaf morphisms

$$\Delta[g] := \operatorname{div}_\eta, g^\sharp, \check{d} : \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \to \operatorname{map}(\boldsymbol{E}, \mathbb{L}^{-2} \otimes \mathbb{R}) : f \mapsto \operatorname{div}_\eta\big(g^\sharp(\check{d}f)\big),$$
$$\Delta[G] := \operatorname{div}_\eta, G^\sharp, \check{d} : \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \to \operatorname{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}) : f \mapsto \operatorname{div}_\eta\big(G^\sharp(\check{d}f)\big),$$

with coordinate expressions

$$\Delta[g]f = \left(g^{ij}\,\partial_{ij} f + \frac{\partial_i (g^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j f \right),$$
$$\Delta[G]f = \left(G_0^{ij}\,\partial_{ij} f + \frac{\partial_i (G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j f \right) u^0. \quad \square$$

3.2.9 Rigid Observers

We discuss the *Lie derivative of the metric* G with respect to the velocity $д[o]$ of an observer o and define the *rigid observers*.

Let us consider an observer o and its associated velocity $д[o]$ (see Proposition 2.7.3).

Proposition 3.2.21 *Being* $д[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ *a projectable scaled spacetime vector field, the observed Lie derivative*

$$L_{д[o]}G \in \sec(\boldsymbol{E},\ V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}).$$

is well defined, in spite of the fact that G is a spacelike covariant tensor (see Lemma 2.2.8).

We have the coordinate expression

$$L_{д[o]}G = (\partial_0 G^0_{ij} + G_{hj}\,\partial_i o^h_0 + G^0_{ih}\,\partial_j o^h_0)\, \check{d}^i \otimes \check{d}^j. \quad \square$$

Definition 3.2.22 An observer o is said to be *rigid* if

$$L_{д[o]}G = 0,$$

i.e., in coordinates, if

$$\partial_0 G^0_{ij} + G^0_{hj}\,\partial_i o^h_0 + G^0_{ih}\,\partial_j o^h_0 = 0. \quad \square$$

Thus, in intuitive words, the observer o is rigid if the "infinitesimal distance" of neighbouring particles of its flow $\mathscr{F}[o]$ does not change along the time evolution.

Chapter 4
Galilean Gravitational Field

We introduce the *galilean gravitational field*, which is represented by a *galilean spacetime connection* $K^{\natural}$. We discuss this concept step by step, through the language of general connections (see Appendix F).

We start by analysing the generic *spacetime connections* K, along with their *vertical restrictions* $\check{K}$, $\overset{\vee}{K}$, $\overset{\shortparallel}{K}$. In particular, we consider the *time preserving spacetime connections* (Sect. 4.1.1). Moreover, we study their *curvature tensor* and *torsion tensors* via the Frölicher–Nijenhuis bracket (Sects. 4.1.2–4.1.4).

Next, we analyse the *special spacetime connections* (i.e. time preserving, linear and torsion free spacetime connections) (Sect. 4.1).

Further, we study in detail the *metric preserving special spacetime connections* K and classify them by showing that every K splits, with reference to an observer o, into a term determined by the metric and a term generated by a *closed observed spacetime 2-form* $\Phi[o]$ (Sect. 4.2). Particular attention is devoted to the curvature tensor of these connections.

Eventually, we define a *galilean spacetime connection* as a special spacetime connection fulfilling the additional symmetry property $R_{i\mu}{}^{0}{}_{j\nu} = R_{j\nu}{}^{0}{}_{i\mu}$ (Sect. 4.3.1). We stress that this property is equivalent to the exactness of $\Phi[o]$; in this way, we achieve the spacetime potential $A[o]$ of $\Phi[o]$. Indeed, later in the quantum theory, this property turns out to be an essential requirement for the existence of the upper quantum connection (see Lemma 9.2.14 and Remark 15.2.3).

Eventually, we postulate the gravitational field $K^{\natural}$ (Sect. 4.3.4).

We conclude this chapter by studying the spacetime differential operators induced by the gravitational field (Sect. 4.4).

In einsteinian General Relativity, the gravitational connection is a Levi–Civita connection determined by the lorentzian metric; for this reason, one is inclined to identify the gravitational field with the metric field. However, in our context, the galilean metric determines the galilean spacetime connection only partially; for this

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reason we are led to maintain a clear distinction between the galilean metric field and the galilean spacetime connection, by saying that only the latter represents the gravitational field.

4.1 Special Spacetime Connections

We discuss the *special spacetime connections*, i.e. the linear, time preserving, torsion free spacetime connections $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ of the vector bundle $T\boldsymbol{E} \to \boldsymbol{E}$, according to the general theory of connections (see Appendix: Note F.1.1 and, for instance, [246, 311]).

In particular, we study the *vertical restrictions*, the *curvature* and the *torsion* of these spacetime connections.

4.1.1 Spacetime Connections

We discuss the *spacetime connections* $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, along with the associated *projection* $\nu[K] : TT\boldsymbol{E} \to T\boldsymbol{E}$ and the linear splitting $TT\boldsymbol{E} = H_K T\boldsymbol{E} \oplus V_{\boldsymbol{E}} T\boldsymbol{E}$.

Moreover, we analyse the three distinguished *vertical restrictions* of K.

Eventually, we analyse the *linear* spacetime connections and the *time preserving* spacetime connections, which are defined by the condition $\nabla dt = 0$, i.e., in coordinates, $K_\lambda{}^0 = 0$.

Definition 4.1.1 A *spacetime connection* is defined to be a section (see Appendix: Definition F.1.1 and, for instance, [246, 311])

$$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

projectable on the section $\mathbf{1}_{\boldsymbol{E}} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, according to the commutative diagram

$$\begin{array}{ccc} T\boldsymbol{E} & \xrightarrow{K} & T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \\ {\scriptstyle \tau_{\boldsymbol{E}}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes T\tau_{\boldsymbol{E}}} \\ \boldsymbol{E} & \xrightarrow{\mathbf{1}_{\boldsymbol{E}}} & T^*\boldsymbol{E} \otimes T\boldsymbol{E} \end{array} .$$

The coordinate expression of a spacetime connection is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^\nu \, \dot{\partial}_\nu), \quad \text{with } K_\lambda{}^\nu \in \mathrm{map}(T\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 4.1.2 *A* spacetime connection *can be also regarded as a fibred morphism over* $\boldsymbol{E}$

$$K : T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TT\boldsymbol{E},$$

which makes the following diagram commutative

$$\begin{array}{ccc}
T\boldsymbol{E} & \xrightarrow{\ \mathrm{id}\ } & T\boldsymbol{E} \\
{\scriptstyle \mathrm{pro}_2}\uparrow & & \uparrow{\scriptstyle T\tau_{\boldsymbol{E}}} \\
T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\ K\ } & TT\boldsymbol{E} \\
{\scriptstyle \mathrm{pro}_1}\downarrow & & \downarrow{\scriptstyle \tau_{T\boldsymbol{E}}} \\
T\boldsymbol{E} & \xrightarrow{\ \mathrm{id}\ } & T\boldsymbol{E}
\end{array} \quad ,$$

and which is a linear fibred morphism with respect to the 2nd factor.

A spacetime connection K is characterised by the surjective linear fibred morphism $\nu[K] : TT\boldsymbol{E} \to T\boldsymbol{E}$ over $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$, which makes the following diagram commutative (see Appendix: Note B.4.2)

$$\begin{array}{ccc}
T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\ K\ } & TT\boldsymbol{E} \\
{\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle \nu[K]} \\
T\boldsymbol{E} & \xrightarrow{\ \mathrm{id}\ } & T\boldsymbol{E}
\end{array} \quad .$$

We have the coordinate expression

$$\nu[K] = (\dot{d}^\lambda - K_\mu{}^\lambda \, d^\mu) \otimes \partial_\lambda.$$

The maps K and $\nu[K]$ yield a linear splitting of the iterated tangent bundle into K-horizontal and vertical components

$$TT\boldsymbol{E} = H_K T\boldsymbol{E} \oplus V_{\boldsymbol{E}} T\boldsymbol{E}.$$

Hence, let us consider a projectable vector field $X : T\boldsymbol{E} \to TT\boldsymbol{E}$ and its projection $\underline{X} : \boldsymbol{E} \to T\boldsymbol{E}$, with coordinate expressions $X = \underline{X}^\lambda \partial_\lambda + X^\lambda \, \dot{\partial}_\lambda$ and $\underline{X} = \underline{X}^\lambda \partial_\lambda$ (see Fig. 4.1). Then, the horizontal lift $K(\underline{X}) : T\boldsymbol{E} \to H_K T\boldsymbol{E}$ and the vertical projection $\nu[K](X) : T\boldsymbol{E} \to V_{\boldsymbol{E}} T\boldsymbol{E}$ have coordinate expressions $K(\underline{X}) = \underline{X}^\lambda \, (\partial_\lambda + K_\lambda{}^\nu \, \dot{\partial}_\nu)$ and $\nu[K](X) = (X^\nu - \underline{X}^\lambda \, K_\lambda{}^\nu) \, \dot{\partial}_\nu$.

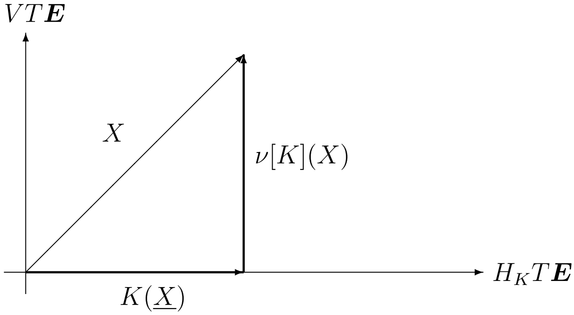

Fig. 4.1 Spacetime connection

So, we obtain the splitting

$$X = K(\underline{X}) + \nu[K](X),$$

with coordinate expression

$$X = \underline{X}^{\lambda}\,(\partial_{\lambda} + K_{\lambda}{}^{\nu}\,\dot{\partial}_{\nu}) + \big(X^{\nu} - \underline{X}^{\lambda}\,K_{\lambda}{}^{\nu}\big)\,\dot{\partial}_{\nu}. \quad \square$$

Proposition 4.1.3 *Given a spacetime connection* $K : \boldsymbol{TE} \underset{\boldsymbol{E}}{\times} \boldsymbol{TE} \to \boldsymbol{TTE}$, *we consider the following three* vertical restrictions *(see Proposition 2.3.2).*

- *The "*1st vertical restriction*" is defined to be the vertical restriction of* K with respect to the 1st factor

$$\overset{\shortparallel}{K} : \boldsymbol{VE} \underset{\boldsymbol{E}}{\times} \boldsymbol{TE} \to \boldsymbol{TVE},$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{VE} \underset{\boldsymbol{E}}{\times} \boldsymbol{TE} & \xrightarrow{\ \overset{\shortparallel}{K}\ } & \boldsymbol{TVE} \\ {\scriptstyle j_{\boldsymbol{E}} \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \cap} \\ \boldsymbol{TE} \underset{\boldsymbol{E}}{\times} \boldsymbol{TE} & \xrightarrow{\ K\ } & \boldsymbol{TTE} \end{array} .$$

Indeed, $\overset{\shortparallel}{K}$ *turns out to be a connection on the vector bundle* $\tau_{\boldsymbol{E}} : \boldsymbol{VE} \to \boldsymbol{E}$. *Its coordinate expression is*

$$\overset{\shortparallel}{K} = d^{\lambda} \otimes (\partial_{\lambda} + \overset{\shortparallel}{K}_{\lambda}{}^{i}\,\dot{\partial}_{i}), \quad \text{where } \overset{\shortparallel}{K}_{\lambda}{}^{i} = K_{\lambda}{}^{i} \circ j_{\boldsymbol{VE}} \in \mathrm{map}(\boldsymbol{VE}, \mathbb{R}).$$

- *The* "2nd vertical restriction" *is defined to be the vertical restriction of* K with respect to the 2nd factor

$$\check{K} : T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} \to V_{\boldsymbol{T}} T\boldsymbol{E},$$

according to the following commutative diagram

$$\begin{array}{ccc} T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} & \xrightarrow{\check{K}} & V_{\boldsymbol{T}} T\boldsymbol{E} \\ {\scriptstyle \mathrm{id} \times j_{\boldsymbol{E}}} \downarrow & & \downarrow {\scriptstyle \cap} \\ T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{K} & TT\boldsymbol{E} \end{array} .$$

Indeed, $\check{K}$ *is not a connection on the vector bundle* $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$.
Its coordinate expression is

$$\check{K} = d^i \otimes (\partial_i + K_i{}^\nu \, \dot{\partial}_\nu), \quad \text{where } K_i{}^\nu \in \mathrm{map}(T\boldsymbol{E}, I\!R).$$

- *The* "full vertical restriction" *is defined to be the vertical restriction of* K with respect to both 1st and 2nd factors

$$\overset{\vee}{K} : V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} \to V_{\boldsymbol{E}} V\boldsymbol{E},$$

according to the following commutative diagram

$$\begin{array}{ccc} V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} & \xrightarrow{\overset{\vee}{K}} & V_{\boldsymbol{E}} V\boldsymbol{E} \\ {\scriptstyle j_{\boldsymbol{E}} \times j_{\boldsymbol{E}}} \downarrow & & \downarrow {\scriptstyle \cap} \\ T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{K} & TT\boldsymbol{E} \end{array} .$$

Indeed, $\overset{\vee}{K}$ *turns out to be a smooth family of connections on the fibres of spacetime.*
Its coordinate expression is

$$\overset{\vee}{K} = d^i \otimes (\partial_i + \check{K}_i{}^h \, \dot{\partial}_h), \quad \text{where } \overset{\shortparallel}{K}_i{}^h = \check{K}_i{}^h = K_i{}^h \circ j_{V\boldsymbol{E}} \in \mathrm{map}(V\boldsymbol{E}, I\!R). \quad \square$$

Definition 4.1.4 A spacetime connection K is said to be *linear* if it is a linear fibred morphism over $\mathbf{1}_{\boldsymbol{E}}$ (see Appendix: Definition F.2.1) i.e., in coordinates, if and only if

$$K_\lambda{}^\mu = K_\lambda{}^\nu{}_\mu \, \dot{x}^\mu, \quad \text{where } K_\lambda{}^\nu{}_\mu \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

Thus, the coordinate expression of a linear spacetime connection K is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^\nu{}_\mu \, \dot{x}^\mu \, \dot{\partial}_\nu), \quad \text{with } K_\lambda{}^\nu{}_\mu \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}). \quad \square$$

Definition 4.1.5 A spacetime connection K is said to be *time preserving* if, for each $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ and $Y \in \mathrm{tim}\sec(\boldsymbol{E}, T\boldsymbol{E})$, we have

$$\nabla_X Y \in \sec(\boldsymbol{E}, V\boldsymbol{E}).$$

In other words, the spacetime connection K is time preserving if the covariant derivative, with respect to any spacetime vector field, of a spacetime vector field whose time component is constant, turns out to be tangent to the fibres of spacetime (see Definition 2.2.6).

Thus, a spacetime connection K turns out to be time preserving if and only if, in coordinates,

$$K_\lambda{}^0 = 0,$$

i.e., if and only if its coordinate expression is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i \, \dot{\partial}_i), \quad \text{with } K_\lambda{}^i \in \mathrm{map}(T\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 4.1.6 *A linear spacetime connection K turns out to be time preserving if and only if*

$$\nabla dt = 0.$$

Thus, the coordinate expression of a time preserving, linear spacetime connection K is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu \, \dot{x}^\mu \, \dot{\partial}_i), \quad \text{where } K_\lambda{}^i{}_\mu \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}). \quad \square$$

4.1.2 Curvature of Spacetime Connections

In Appendix F, given a general connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ on a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, we define the *curvature tensor* to be the tangent valued 2-form $R[c] := -[c, c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$, where $[,]$ is the Frölicher–Nijenhuis bracket (see [246, 248, 311] and Appendix: Definition F.1.9, Theorem E.2.3). This approach turns out to be very convenient in many respects.

Here, we specify the above general notion in the particular case of a time preserving, linear spacetime connection K.

We recall that our symbols of the connection are conventionally defined as the negatives of the usual ones in the standard literature; therefore some apparent changes of sign appear in our coordinate formulas.

Proposition 4.1.7 *The curvature*

$$R[K] := -[K, K]$$

of a time preserving, linear spacetime connection K can be naturally regarded as a spacetime tensor (see Definition F.1.9)

$$R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

and turns out to be given by the standard equality (see Appendix: Note F.2.8)

$$(R[K](X, Y))(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for each $X, Y, Z \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.

We have the coordinate expression

$$\begin{aligned} R[K] &= -2\,(\partial_\lambda K_\mu{}^i{}_\nu + K_\lambda{}^r{}_\nu\, K_\mu{}^i{}_r)\, d^\lambda \wedge d^\mu \otimes \partial_i \otimes d^\nu \\ &= -(\partial_\lambda K_\mu{}^i{}_\nu + K_\lambda{}^r{}_\nu\, K_\mu{}^i{}_r - \partial_\mu K_\lambda{}^i{}_\nu - K_\mu{}^r{}_\nu\, K_\lambda{}^i{}_r)\, d^\lambda \otimes d^\mu \otimes \partial_i \otimes d^\nu. \quad \square \end{aligned}$$

Definition 4.1.8 We define the *Ricci tensor* of a spacetime connection K to be the spacetime tensor

$$r[K] := C^1_2 R[K] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

which, in the case of a time preserving, linear spacetime connection has coordinate expression

$$r[K] = -(\partial_\lambda K_i{}^i{}_\nu + K_\lambda{}^r{}_\nu\, K_i{}^i{}_r - \partial_i K_\lambda{}^i{}_\nu - K_i{}^r{}_\nu\, K_\lambda{}^i{}_r)\, d^\lambda \otimes d^\nu.$$

We define the *scalar curvature* of a spacetime connection K to be each of the scaled functions (see Definition 3.2.2)

$$\begin{aligned} C[g] &\equiv C[g, K] := \bar{g} \lrcorner\, r[K] : \boldsymbol{E} \to \mathbb{L}^2 \otimes \mathbb{R}, \\ C[G] &\equiv C[G, K] := \bar{G} \lrcorner\, r[K] : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}, \end{aligned} \tag{4.1}$$

which, in the case of a time preserving, linear spacetime connection have coordinate expressions (see, also, Proposition 3.2.13)

$$C[g] = -g^{hk}\,(\partial_h K_i{}^i{}_k + K_h{}^r{}_k\, K_i{}^i{}_r - \partial_i K_h{}^i{}_k - K_i{}^r{}_k\, K_h{}^i{}_r),$$
$$C[G] = -G_0^{hk}\,(\partial_h K_i{}^i{}_k + K_h{}^r{}_k\, K_i{}^i{}_r - \partial_i K_h{}^i{}_k - K_i{}^r{}_k\, K_h{}^i{}_r)\, u^0. \quad \square$$

We can apply the above procedure also to the full vertical restriction (see Proposition 4.1.3) $\overset{\vee}{K} : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} V\boldsymbol{E}$ of a time preserving spacetime connection K, by considering it as a smooth family of connections on the fibres of spacetime.

Proposition 4.1.9 *The curvature*

$$R[\overset{\vee}{K}] := -\,[\overset{\vee}{K}, \overset{\vee}{K}]$$

of a time preserving, linear spacetime connection K can be naturally regarded as a spacetime tensor (see Appendix: Definition F.1.9)

$$R[\overset{\vee}{K}] : \boldsymbol{E} \to \Lambda^2\, V^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

and turns out to be given by the standard equality (see Appendix: Note F.2.8)

$$\big(R[\overset{\vee}{K}](X, Y)\big)(Z) = \overset{\vee}{\nabla}_X \overset{\vee}{\nabla}_Y Z - \overset{\vee}{\nabla}_Y \overset{\vee}{\nabla}_X Z - \overset{\vee}{\nabla}_{[X,Y]} Z,$$

for each $X, Y, Z \in \sec(\boldsymbol{E}, V\boldsymbol{E})$.

We have the coordinate expression

$$R[\overset{\vee}{K}] = -2\,(\partial_h K_k{}^i{}_l + K_h{}^r{}_l\; K_k{}^i{}_r)\; d^h \wedge d^k \otimes \partial_i \otimes d^l$$
$$= -(\partial_h K_k{}^i{}_l + K_h{}^r{}_l\; K_k{}^i{}_r - \partial_k K_h{}^i{}_l - K_k{}^r{}_l\; K_h{}^i{}_r)\; d^h \otimes d^k \otimes \partial_i \otimes d^l. \quad \square$$

Definition 4.1.10 We define the *Ricci tensor* of the full vertical restriction $\overset{\vee}{K}$ of a spacetime connection K to be the spacetime tensor (see Proposition 4.1.3)

$$r[\overset{\vee}{K}] := C^1_2 R[\overset{\vee}{K}] : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

which, in the case of a time preserving, linear spacetime connection has coordinate expression

$$r[\overset{\vee}{K}] = -(\partial_h K_i{}^i{}_k + K_h{}^r{}_k\, K_i{}^i{}_r - \partial_i K_h{}^i{}_k - K_i{}^r{}_k\, K_h{}^i{}_r)\, \check{d}^h \otimes \check{d}^k.$$

We define the *scalar curvature* of the full vertical restriction $\overset{\vee}{K}$ of a spacetime connection K to be each of the scaled functions (see Definition 3.2.2)

$$C[g, \overset{\vee}{K}] := \bar{g} \lrcorner r[\overset{\vee}{K}] : \boldsymbol{E} \to \mathbb{L}^2 \otimes \mathbb{R},$$
$$C[G, \overset{\vee}{K}] := \bar{G} \lrcorner r[\overset{\vee}{K}] : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R},$$

which, in the case of a time preserving, linear spacetime connection, have coordinate expressions (see, also, Definition 4.1.8 and Proposition 3.2.13)

$$C[g] = -g^{hk} \, (\partial_h K_i{}^i{}_k + K_h{}^r{}_k \, K_i{}^i{}_r - \partial_i K_h{}^i{}_k - K_i{}^r{}_k \, K_h{}^i{}_r),$$
$$C[G] = -G_0^{hk} \, (\partial_h K_i{}^i{}_k + K_h{}^r{}_k \, K_i{}^i{}_r - \partial_i K_h{}^i{}_k - K_i{}^r{}_k \, K_h{}^i{}_r) \, u^0. \quad \square$$

Proposition 4.1.11 *The Ricci tensor of a time preserving spacetime connection K equals the vertical restriction of the Ricci tensor of K, according to the equality (see Proposition 2.2.4)*

$$r[\overset{\vee}{K}] = \vee \circ r[K].$$

The scalar curvature of a time preserving spacetime connection K factorises through its full vertical restriction $\overset{\vee}{K}$, according to the following equalities

$$C[g, K] = C[g, \overset{\vee}{K}] \qquad \text{and} \qquad C[G, K] = C[G, \overset{\vee}{K}]. \quad \square$$

4.1.3 Torsion of Spacetime Connections

In Appendix F, given a general connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ and a soldering form $\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ on a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, we define the *torsion* to be the tangent valued 2–form $T[c] := 2\,[c, \sigma] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$, where [,] is the Frölicher–Nijenhuis bracket (see [246, 248, 311] and Note F.1.21, Theorem E.2.3). This approach turns out to be very convenient in many respects.

We stress that, in general, according to this general approach, the torsion needs not to be a purely algebraic operator acting on the connection; such a property holds only in some particular cases which depend on the chosen soldering form.

Here, we specify the above general notion in the particular case of a time preserving, linear spacetime connection K.

We recall that our symbols of the connection are conventionally defined as the negatives of the ones in the standard literature; therefore some apparent changes of sign appear in our coordinate formulas.

Lemma 4.1.12 *In the particular case of connections of the manifold $\boldsymbol{E}$, there is a natural choice for the* soldering form σ *to be used for the definition of the torsion, namely (see Appendix: Lemma F.4.4)*

$$\sigma := \nu_{T\boldsymbol{E}} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} T\boldsymbol{E},$$

with coordinate expression

$$\nu_{T\boldsymbol{E}} = d^{\lambda} \otimes \dot{\partial}_{\lambda}. \quad \square$$

Proposition 4.1.13 *The torsion*

$$T[K] := 2\,[K, \nu_{T\boldsymbol{E}}]$$

of a time preserving, linear spacetime connection K can be naturally regarded as a spacetime tensor (see Appendix: Note F.1.21)

$$T[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

and turns out to be given by the standard equality (see Appendix: Note F.4.5)

$$T[K](X, Y) = \nabla_X Y - \nabla_Y X - [X, Y].$$

We have the coordinate expression

$$T[K] = -K_{\lambda}{}^{i}{}_{\mu}\, d^{\lambda} \wedge d^{\mu} \otimes \partial_i. \quad \square$$

4.1.4 Bianchi Identities for Spacetime Connections

In Appendix F, given a general connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ on a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, we achieve the *Bianchi identities* via the definition of curvature and torsion in terms of the FN-bracket and the Jacobi property of this bracket (see Appendix: Theorems E.1.1, E.2.3, F.1.14, and F.1.23, Definition F.1.9, Note F.1.21 and see also [246, 311]).

Here, we specify the above results in the particular case of a time preserving, linear spacetime connection K (for the standard Bianchi identities, see, for instance, [51, 241]).

Indeed, our approach to curvature and torsion via the FN-bracket turns out to be convenient for the achievement of the Bianchi identities. In fact, our procedure emphasises the true deep nature of Bianchi identities, which are an immediate consequence of the Jacobi property of FN-bracket of tangent valued forms and has nothing to do with riemannian geometry. Actually, our spacetime connections cannot be regarded as riemannian connections, because they are not determined by the metric.

Lemma 4.1.14 *If K is a time preserving, linear spacetime connection, then we have the coordinate expression (see Lemma 4.1.12 and Appendix: Theorem E.2.3 and Note B.4.2)*

$$[\nu_{\boldsymbol{TE}}, R] = R_{\lambda\mu}{}^{i}{}_{\nu}\, d^{\lambda} \wedge d^{\mu} \wedge d^{\nu} \otimes \partial_i . \quad \square$$

Lemma 4.1.15 *We have*

$$[K, R] = \mathrm{Alt}_{123}\big(\nabla R + C_3^1(T \otimes R)\big). \quad \square$$

Theorem 4.1.16 (2nd Bianchi identity) *If K is a linear spacetime connection, then the 2nd Bianchi identity becomes (see Appendix: Theorem F.1.14)*

$$[K, R] = \mathrm{Alt}_{123}\big(\nabla R + C_3^1(T \otimes R)\big) = 0.$$

Hence, if K is a linear, time preserving spacetime connection, then the coordinate expression of the 2nd Bianchi identity becomes

$$\begin{aligned}(\partial_\lambda R_{\mu\nu}{}^{i}{}_{\beta} + K_\lambda{}^{j}{}_{\beta}\, R_{\mu\nu}{}^{i}{}_{j} - K_\lambda{}^{i}{}_{j}\, R_{\mu\nu}{}^{j}{}_{\beta})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \otimes d^\beta \\ = (\nabla_\lambda R_{\mu\nu}{}^{i}{}_{\beta} + T_{\lambda\mu}{}^{j}\, R_{j\nu}{}^{i}{}_{\beta})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \otimes d^\beta = 0.\end{aligned}$$

In particular, if the torsion vanishes, then the above coordinate expression becomes

$$\begin{aligned}&\nabla_\lambda R_{\mu\nu}{}^{i}{}_{\beta}\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \otimes d^\beta \\ &\quad = (\partial_\lambda R_{\mu\nu}{}^{i}{}_{\beta} + K_\lambda{}^{j}{}_{\beta}\, R_{\mu\nu}{}^{i}{}_{j} - K_\lambda{}^{i}{}_{j}\, R_{\mu\nu}{}^{j}{}_{\beta})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \otimes d^\beta = 0. \quad \square\end{aligned}$$

Corollary 4.1.17 (Contracted 2nd Bianchi identity) *If K is a linear spacetime connection, then the contacted Bianchi identity becomes*

$$C_1^1\big([K, R]\big) = \mathrm{div}\, R + C_3^1(T \otimes r) + 2\ \mathrm{Alt}_{12}\big(\nabla r + C_{31}^{12}(T \otimes R)\big) = 0,$$

where $\mathrm{div}\, R := C_1^1 \nabla R$.

Hence, if K is a time preserving, linear spacetime connection, then the coordinate expression of the contracted 2nd Bianchi identity becomes

$$\begin{aligned}0 &= (\nabla_i R_{\mu\nu}{}^{i}{}_{\beta} + 2\,\nabla_\mu r_{\nu\beta} + 2\, T_{i\mu}{}^{j}\, R_{j\nu}{}^{i}{}_{\beta} + T_{\mu\nu}{}^{j}\, r_{j\beta})\, d^\mu \wedge d^\nu \otimes d^\beta \\ &= (\nabla_i R_{\mu\nu}{}^{i}{}_{\beta} + \nabla_\mu r_{\nu\beta} - \nabla_\nu r_{\mu\beta} + T_{i\mu}{}^{j}\, R_{j\nu}{}^{i}{}_{\beta} - T_{i\nu}{}^{j}\, R_{j\mu}{}^{i}{}_{\beta} + T_{\mu\nu}{}^{j}\, r_{j\beta})\, d^\mu \otimes d^\nu \otimes d^\beta.\end{aligned}$$

In particular, if the torsion vanishes, then the above coordinate expression becomes

$$\begin{aligned}0 &= (\nabla_i R_{\mu\nu}{}^{i}{}_{\beta} + 2\,\nabla_\mu r_{\nu\beta})\, d^\mu \wedge d^\nu \otimes d^\beta \\ &= (\nabla_i R_{\mu\nu}{}^{i}{}_{\beta} + \nabla_\mu r_{\nu\beta} - \nabla_\mu r_{\nu\beta})\, d^\nu \otimes d^\mu \otimes d^\beta. \quad \square\end{aligned}$$

Theorem 4.1.18 (1st Bianchi identity) *If K is a linear spacetime connection, then the 1st Bianchi identity becomes (see Appendix: Theorem F.1.23)*

$$[\nu_{\boldsymbol{T}\boldsymbol{E}}, R] - [K, T] = \mathrm{Alt}_{123}\big(R - \nabla T - C^2_1(T \otimes T)\big) = 0.$$

Hence, if K is a time preserving, linear spacetime connection, then the coordinate expression of the 1st Bianchi identity becomes

$$(R_{\lambda\mu}{}^{i}{}_{\nu} - \nabla_\lambda T_{\mu\nu}{}^{i} - T_{j\lambda}{}^{i}\, T_{\mu\nu}{}^{j})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i$$
$$= (R_{\lambda\mu}{}^{i}{}_{\nu} - \partial_\lambda T_{\mu\nu}{}^{i} + K_\lambda{}^{i}{}_{j}\, T_{\mu\nu}{}^{j})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0.$$

In particular, if the torsion vanishes, then the coordinate expression of the identity $[\nu_{\boldsymbol{T}\boldsymbol{E}}, R] = 0$ becomes

$$R_{\lambda\mu}{}^{i}{}_{\nu}\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0$$

and, if the curvature vanishes, then the coordinate expression of the identity $[K, T] = 0$ becomes

$$(\partial_\lambda T_{\mu\nu}{}^{i} - K_\lambda{}^{i}{}_{j}\, T_{\mu\nu}{}^{j})\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0. \quad \square$$

4.1.5 Special Spacetime Connections

The time preserving, linear and torsion free spacetime connections are said to be *special*.

Later, we shall find a natural bijection between special spacetime connections and special phase connections (see Theorem 9.2.1).

Definition 4.1.19 A spacetime connection K is said to be *special* if it is time preserving, linear and torsion free.

Thus, the coordinate expression of a special spacetime connection is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^{i}{}_{\mu}\, \dot{x}^\mu\, \dot{\partial}_i), \qquad \text{where} \qquad K_\lambda{}^{i}{}_{\mu} = K_\mu{}^{i}{}_{\lambda} \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}). \quad \square$$

4.2 Metric Preserving Special Spacetime Connections

Next, we discuss the "*metric preserving*" special spacetime connections K. We stress that the metric preserving special spacetime connections are not riemannian connections, because the metric is spacelike.

Here, we start by defining the *metric preserving* time preserving linear spacetime connections K via the condition $\overset{\shortparallel}{\nabla} g = 0$.

Further, with reference to an observer o, we define the two observed tensors $\widehat{\Sigma}\,[o]$ and $\Phi[o]$. Then, we classify the metric preserving special spacetime connections K, with reference to an observer o, by splitting them into a special spacetime connection $\mathrm{K}[o]$ and a term provided by the observed spacetime 2-form $\Phi[o]$.

Eventually, in view of the subsequent notion of galilean spacetime connection, we analyse the curvature of metric preserving special spacetime connections K.

4.2.1 Definition of Metric Preserving Spacetime Connection

We define the *metric preserving spacetime connections* by means of the condition $\overset{\shortparallel}{\nabla} g = 0$, with coordinate expression $g_{hj}\, K_\lambda{}^h{}_i + g_{hi}\, K_\lambda{}^h{}_j = -\partial_\lambda g_{ij}$.

Remark 4.2.1 Let us consider any spacetime connection K.

(1) The covariant differential $\nabla \bar{g}$ is well defined, even if $\bar{g}$ is a vertical tensor.
(2) If K is linear, then it yields a linear connection on the dual bundle of spacetime.
(3) If K is linear, then the covariant differential ∇g is not well defined, because g is a covariant vertical tensor (see Remark 2.2.5). But $\overset{\shortparallel}{\nabla} g$ is well defined (see Proposition 4.1.3). □

Lemma 4.2.2 *If K is a linear spacetime connection, then*

$$\nabla \bar{g} = 0 \qquad \Leftrightarrow \qquad \overset{\shortparallel}{\nabla} g = 0. \quad \square$$

Definition 4.2.3 A time preserving, linear spacetime connection is said to be *metric preserving* if (see Proposition 4.1.3)

$$\overset{\shortparallel}{\nabla} g = 0,$$

i.e., in coordinates, if

$$g_{hj}\, K_\lambda{}^h{}_i + g_{hi}\, K_\lambda{}^h{}_j = -\partial_\lambda g_{ij}. \quad \square$$

For our subsequent purposes (when we shall refer to a particle of mass m) it is convenient to discuss the metric preserving spacetime connections in terms of the rescaled metric G. Accordingly, we equivalently say that a time preserving, linear spacetime connection is *metric preserving* if

$$\overset{\shortparallel}{\nabla} G = 0,$$

i.e., in coordinates, if

$$K_\lambda{}^h{}_i \, G^0_{hj} + K_\lambda{}^h{}_j \, G^0_{hi} = -\partial_\lambda G^0_{ij}.$$

Being the galilean metric G spacelike, the characterisation of metric preserving special spacetime connections is more complicated than in riemannian geometry; actually, it can be achieved in the following way, by splitting the spacetime connection K, with reference to an observer o, into an observed component determined by G and an observed component determined by o.

4.2.2 Distinguished Metric Preserving Spacetime Connection

In view of the characterisation of metric preserving special spacetime connections, we start by exhibiting a distinguished metric preserving special spacetime connection $\mathsf{K}[G, o]$ generated by the metric G and an observer o.

Preliminarily we introduce a notation, which will be frequently used throughout the book.

Notation 4.2.4 For each spacetime covariant tensor

$$\Xi : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}, \quad \text{or} \quad \Xi : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

we define the scaled spacetime mixed tensors

$$\widehat{\Xi} := G^{\sharp 2}(\Xi) : \boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E} \otimes V\boldsymbol{E} \quad \text{and} \quad \hat{\Xi} := g^{\sharp 2}(\Xi) : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes T^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

with coordinate expressions

$$\widehat{\Xi} = G^{ih}_0 \, \Xi_{\lambda h} \, u^0 \otimes d^\lambda \otimes \partial_i \qquad \text{and} \qquad \hat{\Xi} = g^{ih} \, \Xi_{\lambda h} \, d^\lambda \otimes \partial_i. \quad \square$$

Then, we achieve $\mathsf{K}[G, o]$ in four steps.

Lemma 4.2.5 *An observer o yields the tensor*

$$\sigma \circ \big(T(д[o])\big) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

where $\sigma : TT\boldsymbol{E} \to TT\boldsymbol{E}$ is the natural fibred involution of $TT\boldsymbol{E}$ (see, Proposition 2.7.3, Appendix: Proposition B.4.5 and Note B.4.6 and also, for instance, [146]).

We have the coordinate expression, in a spacetime chart adapted to o,

$$\sigma \circ \big(T(д[o])\big) = d^0 \otimes \partial_0. \quad \square$$

Lemma 4.2.6 *The galilean metric G yields on the fibres of spacetime the riemannian connection (see Proposition 3.2.13)*

$$\varkappa = \varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} V\boldsymbol{E},$$

with coordinate expression

$$\varkappa = \check{d}^i \otimes (\partial_i + \varkappa_i{}^h{}_k \, \dot{x}^k \, \dot{\partial}_h),$$

where $\varkappa_h{}^i{}_k = -\frac{1}{2} \, G_0^{ij} \, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})$. □

Lemma 4.2.7 *The galilean metric G and an observer o yield the special spacetime connection (see Definition 4.1.19)*

$$\mathrm{k}[G, o] := \sigma \circ \big(T(д[o])\big) + \theta^*[o]\big(\varkappa[G]\big) : \boldsymbol{TE} \to \boldsymbol{T^*E} \otimes \boldsymbol{TTE}.$$

We have the coordinate expression, in a spacetime chart adapted to o,

$$\mathrm{k}[G, o] = d^\lambda \otimes \partial_\lambda + \varkappa_h{}^i{}_k \, \dot{x}^k \, d^h \otimes \dot{\partial}_i. \quad \square$$

Lemma 4.2.8 *The galilean metric G and an observer o yield the spacetime tensor*

$$\Sigma[G, o] := \theta^*[o]\big(L_{д[o]} G\big) : \boldsymbol{E} \to \boldsymbol{T^*E} \otimes \boldsymbol{T^*E},$$

with coordinate expression, in a spacetime chart adapted to o,

$$\Sigma[G, o] = \partial_0 G^0_{ij} \, d^i \otimes d^j. \quad \square$$

Lemma 4.2.9 *The galilean metric G and an observer o yield the spacetime tensor (see Lemma 4.2.8 and Notation 4.2.4)*

$$\widehat{\Sigma}\,[G, o] := \theta^*[o]\big(\widehat{L_{д[o]} G}\,\big) : \boldsymbol{E} \to \mathbb{T}^* \otimes (\boldsymbol{T^*E} \otimes \boldsymbol{VE}),$$

with coordinate expression, in a spacetime chart adapted to o,

$$\widehat{\Sigma}\,[G, o] = G_0^{ih} \, \partial_0 G^0_{jh} \, u^0 \otimes d^j \otimes \partial_i. \quad \square$$

Proposition 4.2.10 *The galilean metric G and an observer o yield the metric preserving special spacetime connection (see Lemmas 4.2.7 and 4.2.8, Propositions 2.7.3, 3.2.21 and Notation 4.2.4)*

$$\mathrm{K}[G, o] := \mathrm{k}[G, o] - \tfrac{1}{2} \Big(dt \otimes \widehat{\Sigma}\,[G, o] + \widehat{\Sigma}\,[G, o] \otimes dt \Big) : \boldsymbol{TE} \to \boldsymbol{T^*E} \otimes \boldsymbol{TTE},$$

with coordinate expression, in a chart adapted to o,

$$\mathrm{K}[G, o] = d^\lambda \otimes \partial_\lambda + \mathrm{K}_\lambda{}^i{}_\mu \, \dot{x}^\mu \, d^\lambda \otimes \dot{\partial}_i,$$

where

$$\begin{aligned}
\mathsf{K}_0{}^i{}_0 &= 0,\\
\mathsf{K}_0{}^i{}_h &= \mathsf{K}_h{}^i{}_0 = -\tfrac{1}{2}\, G_0^{ij}\, \partial_0 G^0_{hj},\\
\mathsf{K}_k{}^i{}_h &= \mathsf{K}_h{}^i{}_k = -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}). \quad \square
\end{aligned}$$

4.2.3 Observed Spacetime 2-form

Next, given a metric preserving special spacetime connection K, in view of the classification of metric preserving special spacetime connections, we discuss the observed spacetime 2-form $\Phi[o] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}$ generated by K and an observer o.

Indeed, this spacetime form $\Phi[o]$ will play a relevant role in several occasions throughout the present book.

Definition 4.2.11 With reference to a metric preserving special spacetime connection K and to an observer o, we define the *observed spacetime 2-form* to be the 2-form

$$\Phi[o] \;\equiv\; \Phi[G, K, o] := 2\ \mathrm{Ant}\big(\theta[o] \overset{2}{\lrcorner} \underline{\nabla} д[o]\big) : \boldsymbol{E} \to \Lambda^2\, T^* \boldsymbol{E},$$

with coordinate expression, in a spacetime chart adapted to o,

$$\Phi[o] = -2\, G^0_{jh}\, (K_0{}^h{}_0\, d^0 \wedge d^j + K_i{}^h{}_0\, d^i \wedge d^j),$$

where we have set $\underline{\nabla} := G^{\flat}(\nabla)$. $\square$

Proposition 4.2.12 *Given a metric preserving special spacetime connection K, with reference to two observers o and $\acute{o} = o + \vec{v}$, we have the following transition rule*

$$\Phi[\acute{o}] = \Phi[o] + 2\ \mathrm{Ant}\big(\theta[o] \overset{2}{\lrcorner} (\underline{\nabla} \vec{v})\big) - 2\, (\vec{v} \overset{2}{\lrcorner} \underline{\nabla} д[o]) \wedge dt - 2\, (\vec{v} \overset{2}{\lrcorner} \underline{\nabla} \vec{v}) \wedge dt,$$

with coordinate expression, in a spacetime chart adapted to o,

$$\begin{aligned}
\Phi[\acute{o}] = 2\, G^0_{ij}\, \big(&\delta^i_k\, \partial_0 v^j_0 + v^i_0\, \partial_k v^j_0 - \delta^i_k\, K_0{}^j{}_0 - \delta^i_h\, K_k{}^j{}_0\, v^h_0 - \delta^i_k\, K_0{}^j{}_h\, v^h_0\\
&- K_k{}^j{}_h\, v^i_0\, v^h_0\big)\, d^0 \wedge d^k + 2\, G^0_{ij}\, \big(\partial_k v^j_0 - K_k{}^j{}_0 - K_k{}^j{}_h\, v^h_0\big)\, d^k \wedge d^i. \quad \square
\end{aligned}$$

4.2.4 Characterisation of Metric Preserving Connections

Then, we characterise the metric preserving special spacetime connections K, via the choice of an observer o, in terms of the observed metric preserving special spacetime connection $\mathsf{K}[G, o]$ and the observed spacetime 2-form $\Phi[o]$.

Moreover, for each time preserving special spacetime connection K and each observer o, we have the splitting $\nabla(д[o]) = \frac{1}{2}\big(\widehat{\Sigma}\,[o] + \widehat{\Phi}\,[o]\big)$.

Theorem 4.2.13 *If K is a metric preserving special spacetime connection, then the following observed splitting holds, for any observer o (see Notation 4.2.4),*

$$K = \mathsf{K}[G, o] - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi}\,[G, K, o] + \widehat{\Phi}\,[G, K, o] \otimes dt),$$

i.e., more explicitly,

$$\begin{aligned} K = {} & \sigma \circ \big(T(д[o])\big) + \theta^*[o]\big(\varkappa[G]\big) - \tfrac{1}{2}\,(dt \otimes \widehat{\Sigma}\,[G, o] + \widehat{\Sigma}\,[G, o] \otimes dt) \\ & - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi}\,[G, K, o] + \widehat{\Phi}\,[G, K, o] \otimes dt). \end{aligned}$$

We have the following coordinate expression, in a spacetime chart adapted to o,

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu\, \dot{x}^\mu\, \dot{\partial}_i),$$

where

$$\begin{aligned} K_0{}^i{}_0 &= -G^{ij}_0\, \Phi_{0j}, \\ K_0{}^i{}_h &= K_h{}^i{}_0 = -\tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{hj} + \Phi_{hj}), \\ K_k{}^i{}_h &= K_h{}^i{}_k = -\tfrac{1}{2}\, G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}). \end{aligned}$$

Indeed, given an observer o, the above observed splitting yields a bijection

$$K \mapsto \Phi$$

between metric preserving special spacetime connections and spacetime 2-forms.

Proof. (1) Given a metric preserving special spacetime connection K, a comparison of coordinate expressions implies the equality (see Definitions 4.1.19 and 4.2.11 and Proposition 4.2.10)

$$K = \mathsf{K}[G, o] - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi}\,[G, K, o] + \widehat{\Phi}\,[G, K, o] \otimes dt).$$

(2) Conversely, any spacetime 2-form Φ yields the metric preserving special spacetime connection

$$K := \mathsf{K}[G, o, \Phi] = \mathsf{K}[G, o] - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi} + \widehat{\Phi} \otimes dt).$$

Moreover, a computation in coordinates proves the equality $\Phi\big[G, [\mathsf{K}[G, o, \Phi], o\big] = \Phi$. □

Corollary 4.2.14 *If K is a metric preserving special spacetime connection, then we obtain the equality (see Propositions 4.1.3, 3.2.13 and 4.1.11)*

$$\overset{\vee}{K} = \varkappa[G]. \quad \square$$

Corollary 4.2.15 *If K is a metric preserving special spacetime connection, then, for each observer o, we obtain the splitting (see Lemma 4.2.8 and Definition 4.2.11)*

$$\nabla(д[o]) = \tfrac{1}{2}\big(\widehat{\Sigma}\,[G, o] + \widehat{\Phi}\,[K, G, o]\big),$$

with coordinate expression, in a spacetime chart adapted to o,

$$-K_\lambda{}^i{}_0\, d^\lambda \otimes \partial_i \otimes d^0 = G_0^{ij}\,(\Phi_{0j}\, d^0 \otimes \partial_i \otimes d^0) + \tfrac{1}{2}\, G_0^{ij}\,(\Sigma_{hj} + \Phi_{hj})$$
$$(d^h \otimes \partial_i \otimes d^0). \quad \square$$

4.2.5 Curvature of Metric Preserving Special Connections

Now, in view of the notion of "galilean spacetime connection" (see Definition 4.3.1), we discuss the *curvature* of metric preserving special spacetime connections K.

We study also their *Ricci tensor* and *scalar curvature*.

Theorem 4.2.16 *Let us consider a metric preserving special spacetime connection K and an observer o. According to the observed splitting (see Theorem 4.2.13),*

$$K = \mathsf{K}[G, o] - \tfrac{1}{2}\,(dt \otimes \widehat{\Phi}\,[G, K, o] + \widehat{\Phi}\,[G, K, o] \otimes dt),$$

the curvature (see Proposition 4.1.7 and Appendix: Proposition F.2.7)

$$R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E}$$

fulfills the following observed splitting $R[K]$

$$R[K] = -\,[\mathsf{K}, \mathsf{K}] - [\mathsf{K},\ dt \otimes \widehat{\Phi} + \widehat{\Phi} \otimes dt] - \tfrac{1}{4}\,[dt \otimes \widehat{\Phi} + \widehat{\Phi} \otimes dt,$$
$$dt \otimes \widehat{\Phi} + \widehat{\Phi} \otimes dt],$$

where $[\,,]$ denotes the FN-bracket (see Appendix: Theorem E.2.3) and where we have set, for short, $\mathsf{K} = \mathsf{K}[G, o]$ and $\widehat{\Phi} = \widehat{\Phi}\,[G, K, o]$. □

The three terms of the above observed splitting of $R[K]$ have a monomial character of order 0, 1 and 2, respectively, with respect to the spacetime 2-form Φ.

Hence, their coordinate expressions can be easily deduced from the full coordinate expression of $R[K]$ computed in the proposition below.

Proposition 4.2.17 *The coordinate expression of the curvature tensor R of a metric preserving, special spacetime connection K is given by the following equalities*

$$\begin{aligned} R_{0j}{}^{i}{}_{0} &= \tfrac{1}{2}\, G^{is}_{0}\, \partial_{00} G^{0}_{sj} + \tfrac{1}{4}\, \partial_{0} G^{is}_{0}\, \partial_{0} G^{0}_{sj} \\ &\quad + G^{is}_{0}\, (\tfrac{1}{2}\, \partial_{0} \Phi_{js} - \partial_{j} \Phi_{0s}) \\ &\quad + \tfrac{1}{2}\, (-\partial_{j} G^{is}_{0} - G^{rs}_{0}\, G^{ip}_{0}\, \partial_{r} G^{0}_{pj} + G^{rs}_{0}\, G^{ip}_{0}\, \partial_{p} G^{0}_{jr})\, \Phi_{0s} \\ &\quad + \tfrac{1}{4}\, \partial_{0} G^{is}_{0}\, \Phi_{js} + \tfrac{1}{4}\, G^{rs}_{0}\, G^{iq}_{0}\, \partial_{0} G^{0}_{sj}\, \Phi_{rq} + \tfrac{1}{4}\, G^{rs}_{0}\, G^{iq}_{0}\, \Phi_{js}\, \Phi_{rq}, \end{aligned}$$

$$\begin{aligned} R_{0j}{}^{i}{}_{h} &= \tfrac{1}{4}\, \partial_{0} G^{ip}_{0}\, (\partial_{j} G^{0}_{ph} + \partial_{h} G^{0}_{pj} - \partial_{p} G^{0}_{jh}) + \tfrac{1}{2}\, G^{ip}_{0}\, (\partial_{0h} G^{0}_{pj} - \partial_{0p} G^{0}_{jh}) \\ &\quad - \tfrac{1}{4}\, \partial_{0} G^{0}_{ph}\, (\partial_{j} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qj} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{jr}) \\ &\quad - \tfrac{1}{2}\, G^{ip}_{0}\, \partial_{j} \Phi_{hp} - \tfrac{1}{4}\, (\partial_{j} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qj} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{jr})\, \Phi_{hp} \\ &\quad + \tfrac{1}{4}\, G^{ip}_{0}\, G^{rq}_{0}\, (\partial_{j} G^{0}_{qh} + \partial_{h} G^{0}_{qj} - \partial_{q} G^{0}_{jh})\, \Phi_{rp}, \end{aligned}$$

$$\begin{aligned} R_{hk}{}^{i}{}_{0} &= \tfrac{1}{2}\, G^{ip}_{0}\, (\partial_{0h} G^{0}_{pk} - \partial_{0k} G^{0}_{ph}) \\ &\quad - \tfrac{1}{4}\, \partial_{0} G^{0}_{ph}\, (\partial_{k} G^{ip} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qk} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{kr}) \\ &\quad + \tfrac{1}{4}\, \partial_{0} G^{0}_{pk}\, (\partial_{h} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qh} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{hr}) \\ &\quad + \tfrac{1}{2}\, G^{ip}_{0}\, (\partial_{h} \Phi_{kp} - \partial_{k} \Phi_{hp}) \\ &\quad - \tfrac{1}{4}\, (\partial_{k} G^{ip} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qk} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{kr})\, \Phi_{hp} \\ &\quad + \tfrac{1}{4}\, (\partial_{h} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qh} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{hr})\, \Phi_{kp}, \end{aligned}$$

$$\begin{aligned} R_{hk}{}^{i}{}_{l} &= \tfrac{1}{2}\, G^{ip}_{0}\, (\partial_{hl} G^{0}_{pk} - \partial_{hp} G^{0}_{kl} - \partial_{kl} G^{0}_{ph} + \partial_{kp} G^{0}_{hl}) \\ &\quad - \tfrac{1}{4}\, (\partial_{h} G^{0}_{pl} + \partial_{l} G^{0}_{ph} - \partial_{p} G^{0}_{hl})\, (\partial_{k} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qk} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{kr}) \\ &\quad + \tfrac{1}{4}\, (\partial_{k} G^{0}_{pl} + \partial_{l} G^{0}_{pk} - \partial_{p} G^{0}_{kl})\, (\partial_{h} G^{ip}_{0} + G^{rp}_{0}\, G^{iq}_{0}\, \partial_{r} G^{0}_{qh} - G^{rp}_{0}\, G^{iq}_{0}\, \partial_{q} G^{0}_{hr}). \quad \square \end{aligned}$$

The coordinate expression of the curvature tensor $R[K]$ yields the coordinate expression of the Ricci tensor $r[K]$.

Corollary 4.2.18 *The coordinate expression of the Ricci tensor $r[K]$ of a metric preserving, special spacetime connection K is given by the following equalities (see Definition 4.1.8)*

$$
\begin{aligned}
r_{00} &= \tfrac{1}{2}\, G^{is}_0\, \partial_{00} G^0_{si} + \tfrac{1}{4}\, \partial_0 G^{is}_0\, \partial_0 G^0_{si} \\
&\quad - G^{is}_0\, \partial_i \Phi_{0s} - (\partial_i G^{is}_0 + \tfrac{1}{2}\, G^{rs}_0\, G^{ip}_0\, \partial_r G^0_{pi})\, \Phi_{0s} + \tfrac{1}{4}\, G^{rs}_0\, G^{iq}_0\, \Phi_{rq}\, \Phi_{is},
\end{aligned}
$$

$$
\begin{aligned}
r_{0h} = r_{h0} &= \tfrac{1}{4}\, \partial_0 G^{ip}_0\, \partial_h G^0_{pi} + \tfrac{1}{2}\, G^{ip}_0\, (\partial_{0h} G^0_{pi} - \partial_{0p} G^0_{ih}) \\
&\quad - \tfrac{1}{4}\, \partial_0 G^0_{ph}\, (2\, \partial_i G^{ip}_0 + G^{rp}_0\, G^{iq}_0\, \partial_r G^0_{qi}) \\
&\quad - \tfrac{1}{2}\, G^{ip}\, \partial_i \Phi_{hp} - \tfrac{1}{4}\, (2\, \partial_i G^{ip}_0 + G^{rp}_0\, G^{iq}_0\, \partial_r G^0_{qi})\, \Phi_{hp} \\
&\quad + \tfrac{1}{4}\, G^{ip}_0\, G^{rq}_0\, (\partial_i G^0_{qh} - \partial_q G^0_{ih})\, \Phi_{rp},
\end{aligned}
$$

$$
\begin{aligned}
r_{hk} &= -\tfrac{1}{2}\, G^{ip}_0\, (\partial_{ik} G^0_{ph} - \partial_{ip} G^0_{hk} - \partial_{hk} G^0_{pi} + \partial_{hp} G^0_{ik}) + \tfrac{1}{4}\, \partial_h G^{ip}_0\, \partial_k G^0_{pi} \\
&\quad - (\tfrac{1}{2}\, \partial_i G^{ip}_0 + \tfrac{1}{4}\, G^{rp}_0\, G^{iq}_0\, \partial_r G^0_{qi})\, (\partial_h G^0_{pk} + \partial_k G^0_{ph} - \partial_p G^0_{hk}) \\
&\quad + \tfrac{1}{4}\, G^{rp}_0\, G^{iq}_0\, (\partial_r G^0_{qh} - \partial_q G^0_{hr})\, (\partial_i G^0_{pk} - \partial_p G^0_{ik}). \quad \square
\end{aligned}
$$

The Ricci tensor turns out to be symmetric like in standard riemannian geometry.

Corollary 4.2.19 *The Ricci tensor $r[K]$ of a metric preserving, special spacetime connection K is symmetric, i.e., in coordinates,*

$$r_{\lambda\mu} = r_{\mu\lambda}.$$

Proof. It follows immediately from the coordinate expression of the Ricci tensor (see the above Corollary 4.2.18). □

Remark 4.2.20 The above result agrees with the standard symmetry of the Ricci tensor of a (pseudo)-riemannian connection. However, we stress that here we cannot avail straightforwardly of the standard result, because the connection K is not determined by the metric, hence the coordinate expression of the Ricci tensor includes the 2-form $\Phi_{\lambda\mu}$ and its partial derivatives, besides of G^0_{ij} and its partial derivatives. □

The coordinate expression of the Ricci tensor $r[K]$ yields the coordinate expression of the scalar curvature $C[G, K]$.

Proposition 4.2.21 *The coordinate expression of the scalar curvature of a metric preserving, special spacetime connection K is given by (see Definition 4.1.8)*

$$
\begin{aligned}
C[G]_0 = G^{kl}_0\, r_{kl} &= -\, G^{kl}_0\, G^{ip}_0\, (\partial_{il} G^0_{pk} - \partial_{ip} G^0_{kl}) + \tfrac{1}{4}\, G^{kl}_0\, \partial_k G^{ip}_0\, \partial_l G^0_{pi} \\
&\quad - G^{kl}_0\, (\tfrac{1}{2}\, \partial_i G^{ip}_0 + \tfrac{1}{4}\, G^{rp}_0\, G^{iq}_0\, \partial_r G^0_{qi})\, (2\, \partial_k G^0_{pl} - \partial_p G^0_{kl}) \\
&\quad + \tfrac{1}{4}\, G^{kl}_0\, G^{rp}_0\, G^{iq}_0\, (\partial_r G^0_{qk} - \partial_q G^0_{kr})\, (\partial_i G^0_{pl} - \partial_p G^0_{il}). \quad \square
\end{aligned}
$$

Corollary 4.2.22 *The vertical restrictions of the curvature and of the Ricci tensor of a metric preserving, special spacetime connection K coincide, respectively, with the curvature and the Ricci tensor of the full vertical restriction $\check{K}$ and of the spacelike riemannian connection $\varkappa[G]$. Thus, we have the equalities*

$$\check{R}[K] = R[\check{K}] = R\big[\varkappa[G]\big] \qquad \text{and} \qquad \check{r}[K] = r[\check{K}] = r\big[\varkappa[G]\big],$$

with coordinate expressions

$$\check{R}[K] = R_{hk}{}^{i}{}_{l}\, \check{d}^h \wedge \check{d}^k \otimes \partial_i \otimes \check{d}^l \qquad \text{and} \qquad \check{r}[K] = r_{hk}\, \check{d}^h \otimes \check{d}^k,$$

where

$$\begin{aligned}
&\check{R}[K]_{hk}{}^{i}{}_{l} \\
&= \tfrac{1}{2}\, G_0^{ip}\, (\partial_{hl} G^0_{pk} - \partial_{hp} G^0_{kl} - \partial_{kl} G^0_{ph} + \partial_{kp} G^0_{hl}) \\
&\quad - \tfrac{1}{4}\, (\partial_h G^0_{pl} + \partial_l G^0_{ph} - \partial_p G^0_{hl})\, (\partial_k G_0^{ip} + G_0^{rp}\, G_0^{iq}\, \partial_r G^0_{qk} - G_0^{rp}\, G_0^{iq}\, \partial_q G^0_{kr}) \\
&\quad + \tfrac{1}{4}\, (\partial_k G^0_{pl} + \partial_l G^0_{pk} - \partial_p G^0_{kl})\, (\partial_h G_0^{ip} + G_0^{rp}\, G_0^{iq}\, \partial_r G^0_{qh} - G_0^{rp}\, G_0^{iq}\, \partial_q G^0_{hr}),
\end{aligned}$$

$$\begin{aligned}
\check{r}[K]_{hk} = &- \tfrac{1}{2}\, G_0^{ip}\, (\partial_{ik} G^0_{ph} - \partial_{ip} G^0_{hk} - \partial_{hk} G^0_{pi} + \partial_{hp} G^0_{ik}) + \tfrac{1}{4}\, \partial_h G_0^{ip}\, \partial_k G^0_{pi} \\
&- (\tfrac{1}{2}\, \partial_i G_0^{ip} + \tfrac{1}{4}\, G_0^{rp}\, G_0^{iq}\, \partial_r G^0_{qi})\, (\partial_h G^0_{pk} + \partial_k G^0_{ph} - \partial_p G^0_{hk}) \\
&+ \tfrac{1}{4}\, G_0^{rp}\, G_0^{iq}\, (\partial_r G^0_{qh} - \partial_q G^0_{hr})\, (\partial_i G^0_{pk} - \partial_p G^0_{ik}).
\end{aligned}$$

Hence, the scalar curvature $C[G, K]$ of a metric preserving, special spacetime connection K coincides with the scalar curvature of the spacelike riemannian connection $\varkappa[G]$. Thus, we have the equalities

$$C[G, K] = C[\check{K}, G] = C\big[\varkappa[G], G\big]. \quad \square$$

Remark 4.2.23 We stress that the scalar curvature of a metric preserving, special spacetime connection K involves only the metric G and does not involve the spacetime 2-form Φ.

This result will agree, later, with the fact that the electromagnetic field F does not effect the scalar curvature of the joined spacetime connection $K := K^{\natural} + K^{\mathfrak{e}}$ (see Theorem 6.3.1 and Corollary 6.5.4). □

4.2.6 The Covariant Curvature

Further, in view of the notion of "galilean spacetime connection" (see Definition 4.3.1), we introduce the scaled covariant curvature $\underline{R}$ of a metric preserving special spacetime connection K. Actually, we find the important equivalence $A\underline{R} = 0 \Leftrightarrow d\Phi[o] = 0$.

Let us consider a metric preserving special spacetime connection K and its curvature $R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E}$ (see Theorem 4.2.16 and Appendix: Proposition F.2.7).

Lemma 4.2.24 *If K is a metric preserving special spacetime connection, then the metric morphism $G^\flat$ yields the scaled section*

$$\underline{R} \equiv \underline{R}[G, K] := G^\flat(R[K]) : \boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with coordinate expression

$$\underline{R} = R_{\lambda\mu}{}^0_j{}_\nu \, u_0 \otimes (d^\lambda \otimes d^\mu \otimes \check{d}^j \otimes d^\nu), \quad \text{where } R_{\lambda\mu}{}^0_j{}_\nu := G^0_{jh} \, R_{\lambda\mu}{}^h{}_\nu .$$

Moreover, a subsequent vertical restriction of the 1st factor, yields the scaled section

$$\underline{R} \equiv \underline{R}[G, K] : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with coordinate expression

$$\underline{R} = R_{i\mu}{}^0_j{}_\nu \, u_0 \otimes (\check{d}^i \otimes d^\mu \otimes \check{d}^j \otimes d^\nu). \quad \square$$

Lemma 4.2.25 *We can split $\underline{R}$ into its symmetric and anti-symmetric components*

$$\underline{R}[G, K] = S\underline{R}[G, K] + A\underline{R}[G, K],$$

with coordinate expressions

$$S\underline{R} = \tfrac{1}{2} \, (R_{i\mu}{}^0_j{}_\nu + R_{j\nu}{}^0_i{}_\mu) \, u_0 \otimes \check{d}^i \otimes d^\mu \otimes \check{d}^j \otimes d^\nu,$$
$$A\underline{R} = \tfrac{1}{2} \, (R_{i\mu}{}^0_j{}_\nu - R_{j\nu}{}^0_i{}_\mu) \, u_0 \otimes \check{d}^i \otimes d^\mu \otimes \check{d}^j \otimes d^\nu. \quad \square$$

Definition 4.2.26 If K is a linear, time preserving spacetime connection, then we say that its covariant curvature $\underline{R}$ is *symmetric* if

$$A\underline{R}[G, K] = 0, \qquad \text{i.e., in coordinates, if} \qquad R_{i\mu}{}^0_j{}_\nu = R_{j\nu}{}^0_i{}_\mu . \quad \square$$

Remark 4.2.27 Clearly, we can rephrase the above Lemma by replacing G with g.

Indeed, the formulas achieved via g would be more natural than those achieved via G, as they do not depend on any mass m and on the Planck constant $\hbar$. However, we have preferred to refer to G, just because in this way we obtain a more convenient normalisation of the involved tensors. □

Next, in view of the characterisation of the galilean spacetime connections (see Theorem 4.3.3), we compute explicit expressions in coordinates, of the antisymmetric component $A\underline{R}[G, K]$ of the covariant curvature $\underline{R}[G, K]$.

Lemma 4.2.28 *If K is a metric preserving special spacetime connection, then we have the following equalities*

$$\begin{aligned} R_{ih}{}^{0}_{j}{}_{k} - R_{jk}{}^{0}_{i}{}_{h} &= 0, \\ R_{ih}{}^{0}_{j}{}_{0} - R_{j0}{}^{0}_{i}{}_{h} &= -\tfrac{1}{2}\,(\partial_h \Phi_{ij} + \partial_j \Phi_{hi} + \partial_i \Phi_{jh}), \\ R_{i0}{}^{0}_{j}{}_{0} - R_{j0}{}^{0}_{i}{}_{0} &= -(\partial_i \Phi_{0j} + \partial_j \Phi_{0i} + \partial_0 \Phi_{ij}). \end{aligned}$$

Proof. The proof follows from the coordinate expressions of $R_{j\mu}{}^{0}_{i}{}_{\lambda} = G^{0}_{ih} R_{j\mu}{}^{h}{}_{\lambda}$, by using Proposition 4.2.17. □

Hence, we obtain the following result, which will play a fundamental role for galilean spacetime connections (see Theorem 4.3.3).

Proposition 4.2.29 *If K is a metric preserving special spacetime connection, then (with reference to an observer o):*

$$A\underline{R}[G, K] = 0 \qquad \Leftrightarrow \qquad d\Phi[G, K, o] = 0,$$

i.e., in coordinates,

$$R_{i\lambda}{}^{0}_{j}{}_{\mu} = R_{j\mu}{}^{0}_{i}{}_{\lambda} \qquad \Leftrightarrow \qquad R_{ih}{}^{0}_{j}{}_{0} = R_{j0}{}^{0}_{i}{}_{h}, \quad R_{i0}{}^{0}_{j}{}_{0} = R_{j0}{}^{0}_{i}{}_{0} \qquad \Leftrightarrow$$
$$\Leftrightarrow \qquad \partial_\lambda \Phi_{\mu\nu} + \partial_\nu \Phi_{\lambda\mu} + \partial_\mu \Phi_{\nu\lambda} = 0. \quad \square$$

4.3 Galilean Spacetime Connections

Eventually, we are in a position to introduce the notion of "*galilean spacetime connection*", as a metric preserving special spacetime connection, which fulfills an additional condition on the symmetry of its curvature.

We stress that, in riemannian geometry, the curvature of a riemannian connection fulfills a well known algebraic symmetry property, but in our framework, the spacelike metric is unable to yield such a symmetry. So, we postulate it as an additional requirement. Actually, this additional requirement has also been considered by many authors dealing with a covariant approach to Quantum Mechanics (see, for instance, [260]).

Indeed, in the phase space framework, this symmetry property will be equivalent to the fact that the dynamical phase 2-form $\Omega[G, K]$ be closed (see, later Lemma 9.2.14 and Theorem 9.2.15).

Moreover, in the quantum theory, this property turns out to be a necessary condition for the existence of the upper quantum connection (see Remark 15.2.3).

So, the symmetry of the curvature of our metric preserving special spacetime connection turns out to be a key point of our approach.

4.3.1 Definition of Galilean Spacetime Connection

We define the *galilean spacetime connections* to be the metric preserving special spacetime connections K such that $A\underline{R}[G, K] = 0$. Indeed, the galilean spacetime connections turn out to be characterised by the condition $d\Phi[o] = 0$.

Definition 4.3.1 A metric preserving special spacetime connection K is said to be *galilean* if it fulfills the additional condition (see, Definition 4.2.26 and also, for instance, [259])

$$A\underline{R}[G, K] = 0, \quad \text{i.e., in coordinates,} \quad R_{i\mu}{}^{0}{}_{j\nu} = R_{j\nu}{}^{0}{}_{i\mu}. \quad \square$$

Remark 4.3.2 The above equality turns out to be an identity in the case of riemannian spacetime connections. Here, this equality should be postulated because the galilean metric is only a spacelike metric. □

Theorem 4.3.3 *A metric preserving special spacetime connection K is galilean if and only if the observed galilean spacetime 2-form $\Phi[G, K, o]$ is closed.*

Hence, in such a case, $\Phi[o] \equiv \Phi[G, K, o]$ can be derived (locally) from a potential, called observed galilean potential*,*

$$A[o] \equiv A[G, K, o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}),$$

according to the equality (for the notation, see also, later, Remark 10.1.6 and Notation 10.1.7)

$$\Phi[o] = 2\, dA[o], \quad \textit{with coordinate expression } \Phi_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda.$$

Accordingly, the coordinate expression of a galilean spacetime connection is of the type

$$K = d^\lambda \otimes (\partial_\lambda + K_\lambda{}^i{}_\mu\, \dot{x}^\mu\, \dot{\partial}_i),$$

where

$$\begin{aligned}
K_0{}^i{}_0 &= -G_0^{ij}\,(\partial_0 A_j - \partial_j A_0),\\
K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_0 G^0_{hj} + \partial_h A_j - \partial_j A_h),\\
K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}).
\end{aligned}$$

Proof. The proof follows from Proposition 4.2.29. □

In our theory, we achieve the observed potential $A[o]$ in three independent but equivalent and interrelated ways:

- the 1st one is provided by the above Theorem 4.3.3,
- we shall discuss the 2nd approach to $A[o]$ later, via the dynamical phase 2-form Ω (see Proposition 10.1.2),
- eventually, we shall discuss the 3rd approach later, via the upper quantum connection $\text{Ч}^{\uparrow}$ (see Note 15.2.12).

Remark 4.3.4 Clearly, for every observer o, the potential $A[o]$ of $\Phi[o]$ is defined locally up to a gauge of the type

$$df \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}), \qquad \text{with } f \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

We stress that, in the classical theory, there is no natural way to parametrise the gauge of the potentials $A[o]$ of $\Phi[o]$.

However, later, after having introduced the horizontal observer independent potentials $A^{\uparrow}$ of the cosymplectic phase 2-form Ω, we shall be able to compare the gauges of the spacetime 2-forms $\Phi[o]$ and $\Phi[\acute{o}]$ associated with two observers o and $\acute{o}$ (see Remark 10.1.6 and Notation 10.1.7).

Even more, later, in the quantum theory, where we postulate a gauge independent and observer independent global upper quantum connection $\text{Ч}^{\uparrow}$, we obtain a bijection between the local quantum bases $\mathfrak{b}$ and the local potentials $A^{\uparrow}$ of Ω (see Theorem 15.2.4 and Note 15.2.12). So, we will be able to parametrise the gauges of the horizontal potential $A^{\uparrow}$ of Ω. As a consequence, we will be able to parametrise also the potentials $A[o]$ associated with a given observer o. □

Remark 4.3.5 In einsteinian General Relativity, the pseudo-riemannian spacetime connection K is locally derived, via a suitable differential operator, from the 10 spacetime functions

$$g_{\lambda\mu} : \boldsymbol{E} \to \mathbb{L}^2 \otimes \mathbb{R}.$$

Analogously, in our framework, the galilean spacetime connection K is locally derived, via a suitable differential operator, from the $10 = 6 + 4$ spacetime functions

$$G^0_{ij} : \boldsymbol{E} \to \mathbb{R} \qquad \text{and} \qquad A_\lambda : \boldsymbol{E} \to \mathbb{R}.$$

Thus, in the galilean theory, the 4 functions A_λ replace the 4 functions $g_{0\lambda}$ of the einsteinian theory, that do not exist in the galilean theory.

However, we stress that, in einsteinian General Relativity, the 10 functions $g_{\lambda\mu}$ are the components of the unique metric tensor g. Analogously, the 10 = 6 + 4 functions G^0_{ij} and A_λ are the components of the two independent tensors G and $A[o]$. □

4.3.2 Remark on Galilean Spacetime Connections

As an excursus, we analyse the coordinate expression of a galilean spacetime connection, by emphasising its apparent analogy with the expression of riemannian connections.

In fact, the observed expression of a galilean spacetime connection K, with reference to an observer o, looks like the expression of the Levi–Civita connection associated with an observed spacetime extension $\widetilde{G}[A, o]$ of the spacelike metric G induced by a potential $A[o]$. Unfortunately, the condition $\nabla\widetilde{G} = 0$ needs not to be fulfilled.

Note 4.3.6 Let us consider a galilean spacetime connection K, an observer o and the observed potential $A[G, K, o]$ (defined up to a gauge).

Let us consider the *observed spacetime extension* of the metric G (see Proposition 2.7.3)

$$\widetilde{G} := dt \otimes A + A \otimes dt + \theta^*[o]\, G : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

with coordinate expression

$$\widetilde{G}[A, o] = u_0 \otimes (d^0 \otimes A_\lambda\, d^\lambda + A_\lambda\, d^\lambda \otimes d^0 + G^0_{ij}\, d^i \otimes d^j).$$

Thus, we have

$$\widetilde{G}^0_{00} = 2\, A_0, \qquad \widetilde{G}^0_{0j} = \widetilde{G}^0_{j0} = A_j, \qquad \widetilde{G}^0_{ij} = G^0_{ij}.$$

Then, the coordinate expression of the galilean spacetime connection K provided by Theorem 4.3.3, can be written, in a spacetime chart adapted to o, in the following way

$$K_\lambda{}^i{}_\mu = -\tfrac{1}{2}\, G^{ij}_0\, (\partial_\lambda \widetilde{G}^0_{j\mu} + \partial_\mu \widetilde{G}^0_{j\lambda} - \partial_j \widetilde{G}^0_{\lambda\mu}).$$

Hence, K looks like the “Levi–Civita connection” associated with the “observed” metric $\widetilde{G}$, but the equality $\nabla\widetilde{G} = 0$ needs not to be true. In other words, the special spacetime connection K needs not to be metric preserving with respect to the extended metric $\widetilde{G}$.

Proof. The equalities

$$
\begin{aligned}
K_0{}^i{}_0 &= -G_0^{ij}\,(\partial_0 A_j - \partial_j A_0),\\
K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_0 G^0_{hj} + \partial_h A_j - \partial_j A_h),\\
K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}),
\end{aligned}
$$

can be written as

$$
\begin{aligned}
K_0{}^i{}_0 &= -G_0^{ij}\,(\partial_0 \widetilde{G}^0_{0j} - \tfrac{1}{2}\,\partial_j \widetilde{G}^0_{00}),\\
K_0{}^i{}_h = K_h{}^i{}_0 &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_0 \widetilde{G}^0_{hj} + \partial_h \widetilde{G}^0_{0j} - \partial_j \widetilde{G}^0_{0h}),\\
K_k{}^i{}_h = K_h{}^i{}_k &= -\tfrac{1}{2}\,G_0^{ij}\,(\partial_h \widetilde{G}^0_{jk} + \partial_k \widetilde{G}^0_{jh} - \partial_j \widetilde{G}^0_{hk}),
\end{aligned}
$$

i.e. as

$$
K_\lambda{}^i{}_\mu = -\tfrac{1}{2}\,G_0^{ij}\,(\partial_\lambda \widetilde{G}^0_{j\mu} + \partial_\mu \widetilde{G}^0_{j\lambda} - \partial_j \widetilde{G}^0_{\lambda\mu}).
$$

However, in general, we have $\nabla\widetilde{G} \neq 0$.

In fact, for instance, we have

$$
\begin{aligned}
(\nabla\widetilde{G})^0_{000} &= \partial_0 \widetilde{G}^0_{00} + 2\,K_0{}^i{}_0\,\widetilde{G}^0_{0i} = \partial_0 \widetilde{G}^0_{00} - 2\,G_0^{ij}\,(\partial_0 A_j - \partial_j A_0)\,\widetilde{G}^0_{0i}\\
&= 2\,\partial_0 A_0 - G_0^{ij}\;A_i\,(\partial_0 A_j - \partial_j A_0)
\end{aligned}
$$

and the above expression needs not to vanish. □

4.3.3 Spacelike Einstein Identity

We have already analysed the curvature $R[K]$ of metric preserving special spacetime connections K (see Sect. 4.2.5). Now, in the present context of galilean spacetime connections, we can avail of the additional symmetry property $R_{i\mu j\nu} = R_{j\nu i\mu}$ of the covariant curvature tensor $\underline{R}[g, K]$ (see Definition 4.3.1).

This property has further consequences. In particular, we exhibit the *spacelike* antisymmetry property $R_{i\mu\, hk} = -R_{i\mu\, kh}$ of the covariant curvature tensor and the *spacelike Einstein identity* $(\nabla_i\, r^i{}_\mu - \frac{1}{2}\,\nabla_\mu\, C)\, d^\mu = 0$.

These galilean identities resemble the standard properties $R_{\lambda\mu\,\alpha\beta} = -R_{\lambda\mu\,\beta\alpha}$ and $(\nabla_\lambda\, r^\lambda{}_\mu - \frac{1}{2}\,\nabla_\mu\, C)\, d^\mu = 0$ holding for (pseudo)-riemannian connections (see, for instance, [44, 51, 172]). Indeed, the last identity can be written as the vanishing divergence of the *Einstein tensor* $g^{\alpha\beta}\,\nabla_\alpha(r_{\beta\mu} - \frac{1}{2}\,C\,g_{\beta\mu}) = 0$. This Einstein identity is used in the einsteinian General Relativity, in the discussion of Einstein equation and associated conservation law for the momentum-energy tensor (see, for instance, [51, 172, 308]).

Unfortunately, in the galilean framework, the above galilean properties have only a spacelike character, due to the degeneracy of the galilean metric g. Moreover, in the galilean framework, we can define only the spacelike Einstein tensor with coordinate expression $r_{ij} - \frac{1}{2} C\, g_{ij}$. Actually, the spacelike restriction of the Einstein identity can be written as the divergence of the spacelike Einstein tensor.

These facts will be reflected later in our formulation of the galilean version of the Einstein equation (see Sect. 8.2).

In the present context, for each metric preserving special spacetime connection K, it is convenient to refer to the covariant curvature tensor

$$\underline{R}[g, K] : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with coordinate expression

$$\underline{R}[g, K] = R_{i\mu j\nu}\, \check{d}^i \otimes d^\mu \otimes \check{d}^j \otimes d^\nu .$$

Let us consider a galilean spacetime connection K.

Lemma 4.3.7 *The covariant curvature tensor $\underline{R}[g, K]$ (see Lemma 4.2.24) is antisymmetric with respect to the last spacelike two indices, i.e., in coordinates,*

$$R_{i\mu\, hk} = -R_{i\mu\, kh}.$$

Proof. The equality $R_{i\mu j\nu} = R_{j\nu i\mu}$ and the antisymmetry of the first two indices of the 2-form $\underline{R}[g, K]$ imply $R_{i\mu\, hk} = R_{hk\, i\mu} = -R_{kh\, i\mu} = -R_{i\mu\, kh}$ (see Definition 4.3.1). □

Lemma 4.3.8 *We obtain the following equality, in coordinates,*

$$r_{\mu i} = -g^{hk}\, R_{\mu kih}.$$

Proof. The equalities (see Definition 4.1.8 and Lemma 4.3.7)

$$r_{\mu\nu} = r_{\nu\mu}, \qquad r_{\mu\nu} = -R_{h\mu}{}^{h}{}_{\nu}, \qquad R_{i\mu\, hk} = -R_{i\mu\, kh}$$

yield the following equalities

$$r_{\mu i} = r_{i\mu} := -\, R_{hi}{}^{h}{}_{\mu} := -\, g^{hk}\, R_{hik\mu} = -g^{hk}\, R_{k\mu hi} = g^{hk}\, R_{\mu khi} = -g^{hk}\, R_{\mu kih}. \quad □$$

Proposition 4.3.9 (Einstein identity) *The contracted 2nd Bianchi identity yields, by further contraction, the identity (see Corollary 4.1.17)*

$$C^{12}_{23}\Big(\bar{g} \otimes \big(\operatorname{div} R + 2\ \mathrm{Alt}_{12}\, \nabla r\big)\Big) := C^{12}_{23}\Big(\bar{g} \otimes \big(C^1_1([K, R])\big)\Big) = 0,$$

with coordinate expression

$$(2\,\nabla_i\, r^i{}_\mu - \nabla_\mu\, C)\, d^\mu = 0.$$

Proof. The contracted 2nd Bianchi identity

$$\operatorname{div} R + 2\ \mathrm{Alt}_{12}\,\nabla r = C^1_1([K, R]) = 0$$

yields, by further contraction, the identity

$$C^{12}_{23}\Big(\bar{g}\otimes\big(\operatorname{div} R + 2\ \mathrm{Alt}_{12}\,\nabla r\big)\Big) = C^{12}_{23}\Big(\bar{g}\otimes\big(C^1_1([K, R])\big)\Big) = 0,$$

with coordinate expression

$$g^{hk}\,\big(\nabla_i\, R_{\mu h}{}^i{}_k + (\nabla_\mu r_{hk} - \nabla_h r_{\mu k})\big)\, d^\mu = 0.$$

Moreover, being K a metric preserving spacetime connection, we obtain

$$\begin{aligned}&\big(\nabla_i(g^{hk}\, R_{\mu h}{}^i{}_k) + \nabla_\mu(g^{hk}\, r_{hk}) - \nabla_h(g^{hk}\, r_{\mu k})\big)\, d^\mu\\ &= (\nabla_i R_{\mu h}{}^{ih} + \nabla_\mu C - \nabla_h r_\mu{}^h)\, d^\mu = 0.\end{aligned}$$

Then, in virtue of the equality $R_{i\mu\, hk} = -R_{i\mu\, kh}$, we obtain $(-\nabla_i\, r_\mu{}^i + \nabla_\mu C - \nabla_h r_\mu{}^h)\, d^\mu = 0$. □

4.3.4 *Postulate on the Gravitational Field*

We postulate the *gravitational field* as a galilean spacetime connection $K^\natural$.

Postulate C.3 *We postulate, as* gravitational field*, a given galilean spacetime connection (see Definition 4.3.1)*

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E}\otimes TT\boldsymbol{E}. \quad \square$$

Notation 4.3.10 From now on, we shall denote by the label "$\natural$" all objects associated with the gravitational connection. □

4.4 Differential Operators

With reference to a generic metric preserving special *spacetime connection* K, we analyse some differential operators.

4.4.1 Spacetime Connections and Volume Forms

We study the relations between metric preserving special spacetime connections K and the volume forms η and υ (see Definitions 4.1.19, 4.2.3 and Proposition 3.2.4).

Indeed, we obtain equalities which resemble standard equalities of riemannian geometry, in spite of the fact that our metric g is spacelike and our connection K is not properly riemannian.

Let us consider a metric preserving special spacetime connection K (see Definitions 4.1.19 and 4.2.3).

Lemma 4.4.1 *The following technical formula, in coordinates, holds*

$$\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} = \tfrac{1}{2}\, g^{hk} \partial_\lambda g_{hk}.$$

Proof. The fact that the matrix (g^{hk}) is the inverse of the matrix (g_{hk}) yields the following formula

$$g^{hk} = \frac{\epsilon^{k_1 \dots \hat{k}_h \dots k_3}_{1 \dots \hat{h} \dots 3}\; g_{1k_1} \cdots \hat{g}_{hk_h} \cdots g_{3k_3}}{|g|}.$$

Hence, we obtain

$$\begin{aligned}\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} &= \frac{\partial_\lambda |g|}{2\,|g|} = \frac{\partial_\lambda (\epsilon^{k_1 \dots k_3}_{1 \dots 3}\; g_{1k_1} \cdots g_{3k_3})}{2\,|g|} \\ &= \frac{\sum_{h=1}^{3} \epsilon^{k_1 \dots k_h \dots k_3}_{1 \dots h \dots 3}\, g_{1k_1} \cdots \partial_\lambda g_{hk_h} \cdots g_{3k_3}}{2|g|} = \tfrac{1}{2}\, g^{hk}\, \partial_\lambda g_{hk}. \qquad \square\end{aligned}$$

Proposition 4.4.2 *We have the following identity, in coordinates (see Proposition 3.2.13 and Corollary 4.2.14)*

$$g^{hk}\, K_h{}^i{}_k = g^{hk}\, \varkappa_h{}^i{}_k = \frac{\partial_j (g^{ji}\, \sqrt{|g|})}{\sqrt{|g|}}.$$

Proof. In virtue of Theorem 4.2.13 and of the above Lemma 4.4.1, we obtain

$$\begin{aligned} & g^{hk}\, K_h{}^i{}_k \\ &\quad = -\tfrac{1}{2}\, g^{hk}\, g^{ij}\, (\partial_h g_{jk} + \partial_k g_{jh} - \partial_j g_{hk}) = -g^{hk}\, g^{ij}\, \partial_h g_{jk} + \tfrac{1}{2}\, g^{hk}\, g^{ij}\, \partial_j g_{hk} \\ &\quad = g^{hk}\, \partial_h g^{ij}\, g_{jk} + g^{ij}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} = \partial_j g^{ij} + g^{ij}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} = \frac{\partial_j (g^{ji}\, \sqrt{|g|})}{\sqrt{|g|}}. \qquad \square \end{aligned}$$

Proposition 4.4.3 *We have the identity*

$$\overset{\shortparallel}{\nabla}\eta = 0, \quad \textit{which, in coordinates, reads as } K_\lambda{}^j{}_j = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} = -\tfrac{1}{2}\, g^{jk}\, \partial_\lambda g_{kj}.$$

Proof. For each $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, the equality $\bar{g}_\wedge(\eta, \eta) = 1$ and the metricity of the connection yield $0 = X.\big(\bar{g}_\wedge(\eta, \eta)\big) = 2\,\bar{g}_\wedge(\overset{\shortparallel}{\nabla}_X \eta, \eta)$, hence $\overset{\shortparallel}{\nabla}_X \eta = 0$, because η is a basis of $\Lambda^3 V^* \boldsymbol{E}$.

Moreover, the coordinate expression $\eta = \sqrt{|g|}\, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3$ yields the equality

$$\nabla\eta = (\partial_\lambda \sqrt{|g|} + \sqrt{|g|}\, K_\lambda{}^j{}_j)\, d^\lambda \otimes \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3,$$

which implies

$$\nabla\eta = 0 \qquad \Leftrightarrow \qquad K_\lambda{}^i{}_i = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}.$$

Moreover, in virtue of Lemma 4.4.1, we have

$$\tfrac{1}{2}\, g^{ij}\, \partial_\lambda g_{ij} = \frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}. \qquad \square$$

Corollary 4.4.4 *We have the identity*

$$\nabla\upsilon = 0, \quad \textit{which, in coordinates, reads as } K_\lambda{}^j{}_j = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} = -\tfrac{1}{2}\, g^{jk}\, \partial_\lambda g_{kj}.$$

Proof. We have $\nabla\upsilon = (\partial_\lambda \sqrt{|g|} + \sqrt{|g|}\, K_\lambda{}^j{}_j)\, u_0 \otimes d^\lambda \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 = 0$, which implies

$$\nabla\upsilon = 0 \qquad \Leftrightarrow \qquad K_\lambda{}^j{}_j = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}. \qquad \square$$

4.4.2 Spacetime Connections and Divergence

We have already defined the divergence $\mathrm{div}_\upsilon X$ of spacetime vector fields induced by the metric g and the volume form υ (see Definition 3.2.17).

Now, by considering a metric preserving special spacetime connection K, we can further define the *divergences* of contravariant spacetime tensors X and covariant spacetime tensors α of any degree r.

The above notions and results are quite standard (see, for instance, [51]); however, the fibring of spacetime over time, the signature of the galilean metric and the scale dimension of the metric deserve some additional attention.

Let us consider a metric preserving special spacetime connection K (see Definition 4.1.19 and Definition 4.2.3).

Then, we can extend the divergence operator $\operatorname{div}_\upsilon$ to spacetime tensors (see Definition 3.2.17).

Definition 4.4.5 We define the following divergence operators.

(1) For each tensor $X : \boldsymbol{E} \to \otimes^r T\boldsymbol{E}$, with $r \geq 1$, we define the *divergence* to be the tensor

$$\operatorname{div} X \equiv \operatorname{div}[K]\, X := C^1_1 \nabla X : \boldsymbol{E} \to \otimes^{r-1} T\boldsymbol{E},$$

with coordinate expression

$$\begin{aligned}\operatorname{div} X = \; & \big(\partial_\mu X^{\mu\lambda_1\ldots\lambda_{r-1}} - K_h{}^h{}_\mu\, X^{\mu\lambda_1\ldots\lambda_{r-1}}\big)\, \partial_{\lambda_1} \otimes \ldots \otimes \partial_{\lambda_{r-1}} \\ & - K_\mu{}^h{}_\nu\, X^{\mu\nu\lambda_2\ldots\lambda_{r-1}}\, \partial_h \otimes \partial_{\lambda_2} \otimes \ldots \otimes \partial_{\lambda_{r-1}} \\ & - \ldots \\ & - K_\mu{}^h{}_\nu\, X^{\mu\lambda_1\ldots\lambda_{r-2}\nu}\, \partial_{\lambda_1} \otimes \ldots \otimes \partial_{\lambda_{r-2}} \otimes \partial_h .\end{aligned}$$

(2) For each tensor $\alpha : \boldsymbol{E} \to \otimes^r T^*\boldsymbol{E}$, with $r \geq 1$, we define the *divergence* to be the scaled tensor

$$\operatorname{div}[g, K]\alpha := C^{12}_{12}(\bar{g} \otimes \nabla\alpha) : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (\otimes^{r-1} T^*\boldsymbol{E}),$$

with coordinate expression

$$\begin{aligned}\operatorname{div}\alpha = \; & g^{ij}\, \big(\partial_i \alpha_{j\lambda_1\ldots\lambda_{r-1}} + K_i{}^h{}_j\, \alpha_{h\lambda_1\ldots\lambda_{r-1}} \\ & + K_i{}^h{}_{\lambda_1}\, \alpha_{jh\lambda_2\ldots\lambda_{r-1}} + \cdots + K_i{}^h{}_{\lambda_{r-1}}\, \alpha_{j\lambda_1\ldots\lambda_{r-2}h}\big)\, d^{\lambda_1} \otimes \ldots \otimes d^{\lambda_{r-1}},\end{aligned}$$

i.e.

$$\begin{aligned}\operatorname{div}\alpha = \; & \big(\partial_i \alpha^i{}_{\lambda_1\ldots\lambda_{r-1}} - K_i{}^h{}_h\, \alpha^i{}_{\lambda_1\ldots\lambda_{r-1}} \\ & + K_i{}^h{}_{\lambda_1}\, \alpha^i{}_{h\lambda_2\ldots\lambda_{r-2}} + \cdots + K_i{}^h{}_{\lambda_{r-1}}\, \alpha^i{}_{\lambda_1\ldots\lambda_{r-2}h}\big)\, d^{\lambda_1} \otimes \ldots \otimes d^{\lambda_{r-1}}. \quad \square\end{aligned}$$

For a spacetime vector field X, we can compare the above divergence $\operatorname{div} X$, which involves the spacetime connection K, with the divergence $\operatorname{div}_\upsilon X$ defined in Definition 3.2.17, which involves only the metric g.

Example 4.4.6 For each vector field $X : \boldsymbol{E} \to T\boldsymbol{E}$, we obtain (see Definitions 4.4.5 and 3.2.17)

$$\operatorname{div} X = \operatorname{div}_\upsilon X.$$

Proof. By recalling Proposition 4.4.3, we obtain

$$\begin{aligned}\operatorname{div}_\upsilon X &= \frac{\partial_\lambda (X^\lambda \sqrt{|g|})}{\sqrt{|g|}} = \partial_\lambda X^\lambda + X^\lambda\, \frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}} = \partial_\lambda X^\lambda + X^\lambda\, g^{ij}\, \partial_\lambda g_{ij} \\ &= \partial_\lambda X^\lambda - K_j{}^j{}_\mu\, X^\mu = \operatorname{div} X. \quad \square\end{aligned}$$

4.4.3 Spacetime Connections and Curl

We have already seen (see Definition 3.2.19) that the metric g and the volume vector $\bar{\eta}$ yield in a natural way the standard *curl operator* curl acting on spacelike vector fields X.

Now, given a metric preserving special spacetime connection K, we can further define a *curl operator* acting on time preserving spacetime vector fields X.

We shall use this extended curl operator in the context of the electromagnetic field for the observed splitting of the 2nd Galilei–Maxwell equation, by computing curl $д[o]$ (see, for instance, Theorem 5.9.6).

Let us consider a metric preserving special spacetime connection K (see Definitions 4.1.19 and 4.2.3). Then, we can extend the curl operator curl to time preserving spacetime vector fields (see Definitions 2.2.6 and 3.2.19).

Proposition 4.4.7 *We define the* extended curl operator *(see Definition 3.2.19) to be the sheaf morphism*

$$\mathrm{curl} \;\equiv\; \mathrm{curl}[g, K] : \mathrm{tim}(\boldsymbol{E}, T\boldsymbol{E}) \to \sec\big(\boldsymbol{E}, \mathbb{L}^{-1} \otimes V\boldsymbol{E})\big) : X \mapsto i_{\mathrm{Ant}\left(g^{\flat}(\check{\nabla} X)\right)}\, \bar{\eta},$$

with coordinate expression

$$\mathrm{curl}\ X = \frac{1}{\sqrt{|g|}}\, \epsilon^{ijh}\, (\partial_i X_j + \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \Phi_{ij}\, X^0)\, \partial_h.$$

Clearly, the extended curl operator curl$[g, K]$ *coincides with the curl operator* curl$[g]$ *on the subsheaf* $\sec(\boldsymbol{E},\ V\boldsymbol{E}) \subset \mathrm{tim}(\boldsymbol{E},\ T\boldsymbol{E})$.

Proof. In virtue of Theorem 4.2.13, we obtain

$$\begin{aligned}
\check{\nabla} X &= (-K_i{}^j{}_0\, X^0 + \partial_i X^j - K_i{}^j{}_k\, X^k)\, \check{d}^i \otimes \partial_j,\\
g^{\flat}(\check{\nabla} X) &= (-K_{ij0}\, X^0 + \partial_i X_j - \partial_i g_{jk}\, X^k - K_{ijk}\, X^k)\, \check{d}^i \otimes \check{d}^j\\
&= \big(\tfrac{1}{2}\, (\tfrac{\hbar_0}{m}\, \Sigma_{ij} + \tfrac{\hbar_0}{m}\, \Phi_{ij})\, X^0 + \partial_i X_j + \tfrac{1}{2}\, (-\partial_i g_{jk} + \partial_k g_{ij}\\
&\qquad - \partial_j g_{ik})\, X^k\big)\, \check{d}^i \otimes \check{d}^j,\\
\mathrm{Ant}\,\big(g^{\flat}(\check{\nabla} X)\big) &= \tfrac{1}{2}\, \big(\tfrac{\hbar_0}{m}\, \Phi_{ij}\, X^0 + \partial_i X_j - \partial_j X_i\big)\, \check{d}^i \otimes \check{d}^j\\
&= (\tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \Phi_{ij}\, X^0 + \partial_i X_j)\, \check{d}^i \wedge \check{d}^j. \qquad \square
\end{aligned}$$

Example 4.4.8 Let us consider an observer o and the associated scaled time preserving spacetime vector field $д[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$.

Then, we obtain the scaled vertical vector field

$$\operatorname{curl} д[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{-1}) \otimes V\boldsymbol{E},$$

with coordinate expression, in a spacetime chart adapted to o,

$$\operatorname{curl}\ д[o] = \tfrac{1}{2}\, \frac{1}{\sqrt{|g|}}\, \epsilon^{ijh}\, \tfrac{\hbar_0}{m}\, \Phi_{ij}\, u^0 \otimes \partial_h . \quad \square$$

Chapter 5
Galilean Electromagnetic Field

As we have pointed out in the introduction, we are forced to consider a galilean version of the *electromagnetic field* in order to achieve a consistent covariant formulation of the quantum theory in a galilean framework (see, also, [270, 350]).

Here, we introduce such an electromagnetic field F (Sect. 5.1) and discuss the *magnetic field* B (Sect. 5.2), the *observed electric field* $E[o]$ (Sect. 5.3) some *distinguished algebraic invariants* (Sect. 5.6), the *Lorentz force* (Sect. 5.7), and the *1st Maxwell equation* (Sect. 5.8).

Eventually, in view of the Galilei–Maxwell equation, we discuss the divergence of the electromagnetic field (Sect. 5.9).

5.1 Electromagnetic Field

We start by defining the electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}$ as a scaled spacetime 2-form. We stress that the scale dimension of F is determined by the conditions discussed in Introduction: Sect. 1.3.5.

Then, by means of a suitable coupling scale, we define the unscaled electromagnetic field. Furthermore, by means of the metric musical morphism, we obtain a mixed tensor, which will be used later for the definition of joined spacetime connection (see Theorem 6.3.1).

In the true Maxwell theory in a lorentzian framework, the electromagnetic field can be essentially defined in an analogous way (see, for instance, [189, 308, 376]).

Definition 5.1.1 We define, the *electromagnetic field*, to be a scaled spacetime 2-form

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E},$$

with coordinate expression

$$F = 2\, F_{0j}\, d^0 \wedge d^j + F_{ij}\, d^i \wedge d^j\,, \quad \text{where } F_{\lambda\mu} \in \text{map}\big(\boldsymbol{E}, (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}\big). \quad \square$$

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Note 5.1.2 By referring to a particle of mass and charge

$$m \in \mathbb{M} \qquad \text{and} \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R},$$

we obtain the *rescaled* and the *unscaled* spacetime electromagnetic field, respectively,

$$\tfrac{q}{m} F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{2}) \otimes \Lambda^2 T^* \boldsymbol{E} \qquad \text{and} \qquad \tfrac{q}{\hbar} F : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E}.$$

Moreover, the metric g and the rescaled metric $G := \frac{m}{\hbar}\, g$ yield the following *scaled spacelike tensors*

$$\begin{aligned} \bar{F} &:= (g^\sharp \otimes g^\sharp)(F) : \boldsymbol{E} \to (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V\boldsymbol{E}, \\ \overline{F} &:= (G^\sharp \otimes G^\sharp)(F) : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V\boldsymbol{E}, \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \bar{F} &= F^{ij}\, \partial_i \otimes \partial_j, \qquad \text{where} \qquad F^{ij} := g^{hi}\, g^{kj}\, F_{hk}, \\ \overline{F} &= F^{ij}_{00}\, (u^0 \otimes u^0) \otimes \partial_i \otimes \partial_j, \qquad \text{where} \qquad F^{ij}_{00} := G^{hi}_0\, G^{kj}_0\, F_{hk}, \end{aligned}$$

and *mixed scaled spacetime tensors* (see Notation 4.2.4):

$$\begin{aligned} \hat{F} &:= g^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}), \\ \widehat{F} &:= G^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}), \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \hat{F} &= F_0{}^j\, d^0 \otimes \partial_j + F_i{}^j\, d^i \otimes \partial_j, \qquad \text{where} \qquad F_\lambda{}^j := g^{hj}\, F_{\lambda h} \\ \widehat{F} &= (F_0{}^j_0\, d^0 + F_i{}^j_0\, d^i) \otimes u^0 \otimes \partial_j, \qquad \text{where} \qquad F_\lambda{}^j_0 := G^{hj}_0\, F_{\lambda h}. \end{aligned}$$

Further, the coupling constants $\frac{q}{m}$ and $\frac{q}{\hbar}$ yield the following mixed rescaled spacetime tensors:

$$\tfrac{q}{m}\, \hat{F} : \boldsymbol{E} \to \mathbb{T}^{-1} \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}) \qquad \text{and} \qquad \tfrac{q}{\hbar}\, \widehat{F} : \boldsymbol{E} \to \mathbb{T}^{-1} \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}). \quad \square$$

5.2 Magnetic Field

We define the *magnetic field* to be the observer independent vertical component of the electromagnetic field $\check{F} : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V^*\boldsymbol{E}$, or, equivalently, the scaled spacelike vector field $\vec{B} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}$ induced via di Hodge star.

Thus, in the present galilean framework, the magnetic field is defined by considering the observer independent projection $T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ (see Proposition 2.2.4). For this reason, the magnetic field turns out observer independent.

The magnetic field of the true Maxwell theory, in the einsteinian framework, can be defined analogously to our procedure, by considering the projection orthogonal to a chosen observer. For this reason, in the einsteinian case, the magnetic field $\check{F}[o]$ turns out to be observer dependent and the associated vector field $\vec{B}[o]$ is valued in the 3-dimensional subbundle $V_\perp \boldsymbol{E}[o] \subset T\boldsymbol{E}$ orthogonal to д$[o]$ (see, for instance, [189, 308, 376]).

In the present galilean framework, we use the speed of light c in the definition of B just as a normalising scale factor suitable for an easy comparison with the true lorentzian case. We stress that, in our framework, such a c has no true physical role because we have no electromagnetic radiation.

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$.

Definition 5.2.1 We define the following equivalent scaled *observer independent spacelike* objects:

(1) the *spacelike magnetic 2-form* and the *spacelike magnetic 2-vector* (see Definition 3.2.2)

$$\begin{aligned} \check{F} &:= \vee(F) : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V^*\boldsymbol{E} \\ \bar{F} &:= (g^\sharp \otimes g^\sharp)(\check{F}) : \boldsymbol{E} \to (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V\boldsymbol{E}, \end{aligned}$$

(2) the *spacelike magnetic 1-form* and the *spacelike magnetic vector field* (see Proposition 3.2.4)

$$\begin{aligned} B &:= \tfrac{c}{2}\, i_{\bar{F}}\, \eta : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\boldsymbol{E}, \\ \vec{B} &:= \tfrac{c}{2}\, i_{\check{F}}\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}. \end{aligned}$$

For short, we shall often say that $\vec{B}$ is the *magnetic field*.

Indeed, we have the equalities

$$B = g^\flat(\vec{B}) \quad \text{and} \quad \vec{B} = g^\sharp(B),$$
$$\check{F} = \tfrac{2}{c}\, i_{\vec{B}}\, \eta \quad \text{and} \quad \bar{F} = \tfrac{2}{c}\, i_B \bar{\eta}.$$

Moreover, we have the coordinate expressions

$$\check{F} = F_{ij}\, \check{d}^i \wedge \check{d}^j \quad \text{and} \quad \bar{F} = F^{ij}\, \partial_i \wedge \partial_j,$$
$$B = B_i\, \check{d}^i \quad \text{and} \quad \vec{B} = B^i\, \partial_i,$$

where

$$B^i = g^{ih}\, B_h = \tfrac{c}{2}\, \frac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, F_{hk}, \qquad B_i = \tfrac{c}{2}\, \sqrt{|g|}\, \epsilon_{hki}\, F^{hk},$$
$$F^{ij} = g^{ih}\, g^{jk}\, F_{hk} = \tfrac{1}{c}\, \frac{1}{\sqrt{|g|}}\, \epsilon^{ijh}\, B_h, \qquad F_{ij} = \tfrac{1}{c}\, \sqrt{|g|}\, \epsilon_{ijh}\, B^h.$$

We observe that the scale dimensions of the above components can be emphasised as follows

$$\begin{aligned} F_{ij} &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}\big), \\ F^{ij} &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}\big), \\ B_i &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}\big), \\ B^i &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}\big). \quad \square \end{aligned}$$

The above observer independent magnetic field, in its covariant versions B and $\check{F}$, is valued in the cotangent vertical bundle $V^*\boldsymbol{E}$.

We recall that we do not have a natural inclusion $V^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ (see Remark 2.2.5). So, if we want to regard B and $\check{F}$ as valued in the cotangent bundle $T^*\boldsymbol{E}$, we need to choose an observer o and we obtain observed forms $B[o]$ and $\check{F}[o]$.

Such an observed version of the magnetic field will be used later in the splitting of the electromagnetic field, where we need to deal with the same space for the electric and magnetic components (see Proposition 5.4.1).

Definition 5.2.2 With reference to an observer o, we define the scaled *observed magnetic 2-form* and the scaled *observed magnetic 1-form*

$$\begin{aligned} \check{F}[o] &:= \theta[o] \lrcorner \check{F} : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}, \\ B[o] &:= \theta[o] \lrcorner \vec{B} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}. \end{aligned}$$

We have the equalities (see Definition 3.2.6)

$$\check{F}[o] = 2\, \tfrac{1}{c}\, i_{\vec{B}}\, \eta[o] \qquad \text{and} \qquad B[o] = \tfrac{1}{2}\, c\, i_{\bar{F}}\, \eta[o],$$

and the coordinate expressions

$$\check{F}[o] = F_{ij}\, (d^i - o^i_0\, d^0) \wedge (d^j - o^j_0\, d^0) \qquad \text{and} \qquad B[o] = B_i\, (d^i - o^i_0\, d^0). \quad \square$$

5.3 Observed Electric Field

With reference to an observer o, we define the observed *electric field* as the scaled spacetime 1-form $E[o] := -\,д[o] \lrcorner F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes V^*_o\boldsymbol{E}$.

The observed electric field of the true Maxwell theory, in the einsteinian framework, can be defined analogously to our procedure. Indeed, in the einsteinian case, $\vec{E}[o]$ turns out to be valued in the 3-dimensional observer dependent subbundle $V_\perp \boldsymbol{E}[o] \subset T\boldsymbol{E}$ orthogonal to $д[o]$ (see, for instance, [189, 308, 376]).

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ and an observer o.

Definition 5.3.1 We define the *observed electric spacetime 1-form* and the *observed electric spacelike vector field* to be, respectively, the scaled sections

$$E[o] := -\,д[o] \lrcorner F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E},$$
$$\vec{E}[o] := g^\sharp(\check{E}[o]) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E},$$

with coordinate expressions, in a spacetime chart adapted to o,

$$E[o] = -F_{0i}\, u^0 \otimes d^i \qquad \text{and} \qquad \vec{E}[o] = -F_0{}^i\, u^0 \otimes \partial_i .$$

For short, we say that $\vec{E}[o]$ is the electric field. □

Remark 5.3.2 The identity $д[o] \lrcorner д[o] \lrcorner F = 0$ implies

$$д[o] \lrcorner E[o] = 0.$$

Hence, $E[o]$ is valued in the subbundle $V_o^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ orthogonal to $д[o]$ (see Proposition 2.7.3).

The vertical restriction

$$\check{E}[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\boldsymbol{E}$$

fulfills the equality

$$E[o] = \theta[o] \lrcorner \big(\check{E}[o]\big). \quad □$$

5.4 Observed Splitting of the Electromagnetic Field

For each observer o, we exhibit an observed splitting of the electromagnetic field into the electric and the magnetic components.

In the true Maxwell theory, in the einsteinian framework, the observed splitting of the electromagnetic field is quite close to the present one. Actually, one has to replace the observed independent time 1-form dt with an observer dependent time 1-form $\tau[o]$ and the observer independent vertical restriction $\check{F}$ with the observer dependent orthogonal component $F_\perp[o]$ of F orthogonal to $д[o]$ (see, for instance, [189, 308, 376]).

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ and an observer o.

Proposition 5.4.1 *We have the following observed splitting of the electromagnetic field F*

$$F = -2\, dt \wedge E[o] + \check{F}[o]$$
$$= -2\, dt \wedge E[o] + 2\, \tfrac{1}{c}\, i_{\vec{B}}\, \eta[o],$$

with coordinate expression

$$-2\, dt \wedge E[o] = 2\, (F_{0j} + F_{hj}\, o_0^h)\, d^0 \wedge d^j,$$
$$\check{F}[o] = (F_{ij}\, d^i \wedge d^j - 2\, F_{hj}\, o_0^h\, d^0 \wedge d^j).$$

Accordingly, we obtain the observed splitting (see Note 5.1.2)

$$\hat{F} = -dt \otimes \vec{E}[o] + 2\, \tfrac{1}{c}\, g^{\sharp 2}\big(i_{\vec{B}}\, \eta[o]\big),$$

with coordinate expression

$$\hat{F} = E_0{}^j\, d^0 \otimes \partial_j + \tfrac{1}{c}\, \sqrt{|g|}\, g^{jh}\, \epsilon_{ihk}\, B^k\, d^i \otimes \partial_j. \quad \square$$

5.5 Transition Rule of the Electric Field

We have already seen that the magnetic field $\vec{B}$ is observer independent by definition.

Conversely, with reference to two observers o and $\acute{o} = o + \vec{v}$, the observer electric field transforms according to the equality $\vec{E}[\acute{o}] = \vec{E}[o] + \frac{1}{c}\, \vec{v} \times \vec{B}$.

The above transition rule yields the observer equivariants $g(\vec{E}[o],\ \vec{B})$ and $\vec{E}[o] \times \vec{B}$.

In the true Maxwell theory, in the einsteinian framework, the transition rules of the observed electric field and of the observer independent magnetic field have a partial analogy with the transition rule of the observed electric field in the present galilean framework (see, for instance, [189, 308, 376]). Indeed, the above galilean transition rule can be regarded as an approximation of the true one, for a small velocity $\vec{v}$.

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}$ and two observers o and $\acute{o} = o + \vec{v}$.

Proposition 5.5.1 *We obtain the transition rule*

$$\vec{E}[\acute{o}] = \vec{E}[o] + \tfrac{1}{c}\, \vec{v} \times \vec{B}.$$

In particular, if $\vec{v}$ and $\vec{B}$ are parallel, then the electric field turns out to be observer independent, i.e.

$$\vec{E}[\acute{o}] = \vec{E}[o].$$

Proof. The equalities (see Proposition 5.4.1, Note 2.7.6 and Remark 5.3.2)

$$F = -2\,dt \wedge E[o] + \check{F}[o], \qquad д[ó] = д[o] + \vec{v}, \qquad i_{д[o]} E[o] = 0$$

yield (see Definition 5.1.1 and Corollary 3.2.8)

$$\begin{aligned} \vec{E}[ó] &= -\tfrac{1}{2}\, g^{\sharp}(i_{д[ó]} F) \\ &= -\tfrac{1}{2}\, g^{\sharp}(i_{д[o]} F) - \tfrac{1}{2}\, g^{\sharp}(i_{\vec{v}} F) = -\tfrac{1}{2}\, g^{\sharp}(i_{д[o]} F) - \tfrac{1}{2}\, g^{\sharp}(i_{\vec{v}} \check{F}[o]) \\ &= \vec{E}[o] - \tfrac{1}{c}\, g^{\sharp}(i_{\vec{v}}\, i_{\vec{B}}\, \eta[o]) = \vec{E}[o] + \tfrac{1}{c}\, \vec{v} \times \vec{B}. \quad \square \end{aligned}$$

Corollary 5.5.2 *The scaled maps (see Corollary 3.2.8 and Proposition 5.5.1)*

$$\begin{aligned} g(\vec{E}[o],\ \vec{B}) &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M}) \otimes \mathbb{R}\big), \\ \vec{E}[o] \times \vec{B} &\in \operatorname{map}\big(\boldsymbol{E}, (\mathbb{T}^{-2} \otimes \mathbb{L}^{-2} \otimes \mathbb{M}) \otimes V\boldsymbol{E}\big) \end{aligned}$$

turn out to be observer equivariant. □

5.6 Algebraic Invariants of the Electromagnetic Field

We discuss four *algebraic invariants* of the electromagnetic field.

In the einsteinian framework, the algebraic invariants of the electromagnetic field can be achieved in an analogous way (see, for instance, [189, 308, 376]).

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ and an observer o.

Definition 5.6.1 We define the *1st electromagnetic algebraic invariant* to be the observer independent scaled spacetime function

$$\mathcal{F} := i_{\bar{v}}(F \wedge F) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-2} \otimes \mathbb{M}) \otimes \mathbb{R},$$

with coordinate expression

$$\mathcal{F} = -\frac{4}{\sqrt{|g|}}\, \epsilon^{jhk}\, F_{0j}\, F_{hk}\, u^0. \quad \square$$

Proposition 5.6.2 *With reference to an observer o, we have the observed expression, which turns out to be observer equivariant, according to Corollary 5.5.2,*

$$\mathcal{F} = -\tfrac{8}{c}\, g(\vec{E}[o],\ \vec{B}). \quad \square$$

Definition 5.6.3 We define the *2nd electromagnetic algebraic invariant* to be the observer independent scaled spacetime function

$$\check{F}^2 := (\bar{g} \otimes \bar{g}) \lrcorner (\check{F} \otimes \check{F}) : \boldsymbol{E} \to (\mathbb{L}^{-3} \otimes \mathbb{M}) \otimes \mathrm{I\!R},$$

with coordinate expression

$$\check{F}^2 = g^{ih} \, g^{jk} \, F_{ij} \, F_{hk}. \quad \square$$

Proposition 5.6.4 *We have the expression*

$$\check{F}^2 = \tfrac{4}{c^2} \, g(\vec{B}, \vec{B}),$$

i.e., in coordinates,

$$\check{F}^2 = \tfrac{4}{c^2} \, g_{ij} \, B^i \, B^j. \quad \square$$

We stress that the above invariant scaled function has the scale dimension of a mass density. Accordingly, we define the following notion, which will be used later in the context of Galilei–Einstein equation (see Proposition 8.2.1).

Definition 5.6.5 We define the *electromagnetic energy tensor* to be the symmetric scaled spacetime 2-tensor

$$\underset{\sim}{\mathcal{T}}^{\mathfrak{e}} := \tfrac{1}{4} \, \check{F}^2 \, dt \otimes dt = \tfrac{1}{c^2} \, g(\vec{B}, \vec{B}) \, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes H^*\boldsymbol{E} \otimes H^*\boldsymbol{E},$$

with coordinate expression

$$\underset{\sim}{\mathcal{T}}^{\mathfrak{e}} = \tfrac{1}{4} \, g^{ih} \, g^{jk} \, F_{ij} \, F_{hk} \, u_0 \otimes u_0 \otimes d^0 \otimes d^0 = \tfrac{1}{c^2} \, g_{ij} \, B^i \, B^j \, u_0 \otimes u_0 \otimes d^0 \otimes d^0. \quad \square$$

Definition 5.6.6 We define the *3rd electromagnetic algebraic invariant* to be the observer independent scaled spacetime tensor

$$\mathsf{M}^{\mathfrak{e}} := C^{12}_{24}(\bar{g} \otimes F \otimes F) : \boldsymbol{E} \to (\mathbb{L}^{-1} \otimes \mathbb{M}) \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with coordinate expression

$$\mathsf{M}^{\mathfrak{e}} = F_{\lambda i} \, F_{\mu}{}^{i} \, d^{\lambda} \otimes d^{\mu}. \quad \square$$

Proposition 5.6.7 *With reference to an observer o, the observed splitting of F (see Proposition 5.4.1) yields the observed expression*

$$\begin{aligned} \mathsf{M}^{\mathfrak{e}} &= g(\vec{E}[o], \vec{E}[o]) \, dt \otimes dt + \tfrac{1}{c} \, dt \otimes \big(\theta[o] \lrcorner g^{\flat}(\vec{E}[o] \times \vec{B})\big) \\ &\quad + \tfrac{1}{c} \big(\theta[o] \lrcorner g^{\flat}(\vec{E}[o] \times \vec{B})\big) \otimes dt + \tfrac{1}{c^2} \, g(\vec{B}, \vec{B}) \, \theta^*[o] \, g_{\perp}[\vec{B}], \end{aligned}$$

where

$$g_{\perp}[\vec{B}] : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*_{\vec{B}}\boldsymbol{E} \otimes V^*_{\vec{B}}\boldsymbol{E})$$

is the restriction of g to the 2-dimensional vector subbundle $V_{\vec{B}}\boldsymbol{E} \subset V\boldsymbol{E}$ orthogonal to $\vec{B}$.

In a spacetime chart adapted to o, we have the coordinate expression

$$\mathrm{M}^{\mathfrak{e}} = g_{ij}\, E_0^i\, E_0^j\, d^0 \otimes d^0 - \tfrac{1}{c}\sqrt{|g|}\,\epsilon_{ihk}\, E_0^h\, B^k\,(d^0 \otimes d^i + d^i \otimes d^0) + \tfrac{2}{c^2}\, g^{hk}\,\epsilon_{ihr}\,\epsilon_{jks}\, B^r\, B^s\, d^i \otimes d^j.$$

Proof. The equalities (see Proposition 5.4.1 and Note 5.1.2)

$$F = -(dt \otimes E[o] + E[o] \otimes dt) + \tfrac{2}{c}\, i_{\vec{B}}\,\eta[o] \text{ and } \hat{F} = -dt \otimes \vec{E}[o] + \tfrac{2}{c}\, g^{\sharp 2}\big(i_{\vec{B}}\,\eta[o]\big)$$

yield

$$\begin{aligned}\mathrm{M}^{\mathfrak{e}} = C_2^1(F \otimes \hat{F}) &= C_2^1(dt \otimes E[o] \otimes dt \otimes \vec{E}[o]) - \tfrac{2}{c}\, C_2^1\big(dt \otimes E[o] \otimes g^{\sharp 2}(i_{\vec{B}}\eta[o])\big)\\ &\quad - \tfrac{2}{c}\, C_2^1\big(i_{\vec{B}}\eta[o] \otimes dt \otimes \vec{E}[o] + \tfrac{4}{c^2}\, C_2^1\big(i_{\vec{B}}\,\eta[o] \otimes g^{\sharp 2}(i_{\vec{B}}\eta[o])\big).\end{aligned}$$

Moreover, we have

$$\begin{aligned}-\tfrac{2}{c}\, C_2^1\,(i_{\vec{B}}\eta[o] \otimes \vec{E}[o]) &= -\tfrac{1}{c}\, i_{\vec{E}[o]} i_{\vec{B}}\eta[o] = \tfrac{1}{c}\, i_{\vec{B}} i_{\vec{E}[o]}\eta[o]\\ &= \tfrac{1}{c}\,\theta[o] \lrcorner \big(g^{\flat}(\vec{E}[o]\times\vec{B})\big).\end{aligned}$$

Furthermore, let us consider a spacetime chart adapted to the observer o, and pointwisely to a spacelike basis (e_i) and its dual (ϵ^i), such that $g\,(e_i, e_j) = \mathfrak{l}^2\,\delta_{ij}$ and $B^2 = B^3 = 0$. Then, we can write

$$\begin{aligned}\eta[o] = \tfrac{1}{3!}\,\mathfrak{l}^3\,\epsilon_{ihr}\,\epsilon^i \otimes \epsilon^h \otimes \epsilon^r,\quad i_{\vec{B}}\eta[o] = \tfrac{1}{2}\,\mathfrak{l}^3\, B^1\,\epsilon_{1hr}\,\epsilon^h \otimes \epsilon^r,\quad g^{\sharp}\big(i_{\vec{B}}\eta[o]\big)\\ = \tfrac{1}{2}\,\mathfrak{l}^3\, B^1\, g^{hk}\,\epsilon_{1hs}\,\epsilon^h \otimes e_k.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\tfrac{4}{c^2}\, C_2^1\big(i_{\vec{B}}\,\eta[o] \otimes g^{\sharp}(i_{\vec{B}}\eta[o])\big) &= \tfrac{1}{c^2}\,\mathfrak{l}^6\,(B^1)^2\, g^{hk}\,\epsilon_{1hr}\,\epsilon_{1ks}\,\epsilon^r \otimes \epsilon^s\\ &= \tfrac{1}{c^2}\,\mathfrak{l}^6\,(B^1)^2\,(g^{22}\,\epsilon^3 \otimes \epsilon^3 + g^{33}\,\epsilon^2 \otimes \epsilon^2)\\ &= \tfrac{1}{c^2}\,\mathfrak{l}^4\,(B^1)^2\,(\epsilon^3 \otimes \epsilon^3 + \epsilon^2 \otimes \epsilon^2)\\ &= \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\, g_{\perp}[\vec{B}].\end{aligned}$$

Therefore, eventually, we obtain

$$\begin{aligned}\mathrm{M}^{\mathfrak{e}} &= C_2^1(F \otimes \hat{F})\\ &= g\,(\vec{E}[o], \vec{E}[o])\, dt \otimes dt + \tfrac{2}{c}\, dt \otimes \Big(\theta[o] \lrcorner \big(g^{\flat}(\vec{E}[o]\times\vec{B})\big)\Big)\\ &\quad + \tfrac{2}{c}\Big(\theta[o] \lrcorner \big(g^{\flat}(\vec{E}[o]\times\vec{B})\big)\Big) \otimes dt + \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\,\theta^*[o]\, g_{\perp}[\vec{B}]. \quad \square\end{aligned}$$

Example 5.6.8 In the particular case when $\vec{B} = 0$, the 3rd electromagnetic invariant tensor becomes

$$\mathrm{M}^{\mathfrak{e}} = g(\vec{E}[o], \vec{E}[o])\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{L}^{-1} \otimes \mathbb{M}) \otimes H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}. \quad \square$$

Proposition 5.6.9 *The vertical restriction of* $\mathrm{M}^{\mathfrak{e}}$ *yields the further spacelike invariant scaled tensor*

$$\check{\mathrm{M}}^{\mathfrak{e}} := {}_{\vee}\,\mathrm{M}^{\mathfrak{e}} = \tfrac{1}{c^2}\, g\,(\vec{B}, \vec{B})\, g_{\perp}[\vec{B}] : \boldsymbol{E} \to (\mathbb{L}^{-1} \otimes \mathbb{M}) \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}),$$

with coordinate expression

$$\check{\mathrm{M}}^{\mathfrak{e}} = F_{hi}\, F_k{}^i\, \check{d}^h \otimes \check{d}^k.$$

Indeed, we have

$$\vec{B} \lrcorner\, \check{\mathrm{M}}^{\mathfrak{e}} = 0,$$

hence, we obtain

$$\check{\mathrm{M}}^{\mathfrak{e}} : \boldsymbol{E} \to (\mathbb{L}^{-1} \otimes \mathbb{M}) \otimes (V^*_{\vec{B}}\boldsymbol{E} \otimes V^*_{\vec{B}}\boldsymbol{E}). \quad \square$$

Definition 5.6.10 We define the *4th electromagnetic invariant* to be the observer independent scaled spacetime tensor

$$\mathrm{C}^{\mathfrak{e}} := C^{123}_{234}(\bar{g} \otimes \bar{g} \otimes \check{F} \otimes \check{F}) : (\mathbb{L}^{-3} \otimes \mathbb{M}) \otimes V^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

with coordinate expression

$$\mathrm{C}^{\mathfrak{e}} = F_j{}^h\, F_h{}^i\, \check{d}^j \otimes \partial_i. \quad \square$$

Proposition 5.6.11 *We have the equality*

$$\mathrm{C}^{\mathfrak{e}} = -\tfrac{1}{c^2}\, g\,(\vec{B}, \vec{B})\, \Pi_{\vec{B}},$$

where

$$\Pi_{\vec{B}} : V^*\boldsymbol{E} \otimes V_{\vec{B}}\boldsymbol{E} \subset V^*\boldsymbol{E} \otimes V\boldsymbol{E}$$

is the orthogonal linear fibred projection onto the vector subbundle $V_{\vec{B}}\boldsymbol{E} \subset V\boldsymbol{E}$.

We have the coordinate expression

$$\mathrm{C}^{\mathfrak{e}} = -\tfrac{1}{c^2}\, \big(g_{hk}\, B^h\, B^k\, \check{d}^i \otimes \partial_i - g_{jk} B^i\, B^k\, \check{d}^j \otimes \partial_i\big).$$

Proof. Let us refer, pointwisely, to a spacelike basis (e_i) and its dual (ϵ^i), such that $g\,(e_i, e_j) = \mathfrak{l}^2\, \delta_{ij}$ and $B^2 = B^3 = 0$. Then, we obtain the equality $g\,(\vec{B}, \vec{B}) = \mathfrak{l}^2\, (B^1)^2$.

Moreover, Definition 5.2.2) yields

$$F_{ij}\, d^i \wedge d^j = \tfrac{1}{c}\sqrt{|g|}\,\epsilon_{ijh}\, B^h\ d^i \wedge d^j = F_{ij}\,\epsilon^i \otimes \epsilon^j = \tfrac{1}{c}\,\mathfrak{l}^3\, B^1\,(\epsilon^2 \otimes \epsilon^3 - \epsilon^3 \otimes \epsilon^2),$$

hence

$$F_i{}^j\,\epsilon^i \otimes e_j = \tfrac{1}{c}\,\mathfrak{l}\, B^1\,(\epsilon^2 \otimes e_3 - \epsilon^3 \otimes e_2).$$

Therefore, we obtain

$$\begin{aligned} F_j{}^h\, F_h{}^i\, d^j \otimes \partial_i &= F_2{}^3\, F_3{}^2\,\epsilon^2 \otimes e_2 + F_3{}^2\, F_2{}^3\,\epsilon^3 \otimes e_3 \\ &= -\tfrac{1}{c^2}\,\mathfrak{l}^2\,(B^1)^2\,(\epsilon^2 \otimes e_2 + \epsilon^3 \otimes e_3) \\ &= -\tfrac{1}{c^2}\, g\,(\vec{B}, \vec{B})\,(\epsilon^2 \otimes e_2 + \epsilon^3 \otimes e_3). \end{aligned}$$

Clearly, $\epsilon^2 \otimes e_2 + \epsilon^3 \otimes e_3$ is the linear projection operator orthogonal to $\vec{B}$.

Eventually, by referring to a generic spacetime chart, we can easily prove the equality

$$g\,(\vec{B}, \vec{B})\,(\epsilon^2 \otimes e_2 + \epsilon^3 \otimes e_3) = g_{hk}\, B^h\, B^k\, \check{d}^i \otimes \partial_i - g_{jk} B^i\, B^k\, \check{d}^j \otimes \partial_i. \quad \square$$

The exterior product of $\mathrm{C}^{\mathfrak{e}}$ with dt yields a further invariant scaled tensor, which will appear, later, in the splitting of the curvature of the joined spacetime connection (see Theorem 6.4.3).

Corollary 5.6.12 *We obtain the scaled vertical valued 2-form*

$$dt \wedge \mathrm{C}^{\mathfrak{e}} = \tfrac{1}{c^2}\, g\,(\vec{B}, \vec{B})\, dt \wedge \Pi_{\vec{B}} : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

with coordinate expression

$$dt \wedge \mathrm{C}^{\mathfrak{e}} = \tfrac{1}{c^2}\, g\,(\vec{B}, \vec{B})\, dt \wedge \Pi_{\vec{B}} = F_j{}^h\, F_h{}^i\, u_0 \otimes (d^0 \wedge d^j) \otimes \partial_i. \quad \square$$

5.7 Lorentz Force

Here, according to the standard procedure, we give a direct definition of the *Lorentz force* acting on a charged particle. Later, we shall independently recover the Lorentz force, in an original way, as a byproduct of the joined spacetime connection (see Lemma 7.2.1 and Theorem 9.2.6).

Thus, we define the *Lorentz force* to be the observer independent scaled spacelike vector field $\vec{\mathfrak{f}} := -\tfrac{1}{2}\, q\, g^\sharp(i_{ds} F)$, with observed splittings $\vec{\mathfrak{f}} = q\,(\vec{E}[o] + \tfrac{1}{c}\,\nabla[o]s \times \vec{B})$.

In an analogous way, se define the *Lorentz force density* acting on a charged continuum.

In the true Maxwell theory in the einsteinian framework, the Lorentz force can be defined in an analogous way (see, for instance, [189, 308, 376]).

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$.

Moreover, let us consider a particle whose motion and charge are

$$s : \boldsymbol{T} \to \boldsymbol{E} \qquad \text{and} \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}.$$

Definition 5.7.1 We define the *Lorentz form* and the *Lorentz force*, associated with the pair (s, q), to be, respectively, the observer independent scaled 1-form and scaled vector field

$$\mathsf{f} \equiv \mathsf{f}[s, q] := -\tfrac{1}{2}\, q\, i_{ds} F \quad : \boldsymbol{T} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{E},$$
$$\vec{\mathsf{f}} \equiv \vec{\mathsf{f}}[s, q] := -\tfrac{1}{2}\, q\, g^{\sharp}(i_{ds} F) : \boldsymbol{T} \to (\mathbb{T}^{-2} \otimes \mathbb{M}) \otimes V\boldsymbol{E},$$

with coordinate expressions

$$\mathsf{f} = q_0\, u^0 \otimes u^0 \otimes \big(F_{0j}\, \partial_0 s^j\, d^0 - (F_{0j} + F_{hj}\, \partial_0 s^h)\, d^j\big),$$
$$\vec{\mathsf{f}} = -q_0\, u^0 \otimes u^0 \otimes (F_0{}^i + F_h{}^i\, \partial_0 s^h)\, \partial_i .$$

Clearly, we have $i_{ds}\mathsf{f} = 0$. □

Next, let us consider an observer o and analyse the observed splitting of the above objects.

Proposition 5.7.2 *We have the observed splittings*

$$\mathsf{f} = -q\, g\big(\nabla[o]s,\ (\vec{E}[o] \circ s)\big)\, dt + q\, \theta^*[o]\big((E[o] \circ s) + \tfrac{1}{c}\, g^{\flat}(\nabla[o]s \times (\vec{B} \circ s))\big),$$
$$\vec{\mathsf{f}} = q\, \big(\vec{E}[o] \circ s + \tfrac{1}{c}\, \nabla[o]s \times (\vec{B} \circ s)\big),$$

with coordinate expression

$$\mathsf{f} = q_0\, u^0 \otimes u^0 \otimes \big(-E_{0j}\, \partial_0 s^j\, d^0 + (E_{0j} + \sqrt{|g|}\, \epsilon_{jhk}\, \partial_0 s^h\, B^k)\, d^j\big),$$
$$\vec{\mathsf{f}} = q_0\, u^0 \otimes u^0 \otimes \big(E_0{}^i + \sqrt{|g|}\, g^{ij}\, \epsilon_{jhk}\, \partial_0 s^h\, B^k\big)\, \partial_i .$$

Moreover, we have the equality (see Proposition 5.4.1 and Definition 2.7.7)

$$g\,(\nabla[o]s, \vec{\mathsf{f}}) = q\, g\,\big(\nabla[o]s, (\vec{E}[o] \circ s)\big).$$

Proof. The proof follows easily from the observed splittings

$$F = -2\, dt \wedge E[o] + \check{F}[o] \qquad \text{and} \qquad ds = д[o] + \nabla[o]s,$$

and from the equalities $X \times Y := g^\sharp(i_Y i_X \eta)$ and $\check{F} = \frac{2}{c}\, i_{\vec{B}}\eta$ (see Corollary 3.2.8 and Definition 5.2.1), which yield

$$-\tfrac{1}{2}\, g^\sharp(i_{\nabla[o]s}\check{F}) = \tfrac{1}{c}\, \nabla[o]s \times \vec{B}. \quad \square$$

Furthermore, let us consider a continuum whose motion, velocity and charge density are (see Definition 2.4.2)

$$\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}, \quad \mathscr{V} := \partial\mathscr{C} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \;\text{ and }\; \rho \in \mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}.$$

Then, we can rephrase the above stuff as follows.

Definition 5.7.3 We define the *Lorentz form density* and the *Lorentz force density*, associated with the pair $(\mathscr{V}, \rho)$, to be, respectively, the observer independent scaled 1-form and scaled vector field

$$\begin{aligned} \mathsf{f} &\equiv \mathsf{f}[\mathscr{V}, \rho] := -\tfrac{1}{2}\, \rho\, i_{\mathscr{V}} F : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M}) \otimes T^*\boldsymbol{E}, \\ \vec{\mathsf{f}} &\equiv \vec{\mathsf{f}}[\mathscr{V}, \rho] := -\tfrac{1}{2}\, \rho\, g^\sharp(i_{\mathscr{V}} F) : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes V\boldsymbol{E}, \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \mathsf{f} &= \rho_0\, u^0 \otimes u^0 \otimes \big(F_{0j}\, \mathscr{V}_0^j\, d^0 + (-F_{0j} + F_{hj}\, \mathscr{V}_0^h)\, d^j\big), \\ \vec{\mathsf{f}} &= \rho_0\, u^0 \otimes u^0 \otimes (-F_0{}^i + F_h{}^i\, \mathscr{V}_0^h)\, \partial_i. \end{aligned}$$

Clearly, we have $i_{\mathscr{V}} \mathsf{f} = 0$. $\square$

Next, let us consider an observer o and analyse the observed splitting of the above objects.

Proposition 5.7.4 *We have the observed splittings*

$$\begin{aligned} \mathsf{f} &= -\rho\, g(\vec{\mathscr{V}}[o], \vec{E}[o])\, dt + \rho\, \theta^*[o](E[o] + \tfrac{1}{c}\, g^\flat(\vec{\mathscr{V}}[o] \times \vec{B})), \\ \vec{\mathsf{f}} &= \rho\, (\vec{E}[o] + \tfrac{1}{c}\, \vec{\mathscr{V}}[o] \times \vec{B}), \end{aligned}$$

with coordinate expression

$$\begin{aligned} \mathsf{f} &= \rho_0\, u^0 \otimes u^0 \otimes (E_{0j}\, \mathscr{V}_0^j\, d^0 + (E_{0j} + \sqrt{|g|}\, \epsilon_{jhk}\, \mathscr{V}_0^h\, B^k)\, d^j), \\ \vec{\mathsf{f}} &= \rho_0\, u^0 \otimes u^0 \otimes (E_0{}^i + \sqrt{|g|}\, g^{ij}\, \epsilon_{jhk}\, \mathscr{V}_0^h\, B^k)\, \partial_i. \quad \square \end{aligned}$$

We have the equality

$$g(\vec{\mathscr{V}}[o], \vec{\mathsf{f}}) = \rho\, g(\vec{\mathscr{V}}[o], \vec{E}[o]). \quad \square$$

5.8 1st Maxwell Equation

We analyse the *1st Maxwell equation* $dF = 0$, by providing its observed splitting into the system of equations $\operatorname{curl} \vec{E}[o] + \frac{1}{c} L_{д[o]} \vec{B} + \frac{1}{c} (\operatorname{div}_\eta д[o]) \vec{B} = 0$ and $\operatorname{div}_\eta \vec{B} = 0$.

In the true Maxwell theory, in a lorentzian framework, the equation $dF = 0$ is the same of the present galilean equation. However, there are mild differences with respect to the associated systems of observed equations, which are due to the differences of definitions of the electric and magnetic fields.

Lemma 5.8.1 *We have the observed splitting*

$$dF = 2\,dt \wedge dE[o] + dt \wedge L_{д[o]} \check{F}[o] + \theta[o]^* \big(\check{d}\check{F}[o]\big).$$

Proof. The equality (see Proposition 5.4.1) $F = -2\,dt \wedge E[o] + \check{F}[o]$ yields $dF = 2\,dt \wedge dE[o] + d\check{F}[o]$.

Moreover, $d\check{F}[o]$ splits uniquely as $d\check{F}[o] = dt \wedge \mu[o] + \theta[o]^* \big(\check{d}\check{F}[o]\big)$,s where $\mu[o]$ is orthogonal to $д[o]$.

Then, we obtain $\mu[o] = i_{д[o]} d\check{F}[o]$.

We have $i_{д[o]} d\check{F}[o] = L_{д[o]} \check{F}[o] - d i_{д[o]} \check{F}[o] = L_{д[o]} \check{F}[o]$.

Therefore, we obtain $dF = 2\,dt \wedge dE[o] + dt \wedge L_{д[o]} \check{F}[o] + \theta[o]^* \big(\check{d}\check{F}[o]\big)$. □

Lemma 5.8.2 *We have the equalities*

$$\check{d}\check{E}[o] = *_\eta (\operatorname{curl} \vec{E}[o])$$
$$L_{д[o]}\, \check{F} = \tfrac{2}{c} *_\eta (L_{д[o]}\, \vec{B}) + \tfrac{2}{c} *_\eta \vec{B} \operatorname{div}_\eta д[o],$$
$$\theta^*[o]\big(\check{d}\check{F}[o]\big) = \tfrac{2}{c}\, \theta^*[o](*_\eta \operatorname{div}_\eta \vec{B}).$$

Proof. The lemma follows from the definitions of div_η, $\times$ and curl, in the following way (see Definitions 3.2.17 and 3.2.19 and Corollary 3.2.8).

In fact, we have

$$\begin{aligned}
\check{d}\check{E}[o] &= i_{\operatorname{curl} \vec{E}[o]} \eta, \\
L_{д[o]} \check{F}[o] &= \tfrac{2}{c} L_{д[o]}\big(\theta[o]^*\, i_{\vec{B}} \eta\big) = \tfrac{2}{c}\, \theta[o]^* \left(L_{д[o]}(i_{\vec{B}} \eta)\right) \\
&= \tfrac{2}{c}\, \theta[o]^* \left(i_{L_{д[o]} \vec{B}} \eta\right) + \tfrac{2}{c}\, \theta[o]^* \left(i_{\vec{B}} L_{д[o]} \eta\right) = \tfrac{2}{c}\, \theta[o]^* \left(i_{L_{д[o]} \vec{B}} \eta\right) \\
&\quad + \tfrac{2}{c}\, \theta[o]^* \left(i_{\vec{B}} \eta \operatorname{div}_\eta д[o]\right), \\
\check{d}\check{F} &= \tfrac{2}{c}\, \check{d} \left(i_{\vec{B}} \eta\right) = \tfrac{2}{c}\, i_{\operatorname{div}_\eta \vec{B}} \eta.
\end{aligned}$$
□

Definition 5.8.3 The *1st Maxwell equation* is defined to be (just as in the true Maxwell theory in an einsteinian framework) the equation

$$dF = 0, \qquad \text{with coordinate expression} \qquad \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0.$$ □

Lemma 5.8.4 *We define the* curl *of the observed electric field to be the scaled spacetime vertical vector field (see Definitions 5.3.1 and 3.2.19)*

$$\operatorname{curl} \vec{E}[o] := i_{\check{d}\check{E}[o]}\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-5/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E},$$

with coordinate expression

$$\operatorname{curl} \vec{E}[o] = \frac{1}{\sqrt{|g|}}\, \epsilon^{ijh}\, \partial_i E_{j0}\, \partial_h = \frac{1}{\sqrt{|g|}}\, g_{jk}\, \epsilon^{ijh}\, \nabla^{\natural}{}_i E^k{}_0\, \partial_h. \quad \square$$

Lemma 5.8.5 *We define the* divergence *of the magnetic field to be the scaled spacetime vertical vector field (see Definitions 5.2.1 and 3.2.17)*

$$\operatorname{div}_\eta \vec{B} := i_\eta L_{\vec{B}}\eta : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R},$$

with coordinate expression

$$\operatorname{div}_\eta \vec{B} = \frac{\partial_i (B^i \sqrt{|g|})}{\sqrt{|g|}}. \quad \square$$

Theorem 5.8.6 *With reference to an observer o, the 1st Maxwell equation is equivalent to the system of equations (see Corollary 3.2.8, Definition 3.2.19 and Proposition 5.4.1)*

$$\begin{aligned} \operatorname{curl} \vec{E}[o] + \tfrac{1}{c}\, L_{д[o]}\, \vec{B} + \tfrac{1}{c}\, (\operatorname{div}_\eta д[o])\, \vec{B} &= 0, \\ \operatorname{div}_\eta \vec{B} &= 0, \end{aligned}$$

with coordinate expression, in a spacetime chart adapted to o,

$$\begin{aligned} \frac{1}{\sqrt{|g|}}\, \epsilon^{hki}\, \partial_h (g_{kr}\, E^r{}_0) + \tfrac{1}{c}\, \partial_0 B^i + \tfrac{1}{c}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, B^i &= 0, \\ \frac{\partial_i (B^i \sqrt{|g|})}{\sqrt{|g|}} &= 0. \end{aligned}$$

In the particular case when the observer is divergence free, the above system becomes the more usual system

$$\begin{aligned} \operatorname{curl} \vec{E}[o] + \tfrac{1}{c}\, L_{д[o]}\, \vec{B} &= 0, \\ \operatorname{div}_\eta \vec{B} &= 0, \end{aligned}$$

with coordinate expression

$$\frac{1}{\sqrt{|g|}}\,\epsilon^{hki}\,\partial_h(g_{kr}\,E^r{}_0) + \tfrac{1}{c}\,\partial_0 B^i = 0,$$
$$\frac{\partial_i(B^i\,\sqrt{|g|})}{\sqrt{|g|}} = 0. \quad \square$$

We observe that the Lie derivative $L_{\underline{\text{д}}[o]}\,\vec{B}$ is just a rigorous geometric translation of the more usual partial derivative $\partial_0 \vec{B}$, referred to the observer o, appearing in standard literature.

Next, by considering an electromagnetic field fulfilling the 1st Maxwell equation, we introduce the unscaled electromagnetic potential.

Proposition 5.8.7 *The closed spacetime 2-form* $\frac{q}{\hbar}\,F$ *can be derived (locally) from an observer independent* electromagnetic potential

$$A^{\mathfrak{e}} \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$$

according to the equality

$$\tfrac{q}{\hbar}\,F = 2\,dA^{\mathfrak{e}}, \quad \textit{with coordinate expression } \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F_{\lambda\mu} = \partial_\lambda A^{\mathfrak{e}}{}_\mu - \partial_\mu A^{\mathfrak{e}}{}_\lambda.$$

Hence, with reference to an observer o and an adapted spacetime chart, we obtain the equalities (see Definitions 3.2.19, 5.1.1 and 5.2.1)

$$\vec{E}[o] = \tfrac{\hbar}{q}\,g^\sharp(i_{\text{д}[o]}dA^{\mathfrak{e}}) = \tfrac{\hbar}{q}\,g^\sharp(L_{\text{д}[o]}A^{\mathfrak{e}} - d i_{\text{д}[o]}A^{\mathfrak{e}}),$$
$$\vec{B} = \tfrac{\hbar}{q}\,c\,\operatorname{curl}\check{A}^{\mathfrak{e}} = \tfrac{\hbar}{q}\,c\,i_{d\check{A}^{\mathfrak{e}}}\bar{\eta},$$

with coordinate expressions

$$E^i{}_0 = 2\,\tfrac{\hbar}{q}\,g^{ij}\,(\partial_j A^{\mathfrak{e}}{}_0 - \partial_0 A^{\mathfrak{e}}{}_j),$$
$$B^i = \tfrac{\hbar}{q}\,c\,\frac{1}{\sqrt{|g|}}\,\epsilon^{hki}\,(\partial_h A^{\mathfrak{e}}{}_k - \partial_k A^{\mathfrak{e}}{}_h). \quad \square$$

Eventually, we postulate a given electromagnetic field.

Postulate C.4 *We postulate, as* electromagnetic field*, a* given *scaled spacetime 2-form*

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E},$$

which fulfills the 1st Maxwell equation

$$dF = 0. \quad \square$$

5.9 Divergence of the Electromagnetic Field

In view of the Galilei–Maxwell equation and the joined Galilei–Einstein equation (see Postulates C.5 and C.6), we discuss the divergence of the electromagnetic field $\operatorname{div}^{\natural} F$ and its observed splitting.

Let us consider the electromagnetic field and the gravitational spacetime connection

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E} \qquad \text{and} \qquad K^{\natural} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

Lemma 5.9.1 *We define the* divergence *of the observed electric fields* $\vec{E}[o]$ *and* $E[o]$ *to be, respectively, the scaled spacetime functions (see Definitions 3.2.17, 4.4.5 and Corollary 3.2.7)*

$$\begin{aligned} \operatorname{div}^{\natural} E[o] &:= \bar{g} \lrcorner \nabla^{\natural} E[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes I\!R, \\ \operatorname{div}_{\eta} \vec{E}[o] &:= *_{\eta} \check{d} *_{\eta} \vec{E}[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes I\!R. \end{aligned}$$

We have the equality

$$\operatorname{div}^{\natural} E[o] = \operatorname{div}_{\eta} \vec{E}[o],$$

with coordinate expressions

$$\begin{aligned} \operatorname{div}^{\natural} E[o] &= g^{hk} \, (\partial_h E_{k0} + K^{\natural}{}_h{}^r{}_k \, E_{r0}) \, u^0, \\ \operatorname{div}_{\eta} \vec{E}[o] &= \frac{\partial_h (E^h{}_0 \sqrt{|g|})}{\sqrt{|g|}} \, u^0. \quad \square \end{aligned}$$

Lemma 5.9.2 *We define the* curl *of the magnetic field to be the scaled spacetime vertical vector field (see Definitions 5.2.1, 3.2.19 and Example 4.4.6)*

$$\operatorname{curl} \vec{B} := i_{\check{d}B} \, \bar{\eta} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-5/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E},$$

with coordinate expression

$$\operatorname{curl} \vec{B} = \frac{1}{\sqrt{|g|}} \, \epsilon^{ijh} \, \partial_i B_j \, \partial_h = \frac{1}{\sqrt{|g|}} \, g_{jk} \, \epsilon^{ijh} \, \nabla^{\natural}{}_i B^k \, \partial_h. \quad \square$$

Lemma 5.9.3 *We define the* spacelike divergence *of the spacelike electromagnetic field* $\check{F}$ *to be the scaled spacelike 1-form*

$$\overset{\vee}{\operatorname{div}}{}^{\natural} \check{F} := \bar{g} \lrcorner \overset{\vee}{\nabla}{}^{\natural} \check{F} : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\boldsymbol{E},$$

where $\overset{\vee}{\nabla}{}^{\natural}$ *is the covariant differential with respect to the connection* $\overset{\vee}{K}{}^{\natural}$ *(see Definition 5.2.1, Proposition 4.1.3, Postulate C.3).*

The above divergence involves only the galilean metric g and can be expressed through the following equality (see Propositions 3.2.13 and 3.2.4 and Corollary 3.2.7)

$$\overset{\vee}{\mathrm{div}}{}^{\natural} \check{F} = -\tfrac{1}{2} *_{\eta} \check{d} *_{\eta} \check{F}.$$

We have the coordinate expression

$$\begin{aligned}\overset{\vee}{\mathrm{div}}{}^{\natural} \check{F} &= g^{hk} \, (\partial_h F_{kj} + K^{\natural}{}_h{}^r{}_k \, F_{rj} + K^{\natural}{}_h{}^r{}_j \, F_{kr}) \, \check{d}^j \\ &= \frac{1}{\sqrt{|g|}} \, \partial_h (\sqrt{|g|} \, F^h{}_j + K^{\natural}{}_h{}^r{}_j \, F^h{}_j) \, \check{d}^j.\end{aligned}$$

We can express $\overset{\vee}{\mathrm{div}}{}^{\natural} \check{F}$ *in terms of the* curl *of the magnetic field* $\vec{B}$ *as follows (see Definition 3.2.19)*

$$\overset{\vee}{\mathrm{div}}{}^{\natural} \check{F} = -\tfrac{1}{c} \, g^{\flat}(\mathrm{curl} \, \vec{B}). \quad \square$$

Lemma 5.9.4 *We define the* divergence *of the observed magnetic field to be the scaled spacetime 1-form*

$$\mathrm{div}\,{}^{\natural} \check{F}[o] := \bar{g} \,\lrcorner\, (\nabla^{\natural} \check{F}[o]) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^* \boldsymbol{E}.$$

We have the equalities

$$\begin{aligned}\mathrm{div}\,{}^{\natural} \check{F}[o] &= \tfrac{1}{c} \, \theta^*[o] \, \check{\mathrm{div}}{}^{\natural} \check{F} - \big((\vec{\nabla}^{\natural} д[o]) \,\lrcorner\, \check{F} \big) \, dt \\ &= -\tfrac{1}{c} \, \theta^*[o] \, g^{\flat}(\mathrm{curl} \, \vec{B}) - \tfrac{1}{c} \, g(\mathrm{curl} \, д[o], \ \vec{B}) \, dt\end{aligned}$$

and the coordinate expressions

$$\begin{aligned}\mathrm{div}\,{}^{\natural} \check{F}[o] &= g^{hk} \, (\partial_h F_{kj} + K^{\natural}{}_h{}^r{}_k \, F_{rj} + K^{\natural}{}_h{}^r{}_j \, F_{kr}) \, d^j \\ &\quad - g^{hk} \, (\partial_h F_{kj} \, o_0^j + F_{rj} \, o_0^j \, K^{\natural}{}_h{}^r{}_k + F_{kj} \, \partial_h o_0^j) \, d^0. \quad \square\end{aligned}$$

Lemma 5.9.5 *We define the* divergence *of electromagnetic field to be the scaled spacetime form (see Definition 4.4.5)*

$$\mathrm{div}\,{}^{\natural} F := \bar{g} \,\lrcorner\, \nabla^{\natural} F : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^* \boldsymbol{E},$$

with coordinate expression

$$\begin{aligned}\mathrm{div}\,{}^{\natural} F &= g^{hk} \, (\partial_h F_{k\lambda} + K^{\natural}{}_h{}^r{}_k \, F_{r\lambda} + K^{\natural}{}_h{}^r{}_\lambda \, F_{kr}) \, d^{\lambda} \\ &= g^{hk} \, (\partial_h F_{k0} + K^{\natural}{}_h{}^r{}_k \, F_{r0} + K^{\natural}{}_h{}^r{}_0 \, F_{kr}) \, d^0 \\ &\quad + g^{hk} \, (\partial_h F_{kj} + K^{\natural}{}_h{}^r{}_k \, F_{rj} + K^{\natural}{}_h{}^r{}_j \, F_{kr}) \, d^j. \quad \square\end{aligned}$$

Theorem 5.9.6 *We have the observed splitting (see Example 4.4.8)*

$$\operatorname{div}^{\natural} F = \big(\operatorname{div}_\eta \vec{E}[o] - \tfrac{1}{c}\, g(\operatorname{curl} д[o],\ \vec{B})\big)\, dt - \tfrac{1}{c}\, \theta^*[o]\big(g^\flat(\operatorname{curl} \vec{B}),$$

with the coordinate expressions

$$\begin{aligned}
\operatorname{div}_\eta \vec{E}[o]\, dt &= \frac{\partial_h (E_0{}^h \sqrt{|g|})}{\sqrt{|g|}}\, d^0, \\
g\left(\operatorname{curl}\ д[o],\ \vec{B}\right) dt &= \left(\frac{1}{\sqrt{|g|}}\, g_{sr}\, \epsilon^{isj}\, K^{\natural}{}_i{}^r{}_0\, B_j\right) d^0, \\
\theta^*[o]\big(g^\flat(\operatorname{curl} \vec{B})\big) &= \frac{1}{\sqrt{|g|}}\, g_{jr}\, g_{hk}\, \epsilon^{ijh}\, (\partial_i B^r - K^{\natural}{}_i{}^r{}_s\, B^s)\, (d^k - o^k_0\, d^0).
\end{aligned}$$

Proof. The observed splitting of the electromagnetic field (see Proposition 5.4.1)

$$F = -2\, dt \wedge E[o] + \check{F}[o] = E[o] \otimes dt - dt \otimes E[o] + \check{F}[o]$$

and the equalities $\nabla^{\natural} dt = 0$ and $dt \lrcorner \bar{g} = 0$ yield the equality

$$\operatorname{div}^{\natural} F = \bar{g} \lrcorner \nabla^{\natural} F = (\bar{g} \lrcorner \nabla^{\natural} E[o]) \otimes dt + \bar{g} \lrcorner \nabla^{\natural} \check{F}[o] = (\operatorname{div}^{\natural} E[o]) \otimes dt + \operatorname{div}^{\natural} \check{F}[o].$$

We have (see Lemmas 5.9.1 and 5.9.4)

$$\operatorname{div}^{\natural} E[o] = \operatorname{div}_\eta \vec{E}[o] \qquad \text{and} \qquad \operatorname{div}^{\natural} \check{F}[o] = -\tfrac{1}{c}\, \theta^*[o]\, g^\flat(\operatorname{curl} \vec{B}) - \tfrac{1}{c}\, g(\operatorname{curl} д[o],\ \vec{B})\, dt.$$

Hence, we obtain $\operatorname{div}^{\natural} F = \big(\operatorname{div}_\eta \vec{E}[o] - \tfrac{1}{c}\, g(\operatorname{curl} д[o],\ \vec{B})\big)\, dt - \tfrac{1}{c}\, \theta^*[o] \big(g^\flat(\operatorname{curl}\ \vec{B})\big)$.

Moreover, the above coordinate expressions follow from Lemmas 5.9.1 and 5.9.3. □

Remark 5.9.7 The two terms

$$\big(\operatorname{div}_\eta \vec{E}[o] - \tfrac{1}{c}\, g(\operatorname{curl} д[o],\ \vec{B})\big)\, dt \qquad \text{and} \qquad \tfrac{1}{c}\, \theta^*[o]\big(g^\flat(\operatorname{curl} \vec{B})\big)$$

in the above Theorem 5.9.6 have values into the two complementary subbundles over $\boldsymbol{E}$ (see Proposition 2.7.3)

$$H^*\boldsymbol{E} \subset T^*\boldsymbol{E} \qquad \text{and} \qquad V^*_o\boldsymbol{E} \subset T^*\boldsymbol{E}. \quad \square$$

Chapter 6
Joined Spacetime Connection

After choosing a suitable constant scale, we exhibit a natural coupling of the galilean gravitational spacetime connection $K^{\natural}$ with the electromagnetic field F, which yields a *joined galilean spacetime connection* K (Sect. 6.3).

For this purpose, we consider two distinguished constant scales: the universal constant scale $\sqrt{\text{r}}$ associated with the gravitational constant scale r and, with reference to a charged particle, the constant scale $\frac{q}{m}$ (Sect. 6.1). The 1st case will be considered in order to achieve a unified version of the "Galilei–Einstein" equation and the 2nd case will be largely used throughout the book to achieve a unified version of Classical and Quantum Mechanics.

Actually, the present joining of the gravitational spacetime connection $K^{\natural}$ with the electromagnetic field F is one of the specific original features of our approach (see, also, Introduction Sect. 1.4.6).

In the standard literature concerning Analytical Mechanics and its developments one can find a well known minimal coupling of kinetic terms and electromagnetic terms (for instance, with respect to the momentum). Here, our procedure, derives in a covariant way all such couplings from a primitive unifying source provided by the joined spacetime connection.

For considerations on coupling gravitational and electromagnetic fields, se also, for instance [174].

We stress that, in the einsteinian General Relativity, we cannot achieve an analogous joined connection defined on the whole tangent space $T\boldsymbol{E}$, because the analogue of the form dt does not live on spacetime, but on the phase space. However, we can define such a dynamical joined connection on the phase space (see [213, 214, 222]).

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6.1 Coupling Scales

We start by discussing the *joining constant scale* $\mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R}$, which is suitable for coupling the gravitational field $K^{\natural}$ (and derived gravitational objects as well) with the electromagnetic field F.

We consider the two distinguished cases $\mathrm{k} := \frac{q}{m}$ and $\mathrm{k} := \sqrt{\mathrm{r}}$, where $\frac{q}{m}$ is the ratio between the charge and the mass of a given charged particle and r is the constant *gravitational coupling scale* (see Introduction: Sect. 1.3.5).

The coupling constant scale $\mathrm{k} = \frac{q}{m}$ will be used later in the context of joined classical dynamical objects (see Assumption C.2, Definition 7.3.1 and also Theorems 9.2.1 and 10.1.8, Corollary 9.2.4) and in the context of joined quantum dynamical objects (see Definition 15.1.5, Theorems 17.2.2, 17.3.2, 17.4.2, 17.5.2, 17.5.10 and 17.6.5).

Moreover, the coupling constant scale $\mathrm{k} = \sqrt{\mathrm{r}}$ will be used later in the context of joined Galilei–Maxwell equation (see Proposition 8.3.3).

We notice that, in the interplay between the galilean metric g and the rescaled galilean metric $G := \frac{m}{\hbar}\, g$, the coupling constant scale $\mathrm{k} := \frac{q}{m}$ turns out to be naturally replaced by the coupling constant scale $\frac{q}{\hbar}$.

Definition 6.1.1 We define a "*galilean joining constant scale*" to be an element of the type

$$\mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R}.$$

In particular, we deal with two distinguished joining scales:

(1) with reference to a particle of mass $m \in \mathbb{M}$ and charge $q \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathrm{I\!R}$, the *electromagnetic joining scale*

$$\mathrm{k} := \tfrac{q}{m} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R};$$

(2) with reference to the *gravitational coupling constant* $\mathrm{r} \in \mathbb{T}^{-2} \otimes \mathbb{L}^{3} \otimes \mathbb{M}^{-1}$, the *gravitational joining scale*

$$\mathrm{k} := \sqrt{\mathrm{r}} \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2} \subset (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R}. \quad \square$$

Note 6.1.2 We shall often use the equalities

$$\tfrac{q}{m}\, G = \tfrac{q}{\hbar}\, g \quad \text{and} \quad \tfrac{q}{\hbar}\, \bar{G} = \tfrac{q}{m}\, \bar{g}. \quad \square$$

6.2 Electromagnetic Terms

In view of joining the gravitational field $K^{\natural}$ with the electromagnetic field F (see Sect. 6), we discuss the *electromagnetic terms* $\mathrm{k}\, dt \otimes \hat{F}$ and $\mathrm{k}\, \hat{F} \otimes dt$.

Let us consider the electromagnetic field and a coupling scale

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E} \quad \text{and} \quad \mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R}.$$

Lemma 6.2.1 *By taking into account the natural isomorphism*

$$T^* \boldsymbol{E} \otimes V \boldsymbol{E} \to V \boldsymbol{E} \otimes T^* \boldsymbol{E},$$

we obtain the scaled sections (see Definition 6.1.1 and Note 5.1.2),

$$\mathrm{k}\, dt \otimes \hat{F} : \boldsymbol{E} \to T^* \boldsymbol{E} \otimes V \boldsymbol{E} \otimes T^* \boldsymbol{E} \quad \textit{and} \quad \mathrm{k}\, \hat{F} \otimes dt : \boldsymbol{E} \to T^* \boldsymbol{E} \otimes V \boldsymbol{E} \otimes T^* \boldsymbol{E},$$

with coordinate expressions

$$\mathrm{k}\, dt \otimes \hat{F} = \mathrm{k}_0\, g^{ih}\, F_{\lambda h}\, d^0 \otimes \partial_i \otimes d^\lambda \quad \textit{and} \quad \mathrm{k}\, \hat{F} \otimes dt = \mathrm{k}_0\, g^{ih}\, F_{\lambda h}\, d^\lambda \otimes \partial_i \otimes d^0.$$

Indeed, by rearranging the order of the tensor factors, the above tensors can also be naturally regarded as linear fibred morphisms over $\boldsymbol{E}$

$$\mathrm{k}\, dt \otimes \hat{F} : T\boldsymbol{E} \to T^* \boldsymbol{E} \otimes V \boldsymbol{E} \quad \textit{and} \quad \mathrm{k}\, \hat{F} \otimes dt : T\boldsymbol{E} \to T^* \boldsymbol{E} \otimes V \boldsymbol{E},$$

with coordinate expressions

$$\mathrm{k}\, dt \otimes \hat{F} = \mathrm{k}_0\, F_\lambda{}^i\, \dot{x}^0\, d^\lambda \otimes \partial_i \quad \textit{and} \quad \mathrm{k}\, \hat{F} \otimes dt = \mathrm{k}_0\, F_\lambda{}^i\, \dot{x}^\lambda\, d^0 \otimes \partial_i .$$

We have the observed splittings (see Proposition 5.4.1)

$$\mathrm{k}\, dt \otimes \hat{F} = \mathrm{k} \Big(- dt \otimes \vec{E}[o] \otimes dt + 2\, \tfrac{1}{c}\, dt \otimes g^{\sharp 2}\big(i_{\vec{B}}\, \eta[o]\big)\Big),$$
$$\mathrm{k}\, \hat{F} \otimes dt = \mathrm{k} \Big(- dt \otimes \vec{E}[o] \otimes dt + 2\, \tfrac{1}{c}\, g^{\sharp 2}\big(i_{\vec{B}}\, \eta[o]\big) \otimes dt \Big). \quad \square$$

Remark 6.2.2 In the context of Classical and Quantum Mechanics of a particle of mass $m \in \mathbb{M}$ and charge $q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathrm{I\!R}$, it is often convenient to refer to the rescaled metric

$$G := \tfrac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes V^* \boldsymbol{E} \otimes V^* \boldsymbol{E}$$

and replace the electromagnetic term $\hat{F} := g^{\sharp 2}(F)$ with the rescaled electromagnetic term (see Notation 4.2.4)

$$\widehat{F} := G^{\sharp 2}(F) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes (T^* \boldsymbol{E} \otimes V \boldsymbol{E}),$$

with coordinate expression

$$\widehat{F} = F_{\lambda 0}{}^{j}\, u^0 \otimes d^\lambda \otimes \partial_i .$$

Accordingly, we shall consider the equality

$$\tfrac{q}{m}\, \hat{F} = \tfrac{q}{\hbar}\, \widehat{F} : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}),$$

where the scale $\frac{q}{\hbar}$ replaces the joining scale $\frac{q}{m}$. □

6.3 Galilean Joined Spacetime Connection

Next, we discuss the *joined spacetime connection* $K := K^\natural - \frac{1}{2}\, \mathrm{k}\,(dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$, which is obtained by a minimal coupling of the gravitational and electromagnetic fields. Indeed, this joined connection turns out to be a fundamental object of our covariant approach to Classical and Quantum Mechanics and will play a role in several steps of the forthcoming developments.

In particular, in the present section, we start by discussing the *joined observed spacetime 2-form* $\Phi[o]$ and its *joined potential* $A[o]$.

Let us consider the electromagnetic field and a coupling scale

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E} \quad \text{and} \quad \mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}.$$

Theorem 6.3.1 *We have the minimal coupling of the gravitational field $K^\natural$ with the electromagnetic field F into the* joined spacetime connection *(see Note 5.1.2 and Lemma 6.2.1)*

$$K \;\equiv\; K^\natural + K^\mathfrak{e} := K^\natural - \tfrac{1}{2}\, \mathrm{k}\,(dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

with coordinate expression

$$\begin{aligned}
K_0{}^i{}_0 &= K^\natural{}_0{}^i{}_0 - \mathrm{k}_0\, F_0{}^i,\\
K_0{}^i{}_j &= K_j{}^i{}_0 = K^\natural{}_0{}^i{}_j - \tfrac{1}{2}\, \mathrm{k}_0\, F_j{}^i = K^\natural{}_j{}^i{}_0 - \tfrac{1}{2}\, \mathrm{k}_0\, F_j{}^i,\\
K_h{}^i{}_k &= K^\natural{}_h{}^i{}_k,
\end{aligned}$$

where (see Proposition 5.4.1)

$$F_0{}^j = -E_0{}^j \quad \textit{and} \quad F_i{}^j = \tfrac{1}{c}\, \sqrt{|g|}\, g^{jh}\, \epsilon_{ihk}\, B^k.$$

Thus, we have the observed expression

$$K = K^{\natural} - \tfrac{1}{2}\,\mathrm{k}\Big(-dt \otimes \vec{E}[o] \otimes dt + 2\,\tfrac{1}{c}\,dt \otimes g^{\sharp 2}\big(i_{\vec{B}}\,\eta[o]\big)\Big)$$
$$-\,\tfrac{1}{2}\,\mathrm{k}\Big(-dt \otimes \vec{E}[o] \otimes dt + 2\,\tfrac{1}{c}\,g^{\sharp 2}\big(i_{\vec{B}}\,\eta[o]\big) \otimes dt\Big).$$

Indeed, the joined spacetime connection K turns out to be galilean (see Definition 4.3.1).

Proof. We can prove that K is time preserving and metric preserving by taking into account the coordinate expression of K and the fact that F is antisymmetric.

Moreover, we can prove the additional property of galilean spacetime connections by recalling Theorem 4.3.3 and taking into account the fact that $K^{\natural}$ is galilean and F is closed. □

Remark 6.3.2 We stress that it would be rather hard to guess the above minimal coupling starting from its coordinate expression or its observed expression in terms of the electric and the magnetic field. The above minimal coupling can be easily obtained starting from the electromagnetic field F. □

From now on, we shall refer to the joined spacetime connection

$$K \;\equiv\; K^{\natural} + K^{\mathfrak{e}}$$

and to the joined derived objects, which split into gravitational and electromagnetic components.

Corollary 6.3.3 *According to Definition 4.2.11, Theorem 4.2.13 and the above Theorem 6.3.1, the joined spacetime connection $K = K^{\natural} + K^{\mathfrak{e}}$ yields the* joined galilean spacetime 2-form

$$\Phi[o] \;\equiv\; \Phi[G, K, o] \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E}),$$

which splits as

$$\Phi[o] = \Phi^{\natural}[o] + \tfrac{m}{\hbar}\,\mathrm{k}\,F.$$

Moreover, in virtue of Theorem 4.3.3 and the above Theorem 6.3.1, the joined observed galilean spacetime 2-form $\Phi[o]$ turns out to be closed. Hence, it can be derived (locally) from a joined spacetime potential

$$A[o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}),$$

according to the equality

$$\Phi[o] = 2\,dA[o], \quad \textit{with coordinate expression} \quad \Phi_{\lambda\mu} = \partial_{\lambda} A_{\mu} - \partial_{\mu} A_{\lambda}.$$

Clearly, the joined spacetime potential A is defined (locally) up to a gauge of the type

$$df \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}), \quad \textit{where } f \in \operatorname{map}(\boldsymbol{E}, I\!R). \quad \square$$

Remark 6.3.4 A given potential $A^\natural[o]$ of $\Phi^\natural[o]$ and a given potential $A^\mathfrak{e}$ of $\frac{m}{\hbar}\,\mathrm{k}\,F$ yield a "*joined potential*" of the joined spacetime 2-form $\Phi[o] = \Phi^\natural[o] + \frac{1}{2}\,\frac{m}{\hbar}\,\mathrm{k}\,F$

$$(*) \qquad A[o] = A^\natural[o] + A^\mathfrak{e}.$$

We stress that the three potentials $A[o]$, $A^\natural[o]$ and $A^\mathfrak{e}$ are defined up to gauges, respectively, of the type df, $df^\natural$ and $df^\mathfrak{e}$, where $f,\ f^\natural,\ f^\mathfrak{e} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$. But, there is no natural way to split the gauge df into a gravitational and an electromagnetic component. So, there is no unique way to split the joined potential $A[o]$ into a gravitational and an electromagnetic component as in $(*)$. □

6.4 Joined Spacetime Curvature Tensor

We discuss a natural splitting of the *curvature tensor* $R[K]$ of the joined spacetime connection $K = K^\natural + K^\mathfrak{e}$ into a gravitational term, an electromagnetic term and a mixed term.

We need two preliminary lemmas.

Lemma 6.4.1 *By regarding $K^\natural$ and $K^\mathfrak{e}$ as linear tangent valued forms*

$$K^\natural : T\boldsymbol{E} \to \boldsymbol{T}^*\boldsymbol{E} \otimes TT\boldsymbol{E} \quad \textit{and} \quad K^\mathfrak{e} : T\boldsymbol{E} \to \boldsymbol{T}^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

with coordinate expressions

$$K^\natural = d^\lambda \otimes \partial_i + K^\natural{}_\lambda{}^i{}_\nu\, \dot{x}^\nu\, d^\lambda \otimes \dot{\partial}_i \quad \textit{and} \quad K^\mathfrak{e} = -\tfrac{1}{2}\,\mathrm{k}_0\,(F_\lambda{}^i\, \dot{x}^0\, d^\lambda + F_\nu{}^i\, \dot{x}^\nu\, d^0) \otimes \dot{\partial}_i,$$

we can compute their FN-bracket, whose expression turns out to be (here Alt_{12} *denotes antisymmetrisation with respect to indices 1–2)*

$$[K^\natural, K^\mathfrak{e}] = -\tfrac{1}{2}\,\mathrm{k}\,\mathrm{Alt}_{12}(\nabla^\natural \hat{F}) \otimes dt \\ +\tfrac{1}{2}\,\mathrm{k}\,\mathrm{Alt}_{12}(dt \otimes \nabla^\natural \hat{F}) : T\boldsymbol{E} \to \Lambda^2 \boldsymbol{T}^*\boldsymbol{E} \otimes V\boldsymbol{E},$$

i.e., in coordinates,

$$[K^\natural, K^\mathfrak{e}] = \mathrm{k}_0\,(-\tfrac{1}{2}\,\nabla^\natural{}_0 F_j{}^h\, \dot{x}^0 + \tfrac{1}{2}\,\nabla^\natural{}_j\, F_k{}^h\, \dot{x}^k + \nabla^\natural{}_j F_0{}^h\, \dot{x}^0)\, d^0 \wedge d^j \otimes \partial_h \\ - \mathrm{k}_0\, \tfrac{1}{2}\, \nabla^\natural{}_i F_j{}^h\, \dot{x}^0\, d^i \wedge d^j \otimes \partial_h.$$

Next, $[K^\natural, K^\mathfrak{e}]$ *can be regarded as a vertical valued linear form*

$$[K^\natural, K^\mathfrak{e}] : \boldsymbol{E} \to \Lambda^2 \boldsymbol{T}^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

with coordinate expression

$$[K^\natural, K^\mathfrak{e}] = -\mathrm{k}_0 \left(\tfrac{1}{2}\, \nabla^\natural{}_0 F_j{}^h - \nabla^\natural{}_j F_0{}^h\right) d^0 \wedge d^j \otimes \partial_h \otimes d^0 \\ + \tfrac{1}{2}\, \mathrm{k}_0\, \nabla^\natural{}_j F_k{}^h\, d^0 \wedge d^j \otimes \partial_h \otimes d^k - \tfrac{1}{2}\, \mathrm{k}_0\, \nabla^\natural{}_i F_j{}^h\, d^i \wedge d^j \otimes \partial_h \otimes d^0. \quad \square$$

Lemma 6.4.2 *By regarding $K^\mathfrak{e}$ as a linear tangent valued form*

$$K^\mathfrak{e} : T\boldsymbol{E} \to \boldsymbol{T}^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

with coordinate expression

$$K^\mathfrak{e} = -\tfrac{1}{2}\, \mathrm{k}_0 \left(F_\lambda{}^i\, \dot{x}^0\, d^\lambda + F_\nu{}^i\, \dot{x}^\nu\, d^0\right) \otimes \dot{\partial}_i,$$

we obtain the FN-bracket

$$[K^\mathfrak{e}, K^\mathfrak{e}] : T\boldsymbol{E} \to \Lambda^2 \boldsymbol{T}^*\boldsymbol{E} \otimes V\boldsymbol{E}.$$

Next, $[K^\mathfrak{e}, K^\mathfrak{e}]$ can be regarded as a vertical valued linear form

$$[K^\mathfrak{e}, K^\mathfrak{e}] : \boldsymbol{E} \to \Lambda^2 \boldsymbol{T}^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

given by the equality

$$[K^\mathfrak{e}, K^\mathfrak{e}] = -\tfrac{1}{2}\, \mathrm{k}^2\, dt \wedge \langle \hat{F}, \hat{F} \rangle \otimes dt,$$

with coordinate expression

$$[K^\mathfrak{e}, K^\mathfrak{e}] = -\tfrac{1}{2}\, (\mathrm{k}_0)^2\, F_j{}^h\, F_h{}^i\, d^0 \wedge d^j \otimes \partial_i \otimes d^0. \quad \square$$

Then, we can compute the splitting of the curvature $R[K]$ of the joined spacetime connection K.

Theorem 6.4.3 *The curvature tensor*

$$R[K] := -\,[K, K] : T\boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E}$$

of the joined spacetime connection K splits as

$$R[K] = -[K^\natural, K^\natural] - 2\,[K^\natural, K^\mathfrak{e}] - [K^\mathfrak{e}, K^\mathfrak{e}],$$

where

$$R[K^\natural] := -\,[K^\natural, K^\natural]$$

is the curvature tensor of the gravitational spacetime connection.

Hence, by regarding $R[K]$ as a vertical valued linear form

$$R[K] := -[K, K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

we have the coordinate expression

$$\begin{aligned} R[K] = {} & R^{\natural}{}_{\lambda\mu}{}^{h}{}_{\nu}\, d^{\lambda} \wedge d^{\mu} \otimes \partial_h \otimes d^{\nu} \\ & + \mathrm{k}_0\, (\nabla^{\natural}{}_0 F_j{}^h - 2\, \nabla^{\natural}{}_j F_0{}^h)\, d^0 \wedge d^j \otimes \partial_h \otimes d^0 \\ & - \mathrm{k}_0\, \nabla^{\natural}{}_j F_k{}^h\, d^0 \wedge d^j \otimes \partial_h \otimes d^k + \mathrm{k}_0\, \nabla^{\natural}{}_i F_j{}^h\, d^i \wedge d^j \otimes \partial_h \otimes d^0 \\ & + \tfrac{1}{2}\, (\mathrm{k}_0)^2\, F_j{}^h\, F_h{}^i\, d^0 \wedge d^j \otimes \partial_i \otimes d^0. \end{aligned}$$

The above expression can be re-written in terms of the electric and magnetic fields as

$$\begin{aligned} R[K] = {} & R^{\natural}{}_{\lambda\mu}{}^{h}{}_{\nu}\, d^{\lambda} \wedge d^{\mu} \otimes \partial_h \otimes d^{\nu} \\ & + \mathrm{k}_0\, g^{hk}\, (\tfrac{1}{c}\, \sqrt{|g|}\, \epsilon_{jkp} \nabla^{\natural}{}_0 B^p + 2\, \nabla^{\natural}{}_j E_{0k})\, d^0 \wedge d^j \otimes \partial_h \otimes d^0 \\ & - \mathrm{k}_0\, \tfrac{1}{c}\, \sqrt{|g|}\, g^{hr}\, \epsilon_{krp}\, \nabla^{\natural}{}_j B^p\, d^0 \wedge d^j \otimes \partial_h \otimes d^k \\ & + \mathrm{k}_0\, \tfrac{1}{c}\, \sqrt{|g|}\, g^{hr}\, \epsilon_{jrp}\, \nabla^{\natural}{}_i B^p\, d^i \wedge d^j \otimes \partial_h \otimes d^0 \\ & + \tfrac{1}{2}\, (\mathrm{k}_0)^2\, \tfrac{1}{c^2}\, |g|\, g^{hr}\, g^{is}\, \epsilon_{jrp}\, \epsilon_{hsq} B^p\, B^q\, d^0 \wedge d^j \otimes \partial_i \otimes d^0. \end{aligned}$$

Proof. The equality $R[K] := -[K, K]$ yields $R = R^{\natural} - 2\,[K^{\natural},\, K^{\mathfrak{e}}] - [K^{\mathfrak{e}},\, K^{\mathfrak{e}}]$.

Then, the coordinate expression of R follows from Proposition 4.1.7 and the above Lemmas 6.4.1 and 6.4.2. Moreover, the coordinate expression in terms of the electric and magnetic fields follows from the equalities (see Definitions 5.1.1 and 5.2.1)

$$F_{ij} = \tfrac{1}{c}\, \sqrt{|g|} \epsilon_{ijh}\, B^h \quad \text{and} \quad F_{0i} = -E_{0i}. \quad \square$$

Corollary 6.4.4 *The coordinate expression of the covariant curvature tensor*

$$\underline{R}[K, g] : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$$

of the joined spacetime connection is

$$\begin{aligned} R_{h0k0} &= R^{\natural}{}_{h0k0} - \tfrac{1}{2}\, \mathrm{k}_0\, (\nabla^{\natural}{}_0 F_{hk} - 2\, \nabla^{\natural}{}_h F_{0k}) + \tfrac{1}{4}\, (\mathrm{k}_0)^2\, g^{rs}\, F_{hr}\, F_{ks}, \\ R_{hjk0} &= R^{\natural}{}_{hjk0} + \tfrac{1}{2}\, \mathrm{k}_0\, (\nabla^{\natural}{}_h F_{jk} - \nabla^{\natural}{}_j F_{hk}), \\ R_{k0hj} &= R^{\natural}{}_{k0hj} + \tfrac{1}{2}\, \mathrm{k}_0\, \nabla^{\natural}{}_k F_{jh}, \\ R_{hikj} &= R^{\natural}{}_{hikj}. \end{aligned}$$

The above expression can be re-written in terms of the electric and magnetic fields as

$$\begin{aligned} R_{h0k0} &= R^{\natural}{}_{h0k0} - \tfrac{1}{2}\,\mathrm{k}_0\,(\tfrac{1}{c}\,\sqrt{|g|}\,\epsilon_{hkp}\,\nabla^{\natural}{}_0 B^p + 2\,\nabla^{\natural}{}_h E_{0k}) \\ &\quad + \tfrac{1}{4}\,(\mathrm{k}_0)^2\,|g|\,\tfrac{1}{c^2}\,g^{rs}\,\epsilon_{hrp}\,\epsilon_{ksq}\,B^p\,B^q, \\ R_{hjk0} &= R^{\natural}{}_{hjk0} + \tfrac{1}{2}\,\mathrm{k}_0\,\tfrac{1}{c}\,\sqrt{|g|}\,(\epsilon_{jkp}\,\nabla^{\natural}{}_h B^p - \epsilon_{hkp}\,\nabla^{\natural}{}_j B^p), \\ R_{k0\,hj} &= R^{\natural}{}_{k0\,hj} + \tfrac{1}{2}\,\mathrm{k}_0\,\tfrac{1}{c}\,\sqrt{|g|}\,\epsilon_{jhq}\,\nabla^{\natural}{}_k B^q, \\ R_{hikj} &= R^{\natural}{}_{hikj}. \end{aligned}$$

Proof. From the coordinate expression of $R[K]$ (see Theorem 6.4.3) we have

$$\begin{aligned} R_{h0}{}^k{}_0 &= R^{\natural}{}_{h0}{}^k{}_0 - \tfrac{1}{2}\,\mathrm{k}_0\big(\nabla^{\natural}{}_0 F_h{}^k - 2\,\nabla^{\natural}{}_h F_0{}^k\big) - \tfrac{1}{4}\,(\mathrm{k}_0)^2\,F_h{}^s\,F_s{}^k, \\ R_{hj}{}^k{}_0 &= R^{\natural}{}_{hj}{}^k{}_0 + \tfrac{1}{2}\,\mathrm{k}_0\big(\nabla^{\natural}{}_h F_j{}^k - \nabla^{\natural}{}_j F_h{}^k\big), \\ R_{k0\,h}j &= R^{\natural}{}_{k0\,h}j + \tfrac{1}{2}\,\mathrm{k}_0 \nabla^{\natural}{}_k F_j{}^h, \\ R_{hi}{}^k{}_j &= R^{\natural}{}_{hi}{}^k{}_j, \end{aligned}$$

which yields

$$\begin{aligned} R_{h0k0} &= R^{\natural}{}_{h0k0} - \tfrac{1}{2}\,\mathrm{k}_0\,(\nabla^{\natural}{}_0 F_{hk} - 2\,\nabla^{\natural}{}_h F_{0k}) + \tfrac{1}{4}\,(\mathrm{k}_0)^2\,g^{rs}\,F_{hr}\,F_{ks}, \\ R_{hjk0} &= R^{\natural}{}_{hjk0} + \tfrac{1}{2}\,\mathrm{k}_0\,(\nabla^{\natural}{}_h F_{jk} - \nabla^{\natural}{}_j F_{hk}), \\ R_{k0\,hj} &= R^{\natural}{}_{k0\,hj} + \tfrac{1}{2}\,\mathrm{k}_0\,\nabla^{\natural}{}_k F_{jh}, \\ R_{hikj} &= R^{\natural}{}_{hikj}. \end{aligned}$$

Moreover, the coordinate expression in terms of the electric and magnetic fields follows from the equalities (see Definitions 5.2.1 and 5.3.1)

$$F_{ij} = \tfrac{1}{c}\,\sqrt{|g|}\epsilon_{ijh}\,B^h \quad \text{and} \quad F_{0i} = -E_{0i}. \qquad \square$$

Remark 6.4.5 We have already seen that the joined spacetime connection K is galilean by taking into account the equality $\Phi[g, K, o] = \Phi[g, K^{\natural}, o] + \mathrm{k}\,F$ (see Theorem 6.3.1), and the further equalities $d\Phi[g, K^{\natural}, o] = 0$ and $dF = 0$, which follow from the fact that $K^{\natural}$ is galilean and F is closed, by postulates.

By definition, the joined spacetime connection K is galilean if the covariant curvature tensor $\underline{R}[g, K]$ is symmetric. Now, we can directly check this property by means of the coordinate expression of $\underline{R}[g, K]$ shown in the above Corollary 6.4.4.

In fact, the coordinate expression of Corollary 6.4.4 and the equality

$$\nabla^{\natural}{}_{\lambda} F_{\mu\nu} + \nabla^{\natural}{}_{\nu} F_{\lambda\mu} + \nabla^{\natural}{}_{\mu} F_{\nu\lambda} = \partial_{\lambda} F_{\mu\nu} + \partial_{\nu} F_{\lambda\mu} + \partial_{\mu} F_{\nu\lambda} = 0,$$

yield

$$\begin{aligned}
R_{h0k0} &= R^{\natural}{}_{h0k0} + \tfrac{1}{2}\,\mathrm{k}_0\,(\nabla^{\natural}{}_k F_{0h} + \nabla^{\natural}{}_h F_{0k}) + \tfrac{1}{4}\,(\mathrm{k}_0)^2\, g^{rs}\, F_{hr}\, F_{ks} \\
&= R_{k0\,h0}, \\
R_{hjk0} &= R^{\natural}{}_{k0\,hj} + \tfrac{1}{2}\,\mathrm{k}_0\,\nabla^{\natural}{}_k F_{jh} \\
&= R_{k0\,hj}, \\
R_{hikj} &= R^{\natural}{}_{hikj} = R^{\natural}{}_{kjhi} = R_{kjhi}.
\end{aligned}$$

The above expression can be re–written in terms of the electric and magnetic fields as

$$\begin{aligned}
R_{h0k0} &= R^{\natural}{}_{h0k0} - \tfrac{1}{2}\,\mathrm{k}_0\,(\nabla^{\natural}{}_k E_{0h} + \nabla^{\natural}{}_h E_{0k}) + \tfrac{1}{4}\,(\mathrm{k}_0)^2\,|g|\,\tfrac{1}{c^2}\, g^{rs}\,\epsilon_{hrp}\,\epsilon_{ksq}\, B^p\, B^q \\
&= R_{k0\,h0}, \\
R_{hjk0} &= R^{\natural}{}_{k0\,hj} + \tfrac{1}{2}\,\mathrm{k}_0\,\tfrac{1}{c}\,\sqrt{|g|}\,\epsilon_{jhp}\,\nabla^{\natural}{}_k B^p \\
&= R_{k0\,hj}. \quad \square
\end{aligned}$$

6.5 Joined Ricci Tensor

Eventually, we compute the *Ricci tensor* $r[K]$ of the joined spacetime connection K and check that it is symmetric.

Indeed, the electromagnetic field F does not effect the spacelike restriction of $r[K]$. A consequence of this fact is that the scalar curvature $C[K]$ of the joined spacetime connection coincides with the scalar curvature $C[K^{\natural}]$ of the gravitational spacetime connection $K^{\natural}$, hence with the scalar curvature $C[\varkappa]$ of the spacelike metric connection $\varkappa$.

It is remarkable the fact that the Ricci tensor $R[K]$ of the joined spacetime connection K splits into the Ricci tensor $R[K^{\natural}]$ of the gravitational connection $K^{\natural}$ and a term involving the divergence $\mathrm{div}^{\natural} F$ of F plus a timelike term involving the scalar product F^2.

The above splitting of $r[K]$ will play a key role in the splitting of the joined field equation into its gravitational component (Galilei–Einstein equation) and its electromagnetic component (Galilei–Maxwell equation) (see Theorem 8.3.4).

Remark 6.5.1 We can define the "joined divergence" of F analogously to $\mathrm{div}^{\natural} F$ (see Theorem 5.9.6):

$$\mathrm{div}\, F := C^{12}_{12}\,(\bar{g} \otimes \nabla F) : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}.$$

The electromagnetic term $K^{\mathfrak{e}}$ does not effect the above joined divergence, hence we obtain

$$\mathrm{div}\, F = \mathrm{div}^{\natural} F. \quad \square$$

Theorem 6.5.2 *The Ricci tensor*

$$r[K] := C^1_2 R[K] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$$

of the joined connection $K := K^\natural + K^\mathfrak{e}$ *splits as follows*

$$r[K] = r[K^\natural] + \tfrac{1}{2}\,\mathrm{k}\, dt \otimes \mathrm{div}^\natural F + \tfrac{1}{2}\,\mathrm{k}\ \mathrm{div}^\natural F \otimes dt - \tfrac{1}{4}\,\mathrm{k}^2\, \check{F}^2\, dt \otimes dt.$$

We have the coordinate expression

$$\begin{aligned} r_{00} &= r^\natural{}_{00} - \mathrm{k}_0\, \nabla^\natural{}_h F_0{}^h + \tfrac{1}{4}\,(\mathrm{k}_0)^2\, F_h{}^s\, F_s{}^h \\ r_{j0} &= r^\natural{}_{j0} - \tfrac{1}{2}\,\mathrm{k}_0\, \nabla^\natural{}_h F_j{}^h \\ r_{0j} &= r^\natural{}_{0j} - \tfrac{1}{2}\,\mathrm{k}_0\, \nabla^\natural{}_h F_j{}^h \\ r_{ij} &= r^\natural{}_{ij} \end{aligned}$$

and, for each observer o, the observed expression (see Proposition 5.6.4 and Theorem 5.9.6)

$$\begin{aligned} r = r^\natural &+ \mathrm{k}\,\big(\mathrm{div}_\eta\, \vec{E}[o] - \tfrac{1}{c}\, g(\mathrm{curl}\, д[o],\ \vec{B})\big)\, dt \otimes dt - \mathrm{k}^2\, \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\, dt \otimes dt \\ &- \tfrac{1}{2}\,\mathrm{k}\, \tfrac{1}{c}\, \theta^*[o]\big(g^\flat(\mathrm{curl}\, \vec{B})\big) \otimes dt - \tfrac{1}{2}\,\mathrm{k}\, \tfrac{1}{c}\, dt \otimes \theta^*[o]\big(g^\flat(\mathrm{curl}\, \vec{B})\big). \end{aligned}$$

Indeed, the Ricci tensor $r[K]$ *of the joined spacetime connection* K *turns out to be symmetric, according to the fact that the joined spacetime connection* K *is galilean (see Theorem 6.3.1 and Definition 4.1.8).* □

Corollary 6.5.3 *The spacelike restriction of the Ricci tensor depends only on the metric spacelike connection according to the following equalities (see Propositions 3.2.13 and 4.1.3)*

$$\check{r}[K] = \check{r}[K^\natural] = r[\overset{\vee}{K}] = r\big[\varkappa[g]\big].\ \ \square$$

Eventually, we compute the scalar curvature $C[K]$ of the joined spacetime joined connection K and prove that it is equal to the scalar curvature of the gravitational connection $K^\natural$, hence it is equal to the scalar curvature of the spacelike connection $\varkappa$ induced by the metric g.

Corollary 6.5.4 *The scalar curvature of the joined connection* $K := K^\natural + K^\mathfrak{e}$ *depends only on the metric spacelike connection* $\varkappa[g]$ *according to the following equalities (see Propositions 3.2.13 and 4.1.3)*

$$C[G, K] = C[K^\natural, G] = C[\overset{\vee}{K}, G] = C\big[\varkappa[G], G\big].\ \ \square$$

Chapter 7
Classical Dynamics

We sketch the *classical kinematics* and *dynamics* of a *particle* and a *continuum* effected by the gravitational and electromagnetic fields, in terms of the joined spacetime connection.

7.1 Particle Kinematics

We complete the preliminary notions of particle motion and its velocity, by defining the *iterated velocity*, the *gravitational acceleration* and the *joined acceleration*.

Moreover, with reference to an observer o, we recall the *observed velocity* and the observed splitting of the velocity.

Further, we define the *observed motion*, along with the *velocity of the observed motion*, and compare them with the motion and the observed velocity of the motion.

Additionally, we provide the observed splitting of the gravitational and joined accelerations into their *observed acceleration* and *observed deformation, Coriolis, dragging* components.

7.1.1 Absolute Particle Kinematics

We recall the notions of particle motion s and of its velocity ds (see Definition 2.4.1).

Then, we discuss the *velocity*, the *iterated velocity*, the *gravitational acceleration* and the *joined acceleration* of a particle motion s.

Thus, let us consider a particle motion $s : \boldsymbol{T} \to \boldsymbol{E}$ and recall its *velocity* (see Definition 2.4.1)

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$$ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expression $ds = u^0 \otimes \big(\partial_0 \circ s + \partial_0 s^i \, (\partial_i \circ s)\big)$.

Then, we add the notion of iterated velocity.

Definition 7.1.1 We define the *iterated velocity* of s to be the scaled section

$$d^2 s : \boldsymbol{T} \to \mathbb{T}^* \otimes T(\mathbb{T}^* \otimes T\boldsymbol{E}),$$

with coordinate expression

$$d^2 s = u^0 \otimes \big(\partial_0 \circ ds + \partial_0 s^i \, (\partial_i \circ ds) + \partial_{00} s^i \, (\partial_i^0 \circ ds)\big),$$

i.e.

$$(x^0, x^i, \ \dot{x}^0, \dot{x}^i; \ \breve{x}^0, \breve{x}^i, \ \ddot{x}^0, \ddot{x}^i) \circ d^2 s = (x^0, s^i, \ 1, \partial_0 s^i; \ 1, \partial_0 s^i, \ 0, \partial_{00} s^i). \quad \square$$

Next, let us discuss the acceleration of the motion. For this purpose, we need a preliminary technical Lemma.

Lemma 7.1.2 *Let us consider any special spacetime connection (see Definition 4.1.19) and a particle motion*

$$K : T\boldsymbol{E} \to T^*\boldsymbol{T} \otimes TT\boldsymbol{E} \quad \text{and} \quad s : \boldsymbol{T} \to \boldsymbol{E}.$$

Let

$$X : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E} \quad \text{and} \quad Y : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$$

be sections projecting over s, and

$$\widetilde{X} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E} \quad \text{and} \quad \widetilde{Y} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$$

any extensions of the above sections, according to the commutative diagrams.

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{\widetilde{X}} & \mathbb{T}^* \otimes T\boldsymbol{E} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \boldsymbol{T} & \xrightarrow{X} & \mathbb{T}^* \otimes T\boldsymbol{E} \end{array} \qquad \begin{array}{ccc} \boldsymbol{E} & \xrightarrow{\widetilde{Y}} & \mathbb{T}^* \otimes T\boldsymbol{E} \\ {\scriptstyle s}\uparrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \boldsymbol{T} & \xrightarrow{Y} & \mathbb{T}^* \otimes T\boldsymbol{E} \end{array} .$$

Then, the restriction to $s(\boldsymbol{T}) \subset \boldsymbol{E}$

$$\nabla_X Y := \big(\nabla_{\widetilde{X}} \widetilde{Y}\big)_{|s(\boldsymbol{T})} : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes T\boldsymbol{E}$$

of the covariant derivative $\nabla_{\widetilde{X}}\widetilde{Y} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes T\boldsymbol{E}$ *does not depend on the choice of the extension, hence the symbol* $\nabla_X Y$ *is well defined.*

In particular, if $X = Y = ds$, *then we obtain the equality (see Proposition 4.1.2)*

$$\nabla_{ds}\, ds = \nu[K] \circ d^2 s$$

and the coordinate expression

$$\nabla_{ds} ds = \big(\partial_{00}s^i - (K_h{}^i{}_k \circ s)\, \partial_0 s^h\, \partial_0 s^k - 2\,(K_h{}^i{}_0 \circ s)\, \partial_0 s^h - (K_0{}^i{}_0 \circ s)\big) \times u^0 \otimes u^0 \otimes (\partial_i \circ s). \quad \square$$

Now, let us consider the gravitational spacetime connection (see Postulate C.3)

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

Further, let us suppose that the particle has mass and charge

$$m \in \mathbb{M} \quad \text{and} \quad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathrm{I\!R}.$$

Accordingly, let us consider the coupling constant scale (see Definition 6.1.1)

$$\mathrm{k} := \tfrac{q}{m} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathrm{I\!R}$$

and the associated joined spacetime connection (see Theorem 6.3.1)

$$K := K^\natural - \tfrac{1}{2}\, \mathrm{k}\, (dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

Definition 7.1.3 We define the *gravitational acceleration* of the particle motion s, with respect to the gravitational spacetime connection $K^\natural$, to be the scaled section (see Lemma 7.1.2)

$$\nabla^\natural{}_{ds} ds := \nu[K^\natural] \circ d^2 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

given by the commutative diagram.

$$\begin{array}{ccc}
\boldsymbol{T} & \xrightarrow{\ \nabla^\natural{}_{ds} ds\ } & (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E} \\
{\scriptstyle d^2 s}\big\downarrow & & \big\uparrow{\scriptstyle \mathrm{id}} \\
\mathbb{T}^* \otimes T(\mathbb{T}^* \otimes T\boldsymbol{E}) & \xrightarrow{\ \nu[K^\natural]\ } & (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}
\end{array}$$

We have the coordinate expression

$$\nabla^{\natural}{}_{ds}\, ds = \big(\partial_{00}s^i - (K^{\natural}{}_h{}^i{}_k \circ s)\, \partial_0 s^h\, \partial_0 s^k - 2\,(K^{\natural}{}_h{}^i{}_0 \circ s)\, \partial_0 s^h - (K^{\natural}{}_0{}^i{}_0 \circ s)\big) \times u^0 \otimes u^0 \otimes (\partial_i \circ s).$$

We can define in an analogous way the *joined acceleration* of the particle motion s, with respect to the joined spacetime connection K,

$$\nabla_{ds} ds := \nu[K] \circ d^2 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}. \quad \square$$

Proposition 7.1.4 *The joined acceleration and the gravitational acceleration are related by the following equality*

$$\nabla_{ds}\, ds = \nabla^{\natural}{}_{ds}\, ds + \tfrac{q}{\hbar}\, G^{\sharp}\,(ds \lrcorner F)\,,$$

where the additional term turns out to be (up to sign) just the rescaled Lorentz force (see Definition 5.7.1)

$$-\tfrac{q}{\hbar}\; G^{\sharp}\,(ds \lrcorner F) \in \sec\big(\boldsymbol{T},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big)\,.$$

Proof. The proof follows from Theorem 6.3.1. □

Remark 7.1.5 The gravitational acceleration and the joined acceleration turn out to be vertical valued vector fields as a consequence of the fact that the spacetime connections $K^{\natural}$ and K are time preserving. □

7.1.2 Observed Particle Kinematics

We consider a particle motion s and, with reference to an observer o, we discuss the observed splitting of the velocity and the joined acceleration.

Thus, let us consider a particle motion and an observer (see Definition 2.4.1 and Proposition 2.7.3)

$$s : \boldsymbol{T} \to \boldsymbol{E} \quad \text{and} \quad o : \boldsymbol{E} \to J_1\boldsymbol{E}.$$

Definition 7.1.6 We define the *observed velocity* of the motion s to be the scaled section (see Proposition 2.7.3 and Definition 2.7.7)

$$d[o]s := \theta[o] \circ ds : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E},$$

with coordinate expression, in an adapted spacetime chart,

$$(x^0, x^i,\ \dot{x}^0_0, \dot{x}^i_0) \circ d[o]s = (x^0, s^i,\ 1, \partial_0 s^i). \quad \square$$

Let us recall the observed space $\boldsymbol{P}[o]$ and the local fibred isomorphism $\boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{P}[o]$ over $\boldsymbol{T}$ (see Note 2.7.5).

Then, we define the observed motion along with the velocity of the observed motion and compare the velocity of the observed motion with the observed velocity.

Definition 7.1.7 We define the *observed motion* to be the map (see Note 2.7.5)

$$s[o] := p[o] \circ s : \boldsymbol{T} \to \boldsymbol{P}[o],$$

with coordinate expression, in an adapted spacetime chart,

$$x^i \circ s[o] = x^i \circ s.$$

Accordingly, we define the *velocity of the observed motion* to be the map

$$d\,(s[o]) : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{P}[o],$$

with coordinate expression, in an adapted spacetime chart,

$$(x^i,\ \dot{x}^i_0) \circ d(s[o]) = (s^i,\ \partial_0 s^i). \quad \square$$

Note 7.1.8 We can compare ds and $d\,(s[o])$, $d[o]s$ by means of the following commutative diagram.

$$\begin{array}{ccccc}
\boldsymbol{T} & \xrightarrow{d[o]s} & \mathbb{T}^* \otimes V\boldsymbol{E} & \longrightarrow & \cdot \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \cap} & & \\
\boldsymbol{T} & \xrightarrow{ds} & \mathbb{T}^* \otimes T\boldsymbol{E} & & \downarrow{\scriptstyle \mathrm{id} \otimes Vp[o]} \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes Tp[o]} & & \\
\boldsymbol{T} & \xrightarrow{d(s[o])} & \mathbb{T}^* \otimes T\boldsymbol{P}[o] & \longleftarrow & \cdot
\end{array} \quad . \quad \square$$

Note 7.1.9 According to the splitting (see Proposition 2.7.3)

$$\mathbb{T}^* \otimes T\boldsymbol{E} = \left(\mathbb{T}^* \otimes T_o\boldsymbol{E}\right) \underset{\boldsymbol{E}}{\oplus} \left(\mathbb{T}^* \otimes V\boldsymbol{E}\right),$$

we have the observed splitting (see Definition 2.7.7)

$$ds = \text{д}[o] \circ s + d[o]s,$$

where the section

$$д[o] \circ s : \boldsymbol{T} \to \mathbb{T}^* \otimes T_o \boldsymbol{E} \subset \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$$

has coordinate expression, in an adapted spacetime chart,

$$(x^0, x^i, \ \dot{x}_0^0, \dot{x}_0^i) \circ (д[o] \circ s) = (x^0, s^i, \ 1, 0). \quad \square$$

Then, we discuss the observed splitting of the acceleration of the particle motion, by taking into account Lemma 7.1.2.

Definition 7.1.10 We define the following *observed gravitational objects* (see Proposition 2.7.3, Lemma 4.2.9, Definition 4.2.11 and Theorem 4.2.13):

observed acceleration	$\mathrm{A}^\natural{}_{\text{obs}}[o] := \nabla^\natural{}_{ds} (d[o]s) : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed relative acceleration	$\mathrm{A}^\natural{}_{\text{rlt}}[o] := \nabla^\natural{}_{d[o]s} (д[o] \circ s) : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed dragging acceleration	$\mathrm{A}^\natural{}_{\text{drg}}[o] := \nabla^\natural{}_{д[o]\circ s} (д[o] \circ s) : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed deformation acceleration	$\mathrm{A}^\natural{}_{\text{dfr}}[o] := -\frac{1}{2} \big(d[o]s \lrcorner \widehat{\Sigma}^\natural[o]\big) : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed Coriolis acceleration	$\mathrm{A}^\natural{}_{\text{crl}}[o] := -\frac{1}{2} \big(d[o]s \lrcorner \widehat{\Phi}^\natural[o]\big) : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$.

Moreover, we define the corresponding *joined objects*, in an analogous way, by replacing

$$\nabla^\natural \quad \text{with } \nabla, \qquad \widehat{\Sigma}^\natural[o] \quad \text{with } \widehat{\Sigma}[o], \qquad \widehat{\Phi}^\natural[o] \quad \text{with } \widehat{\Phi}[o]. \quad \square$$

Proposition 7.1.11 *We have the following observed splitting of the gravitational acceleration*

$$\nabla^\natural{}_{ds}\, ds = \mathrm{A}^\natural{}_{\text{obs}}[o] + \mathrm{A}^\natural{}_{\text{rlt}}[o] + \mathrm{A}^\natural{}_{\text{drg}}[o].$$

Moreover, we obtain the equality

$$\mathrm{A}^\natural{}_{\text{rlt}}[o] = \mathrm{A}^\natural{}_{\text{dfr}}[o] + \mathrm{A}^\natural{}_{\text{crl}}[o].$$

We have the following coordinate expressions, in a spacetime chart adapted to o,

$$\begin{aligned}
\mathrm{A}^\natural{}_{\text{obs}}[o]^i_{00} &= \partial_{00}s^i - K^\natural{}_0{}^i{}_k\, \partial_0 s^k - K^\natural{}_h{}^i{}_k\, \partial_0 s^h\, \partial_0 s^k \\
&= \partial_{00}s^i + \tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{kj} + \Phi^\natural{}_{kj})\, \partial_0 s^k \\
&\quad + \tfrac{1}{2}\, G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, \partial_0 s^h\, \partial_0 s^k, \\
\mathrm{A}^\natural{}_{\text{rlt}}[o]^i_{00} &= -G^{ij}_0\, K^\natural{}_h{}^0_j{}_0\, \partial_0 s^h \\
&= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{hj} + \Phi^\natural{}_{hj})\, \partial_0 s^h, \\
\mathrm{A}^\natural{}_{\text{drg}}[o]^i_{00} &= -K^\natural{}_0{}^i{}_0 \\
&= G^{ij}_0\, \Phi^\natural{}_{0j},
\end{aligned}$$

$$\begin{aligned} \mathrm{A}^{\natural}{}_{\mathrm{dfr}}[o]^i_{00} &= -G^{ij}_0\, (K^{\natural}{}_h{}^0{}_{j0} + K^{\natural}{}_j{}^0{}_{h0})\, \partial_0 s^h \\ &= G^{ih}_0\, \partial_0 G^0_{jh}\, \partial_0 s^h, \\ \mathrm{A}^{\natural}{}_{\mathrm{crl}}[o]^i_{00} &= -G^{ij}_0\, (K^{\natural}{}_h{}^0{}_{j0} - K^{\natural}{}_j{}^0{}_{h0})\, \partial_0 s^h \\ &= G^{ih}_0\, \Phi^{\natural}{}_{jh}\, \partial_0 s^h. \end{aligned}$$

Proof. The observed splitting $ds = д[o] \circ s + d[o]s$ yields the equality

$$\nabla^{\natural}{}_{ds}\, ds = \nabla^{\natural}{}_{ds}(d[o]s) + \nabla^{\natural}{}_{d[o]s}(д[o] \circ s) + \nabla^{\natural}{}_{д[o]\circ s}\, (д[o] \circ s),$$

i.e. $\nabla^{\natural}{}_{ds}\, ds = \mathrm{A}^{\natural}{}_{\mathrm{obs}}[o] + \mathrm{A}^{\natural}{}_{\mathrm{rlt}}[o] + \mathrm{A}^{\natural}{}_{\mathrm{drg}}[o]$.

Moreover, the splitting $\nabla(д[o]) = \frac{1}{2}\big(\widehat{\Sigma}\,[G, o] + \widehat{\Phi}\,[K, o]\big)$ (see Corollary 4.2.15) yields the equality $\mathrm{A}^{\natural}{}_{\mathrm{rlt}}[o] = \mathrm{A}^{\natural}{}_{\mathrm{dfr}}[o] + \mathrm{A}^{\natural}{}_{\mathrm{crl}}[o]$ (see Corollary 4.2.15). □

Corollary 7.1.12 *We obtain an analogous splitting for the joined acceleration $\nabla_{ds}\, ds$, by taking into account the following equalities (see Proposition 7.1.11 and Definition 5.7.1)*

$$\begin{aligned} \mathrm{A}_{\mathrm{dfr}}[o] &= \mathrm{A}^{\natural}{}_{\mathrm{dfr}}[o], \\ \mathrm{A}_{\mathrm{crl}}[o] &= \mathrm{A}^{\natural}{}_{\mathrm{crl}}[o] - \tfrac{1}{c}\, \tfrac{q}{m}\, d[o]s \times B, \\ \mathrm{A}^{\natural}[o] &= \mathscr{A}^{\natural}{}_{\mathrm{drg}}[o] - \tfrac{q}{m}\, \vec{E}[o]. \end{aligned}$$

Proof. The corollary follows from the above Propositions 7.1.4 and 7.1.11. □

7.2 Particle Dynamics

We postulate the *Newton law of motion* for a charged particle effected by the gravitational and electromagnetic fields. Moreover, we express, more synthetically, the above equation in terms of the joined spacetime connection.

Furthermore, with reference to an observer o, we split the above equation into its observed components.

7.2.1 Absolute Particle Dynamics

We postulate the *Newton law of motion* $m\, \nabla^{\natural}{}_{ds}\, ds = \vec{\mathrm{f}}$ for a charged particle effected by the gravitational and electromagnetic fields.

We stress that this law can be written, more synthetically, in terms of the joined spacetime connection K, as $m\, \nabla_{ds}\, ds = 0$.

Thus, let us consider a charged particle of mass m and charge q, effected by the gravitational field $K^{\natural}$ and the electromagnetic field F (see Postulates C.3 and C.4).

Assumption C.2 We postulate as law of motion the *Newton law of motion*

$$m \, \nabla^{\natural}{}_{ds} \, ds = \vec{\mathrm{f}},$$

where

$$\nabla^{\natural}{}_{ds} \, ds : \boldsymbol{T} \to \mathbb{T}^{-2} \otimes V\boldsymbol{E} \quad \text{and} \quad \vec{\mathrm{f}} := -\, q \, g^{\sharp}(ds \lrcorner F) : \boldsymbol{T} \to (\mathbb{T}^{-2} \otimes \mathbb{M}) \otimes V\boldsymbol{E}$$

are, respectively, the gravitational acceleration of the motion and the Lorentz force (see Definitions 5.7.1 and 7.1.3). □

Lemma 7.2.1 *The Newton law of motion $m \, \nabla^{\natural}{}_{ds} \, ds = \vec{\mathrm{f}}$ can be written as*

$$G^{\flat}(\nabla^{\natural}{}_{ds} \, ds) = -\tfrac{q}{\hbar} \, (ds \lrcorner F),$$

i.e., in coordinates,

$$\begin{aligned} &G^{0}_{ij} \, \big(\partial_{00} s^{i} - (K^{\natural}{}_{0}{}^{i}{}_{0} \circ s) - 2\,(K^{\natural}{}_{h}{}^{i}{}_{0} \circ s)\, \partial_0 s^{h} - (K^{\natural}{}_{h}{}^{i}{}_{k} \circ s)\, \partial_0 s^{h} \, \partial_0 s^{k}\big) \\ &\quad = -\tfrac{q}{\hbar} \, (F_{0j} + F_{ij} \, \partial_0 s^{i}). \quad \square \end{aligned}$$

Proposition 7.2.2 *The Newton law of motion $m \, \nabla^{\natural}{}_{ds} \, ds = \vec{\mathrm{f}}$ can be expressed, more synthetically, in terms of the joined spacetime connection K, as*

$$m \, \nabla_{ds} \, ds = 0. \quad \square$$

7.2.2 Observed Particle Dynamics

By recalling the observed splittings of the acceleration and the Lorentz force, we show the observed splitting of the Newton law of motion.

Note 7.2.3 With reference to an observer o, the Newton law of motion has the observed expression (see Proposition 4.2.10 and Corollary 4.2.15)

$$\begin{aligned} &m \, \nabla^{\natural}{}_{ds} \, \big(d[o]s\big) + m \, (d[o]s) \lrcorner \big(\widehat{\Sigma}{}^{\natural}[o] \circ s\big) + m \, (d[o]s) \lrcorner \big(\widehat{\Phi}{}^{\natural}[o] \circ s\big) \\ &\quad + m \, \big(\nabla^{\natural}{}_{д[o]} д[o]\big) \circ s \\ &= q \, (\vec{E}[o] \circ s + \tfrac{1}{c} \, d[o]s \times \big(\vec{B} \circ s)\big). \quad \square \end{aligned}$$

7.3 Fluid Kinematics

We complete the preliminary notions of *continuum motion* and its *velocity* (see Definition 2.4.2) by considering a *charged fluid* equipped with a *mass density*, a charge density and a pressure density.

Then, we define the *mass density current*, the *charge density current*, the *contravariant energy–momentum tensor* and the *covariant energy tensor*.

Moreover, we define the *gravitational acceleration* and *joined acceleration*.

Further, with reference to an observer o, we define the *observed continuum motion* and its *velocity*. Additionally, we provide the *observed splitting* of the velocity and the acceleration into their observed components.

7.3.1 Absolute Fluid Kinematics

We recall the notions of *continuum motion* and its *velocity* (see Definition 2.4.2).

Then, we introduce the notion of a *charged fluid* $(\mathscr{C}, \mu, \rho, \mathrm{p})$, equipped with a *mass density current*, a *charge density current*.

Moreover, we define the *gravitational acceleration* and the *joined acceleration* (see Postulate C.3 and Theorem 6.3.1). Further, we define the *contravariant energy–momentum tensor* and *covariant energy tensor*.

Definition 7.3.1 We define a *classical charged fluid* to be a 4-plet $(\mathscr{C}, \mu, \rho, \mathrm{p})$, consisting of a *continuum motion* (see Definition 2.4.2), a *mass density*, a *charge density* and a *pressure density* ("per mass unit"), where, respectively,

$$\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E},$$

is a (local) 1-parameter group of fibred diffeomorphisms, over the time translation $\tau : \bar{\mathbb{T}} \times \boldsymbol{T} \to \boldsymbol{T}$, and

$$\begin{aligned}
&\mu \in \mathrm{map}(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes \mathbb{M} \otimes \mathbb{R}),\\
&\rho \in \mathrm{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big),\\
&\mathrm{p} \in \mathrm{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{2}) \otimes \mathbb{R}\big).
\end{aligned}$$

We suppose the charge density to be *proportional* to the mass density according to the equality

$$\rho = \mathrm{k}\,\mu, \quad \text{with}\ \mathrm{k} := \rho/\mu \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}.$$

Moreover, we suppose the mass density (and the charge density) to be integrable on the fibres of spacetime. Accordingly, we define the *total mass* and *total charge* to be the maps

$$m : \boldsymbol{T} \to \mathbb{M} : \mathrm{t} \mapsto \int_{\boldsymbol{E}_{\mathrm{t}}} \mu_{\mathrm{t}}\, \eta_{\mathrm{t}} \quad \text{and} \quad q : \boldsymbol{T} \to \mathbb{Q} : \mathrm{t} \mapsto \int_{\boldsymbol{E}_{\mathrm{t}}} \rho_{\mathrm{t}}\, \eta_{\mathrm{t}}.$$

Hence, we can write $\mathrm{k} = \frac{q}{m}$. □

Thus, let us consider a charged fluid $(\mathscr{C}, \mu, \rho, \mathrm{p})$.

Definition 7.3.2 We define the *velocity*, the *mass density current* and the *charge density current* to be respectively, the maps (here, ∂ denotes the differential with respect to the parameter, evaluated at the 0 value of the parameter)

$$\begin{aligned}
\mathscr{V} &:= \partial \mathscr{C} \in \sec(\boldsymbol{E},\ \mathbb{T}^* \otimes T\boldsymbol{E}),\\
\mathscr{P} &:= \mu\, \mathscr{V} \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes T\boldsymbol{E}\big).\\
\mathscr{J} &:= \rho\, \mathscr{V} \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T\boldsymbol{E}\big).
\end{aligned}$$

We have the coordinate expression (here, δ_0 denotes the derivative with respect to the parameter, evaluated at the 0 value of the parameter)

$$\mathscr{V} \equiv u^0 \otimes (\partial_0 + \mathscr{V}_0^i\, \partial_i) = u^0 \otimes (\partial_0 + \delta_0 \mathscr{C}^i\, \partial_i),$$

where

$$\mathscr{C}^\lambda := x^\lambda \circ \mathscr{C} \in \operatorname{map}\big((\mathbb{T} \times \mathbb{R}) \times \boldsymbol{E},\ \mathbb{R}^4\big). \quad \square$$

Definition 7.3.3 We define the *contravariant energy–momentum tensor* and *covariant energy tensor* associated with the charged fluid $(\mathscr{C}, \mu, \rho, \mathrm{p})$ to be the scaled tensors (see Definition 3.1.1)

$$\begin{aligned}
\mathscr{T}^{\mathrm{m}} &:= \mu\, \mathscr{V} \otimes \mathscr{V} - \mu\, \mathrm{p}\, \bar{g} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E}),\\
\underline{\mathscr{T}}^{\mathrm{m}} &:= \mathbf{g}^\flat(\mathscr{T}^{\mathrm{m}}) = \mu\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes H^*\boldsymbol{E} \otimes H^*\boldsymbol{E},
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\mathscr{T}^{\mathrm{m}} &= \mu\, (\partial_0 + \mathscr{V}_0^i\, \partial_i) \otimes (\partial_0 + \mathscr{V}_0^i\, \partial_i) \otimes u^0 \otimes u^0 - \mu\, \mathrm{p}_{00}\, g^{ij}\, \partial_i \otimes \partial_j \otimes u^0 \otimes u^0,\\
\underline{\mathscr{T}}^{\mathrm{m}} &= \mu\, (u_0 \otimes u_0) \otimes (d^0 \otimes d^0). \quad \square
\end{aligned}$$

Next, let us consider the coupling constant scale (see Postulate C.3, Definitions 6.1.1 and 7.3.1, Theorem 6.3.1)

$$\mathrm{k} := \frac{\rho}{\mu} = \frac{q}{m} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}$$

and the associated joined spacetime connection (see Theorem 6.3.1)

$$K := K^\natural - \tfrac{1}{2}\, \mathrm{k}\, (dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

Definition 7.3.4 We define the *gravitational acceleration* and the *joined acceleration* of the continuum motion to be the sections

$$\mathscr{A}^{\natural} := \nabla^{\natural}{}_{\mathscr{V}} \mathscr{V} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A} := \nabla_{\mathscr{V}} \mathscr{V} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

with coordinate expressions

$$\mathscr{A}^{\natural} = \partial_0 \mathscr{V}_0^i + \mathscr{V}_0^j\, \partial_j \mathscr{V}_0^i - K^{\natural}{}_h{}^i{}_k\, \mathscr{V}_0^h\, \mathscr{V}_0^k - 2\, K^{\natural}{}_h{}^i{}_0\, \mathscr{V}_0^h - K^{\natural}{}_0{}^i{}_0,$$
$$\mathscr{A} = \partial_0 \mathscr{V}_0^i + \mathscr{V}_0^j\, \partial_j \mathscr{V}_0^i - K_h{}^i{}_k\, \mathscr{V}_0^h\, \mathscr{V}_0^k - 2\, K_h{}^i{}_0\, \mathscr{V}_0^h - K_0{}^i{}_0. \quad \square$$

Proposition 7.3.5 *The joined acceleration and gravitational acceleration are related by the following equality*

$$\mathscr{A} = \mathscr{A}^{\natural} + \tfrac{q}{\hbar}\, G^{\sharp}\,(\mathscr{V} \lrcorner F)\,,$$

where the additional term turns out to be (up to sign) the rescaled Lorentz force density (see Definition 5.7.3)

$$-\tfrac{q}{\hbar}\, G^{\sharp}\,(\mathscr{V} \lrcorner F) \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big)\,.$$

Proof. The proof follows from Theorem 6.3.1. □

Remark 7.3.6 Clearly, the velocity, the gravitational acceleration and the joined acceleration of the continuum motion $\mathscr{C}$ coincide with the velocity and the acceleration of its particles. In other words, for each $e \in \boldsymbol{E}$, we have (see Definitions 2.4.1 and 7.1.3)

$$\partial\mathscr{C}(e) = d(\mathscr{C}_e), \qquad \mathscr{A}^{\natural}(e) = \nabla^{\natural}{}_{d(\mathscr{C}_e)} d(\mathscr{C}_e), \qquad \mathscr{A}(e) = \nabla_{d(\mathscr{C}_e)} d(\mathscr{C}_e). \quad \square$$

7.3.2 Observed Continuum Kinematics

We consider a continuum motion $\mathscr{C}$ and, with reference to an observer o, we define the *observed velocity* and show its *observed splitting*.

Moreover, we define the *observed continuum motion*, along with its *velocity of the observed continuum motion*.

Further, we discuss the observed splitting of the gravitational and joined accelerations into their observed components: *observed lagrangian acceleration, observed eulerian acceleration, observed spacelike acceleration, observed deformation acceleration, observed Coriolis acceleration, observed dragging acceleration*.

Thus, let us consider a continuum motion $\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$ and an observer o (see Definition 2.4.2 and Proposition 2.7.3).

First of all, we recall the *observed velocity* of the continuum motion (see Definition 2.7.8)

$$\vec{\mathscr{V}}[o] := \theta[o] \circ \partial\mathscr{C} : \boldsymbol{T} \times \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E},$$

with coordinate expression

$$\vec{\mathscr{V}}[o] = \mathscr{V}_0^i \, u^0 \otimes \partial_i .$$

Indeed, we have the observed splitting

$$\partial\mathscr{C} = \text{д}[o] + \vec{\mathscr{V}}[o].$$

Next, we recall the observed space and related maps (see Note 2.7.5)

$$\boldsymbol{P}[o] := \boldsymbol{E}/\simeq, \qquad p[o] : \boldsymbol{E} \to \boldsymbol{P}[o], \qquad \boldsymbol{E} \simeq \boldsymbol{T} \times \boldsymbol{P}[o].$$

Definition 7.3.7 We define the *observed continuum motion* to be the map

$$\mathscr{C}[o] : \bar{\mathbb{T}} \times (\boldsymbol{T} \times \boldsymbol{P}[o]) \to \boldsymbol{P}[o],$$

given by the commutative diagram

$$\begin{array}{ccc}
\bar{\mathbb{T}} \times \big(\boldsymbol{T} \times \boldsymbol{P}[o]\big) & \xrightarrow{\mathscr{C}[o]} & \boldsymbol{P}[o] \\
\downarrow & & \uparrow p[o] \\
\bar{\mathbb{T}} \times \boldsymbol{E} & \xrightarrow{\mathscr{C}} & \boldsymbol{E}
\end{array} .$$

We have the coordinate expression, in an adapted spacetime chart,

$$x^i \circ \mathscr{C}[o] = \mathscr{C}^i .$$

Accordingly, we define the *velocity of the observed continuum motion* to be the map

$$\partial\mathscr{C}[o] : \boldsymbol{T} \times \boldsymbol{P}[o] \to \mathbb{T}^* \otimes T\boldsymbol{P}[o],$$

with coordinate expression

$$\partial\mathscr{C}[o] = u^0 \otimes (\delta_0 \mathscr{C}^i \, \partial_i). \quad \square$$

Then, we discuss the observed splitting of the gravitational and joined accelerations of the continuum motion.

Definition 7.3.8 We define the following *observed gravitational objects* (see Proposition 2.7.3, Lemma 4.2.9, Definition 4.2.11 and Theorem 4.2.13):

observed lagrangian acceleration	$\mathscr{A}^{\natural}{}_{\text{lag}}[o] := \nabla^{\natural}{}_{\mathscr{V}}\, \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed eulerian acceleration	$\mathscr{A}^{\natural}{}_{\text{e}}[o] := \nabla^{\natural}{}_{Д[o]} \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed spacelike acceleration	$\mathscr{A}^{\natural}{}_{\text{spc}}[o] := \nabla^{\natural}{}_{\vec{\mathscr{V}}[o]} \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed relative acceleration	$\mathscr{A}^{\natural}{}_{\text{rlt}}[o] := \nabla^{\natural}{}_{\vec{\mathscr{V}}[o]} Д[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed dragging acceleration	$\mathscr{A}^{\natural}{}_{\text{drg}}[o] := \nabla^{\natural}{}_{Д[o]} Д[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed deformation acceleration	$\mathscr{A}^{\natural}{}_{\text{dfr}}[o] := -\frac{1}{2}\big(\vec{\mathscr{V}}[o] \lrcorner \widehat{\Sigma}^{\natural}[o]\big) : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$,
observed Coriolis acceleration	$\mathscr{A}^{\natural}{}_{\text{crl}}[o] := -\frac{1}{2}\big(\vec{\mathscr{V}}[o] \lrcorner \widehat{\Phi}^{\natural}[o]\big) : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$.

Moreover, we define the corresponding *joined objects*, in an analogous way, by replacing

$$\nabla^{\natural} \quad \text{with } \nabla, \qquad \widehat{\Sigma}^{\natural}[o] \quad \text{with } \widehat{\Sigma}[o], \qquad \widehat{\Phi}^{\natural}[o] \quad \text{with } \widehat{\Phi}[o]. \quad \square$$

Proposition 7.3.9 *We have the following observed splitting of the gravitational acceleration, in terms, respectively, of the lagrangian and eulerian schemes,*

$$\begin{aligned}\mathscr{A}^{\natural} &= \mathscr{A}^{\natural}{}_{\text{lag}}[o] + \mathscr{A}^{\natural}{}_{\text{rlt}}[o] + \mathscr{A}^{\natural}{}_{\text{drg}}[o], \\ &= (\mathscr{A}^{\natural}{}_{\text{e}}[o] + \mathscr{A}^{\natural}{}_{\text{spc}}[o]) + \mathscr{A}^{\natural}{}_{\text{rlt}}[o] + \mathscr{A}^{\natural}{}_{\text{drg}}[o].\end{aligned}$$

Thus, the observed lagrangian and eulerian accelerations are related by the equality

$$\mathscr{A}^{\natural}{}_{\text{lag}}[o] = \mathscr{A}^{\natural}{}_{\text{e}}[o] + \mathscr{A}^{\natural}{}_{\text{spc}}[o].$$

Moreover, we obtain the equality

$$\mathscr{A}^{\natural}{}_{\text{rlt}}[o] = \mathscr{A}^{\natural}{}_{\text{dfr}}[o] + \mathscr{A}^{\natural}{}_{\text{crl}}[o].$$

We have the following coordinate expressions, in a spacetime chart adapted to o,

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{lag}}[o]^i_{00} &= \partial_0 \mathscr{V}^i_0 - K^{\natural}{}_0{}^i{}_k\, \mathscr{V}^k_0 + (\partial_h \mathscr{V}^i_0 - K^{\natural}{}_h{}^i{}_k\, \mathscr{V}^k_0)\, \mathscr{V}^h_0 \\ &= \partial_0 \mathscr{V}^i_0 + \tfrac{1}{2} G^{ij}_0\, (\partial_0 G^0_{kj} + \Phi^{\natural}{}_{kj})\, \mathscr{V}^k_0 \\ &\quad + \big(\partial_h \mathscr{V}^i_0 + \tfrac{1}{2} G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, \mathscr{V}^k_0\big)\, \mathscr{V}^h_0, \\ \mathscr{A}^{\natural}{}_{\text{e}}[o]^i_{00} &= \partial_0 \mathscr{V}^i_0 - K^{\natural}{}_0{}^i{}_k\, \mathscr{V}^k_0 \\ &= \partial_0 \mathscr{V}^i_0 + \tfrac{1}{2} G^{ij}_0\, (\partial_0 G^0_{kj} + \Phi^{\natural}{}_{kj})\, \mathscr{V}^k_0, \\ \mathscr{A}^{\natural}{}_{\text{spc}}[o]^i_{00} &= (\partial_h \mathscr{V}^i_0 - K^{\natural}{}_h{}^i{}_k\, \mathscr{V}^k_0)\, \mathscr{V}^h_0 \\ &= \big(\partial_h \mathscr{V}^i_0 + \tfrac{1}{2} G^{ij}_0\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk})\, \mathscr{V}^k_0\big)\, \mathscr{V}^h_0,\end{aligned}$$

$$\begin{aligned}
\mathscr{A}^{\natural}{}_{\text{rlt}}[o]^i_{00} &= -G^{ij}_0\, K^{\natural}{}_h{}^0{}_{j0}\, \mathscr{V}^h_0 \\
&= -\tfrac{1}{2}\, G^{ij}_0\, (\partial_0 G^0_{hj} + \Phi^{\natural}{}_{hj})\, \mathscr{V}^h_0, \\
\mathscr{A}^{\natural}{}_{\text{drg}}[o]^i_{00} &= -K^{\natural}{}_0{}^i{}_0 \\
&= G^{ij}_0\, \Phi^{\natural}{}_{0j}, \\
\mathscr{A}^{\natural}{}_{\text{dfr}}[o]^i_{00} &= -G^{ij}_0\, (K^{\natural}{}_h{}^0{}_{j0} + K^{\natural}{}_j{}^0{}_{h0})\, \mathscr{V}^h_0 \\
&= G^{ih}_0\, \partial_0 G^0_{jh}\, \mathscr{V}^h_0, \\
\mathscr{A}^{\natural}{}_{\text{crl}}[o]^i_{00} &= -G^{ij}_0\, (K^{\natural}{}_h{}^0{}_{j0} - K^{\natural}{}_j{}^0{}_{h0})\, \mathscr{V}^h_0 \\
&= G^{ih}_0\, \Phi^{\natural}{}_{jh}\, \mathscr{V}^h_0.
\end{aligned}$$

Proof. The observed splitting $\mathscr{V} = д[o] + \vec{\mathscr{V}}[o]$ (see Definition 7.3.7) yields the equality $\nabla^{\natural}{}_{\mathscr{V}}\mathscr{V} = \nabla^{\natural}{}_{д[o]}д[o] + \nabla^{\natural}{}_{\vec{\mathscr{V}}[o]}д[o] + \nabla^{\natural}{}_{д[o]}\vec{\mathscr{V}}[o] + \nabla^{\natural}{}_{\vec{\mathscr{V}}[o]}\vec{\mathscr{V}}[o]$.

Moreover, the splitting $\nabla^{\natural}(д[o]) = \frac{1}{2}\big(\widehat{\Sigma}{}^{\natural}[G, o] + \widehat{\Phi}{}^{\natural}[K, o]\big)$ (see Corollary 4.2.15) yields the equality $\mathscr{A}^{\natural}{}_{\text{rlt}}[o] = \mathscr{A}^{\natural}{}_{\text{dfr}}[o] + \mathscr{A}^{\natural}{}_{\text{crl}}[o]$. □

Corollary 7.3.10 *We obtain an analogous splitting for the joined acceleration $\mathscr{A}$, by taking into account the following equalities (see Proposition 7.3.9 and Definition 5.7.3)*

$$\mathscr{A}_{\text{dfr}}[o] = \mathscr{A}^{\natural}{}_{\text{dfr}}[o], \qquad \mathscr{A}_{\text{spc}}[o] = \mathscr{A}^{\natural}{}_{\text{spc}}[o]$$

$$\mathscr{A}_{\text{crl}}[o] = \mathscr{A}^{\natural}{}_{\text{crl}}[o] - \tfrac{1}{c}\,\tfrac{q}{m}\,\vec{\mathscr{V}}[o]\times B, \qquad \mathscr{A}_{\text{drg}}[o] = \mathscr{A}^{\natural}{}_{\text{crl}}[o] - \tfrac{q}{m}\,\vec{E}[o].$$

Proof. The corollary follows from the above Propositions 5.7.4, 7.3.5 and 7.3.9. □

Remark 7.3.11 We stress that our definitions of lagrangian and eulerian accelerations turn out to express, in our rigorous geometric language, the more standard notions of "*total* derivative with respect to time" and the "*partial* derivative with respect to time" of the observed velocity, respectively,

$$\mathscr{A}[o]_{\text{lag}} \simeq \frac{d}{dt}\vec{\mathscr{V}}[o] \quad \text{and} \quad \mathscr{A}[o]\text{e} \simeq \frac{\partial}{\partial t}\vec{\mathscr{V}}[o]. \quad \square$$

7.4 Fluid Dynamics

We postulate the *continuity equations* and the *Newton law of motion* for a charged fluid effected by the gravitational and electromagnetic fields.

Moreover, we express, more synthetically, the above equations in terms of the joined spacetime connection and of the energy momentum tensor.

Furthermore, with reference to an observer o, we split the above equations into their *observed components*.

7.4.1 Absolute Fluid Dynamics

We postulate the *continuity equations* and *Newton law of motion* for a charged fluid effected by the gravitational and electromagnetic fields and express the Newton law of motion, more synthetically, in terms of the joined spacetime connection K.

Moreover, we show that the system of equations $\operatorname{div}_\upsilon \mathscr{P} = 0$ and $\mu\, \mathscr{A} = \mu\, \vec{d}\, \mathfrak{p}$ is equivalent to the equation $C_1^1 \nabla(\mathscr{T}^{\mathfrak{m}}) = \mathfrak{p}\, \vec{d} \mu$.

Thus, let us consider a charged fluid $(\mathscr{C}, \mu, \rho, \mathfrak{p})$.

Assumption C.3 We suppose the charged fluid to fulfill the *continuity equations* and the *Newton law of motion* (see Definitions 3.2.17 and 7.3.2)

$$\operatorname{div}_\upsilon \mathscr{P} = 0, \qquad \operatorname{div}_\upsilon \mathscr{J} = 0,$$
$$\mu\, \mathscr{A}^\natural = -\rho\, g^\sharp(\mathscr{V} \lrcorner F) + \mu\, \vec{d}\, \mathfrak{p},$$

where $-\rho\, g^\sharp(\mathscr{V} \lrcorner F)$ is the *Lorentz force density* and $\mu\, \vec{d}\, \mathfrak{p}$ is the *pressure force density* ("per mass unit") (see Definitions 5.7.3 and 3.2.16). □

Note 7.4.1 The continuity equations yield the *conservation of mass* and the *conservation of charge* (see Definition 7.3.1)

$$Dm = 0 \quad \text{and} \quad Dq = 0. \quad \square$$

Proposition 7.4.2 *The Newton law of motion* $\mu\, \mathscr{A}^\natural = -\rho\, g^\sharp(\mathscr{V} \lrcorner F) + \mu\, \vec{d}\, \mathfrak{p}$ *can be expressed, more synthetically, in terms of the joined spacetime connection* K *as*

$$\mu\, \mathscr{A} = \mu\, \vec{d}\, \mathfrak{p}. \quad \square$$

Lemma 7.4.3 *We have the equality*

$$\operatorname{div}_\upsilon \mathscr{P} = C_1^1 \nabla \mathscr{P}. \quad \square$$

Proposition 7.4.4 *The system of equations*

$$\operatorname{div}_\upsilon \mathscr{P} = 0 \quad \text{and} \quad \mu\, \mathscr{A} = \mu\, \vec{d}\, \mathfrak{p}$$

is equivalent to the equation

$$C_1^1 \nabla(\mathscr{T}^{\mathfrak{m}}) = \mathfrak{p}\, \vec{d} \mu.$$

Proof. We have

$$\begin{aligned} C_1^1 \nabla(\mathscr{T}^{\mathfrak{m}})_{00} &= \nabla_\lambda(\mu\, \mathscr{V}_0^\lambda\, \mathscr{V}_0^\nu)\, \partial_\nu - \nabla_i(\mu\, \mathfrak{p}_{00})\, g^{ij}\, \partial_j \\ &= \nabla_\lambda(\mu\, \mathscr{V}_0^\lambda)\, \mathscr{V}_0^\nu\, \partial_\nu + \mu\, \mathscr{V}_0^\lambda\, \nabla_\lambda \mathscr{V}_0^\nu\, \partial_\nu \\ &\quad - \mu\, \nabla_i \mathfrak{p}_{00}\, g^{ij}\, \partial_j - \mathfrak{p}_{00}\, \nabla_i \mu\, g^{ij}\, \partial_j . \end{aligned}$$

Then, by taking into account the natural linear splitting of $T\boldsymbol{E}$ into the component $T_{\mathscr{V}}\boldsymbol{E}$ generated by $\mathscr{V}$ and the vertical component $V\boldsymbol{E}$, the equation $C_1^1 \nabla(\mathscr{T}^{\mathfrak{m}}) = 0$ splits into the system

$$\nabla_\lambda(\mu\, \mathscr{V}_0^\lambda) = 0 \quad \text{and} \quad \mu\, \mathscr{V}_0^\lambda\, \nabla_\lambda \mathscr{V}_0^\nu\, \partial_\nu - \mu\, g^{ij}\, \nabla_i \mathfrak{p}_{00}\, \partial_j = \mathfrak{p}_{00}\, \nabla_i \mu\, g^{ij}\, \partial_j . \quad \square$$

Remark 7.4.5 (a) The standard way to define the pressure and express the equation of motion for a classical fluid is the following.

(1) One defines the pressure as a scaled spacetime function

$$\boldsymbol{p} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M}) \otimes \mathrm{I\!R}$$

and correspondingly writes:

(2) the equation of motion as

$$\mu\, \mathscr{A} = \vec{d}\, \boldsymbol{p},$$

(3) the contravariant energy–momentum tensor as

$$\mathscr{T}^{\mathfrak{m}} := \mu\, \mathscr{V} \otimes \mathscr{V} - \boldsymbol{p}\, \bar{g} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E}),$$

(4) a synthetic expression of the continuity equation and law of motion as

$$C_1^1 \nabla(\mathscr{T}^{\mathfrak{m}}) = 0.$$

(b) Let us consider a “barotropic fluid”. In other words, let us suppose that the scaled spacetime function μ factorises pointwisely through the scaled spacetime function $\boldsymbol{p}$, through a relation of the type

$$\mu = f \circ \boldsymbol{p} \in \operatorname{map}(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes \mathbb{M}),$$

where

$$f \in \operatorname{map}\left(\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M} \otimes \mathrm{I\!R},\ \ \mathbb{L}^{-3} \otimes \mathbb{M}\right).$$

(1) Then, in this case, we can make a change of variable, by redefining the pressure as

$$\mathrm{p} := \mathrm{s} \circ \boldsymbol{p} \in \operatorname{map}\left(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{2}) \otimes \mathrm{I\!R}\right),$$

where the scaled function

$$s : (\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M}) \otimes \mathrm{I\!R} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{2}) \otimes \mathrm{I\!R}$$

is defined by

$$s(\xi) := \int_{[1,\xi]} \frac{1}{f(\lambda)} \, d\lambda.$$

Correspondingly, one can write:

(1) the equation of motion as

$$\mu \mathscr{A} = \mu \vec{d}\, p,$$

(2) the contravariant energy–momentum tensor as

$$\mathscr{T}^{\mathrm{m}} := \mu \mathscr{V} \otimes \mathscr{V} - \mu \, p \, \bar{g} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (\boldsymbol{TE} \otimes \boldsymbol{TE}),$$

(3) a synthetic expression of the continuity equation and law of motion as

$$C^1_1 \nabla(\mathscr{T}^{\mathrm{m}}) = \mu \vec{d}\, p.$$

Indeed, in the case of "barotropic fluids", the above formulations (a) and (b) of the dynamical equation are equivalent.

Actually, formulation (a) is more natural, and holds also for non barotropic fluids.

Nevertheless, in the present Section, we have been dealing with formulation (b) because it fits better the hydrodynamical picture of Quantum Mechanics, that will be discussed later (see Theorems 18.1.1, 18.1.17, 18.2.1 and 18.2.2). □

7.4.2 Observed Fluid Dynamics

By recalling the observed splitting of the velocity, the acceleration and the Lorentz force (see Propositions 5.7.2 and 7.3.9), we show the observed splitting of the continuity equation and the Newton law of motion.

Let us consider a continuum motion $\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$ and an observer o.

Note 7.4.6 The observed splitting of the continuity equation $\mathrm{div}_\upsilon \mathscr{P} = 0$ is

$$\begin{aligned} \mathrm{div}_\upsilon \mathscr{P} &= \mathrm{div}_\upsilon(\mu \, д[o]) + \mathrm{div}_\upsilon(\mu \vec{\mathscr{V}}[o]) \\ &= д[o].\mu + \vec{\mathscr{V}}[o].\mu + \mu \, (\mathrm{div}_\upsilon \, д[o] + \mathrm{div} \, \vec{\mathscr{V}}[o]), \end{aligned}$$

with coordinate expression

$$\begin{aligned}\operatorname{div}_\upsilon \mathscr{P} &= \left(\partial_0\mu + \partial_j\mu\, \mathscr{V}_0^j + \mu\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \frac{\partial_j(\mathscr{V}_0^j\,\sqrt{|g|})}{\sqrt{|g|}})\right) u^0 \\ &= \left(\partial_0\mu + \partial_j\mu\, \mathscr{V}_0^j + \mu\,(\partial_j \mathscr{V}_0^j - K_0{}^h{}_h - K_j{}^h{}_h\, \mathscr{V}_0^j)\right) u^0. \quad \square\end{aligned}$$

Note 7.4.7 The observed splitting of the Newton law

$$\mu\, \mathscr{A}^\natural = -\rho\, g^\sharp(\mathscr{V} \lrcorner F) + \mu\, \vec{d}\mathfrak{p},$$

in *lagrangian form* and *eulerian form*, is respectively

$$\begin{aligned}&\mu\,(\mathscr{A}^\natural{}_{\text{lag}}[o] + \mathscr{A}^\natural{}_{\text{dfr}}[o] + \mathscr{A}^\natural{}_{\text{crl}}[o] + \mathscr{A}^\natural{}_{\text{drg}}[o]) \\ &\quad = \mu\,\vec{d}\,\mathfrak{p} + \rho\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[o]\times\vec{B}), \\ &\mu\,(\mathscr{A}^\natural{}_{\text{e}}[o] + \mathscr{A}^\natural{}_{\text{spc}}[o] + \mathscr{A}^\natural{}_{\text{dfr}}[o] + \mathscr{A}^\natural{}_{\text{crl}}[o] + \mathscr{A}^\natural{}_{\text{drg}}[o]) \\ &\quad = \mu\,\vec{d}\,\mathfrak{p} + \rho\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[o]\times\vec{B}).\end{aligned}$$

Moreover, we have the coordinate expression

$$\begin{aligned}&\mu\,(\partial_0\mathscr{V}_0^i + \mathscr{V}_0^h\,\partial_h\mathscr{V}_0^i - K^\natural{}_h{}^i{}_k\,\mathscr{V}_0^h\,\mathscr{V}_0^k - 2\,K^\natural{}_h{}^i{}_0\,\mathscr{V}_0^h - K^\natural{}_0{}^i{}_0) \\ &\quad = \mu\,\partial_j\mathfrak{p}_{00}\,g^{ij} + \rho_0\,(E_0{}^i + \sqrt{|g|}\,g^{ij}\,\epsilon_{jhk}\,\mathscr{V}_0^h\,B^k). \quad \square\end{aligned}$$

Chapter 8
Sources of Gravitational and Electromagnetic Fields

In the present book, we mainly deal with a given gravitational field $K^\natural$ and a given electromagnetic field F (Sects. 4 and 5).

Nevertheless, for the sake of completeness, the reader might be interested in the following question: how would look like in our galilean framework consistent equations linking the gravitational and electromagnetic fields with their sources? We answer this question by exhibiting the *Galilei–Maxwell equation* and the *Galilei–Einstein equation*, (Sects. 8.1.1 and 8.2.1), which provide a reduced version in our galilean framework of the corresponding true equations (see, also [270]).

We conclude this chapter by showing that the Galilei–Einstein equation and the Galilei–Maxwell equation can be written together, in a compact way, as a "*joined Galilei–Einstein equation*" in terms of the "joined spacetime connection" K defined through the gravitational coupling constant $\sqrt{\mathrm{r}}$ (Sect. 8.3).

8.1 Galilean Version of 2nd Maxwell Equation

In the present approach to Covariant Quantum Mechanics, we mainly deal with a *given* electromagnetic field. Accordingly, in the present context, our interest in the source of the electromagnetic field is not primary. Nevertheless, just for the sake of completeness, we sketch a galilean version of the 2nd Maxwell equation, which is consistent with our covariant galilean framework (see, also, the paper [270], which contains a broad discussion of the galilean version of electromagnetism).

Let us have in mind, as pattern equation, the true *2nd Maxwell equation* $\delta F = g^\flat(\mathscr{J})$, where δ is the lorentzian codifferential, $\mathscr{J}$ is the spacetime vector field representing the charge current, taken as the source of the electromagnetic field F, and g is the lorentzian metric. Then, we look for the galilean version of this equation by a heuristic procedure.

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In the galilean framework, we can define the spacetime *velocity* $\mathscr{V} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ and the *charge density* of a charged fluid $\rho : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}$ (see Definition 7.3.1). Then, we can define the charge current density scaled vector field $\mathscr{J} := \rho\, \mathscr{V} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T\boldsymbol{E}$ (see Definition 7.3.2) and we can postulate the continuity equation $\operatorname{div} \mathscr{J} = 0$, which ensures the charge conservation.

But, the only reasonable scaled 1-form that we can derive in a covariant way from this scaled vector field is the timelike form $\underline{\mathscr{J}} := \mathbf{g}^\flat(\mathscr{J}) = \rho\, dt : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$, which, unfortunately, does not account for the movement of charges (see Definition 3.1.1).

Moreover, in the galilean framework, we cannot define in a covariant way the codifferential δF because of the signature of the galilean metric g. We can define the divergence of the electromagnetic field as $\operatorname{div}^\natural F := \bar{g} \lrcorner \nabla^\natural F : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$ (see Definition 4.4.5 and Theorem 5.9.6). Indeed, this galilean divergence loses some terms with respect to its lorentzian pattern because $\bar{g}$ is spacelike.

Then, we can write the "reduced galilean version" of the 2nd Maxwell equation (which will be conventionally called *Galilei–Maxwell equation*) $\operatorname{div}^\natural F = \rho\, dt$. With reference to an observer o, this equations splits into the system of equations $\operatorname{div}_\eta \vec{E}[o] - \frac{1}{c}\, g(\operatorname{curl} \text{д}[o],\ \vec{B}) = \rho$ and $\operatorname{curl} \vec{B} = 0$.

Thus, we see that this system is unable to account for the effect of the movement of charges. Moreover, we cannot derive the continuity equation $\delta \mathscr{J} = 0$ from the Galilei–Maxwell equation, as it can be done for the true 2nd Maxwell equation.

We stress that, in a flat newtonian spacetime and with reference to an inertial observer (Sect. 24.1), the Galilei–Maxwell equation turns out to be the standard 2nd Maxwell equation for the electrostatic case.

8.1.1 Galilei–Maxwell Equation

We postulate, as a reduced galilean version of the true 2nd Maxwell equation, suitable for our covariant galilean framework, the *Galilei–Maxwell equation* $\operatorname{div}^\natural F = \rho\, dt$.

Unfortunately, the above Galilei–Maxwell equation loses the electromagnetic effect of the movement of the charges, which is accounted by the true 2nd Maxwell equation; indeed this is an unavoidable feeble feature of the galilean scheme.

Definition 8.1.1 Given a charged fluid with *charge density* and *velocity* (see Definitions 7.3.1 and 7.3.2)

$$\rho \in \operatorname{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big) \quad \text{and} \quad \mathscr{V} \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}),$$

we define the *contravariant electric density current* to be the scaled vector field

$$\mathscr{J} := \rho\, \mathscr{V} \in \big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T\boldsymbol{E}\big).$$

Then, by taking into account the fact that the galilean metric g is spacelike, the only reasonable way to define the *covariant electric density current* is to avail of the scaled spacetime metric $\mathbf{g} := dt \otimes dt \in \sec\big(\boldsymbol{E}, \mathbb{T}^2 \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E})\big)$ and set (see Definition 3.1.1)

$$\underline{\mathscr{J}} := \mathbf{g}^{\flat}(\mathscr{J}) = \rho\, dt \in \sec\big(\boldsymbol{E},\ \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes H^*\boldsymbol{E}\big)\,,$$

with coordinate expression

$$\underline{\mathscr{J}} = \rho_0\, d^0. \quad \square$$

Postulate C.5 *We postulate that the electromagnetic field F, whose "source" is the charge density ρ, fulfills the "*Galilei–Maxwell equation*" (see Theorem 5.9.6)*

$$\operatorname{div}^{\natural} F = \underline{\mathscr{J}}^{\,\text{ꟼ}} \equiv \rho\, dt. \quad \square$$

Proposition 8.1.2 *With reference to an observer o, the Galilei–Maxwell equation* $\operatorname{div}^{\natural} F = \rho\, dt$ *splits into the system of equations*

$$\operatorname{div}_\eta \vec{E}[o] - \tfrac{1}{c}\, g(\operatorname{curl}^{\natural} \text{д}[o],\ \vec{B}) = \rho \quad \textit{and} \quad \operatorname{curl} \vec{B} = 0,$$

with coordinate expression

$$\frac{\partial_h (E_0{}^h \sqrt{|g|})}{\sqrt{|g|}} - \frac{1}{c}\, \frac{1}{\sqrt{|g|}}\, g_{hj}\, g_{sr}\, \epsilon^{isj}\, K^{\natural}{}_i{}^r{}_0\, B^h = \rho_0 \quad \textit{and} \quad \epsilon^{hki}\, \partial_h B_k = 0.$$

In the particular case when $\operatorname{curl}^{\natural} \text{д}[o] = 0$*, the Galilei–Maxwell equation splits into the system of more standard equations*

$$\operatorname{div}_\eta \vec{E}[o] = \rho \quad \textit{and} \quad \operatorname{curl} \vec{B} = 0,$$

with coordinate expression

$$\frac{\partial_h (E_0{}^h \sqrt{|g|})}{\sqrt{|g|}} = \rho_0 \quad \textit{and} \quad \epsilon^{hki}\, \partial_h B_k = 0. \quad \square$$

Remark 8.1.3 The true 2nd Maxwell's equation in a lorentzian framework implies the continuity equation for the charge current. Unfortunately, the Galilei–Maxwell equation in a galilean framework has not this fine feature. In our galilean framework, the continuity equation for the charge current $\mathscr{J} = \rho\, \mathscr{V}$ implies the equation

$$L_{\mathscr{V}}\ \operatorname{div}^{\natural} F = \mathscr{V}.\rho\, dt. \quad \square$$

8.2 Galilean Version of Einstein Equation

In the present approach to Covariant Quantum Mechanics, we mainly deal with a *given* gravitational field. Accordingly, in the present context, our interest in the source of the gravitational field is not primary. Nevertheless, just for the sake of completeness, we sketch a galilean version of the Einstein equation, which is consistent with our covariant galilean framework.

Let us have in mind, as pattern equation, the true *Einstein equation* (see, for instance [308])

$$r^{\natural} - \tfrac{1}{2}\, C^{\natural}\, g = \mathrm{r}\, \underline{\mathscr{T}} \equiv \mathrm{r}\, \tfrac{1}{c^4}\, (g^{\flat} \otimes g^{\flat})(\mathscr{T}),$$

where

- $g : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ is the *lorentzian metric*,
- $r^{\natural} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$ is the *Ricci tensor* of the Levi–Civita spacetime connection $K^{\natural}$ associated with g,
- $C^{\natural} := \bar{g} \lrcorner r^{\natural} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes \mathrm{I\!R}$ is the *scalar curvature*,
- $\mathrm{r} \in \mathbb{T}^{-2} \otimes \mathbb{L}^{3} \otimes \mathbb{M}^{-1}$ is the constant *gravitational coupling scale*,
- $\mathscr{T} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T\boldsymbol{E} \otimes T\boldsymbol{E})$ is the *contravariant energy–momentum tensor* associated with the source fields,
- $\underline{\mathscr{T}} : \boldsymbol{E} \to (\mathbb{T}^{2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ is the *covariant energy–momentum tensor* associated with the source fields.

The energy–momentum tensor $\mathscr{T}$ is provided by the constitutive laws of the source fields, hence is supposed to be known a priori.

The left hand side of the Einstein equation fulfills the identity $\operatorname{div}(r^{\natural} - \frac{1}{2}\, C^{\natural}\, g) = 0$. Hence, for consistency reasons, also the covariant energy momentum tensor $\underline{\mathscr{T}}$ should fulfill the identity $\operatorname{div} \underline{\mathscr{T}} = 0$.

Now, we focus our attention on a fluid $(\mathscr{C}, \mu, \mathrm{p})$ and the electromagnetic field F, as source of the gravitational field $K^{\natural}$ (see Definitions 5.1.1 and 7.3.1).

The electromagnetic field F is supposed to be a source of the gravitational field according to a general criterion of General Relativity; however, its effect is very small and nowadays it seems to be beyond a possible experimental measurement. For this reason, in order to argue the appropriate normalisation factor of the energy–momentum tensor $\mathscr{T}^{\mathfrak{e}}$ of the electromagnetic field, we could avail of a lagrangian procedure. Namely, we consider a coupled lagrangian of the gravitational field, the electromagnetic field and the charged fluid, whose terms are normalised in such a way to reproduce the Einstein equation, the 2nd Maxwell equation and the law of motion of the charged fluid. In this way, we obtain, as by product, a distinguished normalisation of the energy–momentum tensor $\mathscr{T}^{\mathfrak{e}}$ (see, for instance [308]). So doing, we obtain the Einstein equation

$$r^{\natural} - \tfrac{1}{2}\, C^{\natural}\, g = \mathrm{r}\, \underline{\mathscr{T}},$$

where the energy momentum tensor $\mathscr{T} = \mathscr{T}^{\mathfrak{m}} + \mathscr{T}^{\mathfrak{e}}$ is the sum of the momentum–energy tensor of the fluid $\mathscr{T}^{\mathfrak{m}} = \mu\, \mathscr{V} \otimes \mathscr{V} - \mathrm{p}\, \bar{g}$ and the energy–momentum tensor of the electromagnetic field with coordinate expressions

$$\mathscr{T}^{\mathfrak{m}\alpha\beta} = \mu\, \mathscr{V}^{\alpha}\, \mathscr{V}^{\beta} - \mathrm{p}\, g^{\alpha\beta} \quad \text{and} \quad \mathscr{T}^{\mathfrak{e}\alpha\beta} = c^2 \left(\tfrac{1}{4}\, F^{\lambda\mu}\, F_{\lambda\mu}\, g^{\alpha\beta} - F^{\alpha\mu}\, F^{\beta}{}_{\mu}\right).$$

The covariant versions of the above tensors, to be coupled with the gravitational field, obtained via the isomorphism $\frac{1}{c^2}\, g^{\flat}$ have the coordinate expressions

$$\underline{\mathscr{T}}^{\mathfrak{m}}{}_{\alpha\beta} = \tfrac{1}{c^4}\left(\mu\, \nu_{\alpha}\, \nu_{\beta} - \mathrm{p}\, g_{\alpha\beta}\right) \quad \text{and} \quad \underline{\mathscr{T}}^{\mathfrak{e}}{}_{\alpha\beta} = \tfrac{1}{c^2}\left(\tfrac{1}{4}\, F^{\lambda\mu}\, F_{\lambda\mu}\, g_{\alpha\beta} - F_{\alpha\mu}\, F_{\beta}{}^{\mu}\right).$$

Indeed, the tensor $\underline{\mathscr{T}} := \underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{e}}$ turns out to be divergence free in virtue of the equation of motion of the fluid, the continuity equation of the fluid and the 1st Maxwell equation.

Now, we look for a suitable galilean version of the Einstein equation, by means of heuristic reasonings, having the true Einstein equation in einsteinian General Relativity as touchstone.

Again, by focusing our attention on a fluid $(\mathscr{C}, \mu, \mathrm{p})$ and the electromagnetic field F, as sources of the gravitational field $K^{\natural}$, let us examine what happens in the galilean framework.

First of all, we observe that the Ricci tensor $r^{\natural}$ is derived from the gravitational connection $K^{\natural}$ exactly as in the lorentzian framework.

A minor problem is that the scalar curvature $C^{\natural} := \bar{g} \lrcorner\, r^{\natural}$ is fully determined by the galilean metric g, hence it accounts only for the spacelike restriction $\check{K}^{\natural} = \varkappa[g]$ of the gravitational connection $K^{\natural}$ (see Proposition 4.1.3).

A major problem is due to the fact that the sum $r^{\natural} - \frac{1}{2}\, C^{\natural}\, g$ of a spacetime tensor with a spacelike tensor has no meaning in the galilean framework. Then, one might be tempted to consider the well defined spacelike restriction of the above sum $\check{r}^{\natural} - \frac{1}{2}\, C^{\natural}\, g$. But, so doing, he would involve only the spacelike restriction $\check{K}^{\natural}$ of the gravitational connection $K^{\natural}$ and, indeed, this is not sufficient for our purposes. Even more, we do not have a reasonable spacelike energy–momentum tensor of our source fields. Therefore, we are led to discard such a possible choice.

We have already seen that, in the galilean framework, we have two distinguished metric tensors, one spacelike and the other one timelike (see Postulate C.2 and Definition 3.1.1),

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}) \quad \text{and} \quad \mathfrak{g} := c^2\, dt \otimes dt : \boldsymbol{E} \to \mathbb{L}^2 \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}).$$

Actually, the following sum is meaningful

$$r^{\natural} - \tfrac{1}{2}\, C^{\natural}\, \mathfrak{g} := r^{\natural} - \tfrac{1}{2}\, c^2\, C^{\natural}\, dt \otimes dt : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}.$$

We recall that, in the einsteinian framework, the normalising factor $-\frac{1}{2}$ of the term $-\frac{1}{2}\, C^{\natural}\, g$ is determined by the requirement that the left hand side of the Einstein equation $r^{\natural} - \frac{1}{2}\, C^{\natural}\, g = \mathrm{r}\, \mathscr{T}^{\natural}$ be divergence free and this is true, in virtue of the

Einstein identity (see [308]). Unfortunately, in the galilean framework, the Einstein identity does not hold in the needed way. So, we choose the above normalisation just by arguing it from the true Einstein equation. Eventually, we shall soon see that such a normalisation turns out to be not relevant in our context.

In the galilean framework, we can define the energy momentum tensor $\mathscr{T}^{\mathfrak{m}}$ rather analogously to the einsteinian General relativity

$$\mathscr{T}^{\mathfrak{m}} := \mu \, \mathscr{V} \otimes \mathscr{V} - \mathrm{p} \, \bar{g}.$$

Indeed, $\mathscr{V} \otimes \mathscr{V}$ is a spacetime tensor and $\bar{g}$ is a spacelike tensor; but, in this case the above sum is well defined because we are adding contravariant tensors.

In order to derive the covariant energy momentum tensor $\underline{\mathscr{T}}^{\mathfrak{m}}$ to be coupled with the left hand side of the reduced Einstein equation, we cannot use the map $g^{\flat}$. So, again, we are led to use the spacetime metric tensor $\mathbf{g} := dt \otimes dt$ (see Definition 3.1.1). Then, we obtain the covariant tensor

$$\underline{\mathscr{T}}^{\mathfrak{m}} := \mu \, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^* \boldsymbol{E} \otimes H^* \boldsymbol{E}).$$

We stress that this tensor is timelike and it does not account neither for the movement of the fluid, nor for its pressure. Again, this is a typical feeble feature of the galilean scheme. Indeed, we have found an analogous result concerning the covariant charge current density (see Definition 8.1.1).

Further, the galilean analogous of the electromagnetic energy–momentum tensor of the true einsteinian theory might be the spacelike tensor with coordinate expression

$$\mathscr{T}^{\mathfrak{e}ij} = \tfrac{1}{4} \, \tfrac{1}{c^2} \, F^{hk} \, F_{hk} \, g^{ij} - \tfrac{1}{c^2} \, F^{ih} \, F^{j}{}_{h}.$$

But, this tensor, being spacelike, is not suitable for our purposes, So, we are led to consider directly the covariant electromagnetic *energy tensor* (see Definition 5.6.3)

$$\underline{\mathscr{T}}^{\mathfrak{e}} := \tfrac{1}{4} \, \check{F}^2 \, \mathbf{g} = \tfrac{1}{c^2} \, g(\vec{B}, \vec{B}) \, dt \otimes dt.$$

Again, we stress that this tensor involves only the magnetic field and not the electric field; once more, this is a feeble feature of the galilean scheme.

In the present book, we do not discuss a lagrangian approach to the reduced Einstein equation; so, in order to determine the appropriate normalisation factor of the above electromagnetic energy tensor, we shall use an argument that arises later in the context of the "*joined Galilei–Einstein equation*" (see Theorem 8.3.4).

Therefore, after these preliminary heuristic arguments, we are led to postulate for the galilean framework the following reduced Einstein equation, which will be conventionally called *Galilei–Einstein equation*,

$$r^{\natural} - \tfrac{1}{2} \, C^{\natural} \, c^2 \, dt \otimes dt = \mathrm{r} \, (\underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{e}}),$$

where $\mathrm{r} \in \mathbb{T}^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^{-1}$ is the constant *gravitational coupling scale* (see Sect. 1.3.5) and

$$\underline{\mathscr{T}}^{\mathfrak{m}} := \mu\, dt \otimes dt \quad \text{and} \quad \underline{\mathscr{T}}^{\mathfrak{e}} := \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\, dt \otimes dt .$$

By taking into account the fact that the right hand side is timelike, the above equation implies $\check{r}^{\natural} = 0$, hence $C^{\natural} = 0$.

So, the above equation turns out to be equivalent to the equation

$$r^{\natural} = \mathrm{r}\,(\underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{e}}).$$

Thus, summing up, the above equation, which is meaningful in a galilean framework, resembles the true Einstein equation, but loses, with respect to its touchstone, the gravitational effect of the movement of masses, of the electric field and other beautiful formal features of the Einstein equation. Indeed, this is an unavoidable price that we have to pay for the consistency of galilean Classical Mechanics and its fitting with Quantum Mechanics.

Nevertheless, we shall see that the above Galilei–Einstein equation yields the standard Newton law of gravitation in a rigorous mathematical way (see Sect. 28.1).

8.2.1 Galilei–Einstein Equation

We postulate the Galilei–Einstein equation $r^{\natural} - \frac{1}{2}\, C^{\natural}\, \mathfrak{g} = \mathrm{r}\,(\mu + \frac{1}{4}\, \check{F}^2)\, dt \otimes dt$ and show that it is equivalent to the equation $r^{\natural} = \mathrm{r}\,(\mu + \frac{1}{4}\, \check{F}^2)\, dt \otimes dt$.

Postulate C.6 *We postulate that the source of the gravitational field $K^{\natural}$ be a fluid $(\mathscr{C}, \mu)$ and an electromagnetic field F, according to the* Galilei-Einstein *equation (see Note 3.1.2)*

$$r^{\natural} - \tfrac{1}{2}\, C^{\natural}\, \mathfrak{g} = \mathrm{r}\,(\underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{e}}),$$

where

$$\underline{\mathscr{T}}^{\mathfrak{m}} := \mu\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}),$$
$$\underline{\mathscr{T}}^{\mathfrak{e}} := \tfrac{1}{4}\, \check{F}^2\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E})$$

are the matter energy density tensor *and the* electromagnetic energy density tensor *(see Definitions 5.6.3 and 7.3.3).* □

Proposition 8.2.1 *The above Galilei–Einstein equation*

(1) $$r^{\natural} - \tfrac{1}{2}\, C^{\natural}\, \mathfrak{g} = \mathrm{r}\,(\mu + \tfrac{1}{4}\, \check{F}^2)\, dt \otimes dt,$$

i.e., in coordinates,

$$r^{\natural}{}_{\lambda\mu}\, d^{\lambda} \otimes d^{\mu} - \tfrac{1}{2}\,(c_0)^2\, C^{\natural}\, d^0 \otimes d^0 = \mathrm{r}_{00}\,(\mu + \tfrac{1}{4}\,\check{F}^2)\, d^0 \otimes d^0,$$

is equivalent to the equation

(2) $$r^{\natural} = \mathrm{r}\,(\mu + \tfrac{1}{4}\,\check{F}^2)\, dt \otimes dt,$$

i.e., in coordinates,

$$r^{\natural}{}_{\lambda\mu}\, d^{\lambda} \otimes d^{\mu} = \mathrm{r}_{00}\,(\mu + \tfrac{1}{4}\,\check{F}^2)\, d^0 \otimes d^0.$$

Proof. (1) implies $\check{r}^{\natural} = 0$, hence $C^{\natural} := \bar{g} \lrcorner \check{r}^{\natural} = 0$. Therefore, (1) implies (2).
(2) implies $\check{r}^{\natural} = 0$, hence $C^{\natural} := \bar{g} \lrcorner \check{r}^{\natural} = 0$. Therefore, (2) implies (1). □

8.3 Joined Galilei–Einstein Equation

We have already shown (see Proposition 7.2.2) that the Newton law of motion for a charged particle effected by the gravitational and electromagnetic field can be written, in a compact way, as the "joined Newton law of motion" $m\, \nabla_{ds} ds = 0$. Moreover, we have already shown an analogous "joined Newton law of motion" for a charged fluid (see Proposition 7.4.2) $\mu\, \nabla_{\mathcal{V}} \mathcal{V} = \mu\, \vec{d}\, \mathrm{p}$.

Now, we show that the Galilei-Einstein equation and the Galilei–Maxwell equation can be written together, in a compact way, as a "joined Galilei–Einstein equation".

8.3.1 The Joined Galilei–Einstein Equation

We have seen that the Ricci tensor r of the joined spacetime connection splits into gravitational and electromagnetic components as (see Theorem 6.5.2)

$$r[K] = r[K^{\natural}] + \tfrac{1}{2}\,\mathrm{k}\, dt \otimes \operatorname{div}{}^{\natural} F + \tfrac{1}{2}\,\mathrm{k}\, \operatorname{div}{}^{\natural} F \otimes dt - \tfrac{1}{4}\,\mathrm{k}^2\, \check{F}^2\, dt \otimes dt.$$

Then, it is natural to investigate the analogue of the Galilei–Einstein equation obtained by replacing the gravitational connection $K^{\natural}$ with the joined connection K. Indeed, it is remarkable the fact that the gravitational Galilei–Einstein equation and the Galilei–Maxwell equation $r^{\natural} = \mathrm{r}\,(\mu + \frac{1}{4}\,\check{F}^2)\, dt \otimes dt$ and $\operatorname{div}{}^{\natural} F = \rho\, dt$ imply the joined Galilei–Einstein equation $r - \frac{1}{2}\, C\, \mathfrak{g} = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\, dt \otimes dt$, where $\mathcal{T} := (\mu + \mathrm{r}^{-1/2}\,\rho)\, dt \otimes dt$ plays the role of "covariant energy density tensor" of the charged fluid. Here, the electromagnetic term does not appear on the right hand side, as it is automatically included in the left hand side via the definition of the joined spacetime connection.

Unfortunately, the inverse implication is not true. But, it would be true if we consider the above joined equation valid for any value of the gravitational coupling scale r.

Let us consider an electromagnetic field $F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^* \boldsymbol{E}$, and a charged fluid with mass density $\mu \in \mathbb{L}^{-3} \otimes \mathbb{M}$ and charge density $\rho \in \mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R}$.

We recall the covariant *matter energy density tensor* and the covariant *electromagnetic energy density tensor* (see Definitions 5.6.3 and 7.3.3)

$$\underline{\mathscr{T}}^{\mathfrak{m}} := \mu \otimes dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^* \boldsymbol{E} \otimes H^* \boldsymbol{E}),$$
$$\underline{\mathscr{T}}^{\mathfrak{e}} := \tfrac{1}{4} \check{F}^2 \, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^* \boldsymbol{E} \otimes H^* \boldsymbol{E}),$$

where $\underline{\mathscr{T}}^{\mathfrak{e}} = \frac{1}{c^2} \, g(\vec{B}, \vec{B}) \, dt \otimes dt$.

Let us consider the joining gravitational constant scale $\mathrm{k} = \sqrt{\mathrm{r}} \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}$.

Definition 8.3.1 We define the covariant *charge energy density tensor*

$$\underline{\mathscr{T}}^{\mathfrak{q}} := \mathrm{k}^{-1} \, \rho \, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^* \boldsymbol{E} \otimes H^* \boldsymbol{E}).$$

Moreover, we define the *charged fluid energy density tensor* to be the scaled section

$$\underline{\mathscr{T}} := \underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{q}} = (\mu + \mathrm{k}^{-1} \, \rho) \, dt \otimes dt. \quad \square$$

Then, by imitating the gravitational Einstein equation, we consider the following equation.

Definition 8.3.2 We define the *joined Galilei–Einstein equation* to be the equation

$$r - \tfrac{1}{2} \, C \, \mathfrak{g} = \mathrm{r} \, \underline{\mathscr{T}} := \mathrm{r} \, (\mu + \mathrm{r}^{-1/2} \, \rho) \, dt \otimes dt,$$

where $\mathfrak{g} := c^2 \, dt \otimes dt$ is the timelike metric tensor (see Definition 3.1.1), r is the Ricci tensor of the joined spacetime connection $K = K^{\natural} + K^{\mathfrak{e}}$ (see Theorem 6.5.2) and C is the associated scalar curvature (see Corollary 6.5.4). $\square$

Proposition 8.3.3 *The above joined Galilei–Einstein equation*

(1) $$r - \tfrac{1}{2} \, C \, \mathfrak{g} = \mathrm{r} \, (\mu + \mathrm{r}^{-1/2} \, \rho) \, dt \otimes dt$$

is equivalent to the equation

(2) $$r = \mathrm{r} \, (\mu + \mathrm{r}^{-1/2} \, \rho) \, dt \otimes dt.$$

Proof. The proof is analogous to that of Proposition 8.2.1. $\square$

So, let us consider the joined Galilei-Einstein equation in the form

$$r = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\,dt \otimes dt.$$

Theorem 8.3.4 *Let us suppose that*

(1) *the gravitational field $K^\natural$ fulfills the gravitational Galilei–Einstein equation (see Postulate C.6)*

$$\text{(E)} \qquad r^\natural = \mathrm{r}\,(\mu + \tfrac{1}{4}\,\check{F}^2)\,dt \otimes dt,$$

whose source fields are the classical fluid and the electromagnetic field,

(2) *the electromagnetic field F fulfills the Galilei–Maxwell equation (see Postulate C.5)*

$$\text{(M)} \qquad \operatorname{div}^\natural F = \rho\,dt,$$

whose source is the charged fluid.

Then, the joined spacetime connection $K := K^\natural - \frac{1}{2}\,\mathrm{r}^{1/2}\,(dt \otimes \hat{F} + \hat{F} \otimes dt)$s fulfills the joined–Galilei–Einstein equation

$$r = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\,dt \otimes dt,$$

where r is the Ricci tensor of the joined spacetime connection.

Proof. In virtue of Theorem 6.5.2, the joined Ricci tensor splits as

$$r[K] = r[K^\natural] + \tfrac{1}{2}\,\mathrm{k}\,dt \otimes \operatorname{div}^\natural F + \tfrac{1}{2}\,\mathrm{k}\,\operatorname{div}^\natural F \otimes dt - \tfrac{1}{4}\,\mathrm{k}^2\,\check{F}^2\,dt \otimes dt.$$

Hence, by taking into account equation (E) and equation (M), we obtain the equality

$$\begin{aligned} r &= r^\natural - \tfrac{1}{4}\,\mathrm{r}\,\check{F}^2\,dt \otimes dt + \tfrac{1}{2}\,\mathrm{r}^{1/2}\,dt \otimes \operatorname{div}^\natural F + \tfrac{1}{2}\,\mathrm{r}^{1/2}\,\operatorname{div}^\natural F \otimes dt \\ &= \mathrm{r}\,\mu\,dt \otimes dt + \tfrac{1}{2}\,\mathrm{r}^{1/2}\,\rho\,dt \otimes dt + \tfrac{1}{2}\,\mathrm{r}^{1/2}\,\rho\,dt \otimes dt \\ &= \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\,dt \otimes dt. \quad \square \end{aligned}$$

Remark 8.3.5 The joined Galilei–Einstein equation

$$r = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\,dt \otimes dt$$

does not imply the system of the gravitational Galilei–Einstein equation and the Galilei–Maxwell equation

$$r^\natural = \mathrm{r}\,(\mu + \tfrac{1}{4}\,\check{F}^2)\,dt \otimes dt \quad \text{and} \quad \operatorname{div}^\natural F = \rho\,dt.$$

However, this implication would be true if we start from the joined Galilei–Einstein equation for a joined spacetime connection

$$K := K^{\natural} - \frac{1}{2}\,\Gamma^{1/2}\,(dt \otimes \hat{F} + \hat{F} \otimes dt)$$

associated with any value of the gravitational coupling scale Γ. □

Chapter 9
Fundamental Fields of Phase Space

So far, in the above chapters, we have been dealing with spacetime, along with its fundamental fields, i.e. the *galilean metric field*, the *galilean gravitational field* and the *galilean electromagnetic field* (Sects. 2, 3, 4 and 5). Moreover, we have postulated the *Newton law of motion* in terms of the joined spacetime connection K on the tangent space $\boldsymbol{TE}$ (see Proposition 7.2.2).

Now, we revisit Classical Mechanics in the framework of the classical phase space $J_1\boldsymbol{E}$.

For this purpose, we introduce the *phase connection* Γ, the *2nd order phase connection* γ, the *cosymplectic phase 2-form* Ω and the *coPoisson phase 2-vector* Λ, along with their properties and relations (Sect. 9.1).

Next, we show that the joined spacetime connection K yields in a covariant way the above phase fields along with their splitting into the gravitational and electromagnetic components (Sect. 9.2). These objects will be largely used in Classical and Quantum Mechanics.

9.1 Fundamental Fields of Phase Space

We define the fundamental fields Γ, γ, Ω, Λ of phase space and discuss their preliminary properties and mutual relations.

9.1.1 The Fundamental Fields of Phase Space

We start by introducing the following fundamental fields of phase space, which will play a relevant role throughout the book, both for Classical Mechanics and Quantum Mechanics:

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*,
Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1_9

the *phase connections*	$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$,
the *dynamical phase connections*	$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$,
the *dynamical phase* 2*-forms*	$\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^*J_1\boldsymbol{E}$,
the *dynamical phase* 2*-vectors*	$\Lambda : J_1\boldsymbol{E} \to \Lambda^2 V_{\boldsymbol{T}}J_1\boldsymbol{E}$.

Definition 9.1.1 A *phase connection* is defined to be a connection of the affine phase bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ (see Proposition 2.5.1 and Appendix: Definition F.1.1)

$$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}.$$

Its coordinate expression is of the type

$$\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i{}_0\, \partial^0_i), \quad \text{with } \Gamma_\lambda{}^i{}_0 \in \text{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

A phase connection Γ is said to be *affine* if it is an affine fibred morphism over $\mathbf{1}_{\boldsymbol{E}} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, i.e., in coordinates, if it is of the type (see Appendix: Definition F.3.1)

$$\Gamma = d^\lambda \otimes \big(\partial_\lambda + (\Gamma_\lambda{}^i{}_0{}^0{}_0 + \Gamma_\lambda{}^i{}_0{}^0{}_j\, x^j_0)\, \partial^0_i\big), \quad \text{with } \Gamma_\lambda{}^i{}_0{}^0{}_0,\ \Gamma_\lambda{}^i{}_0{}^0{}_j \in \text{map}(\boldsymbol{E}, \mathbb{R}).$$

We define the *torsion* of an affine phase connection Γ to be the spacetime section (see Appendix: Note F.1.21 and [311])

$$T[\Gamma] := 2\,[\Gamma, \theta] : \boldsymbol{E} \to \mathbb{T} \otimes \big(\Lambda^2 T^*\boldsymbol{E} \otimes \mathbb{T}^* \otimes V\boldsymbol{E}\big) \simeq \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E}$$

where the complementary contact map θ can be regarded in a natural way as the scaled soldering form (see Corollary 2.6.2)

$$\theta : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} \simeq \mathbb{T} \otimes (T^*\boldsymbol{E} \otimes V_{\boldsymbol{E}}J_1\boldsymbol{E}),$$

and [,] is the Frölicher–Nijenhuis bracket (see, for instance [246, 311]).

The coordinate expression of the torsion of an affine phase connection is

$$T[\Gamma] = -2\,(\Gamma_0{}^i{}_0{}^0{}_k - \Gamma_k{}^i{}_0{}^0{}_0)\, d^0 \wedge d^k \otimes \partial_i - (\Gamma_h{}^i{}_0{}^0{}_k - \Gamma_k{}^i{}_0{}^0{}_h)\, d^h \wedge d^k \otimes \partial_i.$$

Hence, an affine phase connection Γ is *torsion free* if $\Gamma_\lambda{}^i{}_0{}^0{}_\mu = \Gamma_\mu{}^i{}_0{}^0{}_\lambda$.

A phase connection Γ is said to be *special* if it is affine and torsion free.

The *vertical projection* associated with a phase connection Γ

$$\nu[\Gamma] : TJ_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$$

has coordinate expression

$$\nu[\Gamma] = u^0 \otimes (d^i_0 - \Gamma_\lambda{}^i{}_0\, d^\lambda)\, \partial_i.$$

The *vertical restriction* of the phase connection Γ (see Proposition 2.2.4)

$$\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E},$$

has coordinate expression

$$\check{\Gamma} = \check{d}^i \otimes (\partial_i + \Gamma_{i0}^{\;h}\, \partial_h^0).$$

We shall use the notation

$$\Gamma_{00}^{ij} := G_0^{ih}\, \Gamma_{h0}^{\;j} \quad \text{and} \quad \Gamma_{ij} := G_{jh}^0\, \Gamma_{i0}^{\;h}. \quad \square$$

Definition 9.1.2 We define a *dynamical phase connection* to be a connection of the fibred manifold $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ (see Proposition 2.5.1 and Definition F.1.1)

$$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E},$$

which is projectable on д according to the commutative diagram (see Proposition 2.6.1)

$$\begin{array}{ccc} J_1\boldsymbol{E} & \xrightarrow{\;\;\gamma\;\;} & \mathbb{T}^* \otimes TJ_1\boldsymbol{E} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes Tt_0^1} \\ J_1\boldsymbol{E} & \xrightarrow{\;\;\text{д}\;\;} & \mathbb{T}^* \otimes T\boldsymbol{E} \end{array} \quad .$$

Its coordinate expression is of the type

$$\gamma = u^0 \otimes (\partial_0 + x_0^i\, \partial_i + \gamma_{00}^{\;i}\, \partial_i^0), \qquad \text{with } \gamma_{00}^{\;i} \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

A dynamical phase connection γ is said to be *special* if it fulfills the coordinate equivariant condition

$$\gamma_{00}^{\;i} = \gamma_{00\,hk}^{\;i\,00}\, x_0^h\, x_0^k + 2\, \gamma_{00\,h0}^{\;i\,00}\, x_0^h + \gamma_{00\,00}^{\;i\,00}, \qquad \text{with } \gamma_{00\,hk}^{\;i\,00} = \gamma_{00\,kh}^{\;i\,00}, \quad \gamma_{00\,\lambda\mu}^{\;i\,00} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

The *vertical projection* associated with a dynamical phase connection γ

$$\nu[\gamma] : TJ_1\boldsymbol{E} \to V_{\boldsymbol{T}}J_1\boldsymbol{E}$$

has coordinate expression

$$\nu[\gamma] = (d^i - x_0^i\, d^0) \otimes \partial_i + (d_0^i - \gamma_{00}^{\;i}\, d^0) \otimes \partial_i^0.$$

For each motion s, the *covariant differential* associated with a dynamical phase connection γ turns out to be the section

$$\nabla[\gamma]j_1 s := \nu[\gamma] \circ dj_1 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

with coordinate expression

$$\nabla[\gamma]j_1 s = (\partial_{00}s^i - \gamma_0{}^i{}_0 \circ j_1 s)\,(u^0 \otimes u^0) \otimes (\partial_i \circ s). \quad \square$$

Definition 9.1.3 A *dynamical phase 2-form* is defined to be a phase 2-form associated with a phase connection Γ (see Definition 9.1.1 and Corollary 2.6.2)

$$\Omega \;\equiv\; \Omega[G, \Gamma] := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \Lambda^2\, T^* J_1\boldsymbol{E}.$$

Its coordinate expression is of the type

$$\begin{aligned}\Omega[G, \Gamma] &= G^0_{ij}\,(d^i_0 - \Gamma_\lambda{}^i{}_0\, d^\lambda) \wedge \theta^j \\ &= -(G^0_{ij}\,\Gamma_0{}^i{}_0 + G^0_{ih}\,\Gamma_j{}^i{}_0\, x^h_0)\, d^0 \wedge d^j - \tfrac{1}{2}\,(G^0_{hj}\,\Gamma_i{}^h{}_0 - G^0_{hi}\,\Gamma_j{}^h{}_0)\, d^i \wedge d^j \\ &\quad + G^0_{hj}\, x^h_0\, d^0 \wedge d^j_0 - G^0_{ij}\, d^i \wedge d^j_0.\end{aligned}$$

A dynamical phase 2-form $\Omega \;\equiv\; \Omega[G, \Gamma]$ is said to be:

- *quadratic* if it is generated by an affine phase connection Γ,
- *special* if it is generated by a special phase connection Γ. □

Definition 9.1.4 We define a *dynamical phase 2-vector* to be the $\boldsymbol{T}$-vertical phase 2-vector associated with a phase connection Γ (see Definition 9.1.1)

$$\Lambda \;\equiv\; \Lambda[G, \Gamma] := \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 V_{\boldsymbol{T}} J_1\boldsymbol{E},$$

where the tensor

$$\nu : J_1\boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} J_1\boldsymbol{E}) = \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes \mathbb{T}^* \otimes V\boldsymbol{E})$$

with coordinate expression $\nu = u_0 \otimes (\check{d}^i \otimes \partial^0_i)$, is the natural tensor induced by $\mathrm{id}_{V\boldsymbol{E}}$.

Its coordinate expression is of the type

$$\Lambda[G, \Gamma] = G^{ij}_0\,(\partial_i + \Gamma_i{}^h{}_0\, \partial^0_h) \wedge \partial^0_j.$$

A dynamical phase 2-vector $\Lambda \;\equiv\; \Lambda[G, \Gamma]$ is said to be:

- *quadratic* if it is generated by an affine phase connection Γ,
- *special* if it is generated by a special phase connection Γ. □

9.1.2 Phase Volumes

A dynamical phase 2-form Ω, a dynamical phase 2-vector Λ and a dynamical phase connection γ, along with the time form dt, naturally yield a distinguished *volume form* of phase space $dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1\boldsymbol{E}$ and *volume vector* of phase space $\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ_1\boldsymbol{E}$.

It is remarkable the fact that, eventually, these volumes depend only on the fibred structure of spacetime and the galilean metric (i.e., eventually, they do not involve Γ).

Proposition 9.1.5 *A dynamical phase 2-form $\Omega \equiv \Omega[G, \Gamma]$, along with the time form dt, yield, in a natural way, the scaled* volume form *of phase space*

$$dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1\boldsymbol{E},$$

with coordinate expression

$$dt \wedge \Omega \wedge \Omega \wedge \Omega = 3!\, \det(G^0_{ij})\, u_0 \otimes \big(d^0 \wedge d^1 \wedge d^2 \wedge d^3 \wedge d^1_0 \wedge d^2_0 \wedge d^3_0\big).$$

A dynamical phase 2-vector $\Lambda \equiv \Lambda[G, \Gamma]$ yields, in a natural way, the scaled spacelike volume vector *of phase space*

$$\Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \Lambda^6 V_{\boldsymbol{T}} J_1\boldsymbol{E},$$

with coordinate expression

$$\Lambda \wedge \Lambda \wedge \Lambda = -3!\, \det(G^{ij}_0)\, \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial^0_1 \wedge \partial^0_2 \wedge \partial^0_3.$$

A dynamical phase connection γ and a dynamical phase 2-vector $\Lambda \equiv \Lambda[G, \Gamma]$ yield, in a natural way, the scaled volume vector *of phase space*

$$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ_1\boldsymbol{E},$$

with coordinate expression

$$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda = -3!\, \det(G^{ij}_0)\, u^0 \otimes \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3 \wedge \partial^0_1 \wedge \partial^0_2 \wedge \partial^0_3.$$

Indeed, we have the equality $\langle dt \wedge \Omega \wedge \Omega \wedge \Omega \;,\; \gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda\rangle = -3!\,3!$. □

Remark 9.1.6 It is worth observing that the coordinate expressions of the scaled volume form and volume vector

$$dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1\boldsymbol{E} \quad \text{and}$$
$$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ_1\boldsymbol{E}$$

involve only the rescaled metric G.

In fact, by means of a covariant jet technique, we can prove that they are generated by the scaled spacetime volumes and forms υ, η, $\bar{\upsilon}$, $\bar{\eta}$ (see Proposition 3.2.4). □

9.1.3 Relations Between the Fundamental Fields of Phase Space

We discuss the maps which link the fundamental phase fields Γ, γ, Ω, Λ.

Indeed, we have natural bijections $\Gamma \mapsto \gamma$, $\Gamma \mapsto \Omega$, $\Omega \mapsto \gamma$ and a surjective map $\Gamma \mapsto \Lambda$.

Theorem 9.1.7 (1) *The map (see Definitions 9.1.1 and 9.1.2)*

$$\Gamma \mapsto \gamma[\Gamma] := д \lrcorner \Gamma$$

between special phase connections and special dynamical phase connections is bijective.

(2) *The map (see Definitions 9.1.1 and 9.1.3)*

$$\Gamma \mapsto \Omega[\Gamma, G] := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big)$$

between special phase connections and special dynamical phase 2-forms is bijective.

(3) *The map (see Definitions 9.1.1 and 9.1.4)*

$$\Gamma \mapsto \Lambda[G, \Gamma] := \bar{G} \lrcorner (\check{\Gamma} \wedge \nu)$$

between special phase connections and special dynamical phase 2-vectors is surjective, but not injective.

Proof. The proof can be achieved by a comparison of the polynomial coordinate expressions of the above objects (see Definitions 9.1.1, 9.1.2, 9.1.3 and 9.1.4). □

Theorem 9.1.8 *Let us consider a special dynamical phase 2-form $\Omega[G, \Gamma]$. Then, there is a unique dynamical phase connection denoted by $\gamma := \gamma\big[\Omega[G, \Gamma]\big]$, such that*

$$i_\gamma \, \Omega[G, \Gamma] = 0.$$

Indeed, we have $\gamma\big[\Omega[G, \Gamma]\big] = \gamma[\Gamma]$.

Hence, the above condition $i_\gamma \Omega[G, \Gamma] = 0$ yields a natural bijection between special dynamical phase 2-forms and special dynamical phase connections.

The coordinate expression of a special dynamical phase 2-form $\Omega[G, \Gamma]$ can be expressed, by involving $\gamma[\Gamma]$, as

$$\Omega[G, \Gamma] = G^0_{ij}\,(d^i_0 - \gamma_0{}^i_0\, d^0 - \Gamma_h{}^i_0\, \theta^h) \wedge \theta^j.$$

Proof. The coordinate expressions $\Omega = G^0_{ij}\,(d^i_0 - \Gamma_\lambda{}^i_0\, d^\lambda) \wedge \theta^j$ and $\gamma = u^0 \otimes (\partial_0 + x^i_0\,\partial_i + \gamma_0{}_0\partial^0_i)$ yield $i_\gamma \Omega = G^0_{ij}\,(\gamma_0{}^i_0 - \Gamma_0{}^i_0 - \Gamma_j{}^i_0\, x^j_0)\,\theta^j$.

Hence, we have $i_\gamma \Omega = 0$ if and only if $\gamma_0{}^i_0 = \Gamma_0{}^i_0 + \Gamma_j{}^i_0\, x^j_0$, i.e. if and only if $\gamma = д \lrcorner \Gamma$. □

9.2 Spacetime Connection and Phase Fields

In Chap. 4, we have discussed the spacetime connections and postulated the joined galilean spacetime connection K, which encodes the gravitational field $K^\natural$ and the electromagnetic field F.

Now, we show that there is a natural bijection between special spacetime connections K and special phase connections Γ (see [214, 222, 293, 393]).

We stress that in einsteinian General Relativity we can achieve an analogous map $K \mapsto \Gamma$, which, unfortunately, turns out to be non injective (see [214, 222]). Actually, in that framework, we can directly define, in a natural way, a joined phase connection and associated joined fields, but we cannot derive these objects from a joined spacetime connection. However, we can achieve a bijection $K \mapsto \Gamma$ by restricting K to the timelike cone $T^-\boldsymbol{E} \subset T\boldsymbol{E}$.

9.2.1 Spacetime Connections and the Phase Fields

We start by exhibiting the natural bijection $K \mapsto \Gamma[K]$ between special spacetime connections K and special phase connections Γ (see Definitions 4.1.19 and 9.1.1).

As a consequence, a special spacetime connection naturally yields the special dynamical phase connection, the special dynamic phase 2-form and the special dynamical phase 2-vector $\gamma[K] \equiv \gamma\big[\Gamma[K]\big]$, $\Omega[G, K] \equiv \Omega\big[G, \Gamma[K]\big]$, $\Lambda[G, K] \equiv \Lambda\big[G, \Gamma[K]\big]$.

Theorem 9.2.1 ([214]) *If $K : T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TT\boldsymbol{E}$ is a time preserving, linear spacetime connection, then there exists a unique fibred morphism over $J_1\boldsymbol{E}$*

$$\Gamma \equiv \Gamma[K] : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TJ_1\boldsymbol{E},$$

which makes the following diagram commutative (see Proposition 2.6.1)

$$\begin{array}{ccccc}
J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & & \xrightarrow{\quad\Gamma\quad} & & TJ_1\boldsymbol{E} \\
\downarrow {\scriptstyle д \times \mathrm{id}} & & & & \downarrow {\scriptstyle Tд} \\
\mathbb{T}^* \otimes (T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) & \xrightarrow{\widetilde{K}} & \mathbb{T}^* \underset{T\tau}{\otimes} TT\boldsymbol{E} & \xrightarrow{\iota} & T(\mathbb{T}^* \otimes T\boldsymbol{E})
\end{array} \quad ,$$

where $\widetilde{K}$ denotes the natural scaled extension of K.

Actually, this map Γ turns out to be an affine phase connection.

Indeed, the map induced by the above diagram

$$\chi : K \mapsto \Gamma \;\equiv\; \Gamma[K]$$

turns out to be a bijection between special spacetime connections and special phase connections. The coordinate expression of χ is given by the equality

$$\Gamma_{\lambda}{}^{i}{}_{0}{}^{0}{}_{\mu} = K_{\lambda}{}^{i}{}_{\mu}.$$

Proof. The coordinate expressions

$$(x^\lambda,\ \dot{x}^0_0, \dot{x}^i_0;\ \ \overset{\shortparallel}{x}{}^\lambda,\ \ddot{x}^0_0, \ddot{x}^i_0) \circ Tд = (x^\lambda,\ 1, x^i_0;\ \ \overset{\shortparallel}{x}{}^\lambda,\ 0, \dot{x}^i_0),$$
$$(x^\lambda, \dot{x}^0_0, \dot{x}^i_0;\ \grave{x}^\lambda, \ddot{x}^0_0, \ddot{x}^i_0) \circ \iota \circ \widetilde{K} \circ (д \times \mathrm{id}) = \big(x^\lambda,\ 1,\ x^i_0;\ \dot{x}^\lambda, 0,\ (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_j\, x^j_0)\, \dot{x}^\lambda\big)$$

imply that the image of the map $\iota \circ \widetilde{K} \circ (д \times \mathrm{id}) : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T(\mathbb{T}^* \otimes T\boldsymbol{E})$ is contained in the image of the map $Tд : TJ_1\boldsymbol{E} \to T(\mathbb{T}^* \otimes T\boldsymbol{E})$.

Moreover, the map $д : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ is injective, hence the map $Tд : TJ_1\boldsymbol{E} \to T(\mathbb{T}^* \otimes T\boldsymbol{E})$ is also injective. This fact implies that the map $\iota \circ \widetilde{K} \circ (д \times \mathrm{id}) : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T(\mathbb{T}^* \otimes T\boldsymbol{E})$ factorises uniquely through a map $\Gamma : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TJ_1\boldsymbol{E}$.

Furthermore, the above coordinate expressions imply the coordinate expression

$$(x^\lambda, x^i_0\,;\ \dot{x}^\lambda, \dot{x}^i_0) \circ \Gamma = \big(x^\lambda, x^i_0\,;\ \dot{x}^\lambda, (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_j\, x^j_0)\, \dot{x}^\lambda\big).$$

Hence, the map $\Gamma : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TJ_1\boldsymbol{E}$ turns out to be an affine phase connection, with coordinate expression given by

$$\Gamma_{\lambda}{}^{i}_{0} = \Gamma_{\lambda}{}^{i}{}_{0}{}^{0}{}_{0} + \Gamma_{\lambda}{}^{i}{}_{0}{}^{0}{}_{j}\, x^j_0 \quad \text{and} \quad \Gamma_{\lambda}{}^{i}{}_{0}{}^{0}{}_{\mu} = K_\lambda{}^i{}_\mu. \quad \square$$

Corollary 9.2.2 *If K is a special spacetime connection, then we obtain the coordinate expression*

$$\begin{aligned}\Gamma[K] &= d^\lambda \otimes \partial_\lambda \\ &\quad - G^{ij}_0 \left(\tfrac{1}{2} \, (\partial_0 G^0_{hj} + \Phi_{hj}) \, x^h_0) + \Phi_{0j}\right) d^0 \otimes \partial^0_i \\ &\quad - G^{ij}_0 \, \tfrac{1}{2} \left((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \, x^h_0)\right) d^k \otimes \partial^0_i .\end{aligned}$$

Proof. The coordinate expression of $\Gamma[K]$ follows from Theorem 4.2.13. □

Corollary 9.2.3 *For each special spacetime connection K, the curvature of the associated phase connection* $\Gamma \equiv \chi(K)$ *can be expressed through the curvature of K according to the equality*

$$R[\Gamma] = -[\Gamma, \Gamma] = R_{\lambda\mu}{}^{i\,0}_{00} \, d^\lambda \wedge d^\mu \otimes \partial^0_i + R_{\lambda\mu}{}^{i\,0}_{0j} \, x^j_0 \, d^\lambda \wedge d^\mu \otimes \partial^0_i ,$$

where

$$R_{\lambda\mu}{}^{0i}_{00} = R_{\lambda\mu}{}^i{}_0 \quad \text{and} \quad R_{\lambda\mu}{}^{0i}_{j\,0} = R_{\lambda\mu}{}^i{}_j . \square$$

Corollary 9.2.4 *Each special spacetime connection K yields, respectively, the special phase connection, dynamical affine connection, dynamical phase 2-form, dynamical phase 2-vector*

$$\Gamma \equiv \Gamma[K], \quad \gamma \equiv \gamma[K], \quad \Omega \equiv \Omega[G, K], \quad \Lambda \equiv \Lambda[G, K],$$

with coordinate expressions

$$\begin{aligned}\Gamma[K] &= d^\lambda \otimes \partial_\lambda + (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_h \, x^h_0) \, d^\lambda \otimes \partial^0_i \\ &= d^\lambda \otimes \partial_\lambda - G^{ij}_0 \left(\Phi_{0j} + \tfrac{1}{2} \, (\partial_0 G^0_{hj} + \Phi_{hj}) \, x^h_0)\right) d^0 \otimes \partial^0_i \\ &\quad - \tfrac{1}{2} \, G^{ij}_0 \left((\partial_0 G^0_{kj} + \Phi_{kj}) + (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}) \, x^h_0)\right) d^k \otimes \partial^0_i , \\ \gamma[K] &= u^0 \otimes \left(\partial_0 + x^i_0 \, \partial_i + (K_0{}^i{}_0 + 2 \, K_0{}^i{}_k \, x^k_0 + K_h{}^i{}_k \, x^h_0 \, x^k_0) \, \partial^0_i\right) \\ &= u^0 \otimes \left(\partial_0 + x^i_0 \, \partial_i\right. \\ &\quad \left. - G^{ij}_0 \left(\Phi_{0j} + (\partial_0 G^0_{hj} + \Phi_{hj}) \, x^h_0 + (\partial_h G^0_{jk} - \tfrac{1}{2} \, \partial_j G^0_{hk}) \, x^h_0 \, x^k_0\right) \partial^0_i\right), \\ \Omega[K, G] &= G^0_{ij} \left(d^i_0 - (K_\lambda{}^i{}_0 + K_\lambda{}^i{}_h \, x^h_0) \, d^\lambda\right) \wedge (d^j - x^j_0 \, d^0) \\ &= (\partial_0 G^0_{hj} \, x^h_0 + \tfrac{1}{2} \, \partial_j G^0_{hk} \, x^h_0 \, x^k_0) \, d^0 \wedge d^j + (\partial_i G^0_{jh} \, x^h_0) \, d^i \wedge d^j \\ &\quad + G^0_{hj} \, x^h_0 \, d^0 \wedge d^j_0 - G^0_{ij} \, d^i \wedge d^j_0 + \tfrac{1}{2} \, \Phi_{\lambda\mu} \, d^\lambda \wedge d^\mu , \\ \Lambda[K, G] &= G^{ij}_0 \, \partial_i \wedge \partial^0_j + \tfrac{1}{2} \left(G^{jh}_0 \, (K_h{}^i{}_0 + K_h{}^i{}_k \, x^k_0) - G^{ih}_0 \, (K_h{}^j{}_0 + K_h{}^j{}_k \, x^k_0)\right) \partial^0_i \wedge \partial^0_j \\ &= G^{ij}_0 \, \partial_i \wedge \partial^0_j + G^{ih}_0 \, G^{jk}_0 \left(\partial_h G^0_{kr} \, x^r_0 + \tfrac{1}{2} \, \Phi_{hk}\right) \partial^0_i \wedge \partial^0_j .\end{aligned}$$

Moreover, if K is a galilean spacetime connection, then the spacetime 2-form $\Phi[o]$ *is closed and we can write (see Definition 4.2.11 and Theorem 4.3.3)*

$$\Phi_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda . \quad \square$$

9.2.2 Joined Phase Objects

Let us consider the galilean gravitational spacetime connection, the electromagnetic field and the joined spacetime connection (see Postulates C.3 and C.4, and Theorem 6.3.1) Then, the phase fields associated with the galilean joined spacetime connection K (see Theorem 9.2.1 and Corollary 9.2.4) naturally split into their gravitational and electromagnetic components.

Actually, we prove the following equalities

$$\begin{aligned}
\Gamma^{\mathfrak{e}} &= -\tfrac{1}{2}\,\tfrac{q}{m}\,g^{\sharp 2}\,\big(i_{\text{д}} F + \theta \lrcorner F\big) : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V\boldsymbol{E}),\\
\gamma^{\mathfrak{e}} &= -\tfrac{q}{m}\,g^{\sharp}\,(d \lrcorner F) : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},\\
\Omega^{\mathfrak{e}} &= \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F : \boldsymbol{E} \to \Lambda^2\, T^*\boldsymbol{E},\\
\Lambda^{\mathfrak{e}} &= \tfrac{1}{2}\,\tfrac{q}{m}\,\tfrac{\hbar}{m}\,\bar{F} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \Lambda^2\, V\boldsymbol{E}.
\end{aligned}$$

In particular, we stress that:

- the electromagnetic term of $\gamma[K]$ turns out to be just the *Lorentz force* (see Definition 5.7.1),
- the electromagnetic term of $\Omega[K]$ turns out to be just the *electromagnetic field* (up to a coupling constant),
- the electromagnetic term of $\Lambda[K]$ turns out to be just the *magnetic field* (up to a coupling constant).

Unfortunately, the electromagnetic term of $\Gamma[K]$ does not have a simple intuitive meaning.

Let us recall the natural splitting $V_{\boldsymbol{E}} J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E})$ (see Propositions 2.5.1 and B.3.9).

Theorem 9.2.5 *Let us define the* gravitational phase connection *and the* joined phase connection *to be the phase connections*

$$\Gamma^{\natural} := \chi(K^{\natural}) \quad \textit{and} \quad \Gamma := \chi(K)$$

induced by the gravitational spacetime connection $K^{\natural}$ and the joined spacetime connection K, respectively, according to Theorem 9.2.1.

The joined phase connection $\Gamma := \Gamma[K]$ splits into the gravitational and electromagnetic components *as*

$$\Gamma = \Gamma^{\natural} + \Gamma^{\mathfrak{e}},$$

where

$$\Gamma^{\mathfrak{e}} := \Gamma - \Gamma^{\natural} : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \big(T^*\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})\big)$$

turns out to be the map

$$\Gamma^{\mathfrak{e}} : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V\boldsymbol{E})$$

given by the equality (see Proposition 2.6.1 and Corollary 2.6.1)

$$\Gamma^{\mathfrak{e}} = -\tfrac{1}{2}\,\tfrac{q}{m}\, g^{\sharp 2}\,\big(i_\partial F + \theta \lrcorner F\big).$$

With reference to an observer o, we have the observed expression

$$\Gamma^{\mathfrak{e}} = \tfrac{q}{m}\,\big(dt \otimes (\vec{E}[o] + \tfrac{1}{2}\,\nabla[o]\times\vec{B}) - \nu^*[o]\,(g^{\sharp 2}(*\vec{B})\big).$$

Moreover, we have the following coordinate expression

$$\begin{aligned}\Gamma^{\mathfrak{e}} &= -\tfrac{q}{\hbar}\,\big((F_{0\,0}^{\;i} + \tfrac{1}{2}\,F_{j\,0}^{\;i}\,x_0^j)\,d^0 + \tfrac{1}{2}\,F_{j\,0}^{\;i}\,d^j\big) \otimes \partial_i^0\\ &= -\tfrac{q_0}{m}\,\big((F_0{}^i + \tfrac{1}{2}\,F_j{}^i\,x_0^j)\,d^0 + \tfrac{1}{2}\,F_j{}^i\,d^j\big) \otimes (u^0 \otimes \partial_i).\end{aligned}$$

Proof. The proof follows from the coordinate expressions of K and $\Gamma[K]$. □

Theorem 9.2.6 *Let us define the* gravitational dynamical phase connection *and the* joined dynamical phase connection *to be the dynamical phase connections*

$$\gamma^\natural := \gamma[K^\natural] = д \lrcorner \Gamma[K^\natural] \quad \textit{and} \quad \gamma := \gamma[K] = д \lrcorner \Gamma[K]$$

induced by the gravitational spacetime connection $K^\natural$ and the joined spacetime connection K, respectively, according to Corollary 9.2.4.

The joined dynamical phase connection $\gamma := \gamma[K]$ splits into the gravitational and electromagnetic components *as*

$$\gamma = \gamma^\natural + \gamma^{\mathfrak{e}},$$

where

$$\gamma^{\mathfrak{e}} := \gamma - \gamma^\natural : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V_{\boldsymbol{E}}J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \big((\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big)$$

turns out to be the Lorentz force (see Definition 5.7.1)

$$\gamma^{\mathfrak{e}} = д \lrcorner \Gamma^{\mathfrak{e}} = -\tfrac{q}{\hbar}\, G^\sharp\,(д \lrcorner F) : J_1\boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}.$$

With reference to an observer o, we have the observed expression

$$\gamma^{\mathfrak{e}} = \tfrac{q}{m}\,(\vec{E}[o] + \nabla[o]\times\vec{B}).$$

Moreover, we have the following coordinate expression

$$\gamma^{\mathfrak{e}} = -\tfrac{q}{\hbar}\,\big((F_{0}{}^{i}{}_{0} + \tfrac{1}{2}\,F_{j}{}^{i}{}_{0}\,x^{j}_{0})\,d^{0} + \tfrac{1}{2}\,F_{j}{}^{i}{}_{0}\,d^{j}\big) \otimes \partial^{0}_{i}$$
$$= -\tfrac{q_0}{m}\,\big((F_{0}{}^{i} + \tfrac{1}{2}\,F_{j}{}^{i}\,x^{j}_{0})\,d^{0} + \tfrac{1}{2}\,F_{j}{}^{i}\,d^{j}\big) \otimes (u^{0} \otimes \partial_{i}).$$

Proof. The proof follows from the coordinate expressions of K, $K^{\mathfrak{e}}$, $\gamma[K]$ and д (see Theorem 6.3.1, Corollary 9.2.4 and Proposition 2.6.1). □

Corollary 9.2.7 *Let us consider a motion $s : \boldsymbol{T} \to \boldsymbol{E}$ and an observer o. Then, we obtain the section*

$$\gamma^{\mathfrak{e}} \circ j_1 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

which turns out to be just the Lorentz force, whose observed expression is (see Definition 5.7.1 and Proposition 2.7.3)

$$\gamma^{\mathfrak{e}} \circ j_1 s = \tfrac{q}{m}\,\big(\vec{E}[o] \circ s + \vec{v}[o] \times (\vec{B} \circ s)\big),$$

where $\vec{v}[o] := \nabla[o]s = j_1 s - o \circ s : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ is the observed velocity of s. □

Theorem 9.2.8 *Let us define the* gravitational dynamical phase 2-form *and the* joined dynamical phase 2-form *to be the dynamical phase 2-forms*

$$\Omega^{\natural} := \Omega[G, K^{\natural}] := G \lrcorner \big(\nu\big[\Gamma[K^{\natural}]\big] \wedge \theta\big) \text{ and } \Omega := \Omega[G, K] := G \lrcorner \big(\nu\big[\Gamma[K]\big] \wedge \theta\big)$$

induced by the gravitational spacetime connection $K^{\natural}$ and the joined spacetime connection K, respectively, according to Corollary 9.2.4.

The joined dynamical phase 2-form $\Omega := \Omega[G, K]$ splits into the gravitational and electromagnetic components *as*

$$\Omega = \Omega^{\natural} + \Omega^{\mathfrak{e}},$$

where

$$\Omega^{\mathfrak{e}} := \Omega - \Omega^{\natural} = \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}.$$

Proof. The proof follows from the coordinate expressions of K and $\Omega[G, K]$ (see Theorem 6.3.1 and Corollary 9.2.4). □

Exercise 9.2.9 By recalling that a dynamical phase connection γ associated with a given special phase connection Γ is determined by the equality (see Theorem 9.1.8)

$$i_{\gamma}\Omega = 0,$$

we can check that the splittings $\gamma = \gamma^{\natural} + \gamma^{\mathfrak{e}}$ and $\Omega = \Omega^{\natural} + \Omega^{\mathfrak{e}}$ are consistent.

First, we observe that

$$i_{\gamma^{\natural}}F = i_{д}F \quad \text{and} \quad i_{\gamma^{\mathfrak{e}}}F = 0.$$

Moreover, a computation in coordinates shows that

$$\tfrac{q}{\hbar}\, i_{\text{д}} F = -2\,\theta \lrcorner \big(G^{\flat}(\gamma^{\mathfrak{e}})\big) \quad \text{and} \quad i_{\gamma^{\mathfrak{e}}} \Omega^{\natural} = \theta \lrcorner \big(G^{\flat}(\gamma^{\mathfrak{e}})\big).$$

Then, we obtain

$$i_{\gamma}\Omega = i_{(\gamma^{\natural}+\gamma^{\mathfrak{e}})}(\Omega^{\natural} + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F) = -\theta \lrcorner \big(G^{\flat}(\gamma^{\mathfrak{e}})\big) + \theta \lrcorner \big(G^{\flat}(\gamma^{\mathfrak{e}})\big) = 0. \quad \square$$

Remark 9.2.10 The electromagnetic field does not contribute to the volume of the phase space. In fact, we have (see Proposition 9.1.5)

$$dt \wedge \Omega \wedge \Omega \wedge \Omega = dt \wedge \Omega^{\natural} \wedge \Omega^{\natural} \wedge \Omega^{\natural} : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1 \boldsymbol{E}.$$

Theorem 9.2.11 *Let us define the* gravitational dynamical phase 2-vector *and the* joined dynamical phase 2-vector *to be the dynamical phase 2-vectors*

$$\Lambda^{\natural} := \Lambda[G, K^{\natural}] := \bar{G} \lrcorner \big(\check{\Gamma}[K^{\natural}] \wedge \nu\big) \quad \textit{and} \quad \Lambda := \Lambda[G, K] := \bar{G} \lrcorner \big(\check{\Gamma}[K] \wedge \nu\big)$$

induced by the gravitational spacetime connection $K^{\natural}$ and the joined spacetime connection K, respectively, according to Corollary 9.2.4.

The joined dynamical phase 2-vector $\Lambda := \Lambda[G, K]$ splits into the gravitational and electromagnetic components *as*

$$\Lambda = \Lambda^{\natural} + \Lambda^{\mathfrak{e}},$$

where

$$\Lambda^{\mathfrak{e}} := \Lambda - \Lambda^{\natural} : J_1\boldsymbol{E} \to \Lambda^2 V_{\boldsymbol{E}} J_1 \boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \big((\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \Lambda^2 V\boldsymbol{E}\big)$$

turns out to be the section $\Lambda^{\mathfrak{e}} = \frac{1}{2}\,\frac{q}{m}\,\frac{\hbar}{m}\,\bar{F} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes \Lambda^2 V\boldsymbol{E}$, where (see Note 5.1.2) $\bar{F} := (g^{\sharp} \otimes g^{\sharp})(F) : \boldsymbol{E} \to (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V\boldsymbol{E}$.

We can write

$$\Lambda^{\mathfrak{e}} = \tfrac{q}{\hbar}\,\tfrac{\hbar}{m}\,\tfrac{1}{c}\, i_B \bar{\eta}.$$

Moreover, we have the following coordinate expression

$$\Lambda^{\mathfrak{e}} = \tfrac{1}{2}\,\tfrac{q}{\hbar}\, F^{ij}_{00}\, \partial^0_i \wedge \partial^0_j = \tfrac{1}{2}\,\tfrac{q}{m}\,\tfrac{\hbar}{m}\, F^{ij}\, \partial_i \wedge \partial_j.$$

Proof. The proof follows from the coordinate expressions of K and $\Lambda[G, K]$ (see Theorem 6.3.1 and Corollary 9.2.4). □

Remark 9.2.12 It is remarkable that $\Gamma[K]$, $\gamma[K]$ and $\Omega[K]$ account for the full electromagnetic field, while $\Lambda[K]$ accounts only for the magnetic field. □

Remark 9.2.13 The electromagnetic field does not contribute to the volume of the phase space. In fact, we have (see Proposition 9.1.5)

$$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda = \gamma \wedge \Lambda^{\natural} \wedge \Lambda^{\natural} \wedge \Lambda^{\natural} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ_1\boldsymbol{E}. \quad \square$$

9.2.3 Identities for Fundamental Phase Fields

If the spacetime connection K is galilean, then we obtain the following fundamental identities, for the associated phase fields

$$d\Omega = 0, \quad [\gamma, \Lambda] = 0, \quad [\Lambda,\ \Lambda] = 0,$$

where [,] is the Schouten bracket (see [252, 275] and Note I.1.1).

Even more, we can conversely prove that such identities imply that K be galilean.

Lemma 9.2.14 *For each observer o, the dynamical phase 2-form $\Omega[G, K]$ generated by a special spacetime connection K splits as (see Definition 3.2.9 and Theorem 4.3.3)*

$$\Omega[G, K] = d\mathcal{C}[G, o] + \tfrac{1}{2}\, \Phi[G, K, o].$$

Proof. The proof can be easily obtained by comparing the coordinate expressions of the above terms provided by Corollary 9.2.4, Definition 3.2.9 and Theorem 4.2.13. □

Indeed, the following result plays a fundamental role throughout the book, both in Classical Mechanics and Quantum Mechanics.

Theorem 9.2.15 *Let K be a metric preserving special spacetime connection (see Definitions 4.1.19 and 4.2.3). Then, the following facts are equivalent (see also, for instance [259, 263])*

$$d\Omega[G, K] = 0 \quad \Leftrightarrow \quad K \textit{ is galilean}.$$

Proof. The proof follows from the above Lemma 9.2.14 and Theorem 4.3.3. □

Further, we need two preliminary technical Lemmas.

Lemma 9.2.16 *If K is a metric preserving special spacetime connection, then we have the equality*

$$\begin{aligned} [\gamma, \Lambda] &\equiv [\gamma[K], \Lambda[G, K]] \\ &= -u^0 \otimes \big(R^i{}_h{}^j{}_k\, x^h_0\, x^k_0 + (R^i{}_h{}^j{}_0 + R^i{}_0{}^j{}_h)\, x^h_0 + R^i{}_0{}^j{}_0\big)\, \partial^0_i \wedge \partial^0_j, \end{aligned}$$

where [,] denotes the Schouten bracket (see Note I.1.1).

Proof. Let us consider the coordinate expressions (see Corollary 9.2.4)

$$\begin{aligned}\gamma[K] &= u^0 \otimes \big(\partial_0 + x_0^i\,\partial_i + (K_h{}^i{}_k\,x_0^h\,x_0^k + 2\,K_h{}^i{}_0\,x_0^h + K_0{}^i{}_0)\,\partial_i^0\big),\\ \Lambda[G,K] &= G_0^{ij}\,\partial_i \wedge \partial_j^0 + G_0^{jh}\,(K_h{}^i{}_0 + K_h{}^i{}_k\,x_0^k)\,\partial_i^0 \wedge \partial_j^0.\end{aligned}$$

Then, for each closed phase 2-form $\beta : J_1\boldsymbol{E} \to \Lambda^2 T^*J_1\boldsymbol{E}$, we obtain $i_{[\gamma,\Lambda]}\beta = i_\gamma\, di_\Lambda\,\beta - i_\Lambda\, di_\gamma\,\beta$.

Accordingly, we obtain the following vanishing terms

$$\begin{aligned}i_{[\gamma,\Lambda]}(d^0 \wedge d^i) &= u^0 \otimes i_\Lambda\,(d_0^i \wedge d^0) = 0,\\ i_{[\gamma,\Lambda]}(d^i \wedge d^j) &= i_\gamma\, di_\Lambda\,(d^i \wedge d^j) - i_\Lambda\, di_\gamma\,(d^i \wedge d^j) = 0,\\ i_{[\gamma,\Lambda]}(d^0 \wedge d_0^j) &= i_\gamma\, di_\Lambda\,(d^0 \wedge d_0^j) - i_\Lambda\, di_\gamma\,(d^0 \wedge d_0^j) = 0,\end{aligned}$$

and the following further terms

$$\begin{aligned}i_{[\gamma,\Lambda]}(d^i \wedge d_0^j) &= i_\gamma\, di_\Lambda\,(d^i \wedge d_0^j) - i_\Lambda\, di_\gamma\,(d^i \wedge d_0^j)\\ &= u^0 \otimes \big(\partial_0 G_0^{ij} + x_0^h\,\partial_h G_0^{ij} - G_0^{ih}\,(K_h{}^j{}_l\,x_0^l + K_h{}^j{}_0)\\ &\quad - G_0^{jh}(K_h{}^i{}_l\,x_0^l + K_h{}^i{}_0)\big)\\ &= u^0 \otimes (\nabla_0 G_0^{ij} + x_0^h\,\nabla_h G_0^{ij}).\end{aligned}$$

Then, by considering the equalities

$$\Gamma_{\lambda}{}^i{}_0 = K_\lambda{}^i{}_j\,x_0^j + K_\lambda{}^i{}_0, \quad \gamma_{00}^i = \Gamma_{j}{}^i{}_0\,x_0^j + \Gamma_0{}^i{}_0, \quad \partial_h^0\gamma_{00}^i = 2\,\Gamma_h{}^i{}_0,$$

we obtain (after a long algebraic manipulation)

$$\begin{aligned}i_{[\gamma,\Lambda]}(d_0^i \wedge d_0^j) &= i_\gamma\, di_\Lambda\,(d_0^i \wedge d_0^j) - i_\Lambda\, di_\gamma\,(d_0^i \wedge d_0^j)\\ &= u^0\,\big(\Gamma_h{}^i{}_0(\nabla_0 G_0^{jh} + x_0^r\,\nabla_r G_0^{jh}) - \Gamma_h{}^j{}_0(\nabla_0 G_0^{ih} + x_0^r\,\nabla_r G_0^{jm})\\ &\quad + (R^j{}_r{}^i{}_s - R^i{}_r{}^j{}_s)\,x_0^r\,x_0^s + (R^j{}_r{}^i{}_0 - R^i{}_r{}^j{}_0\\ &\quad + R^j{}_0{}^i{}_r - R^i{}_0{}^j{}_r)\,x_0^r + (R^j{}_0{}^i{}_0 - R^i{}_0{}^j{}_0)\big).\end{aligned}$$

So, eventually, we obtain

$$\begin{aligned}[\gamma,\Lambda] &= u^0 \otimes \big(\nabla_0 G_0^{ij} + x_0^h\,\nabla_h G_0^{ij}\big)\,\partial_i \wedge \partial_j^0 + u^0 \otimes \big(\Gamma_h{}^i{}_0(\nabla_0 G_0^{jh} + x_0^r\,\nabla_r G_0^{jh})\\ &\quad + R^j{}_r{}^i{}_s\,x_0^r\,x_0^s + (R^j{}_r{}^i{}_0 + R^j{}_0{}^i{}_r)\,x_0^r + R^j{}_0{}^i{}_0\big)\,\partial_i^0 \wedge \partial_j^0. \quad \square\end{aligned}$$

Lemma 9.2.17 *If K is a metric preserving special spacetime connection, then we have the equality*

$$[\Lambda,\Lambda] \equiv \big[\Lambda[G,K],\Lambda[G,K]\big] = G_0^{ir}\,G_0^{js}\,(R_{rs}{}^h{}_k\,x_0^k + R_{rs}{}^h{}_0)\,\partial_i^0 \wedge \partial_j^0 \wedge \partial_h^0,$$

where $[,]$ *denotes the Schouten bracket (see Note I.1.1).*

Proof. Let us consider the coordinate expression (see Corollary 9.2.4)

$$\Lambda[G,K] = G_0^{ij}\,\partial_i \wedge \partial_j^0 - G_0^{ih}\,(K_h{}^j{}_0 + K_h{}^j{}_k\,x_0^k)\,\partial_i^0 \wedge \partial_j^0.$$

Then, for each closed phase 3-form $\beta : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$, we obtain $i_{[\Lambda,\Lambda]}\beta = 2\,i_\Lambda\,di_\Lambda\,\beta$.

Accordingly, we obtain the following vanishing terms

$$\begin{aligned}
i_{[\Lambda,\Lambda]}(d^0 \wedge d^i \wedge d^j) &= 2\,i_\Lambda di_\Lambda(d^0 \wedge d^i \wedge d^j) = 0,\\
i_{[\Lambda,\Lambda]}(d^0 \wedge d^i \wedge d_0^j) &= 2\,i_\Lambda di_\Lambda(d^0 \wedge d^i \wedge d_0^j) = 0,\\
i_{[\Lambda,\Lambda]}(d^0 \wedge d_0^i \wedge d_0^j) &= 2\,i_\Lambda di_\Lambda(d^0 \wedge d_0^i \wedge d_0^j) = 0,\\
i_{[\Lambda,\Lambda]}(d^h \wedge d^i \wedge d^j) &= 2\,i_\Lambda di_\Lambda(d^h \wedge d^i \wedge d^j) = 0,\\
i_{[\Lambda,\Lambda]}(d^h \wedge d^i \wedge d_0^j) &= 2\,i_\Lambda di_\Lambda(d^h \wedge d^i \wedge d_0^j) = 0,
\end{aligned}$$

and (after a long algebraic manipulation) the following further terms

$$\begin{aligned}
i_{[\Lambda,\Lambda]}(d^h \wedge d_0^i \wedge d_0^j) &= 2\,i_\Lambda di_\Lambda(d^h \wedge d_0^i \wedge d_0^j)\\
&= 2\big(G_0^{ri}\,(\nabla_r G_0^{hs}\,K_s{}^j{}_0 - \nabla_r G_0^{js}\,K_s{}^h{}_0)\\
&\quad + G_0^{rj}\,(\nabla_r G_0^{is}\,K_s{}^h{}_0 - \nabla_r G_0^{hs}\,K_s{}^i{}_0)\\
&\quad + G_0^{rh}\,(\nabla_r G_0^{js}\,K_s{}^i{}_0 - \nabla_r G_0^{is}\,K_s{}^j{}_0)\\
&\quad + G_0^{ri}\,G_0^{js}\,R_{rs}{}^h{}_0 + G_0^{rj}\,G_0^{hs}\,R_{rs}{}^i{}_0 + G_0^{rh}\,G_0^{is}\,R_{rs}{}^j{}_0\big)\\
&\quad + 2\,x_0^k\,\big(G_0^{ri}\,(\nabla_r G_0^{hs}\,K_s{}^j{}_k - \nabla_r G_0^{js}\,K_s{}^h{}_k)\\
&\quad + G_0^{rj}\,(\nabla_r G_0^{is}\,K_s{}^h{}_k - \nabla_r G_0^{hs}\,K_s{}^i{}_k)\\
&\quad + G_0^{rh}\,(\nabla_r G_0^{js}\,K_s{}^i{}_k - \nabla_r G_0^{is}\,K_s{}^j{}_k)\\
&\quad + G_0^{ri}\,G_0^{js}\,R_{rs}{}^h{}_k + G_0^{rj}\,G_0^{hs}\,R_{rs}{}^i{}_k + G_0^{rh}\,G_0^{is}\,R_{rs}{}^j{}_k\big).
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\big[\Lambda,\Lambda\big] &= 2\,G_0^{kj}\,\nabla_k G_0^{hi}\,\partial_h \wedge \partial_i^0 \wedge \partial_j^0\\
&\quad + \big(2\,G_0^{ri}\,\nabla_r G_0^{sh}\,\Gamma_{s0}^{\ j} + G_0^{ir}\,G_0^{js}\,(R_{rs}{}^h{}_k\,x_0^k + R_{rs}{}^h{}_0)\big)\,\partial_i^0 \wedge \partial_j^0 \wedge \partial_h^0. \qquad \square
\end{aligned}$$

Then, we obtain the following main result.

Theorem 9.2.18 *Let us consider a metric preserving special spacetime connection* K *(see Definitions 4.1.19 and 4.2.3).*

Then, the following conditions are equivalent:

(1) *the spacetime connection* K *is galilean (see Definition 4.3.1),*

(2) $$\Big[\gamma[K],\, \Lambda[G,K]\Big] = 0 \quad \textit{and} \quad \Big[\Lambda[G,K],\; \Lambda[G,K]\Big] = 0,$$

where [,] *denotes the Schouten bracket (see Note I.1.1).*

Proof. It follows from the above Lemmas 9.2.16 and 9.2.17 and from the Definition of galilean spacetime connection (see Definition 4.3.1). □

The above joined phase objects fulfill distinguished identities, which will play an important role in Covariant Classical Mechanics and Covariant Quantum Mechanics (see, for instance, Theorems 10.1.1 and 12.5.3, Proposition 12.6.6, Note 15.2.2).

Thus, summing up, we have the following identities.

Theorem 9.2.19 *The phase fields generated by the postulated joined galilean spacetime connection K fulfill the algebraic identities*

$$i_\gamma\, dt = 1,$$
$$i_\gamma\, \Omega = 0, \qquad\qquad i_{dt}\, \Lambda = 0,$$
$$dt \wedge \Omega^3 \not\equiv 0, \qquad\qquad \gamma \wedge \Lambda^3 \not\equiv 0.$$
$$\langle \Omega,\; \Lambda \rangle = -3,$$
$$\langle dt \wedge \Omega \wedge \Omega \wedge \Omega \;\;,\;\; \gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda \rangle = -(3!)^2,$$

and the differential identities

$$d\Omega = 0, \quad [\Lambda,\; \Lambda] = 0, \quad L_\gamma\, \Omega = 0, \quad L_\gamma\, \Lambda = 0.$$

Proof. The proof follows from Definitions 9.1.2, and 9.1.3, Theorems 9.1.8, 9.2.15 and 9.2.18, Proposition 9.1.5. □

Chapter 10
Geometric Structures of Phase Space

We discuss the *cosymplectic structure* (dt, Ω) and the *coPoisson structure* (γ, Λ) induced by the joined spacetime connection K on the phase space $J_1\boldsymbol{E}$ (Sects. 10.1 and 10.2).

In this context, we discuss the *phase upper potential* $A^{\uparrow}[\mathfrak{b}]$ of the cosymplectic phase 2-form Ω and its close relation with the *observed spacetime potential* $A[\mathfrak{b}, o]$ of the spacetime 2-form $\Phi[o]$ (Sect. 10.1.2).

Moreover, we study the distinguished phase 1-forms induced by the cosymplectic 2-form Ω, namely, the *classical lagrangian* $\mathcal{L}[\mathfrak{b}]$, the *classical momentum* $\mathcal{M}\mathfrak{b}]$, the *classical observed hamiltonian* $\mathcal{H}[\mathfrak{b}, o]$, the *classical observed momentum* $\mathcal{P}[\mathfrak{b}, o]$ (Sect. 10.1.3).

10.1 Cosymplectic Structure of Phase Space

The cosymplectic structure of phase space, which replaces the more usual symplectic structure in standard literature, turns out to be one of the main features of our covariant approach to Classical Mechanics and Quantum Mechanics and reflects the fundamental role of time and the covariance of the theory.

Let us consider the joined galilean spacetime connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Postulates C.3 and C.4 and Theorem 6.3.1) and the associated dynamical phase 2-form $\Omega \equiv \Omega[G, K]$ (see Corollary 9.2.4).

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10.1.1 The Cosymplectic Pair of Phase Space

The pair (dt, Ω) equips phase space with a cosymplectic structure (see, for instance [70, 74–76, 223]), which tuns out to play a key role in our covariant approach to Classical and Quantum Mechanics (see, Appendix: Definition I.1.10).

Theorem 10.1.1 *In virtue of Proposition 9.1.5 and Theorem 9.2.15, the pair*

$$(dt, \Omega) \equiv (dt, \Omega[G, K])$$

turns out to be a scaled cosymplectic structure *of phase space (see Definition I.1.10).*

In other words, $dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7 T^* J_1\boldsymbol{E}$ is a scaled volume form of phase space and $d\Omega = 0$. □

The phase 2-form Ω is gauge independent and observer independent, by definition.

For every observer o, we have the following important relation between Ω and $\Phi[o]$ (see Definition 4.2.11).

Proposition 10.1.2 *For each observer o, we obtain the equality (see Theorem 4.3.3)*

$$\Phi[o] = 2\, o^* \Omega,$$

which links objects of spacetime and of phase space.

Proof. The proof can be achieved from the coordinate expression of Ω provided by Corollary 9.2.4, by referring to a spacetime chart adapted to the observer. □

10.1.2 Upper Potential and Observed Potential

We discuss the (local) gauge dependent "*upper*" *potential* $A^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ of the cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ and its relation with the (local) gauge dependent and observer dependent *potential* $A[o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ of the observed spacetime 2-form $\Phi[o] \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E})$ (see Definition 4.2.11 and Theorem 4.3.3).

We stress that the potential $A^{\uparrow}$ is "horizontal". Later, in the quantum theory, we shall see that the existence of horizontal potentials of Ω fits our choice of a quantum bundle $\boldsymbol{Q}$ based on spacetime and of a reducible upper quantum connection $Ч^{\uparrow}$ (see Postulates Q.1 and Q.2, Definitions 15.1.5 and 15.1.6).

In general, throughout the book, the word "upper" will label objects of the phase space $J_1\boldsymbol{E}$ in order to distinguish them with respect to associated objects of the spacetime $\boldsymbol{E}$.

Definition 10.1.3 We define an "*upper*" *potential* to be a potential of the cosymplectic 2-form Ω

$$A^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^* J_1\boldsymbol{E}),$$

according to the equality (for the notation, see also Remark 10.1.6 and Notation 10.1.7)

$$\Omega = dA^{\uparrow}.$$

Thus, we have a coordinate expression of the type

$$A^{\uparrow} = A_{\lambda}\, d^{\lambda} + A_{i}^{0}\, d_{0}^{i}, \quad \text{with } A_{\lambda},\ A_{0}^{i} \in \text{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

and the equalities

$$\Omega_{\lambda\mu} = \tfrac{1}{2}\,(\partial_{\lambda} A^{\uparrow}{}_{\mu} - \partial_{\mu} A^{\uparrow}{}_{\lambda}) \qquad \text{and} \qquad \Omega_{\lambda}{}_{j}^{0} = \tfrac{1}{2}\,(\partial_{\lambda} A_{j}^{0} - \partial_{j}^{0} A^{\uparrow}{}_{\lambda}).$$

Clearly, the potentials $A^{\uparrow}$ of Ω are defined (locally) up to a gauge of the type

$$df \in \sec(J_1\boldsymbol{E}, T^{*}J_1\boldsymbol{E}), \quad \text{with } f \in \text{map}(\boldsymbol{E}, \mathbb{R}). \quad \square$$

Theorem 10.1.4 *The cosymplectic 2-form Ω admits (locally) "*horizontal*" potentials of the type*

$$A^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^{*}\boldsymbol{E}) \subset \sec(J_1\boldsymbol{E}, T^{*}J_1\boldsymbol{E}),$$

i.e., in coordinates, which fulfills a condition of the type

$$A_{i}^{0} = 0.$$

Accordingly, the coordinate expression of the equality $\Omega = dA^{\uparrow}$ becomes

$$\Omega_{\lambda\mu} = \tfrac{1}{2}\,(\partial_{\lambda} A^{\uparrow}{}_{\mu} - \partial_{\mu} A^{\uparrow}{}_{\lambda}) \qquad \textit{and} \qquad \Omega_{\lambda}{}_{j}^{0} = -\tfrac{1}{2}\,\partial_{j}^{0} A^{\uparrow}{}_{\lambda}.$$

Actually, the horizontal potentials $A^{\uparrow}$ of Ω are defined up to a gauge of the type

$$df \in \sec(\boldsymbol{E}, T^{*}\boldsymbol{E}), \quad \text{with } f \in \text{map}(\boldsymbol{E}, \mathbb{R}).$$

With reference to an observer o, each horizontal upper potential $A^{\uparrow}$ turns out to be of the type (see Definition 3.2.9 and Theorem 4.3.3; moreover, for the notation, see also Remark 10.1.6 and Notation 10.1.7)

$$A^{\uparrow} = \mathcal{C}[o] + A[o] = -(\mathcal{K}[o] - \text{д}[o] \lrcorner A[o]) + (\mathcal{Q}[o] + \theta[o] \lrcorner A[o]),$$

where, according to Proposition 10.1.2,

$$A[o] := o^{*} A^{\uparrow}$$

turns out to be a distinguished observed potential of the gauge independent spacetime 2-form $\Phi[o]$.

Thus, the above equality selects the gauge of $A[o]$ *from the chosen gauge of* $A^{\uparrow}$.

In other words, the coordinate expression of the horizontal upper potential $A^{\uparrow}$, *in a chart adapted to o, is given by the equality*

$$A^{\uparrow} = -(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i . \quad \square$$

From now on we shall only deal with horizontal upper potentials $A^{\uparrow}$.

Corollary 10.1.5 *Chosen a horizontal upper potential* $A^{\uparrow}$, *if o and* $\acute{o} = o + \vec{v}$ *are two observers, then we obtain the transition rule (see also the more general Theorem 15.2.26 and, for the notation, Remark 10.1.6, Notation 10.1.7)*

$$A[\acute{o}] = A[o] + \theta[o] \lrcorner\, G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v}),$$

where the observed potentials $A[o]$ *and* $A[\acute{o}]$ *are derived from* $A^{\uparrow}$, *according to the above Theorem 10.1.4.*

In other words, in a spacetime chart adapted to o, we have the equality

$$A[\acute{o}] = A_{\lambda}\, d^{\lambda} - \tfrac{1}{2}\; G^0_{ij}\; v^i_0\, v^j_0\, d^0 + G^0_{ij}\, v^j_0\, d^i .$$

Indeed, for each observer o and scaled vector fields $\vec{u}, \vec{v} \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$, *we obtain the group property*

$$A\left[o + (\vec{u} + \vec{v})\right] = A\left[(o + \vec{u}) + \vec{v}\right] . \quad \square$$

Next, we introduce a convenient notation aimed at emphasising the gauge of $A^{\uparrow}$ and $A[o]$.

Remark 10.1.6 We stress that, in the classical theory, there is no natural way to parametrise the gauge of the horizontal potentials $A^{\uparrow}$ of Ω.

We can only say that, given two horizontal upper potentials $A^{\uparrow}$ and $\acute{A}^{\uparrow}$, their difference turns out to be (locally) an exact spacetime form of the type

$$\acute{A}^{\uparrow} - A^{\uparrow} = d\vartheta \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}), \quad \text{with } \vartheta \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}.)$$

Moreover, with reference to any observer o, for the associated observed potentials $A[o]$ and $\acute{A}[o]$, we have a similar equality

$$\acute{A}[o] - A[o] = d\vartheta \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}), \quad \text{with } \vartheta \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}.)$$

In the quantum theory, where we deal with a chosen, global gauge independent and observer independent, upper quantum connection $\text{Ч}^{\uparrow}$, we obtain a bijection between the local quantum bases b and the local potentials $A^{\uparrow}$ of Ω (see Theorem 15.2.4, Remark 15.2.5 and Note 15.2.12). So, in the quantum theory, we can parametrise the gauge dependent upper potentials $A^{\uparrow}$ via quantum bases b, by means of this bijection.

Then, also in the classical theory, we shall avail of the opportunity of such a notation offered by the quantum theory and parametrise the gauge dependent potentials $A^{\uparrow}$ of Ω. Actually, in the limited framework of classical theory, the parameter $\mathfrak{b}$, which will be called *classical gauge*, plays just the role of a label notation. Its geometric meaning can be emphasised only later, in the quantum context, after having chosen the upper quantum connection $Ч^{\uparrow}$. □

Therefore, we adopt the following notation.

Note 10.1.7 We shall write, also in the classical theory, (see Theorems 4.3.3 and 10.1.4 and Corollary 10.1.5)

$$\Omega = dA^{\uparrow}[\mathfrak{b}], \quad \text{with } A^{\uparrow} = A^{\uparrow}[\mathfrak{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),$$
$$\Phi[o] = 2\, dA[\mathfrak{b}, o], \quad \text{with } A[\mathfrak{b}, o] = o^* A^{\uparrow}[\mathfrak{b}] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}),$$

$$A^{\uparrow}[\mathfrak{b}] = \mathcal{C}[o] + A[\mathfrak{b}, o] = -(\mathcal{K}[o] - д[o] \lrcorner A[\mathfrak{b}, o]) + (\mathcal{Q}[o] + \theta[o] \lrcorner A[\mathfrak{b}, o]),$$

Thus, from now on, all formulas involving $A^{\uparrow}$ and $A[o]$ will be written in this way, by emphasising the classical gauge $\mathfrak{b}$. □

10.1.3 Dynamical Phase 1-Forms

The cosymplectic phase 2-form Ω yields, in a covariant way, the following *dynamical phase 1-forms*

$$\begin{aligned}
&\textit{the classical lagrangian} && \mathcal{L}[\mathfrak{b}] && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}),\\
&\textit{the classical momentum} && \mathcal{M}[\mathfrak{b}] && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),\\
&\textit{the classical observed hamiltonian} && \mathcal{H}[\mathfrak{b}, o] && \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}),\\
&\textit{the classical observed momentum} && \mathcal{P}[\mathfrak{b}, o] && \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),
\end{aligned}$$

which will play a relevant role in both in Classical Mechanics and Quantum Mechanics.

In most lagrangian theories, one usually starts from a given lagrangian $\mathcal{L}$ and derives from it the momentum $\mathcal{M}$ and the Poincaré–Cartan form $\mathcal{L} + \mathcal{M}$.

Analogously, in most hamiltonian theories, one usually starts from a hamiltonian $\mathcal{H}$.

Conversely, in the present theory, we start from the global, gauge independent and observer independent cosymplectic 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ and derive from it the local gauge dependent and observer independent horizontal upper potential $A^{\uparrow}[\mathfrak{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$.

Then, the covariant splitting of $A^{\uparrow}[\mathrm{b}]$, into its $\boldsymbol{E}$-horizontal and $\boldsymbol{E}$-vertical components, yields the local gauge dependent and observer independent lagrangian $\mathcal{L}[\mathrm{b}]$ and momentum $\mathcal{M}[\mathrm{b}]$. Accordingly, the *classical Poincaré–Cartan form* associated with the lagrangian $\mathcal{L}[\mathrm{b}]$ turns out to be just the potential $A^{\uparrow}[\mathrm{b}]$.

Moreover, the observed splitting of $A^{\uparrow}[\mathrm{b}]$, into its observed $\boldsymbol{E}$-horizontal and observed $\boldsymbol{E}$-vertical components, yields the local gauge dependent and observer dependent hamiltonian $\mathcal{H}[\mathrm{b}, o]$ and momentum $\mathcal{P}[\mathrm{b}, o]$.

Thus, we do not postulate a classical lagrangian $\mathcal{L}[\mathrm{b}]$, or a classical hamiltonian $\mathcal{H}[\mathrm{b}, o]$, but we start with the cosymplectic phase 2-form Ω (derived in a covariant way from the gravitational and electromagnetic fields $K^{\natural}$ and F) and derive from Ω the lagrangian $\mathcal{L}[\mathrm{b}]$ and the hamiltonian $\mathcal{H}[\mathrm{b}, o]$. So, we can say that the classical lagrangian $\mathcal{L}[\mathrm{b}]$ and classical observed hamiltonian $\mathcal{H}[\mathrm{b}, o]$ are encoded in the cosymplectic phase 2-form Ω.

Theorem 10.1.8 *The gauge dependent choice of a local upper horizontal potential*

$$A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$$

yields, in a covariant way, the following distinguished phase 1-forms, which are defined, respectively, to be the horizontal and vertical components and the observed horizontal and vertical components of $A^{\uparrow}[\mathrm{b}]$, (see Theorem 10.1.4, Proposition 2.6.1 and Corollary 2.6.2)

the classical lagrangian	$\mathcal{L}[\mathrm{b}] := д \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$,
the classical momentum	$\mathcal{M}[\mathrm{b}] := \theta \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$,
the classical observed hamiltonian	$\mathcal{H}[\mathrm{b}, o] := -д[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$,
the classical observed momentum	$\mathcal{P}[\mathrm{b}, o] := \theta[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$,

with coordinate expressions

$$\mathcal{L}[\mathrm{b}] = \mathcal{L}_0\, d^0 = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_j\, x^j_0 + A_0)\, d^0,$$
$$\mathcal{M}[\mathrm{b}] = \mathcal{M}_0\, d^0 + \mathcal{M}_i\, d^i = (G^0_{ij}\, x^j_0 + A_i)\,(d^i - x^i_0\, d^0),$$

and, in a spacetime chart adapted to o,

$$\mathcal{H}[\mathrm{b}, o] = \mathcal{H}_0\, d^0 = (\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0,$$
$$\mathcal{P}[\mathrm{b}, o] = \mathcal{P}_i\, d^i = (G^0_{ij}\, x^j_0 + A_i)\, d^i.$$

We have the equalities (see Definition 3.2.9)

$$\begin{aligned}
\mathcal{M}[\mathfrak{b}] &= \theta \lrcorner V_{\boldsymbol{E}}\mathcal{L}[\mathfrak{b}],\\
A^{\uparrow}[\mathfrak{b}] &= \mathcal{L}[\mathfrak{b}] + \mathcal{M}[\mathfrak{b}] = -\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o],\\
\mathcal{P}[\mathfrak{b}, o] &= \theta[o] \lrcorner \mathcal{M}[\mathfrak{b}],\\
\check{\mathcal{P}}[\mathfrak{b}, o] &= \check{\mathcal{M}}[\mathfrak{b}],\\
\mathcal{H}[\mathfrak{b}, o] &= д \lrcorner \mathcal{P}[\mathfrak{b}, o] - \mathcal{L}[\mathfrak{b}].
\end{aligned}$$

Thus, according to the equality $A^{\uparrow}[\mathfrak{b}] = \mathcal{L}[\mathfrak{b}] + \theta \lrcorner V_{\boldsymbol{E}}\mathcal{L}[\mathfrak{b}]$, *the phase 1-form* $A^{\uparrow}[\mathfrak{b}]$ *turns out to be the* "Poincaré–Cartan form" *associated with the lagrangian* $\mathcal{L}[\mathfrak{b}]$ *(see, for instance [147, 243, 283, 411]).*

Moreover, we have the observed splitting

$$A^{\uparrow}[\mathfrak{b}] = -\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o].$$

Further, we can express the above dynamical phase 1-forms in terms of the kinetic objects and of the observed potential as follows (see Theorems 10.1.4 and 10.1.8, Definition 3.2.9)

$$\begin{aligned}
A^{\uparrow}[\mathfrak{b}] &= \mathcal{C}[o] + A[\mathfrak{b}, o],\\
\mathcal{L}[\mathfrak{b}] &= \mathcal{K}[o] + д \lrcorner A[\mathfrak{b}, o],\\
\mathcal{M}[\mathfrak{b}] &= \mathcal{N}[o] + \theta \lrcorner A[\mathfrak{b}, o],\\
\mathcal{H}[\mathfrak{b}, o] &= \mathcal{K}[o] - д[o] \lrcorner A[\mathfrak{b}, o],\\
\mathcal{P}[\mathfrak{b}, o] &= \mathcal{Q}[o] + \theta[o] \lrcorner A[\mathfrak{b}, o]. \quad \square
\end{aligned}$$

Remark 10.1.9 We stress that, by definition, $A^{\uparrow}[\mathfrak{b}]$ is observer independent, while $\mathcal{H}[\mathfrak{b}, o]$ and $\mathcal{P}[\mathfrak{b}, o]$ are observer dependent.

We notice that the pair of signs $(-, +)$ appearing in the observed splittings of $A^{\uparrow}[\mathfrak{b}]$ resembles the signature $(- + ++)$ of the Lorentz metric. □

Corollary 10.1.10 *The* square of the classical observed momentum *turns out to be observer independent (see Corollary 15.2.28) and fulfills the equality*

$$\tfrac{1}{2}\,\mathcal{P}^2[\mathfrak{b}] := \tfrac{1}{2}\,\bar{G}\,\left(\mathcal{P}[\mathfrak{b}, o], \mathcal{P}[\mathfrak{b}, o]\right) = \mathcal{L}[\mathfrak{b}] - \alpha[\mathfrak{b}],$$

with coordinate expression

$$\tfrac{1}{2}\,\mathcal{P}_0^2 = \tfrac{1}{2}\,G_0^{ij}\,\mathcal{P}_i\,\mathcal{P}_j = \tfrac{1}{2}\,G^0_{ij}\,x_0^i\,x_0^j + A_i\,x_0^i + \tfrac{1}{2}\,A_i\,A_0^i = \mathcal{L}_0 - (A_0 - \tfrac{1}{2}\,A_0^i\,A_i). \quad \square$$

Note 10.1.11 The above Corollary 10.1.10 is the first occasion when we meet the gauge dependent and observer independent timelike 1-form

$$\alpha[\mathfrak{b}] = д[o] \lrcorner A[\mathfrak{b}, o] - \tfrac{1}{2}\,\bar{G}\big(A[\mathfrak{b}, o], A[\mathfrak{b}, o]\big) \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}),$$

with coordinate expression

$$\alpha = (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, d^0 .$$

Later, in Propositions 12.2.5 and 15.2.29, we shall provide an explicitly observer invariant definition of α.

Moreover, we shall further discuss the observer invariance of $\alpha[\mathrm{b}]$ in Corollary 15.2.28 and its physical meaning in Theorem 15.2.31.

Actually, this spacetime 1-form will often appear later, for instance, in Propositions 12.2.5 and 15.2.29, Theorems 17.5.2 and 17.6.5. □

Remark 10.1.12 The square of the observed momentum fulfills the equality

$$\tfrac{1}{2}\, \mathcal{P}^2_0 = \mathcal{H}_0 + A_0 + A_i\, (x^i_0 + \tfrac{1}{2}\, A^i_0).$$

Hence, only in the particular case when $\check{A}[\mathrm{b}, o] = 0$, we obtain

$$\mathcal{H}_0 = \tfrac{1}{2}\, \mathcal{P}^2_0 - A_0. \quad \square$$

10.1.4 Cosymplectic Versus Symplectic Structures

We have seen that the joined galilean spacetime connection K yields the cosymplectic phase 2-form $\Omega \equiv \Omega[G, K] : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ of the odd dimensional phase space $J_1\boldsymbol{E}$ (see Corollary 9.2.4).

The standard literature deals with more popular even dimensional phase spaces equipped with a symplectic 2-form. So, it might be interesting to see whether in our framework there is a “reduction” of $J_1\boldsymbol{E}$ to a suitable even dimensional space, such that Ω reduces to be a symplectic 2-form. Actually, we do find such a reduction, but discover that it carries a too poor information, hence it turns out to be unsuitable for our dynamical purposes.

Moreover, in standard classical dynamics, one often postulates the lagrangian and hamiltonian functions as additional objects with respect to a given symplectic structure. Conversely, we have seen that the lagrangian and hamiltonian functions are encoded in our cosymplectic 2-form Ω. Even more, we prove that the standard hamiltonian procedure usually followed in symplectic dynamics to derive the equation of motion turns out meaningless in our cosymplectic framework.

Such deep differences between cosymplectic and symplectic frameworks are reflected in all steps of our approach to Classical and Quantum Mechanics.

Proposition 10.1.13 *The dynamical phase connection*

$$\gamma \equiv \gamma[K] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$$

associated with a galilean spacetime connection K turns out to be a scaled infinitesimal symmetry of the cosymplectic structure $(dt, \Omega) \equiv (dt, \Omega[G, K])$, i.e. we have

$$L_\gamma dt = 0 \quad \textit{and} \quad L_\gamma \Omega = 0.$$

Proof. By recalling Theorem 9.2.19, the equality $i_\gamma dt = 1$ implies $L_\gamma dt = 0$ and the equality $i_\gamma \Omega = 0$ implies $L_\gamma \Omega = 0$. □

Remark 10.1.14 In symplectic dynamics, one often deals, for instance, with a manifold $\boldsymbol{M}$ equipped with a symplectic 2-form $\omega \in \Lambda^2 T^* T\boldsymbol{M}$.

Then, one usually looks for the 2nd order differential equations $X : T\boldsymbol{M} \to TT\boldsymbol{M}$, which fulfill the condition $L_X \omega = 0$, i.e. such that $i_X \omega = -d\mathcal{H}$, where $\mathcal{H} \in \operatorname{map}(T\boldsymbol{M}, \mathbb{R})$ is a function, which plays the role of hamiltonian. Often, such hamiltonian function is postulated.

Now, in our framework, the dynamical phase connections γ are essentially the scaled analogues on $J_1\boldsymbol{E}$ of 2nd order differential equations X on $T\boldsymbol{M}$.

So, keeping in mind this analogy, we could look for dynamical phase connections $\acute{\gamma}$ such that

$$(*) \qquad i_{\acute{\gamma}} \Omega = -d\mathcal{H}, \quad \text{where } \mathcal{H} \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}).$$

Actually, every dynamical phase connection $\acute{\gamma}$ can be uniquely written as $\acute{\gamma} = \gamma + \delta$, where γ is the dynamical phase connection associated with Ω (see Theorem 9.1.8) and $\delta \in \sec(J_1\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$.

Then, equation $(*)$ can be written in coordinates, as

$$G^0_{ij}\, \delta^i_{00}\, (d^j - x^j_0\, d^0) = -\partial_0 \mathcal{H}_0\, d^0 - \partial_j \mathcal{H}_0\, d^j - \partial^0_h \mathcal{H}_0\, d^h_0.$$

Indeed, the only solution of this equation is

$$\acute{\gamma} = \gamma \qquad \text{and} \qquad \mathcal{H} \in \mathbb{T}^* \otimes \mathbb{R}.$$

So, in our cosymplectic framework a hamiltonian equation of the type $(*)$ is physically uninteresting.

Actually, we have seen that the cosymplectic phase 2-form Ω encodes the hamiltonian function $\mathcal{H}$, up to a gauge and the choice of an observer; further, we have seen that the cosymplectic phase 2-form Ω encodes the lagrangian function $\mathcal{L}$, up to a gauge (see Theorem 10.1.8).

So, in our context, we do not postulate a hamiltonian function and a lagrangian function, but we derive them from the cosymplectic 2-form Ω. □

Remark 10.1.15 We have seen that the galilean metric G equips the fibres of spacetime with a symplectic structure, which is characterised by the fibrewise scaled symplectic 2-form (see Proposition 3.2.14)

$$\omega[G] : V\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^2 V_{\boldsymbol{E}} V\boldsymbol{E},$$

with coordinate expression

$$\omega[G] = u_0 \otimes (\partial_i G^0_{jh}\, \dot{x}^h\, \check{d}^i \wedge \check{d}^j - G^0_{ij}\, \check{d}^i \wedge \check{\dot{d}}^j).$$

Indeed, we can prove that, by a tricky covariant procedure, we can "reduce" the cosympletic 2-form to the fibrewise symplectic 2-form

$$\check{\Omega}[G, K] : \mathbb{T}^* \otimes V\boldsymbol{E} \to \Lambda^2\, V^*_{\boldsymbol{E}}(\mathbb{T}^* \otimes V\boldsymbol{E}),$$

with coordinate expression

$$\check{\Omega}[G, K] = G^0_{ij}\, \check{\dot{d}}{}^i_0 \wedge \check{d}^j - G^0_{kj}\, K_i{}^k{}_h\, \dot{x}^h_0\, \check{d}^i \wedge \check{d}^j.$$

Moreover, we can make the natural identification

$$\check{\Omega}[G, K] : \mathbb{T}^* \otimes V\boldsymbol{E} \to \Lambda^2\, V^*_{\boldsymbol{E}}(\mathbb{T}^* \otimes V\boldsymbol{E}) \simeq \omega[G] : V\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^2 V_{\boldsymbol{E}} V\boldsymbol{E},$$

according to the equality

$$G^0_{ij}\, \check{\dot{d}}{}^i_0 \wedge \check{d}^j - G^0_{kj}\, K_i{}^k{}_h\, \dot{x}^h_0\, \check{d}^i \wedge \check{d}^j \simeq u_0 \otimes (\partial_i G^0_{jh}\, \dot{x}^h\, \check{d}^i \wedge \check{d}^j - G^0_{ij}\, \check{d}^i \wedge \check{\dot{d}}^j).$$

Actually, the above restriction of $\Omega[G, K]$ deletes the information carried by $\Phi[G, K, o]$, hence by the observed potential $A[G, K, o]$.

Thus, the above fibrewise symplectic 2-form $\check{\Omega}$ carries only information of the galilean metric and forgets about the gravitational and electromagnetic fields. □

10.2 coPoisson Structure of Phase Space

The coPoisson structure (γ, Λ) of phase space, which replaces the more usual Poisson structure in standard literature, turns out to be one of the main features of our covariant approach to Classical Mechanics and Quantum Mechanics and reflects the fundamental role of time and the covariance of the theory.

Let us consider the joined galilean spacetime connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Postulates C.3 and C.4 and Theorem 6.3.1) and the associated dynamical phase pair $(\gamma, \Lambda) \equiv (\gamma[K], \Lambda[G, K])$ (see Corollary 9.2.4).

10.2.1 The coPoisson Pair of Phase Space

The pair (γ, Λ) equips phase space with a coPoisson structure, which tuns out to play a key role in our covariant approach to Classical and Quantum Mechanics (see Appendix: Definition I.1.10).

Theorem 10.2.1 *In virtue of Proposition 9.1.5 and Theorem 9.2.18, the joined pair*

$$(\gamma, \Lambda) \equiv (\gamma[K], \Lambda[G, K])$$

turns out to be a scaled coPoisson structure *of the phase space (see Definition I.1.10).*

In other words,

$$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 T J_1\boldsymbol{E}$$

is a scaled volume form of the phase space and

$$[\gamma, \Lambda] = 0 \quad \textit{and} \quad [\Lambda, \Lambda] = 0. \;\square$$

Remark 10.2.2 We stress that the above phase structure (γ, Λ) is not the more standard "*Jacobi structure*" [275], which would fulfill the identities

$$[\gamma, \Lambda] = 0 \quad \text{and} \quad [\Lambda, \Lambda] = 2\,\gamma \wedge \Lambda.$$

Even more, we stress that in our context the 2nd identity would be inconsistent, with respect to the view point of scales, as Λ is an unscaled object, while γ is a scaled vector field. Actually, there is no way to replace γ with an unscaled vector field in a covariant way. This is a typical example of how a covariance requirement forces our theory. □

Remark 10.2.3 We have already observed that the dynamical phase 2-vector Λ is a spacelike object, hence it encodes less information with respect to the dynamical phase 2-form Ω.

The two pairs (dt, Ω) and (γ, Λ) are equivalent, in the sense that, in virtue of Theorems 9.1.7 and 9.1.8, we can recover one from the other one. □

Chapter 11
Hamiltonian Formalism

We discuss the natural *phase isomorphisms* $\Omega^\flat$ and $\Lambda^\sharp$ associated with the cosymplectic and coPoisson structures (Sect. 11.2).

In this context, we introduce the *affine hamiltonian lift* $X^\uparrow{}_{\text{ham}}[\tau, f]$ of phase functions and discuss the *Poisson bracket* (Sects. 11.3 and 11.4).

Further, we express the *Newton law of motion* by means of the joined dynamical phase connection γ and by means of the joined *Euler–Lagrange equation* $\mathcal{E}$ (Sect. 11.5).

Eventually, we study the Lie algebra of *conserved phase functions* (Sect. 11.6).

11.1 Phase Splittings

A phase connection Γ and a dynamical phase connection γ naturally yield *linear splittings* of the tangent and cotangent spaces of phase space (see Definitions 9.1.1 and 9.1.2). Moreover, if $\gamma = \gamma[\Gamma] = д \lrcorner \Gamma$, then these splitting are compatible.

These splittings will be involved, for instance, in the hamiltonian lift of phase functions (see Definitions 11.3.1 and 11.3.6).

Proposition 11.1.1 *Let us consider the linear fibred maps associated with a dynamical phase connection* $\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$

$$\begin{aligned} &\gamma \;\; : J_1\boldsymbol{E} \times \bar{\mathbb{T}} \to TJ_1\boldsymbol{E}, && \nu[\gamma] \;\; : TJ_1\boldsymbol{E} \to V_{\boldsymbol{T}}J_1\boldsymbol{E}, \\ &\gamma^* : T^*J_1\boldsymbol{E} \to J_1\boldsymbol{E} \times \bar{\mathbb{T}}^*, && \nu[\gamma]^* : V^*_{\boldsymbol{T}}J_1\boldsymbol{E} \to T^*J_1\boldsymbol{E}. \end{aligned}$$

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Then, $\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T J_1\boldsymbol{E}$ *naturally yields the dual splittings of the tangent and cotangent spaces of phase space (see Definition 9.1.2)*

$$T J_1\boldsymbol{E} = H_\gamma J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{T}} J_1\boldsymbol{E}, \quad \text{and} \quad T^* J_1\boldsymbol{E} = H^*_{\boldsymbol{T}} J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V^*_\gamma J_1\boldsymbol{E},$$

where

$$\begin{array}{llll} H_\gamma J_1\boldsymbol{E} & := \mathrm{im}\,(\gamma) = \ker\,(\nu[\gamma]) & \subset T J_1\boldsymbol{E}, & \dim(H_\gamma J_1\boldsymbol{E}) = 7+1, \\ V_{\boldsymbol{T}} J_1\boldsymbol{E} & := \ker dt & \subset T J_1\boldsymbol{E}, & \dim(V_{\boldsymbol{T}} J_1\boldsymbol{E}) = 7+6, \\ H^*_{\boldsymbol{T}} J_1\boldsymbol{E} & := \mathrm{im}\, dt & \subset T^* J_1\boldsymbol{E}, & \dim(H^* J_1\boldsymbol{E}) = 7+1, \\ V^*_\gamma J_1\boldsymbol{E} & := \mathrm{im}\,(\nu[\gamma]^*) = \ker\,(\gamma^*) & \subset T^* J_1\boldsymbol{E}, & \dim(V^*_\gamma J_1\boldsymbol{E}) = 7+6. \end{array}$$

Accordingly, we obtain the following adapted dual bases

$$\begin{array}{lll} e_0 := \partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i, & e_i := \partial_i, & e^0_i := \partial^0_i, \\ \epsilon^0 := d^0, & \epsilon^i := d^i - x^i_0\,d^0, & \epsilon^i_0 := d^i_0 - \gamma_0{}^i_0\,d^0. \quad \square \end{array}$$

Proposition 11.1.2 *Let us consider the linear fibred maps associated with a phase connection* $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes T J_1\boldsymbol{E}$

$$\begin{array}{ll} \Gamma\ : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T J_1\boldsymbol{E}, & \nu[\Gamma]\ : T J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}, \\ \Gamma^* : T^* J_1\boldsymbol{E} \to J_1\boldsymbol{E} \times \bar{\mathbb{T}}^*, & \nu[\Gamma]^* : \mathbb{T} \otimes V^*\boldsymbol{E} \to T^* J_1\boldsymbol{E}. \end{array}$$

Then, Γ *naturally yields the dual splittings of the tangent and cotangent spaces of phase space (see Definition 9.1.1)*

$$T J_1\boldsymbol{E} = H_\Gamma J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{E}} J_1\boldsymbol{E} \quad \text{and} \quad T^* J_1\boldsymbol{E} = H^*_{\boldsymbol{E}} J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V^*_\Gamma J_1\boldsymbol{E},$$

where

$$\begin{array}{llll} V_{\boldsymbol{E}} J_1\boldsymbol{E} & := \ker T t^1_0 & \subset T J_1\boldsymbol{E}, & \dim(V_{\boldsymbol{E}} J_1\boldsymbol{E}) = 7+3, \\ H^*_{\boldsymbol{E}} J_1\boldsymbol{E} & := \mathrm{im}\,(T t^1_0)^* & \subset T^* J_1\boldsymbol{E}, & \dim(H^*_{\boldsymbol{E}} J_1\boldsymbol{E}) = 7+4, \\ H_\Gamma J_1\boldsymbol{E} & := \mathrm{im}\,(\Gamma) = \ker\,(\nu[\Gamma]) & \subset T J_1\boldsymbol{E}, & \dim(H_\Gamma J_1\boldsymbol{E}) = 7+4, \\ V^*_\Gamma J_1\boldsymbol{E} & := \mathrm{im}\,(\nu[\Gamma]^*) = \ker\,(\Gamma^*) & \subset T^* J_1\boldsymbol{E}, & \dim(V^*_\Gamma J_1\boldsymbol{E}) = 7+3. \end{array}$$

Accordingly, we obtain the following adapted dual bases

$$b_\lambda := \partial_\lambda + \Gamma_\lambda{}^i_0\,\partial^0_i, \quad b^0_i := \partial^0_i \quad \text{and} \quad \beta^\lambda := d^\lambda, \quad \beta^i_0 := d^i_0 - \Gamma_\lambda{}^i_0\,d^\lambda. \quad \square$$

Lemma 11.1.3 *For a phase connection* Γ *and the dynamical phase connection* $\gamma[\Gamma]$ *we have natural linear inclusions over* $J_1\boldsymbol{E}$ *(see Theorem 9.1.7)*

$$H_\gamma J_1\boldsymbol{E} \subset H_\Gamma J_1\boldsymbol{E} \quad \text{and} \quad V^*_\Gamma J_1\boldsymbol{E} \subset V^*_\gamma J_1\boldsymbol{E}.$$

Proof. The proof follows from the equalities

$$
\begin{aligned}
&e_0 = b_0 + x_0^i\, b_i + (\gamma_0{}_0^i - \Gamma_0{}_0^i - \Gamma_j{}_0^i\, x_0^j)\, b_i^0, \quad e_i = b_i - \Gamma_i{}_0^j\, b_j^0, \quad e_i^0 = b_i^0,\\
&b_0 = e_0 - x_0^i\,(e_i + \Gamma_i{}_0^j\, e_j^0) - (\gamma_0{}_0^i - \Gamma_0{}_0^i - \Gamma_j{}_0^i\, x_0^j)\, e_i^0,\\
&b_i = e_i + \Gamma_i{}_0^j\, e_j^0, \quad b_i^0 = e_i^0. \quad \square
\end{aligned}
$$

Proposition 11.1.4 *A phase connection $\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ and the dynamical phase connection $\gamma[\Gamma] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ naturally yield the dual linear splittings of the tangent and cotangent spaces of phase space (see Theorem 9.1.7)*

$$
\begin{aligned}
TJ_1\boldsymbol{E} &= H_\gamma J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_\Gamma J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{E}} J_1\boldsymbol{E},\\
T^*J_1\boldsymbol{E} &= H_{\boldsymbol{T}}^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} H_\gamma^* J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_\Gamma^* J_1\boldsymbol{E},
\end{aligned}
$$

where

$$
\begin{aligned}
&\dim(H_\gamma J_1\boldsymbol{E}) = 7+1, \quad &&\dim(V_\Gamma J_1\boldsymbol{E}) = 7+3, \quad &&\dim(V_{\boldsymbol{E}} J_1\boldsymbol{E}) = 7+3,\\
&\dim(H_{\boldsymbol{T}}^* s J_1\boldsymbol{E}) = 7+1, \quad &&\dim(H_\gamma^* J_1\boldsymbol{E}) = 7+3, \quad &&\dim(V_\Gamma^* J_1\boldsymbol{E}) = 7+3.
\end{aligned}
$$

Accordingly, we obtain the following adapted dual bases

$$
\begin{aligned}
&e_0 := \partial_0 + x_0^i\, \partial_i + \gamma_0{}_0^i\, \partial_i^0, \quad &&e_i := \partial_i + \Gamma_i{}_0^j\, \partial_j^0, \quad &&e_i^0 := \partial_i^0,\\
&\epsilon^0 := d^0, &&\epsilon^i := d^i - x_0^i\, d^0, &&\epsilon_0^i := d_0^i - \Gamma_0{}_0^i\, d^0 - \Gamma_j{}_0^i\, d^j. \quad \square
\end{aligned}
$$

11.2 Phase Musical Morphisms

We consider a galilean spacetime connection K and the associated phase fields $\Gamma, \gamma, \Omega, \Lambda$. Then, we discuss the natural *linear musical phase morphisms* $\Omega^\flat$ and $\Lambda^\sharp$; indeed, they are not isomorphisms. But, after having chosen a *phase time scale* τ, these morphisms yield natural isomorphism by means of the additional help of γ.

The above geometric construction will be used later for the definition of hamiltonian lift of phase functions and consequently for the definition of Poisson Lie bracket (see Definition 11.3.6 and Theorem 11.4.6).

Actually, in this context, we shall see that every phase function provides a distinguished time scale, which will be used for the above definitions.

Let us consider a galilean spacetime connection $K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ (see Definition 4.3.1) and the associated phase objects (see Theorem 9.2.1 and Corollary 9.2.4)

$$
\begin{aligned}
&\Gamma[K] : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}, \quad &&\gamma[K] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E},\\
&\Omega[G, K] : J_1\boldsymbol{E} \to \Lambda^2 T^*J_1\boldsymbol{E}, \quad &&\Lambda[G, K] : J_1\boldsymbol{E} \to \Lambda^2 VJ_1\boldsymbol{E}.
\end{aligned}
$$

We recall that $\Omega[G, K]$ turns out to be a cosymplectic phase 2–form; hence, we can derive from it the phase 1-forms (see Theorem 10.1.8)

$$\mathcal{L}[G, K] : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \quad \text{and} \quad \mathcal{M}[G, K] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}.$$

11.2.1 Linear Phase Musical Morphisms

We start by defining the "linear" *phase musical morphisms* $\Omega^\flat$ and $\Lambda^\sharp$. They are not isomorphisms; however their vertical restrictions $\Omega^\flat{}_0$ and $\Lambda^\sharp{}_0$ turn out to be the mutually inverse isomorphisms expressed by the following useful formulas

$$\Omega^\flat(X^\uparrow) = (\Lambda^\sharp{}_0)^{-1}\big(X^\uparrow - \gamma(X^\uparrow)\big) \quad \text{and} \quad \Lambda^\sharp(\alpha^\uparrow) = (\Omega^\flat{}_0)^{-1}\big(\alpha^\uparrow - i_\gamma\, \alpha^\uparrow\big).$$

Definition 11.2.1 The *linear phase musical morphisms* are defined to be the linear fibred morphisms over $J_1\boldsymbol{E}$

$$\Omega^\flat : TJ_1\boldsymbol{E} \to T^*J_1\boldsymbol{E} : X^\uparrow \mapsto i_{X^\uparrow}\,\Omega,$$
$$\Lambda^\sharp : T^*J_1\boldsymbol{E} \to TJ_1\boldsymbol{E} : \alpha^\uparrow \mapsto i_{\alpha^\uparrow}\,\Lambda. \quad \square$$

We show several useful coordinate expressions of the phase musical morphisms.

Proposition 11.2.2 *For each section $X^\uparrow : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$, we have the following coordinate expressions*

$$\begin{aligned}
\Omega^\flat\,(X^\uparrow) &\equiv \Omega^\flat\,(X^0\,\partial_0 + X^i\,\partial_i + X^i_0\,\partial^0_i)\\
&= \big(G^0_{ij}\,(X^j_0 - X^0\,\gamma_0{}^j_0) + (\Gamma_{ij} - \Gamma_{ji})\,(X^j - X^0\,x^j_0)\big)\,(d^i - x^i_0\,d^0)\\
&\quad - G^0_{ij}\,(X^j - X^0\,x^j_0)\,(d^i_0 - \gamma_0{}^i_0\,d^0)\\
&= \big(G^0_{ij}\,X^j_0 - X^0\,(\partial_i\mathcal{L}_0 - \text{\d{d}}_0.\mathcal{P}_i) - (\partial_i\mathcal{P}_j - \partial_j\mathcal{P}_i)\,(X^j - X^0\,x^j_0)\big)\,(d^i - x^i_0\,d^0)\\
&\quad - G^0_{ij}\,(X^j - X^0\,x^j_0)\,(d^i_0 - \gamma_0{}^i_0\,d^0),
\end{aligned}$$

i.e.

$$\begin{aligned}
\Omega^\flat(X^\uparrow) &\equiv \Omega^\flat\,(\tilde{X}^0\,e_0 + \tilde{X}^i\,e_i + \tilde{X}^i_0\,e^0_i)\\
&= \big(G^0_{ij}\,\tilde{X}^j_0 + (\Gamma_{ij} - \Gamma_{ji})\,\tilde{X}^j\big)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0\\
&= (G^0_{ij}\,\tilde{X}^j_0 - \big((\partial_i G^0_{jl} - \partial_j G^0_{il})\,x^l_0 + \Phi_{ij}\big)\,\tilde{X}^j)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0\\
&= \big(G^0_{ij}\,\tilde{X}^j_0 - (\partial_i\mathcal{P}_j - \partial_j\mathcal{P}_i)\,\tilde{X}^j\big)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0,
\end{aligned}$$

where

$$\tilde{X}^0 = X^0, \qquad \tilde{X}^i = X^i - X^0\, x^i_0, \qquad \tilde{X}^i_0 = X^i_0 - X^0\, \gamma_0{}^i{}_0,$$
$$X^0 = \tilde{X}^0, \qquad X^i = \tilde{X}^i + \tilde{X}^0\, x^i_0, \qquad X^i_0 = \tilde{X}^i_0 + \tilde{X}^0\, \gamma_0{}^i{}_0,$$

and

$$\begin{array}{lll} e_0 = \partial_0 + x^i_0\, \partial_i + \gamma_0{}^i{}_0\, \partial^0_i & e_i = \partial_i & e^0_i = \partial^0_i \\ \epsilon^0 = d^0 & \epsilon^i = d^i - x^i_0\, d^0 & \epsilon^i_0 = d^i_0 - \gamma_0{}^i{}_0\, d^0 \\ \partial_0 = e_0 - x^i_0\, e_i - \gamma_0{}^i{}_0\, e^0_i & \partial_i = e_i & \partial^0_i = e^0_i \\ d^0 = \epsilon^0 & d^i = \epsilon^i + x^i_0\, \epsilon^0 & d^i_0 = \epsilon^i_0 + \gamma_0{}^i{}_0\, \epsilon^0. \quad \square \end{array}$$

Proposition 11.2.3 *For each section $\alpha^\uparrow : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E}$, we have the following coordinate expressions*

$$\begin{aligned} \Lambda^\sharp(\alpha^\uparrow) \;&\equiv\; \Lambda^\sharp\,(\alpha_0\, d^0 + \alpha_i\, d^i + \alpha^0_i\, d^i_0) \\ &= -G^{ij}_0\, \alpha^0_j\, \partial_i + \big(G^{ij}_0\, \alpha_j + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})\, \alpha^0_j\big)\, \partial^0_i \\ &= -G^{ij}_0\, \alpha^0_j\, \partial_i + \Big(G^{ij}_0\, \alpha_j - G^{ih}_0\, G^{jk}_0\, \big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \Phi_{hk}\big)\, \alpha^0_j\Big)\, \partial^0_i \\ &= -G^{ij}_0\, \alpha^0_j\, \partial_i + \big(G^{ij}_0\, \alpha_j - G^{ih}_0\, G^{jk}_0\, (\partial_h \mathcal{P}_k - \partial_k \mathcal{P}_h)\, \alpha^0_j\big)\, \partial^0_i, \end{aligned}$$

i.e.

$$\begin{aligned} \Lambda^\sharp\,(\alpha^\uparrow) \;&\equiv\; \Lambda^\sharp\,(\tilde{\alpha}_0\, \epsilon^0 + \tilde{\alpha}_i\, \epsilon^i + \tilde{\alpha}^0_i\, \epsilon^i_0) \\ &= -G^{ij}_0\, \tilde{\alpha}^0_j\, e_i + \big(G^{ij}_0\, \tilde{\alpha}_j + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})\, \tilde{\alpha}^0_j\big)\, e^0_i \\ &= -G^{ij}_0\, \tilde{\alpha}^0_j\, e_i + \big(G^{ij}_0\, \tilde{\alpha}_j - G^{ih}_0\, G^{jk}_0\, (\partial_h \mathcal{P}_k - \partial_k \mathcal{P}_h)\, \tilde{\alpha}^0_j\big)\, e^0_i, \end{aligned}$$

where

$$\tilde{\alpha}_0 := \alpha_0 + \alpha_i\, x^i_0 + \alpha^0_i\, \gamma_0{}^i{}_0, \quad \tilde{\alpha}_i := \alpha_i, \quad \tilde{\alpha}^0_i := \alpha^0_i,$$
$$\alpha_0 = \tilde{\alpha}_0 - \tilde{\alpha}_i\, x^i_0 - \tilde{\alpha}^0_i\, \gamma_0{}^i{}_0, \quad \alpha_i = \tilde{\alpha}_i, \quad \alpha^0_i = \tilde{\alpha}^0_i. \quad \square$$

Corollary 11.2.4 *We have*

$$\begin{array}{lcl} \mathrm{im}\,(\Omega^\flat) \;=\; V^*_\gamma J_1\boldsymbol{E} & \text{and} & \mathrm{im}\,(\Lambda^\sharp) \;=\; V_{\boldsymbol{T}} J_1\boldsymbol{E}, \\ \ker\,(\Omega^\flat) \;=\; H_\gamma J_1\boldsymbol{E} & \text{and} & \ker\,(\Lambda^\sharp) \;=\; H^*_{\boldsymbol{T}} J_1\boldsymbol{E}. \end{array}$$

In other words, for each section $X^\uparrow : J_1\boldsymbol{E} \to T J_1\boldsymbol{E}$ and $\alpha^\uparrow : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E}$, we obtain

$$\langle \gamma,\ \Omega^\flat(X^\uparrow)\rangle = 0,$$
$$\Omega^\flat(X^\uparrow) = 0 \qquad \Rightarrow \qquad X^\uparrow = \gamma(\tau),$$
$$\langle dt,\ \Lambda^\sharp(\alpha^\uparrow)\rangle = 0,$$
$$\Lambda^\sharp(\alpha^\uparrow) = 0 \qquad \Rightarrow \qquad \alpha^\uparrow = dt(\acute{\tau}),$$

with

$$\tau \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{T} \otimes I\!R) \quad \text{and} \quad \acute{\tau} \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{T}^* \otimes I\!R). \quad \square$$

Corollary 11.2.5 *We have*

$$\Omega^\flat \circ \nu[\gamma] = \Omega^\flat \qquad \text{and} \qquad \Lambda^\sharp \circ \Omega^\flat = \nu[\gamma],$$
$$\Lambda^\sharp \circ i_{\nu[\gamma]} = \Lambda^\sharp \qquad \text{and} \qquad \Omega^\flat \circ \Lambda^\sharp = i_{\nu[\gamma]}.$$

In other words, for each section $X^\uparrow : J_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$ *and* $\alpha^\uparrow : J_1\boldsymbol{E} \to T^*J_1\boldsymbol{E}$, *we obtain*

$$(\Lambda^\sharp \circ \Omega^\flat)(X^\uparrow) = X^\uparrow - \gamma(X^\uparrow) \quad \text{and} \quad (\Omega^\flat \circ \Lambda^\sharp)(\alpha^\uparrow) = \alpha^\uparrow - i_\gamma\, \alpha^\uparrow. \quad \square$$

Then, we obtain the following result.

Theorem 11.2.6 *The restrictions of* $\Omega^\flat$ *and* $\Lambda^\sharp$ *to the vector subbundles*

$$V_{\boldsymbol{T}}J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E} \quad \text{and} \quad V^*_\gamma J_1\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$$

yield mutually inverse linear fibred isomorphisms over $J_1\boldsymbol{E}$

$$\Omega^\flat{}_0 : V_{\boldsymbol{T}}J_1\boldsymbol{E} \to V^*_\gamma J_1\boldsymbol{E} \quad \text{and} \quad \Lambda^\sharp{}_0 : V^*_\gamma J_1\boldsymbol{E} \to V_{\boldsymbol{T}}J_1\boldsymbol{E}. \quad \square$$

As a consequence, we obtain the following formulas, which will be frequently used.

Corollary 11.2.7 *We have*

$$\Omega^\flat = (\Lambda^\sharp{}_0)^{-1} \circ \nu[\gamma] \quad \text{and} \quad \Lambda^\sharp = (\Omega^\flat{}_0)^{-1} \circ i_{\nu[\gamma]}.$$

In other words, for each $X^\uparrow \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *and* $\alpha^\uparrow \in \sec(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$, *we obtain*

$$\Omega^\flat(X^\uparrow) = (\Lambda^\sharp{}_0)^{-1}\big(X^\uparrow - \gamma(X^\uparrow)\big) \quad \text{and} \quad \Lambda^\sharp(\alpha^\uparrow) = (\Omega^\flat{}_0)^{-1}\big(\alpha^\uparrow - i_\gamma\, \alpha^\uparrow\big). \quad \square$$

Now, we are in the position to prove the duality of our cosymplectic and coPoisson structures of phase space (see Theorems 10.1.1 and 10.2.1 and Appendix: Definition I.1.5).

Corollary 11.2.8 *In virtue of Theorem I.1.11, the regular pairs*

$$\big(dt, \Omega[G,K]\big) \quad \text{and} \quad \big(\gamma[K], \Omega[G,K]\big)$$

turn out to be mutually dual (see Definition I.1.5). □

11.2.2 Affine Phase Musical Morphisms

Next, by choosing a phase time scale τ and the γ-horizontal vector field $\gamma(\tau)$. we obtain affine mutually inverse isomorphisms.

Thus, let us consider a *phase time scale*

$$\tau = \tau^0 \, u_0 : J_1\boldsymbol{E} \to \mathbb{T} \otimes \mathrm{I\!R}, \quad \text{with } \tau^0 \in \operatorname{map}(J_1\boldsymbol{E}, \mathrm{I\!R}).$$

Definition 11.2.9 We define the *τ-horizontal vector subbundle* to be the affine subbundle over $J_1\boldsymbol{E}$

$$H_\tau J_1\boldsymbol{E} := \{X^\uparrow \in TJ_1\boldsymbol{E}) \mid i_{X^\uparrow}\, dt = \tau\} \subset TJ_1\boldsymbol{E}. \quad \square$$

Note 11.2.10 The phase time scale τ yields the distinguished γ-horizontal vector field (see Proposition 11.1.1)

$$\gamma(\tau) = \tau^0\, \gamma_0 : J_1\boldsymbol{E} \to H_\gamma J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}. \quad \square$$

Theorem 11.2.11 *The restriction to the vector subbundle $H_\tau J_1\boldsymbol{E} \subset TJ_1\boldsymbol{E}$ of the linear musical morphism $\Omega^\flat : TJ_1\boldsymbol{E} \to T^*J_1\boldsymbol{E}$ turns out to be the affine isomorphism*

$$\Omega^\flat{}_\tau : H_\tau J_1\boldsymbol{E} \to V^*_\gamma J_1\boldsymbol{E} : X^\uparrow \mapsto i_{X^\uparrow}\,\Omega,$$

whose inverse isomorphism is

$$\Lambda^\sharp{}_\tau := (\Omega^\flat{}_\tau)^{-1} : V^*_\gamma J_1\boldsymbol{E} \to H_\tau J_1\boldsymbol{E} : \alpha^\uparrow \to \gamma(\tau) + \Lambda^\sharp(\alpha^\uparrow). \quad \square$$

Proposition 11.2.12 *The above affine isomorphisms can be expressed in coordinates in several ways, as follows.*

For each $X^\uparrow \in \operatorname{sec}(J_1\boldsymbol{E}, H_\tau J_1\boldsymbol{E})$, we have the following coordinate expressions

$$\begin{aligned}
\Omega^{\flat}{}_{\tau}\,(X^{\uparrow}) \;\equiv\; & \Omega^{\flat}{}_{\tau}\,(\tau^0\,\partial_0 + X^i\,\partial_i + X^i_0\,\partial^0_i) \\
&= \big(G^0_{ij}\,(X^j_0 - \tau^0\,\gamma_0{}^j_0) + (\Gamma_{ij} - \Gamma_{ji})\,(X^j - \tau^0\,x^j_0)\big)\,(d^i - x^i_0\,d^0) \\
&\quad - G^0_{ij}\,(X^j - \tau^0\,x^j_0)\,(d^i_0 - \gamma_0{}^i_0\,d^0) \\
&= \big(G^0_{ij}\,X^j_0 - \tau^0\,(\partial_i\mathcal{L}_0 - \partial_0.\mathcal{P}_i) - (\partial_i\mathcal{P}_j - \partial_j\mathcal{P}_i)\,(X^j - \tau^0\,x^j_0)\big)\,(d^i - x^i_0\,d^0) \\
&\quad - G^0_{ij}\,(X^j - \tau^0\,x^j_0)\,(d^i_0 - \gamma_0{}^i_0\,d^0),
\end{aligned}$$

i.e.

$$\begin{aligned}
\Omega^{\flat}{}_{\tau}(X^{\uparrow}) \;\equiv\; & \Omega^{\flat}{}_{\tau}\,(\tilde{\tau}^0\,e_0 + \tilde{X}^i\,e_i + \tilde{X}^i_0\,e^0_i) \\
&= \big(G^0_{ij}\,\tilde{X}^j_0 + (\Gamma_{ij} - \Gamma_{ji})\,\tilde{X}^j\big)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0 \\
&= (G^0_{ij}\,\tilde{X}^j_0 - \big((\partial_i G^0_{jl} - \partial_j G^0_{il})\,x^l_0 + \Phi_{ij}\big)\,\tilde{X}^j)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0 \\
&= \big(G^0_{ij}\,\tilde{X}^j_0 - (\partial_i\mathcal{P}_j - \partial_j\mathcal{P}_i)\,\tilde{X}^j\big)\,\epsilon^i - G^0_{ij}\,\tilde{X}^j\,\epsilon^i_0,
\end{aligned}$$

where

$$\begin{aligned}
\tilde{X}^0 &= \tau^0, & \tilde{X}^i &= X^i - \tau^0\,x^i_0, & \tilde{X}^i_0 &= X^i_0 - \tau^0\,\gamma_0{}^i_0, \\
\tau^0 &= \tilde{X}^0, & X^i &= \tilde{X}^i + \tilde{X}^0\,x^i_0, & X^i_0 &= \tilde{X}^i_0 + \tilde{X}^0\,\gamma_0{}^i_0.
\end{aligned}$$

For each $\alpha^{\uparrow} \in \mathrm{sec}(J_1\boldsymbol{E}, V^*_{\gamma}J_1\boldsymbol{E})$, we have the following coordinate expressions

$$\begin{aligned}
\Lambda^{\sharp}{}_{\tau}(\alpha^{\uparrow}) \;\equiv\; & \Lambda^{\sharp}{}_{\tau}\,(\alpha_0\,d^0 + \alpha_i\,d^i + \alpha^0_i\,d^i_0) \\
&= \tau^0\,(\partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i) \\
&\quad - G^{ij}_0\,\alpha^0_j\,\partial_i + \big(G^{ij}_0\,\alpha_j + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})\,\alpha^0_j\big)\,\partial^0_i \\
&= \tau^0\,(\partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i) \\
&\quad - G^{ij}_0\,\alpha^0_j\,\partial_i + \Big(G^{ij}_0\,\alpha_j - G^{ih}_0\,G^{jk}_0\,\big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\,x^l_0 + \Phi_{hk}\big)\,\alpha^0_j\Big)\,\partial^0_i \\
&= \tau^0\,(\partial_0 + x^i_0\,\partial_i + \gamma_0{}^i_0\,\partial^0_i) \\
&\quad - G^{ij}_0\,\alpha^0_j\,\partial_i + \big(G^{ij}_0\,\alpha_j - G^{ih}_0\,G^{jk}_0\,(\partial_h\mathcal{P}_k - \partial_k\mathcal{P}_h)\,\alpha^0_j\big)\,\partial^0_i,
\end{aligned}$$

i.e.

$$\begin{aligned}
\Lambda^{\sharp}{}_{\tau}\,(\alpha^{\uparrow}) \;\equiv\; & \Lambda^{\sharp}{}_{\tau}\,(\tilde{\alpha}_i\,\epsilon^i + \tilde{\alpha}^0_i\,\epsilon^i_0) \\
&= \tau^0\,e_0 - G^{ij}_0\,\tilde{\alpha}^0_j\,e_i + \big(G^{ij}_0\,\tilde{\alpha}_j + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0})\,\tilde{\alpha}^0_j\big)\,e^0_i \\
&= \tau^0\,e_0 - G^{ij}_0\,\tilde{\alpha}^0_j\,e_i + \big(G^{ij}_0\,\tilde{\alpha}_j - G^{ih}_0\,G^{jk}_0\,(\partial_h\mathcal{P}_k - \partial_k\mathcal{P}_h)\,\tilde{\alpha}^0_j\big)\,e^0_i,
\end{aligned}$$

where

$$\tilde{\alpha}_0 := \alpha_0 + \alpha_i \, x^i_0 + \alpha^0_i \, \gamma_0{}^i_0, \qquad \tilde{\alpha}_i := \alpha_i, \qquad \tilde{\alpha}^0_i := \alpha^0_i,$$
$$\alpha_0 = \tilde{\alpha}_0 - \tilde{\alpha}_i \, x^i_0 - \tilde{\alpha}^0_i \, \gamma_0{}^i_0, \qquad \alpha_i = \tilde{\alpha}_i, \qquad \alpha^0_i = \tilde{\alpha}^0_i. \quad \square$$

11.3 Hamiltonian Phase Lift of Phase Functions

Let us consider a galilean spacetime connection K and the associated phase fields $\Gamma, \gamma, \Omega, \Lambda$. Now, we are in a position to discuss the phase lifts of phase functions f.

We start by defining the *scaled hamiltonian lift* of phase functions with respect to an arbitrary phase time scale τ. Further, we show that every phase function f yields a distinguished phase time scale f''. Hence, we obtain a (natural) *hamiltonian lift* of phase functions.

Indeed, the hamiltonian lift of phase functions fulfills the following useful property $i_{X\uparrow_{\text{ham}}[f]} \Omega = df - \gamma.f$ and yields the equivalence $i_{X\uparrow_{\text{ham}}[f]} \Omega = df \Leftrightarrow \gamma.f = 0$.

The hamiltonian lift of phase functions will be largely used throughout to book, in several contexts.

11.3.1 Scaled Hamiltonian Phase Lift of Phase Functions

Let us consider a galilean spacetime connection K and the associated phase objects $\Gamma, \gamma, \Omega, \Lambda$. By recalling the natural splitting of the tangent space of phase space (see Proposition 11.1.1), we introduce the *scaled hamiltonian lift* of phase functions, with respect to a phase time scale τ. Then, we prove some technical statements, which will be frequently used throughout the book.

At a first insight, this definition of scaled hamiltonian phase lift might appear to be rather arbitrary. However, this concept arises, in a natural way, in several steps of this book.

More generally, this concept arises, in a natural way, in a general theorem which classifies the infinitesimal symmetries of a pair (ω, Ω) consisting of a 1-form ω and closed 2-form Ω (see [205]).

In the next section, we shall exhibit a distinguished choice of this time scale associated with each phase function.

Definition 11.3.1 We define the *scaled hamiltonian phase lift* of a phase function $f \in \text{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$, with respect to the *phase time scale* $\tau \in \text{map}(J_1\boldsymbol{E}, \mathbb{T} \otimes \mathrm{I\!R})$, to be the phase vector field

$$X\uparrow_{\text{ham}}[\tau, f] := \tau \lrcorner \gamma + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

Accordingly, we define the sheaf morphism

$$X^{\uparrow}{}_{\text{ham}}[\tau] : \text{map}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[\tau, f].$$

Moreover, we define the subsheaf of scaled hamiltonian lifts (for all phase scale functions) by

$$\text{scl}\,\text{ham}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}). \quad \square$$

Proposition 11.3.2 *For each* $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$ *and* $\tau \in \text{map}(J_1\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R})$*, we have the coordinate expression (see Theorem 10.1.8)*

$$X^{\uparrow}{}_{\text{ham}}[\tau, f] = \tau^0\, \partial_0 + (\tau^0\, x_0^i - G_0^{ij}\, \partial_j^0 f)\, \partial_i + \big(\tau^0\, \gamma_0{}_0^i + G_0^{ij}\, \partial_j f - G_0^{ih}\, G_0^{jk}\, (\partial_h \mathcal{P}_k - \partial_k \mathcal{P}_h)\, \partial_j^0 f\big)\, \partial_i^0. \quad \square$$

The scaled hamiltonian lift fulfills the following property, which will be used frequently.

Proposition 11.3.3 *For each* $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$ *and* $\tau \in \text{map}(J_1\boldsymbol{E}, \bar{\mathbb{T}})$*, we have*

$$i_{X^{\uparrow}{}_{\text{ham}}[\tau, f]}\, \Omega = df - \gamma.f.$$

Hence, we have the following equivalence

$$i_{X^{\uparrow}{}_{\text{ham}}[\tau, f]}\, \Omega = df \qquad \Leftrightarrow \qquad \gamma.f = 0.$$

Proof. In virtue of Theorem 9.2.19 and Corollary 11.2.5, we have

$$\Omega^{\flat}\big(\tau \lrcorner \gamma + \Lambda^{\sharp}(df)\big) = \Omega^{\flat}\big(\tau \lrcorner \gamma\big) + \Omega^{\flat}\big(\Lambda^{\sharp}(df)\big) = 0 + df - i_{\gamma} df. \quad \square$$

Conversely, we have the following result, which characterises our definition of scaled hamiltonian lift of phase functions.

Theorem 11.3.4 *Let* $X^{\uparrow} \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *and* $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$*. Then, the following conditions are equivalent:*

(1) $$i_{X^{\uparrow}}\, \Omega = df,$$

(2) $$X^{\uparrow} = X^{\uparrow}{}_{\text{ham}}\big[dt(X^{\uparrow}), f\big] = \big(dt(X^{\uparrow})\big) \lrcorner \gamma + \Lambda^{\sharp}(df) \quad \text{and} \quad \gamma.f = 0.$$

Proof. (1) $\Rightarrow$ (2).

Corollary 11.2.7 implies $(\Lambda^{\sharp} \circ \Omega^{\flat})(X^{\uparrow}) = X^{\uparrow} - \gamma(X^{\uparrow}) = X^{\uparrow} - dt(X^{\uparrow}) \lrcorner \gamma$.

Hence, the equality $i_{X^{\uparrow}}\, \Omega = df$ yields

$$X^{\uparrow} = dt(X^{\uparrow}) \lrcorner \gamma + \Lambda^{\sharp}\big(\Omega^{\flat}(X^{\uparrow})\big) = dt(X^{\uparrow}) \lrcorner \gamma + \Lambda^{\sharp}(df) = X^{\uparrow}{}_{\text{ham}}\big[dt(X^{\uparrow}), f\big].$$

Moreover, by recalling Theorem 9.2.19, the equality $i_{X^{\uparrow}}\, \Omega = df$ implies

$$\gamma.f = i_\gamma df = i_\gamma i_{X^\uparrow}\Omega = -i_{X^\uparrow} i_\gamma \Omega = 0.$$

(2) $\Rightarrow$ (1).

By recalling Corollary 11.2.7 and Theorem 9.2.19, the equalities $X^\uparrow = \big(dt(X^\uparrow)\big) \lrcorner\, \gamma + \Lambda^\sharp(df)$ and $\gamma.f = 0$ imply $\Omega^\flat(X^\uparrow) = \Omega^\flat\Big(\big(dt(X^\uparrow)\big) \lrcorner\, \gamma + \Lambda^\sharp(df)\Big) = \Omega\big(\Lambda^\sharp(df)\big) = df - i_\gamma df = df.$ □

Later, the above result will be further developed by Theorem 12.6.17 in the context of special phase functions and by Proposition 13.1.2, in the context of symmetries of phase space.

11.3.2 Natural Hamiltonian Phase Lift of Phase Functions

We show that every phase function yields a distinguished phase time scale f'', which can be used to achieve a natural *hamiltonian lift*.

Later, we shall apply these results to special phase functions (see Definition 12.1.1).

Let us consider a particle of mass $m \in \mathbb{M}$ and refer to the rescaled galilean metric $G : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$ (see Definition 3.2.1).

First of all, we show that every phase function yields, in a natural way, a distinguished phase time scale.

Lemma 11.3.5 *For each phase function* $f \in \text{map}(J_1\boldsymbol{E}, I\!R)$*, we can compute the 2nd fibre derivative, with respect to the affine structure of the bundle* $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ *(see Proposition 2.5.1),*

$$D^2 f : J_1\boldsymbol{E} \to \mathbb{T}^2 \otimes V^*\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

with coordinate expression

$$D^2 f = \partial^0_i \partial^0_j f\, u_0 \otimes u_0 \otimes \check{d}^i \otimes \check{d}^j.$$

Then, we obtain the distinguished phase time scale, called the time component *of* f*,*

$$f'' := \tfrac{1}{3}\, \bar{G} \lrcorner\, D^2 f \in \text{map}(J_1\boldsymbol{E}, \mathbb{T} \otimes I\!R),$$

with coordinate expression

$$f'' = f^0\, u_0 = \tfrac{1}{3}\, G^{ij}_0\, \partial^0_i \partial^0_j f\, u_0, \quad \text{where } f^0 = \tfrac{1}{3}\, G^{ij}_0\, \partial^0_i \partial^0_j f. \quad □$$

Then, by replacing, in Definition 11.3.1, the arbitrary phase time scale τ with the distinguished time scale f'', we obtain the following notion.

Definition 11.3.6 We define the *(natural) hamiltonian phase lift* of a phase function $f \in \text{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ to be the vector field

$$X^{\uparrow}{}_{\text{ham}}[f] := X^{\uparrow}_{\text{ham}}[f'', f] = \gamma(f'') + \Lambda^{\sharp}(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

Accordingly, we define the sheaf morphism

$$X^{\uparrow}{}_{\text{ham}} : \text{map}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[f'', f]. \quad \square$$

Corollary 11.3.7 *For each* $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$*, we have the coordinate expression (see Theorem 10.1.8)*

$$X^{\uparrow}{}_{\text{ham}}[f] = f^0\,\partial_0 + (f^0\,x^i_0 - G^{ij}_0\,\partial^0_j f)\,\partial_i + \big(f^0\,\gamma_0{}^i{}_0 + G^{ij}_0\,\partial_j f - G^{ih}_0\,G^{jk}_0\,(\partial_h\mathcal{P}_k - \partial_k\mathcal{P}_h)\,\partial^0_j f\big)\,\partial^0_i.$$

Proof. The proof follows immediately from Proposition 11.3.2 and Lemma 11.3.5. □

The hamiltonian lift fulfills the following property, which will be used frequently.

Theorem 11.3.8 *For each* $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$*, we have*

$$i_{X^{\uparrow}{}_{\text{ham}}[f]}\,\Omega = df - \gamma.f.$$

Hence, we have the following equivalence

$$i_{X^{\uparrow}{}_{\text{ham}}[f]}\,\Omega = df \qquad \Leftrightarrow \qquad \gamma.f = 0.$$

Proof. The theorem follows immediately from Theorem 11.3.4. □

Later, in the context of special phase functions, we shall provide the coordinate expression of the hamiltonian lift of distinguished phase functions (see Example 12.4.3).

Remark 11.3.9 Analogous "hamiltonian" lifts of phase functions are typical of geometrical structures of odd-dimensional manifolds equipped with a Reeb vector field E and a Poisson 2-vector Λ (see, for instance, [75, 223, 224]).

In such theories, as hamiltonian lift of a function f is usually assumed either the vector field (cosymplectic manifolds)

$$X_f = \Lambda^{\sharp}(df),$$

or the vector field (contact or Jacobi manifolds)

$$X_f = \Lambda^{\sharp}(df) + f\,E.$$

Thus, by comparing our hamiltonian lift

$$X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$$

with the above hamiltonian lift

$$X_f = \Lambda^{\sharp}(df) + f\, E,$$

we see that our scaled dynamical phase connection γ replaces the standard Reeb vector field E, and our "unscaled" term $\gamma(f'')$ replaces the standard term $f\, E$.

Actually, we stress that the scaled term $f\,\gamma$ could not be added, in a covariant way, to the unscaled term $\Lambda^{\sharp}(df)$ without the choice of a unit measurements of time ([227, 228, 428]). □

11.4 Poisson Lie Bracket

The Poisson Lie bracket is a standard object of an even dimensional manifold equipped with a Poisson structure Λ (see, for instance, [1, 146]). However, our phase space is odd dimensional; accordingly, our basic 2–form Ω is cosymplectic and the associated 2–vector Λ is not properly its inverse (see Theorems 11.2.6 and 11.2.11).

So, our definition of Poisson bracket is analogous to the standard one, but some differences arise.

Definition 11.4.1 For each $f, \acute{f} \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$, we define their *Poisson bracket* to be the phase function

$$\{f,\ \acute{f}\} := \Lambda^{\sharp}(df \wedge d\acute{f}) = \big(\Lambda^{\sharp}(df)\big).\acute{f} = -\big(\Lambda^{\sharp}(d\acute{f})\big).f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R}),$$

with coordinate expression (see Corollary 9.2.4)

$$\begin{aligned}\{f, \acute{f}\} &= G^{ij}_0\, (\partial_i f\, \partial^0_j \acute{f} - \partial_i \acute{f}\, \partial^0_j f) + G^{ih}_0\, G^{jk}_0\, (\partial_h \mathcal{P}_k - \partial_k \mathcal{P}_h)\, \partial^0_i f\, \partial^0_j \acute{f} \\ &= G^{ij}_0\, (\partial_i f\, \partial^0_j \acute{f} - \partial_i \acute{f}\, \partial^0_j f) + G^{ih}_0 G^{jk}_0 \\ &\quad \times \big((\partial_h G^0_{kl} - \partial_k G^0_{hl})\, x^l_0 + \partial_h A_k - \partial_k A_h\big)\, \partial^0_i f\, \partial^0_j \acute{f}. \quad \square\end{aligned}$$

Note 11.4.2 The Poisson bracket $\{f, \acute{f}\}$ factorises through the vertical differentials $\check{d} f$ and $\check{d} \acute{f}$. More precisely, we can write

$$\{f,\ \acute{f}\} = \bar{G}\,(\check{d}_{\check{\Gamma}} f,\ \check{d}_{\boldsymbol{E}} \acute{f}) - \bar{G}\,(\check{d}_{\check{\Gamma}} \acute{f},\ \check{d}_{\boldsymbol{E}} f),$$

where we have defined the maps (see Definition 9.1.4)

$$\check{d}_{\boldsymbol{E}} f := \nu \circ \check{d} f : J_1\boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E} \quad \text{and} \quad \check{d}_{\check{\Gamma}} f := \check{\Gamma} \circ \check{d} f : J_1\boldsymbol{E} \to V^*\boldsymbol{E},$$

with coordinate expressions

$$\check{d}_{\boldsymbol{E}} f = \partial_i^0 f \, u_0 \otimes \check{d}^i \quad \text{and} \quad \check{d}_{\check{\Gamma}} f = (\partial_i f + \Gamma_i{}^j{}_0 \, \partial_j^0 f) \, \check{d}^i . \quad \square$$

Note 11.4.3 Let $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$. Then, we obtain

$$\{f, \acute{f}\} = 0, \quad \forall \acute{f} \in \text{map}(J_1\boldsymbol{E}, \ \mathbb{R}) \qquad \Leftrightarrow \qquad f \in \text{map}(\boldsymbol{T}, \mathbb{R}).$$

For each $f, \acute{f} \in \text{map}(\boldsymbol{E}, \ \mathbb{R})$, we have $\{f, \ \acute{f}\} = 0$. □

The dynamical phase connection γ turns out to be as a derivation of the Poisson bracket.

Proposition 11.4.4 *For each* $f, \acute{f} \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$*, we have*

$$\gamma.\{f, \acute{f}\} = \{\gamma.f, \acute{f}\} + \{f, \gamma.\acute{f}\}.$$

Proof. The Leibniz rule of the Lie derivative and the equality $L_\gamma \Lambda = 0$ yields (see Theorem 9.2.19)

$$\begin{aligned} L_\gamma\{f, \ \acute{f}\} &= L_\gamma\big(\Lambda(df, \ d\acute{f})\big) \\ &= \Lambda(L_\gamma df, \ d\acute{f}) + \Lambda(df, \ L_\gamma d\acute{f}) = \Lambda(dL_\gamma f, \ d\acute{f}) + \Lambda(df, \ dL_\gamma \ \acute{f}) \\ &= \{\gamma.f, \acute{f}\} + \{f, \gamma.\acute{f}\}. \quad \square \end{aligned}$$

In order to prove that the above Poisson bracket is a Lie bracket, we need a preliminary technical formula.

Lemma 11.4.5 *For each* $f, \acute{f} \in \text{map}(J_1\boldsymbol{E}, \ \mathbb{R})$*, we have*

$$\Lambda^\sharp\big(d\{f, \acute{f}\}\big) = \big[\Lambda^\sharp(df), \ \Lambda^\sharp(d\acute{f})\big].$$

Proof. In virtue of Proposition 11.2.3, we have

$$(1) \quad \Omega^\flat(\Lambda^\sharp(d\acute{f})) = d\acute{f} - \gamma.\acute{f} \quad \text{and} \quad (2) \quad \Omega^\flat\Big(\Lambda^\sharp(d\{f, \ \acute{f}\})\Big) = d\{f, \ \acute{f}\} - \gamma.\{f, \ \acute{f}\}.$$

Hence, in virtue of a standard property of derivations and of Lie derivatives (see [146]) and (1), we obtain

$$\begin{aligned}
\Omega^\flat\Big(\big[\Lambda^\sharp(df),\ \Lambda^\sharp(d\acute{f})\big]\Big) &= i_{\big[\Lambda^\sharp(df),\,\Lambda^\sharp(d\acute{f})\big]}\Omega \\
&= L_{\Lambda^\sharp(df)}\, i_{\Lambda^\sharp(d\acute{f})}\,\Omega - i_{\Lambda^\sharp(d\acute{f})}\, L_{\Lambda^\sharp(df)]}\,\Omega \\
&= L_{[\Lambda^\sharp(df)]}\,(d\acute{f} - \gamma.\acute{f}) - i_{\Lambda^\sharp(d\acute{f})}\, d(df - \gamma.f) \\
&= di_{\Lambda^\sharp(df)}\, d\acute{f} - i_{\Lambda^\sharp(df)}\, d(\gamma.\acute{f}) + i_{\Lambda^\sharp(d\acute{f})}\, d(\gamma.f) \\
&= d\{f, \acute{f}\} - \{f,\ \gamma.\acute{f}\} + \{\acute{f},\ \gamma.f\} \\
&= d\{f, \acute{f}\} - \{f, \gamma.\acute{f}\} - \{\gamma.f,\ \acute{f}\}.
\end{aligned}$$

Therefore, in virtue of (2), we obtain

$$\Omega^\flat\Big(\big[\Lambda^\sharp(df),\ \Lambda^\sharp(d\acute{f})\big]\Big) = d\{f, \acute{f}\} - \gamma.\{f, \acute{f}\}.$$

Hence, we can write

$$\Omega^\flat\Big([\Lambda^\sharp(df),\ \Lambda^\sharp(d\acute{f})]\Big) = \Omega^\flat\big(\Lambda^\sharp(d\{f,\ \acute{f}\})\big).$$

Moreover, we can write

$$\Lambda^\sharp\big(d\{f, \acute{f}\}\big) = \big[\Lambda^\sharp(df),\ \Lambda^\sharp(d\acute{f})\big],$$

because the restriction of $\Omega^\flat$ to the vertical subbundle $V_{\boldsymbol{T}} J_1 \boldsymbol{E} \subset \boldsymbol{T} J_1 \boldsymbol{E}$ is an isomorphism. □

Theorem 11.4.6 *The Poisson bracket is a Lie bracket.*

Proof. We just prove the Jacobi property. In virtue of the above Lemma 11.4.5, we obtain

$$\begin{aligned}
\{\{f, g\}, h\} &= \Lambda^\sharp(d\{f, g\}).h = [\Lambda^\sharp(df), \Lambda^\sharp(dg)].h \\
&= \Lambda^\sharp(df).(\Lambda^\sharp(dg).h) - \Lambda^\sharp(dg).(\Lambda^\sharp(df).h) \\
&= \{f, \{g, h\}\} - \{g, \{f, h\}\} = -\{\{h, f\}, g\} - \{\{g, h\}, f\}. \quad \square
\end{aligned}$$

Example 11.4.7 For each $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$, we have the following distinguished Poisson brackets:

$$
\begin{aligned}
\{x^0,\ f\} &= 0,\\
\{x^i,\ f\} &= G_0^{ij}\,\partial_j^0 f,\\
\{x_0^i,\ f\} &= -G_0^{ih}\,\partial_h f + G_0^{ih}\,G_0^{jk}\,(\partial_h\mathcal{P}_k - \partial_k\mathcal{P}_h)\,\partial_j^0 f,\\
\{\mathcal{P}_h,\ f\} &= G_0^{ij}\,\partial_h\mathcal{P}_i\,\partial_j^0 f - \partial_h f,\\
\{\mathcal{H}_0,\ f\} &= -x_0^i\,\partial_i f - \gamma_{0\,0}^{\ i}\,\partial_i^0 f - G_0^{ij}\,\partial_0\mathcal{P}_i\,\partial_j^0 f\\
&= G_0^{ij}\,\big((\partial_h G_{ik}^0 - \tfrac{1}{2}\,\partial_i G_{hk}^0)\,x_0^h\,x_0^k + (\partial_h A_i - \partial_i A_h)\,x_0^h - \partial_i A_0\big)\\
&\qquad \partial_j^0 f - \partial_i f\,x_0^i.
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
&\{x^\lambda,\ x^\mu\} = 0,\quad \{x^i,\ x_0^j\} = G_0^{ij},\quad \{x_0^i,\ x_0^j\} = G_0^{ih}\,G_0^{jk}\,(\partial_h\mathcal{P}_k - \partial_k\mathcal{P}_h),\\
&\{\mathcal{P}_j,\ x^\lambda\} = -\delta_j^\lambda,\quad \{\mathcal{P}_j,\ x_0^i\} = G_0^{ih}\,\partial_j\mathcal{P}_h,\quad \{\mathcal{P}_i,\ \mathcal{P}_j\} = 0,\\
&\{\mathcal{H}_0,\ x^0\} = 0,\quad \{\mathcal{H}_0,\ x^i\} = -x_0^i,\\
&\{\mathcal{H}_0,\ x_0^i\} = -\gamma_{0\,0}^{\ i} - G_0^{ij}\,\partial_0\mathcal{P}_j,\quad \{\mathcal{H}_0,\ \mathcal{P}_i\} = -\partial_i\mathcal{L}_0.\quad \square
\end{aligned}
$$

We have the following interesting decomposition of the Poisson bracket.

Proposition 11.4.8 *For each* $f, g \in \mathrm{map}(J_1\boldsymbol{E}, I\!R)$*, we can write*

$$\{f,\ g\} = \{x^i,\ x_0^j\}\,(\partial_i f\,\partial_j^0 g - \partial_i g\,\partial_j^0 f) + \{x_0^i,\ x_0^j\}\,\partial_i^0 f\,\partial_j^0 g.\quad \square$$

11.5 Classical Law of Motion

We have already postulated the Newton equation of motion for a charged particle effected by the gravitational field $K^\natural$ and the electromagnetic field F, to be the equation $\nabla_{ds} ds = 0$ expressed in terms of the joined spacetime connection K (see Theorem 6.3.1, Assumption C.2 and Lemma 7.2.1).

Now, we equivalently express the above Newton law of motion
(1) as $\nabla[\gamma]\,j_1 s = 0$, in terms of the joined dynamical phase connection γ,
(2) as $\mathcal{E}\big[\mathcal{L}[\mathrm{b}]\big](j_2 s) = G^\flat\big(\nabla_{ds} ds\big)$ in terms of the Euler–Lagrange operator $\mathcal{E}\big[\mathcal{L}[\mathrm{b}]\big]$ associated with the classical lagrangian $\mathcal{L}[\mathrm{b}]$ derived from the joined dynamical phase 2–form $\Omega = \Omega[G, K]$ (see Theorem 10.1.8).

Proposition 11.5.1 *The solutions* $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$ *of the Newton law of motion*

$$\nabla_{ds} ds = 0,$$

turn out to be the solutions of the equation (see Definition 9.1.2)

$$(*)\qquad \nabla[\gamma]\,j_1 s = 0,$$

with coordinate expression

$$\partial_{00} s^i - \gamma_{0\,0}^{\,i} \circ j_1 s = 0,$$

where

$$\begin{aligned}\gamma_{0\,0}^{\,i} &= K_0{}^i{}_0 + 2\, K_0{}^i{}_k\, x_0^k + K_h{}^i{}_k\, x_0^h\, x_0^k \\ &= -G_0^{ij} \big(\Phi_{0j} + (\partial_0 G^0_{hj} + \Phi_{hj})\, x_0^h + (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x_0^h\, x_0^k.\end{aligned}$$

Indeed, equation $(*)$ can be split into its gravitational and electromagnetic components as (see Lemma 7.2.1 and Theorem 9.2.6)

$$(**) \qquad \nabla[\gamma^\natural] j_1 s = -\tfrac{q}{\hbar}\, G^\sharp\, (д \lrcorner F) \circ j_1 s.$$

Hence, the observed expression of the above equation is (see Corollary 9.2.7)

$$\nabla[\gamma^\natural] j_1 s = \tfrac{q}{m}\, \big(\vec{E}[o] \circ s + \vec{v} \times (\vec{B} \circ s)\big). \quad \square$$

Thus, the motion of a charged particle effected by the gravitational and electromagnetic field follows a geodesic of the joined spacetime connection K.

So we might say that the charged particle "sees" the joined spacetime connection K (associated with the coupling constant $\frac{q}{m}$) rather than the gravitational field $K^\natural$ and the electromagnetic field F separately.

Lemma 11.5.2 *Let us consider the (local) classical lagrangian (see Theorem 10.1.8)*

$$\mathcal{L}[\mathrm{b}] : J_1 \boldsymbol{E} \to \mathbb{T}^* \otimes I\!\!R,$$

with coordinate expression

$$\mathcal{L} = (\tfrac{1}{2}\, G^0_{ij}\, x_0^i\, x_0^j + A_i\, x_0^i + A_0)\, u^0$$

and the associated Euler–Lagrange operator

$$\mathcal{E}\big[\mathcal{L}[\mathrm{b}]\big] : J_2 \boldsymbol{E} \to V^* \boldsymbol{E},$$

with coordinate expression

$$\mathcal{E}\big[\mathcal{L}[\mathrm{b}]\big] = (J_0 \partial_i^0 \mathcal{L} - \partial_i \mathcal{L})\, \check{d}^i,$$

where J_0 denotes the "total time derivative".

Then, for each motion $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$, we have the equality

$$\mathcal{E}\big[\mathcal{L}[\mathrm{b}]\big](j_2 s) = G^\flat\big(\nabla_{ds} ds\big).$$

Proof. Let us consider the coordinate expressions

$$\mathcal{L}_0 = \tfrac{1}{2}\, G^0_{hk}\, x^h_0\, x^k_0 + A_h\, x^h_0 + A_0 \quad \text{and} \quad J_0 = \partial_0 + x^j_0\, \partial_j + x^j_{00}\, \partial^0_j .$$

Then, the equality

$$\begin{aligned} J_0 \partial^0_i \mathcal{L}_0 - \partial_i \mathcal{L}_0 &= \partial_0 G^0_{ik}\, x^k_0 + \partial_0 A_i + \partial_j G^0_{ik}\, x^j_0\, x^k_0 + \partial_j A_i\, x^j_0 + G^0_{ih}\, x^h_{00} \\ &\quad - \tfrac{1}{2}\, \partial_i G^0_{hk}\, x^h_0\, x^k_0 - \partial_i A_h\, x^h_0 - \partial_i A_0) \\ &= G^0_{ih}\, x^h_{00} + (\partial_h G^0_{ik} - \tfrac{1}{2}\, \partial_i G^0_{hk})\, x^h_0\, x^k_0 + \partial_0 G^0_{ih}\, x^h_0 \\ &\quad + (\partial_h A_i - \partial_i A_h)\, x^h_0 + (\partial_0 A_i - \partial_i A_0), \end{aligned}$$

in virtue of Theorem 4.3.3, yields

$$\begin{aligned} (J_0 \partial^0_i \mathcal{L}_0 - \partial_i \mathcal{L}_0) \circ j_2 s &= (G^0_{ij} \circ s)\, \partial_{00} s^j - G^0_{ij}\, \big((K_h{}^j{}_k \circ s)\, \partial_0 s^h\, \partial_0 s^k \\ &\quad + 2\, (K_0{}^j{}_k \circ s)\, \partial_0 s^k + K_0{}^j{}_0 \circ s \big) \\ &= G^0_{ij}\, \big(\nabla_{ds} ds\big)^j_{00} . \quad \square \end{aligned}$$

Proposition 11.5.3 *The solutions* $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$ *of the Newton law of motion*

$$\nabla_{ds} ds = 0$$

turn out to be the solutions of the equation of the Euler–Lagrange equation associated with the classical lagrangian $\mathcal{L}[\mathfrak{b}]$ *(see Theorem 10.1.8)*

$$\mathcal{E}\big[\mathcal{L}[\mathfrak{b}]\big](j_2 s) = 0. \quad \square$$

11.6 Conserved Phase Functions

For each phase function f, the Lie derivative $\gamma.f$ accounts for the time derivative of f along the solutions of the Newton equation of motion. Thus, the knowledge of $\gamma.f$, for all phase functions f, can be regarded as another way of expressing the Newton law of motion.

In particular, here we discuss the "*conserved phase functions*" defined by the condition $\gamma.f = 0$. Indeed, the conserved phase functions will play a relevant role in Covariant Classical Mechanics and Covariant Quantum Mechanics.

Lemma 11.6.1 *Let us consider a phase function* $f \in \mathrm{map}(J_1\boldsymbol{E}, I\!\!R)$*. Then, for each motion* $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$ *solution of the Newton law of motion, we have the equality*

$$d(f \circ j_1 s) = (\gamma.f) \circ j_1 s \in \mathrm{map}(\boldsymbol{T}, \mathbb{T}^* \otimes I\!\!R),$$

i.e., in coordinates,

$$\begin{aligned}&\big((\partial_0 f)\circ j_1 s + (\partial_i f)\circ j_1 s\, \partial_0 s^i + (\partial_i^0 f)\circ j_1 s\, \partial_{00} s^i\big)\, u^0 \\ &\quad = \big((\partial_0 f)\circ j_1 s + (\partial_i f)\circ j_1 s\, \partial_0 s^i + (\partial_i^0 f\, \gamma_{0}{}^{i}{}_{0})\circ j_1 s\big)\, u^0.\end{aligned}$$

Hence, for every phase function $f \in \mathrm{map}(J_1\boldsymbol{E}, I\!R)$*, the scaled function*

$$\gamma.f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{T}^* \otimes I\!R)$$

yields the information of how f *evolves on time along the solutions of the Newton equation of motion.*

Proof. In fact, in virtue of Definition 9.1.2 and Proposition 11.5.1, the motion s is a solution of the Newton law of motion if and only if

$$\partial_{00} s^i = \gamma_{0}{}^{i}{}_{0} \circ s. \quad \square$$

Definition 11.6.2 A phase function $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R})$ is said to be *conserved* if it is conserved along all solutions of the Newton law of motion.

We denote the subsheaf of conserved phase functions by

$$\mathrm{cns}(J_1\boldsymbol{E}, \mathrm{I\!R}) \subset \mathrm{map}(J_1\boldsymbol{E}, \mathrm{I\!R}). \quad \square$$

Proposition 11.6.3 *In virtue of the above Lemma 11.6.1, a phase function* $f \in \mathrm{map}(J_1\boldsymbol{E}, I\!R)$ *is conserved if and only if* $\gamma.f = 0$.

We have the coordinate expressions

$$\gamma.f = (\partial_0 f - \{\mathcal{H}_0,\ f\} - G^{ij}_0\, \partial_0 \mathcal{P}_i\, \partial^0_j f)\, u^0.$$

Hence, a phase function f *is conserved if and only if*

$$\partial_0 f = \{\mathcal{H}_0,\ f\} + G^{ij}_0\, \partial_0 \mathcal{P}_i\, \partial^0_j f,$$

or, equivalently, if and only if

$$(\partial_0 + x^i_0\, \partial_i) f = \big(G^{ij}_0\, \big(\Phi_{0j} + (\partial_0 G^0_{hj} + \Phi_{hj})\, x^h_0 + (\partial_h G^0_{jk} - \tfrac{1}{2}\, \partial_j G^0_{hk})\, x^h_0\, x^k_0\big)\, \partial^0_i f. \quad \square$$

Remark 11.6.4 We stress that in the equation

$$\gamma.f = (\partial_0 f - \{\mathcal{H}_0,\ f\} - G^{ij}_0\, \partial_0 \mathcal{P}_i\, \partial^0_j f)\, u^0$$

the left hand side is gauge independent and observer independent, while every term of the right hand side is gauge dependent and observer dependent. $\square$

Proposition 11.6.5 *In virtue of Proposition 11.4.4, the subsheaf of conserved phase functions is closed with respect to the Poisson bracket.*

In other words, if $f, \acute{f} \in \mathrm{cns}(J_1\boldsymbol{E}, I\!R)$*, then* $\gamma.\{f, \acute{f}\} = 0$*, i.e.* $\{f, \acute{f}\} \in \mathrm{cns}(J_1\boldsymbol{E}, I\!R)$. □

Chapter 12
Lie Algebra of Special Phase Functions

We review the *Lie algebra of special phase functions* (see, for instance, [220, 227, 312] and literature therein), along with some improvements and new results as well.

Thus, we define the *special phase functions* f along with their *tangent lift* $X[f]$, *holonomic phase lift* $X^{\uparrow}{}_{\text{hol}}[f]$ and *hamiltonian phase lift* $X^{\uparrow}{}_{\text{ham}}[f]$. Moreover, we study the *special phase Lie bracket* $[\![f, \acute{f}]\!]$ and analyse *distinguished Lie subalgebras*, which will play a fundamental role in the classical and quantum theories.

Indeed, the Lie algebra of special phase functions is one of the main tools of Covariant Quantum Mechanics. In fact, special phase functions generate, in a covariant way, the infinitesimal symmetries of the classical and quantum theories and, as a consequence, yield the quantum operators, the quantum currents and the quantum expectation forms.

Actually, the Lie algebra of special phase functions is essentially a subject of Covariant Classical Mechanics; however, its full understanding will be better achieved in the context of Covariant Quantum Mechanics, where the classical gauges $\mathfrak{b}$ used for the classification of upper potentials $A^{\uparrow}[\mathfrak{b}]$ can be interpreted in terms of the quantum bases $\mathfrak{b}$.

In our approach, the Lie algebra of special phase functions replaces the more usual Poisson Lie algebra of all phase functions. Clearly, the special phase functions constitute a distinguished subsheaf of the sheaf of all phase functions. The Poisson Lie bracket $\{,\}$ holds for all phase functions and is defined through the dynamical phase 2-vector Λ, which carries only partial information on the postulated gravitational and electromagnetic fields.

Conversely, the special phase Lie bracket $[\![\, , \,]\!]$ holds only for special phase functions and is defined through the pair (γ, Λ), or, equivalently, through the dynamical phase 2-form Ω, which carry full information on the postulated gravitational and electromagnetic fields.

The notion of special phase function f, along with its observed splitting, involves only the spacetime fibring over time $t : \boldsymbol{E} \to \boldsymbol{T}$ and the spacelike metric G. But its gauge splitting involves also the potential $A^{\uparrow}[\mathfrak{b}]$ of the dynamical phase 2-form Ω,

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which coincides with a component of the upper quantum connection $Ч^{\uparrow}$ (see Theorem 15.2.4); so, in this respect, an information is borrowed from Covariant Quantum Mechanics as a very useful tool for the present classical context. The holonomic lift $X^{\uparrow}{}_{\text{hol}}[f]$ of special phase functions involves only the time fibring of spacetime and makes an essential use of the tangent lift $X[f]$, but the hamiltonian lift $X^{\uparrow}{}_{\text{ham}}[f]$ involves only the dynamical phase 2-vector Λ and the dynamical phase connection γ, besides the spacelike metric G.

Indeed, also the special phase Lie bracket $[\![\, ,\,]\!]$ involves only the dynamical phase 2-vector Λ and the dynamical phase connection γ, besides the spacelike metric G. We have already observed that Λ and γ determine the dynamical phase 2-form Ω (see Remark 10.2.3); so it is not strange that the special phase Lie bracket can be expressed also through Ω.

We notice that an analogous sheaf of special phase functions and a special Lie bracket can be achieved in the einsteinian framework (see [220]).

12.1 Special Phase Functions

Several results of our Covariant Classical and Quantum Mechanics emphasise, in a covariant way, the distinguished subsheaf of *special phase functions* $f \in \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{map}(J_1\boldsymbol{E}, \mathbb{R})$, whose coordinate expression is of the type $f = f^0 \frac{1}{2} G^0_{ij} x^i_0 x^j_0 + f^i G^0_{ij} x^j_0 + \breve{f}$, with $f^0,\ f^i,\ \breve{f} \in \text{map}(\boldsymbol{E},\ \mathbb{R})$.

Here, after the definition of special phase functions, we consider further distinguished subsheaves, which are characterised by algebraic conditions and discuss distinguished examples of special phase functions.

Definition 12.1.1 A *special phase function* (s.p.f.) [196, 220] is defined to be a phase function $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$, such that its 2nd fibre derivative with respect to the affine bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$ is of the type

$$D^2 f = f'' \otimes G,$$

where the scaled spacetime function

$$f'' := \tfrac{1}{3}\, \bar{G} \lrcorner D^2 f \in \text{map}(J_1\boldsymbol{E}, \mathbb{T} \otimes \mathbb{R}),$$

(which can be defined for any spacetime function, see Lemma 11.3.5), is called the *time component* of f.

In coordinates, a special phase function is characterised by an expression of the type

$$f = f^0 \tfrac{1}{2} G^0_{ij} x^i_0 x^j_0 + f^i G^0_{ij} x^j_0 + \breve{f}, \quad \text{with } f^0,\ f^i,\ \breve{f} \in \text{map}(\boldsymbol{E},\ \mathbb{R}).$$

Accordingly, we have $f'' = f^0 u_0$, where $f^0 = u^0 \lrcorner f''$.

We denote the subsheaf of special phase functions by

$$\mathrm{spe}(J_1\boldsymbol{E},\ \mathbb{R}) \subset \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{R}). \quad \square$$

Remark 12.1.2 We stress that the product of two special phase functions needs not to be a special phase function. □

We can show several distinguished subsheaves of the sheaf of special phase functions. Here, we start by showing subsheaves which are characterised by algebraic conditions. Later, se shall show further subsheaves, which are characterised by differential conditions (see Sect. 12.6.2).

Definition 12.1.3 We have the following distinguished *"algebraic" subsheaves* of the sheaf $\mathrm{spe}(J_1\boldsymbol{E},\ \mathbb{R})$

$$\begin{array}{llll} \text{the } subsheaf\ of\ projectable\ s.p.f. & := \mathrm{pro\,spe}(J_1\boldsymbol{E},\mathbb{R}) & := \{f \mid \partial_j f^0 = 0\}. \\ \text{the } subsheaf\ of\ time\ preserving\ s.p.f. & := \mathrm{tim\,spe}(J_1\boldsymbol{E},\mathbb{R}) & := \{f \mid \partial_\lambda f^0 = 0\}, \\ \text{the } subsheaf\ of\ affine\ s.p.f. & := \mathrm{aff\,spe}(J_1\boldsymbol{E},\mathbb{R}) & := \{f \mid f^0 = 0\}, \\ \text{the } subsheaf\ of\ spacetime\ s.p.f. & := \mathrm{map}(\boldsymbol{E},\mathbb{R}) & := \{f \mid f^\lambda = 0\}. \quad \square \end{array}$$

Clearly, we have the following inclusions:

$$\mathrm{map}(\boldsymbol{E},\mathbb{R}) \subset \mathrm{aff\,spe}(\boldsymbol{E},\mathbb{R}) \subset \mathrm{tim\,spe}(\boldsymbol{E},\mathbb{R}) \subset \mathrm{pro\,spe}(\boldsymbol{E},\mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E},\mathbb{R}).$$

In Covariant Quantum Mechanics, a distinguished role is played by time preserving special phase functions. In fact, we achieve a covariant procedure to "quantise" these functions (see Theorem 20.1.9).

Example 12.1.4 We have the following distinguished special phase functions, with reference to an observer o and a gauge $\mathfrak{b}$, (see Definition 3.2.9, Theorem 10.1.8, Corollary 10.1.10)

$$f \in \mathrm{map}(\boldsymbol{E},\mathbb{R}), \qquad \mathcal{C}_\lambda[o] \in \mathrm{map}(J_1\boldsymbol{E},\mathbb{R}), \qquad A^{\uparrow}{}_\lambda[\mathfrak{b}] \in \mathrm{map}(J_1\boldsymbol{E},\mathbb{R}).$$

We stress that, even if $A^{\uparrow}[\mathfrak{b}]$ is observer independent, its components $A^{\uparrow}{}_\lambda[\mathfrak{b}]$ depend on the choice of a spacetime chart, hence, implicitly of the associated observer.

Thus, in particular, we have the following special phase functions

$$\begin{aligned} x^\lambda &\in \mathrm{map}(\boldsymbol{E},\mathbb{R}), \\ \mathcal{C}_i[o] = \mathcal{Q}_i[o] &= G^0_{ij}\, x^j_0 \in \mathrm{aff\,spe}(J_1\boldsymbol{E},\mathbb{R}), \\ A^{\uparrow}{}_i[\mathfrak{b},o] = \mathcal{P}_i[\mathfrak{b},o] &= G^0_{ij}\, x^j_0 + A_i \in \mathrm{aff\,spe}(J_1\boldsymbol{E},\mathbb{R}), \\ -\mathcal{C}_0[o] = \mathcal{K}_0[o] &= \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 \in \mathrm{tim\,spe}(J_1\boldsymbol{E},\mathbb{R}), \\ -A^{\uparrow}{}_0[\mathfrak{b},o] = \mathcal{H}_0[\mathfrak{b},o] &= \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0 \in \mathrm{tim\,spe}(J_1\boldsymbol{E},\mathbb{R}). \end{aligned}$$

Moreover, we have the following further distinguished special phase functions

$$\mathcal{L}_0[\mathrm{b}] = \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + A_0 \in \operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),$$
$$\tfrac{1}{2}\,\mathcal{P}^2_0[\mathrm{b}] = \tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 + A_i\, x^i_0 + \tfrac{1}{2}\, A_i\, A^i_0 \in \operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square$$

In the following we shall refer to the above distinguished special phase functions for the illustration of several classical and quantum objects.

12.2 Tangent Lift of Special Phase Functions

In standard Classical Mechanics, a well-known notion is the hamiltonian lift of all phase functions. In the present theory, a relevant property of special phase functions f is due to their "non standard" *tangent lift* $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, whose equivariant coordinate expression is $X[f] = f^0\,\partial_0 - f^i\,\partial_i$.

Indeed, the tangent lift has several remarkable consequences in the classical and quantum theories. We discuss the tangent lift of distinguished phase functions, where unexpectedly we can already perceive a certain "scent" of quantisation rules.

We stress that the definition of special phase function and of its tangent lift involves only the time fibring of spacetime and the galilean metric.

Theorem 12.2.1 *For every* $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$*, we obtain, in a covariant way, the spacetime vector field, called its* tangent lift*, (see Proposition 2.7.3)*

$$X[f] = f'' \lrcorner\, д - G^\sharp(Df) \in \sec(\boldsymbol{E}, T\boldsymbol{E}),$$

with coordinate expression

$$X[f] = f^0\,\partial_0 - f^i\,\partial_i .$$

Thus, we have a surjective sheaf morphism

$$X : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f],$$

whose kernel is the subsheaf $\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$.

In other words, the following sequence is exact

$$0 \longrightarrow \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \xrightarrow{\ \subset\ } \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \xrightarrow{\ X\ } \sec(\boldsymbol{E}, T\boldsymbol{E}) \longrightarrow 0 \quad .$$

Hence, the map X passes to the quotient and we obtain a sheaf isomorphism

$$X : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) / \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \to \sec(\boldsymbol{E}, T\boldsymbol{E}). \quad \square$$

Remark 12.2.2 The minus sign appearing in the above definition of tangent lift is necessary in order to cancel the coordinates x^i_0, so obtaining an object projectable on spacetime. We observe that such a minus sign resembles, some how, the signature of a lorentzian metric. □

Note 12.2.3 If (see Definitions 2.2.6 and 12.1.3)

$$f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}), \qquad f \in \operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}), \qquad f \in \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

then we obtain, respectively,

$$X[f] \in \operatorname{pro}\sec(\boldsymbol{E}, T\boldsymbol{E}), \quad X[f] \in \operatorname{tim}\sec(\boldsymbol{E}, T\boldsymbol{E}), \quad X[f] \in \sec(\boldsymbol{E}, V\boldsymbol{E}). \quad \square$$

Example 12.2.4 We have (see Example 12.1.4):

$$X[x^\lambda] = 0,$$
$$X[\mathcal{C}_i] = X[A^{\uparrow}{}_i] = X[\mathcal{P}_i] = -\partial_i, \qquad X[-\mathcal{C}_0] = X[-A^{\uparrow}{}_0] = X[\mathcal{H}_0] = \partial_0,$$
$$X[\tfrac{1}{2}\check{\mathcal{P}}_0] = X[\mathcal{L}_0] = \partial_0 - A^i_0\,\partial_i.$$

By considering the natural fibred isomorphism $\boldsymbol{E} \times (\mathbb{T} \times \mathbb{R}) \simeq H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ over $\boldsymbol{E}$, we can easily extend the tangent lift to *scaled special phase functions* such as, for instance, $\mathcal{L}[\mathrm{b}], \mathcal{H}[\mathrm{b}, o] \in \operatorname{spe}(J_1\boldsymbol{E}, H^*\boldsymbol{E})$.

Indeed, we obtain

$$X\big[\mathcal{L}[\mathrm{b}]\big], X\big[\mathcal{H}[\mathrm{b}, o]\big] \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}).$$

In particular, the scaled tangent lift of the classical lagrangian $\mathcal{L} \in \operatorname{spe}(J_1\boldsymbol{E}, H^*\boldsymbol{E})$

$$X\big[\mathcal{L}[\mathrm{b}]\big] = (\partial_0 - A^i_0\,\partial_i) \otimes u^0 \in \sec\big(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})\big)$$

yields a distinguished scaled vector field $\nu[\mathrm{b}]$ and a distinguished scaled function $\alpha[\mathrm{b}]$, which are gauge dependent and observer independent.

These objects will frequently appear in the developments of Covariant Quantum Mechanics. Here, we just introduce them; later, in the context of Covariant Quantum Mechanics, we shall explain the deep reason of their equivariant properties (see Corollary 15.2.28).

Proposition 12.2.5 *With reference to a gauge* b*, let us consider the upper quantum potential* $A^{\uparrow}[\mathrm{b}]$*, the associated classical lagrangian* $\mathcal{L}[\mathrm{b}]$ *and its tangent lift* $X\big[\mathcal{L}[\mathrm{b}]\big]$.

Then, we obtain the following distinguished gauge dependent and observed independent scaled spacetime vector field and 1-form

$$\nu[\mathrm{b}] := X\big[\mathcal{L}[\mathrm{b}]\big] \in \sec\big(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})\big),$$
$$\alpha[\mathrm{b}] := \tfrac{1}{2}\Big(\mathcal{L}[\mathrm{b}] + X\big[\mathcal{L}[\mathrm{b}]\big] \lrcorner A^{\uparrow}[\mathrm{b}]\Big) \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}),$$

with coordinate expression

$$\nu[\mathfrak{b}] = (\partial_0 - A^i_0\, \partial_i) \otimes u^0 \quad \text{and} \quad \alpha[\mathfrak{b}] = (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, u^0. \quad \square$$

Corollary 12.2.6 *We have the distinguished gauge dependent and observer dependent scaled spacetime function*

$$A[\mathfrak{b}, o] \lrcorner \nu[\mathfrak{b}] = \alpha[\mathfrak{b}] - \tfrac{1}{2}\, \bar{G}\big(\check{A}[\mathfrak{b}, o]\, \check{A}[\mathfrak{b}, o]\big) \in \operatorname{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}),$$

with coordinate expression

$$A[\mathfrak{b}, o] \lrcorner \nu[\mathfrak{b}] = (A_0 - A_i\, A^i_0)\, u^0.$$

Proof. The above coordinate expressions are observer equivariant because the left hand sides of the above equalities are observer independent. $\square$

12.2.1 Divergence of Special Phase Functions

The tangent lift of projectable special phase functions allows us to define the spacelike divergence of these functions. This concept will play a role in the classification of η-hermitian quantum vector fields, hence in the coordinate expressions of quantum operators (see Theorem 19.1.7 and Example 20.1.12).

Definition 12.2.7 For each projectable special phase function $f \in \operatorname{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$, we define its spacelike *divergence* to be the spacetime function (see Theorem 12.2.1 and Definition 3.2.17)

$$\operatorname{div}_\eta f := \operatorname{div}_\eta X[f] \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}),$$

with coordinate expression

$$\operatorname{div}_\eta f = f^0\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (f^i\, \sqrt{|g|})}{\sqrt{|g|}}. \quad \square$$

Example 12.2.8 We have the following distinguished divergences (see Example 12.1.4):

$$\operatorname{div}_\eta x^\lambda = 0, \qquad \operatorname{div}_\eta x^j_0 = -\frac{\partial_i (G^{ij}_0\, \sqrt{|g|})}{\sqrt{|g|}},$$

$$\operatorname{div}_\eta \mathcal{Q}_j = \operatorname{div}_\eta \mathcal{P}_j = \operatorname{div}_\eta A^\uparrow{}_j = -\frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}},$$

$$\operatorname{div}_\eta \mathcal{K}_0 = \operatorname{div}_\eta \mathcal{H}_0 = -\operatorname{div}_\eta A^\uparrow{}_0 = \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}},$$

$$\operatorname{div}_\eta \mathcal{L}_0 = \operatorname{div}_\eta \tfrac{1}{2}\, |\mathcal{P}|^2_0 = \Big(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0\, \sqrt{|g|})}{\sqrt{|g|}}\Big).$$

Some of the above equalities can be written, more synthetically, as

$$\operatorname{div}_\eta x^\lambda = 0, \qquad \operatorname{div}_\eta A^\uparrow{}_\lambda = -\frac{\partial_\lambda \sqrt{|g|}}{\sqrt{|g|}}. \quad \square$$

12.2.2 Splittings of Special Phase Functions

For each special phase function f, the tangent lift $X[f]$ yields the *observed splitting* $f = (f^0\, \mathcal{K}_0 + f^i\, \mathcal{Q}_i) + \breve{f}$ and the *gauge splitting* $f = (f^0\, \mathcal{H}_0 + f^i\, \mathcal{P}_i) + \hat{f}$, which, later, will play a systematic role in different contexts.

Proposition 12.2.9 *With reference to an observer* o *and to a gauge* $\mathfrak{b}$*, for each* $f \in \operatorname{spe}(J_1\boldsymbol{E}, I\!\!R)$*, we obtain the following*

(1) observed splittings

$$\begin{aligned} f &= \bar{f}[o] + \breve{f}[o] = -X[f] \lrcorner \mathcal{C}[o] + \breve{f}[o] \\ &= f'' \lrcorner \mathcal{K}[o] + f'[o] \lrcorner \mathcal{Q}[o] + \breve{f}[o] \\ &= f''[o] \lrcorner \mathcal{K}[o] + f'[o] \lrcorner \mathcal{Q}[o] + \breve{f}[o] \\ &= (f^0\, \mathcal{K}_0 + f^i\, \mathcal{Q}_i) + \breve{f}, \end{aligned}$$

(2) gauge splittings

$$\begin{aligned} f &= \tilde{f}[\mathfrak{b}] + \hat{f}[\mathfrak{b}] = -X[f] \lrcorner A^\uparrow[\mathfrak{b}] + \hat{f}[\mathfrak{b}] \\ &= f'' \lrcorner \mathcal{H}[\mathfrak{b}, o] + f'[o] \lrcorner \mathcal{P}[\mathfrak{b}, o] + \hat{f}[\mathfrak{b}] \\ &= f''[o] \lrcorner \mathcal{H}[\mathfrak{b}, o] + f'[o] \lrcorner \mathcal{P}[\mathfrak{b}, o] + \hat{f}[\mathfrak{b}] \\ &= (f^0\, \mathcal{H}_0 + f^i\, \mathcal{P}_i) + \hat{f}, \end{aligned}$$

where we have set

$$\begin{aligned} \bar{f}[o] &= -X[f] \lrcorner \mathcal{C}[o], \qquad & \breve{f}[o] &= f + X[f] \lrcorner \mathcal{C}[o] &&= \hat{f} - A_0\, f^0 + A_i\, f^i, \\ \tilde{f}[\mathfrak{b}] &= -X[f] \lrcorner A^\uparrow[\mathfrak{b}], \qquad & \hat{f}[\mathfrak{b}] &= f + X[f] \lrcorner A[\mathfrak{b}, o] &&= \breve{f} + A_0\, f^0 - A_i\, f^i, \end{aligned}$$

and

$$f''[o] := д[o] \lrcorner X[f] = f^0\, \partial_0, \qquad f'[o] := -\,\theta[o] \lrcorner X[f] = f^i\, \partial_i.$$

Hence, we obtain the equality

$$\breve{f} = \breve{f}[o] = f \circ o$$

and the observed splitting of the tangent lift

$$X[f] = f''[o] - f'[o]. \quad \square$$

Corollary 12.2.10 *With reference to two observers o and $\acute{o} = o + \vec{v}$, and to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \mathfrak{b}\, \exp(\mathfrak{i}\, \vartheta)$, we have the following transition rules*

$$\bar{f}[\acute{o}] = \bar{f}[o] - G(\vec{v}, f'[o]) - \tfrac{1}{2}\, f'' \lrcorner\, G(\vec{v}, \vec{v}),$$
$$\breve{f}[\acute{o}] = \breve{f}[o] + G(\vec{v}, f'[o]) + \tfrac{1}{2}\, f'' \lrcorner\, G(\vec{v}, \vec{v}),$$
$$\tilde{f}[\acute{\mathfrak{b}}] = \tilde{f}[\mathfrak{b}] + X[f] \lrcorner\, d\vartheta, \qquad \hat{f}[\acute{\mathfrak{b}}] = \hat{f}[\mathfrak{b}] - X[f] \lrcorner\, d\vartheta. \quad \square$$

Every special phase function f is fully characterised, respectively, by its tangent lift $X[f]$ and the observed spacetime function $\breve{f}[o]$, and by its tangent lift $X[f]$ and the gauged spacetime function $\hat{f}[\mathfrak{b}]$.

This result will be used later for the classification of hermitian and η-hermitian quantum vector fields (see Theorem 19.1.7).

Corollary 12.2.11 *Each $f \in \mathrm{spe}(J_1\boldsymbol{E}, I\!R)$ is characterised by the following "spacetime pairs" (see Proposition 12.2.9):*

- *with reference to an observer o, by the spacetime pair*

$$\big(X[f],\, \breve{f}[o]\big) \in \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R),$$

where $X[f]$ is its observer independent and gauge independent tangent lift and $\breve{f}[o]$ is its observer dependent and gauge independent spacetime component;
- *with reference to a gauge $\mathfrak{b}$, by the spacetime pair*

$$\big(X[f],\, \hat{f}[\mathfrak{b}]\big) \in \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R),$$

where $X[f]$ is its observer independent and gauge independent tangent lift and $\hat{f}[\mathfrak{b}]$ is its gauge dependent and observer independent spacetime component. $\square$

In virtue of the above results, we can characterise special phase functions in four ways via four kinds of *special "transforms"*.

Proposition 12.2.12 *We have the following sheaf isomorphisms:*

(1) *with reference to an observer o, the sheaf isomorphism*

$$\mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R) : f \mapsto \big(X[f],\, \breve{f}[o]\big),$$

whose inverse is the sheaf isomorphism

$$\mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R) \to \mathrm{spe}(J_1\boldsymbol{E}, I\!R) : (Z, \phi) \mapsto -i_Z \mathcal{C}[o] + \phi,$$

(2) *with reference to a quantum basis $\mathfrak{b}$, the sheaf isomorphism*

$$\mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R) : f \mapsto \big(X[f],\, \hat{f}[\mathfrak{b}]\big),$$

whose inverse is the sheaf isomorphism

$$\mathrm{sec}(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, I\!R) \to \mathrm{spe}(J_1\boldsymbol{E}, I\!R) : (Z, \phi) \mapsto -i_Z A^{\uparrow}[\mathfrak{b}] + \phi,$$

(3) *with reference to an observer o and an adapted chart (x^λ), the sheaf isomorphism*

$$\mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{map}(\boldsymbol{E}, I\!R^5) : f \mapsto \big(f^\lambda, \breve{f}\big),$$

whose inverse is the sheaf isomorphism

$$\mathrm{map}(\boldsymbol{E}, I\!R^5) \to \mathrm{spe}(J_1\boldsymbol{E}, I\!R) : (\psi^\lambda, \phi) \mapsto \psi^0 \tfrac{1}{2} G^0_{ij}\, x^i_0\, x^j_0 + \psi^i\, G^0_{ij}\, x^j_0 + \phi,$$

(4) *with reference to a gauge $\mathfrak{b}$, an observer o and an adapted chart (x^λ), the sheaf isomorphism*

$$\mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{map}(\boldsymbol{E}, I\!R^5) : f \mapsto \big(f^\lambda, \hat{f}\big),$$

whose inverse is the sheaf isomorphism

$$\mathrm{map}(\boldsymbol{E}, I\!R^5) \to \mathrm{spe}(J_1\boldsymbol{E}, I\!R) : (\psi^\lambda, \phi) \mapsto \psi^0\, \mathcal{H}_0 + \psi^i\, \mathcal{P}_i + \phi. \quad \square$$

Example 12.2.13 The observed and gauge splittings of distinguished special phase functions are given by the following equalities (see Example 12.1.4)

x^λ	$\bar{x}^\lambda[o] = 0,$	$\breve{x}^\lambda[o] = x^\lambda,$	$\tilde{x}^\lambda[\mathfrak{b}] = 0,$	$\hat{x}^\lambda[\mathfrak{b}] = x^\lambda,$
$\mathcal{P}_j$	$\bar{\mathcal{P}}_j[o] = \mathcal{Q}_j,$	$\breve{\mathcal{P}}_j[o] = A_j,$	$\tilde{\mathcal{P}}_j[\mathfrak{b}] = \mathcal{P}_j,$	$\hat{\mathcal{P}}_j[\mathfrak{b}] = 0,$
$\mathcal{H}_0$	$\bar{\mathcal{H}}_0[o] = \mathcal{K}_0,$	$\breve{\mathcal{H}}_0[o] = -A_0,$	$\tilde{\mathcal{H}}_0[\mathfrak{b}] = \mathcal{H}_0,$	$\hat{\mathcal{H}}_0[\mathfrak{b}] = 0. \quad \square$

12.3 Holonomic Phase Lift of s.p.f.

Every special phase function f, according to a covariant procedure, admits a *holonomic phase lift* $X^{\uparrow}{}_{\mathrm{hol}}[f] := r^1 \circ J_1 X[f] \in \mathrm{sec}(J_1\boldsymbol{E}, T J_1\boldsymbol{E})$.

Indeed, this phase lift is non standard, as it is based on the non standard tangent lift of special phase functions.

We stress that the holonomic phase lift involves only the fibred structure of spacetime.

We start by recalling a general result holding on any fibred manifold (see, Appendix: Sect. G.6).

Proposition 12.3.1 *[246, 282, 283] For each fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, equipped with a fibred chart (x^λ, y^i), we have a distinguished natural fibred morphism*

$$r^1 : J_1 T\boldsymbol{F} \to T J_1 \boldsymbol{F},$$

which yields the natural holonomic prolongation *of each vector field* $Z : \boldsymbol{F} \to T\boldsymbol{F}$ *to the vector field*

$$Z^1 := r^1 \circ J_1 Z : J_1\boldsymbol{F} \to T J_1 \boldsymbol{F},$$

with coordinate expression

$$Z^1 = Z^\lambda \, \partial_\lambda + Z^i \, \partial_i + (\partial_\lambda Z^i + \partial_j Z^i \, y^j_\lambda - \partial_\lambda Z^\mu \, y^i_\mu - \partial_j X^\mu \, y^i_\mu \, y^j_\lambda) \, \partial^\lambda_i.$$

The above expression reduces to the following equality in the case of a projectable vector field Z

$$Z^1 = Z^\lambda \, \partial_\lambda + Z^i \, \partial_i + (\partial_\lambda Z^i + \partial_j Z^i \, y^j_\lambda - \partial_\lambda Z^\mu \, y^i_\mu) \, \partial^\lambda_i.$$

Indeed, the holonomic prolongation turns out to be a Lie algebra morphism with respect to the Lie bracket of vector fields. Actually, the flow of the vector field Z^1 *turns out to coincide with the jet prolongation of the flow of Z.* □

Then, by taking into account the tangent lift $X[f]$ of special phase functions f (see Theorem 12.2.1), we obtain the following result.

Definition 12.3.2 For each $f \in \text{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, we define the *holonomic phase lift* to be the phase vector field

$$X^{\uparrow}{}_{\text{hol}}[f] := r^1 \circ J_1 X[f] \in \text{pro}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}),$$

which is projectable on the tangent lift $X[f] \in \text{sec}(\boldsymbol{E}, T\boldsymbol{E})$.

We denote the holonomic phase lift sheaf morphism and the subsheaf of holonomic phase lifts of all special phase functions, respectively, by

$$X^{\uparrow}{}_{\text{hol}} : \text{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{hol}}[f],$$
$$\text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}). \quad \square$$

Proposition 12.3.3 *For each* $f \in \text{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$*, we have the coordinate expressions*

$$X^{\uparrow}{}_{\text{hol}}[f] = f^0 \, \partial_0 - f^i \, \partial_i - (\partial_0 f^i + \partial_j f^i \, x^j_0 + \partial_0 f^0 \, x^i_0 + \partial_j f^0 \, x^j_0 \, x^i_0) \, \partial^0_i.$$

Accordingly, for each $f \in \text{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$*, we have (see Proposition 12.2.9)*

$$X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{hol}}\big[\bar{f}[o]\big] = X^{\uparrow}{}_{\text{hol}}\big[\tilde{f}[\mathfrak{b}]\big].$$

Indeed, the kernel of the surjective sheaf morphism

$$X^{\uparrow}{}_{\text{hol}} : \text{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{hol}}[f]$$

turns out to be the subsheaf (see Theorem 12.2.1)

$$\operatorname{map}(\boldsymbol{E}, I\!R) \subset \operatorname{spe}(J_1\boldsymbol{E}, I\!R).$$

Hence, the map $X^{\uparrow}{}_{\text{hol}}$ passes to the quotient and we obtain a sheaf isomorphism

$$X^{\uparrow}{}_{\text{hol}} : \operatorname{spe}(J_1\boldsymbol{E}, I\!R)/\operatorname{map}(\boldsymbol{E}, I\!R) \to \operatorname{hol}\sec(\boldsymbol{E}, T\boldsymbol{E}). \quad \square$$

Example 12.3.4 We have the following distinguished holonomic phase lifts (see Example 12.1.4):

$$X^{\uparrow}{}_{\text{hol}}[x^{\lambda}] = 0,$$
$$X^{\uparrow}{}_{\text{hol}}[A^{\uparrow}{}_j] = X^{\uparrow}{}_{\text{hol}}[\mathcal{P}_j] = -\partial_j, \qquad X^{\uparrow}{}_{\text{hol}}[-A^{\uparrow}{}_0] = X^{\uparrow}{}_{\text{hol}}[\mathcal{H}_0] = \partial_0. \quad \square$$

12.4 Hamiltonian Phase Lift of s.p.f.

Let us recall the *hamiltonian phase lift $X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df)$ of a generic phase function $f \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$* (see Definition 11.3.6).

Indeed, we can specialise the above hamiltonian lift of generic phase functions to special phase functions. We notice that, in the particular case of special phase functions, the above time scale f'' (see Lemma 11.3.5), coincides with the time component defined in Definition 12.1.1.

We stress that this phase lift resembles the standard hamiltonian lift in symplectic structures, but involves an additional unusual "horizontal" term which is related to the odd dimension of phase space. Moreover, we emphasise that the hamiltonian phase lift of special phase functions involves essentially the coPoisson structure (γ, Λ) (or, equivalently, the cosymplectic structure (dt, Ω)) of phase space.

Definition 12.4.1 For each $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we define the *hamiltonian phase lift* to be the phase vector field

$$X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) \in \operatorname{pro}\sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}),$$

which is projectable on the tangent lift $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$.

We denote the hamiltonian phase lift sheaf morphism and the subsheaf of hamiltonian phase lifts of all special phase functions, respectively, by

$$X^{\uparrow}{}_{\text{ham}} : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{ham}\sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[f],$$
$$\operatorname{ham}\sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}). \quad \square$$

Proposition 12.4.2 *For each $f \in \operatorname{spe}(J_1\boldsymbol{E}, I\!R)$, we have the coordinate expression (see Corollary 11.3.7, Definition 3.2.9 and Theorem 10.1.8)*

$$X^{\uparrow}{}_{\text{ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + G^{ij}_0\Big(-f^0\,(\partial_0\mathcal{P}_j - \partial_j A_0) + f^h\,(\partial_h\mathcal{P}_j - \partial_j A_h)$$
$$+ \partial_j f^0\,\mathcal{K}_0 + \partial_j f^h\,\mathcal{Q}_h + \partial_j \breve{f}\Big)\,\partial^0_i.$$

In the case of projectable special phase functions, the above expression reduces to the equality (see Definition 12.1.3)

$$X^{\uparrow}{}_{\text{ham}}[f] = f^0\,\partial_0 - f^i\,\partial_i + G_0^{ij}\big(- f^0\,(\partial_0\mathcal{P}_j - \partial_j A_0) + f^h\,(\partial_h\mathcal{P}_j - \partial_j A_h) + \partial_j f^h\,\mathcal{Q}_h + \partial_j \check{f}\big)\,\partial_i^0.$$

Indeed, the kernel of the surjective sheaf morphism

$$X^{\uparrow}{}_{\text{ham}} : \operatorname{spe}(J_1\boldsymbol{E}, I\!R) \to \operatorname{ham}\operatorname{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[f]$$

turns out to be the subsheaf (see Theorem 12.2.1)

$$\operatorname{map}(\boldsymbol{T}, I\!R) \subset \operatorname{spe}(J_1\boldsymbol{E}, I\!R).$$

Hence, the map $X^{\uparrow}{}_{\text{ham}}$ passes to the quotient and we obtain a sheaf isomorphism

$$X^{\uparrow}{}_{\text{ham}} : \operatorname{spe}(J_1\boldsymbol{E}, I\!R)/\operatorname{map}(\boldsymbol{T}, I\!R) \to \operatorname{ham}\operatorname{sec}(\boldsymbol{E}, T\boldsymbol{E}). \quad \square$$

Example 12.4.3 We have the following distinguished phase lifts (see Example 12.1.4):

$$X^{\uparrow}{}_{\text{ham}}[x^0] = 0,$$
$$X^{\uparrow}{}_{\text{ham}}[A^{\uparrow}{}_j] = X^{\uparrow}{}_{\text{ham}}[\mathcal{P}_j] = -\partial_j + G_0^{ih}\,\partial_j\mathcal{P}_h\,\partial_i^0,$$
$$X^{\uparrow}{}_{\text{ham}}[-A^{\uparrow}{}_0] = X^{\uparrow}{}_{\text{ham}}[\mathcal{H}_0] = \partial_0 - G_0^{ih}\,\partial_0\mathcal{P}_h\,\partial_i^0. \quad \square$$

Indeed, our definitions of special phase function and of hamiltonian phase lift of special phase functions are supported by the following result, which will be used later for the classification of infinitesimal symmetries of classical structure (see Proposition 13.1.2).

Theorem 12.4.4 *If $\tau \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{T} \otimes I\!R)$ and $f \in \operatorname{map}(J_1\boldsymbol{E}, I\!R)$, then the following conditions are equivalent:*

(1) $$X^{\uparrow}{}_{\text{ham}}[\tau, f] := \tau \lrcorner \gamma + \Lambda^{\sharp}(df) \in \operatorname{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

projects on a vector field

$$X[\tau, f] := \operatorname{pro}_{\boldsymbol{E}} X^{\uparrow}[\tau, f] \in \operatorname{sec}(\boldsymbol{E}, T\boldsymbol{E}),$$

(2) $$f \in \operatorname{spe}(J_1\boldsymbol{E}, I\!R) \quad \text{and} \quad \tau = f''.$$

Moreover, if the above conditions are fulfilled, then

$$X^{\uparrow}{}_{\mathrm{ham}}[\tau, f] = X^{\uparrow}{}_{\mathrm{ham}}[f] \quad \text{and} \quad X[\tau, f] = X[f].$$

Proof. The vector field

$$X^{\uparrow}{}_{\mathrm{ham}}[\tau, f] = \tau^0 \, \partial_0 + (\tau^0 \, x^i_0 - G^{ij}_0 \, \partial^0_j f) \, \partial_i \\ + \big(\tau^0 \, \gamma_0{}^i{}_0 + G^{ij}_0 \, \partial_j f + (\Gamma^{i\,j}_{0\,0} - \Gamma^{j\,i}_{0\,0}) \, \partial^0_j f\big) \, \partial^0_i$$

(where we have set $\Gamma^{hk}_{0\,0} := G^{hp}_0 \, \Gamma_p{}^k{}_0$) is projectable on $\boldsymbol{T}\boldsymbol{E}$ if and only if

$$(*) \qquad \tau^0 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \quad \text{and} \quad \tau^0 \, G^0_{ij} \, x^i_0 - \partial^0_j f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

Hence, the following implications hold.

(1) If condition $(*)$ is fulfilled, then we obtain

$$(**) \qquad \partial^0_i \partial^0_j f = \tau^0 \, G^0_{ij} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

Integration of equation $(**)$, with respect to the affine fibres of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, yields

$$\partial^0_j f = \tau^0 \, G^0_{ij} \, x^i_0 + f^0_j \qquad \text{with } f^0_j \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}),$$
$$f = \tfrac{1}{2} \, \tau^0 \, G^0_{ij} \, x^i_0 \, x^j_0 + f^0_j \, x^j_0 + \breve{f}, \qquad \text{with } \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

Then, by setting $f^0 := \tau^0$ and $f^i := G^{ij}_0 \, f^0_j$, we obtain

$$f = \tfrac{1}{2} \, f^0 \, G^0_{ij} \, x^i_0 \, x^j_0 + f^i \, G^0_{ij} \, x^j_0 + \breve{f}, \quad \text{with } f^0, \, f^i, \, \breve{f} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

(2) If $f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ is a function of the above type and $\tau^0 = f^0$, then condition $(*)$ is fulfilled. □

12.5 Special Phase Lie Bracket

Several results of our approach suggest a *special Lie bracket* of special phase functions

$$[\![f, \acute{f}]\!] := \{f, \acute{f}\} + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f,$$

which is given by the Poisson bracket plus an additional "horizontal" term (see Definition 11.4.1).

We stress that, in the particular case of affine special phase functions, the special Lie bracket reduces to the Poisson Lie bracket.

Here, we provide a direct definition of the special phase Lie bracket. But it is striking that, later, we might recover it by an independent procedure in a different quantum context, via the classification of η-hermitian quantum vector fields (see Theorem 19.1.7).

We observe that the Poisson Lie bracket of all phase functions does not carry full information of the geometric structure of the phase space, because it is achieved via the vertical phase 2-vector Λ (see Corollary 9.2.4 and Remark 10.2.3).

The *special Lie bracket* is obtained via the pair (γ, Λ), or, equivalently, via the pair (dt, Ω), which carry full information on the geometric structure of phase space (see Theorems 10.1.1 and 10.2.1 and, Appendix: Theorem I.1.11). Clearly, the special Lie bracket $[\![f, \acute{f}]\!]$ carries also full information on gravitational and electromagnetic fields postulated in our theory.

Indeed, the special phase Lie bracket plays a fundamental role in our approach.

We notice that an analogous special phase Lie bracket can be achieved in the einsteinian framework (see [220]).

Definition 12.5.1 For each $f, \acute{f} \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we define their *special phase bracket* to be the special phase function (see Theorems 9.2.6 and 9.2.11, Definition 12.1.1 and Lemma 11.3.5)

$$[\![f, \acute{f}]\!] := \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f. \quad \square$$

Remark 12.5.2 We stress that, in the particular case of affine phase functions, the special phase Lie bracket reduces to the Poisson Lie bracket.

In fact, for each $f, \acute{f} \in \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have $f'' = \acute{f}'' = 0$, hence (see Definition 12.1.3)

$$[\![f, \acute{f}]\!] = \Lambda(df, d\acute{f}) = \{f, \acute{f}\}. \quad \square$$

Theorem 12.5.3 *The special phase bracket* $[\![\, ,]\!]$ *turns out to be a Lie bracket, which can also be expressed by the following equivalent equalities (see Definitions 3.2.9, 4.2.11 and 10.1.3, Theorem 10.1.1, Proposition 12.2.9)*

$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner\, \mathcal{C}[o] + X[f].\breve{\acute{f}} - X[\acute{f}].\breve{f} + \Phi[o]\big(X[f], X[\acute{f}]\big),$$

$$[\![f, \acute{f}]\!] = -\big[X[f], X[\acute{f}]\big] \lrcorner\, A^{\uparrow}[\mathfrak{b}] + X[f].\hat{\acute{f}} - X[\acute{f}].\hat{f},$$

$$[\![f, \acute{f}]\!] = X^{\uparrow}[f].\acute{f} - X^{\uparrow}[\acute{f}].f + 2\,\Omega\big(X^{\uparrow}[f], X^{\uparrow}[\acute{f}]\big),$$

where $X^{\uparrow}[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ *is any phase prolongation of the tangent lift* $X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ *(in particular, the holonomic phase lift* $X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{hol}}[f]$ *and the hamiltonian phase lift* $X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{ham}}[f]$*).*

The coordinate expression of the special Lie bracket is given by the following equalities

$$
\begin{aligned}
[\![f, \acute{f}]\!]^{\lambda} &= X[f]^{\mu}\,\partial_{\mu}X[\acute{f}]^{\lambda} - X[\acute{f}]^{\mu}\,\partial_{\mu}X[f]^{\lambda},\\
[\![f, \check{\acute{f}}]\!] &= X[f]^{\mu}\,\partial_{\mu}\check{\acute{f}} - X[\acute{f}]^{\mu}\,\partial_{\mu}\check{f} + X[f]^{\lambda}\,X[\acute{f}]^{\mu}\,(\partial_{\lambda}A_{\mu} - \partial_{\mu}A_{\lambda}),\\
[\![f, \hat{\acute{f}}]\!] &= X[f]^{\mu}\,\partial_{\mu}\hat{\acute{f}} - X[\acute{f}]^{\mu}\,\partial_{\mu}\hat{f}.
\end{aligned}
$$

Moreover, by taking into account the observed and gauge splittings of special phase functions (see Proposition 12.2.9), we have the following splittings of the special phase Lie bracket

$$
\begin{aligned}
[\![f, \bar{\acute{f}}]\!][o] &= -\big[X[f], X[\acute{f}]\big] \lrcorner\, \mathcal{C}[o]\\
&= \big[X[f], X[\acute{f}]\big]^{0}\,\mathcal{K}_0[o] + \big[X[f], X[\acute{f}]\big]^{i}\,\mathcal{Q}_i[o],\\
\big[\,[\![f, \check{\acute{f}}]\!]\,\big][o] &= X[f].\check{\acute{f}}[o] - X[\acute{f}].\check{f}[o] + \Phi[o]\big(X[f], X[\acute{f}]\big),\\
[\![f, \tilde{\acute{f}}]\!][\mathfrak{b}] &= -\big[X[f], X[\acute{f}]\big] \lrcorner\, A^{\uparrow}[\mathfrak{b}]\\
&= \big[X[f], X[\acute{f}]\big]^{0}\,\mathcal{H}_0[\mathfrak{b}, o] + \big[X[f], X[\acute{f}]\big]^{i}\,\mathcal{P}_i[\mathfrak{b}, o],\\
[\![f, \hat{\acute{f}}]\!][\mathfrak{b}] &= X[f].\hat{\acute{f}}[\mathfrak{b}] - X[\acute{f}].\hat{f}[\mathfrak{b}].
\end{aligned}
$$

Proof. The coordinate expression of the special phase bracket follows from the coordinate expressions of Λ and γ (see Corollary 9.2.4).

The Jacobi property of the special phase bracket can be proved by taking into account the Jacobi property of the Poisson bracket and the identity $L_{\gamma}\Lambda = 0$ (see Theorem 9.2.19). □

Corollary 12.5.4 *For each* $f, \acute{f} \in \operatorname{spe}(J_1\boldsymbol{E}, I\!R)$*, we have the equality*

$$X\big[\,[\![f, \acute{f}]\!]\,\big] = \big[X[f],\ X[\acute{f}]\big].$$

Hence, the tangent lift turns out to be a morphism of Lie algebras, with respect to the special phase Lie bracket and the Lie bracket of vector fields. □

Corollary 12.5.5 *The special Lie bracket vanishes on the subsheaf*

$$\operatorname{map}(\boldsymbol{E}, I\!R) \subset \operatorname{spe}(J_1\boldsymbol{E}, I\!R).$$

Moreover, the special Lie bracket induces a Lie bracket on the quotient sheaf

$$
\begin{aligned}
[\![\,,\,]\!] : \big(\operatorname{spe}(J_1\boldsymbol{E}, I\!R)/\operatorname{map}(\boldsymbol{E}, I\!R)\big) \times \big(\operatorname{spe}(J_1\boldsymbol{E}, I\!R)/\operatorname{map}(\boldsymbol{E}, I\!R)\big) \to\\
\to \operatorname{spe}(J_1\boldsymbol{E}, I\!R)/\operatorname{map}(\boldsymbol{E}, I\!R). \quad \square
\end{aligned}
$$

The special phase Lie bracket can be further re-interpreted in terms of the observed and gauge splittings of special phase functions and of the natural Lie bracket of spacetime pairs as follows.

Note 12.5.6 Let us consider a closed spacetime form Φ. Then, we define the natural *Lie bracket of spacetime pairs* $(X, \phi) \in \sec(\boldsymbol{E}, T\boldsymbol{E}) \times \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ to be the phase pair

$$\big[(X, \phi)\ ,\ (\acute{X}, \acute{\phi})\big]_\Phi = \big([X, X']\ ,\ X.\acute{\phi} - \acute{X}.\phi + \Phi(X, \acute{X})\big). \quad \square$$

Then, by taking into account the observed and gauge splittings of special phase functions (see Corollary 12.2.11)

$$f \simeq \big(X[f], \breve{f}[o]\big) \quad \text{and} \quad f \simeq \big(X[f], \hat{f}[\mathrm{b}]\big),$$

we get the following result.

Proposition 12.5.7 *According to the observed and gauge expressions of the special phase Lie bracket provided by the above Theorem 12.5.3, the special phase Lie bracket can be further expressed, respectively, thorough the equalities (see the above Note 12.5.6)*

$$\begin{aligned}&\big[(X[f], \breve{f}[o])\ ,\ (X[\acute{f}], \breve{\acute{f}}[o])\big]_{\Phi[o]}\\ &\quad = \big(\big[X[f], X[\acute{f}]\big]\ ,\ X[f].\breve{\acute{f}}[o] - X[\acute{f}].\breve{f}[o] + \Phi[o](X[f], X[\acute{f}])\big)\end{aligned}$$

and

$$[\![\, (X[f], \hat{f}[\mathrm{b}])\ ,\ (X[\acute{f}], \hat{\acute{f}}[\mathrm{b}])\,]\!]_0 = \big(\big[X[f], X[\acute{f}]\big]\ ,\ X[f].\hat{\acute{f}}[\mathrm{b}] - X[\acute{f}].\hat{f}[\mathrm{b}]\big). \quad \square$$

Example 12.5.8 We have the following distinguished special phase Lie brackets (see Example 12.1.4):

$$\begin{gathered}[\![\, x^\lambda, x^\mu \,]\!] = 0,\\ [\![\, x^\lambda, A^\uparrow{}_i \,]\!] = [\![\, x^\lambda, \mathcal{P}_i \,]\!] = \delta^\lambda_i, \qquad [\![\, x^\lambda, -A^\uparrow{}_0 \,]\!] = [\![\, x^\lambda, \mathcal{H}_0 \,]\!] = -\delta^\lambda_0,\\ [\![\, A^\uparrow{}_h, A^\uparrow{}_k \,]\!] = [\![\, \mathcal{P}_h, \mathcal{P}_k \,]\!] = 0, \qquad [\![\, A^\uparrow{}_h, -A^\uparrow{}_0 \,]\!] = [\![\, \mathcal{P}_h, \mathcal{H}_0 \,]\!] = 0. \quad \square\end{gathered}$$

Remark 12.5.9 (1) Only apparently, the special phase bracket depends on the 1st jet of the factor functions (see the coordinate expression provided by Theorem 12.5.3).
Indeed, it depends on the 2nd jet. In fact, the time component f'' of a special phase function f depends on the 2nd jet of f (see Lemma 11.3.5).

(2) The special phase bracket has some analogies with the Jacobi bracket

$$[f, \acute{f}] = f\, E.\acute{f} - \acute{f}\, E.f + \Lambda(df \wedge d\acute{f})$$

of a Jacobi structure (E, Λ) (see, for instance, [275, 290]). The differences between the special phase bracket and the Jacobi bracket are the following:

- E is a vector field, while γ is a scaled vector field;

- the identities fulfilled by E and Λ are different from the analogous identities fulfilled by γ and Λ (see Theorem 9.2.19 and [222, 275, 290]);
- in the special phase bracket we have the term $f'' \gamma.\acute{f} - \acute{f}'' \gamma.f$, while in the Jacobi bracket we have the term $f\, E.\acute{f} - \acute{f}\, E.f$;
- the special phase bracket depends on the 2nd jet of the factor functions, while the Jacobi bracket depends on the 1st jet.

(3) The special phase bracket does not contradict the Lichnerowicz theorem on the uniqueness of the Jacobi bracket (see, for instance, [275, 290]) because the uniqueness of the Jacobi bracket is based on the hypothesis of the dependence on the 1st jet.
(4) We could not define the Jacobi bracket in our context, because it would conflict with the covariance, as γ is a scaled vector field, hence we should choose a time scale.
(5) We could define the special bracket for all phase functions, but, in general, we would not obtain the Jacobi property.
(6) The time component f'' of a phase function f and the phase 2-vector Λ have been defined by using the rescaled metric G and the constant Planck scale $\hbar$. Therefore, the special phase bracket is implicitly defined with reference to a particle with mass m and to the constant Planck scale $\hbar$. □

12.6 Lie Subalgebras of Special Phase Functions

We have two families of distinguished Lie subalgebras of the Lie algebra of special phase functions: namely, subalgebras characterised by *algebraic conditions* and subalgebras characterised by *differential conditions*.

12.6.1 Algebraic Lie Subalgebras of Special Phase Functions

We start by discussing distinguished Lie subalgebras of the Lie algebra of special phase functions, which are defined by means of algebraic conditions (see, also, [227]).

Proposition 12.6.1 *The subsheaves of spacetime functions and of the projectable, time preserving and affine special phase functions (see Definition 12.1.3)*

$$\operatorname{map}(\boldsymbol{E}, I\!R) \subset \operatorname{aff}\operatorname{spe}(\boldsymbol{E}, I\!R) \subset \operatorname{tim}\operatorname{spe}(\boldsymbol{E}, I\!R) \subset \operatorname{pro}\operatorname{spe}(\boldsymbol{E}, I\!R) \subset \operatorname{spe}(J_1\boldsymbol{E}, I\!R)$$

turn out to be closed with respect to the special phase Lie bracket. □

Indeed, the following Lie subalgebra of special phase functions will play a role in the classification of infinitesimal symmetries of classical dynamics and in the discussion of classical currents (see Theorem 13.2.6 and Definition 13.3.1)

Now, let us choose a gauge $\mathfrak{b}$.

Definition 12.6.2 With reference to the gauge $\mathfrak{b}$, we define:

- a *short special phase functions* to be a special phase functions $f \in \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ such that $\hat{f}[\mathfrak{b}] = 0$,
- a *quasi-short special phase function* to be a special phase function $f \in \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ such that $\hat{f}[\mathfrak{b}] \in \mathbb{R}$.

The subsheaves of short and quasi-short special phase functions are denoted by

$$\text{srt}_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{srt}'_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 12.6.3 *With reference to the gauge* $\mathfrak{b}$*, the short special phase functions* $f \in \text{srt}_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ *are characterised by their tangent lift through the equality*

$$f = -i_{X[f]} A^{\uparrow}[\mathfrak{b}].$$

Accordingly, the sheaf of short special phase functions is constituted by the special phase functions of the the following type, with reference to any observer o,

$$f = f^0 \, \mathcal{H}_0[\mathfrak{b}, o] + f^i \, \mathcal{P}_i[\mathfrak{b}, o], \quad \text{with } f^{\lambda} \in \text{map}(\boldsymbol{E}, \mathbb{R}).$$

In particular, we have

$$\mathcal{H}_0[\mathfrak{b}, o] \in \text{srt}_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \quad \text{and} \quad \mathcal{P}_j[\mathfrak{b}, o] \in \text{srt}_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

We stress that, for any observer o, we have the equivalence

$$\hat{f}[\mathfrak{b}] = 0 \qquad \Leftrightarrow \qquad \check{f}[o] = -i_{X[f]} A[\mathfrak{b}, o].$$

Thus, the short special phase functions are determined by their tangent lift. $\square$

Proposition 12.6.4 *The subsheaves of short and quasi-short special phase functions*

$$\text{srt}_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{srt}'_{\mathfrak{b}} \, \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$$

turn out to be closed with respect to the special phase Lie bracket.

Proof. The proof follows from the equality (see Theorem 12.5.3)

$$\widehat{[\![f, \acute{f}]\!]} = X[f]^{\mu} \, \partial_{\mu} \hat{\acute{f}} - X[\acute{f}]^{\mu} \, \partial_{\mu} \hat{f}. \quad \square$$

12.6.2 Differential Lie Subalgebras of Special Phase Functions

Next, we discuss distinguished Lie subalgebras of the Lie algebra of special phase functions, which are defined by means of differential conditions (see, also, [227]).

Preliminarily, we show that

- the *holonomic lift* of special phase functions is a surjective Lie algebra morphism,
- the *hamiltonian lift* of projectable special phase functions is a surjective Lie algebra morphism.

Then, we define and characterise the

- *quasi unitary Lie subalgebra* of s.p.f. f, such that $d\,\mathrm{div}_\eta f = 0$,
- *unitary Lie subalgebra* of s.p.f. f, such that $\mathrm{div}_\eta f = 0$,
- *holonomic Lie subalgebra* of s.p.f. f, such that $X^\uparrow{}_{\mathrm{ham}}[f] = X^\uparrow{}_{\mathrm{hol}}[f]$,
- *conserved Lie subalgebra* of s.p.f. f, such that $\gamma.f = 0$.

Proposition 12.6.5 *For each* $f, \acute{f} \in \mathrm{spe}(J_1\boldsymbol{E}, I\!R)$*, we have*

$$\big[X^\uparrow{}_{\mathrm{hol}}[f],\ X^\uparrow{}_{\mathrm{hol}}[\acute{f}]\big] = X^\uparrow{}_{\mathrm{hol}}\big[\,[\![f, \acute{f}]\!]\,\big].$$

Hence, the holonomic lift of special phase functions (see Definition 12.3.2)

$$X^\uparrow{}_{\mathrm{hol}} : \mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to \mathrm{hol\,sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^\uparrow{}_{\mathrm{hol}}[f]$$

turns out to be a surjective Lie algebra sheaf morphism, whose kernel is the subsheaf (see Proposition 12.3.3)

$$\mathrm{map}(\boldsymbol{E}, I\!R) \subset \mathrm{spe}(J_1\boldsymbol{E}, I\!R).$$

Hence, the map $X^\uparrow{}_{\mathrm{hol}}$ *passes to the quotient and we obtain a Lie algebra isomorphism*

$$X^\uparrow{}_{\mathrm{hol}} : \mathrm{spe}(J_1\boldsymbol{E}, I\!R)/\,\mathrm{map}(\boldsymbol{E}, I\!R) \to \mathrm{hol\,sec}(\boldsymbol{E}, T\boldsymbol{E}). \quad \square$$

Proof. In fact, the holonomic lift is the composition $f \mapsto X[f] \mapsto r^1 \circ J_1X[f]$ of two Lie algebra morphisms (see Corollary 12.5.4 and Proposition 12.3.1). $\square$

Let

$$\mathrm{pro\,ham\,sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{ham\,sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \subset \mathrm{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

be the subsheaf of hamiltonian lift of projectable special phase functions.

Proposition 12.6.6 *For each* $f, \acute{f} \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R)$*, we have*

$$\big[X^\uparrow{}_{\mathrm{ham}}[f],\ X^\uparrow{}_{\mathrm{ham}}[\acute{f}]\big] = X^\uparrow{}_{\mathrm{ham}}\big[\,[\![f, \acute{f}]\!]\,\big].$$

Hence, the hamiltonian lift of projectable special phase functions (see Definition 12.4.1)

$$X^{\uparrow}{}_{\text{ham}} : \text{pro}\,\text{spe}(J_1\boldsymbol{E}, I\!R) \to \text{pro}\,\text{ham}\,\text{sec}(J_1\boldsymbol{E}, T\,J_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{ham}}[f]$$

turns out to be a surjective Lie algebra sheaf morphism, whose kernel is the subsheaf (see Proposition 12.4.2)

$$\text{map}(\boldsymbol{T}, I\!R) \subset \text{spe}(J_1\boldsymbol{E}, I\!R).$$

Hence, the map $X^{\uparrow}{}_{\text{ham}}$ passes to the quotient and we obtain a Lie algebra isomorphism

$$X^{\uparrow}{}_{\text{ham}} : \text{pro}\,\text{spe}(J_1\boldsymbol{E}, I\!R)/\,\text{map}(\boldsymbol{T}, I\!R) \to \text{pro}\,\text{ham}\,\text{sec}(\boldsymbol{E}, T\boldsymbol{E}). \quad \square$$

Proof. The proof can be achieved by a long computation, by taking into account the identities holding for dynamical phase objects γ and Λ (see Theorem 9.2.19). □

Proposition 12.6.7 *For each $f, \acute{f} \in \text{pro}\,\text{spe}(J_1\boldsymbol{E}, I\!R)$, we have (see Definition 12.2.7)*

$$\text{div}_\eta [\![f, \acute{f}]\!] = X[f].\,\text{div}_\eta \acute{f} - X[\acute{f}].\,\text{div}_\eta f.$$

Hence, the subsheaves of projectable special phase functions with vanishing divergence and with constant divergence, respectively,

$$\text{uni}_\eta\,\text{spe}(J_1\boldsymbol{E}, I\!R) \subset \text{duni}_\eta\,\text{spe}(J_1\boldsymbol{E}, I\!R) \subset \text{pro}\,\text{spe}(J_1\boldsymbol{E}, I\!R)$$

turn out to be closed with respect to the special phase Lie bracket.

Proof. The proof can be easily achieved from the standard properties of Lie derivatives. □

Definition 12.6.8 A special phase function f is said to be *holonomic* if its holonomic and hamiltonian phase lifts coincide, i.e. if

$$X^{\uparrow}{}_{\text{ham}}[f] = X^{\uparrow}{}_{\text{hol}}[f].$$

We denote the subsheaf of holonomic special phase functions by

$$\text{hol}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{pro}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

For each holonomic special phase function $f \in \text{hol}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})$, the phase lift

$$X^{\uparrow}[f] = X^{\uparrow}{}_{\text{ham}}[f] = X^{\uparrow}{}_{\text{hol}}[f]$$

is called the *special phase lift*. □

Proposition 12.6.9 *A special phase function f turns out to be holonomic if and only if it fulfills the following conditions, with reference to an observer o,*

$$
\begin{aligned}
&(H_1) && f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R),\\
&(H_2) && df'' \otimes G = L_{X[f]} G,\\
&(H_3) && \check{d}\check{f}[o] = -G^{\flat}\big(L_{\text{д}[o]} f'[o]\big) + \tfrac{1}{2}\,\big(i_{X[f]} \check{\Phi}[o]\big),
\end{aligned}
$$

i.e., in coordinates,

$$
\begin{aligned}
&(h_1) && 0 = \partial_i f^0,\\
&(h_2) && 0 = \partial_0 f^0\, G^0_{ij} - (f^0\,\partial_0 - f^h\,\partial_h)\, G^0_{ij} + \partial_j f^h\, G^0_{ih} + \partial_i f^h\, G^0_{jh},\\
&(h_3) && 0 = \partial_i \check{f} + \partial_0 f^h\, G^0_{ih} - f^0\,(\partial_0 A_i - \partial_i A_0) + f^h\,(\partial_h A_i - \partial_i A_h).
\end{aligned}
$$

The subsheaf of holonomic *special phase functions*

$$\operatorname{hol}\operatorname{spe}(J_1\boldsymbol{E}, I\!R) \subset \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$$

turns out to closed with respect to the special phase Lie bracket.

Proof. Conditions (h_1), (h_2), (h_3) follow from Propositions 12.3.3 and 12.4.2.

Moreover, the closure with respect to the special phase Lie bracket follows from Propositions 12.6.5 and 12.6.6. □

Definition 12.6.10 According to the general Definition 11.6.2, a special phase function $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ is said to be *conserved* if $\gamma.f = 0$, i.e. if it is constant along the solutions of the classical equation of motion.

We denote the subsheaf of conserved special phase functions by

$$\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 12.6.11 *The subsheaf of conserved special phase functions is characterised by the conditions, with reference to an observer o,*

$$
\begin{aligned}
&(C_1) = (H_1) && f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R),\\
&(C_2) = (H_2) && df'' \otimes G = L_{X[f]} G,\\
&(C_{3v}) = (H_3) && \check{d}\check{f}[o] = -G^{\flat}\big(L_{\text{д}[o]} f'[o]\big) + \tfrac{1}{2}\,\big(i_{X[f]} \check{\Phi}[o]\big),\\
&(C_{3h}) && \text{д}[o].\check{f}[o] = \tfrac{1}{2}\, i_{f'[o]} i_{\text{д}[o]} \Phi[o].
\end{aligned}
$$

i.e., in coordinates,

$$
\begin{aligned}
&(c_1) = (h_1) && 0 = \partial_i f^0,\\
&(c_2) = (h_2) && 0 = \partial_0 f^0\, G^0_{ij} - (f^0\,\partial_0 - f^h\,\partial_h)\, G^0_{ij} + \partial_j f^h\, G^0_{ih} + \partial_i f^h\, G^0_{jh},\\
&(c_{3v}) = (h_3) && 0 = \partial_i \check{f} + \partial_0 f^h\, G^0_{ih} - f^0\,(\partial_0 A_i - \partial_i A_0) + f^h\,(\partial_h A_i - \partial_i A_h),\\
&(c_{3h}) && 0 = \partial_0 \check{f} - f^h\,(\partial_0 A_h - \partial_h A_0).
\end{aligned}
$$

Indeed, conditions (C_{3v}) *and* (C_{3h}) *can be written together, in a more compact way, as condition*

$$(C_3) \qquad d\breve{f}[o] = -i_{\theta[o]} G^{\flat}\big(L_{Д[o]} f'[o]\big) + \tfrac{1}{2}\, i_{X[f]}\Phi[o].$$

Hence, we have

$$\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, I\!R) \subset \operatorname{hol}\operatorname{spe}(J_1\boldsymbol{E}, I\!R).$$

In particular, the conserved special phase functions turn out to be projectable.

Proof. The conditions (c_1), (c_2), (c_{3v}), (c_{3h}) follow from the coordinate expressions of γ and f. □

Later, we shall deal with the quotient sheaf $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})\big/\operatorname{map}(\boldsymbol{E}, \mathbb{R})$. With reference to subsheaf of conserved special phase functions, we obtain the following reduction.

Note 12.6.12 If $f, \acute{f} \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ fulfill the equality $\acute{f} = f + \phi$, with $\phi \in \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$, then we have $\phi \in \mathbb{R} \subset \operatorname{map}(J_1\boldsymbol{E}, \mathbb{R})$.

Proof. In fact, $\gamma.\phi = 0$ if and only if $\phi \in \mathbb{R}$. □

The following result will be involved later in the discussion of infinitesimal symmetries of classical dynamics and classical and quantum currents (see Theorems 13.2.6 and 21.1.4, Proposition 13.3.2 and Example 21.1.5).

Corollary 12.6.13 *Let us consider the* $I\!R$*-linear sheaf morphism*

$$X^{\uparrow} : \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, I\!R) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f].$$

Then, for each $f, \acute{f} \in \operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$*,*

$$X^{\uparrow}[f] = X^{\uparrow}[\acute{f}] \qquad \Leftrightarrow \qquad \acute{f} = f + \phi, \quad \text{with}\, \phi \in I\!R.$$

Hence, the map $X^{\uparrow}$ *passes to the quotient yielding an injective* $I\!R$*-linear map*

$$\big(\operatorname{cns}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)\big)/I\!R \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : [f + \phi] \mapsto X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{hol}}[f] \\ = X^{\uparrow}{}_{\mathrm{ham}}[f].$$

Proof. The coordinate expression of the holonomic phase lift of special phase functions (see Proposition 12.3.3) shows that, for each $f, \acute{f} \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$,

$$X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{hol}}[\acute{f}] \qquad \Leftrightarrow \qquad \acute{f} = f + \phi, \quad \text{with}\, \phi \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

Hence, the map $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}[f] := X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f]$ passes to the quotient yielding an injective linear map

$$\big(\text{cns}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})\big)/\,\text{map}(\boldsymbol{E}, \mathbb{R}) \to \text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : [f + \phi] \mapsto X^{\uparrow}[f]$$
$$:= X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f].$$

In particular, for each $f, \acute{f} \in \text{cns}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})$,

$$X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{hol}}[\acute{f}] \qquad \Leftrightarrow \qquad \acute{f} = f + \phi, \quad \text{with}\, \phi \in \text{cns}\,\text{map}(\boldsymbol{E}, \mathbb{R}).$$

We have

$$\phi \in \text{cns}\,\text{map}(\boldsymbol{E}, \mathbb{R}) \qquad \Leftrightarrow \qquad \phi \in \mathbb{R}.$$

Hence, for each $f, \acute{f} \in \text{cns}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})$,

$$X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{hol}}[\acute{f}] \qquad \Leftrightarrow \qquad \acute{f} = f + \phi, \quad \text{with}\, \phi \in \mathbb{R}. \quad \square$$

In particular, let us discuss the time preserving conserved special phase functions.

Corollary 12.6.14 *The subsheaf of time preserving conserved special phase functions*

$$\text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R) \subset \text{spe}(J_1\boldsymbol{E}, I\!\!R)$$

is characterised by the conditions

$$\begin{array}{ll}
(C_1') & f \in \text{tim}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R), \\
(C_2') & 0 = L_{X[f]}G, \\
(C_{3v}') = (C_{3v}) & \check{d}\check{f}[o] = -G^{\flat}\big(L_{д[o]}f'[o]\big) + \frac{1}{2}\big(i_{X[f]}\check{\Phi}[o]\big), \\
(C_{3h}') = (C_{3h}) & д[o].\check{f}[o] = \frac{1}{2}\, i_{f'[o]} i_{д[o]}\Phi[o].
\end{array}$$

i.e., in coordinates,

$$\begin{array}{ll}
(c_1') & 0 = \partial_\lambda f^0, \\
(c_2') & 0 = (f^0\,\partial_0 - f^h\,\partial_h)\,G^0_{ij} - \partial_j f^h\,G^0_{ih} - \partial_i f^h\,G^0_{jh}, \\
(c_{3v}') = c_{3v}) & 0 = \partial_i \check{f} + \partial_0 f^h\,G^0_{ih} - f^0\,(\partial_0 A_i - \partial_i A_0) + f^h\,(\partial_h A_i - \partial_i A_h), \\
(c_{3h}') = c_{3h}) & 0 = \partial_0 \check{f} - f^h\,(\partial_0 A_h - \partial_h A_0).
\end{array}$$

Clearly, conditions (C_{3v}') and (C_{3h}') can be written together, in a more compact way, as condition $(C_3') = (C_3)$, as in Proposition 12.6.11. $\square$

Corollary 12.6.15 *For each* $f \in \text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R) \subset \text{spe}(J_1\boldsymbol{E}, I\!\!R)$*, we have* $L_{X[f]}G = 0$*, hence*

$$\text{div}_\eta f = 0.$$

Proof. The proof follows from condition (C_1'). $\square$

Proposition 12.6.16 *The subsheaves*

$$\text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R) \subset \text{cns}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R) \subset \text{spe}(J_1\boldsymbol{E}, I\!\!R)$$

turn out to be closed with respect to the special phase Lie bracket.

Proof. The proof follows from the equalities $\gamma.\{f, \acute{f}\} = 0$ and $[\![f, \acute{f}]\!] = \{f, \acute{f}\}$, for each $f, \acute{f} \in \text{cns}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})$. □

Eventually, we state a characterisation of conserved special phase functions, which will play a relevant role in the following (see Proposition 13.1.2).

Indeed, the results of the present section further support our definition of hamiltonian lift of special phase functions and our definition of holonomic special phase functions.

Theorem 12.6.17 *For each* $f \in \text{spe}(J_1\boldsymbol{E}, I\!\!R)$*, the following equivalences hold*

$$0 = \gamma.f \quad \Leftrightarrow \quad i_{X^{\uparrow}{}_{\text{ham}}[f]}\,\Omega = df \quad \Leftrightarrow \quad i_{X^{\uparrow}{}_{\text{hol}}[f]}\,\Omega = df.$$

Indeed, if the above equivalent conditions are fulfilled, then, according to Proposition 12.6.11,

$$X^{\uparrow}{}_{\text{ham}}[f] = X^{\uparrow}{}_{\text{hol}}[f],$$

i.e.,

$$\text{cns}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R) \subset \text{hol}\,\text{spe}(J_1\boldsymbol{E}, I\!\!R).$$

Proof. The proof follows from Theorem 11.3.4 by taking into account conditions (c_1), (c_2), (c_{3v}), (c_{3h}). □

Chapter 13
Classical Symmetries

We study the *infinitesimal symmetries of the classical structure*, which is encoded in the cosymplectic pair (dt, Ω), and the *infinitesimal symmetries of the classical dynamics*, which is encoded in the dynamical pair $(dt, \mathcal{L}[\mathrm{b}])$ (Sects. 13.1 and 13.2).

Actually, we find that these infinitesimal symmetries are naturally generated by a distinguished subalgebra of the Lie algebra of special phase function.

The original source of our approach to classical infinitesimal symmetries and currents goes back to the Ph.D. thesis of Dirk Saller (see [358]).

The present review includes improvements with respect to further literature and new results as well (see [227, 312, 358, 359] and see, also, [107]).

It is worth comparing the above discussion on classical symmetries and currents with the following discussion on quantum symmetries and currents (see Chaps. 19 and 21).

For further approaches to symmetry, see also [121].

13.1 Symmetries of Classical Structure

We analyse the *infinitesimal symmetries* $X^{\uparrow}$ of the *cosymplectic pair* (dt, Ω), which encodes the basic classical structure.

Actually, we show that such infinitesimal symmetries $X^{\uparrow}$ are of the type

$$X^{\uparrow} = X^{\uparrow}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df),$$

where f is a *conserved time preserving special phase function*.

Definition 13.1.1 We define the *infinitesimal symmetries of classical structure* to be the projectable phase vector fields $X^{\uparrow} \in pro_{\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, which fulfill the conditions (see Theorem 10.1.1)

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$$L_{X^\uparrow} dt = 0 \quad \text{and} \quad L_{X^\uparrow} \Omega = 0. \quad \square$$

Indeed, these symmetries can be classified by the following procedure (see also Theorem 11.3.8).

For this purpose, let us recall the holonomic lift of special phase functions, the hamiltonian lift of special phase functions and the conserved special phase functions (see Definitions 12.3.2, 12.4.1, 12.6.10 and Proposition 12.6.11).

Proposition 13.1.2 *If $X^\uparrow \in pro_{\boldsymbol{E}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, then the following conditions are equivalent (see also [227, 312, 359]):*

(1)
$$L_{X^\uparrow}\, \Omega = 0,$$

(2)
$$X^\uparrow = X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^\sharp(df),$$

with
$$f \in \operatorname{cns\,spe}(J_1\boldsymbol{E}, \mathrm{I\!R}),$$

(3)
$$X^\uparrow = X^\uparrow[f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i,$$

where
$$f^0, f^i, \check{f} \in \operatorname{map}(\boldsymbol{E}, \mathrm{I\!R}),$$

fulfill the conditions (c_1), (c_2), (c_{3v}), (c_{3h}) *(see Proposition 12.6.11) and where*

$$\begin{aligned} X^i_0 &= -\partial_0 f^i - \partial_j f^i\, x^j_0 - \partial_0 f^0\, x^i_0 \\ &= -f^0\,(\partial_0 \mathcal{P}_j - \partial_j A_0) + f^h\,(\partial_h \mathcal{P}_j - \partial_j A_h) + \partial_j f^h\, G^0_{hk}\, x^k_0 + \partial_j \check{f}. \end{aligned}$$

Thus, the IR*-Lie subalgebra of projectable infinitesimal symmetries of* Ω *is constituted by the special phase lifts of conserved special phase functions.*

Proof. For each $X^\uparrow \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, in virtue of the identity $d\Omega = 0$ and of Theorem 11.3.4, we have the following equivalences

$$L_{X^\uparrow}\, \Omega = 0 \quad \Leftrightarrow \quad d i_{X^\uparrow}\, \Omega = 0 \quad \overset{\text{locally}}{\Leftrightarrow} \quad i_{X^\uparrow}\, \Omega = df, \quad f \in \operatorname{map}(J_1\boldsymbol{E}, \mathrm{I\!R}).$$

Moreover, in virtue of Theorem 11.3.4, the infinitesimal symmetries $X^\uparrow$ of Ω, which are projectable on $\boldsymbol{E}$, are the phase vector fields $X^\uparrow$ of the local type

$$X^\uparrow = X^\uparrow{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^\sharp(df), \quad \text{with } f \in \operatorname{cns\,spe}(J_1\boldsymbol{E}, \mathrm{I\!R}).$$

Moreover, Theorem 12.6.17 implies $X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f]$. $\square$

Then, we get the following classification of infinitesimal symmetries of the cosymplectic pair (dt, Ω) in terms of conserved time preserving special phase functions.

Theorem 13.1.3 *If $X^\uparrow \in pro_{\boldsymbol{E}}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$, then the following conditions are equivalent:*

(1) $$L_{X^\uparrow} dt = 0 \quad \text{and} \quad L_{X^\uparrow} \Omega = 0,$$

(2) $$X^\uparrow = X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^\sharp(df),$$

with

$$f \in \operatorname{cns} \operatorname{tim} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

(3) $$X^\uparrow = X^\uparrow[f] = f^0\, \partial_0 - f^i\, \partial_i + X^i_0\, \partial^0_i,$$

where

$$f^0 \in \mathbb{R}, \quad f^i, \breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}),$$

fulfill the conditions (c'_1), (c'_2), (c'_{3v}), (c'_{3h}) provided by Corollary 12.6.14 and where X^i_0 is given by Proposition 12.4.2.

Proof. The proof follows from the above Proposition 13.1.2. □

Next, we discuss the relation between the symmetries of Ω and of $A^\uparrow[\mathfrak{b}]$, by taking into account that Ω is gauge independent, while its potential $A^\uparrow[\mathfrak{b}]$ is gauge dependent.

Proposition 13.1.4 *Let us consider a local potential $A^\uparrow[\mathfrak{b}]$ of Ω and a phase vector field $X^\uparrow \in pro_{\boldsymbol{E}} \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$.*

If $X^\uparrow$ is an infinitesimal symmetry of $A^\uparrow[\mathfrak{b}]$, then it is an infinitesimal symmetry of Ω. But the converse needs not to be true.

Therefore, the Lie algebra of infinitesimal symmetries of $A^\uparrow[\mathfrak{b}]$ is a subalgebra of the Lie algebra of infinitesimal symmetries of Ω.

Proof. (1) If $0 = L_{X^\uparrow} A^\uparrow[\mathfrak{b}]$, then $0 = dL_{X^\uparrow} A^\uparrow[\mathfrak{b}] = L_{X^\uparrow} dA^\uparrow[\mathfrak{b}] = L_{X^\uparrow}\Omega$.

Then, the infinitesimal symmetries of $A^\uparrow[\mathfrak{b}]$ are also infinitesimal symmetries of Ω.

(2) Let us consider another potential $A^\uparrow[\acute{\mathfrak{b}}] = A^\uparrow[\mathfrak{b}] + df$, with $f \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$. If $L_{X^\uparrow}\Omega = 0$ and $L_{X^\uparrow} A^\uparrow[\mathfrak{b}] = 0$, then $L_{X^\uparrow} A^\uparrow[\acute{\mathfrak{b}}] = 0$ if and only if $d(X^\uparrow . f) = 0$.

Then, an infinitesimal symmetry of Ω needs not to be an infinitesimal symmetry of $\acute{A}^\uparrow[\mathfrak{b}]$. □

13.2 Symmetries of Classical Dynamics

We analyse the *infinitesimal symmetries* of the *dynamical pair* $(dt, \mathcal{L}[\mathrm{b}])$, which encodes the basic classical dynamics.

Thus, we say that a spacetime vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$ generates an *infinitesimal symmetry of classical dynamics* if its 1-jet holonomic prolongation X^1 (see, Proposition 12.3.1) fulfills the conditions $L_{X^1} dt = 0$ and $L_{X^1}\mathcal{L}[\mathrm{b}] = 0$.

Actually, we show that such generators X of infinitesimal symmetries are of the type $X = X[f]$, (see Theorem 12.2.1) where f is a *conserved quasi-short time preserving special phase function* (see Definition 12.6.2), which fulfills an *additional condition*.

Moreover, the corresponding infinitesimal symmetries are (see Definition 12.6.8)

$$X^1 = X^{\uparrow}[f] = X^{\uparrow}{}_{\mathrm{hol}}[f] = X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df),$$

Let us recall Lemma H.3.2 (see, for instance, [117, 243, 283, 331, 411]).

Now, let us choose a gauge b and consider the associated classical lagrangian form $\mathcal{L}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$ and Poincaré–Cartan form $A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ (see Theorem 10.1.8).

Proposition 13.2.1 *Let us consider a projectable spacetime vector field and its 1st-jet holonomic prolongation (see Proposition 12.3.1)*

$$X \in \mathit{pro}\sec(\boldsymbol{E}, T\boldsymbol{E}) \quad \text{and} \quad X^1 \in \mathit{pro}_{\boldsymbol{E},\boldsymbol{T}} \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

Then, the following equivalence holds

$$L_{X^1}\mathcal{L}[\mathrm{b}] = 0 \quad \Leftrightarrow \quad L_{X^1}A^{\uparrow}[\mathrm{b}] = 0.$$

Proof. The proof follows immediately from Lemma H.3.2, according to a general result of calculus of variations, and from the fact that $A^{\uparrow}[\mathrm{b}]$ is the Poincaré–Cartan form associated with $\mathcal{L}[\mathrm{b}]$ (see Theorem 10.1.8). □

Now, let us consider a projectable spacetime vector field $X \in \mathrm{pro}\sec(\boldsymbol{E}, T\boldsymbol{E})$ and its 1-jet holonomic prolongation $X^1 \in \mathit{pro}_{\boldsymbol{E},\boldsymbol{T}} \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ (see Proposition 12.3.1).

Corollary 13.2.2 *The following implication holds (see Theorem 10.1.8)*

$$L_{X^1}\mathcal{L}[\mathrm{b}] = 0 \quad \Rightarrow \quad L_{X^1}\mathcal{M}[\mathrm{b}] = 0.$$

Proof. The proof follows easily from from the equality $\mathcal{M}[\mathrm{b}] = A^{\uparrow}[\mathrm{b}] - \mathcal{L}[\mathrm{b}]$ (see Theorem 10.1.8) and the above Proposition 13.2.1. □

Corollary 13.2.3 *The following equivalence holds:*

$$L_{X^1}dt = 0 \quad \textit{and} \quad L_{X^1}A^{\uparrow}[\mathrm{b}] = 0 \quad \Leftrightarrow \quad L_{X^1}dt = 0 \quad \textit{and} \quad L_{X^1}\mathcal{L}[\mathrm{b}] = 0.$$

Proof. The proof follows immediately from the above Proposition 13.2.1. □

Clearly, if the above conditions are fulfilled, then $L_{X^1} dt = 0$ implies

$$X \in \operatorname{tim\,sec}(\boldsymbol{E}, T\boldsymbol{E}) \subset \operatorname{pro\,sec}(\boldsymbol{E}, T\boldsymbol{E}).$$

Now, we are in the position to define and classify the infinitesimal symmetries of classical dynamics.

Definition 13.2.4 We define the *infinitesimal symmetries of the classical dynamics* to be the spacetime vector fields $X \in \operatorname{pro\,sec}(\boldsymbol{E}, T\boldsymbol{E})$, which fulfill the conditions (see Proposition 12.3.1)

$$L_{X^1} dt = 0 \quad \text{and} \quad L_{X^1} \mathcal{L}[\mathfrak{b}] = 0. \quad \square$$

Note 13.2.5 For each gauge $\mathfrak{b}$, the infinitesimal symmetries of the classical dynamics are also infinitesimal symmetries of the classical structure.

Proof. The proof follows from Propositions 13.1.4 and 13.2.1. □

Let us recall the holonomic lift of special phase functions, the hamiltonian lift of special phase functions, the conserved special phase functions and the quasi-short special phase functions (see Definitions 12.3.2, 12.4.1, 12.6.2, 12.6.10 and Proposition 12.6.11).

Then, we get the following classification of infinitesimal symmetries of the pair $(dt, \mathcal{L}[\mathfrak{b}])$ in terms of a distinguished type of conserved time preserving special phase functions.

Theorem 13.2.6 *For each* $X \in \operatorname{pro\,sec}(\boldsymbol{E}, T\boldsymbol{E})$, *the following conditions are equivalent (see Appendix: Lemma H.3.2):*

(1) $$L_{X^1} dt = 0 \quad \text{and} \quad L_{X^1} \mathcal{L}[\mathfrak{b}] = 0,$$

(2) $$X^1 = X^{\uparrow}[f] = X^{\uparrow}{}_{\text{hol}}[f] = X^{\uparrow}{}_{\text{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df),$$

where (see Definition 12.6.2),

$$f \in \operatorname{srt}'_{\mathfrak{b}} \operatorname{cns\,tim\,spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

i.e. where $f \in \operatorname{cns\,tim\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ *fulfills the additional condition*

$$d\breve{f}[o] = -d(i_{X[f]} A[\mathfrak{b}, o]),$$

i.e.

$$\breve{f}[o] = -i_{X[f]} A[\mathfrak{b}, o] + k[\mathfrak{b}], \quad \text{with } k[\mathfrak{b}] \in \mathbb{R}.$$

Thus, the infinitesimal symmetries of the classical dynamics are generated by the special phase functions belonging to the Lie subalgebra (see Proposition 12.6.4)

$$\operatorname{srt}'_{\mathfrak{b}} \operatorname{cns} \operatorname{tim} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{cns} \operatorname{tim} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

Proof. In virtue of Theorem 13.1.3, Proposition 13.1.4 and Corollary 13.2.3, the infinitesimal symmetries of classical dynamics are generated by a subalgebra of conserved time preserving special phase functions; thus, it fulfills conditions (c'_1), (c'_2), (c'_{3v}), (c'_{3h}) (see Corollary 12.6.14).

Actually, for each $f \in \operatorname{cns} \operatorname{tim} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have the equivalence

$$L_{X^{\uparrow}{}_{\text{ham}}[f]}\mathcal{L}[\mathfrak{b}] = 0 \quad \Leftrightarrow \quad f^0\,\partial_0\mathcal{L}_0 - f^i\,\partial_i\mathcal{L}_0 + X^i_0\,\partial^0_i\mathcal{L}_0 + \partial_0 f^0\,\mathcal{L}_0 = 0.$$

Moreover, we have $\mathcal{L}_0 = \frac{1}{2}\,G^0_{ij}\,x^i_0\,x^j_0 + A_j\,x^j_0 + A_0$s and, by taking into account the equality $X^1[f] = X^{\uparrow}{}_{\text{hol}}[f]$, we can write $X^i_0 = -\partial_0 f^i - \partial_j f^i\,x^j_0 - \partial_0 f^0\,x^i_0$.

Then, we obtain $L_{X^1[f]}\mathcal{L}[\mathfrak{b}] = 0$ if and only if

$$\begin{aligned}
0 &= f^0\,\partial_0 G^0_{hk} - f^i\,\partial_i G^0_{hk} - \partial_h f^p\,G^0_{pk} - \partial_k f^p\,G^0_{ph},\\
0 &= f^0\,\partial_0 A_h - f^i\,\partial_i A_h - \partial_h f^i\,A_i - \partial_0 f^i\,G^0_{ih},\\
0 &= f^0\,\partial_0 A_0 - f^i\,\partial_i A_0 - \partial_0 f^i\,A_i + \partial_0 f^0\,A_0.
\end{aligned}$$

The 1st equation is just condition (c'_2) (see Corollary 12.6.14).

By using (c'_{3v}), the 2nd equation can be rewritten as $0 = \partial_h \check{f} + \partial_h(f^0\,A_0 - f^i\,A_i)$.s

Finally, by using (c'_{3h}), the 3rd equation can be rewritten as $0 = \partial_0\check{f} + \partial_0 (f^0\,A_0 - f^i\,A_i)$.

Hence, we obtain $d\check{f} = -d(i_{X[f]}A[\mathfrak{b}, o])$. □

Remark 13.2.7 In virtue of Proposition 12.6.4, the special phase functions considered in the above Theorem 13.2.6 are in bijection with time preserving spacetime vector fields X which are infinitesimal symmetries of G and (in virtue of Theorems 13.1.3, 13.2.6, 13.2.6, Proposition 13.1.4, and Corollary 12.6.14) satisfy the condition

$$L_X A[\mathfrak{b}, o] = i_{\theta[o]} G^{\flat}(L_X \text{Д}[o]). \quad \square$$

13.3 Classical Currents

We apply the general formalism discussed in Appendix: Sect. H.3 to the case of Covariant Classical Mechanics (see, also, [312, 358, 359]).

Thus, we define the *classical current* associated with a special phase function and discuss its splitting and related Lie algebra constructions.

We can implement in the following way the general construction discussed in Sect. H.3.

- We choose a gauge $\mathfrak{b}$,
- the base space is $\boldsymbol{B} := \boldsymbol{T}$ and the total space is $\boldsymbol{F} := \boldsymbol{E}$,
- the scaled volume form is $\upsilon = u_0 \otimes d^0 \in \mathbb{T} \otimes T^*\boldsymbol{T}$,
- the lagrangian form and the Poincaré–Cartan form are, respectively, the local, gauge dependent and observer independent 1-forms (see Theorem 10.1.8)

$$\mathcal{L}[\mathfrak{b}] = (\mathcal{L}_0 \, u^0) \otimes (u_0 \otimes d^0) = \mathcal{L}_0 \, d^0 \in \sec(J_1\boldsymbol{E}, \mathbb{T}^* \otimes H^*\boldsymbol{E}),$$
$$A^{\uparrow}[\mathfrak{b}] = \mathcal{L}[\mathfrak{b}] + \mathcal{M}[\mathfrak{b}] \in \sec(J_1\boldsymbol{E}, \mathbb{T}^* \otimes T^*\boldsymbol{E}).$$

In the present case, we have $m - 1 = 0$, hence the classical currents turn out to be functions.

Definition 13.3.1 We define the *classical current*, associated with a special phase function $f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$, to be the local gauge dependent and observer independent special phase function (see Definition 12.6.2 and Proposition 12.6.3)

$$\mathfrak{c}[f, \mathfrak{b}] := -\, i_{X[f]} A^{\uparrow}[\mathfrak{b}] \in \mathrm{srt}_{\mathfrak{b}}\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})\,. \quad \square$$

Proposition 13.3.2 *For each* $f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$*, we have the equalities (see Proposition 12.2.9)*

$$\mathfrak{c}[f, \mathfrak{b}] := -\, i_{X[f]} A^{\uparrow}[\mathfrak{b}] = \tilde{f}[\mathfrak{b}] = f - \hat{f}[\mathfrak{b}],$$

with coordinate expression

$$\mathfrak{c}[f, \mathfrak{b}] = f^0 \, (\tfrac{1}{2} \, G^0_{ij} \, x^i_0 \, x^j_0 - A_0) + f^i \, (G^0_{ij} \, x^j_0 + A_i) = f - \check{f} - A_0 \, f^0 + A_i \, f^i$$
$$= f - \hat{f}.$$

In other words, the current $\mathfrak{c}[f, \mathfrak{b}]$ *turns out to be just the 1st gauge component of* f *(see Proposition 12.2.9).*

Accordingly, we have

$$X\big[\mathfrak{c}[f, \mathfrak{b}]\big] = X[f].$$

Moreover, the map (see Definition 12.6.2)

$$\mathfrak{c}[\,.\,, \mathfrak{b}] : \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) \to \mathrm{srt}_{\mathfrak{b}}\mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R}) : f \mapsto \mathfrak{c}[f, \mathfrak{b}]$$

turns out to be a $\mathrm{map}(\boldsymbol{E}, \mathrm{I\!R})$*-linear sheaf morphism.*

Hence, for each $f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathrm{I\!R})$*, we can write*

$$f = \mathfrak{c}[f, \mathfrak{b}] + \hat{f}[\mathfrak{b}] = f^0 \, \mathfrak{c}[\mathcal{H}_0, \mathfrak{b}] + f^i \, \mathfrak{c}[\mathcal{P}_i, \mathfrak{b}] + \hat{f}[\mathfrak{b}]. \quad \square$$

Proposition 13.3.3 *For each* $f, \acute{f} \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$*, we have*

$$[\![\mathfrak{c}[f, \mathfrak{b}], \ \mathfrak{c}[\acute{f}, \mathfrak{b}]]\!] = -i_{\left[X[f],\, X[\acute{f}]\right]} A^{\uparrow} = \mathfrak{c}\big[[\![f, \acute{f}]\!], \mathfrak{b}\big].$$

Therefore, the map (see Definition 12.6.2)

$$\mathfrak{c}[\,.\,, \mathfrak{b}] : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{srt}_{\mathfrak{b}}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) : f \mapsto \mathfrak{c}[f, \mathfrak{b}]$$

turns out to be a surjective Lie algebra morphism with respect to the special phase Lie bracket, whose kernel is the subsheaf $\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$.

Hence, the map $\mathfrak{c}[\,.\,, \mathfrak{b}]$ *passes to the quotient and we obtain a sheaf isomorphism*

$$\mathfrak{c}[\,.\,, \mathfrak{b}] : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) / \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \to \operatorname{srt}_{\mathfrak{b}}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 13.3.4 *In virtue of the Noether Theorem (see Theorem H.3.3), for each special phase function* $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$*, which is a generator of classical dynamics (see Definition 13.2.4 and Theorem 13.2.6), the associated classical current is conserved along the motions which are solution of the law of motion, i.e., in other words,*

$$\gamma.\mathfrak{c}[f, \mathfrak{b}] = 0. \quad \square$$

Corollary 13.3.5 *Each* $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$*, which is a generator of classical dynamics (see Definition 13.2.4 and Theorem 13.2.6), can be uniquely written as*

$$f = \mathfrak{c}[f, \mathfrak{b}] + \hat{f}[\mathfrak{b}], \quad \text{with } \hat{f}[\mathfrak{b}] \in \mathbb{R}.$$

Proof. For each $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, in virtue of Proposition 13.3.2, we can uniquely write

$$f = \mathfrak{c}[f, \mathfrak{b}] + \hat{f}[\mathfrak{b}], \quad \text{where } \hat{f}[\mathfrak{b}] \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

Moreover, in virtue of Theorem 13.2.6 and the above Proposition 13.3.4, we have

$$\gamma.f = 0 \quad \text{and} \quad \gamma.\mathfrak{c}[f, \mathfrak{b}] = 0,$$

which imply $\gamma.\hat{f}[\mathfrak{b}] = 0$, hence (see Corollary 9.2.4) $\hat{f}[\mathfrak{b}] \in \mathbb{R}$. $\square$

It is worth comparing the infinitesimal symmetries of classical dynamics with the classical currents.

Corollary 13.3.6 *The surjective Lie algebra sheaf morphism (see Definition 12.3.2)*

$$X^{\uparrow}{}_{\text{hol}} : \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{hol}\sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) : f \mapsto X^{\uparrow}{}_{\text{hol}}[f],$$

whose kernel is the subsheaf $\operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$*, factorises through a Lie algebra sheaf morphism (denoted by the same symbol)*

$$X^{\uparrow}{}_{\text{hol}} : \text{srt}_{\mathfrak{b}} \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}),$$

according to the commutative diagram

$$\begin{array}{ccc} \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) & \xrightarrow{X^{\uparrow}{}_{\text{hol}}} & \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \\ {\scriptstyle \mathfrak{c}[.\,\mathfrak{b}]}\downarrow & & \uparrow{\scriptstyle id} \\ \text{srt}_{\mathfrak{b}}\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) & \xrightarrow{X^{\uparrow}{}_{\text{hol}}} & \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}) \end{array} \quad .$$

Hence, for each special phase function $f \in \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$, *which is a generator of classical dynamics, we have the equality*

$$X^{\uparrow}[f] = X^{\uparrow}\big[\mathfrak{c}[f, \mathfrak{b}]\big].$$

Proof. The proof follows from Proposition 12.3.3 and the above Proposition 13.3.4. □

Indeed, there is a natural bijection between the generators of infinitesimal symmetries of classical dynamics and the associated conserved Noether quantities, according to the following result.

Corollary 13.3.7 *We have a natural injective Lie algebra morphism*

$$X^{\uparrow}{}_{\text{hol}} \circ \mathfrak{c}[., \mathfrak{b}]^{-1} : \text{srt}_{\mathfrak{b}}\,\text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

Proof. The Lie algebra morphism (see Definition 12.3.2)

$$X^{\uparrow}{}_{\text{hol}} : \text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$$

passes to to quotient as an injective Lie algebra morphism (see Propositions 12.3.3, 12.6.5 and Corollary 12.6.13)

$$X^{\uparrow}{}_{\text{hol}} : \text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})\big/\mathbb{R} \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

Moreover, we have a Lie algebra isomorphism (see Proposition 13.3.3)

$$\mathfrak{c}[., \mathfrak{b}] : \text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R})/\mathbb{R} \to \text{srt}_{\mathfrak{b}}\,\text{cns}\,\text{tim}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) : f \mapsto \mathfrak{c}[f, \mathfrak{b}].$$

Then, we obtain an injective Lie algebra morphism

$$X^{\uparrow}{}_{\text{hol}} \circ \mathfrak{c}[., \mathfrak{b}]^{-1} : \text{srt}_{\mathfrak{b}}\,\text{tim}\,\text{cns}\,\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{hol}\,\text{sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}). \quad □$$

Example 13.3.8 We have the following distinguished currents (see Example 12.1.4):

$$\mathfrak{c}[x^\lambda, \mathfrak{b}] = 0,$$
$$\mathfrak{c}[A^\uparrow{}_i, \mathfrak{b}] = \mathfrak{c}[\mathcal{P}_i, \mathfrak{b}] = \mathcal{P}_i, \quad \mathfrak{c}[-A^\uparrow{}_0, \mathfrak{b}] = \mathfrak{c}[\mathcal{H}_0, \mathfrak{b}] = \mathcal{H}_0,$$
$$\mathfrak{c}[\mathcal{L}_0, \mathfrak{b}] = \mathcal{L}_0 - 2\,\alpha_0\,. \quad \square$$

Proposition 13.3.9 *For each* $f \in \operatorname{cns}\operatorname{tim}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ *and each critical motion* $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$*, the time function*

$$\mathfrak{c}[f, \mathfrak{b}](s) := (j_1 s)^* \mathfrak{c}[f, \mathfrak{b}] \in \operatorname{map}(\boldsymbol{T}, \mathbb{R})$$

turns out to be constant.

Proof. The proof follows from the classical Noether theorem (see Theorems H.3.3) and 13.2.6. □

Part II
Covariant Quantum Mechanics

The second part of the book is devoted to an introduction to Covariant Quantum Mechanics of a scalar charged particle.

We discuss the 1 dimensional complex *quantum bundle* over the classical spacetime, equipped with a scaled hermitian metric (Chap. 13). Special attention is devoted to the *proper quantum bundle*, by emphasising the physical meaning of its intrinsic real splitting (Sect. 13.6 and Sect. 13.7).

We analyse in detail the *upper quantum connection* (Chap. 14) and show how it yields in a covariant way the *quantum kinetic tensor*, the *quantum momentum*, the *quantum probability current*, the *quantum lagrangian* and the *Schrödinger equation* (Chap. 16).

We discuss the main *quantum invariants* associated with every quantum section: the *rest observer* and the *timelike potential* "seen by the quantum particle" (Sect. 14.2.6).

We discuss a covariant version of the *hydrodynamical picture of Quantum Mechanics*, by emphasising the role of the distinguished timelike potential (Chap. 17).

We classify the *infinitesimal symmetries* of the quantum structure in terms of the Lie algebra of special phase functions (Chap. 18).

We achieve the *quantum operators* as a byproduct of the classification of hermitian quantum vector fields and show the natural isomorphism between *their Lie algebra* and the *Lie algebra of special phase functions* (Chap. 19).

In order to fulfill the covariance of the quantum theory, we discuss the need of introducing infinite dimensional *sectional quantum bundle* over time (Chap. 21).

Eventually, we derive the *Feynman amplitudes* of the Feynman path integral via the upper quantum connection (Chap. 22).

Chapter 14
Quantum Bundle

In standard Quantum Mechanics, the state of the quantum particle is usually represented by the wave function $\psi : \boldsymbol{E} \to \mathbb{C}$. Here, following a suggestion of Geometric Quantisation (see, for instance [425] and see also [17]), we represent the state of the quantum particle as a section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ of a "*quantum bundle*" $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, whose type fibre is $\mathbb{C}$ (see, for instance [182–186]).

This formalism helps us to manage in a rational geometric way the quantum gauges and to deal with the possible case of a non trivial quantum bundle.

Standard Quantum Mechanics and, more generally, all Quantum Theories, are formulated in terms of a complex mathematical language. It seems that such a complex language be unavoidable; so, one might ask whether there is a deep reason of this fact. Actually, we show that the hermitian 1-dimensional complex structure of the quantum bundle is formally equivalent to a suitable euclidean 2-dimensional real structure. Nevertheless, the complex language provides practical advantages of compactness. So, the standard choice of the complex language for Quantum Mechanics is due more to practical advantages, than to mysterious deep physical reasons. Actually, in several fields of mathematics it happens that the complex language is able to express real concepts in a surprising compact way.

We remark that the base space of our quantum bundle $\boldsymbol{Q}$ is the classical spacetime $\boldsymbol{E}$ and not the classical phase space $J_1\boldsymbol{E}$ (see Postulate Q.1). Later, we shall extend the base space $\boldsymbol{E}$ to the phase space $J_1\boldsymbol{E}$ by pullback (see Definition 14.11.1).

We emphasise that we do not assume any scale dimension for the fibres of the quantum bundle $\boldsymbol{Q}$, but we consider a scaled hermitian quantum metric h (see Proposition 14.3.1).

14.1 Real Quantum Bundle

First, we postulate the quantum bundle as a 2-dimensional real euclidean vector bundle based on spacetime and with oriented fibres.

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Postulate Q.1 *We postulate the* quantum bundle *to be a 2-dimensional real vector bundle over spacetime*

$$\pi : \boldsymbol{Q} \to \boldsymbol{E},$$

equipped with a global orientation of its fibres and a scaled fibred euclidean metric, called real quantum metric*,*

$$\mathrm{g}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*). \quad \square$$

Note 14.1.1 The quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, along with its fibred real euclidean metric and orientation, can be regarded as a bundle associated with a principal bundle over spacetime $\boldsymbol{P}\big[SO(2)\big] \to \boldsymbol{E}$, whose structure group is $SO(2)$. □

Remark 14.1.2 The hypothesis that the fibres of the quantum bundle be smoothly orientable means that there exists a global everywhere non vanishing section $\boldsymbol{E} \to \Lambda^2\boldsymbol{Q}$.

Actually, this means that the bundle $\Lambda^2\boldsymbol{Q} \to \boldsymbol{E}$ be trivial. But, this hypothesis does not imply that the bundle $\boldsymbol{Q} \to \boldsymbol{E}$ be trivial.

We stress that, in general, we do not make any assumption whether the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ be trivial or not (such an hypothesis can be discussed case by case), but, in any case, we do not assume any distinguished trivialisation. □

Proposition 14.1.3 *The real euclidean metric and the orientation of the fibres of quantum bundle yield in a natural way the* quantum norm fibred morphism *over* $\boldsymbol{E}$ *and the global positive oriented scaled* quantum volume vector

$$\|\,\| : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{R} : \Psi_e \mapsto \sqrt{\mathrm{g}_{\boldsymbol{Q}}(\Psi_e, \Psi_e)},$$
$$\bar{\eta}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^2\boldsymbol{Q}.$$

For each (local) quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain its scaled quantum norm*

$$\|\Psi\| := \sqrt{\mathrm{g}_{\boldsymbol{Q}}(\Psi, \Psi)} : \boldsymbol{E} \to \mathbb{L}^{-3/2} \otimes \mathbb{R}.$$

For each (local) quantum sections $\Psi, \acute{\Psi} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain their (local) volume vector*

$$\Psi \wedge \acute{\Psi} : \boldsymbol{E} \to \Lambda^2\boldsymbol{Q}. \quad \square$$

Definition 14.1.4 We define a *real quantum base* to be a (local) scaled, positive oriented, orthonormal basis of the real quantum bundle

$$(\mathrm{b}_\mathrm{a}) \equiv (\mathrm{b}_1, \mathrm{b}_2), \quad \text{with } \mathrm{b}_\mathrm{a} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}, \; \mathrm{g}_{\boldsymbol{Q}}(\mathrm{b}_\mathrm{a}, \mathrm{b}_\mathrm{b}) = \delta_{\mathrm{ab}}.$$

We denote the associated dual (local) scaled *real linear coordinates* on the fibres of quantum bundle by

$$(w^{a}) \equiv (w^{1}, w^{2}), \quad \text{with } w^{a} : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathrm{I\!R}, \quad w^{a}(\mathfrak{b}_{b}) = \delta^{a}_{b}.$$

Indeed, the pairs $(\mathfrak{b}_1, \mathfrak{b}_2)$ and (w^1, w^2) are ordered by the orientation of the fibres of the quantum bundle.

We denote the (local) bases of $\sec(\boldsymbol{Q}, T\boldsymbol{Q})$ and $\sec(\boldsymbol{Q}, T^*\boldsymbol{Q})$ induced by a fibred quantum chart (x^λ, w^a) by

$$(\partial_\lambda, \partial w_a) = (\partial_\lambda, \partial w_1, \partial w_2) \quad \text{and} \quad (d^\lambda, dw^a) = (d^\lambda, dw^1, dw^2).$$

For each (local) quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, with reference to a real quantum basis $(\mathfrak{b}_a)$, we shall write

$$\Psi = \Psi^a\, \mathfrak{b}_a, \quad \text{with } \Psi^a := w^a \circ \Psi \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathrm{I\!R}). \quad \square$$

Proposition 14.1.5 *For each real quantum basis* $(\mathfrak{b}_a)$*, we obtain (locally) (see Proposition 14.1.3)*

$$\bar{\eta}_{\boldsymbol{Q}} = \mathfrak{b}_1 \wedge \mathfrak{b}_2.$$

Hence, for each (local) quantum sections $\Psi, \acute{\Psi} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain the coordinate expressions*

$$\mathfrak{g}_{\boldsymbol{Q}}(\Psi, \acute{\Psi}) = \Psi^1\, \acute{\Psi}^1 + \Psi^2\, \acute{\Psi}^2 \quad \text{and} \quad \Psi \wedge \acute{\Psi} = (\Psi^1\, \acute{\Psi}^2 - \Psi^2\, \acute{\Psi}^1)\, \mathfrak{b}_1 \wedge \mathfrak{b}_2. \quad \square$$

14.2 Complex Structure

The real quantum metric $\mathfrak{g}_{\boldsymbol{Q}}$ and the orientation of the fibres of quantum bundle naturally yield a 1-dimensional complex structure on the fibres of the quantum bundle.

Proposition 14.2.1 *Let us consider the real linear Hodge fibred operator over* $\boldsymbol{E}$

$$\mathfrak{i} : \boldsymbol{Q} \to \boldsymbol{Q} : q \mapsto i_{\mathfrak{g}^\flat_{\boldsymbol{Q}}(q)} \bar{\eta}_{\boldsymbol{Q}},$$

whose coordinate expression is (see Definition 14.1.4)

$$\mathfrak{i}\,(q^a\, \mathfrak{b}_a) = i_{q^a\, \mathfrak{g}^\flat_{\boldsymbol{Q}}(\mathfrak{b}_a)}(\mathfrak{b}_1 \wedge \mathfrak{b}_2) = q^1\, \mathfrak{b}_2 - q^2\, \mathfrak{b}_1.$$

Indeed, we have $\mathfrak{i}^2 = -1$.

In practice, we can regard the imaginary multiplication $\mathfrak{i}$ *of the fibres of quantum bundle as the* positive rotation of the angle $\pi/2$, *with reference to the quantum euclidean metric* $\mathfrak{g}_{\boldsymbol{Q}}$ *and the orientation of the fibres of the quantum bundle.*

Thus, the operator $\mathfrak{i}$ *equips the fibres of the quantum bundle with a 1-dimensional complex structure, via the fibred scalar product over* $\boldsymbol{E}$

$$\varsigma : \mathbb{C} \times \boldsymbol{Q} \to \boldsymbol{Q} : \big((r + \mathrm{i}\, s), \Psi_e\big) \mapsto r\, \Psi_e + \mathrm{i}\,(s\, \Psi_e).$$

The expression of i *in the real quantum basis* (b_a) *is*

$$\mathrm{i}\, \mathrm{b}_1 = \mathrm{b}_2, \quad \mathrm{i}\, \mathrm{b}_2 = -\mathrm{b}_1, \quad \text{i.e.} \begin{pmatrix} \mathrm{i}^1_1 & \mathrm{i}^1_2 \\ \mathrm{i}^2_1 & \mathrm{i}^2_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \det(\mathrm{i}^\mathrm{a}_\mathrm{b}) = 1.$$

Accordingly, the (local) real scaled quantum basis (b_a) *yields the (local) scaled* complex quantum basis *and the associated (local) dual scaled* complex linear coordinate *on the fibres of the quantum bundle*

$$\mathrm{b} := \mathrm{b}_1 : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q} \quad \text{and} \quad z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C},$$

defined by the equalities

$$z = w^1 + \mathrm{i}\, w^2, \quad \bar{z} = w^1 - \mathrm{i}\, w^2, \quad w^1 = \tfrac{1}{2}\,(z + \bar{z}), \quad w^2 = \tfrac{1}{2}\, \mathrm{i}\,(\bar{z} - z).$$

By definition, we have $z(\mathrm{b}) = 1$.

For each (local) quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we shall write, in complex coordinates,*

$$\Psi = \psi\, \mathrm{b}, \quad \text{with}\ \psi := z \circ \Psi \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C}). \quad \square$$

Remark 14.2.2 The type fibre $\mathbb{C}$ of the quantum bundle is equipped with the $\mathbb{R}$-linear *complex conjugation* (which is an output of the splitting of $\mathbb{C}$ into its real and imaginary components)

$$\mathbb{C} \to \mathbb{C} : z = \mathrm{a} + \mathrm{i}\, \mathrm{b} \mapsto \bar{z} = \mathrm{a} - \mathrm{i}\, \mathrm{b}.$$

We stress that this conjugation is *not* inherited in a covariant way by the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, because its (local) splitting into real and imaginary components depends on the choice of a quantum basis. Thus, the conjugation real linear morphism

$$\psi\, \mathrm{b} \ \equiv\ \Psi^1\, \mathrm{b}_1 + \Psi^2\, \mathrm{b}_2 \mapsto \bar{\psi}\, \mathrm{b} \ \equiv\ \Psi^1\, \mathrm{b}_1 - \Psi^2\, \mathrm{b}_2$$

depends on the chosen quantum chart and cannot be extended to an intrinsic morphism, even if the quantum bundle is trivial.

However, we can define the "*conjugate complex structure*" of $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ by considering the conjugate scalar product fibred morphism

$$\bar{\varsigma} : \mathbb{C} \times \boldsymbol{Q} \to \boldsymbol{Q} : (\mathrm{c}, \mathrm{q}) \mapsto \bar{\mathrm{c}}\, \mathrm{q}. \quad \square$$

In several developments of our exposition, we deal with a generic spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$. If spacetime is a bundle with contractible type fibre, then we have the following result.

Remark 14.2.3 Let us recall a classical Theorem of Differential Geometry (see Appendix: Theorem A.2.6), which states that any bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ with a "contractible" base space $\boldsymbol{B}$ is globally trivialisable (see, for instance, [374, Corollary 1.6], [191, p. 488]).

We have already pointed out that, being $\boldsymbol{T}$ contractible, in virtue of the above theorem, if the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ is a bundle, then it is globally trivialisable, hence we have a global bundle isomorphism $\boldsymbol{E} \to \boldsymbol{T} \times \boldsymbol{S}$ (see Remark 2.1.1). Now, if the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ is a bundle and, additionally, the type fibre $\boldsymbol{S}$ is contractible, then, also $\boldsymbol{E}$ turns out to be contractible. Hence, in virtue of the above theorem, the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ turns out to be globally trivialisable, hence we have a global bundle isomorphism $\boldsymbol{Q} \to \boldsymbol{E} \times \mathbb{C}$. But, we stress that, even in the case when spacetime is a bundle with contractible type fibre, there is no distinguished bundle splitting as above. □

14.3 Hermitian Structure

The real quantum euclidean metric $g_{\boldsymbol{Q}}$ and the orientation of the fibres of real quantum bundle further yield in a natural way a scaled hermitian structure h on the fibres of the quantum bundle.

Proposition 14.3.1 *The real quantum metric* $g_{\boldsymbol{Q}}$ *and the volume vector* $\bar{\eta}_{\boldsymbol{Q}}$ *yield, in a covariant way, the scaled* hermitian quantum metric

$$h : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C},$$

which is defined as follows, by splitting $\mathbb{C}$ *into its real and imaginary components (see Proposition 14.1.3),*

$$h(\Psi_e, \acute{\Psi}_e) = g_{\boldsymbol{Q}}(\Psi_e, \acute{\Psi}_e) + i\left(\Psi_e \wedge \acute{\Psi}_e)/\bar{\eta}_{\boldsymbol{Q}}\right).$$

In particular, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain (see Proposition 14.1.3)*

$$h(\Psi, \Psi) = \|\Psi\|^2.$$

Thus, we obtain the usual coordinate expression

$$h = (w_1^1\, w_2^1 + w_1^2\, w_2^2) + i\,(w_1^1\, w_2^2 - w_1^2\, w_2^1) = \bar{z}_1\, z_2,$$

where z_1, z_2 *and* w^a_1, w^a_2 *denote the complex and real coordinates on the 1st and the 2nd factors of* $\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$*, respectively.*

Hence, for each quantum sections Ψ, $\acute{\Psi} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, the coordinate expression of* $\mathfrak{h}$ *turns out to be, as usual,*

$$\mathfrak{h}(\Psi, \acute{\Psi}) = (\Psi^1 \acute{\Psi}^1 + \Psi^2 \acute{\Psi}^2) + \mathfrak{i}\,(\Psi^1 \acute{\Psi}^2 - \Psi^2 \acute{\Psi}^1) = \bar{\psi}\,\acute{\psi}.$$

We can also regard $\mathfrak{h}$ *as a spacetime tensor* $\mathfrak{h} : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q}^* \otimes \mathbb{C}$. □

Later, we show a very intuitive presentation of the hermitian quantum metric, in terms of the polar splitting of the proper quantum bundle (see Proposition 14.7.5).

Note 14.3.2 The quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, along with the fibred hermitian complex structure, can be regarded as a vector bundle associated with a principal bundle over spacetime $\boldsymbol{P}\big[U(1)\big] \to \boldsymbol{E}$, whose structure group is $U(1)$. □

14.4 Complex Versus Real Structures

So far, we have postulated the real structure of quantum bundle, along with the scaled real quantum metric $\mathfrak{g}_{\boldsymbol{Q}}$ and the scaled volume vector of the fibres $\bar{\eta}_{\boldsymbol{Q}}$.

Then, we have seen that the pair $(\mathfrak{g}_{\boldsymbol{Q}}, \bar{\eta}_{\boldsymbol{Q}})$ yields, in a natural way, the hermitian-complex structure $(\mathfrak{i}, \mathfrak{h})$ (see Propositions 14.2.1 and 14.3.1).

Now, we show, conversely, that a hermitian-complex structure of quantum bundle yields a real structure as above. Even more, we show a natural bijection between such real and complex structures.

Proposition 14.4.1 *Let us consider a 1-dimensional complex bundle* $\boldsymbol{Q} \to \boldsymbol{E}$.

Then, the imaginary fibred multiplication $\mathfrak{i} : \boldsymbol{Q} \to \boldsymbol{Q}$ *yields an orientation of the underlying 2-dimensional real fibres, given by the ordered real basis* $(\mathfrak{b}, \mathfrak{i}\,\mathfrak{b})$*, or, equivalently, by the volume vector* $\mathfrak{b} \wedge \mathfrak{i}\,\mathfrak{b}$*, for each complex basis* $\mathfrak{b} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$.

Indeed, this orientation does not depended on the choice of the complex basis $\mathfrak{b}$.

Proof. If we choose another complex basis $(\alpha + \mathfrak{i}\,\beta)\,\mathfrak{b}$, with $\alpha^2 + \beta^2 > 0$, then we obtain the following volume vector (by making a clear distinction between algebraic operations that make sense in the real or in the complex frameworks)

$$\begin{aligned}(\alpha + \mathfrak{i}\,\beta)\,\mathfrak{b} \wedge (\alpha + \mathfrak{i}\,\beta)\,(\mathfrak{i}\,\mathfrak{b}) &= \big(\alpha\,\mathfrak{b} + \beta\,(\mathfrak{i}\,\mathfrak{b})\big) \wedge \big(\alpha\,(\mathfrak{i}\,\mathfrak{b}) - \beta\,\mathfrak{b}\big) \\ &= (\alpha^2 + \beta^2)\,\mathfrak{b} \wedge \mathfrak{i}\,\mathfrak{b}.\end{aligned}$$

Then, the two complex bases yield the same orientation. □

Proposition 14.4.2 *Let us consider a 1-dimensional complex bundle* $\boldsymbol{Q} \to \boldsymbol{E}$.

Then, defining a hermitian metric $\mathfrak{h}$ *is equivalent to choose a volume vector* $\eta_{\boldsymbol{Q}}$ *on the oriented underlying 2-dimensional real bundle* $\boldsymbol{Q} \to \boldsymbol{E}$*. In fact, chosen such an* $\eta_{\boldsymbol{Q}}$*, there exists a unique euclidean metric* $\mathfrak{g}_{\boldsymbol{Q}}$ *on the fibres of this real bundle, such that the associated oriented scaled orthonormal bases* $(\mathfrak{b}_1, \mathfrak{b}_2)$ *fulfill the conditions*

$$\mathfrak{b}_2 = \mathfrak{i}\, \mathfrak{b}_1 \quad \text{and} \quad \mathfrak{b}_1 \wedge \mathfrak{b}_2 = \eta_{\boldsymbol{Q}}.$$

Then, the hermitian metric $\mathfrak{h}$ *is determined by the pair* $(\mathfrak{g}_{\boldsymbol{Q}}, \eta_{\boldsymbol{Q}})$ *according to Proposition 14.3.1.* □

Note 14.4.3 Let us consider a 1-dimensional complex bundle $\boldsymbol{Q} \to \boldsymbol{E}$, equipped with a scaled hermitian quantum metric $\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C}$.

We can split $\mathfrak{h}$ into its real and imaginary components

$$\operatorname{re}\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{R} \quad \text{and} \quad \operatorname{im}\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{R},$$

by writing

$$\mathfrak{h} = \operatorname{re}\mathfrak{h} + \mathfrak{i}\operatorname{im}\mathfrak{h}.$$

Then, the real component turns out to be the scaled euclidean metric

$$\operatorname{re}\mathfrak{h} = \mathfrak{g}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*).$$

Moreover, the imaginary component can be naturally regarded as the global scaled volume form

$$\operatorname{im}\mathfrak{h} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^2 \boldsymbol{Q}^*,$$

which yields an orientation of the fibres of the quantum bundle.

Furthermore, the real component $\operatorname{re}\mathfrak{h}$ yields the mutually inverse real linear fibred isomorphism over $\boldsymbol{E}$

$$(\operatorname{re}\mathfrak{h})^\flat : \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \boldsymbol{Q}^* \quad \text{and} \quad (\operatorname{re}\mathfrak{h})^\sharp : \boldsymbol{Q}^* \to \mathbb{L}^{3} \otimes \boldsymbol{Q},$$

with coordinate expressions

$$(\operatorname{re}\mathfrak{h})^\flat = w^1 \otimes w^1 + w^2 \otimes w^2 \quad \text{and} \quad (\operatorname{re}\mathfrak{h})^\sharp = \mathfrak{b}_1 \otimes \mathfrak{b}_1 + \mathfrak{b}_2 \otimes \mathfrak{b}_2. \quad \square$$

Note 14.4.4 The maps

$$(\mathfrak{g}_{\boldsymbol{Q}}, \bar{\eta}_{\boldsymbol{Q}}) \mapsto (\mathfrak{i}\,, \mathfrak{h}) \quad \text{and} \quad (\mathfrak{i}\,, \mathfrak{h}) \mapsto (\mathfrak{g}_{\boldsymbol{Q}}, \bar{\eta}_{\boldsymbol{Q}}),$$

that we have established in Propositions 14.2.1, 14.3.1 and 14.4.1, yield a bijection.

This bijection agrees with the fact that the real quantum bundle is associated with the principal bundle $\boldsymbol{P}[SO(2)] \to \boldsymbol{E}$, while the hermitian complex bundle is associated with the principal bundle $\boldsymbol{P}[U(1)] \to \boldsymbol{E}$ (see Notes 14.1.1 and 14.3.2) and

$$SO(2) \simeq U(1).$$

Thus, the real structure postulated in Postulate Q.1 is equivalent to the hermitian-complex structure obtained in Propositions 14.2.1 and 14.3.1. □

Note 14.4.5 In virtue of the above results (see Propositions 14.2.1, 14.3.1, 14.4.1 and 14.4.2), the *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, which has been postulated in Postulate Q.1, can be equivalently regarded as a 1-dimensional complex vector bundle equipped with a scaled hermitian quantum metric

$$\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C}. \quad \square$$

14.5 η-Hermitian Quantum Structure

Besides the scaled hermitian quantum metric $\mathfrak{h}$, we define the vector valued hermitian quantum metric $\mathfrak{h}_\eta$, by multiplying $\mathfrak{h}$ with the spacelike volume form η of spacetime (see Proposition 3.2.4). Actually, this vector valued hermitian metric $\mathfrak{h}_\eta$ is suitable for integration on the fibres of spacetime and it allows us to skip semi-quantum sections.

Indeed, $\mathfrak{h}_\eta$ will play a relevant role throughout this book (see, for instance, Theorems 17.5.2, 19.1.7, 19.2.2, 19.3.2, 20.1.9, 21.1.4, 21.1.9 and 21.2.4).

Definition 14.5.1 In view of the integration on the fibres of the quantum bundle, we consider also the vector valued *η-hermitian quantum metric* (see Proposition 3.2.4)

$$\mathfrak{h}_\eta := \mathfrak{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}.$$

The normalisation of complex bases $\mathfrak{b}$, in terms of $\mathfrak{h}_\eta$, becomes $\mathfrak{h}_\eta(\mathfrak{b}, \mathfrak{b}) = \eta$.

We have the coordinate expression

$$\mathfrak{h}_\eta = \big((w_1^1\, w_2^1 + w_1^2\, w_2^2) + \mathfrak{i}\,(w_1^1\, w_2^2 - w_1^2\, w_2^1)\big)\, \eta = \bar{z}_1\, z_2\, \eta.$$

Thus, for each $\Psi, \acute{\Psi} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we can write

$$\mathfrak{h}_\eta(\Psi, \acute{\Psi}) = (\Psi^1\, \acute{\Psi}^1 + \Psi^2\, \acute{\Psi}^2)\, \eta + \mathfrak{i}\,(\Psi^1\, \acute{\Psi}^2 - \Psi^2\, \acute{\Psi}^1)\, \eta = \bar{\psi}\, \acute{\psi}\, \eta.$$

We can also regard $\mathfrak{h}_\eta$ as a spacetime tensor $\mathfrak{h}_\eta : \boldsymbol{E} \to \boldsymbol{Q}^* \otimes \boldsymbol{Q}^* \otimes \Lambda^3 V^* E \otimes \mathbb{C}$. □

14.6 Proper Quantum Bundle

In view of the polar splitting of the quantum bundle (see forthcoming Sect. 14.7), it is convenient to introduce the "*proper quantum bundle*" $\boldsymbol{Q}_{/0} \subset \boldsymbol{Q}$ over $\boldsymbol{E}$, by dropping the zero section. On the fibres of proper quantum bundle we can define the usual polar complex coordinates. Indeed, the proper quantum bundle and the proper quantum sections will play a relevant role throughout the present book.

Definition 14.6.1 We define the *proper quantum bundle* to be the subbundle of the quantum bundle over $\boldsymbol{E}$

$$\pi_{/0} : \boldsymbol{Q}_{/0} \subset \boldsymbol{Q} \to \boldsymbol{E},$$

which is obtained from the quantum bundle $\boldsymbol{Q}$, by dropping the zero section.

A (local) section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, i.e. an everywhere non vanishing quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, is said to be a *proper quantum section*.

The *proper domain* of a (local) quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ is defined to be the subset of the domain of Ψ consisting of points where Ψ is non vanishing. □

Definition 14.6.2 On $\boldsymbol{Q}_{/0}$, each complex coordinate $z : \boldsymbol{Q}_{/0} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$ can be written, as usual, by the following *polar expression*

$$z = \varrho \, e^{i\phi}, \quad \text{where } \varrho : \boldsymbol{Q}_{/0} \to \mathbb{L}^{-3/2} \otimes \mathbb{R}^+, \ \phi : \boldsymbol{Q}_{/0} \to \mathbb{R}/2\pi.$$

Accordingly, for each (local) quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we shall write

$$\Psi = |\psi| \, e^{i\varphi} \, \mathrm{b},$$

where

$$|\psi| := \varrho \circ \Psi \in \operatorname{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{R}^+) \quad \text{and} \quad \varphi := \phi \circ \Psi \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}/2\pi). \quad \square$$

Note 14.6.3 In general, every quantum basis is suitable for the coordinate expression of quantum objects. Actually, each proper quantum section yields a distinguished quantum basis.

In fact, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we obtain the associated scaled quantum basis

$$\mathrm{b}_\Psi \equiv \mathrm{b}[\Psi] := \Psi / \|\Psi\| \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q}).$$

Such a distinguished basis, associated with a given quantum section Ψ, will play later a relevant role (see, for instance, Theorems 15.2.31 and 18.2.2, Proposition 16.4.5, Corollaries 17.3.3 and 17.6.18). □

14.7 Polar Splitting of the Proper Quantum Bundle

Next, we discuss the natural *polar splitting* of the proper quantum bundle $\boldsymbol{Q}_{/0}$ into a real "*norm*" component $\boldsymbol{Q}^{\|\|}_{/0}$, valued in a trivial bundle, and a real "*phase*" component $\boldsymbol{Q}^{(())}_{/0}$, valued in a principal bundle (whose fibres $\boldsymbol{Q}^{(())}_{/0e}$ are 1-dimensional circles $S_1[e]$) associated with the abelian Lie group $U(1) = \mathbb{R}/2\pi$.

This splitting may help understanding quantum sections in intuitive real terms, as it helps regarding, in a covariant way, every quantum particle described by a "*proper quantum section*" in terms of the "*probability to be detected*" and an "*internal angle*".

We stress that an analogous "phase component" valued in $U(1) = \mathbb{R}/2\pi$ is induced by every quantum chart, but it turns out to be gauge dependent.

We shall often use the polar splitting of the proper quantum bundle (see, for instance, Theorems 15.2.31 and 18.2.2, Proposition 16.4.5, Corollaries 17.6.18 and 20.1.19).

Let us recall the notion of "Lie affine space" (see Appendix: Note A.3.8). Moreover, we observe that a *principal bundle* can be regarded as a bundle $p : \boldsymbol{P} \to \boldsymbol{B}$, whose fibres are Lie affine spaces $(\boldsymbol{P}_b, \mathrm{G})$, for each $b \in \boldsymbol{B}$, associated with the same Lie group G (see Appendix: A.3.5).

Proposition 14.7.1 (1) *We define the* "norm quantum bundle" *to be the trivial bundle over* $\boldsymbol{E}$

$$\pi^{\|\|}_{/0} : \boldsymbol{Q}^{\|\|}_{/0} := \{\|\Psi\|_e \mid e \in \boldsymbol{E},\ \Psi_e \in \boldsymbol{Q}_{/0e}\} = \boldsymbol{E} \times \mathbb{L}^{-3/2} \to \boldsymbol{E}.$$

Clearly, we have the surjective fibred morphism over $\boldsymbol{E}$

$$\|\| : \boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} : \Psi_e \mapsto \|\Psi\|_e.$$

(2) *We define the* "phase quantum bundle" *to be the bundle over* $\boldsymbol{E}$

$$\pi^{(())}_{/0} : \boldsymbol{Q}^{(())}_{/0} := \{\Psi/\|\Psi\| \mid \Psi \in \boldsymbol{Q}_{/0}\} \to \boldsymbol{E}.$$

Indeed, the fibres of phase quantum bundle turn out to be Lie affine spaces S_1 *associated with the abelian Lie group* $U(1) = \mathbb{R}/2\pi$ *(see, Appendix: Note A.3.8). Thus, the phase quantum bundle turns out to be a principal bundle associated with this Lie group.*

Each quantum basis b *yields (locally) the Lie affine fibred isomorphisms over* $\boldsymbol{E}$

$$\boldsymbol{Q}^{(())}_{/0} \to \boldsymbol{E} \times \mathbb{R}/2\pi : e^{\mathrm{i}\varphi}\,\mathsf{b} \mapsto \varphi.$$

Clearly, we have the surjective fibred morphism over $\boldsymbol{E}$

$$(()) : \boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{(())}_{/0} : \Psi_e \mapsto ((\Psi))_e := \Psi_e/\|\Psi\|_e.$$

The "Lie affine structure" of the fibres of the phase quantum bundle yields in a natural way the, gauge independent, fibred "difference morphism" *over spacetime*

$$-: \boldsymbol{Q}^{\circledcirc}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{\circledcirc}_{/0} \to U(1) = \mathbb{R}/2\pi : (\Psi, \acute{\Psi}) \equiv \big(e^{i\varphi}\,\mathsf{b},\, e^{i\acute{\varphi}}\,\mathsf{b}\big) \mapsto (\!(\acute{\Psi})\!) - (\!(\Psi)\!) := \acute{\varphi} - \varphi. \quad \square$$

Proposition 14.7.2 *The proper quantum bundle naturally splits into a fibred product, called* polar splitting,

$$\boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{\circledcirc}_{/0} : \Psi_e \mapsto \big(\|\Psi\|_e, (\!(\Psi)\!)_e\big).$$

Thus, we have the gauge independent and observer independent surjective and injective real linear fibred morphisms over $\boldsymbol{E}$

$$p^{\|\|} : \boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} : |\psi|\,e^{i\varphi}\,\mathsf{b} \mapsto |\psi| \quad \text{and} \quad p^{\circledcirc} : \boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\circledcirc}_{/0} : |\psi|\,e^{i\varphi}\,\mathsf{b} \mapsto \varphi,$$
$$i^{\|\|} : \boldsymbol{Q}^{\|\|}_{/0} \to \boldsymbol{Q}_{/0} : |\psi| \mapsto |\psi|\,\mathsf{b} \quad \text{and} \quad i^{\circledcirc} : \boldsymbol{Q}^{\circledcirc}_{/0} \to \boldsymbol{Q}_{/0} : \varphi \mapsto e^{i\varphi}\,\mathsf{b}.$$

Accordingly, each $\Psi_e \in \boldsymbol{Q}_{/0}$, *is characterised by the*

- *intrinsic* "norm" $\|\Psi\|_e \in \mathbb{L}^{-3/2}$,
- *the intrinsic* "phase" $(\!(\Psi)\!)_e \in S_{1\,e}$. □

Remark 14.7.3 The phase quantum bundle $\pi^{\circledcirc}_{/0} : \boldsymbol{Q}^{\circledcirc}_{/0} \to \boldsymbol{E}$ turns out to be trivial if and only if the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ is trivial. □

Remark 14.7.4 It is worth mentioning the following facts and relevant differences:

- for each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the phase $\varphi(\Psi)$ depends on the chosen polar chart;
- if two proper quantum sections $\Psi, \acute{\Psi} \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, have the same phase in a polar quantum chart, then they have the same phase in any polar quantum chart; this is the reason why we say that the fibres of the phase quantum bundle are naturally isomorphic to S_1, not to $U(1)$; all manifolds isomorphic to S_1 are Lie affine spaces associated with the same Lie group $U(1)$;
- we have a *gauge dependent* local splitting of $\boldsymbol{Q}_{/0}$ into the norm component valued in $\mathbb{L}^{-3/2}$ and the phase component valued in the 1-dimensional group $U(1)$ induced by every complex quantum chart z;
- we have a *gauge independent* global splitting of $\boldsymbol{Q}_{/0}$ into the norm component valued in $\mathbb{L}^{-3/2}$ and the phase component valued in the principal bundle $\boldsymbol{Q}^{\circledcirc}_{/0}$, whose type fibre is naturally isomorphic to the 1-dimensional circle S_1 and whose structure group is $U(1)$. □

The hermitian quantum metric can be expressed in a very intuitive way in terms of the polar quantum splitting.

Proposition 14.7.5 *The modulus and the phase of the hermitian quantum metric factorise through the norm and the phase of the quantum sections as follows.*

For each $\Psi,\ \acute{\Psi} \in \sec(\boldsymbol{E},\ \boldsymbol{Q}_{/0})$, *we have the following gauge independent equality*

$$\mathfrak{h}(\Psi,\ \acute{\Psi}) = \|\Psi\|\,\|\acute{\Psi}\| + \mathfrak{i}\,\big((\!(\acute{\Psi})\!) - (\!(\Psi)\!)\big). \quad \square$$

14.8 Quantum Covariance Group

Now, we discuss the transition between fibred charts on the quantum bundle and the *quantum covariance group*.

Note 14.8.1 Given two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}}$, we can uniquely write

$$\acute{\mathfrak{b}} = e^{\mathfrak{i}\,\vartheta}\,\mathfrak{b}, \qquad \text{where } \vartheta \in \text{map}(\boldsymbol{E}, \mathbb{R}/2\pi).$$

The transition rule between two quantum charts (x^λ, z) and $(\acute{x}^\mu, \acute{z})$ is of the type

$$\acute{z} = z\,e^{\mathfrak{i}\,\vartheta}, \qquad \text{with } \vartheta \in \text{map}(\boldsymbol{E}, \mathbb{R}/2\pi),$$
$$\partial_0 \acute{x}^0 \in \mathbb{R}^+, \quad \partial_j \acute{x}^0 = 0, \quad \det(\partial_j \acute{x}^i) > 0. \quad \square$$

Proposition 14.8.2 *The* covariance (local) group *of the quantum bundle is constituted by the (local) hermitian complex linear fibred diffeomorphisms*

$$f_{\boldsymbol{Q}} : \boldsymbol{Q} \to \boldsymbol{Q}$$

over galilean automorphisms (see Proposition 2.1.5)

$$f_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E},$$

according to the commutative diagram

$$\begin{array}{ccc}
\boldsymbol{Q} & \xrightarrow{\ f_{\boldsymbol{Q}}\ } & \boldsymbol{Q} \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\
\boldsymbol{E} & \xrightarrow{\ f_{\boldsymbol{E}}\ } & \boldsymbol{E} \\
{\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\
\boldsymbol{T} & \xrightarrow{\ f_{\boldsymbol{T}}\ } & \boldsymbol{T}
\end{array} \quad .$$

Their coordinate expressions are of the type

$$\begin{aligned} f^z &:= z \circ f_{\boldsymbol{Q}} = e^{\mathrm{i}\,\vartheta}, && \text{with } \vartheta \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}/2\pi), \\ f^i &:= x^i \circ f_{\boldsymbol{E}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}), && \text{with } \det(\partial_j f^i) > 0, \\ f^0 &:= x^0 \circ f_{\boldsymbol{T}} = f^0_0\, x^0 + \check{f}^0, && \text{with } f^0_0 \in \mathbb{R}^+, \quad \check{f}^0 \in \mathbb{R}. \quad \square \end{aligned}$$

14.9 Quantum Sections

We follow the standard formulation of Quantum Mechanics in terms of wave functions and the associated probabilistic interpretation. However, we have to take into account that in our context the wave functions depend on the choice of a quantum basis, as in Geometric Quantisation (see, for instance [1, 149, 153, 425]).

According to the polar splitting of the proper quantum bundle (see Proposition 14.7.2), we can say that each proper quantum section has two "*real degrees of freedom*":

- the 1st one, valued in $\mathbb{L}^{-3/2}$, dealing with the *probability of detecting* the particle,
- the 2nd one, valued in a principal bundle with type fibre S_1, which might be interpreted and an "*internal angle*".

Thus, we have two equivalent formulations of quantum sections in real and complex terms. The 1st one might help some intuitive understanding, while the more usual 2nd one is more suitable for dealing with the linearity features of Quantum Mechanics.

Assumption Q.1 *The* quantum states *are represented by the* quantum sections

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}).$$

Hence, chosen a (local) scaled quantum basis $\mathfrak{b}$*, a quantum section* Ψ *is represented by the usual scaled* wave function

$$\psi := z \circ \Psi \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C}). \quad \square$$

Remark 14.9.1 If the quantum bundle is not trivial, then a globally defined proper quantum section does not exist.

Actually, due to the fact that the fibres of quantum bundle are 1-dimensional complex vector spaces, the quantum bundle is trivial if and only if there exist global proper quantum sections. $\square$

14.10 Quantum Liouville Vector Field

We shall be often involved with the following distinguished vertical vector field of the quantum bundle.

Definition 14.10.1 We define the *quantum Liouville vector field* to be the vertical valued quantum vector field

$$\mathbb{I} : \boldsymbol{Q} \to V_{\boldsymbol{E}} \boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : q \mapsto (q, q),$$

whose coordinate expression is

$$\mathbb{I} = w^{\mathrm{a}} \, \partial w_{\mathrm{a}} = z \, \partial_z . \quad \square$$

The vertical valued vector field $\mathbb{I}$ is the infinitesimal generator of the homotheties of the vector bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ (see [146]).

14.11 Upper Quantum Bundle

In view of the forthcoming postulate concerning the upper quantum connection, we need to extend the base space $\boldsymbol{E}$ of the quantum bundle to the phase space $J_1 \boldsymbol{E}$ (see Definition 15.1.5).

In the present context, we shall interpret the base points of this enlarged quantum bundle as pointwise observers. So, we can say that we enlarge the base space of the quantum bundle by considering, for each base event all, possible pointwise observers at that event.

The fact that the quantum bundle has been originally defined on spacetime and that later the extension to the base space is obtained "by pullback" is a strategic choice of our approach.

Definition 14.11.1 We define, by pullback, the *upper quantum bundle* to be the 1-dimensional complex vector bundle based on phase space

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1 \boldsymbol{E} . \quad \square$$

Remark 14.11.2 In virtue of the above construction by pullback, the fibres of the bundles

$$\pi : \boldsymbol{Q} \to \boldsymbol{E} \quad \text{and} \quad \pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1 \boldsymbol{E}$$

coincide. Hence, we have

$$V_{J_1 \boldsymbol{E}} \boldsymbol{Q}^{\uparrow} = J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} V_{\boldsymbol{E}} \boldsymbol{Q} = J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}) . \quad \square$$

Note 14.11.3 By pullback, we obtain the *real upper quantum metric*, the *imaginary unit*, the *hermitian upper quantum metric*, the η*-hermitian upper quantum metric* and the *upper quantum Liouville vector field*

$$\mathfrak{g}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\boldsymbol{Q}^{\uparrow *} \otimes \boldsymbol{Q}^{\uparrow *}), \quad \mathfrak{i} : \boldsymbol{Q}^{\uparrow} \to \boldsymbol{Q}^{\uparrow},$$
$$\mathfrak{h}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^{\uparrow} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}, \quad \mathfrak{h}^{\uparrow}{}_{\eta} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^{\uparrow} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C},$$
$$\mathbb{I}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to V_{J_1\boldsymbol{E}} \boldsymbol{Q}^{\uparrow}.$$

For the sake of simplicity, the upper quantum objects $\mathfrak{g}_{\boldsymbol{Q}}$ and $\mathfrak{i}$ will be denoted by the same symbols of the corresponding quantum objects. □

Chapter 15
Galilean Upper Quantum Connection

Our covariant quantum framework is based on the two minimal assumptions: the *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ over spacetime and the *galilean upper quantum connection* $Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1 \boldsymbol{E} \otimes T\, \boldsymbol{Q}^{\uparrow}$.

Thus, this chapter is devoted to a broad analysis of the galilean upper quantum connection.

We start by discussing the *quantum connections*, the *systems of quantum connections*, the *systems of observed quantum connections* and the *upper quantum connections*, including the *real linear*, *complex linear* and *hermitian cases* (Sect. 15.1). In particular, we study the *reducible upper quantum connections* and the *universal upper quantum connection* associated with a system of quantum connections.

We discuss the local and global existence of the *galilean upper quantum connections*, which are defined to be hermitian, reducible upper quantum connections, which fulfill the additional condition $R[Ч^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}$ (Sect. 15.2).

Then, we choose, by *postulate*, such a galilean upper connection $Ч^{\uparrow}$, as source of all further quantum developments (see Postulate Q.2).

Further, we analyse the *transition rule for the observed potential* and two distinguished invariants $\nu[\mathfrak{b}]$ and $\alpha[\mathfrak{b}]$ (Sect. 15.2.5).

Eventually, for every proper quantum section Ψ, we exhibit the distinguished *horizontal potential* $A[\Psi]$ and the distinguished "*rest observer*" $o[\Psi]$ (Sect. 15.2.6).

Indeed, the above objects will play a relevant role in the context of the Schrödinger equation and of the hydrodynamical picture of Quantum Mechanics (see, for instance, Corollary 17.6.18 and Theorems 18.2.1 and 18.2.2).

The upper quantum bundle $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ can be naturally regarded also as a fibred manifold $\boldsymbol{Q}^{\uparrow} \to \boldsymbol{T}$ over time. Indeed, the composition of the upper quantum connection $Ч^{\uparrow}$ with the dynamical phase connection γ yields a connection of the fibred manifold $\boldsymbol{Q}^{\uparrow} \to \boldsymbol{T}$ (Sect. 15.3). It is remarkable that the above connection involves the classical lagrangian $\mathcal{L}$. Later, we shall use this *upper quantum connection over time* to introduce the "amplitudes" of the Feynman path integral (see Theorem 23.2.2).

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The concepts of system of observed quantum connections and reducible upper quantum connection are based on a general theory of systems of connections. Here, we refer to the paper [225], where the reader can find an extended general exposition with further details. For further literature on this subject see also [35–37, 141, 143, 285].

15.1 Quantum and Upper Quantum Connections

We define the *quantum connections*, the *systems of quantum connections*, the *observed quantum connections* and the *upper quantum connections*

$$\text{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}, \quad \xi : \boldsymbol{Q}^{\uparrow} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}, \quad \text{Ч}[o] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q},$$
$$\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^{\uparrow}.$$

Moreover, we characterise the *real linear*, the *complex linear* and the *hermitian* cases.

Further, we discuss the *reducible upper quantum connections* and the *universal connection* of a system of quantum connections. Eventually, we study the *curvature* of quantum connections and upper quantum connections.

15.1.1 Quantum Connections

We define the *quantum connections* $\text{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}$, the *upper systems of quantum connections* $\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}$, and the *observed quantum connections* associated with an upper system of quantum connections $\text{Ч}[o] := o^*\xi : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\,\boldsymbol{Q}$ (see, also [225, 304]).

Definition 15.1.1 We define a *quantum connection* to be a connection of the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, i.e. a section (see, Appendix: Definition F.1.1)

$$\text{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q},$$

which is projectable on $\boldsymbol{1}_{\boldsymbol{E}} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{E}$, according to the following commutative diagram

$$\begin{CD}
\boldsymbol{Q} @>{\text{Ч}}>> T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q} \\
@V{\pi}VV @VV{\mathrm{id} \otimes T\pi}V \\
\boldsymbol{E} @>{\boldsymbol{1}_{\boldsymbol{E}}}>> T^*\boldsymbol{E} \otimes T\,\boldsymbol{E}
\end{CD} \quad .$$

Thus, the coordinate expression, in real coordinates, of a quantum connection is of the type

$$\text{Ч} = d^\lambda \otimes (\partial_\lambda + \text{Ч}_\lambda{}^{\mathrm{a}}\, \partial w_{\mathrm{a}}), \quad \text{with } \text{Ч}_\lambda{}^{\mathrm{a}} \in \mathrm{map}(\boldsymbol{Q}, \mathbb{R}). \quad \square$$

Definition 15.1.2 We define an *upper system of quantum connections* to be a family of connections of the quantum bundle parametrised by the classical phase space $J_1\boldsymbol{E}$, i.e. a fibred morphism over $\boldsymbol{Q}$ (see, also [225])

$$\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q},$$

which projects on $\mathbf{1}_{\boldsymbol{E}} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E}$, according to the commutative diagram

$$\begin{array}{ccc}
J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} & \xrightarrow{\ \xi\ } & T^*\boldsymbol{E} \otimes T\boldsymbol{Q} \\
{\scriptstyle \mathrm{pro}}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes T\pi} \\
\boldsymbol{E} & \xrightarrow{\ \mathbf{1}_{\boldsymbol{E}}\ } & T^*\boldsymbol{E} \otimes T\boldsymbol{E}
\end{array} .$$

Thus, the coordinate expression, in real coordinates, of an upper system of quantum connections is of the type

$$\xi = d^\lambda \otimes (\partial_\lambda + \xi_\lambda{}^{\mathrm{a}}\, \partial w_{\mathrm{a}}), \quad \text{with } \xi_\lambda{}^{\mathrm{a}} \in \mathrm{map}(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}, \mathbb{R}). \quad \square$$

Lemma 15.1.3 *Let us consider an upper system of quantum connections ξ. Then, for each observer $o \in \mathrm{sec}(\boldsymbol{E}, J_1\boldsymbol{E})$, we obtain, by pullback, the* "observed quantum connection"

$$\text{Ч}[o] := o^*\xi : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q},$$

according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\ \text{Ч}[o]\ } & T\boldsymbol{Q} \\
{\scriptstyle (\pi \circ \mathrm{pro}_1, \mathrm{id})}\downarrow & & \\
\boldsymbol{E} \times (\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) & & \uparrow{\scriptstyle \mathrm{id}} \\
{\scriptstyle (o, \mathrm{id})}\downarrow & & \\
J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) & \xrightarrow{\ \xi\ } & T\boldsymbol{Q}
\end{array} .$$

Thus, we have the coordinate expression, in real coordinates,

$$\text{Ч}[o] = d^\lambda \otimes \partial_\lambda + (\xi_\lambda{}^{\mathrm{a}} \circ o)\, d^\lambda \otimes \partial w_{\mathrm{a}}. \quad \square$$

Definition 15.1.4 A family of quantum connections, parametrised by classical observers, as in the above Lemma 15.1.3,

$$\{\text{Ч}[o] \ \mid \ o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})\}$$

is called a *system of observed quantum connections*. $\square$

15.1.2 Upper Quantum Connections

We define the *upper quantum connections* $\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^\uparrow$.

Then, we define the *reducible* upper quantum connections $\text{Ч}^\uparrow$, which factorise through an upper system of quantum connections $\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}$, and the *universal connection* $\text{Ч}^\uparrow$ of an upper system of quantum connections, which is characterised by the property that all connections of the system can be obtained by pullback from the universal connection.

Moreover, we exhibit a natural bijection $\text{Ч}^\uparrow \leftrightarrow \xi$ between reducible upper quantum connections and upper systems of quantum connections.

The original notion of “universal principal connection” was introduced by P. L. García on principal bundles, by exploiting the properties of the associated Lie algebra (see [141]). Then, this notion has been generalised, at a more basic level, to general systems of connections on fibred manifolds (see [35, 285]).

Definition 15.1.5 We define an *upper quantum connection* to be a connection of the upper quantum bundle $\pi^\uparrow : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$, i.e. a section (see, Appendix: Definition F.1.1)

$$\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^\uparrow,$$

which is projectable on $\mathbf{1}_{J_1\boldsymbol{E}} : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E} \otimes T J_1\boldsymbol{E}$, according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{Q}^\uparrow & \xrightarrow{\ \text{Ч}^\uparrow\ } & T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^\uparrow \\
{\scriptstyle \pi}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id} \otimes T\pi^\uparrow} \\
J_1\boldsymbol{E} & \xrightarrow{\ \mathbf{1}_{J_1\boldsymbol{E}}\ } & T^* J_1\boldsymbol{E} \otimes T J_1\boldsymbol{E}
\end{array} .$$

Thus, the coordinate expression, in real coordinates, of an upper quantum connection is of the type

$$\text{Ч}^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + \text{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}}\, \partial w_{\mathrm{a}}) + d^{i}_{0} \otimes (\partial^{0}_{i} + \text{Ч}^{\uparrow}{}^{0\,\mathrm{a}}_{i}\, \partial w_{\mathrm{a}}),$$

with

$$\text{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}},\ \text{Ч}^{\uparrow}{}^{0\,\mathrm{a}}_{i} \in \mathrm{map}(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q},\ \mathbb{R}). \quad \square$$

Definition 15.1.6 An upper quantum connection $\text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} \to T\boldsymbol{Q}^{\uparrow}$ is said to be *reducible* if it factorises through a system of quantum connections ξ, according to the following commutative diagram (see, also [225])

$$\begin{array}{ccc} \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} & \xrightarrow{\ \text{Ч}^{\uparrow}\ } & T\boldsymbol{Q}^{\uparrow} \\ \downarrow & & \downarrow \\ J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) & \xrightarrow{\ \xi\ } & T\boldsymbol{Q} \end{array} .$$

Indeed, an upper quantum connection $\text{Ч}^{\uparrow}$ is reducible if and only if, in coordinates,

$$\text{Ч}^{\uparrow}{}^{0\,\mathrm{a}}_{i} = 0.$$

Thus, an upper quantum connection $\text{Ч}^{\uparrow}$ is reducible to the system ξ if and only if its coordinate expression is of the type

$$\text{Ч}^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i} + \text{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}}\, d^{\lambda} \otimes \partial w_{\mathrm{a}}, \quad \text{where } \text{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}} = \xi_{\lambda}{}^{\mathrm{a}}. \quad \square$$

Remark 15.1.7 For a general connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ of a generic fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ a possible condition by which a component of the connection vanishes depends on the chosen chart.

However, the condition $\text{Ч}^{\uparrow}{}^{0\,\mathrm{a}}_{i} = 0$ found in the above Definition 15.1.6 is coordinate equivariant because the upper quantum bundle is obtained by pullback. $\square$

Proposition 15.1.8 *We have a natural bijection between upper systems of quantum connections and reducible upper quantum connections in the following way (see, also [225]).*

(1) *If* $\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) \to T\boldsymbol{Q}$ *is a system of quantum connections, then the map*

$$\begin{aligned} \text{Ч}^{\uparrow} := \xi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} &\to T\boldsymbol{Q}^{\uparrow} \\ = TJ_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{Q} : \big((e_1, q),\ X^{\uparrow}{}_{e_1}\big) &\mapsto \Big(X^{\uparrow}{}_{e_1},\ \xi\big(q,\ Tt^{1}_{0}(X^{\uparrow}{}_{e_1})\big)\Big), \end{aligned}$$

with coordinate expression

$$Ч^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + Ч^{\uparrow}{}_{\lambda}{}^{a}\, d^{\lambda} \otimes \partial w_{a}, \quad \textit{where}\ Ч^{\uparrow}{}_{\lambda}{}^{a} = \xi_{\lambda}{}^{a} \in \mathrm{map}(\boldsymbol{Q}^{\uparrow}, \mathbb{R}),$$

turns out to be a reducible upper quantum connection.

(2) *If* $Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} T J_1\boldsymbol{E} \to T\boldsymbol{Q}^{\uparrow}$ *is a reducible upper quantum connection, then the factor map (see the above Definition 15.1.6)*

$$\xi : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q},$$

with coordinate expression

$$\xi = d^{\lambda} \otimes \partial_{\lambda} + \xi_{\lambda}{}^{a}\, d^{\lambda} \otimes \partial w_{a}, \quad \textit{where}\ \xi_{\lambda}{}^{a} = Ч^{\uparrow}{}_{\lambda}{}^{a} \in \mathrm{map}(\boldsymbol{Q}^{\uparrow}, \mathbb{R}),$$

turns out to be an upper system of quantum connections.

(3) *The above coordinate expressions exhibit a natural bijection*

$$\xi \to Ч^{\uparrow} := \xi^{\uparrow}$$

between upper systems of quantum connections and reducible upper quantum connections. □

Definition 15.1.9 With reference to the above Proposition 15.1.8, we say that the reducible upper quantum connection (see, also [225])

$$Ч^{\uparrow} := \xi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \underset{J_1\boldsymbol{E}}{\times} T J_1\boldsymbol{E} \to T\boldsymbol{Q}^{\uparrow}$$

is the *universal upper quantum connection* associated with the upper system of quantum connections

$$\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$$

and that the quantum connection

$$Ч[o] := o^*\xi := o^*Ч^{\uparrow} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}$$

is the *observed quantum connection* associated with the reducible upper quantum connection $Ч^{\uparrow}$ and the observer o. □

15.1.3 Hermitian Quantum Connections

Next, we discuss the *real linear*, *complex linear* and *hermitian* quantum connections $\mathrm{Ч}$ and upper systems of quantum connections $\mathrm{Ч}[o]$.

Definition 15.1.10 A quantum connection $\mathrm{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\,\boldsymbol{Q}$ is said to be:

- *real linear* if it is a real linear fibred morphism over $\boldsymbol{1}_{\boldsymbol{E}}$, i.e. if its coordinate expression is of the type (see, Appendix: Definition F.2.1)

$$\mathrm{Ч} = d^\lambda \otimes (\partial_\lambda + \mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial w_{\mathrm{a}}), \qquad \text{with } \mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}),$$

- *complex linear* if it is real linear and

$$\nabla \mathfrak{i} = 0,$$

i.e. if its coordinate expression is of the type

$$\mathrm{Ч} = d^\lambda \otimes (\partial_\lambda + \mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial w_{\mathrm{a}}),$$

with $\mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, and

$$\mathrm{Ч}_\lambda{}^1{}_1 = \mathrm{Ч}_\lambda{}^2{}_2, \qquad \mathrm{Ч}_\lambda{}^2{}_1 = -\mathrm{Ч}_\lambda{}^1{}_2,$$

i.e. if its coordinate expression is of the type (see Definition 14.10.1)

$$\begin{aligned}\mathrm{Ч} &= d^\lambda \otimes \big(\partial_\lambda + \mathrm{Ч}_\lambda{}^1{}_1\,(w^1\,\partial w_1 + w^2\,\partial w_2) + \mathrm{Ч}_\lambda{}^2{}_1\,(w^1\,\partial w_2 - w^2\,\partial w_1)\big)\\ &= d^\lambda \otimes \big(\partial_\lambda + (\mathrm{Ч}_\lambda{}^1{}_1 + \mathfrak{i}\,\mathrm{Ч}_\lambda{}^2{}_1)\,\mathbb{I}\big),\end{aligned}$$

with $\mathrm{Ч}_\lambda{}^1{}_1,\ \mathrm{Ч}_\lambda{}^2{}_1 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$,

- *hermitian* if it is real linear and

$$\nabla \mathfrak{h} = 0,$$

i.e. if its coordinate expression is of the type

$$\mathrm{Ч} = d^\lambda \otimes (\partial_\lambda + \mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial w_{\mathrm{a}}),$$

with $\mathrm{Ч}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$, and

$$\mathrm{Ч}_\lambda{}^1{}_1 = \mathrm{Ч}_\lambda{}^2{}_2 = 0, \qquad \mathrm{Ч}_\lambda{}^2{}_1 = -\mathrm{Ч}_\lambda{}^1{}_2,$$

i.e. if its coordinate expression is of the type (see Definition 14.10.1)

$$\begin{aligned} \text{Ч} &= d^\lambda \otimes \big(\partial_\lambda + \text{Ч}_\lambda{}^2{}_1\,(w^1\,\partial w_2 - w^2\,\partial w_1)\big) \\ &= d^\lambda \otimes \big(\partial_\lambda + \mathrm{i}\,\text{Ч}_\lambda{}^2{}_1\,\mathbb{I}\big), \end{aligned}$$

with $\text{Ч}_\lambda{}^2{}_1 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

A hermitian quantum connection Ч turns out to be complex linear. □

Definition 15.1.11 An upper system of quantum connections ξ is said to be:

- *real linear* if all quantum connections of the system are real linear, i.e., if its coordinate expression is of the type

$$\xi = d^\lambda \otimes (\partial_\lambda + \xi_\lambda{}^a{}_b\,w^b\,\partial w_a), \quad \text{with}\, \xi_\lambda{}^a{}_b \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{R}),$$

- *complex linear* if all quantum connections of the system are complex linear, i.e., if its coordinate expression is of the type

$$\xi = d^\lambda \otimes (\partial_\lambda + \xi_\lambda{}^a{}_b\,w^b\,\partial w_a), \quad \text{with}\, \xi_\lambda{}^a{}_b \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{R}),$$

  and

$$\xi_\lambda{}^1{}_1 = \xi_\lambda{}^2{}_2, \quad \xi_\lambda{}^2{}_1 = -\xi_\lambda{}^1{}_2,$$

  i.e. if its coordinate expression is of the type (see Definition 14.10.1)

$$\begin{aligned} \xi &= d^\lambda \otimes \big(\partial_\lambda + \xi_\lambda{}^1{}_1\,(w^1\,\partial w_1 + w^2\,\partial w_2) + \xi_\lambda{}^2{}_1\,(w^1\,\partial w_2 - w^2\,\partial w_1)\big) \\ &= d^\lambda \otimes \big(\partial_\lambda + (\xi_\lambda{}^1{}_1 + \mathrm{i}\,\xi_\lambda{}^2{}_1)\,\mathbb{I}\big), \quad \text{with}\, \xi_\lambda{}^1{}_1,\ \xi_\lambda{}^2{}_1 \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}) \end{aligned}$$

- *hermitian* if all quantum connections of the system are hermitian, i.e., if its coordinate expression is of the type

$$\xi = d^\lambda \otimes (\partial_\lambda + \xi_\lambda{}^a{}_b\,w^b\,\partial w_a), \quad \text{with}\, \xi_\lambda{}^a{}_b \in \mathrm{map}(J_1\boldsymbol{E},\ \mathbb{R}),$$

  and

$$\xi_\lambda{}^1{}_1 = \xi_\lambda{}^2{}_2 = 0, \quad \xi_\lambda{}^2{}_1 = -\xi_\lambda{}^1{}_2,$$

  i.e. if its coordinate expression is of the type (see Definition 14.10.1)

$$\begin{aligned} \xi &= d^\lambda \otimes \big(\partial_\lambda + \xi_\lambda{}^2{}_1\,(w^1\,\partial w_2 - w^2\,\partial w_1)\big) \\ &= d^\lambda \otimes \big(\partial_\lambda + \mathrm{i}\,\xi_\lambda{}^2{}_1\,\mathbb{I}\big), \quad \text{with}\, \xi_\lambda{}^2{}_1 \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}). \end{aligned}$$

A hermitian upper system of quantum connections ξ turns out to be complex linear.

Clearly, if an upper system of quantum connections ξ is real linear (hermitian), then all associated observed quantum connections $\text{Ч}[o]$ turn out to be real linear (hermitian). □

15.1.4 Hermitian Upper Quantum Connections

We characterise the *real linear*, *complex linear* and *hermitian* upper quantum connections $\mathrm{Ч}^{\uparrow}$.

Definition 15.1.12 An upper quantum connection $\mathrm{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1 \boldsymbol{E} \otimes T \boldsymbol{Q}^{\uparrow}$ is said to be:

(1) *real linear* if it is a real linear fibred morphism over $\mathbf{1}_{J_1 \boldsymbol{E}}$, i.e. if its coordinate expression is of the type (see, Appendix: Definition F.2.1)

$$\mathrm{Ч}^{\uparrow} = d^{\lambda} \otimes (\partial_{\lambda} + \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial w_{\mathrm{a}}) + d^{i}_{0} \otimes (\partial^{0}_{i} + \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, \partial w_{\mathrm{a}}),$$

with

$$\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{\mathrm{a}}{}_{\mathrm{b}},\ \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{\mathrm{a}}{}_{\mathrm{b}} \in \mathrm{map}(J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times}, \mathrm{I\!R}),$$

(2) *complex linear* if it is real linear and

$$\nabla^{\uparrow} \mathrm{i}^{\uparrow} = 0,$$

i.e. if its is real linear and

$$\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{1} = \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{2}, \quad \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} = -\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{2}, \quad \mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{1} = \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{2}, \quad \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{1} = -\mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{2},$$

i.e. if its coordinate expression is of the type (see Note 14.11.3)

$$\begin{aligned}
\mathrm{Ч}^{\uparrow} &= d^{\lambda} \otimes \big(\partial_{\lambda} + \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{1}\, (w^{1}\, \partial w_{1} + w^{2}\, \partial w_{2}) + \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1}\, (w^{1}\, \partial w_{2} - w^{2}\, \partial w_{1})\big) \\
&\quad + d^{i}_{0} \otimes (\partial^{0}_{i} + \big(\mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{1}\, (w^{1}\, \partial w_{1} + w^{2}\, \partial w_{2}) + \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{1}\, (w^{1}\, \partial w_{2} - w^{2}\, \partial w_{1}) \\
&= d^{\lambda} \otimes \big(\partial_{\lambda} + (\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{1} + \mathrm{i}\, \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1})\, \mathbb{I}^{\uparrow}\big) + d^{i}_{0} \otimes \big(\partial^{0}_{i} + \big(\mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{1} + \mathrm{i}\, \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{1}\big)\, \mathbb{I}^{\uparrow}\big),
\end{aligned}$$

with

$$\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{1},\ \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1},\ \mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{1},\ \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{1} \in \mathrm{map}(J_1 \boldsymbol{E}, \mathrm{I\!R}),$$

(3) *hermitian* if it is real linear and

$$\nabla \mathrm{h}^{\uparrow} = 0,$$

i.e. if its is real linear and

$$\begin{aligned}
\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{1} &= \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{2} = 0, \quad \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} = -\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{1}{}_{2}, \\
\mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{1} &= \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{2} = 0, \quad \mathrm{Ч}^{\uparrow}{}^{02}_{i}{}_{1} = -\mathrm{Ч}^{\uparrow}{}^{01}_{i}{}_{2},
\end{aligned}$$

i.e. if its coordinate expression is of the type (see Note 14.11.3)

$$\begin{aligned} \mathrm{Ч}^{\uparrow} &= d^{\lambda} \otimes \big(\partial_{\lambda} + \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} \, (w^{1} \, \partial w_{2} - w^{2} \, \partial w_{1})\big) \\ &\quad + d^{i}_{0} \otimes (\partial^{0}_{i} + \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{2}{}_{1} \, (w^{1} \, \partial w_{2} - w^{2} \, \partial w_{1}) \\ &= d^{\lambda} \otimes \big(\partial_{\lambda} + \mathrm{i} \, \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} \, \mathbb{I}^{\uparrow}\big) + d^{i}_{0} \otimes (\partial^{0}_{i} + \mathrm{i} \, \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{2}{}_{1} \, \mathbb{I}^{\uparrow}), \end{aligned}$$

with

$$\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1}, \ \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{2}{}_{1} \in \mathrm{map}(J_{1}\boldsymbol{E}, \mathbb{R}).$$

A hermitian upper quantum connection $\mathrm{Ч}^{\uparrow}$ turns out to be complex linear, i.e.

$$\nabla^{\uparrow} \mathrm{i}^{\uparrow} = 0.$$

Thus, the coordinate expression of a *reducible hermitian* upper quantum connection $\mathrm{Ч}^{\uparrow}$ is of the type

$$\begin{aligned} \mathrm{Ч}^{\uparrow} &= d^{\lambda} \otimes \big(\partial_{\lambda} + \mathrm{i} \, \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} \, (w^{1} \, \partial w_{2} - w^{2} \, \partial w_{1})\big) + d^{i}_{0} \otimes \partial^{0}_{i} \\ &= d^{\lambda} \otimes \big(\partial_{\lambda} + \mathrm{i} \, \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} \, \mathbb{I}^{\uparrow}\big) + d^{i}_{0} \otimes \partial^{0}_{i}, \end{aligned}$$

with

$$\mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1} \in \mathrm{map}(J_{1}\boldsymbol{E}, \mathbb{R}). \quad \square$$

Proposition 15.1.13 *Let us consider a reducible upper quantum connection* $\mathrm{Ч}^{\uparrow}$ *and the associated system* $\{\mathrm{Ч}[o]\}$ *of observed quantum connections. Then, we have the following equivalences:*

$\mathrm{Ч}^{\uparrow}$ *is real linear iff all* $\mathrm{Ч}[o]$ *are real linear,*
$\mathrm{Ч}^{\uparrow}$ *is complex linear iff all* $\mathrm{Ч}[o]$ *are complex linear,*
$\mathrm{Ч}^{\uparrow}$ *is hermitian iff all* $\mathrm{Ч}[o]$ *are hermitian.* □

15.1.5 Splitting of Quantum and Upper Quantum Connection

A quantum basis $\mathfrak{b}$ yields (locally) a splitting of the quantum bundle $\boldsymbol{Q}$, hence a hermitian flat quantum connection $\chi[\mathfrak{b}]$ of the quantum bundle.

Then, this connection $\chi[\mathfrak{b}]$ can be used to split every quantum connection $\mathrm{Ч}$ into the quantum connection $\chi[\mathfrak{b}]$ and a gauge dependent tensor $\check{\mathrm{Ч}}[\mathfrak{b}]$. In particular, it is remarkable that a hermitian quantum connection $\mathrm{Ч}$ splits as $\mathrm{Ч} = \chi[\mathfrak{b}] + \mathrm{i}\, A[\mathfrak{b}]\, \mathbb{I}$, where $A[\mathfrak{b}] \in \sec(\boldsymbol{E}, T^{*}\boldsymbol{E})$.

Analogous gauge splittings hold for upper systems ξ of quantum connections and for observed quantum systems of quantum connections $\{\mathrm{Ч}[o]\}$.

In a similar way, a quantum basis b yields (locally) a splitting of the upper quantum bundle $\boldsymbol{Q}^{\uparrow}$, hence a reducible hermitian flat upper quantum connection $\chi^{\uparrow}[\mathsf{b}]$ of the upper quantum bundle.

Then, this connection $\chi^{\uparrow}[\mathsf{b}]$ can be used to split every upper quantum connection $\mathrm{Ч}^{\uparrow}$ into the upper quantum connection $\chi^{\uparrow}[\mathsf{b}]$ and a tensor. In particular, it is remarkable that a reducible hermitian upper quantum connection $\mathrm{Ч}^{\uparrow}$ splits as $\mathrm{Ч}^{\uparrow} = \chi^{\uparrow}[\mathsf{b}] + \mathrm{i}\, \mathrm{A}^{\uparrow}[\mathsf{b}]\, \mathbb{I}^{\uparrow}$, where $\mathrm{A}^{\uparrow}[\mathsf{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$.

Lemma 15.1.14 *A quantum basis* $\mathsf{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$ *yields (locally) the bundle splittings*

$$\boldsymbol{Q} \to \boldsymbol{E} \times (\mathbb{L}^{-3/2} \otimes \mathbb{C}) \quad \textit{and} \quad \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E} \times (\mathbb{L}^{-3/2} \otimes \mathbb{C}),$$

hence it yields locally a hermitian flat quantum connection and a reducible hermitian upper quantum connection

$$\chi[\mathsf{b}] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q} \quad \textit{and} \quad \chi^{\uparrow}[\mathsf{b}] : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow},$$

with coordinate expression in any adapted chart

$$\chi[\mathsf{b}] = d^{\lambda} \otimes \partial_{\lambda} \quad \textit{and} \quad \chi^{\uparrow}[\mathsf{b}] = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i}. \quad \square$$

Proposition 15.1.15 *Let us consider a quantum connection* $\mathrm{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ *and a quantum basis* $\mathsf{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$. *Then, the following facts hold.*

(1) *The quantum connection* $\mathrm{Ч}$ *splits as (see Lemma 15.1.14)*

$$\mathrm{Ч} = \chi[\mathsf{b}] + \check{\mathrm{Ч}}[\mathsf{b}], \quad \textit{where } \check{\mathrm{Ч}}[\mathsf{b}] \in \sec(\boldsymbol{Q}, T^*\boldsymbol{E} \otimes \boldsymbol{Q}).$$

In any adapted chart, we have the coordinate expression (see Definition 15.1.1)

$$\check{\mathrm{Ч}}[\mathsf{b}] = \mathrm{Ч}_{\lambda}{}^{\mathsf{a}}\, d^{\lambda} \otimes \mathsf{b}_{\mathsf{a}} \quad \textit{with } \mathrm{Ч}_{\lambda}{}^{\mathsf{a}} \in \mathrm{map}(\boldsymbol{Q}, \mathbb{R}).$$

(2) *If the quantum connection* $\mathrm{Ч}$ *is real linear, then* $\check{\mathrm{Ч}}[\mathsf{b}]$ *can be regarded as a section*

$$\check{\mathrm{Ч}}[\mathsf{b}] \in \sec(\boldsymbol{E},\ T^*\boldsymbol{E} \otimes \boldsymbol{Q} \otimes \boldsymbol{Q}^*),$$

with coordinate expression (see Definition 15.1.10)

$$\check{\mathrm{Ч}}[\mathsf{b}] = \mathrm{Ч}_{\lambda}{}^{\mathsf{a}}{}_{\mathsf{b}}\, w^{\mathsf{b}}\, d^{\lambda} \otimes \mathsf{b}_{\mathsf{a}}, \quad \textit{with } \mathrm{Ч}_{\lambda}{}^{\mathsf{a}}{}_{\mathsf{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

(3) *If the quantum connection* $\mathrm{Ч}$ *is complex linear, then* $\check{\mathrm{Ч}}[\mathsf{b}]$ *can be regarded as a section*

$$\check{\mathrm{Ч}}[\mathsf{b}] \in \sec(\boldsymbol{E},\ T^*\boldsymbol{E} \otimes \boldsymbol{Q} \otimes \boldsymbol{Q}^*),$$

with coordinate expression

$$\begin{aligned}\check{\text{Ч}}[\mathrm{b}] &= d^\lambda \otimes \big(\text{Ч}_\lambda{}^1{}_1\,(w^1\,\mathrm{b}_1 + w^2\,\mathrm{b}_2) + \text{Ч}_\lambda{}^2{}_1\,(w^1\,\mathrm{b}_2 - w^2\,\mathrm{b}_1)\big)\\ &= d^\lambda \otimes \big(\text{Ч}_\lambda{}^1{}_1 + \mathrm{i}\,\text{Ч}_\lambda{}^2{}_1)\,\mathbb{I}\big).\end{aligned}$$

(4) *If the quantum connection* Ч *is hermitian, then* $\check{\text{Ч}}[\mathrm{b}]$ *can be regarded as a section*

$$\check{\text{Ч}}[\mathrm{b}] \equiv \mathrm{i}\,A[\mathrm{b}] \otimes \mathbb{I} \in \sec(\boldsymbol{E},\ T^*\boldsymbol{E} \otimes \boldsymbol{Q} \otimes \boldsymbol{Q}^*),$$

where

$$A[\mathrm{b}] = \text{Ч}_\lambda{}^2{}_1\, d^\lambda \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}). \quad \square$$

Proposition 15.1.16 *Let us consider an upper quantum connection*

$$\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^\uparrow$$

and a quantum basis $\mathrm{b} \in \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$. *Then, the following facts hold.*

(1) *The upper quantum connection* $\text{Ч}^\uparrow$ *splits as (see Lemma 15.1.14)*

$$\text{Ч}^\uparrow = \chi^\uparrow[\mathrm{b}] + \check{\text{Ч}}^\uparrow[\mathrm{b}], \quad \textit{where}\ \check{\text{Ч}}[\mathrm{b}] \in \sec(\boldsymbol{Q}^\uparrow, T^* J_1\boldsymbol{E} \otimes \boldsymbol{Q}^\uparrow).$$

In any adapted chart, we have the coordinate expression (see Definition 15.1.1)

$$\check{\text{Ч}}^\uparrow[\mathrm{b}] = \text{Ч}_\lambda{}^{\mathrm{a}}\, d^\lambda \otimes \mathrm{b}_{\mathrm{a}} + \text{Ч}^{0\,\mathrm{a}}_i\, d^i_0 \otimes \mathrm{b}_{\mathrm{a}} \quad \textit{with}\ \text{Ч}_\lambda{}^{\mathrm{a}},\ \text{Ч}^{0\,\mathrm{a}}_i \in \mathrm{map}(\boldsymbol{Q}^\uparrow, \mathbb{R}).$$

(2) *If the quantum connection* $\text{Ч}^\uparrow$ *is real linear, then* $\check{\text{Ч}}^\uparrow[\mathrm{b}]$ *can be regarded as a section*

$$\check{\text{Ч}}^\uparrow[\mathrm{b}] \in \sec(J_1\boldsymbol{E},\ T^* J_1\boldsymbol{E} \otimes \boldsymbol{Q}^\uparrow \otimes \boldsymbol{Q}^{\uparrow *}),$$

with coordinate expression (see Definition 15.1.10)

$$\check{\text{Ч}}^\uparrow[\mathrm{b}] = \text{Ч}^\uparrow{}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}}\, w^{\mathrm{b}}\, d^\lambda \otimes \mathrm{b}_{\mathrm{a}} + \text{Ч}^{\uparrow 0\mathrm{a}}_{i\,\mathrm{b}}\, w^{\mathrm{b}}\, d^i_0 \otimes \mathrm{b}_{\mathrm{a}},$$

with

$$\text{Ч}^\uparrow{}_\lambda{}^{\mathrm{a}}{}_{\mathrm{b}},\ \text{Ч}^{\uparrow 0\mathrm{a}}_{i\,\mathrm{b}} \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R}).$$

(3) *If the quantum connection* Ч *is complex linear, then* $\check{\text{Ч}}[\mathrm{b}]$ *can be regarded as a section*

$$\check{\text{Ч}}^\uparrow[\mathrm{b}] \in \sec(J_1\boldsymbol{E},\ T^* J_1\boldsymbol{E} \otimes \boldsymbol{Q}^\uparrow \otimes \boldsymbol{Q}^{\uparrow *}),$$

with coordinate expression

$$\begin{aligned}\check{\mathrm{Ч}}^{\uparrow}[\mathrm{b}] &= \mathrm{Ч}_{\lambda}{}^{1}{}_{1}\, d^{\lambda} \otimes (w^{1}\, \mathrm{b}_{1} + w^{2}\, \mathrm{b}_{2}) + \mathrm{Ч}_{\lambda}{}^{2}{}_{1}\, d^{\lambda} \otimes (w^{1}\, \mathrm{b}_{2} - w^{2}\, \mathrm{b}_{1}) \\ &\quad + \mathrm{Ч}^{0}_{i}{}^{1}_{1}\, d^{i}_{0} \otimes (w^{1}\, \mathrm{b}_{1} + w^{2}\, \mathrm{b}_{2}) + \mathrm{Ч}^{0}_{i}{}^{1}_{1}\, d^{i}_{0} \otimes (w^{1}\, \mathrm{b}_{2} - w^{2}\, \mathrm{b}_{1}) \\ &= d^{\lambda} \otimes (\mathrm{Ч}_{\lambda}{}^{1}{}_{1} + \mathfrak{i}\, \mathrm{Ч}_{\lambda}{}^{2}{}_{1})\, \mathbb{I}^{\uparrow} + d^{i}_{0} \otimes (\mathrm{Ч}^{0}_{i}{}^{1}_{1} + \mathrm{Ч}^{0}_{i}{}^{1}_{1})\mathbb{I}^{\uparrow}.\end{aligned}$$

(4) *If the upper quantum connection* $\mathrm{Ч}^{\uparrow}$ *is hermitian, then* $\check{\mathrm{Ч}}^{\uparrow}[\mathrm{b}]$ *can be regarded as a section*

$$\check{\mathrm{Ч}}^{\uparrow}[\mathrm{b}] \;\equiv\; \mathfrak{i}\, \mathrm{A}^{\uparrow}[\mathrm{b}] \otimes \mathbb{I}^{\uparrow} \in \sec(J_1\boldsymbol{E},\; T^{*}J_1\boldsymbol{E} \otimes \boldsymbol{Q}^{\uparrow} \otimes \boldsymbol{Q}^{\uparrow *}),$$

where

$$\mathrm{A}^{\uparrow}[\mathrm{b}] = \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1}\, d^{\lambda} + \mathrm{Ч}^{\uparrow}{}^{0}_{i}{}^{2}_{1}\, d^{i}_{0} \in \sec(J_1\boldsymbol{E}, T^{*}J_1\boldsymbol{E}).$$

(5) *If the upper quantum connection* $\mathrm{Ч}^{\uparrow}$ *is reducible and hermitian, then* $\check{\mathrm{Ч}}^{\uparrow}[\mathrm{b}]$ *can be regarded as a section*

$$\check{\mathrm{Ч}}^{\uparrow}[\mathrm{b}] \;\equiv\; \mathfrak{i}\, \mathrm{A}^{\uparrow}[\mathrm{b}] \otimes \mathbb{I}^{\uparrow} \in \sec(J_1\boldsymbol{E},\; T^{*}\boldsymbol{E} \otimes \boldsymbol{Q}^{\uparrow} \otimes \boldsymbol{Q}^{\uparrow *}),$$

where

$$\mathrm{A}^{\uparrow}[\mathrm{b}] = \mathrm{Ч}^{\uparrow}{}_{\lambda}{}^{2}{}_{1}\, d^{\lambda} \in \sec(J_1\boldsymbol{E}, T^{*}\boldsymbol{E}). \quad \square$$

15.1.6 Curvature of Quantum and Upper Quantum Connection

We discuss the curvature $R[\mathrm{Ч}]$ of a quantum connection $\mathrm{Ч}$ and the curvature $R[\mathrm{Ч}^{\uparrow}]$ of an upper quantum connection $\mathrm{Ч}^{\uparrow}$. In particular, we analyse the real linear, complex linear and hermitian linear cases.

Moreover, we analyse the curvature of reducible hermitian upper quantum connections.

Proposition 15.1.17 *Let us consider a quantum connection* $\mathrm{Ч} : \boldsymbol{Q} \to T^{*}\boldsymbol{E} \otimes T\boldsymbol{Q}$.

(1) *The* curvature *of* $\mathrm{Ч}$ *can be regarded as a section (see, Appendix: Definition F.1.9)*

$$R[\mathrm{Ч}] : \boldsymbol{Q} \to \Lambda^{2}T^{*}\boldsymbol{E} \otimes \boldsymbol{Q},$$

with coordinate expression

$$R[\mathrm{Ч}] = -2\, (\partial_{\lambda}\mathrm{Ч}_{\mu}{}^{\mathrm{a}} + \mathrm{Ч}_{\lambda}{}^{\mathrm{c}}\, \partial_{\mathrm{c}}\mathrm{Ч}_{\mu}{}^{\mathrm{a}})\, d^{\lambda} \wedge d^{\mu} \otimes \mathrm{b}_{\mathrm{a}}.$$

(2) *If* Ψ *is real linear, then its* curvature *can be regarded as a section (see, Appendix: Proposition F.2.7)*

$$R[\Psi] : \boldsymbol{E} \to \Lambda^2 T^* \boldsymbol{E} \otimes (\boldsymbol{Q} \otimes \boldsymbol{Q}^*),$$

with coordinate expression

$$R[\Psi] = -2\,(\partial_\lambda \Psi_\mu{}^{\mathrm{a}}{}_{\mathrm{b}} + \Psi_\lambda{}^{\mathrm{c}}{}_{\mathrm{b}}\, \Psi_\mu{}^{\mathrm{a}}{}_{\mathrm{c}})\, w^{\mathrm{b}}\, d^\lambda \wedge d^\mu \otimes \mathrm{b}_{\mathrm{a}}.$$

(3) *If* Ψ *is complex linear, then its curvature has coordinate expression*

$$\begin{aligned} R[\Psi] &= -2\,\partial_\lambda \Psi_\mu{}^1{}_1\, d^\lambda \wedge d^\mu \otimes (w^1\, \partial w_1 + w^2\, \partial w_2) \\ &\quad - 2\,\partial_\lambda \Psi_\mu{}^2{}_1\, d^\lambda \wedge d^\mu \otimes (w^1\, \partial w_2 - w^2\, \partial w_1) \\ &= -2\,\partial_\lambda \Psi_\mu{}^1{}_1\, d^\lambda \wedge d^\mu \otimes \mathbb{I} - 2\,\mathrm{i}\,\partial_\lambda \Psi_\mu{}^2{}_1\, d^\lambda \wedge d^\mu \otimes \mathbb{I}. \end{aligned}$$

(4) *If* Ψ *is hermitian, then its curvature turns out to be the section (see Proposition 15.1.15)*

$$R[\Psi] = -2\,\mathrm{i}\, d\mathrm{A}[\mathrm{b}] \otimes \mathbb{I}, \quad \textit{where}\; d\mathrm{A}[\mathrm{b}] \in \sec(\boldsymbol{E}, \Lambda^2 T^* \boldsymbol{E}),$$

with coordinate expression

$$R[\Psi] = -2\,\mathrm{i}\,\partial_\lambda \mathrm{A}_\mu\, d^\lambda \wedge d^\mu \otimes \mathbb{I}.$$

Thus, the curvature $R[\Psi]$ *of a hermitian quantum connection* Ψ *is determined by the spacetime 2-form*

$$d\mathrm{A}[\mathrm{b}] \in \sec(\boldsymbol{E}, \Lambda^2 T^* \boldsymbol{E}). \quad \square$$

Proposition 15.1.18 *Let us consider a reducible hermitian upper quantum connection* $\Psi^\uparrow$ *(see Definition 15.1.12).*

Then, the curvature $R[\Psi^\uparrow]$ *of* $\Psi^\uparrow$ *can be regarded as a section (see Proposition 15.1.16)*

$$R[\Psi^\uparrow] = -2\,\mathrm{i}\, d\mathrm{A}^\uparrow[\mathrm{b}] \otimes \mathbb{I}^\uparrow, \quad \textit{where}\; d\mathrm{A}^\uparrow[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, \Lambda^2 T^* J_1\boldsymbol{E}),$$

with coordinate expression

$$R[\Psi^\uparrow] = -2\,\mathrm{i}\,(\partial_\lambda \mathrm{A}^\uparrow{}_\mu\, d^\lambda \wedge d^\mu + \partial^0_i \mathrm{A}^\uparrow{}_\mu\, d^i_0 \wedge d^\mu) \otimes \mathbb{I}^\uparrow.$$

Thus, the curvature $R[\Psi^\uparrow]$ *of a reducible hermitian quantum connection* $\Psi^\uparrow$ *is determined by the phase 2-form*

$$d\mathrm{A}^\uparrow[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, \Lambda^2 T^* J_1\boldsymbol{E}). \quad \square$$

Remark 15.1.19 We stress that, in Propositions 15.1.17 and 15.1.18, the curvature $R[\mathrm{Ч}^{\uparrow}]$ is gauge independent, while the spacetime 1-form $A^{\uparrow}[\mathrm{b}]$ is gauge dependent. However, the exterior differential d cancels the gauge dependence of $A^{\uparrow}[\mathrm{b}]$. □

Proposition 15.1.20 *The curvature* $R[\mathrm{Ч}^{\uparrow}]$ *of a reducible upper quantum connection* $\mathrm{Ч}^{\uparrow}$ *has a* universal property *with respect to the curvatures* $R[\mathrm{Ч}[o]]$ *of the associated observed quantum connections* $\mathrm{Ч}[o]$ *(see Lemma 15.1.3 and Definition 15.1.4 and [35, 141]). In fact,* $R[\mathrm{Ч}^{\uparrow}]$ *is characterised by the equalities*

$$R\big[\mathrm{Ч}[o]\big] = o^* R[\mathrm{Ч}^{\uparrow}], \quad \textit{for each } o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E}). \quad \square$$

15.2 Galilean Upper Quantum Connections

We define the *galilean upper quantum connections* by means of three conditions and discuss their local and global existence.

Then, after having possibly checked a cohomological condition on the cosymplectic phase 2-form Ω, we *postulate* a given *galilean upper quantum connection* $\mathrm{Ч}^{\uparrow}$, which will be the source of all further quantum developments.

Further, we analyse the transition rule for the potential and the distinguished invariants $\alpha[\mathrm{b}]$ and $\nu[\mathrm{b}]$, which arise from the chosen galilean upper quantum potential.

Eventually, we discuss the distinguished (local) *horizontal potential* $A[\Psi] : \boldsymbol{E} \to H^*\boldsymbol{E}$ and the distinguished "*rest observer*" $o[\Psi] : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ associated with any proper quantum section Ψ.

15.2.1 Definition

We define the *galilean upper quantum connections* as the upper quantum connections $\mathrm{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$, which are *hermitian*, *reducible* and whose *curvature is proportional* to the classical cosymplectic phase 2-form Ω.

Indeed, this definition resembles a choice of Geometric Quantisation (see, for instance [425]), with two essential changes: in our approach we require the reducibility of $\mathrm{Ч}^{\uparrow}$ (with relation to the definition of the quantum bundle based over spacetime) and the proportionality to the cosymplectic phase 2-form Ω, instead to the symplectic phase 2-form ω (with relation to the odd dimension of phase space).

For a covariant approach to Quantum Mechanics based on a reducible upper quantum connection, see also, Künzle [263].

Definition 15.2.1 We define a *galilean upper quantum connection* (see Definition 15.1.5)

$$\mathrm{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$$

to be a connection of the upper quantum bundle, such that:

(a) it is hermitian (see Definition 15.1.12),
(b) it is reducible (see Definition 15.1.6),
(c) its curvature fulfills the condition (see Proposition 15.1.18)

$$R[\Psi^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow},$$

where

$$\Omega \equiv \Omega[G, K] = \Omega^{\natural} + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F$$

is the classical joined cosymplectic phase 2-form (see Corollary 9.2.4). □

Remark 15.2.2 We recall that the joined dynamical phase 2-form Ω encodes both the classical gravitational field $K^{\natural}$ and electromagnetic field F, which effect to quantum particle.

Moreover, we recall that Ω is normalised through the factor $\frac{m}{\hbar}$, via the rescaled metric $G := \frac{m}{\hbar}\,g$. Thus, we stress that the mass m and the charge q of the quantum particle and the Planck constant $\hbar$ are encoded in the above Definition 15.2.1, via the normalisation and the joined cosymplectic 2-form Ω. □

Remark 15.2.3 The Bianchi identities for the galilean upper quantum connection $\Psi^{\uparrow}$ imply (see, Appendix: Proposition F.2.10)

$$0 = \big[\Psi^{\uparrow}, R[\Psi^{\uparrow}]\big] = -2\,\mathfrak{i}\,d\Omega\,\mathbb{I}^{\uparrow}.$$

Hence, we stress that the closure of Ω, that we have achieved in Theorem 10.1.1, turns out to be a necessary integrability condition for the local existence of $\Psi^{\uparrow}$.

We recall also that the closure of Ω is equivalent to the symmetry property of the curvature tensor $R[K]$ of the joined galilean connection K, hence to the symmetry property of the curvature tensor $R[K^{\natural}]$ of the gravitational galilean connection $K^{\natural}$ and the closure of the electromagnetic field F. □

15.2.2 Local Existence

The local existence of galilean upper quantum connections can be easily proved.

Indeed, we show gauge dependent and observer dependent local expressions of any galilean upper quantum connection $\Psi^{\uparrow}$ in terms of the classical phase dynamical 1-forms $\mathcal{H}[\mathrm{b}, o]$ and $\mathcal{P}[\mathrm{b}, o]$ (see [316]).

These expressions will be largely used throughout the book.

Theorem 15.2.4 *A galilean upper quantum connection* $\Psi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^{\uparrow}$ *exists locally.*

Indeed, with reference to a quantum basis $\mathfrak{b}$ *and an observer* o*, the expression of a galilean upper quantum connection* $\mathcal{Y}^{\uparrow}$ *is locally of the type (see Lemma 15.1.14, Definition 3.2.9 and Theorem 10.1.8)*

$$\begin{aligned}\mathcal{Y}^{\uparrow} &= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow} \\ &= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(\mathcal{C}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^{\uparrow} \\ &= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{K}[o] + \mathcal{Q}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^{\uparrow} \\ &= \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\,\big(-\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o]\big) \otimes \mathbb{I}^{\uparrow},\end{aligned}$$

i.e., in adapted coordinates,

$$\mathcal{Y}^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^{i}_{0} \otimes \partial^{0}_{i} + \mathfrak{i}\,\big(-(\tfrac{1}{2}\, G^{0}_{ij}\, x^{i}_{0}\, x^{j}_{0} - A_{0})\, d^{0} + (G^{0}_{ij}\, x^{j}_{0} + A_{i})\, d^{i}\big) \otimes \mathbb{I}^{\uparrow},$$

where

(1) $\chi^{\uparrow}[\mathfrak{b}] : \boldsymbol{Q}^{\uparrow} \to T^{*}J_{1}\boldsymbol{E} \otimes T\,\boldsymbol{Q}^{\uparrow}$ *is the flat hermitian upper quantum connection induced by the quantum basis* $\mathfrak{b}$ *(see Lemma 15.1.14),*
(2) $A^{\uparrow}[\mathfrak{b}] : J_{1}\boldsymbol{E} \to T^{*}\boldsymbol{E}$ *is an upper potential of the cosymplectic phase 2-form* $\Omega : J_{1}\boldsymbol{E} \to \Lambda^{2}T^{*}J_{1}\boldsymbol{E}$ *(see Theorem 10.1.4),*
(3) $A[\mathfrak{b}, o] : \boldsymbol{E} \to T^{*}\boldsymbol{E}$ *is a potential of the observed spacetime 2-form* $\Phi[o] = 2\, o^{*}\Omega : \boldsymbol{E} \to \Lambda^{2}T^{*}\boldsymbol{E}$ *(see Theorem 4.3.3).*

Actually, the galilean upper quantum connections $\mathcal{Y}^{\uparrow}$ *are defined locally up to a gauge of the type*

$$\mathfrak{i}\, df \otimes \mathbb{I}^{\uparrow}, \quad \textit{where } f \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

Proof. In virtue of Proposition 15.1.16, the upper quantum connection $\mathrm{Y}^{\uparrow}$, whose local expression is

$$\mathrm{Y}^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow}, \quad \text{with}\, dA^{\uparrow}[\mathfrak{b}] = \Omega,$$

is hermitian, reducible and, in virtue of Proposition 15.1.18, its curvature is

$$R[\mathrm{Y}^{\uparrow}] = -2\,\mathfrak{i}\, dA[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow} = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}.$$

Moreover, the other local expression of $\mathrm{Y}^{\uparrow}$ follow from Theorem 10.1.8. □

Remark 15.2.5 In the above Theorem 15.2.4, we deal with a given galilean upper quantum connection $\mathrm{Y}^{\uparrow}$ and choose a quantum basis $\mathfrak{b}$ along with the associated flat connection $\chi^{\uparrow}[\mathfrak{b}]$.

Then, the selection of the gauge dependent upper potential $A^{\uparrow}[\mathfrak{b}]$ and of the gauge dependent observed spacetime potential $A[\mathfrak{b}, o]$ are determined by the above choice.

Indeed, this is the phenomenon (which has been anticipated in Remark 10.1.6), by which, given a galilean quantum connection $\mathrm{Y}^{\uparrow}$, we can classify the gauge dependent classical potentials through quantum bases. In a sense, we might say that the quantum bundle and a galilean upper quantum connection is a way to classify gauge dependent classical potentials.

Conversely, according to Theorem 15.2.4, given a quantum basis $\mathfrak{b}$, the knowledge of the local upper potential $A^{\uparrow}$, or of the local observed spacetime potential $A[o]$, determines locally the galilean upper quantum connection $Ч^{\uparrow}$. □

Corollary 15.2.6 *A galilean upper quantum connection* $Ч^{\uparrow}$*, with reference to an observer o, yields the observer dependent and gauge independent* observed quantum connection

$$Ч[o] := o^* Ч^{\uparrow} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}.$$

With further reference to a quantum basis $\mathfrak{b}$*, we have the splitting*

$$Ч[o] = \chi[\mathfrak{b}] + \mathrm{i}\, A[\mathfrak{b}, o]\, \mathbb{I}, \quad \textit{where } A[\mathfrak{b}, o] = o^* A^{\uparrow}[\mathfrak{b}],$$

i.e., in adapted coordinates,

$$Ч[o] = d^{\lambda} \otimes (\partial_{\lambda} + \mathrm{i}\, A_{\lambda}[\mathfrak{b}, o]\, \mathbb{I}),$$

where (see Lemma 15.1.14 and Theorem 10.1.4)

$$\chi[\mathfrak{b}] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q} \quad \textit{and} \quad A[\mathfrak{b}, o] := o^* A^{\uparrow}[\mathfrak{b}]$$

are the flat hermitian quantum connection induced by the quantum basis $\mathfrak{b}$ *and the observed spacetime potential.* □

Indeed, the choice of a galilean upper quantum connection $Ч^{\uparrow}$ turns out to be equivalent to the choice of a system of hermitian observed quantum connections, which fulfill a certain transition rule stated in the following Theorem.

Theorem 15.2.7 *Let us consider a galilean upper quantum connection* $Ч^{\uparrow}$ *and the associated observed system of quantum connections* $\{Ч[o]\}$.

Then, we have the following transition rule, for each observers o and $\acute{o} = o + \vec{v}$*, (see Proposition 2.7.3)*

$$Ч[\acute{o}] = Ч[o] + \mathrm{i}\big(\theta[o] \lrcorner G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v})\big) \otimes \mathbb{I},$$

with coordinate expression, in a chart adapted to o,

$$Ч[\acute{o}] = Ч[o] + \mathrm{i}\,(G^{0}_{ij}\, v^{i}_{0}\, d^{j} - \tfrac{1}{2}\, G^{0}_{ij}\, v^{i}_{0}\, v^{j}_{0}\, d^{0}) \otimes \mathbb{I}.$$

Proof. In virtue of Theorem 15.2.4 and Lemma 15.1.3, we have

$$\begin{aligned}(o+\vec{v})^* \text{Ч}^{\uparrow} &= (o+\vec{v})^* \big(\chi^{\uparrow}[\mathfrak{b}] + \mathrm{i}\,\big(\mathcal{C}[o] + A[\mathfrak{b}, o]\big) \otimes \mathbb{I}^{\uparrow}\big) \\ &= \chi[\mathfrak{b}] + \mathrm{i}\,\big(\mathcal{C}[o] \circ (o+\vec{v}) + A[\mathfrak{b}, o]\big) \otimes \mathbb{I} \\ &= \text{Ч}[o] + \mathrm{i}\,\big(\theta[o] \lrcorner G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v})\big) \otimes \mathbb{I}. \quad \square\end{aligned}$$

Remark 15.2.8 In virtue of Definition 15.1.10, the transition rule for the observed quantum connections $\text{Ч}[\acute{o}] = \text{Ч}[o] + \mathrm{i}\,\big(\theta[o] \lrcorner G^{\flat}(v) - \frac{1}{2}\, G(v, v)\big) \otimes \mathbb{I}$, which has been achieved in the above Theorem 15.2.7 turns out to be consistent with the fact that all quantum connections of the system are hermitian. □

The curvature of the observed quantum connections $\text{Ч}[o]$ can be expressed via the joined observed spacetime 2-form $\Phi[o]$.

Corollary 15.2.9 *Let us consider a galilean upper quantum connection* $\text{Ч}^{\uparrow}$ *and the associated observed system of quantum connections* $\{\text{Ч}[o]\}$ *(see Definition 15.1.9).*

Then, for each observer o*, we have (see Definition 4.2.11 and Corollary 6.3.3)*

$$R\big[\text{Ч}[o]\big] = -\mathrm{i}\, \Phi[o] \otimes \mathbb{I}.$$

Indeed, the above equality agrees with the equalities

$$R\big[\text{Ч}[o]\big] = -2\,\mathrm{i}\, dA[\mathfrak{b}, o] \otimes \mathbb{I} \quad \textit{and} \quad \Phi[o] = 2\, dA[\mathfrak{b}, o].$$

Proof. We have $R\big[\text{Ч}[o]\big] = R\big[o^* \text{Ч}^{\uparrow}\big] = -2\,\mathrm{i}\, o^* \Omega \otimes \mathbb{I} = -\mathrm{i}\, \Phi[o] \otimes \mathbb{I}$. □

Note 15.2.10 Let us consider a galilean upper quantum connection $\text{Ч}^{\uparrow}$, the associated observed system of quantum connections $\{\text{Ч}[o]\}$ and two observers o and $\acute{o}$.

Then, we have a consistent transition between observers and associated objects, according to the following commutative diagram

$$\begin{array}{ccc} o & \xrightarrow{\;\mapsto\;} & \acute{o} \\ \downarrow & & \downarrow \\ \text{Ч}[o] & \xrightarrow{\;\mapsto\;} & \text{Ч}[\acute{o}] \\ \downarrow & & \downarrow \\ R\big[\text{Ч}[o]\big] & \xrightarrow{\;\mapsto\;} & R\big[\text{Ч}[\acute{o}]\big] \\ \downarrow & & \downarrow \\ \Phi[o] & \xrightarrow{\;\mapsto\;} & \Phi[\acute{o}]\,. \end{array} \quad \square$$

Exercise 15.2.11 Let us consider a galilean upper quantum connection $Ч^{\uparrow}$, the associated observed system of quantum connections $\{Ч[o]\}$ and two observers o and $\acute{o} = o + \vec{v}$.

Then, in virtue of Theorem 15.2.7, Corollary 15.2.9 and Proposition 4.2.12, we have the following transition rules

$$R\big[Ч[\acute{o}]\big] = R\big[Ч[o]\big] + 2\,\mathrm{i}\, d\big(\theta[o] \lrcorner (G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G\,(\vec{v}, \vec{v})\big) \otimes \mathbb{I},$$
$$\Phi[\acute{o}] = \Phi[o] + 2\ \mathrm{Ant}\,\big(\theta[o] \overset{2}{\lrcorner} (\underline{\nabla}\vec{v})\big) - 2\,(\vec{v} \overset{2}{\lrcorner} \underline{\nabla}д[o]) \wedge dt - 2\,(\vec{v} \overset{2}{\lrcorner} \underline{\nabla}\vec{v}) \wedge dt.$$

It is instructive to check directly that the above transition rules fit together.

Solution In fact, in a spacetime chart adapted to o, we have the coordinate expressions

$$\begin{aligned}\Phi[\acute{o}] - \Phi[o] &= 2\ \mathrm{Ant}\,\big(\theta[o] \overset{2}{\lrcorner} (\underline{\nabla}\vec{v})\big) - 2\,(\vec{v} \overset{2}{\lrcorner} \underline{\nabla}д[o]) \wedge dt - 2\,(\vec{v} \overset{2}{\lrcorner} \underline{\nabla}\vec{v}) \wedge dt \\ &= 2\, G^0_{hj}\,(\partial_\lambda v^h_0 - K_\lambda{}^h{}_k\, v^k_0)\, d^\lambda \wedge d^j - 2\, G^0_{ih}\, K_j{}^h{}_0\, v^i_0\, d^0 \wedge d^j \\ &\quad + 2\, G^0_{ih}\,(\partial_j v^h_0 - K_j{}^h{}_k\, v^k_0)\, v^i_0\, d^0 \wedge d^j\end{aligned}$$

and

$$\begin{aligned}d\big(2\,\theta[o] \lrcorner G^{\flat}(\vec{v}) - G\,(\vec{v}, \vec{v})\big) &= d(2\, G^0_{ij}\, v^j_0\, d^i - G^0_{ij}\, v^i_0\, v^j_0\, d^0) \\ &= 2\,(\partial_0 G^0_{ij}\, v^j_0 + G^0_{ij}\, \partial_0 v^j_0)\, d^0 \wedge d^i \\ &\quad + 2\,(\partial_h G^0_{ij}\, v^j_0 + G^0_{ij}\, \partial_h v^j_0)\, d^h \wedge d^i \\ &\quad + (\partial_h G^0_{ij}\, v^i_0\, v^j_0 + 2\, G^0_{ij}\, \partial_h v^i_0\, v^j_0)\, d^0 \wedge d^h.\end{aligned}$$

Then, the above equalities agree in virtue of the coordinate expression of $K_\lambda{}^h{}_k$ provided by Theorem 4.3.3. □

Note 15.2.12 Summing up the occurrences of the joined upper potential $A^{\uparrow}[\mathfrak{b}]$ and of the joined observed potential $A[\mathfrak{b}, o]$, we observe that they appear independently in the following contexts.

The gauge dependent upper potential $A^{\uparrow}[\mathfrak{b}] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$ appears as:

(1) the (local) potential of the cosymplectic dynamical phase 2-form

$$\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E},$$

according to the equality (see Theorem 10.1.4, Remark 10.1.6 and Notation 10.1.7)

$$\Omega = d A^{\uparrow}[\mathfrak{b}],$$

(2) the (local) potential of the galilean upper quantum connection

$$Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^{\uparrow},$$

according to the gauge dependent expression (see Theorem 15.2.4)

$$Ч^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow}.$$

The link which yields the identification of the two occurrences above of $A^{\uparrow}[\mathfrak{b}]$ is given by the postulated equality (see Definition 15.1.5)

$$R[Ч^{\uparrow}] = -2\,\mathfrak{i}\,\Omega \otimes \mathbb{I}^{\uparrow}.$$

The observer dependent and gauge dependent potential $A[\mathfrak{b}, o] : \boldsymbol{E} \to T^*\boldsymbol{E}$ appears as:

(1) the (local) potential of the (local) observed spacetime 2-form

$$\Phi[o] \;\equiv\; \Phi[G, K, o] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E},$$

according to the equality (see Definition 4.2.11 and Theorem 4.3.3)

$$\Phi[o] = 2\, d A[\mathfrak{b}, o],$$

(2) the observed pullback of the upper potential (see Theorem 10.1.4)

$$A[\mathfrak{b}, o] = o^* A^{\uparrow}[\mathfrak{b}].$$

The link which yields the identification of the two occurrences above of $A[\mathfrak{b}, o]$ is given by the equality (see Proposition 10.1.2)

$$\Phi[o] = 2\, o^*\Omega.$$

As we have already anticipated (see Remark 10.1.6 and Notation 10.1.7), we observe that, in virtue of the above identifications, the quantum bases $\mathfrak{b}$ allow us to parametrise, also in the classical theory, the upper quantum potentials $A^{\uparrow}[\mathfrak{b}]$, hence the *observed quantum potentials* $A[\mathfrak{b}, o]$. □

Remark 15.2.13 We have defined the concept of galilean upper quantum connection and of the associated galilean system of observed quantum connections. But, it would not make sense to say that a single quantum connection be galilean.

Actually, such a specification can be applied to a system of observed quantum connections as a whole, according to the above Theorem 15.2.7. □

15.2.3 Global Existence

Next, we ask whether galilean upper quantum connections exist globally.

More precisely, we define a *quantum structure* to be a 3-plet $(\boldsymbol{Q}, \mathrm{h}, \text{Ч}^{\uparrow})$ consisting of a quantum bundle, a hermitian quantum metric and a galilean upper quantum connection, and arise the following natural questions:

- do quantum structures exist?
- how many equivalent quantum structures exist?

Indeed, in our covariant galilean framework, the answer to these questions (see [410]) is similar to the answer given to an analogous questions in Geometric Quantisation (see, for instance [143, 250, 373, 425]).

Here, we follow the paper [410], where the reader can find the proofs of theorems and further details.

Preliminarily, we denote the Čech cohomology of $\boldsymbol{E}$ with values in $\mathbb{Z}$ and $\mathbb{R}$, respectively, by

$$H^*(\boldsymbol{E}, \mathbb{Z}) \quad \text{and} \quad H^*(\boldsymbol{E}, \mathbb{R}).$$

We recall that the natural inclusion $i : \mathbb{Z} \to \mathbb{R}$ yields a group morphism (which needs not to be injective)

$$i : H^2(\boldsymbol{E}, \mathbb{Z}) \to H^2(\boldsymbol{E}, \mathbb{R}),$$

hence a subgroup

$$i\big(H^2(\boldsymbol{E}, \mathbb{Z})\big) \subset H^2(\boldsymbol{E}, \mathbb{R}).$$

Moreover, we recall the natural isomorphism

$$H^*(\boldsymbol{E}, \mathbb{R}) \to H^*_{\text{de Rham}}(\boldsymbol{E}).$$

We start the present discussion on the global existence of a galilean upper quantum connection by summarising some notions discussed before, and introducing the following definition (see Postulate Q.1, Propositions 14.2.1, 14.3.1 14.4.1, 14.4.2, and Definition 15.2.1).

Definition 15.2.14 We define:

- a *quantum bundle* to be a 1-dimensional complex bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$,
- a *hermitian quantum bundle* to be pair $(\boldsymbol{Q}, \mathrm{h})$ consisting of a quantum bundle and a hermitian quantum metric $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and $\mathrm{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C}$,
- a *quantum structure* to be a 3-plet $(\boldsymbol{Q}, \mathrm{h}, \text{Ч}^{\uparrow})$ consisting of a quantum bundle, a hermitian quantum metric and a galilean upper quantum connection

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}, \quad \mathrm{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C}, \quad \text{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1 \boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}. \quad \square$$

Definition 15.2.15 We define the following equivalences:

- $(\boldsymbol{Q}, \mathrm{h})$ and $(\acute{\boldsymbol{Q}}, \acute{\mathrm{h}})$ are said to be *equivalent* if there exists a complex linear bundle isomorphism $f : \boldsymbol{Q} \to \acute{\boldsymbol{Q}}$ over $\boldsymbol{E}$, which maps h onto $\acute{\mathrm{h}}$;
- $(\boldsymbol{Q}, \mathrm{h}, \Psi^{\uparrow})$ and $(\acute{\boldsymbol{Q}}, \acute{\mathrm{h}}, \acute{\Psi}^{\uparrow})$ are said to be *equivalent* if there exists a complex linear bundle isomorphism $f : \boldsymbol{Q} \to \acute{\boldsymbol{Q}}$ over $\boldsymbol{E}$, which maps h onto $\acute{\mathrm{h}}$ and (via the natural 1-st jet prolongation) $\Psi^{\uparrow}$ onto $\acute{\Psi}^{\uparrow}$. □

We stress that not all hermitian quantum bundles admit a galilean upper quantum connection.

Definition 15.2.16 ([410]) A hermitian quantum bundle $\boldsymbol{Q}$ is said to be *admissible* if it admits a galilean upper quantum connection $\Psi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$.

We denote the set of *equivalence classes of hermitian quantum bundles* and the subset of *equivalence classes of admissible hermitian quantum bundles* by

$$\mathfrak{L}[\boldsymbol{E}] \equiv \{[(\boldsymbol{Q}, \mathrm{h})]\} \quad \text{and} \quad \mathfrak{A}[\boldsymbol{E}] \equiv \{[(\boldsymbol{Q}, \mathrm{h})]\}_{\mathrm{adm}} \subset \mathfrak{L}[\boldsymbol{E}]. \quad \square$$

Note 15.2.17 ([410]) The set $\mathfrak{L}[\boldsymbol{E}]$ has a natural structure of abelian group with respect to the complex tensor product, and there exists a natural abelian group isomorphism

$$\mathfrak{L}[\boldsymbol{E}] \to H^2(\boldsymbol{E}, \mathbb{Z}). \quad \square$$

Note 15.2.18 ([410]) In virtue of the topological triviality of the fibres of the bundle $J_1\boldsymbol{E} \to \boldsymbol{E}$, there exist isomorphisms (but no distinguished one)

$$H^*(J_1\boldsymbol{E}, \mathbb{R}) \to H^*(\boldsymbol{E}, \mathbb{R}). \quad \square$$

Now, let us consider the cosymplectic phase 2-form $\Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}$ (see Theorem 9.2.8). Even if there is no distinguished isomorphism as above, the cosymplectic phase 2-form Ω yields naturally a class

$$[\Omega] \in H^2(\boldsymbol{E}, \mathbb{R}).$$

Lemma 15.2.19 ([410]) *The class* $[\Omega] \in H^2_{de\ Rham}(J_1\boldsymbol{E})$ *yields naturally a class*

$$[\Omega] \in H^2(\boldsymbol{E}, \mathbb{R}). \quad \square$$

Then, we can state the following theorem concerning the existence of quantum structures.

Theorem 15.2.20 (Vitolo [410]) *The following conditions are equivalent:*

(1) *there exists a quantum structure* $(\boldsymbol{Q}, \mathrm{h}, \Psi^{\uparrow})$,
(2) *the cohomology class* $[\Omega] \in H^2[\boldsymbol{E}, \mathbb{R}]$ *determined by the de Rham class of the closed phase 2-form* Ω *lies in the subgroup*

$$[\Omega] \in i(H^2[\boldsymbol{E}, \mathbb{Z}]) \subset H^2[\boldsymbol{E}, \mathbb{R}]. \quad \square$$

In other words, there exists a quantum structure if and only if the Čech cohomology class $[\Omega]$ determined by the de Rham class of the closed cosymplectic phase 2-form Ω is integer.

Then, we can classify the quantum structures.

Theorem 15.2.21 (Vitolo [410]) *If* $[\Omega] \in i(H^2[\boldsymbol{E}, \mathbb{Z}]) \subset H^2[\boldsymbol{E}, \mathbb{R}]$*, then we have*

$$\mathfrak{A}[\boldsymbol{E}] = i^{-1}([\Omega]) \subset H^2(\boldsymbol{E}, \mathbb{Z}).$$

Hence, $\mathfrak{A}[\boldsymbol{E}]$ *turns out to be an affine space associated with the abelian group*

$$\ker i \subset H^2(\boldsymbol{E}, \mathbb{Z}). \quad \square$$

Theorem 15.2.22 (Vitolo [410]) *If* $[\Omega] \in i(H^2[\boldsymbol{E}, \mathbb{Z}]) \subset H^2[\boldsymbol{E}, \mathbb{R}]$*, then we have*

$$\mathfrak{A}[\boldsymbol{E}] = i^{-1}([\Omega]) \subset H^2(\boldsymbol{E}, \mathbb{Z}).$$

Hence, $\mathfrak{A}[\boldsymbol{E}]$ *turns out to be an affine space associated with the abelian group*

$$\ker i \subset H^2(\boldsymbol{E}, \mathbb{Z}). \quad \square$$

Corollary 15.2.23 (Vitolo [410]) *We have a natural bijection*

$$\mathfrak{L}[\boldsymbol{E}] \to H^1[\boldsymbol{E}, U(1)].$$

In particular, if $\boldsymbol{E}$ *is simply connected, then there exists only one equivalence class of quantum structures.* $\square$

Indeed, the condition on the global existence of a galilean upper quantum connection stated in Theorem 15.2.20 is fulfilled by several spacetimes of physical interest.

First of all, we mention the quantum structure of standard Quantum Mechanics in a flat galilean spacetime equipped with a suitable given electromagnetic field (see, for instance, Sects. 25.2, 26.2 and 27.2).

Another, non trivial example is given by the Dirac monopole in a galilean framework (see [410]).

Further, we would like to mention a different kind of quantum model which fits our setting and where the 4-dimensional galilean spacetime is replaced by a 7-dimensional configuration space. This is the case of our model of classical and quantum top discussed in [313, 317].

In this model, we show that:

- the rigid system can be regarded as a "representative particle" moving in a configuration space which fulfills the basic axioms of our galilean spacetime,

- the electromagnetic field acting on the particles of the system can be regarded as a “representative electromagnetic field” acting on the representative particle,
- the quantum description of the system fits the basic axioms of our formulation of the quantum theory.

15.2.4 Postulate on Galilean Upper Quantum Connection

Now, we are in the position to state our 2nd fundamental Postulate of Covariant Quantum Mechanics concerning the galilean upper quantum connection $Ч^{\uparrow}$.

Postulate Q.2 *We suppose the cohomology class of* Ω *to be integer, according to the following integrality condition (see Theorem 15.2.20)*

$$\Omega \in i(H^2[\boldsymbol{E}, \mathbb{Z}]) \subset H^2[\boldsymbol{E}, \mathrm{I\!R}],$$

and postulate a given galilean upper quantum connection

$$Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* J_1 \boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow},$$

as source of all further quantum developments. □

So, from now on, we shall refer to this postulated galilean upper quantum connection $Ч^{\uparrow}$.

Remark 15.2.24 If $Ч^{\uparrow}$ is a (global) galilean upper quantum connection, then any other galilean upper quantum connection $Ч^{\uparrow}$ is of the type

$$\acute{Ч}^{\uparrow} = Ч^{\uparrow} + \mathfrak{i}\,\beta \otimes \mathbb{I}^{\uparrow},$$

where β is a global closed spacetime 1-form

$$\beta : \boldsymbol{E} \to T^*\boldsymbol{E}. \quad \square$$

Remark 15.2.25 According to Remark 15.2.5, the above postulate on a galilean upper quantum connection $Ч^{\uparrow}$ (see the above Postulate Q.2) turns out to be an implicit choice, for each quantum basis $\mathfrak{b}$, of the upper potential $A^{\uparrow}[\mathfrak{b}]$ of the phase 2-form Ω, according to the equality (see Note 15.2.12)

$$Ч^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow}.$$

We know that, if $A^{\uparrow}[\mathfrak{b}]$ is a potential of Ω, then also

$$\acute{A}^{\uparrow} := A^{\uparrow}[\mathfrak{b}] + d\vartheta, \quad \text{with } \vartheta : \boldsymbol{E} \to \mathrm{I\!R},$$

is a potential of Ω.

Then, it is natural to ask the following question: what would happen if we postulate the upper quantum connection

$$\acute{\mathrm{Ч}}^{\uparrow} := \mathrm{Ч}^{\uparrow} + \mathfrak{i}\, d\vartheta \otimes \mathbb{I}^{\uparrow} = \chi^{\uparrow}[\mathfrak{b}] + \mathfrak{i}\, A^{\uparrow}[\mathfrak{b}] \otimes \mathbb{I}^{\uparrow} + \mathfrak{i}\, d\vartheta \otimes \mathbb{I}^{\uparrow},$$

instead of $\mathrm{Ч}^{\uparrow}$?

Well, the answer is that we would obtain a different, but physically equivalent, formulation of the quantum theory. In fact, it is as if we had performed the following global complex linear fibred transformation of the quantum bundle

$$\boldsymbol{Q} \to \boldsymbol{Q} : \mathfrak{q} \mapsto \exp(\mathfrak{i}\,\vartheta)\, \mathfrak{q}. \quad \square$$

15.2.5 Transition Rule for the Potential and Invariants

Now, with reference to the postulated global galilean upper quantum connection $\mathrm{Ч}^{\uparrow}$, we are able to provide a definite transition rule for the observed potential, with respect to the change of quantum bases and observers.

Moreover, we exhibit two distinguished observer invariants $\nu[\mathfrak{b}]$ and $\alpha[\mathfrak{b}]$, which will frequently play a role in our formulation of Covariant Quantum Mechanics (see, for instance, Propositions 12.2.5 and 15.2.29, Corollary 15.2.30 Lemma 19.1.2 and Remark 19.1.11).

Thus, let us consider the postulated global galilean upper quantum connection $\mathrm{Ч}^{\uparrow}$.

Theorem 15.2.26 ([224]) *With reference to two quantum bases* $\mathfrak{b}$ *and* $\acute{\mathfrak{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$ *and two observers and* $\acute{o} = o + \vec{v}$*, with* $\vartheta \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}/2\pi)$ *and* $\vec{v} \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E})$*, we have the transition rule, which extends the transition rule shown in Theorem 10.1.4,*

$$A[\acute{\mathfrak{b}}, \acute{o}] = A[\mathfrak{b}, o] - d\vartheta + \theta[o] \lrcorner G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v}),$$

i.e., in a spacetime chart adapted to o,

$$A[\acute{\mathfrak{b}}, \acute{o}] = A_{\lambda}\, d^{\lambda} - \partial_{\lambda}\vartheta\, d^{\lambda} + G^{0}_{ij}\, v^{i}_{0}\, d^{j} - \tfrac{1}{2}\, G^{0}_{ij}\, v^{i}_{0}\, v^{j}_{0}\, d^{0}.$$

Proof. This transition rule follows from the expression of the gauge invariant and observer invariant upper quantum connection $\mathrm{Ч}^{\uparrow}$ (see Theorem 15.2.4). $\square$

Exercise 15.2.27 Check that the above transition rule agrees with the transition rule for $\Phi[o]$ provided by Proposition 4.2.12. $\square$

Indeed, the above transition rule yields two observer invariants, which have already been found via tangent lift of $\mathcal{L}[\mathfrak{b}]$ (see Proposition 12.2.5).

Corollary 15.2.28 *Let us consider a quantum basis* $\mathfrak{b}$. *Then, the scaled vector field and the timelike 1-form (see Proposition 12.2.5)*

$$\nu[\mathfrak{b}] := д[o] - G^{\sharp}(\check{A}[\mathfrak{b}, o]) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$$
$$\alpha[\mathfrak{b}] := д[o] \lrcorner A[\mathfrak{b}, o] - \tfrac{1}{2}\bar{G}\big(\check{A}[\mathfrak{b}, o], \check{A}[\mathfrak{b}, o]\big) \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$$

depend only on $\mathfrak{b}$, *hence they are equivariant with respect to the observer* o.

The coordinate expressions of the above objects, in a spacetime chart adapted to o, *are*

$$\nu[\mathfrak{b}] = u^0 \otimes (\partial_0 - A^i_0\,\partial_i) \quad \textit{and} \quad \alpha[\mathfrak{b}] = (A_0 - \tfrac{1}{2}\,A_i\,A^i_0)\,u^0. \quad \square$$

In Proposition 12.2.5 we have provided an explicit observer independent definition of $\nu[\mathfrak{b}]$ and $\alpha[\mathfrak{b}]$.

15.2.6 Distinguished Observer and Potential

By following the paper [224], we discuss the condition by which the spacelike observed potential vanishes and we exhibit the distinguished "rest observer" $o[\Psi]$ and the distinguished "timelike observed potential" $A[\Psi]$ associated with a proper quantum section Ψ.

The rest observer $o[\Psi]$, which can be regarded as the observer at rest with respect to the hydrodynamic classical fluid associated with the quantum section, will play a relevant role in different occasions (see, for instance, Theorems 17.2.2, 17.4.2 and 18.1.1, Corollaries 17.3.3 and 17.6.18).

Moreover the timelike potential $A[\Psi]$ can be regarded as the potential "*intrinsically seen*" by the quantum particle regardless of any gauge choice and will play a relevant role in different occasions (see, for instance, Corollary 15.2.33, Proposition 16.1.12). Propositions 16.1.17, 16.2.7, 16.2.11, Corollaries 16.1.24, 17.5.3, 17.6.12, 17.6.18 and 18.1.10, Remark 18.1.11, Theorems 18.1.17 and 18.2.1).

Proposition 15.2.29 *With reference to a quantum basis* $\mathfrak{b}$, *there exists a unique* distinguished observer $o[\mathfrak{b}]$, *such that*

$$\check{A}\big[\mathfrak{b}, o[\mathfrak{b}]\big] = 0.$$

Indeed, this observer $o[\mathfrak{b}]$ *is characterised by the condition (see Corollary 15.2.28)*

$$д\big[o[\mathfrak{b}]\big] = \nu[\mathfrak{b}],$$

i.e., with reference to any observer o, *by the condition*

$$o^i_0[\mathfrak{b}] = -A^i_0[\mathfrak{b}, o].$$

Thus, for every quantum basis $\mathfrak{b}$*, we obtain the* distinguished timelike observed potential *determined by* $\mathfrak{b}$*, which turns out to be given by the equality (see Corollary 15.2.28)*

$$A[\mathfrak{b}] := A\big[\mathfrak{b}, o[\mathfrak{b}]\big] = \alpha[\mathfrak{b}] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}),$$

i.e., with reference to any observer o, by the condition

$$A[\mathfrak{b}] = (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\, d^0. \quad \square$$

According to the following corollary, the observer invariant horizontal form $\alpha[\mathfrak{b}]$ can be interpreted via the condition by which there exists an observer o such that $A[\mathfrak{b}, o] = 0$.

Corollary 15.2.30 *With reference to a quantum basis* $\mathfrak{b}$*, there exists an observer o such that the potential* $A[\mathfrak{b}, o] = 0$ *if and only if*

$$\alpha[\mathfrak{b}] = 0. \quad \square$$

Indeed, the following theorem will play a relevant role throughout the book.

Theorem 15.2.31 *For every* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, we obtain the* distinguished observer *and the* distinguished potential*, which are determined only by* Ψ*, (see the above Proposition 15.2.29)*

$$o_\Psi \equiv o[\Psi] := o\big[\mathfrak{b}_\Psi\big] \in \sec(\boldsymbol{E}, J_1\boldsymbol{E}),$$
$$A_\Psi \equiv A[\Psi] := A\big[\mathfrak{b}_\Psi\big] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}).$$

Indeed, with reference to any quantum basis $\mathfrak{b}$ *and any observer o, we have the coordinates expressions*

$$o^i_0[\Psi] = G^{ij}_0\, (\partial_j\varphi - A_j),$$
$$A[\Psi] = -\big((\partial_0\varphi - A_0\big) + \tfrac{1}{2}\, G^{ij}_0\, (\partial_i\varphi - A_i)\, (\partial_j\varphi - A_j)\big)\, d^0.$$

In particular, with reference to the quantum basis $\mathfrak{b}_\Psi$ *and any observer o, we obtain the equalities*

$$o[\Psi] = o - G^\sharp\big(A\big[\mathfrak{b}_\Psi, o\big]\big) \quad \textit{and} \quad A[\Psi] = \alpha[\mathfrak{b}_\Psi]. \quad \square$$

Remark 15.2.32 Given a proper quantum section Ψ, we are led to regard $o[\Psi]$ as the distinguished *observer at rest* with respect to Ψ and $A[\Psi]$ as the distinguished *observed potential "seen"* by Ψ, regardless of any arbitrary gauge. □

Corollary 15.2.33 *Let us consider a quantum particle, with mass* $m \in \mathbb{M}$ *and charge* $q \in \mathbb{Q}$*, described by a proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*. Moreover, let us consider a generic classical particle, with the same mass and charge.*

Then, we obtain the following expressions of the distinguished phase maps $\mathcal{H}$, $\mathcal{P}$ and $\mathcal{C}$ related to such a generic particle and referred to the distinguished quantum gauge b_Ψ *and the distinguished observer* o_Ψ *(see Theorem 10.1.8 and Definition 3.2.9)*

$$\mathcal{H}[\mathrm{b}_\Psi, o_\psi] = \mathcal{K}[o_\Psi] - A[\Psi], \quad \mathcal{P}[\mathrm{b}_\Psi, o_\psi] = \mathcal{Q}[o_\Psi], \quad \mathcal{C}[o_\Psi] = A^\uparrow[\mathrm{b}_\Psi] - A[\Psi]. \quad \square$$

Later, we shall further discuss the above maps in Corollary 18.1.10.

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the invariants of a plane wave.

Example 15.2.34 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda\, x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}, \;\; |\psi| \neq 0.$$

Then, we obtain

$$\begin{aligned} o[\Psi] &= u^0 \otimes \big(\partial_0 + \tfrac{\hbar_0}{m}\, g^{ij}\,(\mathrm{k}_j - A_j)\,\partial_i\big), \\ A[\Psi] &= \big(A_0 - \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\, g^{ij}\,(\mathrm{k}_i - A_i)\,(\mathrm{k}_j - A_j)\big)\, d^0. \quad \square \end{aligned}$$

15.3 Upper Quantum Connection Over Time

The quantum space $\boldsymbol{Q}$ is not only a bundle over the spacetime $\boldsymbol{E}$, but it turns out to be also a fibred manifold $\boldsymbol{Q} \to \boldsymbol{T}$ over time, via the double fibred manifold $\boldsymbol{Q} \to \boldsymbol{E} \to \boldsymbol{T}$.

Analogously, the upper quantum bundle $\boldsymbol{Q}^\uparrow$ is not only a bundle over the classical phase space $J_1\boldsymbol{E}$, but it turns out to be also a fibred manifold $\boldsymbol{Q}^\uparrow \to \boldsymbol{T}$ over time, via the triple fibred manifold $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E} \to \boldsymbol{E} \to \boldsymbol{T}$.

Now, we have two distinguished connections (see Definitions 9.1.2 and 15.2.1):

- the *upper quantum connection* $\text{Ч}^\uparrow$ of the upper quantum bundle $\boldsymbol{Q}^\uparrow \to J_1\boldsymbol{E}$,
- the *dynamical phase connection* γ of the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$.

Actually, the composition of these two connections naturally yields a connection $\underline{\text{Ч}}^\uparrow : \boldsymbol{Q}^\uparrow \to \mathbb{T}^* \otimes T\boldsymbol{Q}^\uparrow$ of the fibred manifold $\boldsymbol{Q}^\uparrow \to \boldsymbol{T}$.

It is remarkable that this composed connection $\underline{\text{Ч}}^\uparrow$ turns out to be the natural object suitable to define the "Feynman amplitudes" arising in the Feynman path integral (see Sect. 23.2).

Theorem 15.3.1 *The galilean upper quantum connection*

$$\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \underset{J_1\boldsymbol{E}}{\times} TJ_1\boldsymbol{E} \to TJ_1\boldsymbol{E}$$

and the dynamical phase connection

$$\gamma : J_1 \boldsymbol{E} \underset{\boldsymbol{T}}{\times} T\boldsymbol{T} \to T J_1 \boldsymbol{E}$$

yield the non linear connection of the fibred manifold $\boldsymbol{Q}^\uparrow \to \boldsymbol{T}$*,*

$$\underline{\text{Ч}}^\uparrow := \gamma \lrcorner \text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to \mathbb{T}^* \otimes T\boldsymbol{Q}^\uparrow,$$

via the composition

$$\begin{array}{ccccc} \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} (J_1 \boldsymbol{E} \times T\boldsymbol{T}) & \xrightarrow{\mathrm{id}_{\boldsymbol{Q}} \times \gamma} & \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T J_1 \boldsymbol{E} & \xrightarrow{\text{Ч}^\uparrow} & T\boldsymbol{Q}^\uparrow \\ \mathrm{id} \downarrow & & & & \uparrow \mathrm{id} \\ \boldsymbol{Q}^\uparrow \times T\boldsymbol{T} & & \xrightarrow{\underline{\text{Ч}}^\uparrow} & & T\boldsymbol{Q}^\uparrow \end{array} .$$

With reference to a quantum basis $\mathfrak{b}$*, its expression is*

$$\underline{\text{Ч}}^\uparrow = u^0 \otimes \big(\gamma_0 \lrcorner \chi[\mathfrak{b}] + \mathrm{i}\, \mathcal{L}_0[\mathfrak{b}]\, \mathbb{I}^\uparrow\big),$$

i.e., in coordinates,

$$\underline{\text{Ч}}^\uparrow = u^0 \otimes \big(\partial_0 + x_0^i\, \partial_i + \gamma_{0\,0}^{\ i}\, \partial_i^0 + \mathrm{i}\, \mathcal{L}_0\, \mathbb{I}^\uparrow\big),$$

where

$$\mathcal{L}_0 = \tfrac{1}{2}\, G_{ij}^0\, x_0^i\, x_0^j + A_j\, x_0^j + A_0. \quad \square$$

Definition 15.3.2 The connection

$$\underline{\text{Ч}}^\uparrow := \gamma \lrcorner \text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to \mathbb{T}^* \otimes T\boldsymbol{Q}^\uparrow$$

of the fibred manifold $\boldsymbol{Q}^\uparrow \to \boldsymbol{T}$ is called *upper quantum connection over time*. □

Remark 15.3.3 It is remarkable that the classical lagrangian $\mathcal{L}$ appears in the above expression of the upper quantum connection over time $\underline{\text{Ч}}^\uparrow$.

This fact shows a direct link between the classical and quantum dynamics, in agreement with the approach to Quantum Mechanics based on the Feynman path integral (see Theorem 23.2.2). □

We need a preliminary technical observation.

Lemma 15.3.4 *The* $\boldsymbol{T}$*-vertical subbundle* $V_{\boldsymbol{T}}\boldsymbol{Q}^\uparrow \subset T\boldsymbol{Q}^\uparrow$ *splits as*

$$V_{\boldsymbol{T}}\, \boldsymbol{Q}^{\uparrow} = V_{\boldsymbol{T}}\, J_1 \boldsymbol{E} \underset{V\boldsymbol{E}}{\times} V_{\boldsymbol{T}}\, \boldsymbol{Q},$$

and the $\boldsymbol{E}$*-vertical subbundle* $V_{\boldsymbol{E}}\, \boldsymbol{Q}^{\uparrow} \subset V_{\boldsymbol{T}}\, \boldsymbol{Q}^{\uparrow}$ *splits as*

$$V_{\boldsymbol{E}}\, \boldsymbol{Q}^{\uparrow} = \boldsymbol{Q}^{\uparrow} \underset{\boldsymbol{E}}{\times} \big((\mathbb{T}^* \otimes V\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}\big). \quad \square$$

Now, let us consider a section $\underline{\Psi} \in \sec(\boldsymbol{T}, \boldsymbol{Q})$, along with the associated motion

$$s := \pi \circ \underline{\Psi} \in \sec(\boldsymbol{T}, \boldsymbol{E})$$

and the associated natural lift to the upper quantum bundle

$$\underline{\Psi}^{\uparrow} := (j_1 s, \underline{\Psi}) \in \sec(\boldsymbol{T}, \boldsymbol{Q}^{\uparrow}),$$

according to the following commutative diagram

$$\begin{array}{ccc}
J_1\boldsymbol{E} & \xrightarrow{\ \underline{\Psi}^{\uparrow}\ } & \boldsymbol{Q}^{\uparrow} \\
{\scriptstyle j_1 s}\uparrow & & \downarrow \\
\boldsymbol{T} & \xrightarrow{\ \underline{\Psi}\ } & \boldsymbol{Q} \\
{\scriptstyle \mathrm{id}_{\boldsymbol{T}}}\uparrow & & \downarrow{\scriptstyle \pi} \\
\boldsymbol{T} & \xrightarrow{\ s\ } & \boldsymbol{E}
\end{array} \quad .$$

Lemma 15.3.5 *The* upper quantum covariant differential *of* $\underline{\Psi}$*, with respect to the connection* $\underline{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^* \otimes T\,\boldsymbol{Q}^{\uparrow}$ *is the section*

$$\underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow} \in \sec(\boldsymbol{T}, \mathbb{T}^* \otimes V_{\boldsymbol{T}}\, \boldsymbol{Q}^{\uparrow}),$$

which factorises as a section

$$\underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow} \in \sec(\boldsymbol{T}, \mathbb{T}^* \otimes V_{\boldsymbol{E}}\, \boldsymbol{Q}^{\uparrow}),$$

according to the following commutative diagram

$$\begin{array}{ccc}
\boldsymbol{T} & \xrightarrow{\ \underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow}\ } & \mathbb{T}^* \otimes V_{\boldsymbol{T}}\, \boldsymbol{Q}^{\uparrow} \\
{\scriptstyle \mathrm{id}}\downarrow & & \uparrow{\scriptstyle \cup} \\
\boldsymbol{T} & \xrightarrow{\ \underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow}\ } & \mathbb{T}^* \otimes V_{\boldsymbol{E}}\, \boldsymbol{Q}^{\uparrow}.
\end{array} \quad \square$$

Now, let us consider the natural splitting

$$\mathbb{T}^* \otimes (V_{\boldsymbol{E}}\, \boldsymbol{Q}^{\uparrow}) = \mathbb{T}^* \otimes (V_{\boldsymbol{E}}\, J_1 \boldsymbol{E} \underset{\boldsymbol{E}}{\times} V_{\boldsymbol{E}}\, \boldsymbol{Q}) \;\simeq\; \big((\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V \boldsymbol{E}\big) \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes \boldsymbol{Q}).$$

Proposition 15.3.6 *According to the above Lemmas 15.3.4 and 15.3.5 and the above bundle splitting, the upper quantum covariant differential* $\underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow}$ *splits into two components as*

$$\underline{\nabla}^{\uparrow}\underline{\Psi}^{\uparrow} = \big(\nabla j_1 s,\; \underline{\nabla\Psi}\big),$$

where

$$\nabla j_1 s \in \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \mathbb{T}^* \otimes V \boldsymbol{E}) \quad \text{and} \quad \underline{\nabla\Psi} \in \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

Actually, the 1st component turns out to be just the covariant differential induced by γ *(see Definition 9.1.2)*

$$\underline{\nabla} j_1 s = \nabla[\gamma] j_1 s$$

and the 2nd component has coordinate expression

$$\underline{\nabla\Psi} = \big(\partial_0 \underline{\psi} - \mathfrak{i}\,(\mathcal{L}_0 \circ j_1 s)\,\underline{\psi}\big)\,(\mathfrak{b} \circ s),$$

where $\underline{\psi} := z \circ \underline{\Psi} \in \mathrm{map}(\boldsymbol{T}, \mathbb{C})$. □

Exercise 15.3.7 The above equality

$$\underline{\nabla\Psi} = \big(\partial_0 \underline{\psi} - \mathfrak{i}\,(\mathcal{L}_0 \circ j_1 s)\,\underline{\psi}\big)\,(\mathfrak{b} \circ s)$$

is gauge invariant because it has been derived by an intrinsic procedure.

It is instructive to check explicitly its invariance as follows.

Let us consider to quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} := \exp(\mathfrak{i}\,\vartheta)\,\mathfrak{b}$, with $\vartheta \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$.

Then, the transition rule $A[\acute{\mathfrak{b}}] = A[\mathfrak{b}] - d\vartheta$ yields

$$\mathcal{L}_0[\acute{\mathfrak{b}}] \circ j_1 s = \mathcal{L}_0[\mathfrak{b}] \circ j_1 s - ds.\vartheta$$

and the transition rule $\underline{\acute{\psi}} = \exp(-\mathfrak{i}\,\vartheta)\,\underline{\psi}$ yields

$$\partial_0 \underline{\acute{\psi}} = \exp(-\mathfrak{i}\,\vartheta)\,\big(\partial_0 \underline{\psi} - (\mathfrak{i}\, ds.\vartheta)\,\underline{\psi}\big).$$

Hence, we obtain

$$\begin{aligned}
\underline{\nabla\Psi} &= \big(\partial_0 \underline{\acute{\psi}} - \mathfrak{i}\,(\mathcal{L}_0[\acute{\mathfrak{b}}] \circ j_1 s)\,\underline{\acute{\psi}}\big)\,(\acute{\mathfrak{b}} \circ s) \\
&= \exp(-\mathfrak{i}\,\vartheta)\,\Big(\big(\partial_0 \underline{\psi} - (\mathfrak{i}\, ds.\vartheta)\,\underline{\psi}\big) - \mathfrak{i}\,(\mathcal{L}_0[\mathfrak{b}] \circ j_1 s)\,\underline{\psi} + (\mathfrak{i}\, ds.\vartheta)\,\underline{\psi}\big)\Big)\,(\acute{\mathfrak{b}} \circ s) \\
&= \big(\partial_0 \underline{\psi} - \mathfrak{i}\,(\mathcal{L}_0 \circ j_1 s)\,\underline{\psi}\big)\,(\mathfrak{b} \circ s). \quad \square
\end{aligned}$$

Chapter 16
Quantum Differentials

We discuss in detail the covariant differentials induced by the galilean upper quantum connection and the associated system of observed quantum connections: the *1st order observed quantum covariant differential*, the *1st order observed phase quantum covariant differential* (Sect. 16.1), the *2nd order observed quantum covariant differential*, the *2nd order observed phase quantum covariant differential* (Sect. 16.2), the *observed quantum laplacian*, the *observed phase quantum laplacian* (Sect. 16.3), the *upper quantum covariant differential* and the *upper phase quantum covariant differential* (Sect. 16.4).

On the way, we obtain relevant interpretations of the distinguished observed potential $A[\Psi]$ and upper potential $A^{\uparrow}[\mathfrak{b}_{\Psi}]$ through the above quantum differentials (see Theorem 15.2.31 and Note 14.6.3).

16.1 1st Order Quantum Covariant Differentials

We discuss the 1st order *observed quantum covariant differential* $\nabla[o]\Psi$ and the 1st order *observed phase quantum covariant differential* $\nabla^{(())}[o](\!(\Psi)\!)$ and the real polar splitting of $\nabla[o]\Psi$.

We stress that the quantum phase $(\!(\Psi)\!)$ and the 1st order observed phase quantum covariant differential $\nabla^{(())}[o](\!(\Psi)\!)$ are purely real objects, according to our viewpoint on real degrees of freedom of quantum particles (see Proposition 14.7.1).

16.1.1 1st Observed Quantum Covariant Differential

We start by discussing the 1st order *observed quantum covariant differential* $\nabla[o]\Psi$ of a quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$.

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In particular, for each proper quantum section Ψ, we obtain the distinguished splitting $\nabla[o_\Psi]\Psi = \big(d|\psi| - \mathrm{i}\, A[\Psi]\big) \otimes \mathrm{b}_\Psi$, with reference to the distinguished observer o_Ψ.

Let us consider an observer o along with the associated observed quantum connection $\text{Ч}[o] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ (see Corollary 15.2.6).

Proposition 16.1.1 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the* 1st order observed quantum covariant differential, *with respect to the observed quantum connection* $\text{Ч}[o]$, *turns out to be the gauge independent section*

$$\nabla[o]\Psi := \nabla\big[\text{Ч}[o]\big]\Psi \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes \boldsymbol{Q}),$$

with coordinate expression

$$\nabla[o]\Psi = \nabla_\lambda\psi\, d^\lambda \otimes \mathrm{b}, \quad \textit{where } \nabla_\lambda\psi := \partial_\lambda\psi - \mathrm{i}\, A_\lambda\, \psi. \quad \square$$

Remark 16.1.2 We stress that in the classical theory we have already used the symbol $\nabla[o]$, for denoting the covariant differential of spacetime motions $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$ with respect to the connection o of the fibred manifold $\boldsymbol{E} \to \boldsymbol{T}$ (see Definition 2.7.7).

Here, we use the same symbol $\nabla[o]$, as a simplified version of the symbol $\nabla\big[\text{Ч}[o]\big]$, for denoting the covariant differential of quantum sections $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, with respect to the observed quantum connection $\text{Ч}[o]$ of the bundle $\boldsymbol{Q} \to \boldsymbol{E}$.

Indeed, there is no relation between the two covariant differentials above. Although the same symbol is used in two quite different contexts, in practice, there is no risk of confusion. $\square$

Definition 16.1.3 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and each observer o, we obtain the scaled spacelike section

$$\vec{\nabla}[o]\Psi := G^\sharp \circ \check{\nabla}[o]\Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E} \otimes \boldsymbol{Q}),$$

with coordinate expression

$$\vec{\nabla}[o]\Psi = \vec{\nabla}^i_0\psi\, u^0 \otimes \partial_i \otimes \mathrm{b}, \quad \text{where } \vec{\nabla}^i_0\,\psi := G^{ij}_0\, \partial_j\psi - \mathrm{i}\, A^i_0\, \psi. \quad \square$$

Proposition 16.1.4 *Given two observers o and $\acute{o} = o + v$, we obtain the following transition rule*

$$\nabla[\acute{o}] = \nabla[o] + \mathrm{i}\,\big(\tfrac{1}{2}\, G\,(v, v) - \theta[o] \lrcorner\, G^\flat(v)\big).$$

Proof. The proof follows from the transition rule (see Theorem 15.2.26)

$$A[\mathrm{b}, \acute{o}] = A[\mathrm{b}, o] + \theta[o] \lrcorner\, G^\flat(v) - \tfrac{1}{2}\, G(v, v). \quad \square$$

The following equality emphasises once more the physical meaning of the distinguished observer $o[\Psi]$ and of the distinguished observed potential $A[\Psi]$ (see Theorem 15.2.31).

Proposition 16.1.5 *For each proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* 1st order observed quantum covariant differential, *with respect to the distinguished observed quantum connection* $Ч[o_\Psi]$, *turns out to be the section*

$$\nabla[o_\Psi]\Psi = \big(d|\psi| - \mathrm{i}\, A[\Psi]\, |\psi|\big) \otimes \mathrm{b}_\Psi ,$$

with coordinate expression

$$\nabla[o_\Psi]\Psi = \big((\partial_0|\psi| - \mathrm{i}\, A_0\, |\psi|)\, d^0 + \ \partial_i |\psi|\, d^i\big) \otimes \mathrm{b}_\Psi .$$

Proof. The equality $\nabla[o_\Psi]\Psi = (d\psi - \mathrm{i}\, A[\mathrm{b}, o_\Psi]\, \psi)\, d^\lambda \otimes \mathrm{b}$ is gauge independent. Hence, we can compute it with reference to any quantum basis b. In particular, with reference to the distinguished quantum basis b_Ψ, we have $\psi = |\psi|$ and $A[\mathrm{b}_\Psi, o_\Psi] = A[\Psi]$, hence

$$\nabla[o_\Psi]\Psi = \big(d|\psi| - \mathrm{i}\, A[\Psi]|\psi|\big) \otimes \mathrm{b}_\Psi . \quad \square$$

Exercise 16.1.6 By definition, the section $\nabla[o]\Psi$ is gauge independent.

It is instructive to check this fact by means of the transition rule stated by Theorem 15.2.26. In fact, if $\acute{\mathrm{b}} = \exp(\mathrm{i}\,\vartheta)\, \mathrm{b}$ is another quantum basis, then we have

$$\begin{aligned}(d\acute{\psi} - \mathrm{i}\, \acute{A}\, \acute{\psi}) \otimes \acute{\mathrm{b}} &= \exp(-\mathrm{i}\,\vartheta)\, (-\mathrm{i}\, d\vartheta\, \psi + d\psi) \otimes \big((\exp(\mathrm{i}\,\vartheta)\, \mathrm{b}\big) \\ &\quad - \exp(-\mathrm{i}\,\vartheta)\, \mathrm{i}\, \big((A - d\vartheta)\, \psi\big) \otimes \big((\exp(\mathrm{i}\,\vartheta)\, \mathrm{b}\big) \\ &= (d\psi - \mathrm{i}\, A\, \psi) \otimes \mathrm{b}. \quad \square\end{aligned}$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the 1st order observed quantum differential of a plane wave.

Example 16.1.7 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with} \ \ \varphi = \mathrm{k}_\lambda\, x^\lambda, \ \mathrm{k}_\lambda \in \mathbb{R}, \ |\psi| \neq 0.$$

Then, we obtain

$$\nabla[o]\Psi = \big(\partial_\lambda \log|\psi| + \mathrm{i}\, (\mathrm{k}_\lambda - A_\lambda)\big)\, d^\lambda \otimes \Psi. \quad \square$$

16.1.2 1st Observed Quantum Covariant Differential of Quantum Bases

Given an observer o and a quantum basis $\mathfrak{b}$, we show the equality $\nabla[o]\mathfrak{b} = -\mathfrak{i}\, A[\mathfrak{b}, o] \otimes \mathfrak{b}$ and prove that there exists a quantum basis $\mathfrak{b}$ such that $\nabla[o]\mathfrak{b} = 0$ if and only if $\Phi[o] = 0$.

In particular, for each proper quantum section Ψ, we show the distinguished equality $\nabla[o_\Psi]\mathfrak{b}_\Psi = -\mathfrak{i}\, A[\Psi] \otimes \mathfrak{b}_\Psi$, with reference to the distinguished observer o_Ψ and the distinguished quantum basis $\mathfrak{b}_\Psi$ (see Theorem 15.2.31).

Proposition 16.1.8 *Let us consider a quantum basis* $\mathfrak{b}$ *and an observer* o.

Then, we obtain the scaled complex linear quantum 1-form

$$\nabla[o]\mathfrak{b} = -\mathfrak{i}\, A[\mathfrak{b}, o] \otimes \mathfrak{b} \in \sec\big(\boldsymbol{Q},\, \mathbb{L}^{3/2} \otimes (T^*\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$

with coordinate expression

$$\nabla[o]\mathfrak{b} = -\mathfrak{i}\, A_\lambda\, d^\lambda \otimes \mathfrak{b}. \quad \square$$

Corollary 16.1.9 *Let us consider a quantum basis* $\mathfrak{b}$. *If for a certain observer* o *we have*

$$\nabla[o]\mathfrak{b} = 0,$$

then, for any other observer $\acute{o} = o + v$, *we have*

$$\nabla[\acute{o}]\mathfrak{b} = \mathfrak{i}\,\big(\tfrac{1}{2}\, G(v, v) - \theta[o] \lrcorner G^\flat(v)\big) \otimes \mathfrak{b}.$$

Proof. It follows from the above Proposition 16.1.8 and from Theorem 15.2.26 and Proposition 16.1.4. $\square$

Corollary 16.1.10 *Let us consider an observer* o. *For each quantum basis* $\mathfrak{b}$, *the observed covariant exterior differential of the observed covariant differential of* $\mathfrak{b}$ *turns out to be the gauge independent vector valued form (see Theorem 4.3.3 and, Appendix: Note F.2.4)*

$$d_{\text{Ч}[o]}\nabla[o]\mathfrak{b} = -\mathfrak{i}\, d\, A[\mathfrak{b}, o] \otimes \mathfrak{b} = -\tfrac{1}{2}\,\mathfrak{i}\,\Phi[o] \otimes \mathfrak{b}. \quad \square$$

Proposition 16.1.11 *Let us consider an observer* o.

Then, there exists (locally) a quantum basis $\mathfrak{b}$ *such that* $\nabla[o]\mathfrak{b} = 0$ *if and only if* $\Phi[o] = 0$.

Proof. (1) Let $\Phi[o] = 0$. Then, in virtue of Corollary 15.2.9, we have $R\big[\text{Ч}[o]\big] = 0$.

Further, let us consider an element $\mathfrak{b}_e \in \boldsymbol{Q}_e$, such that $\mathfrak{h}(\mathfrak{b}_e, \mathfrak{b}_e) = 1$. Then, in virtue of Proposition F.1.12, in a neighbourhood of $e \in \boldsymbol{E}$, there exists locally a section $\mathfrak{b} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, such that $\nabla[o]\mathfrak{b} = 0$ and $\mathfrak{b}(e) = \mathfrak{b}_e$.

Being $\mathfrak{h}$ hermitian, for any vector field $X \in \sec(\boldsymbol{E}, T\boldsymbol{E})$, we have

$$\nabla_X\big(\mathfrak{h}(\mathfrak{b}, \mathfrak{b})\big) = \mathfrak{h}(\nabla_X[o]\mathfrak{b}, \mathfrak{b}) + \mathfrak{h}(\mathfrak{b}, \nabla_X[o]\mathfrak{b}) = 0.$$

Hence, we have locally $\mathfrak{h}(\mathfrak{b}, \mathfrak{b}) = 1$.

Thus, if $\Phi[o] = 0$ then there exists locally a quantum section $\mathfrak{b} \in \mathrm{her}(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ such that $\nabla[o]\mathfrak{b} = 0$.

(2) If $\nabla[o]\mathfrak{b} = 0$, then, in virtue of the above Proposition 16.1.8, we have

$$0 = d_{\text{Ч}[o]}\nabla[o]\mathfrak{b} = -\mathfrak{i}\, d\, A[\mathfrak{b}, o] \otimes \mathfrak{b} = -\tfrac{1}{2}\,\mathfrak{i}\,\Phi[o] \otimes \mathfrak{b},$$

hence $\Phi[o] = 0$. □

The following equality emphasises once more the physical meaning of the distinguished observer $o[\Psi]$ and of the distinguished observed potential $A[\Psi]$ (see Theorem 15.2.31).

Proposition 16.1.12 *Let us consider a proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ *and the associated quantum basis* $\mathfrak{b}_\Psi$. *Then, we have the equality*

$$\nabla[o_\Psi]\mathfrak{b}_\Psi = -\mathfrak{i}\, A[\Psi] \otimes \mathfrak{b}_\Psi,$$

with coordinate expression (see Definition 14.10.1)

$$\nabla[o_\Psi]\mathfrak{b}_\Psi = -\mathfrak{i}\, A_0\, d^0 \otimes \mathfrak{b}_\Psi. \quad \square$$

16.1.3 1st Observed Phase Quantum Covariant Differential

In order to emphasise the quantum phase as a real degree of freedom of the quantum particle, we analyse the *1st order observed phase quantum covariant differential* $\nabla^{(\!)}[o]\langle\!\langle\Psi\rangle\!\rangle$ of a proper quantum section Ψ.

In particular, with reference to the distinguished observer o_Ψ, we have the distinguished equality $\nabla^{(\!)}[o_\Psi]\langle\!\langle\Psi\rangle\!\rangle = -A[\Psi]$ (see Theorem 15.2.31).

Indeed, the above equality emphasises a further physical meaning of the distinguished potential $A[\Psi]$ "*seen*" by the proper quantum section Ψ.

Lemma 16.1.13 *For each observer* o, *the observed quantum connection* $\text{Ч}[o]$ *yields, in a covariant way, the real linear connection of the phase quantum bundle* $\pi^{(\!)} : \boldsymbol{Q}^{(\!)}_{/0} \to \boldsymbol{E}$, *called* observed phase quantum connection *(see Proposition 14.7.1)*

$$\text{Ч}^{(\!)}[o] : \boldsymbol{Q}^{(\!)}_{/0} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}^{(\!)}_{/0}$$

through the commutative diagram

$$\begin{array}{ccc} \boldsymbol{Q}_{/0} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\text{Ч}[o]} & T\boldsymbol{Q}_{/0} \\ {\scriptstyle i^{\circledcirc} \times \mathrm{id}_{T\boldsymbol{E}}} \uparrow & & \downarrow {\scriptstyle Tp^{\circledcirc}} \\ \boldsymbol{Q}^{\circledcirc}_{/0} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\text{Ч}^{\circledcirc}[o]} & T\boldsymbol{Q}^{\circledcirc}_{/0} \end{array} . \square$$

Proposition 16.1.14 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* 1st order observed phase quantum covariant differential*, with respect to the observed phase quantum connection* $\text{Ч}^{\circledcirc}[o]$, *is the (gauge independent) spacetime form*

$$\nabla^{\circledcirc}[o]((\Psi)) \in \sec(\boldsymbol{E}, T^*\boldsymbol{E}),$$

with coordinate expression

$$\nabla^{\circledcirc}[o]((\Psi)) = \nabla^{\circledcirc}_{\lambda} \varphi \, d^{\lambda}, \qquad \textit{where } \nabla^{\circledcirc}_{\lambda} \varphi := \partial_{\lambda}\varphi - A_{\lambda}. \quad \square$$

After having introduced of the 1st order observed phase quantum covariant differential $\nabla^{\circledcirc}[o]((\Psi))$, the main formula of Theorem 15.2.31 can be rewritten in the following interesting way, which involves only the phase component $((\Psi))$ of the proper quantum section Ψ.

Corollary 16.1.15 *Let us consider a proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$.

Then, the distinguished observer and the distinguished potential introduced in Theorem 15.2.31 can be expressed, for each observer o, *by the equalities*

$$\begin{aligned} o[\Psi] &= o - \overrightarrow{\nabla}^{\circledcirc}[o]((\Psi)), \\ A[\Psi] &= -\text{д}[o] \lrcorner \nabla^{\circledcirc}[o]((\Psi)) - \tfrac{1}{2} \, G\Big(\big(\overrightarrow{\nabla}^{\circledcirc}[o]((\Psi))\big), \big(\overrightarrow{\nabla}^{\circledcirc}[o]((\Psi))\big)\Big). \quad \square \end{aligned}$$

The above Proposition 16.1.14 yields, as consequence, a further relevant interpretation of the distinguished quantum potential $A[\Psi]$ exhibited in Theorem 15.2.31.

Note 16.1.16 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, with reference to the two observers o and $\acute{o} = o + v$, we obtain the transition rule

$$\nabla^{\circledcirc}[\acute{o}]((\Psi)) = \nabla^{\circledcirc}[o]((\Psi)) - \theta[o] \lrcorner G^{\flat}(v) + \tfrac{1}{2} \, G(v, v).$$

Proof. The proof follows from the equalities (see Theorem 15.2.26)

$$\nabla^{\circledcirc}[o]((\Psi)) = d\varphi - A[\mathrm{b}, o] \quad \text{and} \quad A[\mathrm{b}, \acute{o}] = A[\mathrm{b}, o] + \theta[o] \lrcorner G^{\flat}(v) - \tfrac{1}{2} \, G(v, v). \quad \square$$

As a particular case of the above Proposition 16.1.14, we obtain the following relevant intrinsic characterisation of the distinguished potential $A[\Psi]$ through the phase covariant quantum differential (see Theorem 15.2.31).

Proposition 16.1.17 *With reference to the distinguished observer o_Ψ, we obtain the equality*

$$\nabla^{\circledcirc}[o_\Psi](\!(\Psi)\!) = -A[\Psi].$$

Proof. The equality $\nabla^{\circledcirc}[o_\Psi](\!(\Psi)\!) = (\partial_\lambda \varphi - A_\lambda)\, d^\lambda$ is gauge independent. Hence, we can compute it with reference to any quantum basis b. In particular, with reference to the distinguished quantum basis b_Ψ, we have $\varphi = 0$ and $A[\mathrm{b}_\Psi, o_\Psi] = A[\Psi]$, $\nabla^{\circledcirc}[o_\Psi](\!(\Psi)\!) = -A[\Psi]$. □

Exercise 16.1.18 By definition, the section $\nabla^{\circledcirc}[o](\!(\Psi)\!)$ is gauge independent.

It is instructive to check this fact by means of the transition rule stated by Theorem 15.2.26. In fact, if $\acute{\mathrm{b}} = \exp(\mathfrak{i}\,\vartheta)\,\mathrm{b}$ is another quantum basis, then we have

$$\begin{aligned} d(\varphi[\Psi, \acute{\mathrm{b}}]) - A[o, \acute{\mathrm{b}}] &= \big(d(\varphi[\Psi, \mathrm{b}]) - d\vartheta\big) - (A[o, \mathrm{b}] - d\vartheta) \\ &= d(\varphi[\Psi, \mathrm{b}]) - A[o, \mathrm{b}]. \quad \square \end{aligned}$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the 1st order observed phase quantum covariant differential of a plane wave.

Example 16.1.19 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathfrak{i}\,\varphi}\, \mathrm{b}, \quad \text{with} \ \ \varphi = \mathrm{k}_\lambda\, x^\lambda, \ \ \mathrm{k}_\lambda \in \mathbb{R}, \ \ |\psi| \neq 0.$$

Then, we obtain

$$\nabla^{\circledcirc}[o](\!(\Psi)\!) = \mathrm{k} - A[\mathrm{b}, o], \quad \text{where} \ \ \mathrm{k} = \mathrm{k}_\lambda\, d^\lambda. \quad \square$$

16.1.4 Polar Splitting of 1st Observed Quantum Differential

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and find the equality $\nabla[o]\Psi = \big(d \log \|\Psi\| + \mathfrak{i}\, \nabla^{\circledcirc}[o](\!(\Psi)\!)\big) \otimes \Psi$.

In particular, with reference to the distinguished observer o_Ψ, we have the distinguished equality $\nabla[o_\Psi]\Psi = (d \log \|\Psi\| - \mathfrak{i}\, A[\Psi]) \otimes \Psi$ (see Theorem 15.2.31).

Indeed, the above equality emphasises a further physical meaning of the distinguished potential $A[\Psi] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$ "*seen*" by the proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$.

Lemma 16.1.20 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we can define the (gauge independent) complex spacetime form given by the ratio*

$$\nabla[o]\Psi / \Psi \in \sec(\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E}),$$

by taking into account the fact that the complex fibres of the quantum bundle are 1-dimensional. Moreover, the above ratio characterises $\nabla[o]\Psi$ through the equality

$$\nabla[o]\Psi = (\nabla[o]\Psi / \Psi)\, \Psi.$$

Indeed, the following equality holds

$$\nabla[o]\Psi / \Psi = \mathfrak{h}\big(\Psi,\ \nabla[o]\Psi\big) / \|\Psi\|^2.$$

Proof. We have

$$\frac{\mathfrak{h}\big(\Psi,\ \nabla[o]\Psi\big)}{\|\Psi\|^2} = \mathfrak{h}\left(\Psi,\ \frac{\nabla[o]\Psi}{\Psi}\ \Psi\right) / \|\Psi\|^2 = \frac{\nabla[o]\Psi}{\Psi}\ \frac{\|\Psi\|^2}{\|\Psi\|^2} = \frac{\nabla[o]\Psi}{\Psi}\,. \quad \square$$

Proposition 16.1.21 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the (gauge independent) real and imaginary components of $\nabla[o]\Psi / \Psi$ can be expressed through the polar components of Ψ as follows (see Remark K.2.8)*

$$\mathrm{re}\,(\nabla[o]\Psi / \Psi) = d \log \|\Psi\| \qquad \textit{and} \qquad \mathrm{im}\,(\nabla[o]\Psi / \Psi) = \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!).$$

We have the coordinate expression

$$\nabla[o]\Psi / \Psi = \big(\partial_\lambda(\log|\psi|) + \mathfrak{i}\,(\partial_\lambda \varphi - A_\lambda)\big)\, d^\lambda,$$

where

$$\partial_\lambda(\log|\psi|) = \tfrac{1}{2}\, \frac{\bar{\psi}\, \partial_\lambda \psi + \psi\, \partial_\lambda \bar{\psi}}{|\psi|^2} \qquad \textit{and} \qquad \partial_\lambda \varphi = \tfrac{1}{2}\, \mathfrak{i}\, \frac{\psi\, \partial_\lambda \bar{\psi} - \bar{\psi}\, \partial_\lambda \psi}{|\psi|^2}\,.$$

Hence, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the following equality holds (see Proposition 16.1.14)

$$\nabla[o]\Psi = \big(d \log \|\Psi\| + \mathfrak{i}\, \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!)\big) \otimes \Psi. \quad \square$$

Remark 16.1.22 We stress the fact that the real and the imaginary components of the complex spacetime form $\nabla[o]\Psi / \Psi$ are expressed by the covariant differentials of the hermitian norm and of the quantum phase of the quantum section, respectively.

This is a further interesting feature of the polar picture of the quantum particle.□

Corollary 16.1.23 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ and each observer o, the scaled real function (see Definition 16.1.3)*

$$(G \otimes \hbar)(\vec{\nabla}[o]\Psi, \vec{\nabla}[o]\Psi) \in \sec(\boldsymbol{E}, \mathbb{L}^{-3} \otimes \mathbb{T}^* \otimes I\!R)$$

is expressed, in polar form, by the equality

$$(G \otimes \hbar)(\vec{\nabla}[o]\Psi, \vec{\nabla}[o]\Psi) = \Big(G\,(\vec{d}\,\|\Psi\|, \vec{d}\,\|\Psi\|) + G\,(\vec{\nabla}^{\circledcirc}[o]((\Psi)), \vec{\nabla}^{\circledcirc}[o]((\Psi)))\Big) \otimes \|\Psi\|^2,$$

with coordinate expression

$$(G \otimes \hbar)(\vec{\nabla}[o]\Psi, \vec{\nabla}[o]\Psi)$$
$$= G_0^{ij}\big(\partial_i \log|\psi|\partial_j \log|\psi|\big)|\psi|^2 u^0 + G_0^{ij}\big(\partial_i\varphi\partial_j\varphi - 2\,A_i\partial_j\varphi + A_i\,A_j\big)|\psi|^2\,u^0. \quad \square$$

The following equality emphasises once more the physical meaning of the distinguished observer $o[\Psi]$ and of the distinguished observed potential $A[\Psi]$ (see Theorem 15.2.31).

Corollary 16.1.24 *For each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, with reference to the distinguished observer o_Ψ, we obtain the polar splitting*

$$\operatorname{re}(\nabla[o_\Psi]\Psi/\Psi) = d\log\|\Psi\| \qquad \textit{and} \qquad \operatorname{im}(\nabla[o_\Psi]\Psi/\Psi) = -A[\Psi].$$

Proof. The proof follows from Proposition 16.1.5. $\square$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the 1st order observed quantum covariant differential of a plane wave.

Example 16.1.25 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\,x^\lambda, \;\mathrm{k}_\lambda \in I\!R, \;|\psi| \neq 0.$$

Then, we obtain

$$\operatorname{re}(\nabla[o]\Psi/\Psi) = \partial_\lambda \log|\psi|\,d^\lambda \qquad \text{and} \qquad \operatorname{im}(\nabla[o]\Psi/\Psi) = (\mathrm{k}_\lambda - A_\lambda)\,d^\lambda. \quad \square$$

16.2 2nd Order Quantum Covariant Differentials

We discuss the 2nd order *observed quantum covariant differential* $\nabla^2[o]$ and the 2nd order *observed phase quantum covariant differential* $\nabla^{⦿2}[o]((\Psi))$, along with their polar splittings.

We stress that the quantum phase $((\Psi))$ and the 2nd order observed phase quantum covariant differential $\nabla^{⦿2}[o]((\Psi))$ are purely real objects, according to our viewpoint on real degrees of freedom of quantum particles (see Proposition 14.7.1).

16.2.1 2nd Observed Quantum Covariant Differential

With reference to the two connections $Ч[o]$ and K, we discuss the 2nd order observed quantum covariant differential $\nabla^2[o]\Psi$ of a quantum section Ψ.

Proposition 16.2.1 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the* 2nd order observed quantum covariant differential*, with respect to the observed quantum connection* $Ч[o]$ *and to the galilean spacetime connection* K, *is the (gauge independent) section*

$$\nabla^2[o]\Psi := (\nabla[K] \otimes \nabla[o])\nabla[o]\Psi \in \sec\big(\boldsymbol{E},\ (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes \boldsymbol{Q}\big),$$

with coordinate expression

$$\nabla^2[o]\Psi = \nabla_{\lambda\mu}\psi\, d^\lambda \otimes d^\mu \otimes \mathrm{b},$$

where we have set

$$\begin{aligned}\nabla_{\lambda\mu}\,\psi &= (\nabla_\lambda\nabla_\mu + K_\lambda{}^j{}_\mu\,\nabla_j)\,\psi\\ &= (\partial_\lambda - \mathfrak{i}\,A_\lambda)(\partial_\mu - \mathfrak{i}\,A_\mu)\,\psi + K_\lambda{}^j{}_\mu\,(\partial_j - \mathfrak{i}\,A_j)\,\psi\\ &= (\partial_{\lambda\mu} - \mathfrak{i}\,A_\lambda\,\partial_\mu - \mathfrak{i}\,A_\mu\,\partial_\lambda - \mathfrak{i}\,\partial_\lambda A_\mu - A_\lambda\,A_\mu)\,\psi + K_\lambda{}^j{}_\mu\,(\partial_j - \mathfrak{i}\,A_j)\,\psi.\end{aligned}\quad\square$$

Remark 16.2.2 We stress that $\nabla^2[o]\Psi$ is not symmetric. In fact, this 2nd quantum covariant differential is not related to a single torsion free connection, but to two different connections. □

Proposition 16.2.3 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* 2nd order observed quantum covariant differential*, with respect to the observed quantum connection* $Ч[o_\Psi]$ *and to the galilean spacetime connection* K, *has the coordinate expression, with reference to the distinguished quantum basis* b_Ψ,

$$\nabla^2[o_\Psi]\Psi = \nabla_{\lambda\mu}|\psi|\, d^\lambda \otimes d^\mu \otimes \mathrm{b}_\Psi,$$

where

$$\begin{aligned}
\nabla_{00}\,|\psi| &= (\partial_{00} - 2\,\mathrm{i}\,A_0\,\partial_0 - \mathrm{i}\,\partial_0 A_0 - A_0\,A_0)\,|\psi| + K_0{}^j{}_0\,\partial_j\,|\psi|,\\
\nabla_{0k}\,\psi &= (\partial_{0k} - \mathrm{i}\,A_0\,\partial_k)\,|\psi| - K_0{}^j{}_k\,\partial_j|\psi|,\\
\nabla_{h0}\,\psi &= (\partial_{h0} - \mathrm{i}\,A_0\,\partial_h - \mathrm{i}\,\partial_h A_0)\,|\psi| + K_h{}^j{}_0\,\partial_j|\psi|,\\
\nabla_{hk}\,\psi &= \partial_{hk}|\psi| + K_h{}^j{}_k\,\partial_j|\psi|.
\end{aligned}$$

Proof. Being $\nabla^2[o_\Psi]\Psi$ gauge independent, we can compute it with reference to the distinguished quantum basis b_Ψ. □

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the 2nd order observed quantum covariant differential of a plane wave.

Example 16.2.4 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi = |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\,x^\lambda,\;\; \mathrm{k}_\lambda \in \mathbb{R},\;\; |\psi| \neq 0.$$

Then, we obtain

$$\begin{aligned}
\nabla^2[o]\Psi = &\Big(\frac{\partial_{\lambda\mu}|\psi|}{|\psi|} - (\mathrm{k}_\lambda - A_\lambda)\,(\mathrm{k}_\mu - A_\mu)\Big)\,d^\lambda \otimes d^\mu \otimes \Psi\\
&+ \mathrm{i}\,\Big(\frac{\partial_\lambda|\psi|}{\psi}\,(\mathrm{k}_\mu - A_\mu) + \frac{\partial_\mu|\psi|}{\psi}\,(\mathrm{k}_\lambda - A_\lambda)\Big) - \partial_\lambda A_\mu\big)\,d^\lambda \otimes d^\mu \otimes \Psi. \quad \square
\end{aligned}$$

16.2.2 2nd Observed Phase Quantum Covariant Differential

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and study the *2nd order observed phase quantum covariant differential* $\nabla^{(\!(\!)\!)2}[o](\!(\Psi)\!)$.

In particular, with reference to the distinguished observer o_Ψ, we have the distinguished equality $\nabla^{(\!(\!)\!)2}[o_\Psi](\!(\Psi)\!) = -\nabla[K]A[\Psi]$ (see Theorem 15.2.31).

Indeed, the above equality emphasises a further physical meaning of the distinguished potential $A[\Psi]$ "*seen*" by the proper quantum section Ψ.

Proposition 16.2.5 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* 2nd order observed phase quantum covariant differential, *with respect to the observed upper quantum connection* $\mathrm{Ч}^{(\!(\!)\!)}[o]$, *is the (gauge independent) spacetime tensor (see Lemma 16.1.13 and Proposition 16.1.14)*

$$\nabla^{(\!(\!)\!)2}[o](\!(\Psi)\!) \in \sec(\boldsymbol{E},\; T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

with coordinate expression

$$\nabla^{\circledcirc 2}[o](\!(\Psi)\!) = \nabla^{\circledcirc}_{\lambda\mu}\varphi\, d^\lambda \otimes d^\mu,$$

where we have set

$$\begin{aligned}\nabla^{\circledcirc}_{\lambda\mu}\varphi &= (\partial_\lambda \nabla^{\circledcirc}_{\mu} + K_\lambda{}^j{}_\mu\, \nabla^{\circledcirc}_{j})\,\varphi \\ &= \partial_\lambda(\partial_\mu\varphi - A_\mu) + K_\lambda{}^j{}_\mu\,(\partial_j\varphi - A_j) \\ &= \partial_{\lambda\mu}\varphi - \partial_\lambda A_\mu + K_\lambda{}^j{}_\mu\,(\partial_j\varphi - A_j). \quad \square\end{aligned}$$

Remark 16.2.6 We stress that $\nabla^{\circledcirc 2}[o](\!(\Psi)\!)$ is not symmetric. In fact, this 2nd covariant differential is not related to a single torsion free connection, but to two different connections. $\square$

As a particular case of the above Proposition 16.2.5, we obtain the following relevant intrinsic characterisation of the covariant differential of the distinguished potential $A[\Psi]$ (see Theorem 15.2.31).

Proposition 16.2.7 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* 2nd observed upper quantum covariant differential*, with respect to the distinguished observed upper quantum connection* $\text{Ч}^{\circledcirc}[o_\Psi]$, *is the tensor*

$$\nabla^{\circledcirc 2}[o_\Psi](\!(\Psi)\!) = -\nabla[K]A[\Psi] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}). \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the 2nd order observed phase quantum covariant differential of a plane wave.

Example 16.2.8 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) the plane wave

$$\Psi = |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with } \ \varphi = \mathrm{k}_\lambda\, x^\lambda,\ \ \mathrm{k}_\lambda \in \mathbb{R},\ \ |\psi| \neq 0.$$

Then, we obtain

$$\nabla^{\circledcirc}_{\lambda\mu}\varphi = -\partial_\lambda A_\mu + K_\lambda{}^j{}_\mu\,(\mathrm{k}_j - A_j). \quad \square$$

16.2.3 Polar Splitting of the 2nd Quantum Differential

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and analyse the splittings of $\mathrm{re}\,(\nabla^2[o]\Psi/\Psi)$ and $\mathrm{im}\,(\nabla^2[o]\Psi/\Psi)$. In particular, we obtain remarkable splittings with reference to the distinguished observer o_Ψ.

Lemma 16.2.9 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we can define the (gauge independent) complex spacetime tensor given by the ratio*

$$\nabla^2[o]\Psi/\Psi \in \sec\big(\boldsymbol{E}, \mathbb{C} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})\big),$$

by taking into account the fact that the complex fibres of the quantum bundle are 1–dimensional. Moreover, the above ratio characterises $\nabla^2[o]\Psi$ through the equality

$$\nabla^2[o]\Psi = \big(\nabla^2[o]\Psi/\Psi\big) \otimes \Psi.$$

Indeed, the following equality holds

$$\nabla^2[o]\Psi/\Psi = \mathfrak{h}\big(\Psi,\ \nabla^2[o]\Psi\big)/\|\Psi\|^2. \quad \square$$

Proposition 16.2.10 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the (gauge independent and observer independent) real and imaginary components of $\nabla^2[o]\Psi/\Psi$ can be expressed through the polar components of Ψ as follows (see Remark K.2.8)*

$$\operatorname{re}(\nabla^2[o]\Psi/\Psi) = \nabla d \log\|\Psi\| + d\log\|\Psi\| \otimes d\log\|\Psi\| - \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!) \otimes \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!),$$
$$\operatorname{im}(\nabla^2[o]\Psi/\Psi) = \nabla^{(\!(\!)\!)2}[o](\!(\Psi)\!) + d\log\|\Psi\| \otimes \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!) + \nabla^{(\!(\!)\!)}[o](\!(\Psi)\!) \otimes d\log\|\Psi\|.$$

We have the following coordinate expression

$$\begin{aligned}
&\nabla^2[o]\Psi/\Psi \\
&\quad = \big(\partial_{\lambda\mu}\log|\psi| + \partial_\lambda\log|\psi|\,\partial_\mu\log|\psi| - \partial_\lambda\varphi\,\partial_\mu\varphi \\
&\qquad + \partial_\lambda\varphi\,A_\mu + \partial_\mu\varphi\,A_\lambda - A_\lambda\,A_\mu + K_\lambda{}^h{}_\mu\,\partial_h(\log|\psi|)\big)\,d^\lambda \otimes d^\mu \\
&\qquad + \mathfrak{i}\,\big(\partial_{\lambda\mu}\varphi + \partial_\lambda\log|\psi|\,\partial_\mu\varphi + \partial_\mu\log|\psi|\,\partial_\lambda\varphi \\
&\qquad - \partial_\lambda\log|\psi|\,A_\mu - \partial_\mu\log|\psi|\,A_\lambda - \partial_\lambda A_\mu + K_\lambda{}^h{}_\mu\,(\partial_h\varphi - A_h)\big)\,d^\lambda \otimes d^\mu,
\end{aligned}$$

where

$$\begin{aligned}
\partial_\lambda(\log|\psi|) &= \tfrac{1}{2}\,\frac{\bar\psi\,\partial_\lambda\psi + \psi\,\partial_\lambda\bar\psi}{|\psi|^2}, \\
\partial_\lambda\varphi &= \tfrac{1}{2}\,\mathfrak{i}\,\frac{\psi\,\partial_\lambda\bar\psi - \bar\psi\,\partial_\lambda\psi}{|\psi|^2}, \\
\partial_{\lambda\mu}(\log|\psi|) &= \tfrac{1}{2}\,\frac{\bar\psi\,\partial_{\mu\lambda}\psi + \psi\,\partial_{\mu\lambda}\bar\psi}{|\psi|^2} - \tfrac{1}{2}\,\frac{\bar\psi\,\bar\psi\,\partial_\mu\psi\,\partial_\lambda\psi + \psi\,\psi\,\partial_\mu\bar\psi\,\partial_\lambda\bar\psi}{|\psi|^4}, \\
\partial_{\lambda\mu}\varphi &= \tfrac{1}{2}\,\mathfrak{i}\,\frac{\psi\,\partial_{\mu\lambda}\bar\psi - \bar\psi\,\partial_{\mu\lambda}\psi}{|\psi|^2} - \tfrac{1}{2}\,\mathfrak{i}\,\frac{\psi\,\psi\,\partial_\mu\bar\psi\,\partial_\lambda\bar\psi - \bar\psi\,\bar\psi\,\partial_\mu\psi\,\partial_\lambda\psi}{|\psi|^4}. \quad \square
\end{aligned}$$

The following equality emphasises once more the physical meaning of the distinguished observer $o[\Psi]$ and of the distinguished observed potential $A[\Psi]$ (see Theorem 15.2.31).

Proposition 16.2.11 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the (gauge independent and observer independent) real and imaginary components of* $\nabla^2[o_\Psi]\Psi/\Psi$ *can be expressed through the polar components of* Ψ *as follows (see Theorem 15.2.31, Propositions 16.2.10, 16.1.17 and, Appendix: Remark K.2.8)*

$$\begin{aligned}
\mathrm{re}\,(\nabla^2[o_\Psi]\Psi/\Psi) &= \nabla d \log \|\Psi\| + d \log \|\Psi\| \otimes d \log \|\Psi\| - A[\Psi] \otimes A[\Psi], \\
\mathrm{im}\,(\nabla^2[o_\Psi]\Psi/\Psi) &= -\nabla A[\Psi] - d \log \|\Psi\| \otimes A[\Psi] - A[\Psi] \otimes d \log \|\Psi\|. \quad \square
\end{aligned}$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the polar splitting of the 2nd order quantum differential of a plane wave.

Example 16.2.12 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi = |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathsf{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\, x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}, \;\; |\psi| \neq 0.$$

Then, we obtain

$$\begin{aligned}
\mathrm{re}\,(\nabla^2[o]\Psi/\Psi) &= \frac{\partial_{\lambda\mu}|\psi|}{|\psi|} - (\mathrm{k}_\lambda - A_\lambda)\,(\mathrm{k}_\mu - A_\mu) - \partial_\lambda A_\mu + K_\lambda{}^j{}_\mu\, \frac{\partial_j|\psi|}{|\psi|}, \\
\mathrm{im}\,(\nabla^2[o]\Psi/\Psi) &= \frac{\partial_\lambda|\psi|}{\psi}\,(\mathrm{k}_\mu - A_\mu) + \frac{\partial_\mu|\psi|}{\psi}\,(\mathrm{k}_\lambda - A_\lambda) + K_\lambda{}^j{}_\mu\,(\mathrm{k}_j - A_j). \quad \square
\end{aligned}$$

16.3 Observed Quantum Laplacian

With reference to the two connections $\text{Ч}[o]$ and K, we discuss the *observed quantum covariant laplacian* $\Delta[o]\Psi := \bar{G} \lrcorner \check{\nabla}^2[o]\Psi$ of a quantum section Ψ.

In particular, for each proper quantum section Ψ, we obtain the coordinate expression

$$\Delta_0\psi = G_0^{ij}\,\partial_{ij}|\psi| + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h|\psi|,$$

with reference to the distinguished observer o_Ψ and quantum basis b_Ψ.

Definition 16.3.1 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and each observer o, the gauge independent *observed quantum laplacian* (related to the observed quantum connection $\text{Ч}[o]$ and to the spacelike metric G) is defined to be the scaled section

$$\Delta[o]\Psi := \bar{G} \lrcorner \check{\nabla}^2[o]\Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}). \quad \square$$

Proposition 16.3.2 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and each observer o, the expression of the observed quantum laplacian in terms of the scalar complex component ψ (with respect to a quantum basis b) is*

$$\Delta[G, o]\Psi = \Delta_0 \psi \, u^0 \otimes \mathrm{b},$$

where

$$\Delta_0 \psi = G_0^{ij} \, \partial_{ij} \psi + \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}} \, \partial_h \psi - 2 \, \mathrm{i} \, A_0^i \, \partial_i \psi - \mathrm{i} \, \frac{\partial_i (A_0^i \sqrt{|g|})}{\sqrt{|g|}} \, \psi - A_0^i \, A_i \, \psi.$$

Proof. The equalities (see Propositions 16.2.1 and 4.4.2)

$$\nabla_{\lambda\mu} \, \psi = (\partial_{\lambda\mu} - \mathrm{i} \, A_\lambda \, \partial_\mu - \mathrm{i} \, A_\mu \, \partial_\lambda - \mathrm{i} \, \partial_\lambda A_\mu - A_\lambda \, A_\mu) \, \psi + K_\lambda{}^j{}_\mu \, (\partial_j - \mathrm{i} \, A_j) \, \psi,$$
$$G_0^{ij} \, \varkappa_i{}^h{}_j = \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}}$$

yield

$$\check{\nabla}_{ij} \psi = (\partial_{ij} - \mathrm{i} \, A_i \, \partial_j - \mathrm{i} \, A_j \, \partial_i - \mathrm{i} \, \partial_i A_j - A_i \, A_j) \, \psi + \varkappa_i{}^h{}_j \, (\partial_h - \mathrm{i} \, A_h) \, \psi,$$

hence

$$\Delta_0 \psi = G_0^{ij} \, \partial_{ij} \psi + \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}} \, \partial_h \psi - 2 \, \mathrm{i} \, A_0^i \, \partial_i \psi - \mathrm{i} \, \frac{\partial_i (A_0^i \sqrt{|g|})}{\sqrt{|g|}} \, \psi - A_0^i \, A_i \, \psi. \quad \square$$

Clearly, the observed quantum laplacian Δ_0 is a $\mathbb{C}$-linear differential operator. Moreover, we have the following property.

Corollary 16.3.3 *For each spacetime function $f \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ and each quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we have the equality*

$$\Delta[G, o](f \, \Psi) = f \, \Delta[G, o]\Psi + (\Delta[G] f) \, \Psi + 2 \, \vec{d} \, f \lrcorner \check{\nabla}[o]\Psi,$$

with coordinate expression

$$\Delta[G, o](f \, \psi) = f \, \Delta[G, o]\psi + \left(G_0^{ij} \, \partial_{ij} f + \frac{\partial_i (G_0^{ij} \sqrt{|g|})}{\sqrt{|g|}} \, \partial_j f \right) \psi + 2 \, G_0^{ij} \, \partial_i f \, \partial_j \psi - 2 \, \mathrm{i} \, G_0^{ih} \, A_h \, \partial_i f \, \psi. \quad \square$$

Remark 16.3.4 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the component $\psi := z \circ \Psi$ depends only on Ψ and on the quantum basis b. Hence, the observer o is involved in the observed laplacian just through the spacelike observed potential $\vec{A} := \vec{A}[\mathrm{b}, o]$. $\square$

Corollary 16.3.5 *If, for a certain observer o, the spacelike observed potential $\vec{A}[\mathrm{b}, o]$ vanishes, then we obtain the equality*

$$\Delta[G, o]\,\psi := \Delta[G]\,\psi,$$

with coordinate expression

$$\Delta_0[G]\,\psi = G_0^{ij}\,\partial_{ij}\psi + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\psi. \quad \square$$

Corollary 16.3.6 *Let us consider two observers o and $\acute{o} := o + v$.*

Then, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the following transition rule (see Definition 3.2.17)

$$\Delta[G, \acute{o}]\,\Psi = \Delta[G, o]\,\Psi - 2\,\mathrm{i}\,\nabla_v[o]\Psi - \big(\mathrm{i}\,\mathrm{div}_\eta v + G\,(v, v)\big)\,\Psi,$$

where

$$\nabla_v[o]\Psi = v_0^j\,(\partial_j\psi - \mathrm{i}\,A_j\,\psi)\,\mathrm{b}. \quad \square$$

Corollary 16.3.7 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the coordinate expression of the observed quantum laplacian associated with the distinguished observer o_Ψ, can be written as*

$$\Delta_0\psi = G_0^{ij}\,\partial_{ij}|\psi| + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h|\psi|.$$

Proof. In fact, the observed quantum laplacian is gauge independent and, in the proper domain of Ψ, with reference to the distinguished quantum basis b_Ψ, we obtain $\psi = |\psi|$ and $A_0^i = 0$. $\square$

Exercise 16.3.8 The observed quantum laplacian is gauge independent by definition. We can check this property just as an exercise, in order to see in detail how the spacelike observed potential compensates the change of quantum gauge.

Indeed, with reference to two quantum bases b and $\acute{\mathrm{b}} = \exp(\mathrm{i}\,\vartheta)\,\mathrm{b}$, the equalities (see Theorem 15.2.26)

$$\acute{\psi} = \psi\,\exp(-\mathrm{i}\,\vartheta) \qquad \text{and} \qquad \vec{A}[\acute{\mathrm{b}}, o] = \vec{A}[\mathrm{b}, o] - \vec{d}\,\vartheta$$

yield

$$\Delta[G, o]\,(\acute{\psi}\,\acute{\mathrm{b}}) = \Delta[G, o]\,(\psi\,\mathrm{b}).$$

In fact, we have

$$\begin{aligned}
\Delta[G,o]\,\acute{\psi} &= \Delta[G]\,\acute{\psi} - 2\,\mathrm{i}\,\vec{A}[\acute{\mathrm{b}},o].\acute{\psi} - \mathrm{i}\,(\mathrm{div}_\eta \vec{A}[\acute{\mathrm{b}},o])\,\acute{\psi} - G\,(\vec{A}[\acute{\mathrm{b}},o],\,\vec{A}[\acute{\mathrm{b}},o])\,\acute{\psi} \\
&= \Big(\Delta[G]\,\psi - \psi\,\big(\mathrm{i}\,\Delta[G]\,\vartheta + \bar{G}\,(\check{d}\vartheta,\check{d}\vartheta)\big) - 2\,\mathrm{i}\,\bar{G}(\check{d}\psi,\check{d}\vartheta)\Big)\,\exp(-\mathrm{i}\,\vartheta) \\
&\quad - 2\,\big(\mathrm{i}\,\vec{A}[\mathrm{b},o].\psi - \mathrm{i}\,\vec{d}\,\vartheta.\psi + \psi\,(\vec{A}[\mathrm{b},o].\vartheta - \vec{d}\,\vartheta.\vartheta)\big)\,\exp(-\mathrm{i}\,\vartheta) \\
&\quad - \mathrm{i}\,\psi\,(\mathrm{div}_\eta \vec{A}[\mathrm{b},o])\,\psi - \mathrm{div}_\eta \vec{d}\,\vartheta)\,\exp(-\mathrm{i}\,\vartheta) \\
&\quad - \big(G\,(\vec{A}[\mathrm{b},o],\,\vec{A}[\mathrm{b},o]) - 2\,\vec{A}[\mathrm{b},o].\vartheta + \vec{d}\,\vartheta.\vartheta\big)\,\psi\,\exp(-\mathrm{i}\,\vartheta) \\
&= (\Delta[G]\,\psi)\exp(-\mathrm{i}\,\vartheta) - 2\,(\mathrm{i}\,\vec{A}[\mathrm{b},o].\psi)\,\exp(-\mathrm{i}\,\vartheta) \\
&\quad - \mathrm{i}\,\psi\,(\mathrm{div}_\eta \vec{A}[\mathrm{b},o])\,\exp(-\mathrm{i}\,\vartheta) \\
&\quad - \big(G\,(\vec{A}[\mathrm{b},o],\,\vec{A}[\mathrm{b},o])\,\exp(-\mathrm{i}\,\vartheta) \\
&= \Big(\Delta[G]\,\psi - 2\,\mathrm{i}\,\vec{A}[\mathrm{b},o].\psi - \mathrm{i}\,(\mathrm{div}_\eta \vec{A}[\mathrm{b},o])\,\psi \\
&\quad - G\,(\vec{A}[\mathrm{b},o],\vec{A}[\mathrm{b},o])\,\psi\Big)\,\exp(-\mathrm{i}\,\vartheta). \quad \square
\end{aligned}$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the observed quantum laplacian of a plane wave.

Example 16.3.9 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi = |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with } \ \varphi = \mathrm{k}_\lambda\,x^\lambda,\ \mathrm{k}_\lambda \in \mathbb{R},\ |\psi| \neq 0.$$

Then, we obtain

$$\begin{aligned}
\Delta_0\psi = G_0^{ij}\,\Big(&\frac{\partial_{ij}|\psi|}{|\psi|} - (\mathrm{k}_i - A_i)\,(\mathrm{k}_j - A_j) \\
&+ \mathrm{i}\,\big(\frac{\partial_i|\psi|}{\psi}\,(\mathrm{k}_j - A_j) + \frac{\partial_j|\psi|}{\psi}\,(\mathrm{k}_i - A_i)\big) - \partial_i A_j\Big)\,\psi. \quad \square
\end{aligned}$$

16.3.1 Observed Phase Quantum Laplacian

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and study the *observed phase quantum laplacian* $\Delta^{(\!(\,)\!)}[G,o](\!(\Psi)\!) = \bar{G} \lrcorner \check{\nabla}^{(\!(\,)\!)2}[o](\!(\Psi)\!) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R})$.

In particular, with reference to the distinguished observer o_Ψ, we have $\Delta^{(\!(\,)\!)}[G,o_\Psi]$ $(\!(\Psi)\!) = 0$ (see Theorem 15.2.31).

We stress that the quantum phase $(\!(\Psi)\!)$ and the observed phase quantum laplacian $\Delta^{(\!)}[G, o](\!(\Psi)\!)$ are purely real objects, according to our viewpoint on real degrees of freedom of quantum particles (see Proposition 14.7.1).

Definition 16.3.10 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ and each observer o, the *observed phase quantum laplacian* is defined to be the scaled section (see Proposition 16.1.14)

$$\Delta^{(\!)}[G, o](\!(\Psi)\!) = \bar{G} \lrcorner \check{\nabla}^{(\!)2}[o](\!(\Psi)\!) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}). \quad \square$$

Proposition 16.3.11 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ and each observer o, we have the coordinate expression*

$$\Delta^{(\!)}[G, o](\!(\Psi)\!) = \Delta_0^{(\!)} \varphi \, u^0,$$

where

$$\Delta_0^{(\!)} \varphi = G_0^{ij} \, \partial_{ij} \varphi + \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}} \, \partial_h \varphi - \frac{\partial_i (A_0^i \sqrt{|g|})}{\sqrt{|g|}}.$$

Proof. The equalities

$$\nabla^{(\!)}[o](\!(\Psi)\!) = (\partial_\lambda \varphi - A_\lambda) \, d^\lambda \qquad \text{and} \qquad G_0^{ij} \, \varkappa_i{}^h{}_j = \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}}$$

yield

$$\check{\nabla}[\varkappa] \, \check{\nabla}^{(\!)}[o](\!(\Psi)\!) = \big(\partial_{ij} \varphi - \partial_i A_j + \varkappa_i{}^h{}_j \, (\partial_h \varphi - A_h)\big) \, \check{d}^i \otimes \check{d}^j,$$

hence

$$\Delta_0^{(\!)} \varphi = G_0^{ij} \, \partial_{ij} \varphi + \frac{\partial_i (G_0^{ih} \sqrt{|g|})}{\sqrt{|g|}} \, \partial_h \varphi - \frac{\partial_i (A_0^i \sqrt{|g|})}{\sqrt{|g|}}. \quad \square$$

Corollary 16.3.12 *Let us consider two observers o and $\acute{o} := o + v$.*

Then, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain the following equality (see Definition 3.2.17)

$$\Delta^{(\!)}[G, \acute{o}](\!(\Psi)\!) = \Delta^{(\!)}[G, o](\!(\Psi)\!) - \operatorname{div}_\eta v.$$

Proof. The proof follows from the transition rule Theorem 15.2.26

$$\vec{A}[\mathrm{b}, \acute{o}] = \vec{A}[\mathrm{b}, o] + v. \quad \square$$

Exercise 16.3.13 The observed phase quantum laplacian is gauge independent by definition. We can check this property just as an exercise, in order to see in detail how the spacelike observed potential compensates the change of quantum gauge.

Indeed, with reference to two quantum bases $\mathfrak{b}$ and $\acute{\mathfrak{b}} = \exp(\mathrm{i}\,\vartheta)\,\mathfrak{b}$, the equalities (see Theorem 15.2.26)

$$\acute{\varphi} = \varphi - \vartheta \qquad \text{and} \qquad \vec{A}[\acute{\mathfrak{b}}, o] = \vec{A}[\mathfrak{b}, o] - \vec{d}\,\vartheta$$

yield

$$\Delta^{(0)}\acute{\varphi} = \Delta^{(0)}\varphi.$$

In fact, by taking into account the above Proposition 16.3.11 and the equality

$$\Delta[G]\,\vartheta = \operatorname{div}_\eta G^\sharp(\check{d}\vartheta),$$

we obtain

$$\begin{aligned}
\Delta^{(0)}[G, o]\,\acute{\varphi} &= \Delta[G]\,\acute{\varphi} - \operatorname{div}_\eta \vec{A}[\acute{\mathfrak{b}}, o] \\
&= \Delta[G, o]\,\varphi - \Delta[G, o]\,\vartheta - \operatorname{div}_\eta \vec{A}[\mathfrak{b}, o] + \operatorname{div}_\eta \vec{d}\,\vartheta \\
&= \Delta[G, o]\,\varphi - \operatorname{div}_\eta \vec{A}[\mathfrak{b}, o] \\
&= \Delta^{(0)}[G, o]\,\varphi. \quad \square
\end{aligned}$$

Corollary 16.3.14 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the observed quantum laplacian associated with the distinguished observer* o_Ψ *vanishes:*

$$\Delta^{(0)}[G, o_\Psi](\!(\Psi)\!) = 0.$$

Proof. Being $\Delta^{(0)}[G, o_\Psi](\!(\Psi)\!)$ gauge independent, we can compute it by referring to the distinguished quantum basis $\mathfrak{b}_\Psi$. Then, the equalities $\varphi[\Psi, \mathfrak{b}_\Psi] = 0$, $\vec{A}[\mathfrak{b}_\Psi, o_\Psi] = 0$ and the above Proposition 16.3.2 yield the equality $\Delta^{(0)}[G, o_\Psi]$ $(\!(\Psi)\!) = \Delta[G, o]\,\varphi[\Psi, \mathfrak{b}_\Psi] = \Delta[G]\,\varphi[\Psi, \mathfrak{b}_\Psi] - \operatorname{div}_\eta \vec{A}[\mathfrak{b}_\Psi, o_\Psi] = 0.$ □

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the observed phase quantum laplacian of a plane wave.

Example 16.3.15 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi = |\psi|\, e^{\mathrm{i}\,\varphi}\,\mathfrak{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\, x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}, \;\; |\psi| \neq 0.$$

Then, we obtain

$$\Delta^{(0)}\,\varphi = -G^{ij}_0\,\partial_i A_j\, u^0. \quad \square$$

16.3.2 Polar Splitting of the Observed Quantum Laplacian

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and study the splittings of $\operatorname{re}(\Delta[G, o]\Psi/\Psi)$ and $\operatorname{im}(\Delta[G, o]\Psi/\Psi)$.

In particular, with reference to the distinguished observer o_Ψ the observed quantum laplacian turns out to be *real*, according to the equality $\Delta[G, o_\Psi]\Psi/\Psi = \Delta[G]\|\Psi\|/\|\Psi\|$.

Lemma 16.3.16 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ *and each observer* o, *we can define the (gauge independent)* scaled complex function *given by the ratio*

$$\Delta[G, o]\Psi/\Psi \in \operatorname{map}(\boldsymbol{E}, \mathbb{C} \otimes \mathbb{T}^*),$$

by taking into account the fact that the complex fibres of the quantum bundle are 1–dimensional. Moreover, the above ratio characterises the observed quantum laplacian through the equality

$$\Delta[G, o]\Psi = (\Delta[G, o]\Psi/\Psi) \otimes \Psi,$$

Indeed, the following equality holds

$$\Delta[G, o]\Psi/\Psi = \mathrm{h}\big(\Psi,\ \Delta[G, o]\Psi\big)/\|\Psi\|^2. \quad \square$$

Proposition 16.3.17 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ *and each observer* o, *the (gauge independent) real and imaginary components of* $\Delta[G, o]\Psi/\Psi$ *can be expressed through the polar components of* Ψ *in the following way*

$$\operatorname{re}(\Delta[G, o]\Psi/\Psi) = \frac{\Delta[G]\|\Psi\|}{\|\Psi\|} - G\left(\vec{\nabla}^{\circledcirc}[o]\langle\!\langle\Psi\rangle\!\rangle, \vec{\nabla}^{\circledcirc}[o]\langle\!\langle\Psi\rangle\!\rangle\right),$$

$$\operatorname{im}(\Delta[G, o]\Psi/\Psi) = \Delta^{\circledcirc}[G, o]\langle\!\langle\Psi\rangle\!\rangle + 2\,G\left(\frac{\vec{d}\,\|\Psi\|}{\|\Psi\|}, \vec{\nabla}^{\circledcirc}[o]\langle\!\langle\Psi\rangle\!\rangle\right),$$

where we have set $\vec{\nabla}^{\circledcirc}[o] := G^\sharp(\check{\nabla}^{\circledcirc}[o])$ *and* $\vec{d} := G^\sharp(\check{d})$.

Accordingly, we have the following coordinate expressions

$$\begin{aligned}
\operatorname{re}\frac{\Delta_0[G,o]\Psi}{\Psi} &= \frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|} \\
&\quad - G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi + 2\,A_0^i\,\partial_i\varphi - A_0^i\,A_i \\
&= G_0^{ij}\,\big(\tfrac{1}{2}\,\frac{\partial_{ij}|\psi|^2}{|\psi|^2} - \tfrac{1}{4}\,\frac{\partial_i|\psi|^2}{|\psi|^2}\,\frac{\partial_j|\psi|^2}{|\psi|^2} - \partial_i\varphi\,\partial_j\varphi\big) \\
&\quad + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|^2}{|\psi|^2} \\
&\quad + 2\,A_0^i\,\partial_i\varphi - A_0^i\,A_i,
\end{aligned}$$

$$\begin{aligned}
\operatorname{im}\frac{\Delta_0[G,o]\Psi}{\Psi} &= G_0^{ij}\,(\partial_{ij}\varphi + 2\,\frac{\partial_i|\psi|}{|\psi|}\,\partial_j\varphi) + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi - 2\,A_0^i\,\frac{\partial_i|\psi|}{|\psi|} \\
&\quad - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}} \\
&= G_0^{ij}\,\big(\partial_{ij}\varphi + \frac{\partial_i|\psi|^2}{|\psi|^2}\,\partial_j\varphi\big) + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi - A_0^i\,\frac{\partial_i|\psi|^2}{|\psi|^2} \\
&\quad - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}.
\end{aligned}$$

In particular, with reference to the distinguished basis b_Ψ, *the above coordinate expressions become*

$$\begin{aligned}
\operatorname{re}(\Delta_0[G,o]\Psi/\Psi) &= \frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|} - A_0^i\,A_i, \\
\operatorname{im}(\Delta_0[G,o]\Psi/\Psi) &= -2\,A_0^i\,\frac{\partial_i|\psi|}{|\psi|} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}. \quad \square
\end{aligned}$$

Corollary 16.3.18 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the scaled complex function*

$$\Delta[G, o_\Psi]\Psi/\Psi \in \operatorname{map}(\boldsymbol{E}, \mathbb{C}\otimes\mathbb{T}^*)$$

associated with the distinguished observer o_Ψ *turns out to be* real. *Indeed, it is given by the following equality*

$$\Delta[G, o_\Psi]\Psi/\Psi = \Delta[G]\|\Psi\|/\|\Psi\|,$$

with coordinate expression

$$\Delta_0[G, o_\Psi]\Psi/\Psi = G_0^{ij}\,\frac{\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|}{|\psi|}\,. \quad \square$$

Remark 16.3.19 The above Corollary 16.3.18 shows that apparently $\Delta[G, o_\Psi]\Psi/\Psi$ does not depend on the quantum phase of the quantum particle.

However, we stress that the distinguished observer o_Ψ involved in the above laplacian depends essentially on the quantum phase of the quantum particle. □

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the polar splitting of the observed quantum laplacian of a plane wave.

Example 16.3.20 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1)

$$\Psi = |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\, x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}, \;\; |\psi| \neq 0.$$

Then, we obtain

$$\mathrm{re}\,(\Delta[G, o]\Psi/\Psi)_0 = G_0^{ij}\left(\frac{\partial_{ij}|\psi|}{|\psi|} - (\mathrm{k}_i - A_i)\,(\mathrm{k}_j - A_j)\right),$$
$$\mathrm{im}\,(\Delta[G, o]\Psi/\Psi)_0 = 2\, G_0^{ij}\,\frac{\partial_i|\psi|}{|\psi|}\,(\mathrm{k}_j - A_j) - G_0^{ij}\,\partial_i A_j. \quad \square$$

16.4 Upper Quantum Covariant Differentials

We discuss the *upper quantum covariant differential* $\nabla^\uparrow\Psi$ and the *upper phase quantum covariant differential* $\nabla^{\uparrow(\!(\,)\!)}(\!(\Psi)\!)$, along with their polar splittings in the case of a proper quantum section.

We stress that the quantum phase $(\!(\Psi)\!)$ and the upper phase quantum covariant differential $\nabla^{\uparrow(\!(\,)\!)}(\!(\Psi)\!)$ are purely real objects, according to our viewpoint on real degrees of freedom of quantum particles (see Proposition 14.7.1).

16.4.1 Upper Quantum Covariant Differential

The upper quantum connection $\text{Ч}^\uparrow$ yields the *upper quantum covariant differential* $\nabla^\uparrow\Psi$ of a quantum section Ψ, by taking into account its natural pullback to $\boldsymbol{Q}^\uparrow$.

Let us consider a quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and its natural phase pullback $\Psi^\uparrow \in \sec(J_1\boldsymbol{E}, \boldsymbol{Q}^\uparrow)$.

Proposition 16.4.1 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the* upper quantum covariant differential*, with respect to the upper quantum connection* $\mathcal{Y}^{\uparrow}$*, is the (gauge independent and observer independent)* horizontal *section*

$$\nabla^{\uparrow}\Psi := \nabla[\mathcal{Y}^{\uparrow}]\Psi \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E} \otimes \boldsymbol{Q}),$$

with coordinate expression

$$\nabla^{\uparrow}\Psi = \nabla^{\uparrow}{}_{\lambda}\,\psi\, d^{\lambda} \otimes \mathsf{b}, \quad \textit{where } \nabla^{\uparrow}{}_{\lambda}\,\psi := \partial_{\lambda}\psi - \mathfrak{i}\, A^{\uparrow}{}_{\lambda}\,\psi. \quad \square$$

Corollary 16.4.2 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the coordinate expression of* $\nabla^{\uparrow}\Psi$ *can be written as*

$$\nabla^{\uparrow}\Psi = \nabla^{\uparrow}{}_{\lambda}\,|\psi|\, d^{\lambda} \otimes \mathsf{b}_{\Psi}, \quad \textit{where } \nabla^{\uparrow}{}_{\lambda}\,|\psi| := \partial_{\lambda}|\psi| - \mathfrak{i}\, A^{\uparrow}{}_{\lambda}\,|\psi|. \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the upper quantum covariant differential of a plane wave.

Example 16.4.3 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathfrak{i}\,\varphi}\, \mathsf{b}, \quad \text{with } \varphi = \mathsf{k}_{\lambda}\, x^{\lambda}, \ \mathsf{k}_{\lambda} \in \mathbb{R}, \ |\psi| \neq 0.$$

Then, we obtain

$$\nabla^{\uparrow}\Psi = \big(\partial_{\lambda} \log|\psi| + \mathfrak{i}\,(\mathsf{k}_{\lambda} - A^{\uparrow}{}_{\lambda})\, d^{\lambda} \otimes \Psi. \quad \square$$

16.4.2 Phase Upper Quantum Covariant Differential

In order to emphasise the quantum phase as a real degree of freedom of the quantum particle, we analyse the *phase upper quantum covariant differential* $\nabla^{\uparrow\circledcirc}(\!(\Psi)\!)$ of a proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$.

In particular, with reference to the distinguished quantum basis b_{Ψ}, we obtain the equality $\nabla^{\uparrow\circledcirc}(\!(\Psi)\!) = -A^{\uparrow}[\Psi]$ (see Note 14.6.3).

Indeed, the above equality emphasises a further physical meaning of the distinguished upper quantum potential $A^{\uparrow}[\mathsf{b}_{\Psi}]$ "*seen*" by the proper quantum section Ψ.

Lemma 16.4.4 *The upper quantum connection* $\mathcal{Y}^{\uparrow}$ *yields, in a covariant way, the real linear connection of the phase upper quantum bundle* $\pi^{\circledcirc} : \boldsymbol{Q}^{\circledcirc}_{/0} \to \boldsymbol{E}$, *called* phase upper quantum connection *(see Proposition 14.7.1)*

$$\mathcal{Y}^{\uparrow} : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{\circledcirc}_{/0} \to T^*J_1\boldsymbol{E} \otimes T\,\boldsymbol{Q}^{\circledcirc}_{/0}$$

through the commutative diagram

$$\begin{array}{ccc}
\boldsymbol{Q}^{\uparrow}{}_{/0} \underset{J_1\boldsymbol{E}}{\times} T J_1\boldsymbol{E} & \xrightarrow{\;\text{Ч}^{\uparrow}\;} & T\,\boldsymbol{Q}_{/0} \\
{\scriptstyle i^{\uparrow\circledcirc} \times \mathrm{id}_{TJ_1\boldsymbol{E}}} \Big\uparrow & & \Big\downarrow {\scriptstyle Tp^{\circledcirc}} \\
\boldsymbol{Q}^{\uparrow\circledcirc}_{/0} \underset{J_1\boldsymbol{E}}{\times} T\boldsymbol{E} & \xrightarrow{\;\text{Ч}^{\uparrow\circledcirc}\;} & T\,\boldsymbol{Q}^{\circledcirc}_{/0}
\end{array} \quad . \square$$

Then, we obtain the following relevant intrinsic characterisation of the distinguished upper potential $A^{\uparrow}[\mathrm{b}_{\Psi}]$ (see Note 14.6.3).

Proposition 16.4.5 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the* upper phase quantum covariant differential, *with respect to the upper quantum connection* $\text{Ч}^{\uparrow\circledcirc}$, *is the (gauge independent and observer independent)* horizontal *form*

$$\nabla^{\uparrow\circledcirc}(\!(\Psi)\!) := \nabla[\text{Ч}^{\uparrow\circledcirc}](\!(\Psi)\!)^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}),$$

where $(\!(\Psi)\!)^{\uparrow} \in \sec(J_1\boldsymbol{E},\; J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{\circledcirc}_{/0})$ *is the pullback of* $(\!(\Psi)\!) \in \sec(\boldsymbol{E},\, \boldsymbol{Q}^{\circledcirc}_{/0})$.

We have the coordinate expression

$$\nabla^{\uparrow\circledcirc}(\!(\Psi)\!) = \nabla^{\uparrow\circledcirc}_{\lambda}\,\varphi\, d^{\lambda}, \quad \textit{where} \;\; \nabla^{\uparrow\circledcirc}_{\lambda}\,\varphi := \partial_{\lambda}\varphi - A^{\uparrow}{}_{\lambda}.$$

In particular, with reference to the distinguished quantum basis b_{Ψ}, *we obtain the equality (see Note 14.6.3)*

$$\nabla^{\uparrow\circledcirc}(\!(\Psi)\!) = -A^{\uparrow}[\mathrm{b}_{\Psi}]. \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the phase upper quantum covariant differential of a plane wave.

Example 16.4.6 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_{\lambda}\, x^{\lambda},\; \mathrm{k}_{\lambda} \in \mathbb{R},\; |\psi| \neq 0.$$

Then, we obtain

$$\nabla^{\uparrow\circledcirc}(\!(\Psi)\!) = (\mathrm{k}_{\lambda} - A^{\uparrow}{}_{\lambda})\, d^{\lambda}. \quad \square$$

16.4.3 Polar Splitting of the Upper Quantum Differential

In order to emphasise the quantum norm and the quantum phase as two real degrees of freedom of the quantum particle, we consider a proper quantum section Ψ and find analyse $\mathrm{re}\,(\nabla^{\uparrow}\Psi/\Psi)$ and $\mathrm{im}\,(\nabla^{\uparrow}\Psi/\Psi)$.

In particular, with reference to the distinguished quantum basis b_Ψ the upper quantum differential becomes $\nabla^{\uparrow}\Psi = (d\log\|\Psi\| - \mathfrak{i}\,A^{\uparrow}[\mathsf{b}_\Psi]) \otimes \Psi$.

Lemma 16.4.7 *For each $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we can define the (gauge independent and observer independent) complex spacetime form given by the ratio*

$$\nabla^{\uparrow}\Psi/\Psi \in \mathrm{sec}(J_1\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E}),$$

by taking into account the fact that the complex fibres of the quantum bundle are 1–dimensional. Moreover, the above ratio characterises $\nabla^{\uparrow}\Psi$ through the equality

$$\nabla^{\uparrow}\Psi = (\nabla^{\uparrow}\Psi/\Psi)\,\Psi.$$

Indeed, the following equality holds

$$\nabla^{\uparrow}\Psi/\Psi = \mathsf{h}\big(\Psi,\, \nabla^{\uparrow}\Psi\big)/\|\Psi\|^2. \quad \square$$

Proposition 16.4.8 *For each $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the (gauge independent and observer independent) real and imaginary components of $\nabla^{\uparrow}\Psi/\Psi$ can be expressed through the polar components of Ψ as follows (see Remark K.2.8)*

$$\mathrm{re}\,(\nabla^{\uparrow}\Psi/\Psi) = d\log\|\Psi\| \qquad \textit{and} \qquad \mathrm{im}\,(\nabla^{\uparrow}\Psi/\Psi) = \nabla^{\uparrow(\!(\!)\!)}(\!(\Psi)\!).$$

We have the coordinate expression

$$\nabla^{\uparrow}\Psi/\Psi = \big(\partial_\lambda(\log|\psi|) + \mathfrak{i}\,(\partial_\lambda\varphi - A^{\uparrow}{}_\lambda)\big)\,d^\lambda,$$

where

$$\partial_\lambda(\log|\psi|) = \tfrac{1}{2}\,\frac{\bar\psi\,\partial_\lambda\psi + \psi\,\partial_\lambda\bar\psi}{|\psi|^2} \qquad \textit{and} \qquad \partial_\lambda\varphi = \tfrac{1}{2}\,\frac{\bar\psi\,\partial_\lambda\psi - \psi\,\partial_\lambda\bar\psi}{|\psi|^2}\,.$$

In particular, with reference to the distinguished basis b_Ψ, *the above coordinate expression becomes*

$$\nabla^{\uparrow}\Psi/\Psi = \big(\partial_\lambda(\log|\psi|) - \mathfrak{i}\,A^{\uparrow}{}_\lambda[\mathsf{b}_\Psi]\big)\,d^\lambda. \quad \square$$

Remark 16.4.9 We stress the fact that the real and the imaginary components of the complex spacetime form $\nabla^{\uparrow}\Psi/\Psi$ are expressed by the differentials of the hermitian norm and of the quantum phase of the quantum section, respectively.

This is a further interesting feature of the polar picture of the quantum particle.□

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the polar splitting of the upper quantum covariant differential of a plane wave.

Example 16.4.10 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi| \, e^{\mathrm{i}\varphi} \, \mathfrak{b}, \quad \text{with} \ \ \varphi = \mathrm{k}_\lambda \, x^\lambda, \ \mathrm{k}_\lambda \in \mathbb{R}, \ |\psi| \neq 0.$$

Then, we obtain

$$\mathrm{re}\,(\nabla^{\uparrow}\Psi/\Psi) = \partial_\lambda(\log|\psi|)\, d^\lambda \qquad \text{and} \qquad \mathrm{im}\,(\nabla^{\uparrow}\Psi/\Psi) = (\mathrm{k}_\lambda - A^{\uparrow}{}_\lambda)\, d^\lambda. \quad \square$$

16.5 Remarks on Notation

We emphasise some partially non standard features of our notation concerning quantum covariant differential operators.

Remark 16.5.1 We must pay attention to our (partially non standard) notation concerning symbols of covariant differential operators.

(1) According to our notation, the differential operators

$$\nabla^{\uparrow}, \qquad \nabla^{\uparrow 2}, \qquad \nabla[o], \qquad \nabla[o]^2$$

act on quantum sections Ψ, but the differential operators

$$\nabla^{\uparrow}{}_\lambda, \qquad \nabla^{\uparrow}{}_\lambda \nabla^{\uparrow}{}_\mu, \qquad \nabla^{\uparrow}{}_{\lambda\mu}, \qquad\qquad \nabla_\lambda, \qquad \nabla_\lambda \nabla_\mu, \qquad \nabla_{\lambda\mu}$$

act *only* on complex valued functions ψ and *not* on quantum sections Ψ.

(2) According to our notation, we set

$$\nabla^{\uparrow}{}_\lambda \Psi := (\nabla^{\uparrow}\Psi)_\lambda (\nabla^{\uparrow}{}_\lambda \psi)\, \mathfrak{b}, \qquad \nabla^{\uparrow 2}_{\lambda\mu}\Psi := (\nabla^{\uparrow 2}\Psi)_{\lambda\mu} := (\nabla^{\uparrow}{}_{\lambda\mu}\psi)\, \mathfrak{b},$$

$$\nabla[o]_\lambda \Psi := (\nabla[o]\Psi)_\lambda := (\nabla_\lambda \psi)\, \mathfrak{b}, \qquad \nabla^2[o]_{\lambda\mu}\Psi := (\nabla^2[o]\Psi)_{\lambda\mu} := (\nabla_{\lambda\mu}\psi)\, \mathfrak{b},$$

and obtain the equalities

$$\begin{aligned}
\nabla^{\uparrow 2}\Psi &= (\nabla^{\uparrow 2}_{\lambda\mu}\Psi) \otimes d^\lambda \otimes d^\mu \\
&= (\nabla^{\uparrow}{}_{\lambda\mu}\,\psi)\, d^\lambda \otimes d^\mu \otimes \mathfrak{b} - \mathrm{i}\, G^0_{ij}\, \psi\, d^i_0 \otimes (d^j - x^j_0\, d^0) \otimes \mathfrak{b} \\
&= \big((\nabla^{\uparrow}{}_\lambda \nabla^{\uparrow}{}_\mu + K_\lambda{}^j{}_\mu\, \nabla^{\uparrow}{}_j)\,\psi\big)\, d^\lambda \otimes d^\mu \otimes \mathfrak{b} - \mathrm{i}\, G^0_{ij}\, \psi\, d^i_0 \otimes (d^j - x^j_0\, d^0) \otimes \mathfrak{b}, \\
\nabla^2[o]\Psi &= (\nabla^2[o]_{\lambda\mu}\Psi) \otimes d^\lambda \otimes d^\mu \\
&= (\nabla_{\lambda\mu}\,\psi)\, d^\lambda \otimes d^\mu \otimes \mathfrak{b} \\
&= \big((\nabla_\lambda \nabla_\mu + K_\lambda{}^j{}_\mu\, \nabla_j)\,\psi\big)\, d^\lambda \otimes d^\mu \otimes \mathfrak{b}.
\end{aligned}$$

(3) Thus, according to our notation, we have

$$\nabla^{\uparrow}{}_{\lambda\mu} \neq \nabla^{\uparrow}{}_\lambda\, \nabla^{\uparrow}{}_\mu \qquad \text{and} \qquad \nabla_{\lambda\mu} \neq \nabla_\lambda\, \nabla_\mu,$$

i.e.

$$\nabla^{\uparrow}{}_{\lambda\mu}\Psi \neq (\nabla^{\uparrow}{}_\lambda \nabla^{\uparrow}{}_\mu \psi)\, \mathfrak{b}, \qquad \nabla^2[o]_{\lambda\mu}\Psi \neq (\nabla_\lambda \nabla_\mu \psi)\, \mathfrak{b}. \quad \square$$

Remark 16.5.2 We stress that the differential operators $\nabla^{\uparrow}{}_\lambda\, \nabla^{\uparrow}{}_\mu$ and $\nabla_\lambda\, \nabla_\mu$ are not symmetric with respect to the indices λ and μ.

Indeed, we have

$$\begin{aligned}
(\nabla^{\uparrow}{}_\lambda\, \nabla^{\uparrow}{}_\mu - \nabla^{\uparrow}{}_\mu\, \nabla^{\uparrow}{}_\lambda)\,\psi &= -2\,\mathrm{i}\,\Omega_{\lambda\mu}\,\psi = -\mathrm{i}\,(\partial_\lambda A^{\uparrow}{}_\mu - \partial_\mu A^{\uparrow}{}_\lambda)\,\psi \\
(\nabla_\lambda\, \nabla_\mu - \nabla_\mu\, \nabla_\lambda)\,\psi &= -\mathrm{i}\,\Phi_{\lambda\mu}\,\psi = -\mathrm{i}\,(\partial_\lambda A_\mu - \partial_\mu A_\lambda)\,\psi. \quad \square
\end{aligned}$$

Chapter 17
Quantum Dynamics

We derive, in a covariant constructive way, the main dynamical objects of Covariant Quantum Mechanics from the upper quantum connection $\text{Ч}^{\uparrow}$.

For this purpose, we follow a "*criterion of projectability*", with the aim of searching objects based on the spacetime $\boldsymbol{E}$, by getting rid of the family $\{o\}$ of observers which is encoded in the base space $J_1\boldsymbol{E}$ of the upper quantum bundle $\boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$ (Sect. 17.1). Actually, this is a way to implement the covariance of the theory (see, for instance [219]).

Thus, we obtain the "*quantum velocity*", the "*kinetic quantum tensor*", the "*quantum probability current*", the "*quantum lagrangian*" and the "*Schrödinger operator*" (Sects. 17.2, 17.3, 17.4, 17.5 and 17.6).

In particular, for each proper quantum section Ψ, by taking into account the distinguished observer $o[\Psi]$ and the associated distinguished timelike potential $A[\Psi]$, we get compact expressions of these quantum objects (see Theorem 15.2.31).

Accordingly, the Schrödinger equation splits into the system consisting of the global continuity equation and a new equation involving the distinguished timelike potential (see Theorem 15.2.31).

In the flat case the Schrödinger operator, the quantum probability current and the quantum lagrangian turn out to fit the standard expressions.

Further, from the quantum lagrangian we derive the "*quantum momentum form*" and the "*quantum Poincaré–Cartan form*" (Sects. 17.5.2 and 17.5.3).

Indeed, all above dynamical quantum objects are global, gauge independent and observer independent. We recall that, conversely, the classical lagrangian, the classical momentum and the classical Poincaré–Cartan form are local and gauge dependent (see Theorem 10.1.8).

After the above constructive achievement of covariant dynamical quantum objects, we arise the question whether there exist other covariant dynamical objects of the above type (Sect. 17.7). Actually, by means of the methods of "natural geometry" (see [246]) we can prove that the only requirement of covariance determines essentially the above objects (this result was achieved in [219]). Indeed, this discus-

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sion is very technical, hence the reader might wish just to grasp the main ideas and skip the details. We discuss explicitly the cases of the quantum lagrangian and the Schrödinger operator; the other dynamical quantum objects can be approached in a similar way.

17.1 Criterion of Projectability

The theoretical developments of the present Covariant Quantum Mechanics is fully determined by the two Postulates Q.1 and Q.2.

Thus, we derive all other quantum objects (including the dynamical quantum objects and the quantum operators) from the hermitian quantum metric h and the galilean upper quantum connection $\text{Ч}^{\uparrow}$ by means of covariant procedures.

In the Introduction, we have explained our strategy concerning the quantum bundle $\boldsymbol{Q}$, the upper quantum bundle $\boldsymbol{Q}^{\uparrow}$, the cosymplectic phase 2-form Ω and our goal to derive observer independent dynamical quantum objects from $\text{Ч}^{\uparrow}$ (see, Introduction, Sect. 1.5.8),

We have mentioned the fact that our quantum bundle $\boldsymbol{Q}$ lives on the spacetime $\boldsymbol{E}$, but the classical cosymplectic phase 2-form Ω lives on the classical phase space $J_1\boldsymbol{E}$. Therefore, in order to link the quantum theory with the cosymplectic phase 2-form Ω, we have been led, according to a "minimality criterion", to define *by pullback* the upper quantum bundle $\boldsymbol{Q}^{\uparrow}$ over the classical phase space $J_1\boldsymbol{E}$, and to postulate a *reducible* galilean upper quantum connection $\text{Ч}^{\uparrow}$, which is suitable to be linked to the cosymplectic phase 2-form Ω.

We have also emphasised the fact that the galilean upper quantum connection $\text{Ч}^{\uparrow}$ encodes all observers o. Therefore, in view of the covariance of the theory, we look for dynamical quantum objects derived from the upper quantum connection $\text{Ч}^{\uparrow}$, which are observer independent, i.e. projectable on the spacetime $\boldsymbol{E}$.

Such a procedure is called "*criterion of projectability*".

Our postulate concerning the upper quantum connection $\text{Ч}^{\uparrow}$ (see Postulate Q.2 and, for instance [425]) has evident analogies with Geometric Quantisation, but essential differences as well (see Introduction Sect. 1.5.8).

Note 17.1.1 By "*criterion of projectability*" we mean a procedure aimed at searching quantum objects which,

(1) on the one hand, are *naturally* derived from $\text{Ч}^{\uparrow}$, hence live on the upper quantum bundle $\boldsymbol{Q}^{\uparrow}$, whose base space $J_1\boldsymbol{E}$ is "parametrised" by all observers o,
(2) on the other hand, are *projectable* on the quantum bundle $\boldsymbol{Q}$, whose base space is the spacetime $\boldsymbol{E}$, so getting rid of all observers o.

In practice, quite often (as we shall see in the forthcoming sections of this chapter), the theory exhibits in a natural way two distinguished intrinsic objects, which are derived by a covariant procedure from the upper quantum connection $\text{Ч}^{\uparrow}$ and which have the same source and the same target spaces. The source space of these objects involves the phase space $J_1\boldsymbol{E}$, i.e. involves all observers o. However, we can prove that there is a unique combination of these objects which factorises through the spacetime $\boldsymbol{E}$, so getting rid of all observers o.

In this way, our formalism offers in a covariant way observer independent objects, which are natural candidates to represent fundamental dynamical objects of Quantum Mechanics (see Theorems 17.2.2, 17.3.2, 17.4.2, 17.5.2 and 17.6.5). □

17.2 Quantum Velocity

For every proper quantum section Ψ, we introduce, in a covariant way, via the criterion of projectability, the *quantum velocity* field $\mathrm{V}[\Psi] := д + \vec{\nabla}^{\uparrow\circledcirc}(\!(\Psi)\!)$ (see Propositions 2.6.1 and 16.4.5, Definition 14.6.1).

In particular, with reference to the distinguished quantum basis b_Ψ and to the distinguished observer o_Ψ, the above equality becomes $\mathrm{V}[\Psi] = д[o_\Psi] = u^0 \otimes \partial_0$ (see Note 14.6.3 and Theorem 15.2.31). Thus, we have recovered, by a different approach, the distinguished observer o_Ψ associated with a proper quantum section Ψ (see Theorem 15.2.31).

Later, the velocity field $\mathrm{V}[\Psi]$ will appear as a real component of the kinetic quantum tensor and of the quantum probability current (see Theorems 17.3.2 and 17.4.2). Moreover, the velocity field $\mathrm{V}[\Psi]$ will be the source of the hydrodynamic picture of Quantum Mechanics (see Theorem 18.1.1).

Lemma 17.2.1 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we obtain, in a covariant way, the following distinguished maps, whose source is the classical phase space $J_1\boldsymbol{E}$ (which here accounts pointwisely for all classical observers) and whose target is the scaled tangent space of spacetime, (see Propositions 2.6.1 and 16.4.5, Definition 14.6.1)*

$$д \in \mathrm{fib}_{\boldsymbol{E}}(J_1\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}),$$

$$\vec{\nabla}^{\uparrow\circledcirc}(\!(\Psi)\!) \in \mathrm{fib}_{\boldsymbol{E}}(J_1\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}).$$

We have the coordinate expressions

$$д = u^0 \otimes (\partial_0 + x_0^i\, \partial_i),$$

$$\vec{\nabla}^{\uparrow\circledcirc}(\!(\Psi)\!) = (G_0^{ij}\, \partial_j \varphi - x_0^i - A_0^i)\, u^0 \otimes \partial_i. \quad \square$$

Theorem 17.2.2 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the sum of the two maps above (see the above Lemma 17.2.1) factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled spacetime vector field*

$$\mathrm{V}[\Psi] := д + \vec{\nabla}^{\uparrow\circledcirc}(\!(\Psi)\!) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}),$$

which fulfills the property

$$dt \lrcorner (\mathrm{V}[\Psi]) = 1.$$

With reference to an observer o, we have the observed splitting (which turns out to be observer equivariant)

$$\mathrm{V}[\Psi] = д[o] + \vec{\nabla}^{(\!)}[o](\!(\Psi)\!)$$

and the coordinate expression

$$\mathrm{V}[\Psi] = u^0 \otimes \Big(\partial_0 + \big(G_0^{ij}\,\partial_j\varphi - A_0^i\big)\,\partial_i\Big).$$

In particular, with reference to the distinguished observer o_Ψ, the observed splitting reduces to the equality (see Theorem 15.2.31)

$$\mathrm{V}[\Psi] = д[o_\Psi]$$

with coordinate expression

$$\mathrm{V}[\Psi] = \partial_0 \otimes u^0.$$

In other words, the quantum velocity $\mathrm{V}[\Psi]$ *turns out to be just the velocity of the distinguished observer o_Ψ determined by Ψ. Thus,* $\mathrm{V}[\Psi]$ *turns out to be a* classical velocity field *naturally associated with the proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$. □

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the velocity vector field of a plane wave.

Example 17.2.3 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda\, x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}.$$

Then, we obtain

$$\mathrm{V}[\Psi] = u^0 \otimes \big(\partial_0 + \tfrac{\hbar_0}{m}\, g^{ij}\,(\mathrm{k}_j - A_j)\,\partial_i\big). \quad \square$$

17.3 Kinetic Quantum Tensor

For each quantum section Ψ, we discuss the *kinetic quantum tensor* $\mathrm{Q}[\Psi]$ and the associated *kinetic quantum vector field* $\mathrm{Q}[\Psi]/\Psi$.

Later, we shall see that the quantum kinetic tensor is a source of the Schrödinger operator (see Theorem 17.6.5).

17.3.1 Definition of Kinetic Quantum Tensor

We introduce, in a covariant way, via the criterion of projectability (see Note 17.1.1), the *kinetic quantum tensor* $\mathrm{Q}[\Psi] := д \otimes \Psi - \mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi$ associated with every quantum section Ψ (see Propositions 2.6.1 and 16.4.1).

In particular, for each proper quantum section Ψ, with reference to the distinguished quantum basis b_Ψ and to the distinguished observer o_Ψ, the above equality becomes $\mathrm{Q}[\Psi] = \big(д[o_\Psi] - \mathrm{i}\,\vec{d}\,\log\|\Psi\|\big) \otimes \Psi$ (see Note 14.6.3 and Theorem 15.2.31).

Lemma 17.3.1 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain, in a covariant way, the following distinguished maps, whose source space is the classical phase space $J_1\boldsymbol{E}$ (which here accounts pointwisely for all classical observers) and whose target space is the scaled tensor product $\mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})$ (see Propositions 2.6.1 and 16.4.1)*

$$д \otimes \Psi \in \mathrm{fib}_{\boldsymbol{E}}\big(J_1\boldsymbol{E},\ \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$
$$\mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi \in \mathrm{fib}_{\boldsymbol{E}}\big(J_1\boldsymbol{E},\ \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes \boldsymbol{Q})\big).$$

We have the coordinate expressions

$$д \otimes \Psi = (\psi\,\partial_0 + x_0^i\,\psi\,\partial_i) \otimes u^0 \otimes \mathrm{b},$$
$$\mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi = \big(\mathrm{i}\,G_0^{ij}\,\partial_j\psi + (x_0^i + A_0^i)\,\psi\big)\,u^0 \otimes \partial_i \otimes \mathrm{b}. \quad \square$$

Theorem 17.3.2 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the difference of the two maps above (see the above Lemma 17.3.1) factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime*

$$\mathrm{Q}[\Psi] := д \otimes \Psi - \mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi \in \sec\big(\boldsymbol{E},\ \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big).$$

With reference to any observer o, we have the observed splitting (which turns out to be observer equivariant)

$$\mathrm{Q}[\Psi] = д[o] \otimes \Psi - \mathrm{i}\,\vec{\nabla}[o]\Psi.$$

Moreover, we have the coordinate expression

$$\mathrm{Q}[\Psi] = \big(\psi\,\partial_0 - \mathrm{i}\,G_0^{ij}\,(\partial_j\psi - \mathrm{i}\,A_j\,\psi)\,\partial_i\big) \otimes u^0 \otimes \mathrm{b}. \quad \square$$

Corollary 17.3.3 *For each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, with reference to the distinguished quantum basis b_Ψ and to the distinguished observer o_Ψ (see Note 14.6.3 and Theorem 15.2.31), the above coordinate expression becomes*

$$Q[\Psi] = \big(д[o_\Psi] - \mathrm{i}\,\vec{d}\,\log\|\Psi\|\big) \otimes \Psi, \quad \textit{where}\ \vec{d} := G^\sharp \circ \check{d}.$$

Thus, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, we can write (see Theorem 17.2.2)*

$$Q[\Psi] = \big(V[\Psi] - \mathrm{i}\,\vec{d}\,\log\|\Psi\|\big) \otimes \Psi. \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the quantum kinetic tensor of a plane wave.

Example 17.3.4 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\, \mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda\, x^\lambda,\ \mathrm{k}_\lambda \in \mathbb{R},\ |\psi| \neq 0.$$

Then, we obtain

$$Q[\Psi] = u^0 \otimes \Big((\partial_0 + \tfrac{\hbar_0}{m}\, g^{ij}\, (\mathrm{k}_j - A_j)\, \partial_i) - \mathrm{i}\, \tfrac{\hbar_0}{m}\, g^{ij}\, \partial_j \log|\psi|\, \partial_i \Big) \otimes \Psi. \quad \square$$

17.3.2 Kinetic Quantum Vector Field

For every proper quantum section Ψ, we obtain the complex spacetime vector field $Q[\Psi]/\Psi$. We have $\mathrm{re}\,(Q[\Psi]/\Psi) = V[\Psi]$ (see Theorem 17.2.2).

In particular, with reference to the rest observer o_Ψ, we have the equality $Q[\Psi]/\Psi = V[\Psi] - \mathrm{i}\,\vec{d}\,\log\|\Psi\|$.

Corollary 17.3.5 *For each proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, we obtain the complexified spacetime vector field (see Definition 14.6.1)*

$$Q[\Psi]/\Psi \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes T\boldsymbol{E} \otimes \mathbb{C})\big),$$

with coordinate expression

$$Q[\Psi]/\Psi = \big(\partial_0 - (\mathrm{i}\, G^{ij}_0\, \partial_j\psi/\psi + A^i_0)\,\partial_i\big) \otimes u^0,$$

i.e., in polar coordinates,

$$Q[\Psi]/\Psi = u^0 \otimes \Big(\partial_0 + G^{ij}_0\, \big((\partial_j\varphi - A_j) - \mathrm{i}\,\partial_j(\log|\psi|)\big)\,\partial_i\Big).$$

Indeed, the above complex spacetime tensor can be split into real and imaginary components, whose coordinate expressions are

$$\mathrm{re}\,(\mathrm{Q}[\Psi]/\Psi) = u^0 \otimes \big(\partial_0 + (\mathrm{i}\,\tfrac{1}{2}\,G_0^{ij}\,\frac{\psi\,\partial_j\bar\psi - \bar\psi\,\partial_j\psi}{|\psi|^2} - A_0^i)\,\partial_i\big)$$
$$= u^0 \otimes \big(\partial_0 + G_0^{ij}\,(\partial_j\varphi - A_j)\,\partial_i\big),$$
$$\mathrm{im}\,(\mathrm{Q}[\Psi]/\Psi) = -\tfrac{1}{2}\,G_0^{ij}\,\frac{\bar\psi\,\partial_j\psi + \psi\,\partial_j\bar\psi}{|\psi|^2}\,u^0 \otimes \partial_i$$
$$= -G_0^{ij}\,\partial_j(\log|\psi|)\,u^0 \otimes \partial_i .$$

Thus, we obtain the equalities (see Theorem 17.2.2)

$$\mathrm{re}\,(\mathrm{Q}[\Psi]/\Psi) = \mathrm{V}[\Psi] \quad \textit{and} \quad \mathrm{im}\,(\mathrm{Q}[\Psi]/\Psi) = -\overrightarrow{d}\,\log\|\Psi\|.$$

In other words, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, with reference to the distinguished observer* o_ψ *and quantum basis* b_Ψ*, we have the equality (see Theorem 15.2.31 and Note 14.6.3)*

$$\mathrm{Q}[\Psi]/\Psi = \mathrm{V}[\Psi] - \mathrm{i}\,\overrightarrow{d}\,\log\|\Psi\|. \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the quantum kinetic vector field of a plane wave.

Example 17.3.6 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda\,x^\lambda,\ \mathrm{k}_\lambda \in \mathbb{R},\ |\psi| \neq 0.$$

Then, we obtain

$$\mathrm{Q}[\Psi]/\Psi = u^0 \otimes \big((\partial_0 + \tfrac{\hbar_0}{m}\,g^{ij}\,(\mathrm{k}_i - A_i)\,\partial_j) - \mathrm{i}\,\tfrac{\hbar_0}{m}\,g^{ij}\,\partial_j\,\log|\psi|\,\partial_i\big). \quad \square$$

17.4 Quantum Probability Current

We discuss the *quantum probability current* $\mathrm{J}[\Psi]$ and the associated *quantum probability current form* $\underline{\mathrm{J}}[\Psi]$.

17.4.1 Definition of Quantum Probability Current

We introduce, in a covariant way, via the criterion of projectability (see Note 17.1.1), the *quantum probability current* $\mathrm{J}[\Psi] := д \otimes \|\Psi\|^2 - \mathrm{re}\,\mathrm{h}(\Psi,\ \mathrm{i}\,\overrightarrow{\nabla}{}^{\uparrow}\Psi)$ associated with every quantum section Ψ (see Propositions 2.6.1 and 14.3.1).

In particular, for each proper quantum section Ψ, with reference to the distinguished quantum basis $\mathfrak{b}_\Psi$ and to the distinguished observer o_Ψ, the above equality becomes $\mathrm{J}[\Psi] = \|\Psi\|^2\, д[o_\Psi] = \|\Psi\|^2\, \mathrm{V}[\Psi]$ (see Note 14.6.3 and Theorem 15.2.31).

Lemma 17.4.1 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain, in a covariant way, the following distinguished maps, whose source space is the classical phase space $J_1\boldsymbol{E}$ (which here accounts pointwisely for all classical observers), (see Propositions 2.6.1, 14.3.1 and 16.4.1),*

$$д \otimes \|\Psi\|^2 \in \mathrm{fib}_{\boldsymbol{E}}\big(J_1\boldsymbol{E},\ \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big),$$
$$\mathrm{re}\,\mathrm{h}(\Psi,\ \mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi) \in \mathrm{fib}_{\boldsymbol{E}}\big(J_1\boldsymbol{E},\ \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big).$$

We have the coordinate expressions

$$д \otimes \|\Psi\|^2 = |\psi|^2\, u^0 \otimes (\partial_0 + x^i_0\, \partial_i),$$
$$\mathrm{re}\,\mathrm{h}(\Psi,\ \mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi) = \big(\mathrm{i}\, G^{ij}_0\, (\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi}) + |\psi|^2\, (x^i_0 + A^i_0[\mathfrak{b}, o])\big)\, u^0 \otimes \partial_i. \quad \square$$

Theorem 17.4.2 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the difference of the two maps above (see the above Lemma 17.4.1) factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled spacetime vector field, called* quantum probability vector field*,*

$$\mathrm{J}[\Psi] := д \otimes \|\Psi\|^2 - \mathrm{re}\,\mathrm{h}(\Psi,\ \mathrm{i}\,\vec{\nabla}^{\uparrow}\Psi) \in \sec\big(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big),$$

which fulfills the property

$$dt \lrcorner \mathrm{J}[\Psi] = \|\Psi\|^2.$$

With reference to an observer o, we have the observed splitting (which turns out to be observer equivariant)

$$\mathrm{J}[\Psi] = \|\Psi\|^2\, д[o] - \mathrm{re}\,\mathrm{h}(\Psi, \mathrm{i}\,\vec{\nabla}[o]\Psi),$$

and the coordinate expression

$$\mathrm{J}[\Psi] = \big(|\psi|^2\,\partial_0 + (\mathrm{i}\,\tfrac{1}{2}\, G^{ij}_0\, (\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) - A^i_0\, |\psi|^2)\,\partial_i\big) \otimes u^0.$$

In the proper domain of Ψ, the above observed splitting and coordinate expression can be written as

$$\mathrm{J}[\Psi] = \|\Psi\|^2\, \mathrm{V}[\Psi] = |\psi|^2\, u^0 \otimes \big(\partial_0 + G^{ij}_0\, (\partial_j\varphi - A_j)\,\partial_i\big).$$

In particular, with reference to the distinguished observer o_Ψ (see Theorem 15.2.31), in the proper domain of Ψ, the observed splitting and coordinate expression can be written as

$$\mathrm{J}[\Psi] = \|\Psi\|^2\, д[o_\Psi] = \|\Psi\|^2\, \mathrm{V}[\Psi] = |\psi|^2\, u^0 \otimes \partial_0. \quad \square$$

Remark 17.4.3 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and $e \in \boldsymbol{E}$, we have

$$(\mathrm{J}[\Psi])(e) = 0 \qquad \Leftrightarrow \qquad \Psi(e) = 0.$$

Moreover, for each proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the associated $\mathrm{J}[\Psi]$ characterises Ψ up to an additive phase $\vartheta \in \mathrm{map}(\boldsymbol{T}, \mathbb{R}/2\pi)$, which factorises through time, i.e. which is constant along the fibres of spacetime. $\square$

Exercise 17.4.4 According to Theorem 17.4.2, the probability vector field is gauge independent and observer independent, by definition.

It is instructive to further analyse this invariance by discussing in detail the terms of its expressions.

(1) Let us consider two quantum bases b and $\acute{\mathrm{b}} = \exp(\mathrm{i}\,\vartheta)\,\mathrm{b}$.
Then, in virtue of Theorem 15.2.26, we obtain

$$\begin{aligned}
\mathrm{J}[\Psi] &= \big(|\acute{\psi}|^2\,\partial_0 + (\mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\acute{\psi}\,\partial_j\bar{\acute{\psi}} - \bar{\acute{\psi}}\,\partial_j\acute{\psi}) - \acute{A}^i_0\,|\acute{\psi}|^2)\,\partial_i\big) \otimes u^0 \\
&= \big(|\psi|^2\,\partial_0 + (\mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) - A^i_0\,|\psi|^2)\,\partial_i\big) \otimes u^0 \\
&\quad - \big(G^{ij}_0\,\partial_j\vartheta\,|\psi|^2 - G^{ij}_0\,\partial_j\vartheta\,|\psi|^2\big)\,\partial_i \otimes u^0 \\
&= \big(|\psi|^2\,\partial_0 + (\mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) - A^i_0\,|\psi|^2)\,\partial_i\big) \otimes u^0.
\end{aligned}$$

(2) Let us consider two observers o and $\acute{o} = o + v$.

Then, in virtue of Theorem 15.2.26, we obtain

$$\begin{aligned}
\mathrm{J}[\Psi] &= \|\Psi\|^2\, д[\acute{o}] - \mathrm{re}\,\mathrm{h}(\Psi, \mathrm{i}\,\vec{\nabla}[\acute{o}]\Psi) \\
&= \|\Psi\|^2\, д[o] - \mathrm{re}\,\mathrm{h}(\Psi, \mathrm{i}\,\vec{\nabla}[o]\Psi) + \|\Psi\|^2\, v - \mathrm{re}\,\mathrm{h}(\Psi, \Psi)\, v \\
&= \|\Psi\|^2\, д[o] - \mathrm{re}\,\mathrm{h}(\Psi, \mathrm{i}\,\vec{\nabla}[o]\Psi). \quad \square
\end{aligned}$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the quantum probability current of a plane wave.

Example 17.4.5 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \qquad \text{with } \varphi = \mathrm{k}_\lambda\, x^\lambda,\ \ \mathrm{k}_\lambda \in \mathbb{R},\ \ |\psi| \neq 0.$$

Then, we obtain

$$\mathrm{J}[\Psi] = |\psi|^2 \, u^0 \otimes \big(\partial_0 + G^{ij}_0 \, (\mathrm{k}_j - A_j) \, \partial_i\big). \quad \square$$

17.4.2 Quantum Probability Current Form

It is convenient to introduce also the *quantum probability form* $\underline{\mathrm{J}}[\Psi] := i_{\mathrm{J}[\Psi]} \, \upsilon$ (see Proposition 3.2.4).

Indeed, later, we shall recover this quantum probability current form as a particular case of our general classification of quantum currents by means of the special phase functions: indeed, the quantum probability form turns out to be associated with the distinguished function $f = 1$ (see Definitions 21.1.3 and 21.2.1, Example 21.1.5).

Proposition 17.4.6 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain, in a covariant way, the (unscaled) spacetime 3-form, called* quantum probability current form*, (see Proposition 3.2.4)*

$$\underline{\mathrm{J}}[\Psi] := i_{\mathrm{J}[\Psi]} \, \upsilon \in \sec(\boldsymbol{E}, \Lambda^3 \, T\boldsymbol{E}),$$

with coordinate expression (see Proposition 3.2.4)

$$\underline{\mathrm{J}}[\Psi] = |\psi|^2 \, \upsilon^0_0 + \big(\mathrm{i} \, \tfrac{1}{2} \, G^{ij}_0 \, (\psi \, \partial_j \bar{\psi} - \bar{\psi} \, \partial_j \psi) - A^i_0 \, |\psi|^2\big) \, \upsilon^0_i .$$

In the proper domain of Ψ*, the above expression can be written as*

$$\underline{\mathrm{J}}[\Psi] = |\psi|^2 \, \big(\upsilon^0_0 + G^{ij}_0 \, (\partial_j \varphi - A_j) \, \upsilon^0_i\big). \quad \square$$

Eventually, in order to make a comparison with standard Quantum Mechanics, we consider an example concerning the quantum probability current of a plane wave.

Example 17.4.7 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi| \, e^{\mathrm{i} \, \varphi} \, \mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda \, x^\lambda, \ \mathrm{k}_\lambda \in \mathbb{R}.$$

Then, we obtain

$$\underline{\mathrm{J}}[\Psi] = |\psi|^2 \, \big(\upsilon^0_0 + G^{ij}_0 \, (\mathrm{k}_j - A_j) \, \upsilon^0_i\big). \quad \square$$

17.5 Quantum Lagrangian

We discuss the *quantum lagrangian* $\mathsf{L}[\Psi]$, along with the associated *quantum momentum* P and *quantum Poincaré–Cartan form* C.

17.5.1 Definition of Quantum Lagrangian

We introduce, in a covariant way, via the criterion of projectability (see Note 17.1.1), the *quantum lagrangian* $\mathsf{L}[\Psi]$ associated with every quantum section Ψ (see Propositions 2.6.1, 14.7.5 and 16.4.1).

In particular, for each proper quantum section Ψ, the expression of the quantum lagrangian becomes $\mathsf{L}[\Psi] = (\frac{1}{2}\,\bar{G}\,(\check{d}\|\Psi\|,\ \check{d}\|\Psi\|) + A[\Psi]\,\|\Psi\|^2) \otimes \upsilon$ (see Note 14.6.3 and Theorem 15.2.31).

Later, the quantum lagrangian will be the source of the lagrangian approach to the Schrödinger operator (see Theorem 17.6.23).

For a comparison with the quantum lagrangian in standard Quantum Mechanics (see, for instance [362, p. 348], [178]).

Lemma 17.5.1 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we obtain, in a covariant way, the following distinguished sections, with the same source space* $J_1\boldsymbol{E}$ *(which here accounts pointwisely for all classical observers) and the same target space* $\Lambda^4 T^*\boldsymbol{E}$*, (see Propositions 2.6.1 and 16.4.1, Definition 14.5.1)*

$$-dt \wedge \operatorname{im} \mathrm{h}_\eta(\Psi,\ д \lrcorner \nabla^{\uparrow}\Psi) : J_1\boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E},$$
$$\tfrac{1}{2}\, dt \wedge (\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\ \check{\nabla}^{\uparrow}\Psi) : J_1\boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E},$$

with coordinate expressions (see Proposition 3.2.4 and Theorem 10.1.8)

$$\begin{aligned}
&- dt \wedge \operatorname{im} \mathrm{h}_\eta(\Psi,\ д \lrcorner \nabla^{\uparrow}\Psi) \\
&\quad = \big(\tfrac{1}{2}\,\mathfrak{i}\,(\bar{\psi}\,\partial_0\psi - \psi\,\partial_0\bar{\psi}) + \tfrac{1}{2}\,\mathfrak{i}\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\,x^i_0 + \mathcal{L}_0\,\bar{\psi}\,\psi\big)\,\upsilon^0, \\
&\quad\ \ \times \tfrac{1}{2}\, dt \wedge (\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\ \check{\nabla}^{\uparrow}\Psi) \\
&\quad = \big(\tfrac{1}{2}\,G^{ij}_0\,(\partial_i\bar{\psi}\,\partial_j\psi + \mathfrak{i}\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\,A_j) + \tfrac{1}{2}\,\mathfrak{i}\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\,x^i_0 \\
&\quad\ \ + \tfrac{1}{2}\,G^{ij}_0\,\mathcal{P}_i\,\mathcal{P}_j\,\bar{\psi}\,\psi\big)\,\upsilon^0. \quad \square
\end{aligned}$$

Theorem 17.5.2 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, the difference of the two maps above (see the above Lemma 17.5.1) factorises through spacetime, so yielding, in a covariant way, the (gauge independent and observer independent) scaled spacetime 4-form, called* quantum lagrangian*,*

$$\mathsf{L}[\Psi] := -\,dt \wedge \Big(\operatorname{im} \mathrm{h}_\eta(\Psi,\ д \lrcorner \nabla^{\uparrow}\Psi) + \tfrac{1}{2}\,(\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow}\Psi,\ \check{\nabla}^{\uparrow}\Psi)\Big) : \boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E}.$$

With reference to an observer o, we have the observed expression (which turns out to be observer equivariant)

$$\mathrm{L}[\Psi] = -dt \wedge \big(\mathrm{im}\, \mathrm{h}_\eta(\Psi,\ \nabla_{Д[o]}[o]\Psi) + \tfrac{1}{2}\,(\bar{G} \otimes \mathrm{h}_\eta)\,(\nabla[o]\Psi,\ \nabla[o]\Psi)\big),$$

and the coordinate expression (see Note 10.1.11)

$$\mathrm{L}[\Psi] = \tfrac{1}{2}\,\big(-G_0^{ij}\,\partial_i\bar{\psi}\,\partial_j\psi + \mathrm{i}\,(\bar{\psi}\,\partial_0\psi - \psi\,\partial_0\bar{\psi}) \\ -\mathrm{i}\,A_0^j\,(\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi}) + 2\,\alpha_0\,\bar{\psi}\,\psi\big)\,\upsilon^0.$$

Thus, the quantum lagrangian operator

$$\mathrm{L} \,\equiv\, \mathrm{L}_0\,\upsilon^0 : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{E}$$

has coordinate expression, in real coordinates,

$$\mathrm{L}_0 = (w^2\,w_0^1 - w^1\,w_0^2) + A_0\,(w^1\,w^1 + w^2\,w^2) - \tfrac{1}{2}\,G_0^{ij}\,(w_i^1\,w_j^1 + w_i^2\,w_j^2) \\ -\tfrac{1}{2}\,A_0^i\,A_i\,(w^1\,w^1 + w^2\,w^2) - A_0^i\,(w^2\,w_i^1 - w^1\,w_i^2). \quad \square$$

With reference to the proper quantum sections Ψ and the associated distinguished quantum basis b_Ψ, the quantum lagrangian can be expressed in the following distinguished way.

Corollary 17.5.3 *For each proper quantum section $\Psi \in \sec(\boldsymbol{E},\ \boldsymbol{Q}_{/0})$, we have the following coordinate expression*

$$\mathrm{L}[\Psi] = \Big(\tfrac{1}{2}\,G_0^{ij}\,\big(\partial_i|\psi|\,\partial_j|\psi| + |\psi|^2\,\partial_i\varphi\,\partial_j\varphi\big) - \mathrm{i}\ |\psi|^2\,\partial_0\varphi \\ +\mathrm{i}\ A_0^j\,|\psi|^2\,\partial_j\varphi + \alpha_0\,|\psi|^2\Big)\,\upsilon^0.$$

Hence, with reference to the distinguished quantum basis b_Ψ, the distinguished observer o_Ψ, and the potential $A[\Psi]$ "observed by" Ψ, the above expression can be written in the following remarkable way (see Theorem 15.2.31)

$$\mathrm{L}[\Psi] = (\tfrac{1}{2}\,\bar{G}\,(\check{d}\|\Psi\|,\ \check{d}\|\Psi\|) + A[\Psi]\,\|\Psi\|^2) \otimes \upsilon. \quad \square$$

17.5.2 Quantum Momentum Form

According to a standard procedure of lagrangian theories (see, for instance [1, 141, 147, 360, 411]), the quantum Lagrangian $\mathrm{L} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{E}$ (see Theorem 17.5.2) yields the *quantum tensor form* $\mathrm{P} := \vartheta \bar{\wedge} V_{\boldsymbol{Q}}\mathrm{L}$.

Later, we shall compare the quantum vertical differential $V_{\boldsymbol{Q}}\mathrm{L}$ with the quantum tensor Q (see Theorem 17.3.2).

Let us consider the 1st jet bundle $\pi^1_0 : J_1\boldsymbol{Q} \to \boldsymbol{Q}$ of sections of the quantum bundle, along with the fibred charts $(x^\lambda, z, z_\lambda)$ (see, Appendix: Definition G.1.4 and Theorem G.1.9).

Let $\vartheta_{\boldsymbol{Q}} : J_1\boldsymbol{Q} \underset{\boldsymbol{Q}}{\times} T\boldsymbol{Q} \to V\boldsymbol{Q}$ be the *quantum contact map* with coordinate expression (see, Appendix: Propositions G.3.8 and G.3.9)

$$\vartheta_{\boldsymbol{Q}} = (dw^{\mathrm{a}} - w^{\mathrm{a}}_\lambda\, d^\lambda) \otimes \partial w_{\mathrm{a}} = (d^z - z_\lambda\, d^\lambda) \otimes \partial_z .$$

Moreover, let us denote by $\bar{\wedge}$ the exterior product followed by contraction.

Let us consider the *quantum lagrangian operator* (see Theorem 17.5.2)

$$\mathrm{L} \equiv \mathrm{L}_0\, \upsilon^0 : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{E} \otimes \mathbb{C},$$

with coordinate expression, in real and complex coordinates,

$$\begin{aligned}\mathrm{L}_0 &= (A_0 - \tfrac{1}{2}\, A_i\, A^i_0)\,(w^1\, w^1 + w^2\, w^2)\\ &\quad + (w^2\, w^1_0 - w^1\, w^2_0) - A^i_0\,(w^2\, w^1_i - w^1\, w^2_i) - \tfrac{1}{2}\, G^{ij}_0\,(w^1_i\, w^1_j + w^2_i\, w^2_j)\\ &= (A_0 - \tfrac{1}{2}\, A_0\, A^i_0)\,\bar{z}\, z + \tfrac{1}{2}\,\mathrm{i}\,(\bar{z}\, z_0 - z\,\bar{z}_0) - \tfrac{1}{2}\,\mathrm{i}\, A^i_0\,(\bar{z}\, z_i - z\,\bar{z}_i) - \tfrac{1}{2}\, G^{ij}_0\,\bar{z}_i\, z_j ,\end{aligned}$$

and

$$\upsilon^0 = \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 .$$

We start with a couple of preliminary technical Lemmas.

Lemma 17.5.4 *We have a natural linear fibred isomorphism over* $\boldsymbol{E}$

$$T\boldsymbol{E} \otimes \Lambda^4 T^*\boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E} : X \otimes \alpha \mapsto i_X\alpha .$$

The inverse isomorphism is

$$\Lambda^3 T^*\boldsymbol{E} \to T\boldsymbol{E} \otimes \Lambda^4 T^*\boldsymbol{E} : \beta \mapsto (i_{\bar{\upsilon}}\beta) \otimes \upsilon .$$

For each

$$\alpha = \alpha_{0123}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \quad \textit{and} \quad \beta = \beta_{\lambda_1\lambda_2\lambda_3}\, d^{\lambda_1} \wedge d^{\lambda_2} \wedge d^{\lambda_3},$$

we have the coordinate expressions

$$\begin{aligned} i_X\alpha &= \alpha_{0123}\,(X^0\, d^1 \wedge d^2 \wedge d^3 - X^1\, d^0 \wedge d^2 \wedge d^3\\ &\qquad + X^2\, d^0 \wedge d^1 \wedge d^3 - X^3\, d^0 \wedge d^1 \wedge d^2)\\ (i_{\bar{\upsilon}}\beta) \otimes \upsilon &= -\,3!\,(\beta_{123}\,\partial_0 - \beta_{023}\,\partial_1 + \beta_{013}\,\partial_2 - \beta_{012}\,\partial_3) \otimes d^0 \wedge d^1 \wedge d^2 \wedge d^3 . \end{aligned}$$

□

Lemma 17.5.5 *In virtue of the above Lemma 17.5.4, we can regard the vertical differential of the quantum Lagrangian as the fibred morphism over* $\boldsymbol{Q}$

$$V_{\boldsymbol{Q}}\mathsf{L} : J_1\boldsymbol{Q} \to V^*_{\boldsymbol{E}}\boldsymbol{Q} \otimes \Lambda^3 T^*\boldsymbol{E},$$

with coordinate expression

$$V_{\boldsymbol{Q}}\mathsf{L} = \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0 \, \check{d} w^{\mathrm{a}} \otimes \upsilon^0_{\lambda},$$

i.e., more explicitly, in real and complex coordinates, (see Proposition 3.2.4)

$$\begin{aligned} V_{\boldsymbol{Q}}\mathsf{L} &= (w^2\,\check{d}w^1 - w^1\,\check{d}w^2) \otimes \upsilon^0_0 - \big(G^{ij}_0\,(w^1_i\,\check{d}w^1 + w^2_i\,\check{d}w^2) \\ &\quad + A^j_0\,(w^2\,\check{d}w^1 - w^1\,\check{d}w^2)\big) \otimes \upsilon^0_j \\ &= \tfrac{1}{2}\Big(\mathrm{i}\,(\bar{z}\,\check{d}z - z\,\check{d}\bar{z}) \otimes \upsilon^0_0 - \big(G^{ij}_0\,(z_i\,\check{d}\bar{z} + \bar{z}_i\,\check{d}z) \\ &\quad + \mathrm{i}\,A^j_0\,(z\,\check{d}\bar{z} - \bar{z}\,\check{d}z)\big) \otimes \upsilon^0_j\Big). \quad \square \end{aligned}$$

Next, it is convenient to regard the vertical differential $V_{\boldsymbol{Q}}\mathsf{L}$, in a natural way, as a form.

Lemma 17.5.6 *We have the fibred morphism over* $\boldsymbol{Q}$

$$\vartheta_{\boldsymbol{Q}} \lrcorner V_{\boldsymbol{Q}}\mathsf{L} : J_1\boldsymbol{Q} \to T^*\boldsymbol{Q} \otimes \Lambda^3 T^*\boldsymbol{E},$$

with coordinate expression

$$\vartheta_{\boldsymbol{Q}} \lrcorner V_{\boldsymbol{Q}}\mathsf{L} = \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0 \, (dw^{\mathrm{a}} - w^{\mathrm{a}}_{\mu}\, d^{\mu}) \otimes \upsilon^0_{\lambda},$$

i.e., more explicitly, in real and complex coordinates,

$$\begin{aligned} \vartheta_{\boldsymbol{Q}} \lrcorner V_{\boldsymbol{Q}}\mathsf{L} &= (w^2\,dw^1 - w^1\,dw^2) \otimes \upsilon^0_0 - \big(G^{ij}_0\,(w^1_i\,dw^1 + w^2_i\,dw^2) \\ &\quad + A^j_0\,(w^2\,dw^1 - w^1\,dw^2)\big) \otimes \upsilon^0_j - (w^2\,w^1_{\lambda}\,d^{\lambda} - w^1\,w^2_{\lambda}\,d^{\lambda}) \otimes \upsilon^0_0 \\ &\quad + \big(G^{ij}_0\,(w^1_i\,w^1_{\lambda}\,d^{\lambda} + w^2_i\,w^2_{\lambda}\,d^{\lambda}) + A^j_0\,(w^2\,w^1_{\lambda}\,d^{\lambda} - w^1\,w^2_{\lambda}\,d^{\lambda})\big) \otimes \upsilon^0_j \\ &= \tfrac{1}{2}\,\mathrm{i}\,(\bar{z}\,dz - z\,d\bar{z}) \otimes \upsilon^0_0 - \tfrac{1}{2}\big(G^{ij}_0\,(\bar{z}_i\,dz + z_i\,d\bar{z}) \\ &\quad + \mathrm{i}\,A^j_0\,(z\,d\bar{z} - \bar{z}\,dz)\big) \otimes \upsilon^0_0 \\ &\quad - \tfrac{1}{2}\,\mathrm{i}\,(\bar{z}\,z_{\lambda} - z\,\bar{z}_{\lambda})\,d^{\lambda} \otimes \upsilon^0_0 + \tfrac{1}{2}\big(G^{ij}_0\,(\bar{z}_i\,z_{\lambda} + z_i\,\bar{z}_{\lambda})\,d^{\lambda} \\ &\quad + \mathrm{i}\,A^j_0\,(z\,\bar{z}_{\lambda} - \bar{z}\,z_{\lambda})\,d^{\lambda}\big) \otimes \upsilon^0_0. \quad \square \end{aligned}$$

Further, let us consider the natural linear fibred inclusion over $\boldsymbol{Q}$

$$T^*\boldsymbol{Q} \otimes \Lambda^3 T^*\boldsymbol{E} \to T^*\boldsymbol{Q} \otimes \Lambda^3 T^*\boldsymbol{Q}.$$

Proposition 17.5.7 *By antisymmetrising the map* $\vartheta \lrcorner V_{\boldsymbol{Q}} \mathsf{L}$*, we obtain the section, called* quantum momentum form

$$\mathsf{P} := \vartheta_{\boldsymbol{Q}} \bar{\wedge} V_{\boldsymbol{Q}} \mathsf{L} : J_1 \boldsymbol{Q} \to \Lambda^4 T^* \boldsymbol{Q},$$

with coordinate expression

$$\mathsf{P} = \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0 \, (d w^{\mathrm{a}} - w^{\mathrm{a}}_{\mu} \, d^{\mu}) \wedge \upsilon^0_{\lambda},$$

i.e., more explicitly, in real and complex coordinates,

$$\begin{aligned}
\mathsf{P} &= (w^2 \, dw^1 - w^1 \, dw^2) \wedge \upsilon^0_0 - \big(G^{ij}_0 \, (w^1_i \, dw^1 + w^2_i \, dw^2) \\
&\quad + A^j_0 \, (w^2 \, dw^1 - w^1 \, dw^2) \big) \wedge \upsilon^0_j + \Big(- (w^2 \, w^1_0 - w^1 \, w^2_0) \\
&\quad + \big(G^{ij}_0 \, (w^1_i \, w^1_j + w^2_i \, w^2_j) + A^j_0 \, (w^2 \, w^1_j - w^1 \, w^2_j) \big) \Big) \, \upsilon^0 \\
&= \tfrac{1}{2} \, \mathfrak{i} \, (\bar{z} \, dz - z \, d\bar{z}) \wedge \upsilon^0_0 - \tfrac{1}{2} \big(G^{ij}_0 \, (\bar{z}_i \, dz + z_i \, d\bar{z}) + \mathfrak{i} \; A^j_0 \, (z \, d\bar{z} - \bar{z} \, dz) \big) \wedge \upsilon^0_j \\
&\quad + \Big(- \tfrac{1}{2} \, \mathfrak{i} \, (\bar{z} \, z_0 - z \, \bar{z}_0) + \tfrac{1}{2} \big(G^{ij}_0 \, (\bar{z}_i \, z_j + z_i \, \bar{z}_j) + \mathfrak{i} \; A^j_0 \, (z \, \bar{z}_j - \bar{z} \, z_j) \big) \Big) \, \upsilon^0. \qquad \square
\end{aligned}$$

Remark 17.5.8 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the pullback spacetime section

$$\mathsf{P}[\Psi] := \Psi^* \mathsf{P} : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}$$

vanishes. □

We can compare the quantum kinetic tensor and the quantum kinetic momentum as follows.

Proposition 17.5.9 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we have (see Note 14.4.3 and Theorem 17.3.2)*

$$\mathsf{Q}[\Psi] = -\mathfrak{i} \, (\mathrm{re} \, \mathrm{h})^{\sharp} (i_{\bar{\upsilon}} V_{\boldsymbol{Q}} \mathsf{L})[\Psi].$$

Proof. The equalities

$$\begin{aligned}
V_{\boldsymbol{Q}} \mathsf{L} &= (w^2 \, \check{d} w^1 - w^1 \, \check{d} w^2) \otimes \upsilon^0_0 - G^{ij}_0 \, \big((w^1_i \, \check{d} w^1 + w^2_i \, \check{d} w^2) \\
&\qquad - A_i \, (w^1 \, \check{d} w^2 - w^2 \, \check{d} w^1) \big) \otimes \upsilon^0_j, \\
i_{\upsilon^0_\lambda} \bar{\upsilon}_0 &= (-1)^{\lambda} \, \partial_{\lambda}
\end{aligned}$$

yield

$$\begin{aligned}
i_{\bar{\upsilon}} V_{\boldsymbol{Q}} \mathsf{L} &= -(w^2 \, \check{d} w^1 - w^1 \, \check{d} w^2) \otimes u^0 \otimes \partial_0 + G^{ij}_0 \, \big((w^1_i \, \check{d} w^1 + w^2_i \, \check{d} w^2) \\
&\qquad - A_i \, (w^1 \, \check{d} w^2 - w^2 \, \check{d} w^1) \big) \otimes u^0 \otimes \partial_j,
\end{aligned}$$

hence, in virtue of Note 14.4.3,

$$\mathrm{i}\,(\mathrm{re}\,\mathrm{h})^\sharp(i_{\bar{v}}V_{\boldsymbol{Q}}\mathsf{L}) = -\mathrm{i}\,(w^2\,\mathrm{b} - w^1\,\mathrm{i}\,\mathrm{b})\otimes u^0\otimes\partial_0 + \mathrm{i}\,G_0^{ij}\,\big((w_i^1\,\mathrm{b} + w_i^2\,\mathrm{i}\,\mathrm{b}) - A_i\,(w^1\,\mathrm{i}\,\mathrm{b} - w^2\,\mathrm{b})\big)\otimes u^0\otimes\partial_j$$
$$= \big(-z\,\partial_0 + \mathrm{i}\,G_0^{ij}\,(z_i - \mathrm{i}\,A_i\,z)\,\partial_j\big)\otimes\mathrm{b}\otimes u^0.$$

Therefore, a comparison of the above equality with Theorem 17.3.2 yields the result. □

17.5.3 Quantum Poincaré–Cartan Form

According to a standard procedure of lagrangian field theory, the quantum lagrangian form yields the *quantum Poincaré–Cartan form* $\mathsf{C} := \mathsf{L} + \mathsf{P}$ (see, for instance [147]).

Later, the quantum Poincaré–Cartan form will be a key tool for the definition of quantum currents, via the Noether theorem (see Theorem H.3.3 and Definition 21.1.3)

Theorem 17.5.10 *The quantum lagrangian form (see Theorem 17.5.2) yields, in a covariant way, the gauge invariant and observer invariant* quantum Poincaré–Cartan form

$$\mathsf{C} := \mathsf{L} + \mathsf{P} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{Q},$$

with coordinate expression

$$\mathsf{C} = \tfrac{1}{2}\,\mathrm{i}\,(\bar{z}\,dz - z\,d\bar{z})\wedge\upsilon_0^0 - \tfrac{1}{2}\,\big(G_0^{ij}\,(\bar{z}_i\,dz + z_i\,d\bar{z}) + \mathrm{i}\,A_0^i\,(\bar{z}\,dz - z\,d\bar{z})\big)\wedge\upsilon_j^0 + \big(\tfrac{1}{2}\,G_0^{ij}\,\bar{z}_i\,z_j + \alpha_0\,\bar{z}\,z\big)\,\upsilon^0$$
$$= \mathrm{i}\,(w^1\,dw^2 - w^2\,dw^1)\wedge\upsilon_0^0 - \big(G_0^{ij}\,(w_i^1\,dw^1 + w_i^2\,dw^2) + \mathrm{i}\,(w^1\,dw^2 - w^2\,dw^1)\big)\wedge\upsilon_j^0 + \big(\tfrac{1}{2}\,G_0^{ij}\,(w_i^1\,w_j^1 + w_i^2\,w_j^2) + \alpha_0\,(w_i^1\,w_j^1 + w_i^2\,w_j^2)\big)\,\upsilon^0. \quad \square$$

17.6 Schrödinger Operator

We introduce, in a covariant way, the Schrödinger operator by a differential procedure analogous to that of the other dynamical quantum objects above (see [219]).

Moreover, we can see that the above Schrödinger operator coincides with the Euler–Lagrange operator associated with the quantum lagrangian L.

Additionally, we prove that the probability current turns out to be a conserved form along the solutions of the Schrödinger equation.

Later, we shall see that this operator becomes the usual one in the flat case (see Propositions 25.2.4, 26.2.2 and 27.2.2).

For all these reasons, we feel to be entitled to consider the above Schrödinger equation as dynamical equation of our covariant approach to Quantum Mechanics.

17.6.1 Codifferential of the Kinetic Quantum Tensor

In view of the Schrödinger operator, we start by computing, step by step, the *upper quantum covariant codifferential* $\delta^{\uparrow}(\underline{\mathrm{Q}}[\Psi])$ of the kinetic quantum tensor $\mathrm{Q}[\Psi]$, with respect to the upper quantum connection $\mathrm{Ч}^{\uparrow}$, (see Postulate Q.2 and Theorem 17.3.2).

Lemma 17.6.1 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *we define the* kinetic quantum form *to be the scaled vector valued form of spacetime (see Theorem 17.3.2)*

$$\underline{\mathrm{Q}}[\Psi] := i_{\mathrm{Q}[\Psi]} \upsilon \in \sec\big(\boldsymbol{E},\ \mathbb{L}^3 \otimes (\Lambda^3 T\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$

whose coordinate expression is (see Proposition 3.2.4)

$$\begin{aligned}\underline{\mathrm{Q}}[\Psi] &= (\psi\, \upsilon_0^0 - \mathrm{i}\, \vec{\nabla}{}_0^i \psi\, \upsilon_i^0) \otimes \mathsf{b} \\ &= \big(\psi\, \upsilon_0^0 - \mathrm{i}\, G_0^{ij}\, (\partial_j \psi - \mathrm{i}\, A_j\, \psi)\, \upsilon_i^0\big) \otimes \mathsf{b}. \quad \square\end{aligned}$$

Lemma 17.6.2 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the* upper quantum covariant differential, *with respect to the upper quantum connection* $\mathrm{Ч}^{\uparrow}$, *of the kinetic quantum form* $\underline{\mathrm{Q}}[\Psi]$ *is the scaled vector valued 4-form (see, Appendix: Note F.2.4)*

$$d^{\uparrow}(\underline{\mathrm{Q}}[\Psi]) \equiv d_{\mathrm{Ч}^{\uparrow}}(\underline{\mathrm{Q}}[\Psi]) \in \sec\big(J_1\boldsymbol{E},\ \mathbb{L}^3 \otimes (\Lambda^4 T\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$

with coordinate expression

$$\begin{aligned} d^{\uparrow}(\underline{\mathrm{Q}}[\Psi]) = \Big(&\partial_0\psi + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi - \mathrm{i}\, G_0^{ij}\, \partial_{ij}\psi - x_0^j\, \partial_j\psi \\ &- \mathrm{i}\, \frac{\partial_i (G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}} \partial_j\psi - 2\, A_0^j\, \partial_j\psi\Big)\, \upsilon^0 \otimes \mathsf{b} \\ &+ \Big(-\frac{\partial_i (G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}}\, A_j - \mathrm{i}\, A_0 + \tfrac{1}{2}\, \mathrm{i}\, G_{ij}^0\, x_0^i\, x_0^j + \mathrm{i}\, A_j\, x_0^j \\ &+ \mathrm{i}\, A_0^j\, A_j - G_0^{ij}\, \partial_i A_j\Big)\, \psi\, \upsilon^0 \otimes \mathsf{b}. \end{aligned}$$

Proof. The coordinate expressions of the quantum kinetic form and of the upper quantum covariant differential (see Definition 15.2.1, Note 17.6.1 and, Appendix: Note F.2.4)

$$\begin{aligned}\underline{\mathrm{Q}}[\Psi] &= \big(\psi\, \upsilon_0^0 - \mathrm{i}\, G_0^{ij}\, (\partial_j \psi - \mathrm{i}\, A_j\, \psi)\, \upsilon_i^0\big) \otimes \mathsf{b}, \\ d^{\uparrow}(\underline{\mathrm{Q}}[\Psi]) &= (\partial_{\lambda_1} \mathrm{Q}_{\lambda_2\lambda_3\lambda_4} - \mathrm{i}\, A^{\uparrow}{}_{\lambda_1}\, \mathrm{Q}_{\lambda_2\lambda_3\lambda_4})\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_4} \otimes \mathsf{b}\end{aligned}$$

yield the equality

$$
\begin{aligned}
d^{\uparrow}(\underline{Q}[\Psi]) = {} & \frac{1}{\sqrt{|g|}} \big(\partial_0(\sqrt{|g|}\,\psi) - \mathrm{i}\,\partial_i\big(\sqrt{|g|}\,G^{ij}_0\,(\partial_j\psi - \mathrm{i}\,A_j\,\psi)\big)\big)\,\upsilon^0 \otimes \mathrm{b} \\
& - \big(\mathrm{i}\,A^{\uparrow}{}_0\,\psi + A^{\uparrow}{}_i\,G^{ij}_0\,(\partial_j\psi - \mathrm{i}\,A_j\,\psi)\big)\,\upsilon^0 \otimes \mathrm{b} \\
= {} & \big(\partial_0\psi - \mathrm{i}\,G^{ij}_0\,\partial_{ij}\psi - x^j_0\,\partial_j\psi \\
& - \mathrm{i}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\partial_j\psi - 2\,A^j_0\,\partial_j\psi\big)\,\upsilon^0 \otimes \mathrm{b} \\
& + \big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,A_j - \mathrm{i}\,A_0 + \tfrac{1}{2}\,\mathrm{i}\,G^0_{ij}\,x^i_0\,x^j_0 \\
& + \mathrm{i}\,A_j\,x^j_0\, + \mathrm{i}\,A^j_0\,A_j - G^{ij}_0\,\partial_i A_j\big)\,\psi\,\upsilon^0 \otimes \mathrm{b}. \quad \square
\end{aligned}
$$

Lemma 17.6.3 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the* upper quantum covariant codifferential, *with respect to the upper quantum connection* $Ч^{\uparrow}$, *of the kinetic quantum tensor* $\mathrm{Q}[\Psi]$ *is the scaled fibred morphism over* $\boldsymbol{E}$ *(see the above Lemma 17.6.2)*

$$\delta^{\uparrow}(\mathrm{Q}[\Psi]) \;\equiv\; \delta^{\uparrow}{}_{Ч^{\uparrow}}(\mathrm{Q}[\Psi]) := \bar{\upsilon} \lrcorner\, d^{\uparrow}\underline{Q}[\Psi] \in \sec(J_1\boldsymbol{E},\, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

With reference to each observer o, we have the equality

$$\delta^{\uparrow}(\mathrm{Q}[\Psi]) = -д \lrcorner \nabla^{\uparrow}\Psi + 2\,д[o] \lrcorner \nabla[o]\Psi + \mathrm{div}_{\eta}\,д[o] - \mathrm{i}\,\Delta[o]\Psi.$$

We have the coordinate expression

$$
\begin{aligned}
\delta^{\uparrow}(\mathrm{Q}[\Psi]) &= \big(\partial_0 - 2\,\mathrm{i}\,A_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \mathrm{i}\,\Delta[o]_0 - x^h_0\,\partial_h + \mathrm{i}\,\mathcal{L}_0\big)\,\psi\,u^0 \otimes \mathrm{b} \\
&= (-\nabla^{\uparrow}{}_0 + 2\,\nabla[o]_0 - \mathrm{i}\,\Delta[o]_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}})\,\psi\,u^0 \otimes \mathrm{b}.
\end{aligned}
$$

Proof. The above Lemma 17.6.2 and Proposition 16.3.2 yield

$$
\begin{aligned}
\delta^{\uparrow}(\underline{Q}[\Psi]) = {} & \big(-\partial_0\psi - x^j_0\,\partial_j\psi + 2\,\partial_0\psi - \mathrm{i}\,G^{ij}_0\,\partial_{ij}\psi \\
& - \mathrm{i}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j\psi - 2\,A^j_0\,\partial_j\psi\big)\,u^0 \otimes \mathrm{b} \\
& + \big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,A_j - 2\,\mathrm{i}\,A_0 + \mathrm{i}\,\mathcal{L}_0[\mathrm{b}] \\
& + \mathrm{i}\,A^j_0\,A_j - G^{ij}_0\,\partial_i A_j\big)\,\psi\,u^0 \otimes \mathrm{b} \\
= {} & (-\nabla^{\uparrow}{}_0 + 2\,\nabla[o]_0 - \mathrm{i}\,\Delta[o]_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}})\,\psi\,u^0 \otimes \mathrm{b}. \quad \square
\end{aligned}
$$

17.6.2 Definition of the Schrödinger Operator

We introduce, in a covariant way, via the criterion of projectability (see Note 17.1.1), the *Schrödinger operator* $\mathrm{S}[\Psi] = \nabla_{д[o]}[o]\Psi + \frac{1}{2}\,\mathrm{div}_\eta д[o]\,\Psi - \mathrm{i}\,\frac{1}{2}\,\Delta[G, o]\,\Psi$ associated with every quantum section Ψ (see Propositions 2.6.1 and 16.4.1, Definition 14.5.1).

In particular, for each proper quantum section Ψ, with reference to the distinguished quantum basis b_Ψ and to the distinguished observer o_Ψ, the above expression becomes $\mathrm{S}[\Psi] = \big(д[o_\Psi].\|\Psi\| + \frac{1}{2}\,\|\Psi\|\,\mathrm{div}_\eta д[o_\Psi] - \mathrm{i}\,(\frac{1}{2}\,\Delta[G]\|\Psi\| + A[\Psi])\,\|\Psi\|\big) \otimes \mathrm{b}_\Psi$ (see Note 14.6.3 and Theorem 15.2.31).

For a comparison with the quantum Schrödinger operator in standard Quantum Mechanics (see, for instance [362, p. 138]).

Lemma 17.6.4 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we obtain, in a covariant way, the following distinguished maps, with the same source space $J_1\boldsymbol{E}$ (which here accounts pointwisely for all classical observers) and the same target space $\mathbb{T}^* \otimes \boldsymbol{Q}$, (see Propositions 2.6.1 and 16.4.1)*

$$д \lrcorner \nabla^\uparrow \Psi \in \sec(J_1\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}) \quad \textit{and} \quad \delta^\uparrow(\mathrm{Q}[\Psi]) \in \sec(J_1\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

We have the coordinate expressions (see Theorem 10.1.8, Proposition 16.4.1 and Lemma 17.6.3)

$$д \lrcorner \nabla^\uparrow \Psi = \big(\partial_0 + x_0^j\,\partial_j - \mathrm{i}\,\mathcal{L}_0\big)\,\psi\,u^0 \otimes \mathrm{b},$$
$$\delta^\uparrow(\mathrm{Q}[\Psi]) = \Big(\partial_0 - x_0^j\,\partial_j - 2\,\mathrm{i}\,A_0 + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \mathrm{i}\,\Delta_0 + \mathrm{i}\,\mathcal{L}_0\Big)\,\psi\,u^0 \otimes \mathrm{b}. \quad \square$$

Theorem 17.6.5 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the sum of the above maps (see the above Lemma 17.6.4) factorises through the spacetime $\boldsymbol{E}$ (thus cancelling the dependence on classical observers), so yielding, in a covariant way, the gauge independent and observer independent scaled section defined on spacetime*

$$\mathrm{S}[\Psi] := \tfrac{1}{2}\,\big(д \lrcorner \nabla^\uparrow \Psi + \delta^\uparrow(\mathrm{Q}[\Psi])\big) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

With reference to any observer o, we have the observed expression (which turns out to be observer equivariant) (see Propositions 2.7.3, 16.1.1 and 16.3.2, Example 3.2.18)

$$\mathrm{S}[\Psi] = \nabla_{д[o]}[o]\Psi + \tfrac{1}{2}\,\mathrm{div}_\eta д[o]\,\Psi - \mathrm{i}\,\tfrac{1}{2}\,\Delta[G, o]\,\Psi,$$

where, in coordinates,

$$\nabla_{\text{д}[o]}[o]\,\Psi = (\partial_0\psi - \mathrm{i}\,A_0\,\psi)\,u^0 \otimes \mathfrak{b},$$

$$\operatorname{div}_\eta \text{д}[o]\,\Psi = \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi\,u^0 \otimes \mathfrak{b},$$

$$\Delta[o]\,(\Psi) = \big(G_0^{ij}\,\nabla_i\,\nabla_j\,\psi + G_0^{ij}\,K_i{}^h{}_j\,\nabla_h\,\psi\big)\,u^0 \otimes \mathfrak{b}, \quad \textit{with}\ \nabla_h\psi = \partial_h\psi - \mathrm{i}\,A_h\,\psi.$$

Thus, we have the coordinate expression (see Corollary 15.2.28)

$$\mathrm{S}[\Psi] = \Big(\partial_0\psi - \tfrac{1}{2}\,\mathrm{i}\,G_0^{ij}\,\partial_{ij}\psi - (A_0^j + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G_0^{ij}\,\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi$$
$$+ \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\,\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\,2\,\alpha_0)\,\psi\Big)\,u^0 \otimes \mathfrak{b}.$$

Proof. In virtue of Lemma 17.6.4, we have the coordinate expression

$$\tfrac{1}{2}\,\big(\text{д} \lrcorner \nabla^\uparrow\Psi + \delta^\uparrow(\mathrm{Q}[\Psi])\big)$$
$$= \tfrac{1}{2}\,\big(\partial_0\psi + x_0^j\,\partial_j\psi - \mathrm{i}\,\mathcal{L}_0[\mathfrak{b}]\,\psi + \partial_0\psi - x_0^j\,\partial_j\psi - 2\,\mathrm{i}\,A_0\,\psi + \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi$$
$$- \mathrm{i}\,\Delta_0[G,o]\psi + \mathrm{i}\,\mathcal{L}_0[\mathfrak{b}]\,\psi\big)\,u^0 \otimes \mathfrak{b}$$
$$= \big(\partial_0\psi - \mathrm{i}\,A_0\,\psi + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi - \mathrm{i}\,\tfrac{1}{2}\,\Delta_0[G,o]\psi\big)\,u^0 \otimes \mathfrak{b}$$
$$= \Big(\nabla_0\psi + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,\psi - \mathrm{i}\,\tfrac{1}{2}\,\big(G_0^{ij}\,\nabla_i\,\nabla_j\psi + \frac{\partial_i(G_0^{ij}\,\sqrt{|g|})}{\sqrt{|g|}}\,\nabla_j\psi\big)\Big)\,u^0 \otimes \mathfrak{b}$$
$$= \Big(\partial_0\psi - \mathrm{i}\,\tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}\psi - (A_0^j + \mathrm{i}\,\tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\,\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi$$
$$+ \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\,\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\,\alpha_0)\,\psi\Big)\,u^0 \otimes \mathfrak{b},$$

where the coordinates x_0^i of the fibre of the classical phase space have been cancelled.

Hence, the map $\tfrac{1}{2}\,\big(\text{д} \lrcorner \nabla^\uparrow\Psi + \delta^\uparrow(\mathrm{Q}.\Psi)\big)$ factorises through the spacetime $\boldsymbol{E}$. □

Exercise 17.6.6 According to Theorem 17.6.5, the Schrödinger operator is gauge independent and observer independent, by definition.

It is instructive to further analysing this invariance by discussing in detail the terms of its expressions.

(1) The observed splitting $\mathrm{S}[\Psi] = \nabla_{\text{д}[o]}[o]\,\Psi + \tfrac{1}{2}\operatorname{div}_\eta \text{д}[o]\,\Psi - \mathrm{i}\,\tfrac{1}{2}\,\Delta[G,o]\,\Psi$ proves that the Schrödinger operator is gauge independent.
(2) Let us consider two observers o and $\acute{o} + v$. Then, by taking into account the transition rules

$$\begin{aligned}
A[\mathfrak{b}, \acute{o}] &= A[\mathfrak{b}, o] + \theta[o] \lrcorner G^{\flat}(v) - \tfrac{1}{2}\, G(v, v), \\
\nabla_{\text{д}[\acute{o}]}[\acute{o}]\, \Psi &= \nabla_{\text{д}[o]}[o]\, \Psi + \nabla_v[o]\Psi - \tfrac{1}{2}\, \mathfrak{i}\, G(v, v)\, \Psi, \\
\operatorname{div}_\eta \text{д}[\acute{o}] &= \operatorname{div}_\eta \text{д}[o] + \operatorname{div}_\eta v, \\
\Delta[G, \acute{o}]\, \Psi &= \Delta[G, o]\, \Psi - 2\, \mathfrak{i}\, \nabla_v[o]\Psi - \big(\mathfrak{i} \operatorname{div}_\eta v + G\,(v, v)\big)\, \Psi,
\end{aligned}$$

we obtain the equality

$$\begin{aligned}
&\nabla_{\text{д}[\acute{o}]}[\acute{o}]\, \Psi - \mathfrak{i}\, \tfrac{1}{2}\, \Delta[G, \acute{o}]\, \Psi + \tfrac{1}{2} \operatorname{div}_\eta \text{д}[\acute{o}]\, \Psi \\
&\quad = \nabla_{\text{д}[o]}[o]\, \Psi - \mathfrak{i}\, \tfrac{1}{2}\, \Delta[G, o]\, \Psi + \tfrac{1}{2} \operatorname{div}_\eta \text{д}[o]\, \Psi.
\end{aligned}$$

Thus, the invariance with respect to the observer is ensured by the transition rule of the observed potential $A[\mathfrak{b}, o]$ (see Theorem 15.2.26). □

Corollary 17.6.7 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, with reference to any quantum basis $\mathfrak{b}$, we have the following splitting, in terms of the observer invariants $\alpha[\mathfrak{b}]$ and $\nu[\mathfrak{b}]$ (see Corollary 15.2.28),*

$$\mathsf{S}[\Psi] = \Big(\nu_0[\mathfrak{b}].\psi - \tfrac{1}{2}\, \mathfrak{i}\, \big(\Delta[G]_0\psi - 2\, \alpha_0[\mathfrak{b}]\, \psi\big) + \tfrac{1}{2}\, \big(\operatorname{div}_\eta \nu[\mathfrak{b}]_0\, \psi\big)\Big)\, u^0 \otimes \mathfrak{b},$$

where $\Delta[G]_0\psi$ is the metric laplacian of ψ (see Definition 3.2.20).

Accordingly, we have the following coordinate expression, with reference to the quantum basis $\mathfrak{b}$ and to any observer o,

$$\begin{aligned}
\mathsf{S}[\Psi] = \Big(&(\partial_0\psi - A_0^j\, \partial_j\psi) + \tfrac{1}{2}\, \Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i\, (A_0^i\, \sqrt{|g|})}{\sqrt{|g|}}\Big)\, \psi \\
&- \tfrac{1}{2}\, \mathfrak{i}\, \Big(G_0^{ij}\, \partial_{ij}\psi - \frac{\partial_i\, (G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}}\, \partial_j\psi + 2\, \alpha_0\, \psi\Big)\Big)\, u^0 \otimes \mathfrak{b}.
\end{aligned}$$

Moreover, with reference to any quantum basis $\mathfrak{b}$ and to the associated distinguished observer $o[\mathfrak{b}]$ (see Proposition 15.2.29), the above coordinate expression becomes

$$\mathsf{S}[\Psi] = \Big(\partial_0\psi + \tfrac{1}{2}\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, \psi - \tfrac{1}{2}\, \mathfrak{i}\, \Big(G_0^{ij}\, \partial_{ij}\psi - \frac{\partial_i\, (G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}}\, \partial_j\psi + 2\, A_0\, \psi\Big)\Big)\, u^0 \otimes \mathfrak{b}.$$

Proof. The 1st coordinate expression of the corollary follows from Theorem 17.6.5.

The 2nd coordinate expression follows from the 1st one by taking into account the equality $A_0^i = 0$. □

Proposition 17.6.8 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, the coordinate expression of the Schrödinger operator can be written as*

$$\begin{aligned}
\mathrm{S}[\Psi] &= (\partial_0|\psi| + \mathrm{i}\,|\psi|\,\partial_0\varphi)\, e^{\mathrm{i}\,\varphi}\, u^0 \otimes \mathrm{b} \\
&\quad - \tfrac{1}{2}\, G^{ij}_0\, (\mathrm{i}\,\partial_{ij}|\psi| - 2\,\partial_i|\psi|\,\partial_j\varphi - |\psi|\,\partial_{ij}\varphi - \mathrm{i}\,|\psi|\,\partial_i\varphi\,\partial_j\varphi)\, e^{\mathrm{i}\,\varphi}\, u^0 \otimes \mathrm{b} \\
&\quad - (A^j_0 + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}})\,(\partial_j|\psi| + \mathrm{i}\,|\psi|\,\partial_j\varphi)\, e^{\mathrm{i}\,\varphi}\, u^0 \otimes \mathrm{b} \\
&\quad + \tfrac{1}{2}\,\Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\,\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\,2\,\alpha_0\Big)\,|\psi|\, e^{\mathrm{i}\,\varphi}\, u^0 \otimes \mathrm{b}.
\end{aligned}$$

In particular, with reference to the distinguished quantum basis b_Ψ*, the above expression becomes*

$$\begin{aligned}
\mathrm{S}[\Psi] &= \big(\partial_0|\psi| - \tfrac{1}{2}\,\mathrm{i}\,G^{ij}_0\,\partial_{ij}|\psi|\big)\, u^0 \otimes \mathrm{b}_\Psi \\
&\quad - \Big((A^j_0 + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j|\psi| + \tfrac{1}{2}\,\big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} \\
&\quad - \frac{\partial_i(A^i_0\,\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\,2\,\alpha_0\big)\,|\psi|\Big)\, u^0 \otimes \mathrm{b}_\Psi.
\end{aligned}$$

Proof. The proposition follows from Theorem 17.6.5. □

Corollary 17.6.9 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, with reference to the distinguished observer* o_Ψ *and the distinguished quantum basis* b_Ψ*, the Schrödinger operator can be written as*

$$\mathrm{S}[\Psi] = \Big(д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\,\mathrm{div}_\eta д[o_\Psi] - \mathrm{i}\,\big(\tfrac{1}{2}\,\Delta[G]\|\Psi\| + A[\Psi]\,\|\Psi\|\big)\Big) \otimes \mathrm{b}_\Psi.$$

Proof. The corollary follows from the above Proposition 17.6.8, by taking into account Definition 14.6.2 and Theorem 15.2.31. □

17.6.3 Polar Splitting of the Schrödinger Operator

For each proper quantum section Ψ, the Schrödinger operator $\mathrm{S}[\Psi]$ yields a *scaled complex quantum function* $\mathrm{S}[\Psi] = (\mathrm{S}[\Psi]/\Psi) \otimes \Psi$ (see Definition 14.6.1 and Proposition 14.7.2).

This complex function can be split into its real and imaginary components; in particular, with reference to the distinguished observer o_Ψ and the distinguished quantum basis b_Ψ, the above splitting can be written as (see Theorem 15.2.31)

$$\begin{aligned}
\mathrm{re}\,(\mathrm{S}[\Psi]/\Psi) &= \frac{д[o_\Psi].\|\Psi\|}{\|\Psi\|} + \tfrac{1}{2}\,\mathrm{div}_\eta д[o_\Psi] \quad \text{and} \\
\mathrm{im}\,(\mathrm{S}[\Psi]/\Psi) &= -\tfrac{1}{2}\,\frac{\Delta[G]\|\Psi\|}{\|\Psi\|} - A[\Psi].
\end{aligned}$$

Lemma 17.6.10 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, we can define the (gauge independent and observer independent) scaled complex function given by the ratio (see Definition 14.6.1)*

$$\mathrm{S}[\Psi]/\Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{C}),$$

by taking into account the fact that the complex fibres of the quantum bundle are 1-dimensional. Moreover, the above ratio characterises $\mathrm{S}[\Psi]$ *through the equality*

$$\mathrm{S}[\Psi] = (\mathrm{S}[\Psi]/\Psi) \otimes \Psi.$$

Indeed, the following equality holds

$$\mathrm{S}[\Psi]/\Psi = \mathrm{h}\big(\Psi, \mathrm{S}[\Psi]\big)/\|\Psi\|^2.$$

Proof. We have

$$\frac{\mathrm{h}\big(\Psi, \mathrm{S}[\Psi]\big)}{\|\Psi\|^2} = \mathrm{h}\big(\Psi, \frac{\mathrm{S}[\Psi]}{\Psi}\, \Psi\big)/\|\Psi\|^2 = \frac{\mathrm{S}[\Psi]}{\Psi}\, \frac{\|\Psi\|^2}{\|\Psi\|^2} = \frac{\mathrm{S}[\Psi]}{\Psi}. \quad \square$$

Theorem 17.6.11 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$*, the real and imaginary components of the scaled complex function* $\mathrm{S}[\Psi]/\Psi$*, with reference to an observer* o*, are given by the following equalities*

$$\begin{aligned}
\mathrm{re}\,(\mathrm{S}[\Psi]/\Psi) &= \frac{д[o].\|\Psi\|}{\|\Psi\|} + \tfrac{1}{2}\,\Delta^{\circledcirc}[G,o]((\Psi)) \\
&\quad + G\big(\frac{\vec{d}\,\|\Psi\|}{\|\Psi\|}, \vec{\nabla}^{\circledcirc}[o]((\Psi))\big) + \tfrac{1}{2}\,\mathrm{div}_\eta д[o] \\
&= \tfrac{1}{2}\,\frac{д[o].\|\Psi\|^2}{\|\Psi\|^2} + \tfrac{1}{2}\,\Delta^{\circledcirc}[G,o]((\Psi)) \\
&\quad + \tfrac{1}{2}\,G\big(\frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}, \vec{\nabla}^{\circledcirc}[o]((\Psi))\big) + \tfrac{1}{2}\,\mathrm{div}_\eta д[o] \\
\mathrm{im}\,(\mathrm{S}[\Psi]/\Psi) &= д[o] \lrcorner \nabla^{\circledcirc}[o]((\Psi)) - \tfrac{1}{2}\,\frac{\Delta[G]\|\Psi\|}{\|\Psi\|} \\
&\quad + \tfrac{1}{2}\,G\big(\vec{\nabla}^{\circledcirc}[o]((\Psi)), \vec{\nabla}^{\circledcirc}[o]((\Psi))\big) \\
&= д[o] \lrcorner \nabla^{\circledcirc}[o]((\Psi)) - \tfrac{1}{4}\,\frac{\Delta[G]\|\Psi\|^2}{\|\Psi\|^2} + \tfrac{1}{8}\,G\big(\frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}, \frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}\big) \\
&\quad + \tfrac{1}{2}\,G\big(\vec{\nabla}^{\circledcirc}[o]((\Psi)), \vec{\nabla}^{\circledcirc}[o]((\Psi))\big),
\end{aligned}$$

i.e., in coordinates, with reference to an observer o *and a quantum basis* $\mathfrak{b}$*,*

$$\begin{aligned}
\operatorname{re}\,(\mathrm{S}[\Psi]/\Psi)_0 &= \frac{\partial_0|\psi|}{|\psi|} + \tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}\varphi + G_0^{ij}\,\frac{\partial_i|\psi|}{|\psi|}\,(\partial_j\varphi - A_j) \\
&\quad - \tfrac{1}{2}\,G_0^{ij}\,\partial_i A_j + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\varphi - A_j) \\
&= \tfrac{1}{2}\,\frac{\partial_0|\psi|^2}{|\psi|^2} + \tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}\varphi + \tfrac{1}{2}\,G_0^{ij}\,\frac{\partial_i|\psi|^2}{|\psi|^2}\,(\partial_j\varphi - A_j) \\
&\quad - \tfrac{1}{2}\,G_0^{ij}\,\partial_i A_j + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\varphi - A_j), \\
\operatorname{im}\,(\mathrm{S}[\Psi]/\Psi)_0 &= \partial_0\varphi + \tfrac{1}{2}\,G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi - A_0^i\,\partial_i\varphi - A_0 + \tfrac{1}{2}\,A_i\,A_0^i \\
&\quad - \tfrac{1}{2}\,G_0^{ij}\,\frac{\partial_{ij}|\psi|}{|\psi|} - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|}{|\psi|} \\
&= \partial_0\varphi + \tfrac{1}{2}\,G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi - A_0^i\,\partial_i\varphi - A_0 + \tfrac{1}{2}\,A_i\,A_0^i \\
&\quad - \tfrac{1}{4}\,G_0^{ij}\,\frac{\partial_{ij}|\psi|^2}{|\psi|^2} + \tfrac{1}{8}\,G_0^{ij}\,\frac{\partial_i|\psi|^2}{|\psi|^2}\,\frac{\partial_j|\psi|^2}{|\psi|^2} \\
&\quad - \tfrac{1}{4}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|^2}{|\psi|^2}\,.
\end{aligned}$$

Proof. The equality (see Theorem 17.6.5)

$$\mathrm{S}[\Psi] = д[o] \lrcorner \nabla[o]\Psi - \mathrm{i}\,\tfrac{1}{2}\,\Delta[G,o]\Psi + \tfrac{1}{2}\,\mathrm{div}_\eta д[o]\,\Psi$$

and the polar splittings (see Propositions 16.1.21 and 16.3.17)

$$\begin{aligned}
\operatorname{re}\,(\nabla[o]\Psi/\Psi) &= \frac{d\|\Psi\|}{\|\Psi\|} \\
&= \tfrac{1}{2}\,\frac{d\|\Psi\|^2}{\|\Psi\|^2} \\
\operatorname{im}\,(\nabla[o]\Psi/\Psi) &= \nabla^{\circledcirc}[o](\!(\Psi)\!), \\
\operatorname{re}\,(\Delta[G,o]\Psi/\Psi) &= \frac{\Delta[G]\|\Psi\|}{\|\Psi\|} - G\,\big(\vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!), \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!)\big) \\
&= \tfrac{1}{2}\,\frac{\Delta[G]\|\Psi\|^2}{\|\Psi\|^2} - \tfrac{1}{4}\,G\,(\frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}, \frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}) \\
&\quad - G\,\big(\vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!), \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!)\big) \\
\operatorname{im}\,(\Delta[G,o]\Psi/\Psi) &= \Delta^{\circledcirc}[G,o](\!(\Psi)\!) + 2\,G\,(\frac{\vec{d}\,\|\Psi\|}{\|\Psi\|}, \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!)) \\
&= \Delta^{\circledcirc}[G,o](\!(\Psi)\!) + G\,\big(\frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}, \vec{\nabla}^{\circledcirc}[o](\!(\Psi)\!)\big)
\end{aligned}$$

yield

$$\operatorname{re}(S[\Psi]/\Psi) = \frac{д[o].\|\Psi\|}{\|\Psi\|} + \tfrac{1}{2}\,\Delta^{(\!)}[o](\!(\Psi)\!) + G\Big(\frac{\vec{d}\,\|\Psi\|}{\|\Psi\|}, \vec{\nabla}^{(\!)}[o](\!(\Psi)\!)\Big) + \tfrac{1}{2}\operatorname{div}_\eta д[o]$$

$$\operatorname{im}(S[\Psi]/\Psi) = д[o] \lrcorner \nabla^{(\!)}[o](\!(\Psi)\!) - \tfrac{1}{2}\,\frac{\Delta[G]\|\Psi\|}{\|\Psi\|} + \tfrac{1}{2}\,G\big(\vec{\nabla}^{(\!)}[o](\!(\Psi)\!), \vec{\nabla}^{(\!)}[o](\!(\Psi)\!)\big).$$

Then, the coordinate expressions

$$\Delta_0[G]\|\Psi\| = G_0^{ij}\,\partial_{ij}|\psi| + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h|\psi|,$$

$$\Delta[G,o](\!(\Psi)\!) = G_0^{ij}\,\partial_{ij}\varphi + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}$$

yield

$$\begin{aligned}\operatorname{re}(S[\Psi]/\Psi)_0 = {} & \frac{\partial_0|\psi|}{|\psi|} + \tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}\varphi \\ & + G_0^{ij}\,\frac{\partial_i|\psi|}{|\psi|}\,(\partial_j\varphi - A_j) \\ & - \tfrac{1}{2}\,G_0^{ij}\,\partial_i A_j + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\varphi - A_j),\\ \operatorname{im}(S[\Psi]/\Psi)_0 = {} & \partial_0\varphi - \tfrac{1}{2}\,G_0^{ij}\,\frac{\partial_{ij}|\psi|}{|\psi|} + \tfrac{1}{2}\,G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi \\ & - A_0^i\,\partial_i\varphi - A_0 \\ & + \tfrac{1}{2}\,A_i\,A_0^i - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|}{|\psi|}\big). \quad \square\end{aligned}$$

The Schrödinger operator is observer independent; then, we obtain a remarkable expression in terms of the distinguished observer o_Ψ, for each proper quantum section Ψ.

Corollary 17.6.12 *For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, with reference to the distinguished observer o_Ψ (see Theorem 15.2.31), we obtain the equalities*

$$\begin{aligned}
\operatorname{re}(\mathrm{S}[\Psi]/\Psi) &= \frac{д[o_\Psi].\|\Psi\|}{\|\Psi\|} + \tfrac{1}{2}\operatorname{div}_\eta д[o_\Psi] \\
&= \tfrac{1}{2}\,\frac{д[o].\|\Psi\|^2}{\|\Psi\|^2} + \tfrac{1}{2}\operatorname{div}_\eta д[o], \\
\operatorname{im}(\mathrm{S}[\Psi]/\Psi) &= -\tfrac{1}{2}\,\frac{\Delta[G]\|\Psi\|}{\|\Psi\|} - A[\Psi] \\
&= -\tfrac{1}{4}\,\frac{\Delta[G]\|\Psi\|^2}{\|\Psi\|^2} + \tfrac{1}{8}\,G\Big(\frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}, \frac{\vec{d}\,\|\Psi\|^2}{\|\Psi\|^2}\Big) - A[\Psi],
\end{aligned}$$

i.e., in coordinates, with reference to the distinguished observer o_Ψ and the distinguished quantum basis b_Ψ*,*

$$\begin{aligned}
\operatorname{re}(\mathrm{S}[\Psi]/\Psi)_0 &= \frac{\partial_0|\psi|}{|\psi|} + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} \\
&= \tfrac{1}{2}\,\frac{\partial_0|\psi|^2}{|\psi|^2} + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}, \\
\operatorname{im}(\mathrm{S}[\Psi]/\Psi)_0 &= -\tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}\frac{|\psi|}{\psi} - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|}{\psi} - A_0 \\
&= -\tfrac{1}{4}\,G_0^{ij}\,\Big(\frac{\partial_{ij}|\psi|^2}{|\psi|^2} - \tfrac{1}{2}\,\frac{\partial_i|\psi|^2}{|\psi|^2}\,\frac{\partial_j|\psi|^2}{|\psi|^2}\Big) \\
&\quad - \tfrac{1}{4}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_j|\psi|^2}{|\psi|^2} - A_0.
\end{aligned}$$

Proof. The corollary follows from the above Theorem 17.6.11, by taking into account the equalities (see Proposition 16.4.5 and Corollary 16.3.14)

$$\nabla^{⦅⦆}[o_\Psi]⦅\Psi⦆ = -A[\Psi] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}) \quad \text{and} \quad \Delta^{⦅⦆}[o_\Psi]⦅\Psi⦆ = 0. \quad \square$$

Remark 17.6.13 The two real and imaginary components of the scaled function $\mathrm{S}[\Psi]/\Psi$ appearing in the above Corollary 17.6.12 exhibit openly the norm component $\|\Psi\|$ of Ψ, while the phase component $⦅\Psi⦆$ of Ψ is involved implicitly in the choice of the distinguished observer o_Ψ and of the distinguished quantum basis b_Ψ through the distinguished quantum potential $A[\Psi]$. □

17.6.4 Schrödinger Equation

The Schrödinger operator yields the Schrödinger equation.

Indeed, as in standard Quantum Mechanics, the quantum probability current turns out to be conserved along the solutions of the Schrödinger equation.

The polar splitting of the Schrödinger equation will play a role in the hydrodynamical picture of Quantum Mechanics (see Theorem 18.2.2).

Assumption Q.2 We assume, as "*quantum dynamical equation*", the *(generalised) Schrödinger equation* (see Theorem 17.6.5)

$$\mathrm{S}[\Psi] = 0. \quad \square$$

Now, we exhibit a direct proof of the conservation of the quantum probability current along the solutions of the Schrödinger equation. Later, we shall find the same result as a consequence of a symmetry property of the quantum lagrangian (see Definitions 19.3.1 and 21.1.3, Theorems 19.3.2 and H.3.3, Example 21.1.5).

Lemma 17.6.14 *We have the equality*

$$\delta\big(\underline{\mathrm{J}}[\Psi]\big) := i_{\bar{v}} d\big(\underline{\mathrm{J}}[\Psi]\big) = 2\,\mathrm{re}\,\mathrm{h}(\Psi, \mathrm{S}[\Psi]).$$

Proof. The lemma follows from the coordinate expression (see Proposition 17.4.6)

$$\begin{aligned}
\delta\big(\underline{\mathrm{J}}[\Psi]\big)_0 &= |\psi|^2\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + (\bar{\psi}\,\partial_0\psi + \psi\,\partial_0\bar{\psi}) + \mathrm{i}\,\tfrac{1}{2}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) \\
&\quad + \mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\partial_i\psi\,\partial_j\bar{\psi} - \partial_i\bar{\psi}\,\partial_j\psi) + \mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\psi\,\partial_{ij}\bar{\psi} - \bar{\psi}\,\partial_{ij}\psi) \\
&\quad - \partial_i A^i_0\,|\psi|^2 - A^i_0\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,|\psi|^2 - A^i_0\,(\bar{\psi}\,\partial_i\psi + \psi\,\partial_i\bar{\psi}) \\
&= (\bar{\psi}\,\partial_0\psi + \psi\,\partial_0\bar{\psi}) + \mathrm{i}\,\tfrac{1}{2}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi) \\
&\quad + \mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\partial_i\psi\,\partial_j\bar{\psi} - \partial_i\bar{\psi}\,\partial_j\psi) + \mathrm{i}\,\tfrac{1}{2}\,G^{ij}_0\,(\psi\,\partial_{ij}\bar{\psi} - \bar{\psi}\,\partial_{ij}\psi) \\
&\quad + \Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A^i_0\,\sqrt{|g|})}{\sqrt{|g|}}\Big)\,|\psi|^2 - A^i_0\,(\bar{\psi}\,\partial_i\psi + \psi\,\partial_i\bar{\psi}). \quad \square
\end{aligned}$$

Proposition 17.6.15 *The probability current form is conserved along the solutions of the Schrödinger equation:*

$$\mathrm{S}[\Psi] = 0 \qquad \Rightarrow \qquad \delta\big(\underline{\mathrm{J}}[\Psi]\big) = 0.$$

Proof. The proposition follows immediately from the above Lemma 17.6.14. $\square$

Example 17.6.16 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave

$$\Psi := |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with } \varphi = \mathrm{k}_\lambda\,x^\lambda, \;\; \mathrm{k}_\lambda \in \mathbb{R}, \;\; |\psi| \neq 0.$$

Then, we obtain

$$\begin{aligned}
\mathrm{S}[\Psi] = \Big(&\partial_0|\psi| + \mathrm{i}\,|\psi|\,\mathrm{k}_0 - \tfrac{1}{2}\,G_0^{ij}\,(\mathrm{i}\,\partial_{ij}|\psi| - 2\,\partial_i|\psi|\,\mathrm{k}_j - \mathrm{i}\,|\psi|\,\mathrm{k}_i\,\mathrm{k}_j) \\
&- A_0^j\,(\partial_j|\psi| + \mathrm{i}\,|\psi|\,\mathrm{k}_j) - \tfrac{1}{2}\left(\partial_i A_0^i + \mathrm{i}\;2\,\alpha_0\right)|\psi|\Big)\,e^{\mathrm{i}\,\varphi}\,u^0 \otimes \mathrm{b} \\
= \Big(&\partial_0|\psi| + G_0^{ij}\,\partial_i|\psi|\,\mathrm{k}_j - A_0^j\,\partial_j|\psi| - \tfrac{1}{2}\,\partial_i A_0^i\,|\psi| \\
&+ \mathrm{i}\left(|\psi|\,\mathrm{k}_0 - \tfrac{1}{2}\,G_0^{ij}\,(\partial_{ij}|\psi| - |\psi|\,\mathrm{k}_i\,\mathrm{k}_j) - A_0^j\,\mathrm{k}_j\,|\psi| - \alpha_0\,|\psi|\right)\Big)\,e^{\mathrm{i}\,\varphi}\,u^0 \otimes \mathrm{b}.
\end{aligned}$$

Hence, $\mathrm{S}[\Psi] = 0$ if and only if, in cartesian coordinates,

$$\begin{aligned}
0 &= \partial_0 \log|\psi| + \tfrac{\hbar_0}{m}\,\delta^{ij}\left(\partial_i \log|\psi|\,(\mathrm{k}_j - A_j) - \tfrac{1}{2}\,\partial_i A_j\right), \\
0 &= (\mathrm{k}_0 - A_0) - \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{ij}\left(\frac{\partial_{ij}|\psi|}{|\psi|} - (\mathrm{k}_i - A_i)\,(\mathrm{k}_j - A_j)\right). \quad \square
\end{aligned}$$

17.6.5 Polar Splitting of the Schrödinger Equation

The polar splitting of the Schrödinger operator yields a remarkable polar splitting of the Schrödinger equation into the system of equations,

$$Д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\,\mathrm{div}_\eta Д[o_\Psi] = 0 \quad \text{and} \quad \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi] = 0,$$

in terms of the distinguished observer o_Ψ and potential $A[\Psi]$, (see Theorem 15.2.31).

Indeed, later, the above system will play a relevant role in the hydrodynamical picture of quantum dynamics (see Theorem 18.2.2).

Proposition 17.6.17 *A proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ is a solution of the Schrödinger equation $\mathrm{S}[\Psi] = 0$ if and only if it is a solution of the following system of real equations (see Definition 14.6.1)*

$$\begin{aligned}
0 = {} & \partial_0|\psi| + \tfrac{1}{2}\,|\psi|\,G_0^{ij}\,\partial_{ij}\varphi + G_0^{ij}\,\partial_i|\psi|\,(\partial_j\varphi - A_j) \\
& - \tfrac{1}{2}\,G_0^{ij}\,|\psi|\,\partial_i A_j + \tfrac{1}{2}\,|\psi|\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} + \tfrac{1}{2}\,|\psi|\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,(\partial_j\varphi - A_j), \\
0 = {} & -\tfrac{1}{2}\,G_0^{ij}\,\partial_{ij}|\psi| - \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi| \\
& + |\psi|\left((\partial_0\varphi - A_0) + \tfrac{1}{2}\,G_0^{ij}\,(\partial_i\varphi - A_i)\,(\partial_j\varphi - A_j)\right). \quad \square
\end{aligned}$$

By taking into account Theorem 15.2.31, we obtain an "intrinsic" polar splitting of the Schrödinger equation, as follows. Indeed, the following result plays a role in the hydrodynamical description of Covariant Quantum Mechanics (see Theorem 18.2.2).

Corollary 17.6.18 *A proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ *is a solution of the Schrödinger equation* $\mathsf{S}[\Psi] = 0$ *if and only if the pair* $\big(\|\Psi\|, A[\Psi]\big)$ *is a solution of the following system of equations, related to the distinguished quantum basis* $\mathfrak{b}_\Psi$ *and to the distinguished observer* o_Ψ *(see Theorem 15.2.31, Corollary 17.6.12 and Example 3.2.18)*

$$0 = д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\,\|\Psi\|\,\mathrm{div}_\eta д[o_\Psi],$$
$$0 = \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi].$$

The coordinate expression of the above system, with reference to the distinguished observer o_Ψ *and to the distinguished quantum basis* $\mathfrak{b}_\Psi$*, is*

$$0 = \partial_0|\psi| + \tfrac{1}{2}\,|\psi|\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}},$$
$$0 = G^{ij}_0\,\partial_{ij}|\psi| + \frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi| + 2\,A_0\,|\psi|. \quad \square$$

17.6.6 Lagrangian Approach to Schrödinger Equation

The Schrödinger equation can be also derived from the quantum lagrangian, via a lagrangian approach, as in standard Quantum Mechanics (see, for instance [84], [362, p. 248]).

Thus, we show that the Schroedinger operator S (see Theorem 17.6.5) can be derived from the quantum lagrangian L (see Theorem 17.5.2).

Actually, according to a standard geometric procedure of lagrangian theories (see, for instance [1, 147, 360, 411]), the quantum Lagrangian L yields the *quantum Euler–Lagrange* fibred morphism $\mathsf{E}[\mathsf{L}] := d_V\,\mathsf{P}[\mathsf{L}] - d\,\mathsf{C}[\mathsf{L}]$ over $\boldsymbol{Q}$ (see Proposition 17.5.7 and Theorem 17.5.10).

Indeed, we obtain the equality $\mathsf{S}[\Psi] = \frac{1}{2}\,\mathfrak{i}\,i_{\bar{v}}(\mathrm{re}\,\mathrm{h})^\sharp\big(\mathsf{E}[\mathsf{L}]\big) \circ j_2\Psi$.

Definition 17.6.19 The *vertical differential* of $\mathsf{P}[\mathsf{L}]$ is defined to be the form (see [411])

$$d_V\,\mathsf{P}[\mathsf{L}] := i_{\vartheta_2} d\mathsf{P}[\mathsf{L}] - d i_{\vartheta_1}\mathsf{P}[\mathsf{L}] : J_2\boldsymbol{Q} \to \Lambda^5 T^* J_1\boldsymbol{Q},$$

where

$$\vartheta_1 : J_1\boldsymbol{Q} \to T^*\boldsymbol{E} \otimes \boldsymbol{Q} \quad \text{and} \quad \vartheta_2 : J_2\boldsymbol{Q} \to T^* J_1\boldsymbol{E} \otimes (T^*\boldsymbol{E} \otimes \boldsymbol{Q})$$

are the 1st and 2nd order quantum complementary contact maps (see, Appendix: Proposition G.3.8).

Proposition 17.6.20 *We have the coordinate expression*

$$d_V \mathsf{P}[\mathsf{L}] = \partial w_{\mathrm{b}}\, \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0\, \vartheta^{\mathrm{b}} \wedge \vartheta^{\mathrm{a}} \wedge \upsilon^0_{\lambda} + \partial w^{\mu}_{\mathrm{b}}\, \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0\, \vartheta^{\mathrm{b}}_{\mu} \wedge \vartheta^{\mathrm{a}} \wedge \upsilon^0_{\lambda}.$$

Proof. The proof follows from the equalities (see Proposition 17.5.7 and, Appendix: Proposition G.3.8

$$\begin{aligned}\mathsf{P}[\mathsf{L}] = {} & (w^2\, dw^1 - w^1\, dw^2) \wedge \upsilon^0_0 - G^{ij}_0 \big((w^1_i\, dw^1 + w^2_i\, dw^2) \\ & + A_i\, (w^2\, dw^1 - w^1\, dw^2) \big) \wedge \upsilon^0_j + \Big(- (w^2\, w^1_0 - w^1\, w^2_0) \\ & + G^{ij}_0 \big((w^1_i\, w^1_j + w^2_i\, w^2_j) + A_i\, (w^2\, w^1_j - w^1\, w^2_j) \big) \Big)\, \upsilon^0, \\ \vartheta_1 = {} & (dw^{\mathrm{a}} - w^{\mathrm{a}}_{\lambda}\, d^{\lambda}) \otimes \partial w_{\mathrm{a}} \quad \text{and} \quad \vartheta_2 = (dw^{\mathrm{a}}_{\mu} - w^{\mathrm{a}}_{\lambda\mu}\, d^{\lambda}) \otimes \partial w^{\mu}_{\mathrm{a}}. \quad \square\end{aligned}$$

Definition 17.6.21 We define the *quantum Euler–Lagrange operator* to be the fibred morphism over $\boldsymbol{Q}$ (see, Theorem 17.5.10 and, for instance [360, 411])

$$\mathsf{E}[\mathsf{L}] := d_V \mathsf{P}[\mathsf{L}] - d\mathsf{C}[\mathsf{L}] : J_2\boldsymbol{Q} \to \Lambda^5 T^* J_1 \boldsymbol{Q}. \quad \square$$

Proposition 17.6.22 *We can naturally regard* $\mathsf{E}[\mathsf{L}]$ *as a fibred morphism over* $\boldsymbol{E}$

$$\mathsf{E}[\mathsf{L}] : J_2\boldsymbol{Q} \to \Lambda^4\, T^*\boldsymbol{E} \otimes V^*_{\boldsymbol{E}}\, \boldsymbol{Q},$$

with coordinate expression,

$$\mathsf{E}[\mathsf{L}] = \Big((\partial_{\lambda} + w^{\mathrm{b}}_{\lambda}\, \partial w_{\mathrm{b}} + w^{\mathrm{b}}_{\lambda\mu}\, \partial w^{\mu}_{\mathrm{b}})\, \partial w^{\lambda}_{\mathrm{a}} \mathsf{L}_0 - \partial w_{\mathrm{a}} \mathsf{L}_0 \Big)\, \check{d}w^{\mathrm{a}} \otimes \upsilon^0,$$

i.e., more explicitly, in real coordinates,

$$\begin{aligned}\mathsf{E}[\mathsf{L}] = 2 \Big(& w^2_0 - A_0\, w^1 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, w^2 \\ & - \tfrac{1}{2}\, G^{ij}_0\, \big(w^1_{ij} + 2\, A_i\, w^2_j + \partial_i A_j\, w^2 - A_i\, A_j\, w^1 \big) \\ & - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^1_j + A_j\, w^2) \Big)\, \upsilon^0 \otimes \check{d}w^1 \\ & - 2 \Big(w^1_0 + A_0\, w^2 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, w^1 \\ & - \tfrac{1}{2}\, G^{ij}_0\, \big(- w^2_{ij} + 2\, A_i\, w^1_j + \partial_i A_j\, w^1 + A_i\, A_j\, w^2 \big) \\ & + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^2_j - A_j\, w^1) \Big)\, \upsilon^0 \otimes \check{d}w^2.\end{aligned}$$

Proof. The coordinate expression (see Theorem 17.5.2)

$$\mathsf{L} = \big(-\tfrac{1}{2}\, G_0^{ij}\, (w_i^1\, w_j^1 + w_i^2\, w_j^2) - (w^1\, w_0^2 - w^2\, w_0^1) + G_0^{ij}\, (w^1\, w_i^2 - w^2\, w_i^1)\, A_j \\ + \alpha_0\, ((w^1)^2 + (w^2)^2) \big)\, \upsilon^0$$

yields the following equalities

$$\partial w_1^0 \mathsf{L} = w^2\, \upsilon^0, \qquad \partial w_2^0 \mathsf{L} = -w^1\, \upsilon^0,$$
$$\partial w_1^i \mathsf{L} = -G_0^{ij}\, (w_j^1 + A_j\, w^2)\, \upsilon^0, \qquad \partial w_2^i \mathsf{L} = -G_0^{ij}\, (w_j^2 - A_j\, w^1)\, \upsilon^0,$$

$$\partial_\lambda \partial w_1^\lambda \mathsf{L} = \Big(w^2\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \partial_i G_0^{ij} (w_j^1 + A_j\, w^2) - G_0^{ij}\, \partial_i A_j\, w^2 \\ - G_0^{ij}\, (w_j^1 + A_j\, w^2)\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}} \Big)\, \upsilon^0,$$
$$\partial_\lambda \partial w_2^\lambda \mathsf{L} = -\Big(w^1\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} + \partial_i G_0^{ij} (w_j^2 - A_j\, w^1) - G_0^{ij}\, \partial_i A_j\, w^1 \\ + G_0^{ij}\, (w_j^2 - A_j\, w^1)\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}} \Big)\, \upsilon^0,$$

$$(\partial w_{\mathrm{a}} \partial w_1^\lambda \mathsf{L})\, w_\lambda^{\mathrm{a}} = (w_0^2 - G_0^{ij}\, A_j\, w_i^2)\, \upsilon^0,$$
$$(\partial w_{\mathrm{a}} \partial w_2^\lambda \mathsf{L})\, w_\lambda^{\mathrm{a}} = -(w_0^1 - G_0^{ij}\, A_j\, w_i^1)\, \upsilon^0,$$
$$(\partial w_{\mathrm{a}}^\mu \partial w_1^\lambda \mathsf{L})\, w_{\lambda\mu}^{\mathrm{a}} = -G_0^{ij}\, w_{ij}^1\, \upsilon^0, \quad (\partial w_{\mathrm{a}}^\mu \partial w_2^\lambda \mathsf{L})\, w_{\lambda\mu}^{\mathrm{a}} = -G_0^{ij}\, w_{ij}^2\, \upsilon^0,$$

$$\partial w_1 \mathsf{L} = \big(-w_0^2 + G_0^{ij}\, A_j\, w_i^2 + 2\, (A_0 - \tfrac{1}{2}\, G_0^{ij}\, A_i\, A_j)\, w^1 \big)\, \upsilon^0,$$
$$\partial w_2 \mathsf{L} = \big(w_0^1 - G_0^{ij}\, A_j\, w_i^1 + 2\, (A_0 - \tfrac{1}{2}\, G_0^{ij}\, A_i\, A_j)\, w^2 \big)\, \upsilon^0.$$

Hence, we obtain

$$\begin{aligned}
\mathsf{E}_1 &= \partial_\lambda(\partial w_1^\lambda \mathsf{L}) + \partial w_{\mathrm{a}}(\partial w_1^\lambda \mathsf{L})\, w_\lambda^{\mathrm{a}} + \partial w_{\mathrm{a}}^\mu(\partial w_1^\lambda \mathsf{L})\, w_{\lambda\mu}^{\mathrm{a}} - \partial w_1 \mathsf{L} \\
&= \big(w^2\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \partial_i G_0^{ij}\,(w_j^1 + A_j\, w^2) - G_0^{ij}\, \partial_i A_j\, w^2 \\
&\quad - G_0^{ij}\,(w_j^1 + A_j\, w^2)\, \frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}} \\
&\quad + w_0^2 - G_0^{ij}\, A_j\, w_i^2 - G_0^{ij}\, w_{ij}^1 + w_0^2 - G_0^{ij}\, A_j\, w_i^2 - 2\,\alpha_0\, w^1\big)\, \upsilon^0 \\
&= 2\Big(w_0^2 - \alpha_0\, w^1 + \tfrac{1}{2}\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, w^2 - \tfrac{1}{2}\, G_0^{ij}\, \big(w_{ij}^1 + 2\, A_i\, w_j^2 + \partial_i A_j\, w^2\big) \\
&\quad - \tfrac{1}{2}\, \frac{\partial_i(G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}}\, (w_j^1 + A_j\, w^2)\Big)\, \upsilon^0,
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{E}_2 &= \partial_\lambda(\partial w_2^\lambda \mathsf{L}) + \partial w_{\mathrm{a}}(\partial w_2^\lambda \mathsf{L})\, w_\lambda^{\mathrm{a}} + \partial w_{\mathrm{a}}^\mu(\partial w_2^\lambda \mathsf{L})\, w_{\lambda\mu}^{\mathrm{a}} - \partial w_2 \mathsf{L} \\
&= \big(- w^1\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \partial_i G_0^{ij}\,(w_j^2 - A_j\, w^1) + G_0^{ij}\, \partial_i A_j\, w^1 \\
&\quad - G_0^{ij}\,(w_j^2 - A_j\, w^1)\, \frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}} \\
&\quad - w_0^1 + G_0^{ij}\, A_j\, w_i^1 - G_0^{ij}\, w_{ij}^2 - w_0^1 + G_0^{ij}\, A_j\, w_i^1 - 2\,\alpha_0\, w^2\big)\, \upsilon^0 \\
&= -2\Big(w_0^1 + \alpha_0\, w^2 + \tfrac{1}{2}\, \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, w^1 - \tfrac{1}{2}\, G_0^{ij}\, \big(- w_{ij}^i + 2\, A_i\, w_j^1 + \partial_i A_j\, w^1\big) \\
&\quad + \tfrac{1}{2}\, \frac{\partial_i(G_0^{ij}\, \sqrt{|g|})}{\sqrt{|g|}}\, (w_j^2 - A_j\, w^1)\Big)\, \upsilon^0. \quad \square
\end{aligned}$$

Now, we can prove that the Schrödinger operator S exhibited by Theorem 17.6.5 can be obtained from the quantum lagrangian L provided by Theorem 17.5.2, through the Euler-Lagrange operator $\mathsf{E}[\mathsf{L}]$.

Theorem 17.6.23 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we have*

$$\mathsf{S}[\Psi] = \tfrac{1}{2}\, \mathfrak{i}\, i_{\bar{\upsilon}}(\mathrm{re}\, \mathfrak{h})^\sharp\big(\mathsf{E}[\mathsf{L}]\big) \circ j_2\Psi.$$

Proof. In virtue of Proposition 17.6.22 and of the natural fibred isomorphism $V\boldsymbol{Q} \simeq \boldsymbol{Q}$, we can write $\partial w_{\mathrm{a}} \simeq \mathsf{b}_{\mathrm{a}}$ and obtain

$$
\begin{aligned}
i_{\bar{v}}(\mathrm{re}\,\mathsf{h})^\sharp(\mathsf{E}) = 2\Big(& w^2_0 - A_0\, w^1 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, w^2 \\
& - \tfrac{1}{2}\, G^{ij}_0\, \big(w^1_{ij} + 2\, A_i\, w^2_j + \partial_i A_j\, w^2 - A_i\, A_j\, w^1\big) \\
& - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^1_j + A_j\, w^2)\Big)\, u^0 \otimes \mathsf{b}_1 \\
& - 2\Big(w^1_0 + A_0\, w^2 + \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, w^1 \\
& - \tfrac{1}{2}\, G^{ij}_0\, \big(- w^2_{ij} + 2\, A_i\, w^1_j + \partial_i A_j\, w^1 + A_i\, A_j\, w^2\big) \\
& + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^2_j - A_j\, w^1)\Big)\, u^0 \otimes \mathsf{b}_2 \\
= 2\Big(& - \mathrm{i}\, (w^1_0 + \mathrm{i}\, w^2_0) + A_0\, (w^1 + \mathrm{i}\, w^2) - \tfrac{1}{2}\, \mathrm{i}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, (w^1 + \mathrm{i}\, w^2) \\
& - \tfrac{1}{2}\, G^{ij}_0\, (w^1_{ij} + \mathrm{i}\, w^2_{ij}) + \mathrm{i}\, G^{ij}_0\, A_i\, (w^1_j + \mathrm{i}\, w^2_j) \\
& + \tfrac{1}{2}\, \mathrm{i}\, G^{ij}_0\, \partial_i A_j\, (w^1 + \mathrm{i}\, w^2) + \tfrac{1}{2}\, G^{ij}_0\, A_i\, A_j\, (w^1 + \mathrm{i}\, w^2) \\
& - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^1_j + \mathrm{i}\, w^2_j) \\
& + \tfrac{1}{2}\, \mathrm{i}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, A_j\, (w^1 + \mathrm{i}\, w^2)\Big)\, u^0 \otimes \mathsf{b}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
i_{\bar{v}}(\mathrm{re}\,\mathsf{h})^\sharp(\mathsf{E}) \circ j_2\Psi = 2\,\mathrm{i}\Big(& - \partial_0\psi + \mathrm{i}\, A_0\, \psi - \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, \psi + \tfrac{1}{2}\, \mathrm{i}\, G^{ij}_0\, \partial_{ij}\psi \\
& + G^{ij}_0\, A_i\, \partial_j \psi + \tfrac{1}{2}\, G^{ij}_0\, \partial_i A_j\, \psi - \tfrac{1}{2}\, \mathrm{i}\, G^{ij}_0\, A_i\, A_j\, \psi \\
& + \tfrac{1}{2}\, \mathrm{i}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, \partial_j \psi + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, A_j\, \psi\Big)\, u^0 \otimes \mathsf{b} \\
= -2\,\mathrm{i}\,\mathsf{S}[\Psi]. \quad \square &
\end{aligned}
$$

Then, the Schrödinger equation can be written in lagrangian language as follows.

Corollary 17.6.24 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$*, we have*

$$
\mathsf{S}[\Psi] = 0 \qquad \Leftrightarrow \qquad \mathsf{E}[\mathsf{L}] \circ j_2\Psi = 0. \quad \square
$$

17.6.7 Quantum Noether Theorem

We start by translating in our quantum framework a classical Theorem of lagrangian theory, which characterises the critical sections of the Euler–Lagrange equation via the Poincaré–Cartan form (see, for instance [360, 411]).

Then, we obtain the quantum Noether theorem as a direct consequence of the above result (see, also later, Theorem 21.2.4).

Indeed, the conservation of the quantum probability current can be regarded as a particular case of the above theorem (see Proposition 17.6.15).

Theorem 17.6.25 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *the following equivalence holds:*

$$\mathsf{E}[\mathsf{L}] \circ j_2\Psi = 0$$

if and only if

$$(j_1\Psi)^*(i_{Y_1} d\mathsf{C}[\mathsf{L}]) = 0, \quad \text{for each } Y \in \mathrm{pro}(\boldsymbol{Q}, T\boldsymbol{Q}),$$

where $Y_1 := r_1 \circ TY \in \sec(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q})$ *is the 1st holonomic prolongation of* Y *(see, Appendix: Theorem G.6.3 and Example G.6.5, and Proposition 12.3.1).* □

Indeed, the above Theorem 17.6.25 yields the Noether Theorem, which deals with the link between symmetries of the quantum Poincaré–Cartan form C and conserved quantum currents (see, also, Theorem 21.2.4).

Theorem 17.6.26 (Quantum Noether theorem) *For each* $Y \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *we have the implication*

$$\mathsf{E}[\mathsf{L}] \circ j_2\Psi = 0, \quad L_{Y_1}\mathsf{C}[\mathsf{L}] = 0 \qquad \Rightarrow \qquad d\Big((j_1\Psi)^*(i_{Y_1}\mathsf{C})\Big) = 0.$$

Proof. The implication, for each $Y \in \mathrm{pro}(\boldsymbol{Q}, TJ_1\boldsymbol{Q})$, (see the above Theorem 17.6.25),

$$\mathsf{E} \circ j_2\Psi = 0 \qquad \Rightarrow \qquad (j_1\Psi)^*(i_{Y_1} d\mathsf{C}) = 0,$$

can be written as

$$\mathsf{E} \circ j_2\Psi = 0 \qquad \Rightarrow \qquad (j_1\Psi)^*(L_{Y_1}\mathsf{C} - d i_{Y_1}\mathsf{C}) = 0.$$

Hence, we obtain the implication

$$\mathsf{E} \circ j_2\Psi = 0, \quad L_{Y_1}\mathsf{C} = 0 \qquad \Rightarrow \qquad (j_1\Psi)^*(d i_{Y_1}\mathsf{C}) = 0,$$

i.e., by exchanging the exterior differential and the pullback,

$$\mathsf{E} \circ j_2\Psi = 0, \quad L_{Y_1}\mathsf{C} = 0 \qquad \Rightarrow \qquad d\Big((j_1\Psi)^*(i_{Y_1}\mathsf{C})\Big) = 0. \quad \square$$

Eventually, as an example, we show that the conservation of the probability current (see Proposition 17.6.15) can be derived from the quantum Noether theorem.

Later, we shall discuss more extensively, this subject in the context of quantum currents (see Sect. 21.1.1).

Example 17.6.27 Later, we shall see that the distinguished "conserved time preserving special phase function" $1 \in \text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R})$ (see Definitions 12.1.3 and 12.6.10) yields in a natural way the distinguished vector field (see Lemma 21.1.2, Examples 19.1.8 and 21.1.5)

$$Y_1 : J_1\boldsymbol{Q} \to T J_1\boldsymbol{Q},$$

with coordinate expression

$$Y_1 = w^1\,\partial w_2 - w^2\,\partial w_1 + w^1_\mu\,\partial w^\mu_2 - w^2_\mu\,\partial w^\mu_1.$$

Indeed, we have

$$L_{Y_1}\mathrm{C} = 0.$$

Hence, in virtue of the above Noether Theorem 17.6.26, we obtain, for each solution Ψ of the quantum Euler–Lagrange equation, the conserved form (see, also, Proposition 17.6.15)

$$(j_1\Psi)^*\big(i_{Y_1}\mathrm{C}\big) : \boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E}.$$

Actually, this form turns out to be just the probability current (see Theorem 17.4.2). □

17.7 Purely Covariant Approach

In the previous sections of this chapter, we have presented a constructive approach to the covariant quantum dynamical objects $\mathsf{V}[\Psi]$, $\mathsf{Q}[\Psi]$, $\mathsf{J}[\Psi]$, $\mathsf{L}[\Psi]$, $\mathsf{S}[\Psi]$ (see Theorems 17.2.2, 17.3.2, 17.4.2, 17.5.2, and 17.6.5).

Actually, we have shown that the upper quantum connection $\mathrm{Ч}^\uparrow$ yields in a natural way distinguished objects based on the classical phase space. Then, according to the criterion of projectability (see Note 17.1.1), we have derived distinguished objects which factorise through spacetime, in other words, which turn out to be observer independent.

This procedure turns out to be an implementation of covariance in Quantum Mechanics.

In particular, by the above procedure, we have found a Schrödinger operator and a quantum lagrangian, by a covariant geometric procedure and have shown that the Schrödinger operator can be derived from the quantum lagrangian.

Then, the following natural questions arise:

- There exist other possible covariant Schrödinger operators?
- There exist other possible covariant quantum lagrangians?
- Are all covariant Schrödinger operators derived from the quantum lagrangian?
- Can we derive the Schrödinger operator and the quantum lagrangian by means of the only requirement of covariance?

These questions have been answered in [219] by proving that, under reasonable weak conditions concerning the order of derivatives, our Schrödinger operator and our quantum lagrangian are essentially unique, provided they are covariant with respect to the quantum covariance group and to the change of units of measurement (see, Appendix: Definition J.4.7).

We stress that if, in the above proof of uniqueness, we release the invariance of the Schrödinger operator and of the quantum lagrangian with respect to the change of units of measurement, then other solutions are admissible. For instance, under such released hypotheses, non linear Schrödinger equation is admissible (see, for instance [86–91]).

Here, for the convenience of the reader, we recall the content of the paper [219], inserting it in a consistent way in the framework of the present book. Unfortunately, this discussion is very technical, hence the reader might wish to grasp the main results, focusing his attention on the claims and skipping the details of technicalities.

17.7.1 Covariant Operators

First of all, we recall the notion of *covariance* and *gauge covariance* and formulate our general problem (see, Appendix: Definitions J.4.6 and J.4.10).

In order to prove uniqueness theorems, we need to consider the classical and quantum objects G, K, h, $\Psi^{\uparrow}$ as variable sections of natural or gauge natural bundles (see, Appendix: Definitions J.2.1 and J.3.1).

For this purpose, we observe the following facts.

Note 17.7.1 In our theory, we deal with the following objects (see, Appendix: Proposition J.3.12):

- the *galilean metrics* can be regarded as sections (see Definition 3.2.1)

$$G : \boldsymbol{E} \to \mathrm{Met}(\boldsymbol{E}) := \mathbb{T} \otimes (V^* \boldsymbol{E} \otimes V^* \boldsymbol{E})$$

  of a 1st order natural scaled vector bundle $\mathrm{Met}(\boldsymbol{E}) \to \boldsymbol{E}$,
- the *contravariant rescaled galilean metrics* can be regarded as sections (see Definition 3.2.2)

$$\bar{G} : \boldsymbol{E} \to \overline{\mathrm{Met}}(\boldsymbol{E}) := \mathbb{T}^* \otimes (V \boldsymbol{E} \otimes V \boldsymbol{E})$$

  of a 1st order natural scaled vector bundle $\overline{\mathrm{Met}}(\boldsymbol{E}) \to \boldsymbol{E}$,

- the *special spacetime connections* can be regarded as sections (see Definition 4.1.19)

$$K : \boldsymbol{E} \to \text{Con}(\boldsymbol{E})$$

of a 2nd order natural bundle $\text{Con}(\boldsymbol{E}) \to \boldsymbol{E}$,
- the *hermitian quantum metrics* can be regarded as sections (see Proposition 14.3.1)

$$\mathfrak{h} : \boldsymbol{E} \to \text{Met}(\boldsymbol{Q}) := \mathbb{L}^{-3} \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*)$$

of a 0-order $U(1, \mathbb{C})$-gauge natural scaled vector bundle $\text{Met}(\boldsymbol{Q}) \to \boldsymbol{E}$
- the reducible $\mathbb{R}$-*linear upper quantum connections* can be regarded as sections (see Definition 15.1.5)

$$Ч^{\uparrow} : J_1\boldsymbol{E} \to \text{Con}^{\uparrow}(\boldsymbol{Q}^{\uparrow})$$

of a $(1, 1)$-order $U(1, \mathbb{C})$-gauge natural bundle. □

Note 17.7.2 We recall that, according to our postulates, the above variables fulfill the following *interaction constraints*, which imply that the spacetime connections K and that the upper quantum connections $Ч^{\uparrow}$ are galilean (see Definitions 3.2.2, 4.2.3 and 15.1.5, Theorem 9.2.15)

$$\bar{G} = (G^{\sharp} \otimes G^{\sharp})(G), \quad \nabla[K]\, G = 0, \quad d\,\Omega[G, K] = 0,$$
$$\nabla[Ч^{\uparrow}]\,\mathfrak{h} = 0, \quad R[Ч^{\uparrow}] = -2\,\mathfrak{i}\,\Omega[G, K] \otimes \mathbb{I}. \quad \square$$

Remark 17.7.3 We stress that the fields $G,\ \bar{G},\ K, \mathfrak{h}$ are based on the spacetime $\boldsymbol{E}$, while the field $Ч^{\uparrow}$ is based on the phase space $J_1\boldsymbol{E}$. Indeed, such different base spaces would arise cumbersome constructions.

Therefore, it is more convenient to replace the upper quantum connections $Ч^{\uparrow}$ with the equivalent systems of observed quantum connections $\{Ч[o]\} := \{o^* Ч^{\uparrow}\}$, whose elements $\{Ч[o]\}$ are based on the spacetime $\boldsymbol{E}$ (see Corollary 15.2.6). □

Note 17.7.4 We recall that, the quantum connections of a given system of observed quantum connections fulfill the following transitions rule, for each o and $\acute{o} = o + v$ (see Theorem 15.2.7)

$$Ч[\acute{o}] = Ч[o] + \mathfrak{i}\,\big(\theta^*[o](G^{\flat}(v)) - \tfrac{1}{2}\,G(v, v)\big) \otimes \mathbb{I}.$$

To be precise, the notation $\{Ч[o]\}$ is appropriate when we deal with a *given* system of observed quantum connections, but it is ambiguous when we deal with a *variable* system of observed quantum connections.

In order to set this problem, we observe that the $\mathbb{R}$-*linear quantum connections* can be regarded as sections (see Definition 15.1.1)

$$Ч : \boldsymbol{E} \to \text{Con}(\boldsymbol{Q})$$

of a (1, 1)-order $U(1, \mathbb{C})$-gauge natural bundle.

Then, we can say that a *system of observed quantum connections* is defined to be an equivalent class of pairs (see, Appendix: Definition F.2.1)

$$\big[(o, Ч)\big], \quad \text{where } o \in \sec(\boldsymbol{E}, J_1\boldsymbol{E}), \;\; Ч \in \sec\big(\boldsymbol{E}, \operatorname{Con}(\boldsymbol{Q})\big),$$

obtained by quotient with respect to the equivalence relation (see Theorem 15.2.7)

$$(\acute{o}, \acute{Ч}) \sim (o, Ч) \quad \text{iff } \acute{Ч} = Ч + \mathrm{i}\big(\theta^*[o](G^\flat(v)) - \tfrac{1}{2}\, G(v, v)\big) \otimes \mathbb{I},$$

where $v := \acute{o} - o$.

Accordingly, by considering the quotient bundle

$$\operatorname{Sys}(\boldsymbol{Q}) := \big(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \operatorname{Con}(\boldsymbol{Q})\big)_\sim \to \boldsymbol{E},$$

each system of observed quantum connections can be regarded as a section

$$\big[(o, Ч)\big] : \boldsymbol{E} \to \operatorname{Sys}(\boldsymbol{Q}) := \big(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \operatorname{Con}(\boldsymbol{Q})\big)_\sim$$

of a (1, 1)-order $U(1, \mathbb{C})$-gauge natural bundle.

By considering the curvature $R[Ч]$ of the quantum connection $Ч$, each equivalent class of pairs $[(o, Ч)]$ yields the equivalent class of pairs $\big[(o, R[Ч])\big]$, obtained by quotient with respect to the equivalence relation (see Exercise 15.2.11)

$$\big[(\acute{o},\; R[\acute{Ч}])\big] = \Big[\Big(o + v, \;\; R[Ч] + 2\,\mathrm{i}\, d\big(\theta^*[o]\,(G^\flat(v)) - \tfrac{1}{2}\, G\,(v, v)\big) \otimes \mathbb{I}\Big)\Big]. \qquad \square$$

Therefore, we are led to introduce the following notion.

Definition 17.7.5 We define:

- the *source bundle* to be the bundle

$$\boldsymbol{F} := \overline{\operatorname{Met}}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \operatorname{Con}(\boldsymbol{E}) \underset{\boldsymbol{E}}{\times} \operatorname{Met}(\boldsymbol{Q}) \underset{\boldsymbol{E}}{\times} \operatorname{Sys}(\boldsymbol{Q}) \to \boldsymbol{E},$$

- the sheaf of *source fields* to be the subsheaf

$$\operatorname{fields}(\boldsymbol{E}, \boldsymbol{F}) \subset \sec\big(\boldsymbol{E}, \boldsymbol{F}\big),$$

which is constituted by all sections

$$\big(\bar{G},\; K, \mathfrak{h},\; [(o, Ч)]\big) \in \sec\big(\boldsymbol{E}, \boldsymbol{F}\big)$$

fulfilling the *fundamental interaction conditions*

$$\bar{G} = (G^\sharp \otimes G^\sharp)(G), \quad \nabla[K]\, G = 0, \quad d\,\Omega[G, K] = 0,$$
$$[(o, \text{Ч})] = \Big[\Big(o + v, \quad \text{Ч} + \mathrm{i}\,\big(\theta^*[o](G^\flat(v)) - \tfrac{1}{2}\, G(v, v)\big) \otimes \mathbb{I}\Big)\Big],$$
$$\nabla[\text{Ч}]\, \mathfrak{h} = 0, \quad \big[\big(o, R[\text{Ч}]\big)\big] = \big[\big(o, \ -2\,\mathrm{i}\, o^*\Omega[G, K] \otimes \mathbb{I}\big)\big],$$

which imply that the spacetime connection K is galilean and that the equivalence classes of pairs $\big[(o, \ \text{Ч})\big]$ determine a galilean system of observed connections.

Moreover, for each integer $0 \leq r$, we define the sheaf of sections

$$\text{fields}_r(\boldsymbol{E}, \boldsymbol{F}) := \Big\{\Big(j_r\bar{G}, \ j_{r-1}K, \ j_r\mathfrak{h}, \ \big[(j_{r-1}o, \ j_{r-1}\text{Ч})\big]\Big) \ \mid \ \Big(\bar{G}, \ K, \ \mathfrak{h}, \ \big[(o, \ \text{Ч})\big]\Big) \in \text{fields}(\boldsymbol{E}, \boldsymbol{F})\Big\}$$

and denote the corresponding jet prolongation by

$$j_r : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \to \text{fields}_r(\boldsymbol{E}, \boldsymbol{F}). \quad \square$$

Now, let us consider:

- a $U(1, \mathbb{C})$-gauge natural vector bundle

$$\chi : \boldsymbol{G} \to \boldsymbol{E}$$

of order $(1, 0)$,
- a positive space of scales (see, Appendix: Definition K.2.1)

$$\mathbb{S} := \mathbb{T}^{k_1} \otimes \mathbb{L}^{k_2} \otimes \mathbb{M}^{k_1},$$

where $k_1, \ k_2, \ k_3$ are any rational numbers,
- a sheaf morphism

$$\mathfrak{F} : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}) :$$
$$: \big(\bar{G}, \ K, \ \mathfrak{h}, \ [(o, \text{Ч})]; \ \Psi\big) \mapsto \mathfrak{F}_{\left[\bar{G}, K, \mathfrak{h}, [(o, \text{Ч})]\right]}(\Psi).$$

Thus, for each choice $\big(\bar{G}, \ K, \ \mathfrak{h}, \ [(o, \text{Ч})]\big) \in \text{fields}(\boldsymbol{E}, \boldsymbol{F})$ of the source fields, we obtain the sheaf morphism

$$\mathsf{F} := \mathfrak{F}_{\left[\bar{G}, K, \mathfrak{h}, [(o, \text{Ч})]\right]} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}).$$

Let us define the sheaf of sections

$$\sec_r(\boldsymbol{E}, \boldsymbol{Q}) := \big\{ j_r\Psi \ \mid \ \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}) \big\}$$

and denote the corresponding jet prolongation by

$$j_r : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec_r(\boldsymbol{E}, \boldsymbol{Q}).$$

Definition 17.7.6 We say that the sheaf morphism $\mathfrak{F}$ is of *order r*, if it factorises through a sheaf morphism

$$\mathfrak{F}_r : \text{fields}_r(\boldsymbol{E}, \boldsymbol{F}) \times \sec_r(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}),$$

according to the following commutative diagram

$$\begin{array}{ccc} \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathfrak{F}} & \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}) \\ {\scriptstyle j_r}\downarrow & & \uparrow{\scriptstyle \text{id}} \\ \text{fields}_r(\boldsymbol{E}, \boldsymbol{F}) \times \sec_r(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathfrak{F}_r} & \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}) \end{array} \quad . \quad \square$$

Definition 17.7.7 We say that the sheaf morphism

$$\mathfrak{F} : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G})$$

is *covariant* if

- it is a natural operator in the sense of Definition J.3.18,
- it is invariant with respect to the change of scale units (see, Appendix: Definition K.2.1).

In other words, we say that the sheaf morphism $\mathfrak{F}$ is *covariant* if, for each automorphisms

$$\phi_{\boldsymbol{E}} : \boldsymbol{E} \to \boldsymbol{E}, \quad \phi_{\boldsymbol{Q}} : \boldsymbol{Q} \to \boldsymbol{Q}, \quad \phi_{\boldsymbol{G}} : \boldsymbol{G} \to \boldsymbol{G}, \quad \phi_{\mathbb{S}} : \mathbb{S} \to \mathbb{S}$$

of the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$, of the complex vector bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, of the $U(1, \mathbb{C})$-gauge natural scaled vector bundle $\chi : \boldsymbol{G} \to \boldsymbol{E}$, of the positive space $\mathbb{S}$, respectively, the following diagram commutes

$$\begin{array}{ccc} \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathfrak{F}} & \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi'} \\ \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathfrak{F}} & \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}) \end{array} \quad ,$$

where ϕ and ϕ' are the sheaf automorphisms induced by the above automorphisms. □

Thus, we have the following result.

Proposition 17.7.8 *Given a* covariant operator $\mathfrak{F}$, *if we postulate the* fundamental fields

$$\big(\bar{G},\ K,\ \hbar,\ [(o,\ \Psi)]\big) \in \text{fields}(\boldsymbol{E}, \boldsymbol{F}),$$

then we obtain the distinguished sheaf morphism

$$\mathsf{F} := \mathfrak{F}_{\left[\bar{G},\, K,\, \hbar,\, [(o, \Psi)]\right]} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G}),$$

which turns out to be observer independent and unit of measurement independent.

Proof. Indeed, the fact that F is observer independent follows from the fact that the observer o appears, among the variables of $\mathfrak{F}$, only through the equivalence classes $[(o, \Psi)]$ through a quotient. □

Then, we introduce also the following concept.

Definition 17.7.9 Let

$$\mathfrak{F} : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G})$$

be a covariant operator in the sense of Definition 17.7.7.

Then, for each choice of the source fields

$$\big(\bar{G},\ K,\ \hbar,\ [(o,\ \Psi)]\big) \in \text{fields}(\boldsymbol{E}, \boldsymbol{F}),$$

we also say that the sheaf morphism

$$\mathsf{F} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{S} \otimes \boldsymbol{G})$$

is *covariant*, in the sense that it is *observer independent* and *unit of measurement independent*. □

Example 17.7.10 For instance,

- the *quantum velocity* is a a 1st order covariant sheaf morphism (see Theorem 17.2.2)
$$\mathsf{V} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}),$$
- the *quantum kinetic tensor* is a 1st order covariant sheaf morphism (see Theorem 17.3.2)
$$\mathsf{Q} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec\big(\boldsymbol{E}, \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$
- the *quantum probability current* is a 1st order covariant sheaf morphism (see Theorem 17.4.2)
$$\mathsf{J} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{L}^{-3} \otimes \mathbb{T}^* \otimes T\boldsymbol{E}),$$

- the *Schrödinger operator* is a 2nd order covariant sheaf morphism (see Theorem 17.6.5)

$$\mathsf{S} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}),$$

- the *quantum lagrangian* is a 1st order covariant sheaf morphism (see Theorem 17.5.2)

$$\mathsf{L} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{map}(\boldsymbol{E}, \Lambda^4 T^* \boldsymbol{E}). \quad \square$$

In the following sections (see Sects. 17.7.2 and 17.7.3), we prove uniqueness theorems for the Schrödinger operator and the quantum lagrangian.

The other cases are left to the reader as exercises to be proved on the same line.

Our proofs are based on the "orbit reduction theorem" (see Theorem J.2.28 [203] and [246, p. 233]) and the "homogeneous function theorem" (see Theorem J.3.26 and [246, p. 213]).

The orbit reduction theorem applied to our case turns out to be a 2nd order "Utiyama like theorem" (see, Appendix: Example J.3.25 and [203, 204, 397]) applied to natural differential operators of linear galilean connections, system of quantum connections, quantum sections and the others fundamental fields.

Then, according to the orbit reduction theorem, such operators are expressed through 2nd order covariant differentials of the quantum fields and the other fundamental fields and the curvature tensors of the linear galilean connection and the curvature tensor of the system of quantum connections.

17.7.2 Schrödinger Operator by Covariance

By following the scheme proposed in [219], we prove that all 2nd order *covariant* Schrödinger operators are essentially proportional to the Schrödinger operator exhibited in Theorem 17.6.5.

Definition 17.7.11 According to the general Definition 17.7.7 of covariant operator, we define a *Schrödinger operator* to be a sheaf morphism of the type (see Definition 17.7.5)

$$\mathfrak{F} : \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

Thus, for each choice $\big(\bar{G},\ K,\ \hbar,\ [(o,\ \Psi)]\big) \in \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F})$ of the source fields, we obtain the Schrödinger operator to be the sheaf morphism

$$\mathsf{F} := \mathfrak{F}_{\left[\bar{G},\, K,\, \hbar,\, [(o, \Psi)]\right]} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}). \quad \square$$

Theorem 17.7.12 *All 2nd order covariant Schrödinger operators (see Definitions 17.6.6 and 17.7.7)*

$$\mathfrak{F} : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$$

yield, for each choice of the source fields

$$\big(\bar{G}, K, \mathrm{h}, [(o, \text{Ч})]\big) \in \text{fields}(\boldsymbol{E}, \boldsymbol{F}),$$

covariant Schrödinger sheaf morphisms

$$\mathsf{F} := \mathfrak{F}_{\left[\bar{G}, K, \mathrm{h}, [(o, \text{Ч})]\right]} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$$

of the type

$$\mathsf{F}(\Psi) = \alpha\, \mathsf{S}(\Psi) + \beta\, C\, \Psi, \quad \textit{with}\ \alpha, \beta \in \mathbb{C},$$

where $\mathsf{S} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$ *is the Schrödinger operator achieved in Theorem 17.6.5 and* $C : \boldsymbol{E} \to \mathbb{T}^* \otimes I\!R$ *is the spacelike scalar curvature of the spacetime connection* K*, with respect to the rescaled metric* G*, (see Proposition 3.2.13).*

Proof. Let us consider a 2nd order covariant Schrödinger operator

$$\mathfrak{F} : \text{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}).$$

(1) Let us chose a pair $(o, \text{Ч})$ as representative of an equivalence class $[(o, \text{Ч})]$.

Then, in virtue of Theorem J.2.28 and Example J.3.25, we can express $\mathfrak{F}$ through the covariant differentials and curvatures, as

$$\mathfrak{F}\big(\bar{G}, K, \mathrm{h}, (o, \text{Ч}); \Psi\big) = \mathfrak{F}'\big(\nabla^{(2)} \bar{G}, R[K], \nabla^{(2)} \mathrm{h}, \big(\nabla^{(1)} \text{д}[o], R[\text{Ч}]\big); \nabla^{(2)} \Psi\big),$$

where we have set

$$\begin{aligned}
\nabla^{(2)} \bar{G} &:= \big(\bar{G}, \ \nabla[K]\bar{G}, \ \nabla[K]\,\nabla[K]\bar{G}\big), \\
\nabla^{(2)} \mathrm{h} &:= (\mathrm{h}, \ \nabla[\text{Ч}]\,\mathrm{h}, \ \nabla[K \otimes \text{Ч}]\,\nabla[\text{Ч}]\,\mathrm{h}), \\
\nabla^{(1)} \text{д}[o] &:= (o, \ \nabla[K]\text{д}[o]), \\
\nabla^{(2)} \Psi &:= \Big(\Psi, \ \nabla[\text{Ч}]\Psi, \ \nabla[K \otimes \text{Ч}]\,\nabla[\text{Ч}]\Psi)\Big).
\end{aligned}$$

By hypothesis, $\mathfrak{F}$ passes to the quotient with respect to the equivalence relation in the set of pairs $(o, \text{Ч})$ given by their transition rule (see Theorem 15.2.7).
Hence, also $\mathfrak{F}'$ passes to the quotient with respect to the equivalence relation

$$\big[\big(\nabla^{(1)} \acute{o}, R[\acute{\text{Ч}}]\big)\big] = \Big[\Big(\nabla^{(1)} \text{д}[o] + \nabla^{(1)} v, \ R[\text{Ч}] + 2\,\mathrm{i}\, d\big(\theta^*[o]\,(G^\flat(v)) - \tfrac{1}{2}\, G\,(v, v)\big) \otimes \mathbb{I}\Big)\Big].$$

(2) By considering the fundamental interaction conditions, we can express $\mathfrak{F}$ as

$$\mathfrak{F}\big(\bar{G},\ K,\ \mathrm{h},\ (o,\text{Ч})\,;\ \Psi\big) = \mathfrak{F}'\Big(\bar{G},\ R[K],\ \mathrm{h},\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\Psi\Big),$$

(3) The operator $\mathfrak{F}'$ is invariant with respect of the change of length scale and we get that $\mathfrak{F}'$ satisfies

$$\begin{aligned}&\mathfrak{F}'\Big(\bar{G},\ R[K],\ \mathrm{h},\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\Psi\Big)\\ &\quad = \mathfrak{F}'\Big(\bar{G},\ R[K],\ k^3\,\mathrm{h},\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\Big),\end{aligned}$$

with $k\in\mathbb{R}^+$.
Then, in virtue of Theorem J.3.26, $\mathfrak{F}'$ is independent of h, i.e.

$$\begin{aligned}&\mathfrak{F}\big(\bar{G},\ K,\ \mathrm{h},\ (o,\text{Ч})\,;\ \Psi\big)\\ &\quad = \mathfrak{F}'\Big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\Psi\Big).\end{aligned}$$

(4) By considering the subgroup $\mathbb{R}^+\subset U(\mathbb{C},1)$ generating the real homotheties of fibres of $\boldsymbol{Q}$, the operator $\mathfrak{F}'$ satisfies

$$\begin{aligned}&k\,\mathfrak{F}'\Big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\Big)\\ &\quad = \mathfrak{F}'\Big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ k\,\Psi,\ k\,\nabla\Psi,\ k\,\nabla^2\Psi\Big),\end{aligned}$$

with $k\in\mathbb{R}^+$.

Then, in virtue of Theorem J.3.26, the operator $\mathfrak{F}$ can be expressed as a polynomial of the type

$$\mathfrak{F}\big(\bar{G},\ K,\ \mathrm{h},\ (o,\text{Ч})\,;\ \Psi\big) = A_1\,\Psi + A_2\,\nabla\Psi + A_3\,\nabla^2\Psi,$$

where

- $A_1 := A_1\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\big)$ is a $\mathbb{T}^*$-valued natural operator,
- $A_2 := A_2\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\big)$ is a $\mathbb{T}^*\otimes T\boldsymbol{E}$-valued natural operator,
- $A_3 := A_3\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\big)$ is a $\mathbb{T}^*\otimes T\boldsymbol{E}\otimes T\boldsymbol{E}$-valued natural operator.

(5) The operators A_i, with $i=1,2,3$, are invariant with respect of the change of time scale and we get that A_i satisfy

$$\begin{aligned}&k\,A_i\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\big)\\ &\quad = A_i\big(k\,\bar{G},\ R[K],\ \big(k\,o,\ k\,\nabla\text{д}[o],\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\big),\end{aligned}$$

for $k \in \mathbb{R}^+$. Then, in virtue of Theorem J.3.26, A_i are sums of three polynomials linear in o, $\nabla Д[o]$, $\bar{G}$ with coefficients being natural operators on $R[K]$, $o^*\Omega[G, K] \otimes \mathbb{I}$.

So we obtain the following results.

(a) The operator A_1 is of the form

$$A_1 \left(\bar{G},\ R[K],\ \left(\nabla^{(1)} Д[o],\ o^*\Omega[G, K] \otimes \mathbb{I}\right)\right) = a_1\, Д[o] + b_1\, \nabla Д[o] + c_1\, \bar{G},$$

where

- $a_1 := a_1(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $T^*\boldsymbol{E}$-valued natural operator,
- $b_1 := b_1(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $T\boldsymbol{E} \otimes T^*\boldsymbol{E}$-valued natural operator,
- $c_1 := c_1(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$-valued natural operator.

If we assume the action of the positive multiple of the unit in the differential group G^1_4 acting on the standard fibres of the source and the target bundles of the operators $a_1,\ b_1,\ c_1$, we get that these operators satisfy

$$\begin{aligned} k\, a_1\, (R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}) &= a_1\, (k^2\, R[K],\ k^2\, o^*\Omega[G, K] \otimes \mathbb{I}), \\ b_1\, (R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}) &= b_1\, (k^2\, R[K],\ k^2\, o^*\Omega[G, K] \otimes \mathbb{I}), \\ k^2\, c_1\, (R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}) &= c_1\, (k^2\, R[K],\ k^2\, o^*\Omega[G, K] \otimes \mathbb{I}), \end{aligned}$$

with $k \in \mathbb{R}^+$.

Then, in virtue of Theorem J.3.26, $a_1,\ b_1,\ c_1$ are polynomials of degrees a in $R[K]$ and b in $o^*\Omega[G, K]$, such that, respectively,

$$1 = 2\,a + 2\,b, \quad 0 = 2\,a + 2\,b, \quad 2 = 2\,a + 2\,b.$$

Hence, according to Theorem J.3.26, a_1 is the zero operator.

Further, b_1 is independent of $R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}$ and it is an invariant tensor and, by Example J.4.8, it is a constant complex multiple of $\mathbf{1}_{T\boldsymbol{E}}$ and the corresponding operator is a constant complex multiple of $C^1_1(\nabla Д[o])\, \Psi$.

Eventually, c_1 is a sum of polynomials linear in $R[K]$ and $o^*\Omega[G, K] \otimes \mathbb{I}$.

The coefficients in these polynomials are invariant tensors (see Example J.4.8) and, from the properties of $R[K]$ and $o^*\Omega[G, K] \otimes \mathbb{I}$, the unique operator which is a constant complex multiple of the operator $C[G]\, \Psi$, where $C[G]$, turns out to be the scalar curvature of K.

Summarising

$$A_1\, \Psi = (a\, C^1_1(\nabla Д[o]) + d\, C[G])\, \Psi, \quad \text{with}\, a,\ d \in \mathbb{C}.$$

(b) The operator A_2 is of the form

$$A_2 \left(\bar{G},\ R[K],\ \left(\nabla^{(1)} Д[o],\ o^*\Omega[G, K] \otimes \mathbb{I}\right)\right) = a_2\, Д[o] + b_2\, \nabla Д[o] + c_2\, \bar{G},$$

where

- $a_2 := a_2(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes T^*\boldsymbol{E}$-valued natural operator,
- $b_2 := b_2(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes \boldsymbol{TE} \otimes T^*\boldsymbol{E}$-valued natural operator,
- $c_2(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$-valued natural operator.

The operator a_2 is of the same type as the operator b_1 in the case (a) and it is a constant multiple of $\mathbf{1}_{\boldsymbol{TE}}$ and the corresponding operator is a constant complex multiple of the operator $o \lrcorner \nabla\Psi$.

If we assume the action of the positive multiple of the unit in the differential group G^1_4 acting on the standard fibres of the source and the target bundles of the operators b_2 and c_2, then these operators satisfy the equalities

$$k^{-1}\, b_2(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}) = b_2\,(k^2\, R[K],\ k^2\, o^*\Omega[G, K] \otimes \mathbb{I}),$$
$$k\, c_2(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I}) = c_2\,(k^2\, R[K],\ k^2\, o^*\Omega[G, K] \otimes \mathbb{I}),$$

with $k \in \mathbb{R}^+$.

Then, in virtue of Theorem J.3.26, b_2 and c_2 are polynomials of degrees a in $R[K]$ and b in $o^*\Omega[G, K]$, such that, respectively,

$$-1 = 2\,a + 2\,b \quad \text{and} \quad 1 = 2\,a + 2\,b.$$

Then, in virtue of Theorem J.3.26, b_2 and c_2 turn out to be the zero operators. Hence,

$$A_2\, \nabla\Psi = b\, д[o] \lrcorner \nabla\Psi, \qquad \text{with}\, b \in \mathbb{C}.$$

(c) Eventually, the operator A_3 is of the form

$$A_3\left(\bar{G},\ R[K],\ \left(\nabla^{(1)} д[o],\ o^*\Omega[G, K] \otimes \mathbb{I}\right)\right) = a_3\, д[o] + b_3\, \nabla д[o] + c_3\, \bar{G},$$

where

- $a_3 := a_3(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes \boldsymbol{TE} \otimes T^*\boldsymbol{E}$-valued natural operator,
- $b_3 := b_3\,(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes \boldsymbol{TE} \otimes \boldsymbol{TE} \otimes T^*\boldsymbol{E}$-valued natural operator,
- $c_3 := c_3\,(R[K],\ o^*\Omega[G, K] \otimes \mathbb{I})$ is a $\boldsymbol{TE} \otimes \boldsymbol{TE} \otimes T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$-valued natural operator.

The operator a_3 is of the same type of the operator b_2 in the case (b) and it is the zero operator.

If we assume the action of the positive multiple of the unit in the differential group G^1_4 acting on the standard fibres of the source and the target bundles of the operators b_3 and c_3, then these operators satisfy

$$k^{-2}\, b_3(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = b_3\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$
$$c_3\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = c_3\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$

with $k \in \mathbb{R}^+$.

Then, in virtue of Theorem J.3.26, b_3 and c_3 turn out to be polynomials of degrees a in $R[K]$ and b in $o^*\Omega[G,K]$, such that, respectively,

$$-2 = 2\,a + 2\,b, \qquad 0 = 2\,a + 2\,b.$$

Hence, in virtue of Theorem J.3.26, b_3 turns out to be the zero operator and c_3 turns out to be an invariant tensor.

By Example J.4.8, the corresponding operator is constant complex multiple of $\bar{G} \lrcorner \nabla^2\Psi$, i.e.

$$A_3\, \nabla^2\Psi = c\, \bar{G} \lrcorner \nabla^2\Psi, \qquad \text{with}\, c \in \mathbb{C}.$$

Summarising the above results, we can express $\mathfrak{F}$ as

$$\mathfrak{F}\big(\bar{G},\ K,\ \hbar,\ (д[o], Ч)\,;\ \Psi\big) = a\,(C^1_1 \nabla д[o])\,\Psi + b\, д[o] \lrcorner \nabla\Psi$$
$$+c\,\bar{G} \lrcorner \nabla^2\Psi + d\, C[G]\,\Psi, \qquad \text{with}\, a, b, c, d \in \mathbb{C}.$$

And, by observing that $C^1_1\, \nabla д[o] = \mathrm{div}_\eta д[o]$, $o \lrcorner \nabla\Psi = \nabla_{д[o]}\Psi$ and $\bar{G} \lrcorner \nabla^2\Psi = \Delta[G,o]\Psi$, we obtain

$$\mathfrak{F}\big(\bar{G},\ K,\ \hbar,\ (o, Ч)\,;\ \Psi\big) = a\, \mathrm{div}_\eta д[o]\,\Psi + b\, \nabla_{д[o]}\Psi + c\, \Delta[G,o]\Psi + d\, C[G]\,\Psi.$$

(6) Given two observers o and $\acute{o} = o + v$, then, by considering the transition rules,

$$\mathrm{div}_\eta д[\acute{o}] = \mathrm{div}_\eta д[o] + \mathrm{div}_\eta v,$$
$$\nabla_{д[\acute{o}]} = \nabla_{д[o]} + \nabla_v - \mathrm{i}\,\tfrac{1}{2}\, G(v,v),$$
$$\Delta[G,\acute{o}] = \Delta[G,o] - 2\,\mathrm{i}\,\nabla_v - \mathrm{i}\,\mathrm{div}_\eta v - G(v,v),$$

we get

$$\mathfrak{F}\big(\bar{G},\ K,\ \hbar,\ (\acute{o},\ \acute{Ч});\ \Psi\big)$$
$$= a\,\big(\mathrm{div}_\eta д[o] + \mathrm{div}_\eta v\big)\,\Psi + b\,\big(\nabla_{д[o]} + \nabla_v - \mathrm{i}\,\tfrac{1}{2}\,G(v,v)\big)\,\Psi$$
$$+\, c\,\big(\Delta[G,o] - 2\,\mathrm{i}\,\nabla_v - \mathrm{i}\,\mathrm{div}_\eta v - G(v,v)\big)\,\Psi + d\, C[G]\,\Psi$$
$$= \mathfrak{F}\big(\bar{G},\ K,\ \hbar,\ (o,\ Ч);\ \Psi\big) + (a - \mathrm{i}\,c)\,\mathrm{div}_\eta v\,\Psi + (b - 2\,\mathrm{i}\,c)\,\nabla_v\Psi$$
$$-\,\big(\tfrac{1}{2}\,\mathrm{i}\,b + c\big)\,G(v,v)\,\Psi$$

and the operator is observer independent if and only if $a = \mathrm{i}\,c$ and $b = 2\,\mathrm{i}\,c$.

Hence, in virtue of Theorem 17.6.5,

$$\mathfrak{F}\big(\bar{G},\ K,\ \mathrm{h},\ (o,\ Ч);\ \Psi\big) = c\,\big(\mathrm{i}\,\mathrm{div}_{\eta}Д[o] + 2\,\mathrm{i}\ \nabla_{Д[o]} + \Delta[G, o]\big)\,\Psi + d\,C[G]\,\Psi$$
$$= 2\,\mathrm{i}\,c\,\mathrm{S}(\Psi) + d\,C[G]\,\Psi, \qquad \text{with}\, c,\ d \in \mathbb{C},$$

and putting $\alpha = 2\,\mathrm{i}\,c, \quad \beta = d$, we get our claim. □

Remark 17.7.13 If our quantum system would be involved with a "fundamental" distinguished time scale and a "fundamental" distinguished length scale, then we could not require the covariance of the Schrödinger operator with respect to the change of units of scales.

In such a case, the steps (3) and (5) of the above proof would be weaker and we would obtain many more solutions of our problem, including non linear Schrödinger operators.

For instance, if we would postulate a distinguished fundamental time scale $\tau \in \mathbb{T}^*$ and a distinguished fundamental length scale $\ell \in \mathbb{L}$, then any additional term of the type

$$f \circ \big(\ell^3 \otimes \mathrm{h}(\Psi, \Psi)\big)\,\tau \otimes \Psi \in \sec(\boldsymbol{E},\ \mathbb{T}^* \otimes \boldsymbol{Q}),$$

where $f : \mathbb{C} \to \mathbb{C}$ is any function, would still yield a covariant Schrödinger operator, according to the weakened definition of covariance. □

Remark 17.7.14 If the gravitational field $K^{\natural}$ is generated by a fluid and an electromagnetic fields, then we obtain $C = C^{\natural} = 0$ (see Proposition 8.2.1). Then, in such a case, we can disregard the additional term of the Schrödinger operator generated by the scalar curvature C and consider just the Schrödinger operator defined in Theorem 17.6.5. □

17.7.3 Quantum Lagrangian by Covariance

By following the scheme proposed in [219], we prove that all physically relevant 2nd order *covariant* quantum lagrangians are proportional to the quantum lagrangian L exhibited in Theorem 17.5.2 (see, also [200–202]).

Definition 17.7.15 According to the general Definition 17.7.7 of covariant operator, we define a *quantum lagrangian* to be a sheaf morphism of the type (see Definition 17.7.5)

$$\mathfrak{L} : \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E},\ \Lambda^4 T^* \boldsymbol{E}).$$

Thus, for each choice $\big(\bar{G},\ K,\ \mathrm{h},\ [(o,\ Ч)]\big) \in \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F})$ of the source fields, we obtain the quantum lagrangian to be the sheaf morphism

$$L := \mathfrak{L}_{\big[\bar{G},\, K,\, \mathrm{h},\, [(o, Ч)]\big]} : \sec(\boldsymbol{E},\ \boldsymbol{Q}) \to \sec(\boldsymbol{E},\ \Lambda^4 T^* \boldsymbol{E}). \quad □$$

Theorem 17.7.16 *All 2nd order covariant quantum lagrangians (see Definitions 17.7.6 and 17.7.7)*

$$\mathfrak{L} : \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F}) \times \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \ \Lambda^4 T^* \boldsymbol{E})$$

yield, for each choice of the source fields

$$\big(\bar{G}, \ K, \ \mathfrak{h}, \ [(o, Ч)]\big) \in \mathrm{fields}(\boldsymbol{E}, \boldsymbol{F}),$$

covariant quantum lagrangian sheaf morphisms

$$L := \mathfrak{L}_{\left[\bar{G}, K, \mathfrak{h}, [(o, Ч)]\right]} : \mathrm{sec}(\boldsymbol{E}, \ \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \ \Lambda^4 T^* \boldsymbol{E})$$

of the type

$$L(\Psi) = a\,\mathsf{L}[\Psi] + b\,\mathsf{L}_i[\Psi] + c\,\mathsf{L}_r[\Psi] + d\,\mathsf{L}_s[\Psi], \quad \textit{with}\, a, b, c, d \in \mathbb{R},$$

where

(1) $$\mathsf{L}[\Psi] : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}$$

is the 1st order quantum lagrangian defined in Theorem 17.5.2,

(2) $$\mathsf{L}_i[\Psi] \equiv \mathsf{L}_{i\,0}[\Psi]\,\upsilon^0 := dt \wedge \mathrm{im}\,\mathfrak{h}_\eta(\Psi, \mathsf{S}[\Psi]) : \boldsymbol{E} \to \Lambda^4\, T^* \boldsymbol{E}$$

and

(3) $$\mathsf{L}_r[\Psi] \equiv \mathsf{L}_{r\,0}[\Psi]\,\upsilon^0 := dt \wedge \mathrm{re}\,\mathfrak{h}_\eta(\Psi, \mathsf{S}[\Psi]) : \boldsymbol{E} \to \Lambda^4\, T^* \boldsymbol{E},$$

are 2nd order quantum lagrangians derived from the Schrödinger operator S *via the* η*-hermitian quantum product (see Theorem 17.6.5 and Definition 14.5.1),*

(4) $$\mathsf{L}_s[\Psi] := C[G]\,\mathfrak{h}\,(\Psi, \ \Psi)\,\upsilon : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}$$

is a 0-order quantum lagrangian associated with the spacetime scalar curvature $C[G]$ *(see Propositions 3.2.13 and 3.2.4).*

Proof. First, let us recall the volume form $\upsilon : \boldsymbol{E} \to \mathbb{T} \otimes \mathbb{L}^3 \otimes \Lambda^4 T^* \boldsymbol{E}$ (see Proposition 3.2.4) naturally induced by the spacetime metric. Then, any covariant quantum lagrangian is of the form

$$L\big(\bar{G}, \ K, \ \mathfrak{h}, \ \big(o, \ Ч\big); \ \Psi\big) \otimes \upsilon,$$

where L is a 2nd order covariant $\mathbb{T}^{-1} \otimes \mathbb{L}^{-3}$-valued real function.

In the proof we apply a procedure and a notation analogous to that of the proof of Theorem 17.7.12.

(1) In virtue of Theorem J.2.28 and Example J.3.25, we can express L through the covariant differentials and curvatures, as

$$\begin{aligned} &L\big(\bar{G},\ K,\ \hbar,\ \big(o,\ \text{Ч}\big)\,;\ \Psi\big) \\ &\quad = L'\big(\nabla^{(2)}\,\bar{G},\ R[K],\ \nabla^{(2)}\,\hbar,\ \big(\nabla^{(1)}\,o,\ R[\text{Ч}]\big)\,;\ \nabla^{(2)}\,\Psi\big), \end{aligned}$$

where the covariant differentials are performed with respect to K and Ч.

(2) By considering the fundamental interaction conditions (see Definition 17.7.5), we can express L as

$$\begin{aligned} &L\big(\bar{G},\ K,\ \hbar,\ \big(o,\ \text{Ч}\big)\,;\ \Psi\big) \\ &\quad = L'\big(\bar{G},\ R[K],\ \hbar,\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\ \ \nabla^{(2)}\,\Psi\big). \end{aligned}$$

(3) The function L is invariant with respect of the change of length scale, hence L' satisfies the equality

$$\begin{aligned} &k^3\,L'\big(\bar{G},\ R[K],\ \hbar,\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big) \\ &\quad = L'\,\big(\bar{G},\ R[K],\ k^3\,\hbar,\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big), \end{aligned}$$

for $k \in \mathbb{R}^+$.
Then, in virtue of Theorem J.3.26, L' is a linear polynomial in $\hbar$ and the operator L can be expressed as

$$\begin{aligned} &L\big(\bar{G},\ R[K],\ \hbar,\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big) \\ &\quad = H\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big)\lrcorner\,\hbar, \end{aligned}$$

where

$$H := H\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big)$$

is a $\mathbb{T}^*\otimes\boldsymbol{Q}\otimes\boldsymbol{Q}$-valued natural differential operator.

(4) By considering the subgroup $\mathbb{R}^+\subset U(\mathbb{C},1)$ generating the real homotheties of fibres of $\boldsymbol{Q}$, we see that the operator H satisfies the equality

$$\begin{aligned} &k^2\,H\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ \nabla^{(2)}\,\Psi\big) \\ &\quad = H\big(\bar{G},\ R[K],\ \big(\nabla^{(1)}\,o,\ o^*\Omega[G,K]\otimes\mathbb{I}\big)\,;\ k\,\Psi,\ k\,\nabla\,\Psi,\ k\,\nabla^2\,\Psi\big), \end{aligned}$$

with $k \in \mathbb{R}^+$.

Then, in virtue of Theorem J.3.26, the operator H is a polynomial of orders p in Ψ, q in $\nabla\,\Psi$ and r in $\nabla^2\,\Psi$ such that

$$2 = p + q + r.$$

Hence, H can be expressed as a linear combination

$$\begin{aligned} H &= A_1\, \Psi \otimes \Psi + A_2\, \Psi \otimes \nabla\, \Psi + \bar{A}_2\, \nabla\, \Psi \otimes \Psi \\ &\quad + A_3\, \Psi \otimes \nabla^2\, \Psi + \bar{A}_3\, \nabla^2\, \Psi \otimes \Psi + A_4\, \nabla\, \Psi \otimes \nabla\, \Psi \\ &\quad + A_5\, \nabla\, \Psi \otimes \nabla^2\, \Psi + \bar{A}_5\, \nabla^2\, \Psi \otimes \nabla\, \Psi + A_6\, \nabla^2\, \Psi \otimes \nabla^2\, \Psi, \end{aligned}$$

i.e.

$$\begin{aligned} & L\big(\bar{G},\ K,\ \text{h},\ \big(o,\ \text{Ч}\big)\,;\ \Psi\big) \\ &\quad = A_1\, \text{h}\,(\Psi,\, \Psi) + A_2\, \text{h}\,(\Psi,\, \nabla\Psi) + \bar{A}_2\, \text{h}\,(\nabla\Psi,\, \Psi) \\ &\qquad + A_3\, \text{h}\,(\Psi,\, \nabla^2\Psi) + \bar{A}_3\, \text{h}\,(\nabla^2\Psi,\, \Psi) + A_4\, \text{h}\,(\nabla\Psi,\, \nabla\Psi) \\ &\qquad + A_5\, \text{h}\,(\nabla\Psi,\, \nabla^2\Psi) + \bar{A}_5\, \text{h}\,(\nabla^2\Psi,\, \nabla\Psi) + A_6\, \text{h}\,(\nabla^2\Psi,\, \nabla^2\Psi), \end{aligned}$$

where the coefficients

$$\begin{aligned} A_i &:= A_i\,\big(\bar{G},\ \ R[K],\ \ \big(\nabla^{(1)}o,\ \ o^*\Omega[G,K] \otimes \mathbb{I}\big)\big), \quad \text{with}\, i = 1, \ldots, 6, \\ \bar{A}_i &:= \bar{A}_i\,\Big(\bar{G},\ \ R[K],\ \ \big(\nabla^{(1)}o,\ \ o^*\Omega[G,K] \otimes \mathbb{I}\big)\Big), \quad \text{with}\, i = 2, 3, 5, \end{aligned}$$

are natural differential operators of the following types:

- A_1 is a $\mathbb{T}^*$-valued covariant function,
- A_2 and $\bar{A}_2$ are $\mathbb{T}^* \otimes T^*\boldsymbol{E}$-valued natural differential operators,
- A_3, $\bar{A}_3$, A_4 and $\bar{A}_4$ are $\mathbb{T}^* \otimes \otimes^2 T^*\boldsymbol{E}$-valued natural differential operators,
- A_5 and $\bar{A}_5$ are $\mathbb{T}^* \otimes \otimes^3 T^*\boldsymbol{E}$-valued natural differential operators,
- A_6 is a $\mathbb{T}^* \otimes \otimes^4 T^*\boldsymbol{E}$-valued natural differential operator.

(5) The operators A_i, $i = 1, \ldots, 6$ and $\bar{A}_i$, $i = 2, 3, 5$ are invariant with respect of the change of time scale and we get that A_i satisfy the equality

$$\begin{aligned} & k\, A_i\,\big(\bar{G},\ \ R[K],\ \ \big(\nabla^{(1)}o,\ \ o^*\Omega[G,K] \otimes \mathbb{I}\big)\big) \\ &\quad = A_i\,\big(k\,\bar{G},\ \ R[K],\ \ \big(k\,o,\ \ k\,\nabla o,\ \ o^*\Omega[G,K] \otimes \mathbb{I}\big)\big)\big), \quad \text{with}\, k \in \mathbb{R}^+. \end{aligned}$$

Then, in virtue of Theorem J.3.26, A_i are polynomials linear in o, ∇o, $\bar{G}$ with coefficients being natural operators on $R[K]$, $o^*\Omega[G,K] \otimes \mathbb{I}$.

The same result holds for the operators $\bar{A}_i$, $i = 2, 3, 5$.

So we have the following results.

(a) The operator A_1 is of the same type as the operator A_1 in the proof of Theorem 17.7.12, i.e. the corresponding operator is

$$A_1\, \text{h}\,(\Psi,\, \Psi) = (a\, C^1_1 \nabla o + d\, C[G])\, \text{h}\,(\Psi,\, \Psi).$$

This operator is valued in a real tensor bundle, hence $a,\ d \in \mathbb{R}$.

(b) The operators A_2 and $\bar{A}_2$ are of the same type as the operator A_2 in the proof of Theorem 17.7.12, i.e. the corresponding operator is

$$\begin{aligned} & A_2\, \mathrm{h}\,(\Psi, \nabla\Psi) + \bar{A}_2\, \mathrm{h}\,(\nabla\Psi, \Psi) \\ & \quad = b\, \mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi) + \tilde{b}\, \mathrm{h}\,(o \lrcorner \nabla\Psi, \Psi), \quad \text{with}\, b, \tilde{b} \in \mathbb{C}. \end{aligned}$$

These operators are valued in a real tensor bundle, hence $\tilde{b} = \bar{b}$ and we can rewrite them as

$$\begin{aligned} & (b_1 + b_2\, \mathrm{i}\,)\, \mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi) + (b_1 - b_2\, \mathrm{i}\,)\, \mathrm{h}\,(o \lrcorner \nabla\Psi, \Psi) \\ & \quad = b_1 \big(\mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi) + \mathrm{h}\,(o \lrcorner \nabla\Psi, \Psi)\big) \\ & \qquad + b_2\, \mathrm{i} \big(\mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi) - \mathrm{h}\,(o \lrcorner \nabla\Psi, \Psi)\big) \\ & \quad = 2\, b_1\, \mathrm{re}\, \mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi) - 2\, b_2\, \mathrm{im}\, \mathrm{h}\,(\Psi, o \lrcorner \nabla\Psi), \quad \text{with}\, b_1, b_2 \in \mathbb{R}. \end{aligned}$$

(c) The operators A_3 and $\bar{A}_3$ are of the same type as the operator A_3 in the proof of Theorem 17.7.12, i.e. the corresponding operators

$$\begin{aligned} & A_3\, \mathrm{h}\,(\Psi, \nabla^2\Psi) + \bar{A}_3\, \mathrm{h}\,(\nabla^2\Psi, \Psi) \\ & \quad = c\, \mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi) + \tilde{c}\, \mathrm{h}\,(\bar{G} \lrcorner \nabla^2\Psi,\ \Psi), \quad \text{with}\, c, \tilde{c} \in \mathbb{C}. \end{aligned}$$

These operators are valued in the real tensor bundle which imply $\tilde{c} = \bar{c}$ and we can rewrite them as

$$\begin{aligned} & (c_1 + c_2\, \mathrm{i}\,)\, \mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi) + (c_1 - c_2\, \mathrm{i}\,)\, \mathrm{h}\,(\bar{G} \lrcorner \nabla^2\Psi),\ \Psi) \\ & \quad = c_1 \big(\mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi)) + \mathrm{h}\,(\bar{G} \lrcorner \nabla^2\Psi, \Psi)\big) + c_2\, \mathrm{i} \big(\mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi) \\ & \qquad - \mathrm{h}\,(\bar{G} \lrcorner \nabla^2\Psi, \Psi)\big) \\ & \quad = 2\, c_1\, \mathrm{re}\, \mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi) - 2\, c_2\, \mathrm{im}\, \mathrm{h}\,(\Psi, \bar{G} \lrcorner \nabla^2\Psi), \quad \text{with}\, c_1, c_2 \in \mathbb{R}. \end{aligned}$$

(d) The operator A_4 is the same as the operator A_3 in the proof of Theorem 17.7.12, i.e. the corresponding operator is

$$A_4\, \mathrm{h}\,(\nabla\Psi, \nabla\Psi) = e\, (\bar{G} \otimes \mathrm{h})\, (\nabla\Psi, \nabla\Psi).$$

This operator is valued in a real tensor bundle, hence $e \in \mathbb{R}$.

(e) The operator A_5 is of the form

$$A_5 \big(\bar{G},\ R[K],\ \big(\nabla^{(1)} o,\ o^* \Omega[G, K]\big)\big) = a_5\, (o) + b_5\, (\nabla o) + c_5\, (\bar{G}),$$

where

- $a_5 := a_5(R[K],\ o^* \Omega[G, K] \otimes \mathbb{I})$ is a $\otimes^3 \boldsymbol{T} \boldsymbol{E} \otimes \boldsymbol{T}^* \boldsymbol{E}$ valued natural operator,

- $b_5 := b_5(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I})$ is a $\otimes^4 \boldsymbol{TE} \otimes T^*\boldsymbol{E}$ valued natural operator,
- $c_5 := c_(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I})$ is a $\otimes^3 \boldsymbol{TE} \otimes \otimes^2 T^*\boldsymbol{E}$ valued natural operator.

If we assume the action of the positive multiple of the unit in the differential group G_4^1 acting on the standard fibres of the source and the target bundles of the operators a_5, b_5 and c_5, then these operators satisfy the equalities

$$k^{-2}\, a_5(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = a_5\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$
$$k^{-3}\, b_5\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = b_5\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$
$$k^{-1}\, c_5\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = c_5\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$

with $k \in \mathbb{R}^+$.
Then, in virtue of Theorem J.3.26, $a_5,\ b_5,\ c_5$ are polynomials of degrees a in $R[K]$ and b in $o^*\Omega[G,K]$, such that, respectively,

$$-2 = 2\,a + 2\,b, \qquad -3 = 2\,a + 2\,b, \qquad -1 = 2\,a + 2\,b.$$

Then, in virtue of Theorem J.3.26, all these operators are the zero operators. The same result holds for $\bar{A}_5$.

(f) Finally, the operator A_6 is of the form

$$A_6\left(\bar{G},\ R[K],\ \left(\nabla^{(1)}o,\ o^*\Omega[G,K]\right)\right) = a_6\,(o) + b_6\,(\nabla o) + c_6\,(\bar{G}),$$

where

- $a_6\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I})$ is a $\otimes^4\, \boldsymbol{TE} \otimes T^*\boldsymbol{E}$ valued natural operator,
- $b_6\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I})$ is a $\otimes^5 \boldsymbol{TE} \otimes T^*\boldsymbol{E}$ valued natural operator,
- $c_6\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I})$ is a $\otimes^4 \boldsymbol{TE} \otimes \otimes^2 T^*\boldsymbol{E}$ valued natural operator.

If we assume the action of the positive multiple of the unit in the differential group G_4^1 acting on the standard fibres of the source and the target bundles of the operators a_6, b_6 and c_6, then these operators satisfy the equalities

$$k^{-3}\, a_5\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = a_5\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$
$$k^{-4}\, b_5\,(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = b_5(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}),$$
$$k^{-2}\, c_5(R[K],\ o^*\Omega[G,K] \otimes \mathbb{I}) = c_5\,(k^2\, R[K],\ k^2\, o^*\Omega[G,K] \otimes \mathbb{I}), \quad \text{with}\, k \in \mathbb{R}^+.$$

Then, in virtue of Theorem J.3.26, $a_6,\ b_6,\ c_6$ are polynomials of degrees a in $R[K]$ and b in $o^*\Omega[G,K]$ such that, respectively,

$$-3 = 2\,a + 2\,b, \qquad -4 = 2\,a + 2\,b, \qquad -2 = 2\,a + 2\,b.$$

Then, in virtue of Theorem J.3.26, all these operators are the zero operators.

Summarising the above results, we can express L as

$$\begin{aligned} &L\big(\bar{G},\ K,\ \mathrm{h},\ \big(o,\ Ч\big);\ \Psi\big) \\ &= a\,(C^1_1 \nabla д[o])\, \mathrm{h}\,(\Psi,\ \Psi) + (b_1 + b_2\,\mathrm{i}\,)\,\mathrm{h}\,(\Psi,\ д[o] \lrcorner \nabla\Psi) \\ &\quad + (b_1 - b_2\,\mathrm{i}\,)\,\mathrm{h}\,(д[o] \lrcorner \nabla\Psi,\ \Psi) + (c_1 + c_2\,\mathrm{i}\,)\,\mathrm{h}\,(\Psi,\ \bar{G} \lrcorner \nabla^2\Psi)) \\ &\quad + (c_1 - c_2\,\mathrm{i}\,)\,\mathrm{h}\,(\bar{G} \lrcorner \nabla^2\Psi,\ \Psi) + d\,C[G]\,\mathrm{h}\,(\Psi,\ \Psi) + e\,(\bar{G} \otimes \mathrm{h})\,(\nabla\Psi,\ \nabla\Psi), \end{aligned}$$

with $a, b_1, b_2, c_1, c_2, d, e \in \mathbb{R}$.

Moreover, by observing that

$$C^1_1\,\nabla д[o] = \mathrm{div}_\eta д[o], \qquad д[o] \lrcorner \nabla\Psi = \nabla_{д[o]}\Psi, \qquad \bar{G} \lrcorner \nabla^2\Psi) = \Delta[G, o]\Psi,$$

we obtain

$$\begin{aligned} &L\big(\bar{G},\ K,\ \mathrm{h},\ \big(o,\ Ч\big);\ \Psi\big) \\ &\quad = a\,\mathrm{div}_\eta д[o]\,\mathrm{h}\,(\Psi,\ \Psi) + (b_1 + b_2\,\mathrm{i}\,)\,\mathrm{h}\,(\Psi,\ \nabla_{д[o]}\Psi) \\ &\qquad + (b_1 - b_2\,\mathrm{i}\,)\,\mathrm{h}\,(\nabla_{д[o]}\Psi,\ \Psi) + (c_1 + c_2\,\mathrm{i}\,)\,\mathrm{h}\,(\Psi,\ \Delta[G, o]\Psi,\ \Psi) \\ &\qquad + (c_1 - c_2\,\mathrm{i}\,)\,\mathrm{h}\,(\Delta[G, o]\Psi,\ \Psi) + d\,C[G]\,\mathrm{h}(\Psi,\ \Psi) + e\,(\bar{G} \otimes \mathrm{h})\,(\nabla\Psi,\ \nabla\Psi). \end{aligned}$$

(6) In virtue of the invariance of L with respect to the change of observers, see Note 16.1.16 and the step (6) of Theorem 17.7.12, we obtain the following identities:

$$a + 2\,c_2 = 0, \qquad b_1 + 2\,c_2 = 0, \qquad b_2 - 2\,c_1 + e0.$$

Hence

$$\begin{aligned} L\big(\bar{G},\ K,\ \mathrm{h},\ \big(o,\ Ч\big);\ \Psi\big) &= e\,\big((\bar{G} \otimes \mathrm{h})\,(\nabla\Psi,\ \nabla\Psi) - \mathrm{i}\,(\mathrm{h}\,(\Psi,\ \nabla_{д[o]}\Psi) \\ &\quad - \mathrm{h}\,(\nabla_{д[o]}\Psi,\ \Psi))\big) - 2\,c_1\,\big(\mathrm{h}\,(\Psi,\ \mathrm{S}[\Psi]) + \mathrm{h}(\mathrm{S}[\Psi],\ \Psi)\big) \\ &\quad + 2\,c_2\,\mathrm{i}\,\big(\mathrm{h}\,(\Psi,\ \mathrm{S}[\Psi]) - \mathrm{h}\,(\mathrm{S}[\Psi],\ \Psi)\big) \\ &\quad + d\,C[G]\,\mathrm{h}\,(\Psi,\ \Psi) \\ &= e\,\big((\bar{G} \otimes \mathrm{h})\,(\nabla\Psi,\ \nabla\Psi) + 2\,\mathrm{im}\,\mathrm{h}\,(\Psi,\ \nabla_{д[o]}\Psi) \\ &\quad - 4\,c_1\,\mathrm{re}\,\mathrm{h}\,(\Psi,\ \mathrm{S}[\Psi]) - 4\,c_2\,\mathrm{im}\,\mathrm{h}\,(\Psi,\ \mathrm{S}[\Psi]) \\ &\quad + d\,C[G]\,\mathrm{h}\,(\Psi,\ \Psi), \end{aligned}$$

i.e. by recalling Theorem 17.5.2,

$$\begin{aligned} L[\Psi] = -2\,e\,\mathrm{L}(\Psi) + d\,C[G]\mathrm{h}\,(\Psi,\ \Psi)\,\upsilon - 4\,c_1\,\mathrm{re}\,\mathrm{h}\,(\Psi,\mathrm{S}[\Psi])\,\upsilon \\ - 4\,c_2\,\mathrm{im}\,\mathrm{h}\,(\Psi,\ \mathrm{S}[\Psi])\,\upsilon, \quad \text{with}\, e, d, c_1, c_2 \in \mathbb{R}. \end{aligned}$$

Then, after putting

$$a = -2\,e, \qquad b = -4\,c_2, \qquad c = -4\,c_1,$$

we get our claim by observing that

$$\mathrm{re}\,\mathsf{h}\,(\Psi, \mathsf{S}[\Psi])\,\upsilon = dt \wedge \mathrm{re}\,\mathsf{h}_\eta\,(\Psi, \mathsf{S}(\Psi))$$

and

$$\mathrm{im}\,\mathsf{h}(\Psi, \mathsf{S}(\Psi))\,\upsilon = dt \wedge \mathrm{im}\,\mathsf{h}_\eta(\Psi, \mathsf{S}(\Psi)). \quad \square$$

Note 17.7.17 We have already seen (see Theorem 17.6.23) that the Euler–Lagrange operator associated with the quantum lagrangian $\mathsf{L}[\Psi]$ yields the Schrödinger operator $\mathsf{S}[\Psi]$ defined in Theorem 17.6.5.

Moreover, we can easily see that the Euler–Lagrange operator associated with the quantum lagrangian $\mathsf{L}_s[\Psi]$ yields the additional term $\mathsf{S}_s[\Psi] = \mathfrak{i}\,C[G]\,\Psi$ of the above Schrödinger operator.

Actually, in the present book, we do not pay attention to such an additional term because in our main models of spacetime the spacetime scalar curvature $C[G]$ vanishes (see Theorem 8.3.4). □

Next, we prove that the other two possible covariant lagrangians $\mathsf{L}_i[\Psi]$ and $\mathsf{L}_r[\Psi]$ are physically non relevant.

Note 17.7.18 The Euler–Lagrange operator $\mathsf{E}[\mathsf{L}_i] : J_2\boldsymbol{Q} \to T^*\boldsymbol{Q} \otimes \Lambda^4 T^*\boldsymbol{E}$ associated with the covariant quantum lagrangian $\mathsf{L}_i[\Psi] = dt \wedge \mathrm{im}\,\mathsf{h}_\eta(\Psi, \mathsf{S}[\Psi])$ (see the above Theorem 17.7.16) is proportional to the Schrödinger operator $S[\Psi]$ exhibited by Theorem 17.6.5.

More explicitly we have $i_{\bar{\upsilon}}(\mathrm{re}\,\mathsf{h})^\sharp\big(\mathsf{E}[\mathsf{L}_i]\big) = 2\,\mathfrak{i}\,\mathsf{S}$.

Proof. The real coordinate expression of the lagrangian $\mathsf{L}_i[\Psi] = dt \wedge \mathrm{im}\,\mathsf{h}_\eta(\Psi, \mathsf{S}[\Psi])$ is

$$\begin{aligned}
\mathsf{L}_i &= \mathsf{L}_0\,\upsilon^0 \\
&= \big((w^1\,w^2_0 - w^2\,w^1_0) - \tfrac{1}{2}\,G^{ij}_0\,(w^1\,w^1_{ij} + w^2\,w^2_{ij}) \\
&\quad + A^j_0\,(w^1_j\,w^2 - w^1\,w^2_j) - \frac{1}{2}\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,(w^1\,w^1_j + w^2\,w^2_j) \\
&\quad + \tfrac{1}{2}\,\alpha_0\,((w^1)^2 + (w^2)^2)\big)\,\upsilon^0.
\end{aligned}$$

Hence, the associated Euler–Lagrange operator has coordinate expression, in real coordinates,

$$\begin{aligned}\mathsf{E}[\mathsf{L}_i] = \big(&- 2\,w_0^2 + G_0^{ij}\,w_{ij}^1 + 2\,A_0^j\,w_j^2 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}} w_j^1 - \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} w^2 \\ &+ \frac{\partial_j(A_0^j\sqrt{|g|})}{\sqrt{|g|}} w^2 - \alpha_0\,w^1\big)\,\check{d}w^1 \wedge \upsilon^0 \\ &+ \big(2\,w_0^1 + G_0^{ij}\,w_{ij}^2 - 2\,A_0^j\,w_j^1 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}} w_j^2 \\ &+ \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} w^1 - \frac{\partial_j(A_0^j\sqrt{|g|})}{\sqrt{|g|}} w^1 - \alpha_0\,w^2\big)\,\check{d}w^2 \wedge \upsilon^0 .\end{aligned}$$

Hence, we obtain

$$\begin{aligned}i_{\bar{\upsilon}}(\mathrm{re}\,\mathfrak{h})^\sharp\big(\mathsf{E}[\mathsf{L}_i]\big) = \big(&- 2\,w_0^2 + G_0^{ij}\,w_{ij}^1 + 2\,A_0^j\,w_j^2 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\, w_j^1 - \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, w^2 \\ &+ \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}} w^2 - \alpha_0\,w^1\big)\,u^0 \otimes \partial w_1 \\ &+ \big(2\,w_0^1 + G_0^{ij}\,w_{ij}^2 - 2\,A_0^j\,w_j^1 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\, w_j^2 \\ &+ \frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\, w^1 - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}\, w^1 - \alpha_0\,w^2\big)\,u^0 \otimes \partial w_2 .\end{aligned}$$

The coordinate expression of the Schrödinger operator S is

$$\begin{aligned}\mathsf{S} &= \big((w_0^1 + \mathrm{i}\,w_0^2) - \tfrac{1}{2}\,\mathrm{i}\,G_0^{ij}\,(w_{ij}^1 + \mathrm{i}\,w_{ij}^2) \\ &\quad - \big(A_0^j + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\big)\,(w_j^1 + \mathrm{i}\,w_j^2)\big)\,u^0 \otimes \mathfrak{b} \\ &\quad + \big(\tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}} + \mathrm{i}\,\alpha_0)\,(w^1 + \mathrm{i}\,w^2)\big)\,u^0 \otimes \mathfrak{b} \\ &= \Big((w_0^1 - \tfrac{1}{2}\,G_0^{ij}\,w_{ij}^1 + (A_0^j + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}})\,w_j^2 \\ &\quad + \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}})\,w^1 - \tfrac{1}{2}\,\alpha_0\,w^2\Big)\,u^0 \otimes \mathfrak{b}_1 \\ &\quad + \Big(w_0^2 - \tfrac{1}{2}\,G_0^{ij}\,w_{ij}^2 - (A_0^j + \tfrac{1}{2}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}})\,w_j^1 \\ &\quad + \tfrac{1}{2}\,(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}})\,w^2 + \tfrac{1}{2}\,\alpha_0\,w^1\Big)\,u^0 \otimes \mathfrak{b}_2 .\end{aligned}$$

Then, the claim is obtained by comparison of the coordinate expressions of $i_{\bar{\upsilon}}(\mathrm{re}\,\mathfrak{h})^\sharp\big(\mathsf{E}[\mathsf{L}_i]\big)$ and S and by taking into account the natural fibred isomorphism $V\boldsymbol{Q} \simeq \boldsymbol{Q}$, which yields $\partial w_{\mathrm{a}} \simeq \mathfrak{b}_{\mathrm{a}}$. □

Note 17.7.19 The Euler–Lagrange operator $\mathsf{E}[\mathsf{L}_r] : J_2 \boldsymbol{Q} \to T^* \boldsymbol{Q} \otimes \Lambda^4 T^* \boldsymbol{E}$ associated with the covariant quantum lagrangian $\mathsf{L}_r[\Psi] = dt \wedge \mathrm{re}\, \mathrm{h}_\eta(\Psi, \mathsf{S}[\Psi])$ vanishes (see the above Theorem 17.7.16).

Proof. The real coordinate expression of the lagrangian $\mathsf{L}_r[\Psi] = dt \wedge \mathrm{re}\, \mathrm{h}_\eta(\Psi, \mathsf{S}[\Psi])$ is

$$\begin{aligned}
\mathsf{L}_r = \Big(& (w^1\, w^1_0 + w^2\, w^2_0) + \tfrac{1}{2}\, G^{ij}_0\, (w^1\, w^2_{ij} - w^2\, w^1_{ij}) - A^j_0\, (w^1\, w^1_j + w^2\, w^2_j) \\
& + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, (w^1\, w^2_j - w^2\, w^1_j) \\
& + \tfrac{1}{2}\, \Big(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\Big) \big((w^1)^2 + (w^2)^2\big)\Big)\, dw^1 \otimes \upsilon^0 .
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
\mathsf{E}[\mathsf{L}_r] = \Big(& - w^1_0 - \tfrac{1}{2}\, G^{ij}_0\, w^2_{ij} + A^j_0\, w^1_j - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^2_j \\
& - \Big(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\Big)\, w^1 + \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} w^1 + w^1_0 - \frac{\partial_j (A^j_0 \sqrt{|g|})}{\sqrt{|g|}} w^1 \\
& - A^j_0\, w^1_j - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^2 - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}} w^2_j \\
& + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}},\, w^2 + \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}} w^2_j + \tfrac{1}{2}\, G^{ij}_0\, w^2_{ij}\Big)\, \check{d} w^1 \wedge \upsilon^0 \\
& + \Big(- w^2_0 + \tfrac{1}{2}\, G^{ij}_0\, w^1_{ij} + A^j_0\, w^2_j + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^1_j \\
& - \Big(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\Big)\, w^2 + \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} w^2 + w^2_0 - \frac{\partial_j (A^j_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^2 \\
& - A^j_0\, w^2_j + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^1 + \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^1_j \\
& - \tfrac{1}{2}\, \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^1 - \frac{\partial_i (G^{ij}_0 \sqrt{|g|})}{\sqrt{|g|}}\, w^1_j - \tfrac{1}{2}\, G^{ij}_0\, w^1_{ij}\Big)\, \check{d} w^2 \wedge \upsilon^0 \\
= 0. \quad \square &
\end{aligned}$$

Chapter 18
Hydrodynamical Picture of QM

In Sect. 7.3 we have discussed a suitable setting for kinematics and dynamics of a classical charged fluid fitting our covariant curved galilean framework. Now, we show that we can formally associate, in a covariant way, with every quantum section Ψ, in its proper domain, a classical charged fluid $\mathfrak{F}[\Psi]$, which encodes, in a classical language, information carried by the quantum section Ψ (Sect. 18.1).

We stress that the proper quantum section Ψ determines uniquely the associated classical fluid $\mathfrak{F}[\Psi]$, but, conversely, the proper quantum section can be recovered from the classical fluid only up to a phase transformation which is spacelikely constant.

We consider this classical picture just as a natural "mathematical mate" of the quantum picture, but deliberately we express no opinion about its possible "realistic" interpretation.

Indeed, we express the classical law of motion of the associated fluid and the quantum pressure through the distinguished timelike potential $A[\Psi]$ "seen" by the quantum particle (see Theorems 15.2.31 and 15.2.31). Actually, our interpretation of the quantum pressure $\mathfrak{p}[\Psi]$ in terms of the distinguished potential $A[\Psi]$ can be hardly achieved in standard Quantum Mechanics, as it deals with the possibly non inertial observer o_Ψ (see Theorem 15.2.31).

The above picture generalises, in a covariant way, to our curved galilean spacetime, well known achievements of standard Quantum Mechanics (see, for instance, [58, 230, 280, 340]).

Thus, in the present Chap. 18, we consider a *scalar quantum particle*, of *mass* and *charge*

$$m \in \mathbb{M} \qquad \text{and} \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R},$$

represented by a *proper quantum section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$ (see Definition 14.6.1).

In order to achieve a consistent physical interpretation of the mass and the charge, we suppose that the proper quantum section be normalised on the fibres of spacetime.

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Moreover, we have in mind our general discussion on the kinematics and dynamics of a generic classical continuum motion (see Sects. 7.3 and 7.4).

Actually, the present discussion turns out to be a covariant version, in our curved galilean framework, of the "hydrodynamical picture" of standard Quantum Mechanics (see, for instance, [176, 230, 280]).

Nowadays, the hydrodynamical picture of standard Quantum Mechanics is considered by the Bohmian approach to Quantum Mechanics (see, for instance, [7, 8, 29, 30, 62, 63, 98, 99, 101, 179–181, 394, 426]) and is largely used in condensed matter Physics (see, for instance, [58, 340]).

18.1 Kinematics of the Associated Classical Fluid

We show that we can formally associate, in a covariant way, a *classical charged fluid* $\big(\mathscr{C},\ \mu,\ \rho\big) \equiv \big(\mathscr{C}[\Psi],\ \mu[\Psi],\ \rho[\Psi]\big)$ with a *proper quantum section* Ψ, representing a quantum particle of mass m and charge q (see Definition 7.3.1).

Indeed, this approach is analogous to the usual hydrodynamical picture of standard Quantum Mechanics (see, for instance, [230, 280]), but the present discussion is extended to our curved galilean framework and it is gauge and observer independent (for einsteinian general relativistic hydrodynamics, see, for instance, [276]).

18.1.1 Associated Classical Fluid

The proper quantum section Ψ yields, formally, in a covariant way a, gauge independent and observer independent, continuum motion $\mathscr{C}[\Psi]$, along with its velocity $\mathscr{V}[\Psi]$, mass density $\mu[\Psi]$ and charge density $\rho[\Psi]$.

Indeed, the mass density $\mu[\Psi]$ and charge density $\rho[\Psi]$ depend only on the norm $\|\Psi\|$ of the quantum section Ψ, while the continuum motion $\mathscr{C}[\Psi]$ and its velocity $\mathscr{V}[\Psi]$ depend only on the phase $(\!(\Psi)\!)$ of the quantum section Ψ (see Proposition 14.7.1). Actually, the present hydrodynamical picture of Quantum Mechanics is compatible with the classical addition rule of velocities (see Definition 2.7.7).

We discuss how our convention on the orientation of the phase quantum bundle effects the associated classical fluid. Moreover, we discuss to what extent the hydrodynamical picture characterises the proper quantum section. Further, we emphasise an interesting relation between the associated observed velocity and the operator associated with the kinetic momentum re $\frac{\mathrm{O}[\mathcal{Q}_j](\Psi)}{\Psi} = (\partial_j \varphi - A_j)$ (see Definition 3.2.9, Note 18.1.12 and Example 20.1.20).

Let us consider a *proper quantum section* along with its *mass*, *charge* and *quantum velocity field* (see Definition 14.6.1 and Theorem 17.2.2)

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}),$$
$$m \in \mathbb{M}, \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R} \qquad \mathrm{V}[\Psi] \in \sec(\boldsymbol{E},\ \mathbb{T}^* \otimes T\boldsymbol{E}).$$

Moreover, let us have in mind our general model of a generic classical continuum motion (see Definitions 7.3.1–7.3.3).

Theorem 18.1.1 *The proper quantum section* Ψ *yields* "formally"*, in a covariant way, in its proper domain a* charged classical continuum*, whose observer independent and gauge independent,* mass density*,* charge density *and* velocity field *are defined by (see Definitions 2.4.2, 7.3.1, Proposition 14.3.1 and Theorem 17.2.2)*

$$\begin{aligned}
\mu &\equiv \mu[\Psi] := m\,\|\Psi\|^2 \in \operatorname{map}(\boldsymbol{E}, \mathbb{L}^{-3} \otimes \mathbb{M}),\\
\rho &\equiv \rho[\Psi] := q\,\|\Psi\|^2 \ \in \operatorname{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big),\\
\mathscr{V} &\equiv \mathscr{V}[\Psi] := \mathrm{V}[\Psi] \quad\ \in \sec(\boldsymbol{E},\ \mathbb{T}^* \otimes T\boldsymbol{E}).
\end{aligned}$$

With reference to an observer o*, we have the following observed splitting (see Propositions 2.7.3 and 16.1.14)*

$$\mathscr{V}[\Psi] = д[o] + \vec{\nabla}^{(0)}[o]((\Psi))$$

and the following coordinate expressions (see Theorem 17.2.2)

$$\mu[\Psi] = m\,|\psi|^2, \qquad \rho[\Psi] = q\,|\psi|^2, \qquad \mathscr{V}[\Psi] = u^0 \otimes \big(\partial_0 + G^{ij}_0\,(\partial_j\varphi - A_j)\,\partial_i\big).$$

Clearly, the velocity vector field $\mathscr{V}[\Psi]$ *determines (locally) a* continuum motion *(see Definition 7.3.1)*

$$\mathscr{C}[\Psi] : (\mathbb{T} \times \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E},$$

which is characterised by the equation

$$\partial\mathscr{C}[\Psi] = \mathscr{V}[\Psi],$$

with coordinate expression

$$\delta_0\mathscr{C}^0 = 1 \qquad \textit{and} \qquad \delta_0\mathscr{C}^i = G^{ij}_0\,(\partial_j\varphi - A_j).$$

Indeed, the continuum motion $\mathscr{C}[\Psi]$ *and its velocity* $\mathscr{V}[\Psi]$ *coincide, respectively, with the motion* $\mathscr{F}[o_\Psi]$ *and the velocity* $д[o_\Psi]$ *of the distinguished observer* o_Ψ*, according to the equalities (see Theorem 15.2.31)*

$$\mathscr{C}[\Psi] = \mathscr{F}[o_\Psi] \qquad \textit{and} \qquad \mathscr{V}[\Psi] = д[o_\Psi].$$

Proof. The expression of $\mathscr{V}[\Psi]$ follows from Theorems 17.2.2 and 15.2.31. □

We can equivalently define the velocity $\mathscr{V}[\Psi]$ in terms of jets as $\mathrm{j}_1\mathscr{C}$, as follows.

Remark 18.1.2 The map

$$\mathrm{j}_1\mathscr{C} : \boldsymbol{E} \to J_1\boldsymbol{E}$$

is uniquely defined by the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{\ j_1\mathscr{C}[\Psi]\ } & J_1\boldsymbol{E} \\ & {\scriptstyle \mathscr{V}[\Psi]}\searrow \quad \swarrow{\scriptstyle \text{д}} & \\ & \mathbb{T}\otimes \boldsymbol{TE} & \end{array} \quad . \square$$

Definition 18.1.3 We say the charged classical fluid $\big(\mathscr{C}[\Psi],\ \mu[\Psi],\ \rho[\Psi]\big)$ defined by the above Theorem 18.1.1 to be the classical *fluid associated* with the proper quantum section Ψ. For the clear reasons arising from the above theorem, the distinguished observer o_Ψ will be called the "*rest observer*" associated with Ψ. □

Thus, the above Theorem 18.1.1 provides an intuitive interpretation of the distinguished observer o_Ψ and of the quantum velocity field $\mathsf{V}[\Psi]$, that we have already introduced by other means in Theorems 15.2.31 and 17.2.2.

Corollary 18.1.4 *For each observer o, the gauge independent observed velocity is given by the equality (see Definition 2.7.8 and Proposition 16.1.14)*

$$\vec{\mathscr{V}}[\Psi, o] = \vec{\nabla}^{(\!(\!)\!)}[o](\!(\Psi)\!),$$

with coordinate expression

$$\vec{\mathscr{V}}[\Psi, o] = G^{ij}_0 \left(\partial_j \varphi - A_j\right) u^0 \otimes \partial_i .$$

In particular, the observed velocity $\vec{\mathscr{V}}[\Psi, o_\Psi]$ associated with the distinguished observer o_Ψ vanishes:

$$\vec{\mathscr{V}}[\Psi, o_\Psi] = 0.$$

Proof. In virtue of the above Theorem 18.1.1, the proof follows from the equalities

$$\mathscr{V}[\Psi] = \text{д}[o] + \vec{\nabla}^{(\!(\!)\!)}[o](\!(\Psi)\!) \qquad \text{and} \qquad \mathscr{V}[\Psi] = \text{д}[o_\Psi]. \quad \square$$

The basic objects of the associated classical fluid factorise through the two real degrees of freedom of the proper quantum section in the following way (see Proposition 14.7.2).

Remark 18.1.5 The mass density $\mu[\Psi]$, the charge density $\rho[\Psi]$, the continuum motion $\mathscr{C}[\Psi]$ and the velocity $\mathscr{V}[\Psi]$ defined in the above Theorem 18.1.1 factorise separately through the two, gauge independent and observer independent, real components of Ψ

$$\|\Psi\| \in \sec\big(\boldsymbol{E}, \boldsymbol{Q}^{\|\,\|}_{/0}\big) = \sec\big(\boldsymbol{E},\ \boldsymbol{E} \times \mathbb{L}^{-3/2}\big),$$
$$(\!(\Psi)\!) \in \sec\big(\boldsymbol{E}, \boldsymbol{Q}^{(\!(\,)\!)}_{/0}\big) \simeq \sec\big(\boldsymbol{E},\ \boldsymbol{E} \times \mathbb{R}/2\pi\big),$$

according to the following scheme (see Propositions 14.7.2 and 14.3.1 and Theorem 17.2.2)

$$\begin{array}{ccccccc}
 & & \|\Psi\| & \xleftarrow{\ \|\ \|\ } & \Psi & \xrightarrow{\ (\!(\ )\!)\ } & (\!(\Psi)\!) & & \\
 & & {\scriptstyle m\,(.)^2}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathscr{V}} & & \\
\rho[\Psi] & \xleftarrow{\ \frac{q}{m}\ } & \mu[\Psi] & & & & \mathscr{V}[\Psi] & \xrightarrow{\ \int\ } & \mathscr{C}[\Psi]\ .\square
\end{array}$$

It is worth discussing the gauge independence and observer independence of $\mathscr{V}[\Psi]$.

Remark 18.1.6 Having in mind the equalities

$$\mathscr{V}[\Psi] = \text{д}[o] + \vec{\mathscr{V}}[\Psi, o] = u^0 \otimes (\partial_0 + o^i_0\, \partial_i) + G^{ij}_0\, \big(\partial_j \varphi - A_j\big)\, u^0 \otimes \partial_i,$$

we stress that

- $\mathscr{V}[\Psi]$ is gauge independent and observer independent by definition,
- $\text{д}[o]$ is gauge independent and observer dependent,
- $\vec{\mathscr{V}}[\Psi, o]$ is gauge independent and observer dependent,
- $\partial_j \varphi$ is gauge dependent and observer independent,
- A_j is gauge dependent and observer dependent.

Thus, the gauge dependence of $\varphi[\mathrm{b}]$ and $A[\mathrm{b}, o]$ compensate each other, yielding the gauge independence of $\mathscr{V}[\Psi]$.

Moreover, the observer dependence of $\text{д}[o]$ and $A[\mathrm{b}, o]$ compensate each other, yielding the observer independence of $\mathscr{V}[\Psi]$.

We stress that the observed potential $A[\mathrm{b}, o]$ involved in the above formulas is joined. □

We can compare the observer dependence of $\vec{\mathscr{V}}[\Psi, o]$ with the classical law of composition of velocities.

Exercise 18.1.7 Let us consider two observers o and $\acute{o} = o + v$. Then, the equalities

$$\vec{\mathscr{V}}[\Psi, o] := \vec{\nabla}^{(\!(\,)\!)}[o](\!(\Psi)\!) \qquad \text{and} \qquad \vec{\mathscr{V}}[\Psi, \acute{o}] := \vec{\nabla}^{(\!(\,)\!)}[\acute{o}](\!(\Psi)\!)$$

and the transition rule of observed quantum potential (see Theorem 15.2.26)

$$A[\mathrm{b}, \acute{o}] = A[\mathrm{b}, o] + \theta[o] \lrcorner G^{\flat}(\vec{v}) - \tfrac{1}{2}\, G(\vec{v}, \vec{v}),$$

yield the transition rule (see Theorem 18.1.1)

$$\vec{\mathscr{V}}[\Psi, \acute{o}] = \vec{\mathscr{V}}[\Psi, o] - \vec{v},$$

which agrees with the standard classical law of composition of velocities (see Definition 2.7.8). □

Next, we discuss how the convention on the quantum phase orientation effects the associated fluid.

Remark 18.1.8 The directions of the velocity $\mathscr{V}[\Psi]$ and of the observed velocity $\vec{\mathscr{V}}[\Psi, o]$ of a continuum motion do not depend on any convention, up to the "future direction of time", i.e. the orientation of $\boldsymbol{T}$.

If we change the orientation of the fibres of the phase quantum bundle $\pi^{(\!)} : \boldsymbol{Q}^{(\!)} \to \boldsymbol{E}$, (i.e. if we change the sign of the phase of all quantum sections), then we obtain a quantum model which is equivalent to the present one. But, after such a transformation, the above Theorem 18.1.1 would yield, for each observer o, an observed velocity $\mathscr{V}[\Psi, o]$ with opposite sign.

So, in order to obtain the same observer independent associated velocity $\mathscr{V}[\Psi]$, we should change, at the same time, the orientation of the fibres of the phase quantum bundle $\pi^{(\!)} : \boldsymbol{Q}^{(\!)} \to \boldsymbol{E}$ and the orientation of $\boldsymbol{T}$. □

It is worth discussing at what extent the associated classical fluid characterises the proper quantum section.

Remark 18.1.9 The following correspondences occur:

(1) The map

$$\sec(\boldsymbol{E}, \boldsymbol{Q}^{\|\,\|}_{/0}) \to \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3} \otimes \mathbb{M}) : \|\Psi\| \mapsto \mu[\Psi]$$

is bijective.

(2) The map

$$\sec(\boldsymbol{E}, \boldsymbol{Q}^{(\!)}_{/0}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{TE}) : ((\Psi)) \mapsto \mathscr{V}[\Psi]$$

is surjective. Indeed, it is injective only up to a phase transformation of the quantum bundle

$$\boldsymbol{Q} \to \boldsymbol{Q} : \mathrm{q} \mapsto \mathrm{q}\, \exp(\mathrm{i}\,\vartheta), \quad \text{with } \vartheta \in \mathrm{map}(\boldsymbol{T}, \mathbb{R}/2\pi). \quad □$$

In Corollary 15.2.33 we have calculated the distinguished phase maps $\mathcal{H}$, $\mathcal{P}$, $\mathcal{C}$ related to a generic classical particle and referred to the distinguished gauge b_Ψ and the distinguished observer o_Ψ associated with a proper quantum section Ψ.

Now, by replacing the generic classical particle with every classical particle associated with the classical flow $\mathscr{C}[\Psi]$ of the quantum particle itself, we obtain the following equalities.

Corollary 18.1.10 *With reference to every particle of the flow* $\mathscr{C}[\Psi]$, *we have the following equalities (see Remark 18.1.2)*

$$\begin{aligned} \mathcal{H}[\mathrm{b}_\Psi, o_\psi], \mathrm{j}_1\mathscr{C}[\Psi] &= \mathcal{K}[o_\Psi], \mathrm{j}_1\mathscr{C}[\Psi] - A[\Psi] = -A[\Psi], \\ \mathcal{P}[\mathrm{b}_\Psi, o_\psi], \mathrm{j}_1\mathscr{C}[\Psi] &= \mathcal{Q}[o_\Psi], \mathrm{j}_1\mathscr{C}[\Psi] = 0, \\ \mathcal{C}[o_\Psi], \mathrm{j}_1\mathscr{C}[\Psi] &= A^{\uparrow}[\mathrm{b}_\Psi], \mathrm{j}_1\mathscr{C}[\Psi] - A[\Psi] = -A[\Psi]. \quad \square \end{aligned}$$

Remark 18.1.11 We stress that the above formulas implicitly assign the same mass m and charge q of the quantum particle to every classical particle of the flow $\mathscr{C}[\Psi]$.

In fact, these values are included in the definition of $A[\Psi]$. □

The above formulas emphasise a further physical meaning of the distinguished timelike potential $A[\Psi]$ "seen by the quantum particle".

We emphasise the link between the associated velocity $\mathscr{V}[\Psi]$ and the quantum operator $\mathrm{O}[\mathcal{Q}_j]$.

Note 18.1.12 Later, we shall find the quantum operators associated with distinguished phase functions (see Example 20.1.20).

In particular, we shall find the real component of the quantum operator associated with the phase functions $\mathcal{Q}_j$

$$\mathrm{re}\,\frac{\mathrm{O}[\mathcal{Q}_j](\Psi)}{\Psi} = (\partial_j \varphi - A_j).$$

Indeed, this quantum operator fits the covariant observed velocity of the associated fluid, according to the equality,

$$\big(G^\flat(\vec{\mathscr{V}}[\Psi, o])\big)_j = G^0_{ij}\mathscr{V}^i_0 = \partial_j\varphi - A_j. \quad \square$$

Eventually, in view of a comparison with standard Quantum Mechanics, we analyse the example of the classical velocity associated with a quantum plane wave.

Example 18.1.13 Let us consider a flat spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave (with vanishing electromagnetic field and with reference to an inertial observer and cartesian coordinates)

$$\Psi := |\psi|\, e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with } \ \varphi = \mathrm{k}_\lambda\, x^\lambda,\ \mathrm{k}_\lambda \in \mathbb{R},\ |\psi| \neq 0.$$

Then, we obtain

$$\mathscr{V}[\Psi] = u^0 \otimes \big(\partial_0 + \tfrac{\hbar_0}{m}\, g^{ij}\mathrm{k}_j\,\partial_i\big) \qquad \text{and} \qquad \vec{\mathscr{V}}[\Psi, o] = u^0 \otimes \big(\tfrac{\hbar_0}{m}\, g^{ij}\mathrm{k}_j\,\partial_i\big). \square$$

18.1.2 Associated Mass and Charge Density Currents

Given a proper quantum section Ψ, the associated velocity $\mathscr{V}[\Psi]$, along with the mass m and the charge q, yield the associated *mass density current* $\mathscr{P}[\Psi]$ and *charge density current* $\mathscr{J}[\Psi]$.

Let us consider a *normalised proper quantum section* along with the associated *velocity field*, *mass density* and *charge density* (see Theorem 18.1.1)

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}), \qquad \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ \mathbb{T}^* \otimes T\boldsymbol{E}\big),$$
$$\mu[\Psi] \in \operatorname{map}(\boldsymbol{E}, \mathbb{L}^{-3} \otimes \mathbb{M}), \qquad \rho[\Psi] \in \operatorname{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big).$$

Moreover, let us have in mind our general discussion on the kinematics of a generic classical continuum motion (see Definitions 7.3.1–7.3.3).

Definition 18.1.14 We define the, gauge independent and observer independent, *mass density current* and *charge density current* associated with the proper quantum section Ψ to be the scaled sections (see Definition 7.3.2)

$$\mathscr{P} \equiv \mathscr{P}[\Psi] := \mu[\Psi]\, \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes T\boldsymbol{E}\big),$$
$$\mathscr{J} \equiv \mathscr{J}[\Psi] := \rho[\Psi]\, \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T\boldsymbol{E}\big).$$

Indeed, we have

$$\mathscr{J}[\Psi] = \tfrac{q}{m}\, \mathscr{P}[\Psi],$$
$$dt \lrcorner \mathscr{P}[\Psi] = \mu[\Psi] \qquad \text{and} \qquad dt \lrcorner \mathscr{J}[\Psi] = \rho[\Psi]. \quad \square$$

Proposition 18.1.15 *If* Ψ *is normalised by the condition*

$$\int_{\boldsymbol{E}_t} \|\Psi\|^2\, \eta = 1, \qquad \forall t \in \boldsymbol{T},$$

then we obtain

$$m = \int_{\boldsymbol{E}_t} \mu[\Psi]\, \eta \qquad \textit{and} \qquad q = \int_{\boldsymbol{E}_t} \rho[\Psi]\, \eta, \qquad \forall t \in \boldsymbol{T}. \quad \square$$

18.1.3 Associated Acceleration

Next, in view of the law of motion for the associated fluid, we prove the equality $\mathscr{A}[\Psi] = -\vec{d} A[\Psi]$, which expresses the associated joined acceleration through the spacelike gradient of the distinguished timelike potential “seen” by the particle (see Theorem 15.2.31).

Indeed, this result will provide our interpretation of the quantum pressure (see Theorem 18.2.1). Actually, this result can be hardly achieved in standard Quantum Mechanics, as it deals with the possibly non inertial observer o_Ψ.

Then, we discuss the observed splitting of the gravitational acceleration $\mathscr{A}^\natural[\Psi]$ and of the joined acceleration $\mathscr{A}[\Psi]$ of the associated fluid, by translating to the present case the general results holding for a generic fluid (see Proposition 7.3.9 and Corollary 7.3.10).

Let us consider a proper quantum section along with the associated, gauge independent and observer independent, velocity field (see Theorem 18.1.1)

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}) \qquad \text{and} \qquad \mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big).$$

With reference to a generic quantum basis b and a generic observer o, we recall the coordinate expression, in an adapted quantum chart,

$$\mathscr{V}[\Psi] = д[o] + \vec{\mathscr{V}}[o] = u^0 \otimes \big(\partial_0 + G^{ij}_0\,(\partial_j\varphi - A_j)\,\partial_i\big),$$

where $A[\mathsf{b}, o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ is the joined observed potential (see Theorem 10.1.4).

In particular, with reference to the distinguished basis b_Ψ and the distinguished observer o_Ψ, the above coordinate expression, in an adapted quantum chart, becomes (see Note 14.6.3 and Theorem 15.2.31)

$$\mathscr{V}[\Psi] = д[o_\Psi] = u^0 \otimes \partial_0.$$

Definition 18.1.16 We define the associated *gravitational acceleration* and *joined acceleration* to be the, gauge independent and observer independent, scaled sections (see Postulate C.3 and Theorem 6.3.1)

$$\begin{aligned}
\mathscr{A}^\natural \ &\equiv\ \mathscr{A}^\natural[\Psi] := \nabla^\natural{}_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi] \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big),\\
\mathscr{A} \ &\equiv\ \mathscr{A}[\Psi] := \nabla_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi] \quad\ \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big). \qquad \square
\end{aligned}$$

First of all, we compute, in the present framework, the associated joined acceleration $\mathscr{A}[\Psi]$ and express it in terms of the distinguished potential $A[\Psi]$, which is encoded (in a hidden way) in the associated velocity $\mathscr{V}[\Psi]$ (see Theorems 18.1.1 and 15.2.31).

Theorem 18.1.17 *The gauge independent and observer independent joined acceleration $\mathscr{A}[\Psi]$ of the associated fluid turns out to be given by the following equality*

$$\mathscr{A}[\Psi] = -\vec{d} A[\Psi],$$

with coordinate expression, in a spacetime chart adapted to o_Ψ,

$$\mathscr{A}[\Psi] = -G_0^{ij}\,\partial_j A_0[\Psi]\,u^0 \otimes u^0 \otimes \partial_i,$$

where

$$A[\Psi] := A[\mathrm{b}_\Psi, o_\Psi] \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes I\!\!R)$$

is the distinguished gauge independent and observed independent timelike observed joined potential $A[\Psi]$ *"seen" by the quantum particle (see Theorem 15.2.31).*

Proof. As $\mathscr{A}[\Psi]$ is observer independent, we can compute it by referring to the distinguished observer o_Ψ (and to the distinguished basis b_Ψ).

Then, in virtue of Theorem 18.1.1, with reference to the observer o_Ψ and to an adapted spacetime chart, we obtain

$$\mathscr{A}[\Psi] = \nabla_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi] = \nabla_{Д[o_\Psi]}Д[o_\Psi] = -K_0{}^i{}_0\,u^0 \otimes u^0 \otimes \partial_i = G_0^{ij}\,\Phi_{0j}\,u^0 \otimes u^0 \otimes \partial_i.$$

With reference to the distinguished basis b_Ψ, we have $A_j = 0$, hence (see Theorem 15.2.31)

$$\Phi[o_\Psi]_{0j} = \partial_0 A_j - \partial_j A_0 = -\partial_j A_0. \quad \square$$

Thus, the above expression $\mathscr{A}[\Psi] = -\vec{d}A[\Psi]$ arises from the joined acceleration of the rest observer o_Ψ.

Next, we can compare the joined and gravitational acceleration according to a general formula holding for any classical fluid.

Proposition 18.1.18 *The associated joined acceleration and gravitational acceleration are related by the following equality (see Proposition 7.3.5)*

$$\mathscr{A}[\Psi] = \mathscr{A}^{\natural}[\Psi] + \tfrac{q}{\hbar}\,G^{\sharp}\big(\mathscr{V}[\Psi] \lrcorner F\big),$$

where the additional term arises from the Lorentz force density (see Definition 5.7.3)

$$-\tfrac{q}{\hbar}\,G^{\sharp}\big(\mathscr{V}[\Psi] \lrcorner F\big) \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big). \quad \square$$

Next, in view of a comparison with standard literature, we analyse, in the present framework, the observed splitting of the above gravitational acceleration, according to our general lagrangian and eulerian schemes achieved for a generic fluid (see Proposition 7.3.9).

Proposition 18.1.19 *We have the following observed splitting of the associated gravitational acceleration, in terms, respectively, of the lagrangian and eulerian schemes,*

$$\mathscr{A}^\natural[\Psi] = \mathscr{A}^\natural{}_{\text{lag}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{rlt}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{drg}}[\Psi, o],$$
$$= (\mathscr{A}^\natural{}_{\text{eul}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{spc}}[\Psi, o]) + \mathscr{A}^\natural{}_{\text{rlt}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{drg}}[\Psi, o],$$

where we have set

$$\mathscr{A}^\natural{}_{\text{lag}}[\Psi, o] := \nabla^\natural{}_{\mathscr{V}[\Psi]}\vec{\mathscr{V}}[\Psi, o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A}^\natural{}_{\text{eul}}[\Psi, o] := \nabla^\natural{}_{\text{Д}[o]}\vec{\mathscr{V}}[\Psi, o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A}^\natural{}_{\text{spc}}[\Psi, o] := \nabla^\natural{}_{\vec{\mathscr{V}}[\Psi,o]}\vec{\mathscr{V}}[\Psi, o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A}^\natural{}_{\text{rlt}}[\Psi, o] := \nabla^\natural{}_{\vec{\mathscr{V}}[\Psi,o]}\text{Д}[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A}^\natural{}_{\text{drg}}[\Psi, o] := \nabla^\natural{}_{\text{Д}[o]}\text{Д}[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}.$$

Thus, the lagrangian and eulerian accelerations are related by the equality

$$\mathscr{A}^\natural{}_{\text{lag}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{eul}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{spc}}[\Psi, o].$$

Moreover, we obtain the equality

$$\mathscr{A}^\natural{}_{\text{rlt}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{dfr}}[\Psi, o] + \mathscr{A}^\natural{}_{\text{crl}}[\Psi, o],$$

by setting

$$\mathscr{A}^\natural{}_{\text{dfr}}[\Psi, o] := -\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \widehat{\Sigma}^\natural[o]\big) : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$
$$\mathscr{A}^\natural{}_{\text{crl}}[\Psi, o] := -\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \widehat{\Phi}^\natural[o]\big) : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}. \qquad \square$$

Corollary 18.1.20 *We obtain an analogous splitting of the joined acceleration $\mathscr{A}[\Psi]$, by taking into account the following equalities (see Proposition 7.4.2 and Definition 5.7.3)*

$$\mathscr{A}_{\text{dfr}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{dfr}}[\Psi, o], \qquad \mathscr{A}_{\text{spc}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{spc}}[\Psi, o]$$

and

$$\mathscr{A}_{\text{crl}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{crl}}[\Psi, o] - \tfrac{1}{c}\,\tfrac{q}{m}\,\vec{\mathscr{V}}[\Psi, o] \times B,$$
$$\mathscr{A}_{\text{drg}}[\Psi, o] = \mathscr{A}^\natural{}_{\text{drg}}[\Psi, o] - \tfrac{q}{m}\,\vec{E}[o],$$

where the additional terms arise from the Lorentz force density

$$-\tfrac{q}{\hbar}\,G^\sharp\big(\mathscr{V}[\Psi] \lrcorner F\big) \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}\big). \qquad \square$$

Eventually, we provide two explicit expressions of the gravitational relative, dragging and spacelike observed acceleration.

Proposition 18.1.21 *For each observer o, the relative and the dragging gravitational accelerations are expressed by the equalities*

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi, o] &= \tfrac{1}{2}\, G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}^{\natural}[o]\big) + \tfrac{1}{2}\, G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner L_{\text{д}[o]} G\big),\\ \mathscr{A}^{\natural}{}_{\text{drg}}[\Psi, o] &= G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big),\end{aligned}$$

with coordinate expressions, in a spacetime chart adapted to o,

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi, o]^{i}_{00} &= \tfrac{1}{2}\; G^{ij}_{0}\,\big(\partial_0 G^{0}_{hj} + \Phi^{\natural}{}_{hj}\big)\, \mathscr{V}^{h}_{0},\\ \mathscr{A}^{\natural}{}_{\text{drg}}[\Psi, o]^{i}_{00} &= G^{ij}_{0}\; \Phi^{\natural}{}_{0j}.\end{aligned}$$

In particular, with reference to the distinguished rest observer o_{Ψ}, the above equality becomes

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi, o_{\Psi}] &:= \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi, o_{\Psi}]}\text{д}[o_{\Psi}] = 0,\\ \mathscr{A}^{\natural}{}_{\text{drg}}[\Psi, o_{\Psi}] &:= \nabla^{\natural}{}_{\text{д}[o_{\Psi}]}\text{д}[o_{\Psi}].\end{aligned}$$

Proof. We have (see Theorem 4.2.13)

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi, o_{\Psi}] &= \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi, o]}\text{д}[o]\\ &= -K^{\natural}{}_{h}{}^{i}{}_{0}\, \mathscr{V}^{h}_{0}\, u^0 \otimes u^0 \otimes \partial_i\\ &= \tfrac{1}{2}\, G^{ij}\,\big(\partial_0 G^{0}_{hj} + \Phi^{\natural}{}_{hj}\big)\, \mathscr{V}^{h}_{0}\, u^0 \otimes u^0 \otimes \partial_i\\ &= \tfrac{1}{2}\, G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}^{\natural}[o]\big) + \tfrac{1}{2}\, G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner L_{\text{д}[o]} G\big)\end{aligned}$$

and

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\text{drg}}[\Psi, o_{\Psi}] &= \nabla^{\natural}{}_{\text{д}[o]}\text{д}[o] = -K^{\natural}{}_{0}{}^{i}{}_{0}\, u^0 \otimes u^0 \otimes \partial_i = G^{ij}\, \Phi^{\natural}{}_{0j}\, u^0 \otimes u^0 \otimes \partial_i\\ &= G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big).\end{aligned}$$

Eventually, we recall that $\vec{\mathscr{V}}[\Psi, o_{\Psi}] = 0$ (see Theorem 18.1.1). □

Proposition 18.1.22 *For each observer o, the spacelike gravitational acceleration is expressed by the equalities (see Corollary 18.1.4)*

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o] &:= \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi,o]}\vec{\mathscr{V}}[\Psi, o]\\ &= G^{\sharp}\big(L_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}(\vec{\mathscr{V}}[\Psi, o])\big) - \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big)\\ &= \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big) - G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}[o]\big),\end{aligned}$$

with coordinate expression, in a spacetime chart adapted to o,

$$\begin{aligned}\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o]^i_{00} &= \mathscr{V}_0^h\,\partial_h\mathscr{V}_0^i + G_0^{ij}\,\partial_h G^0_{jk}\,\mathscr{V}_0^k\,\mathscr{V}_0^h - \tfrac{1}{2}\,G_0^{ij}\,\partial_j G^0_{hk}\,\mathscr{V}_0^k\,\mathscr{V}_0^h\\ &= \tfrac{1}{2}\,G_0^{ih}\,\partial_h(G^0_{rs}\,\mathscr{V}_0^r\,\mathscr{V}_0^s) - G_0^{ij}\,\mathscr{V}_0^h\,\Phi_{hj}.\end{aligned}$$

In particular, with reference to the distinguished rest observer o_Ψ, the above equality becomes

$$\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o_\Psi] := \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi,o_\Psi]}\vec{\mathscr{V}}[\Psi, o_\Psi] = 0.$$

Proof. We have (see Theorem 4.2.13)

$$\begin{aligned}\nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi,o]}\vec{\mathscr{V}}[\Psi, o] &= \big(\partial_h\mathscr{V}_0^i - K^{\natural}{}_h{}^i{}_j\,\mathscr{V}_0^j\big)\,\mathscr{V}_0^h\,u^0 \otimes u^0 \otimes \partial_i\\ &= \big(\partial_h\mathscr{V}_0^i + \tfrac{1}{2}\,G_0^{ir}\,(\partial_h G^0_{rj} + \partial_j G^0_{rh} - \partial_r G^0_{jh})\,\mathscr{V}_0^j\big)\,\mathscr{V}_0^h\,u^0 \otimes u^0 \otimes \partial_i\\ &= \big(\partial_h\mathscr{V}_0^i\,\mathscr{V}_0^h + G_0^{ij}\partial_h G^0_{jk}\,\mathscr{V}_0^k\,\mathscr{V}_0^h - \tfrac{1}{2}\,G_0^{ij}\partial_j G^0_{hk}\,\mathscr{V}_0^k\,\mathscr{V}_0^h\big)\,u^0 \otimes u^0 \otimes \partial_i.\end{aligned}$$

Moreover, we have

$$\begin{aligned}&G^{\sharp}\Big(L_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) - \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big)\\ &= G^{\sharp}\Big(i_{\vec{\mathscr{V}}[\Psi,o]}\,\check{d}\,G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big) + \check{d}\,i_{\vec{\mathscr{V}}[\Psi,o]}\,G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) - \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big)\\ &= G^{\sharp}\big(i_{\vec{\mathscr{V}}[\Psi,o]}\,\check{d}\,G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big) + \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big)\\ &= G^{\sharp}\big(i_{\vec{\mathscr{V}}[\Psi,o]}\,\partial_j(G^0_{hi}\,\mathscr{V}_0^h)\,\check{d}^j \wedge \check{d}^i\big) + \tfrac{1}{2}\,G_0^{ij}\,\partial_j(G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^k)\,u^0 \otimes u^0 \otimes \partial_i\\ &= G^{\sharp}\Big(\partial_j(G^0_{hi}\,\mathscr{V}_0^h)\,\mathscr{V}_0^j - \partial_i(G^0_{hj}\,\mathscr{V}_0^h)\,\mathscr{V}_0^j\big)\,u^0 \otimes \check{d}^i\Big)\\ &\quad + \big(\tfrac{1}{2}\,G_0^{ij}\,\partial_j G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^k + G_0^{ij}\,G^0_{hk}\,\mathscr{V}_0^h\,\partial_j\mathscr{V}_0^k\big)\,u^0 \otimes u^0 \otimes \partial_i\\ &= G_0^{ik}\,\big(\partial_j G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^j + G^0_{hk}\,\mathscr{V}_0^j\,\partial_j\mathscr{V}_0^h - \partial_k G^0_{hj}\,\mathscr{V}_0^h\,\mathscr{V}_0^j - G^0_{hj}\,\mathscr{V}_0^j\,\partial_k\mathscr{V}_0^h\big)\,u^0 \otimes u^0 \otimes \partial_i\\ &\quad + \big(\tfrac{1}{2}\,G_0^{ij}\,\partial_j G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^k + G_0^{ij}\,G^0_{hk}\,\mathscr{V}_0^h\,\partial_j\mathscr{V}_0^k\big)\,u^0 \otimes u^0 \otimes \partial_i\\ &= \big(\partial_j\mathscr{V}_0^i\,\mathscr{V}_0^j + G_0^{ik}\,\partial_j G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^j - \tfrac{1}{2}\,G_0^{ij}\,\partial_j G^0_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^k\big)\,u^0 \otimes u^0 \otimes \partial_i.\end{aligned}$$

Hence, by comparing the above coordinate expressions, we obtain the equality

$$\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o] = G^{\sharp}\big(L_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}(\vec{\mathscr{V}}[\Psi, o])\big) - \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big).$$

Further, let us prove the equality

$$\begin{aligned} &G^{\sharp}\Big(L_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) - \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big) \\ &\quad = -G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}[o]\big) + \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big). \end{aligned}$$

We have

$$\begin{aligned} &G^{\sharp}\Big(L_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) - \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big) \\ &\quad = G^{\sharp}\Big(i_{\vec{\mathscr{V}}[\Psi,o]}\,\check{d}\,G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big) + d i_{\vec{\mathscr{V}}[\Psi,o]} G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) - \tfrac{1}{2}\,\vec{d}\,\Big(G\,\big(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o]\big)\Big) \\ &\quad = G^{\sharp}\big(i_{\vec{\mathscr{V}}[\Psi,o]}\,\check{d}\,G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big) + \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big). \end{aligned}$$

Moreover, the coordinate expression $\mathscr{V}^{i}_{0} = G^{ir}_{0}\,(\partial_r \varphi - A_r)$ (see Corollary 18.1.4) yields

$$\begin{aligned} G^{\sharp}\Big(i_{\vec{\mathscr{V}}[\Psi,o]}\check{d} G^{\flat}\big(\vec{\mathscr{V}}[\Psi, o]\big)\Big) &= G^{\sharp}\big(i_{\vec{\mathscr{V}}[\Psi,o]}(\partial_{ji}\varphi - \partial_j A_i)\, d^j \wedge d^i\big) \\ &= -G^{\sharp}\big(\mathscr{V}^{j}_{0}\,(\partial_j A_i - \partial_i A_j)\big)\, u^0 \otimes d^i\big) \\ &= -G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}[o]\big). \end{aligned}$$

Therefore, we also obtain

$$\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o] = \tfrac{1}{2}\,\vec{d}\,\big(G\,(\vec{\mathscr{V}}[\Psi, o], \vec{\mathscr{V}}[\Psi, o])\big) - G^{\sharp}\big(\vec{\mathscr{V}}[\Psi, o] \lrcorner \check{\Phi}[o]\big).$$

Eventually, we recall that $\vec{\mathscr{V}}[\Psi, o_{\Psi}] = 0$ (see Theorem 18.1.1). □

18.2 Dynamics of the Associated Classical Fluid

Eventually, we discuss the continuity equation and the equation of motion for the associated classical fluid. Moreover, we show that the two polar components of the Schrödinger equation imply, respectively, the continuity equation and the constitutive law for the quantum pressure. This procedure reflects an analogous one of standard Quantum Mechanics and generalises it, in a covariant way, to our curved galilean framework (see, for instance, [230, 280]).

18.2.1 Law of Motion of the Associated Fluid

We start by achieving the law of motion of the fluid associated with a proper quantum section Ψ, by a purely kinematical argument, *regardless of the hypothesis that* Ψ *be a solution of the Schrödinger equation or not*,

$$\mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] = -\rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\mathfrak{p}[\Psi], \quad \text{where } \mathfrak{p}[\Psi] = -\tfrac{\hbar}{m} A[\Psi].$$

It is remarkable the fact that we express the quantum pressure $\mathfrak{p}[\Psi]$ via the distinguished potential $A[\Psi]$ (see Theorem 15.2.31).

In Sect. 18.2.2, we shall complete this result for a proper quantum section which is a solution of the Schrödinger equation.

Let us consider a proper quantum section and the associated joined acceleration field (see Definition 18.1.16)

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}) \qquad \text{and} \qquad \mathscr{A}[\Psi] \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}).$$

Theorem 18.2.1 *The associated fluid fulfills the following gauge independent and observer independent law of motion (see Assumption C.3 and Remark 7.4.5)*

$$\mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] = -\rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\mathfrak{p}[\Psi],$$

where the gauge independent and observer independent "quantum pressure" $\mathfrak{p}[\Psi]$ *is given, up to an arbitrary time function, by the following equality*

$$\mathfrak{p}[\Psi] = -\tfrac{\hbar}{m} A[\Psi] \in \operatorname{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^2) \otimes \mathbb{R}\big).$$

Proof. In virtue of the kinematical Theorem 18.1.17, we have the equality

$$\mu[\Psi]\,\mathscr{A}[\Psi] = -\mu[\Psi]\,\vec{d} A[\Psi].$$

Then, Proposition 18.1.18 yields $\mu\,\mathscr{A}^{\natural} = -\rho\,g^{\sharp}(\mathscr{V} \lrcorner F) - \mu\,\frac{\hbar}{m}\,\vec{d}A[\Psi])$. □

18.2.2 Dynamical Equations of the Associated Fluid

Next, we consider a normalised proper quantum section Ψ, *which is a solution of the Schrödinger equation* $\mathsf{S}[\Psi] = 0$ (see Theorem 17.6.5) and take into account the polar splitting of this equation into a system of two real equations (see Corollary 17.6.18)

$$д[o_\Psi].\|\Psi\|^2 + \|\Psi\|^2\,\operatorname{div}_\eta д[o_\Psi] = 0 \qquad \text{and} \qquad \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi] = 0.$$

Then, we show the following facts.

(1) The 1st real Schrödinger equation is equivalent to the continuity equation of the classical fluid.
(2) The 2nd real Schrödinger equation yields an expression of the quantum pressure $\mathrm{p}[\Psi]$ in terms of the hermitian quantum norm $\|\Psi\|$, hence in terms of a spacelike differential operator acting on the mass density $\mu[\Psi]$. In this way, we obtain a further specification of the law of motion of the associated fluid, which has been preliminary discussed in Theorem 18.2.1.

Besides the intrinsic way, we express the classical law of motion in observed eulerian form and in observed lagrangian form, as well.

We observe the following facts.

(1) The continuity equation $\delta\mathscr{P}[\Psi] = 0$ is equivalent to the conservation law for the probability current $\mathrm{J}[\Psi]$ (see Proposition 17.6.15).
(2) The law of motion of the associated fluid is obtained from the 2nd real Schrödinger equation via a spacelike differentiation, hence these equations are not fully equivalent. Indeed, the law of motion of the associated fluid characterises the 2nd real Schrödinger equation up to a phase coefficient which is spacelikely constant.

Indeed, it is a well known result of standard Quantum Mechanics that the Schrödinger equation implies the conservation law for the probability current and a dynamical equation for the associated fluid (see, for instance, [302] and also [58, 340]).

Here, our results agree with the above standard results in the flat case and provide a covariant generalisation of the subject to a curved spacetime and a link between the quantum pressure $\mathrm{p}[\Psi]$ and the distinguished observed quantum potential $A[\Psi]$ "seen" by the quantum particle (see Theorem 15.2.31).

Let us consider a normalised proper quantum section and the associated fluid (see Theorems 18.1.1 and 18.2.1)

$$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}) \qquad \text{and} \qquad (\mathscr{C}[\Psi], \mu[\Psi], \rho[\Psi], \mathrm{p}[\Psi]),$$

and suppose that Ψ be a solution of the Schrödinger equation $\mathrm{S}[\Psi] = 0$ (see Theorem 17.6.5).

Theorem 18.2.2 *The classical fluid associated with Ψ fulfills the following classical dynamical equations (see Assumption C.3):*

(1) the continuity equation

$$\mathrm{div}_\upsilon \mathscr{P}[\Psi] \equiv \mathscr{V}[\Psi].\mu[\Psi] + \mu[\Psi]\,\mathrm{div}_\eta \mathscr{V}[\Psi] = 0,$$

(2) the equation of motion

$$\mu[\Psi]\,\mathscr{A}^\natural[\Psi] = -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\mathrm{p}[\Psi],$$

where the "quantum pressure" $\mathrm{p}[\Psi]$ *is given, in terms of the mass density* $\mu[\Psi]$, *by the following* "constitutive equation"

$$\mathrm{p}[\Psi] = \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\,,$$

where $\Delta[g]$ *is the laplacian operator associated with the metric* g.

Proof. Let us take into account the equalities (see Theorem 18.1.1 and Corollary 18.1.4)

$$\mu[\Psi] = m\,|\psi|^2, \quad \vec{\mathscr{V}}[\Psi, o] = \vec{\nabla}^{(0)}[o](\!(\Psi)\!), \quad \mathscr{V}[\Psi] = д[o] + \vec{\mathscr{V}}[\Psi, o] = д[o_\Psi],$$

In virtue of Corollary 17.6.18, the Schrödinger equation splits into the system of two equations, with respect to the distinguished observer o_Ψ,

$$д[o_\Psi].\|\Psi\|^2 + \|\Psi\|^2\ \mathrm{div}_\eta\, д[o_\Psi] = 0, \qquad \Delta[G]\|\Psi\| + 2\,\|\Psi\|\,A[\Psi] = 0.$$

(1) The 1st one of the above equations can be rewritten as $\mathscr{V}[\Psi].\mu[\Psi] + \mu[\Psi]\ \mathrm{div}_\eta\ \mathscr{V}[\Psi] = 0$.
(2) The 2nd one of the above equations can be rewritten as $A[\Psi] = -\frac{1}{2}\,\frac{\Delta[G]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}$.

Then, the law of motion (see Theorem 18.2.1)

$$\mu[\Psi]\,\mathscr{A}^\natural[\Psi] = -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi] \lrcorner F) + \mu[\Psi]\,\vec{d}\,\mathrm{p}[\Psi], \quad \text{with } \mathrm{p}[\Psi] = -\tfrac{\hbar}{m}\,A[\Psi],$$

which holds for any proper quantum section Ψ, becomes

$$\begin{aligned}\mu[\Psi]\,\mathscr{A}^\natural[\Psi] &= -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi] \lrcorner F) - \tfrac{\hbar}{m}\,\mu[\Psi]\,\vec{d}\,A[\Psi] \\ &= -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi] \lrcorner F) + \tfrac{1}{2}\,\tfrac{\hbar}{m}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[G]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right) \\ &= -\rho[\Psi]\,g^\sharp(\mathscr{V}[\Psi] \lrcorner F) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right). \qquad \square\end{aligned}$$

The observed expression of the equation of motion of the associated fluid achieved in the above Theorem 18.2.2 can be written equivalently in *lagrangian form* and in *eulerian form* as follows.

Corollary 18.2.3 *For each observer o, the observed expressions of the equation of motion of the associated fluid in lagrangian form and eulerian form are, respectively,*

$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{lag}}[\Psi,o] = \rho[\Psi]\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\,\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right)$$
$$- \mu[\Psi]\,G^{\sharp}\big(\text{д}[o]\lrcorner\,\Phi^{\natural}[o]\big) - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\,\Phi^{\natural}[o]\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\,L_{\text{д}[o]}G\big),$$
$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{eul}}[\Psi,o] = \rho[\Psi]\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\,\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right)$$
$$- \mu[\Psi]\,G^{\sharp}\big(\text{д}[o]\lrcorner\,\Phi^{\natural}[o]\big) + \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\,\Phi^{\natural}[o]\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\,L_{\text{д}[o]}G\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,\vec{d}\big(G\,(\vec{\mathscr{V}}[\Psi,o],\vec{\mathscr{V}}[\Psi,o])\big),$$

with coordinate expression

$$\mu\,\mathscr{A}^{\natural}{}_{\text{lag}}[o]^i_{00} = \rho_0\,\big(E_0{}^i + \tfrac{1}{c}\,\sqrt{|g|}\,g^{ij}\,\epsilon_{jhk}\,\mathscr{V}_0^h\,B^k\big) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu\,g^{ij}\,\partial_j\left(\frac{\Delta[g]\sqrt{\mu}}{\sqrt{\mu}}\right)$$
$$- \mu\,G_0^{ij}\,(\partial_0 A^{\natural}{}_j - \partial_j A^{\natural}{}_0) - \tfrac{1}{2}\,\mu\,G_0^{ij}\,\mathscr{V}_0^h\,(\partial_h A^{\natural}{}_j - \partial_j A^{\natural}{}_h)$$
$$- \tfrac{1}{2}\,\mu\,G_0^{ij}\,\mathscr{V}_0^h\,\partial_0 G^0_{hj},$$
$$\mu\,\mathscr{A}^{\natural}{}_{\text{eul}}[o]^i_{00} = \rho_0\,\big(E_0{}^i + \tfrac{1}{c}\,\sqrt{|g|}\,g^{ij}\,\epsilon_{jhk}\,\mathscr{V}_0^h\,B^k\big) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu\,g^{ij}\,\partial_j\left(\frac{\Delta[g]\sqrt{\mu}}{\sqrt{\mu}}\right)$$
$$- \mu\,G_0^{ij}\,(\partial_0 A^{\natural}{}_j - \partial_j A^{\natural}{}_0) + \tfrac{1}{2}\,\mu\,G_0^{ij}\,\mathscr{V}_0^h\,(\partial_h A^{\natural}{}_j - \partial_j A^{\natural}{}_h)$$
$$- \tfrac{1}{2}\,\mu\,G_0^{ij}\,\mathscr{V}_0^h\,\partial_0 G^0_{hj} - \tfrac{1}{2}\,\mu\,g^{ij}\,\partial_j(g_{hk}\,\mathscr{V}_0^h\,\mathscr{V}_0^k),$$

where

$$\Delta[g]\sqrt{\mu} = g^{hk}\,\partial_{hk}\sqrt{\mu} + \frac{\partial_h(g^{hk}\,\sqrt{|g|})}{\sqrt{|g|}}\,\partial_k\sqrt{\mu}.$$

Proof. Propositions 18.1.19 and 18.1.21 and Theorem 18.2.2 yield

$$\begin{aligned}
\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{lag}}[\Psi,o] &= \mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] - \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi,o] - \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{drg}}[\Psi,o] \\
&= \mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] - \mu[\Psi]\,G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big) \\
&\quad - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner \check{\Phi}^{\natural}[o]\big) - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner L_{\text{д}[o]}G\big) \\
&= -\rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right) \\
&\quad - \mu[\Psi]\,G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big) - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner \check{\Phi}^{\natural}[o]\big) \\
&\quad - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner L_{\text{д}[o]}G\big), \\
\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{eul}}[\Psi,o] &= \mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] - \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{spc}}[\Psi,o] - \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{rlt}}[\Psi,o] \\
&\quad - \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{drg}}[\Psi,o] = \mu[\Psi]\,\mathscr{A}^{\natural}[\Psi] - \mu[\Psi]\,G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big) \\
&\quad + \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner \check{\Phi}^{\natural}[o]\big) - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner L_{\text{д}[o]}G\big) \\
&\quad - \tfrac{1}{2}\,\mu[\Psi]\,\vec{d}\big(G\,(\vec{\mathscr{V}}[\Psi,o],\,\vec{\mathscr{V}}[\Psi,o])\big) \\
&= -\rho[\Psi]\,g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right) \\
&\quad - \mu[\Psi]\,G^{\sharp}\big(\text{д}[o] \lrcorner \Phi^{\natural}[o]\big) + \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner \check{\Phi}^{\natural}[o]\big) \\
&\quad - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o] \lrcorner L_{\text{д}[o]}G\big) \\
&\quad - \tfrac{1}{2}\,\mu[\Psi]\,\vec{d}\big(G\,(\vec{\mathscr{V}}[\Psi,o],\,\vec{\mathscr{V}}[\Psi,o])\big).
\end{aligned}$$

Moreover, the coordinate expressions (see Definitions 5.2.1 and 5.1.1)

$$F_{0j} = -E_{0j} \qquad \text{and} \qquad F_{ij} = \tfrac{1}{c}\,\sqrt{|g|}\,\epsilon_{ijh}\,B^h$$

yield

$$\begin{aligned}
g^{\sharp}(\mathscr{V}[\Psi] \lrcorner F) &= g^{\sharp}(\text{д}[o] \lrcorner F + \vec{\mathscr{V}}[\Psi,o] \lrcorner F) \\
&= g^{ij}\,\big(F_{0j} + \mathscr{V}_0^h\,F_{hj}\big)\,u^0 \otimes \partial_i \\
&= \big(-E_0^i + \mathscr{V}_0^h\,\tfrac{1}{c}\,\sqrt{|g|}g^{ij}\,\epsilon_{hjk}\,B^k\big)\,u^0 \otimes \partial_i \\
&= -\vec{E}[o] - \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o] \times \vec{B}.
\end{aligned}$$

Hence, we obtain

$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{lag}}[\Psi,o] = \rho[\Psi]\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right)$$
$$- \mu[\Psi]\,G^{\sharp}\big(\text{д}[o]\lrcorner\Phi^{\natural}[o]\big) - \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\check{\Phi}^{\natural}[o]\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner L_{\text{д}[o]}G\big),$$
$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{eul}}[\Psi,o] = \rho[\Psi]\,(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right)$$
$$- \mu[\Psi]\,G^{\sharp}\big(\text{д}[o]\lrcorner\Phi^{\natural}[o]\big) + \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner\check{\Phi}^{\natural}[o]\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,G^{\sharp}\big(\vec{\mathscr{V}}[\Psi,o]\lrcorner L_{\text{д}[o]}G\big)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,\vec{d}\big(G\,(\vec{\mathscr{V}}[\Psi,o],\vec{\mathscr{V}}[\Psi,o])\big). \quad \square$$

Corollary 18.2.4 *With reference to the distinguished observer o_Ψ (see Theorem 15.2.31), the equations of motion of the associated fluid in eulerian and lagrangian form coincide.*

In fact, the observed expressions of the equation of motion in lagrangian and eulerian form are

$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{lag}}[\Psi,o_\Psi] = \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{eul}}[\Psi,o_\Psi]$$
$$= \rho[\Psi]\,\vec{E}[o_\Psi] + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right) - \mu[\Psi]\,G^{\sharp}\big(\text{д}[o_\Psi]\lrcorner\Phi^{\natural}[o_\Psi]\big). \quad \square$$

In view of a comparison with standard Quantum Mechanics, we consider the following particular cases.

Example 18.2.5 Let us consider an observer o such that $\nabla^{\natural}\text{д}[o] = 0$.

Then, the equations of motion in lagrangian and eulerian form become:

$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{lag}}[\Psi,o] = \rho[\Psi]\,\big(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}\big) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right),$$
$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\mathrm{eul}}[\Psi,o] = \rho[\Psi]\,\big(\vec{E}[o] + \tfrac{1}{c}\,\vec{\mathscr{V}}[\Psi,o]\times\vec{B}\big) + \tfrac{1}{2}\,\tfrac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right)$$
$$- \tfrac{1}{2}\,\mu[\Psi]\,\vec{d}\big(G\,(\vec{\mathscr{V}}[\Psi,o],\vec{\mathscr{V}}[\Psi,o])\big). \quad \square$$

Example 18.2.6 Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 24.1) and the plane wave (with vanishing electromagnetic field), expressed with reference to an inertial observer and cartesian coordinates,

$$\Psi := |\psi|\,e^{\mathrm{i}\,\varphi}\,\mathrm{b}, \quad \text{with} \;\; \varphi = \mathrm{k}_\lambda\,x^\lambda,\; \mathrm{k}_\lambda \in \mathbb{R},\; |\psi| \neq 0.$$

Then, we have $\Phi[o] = 0$ and $L_{\text{д}[o]}G = 0$.

Hence, the lagrangian and eulerian expressions of the law of motion of the associated fluid become

$$\mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{lag}}[\Psi, o] = \mu[\Psi]\,\mathscr{A}^{\natural}{}_{\text{eul}}[\Psi, o] = \frac{1}{2}\,\frac{\hbar^2}{m^2}\,\mu[\Psi]\,\vec{d}\,\left(\frac{\Delta[g]\sqrt{\mu[\Psi]}}{\sqrt{\mu[\Psi]}}\right). \square$$

Remark 18.2.7 In Remark 7.4.5 we have discussed two equivalent formulations of the pressure and of the dynamical law holding for a barotropic fluid.

We have also added that, in that classical context, we have preferred to deal with the less natural formulation just because it fits better the hydrodynamical picture of Quantum Mechanics.

However, now we must observe that, in the present quantum context, the associated fluid is not properly barotropic, because the mass density does not depend pointwisely on the pressure. Actually, the pressure depends on the mass density through a spacelike differential operator. So, the interpretation of p as a classical pressure, to some extent, does not hold fully. □

Chapter 19
Quantum Symmetries

We start by studying the Lie algebra $\text{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q})$ of *infinitesimal symmetries of the η-hermitian quantum metric* h_η (Sect. 19.1).

Actually, we exhibit a distinguished isomorphism between the Lie algebra $\text{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q})$ and the Lie algebra $\text{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$ of projectable special phase functions (see Theorem 19.1.7).

Then, we observe that

(1) the quantum structure is encoded in the gauge independent and observer independent quantum triplet $(dt, \mathrm{h}_\eta, \mathrm{Ч}^\uparrow)$,
(2) the quantum dynamics is encoded in the gauge independent and observer independent quantum dynamical pair (dt, L).

Therefore, we define

(1) the *infinitesimal symmetries of quantum structure* to be the projectable upper quantum vector fields which preserve the 3plet $(dt, \mathrm{h}_\eta, \mathrm{Ч}^\uparrow)$ (Sect. 19.2),
(2) the generators of *infinitesimal symmetries of quantum dynamics* to be the projectable quantum vector fields which preserve the pair (dt, L) (Sect. 19.3).

Actually, we exhibit two distinguished isomorphisms between the above Lie algebras of infinitesimal symmetries and the Lie algebra $\text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R})$ of conserved time preserving special phase functions (see Theorems 19.2.2 and 19.3.2).

It is remarkable that all above quantum infinitesimal symmetries be classified, in a covariant way, by the same Lie subalgebra of conserved time preserving special phase functions.

It is worth comparing the above discussion on quantum symmetry with the previous discussion on classical symmetries (see Chap. 13). Actually, the infinitesimal symmetries of classical structure, quantum structure and quantum dynamics are generated by the same Lie subalgebra $\text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R})$ (see Theorems 13.1.3, 19.2.2 and 19.3.2) while the infinitesimal symmetries of classical dynamics are generated by a distinguished further Lie subalgebra (see Theorem 13.2.6).

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The original source of our approach to quantum infinitesimal symmetries goes back to the Ph.D. thesis of Dirk Saller (see [358]).

The present review includes improvements with respect to further literature and new results as well (see, also, [226, 227, 227, 312, 358, 359]).

For further approaches to symmetry, see also [121].

19.1 Symmetries of the Hermitian Quantum Metric

We start by defining the *quantum lifts* $Y[f]$ and the η-*quantum lifts* $Y_\eta[f]$ of special phase functions and projectable special phase functions f, respectively.

Then, we classify, in a covariant way, the Lie algebras of *hermitian quantum vector fields* Y and η-*hermitian quantum vector fields* Y_η defined, respectively, by the conditions $L_Y \mathfrak{h} = 0$ and $L_{Y_\eta} \mathfrak{h}_\eta = 0$, via the Lie algebras of special phase functions $\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ and projectable special phase functions $\operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ (see also [220]).

In fact, we show that the hermitian quantum vector fields and η-hermitian quantum vector fields are, respectively, of the type $Y = Y[f]$ and $Y_\eta = Y_\eta[f]$.

Indeed, this result further supports our definitions of special phase functions and of special phase Lie bracket.

Actually, we are mainly interested in η-hermitian quantum vector fields because they yield quantum differential operators, which turn out to be hermitian on the Hilbert quantum bundle (see Proposition 22.5.3).

19.1.1 Quantum Lifts of Special Phase Functions

We introduce the concepts of *quantum lift* of special phase functions f and η-*quantum lift* of projectable special phase function f

$$Y[f] = X[f] \lrcorner \text{Ч}[o] + \mathrm{i}\, \check{f}[o]\, \mathbb{I} = X[f] \lrcorner \chi[\mathfrak{b}] + \mathrm{i}\, \hat{f}[\mathfrak{b}]\, \mathbb{I},$$
$$Y_\eta[f] = X[f] \lrcorner \text{Ч}[o] + (\mathrm{i}\, \check{f}[o] - \tfrac{1}{2} \operatorname{div}_\eta f)\, \mathbb{I} = X[f] \lrcorner \chi[\mathfrak{b}] + (\mathrm{i}\, \hat{f}[\mathfrak{b}] - \tfrac{1}{2} \operatorname{div}_\eta f)\, \mathbb{I}.$$

Indeed, these lifts will appear in the forthcoming classification of hermitian and η-hermitian quantum vector fields.

Note 19.1.1 The subsheaves of $\mathbb{C}$-linear and $\mathbb{R}$-linear quantum vector fields

$$\operatorname{lin}_{\mathbb{C}} \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \operatorname{lin}_{\mathbb{R}} \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \operatorname{pro}_{\boldsymbol{E}} \sec(\boldsymbol{Q}, T\boldsymbol{Q})$$

turn out to be closed with respect to the Lie bracket of vector fields. □

The following lemma turns out to be an essential tool for the notions of quantum lift and η-quantum lift of special phase functions.

Lemma 19.1.2 *If $f \in \operatorname{spe}(J_1\boldsymbol{E}, I\!R)$, then (see Theorem 12.2.1, Proposition 12.2.9, Lemma 15.1.3 and Definition 14.10.1):*

- *for each observer o, the vector field*

$$Y[f, o] := X[f] \lrcorner \, \Psi[o] + \mathrm{i} \, \breve{f}[o] \, \mathbb{I} \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$$

is, by definition, gauge independent;

- *for each quantum basis b, the vector field*

$$Y[f, \mathsf{b}] := X[f] \lrcorner \, \chi[\mathsf{b}] + \mathrm{i} \, \hat{f}[\mathsf{b}] \, \mathbb{I} \in \sec(\boldsymbol{Q}, T\boldsymbol{Q})$$

is, by definition, observer independent.

Moreover, we have

$$X[f] \lrcorner \, \Psi[o] + \mathrm{i} \, \breve{f}[o] \, \mathbb{I} = X[f] \lrcorner \, \chi[\mathsf{b}] + \mathrm{i} \, \hat{f}[\mathsf{b}] \, \mathbb{I}.$$

So, for each $f \in \operatorname{spe}(J_1\boldsymbol{E}, I\!R)$, we obtain, in a covariant way, the observer independent and gauge independent quantum vector field

$$Y[f] := X[f] \lrcorner \, \Psi[o] + \mathrm{i} \, \breve{f}[o] \, \mathbb{I} = X[f] \lrcorner \, \chi[\mathsf{b}] + \mathrm{i} \, \hat{f}[\mathsf{b}] \, \mathbb{I}.$$

Proof. The proof follows from the transition rules of the quantum potential and of the components of the special phase function, which fit very well (see Theorem 15.2.26 and Corollary 12.2.10). □

Definition 19.1.3 We define:

- the *quantum lift* of a special phase function $f \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ to be the quantum vector field

$$Y[f] := X[f] \lrcorner \, \Psi[o] + \mathrm{i} \, \breve{f}[o] \, \mathbb{I} = X[f] \lrcorner \, \chi[\mathsf{b}] + \mathrm{i} \, \hat{f}[\mathsf{b}] \, \mathbb{I},$$

with coordinate expression

$$Y[f] = f^0 \, \partial_0 - f^i \, \partial_i + \mathrm{i} \, (A_0 \, f^0 - A_i \, f^i + \breve{f}) \, \mathbb{I},$$

- the *η-quantum lift* of a projectable special phase function $f \in \operatorname{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ to be the quantum vector field (see Definition 12.2.7)

$$\begin{aligned} Y_\eta[f] &:= Y[f] - \tfrac{1}{2} \operatorname{div}_\eta f \, \mathbb{I} \\ &= X[f] \lrcorner \, \Psi[o] + (\mathrm{i} \, \breve{f}[o] - \tfrac{1}{2} \operatorname{div}_\eta f) \, \mathbb{I} \\ &= X[f] \lrcorner \, \chi[\mathsf{b}] + (\mathrm{i} \, \hat{f}[\mathsf{b}] - \tfrac{1}{2} \operatorname{div}_\eta f) \, \mathbb{I}, \end{aligned}$$

with coordinate expression

$$Y_\eta[f] = f^0\,\partial_0 - f^i\,\partial_i + \big(\mathfrak{i}\,(A_0\,f^0 - A_i\,f^i + \breve{f}) - \tfrac{1}{2}\,\operatorname{div}_\eta f\big)\,\mathbb{I},$$

where

$$\operatorname{div}_\eta f = f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}. \quad \square$$

Remark 19.1.4 The additional term $-\frac{1}{2}\,\operatorname{div}_\eta f\,\mathbb{I}$ in the above Definition 19.1.3 might appear quite arbitrary. Actually, it will be justified by the forthcoming Theorem 19.1.7. □

19.1.2 Classification of Hermitian Quantum Vector Fields

We show that the *hermitian quantum vector fields* Y are just the *quantum lifts* $Y[f]$ of special phase functions f and the η*-hermitian quantum vector fields* Y_η are just the η*-quantum lifts* $Y_\eta[f]$ of projectable special phase functions f.

Even more, the maps

$$f \mapsto Y[f] \qquad \text{and} \qquad f \mapsto Y_\eta[f]$$

turn out to be Lie algebra isomorphisms with respect to the special phase Lie bracket and the Lie bracket of quantum vector fields.

Definition 19.1.5 We define the *hermitian quantum vector fields* to be the infinitesimal symmetries of the quantum metric h and the η*-hermitian quantum vector fields* to be the infinitesimal symmetries of the η-quantum metric h_η, i.e. the vector fields (see, Appendix: Definition D.3.1 and Proposition D.3.2)

$$Y \in \operatorname{lin}_{\mathbb{R}} \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\,\boldsymbol{Q}) \qquad \text{and} \qquad Y_\eta \in \operatorname{lin}_{\mathbb{R}} \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\,\boldsymbol{Q}),$$

such that, respectively,

$$L_Y \mathsf{h} = 0 \qquad \text{and} \qquad L_{Y_\eta} \mathsf{h}_\eta = 0.$$

We denote the Lie algebra subsheaves of hermitian quantum vector fields and η-hermitian quantum vector fields by

$$\operatorname{her}(\boldsymbol{Q}, T\,\boldsymbol{Q}) \subset \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\,\boldsymbol{Q}) \qquad \text{and} \qquad \operatorname{her}_\eta(\boldsymbol{Q}, T\,\boldsymbol{Q}) \subset \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\,\boldsymbol{Q}). \quad \square$$

Remark 19.1.6 In the above Definition 19.1.5 we have considered projectable quantum vector fields, because the projectability is a necessary requirement for a consistent definition of the Lie derivative of the *vertical valued* form $\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}$ (see, Appendix: Proposition D.2.1). □

Theorem 19.1.7 [220] *We have the following results:*

(1) The hermitian quantum vector fields are of the type (see Definition 19.1.3)

$$Y = Y[f] = X[f] \lrcorner \chi[\mathfrak{b}] + \mathfrak{i}\, \hat{f}[\mathfrak{b}]\, \mathbb{I}$$
$$= X[f] \lrcorner Ч[o] + \mathfrak{i}\, \check{f}[o]\, \mathbb{I}, \qquad \text{with} \qquad f \in \mathrm{spe}(J_1\boldsymbol{E}, I\!R),$$

i.e., in coordinates, of the type

$$Y = Y[f] = f^0\, \partial_0 - f^i\, \partial_i + \mathfrak{i}\,(\check{f} + A_0\, f^0 - A_i\, f^i)\, \mathbb{I}$$
$$= f^0\, \partial_0 - f^i\, \partial_i + \mathfrak{i}\, \hat{f}\, \mathbb{I}.$$

Indeed, the map

$$Y : \mathrm{spe}(J_1\boldsymbol{E}, I\!R) \to her(\boldsymbol{Q}, T\,\boldsymbol{Q}) : f \mapsto Y[f]$$

turns out to be an $I\!R$*-Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.*
Actually, for each $f,\ \acute{f} \in \mathrm{spe}(J_1\boldsymbol{E}, I\!R)$, *we have*

$$(*) \quad \big[Y[f],\ Y[\acute{f}]\big] = \big[X[f],\ X[\acute{f}]\big] \lrcorner \chi[\mathfrak{b}] + \Big(\mathfrak{i}\,(X[f].\hat{\acute{f}}[\mathfrak{b}] - X[\acute{f}].\hat{f}[\mathfrak{b}])\Big)\, \mathbb{I}$$
$$= X\big[\,[\![f, \acute{f}]\!]\,\big] \lrcorner \chi[\mathfrak{b}] + \Big(\mathfrak{i}\,(X[f].\hat{\acute{f}}[\mathfrak{b}] - X[\acute{f}].\hat{f}[\mathfrak{b}])\Big)\, \mathbb{I}.$$

(2) The η*-hermitian quantum vector fields are of the type (see Definitions 19.1.3 and 12.2.7)*

$$Y_\eta = Y_\eta[f] = X[f] \lrcorner \chi[\mathfrak{b}] + \big(\mathfrak{i}\, \hat{f}[\mathfrak{b}] - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}$$
$$= X[f] \lrcorner Ч[o] + \big(\mathfrak{i}\, \check{f}[o] - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}, \ \textit{with}\ f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R),$$

i.e., in coordinates, of the type

$$Y_\eta = Y_\eta[f] = f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\,(\check{f} + A_0\, f^0 - A_i\, f^i) - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}$$
$$= f^0\, \partial_0 - f^i\, \partial_i + \big(\mathfrak{i}\, \hat{f} - \tfrac{1}{2}\, \mathrm{div}_\eta\, f\big)\, \mathbb{I}.$$

Indeed, the map

$$Y_\eta : \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R) \to her_\eta(\boldsymbol{Q}, T\,\boldsymbol{Q}) : f \mapsto Y_\eta[f]$$

turns out to be an $I\!R$*-Lie algebra isomorphism with respect to the special phase Lie bracket and the Lie bracket of vector fields.*
Actually, for each $f,\ \acute{f} \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R)$, *we have*

$$(*)\quad \big[Y_\eta[f],\, Y_\eta[\acute f]\big]$$
$$= \big[X[f],\, X[\acute f]\big] \lrcorner \chi[\mathrm{b}] + \Big(\mathrm{i}\,(X[f].\hat{\acute f}[\mathrm{b}] - X[\acute f].\hat f[\mathrm{b}]) - \tfrac{1}{2}\,\mathrm{div}_\eta \big[X[f],\, X[\acute f]\big]\Big)\,\mathbb{I}$$
$$= X\big[\,[\![f, \acute f]\!]\,\big] \lrcorner \chi[\mathrm{b}] + \Big(\mathrm{i}\,(X[f].\hat{\acute f}[\mathrm{b}] - X[\acute f].\hat f[\mathrm{b}]) - \tfrac{1}{2}\,\mathrm{div}_\eta\, [\![f, \acute f]\!]\Big)\,\mathbb{I}.$$

Proof. We prove the 2nd case; the 1st one is simpler and can be proved in the same way.

Let us consider a projectable quantum vector field, whose coordinate expression is of the type

$$Y_\eta = X^\lambda\,\partial_\lambda + Y^{\mathrm{a}}_{\mathrm{b}}\,w^{\mathrm{b}}\,\partial w_{\mathrm{a}}, \quad \text{with} \quad X^0 \in \mathrm{map}(\boldsymbol{T}, \mathbb{R}), \quad X^i,\; Y^{\mathrm{a}}_{\mathrm{b}} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}).$$

Then, we obtain $L_{Y_\eta}\mathrm{h}_\eta = 0$ if and only if $Y^1_1 = Y^2_2 = -\frac{1}{2}\,\mathrm{div}_\eta\, X$ and $Y^1_2 = -Y^2_1$, i.e. if and only if Y_η is of the type

$$\begin{aligned} Y_\eta &= X^\lambda\,\partial_\lambda + Y^2_1\,(w^1\,\partial w_2 - w^2\,\partial w_1) - \tfrac{1}{2}\,\mathrm{div}_\eta\, X\,(w^1\,\partial w_1 + w^2\,\partial w_2),\\ &= X^\lambda\,\partial_\lambda + (\mathrm{i}\,Y^2_1 - \tfrac{1}{2}\,\mathrm{div}_\eta\, X)\,(w^1\,\partial w_1 + w^2\,\partial w_2)\\ &= X^\lambda\,\partial_\lambda + (\mathrm{i}\,Y^2_1 - \tfrac{1}{2}\,\mathrm{div}_\eta\, X)\,\mathbb{I}. \end{aligned}$$

Further, the bijection between η-hermitian quantum vector fields Y_η and projectable special phase functions f can be achieved by comparing the splitting of Y_η into horizontal and vertical components with respect to a quantum connection $\text{Ч}[o]$ (or $\chi[\mathrm{b}]$) and the splitting the (projectable) special phase functions f into their tangent lift $X[f]$ and observed component $\check f[o]$ (or gauge component $\hat f[\mathrm{b}]$) according to Corollary 12.2.11.

In this way, we obtain the above expression of Y_η, whose covariance is ensured by the above Lemma 19.1.2.

Eventually, a comparison of the coordinate expressions of Lie brackets proves that the map $f \mapsto Y_\eta[f]$ is a Lie algebra isomorphism (see Theorem 12.5.3). □

Example 19.1.8 We have the following distinguished η-hermitian quantum vector fields (see Example 12.1.4):

$$\begin{aligned} Y_\eta[x^\lambda] &= \mathrm{i}\,x^\lambda\,\mathbb{I},\\ Y_\eta[x^j_0] &= -G^{ij}_0\,\partial_i + \left(-\mathrm{i}\,A^j_0 + \tfrac{1}{2}\,\frac{\partial_i(G^{ij}_0\sqrt{|g|})}{\sqrt{|g|}}\right)\mathbb{I},\\ Y_\eta[\mathcal{C}_i] = Y_\eta[\mathcal{Q}_i] &= -\partial_i + \left(-\mathrm{i}\,A_i + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\right)\mathbb{I},\\ Y_\eta[A^\uparrow{}_i] = Y_\eta[\mathcal{P}_i] &= -\partial_i + \tfrac{1}{2}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\mathbb{I}, \end{aligned}$$

$$Y_\eta[-\mathcal{C}_0] = Y_\eta[\mathcal{K}_0] = \partial_0 + \left(\mathrm{i}\, A_0 - \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\right) \mathbb{I},$$
$$Y_\eta[-A^{\uparrow}{}_0] = Y_\eta[\mathcal{H}_0] = \partial_0 - \tfrac{1}{2}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, \mathbb{I}.$$

Moreover, we have

$$Y_\eta[\mathcal{L}_0] = \partial_0 - A^i_0\, \partial_i - \left(\mathrm{i}\, 2\, \alpha_0 - \tfrac{1}{2} \left(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\right)\right) \mathbb{I},$$
$$Y_\eta[\tfrac{1}{2}\, |\mathcal{P}|^2_0] = \partial_0 - A^i_0\, \partial_i - \left(\mathrm{i}\, \alpha_0 - \tfrac{1}{2} \left(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\right)\right) \mathbb{I}. \quad \square$$

Let us recall that the 5 distinguished special phase functions $\mathcal{H}_0,\ \mathcal{P}_i,\ 1$ generate all special phase functions over $\text{map}(\boldsymbol{E}, \mathbb{R})$ (see Proposition 12.2.9). Then, we can split the η-quantum lift of special phase functions according to the following result.

Proposition 19.1.9 *Let us refer to a quantum basis* b, *an observer* o *and a spacetime chart adapted to* o. *Then, for each* $f \in \text{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have the expressions (see Example 12.1.4 and Proposition 12.2.9).*

$$Y_\eta[f] = f^0\, Y_\eta[\mathcal{H}_0] + f^i\, Y_\eta[\mathcal{P}_i] + \hat{f}\, Y_\eta[1] + \tfrac{1}{2}\, \partial_i f^i\, \mathbb{I}.$$

Moreover, for each $f,\ \acute{f} \in \text{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have the equality (see Theorem 12.5.3)*

$$\begin{aligned}\big[Y_\eta[f],\ Y_\eta[\acute{f}]\big] &= Y_\eta\big[\,[\![f,\ \acute{f}]\!]\,\big] \\ &= \big[X[f], X[\acute{f}]\big]^0\, Y_\eta[\mathcal{H}_0] + \big[X[f], X[\acute{f}]\big]^i\, Y_\eta[\mathcal{P}_i] \\ &\quad + \big(X[f].\hat{\acute{f}} - X[\acute{f}].\hat{f}\big)\, Y_\eta[1] \\ &\quad + \tfrac{1}{2}\, \partial_i\big(- f^0\, \partial_0 \acute{f}^i + f^j\, \partial_j \acute{f}^i + \acute{f}^0\, \partial_0 f^i - \acute{f}^j\, \partial_j f^i\big)\, \mathbb{I}.\end{aligned}$$

Proof. The proof follows from Proposition 12.2.9 and Theorem 19.1.7. □

Remark 19.1.10 We emphasise the "odd" term $\tfrac{1}{2}\, \partial_i f^i\, \mathbb{I}$, appearing in the above Proposition 19.1.9. This term cannot be selected by the only choice of the quantum basis b and the observer o, but depends also on the further choice of the spacetime chart adapted to o.

Later, we shall see that a similar odd term appears in the expression of quantum currents, but disappears in the expression of vertical quantum currents (see Corollary 21.1.7 and Theorem 21.1.9). □

Remark 19.1.11 The natural isomorphisms between the Lie algebras of hermitian quantum vector fields and η-hermitian quantum vector fields and the Lie algebras of special phase functions and projectable special phase functions are quite surprising, at a first insight.

In fact, the 1st Lie algebras involve only the hermitian quantum metric h and the η-hermitian quantum metric h_η, while the 2nd Lie algebra involves only the spacetime fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ and the spacelike metric G; clearly, these structures are totally independent. But the explicit expression of the above isomorphism involves the upper quantum connection $\mathrm{Y}^\uparrow$ (so that, if we would change the upper quantum connection, then we would obtain another analogous isomorphism).

Actually, we stress that the hidden deep reason of the above isomorphism lies in the striking similarities of the transition rules of the special phase functions and of the upper quantum potential (see Corollary 12.2.10 and Theorem 15.2.26).

Indeed, both the Lie bracket of special phase functions and the upper quantum connection are related to the same cosymplectic 2-form Ω. □

19.2 Symmetries of Quantum Structure

Then, we show that the *infinitesimal symmetries* $Y^\uparrow{}_\eta$ of the *quantum triplet* $(dt, \mathrm{h}_\eta, \mathrm{Y}^\uparrow)$, are of the type $Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] = \mathrm{Y}^\uparrow\big(X^\uparrow[f]\big) + \mathfrak{i}\, f\, \mathbb{I}^\uparrow$, where f is a *conserved time preserving special phase function.*

Definition 19.2.1 We define the *infinitesimal symmetries of quantum structure* to be the real linear projectable upper quantum vector fields

$$Y^\uparrow{}_\eta \in \mathrm{lin}_{\mathbb{R}}\, \mathrm{pro}_{\mathrm{J}_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T},\boldsymbol{Q}}(\boldsymbol{Q}^\uparrow, T\,\boldsymbol{Q}^\uparrow),$$

which fulfill the conditions

$$L_{Y^\uparrow{}_\eta} dt = 0, \qquad L_{Y^\uparrow{}_\eta} \mathrm{h}^\uparrow_\eta = 0, \qquad L_{Y^\uparrow{}_\eta} \mathrm{Y}^\uparrow = 0. \quad \square$$

Theorem 19.2.2 [358] *The infinitesimal symmetries of quantum structure turn out to be the upper quantum vector fields of the type*

$$Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] = \mathrm{Y}^\uparrow\big(X^\uparrow[f]\big) + \mathfrak{i}\, f\, \mathbb{I}^\uparrow,$$

where (see Definitions 12.1.3, 12.6.10, Propositions 12.3.3 and 12.4.2)

$$f \in \mathrm{cns\,tim\,spe}(J_1\boldsymbol{E},\, \mathbb{R}) \qquad \text{and} \qquad X^\uparrow[f] = X^\uparrow{}_{\mathrm{hol}}[f] = X^\uparrow{}_{\mathrm{ham}}[f].$$

Proof. An explicit computation in coordinates (see [227]) shows that the infinitesimal symmetries $Y^\uparrow{}_\eta \in \mathrm{lin}_{\mathbb{R}}\, \mathrm{pro}_{\mathrm{J}_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}^\uparrow, T\,\boldsymbol{Q}^\uparrow)$ of $\mathrm{h}^\uparrow{}_\eta$ and $\mathrm{Y}^\uparrow$ are of the type (see Proposition 12.6.7)

$$Y^\uparrow{}_\eta = Y^\uparrow{}_\eta[f] := \mathrm{Y}^\uparrow\big(X^\uparrow[f]\big) + (\mathfrak{i}\, f - \tfrac{1}{2}\, \mathrm{div}_\eta\, f)\, \mathbb{I}^\uparrow,$$

with $f \in \text{duni}_\eta \, \text{cns spe}(J_1\boldsymbol{E}, \mathbb{R})$ and $X^\uparrow[f] = X^\uparrow{}_{\text{hol}}[f] = X^\uparrow{}_{\text{ham}}[f]$, i.e. of the type

$$\begin{aligned} Y^\uparrow{}_\eta &= f^0\,\partial_0 - f^i\,\partial_i + X^j_0\,\partial^0_j + (\breve{f} + A_0\,f^0 - A_i\,f^i)\,(w^1\,\partial w_2 - w^2\,\partial w_1) \\ &\quad - \tfrac{1}{2}\,\text{div}_\eta\,f\,(w^1\,\partial w_1 + w^2\,\partial w_2), \end{aligned}$$

where the spacetime functions $f^0 \in \text{map}(\boldsymbol{T}, \mathbb{R})$, f^i, $\breve{f} \in \text{map}(\boldsymbol{E}, \mathbb{R})$ fulfill the conditions

$$\begin{aligned} 0 &= \partial_\lambda f^0, \\ 0 &= -f^0\,\partial_0 G^0_{hk} + f^i\,\partial_i G^0_{hk} + \partial_h f^i\,G^0_{ik} + \partial_k f^i\,G^0_{ih}, \\ 0 &= \partial_h \breve{f} - f^0\,(\partial_0 A_h - \partial_h A_0) + f^i\,(\partial_i A_h - \partial_h A_i) + \partial_0 f^i\,G^0_{ih}, \\ 0 &= \partial_0 \breve{f} - f^i\,\left(\partial_0 A_i - \partial_i A_0\right), \\ 0 &= d\left(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}\right). \end{aligned}$$

Now, the above 1st four conditions coincide with conditions $(c_1,\ c_2,\ c_{3v},\ c_{3h})$ of Proposition 12.6.11. Hence, $f \in \text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R})$. As a consequence, we have also

$$\text{div}_\eta\,f = f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}} = 0.$$

Accordingly, we can write

$$\begin{aligned} X^\uparrow[f] &= f^0\,\partial_0 - f^i\,\partial_i - (\partial_0 f^i + \partial_j f^i\,x^j_0)\,\partial^0_i, \\ &= f^0\,\partial_0 - f^i\,\partial_i \\ &\quad + G^{ij}_0\left(\partial_j \breve{f} + \partial_j f^h\,G^0_{hk}\,x^k_0 - f^0\left(\partial_0 G^0_{hj}\,x^h_0 + (\partial_0 A_j - \partial_j A_0)\right)\right. \\ &\quad \left. + f^h\left(\partial_h G^0_{jk}\,x^k_0 - (\partial_j A_h - \partial_h A_j)\right)\right)\partial^0_i. \end{aligned}$$

Then, we can show that the above coordinate expression proves the claim (see [227]). □

19.3 Symmetries of Quantum Dynamics

We analyse the generators Y_η of *infinitesimal symmetries* Y^1_η of the *dynamical pair* (dt, L) (see [227]).

Actually, we show that these generators are of the type $Y_\eta = Y_\eta[f]$, where f is a *conserved time preserving special phase function* (see Definition 19.1.5).

Definition 19.3.1 We define the generators of *infinitesimal symmetries of quantum dynamics* to be the real linear quantum vector fields

$$Y_\eta \in \operatorname{lin}_{\mathbb{R}} \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$$

such that their 1-jet holonomic prolongations (see Proposition 12.3.1)

$$Y^1_\eta \in \operatorname{pro}_{J_1\boldsymbol{E},\boldsymbol{E},\boldsymbol{T},\boldsymbol{Q}} \sec(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q})$$

fulfill the condition

$$L_{Y^1_\eta} dt = 0 \qquad \text{and} \qquad L_{Y^1_\eta} \mathrm{L} = 0.$$

Theorem 19.3.2 *The generators of infinitesimal symmetries of quantum dynamics are the η-hermitian quantum vector fields (see Definitions 19.1.5 and 12.1.3 and Corollary 12.6.14)*

$$Y_\eta = Y_\eta[f], \quad \text{with} \;\; f \in \operatorname{cns} \operatorname{tim} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

In other words, they are the quantum vector fields of the type

$$Y_\eta = Y_\eta[f] = f^0\,\partial_0 - f^i\,\partial_i + \mathfrak{i}\,(\check{f} + A_0\,f^0 - A_i\,f^i)\,\mathbb{I},$$

where the functions $f^0, f^i, \check{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R})$ fulfill conditions $c'_1, c'_2, c'_{3v}, c'_{3h}$ (see Corollary 12.6.14). □

Proposition 19.3.3 *Let us consider a projectable special phase function, its quantum lift and the induced holonomic lift (see Definition 12.1.3, Lemma 19.1.2, Proposition 12.3.1)*

$$f \in \operatorname{pro} \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}), \quad Y_\eta[f] \in \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\boldsymbol{Q}),$$
$$Y^1_\eta[f] \in \operatorname{pro}_{\boldsymbol{Q},\boldsymbol{E},\boldsymbol{T}}(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q}).$$

Then, we have the equivalence (see Theorems 17.5.2 and 17.5.10):

$$L_{Y^1_\eta[f]}\,\mathrm{L} = 0 \qquad \Leftrightarrow \qquad L_{Y^1_\eta[f]}\mathrm{C} = 0.$$

Proof. The equivalence follows from Lemma H.3.2, according to a general result of calculus of variations. □

Chapter 20
Quantum Differential Operators

We start by showing that, for each projectable special phase function f, the associated η-hermitian quantum vector field $Y_\eta[f]$ naturally acts on quantum sections Ψ as a 1st order differential operator (Sect. 20.1.1). In particular, we compute the quantum differential operators $\mathrm{i}\, Y_\eta[x^\lambda]$, $\mathrm{i}\, Y_\eta[\mathcal{P}_j]$ and $\mathrm{i}\, Y_\eta[\mathcal{H}_0]$ (Sect. 20.1.2).

Indeed, $\mathrm{i}\, Y_\eta[x^\lambda]$ and $\mathrm{i}\, Y_\eta[\mathcal{P}_j]$ agree with the standard corresponding quantum differential operators of standard Quantum Mechanics. In particular, the above quantum differential operator $i Y_\eta[x^0]$ deals with subtle problems of quantum measurement of time, which stand beyond the scope of the present book (see, for instance, [192, 235–237, 320, 413–415, 429]). So, here we keep this operator just for its nice formal expression, but neglect to investigate its true physical meaning.

Moreover, the operator $\mathrm{i}\, Y_\eta[\mathcal{H}_0]$, even if it looks interesting because of its relativistic shape, involves the partial derivative $\partial_0 \psi$. Actually, this fact conflicts with a criterion of standard Quantum Mechanics, by which a measurement performed at a certain time $\mathtt{t} \in \boldsymbol{T}$ should depend on the restriction $\Psi_{\mathtt{t}} \in \sec(\boldsymbol{E}_{\mathtt{t}}, \boldsymbol{Q}_{\mathtt{t}})$ of the quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$. We can overcome the above problem by observing that the quantum differential operators are intended to be applied to solutions of the Schrödinger equation; hence, we can derive the partial derivative $\partial_0 \psi$ from the Schrödinger equation. However, it is more convenient to deal with quantum differential operators acting on all quantum sections and providing the expected value on the solutions of the Schrödinger equation (Sect. 20.1.3). In this way, we achieve in a covariant way an expression for the quantum differential operators $\mathrm{O}[f]$ associated with all projectable special phase functions, including spacetime coordinates, components of the momentum and hamiltonian (see Theorem 20.1.9).

We observe that, in the above heuristic procedure, we did not start from an "a priori" *correspondence principle*, but we have got a link between the projectable special phase functions f and the corresponding quantum differential operators $\mathrm{O}[f]$ as a byproduct of the classification of the Lie algebra of infinitesimal symmetries of the η-quantum metric h_η. Indeed, here we have considered the η-quantum metric

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$\mathfrak{h}_\eta$ and not the simpler quantum metric $\mathfrak{h}$, because we are aimed at obtaining, later, hermitian fibred quantum operators on the Hilbert quantum bundle (see Theorem 22.5.5).

We observe that, in the definition of $\mathrm{O}[f]$, we have considered the imaginary factor $\mathfrak{i}$ again in view of hermitian fibred quantum operators on the Hilbert quantum bundle.

Actually, in order to obtain hermitian quantum operators on the Hilbert quantum bundle, we shall need to further restrict our attention to *time preserving* projectable special phase functions (see Definition 12.1.3). The phase functions x^λ, $\mathcal{P}_j$, $\mathcal{H}_0$ are of this type.

We stress that our procedure treats the spacetime functions x^λ, the components $\mathcal{P}_j$ of the classical momentum and the component $\mathcal{H}_0$ of the classical Hamiltonian function on the same footing. Moreover, no ordering techniques are required by the phase function $\mathcal{H}_0$.

Eventually, we study the *commutator* of the above special quantum differential operators (Sect. 20.1.5).

Thus, we stress that, in the general case, the commutator of two special quantum differential operators is not a special quantum differential operator. Moreover, in the general case, the map $f \mapsto \mathrm{O}[f]$ is not a morphism of Lie algebras, with respect to the special phase Lie bracket and the natural Lie bracket of differential operators.

However, the subsheaf of special quantum differential operators associated with affine special phase functions turns out to be a Lie algebra and the map $f \mapsto \mathrm{O}[f]$ turns out to be a morphism of Lie algebras.

Thus, the above procedure deals with time preserving special phase functions. Indeed, these functions cover most fundamental classical functions considered in standard Quantum Mechanics for quantum measurement. However, a relevant exception is the square of angular momentum. For the angular momentum there are several problems of different nature (see, Introduction, Sect. 1.5.19).

20.1 Quantum Differential Operators and s.p.f

The classification of η-hermitian quantum vector fields in terms of projectable special phase functions and the Schrödinger operator suggest how to achieve, in a covariant way, quantum operators acting on sections of the quantum bundle (see also, for instance, [196, 220] and literature therein).

Later, we shall see that the quantum differential operators acting on the sections of the quantum bundle can be naturally translated into fibrewise operators acting on the infinite dimensional "Hilbert quantum bundle" based on $\boldsymbol{T}$ and we can prove that the quantum operators associated with time preserving special phase functions turn out to be symmetric operators, but details on this subject are beyond the scope of the present book (see Proposition 22.5.3).

20.1.1 Quantum Differential Operators

Preliminarily, we define the *quantum differential operators* (of order k) and, in particular, the *spacelike quantum differential operators* (of order k).

Moreover, we define the Lie bracket of quantum differential operators.

Definition 20.1.1 We define a *quantum differential operator* of order k to be a sheaf morphism

$$\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}),$$

which factorises through a fibred morphism $\mathrm{O}_k : J_k \boldsymbol{Q} \to \boldsymbol{Q}$ over $\boldsymbol{E}$ according to the following commutative diagram, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$,

$$\begin{array}{ccc} \boldsymbol{E} & \xrightarrow{\mathrm{O}(\Psi)} & \boldsymbol{Q} \\ {\scriptstyle j_k\Psi}\downarrow & & \uparrow{\scriptstyle \mathrm{id}_{\boldsymbol{Q}}} \\ J_k \boldsymbol{Q} & \xrightarrow{\mathrm{O}_k} & \boldsymbol{Q} \end{array} .$$

In simple words, a quantum differential operator is a map $\Psi \mapsto \mathrm{O}(\Psi)$, which can be expressed through the partial derivatives ∂_λ, up to order k.

We denote the sheaf of quantum differential operators of undetermined order and the subsheaf of quantum differential operators of order k by

$$\mathrm{ope}_k\big(\sec(\boldsymbol{E}, \boldsymbol{Q}), \sec(\boldsymbol{E}, \boldsymbol{Q})\big) \subset \mathrm{ope}\big(\sec(\boldsymbol{E}, \boldsymbol{Q}), \sec(\boldsymbol{E}, \boldsymbol{Q})\big).$$

A quantum differential operator of order k is said to be "*spacelike*" if, for each $\mathrm{t} \in \boldsymbol{T}$, the spacelike restriction of $\mathrm{O}(\Psi)$ factorises through the spacelike restriction of Ψ, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{E}_{\mathrm{t}} & \xrightarrow{\mathrm{O}(\Psi)_{|\mathrm{t}}} & \boldsymbol{Q}_{\mathrm{t}} \\ {\scriptstyle j_k(\Psi_{\mathrm{t}})}\downarrow & & \uparrow{\scriptstyle \mathrm{id}_{\boldsymbol{Q}_{\mathrm{t}}}} \\ J_k(\boldsymbol{Q}_{\mathrm{t}}) & \xrightarrow{\mathrm{O}_{\mathrm{t},k}} & \boldsymbol{Q}_{\mathrm{t}} \end{array} .$$

In simple words, a quantum differential operator of order k is said to be "*spacelike*" if its coordinate expression contains only the partial derivatives ∂_i, up to order k, and does not contain the partial derivative ∂_0. □

Proposition 20.1.2 *The bracket*

$$[D, \acute{D}] := -\,\mathrm{i}\,\big(\acute{D} \circ D - D \circ \acute{D}\big)$$

of two quantum differential operators turns out to be a Lie bracket.

Hence, the sheaf of quantum differential operators $\operatorname{ope}\big(\sec(\boldsymbol{E}, \boldsymbol{Q}), \sec(\boldsymbol{E}, \boldsymbol{Q})\big)$ *turns out to be a sheaf of Lie algebras.* □

20.1.2 η-Hermitian Quantum Vector Fields as Operators

Then, we regard the η-hermitian quantum lifts of projectable special phase functions as quantum differential operators acting on quantum section via the Lie derivative.

Later, we shall focus our attention to time preserving special phase functions, because, they yield hermitian operators on the Hilbert quantum bundle (see Proposition 22.5.3).

Let us recall a general result, by which a projectable vector field of a vector bundle acts in a natural way on the sections of the bundle (see, Appendix: Proposition D.1.2). Here, we specify this general result to the case of the quantum bundle.

Lemma 20.1.3 *For each projectable quantum vector field* $Y \in \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$, *we can naturally define the "*Lie derivative*" of a section* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ *as a quantum section*

$$Y(\Psi) \equiv Y.\Psi := L_Y \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}).$$

If

$$Y = X^\lambda \, \partial_\lambda + Y^a \, \partial_a, \quad \text{with } X^\lambda \in \operatorname{map}(\boldsymbol{E}, I\!R), \; Y^a \in \operatorname{map}(\boldsymbol{Q}, I\!R),$$

then we obtain the coordinate expression, in real coordinates,

$$Y(\Psi) = (X^\mu \, \partial_\mu \Psi^a - \Psi^b \, \partial_b Y^a) \, \mathrm{b}_a.$$

Thus, each projectable quantum vector field $Y \in \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$ *turns out to be a quantum differential operator of order 1*

$$Y \in \operatorname{ope}_1\big(\sec(\boldsymbol{E}, \boldsymbol{Q}), \sec(\boldsymbol{E}, \boldsymbol{Q})\big).$$

Indeed, for each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ *and* $Y, \acute{Y} \in \operatorname{pro}_{\boldsymbol{E}}(\boldsymbol{Q}, T\boldsymbol{Q})$, *we have*

$$[Y, \acute{Y}](\Psi) = Y\big(\acute{Y}(\Psi)\big) - \acute{Y}\big(Y(\Psi)\big).$$

Proof. Let us recall the natural linear fibred isomorphism $V\boldsymbol{Q} \simeq \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}$ over $\boldsymbol{Q}$.

Then, each quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ can be naturally regarded as a vertical vector field $\tilde{\Psi} \in \sec(\boldsymbol{Q}, V\boldsymbol{Q})$, with coordinate expression $\tilde{\Psi} = \Psi^a \, \partial_a$.

Moreover, its Lie derivative

$$L_Y \tilde{\Psi} = [Y, \, \tilde{\Psi}] \in \sec(\boldsymbol{Q}, V\boldsymbol{Q}),$$

with coordinate expression

$$L_Y \tilde{\Psi} = (X^\lambda \partial_\lambda \Psi^{\rm a} - \partial_{\rm b} X^{\rm a} \Psi^{\rm b}) \partial_{\rm a},$$

can uniquely be regarded as a section, which will be denoted by $L_Y \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$. □

We emphasise that the above quantum differential operator

$$Y : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$$

turns out to be a *natural operator* in the sense of Appendix: Definition J.3.18.

Remark 20.1.4 In the above Lemma 20.1.3 we have used in an essential way the hypothesis that the vector field Y be projectable. In fact, under this hypothesis, we obtain

$$L_Y \tilde{\Psi} \in \sec(\boldsymbol{Q}, V\boldsymbol{Q}) \subset \sec(\boldsymbol{Q}, T\boldsymbol{Q}). \quad \square$$

Remark 20.1.5 We emphasise the minus sign appearing in the following equality (see the above Lemma 20.1.3)

$$Y(\Psi) = (X^\mu \partial_\mu \Psi^{\rm a} - \Psi^{\rm b} \partial_{\rm b} Y^{\rm a}) \, \mathtt{b}_{\rm a}. \quad \square$$

Example 20.1.6 We have the following Lie derivatives of quantum sections associated with η-hermitian quantum vector fields (see Lemma 20.1.3, Examples 19.1.8 and 12.1.4):

$$\begin{aligned}
Y_\eta[x^\lambda](\Psi) &= -\mathrm{i}\, x^\lambda \, \psi \, \mathtt{b}, \\
Y_\eta[\mathcal{C}_j](\Psi) = Y_\eta[\mathcal{Q}_j](\Psi) &= -\Big(\nabla_j \psi - \tfrac{1}{2} \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} \, \psi\Big) \, \mathtt{b}, \\
Y_\eta[A^\uparrow{}_j](\Psi) = Y_\eta[\mathcal{P}_j](\Psi) &= -\Big(\partial_j \psi - \tfrac{1}{2} \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}} \, \psi\Big) \, \mathtt{b}, \\
Y_\eta[-\mathcal{C}_0](\Psi) = Y_\eta[\mathcal{K}_0](\Psi) &= \Big(\nabla_0 \psi + \tfrac{1}{2} \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} \psi\Big) \, \mathtt{b}, \\
Y_\eta[-A^\uparrow{}_0](\Psi) = Y_\eta[\mathcal{H}_0](\Psi) &= \Big(\partial_0 \psi - \tfrac{1}{2} \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} \, \psi\Big) \, \mathtt{b}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
Y_\eta[\mathcal{L}_0](\Psi) = \Big(& (\partial_0 - A^i_0 \, \partial_i) \, \psi \\
& + \Big(\mathrm{i}\, 2\, \alpha_0 - \tfrac{1}{2} \Big(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A^i_0 \sqrt{|g|})}{\sqrt{|g|}}\Big) \, \psi\Big) \, \mathtt{b},
\end{aligned}$$

$$Y_\eta[\tfrac{1}{2}\,|\mathcal{P}|_0^2](\Psi) = \Big((\partial_0 - A_0^i\,\partial_i)\,\psi \\ + \Big(\mathrm{i}\,\alpha_0 - \tfrac{1}{2}\,\Big(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\,\sqrt{|g|})}{\sqrt{|g|}}\Big)\,\psi\Big)\,\mathsf{b}. \quad \square$$

Remark 20.1.7 The special phase functions $\mathcal{Q}_j[o]$ and $\mathcal{K}_0[o]$ are gauge independent.

Accordingly, also the Lie derivatives $Y_\eta[\mathcal{Q}_j](\Psi)$ and $Y_\eta[\mathcal{K}_0](\Psi)$ turn out to be gauge independent, in spite of the fact that the covariant differential $\nabla_\lambda\psi = (\partial_\lambda - \mathrm{i}\,A_\lambda)\,\psi$ involves the gauge dependent spacetime functions A_λ.

In fact, in virtue of Theorem 15.2.26, the gauge transition rules of A_λ, ψ and b compensate each other (see, also, Exercise 16.1.6).

The special phase functions $\mathcal{P}_j[o]$ and $\mathcal{H}_0[o]$ are gauge dependent, by definition. So, the Lie derivatives $Y_\eta[\mathcal{P}_j](\Psi)$ and $Y_\eta[\mathcal{H}_0](\Psi)$ turn out to be gauge dependent.□

20.1.3 Special Quantum Differential Operators

A suitable combination of the Lie derivatives $Y_\eta[f]$ associated with projectable special phase functions f and of the Schrödinger operator S yields, in a covariant way, quantum differential operators $\mathrm{O}[f] = \mathrm{i}\,\big(Y_\eta[f] - \mathrm{S}[f]\big)$ of order 2 whose coordinate expression is

$$\mathrm{O}[f](\Psi) = \Big(\Big(\check{f} - A_i\,f^i - \mathrm{i}\,\Big(f^i\,\partial_i + \tfrac{1}{2}\,\frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}\Big) - \tfrac{1}{2}\,f^0\,\Delta_0\Big)\,\psi\Big)\,\mathsf{b}.$$

We emphasise that the above operators $\mathrm{O}[f]$ turn out to be *natural operators* in the sense of Appendix: Definition J.3.18 (see also [218]).

Indeed, these operators appear to be reasonable candidates as "*quantum differential operators*" for Covariant Quantum Mechanics in any curved spacetime and with reference to any observer and any spacetime chart.

We stress that, in the flat case, these operators agree with the corresponding quantum differential operators of standard Quantum Mechanics associated with several basic phase functions. In fact, this correspondence includes the spacetime functions, the components $\mathcal{P}_i$ of the classical momentum and the component $\mathcal{H}_0$ of the classical hamiltonian (with reference to any observer and any spacetime chart). As far as angular momentum is concerned there are some problems which are discussed in Introduction: Sect. 1.5.19 and, later, in Chap. 25.

Later, we shall focus our attention to time preserving projectable special phase functions, because, we are looking for hermitian operators on the Hilbert quantum bundle. In fact, the hermitianity requires the condition of time preserving (see Proposition 22.5.3).

Lemma 20.1.8 *For each* $f \in \text{pro spe}(J_1\boldsymbol{E}, I\!R)$ *(see Definition 12.1.3),*

(1) the quantum vector field

$$Y_\eta[f] : \boldsymbol{Q} \to T\boldsymbol{Q}$$

acts as the 1st order differential operator (see Lemma 20.1.3 and Theorem 19.1.7)

$$Y_\eta : \text{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E}, \boldsymbol{Q}) : \Psi \mapsto Y_\eta(\Psi) := L_{Y_\eta}\Psi,$$

with coordinate expression

$$Y_\eta[f](\Psi) = \Big((f^0\,\partial_0 - f^i\,\partial_i)\,\psi - \big(\mathfrak{i}\,(\check{f} + f^0\,A_0 - f^i\,A_i) + \tfrac{1}{2}\,(f^0\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}})\big)\,\psi\Big)\,\mathsf{b},$$

(2) the quantum differential operator (see Definition 12.1.1 and Theorem 17.6.5)

$$\mathsf{S}[f] := f'' \,\lrcorner\, \mathsf{S} : \text{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E}, \boldsymbol{Q}) : \Psi \mapsto \mathsf{S}[f](\Psi),$$

is a 2nd order differential operator, with coordinate expression

$$\mathsf{S}[f](\Psi) = f^0\,\Big(\big(\nabla_0 + \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \tfrac{1}{2}\,\mathfrak{i}\,\Delta_0\big)\,\psi\Big)\,\mathsf{b}.$$

Actually, the above operators $Y_\eta[f]$ *and* $\mathsf{S}[f]$ *turn out to be* natural operators *in the sense of Appendix: Definition J.3.18.* □

Then, we obtain the following result.

Theorem 20.1.9 *For each* $f \in \text{pro spe}(J_1\boldsymbol{E}, I\!R)$, *we obtain, in a covariant way, the* "spacelike" *differential operator of order 2*

$$\mathsf{O}[f] = \mathfrak{i}\,\big(Y_\eta[f] - \mathsf{S}[f]\big) : \text{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{E}, \boldsymbol{Q}),$$

with coordinate expression (see Definition 12.1.1)

$$\mathsf{O}[f](\Psi) = \Big(\big(\check{f} - A_i\,f^i - \mathfrak{i}\,(f^i\,\partial_i + \tfrac{1}{2}\,\frac{\partial_i(f^i\,\sqrt{|g|})}{\sqrt{|g|}}) - \tfrac{1}{2}\,f^0\,\Delta_0\big)\,\psi\Big)\,\mathsf{b}. \quad \square$$

Definition 20.1.10 The quantum differential operator $\mathsf{O}[f]$ associated with a time preserving projectable special phase function $f \in \text{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$ is said to be a *special quantum differential operator*.

We denote the subsheaf of special quantum differential operators by

$$\text{spe ope}_2\big(\text{sec}(\boldsymbol{E}, \boldsymbol{Q}),\ \text{sec}(\boldsymbol{E}, \boldsymbol{Q})\big) \subset \text{ope}_2\big(\text{sec}(\boldsymbol{E}, \boldsymbol{Q}),\ \text{sec}(\boldsymbol{E}, \boldsymbol{Q})\big).$$

Moreover, we denote the subsheaves of special quantum differential operators associated with time preserving special phase functions and affine special phase functions by

$$\begin{aligned} \text{aff spe ope}_1 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) &\subset \text{tim spe ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) \\ \subset \text{spe ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) &\subset \text{ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big). \end{aligned}$$

Proposition 20.1.11 *We have the following bijective maps*

$$\begin{aligned} \text{spe}(J_1\boldsymbol{E}, \mathbb{R}) &\to \text{spe ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) &&: f \mapsto \mathrm{O}[f], \\ \text{tim spe}(J_1\boldsymbol{E}, \mathbb{R}) &\to \text{tim spe ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) &&: f \mapsto \mathrm{O}[f], \\ \text{aff spe}(J_1\boldsymbol{E}, \mathbb{R}) &\to \text{aff spe ope}_2 \big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) &&: f \mapsto \mathrm{O}[f]. \end{aligned}$$

Proof. The bijectivity is proved by the coordinate expression of the map. □

Assumption Q.3 We assume the operators

$$\mathrm{O}[f] : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}), \quad \text{for every} \quad f \in \text{pro spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

as quantum differential operators to be implemented on the Hilbert quantum bundle, according to the standard interpretation of Quantum Mechanics. □

Example 20.1.12 We have the distinguished quantum operators (see Example 12.1.4):

$$\begin{aligned} \mathrm{O}[x^\lambda](\Psi) &= x^\lambda\, \psi\, \mathsf{b}, \\ \mathrm{O}[\mathcal{C}_j](\Psi) = \mathrm{O}[\mathcal{Q}_j](\Psi) &= -\mathrm{i}\,\Big(\nabla_j \psi + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathsf{b}, \\ \mathrm{O}[A^{\uparrow}{}_j](\Psi) = \mathrm{O}[\mathcal{P}_j](\Psi) &= -\mathrm{i}\,\Big(\partial_j \psi + \tfrac{1}{2}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, \psi\Big)\, \mathsf{b}, \\ \mathrm{O}[-\mathcal{C}_0](\Psi) = \mathrm{O}[\mathcal{K}_0](\Psi) &= -(\tfrac{1}{2}\, \Delta_0\, \psi)\, \mathsf{b}, \\ \mathrm{O}[-A^{\uparrow}{}_0](\Psi) = \mathrm{O}[\mathcal{H}_0](\Psi) &= -\big(\tfrac{1}{2}\, \Delta_0\, \psi + A_0\, \psi\big)\, \mathsf{b}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \mathrm{O}[\mathcal{L}_0] &= \bigg(\bigg(A_0 - \mathrm{i}\, A^i_0\, \bigg(\nabla_i + \tfrac{1}{2}\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}}\bigg) - \mathrm{i}\, \partial_i A^i_0 - \tfrac{1}{2}\, \Delta_0\bigg)\, \psi\bigg)\, \mathsf{b}, \\ \mathrm{O}[\tfrac{1}{2}\, \mathcal{P}^2_0] &= \bigg(\bigg(-\tfrac{1}{2}\, A_i\, A^i_0 - \mathrm{i}\, A^i_0\, \bigg(\partial_i + \tfrac{1}{2}\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}}\bigg) - \mathrm{i}\, \partial_i A^i_0 - \tfrac{1}{2}\, \Delta_0\bigg)\, \psi\bigg)\, \mathsf{b}. \end{aligned}$$ □

Remark 20.1.13 By the above covariant procedure based on projectable special phase functions and their hermitian representation, we achieve, on the same footing, the quantum differential operators associated with spacetime functions, momentum and energy.

In the particular case of affine special phase functions, the associated quantum operators reduce to the Lie derivatives with respect to hermitian quantum vector fields. □

Remark 20.1.14 Our general procedure, yields formally the quantum operator

$$\mathrm{O}[x^0](\Psi) = x^0\, \psi\, \mathrm{b}$$

associated with the time function x^0.

Of course, this purely formal achievement, would require a deeper physical understanding. However, here we take this result just as a preliminary hint and omit a further analysis.

For a general discussion about time in standard Quantum Mechanics, see, for instance, [26, 192, 235–237, 320] and literature therein. □

Remark 20.1.15 Theorem 20.1.9 deals with a generic special phase function f and the coordinate expression has been referred to a generic observer o and to a generic quantum basis b.

Quite often a given special phase function f is defined by means of a certain quantum gauge b and a certain observer o. This is the case, for instance, of the observed Hamiltonian function and of the observed momentum function

$$f := \mathcal{H}_0[\mathrm{b}, o] \quad \text{and} \quad f := \mathcal{P}_j[\mathrm{b}, o].$$

Usually, the above observer o is the observer adapted to the laboratory. Accordingly, we might say that the above special phase functions are the Hamiltonian and the momentum of the particle "*seen*" by the laboratory.

However, in principle, the quantum basis b and the observer o considered for the definition of the phase function f might be different from the quantum basis and the observer used for the coordinate expression of $\mathrm{O}[f](\Psi)$.

Actually, in order to skip intricate formulas, we usually refer to the same quantum basis and observer that have been used for the definition of f.

If we consider a proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, then we obtain the distinguished special phase functions

$$f := \mathcal{H}_0[\mathrm{b}_\Psi, o_\Psi] \quad \text{and} \quad f := \mathcal{P}_j[\mathrm{b}_\Psi, o_\Psi]$$

and we might say, in a sense, that they are the Hamiltonian and the momentum of the particle "*seen*" by the particle.

By the way, we stress that in most contexts, Ψ is unknown, hence also o_Ψ and b_Ψ are not explicitly known.

We notice that usually standard Quantum Mechanics does not pay attention on these developments because o_Ψ is likely a non inertial observer, hence it is far from its scope. □

Thus, if, given a proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we refer to the distinguished observer o_Ψ and to the distinguished quantum basis b_Ψ, then we obtain the following remarkable formulas.

Corollary 20.1.16 *For each* $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$, *the coordinate expression of the restriction of the quantum operator* $\mathrm{O}[f]$ *to proper quantum sections*

$$\mathrm{O}[f] = \mathrm{i}\big(Y_\eta[f] - \mathrm{S}[f]\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}),$$

with reference to the distinguished observer o_Ψ *and quantum basis* b_Ψ, *becomes (see Theorem 15.2.31)*

$$\mathrm{O}[f](\Psi) = \Big(\big(\check{f} - \mathrm{i}\,\Big(f^i\,\partial_i + \tfrac{1}{2}\,\frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}\Big) - \tfrac{1}{2}\,f^0\,\Delta_0\big)\,|\psi|\Big)\,\mathrm{b}_\Psi. \quad \square$$

Example 20.1.17 With reference to a proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, and to the associated distinguished observer o_Ψ and quantum basis b_Ψ, let us consider the special phase functions

$$\mathcal{H}_0 \;\equiv\; \mathcal{H}_0[\mathrm{b}_\Psi, o_\Psi] \quad \text{and} \quad \mathcal{P}_j \;\equiv\; \mathcal{P}_j[\mathrm{b}_\Psi, o_\Psi].$$

Then, the coordinate expressions of the above Example 20.1.12, become, with reference to the same observer o_Ψ and quantum basis b_Ψ :

$$\mathrm{O}[\mathcal{P}_j](\Psi) = -\mathrm{i}\,\Big(\partial_j|\psi| + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,|\psi|\Big)\,\mathrm{b}_\Psi,$$
$$\mathrm{O}[\mathcal{H}_0](\Psi) = -\big(\tfrac{1}{2}\,\Delta_0\,|\psi| + A_0\,|\psi|\big)\,\mathrm{b}_\Psi. \quad \square$$

20.1.4 Polar Splitting of Quantum Differential Operators

For each time preserving special phase function f, we can split the complex function $\mathrm{O}[f](\Psi)/\Psi$ into its real and imaginary components.

In particular, in the case of affine special phase function f, the above splitting factorises through the polar components of the quantum sections (see Proposition 14.7.2). Thus, in this case, we can say that the quantum operators split into two real operators acting separately on the real polar components of the quantum sections.

Proposition 20.1.18 *For each* $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the spacetime complex function (see Theorem 20.1.9)*

$$\frac{\mathrm{O}[f](\Psi)}{\Psi} \in \sec(\boldsymbol{E}, \mathbb{C})$$

splits, in the proper domain of $\mathrm{O}[f](\Psi)/\Psi$, *as*

$$\frac{\mathrm{O}[f](\Psi)}{\Psi} = \mathrm{re}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} + \mathfrak{i}\,\mathrm{im}\,\frac{\mathrm{O}[f](\Psi)}{\Psi}\,,$$

where

$$\begin{aligned}
\mathrm{re}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} &= -\tfrac{1}{2}\, f^0 \left(\frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|}\right.\\
&\qquad \left. -G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi + 2\,A_0^i\,\partial_i\varphi - A_0^i\,A_i\right)\\
&\qquad + f^i\,\partial_i\varphi + \breve{f} - A_i\,f^i,\\
\mathrm{im}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} &= -\tfrac{1}{2}\, f^0 \left(G_0^{ij}\,(\partial_{ij}\varphi + 2\,\frac{\partial_i|\psi|}{|\psi|}\,\partial_j\varphi) + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi\right.\\
&\qquad \left. -2\,A_0^i\,\frac{\partial_i|\psi|}{|\psi|} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}\right)\\
&\qquad - f^i\,\frac{\partial_i|\psi|}{|\psi|} - \tfrac{1}{2}\,\frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}. \qquad \square
\end{aligned}$$

Corollary 20.1.19 *For each* $f \in \mathrm{aff\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ *and each* $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *the above equalities become*

$$\mathrm{re}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} = f^i\,\partial_i\varphi + \breve{f} - A_i\,f^i, \quad \mathrm{im}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} = -f^i\,\frac{\partial_i|\psi|}{|\psi|} - \tfrac{1}{2}\,\frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}.$$

Thus, the following diagrams commute (see Proposition 14.7.2)

$$\begin{array}{ccc}
\Psi & \xrightarrow{\mathrm{O}[f]/} & \dfrac{\mathrm{O}[f](\Psi)}{\Psi}\\
{\scriptstyle (())}\downarrow & & \downarrow{\scriptstyle \mathrm{re}}\\
((\Psi)) & \longrightarrow & \mathrm{re}\,(\dfrac{\mathrm{O}[f](\Psi)}{\Psi})
\end{array}
\qquad \textit{and} \qquad
\begin{array}{ccc}
\Psi & \xrightarrow{\mathrm{O}[f]/} & \dfrac{\mathrm{O}[f](\Psi)}{\Psi}\\
{\scriptstyle \|\|}\downarrow & & \downarrow{\scriptstyle \mathrm{im}}\\
\|\Psi\| & \longrightarrow & \mathrm{im}\,(\dfrac{\mathrm{O}[f](\Psi)}{\Psi})
\end{array}\,. \square$$

Example 20.1.20 For each $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, we have the following complex splittings of distinguished quantum operators (see Example 12.1.4)

$$\mathrm{re}\,\frac{\mathrm{O}[x^\lambda](\Psi)}{\Psi} = x^\lambda, \qquad \mathrm{im}\,\frac{\mathrm{O}[x^\lambda](\Psi)}{\Psi} = 0,$$

$$\mathrm{re}\,\frac{\mathrm{O}[\mathcal{Q}_j](\Psi)}{\Psi} = (\partial_j\varphi - A_j), \qquad \mathrm{im}\,\frac{\mathrm{O}[\mathcal{Q}_j](\Psi)}{\Psi} = -\partial_j \log|\psi| - \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}},$$

$$\mathrm{re}\,\frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} = \partial_j \varphi, \qquad \mathrm{im}\,\frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} = -\partial_j \log|\psi| - \tfrac{1}{2}\,\frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}},$$

$$\begin{aligned}
\mathrm{re}\,\frac{\mathrm{O}[\mathcal{K}_0](\Psi)}{\Psi} &= -\tfrac{1}{2}\Big(\frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|} \\
&\quad - G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi + 2\,A_0^i\,\partial_i\varphi - A_0^i\,A_i\Big), \\
\mathrm{im}\,\frac{\mathrm{O}[\mathcal{K}_0](\Psi)}{\Psi} &= -\tfrac{1}{2}\,G_0^{ij}\,\Big(\partial_{ij}\varphi + 2\,\frac{\partial_i|\psi|}{|\psi|}\,\partial_j\varphi + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi \\
&\quad -2\,A_0^i\,\frac{\partial_i|\psi|}{|\psi|} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}\Big), \\
\mathrm{re}\,\frac{\mathrm{O}[\mathcal{H}_0](\Psi)}{\Psi} &= -\tfrac{1}{2}\,\Big(\frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|} \\
&\quad -G_0^{ij}\,\partial_i\varphi\,\partial_j\varphi + 2\,A_0^i\,\partial_i\varphi - A_i\,A_0^i + 2\,A_0\Big), \\
\mathrm{im}\,\frac{\mathrm{O}[\mathcal{H}_0](\Psi)}{\Psi} &= -\tfrac{1}{2}\,G_0^{ij}\,\Big(\partial_{ij}\varphi + 2\,\frac{\partial_i|\psi|}{|\psi|}\,\partial_j\varphi + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_h\varphi \\
&\quad -2\,A_0^i\,\frac{\partial_i|\psi|}{|\psi|} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}}\Big). \quad \square
\end{aligned}$$

If, given a proper quantum section Ψ, we refer to the distinguished observer o_Ψ and to the distinguished quantum basis $\mathfrak{b}_\Psi$, then we obtain the following remarkable formulas.

By the way, we stress that in most contexts, Ψ is unknown, hence also o_Ψ and $\mathfrak{b}_\Psi$ are not explicitly known.

Corollary 20.1.21 *For each* $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R)$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, *we obtain the following polar splitting, with reference to the distinguished observer* o_Ψ *and the distinguished quantum basis* $\mathfrak{b}_\Psi$ *(see Theorem 15.2.31)*

$$\begin{aligned}
\mathrm{re}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} &= -\tfrac{1}{2}\,f^0\,\Big(\frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|}\Big) + \breve{f}, \\
\mathrm{im}\,\frac{\mathrm{O}[f](\Psi)}{\Psi} &= -f^i\,\frac{\partial_i|\psi|}{|\psi|} - \tfrac{1}{2}\,\frac{\partial_i(f^i\sqrt{|g|})}{\sqrt{|g|}}. \quad \square
\end{aligned}$$

Example 20.1.22 For each $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, let us consider the distinguished special phase functions $\mathcal{P}_j[\mathfrak{b}_\Psi, o_\Psi]$ and $\mathcal{H}_0[\mathfrak{b}_\Psi, o_\Psi]$ "*seen*" by the quantum section (see Theorem 15.2.31).

Then, we have the following complex splittings of distinguished quantum operators, with reference to the distinguished observer o_Ψ and the distinguished quantum basis b_Ψ,

$$\begin{aligned}
\mathrm{re}\,\frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} &= 0,\\
\mathrm{im}\,\frac{\mathrm{O}[\mathcal{P}_j](\Psi)}{\Psi} &= -\partial_j \log|\psi| - \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}},
\end{aligned}$$

$$\begin{aligned}
\mathrm{re}\,\frac{\mathrm{O}[\mathcal{H}_0](\Psi)}{\Psi} &= -\tfrac{1}{2}\left(\frac{G_0^{ij}\,\partial_{ij}|\psi|}{|\psi|} + \frac{\partial_i(G_0^{ih}\sqrt{|g|})}{\sqrt{|g|}}\,\frac{\partial_h|\psi|}{|\psi|} + 2\,A_0\right),\\
\mathrm{im}\,\frac{\mathrm{O}[\mathcal{H}_0](\Psi)}{\Psi} &= 0. \quad \square
\end{aligned}$$

20.1.5 Commutator of Special Quantum Differential Operators

We define the commutator of two special quantum differential operators as $\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big]$ and find the equality

$$\begin{aligned}
\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] = \mathrm{i}\,Y_\eta\big[\,[\![\,\acute{f},\, f\,]\!]\,\big] &+ \mathrm{i}\,\big(Y_\eta[f]\circ \mathrm{S}[\acute{f}] - \mathrm{S}[\acute{f}]\circ Y_\eta[f]\big)\\
&+ \mathrm{i}\,\big(\mathrm{S}[f]\circ Y_\eta[\acute{f}] - Y_\eta[\acute{f}]\circ \mathrm{S}[f]\big).
\end{aligned}$$

Thus, we stress that, in the general case, the commutator of two special quantum differential operators is not a special quantum differential operator.

Moreover, in the general case, the map $f \mapsto \mathrm{O}[f]$ is not a morphism of Lie algebras, with respect to the special phase Lie bracket and the natural Lie bracket of differential operators. However, the subsheaf of special quantum differential operators associated with affine special phase functions turns out to be a Lie algebra and the map $f \mapsto \mathrm{O}[f]$ turns out to be an isomorphism of Lie algebras.

Definition 20.1.23 According to Definition 20.1.1, for each $f,\ \acute{f} \in \mathrm{pro\,spe}\,(J_1\boldsymbol{E}, \mathrm{I\!R})$, we define the *commutator* of the associated special quantum differential operators to be the quantum differential operator

$$\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] : \sec(\boldsymbol{E},\ \boldsymbol{Q}) \to \sec(\boldsymbol{E},\ \boldsymbol{Q}),$$

given by the equality

$$\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] := -\,\mathrm{i}\,\big((\mathrm{O}[\acute{f}]\circ \mathrm{O}[f]) - (\mathrm{O}[f]\circ \mathrm{O}[\acute{f}])\big). \quad \square$$

Theorem 20.1.24 *For each* $f,\ \acute{f} \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have the equality*

$$\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] = \mathrm{i}\, Y_\eta\big[\,[\![\, \acute{f}, f \,]\!]\,\big] + \mathrm{i}\,\big(Y_\eta[f] \circ \mathrm{S}[\acute{f}] - \mathrm{S}[\acute{f}] \circ Y_\eta[f]\big) + \mathrm{i}\,\big(\mathrm{S}[f] \circ Y_\eta[\acute{f}] - Y_\eta[\acute{f}] \circ \mathrm{S}[f]\big),$$

where $[\![\,,\,]\!]$ *is the special phase Lie bracket of special phase functions (see Theorem 12.5.3).*

Proof. We have

$$\begin{aligned}
\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] &= -\mathrm{i}\Big(\mathrm{i}\,\big(Y_\eta[\acute{f}] - \mathrm{S}[\acute{f}]\big)\Big) \circ \Big(\mathrm{i}\,\big(Y_\eta[f] - \mathrm{S}[f]\big)\Big) \\
&\quad + \mathrm{i}\Big(\mathrm{i}\,\big(Y_\eta[f] - \mathrm{S}[f]\big)\Big) \circ \Big(\mathrm{i}\,\big(Y_\eta[\acute{f}] - \mathrm{S}[\acute{f}]\big)\Big) \\
&= \mathrm{i}\Big(\big(Y_\eta[\acute{f}] - \mathrm{S}[\acute{f}]\big)\Big) \circ \Big(\big(Y_\eta[f] - \mathrm{S}[f]\big)\Big) \\
&\quad - \mathrm{i}\Big(\big(Y_\eta[f] - \mathrm{S}[f]\big)\Big) \circ \Big(\big(Y_\eta[\acute{f}] - \mathrm{S}[\acute{f}]\big)\Big) \\
&= \mathrm{i}\,\big(Y_\eta[\acute{f}] \circ Y_\eta[f] - Y_\eta[f] \circ Y_\eta[\acute{f}]\big) + \mathrm{i}\,\big(\mathrm{S}[\acute{f}] \circ \mathrm{S}[f] - \mathrm{S}[f] \circ \mathrm{S}[\acute{f}]\big) \\
&\quad + \mathrm{i}\,\big(Y_\eta[f] \circ \mathrm{S}[\acute{f}] - \mathrm{S}[\acute{f}] \circ Y_\eta[f]\big) + \mathrm{i}\,\big(\mathrm{S}[f] \circ Y_\eta[\acute{f}] - Y_\eta[\acute{f}] \circ \mathrm{S}[f]\big).
\end{aligned}$$

Then, by taking into account that $\mathbb{T} \otimes \mathbb{R}$ is a 1-dimensional vector space, we have

$$\begin{aligned}
\mathrm{S}[\acute{f}] \circ \mathrm{S}[f] - \mathrm{S}[f] \circ \mathrm{S}[\acute{f}] &= (\acute{f}'' \lrcorner\, \mathrm{S}) \circ (f'' \lrcorner\, \mathrm{S}) - (f'' \lrcorner\, \mathrm{S}) \circ (\acute{f}'' \lrcorner\, \mathrm{S}) \\
&= (\acute{f}'' \otimes f'' - f'' \otimes \acute{f}'') \lrcorner\, (\mathrm{S} \circ \mathrm{S}) = 0 \lrcorner\, (\mathrm{S} \circ \mathrm{S}) = 0.
\end{aligned}$$

Moreover, in virtue of a well known property of Lie derivatives and Theorem 19.1.7, we have

$$\big(Y_\eta[\acute{f}] \circ Y_\eta[f]\big) - \big(Y_\eta[f] \circ Y_\eta[\acute{f}]\big) = \big[Y_\eta[\acute{f}],\ Y_\eta[f]\big] = Y_\eta\big[\,[\![\, \acute{f}, f \,]\!]\,\big].$$

Therefore, we obtain

$$\big[\mathrm{O}[\acute{f}],\ \mathrm{O}[f]\big] = \mathrm{i}\, Y_\eta\big[\,[\![\, \acute{f}, f \,]\!]\,\big] + \mathrm{i}\,\big(Y_\eta[f] \circ \mathrm{S}[\acute{f}] - \mathrm{S}[\acute{f}] \circ Y_\eta[f]\big) + \mathrm{i}\,\big(\mathrm{S}[f] \circ Y_\eta[\acute{f}] - Y_\eta[\acute{f}] \circ \mathrm{S}[f]\big). \quad \square$$

Remark 20.1.25 In general, the commutator of two special quantum differential operators is still a spacelike quantum differential operator $\sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$, but it needs not to be a special quantum differential operator associated with a projectable special phase function.

Hence, in general, it does not make sense asking whether the special quantum differential operators constitute a Lie algebra. □

We have the following particular case. We recall that, in the particular case $f \in \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have

$$\mathrm{O}[f](\Psi) = \mathrm{i}\, Y_\eta[f](\Psi).$$

Corollary 20.1.26 *For each* $f,\ \acute{f} \in \operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have*

$$\big[\mathrm{O}[f],\ \mathrm{O}[\acute{f}]\big] = \mathrm{i}\,\big(Y_\eta[\acute{f}] \circ Y_\eta[f] - Y_\eta[f] \circ Y_\eta[\acute{f}]\big),$$

hence we obtain

$$\big[\mathrm{O}[f],\ \mathrm{O}[\acute{f}]\big] = \mathrm{O}\big[\,[\![\, \acute{f}, f \,]\!]\,\big],$$

where

$$\mathrm{O}\big[\,[\![\, \acute{f}, f \,]\!]\,\big] = \mathrm{i}\, Y_\eta\big[\,[\![\, \acute{f}, f \,]\!]\,\big] = \mathrm{i}\, Y_\eta\big[\{\acute{f}, f\}\big].$$

Thus, the special quantum operators associated with affine special phase functions constitute an $\mathbb{R}$*-Lie algebra*

$$\operatorname{aff}\operatorname{spe}\operatorname{ope}_1\big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big) \subset \operatorname{spe}\operatorname{ope}_1\big(\sec(\boldsymbol{E}, \boldsymbol{Q}),\ \sec(\boldsymbol{E}, \boldsymbol{Q})\big)$$

naturally isomorphic to

(1) the $\mathbb{R}$*-Lie algebra of* η*-hermitian quantum vector fields with vanishing time component* f''

$$\operatorname{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) \subset \operatorname{pro}_{\boldsymbol{E},\boldsymbol{T}}(\boldsymbol{Q}, T\boldsymbol{Q}),$$

with respect to the Lie bracket of vector fields,

(2) the $\mathbb{R}$*-Lie algebra of affine special phase functions*

$$\operatorname{aff}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

with respect to the special phase Lie bracket, which, for these phase functions, coincides with the Poisson bracket. □

In view of examples, we start by discussing a technical Lemma, which is useful for the explicit computation of some special quantum commutators.

Lemma 20.1.27 *For each* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *we have the following equalities*

$$\mathrm{S}_0(x^0\, \Psi) = x^0\, \mathrm{S}_0(\Psi) + \Psi,$$

$$\begin{aligned} \mathrm{S}_0(x^j\, \Psi) &= x^j\, \mathrm{S}_0(\Psi) + \mathrm{i}\, Y_\eta[x^j_0](\Psi) \\ &= x^j\, \mathrm{S}_0(\Psi) - \mathrm{i}\,\Big(G^{ij}_0\, \partial_i \psi + \tfrac{1}{2}\, \frac{\partial_i(G^{ij}_0\, \sqrt{|g|})}{\sqrt{|g|}}\, \psi\Big) - G^{ij}_0\, A_i\, \psi, \end{aligned}$$

$$\begin{aligned}
&\mathrm{S}_0\big(Y_\eta[\mathcal{P}_j](\Psi)\big) - Y_\eta[\mathcal{P}_j]\,\big(\mathrm{S}_0(\Psi)\big) \\
&= -\tfrac{1}{2}\,\mathrm{i}\,\partial_j G_0^{ih}\,\partial_{ih}\psi - \Big(\tfrac{1}{2}\,\mathrm{i}\,\partial_{ji} G_0^{ih} + \tfrac{1}{2}\,\mathrm{i}\,\partial_j G_0^{ih}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}} + \partial_j A_0^h\Big)\,\partial_h\psi \\
&\quad + \tfrac{1}{4}\,\mathrm{i}\,\Big(G_0^{ih}\,\partial_{ih}(\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}) + \partial_i G_0^{ih}\,\partial_h(\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}) + G_0^{ih}\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\,\partial_h(\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}})\Big)\,\psi \\
&\quad - \tfrac{1}{2}\,\Big(\partial_{ij} A_0^i + \partial_j A_0^i\,\frac{\partial_i\sqrt{|g|}}{\sqrt{|g|}}\Big)\,\psi - \mathrm{i}\,\big(\partial_j\alpha_0\big)\,\psi. \quad \square
\end{aligned}$$

Example 20.1.28 For any special phase function $f \in \operatorname{tim\,spe}(J_1\boldsymbol{E}, \mathbb{R})$, we have

$$\big[\mathrm{O}[x^0], \mathrm{O}[f]\big](\Psi) = 0.$$

Moreover, we have

$$\begin{aligned}
&(1) \qquad \big[\mathrm{O}[x^\lambda], \mathrm{O}[x^\mu]\big](\Psi) = \mathrm{O}\big[\,[\![\, x^\lambda, x^\mu \,]\!]\,\big](\Psi) = 0, \\
&(2) \qquad \big[\mathrm{O}[x^\lambda], \mathrm{O}[\mathcal{P}_j]\big](\Psi) = \mathrm{O}\big[\,[\![\, x^\lambda, \mathcal{P}_j \,]\!]\,\big](\Psi) = \delta^\lambda_j\,\Psi, \\
&(3) \qquad \big[\mathrm{O}[\mathcal{P}_i], \mathrm{O}[\mathcal{P}_j]\big](\Psi) = \mathrm{O}\big[\,[\![\, \mathcal{P}_i, \mathcal{P}_j \,]\!]\,\big](\Psi) = 0,
\end{aligned}$$

$$\begin{aligned}
&(4) \qquad \big[\mathrm{O}[x^0], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = 0, \\
&(5) \qquad \big[\mathrm{O}[x^i], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = \mathrm{i}\,Y_\eta[x_0^i](\Psi), \\
&(6) \qquad \big[\mathrm{O}[\mathcal{P}_i], \mathrm{O}[\mathcal{H}_0]\big](\Psi) = \mathrm{i}\,\Big(\mathrm{S}_0\big(Y_\eta[\mathcal{P}_j](\Psi)\big) - Y_\eta[\mathcal{P}_j]\,\big(\mathrm{S}_0(\Psi)\big)\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{i}\,Y_\eta[x_0^i](\Psi) &= -\left(\left(G_0^{ij}\,\partial_j\psi + \tfrac{1}{2}\,\frac{\partial_j(G_0^{ji}\,\sqrt{|g|})}{\sqrt{|g|}}\psi\right) - \mathrm{i}\,G_0^{ij}\,A_j\,\psi\right)\,\mathrm{b}, \\
Y_\eta(\mathcal{P}_j)(\Psi) &= -\left(\partial_j\psi + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,\psi\right)\,\mathrm{b}. \quad \square
\end{aligned}$$

Chapter 21
Quantum Currents and Expectation Forms

According to the general discussion of currents in a generic lagrangian theory (see, Appendix: Sect. H.3), we refer to the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and consider the quantum lagrangian L and the associated Poincaré–Cartan form C (see Theorems 17.5.2 and 17.5.10). Moreover, we focus our attention on the η-quantum lift $Y_\eta[f]$ of projectable special phase functions f (see Definition 19.1.3).

Accordingly, we discuss the *quantum currents* $\mathrm{j}_\eta[f]$ and show that, in the case of conserved time preserving special phase functions, the quantum current $\mathrm{j}_\eta[f]$ turns out to be conserved (see Corollary 12.6.14). Thus, it is remarkable that both the conserved classical and quantum currents be generated by conserved time preserving special phase functions f (Sect. 13.3).

Eventually, we define the *quantum expectation forms* $\epsilon_\eta[f](\Psi)$ associated with projectable special phase functions f and compare them with the corresponding vertical quantum current form $\check{\mathrm{j}}_\eta[f](\Psi)$. We show a distinguished relevant case when they are physically equivalent.

The original source of our approach to classical and quantum currents and to quantum expectation forms goes back to [358].

21.1 Quantum Currents

The notion of quantum currents turn out to be a generalisation of probability current.

Current algebras are well known in Quantum Field Theories (see, for instance, [5, 342] and references therein). However, in Quantum Mechanics, a systematic extension of currents beyond probability current can hardly be found in literature.

In our approach, a Lie algebra of quantum currents is obtained as a byproduct of the classification of η-hermitian quantum vector fields in terms of projectable special phase functions. In this way, we obtain a further correspondence rule between a

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classical Lie algebra and quantum objects. Indeed, quantum currents turn out to be a covariant tool to recover expectation values and integrals of motion in Quantum Mechanics.

21.1.1 Quantum Currents

We apply the general formalism discussed in Sect. H.3 to the case of Covariant Quantum Mechanics (see, also, [358]).

Thus, we define the *quantum current* $\mathrm{j}_\eta[f] := -i_{Y_\eta[f]}\,\mathrm{C}$ associated with a projectable special phase function f and discuss its splitting and related Lie algebra constructions.

We can implement in the following way the general construction discussed in Sect. H.3. In the quantum framework,

- the base space is $\boldsymbol{B} := \boldsymbol{E}$ and the total space is $\boldsymbol{F} := \boldsymbol{Q}$,
- the scaled volume form is $\upsilon \in \sec(\boldsymbol{E}, \mathbb{T} \otimes \Lambda^4 T^*\boldsymbol{E})$,
- the lagrangian form and the Poincaré–Cartan form are, respectively, the global, gauge independent and observer independent 4-forms (see Theorems 17.5.2 and 17.5.10)

$$\mathrm{L} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{E} \qquad \text{and} \qquad \mathrm{C} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{Q}.$$

In the present case, we have $m - 1 = 3$, hence the quantum currents turn out to be 3-forms.

Let us start with a preliminary discussion on the 1-jet quantum lift of projectable special phase functions.

Note 21.1.1 Let us consider the complex linear *1st jet quantum bundle*

$$\pi^1 : J_1\boldsymbol{Q} \to \boldsymbol{E}.$$

We shall refer to the real linear fibred charts and to the associated real linear bases

$$(x^\lambda;\ w^1,\ w^2;\ w^1_\mu,\ w^2_\mu) \qquad \text{and} \qquad (\partial_\lambda;\ \partial w_1,\ \partial w_2;\ \partial w_1^\mu,\ \partial w_2^\mu).$$

Moreover, by a complex notation, we set

$$z_\mu = w^1_\mu + \mathrm{i}\, w^2_\mu, \quad \bar{z}_\mu = w^1_\mu - \mathrm{i}\, w^2_\mu, \qquad \partial w_1^\mu = \mathrm{b}^\mu, \quad \partial w_2^\mu = \mathrm{i}\,\mathrm{b}^\mu. \ \square$$

Lemma 21.1.2 *For each* $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$, *we obtain the vector field (see Proposition 12.3.1)*

$$Y^1_\eta[f] := (r^1 \circ J_1)(Y_\eta[f]) \in \operatorname{pro}_{\boldsymbol{E},\boldsymbol{Q}}(J_1\boldsymbol{Q},\ TJ_1\boldsymbol{Q}) \subset \sec(J_1\boldsymbol{Q},\ TJ_1\boldsymbol{Q}),$$

with coordinate expression, in real and complex coordinates,

$$\begin{aligned} Y^1_\eta[f] = \;& f^0\,\partial_0 - f^i\,\partial_i - \tfrac12\,\mathrm{div}_\eta f\,(w^1\,\partial w_1 + w^2\,\partial w_2) + \hat f\,(w^1\,\partial w_2 - w^2\,\partial w_1)\\ & - \tfrac12\,\partial_\mu\,\mathrm{div}_\eta f\,(w^1\,\partial w^\mu_1 + w^2\,\partial w^\mu_2) + \partial_\mu \hat f\,(w^1\,\partial w^\mu_2 - w^2\,\partial w^\mu_1)\\ & - \tfrac12\,\mathrm{div}_\eta f\,(w^1_\mu\,\partial w^\mu_1 + w^2_\mu\,\partial w^\mu_2) + \hat f\,(w^1_\mu\,\partial w^\mu_2 - w^2_\mu\,\partial w^\mu_1)\\ & - \partial_0 f^0\,(w^1_0\,\partial w^0_1 + w^2_0\,\partial w^0_2) + \partial_\mu f^i\,(w^1_i\,\partial w^\mu_1 + w^2_i\,\partial w^\mu_2)\\ = \;& f^0\,\partial_0 - f^i\,\partial_i + (\mathrm{i}\,\hat f - \tfrac12\,\mathrm{div}_\eta f)\,z\,\mathrm{b}\\ & + (\mathrm{i}\,\partial_\mu \hat f - \tfrac12\,\partial_\mu\,\mathrm{div}_\eta f)\,z\,\mathrm{b}^\mu\\ & + (\mathrm{i}\,\hat f - \tfrac12\,\mathrm{div}_\eta f)\,z_\mu\,\mathrm{b}^\mu\\ & - \partial_0 f^0\,z_0\,\mathrm{b}^0 + \partial_\mu f^i\,z_i\,\mathrm{b}^\mu. \quad \square \end{aligned}$$

Proof. The proof follows from Proposition 12.3.1 and Theorem 19.1.7. □

Now, let us consider the global, gauge independent and observer independent quantum lagrangian form L and the quantum Poincaré–Cartan form C.

Definition 21.1.3 We define the *quantum current* associated with a projectable special phase function $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ to be the horizontal 3-form

$$\mathrm{j}_\eta[f] := -\,i_{Y^1_\eta[f]}\,\mathrm{C} = -i_{Y_\eta[f]}\,\mathrm{C} \in \sec(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}).$$

We denote the subsheaf of quantum currents by

$$\mathrm{cur}_\eta(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}) \subset \sec(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}). \quad \square$$

Theorem 21.1.4 *For each* $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$, *we have the coordinate expression*

$$\begin{aligned} \mathrm{j}_\eta[f] = & -\big(\tfrac12\,G^{hk}_0\,(w^1_h\,w^1_k + w^2_h\,w^2_k) + \alpha_0\,(w^1\,w^1 + w^2\,w^2)\big)\\ & \times \big(f^0\,\upsilon^0_0 - f^j\,\upsilon^0_j\big) + \big(G^{hj}_0\,(w^1_h\,dw^1 + w^2_h\,dw^2)\\ & + A^j_0\,(w^2\,dw^1 - w^1\,dw^2)\big) \wedge \big(f^0\,\upsilon^0_{0j} - f^i\,\upsilon^0_{ij}\big)\\ & - f^j\,(w^2\,dw^1 - w^1\,dw^2) \wedge \upsilon^0_{0j}\\ & + \hat f\,\big(G^{hj}_0\,(w^1\,w^2_h - w^2\,w^1_h)\,\upsilon^0_j + (w^1\,w^1 + w^2\,w^2)(\upsilon^0_0 - A^j_0\,\upsilon^0_j)\big)\\ & - \tfrac12\,\mathrm{div}_\eta f\,G^{hj}_0\,(w^1\,w^1_h + w^2\,w^2_h)\,\upsilon^0_j\\ = & -(\tfrac12\,G^{hk}_0\,z_h\,\bar z_k + \alpha_0\,|z|^2\big)\,\big(f^0\,\upsilon^0_0 - f^j\,\upsilon^0_j)\\ & + \tfrac12\,\big(G^{hj}_0\,(z_h\,d\bar z + \bar z_h\,dz) + \mathrm{i}\,A^j_0\,(\bar z\,dz - z\,d\bar z)\big) \wedge (f^0\,\upsilon^0_{0j} - f^i\,\upsilon^0_{ij})\\ & - \tfrac12\,\mathrm{i}\,f^j\,(\bar z\,dz - z\,d\bar z) \wedge \upsilon^0_{0j}\\ & + \hat f\,\big(\tfrac12\,\mathrm{i}\,G^{hj}_0\,(z\,\bar z_h - \bar z\,z_h)\,\upsilon^0_j + |z|^2\,(\upsilon^0_0 - A^j_0\,\upsilon^0_j)\big)\\ & - \tfrac14\,\mathrm{div}_\eta f\,G^{hj}_0\,(z\,\bar z_h + \bar z\,z_h)\,\upsilon^0_j. \end{aligned}$$

The sheaf morphism

$$\mathrm{j}_\eta : \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R) \to \sec(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}) : f \mapsto \mathrm{j}_\eta[f]$$

is $\mathrm{map}(\boldsymbol{T}, I\!R)$*-linear and injective. Hence, the subsheaf of* $I\!R$*-vector spaces*

$$\mathrm{cur}_\eta(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}) \subset \sec(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q})$$

inherits an $I\!R$*-Lie bracket from the special phase bracket. Indeed, by definition, we have*

$$\big[\mathrm{j}_\eta[f], \mathrm{j}_\eta[\acute{f}]\big] := \mathrm{j}_\eta\big[[\![f, \acute{f}]\!] \big], \qquad \text{for each} \qquad f, \acute{f} \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, I\!R).$$

Proof. The coordinate expression of $\mathrm{j}_\eta[f]$ follows from Theorem 19.1.7 and Proposition 12.3.1. □

By the above approach we can recover, as a particular case, the standard probability current in our curved framework.

Example 21.1.5 In virtue of Theorem 19.3.2, the vector field

$$Y^1_\eta[1] : J_1\boldsymbol{Q} \to T J_1\boldsymbol{Q},$$

with coordinate expression

$$Y^1_\eta[1] = w^1\,\partial w_1 + w^2\,\partial w_2 + w^1_\mu\,\partial w^\mu_1 + w^2_\mu\,\partial w^\mu_2,$$

is an infinitesimal symmetry of the quantum lagrangian.

Indeed, the quantum current associated with the distinguished projectable special phase function $f = 1 \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ turns out to be just the Hodge star of the quantum probability current (see Theorem 17.4.2)

$$\mathrm{j}_\eta[1] = *_\upsilon(\mathrm{J}),$$

with coordinate expressions

$$\mathrm{j}_\eta[1] = G^{hj}_0\,(w^1\,w^2_h - w^2\,w^1_h)\,\upsilon^0_j + (w^1\,w^1 + w^2\,w^2)(\upsilon^0_0 - A^j_0\,\upsilon^0_j). \quad \square$$

Besides the probability current, we can achieve other distinguished quantum currents.

Example 21.1.6 We have the following distinguished quantum currents (see Example 12.1.4):

$$\begin{aligned}
\mathfrak{j}_\eta[x^\lambda] &= x^\lambda \big(G_0^{hj}\,(w^1\,w_h^2 - w^2\,w_h^1)\,\upsilon_j^0 + (w^1\,w^1 + w^2\,w^2)(\upsilon_0^0 - A_0^j\,\upsilon_j^0)\big),\\
\mathfrak{j}_\eta[-A^\uparrow{}_0] = \mathfrak{j}_\eta[\mathcal{H}_0] &= -\big(\tfrac{1}{2}\,G_0^{hk}\,(w_h^1\,w_k^1 + w_h^2\,w_k^2) - \alpha_0\,(w^1\,w^1 + w^2\,w^2)\big)\,\upsilon_0^0\\
&\quad - \big(G_0^{hi}\,(w_h^1\,dw^1 + w_h^2\,dw^2) + A_0^i\,(w^2\,dw^1 - w^1\,dw^2)\big)\wedge\upsilon_{i0}^0\\
&\quad - \tfrac{1}{2}\,\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}}\,G_0^{hk}\,(w^1\,w_h^1 + w^2\,w_h^2)\,\upsilon_k^0,\\
\mathfrak{j}_\eta[A^\uparrow{}_j] = \mathfrak{j}_\eta[\mathcal{P}_j] &= \big(\tfrac{1}{2}\,G_0^{hk}\,(w_h^1\,w_k^1 + w_h^2\,w_k^2) + \alpha_0\,(w^1\,w^1 + w^2\,w^2)\big)\,\upsilon_j^0\\
&\quad - (w^2\,dw^1 - w^1\,dw^2)\wedge\upsilon_{0j}^0\\
&\quad + \big(G_0^{hi}\,(w_h^1\,dw^1 + w_h^2\,dw^2) + A_0^i\,(w^2\,dw^1 - w^1\,dw^2)\big)\wedge\upsilon_{ij}^0\\
&\quad + \tfrac{1}{2}\,\frac{\partial_j\sqrt{|g|}}{\sqrt{|g|}}\,G_0^{hk}\,(w^1\,w_h^1 + w^2\,w_h^2)\,\upsilon_k^0. \quad \square
\end{aligned}$$

The above results provide a further splitting (besides the observed and gauge splittings, see Proposition 12.2.9) of projectable special phase functions in terms of quantum currents.

Corollary 21.1.7 *For each projectable special phase function (see Proposition 13.3.2)*

$$\begin{aligned}
f &= \mathfrak{c}[f, \mathfrak{b}] + \hat{f}[\mathfrak{b}]\\
&= f^0\,\mathfrak{c}[\mathcal{H}_0, \mathfrak{b}] + f^i\,\mathfrak{c}[\mathcal{P}_i, \mathfrak{b}] + \hat{f}[\mathfrak{b}]\\
&= f^0\,\mathcal{H}_0 + f^i\,\mathcal{P}_i + \hat{f},
\end{aligned}$$

we have the splitting

$$\begin{aligned}
\mathfrak{j}_\eta[f] &= \mathfrak{j}_\eta\big[\mathfrak{c}[f, \mathfrak{b}]\big] + \hat{f}[\mathfrak{b}]\,\mathfrak{j}_\eta[1]\\
&= f^0\,\mathfrak{j}_\eta\big[\mathfrak{c}[\mathcal{H}_0, \mathfrak{b}]\big] + f^i\,\mathfrak{j}_\eta\big[\mathfrak{c}[\mathcal{P}_i, \mathfrak{b}]\big] + \hat{f}[\mathfrak{b}]\,\mathfrak{j}_\eta[1] + \omega_\eta[f]\\
&= f^0\,\mathfrak{j}_\eta[\mathcal{H}_0] + f^i\,\mathfrak{j}_\eta[\mathcal{P}_i] + \hat{f}[\mathfrak{b}]\,\mathfrak{j}_\eta[1] + \omega_\eta[f],
\end{aligned}$$

where, according to the equality

$$\omega_\eta[f] = \tfrac{1}{2}\,\partial_i f^i\,G_0^{hj}\,(w^1\,w_h^1 + w^2\,w_h^2)\,\upsilon_j^0,$$

the term $\omega_\eta[f]$ depends not only on the gauge, but also on the spacetime chart involved in the components $\mathcal{H}_0$ and $\mathcal{P}_i$. □

The above “odd term” $\omega_\eta[f]$ will disappear in the analogous splitting of vertical quantum currents (see Theorem 21.1.9).

Note 21.1.8 It is worth comparing the classical currents with the quantum currents.

First of all, we notice that the classical currents are gauge dependent, while the quantum currents are gauge independent. This is due to the fact that the classical lagrangian is gauge dependent, while the quantum lagrangian is gauge independent (see Theorems 10.1.8 and 17.5.2).

Moreover, we stress that the classical currents involve the special phase functions "gauging out" the spacetime functions (see Definition 13.3.1, Proposition 13.3.2, Corollaries 13.3.5 and 13.3.7). Conversely, the quantum currents involve essentially the spacetime functions; even more, according to our procedure, the constant spacetime function $1 \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ turns out to be the generator of quantum probability current (see Definition 21.1.3, Theorem 21.1.4, Examples 21.1.5 and 21.1.6 and Corollary 21.1.7). Indeed, there is no classical counterpart of the quantum probability current. □

21.1.2 Vertical Quantum Currents

In view of integration on the fibres of spacetime, we analyse the $\boldsymbol{T}$-vertical restrictions of quantum currents.

Let us denote the subsheaf of vertical restrictions of quantum currents by

$$\check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}) \subset \mathrm{sec}(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}).$$

Theorem 21.1.9 *The $\boldsymbol{T}$-vertical restrictions of quantum currents*

$$\check{\mathfrak{j}}_\eta[f] \in \check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q})$$

have coordinate expressions

$$\begin{aligned}\check{\mathfrak{j}}_\eta[f] = &- f^0\big(\tfrac{1}{2}G^{hk}_0(w^1_h w^1_k + w^2_h w^2_k) + (A_0 - \tfrac{1}{2}A_i A^i_0)(w^1 w^1 + w^2 w^2)\big)\eta \\ &+ f^0\big(G^{hj}_0(w^1_h dw^1 + w^2_h dw^2) + A^j_0(w^2 dw^1 - w^1 dw^2)\big)\wedge\eta_j \\ &- f^j(w^2 dw^1 - w^1 dw^2)\wedge\eta_j \\ &+ \hat{f}(w^1 w^1 + w^2 w^2)\eta.\end{aligned}$$

Thus, the sheaf morphism

$$(*) \qquad \check{\mathfrak{j}}_\eta : \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}) : f \mapsto \check{\mathfrak{j}}_\eta[f]$$

turns out to be $\mathrm{map}(\boldsymbol{E}, \mathbb{R})$*-linear and injective.*

Accordingly, for each projectable special phase function $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$, which can be written as (see Proposition 13.3.2)

$$\begin{aligned} f &= \mathfrak{c}[f, \mathfrak{b}] + \hat{f}[\mathfrak{b}] \\ &= f^0\,\mathfrak{c}[\mathcal{H}_0, \mathfrak{b}] + f^i\,\mathfrak{c}[\mathcal{P}_i, \mathfrak{b}] + \hat{f}[\mathfrak{b}] \\ &= f^0\,\mathcal{H}_0 + f^i\,\mathcal{P}_i + \hat{f},\end{aligned}$$

the associated vertical quantum current has the following gauge splitting

$$\begin{aligned}\check{\mathfrak{j}}_\eta[f] &= \check{\mathfrak{j}}_\eta\big[\mathfrak{c}[f, \mathfrak{b}]\big] + \hat{f}[\mathfrak{b}]\,\check{\mathfrak{j}}_\eta[1]\\ &= f^0\,\check{\mathfrak{j}}_\eta\big[\mathfrak{c}[\mathcal{H}_0, \mathfrak{b}]\big] + f^i\,\check{\mathfrak{j}}_\eta\big[\mathfrak{c}[\mathcal{P}_i, \mathfrak{b}]\big] + \hat{f}[\mathfrak{b}]\,\check{\mathfrak{j}}_\eta[1]\\ &= f^0\,\check{\mathfrak{j}}_\eta[\mathcal{H}_0] + f^i\,\check{\mathfrak{j}}_\eta[\mathcal{P}_i] + \hat{f}[\mathfrak{b}]\,\check{\mathfrak{j}}_\eta[1].\end{aligned}$$

In virtue of the injectivity of the map $(*)$, *the sheaf* $\check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q})$ *inherits an* $\mathbb{R}$*-Lie bracket from the special phase Lie bracket. Indeed, by definition, we have*

$$\big[\check{\mathfrak{j}}_\eta[f],\ \check{\mathfrak{j}}_\eta[\acute{f}]\big] := \check{\mathfrak{j}}_\eta\big[\,[\![\, f,\ \acute{f}\,]\!]\,\big], \quad \textit{for each}\, f,\ \acute{f} \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}).$$

Moreover the map

$$\check{}\ : \mathrm{cur}_\eta(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}) \to \check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}) : \mathfrak{j}_\eta[f] \mapsto \check{\mathfrak{j}}_\eta[f]$$

turns out to be a Lie algebra isomorphism. □

Thus, in the vertical restriction of quantum current, the contribution of the term $\mathrm{div}_\eta f$ disappears and this fact makes the map $f \mapsto \check{\mathfrak{j}}_\eta[f]$ linear with respect to the coefficients $f^\lambda,\ \check{f}$.

Example 21.1.10 We have the following distinguished vertical quantum currents (see Example 12.1.4):

$$\begin{aligned}\check{\mathfrak{j}}_\eta[x^\lambda] &= x^\lambda\,(w^1\,w^1 + w^2\,w^2)\,\eta,\\ \check{\mathfrak{j}}_\eta[-A^\uparrow{}_0] = \check{\mathfrak{j}}_\eta[\mathcal{H}_0] &= -\big(\tfrac{1}{2}\,G^{hk}_0\,(w^1_h\,w^1_k + w^2_h\,w^2_k)\\ &\quad - (A_0 - \tfrac{1}{2}\,A_i\,A^i_0)\,(w^1\,w^1 + w^2\,w^2)\big)\,\eta\\ &\quad - \big(G^{hi}_0\,(w^1_h\,dw^1 + w^2_h\,dw^2) + A^i_0\,(w^2\,dw^1 - w^1\,dw^2)\big)\wedge\eta_i,\\ \check{\mathfrak{j}}_\eta[A^\uparrow{}_j] = \check{\mathfrak{j}}_\eta[\mathcal{P}_j] &= -(w^2\,dw^1 - w^1\,dw^2)\wedge\eta_j.\quad \square\end{aligned}$$

Remark 21.1.11 It is worth comparing the classical currents and the vertical quantum currents associated with projectable special phase functions.

(1) The classical current map

$$\mathfrak{c}[.,\mathfrak{b}] : \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{srt\,spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

which is originated by the tangent lift $X[f]$ of the phase function f, factorises through the subsheaf $\mathrm{srt\,spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$, as $\mathfrak{c}[f, \mathfrak{b}]$ does not involve $\hat{f}[\mathfrak{b}]$.

(2) Conversely, the vertical quantum current map

$$\check{\mathfrak{j}}_\eta : \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \check{\mathrm{cur}}_\eta(J_1\boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}}\boldsymbol{Q}) : f \mapsto \check{\mathfrak{j}}_\eta[f],$$

which is originated by the η-quantum lift $Y_\eta[f]$, involves the gauge component $\hat{f}[\mathrm{b}]$ essentially. In fact, this component yields the probability current. □

21.2 Quantum Current Forms

We discuss the spacetime forms obtained by taking the pullback of quantum currents and of vertical quantum currents with respect to quantum sections.

21.2.1 Quantum Current Forms

The pullback of quantum currents $\mathrm{j}_\eta[f]$ with respect to quantum sections Ψ provides distinguished 3-forms of spacetime $\mathrm{j}_\eta[f](\Psi) := (j_1\Psi)^*\mathrm{j}_\eta[f]$, which are associated with projectable special phase functions.

Indeed, these quantum current forms $\mathrm{j}_\eta[f](\Psi)$ turn out to be closed in the case of a conserved time preserving special phase function and a quantum solution of the Schrödinger equation.

Definition 21.2.1 We define the *quantum current form* associated with the phase function $f \in \operatorname{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$ and the quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ to be the spacetime 3-form

$$\mathrm{j}_\eta[f](\Psi) := (j_1\Psi)^*\mathrm{j}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3 T^*\boldsymbol{E}). \quad \square$$

Proposition 21.2.2 *For each* $f \in \operatorname{pro spe}(J_1\boldsymbol{E}, \mathbb{R})$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *we have the coordinate expression*

$$\begin{aligned}
\mathrm{j}_\eta[f](\Psi) = \tfrac{1}{2} \big(& G_0^{hk}\, \partial_h\bar{\psi}\, \partial_k\psi + \mathrm{i}\, A_0^h\, (\bar{\psi}\, \partial_h\psi - \psi\, \partial_h\bar{\psi}) \\
& - 2\,(A_0 - \tfrac{1}{2}\, A_i\, A_0^i)\, |\psi|^2\big) \big(f^0\, \upsilon_0^0 - f^j\, \upsilon_j^0\big) \\
& - f^0\, \tfrac{1}{2} \big(G_0^{hj}\, (\partial_h\bar{\psi}\, \partial_0\psi + \partial_h\psi\, \partial_0\bar{\psi}) + \mathrm{i}\, A_0^j\, (\bar{\psi}\, \partial_0\psi - \psi\, \partial_0\bar{\psi})\big)\, \upsilon_j^0 \\
& + f^i\, \tfrac{1}{2} \big(G_0^{jh}\, (\partial_i\bar{\psi}\, \partial_h\psi + \partial_i\psi\, \partial_h\bar{\psi}) + \mathrm{i}\, A_0^j\, (\bar{\psi}\, \partial_i\psi - \psi\, \partial_i\bar{\psi})\big)\, \upsilon_j^0 \\
& - f^j\, \tfrac{1}{2}\, \mathrm{i}\, \big((\bar{\psi}\, \partial_j\psi - \psi\, \partial_j\bar{\psi})\, \upsilon_0^0 - (\bar{\psi}\, \partial_0\psi - \psi\, \partial_0\bar{\psi})\, \upsilon_j^0\big) \\
& - \hat{f}\, \big(\tfrac{1}{2}\, \mathrm{i}\, G_0^{hj}\, (\bar{\psi}\, \partial_h\psi - \psi\, \partial_h\bar{\psi})\, \upsilon_j^0 - |\psi|^2\, (\upsilon_0^0 - A_0^j\, \upsilon_j^0)\big) \\
& - \tfrac{1}{4}\, \mathrm{div}_\eta\, f\, G_0^{hj}\, \partial_h|\psi|^2\, \upsilon_j^0. \quad \square
\end{aligned}$$

Example 21.2.3 We have the following distinguished quantum current forms (see Example 12.1.4):

$$\begin{aligned}
\mathrm{j}_\eta[x^\lambda](\Psi) &= x^\lambda \big(- \tfrac{1}{2}\, \mathrm{i}\, G_0^{hk}\, (\bar\psi\, \partial_h \psi - \psi\, \partial_h \bar\psi)\, \upsilon_k^0 + |\psi|^2\, (\upsilon_0^0 - A_0^h\, \upsilon_h^0)\big),\\
\mathrm{j}_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2} \big(G_0^{hk}\, \partial_h \bar\psi\, \partial_k \psi + \mathrm{i}\, A_0^h\, (\bar\psi\, \partial_h \psi - \psi\, \partial_h \bar\psi)\\
&\quad - 2\, (A_0 - \tfrac{1}{2}\, A_i\, A_0^i)\, |\psi|^2 \big)\, \upsilon_j^0\\
&\quad + \tfrac{1}{2} \big(G_0^{hk}\, (\partial_j \bar\psi\, \partial_h \psi + \partial_j \psi\, \partial_h \bar\psi) + \mathrm{i}\, A_0^k\, (\bar\psi\, \partial_j \psi - \psi\, \partial_j \bar\psi) \big)\, \upsilon_k^0\\
&\quad - \tfrac{1}{2}\, \mathrm{i} \big((\bar\psi\, \partial_j \psi - \psi\, \partial_j \bar\psi)\, \upsilon_0^0 - (\bar\psi\, \partial_0 \psi - \psi\, \partial_0 \bar\psi)\, \upsilon_j^0 \big)\\
&\quad + \tfrac{1}{4}\, \frac{\partial_j \sqrt{|g|}}{\sqrt{|g|}}\, G_0^{hk}\, \partial_h |\psi|^2\, \upsilon_k^0,\\
\mathrm{j}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2} \big(G_0^{ij}\, \partial_i \bar\psi\, \partial_j \psi + \mathrm{i}\, A_0^i\, (\bar\psi\, \partial_i \psi - \psi\, \partial_i \bar\psi)\\
&\quad - 2\, (A_0 - \tfrac{1}{2}\, A_i\, A_0^i)\, |\psi|^2 \big)\, \upsilon_0^0\\
&\quad - \tfrac{1}{2} \big(G_0^{ij}\, (\partial_i \bar\psi\, \partial_0 \psi + \partial_i \psi\, \partial_0 \bar\psi) + \mathrm{i}\, A_0^j\, (\bar\psi\, \partial_0 \psi - \psi\, \partial_0 \bar\psi) \big)\, \upsilon_j^0\\
&\quad - \tfrac{1}{4}\, \frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}}\, G_0^{hj}\, \partial_h |\psi|^2\, \upsilon_j^0. \quad \square
\end{aligned}$$

Eventually, the Noether theorem provides the following conservation law.

Theorem 21.2.4 *For each* $f \in \operatorname{cns}\operatorname{timspe}(J_1\boldsymbol{E}, I\!R)$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *which is a solution of the Schrödinger equation, the associated current form*

$$\mathrm{j}_\eta[f](\Psi) \in \sec(\boldsymbol{E}, \Lambda^3 T^*\boldsymbol{E})$$

turns out to be closed, i.e.

$$d\big(\mathrm{j}_\eta[f](\Psi)\big) = 0.$$

Proof. The proof follows from Theorems H.3.3 and 19.3.2 and Proposition 19.3.3. □

In particular, the η-quantum current form associated with the projectable special phase function $f = 1$ turns out to be just the Hodge star of the conserved quantum probability current form (see Example 21.1.5).

21.2.2 Vertical Quantum Current Forms

In view of integration on the fibres of spacetime, we analyse the $\boldsymbol{T}$-vertical restrictions of quantum current forms.

Lemma 21.2.5 *For each* $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, I\!R)$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *we have*

$$(j_1\Psi)^* \check{\mathrm{j}}_\eta[f] = (\check{j}_1\Psi)^* \check{\mathrm{j}}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3\, V^*\boldsymbol{E}),$$

where $\check{j}_1\Psi \in \sec(\boldsymbol{E}, \check{J}_1\boldsymbol{Q})$ *denotes the "fibrewise" 1-jet prolongation of* Ψ. □

Proposition 21.2.6 *In view of integration on the fibres of spacetime, we consider the vertical restrictions of quantum current forms*

$$\check{\mathrm{j}}_\eta[f](\Psi) := (j_1\Psi)^* \check{\mathrm{j}}_\eta[f] = (\check{j}_1\Psi)^* \check{\mathrm{j}}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3 V^*\boldsymbol{E}),$$

with coordinate expressions

$$\begin{aligned}\check{\mathrm{j}}_\eta[f](\Psi) = \tfrac{1}{2} f^0 \big(&G_0^{hk}\, \partial_h\bar\psi\, \partial_k\psi + \mathrm{i}\, A_0^h\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi)\\ &- 2\,(A_0 - \tfrac{1}{2} A_i\, A_0^i)\, |\psi|^2\big)\, \eta\\ &- \tfrac{1}{2} f^j\, \mathrm{i}\, (\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \eta + \hat f\, |\psi|^2\, \eta. \quad \square\end{aligned}$$

Example 21.2.7 We have the following distinguished vertical quantum current forms (see Example 12.1.4):

$$\begin{aligned}\check{\mathrm{j}}_\eta[x^\lambda](\Psi) &= x^\lambda\, |\psi|^2\, \eta,\\ \check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) &= \tfrac{1}{2}\, \mathrm{i}\, (\psi\, \partial_j\bar\psi - \bar\psi\, \partial_j\psi)\, \eta,\\ \check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2} \big(G_0^{hk}\, \partial_h\bar\psi\, \partial_k\psi + \mathrm{i}\, A_0^h\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi) - 2\,\alpha_0\, |\psi|^2\big)\, \eta,\end{aligned}$$

hence, in polar coordinates, in the proper domain of Ψ,

$$\begin{aligned}\check{\mathrm{j}}_\eta[x^\lambda](\Psi) &= x^\lambda\, |\psi|^2\, \eta,\\ \check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) &= |\psi|^2\, \partial_j\varphi\, \eta,\\ \check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \big(\tfrac{1}{2}\, G_0^{hk}\, (\partial_h|\psi|\, \partial_k|\psi| + |\psi|^2\, \partial_h\varphi\, \partial_k\varphi) - A_0^h\, \partial_h\varphi - \alpha_0\, |\psi|^2\big)\, \eta. \quad \square\end{aligned}$$

21.3 Quantum Expectation Forms

The quantum currents discussed above deal with a theoretic approach to Quantum Mechanics, which links the classical special phase functions with quantum objects.

Now, we discuss, in our covariant context, the quantum expectation forms, which deal with the standard probabilistic interpretation of Quantum Mechanics (see, also, [358]).

Thus, we define the *quantum expectation forms* $\epsilon_\eta[f](\Psi) := \mathrm{re}\, \mathrm{h}_\eta\big(\Psi,\ \mathrm{O}[f](\Psi)\big)$ associated with projectable special phase functions f and compare them with the corresponding vertical quantum current form $\check{\mathrm{j}}[f](\Psi)$.

Actually, we prove that, in the case of a quantum section with compact spacelike support, these forms are equal up to a term whose spacelike integral vanishes.

Definition 21.3.1 We define the *quantum expectation form* associated with the special phase function $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ and the quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ to be the vertical spacetime 3-form (see Postulate Q.1 and Theorem 20.1.9)

$$\epsilon_\eta[f](\Psi) := \operatorname{re} \mathrm{h}_\eta\big(\Psi,\ \mathsf{O}[f](\Psi)\big) \in \sec(\boldsymbol{E}, \Lambda^3 V^* \boldsymbol{E}),$$

given by

$$\epsilon_\eta[f](\Psi) = \tfrac{1}{2}\,\mathrm{i}\Big(\mathrm{h}_\eta\big(\Psi, Y_\eta[f](\Psi)\big) - \mathrm{h}_\eta\big(Y_\eta[f](\Psi), \Psi\big) - \mathrm{h}_\eta\big(\Psi, \mathsf{S}[f](\Psi)\big) + \mathrm{h}_\eta\big(\mathsf{S}[f](\Psi), \Psi\big)\Big). \quad \square$$

Proposition 21.3.2 *We have the coordinate expression*

$$\begin{aligned}
&\epsilon_\eta[f](\Psi)\\
&= \Big((\breve{f} - A_i\, f^i)\,|\psi|^2 - \tfrac{1}{2}\,\mathrm{i}\, f^i\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi}) - \tfrac{1}{4}\, f^0\,(\bar{\psi}\,\Delta_0\psi + \psi\,\overline{\Delta_0\psi})\Big)\,\eta\\
&= -\tfrac{1}{4}\, f^0\,\Big(G^{ij}_0\,(\bar{\psi}\,\partial_{ij}\psi + \psi\,\partial_{ij}\bar{\psi}) + (\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}})\,(\bar{\psi}\,\partial_j\psi + \psi\,\partial_j\bar{\psi})\Big)\,\eta\\
&\quad + \tfrac{1}{2}\,\mathrm{i}\,(f^0\,A^j_0 - f^j)\,(\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi})\,\eta + (\breve{f} + \tfrac{1}{2}\, f^0\,A_i\,A^i_0 - A_i\,f^i)\,|\psi|^2\,\eta.
\end{aligned}$$

Proof. The proof follows from Theorems 20.1.9 and 17.6.5. □

Example 21.3.3 We have the following distinguished quantum expectation forms (see Example 12.1.4):

$$\begin{aligned}
\epsilon_\eta[x^\lambda](\Psi) &= x^\lambda |\psi|^2\,\eta,\\
\epsilon_\eta[\mathcal{P}_j](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_j\bar{\psi} - \bar{\psi}\,\partial_j\psi)\,\eta,\\
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\big(A_0\,|\psi|^2 + \tfrac{1}{4}\,(\psi\,\overline{\Delta[G]_0\psi} + \bar{\psi}\,\Delta[G]_0\psi)\big)\,\eta,
\end{aligned}$$

hence, in polar coordinates, in the proper domain of Ψ,

$$\begin{aligned}
\epsilon_\eta[x^\lambda](\Psi) &= x^\lambda\,|\psi|^2\,\eta,\\
\epsilon_\eta[\mathcal{P}_j](\Psi) &= |\psi|^2\,\partial_j\varphi\,\eta,\\
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\Big((A_0 - \tfrac{1}{2}\,A^i_0\,A_i + A^i_0\,\partial_i\varphi)\,|\psi|^2 + \tfrac{1}{2}\,|\psi|\,G^{ij}_0\,(\partial_{ij}|\psi| - |\psi|\,\partial_i\varphi\,\partial_j\varphi)\\
&\qquad + \tfrac{1}{2}\,|\psi|\,\frac{\partial_i(G^{ij}_0\,\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi|\Big)\,\eta\\
&= -\Big(A_0\,|\psi|^2 + \tfrac{1}{2}\,|\psi|\,\Delta_0|\psi| - \tfrac{1}{2}\,|\psi|^2\,G^{ij}_0\,(\partial_i\varphi - A_i)\,(\partial_j\varphi - A_j)\Big)\,\eta\\
&= -\Big(A_0\,|\psi|^2 + \tfrac{1}{2}\,|\psi|\,\Delta_0|\psi| - \tfrac{1}{2}\,|\psi|^2\,G(\overrightarrow{\nabla}^{(\!\!\;\!\!)}(\!(\Psi)\!), \overrightarrow{\nabla}^{(\!\!\;\!\!)}(\!(\Psi)\!))\Big)\,\eta. \quad \square
\end{aligned}$$

Remark 21.3.4 By taking into account the equality

$$\tfrac{1}{2}\,|\psi|\,\Delta_0|\psi| = \tfrac{1}{4}\,\Delta_0|\psi|^2 - \tfrac{1}{2}\,G^{ij}_0\,\partial_i|\psi|\,\partial_j|\psi|,$$

we can write also

$$\begin{aligned}
&\epsilon_\eta[\mathcal{H}_0](\Psi) \\
&\quad = -\Big(A_0\,|\psi|^2 + \tfrac{1}{4}\;\Delta_0|\psi|^2 - \tfrac{1}{2}\,|\psi|^2\,\big(G\,(\vec{\nabla}^{\circledcirc}(\!(\Psi)\!),\,\vec{\nabla}^{\circledcirc}(\!(\Psi)\!))\\
&\qquad + G\,(\vec{d}\log|\psi|,\;\vec{d}\log|\psi|)\big)\Big)\,\eta. \quad \square
\end{aligned}$$

We notice that the vertical quantum current forms and the quantum expectation forms have the same source and target, even if they have been achieved by very different procedures.

So, it is natural to compare them; actually, we prove that the spacelike integral of their difference vanishes.

Definition 21.3.5 For each $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$ and $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, we define the spacetime vertical form

$$\mathfrak{d}_\eta[f](\Psi) := \check{\jmath}_\eta[f](\Psi) - \epsilon_\eta[f](\Psi) \in \sec(\boldsymbol{E}, \Lambda^3 V^*\boldsymbol{E}),$$

with coordinate expression

$$\begin{aligned}
\mathfrak{d}_\eta[f](\Psi) &= \tfrac{1}{4}\,f'' \lrcorner\,\Delta|\psi|^2\,\eta \\
&= \tfrac{1}{2}\,f'' \lrcorner \Big(|\psi|\,\Delta|\psi| + |\psi|^2 G\,(\vec{d}\log|\psi|,\;\vec{d}\log|\psi|)\Big)\,\eta,
\end{aligned}$$

i.e., more explicitly,

$$\begin{aligned}
&\mathfrak{d}_\eta[f](\Psi) \\
&\quad = \tfrac{1}{4}\,f^0\,\Delta_0|\psi|^2\,\eta \\
&\quad = \tfrac{1}{4}\,f^0\left(G_0^{ij}\,\partial_{ij}|\psi|^2 + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi|^2\right)\eta \\
&\quad = f^0\Big(\tfrac{1}{2}\,G_0^{ij}\,\partial_i\bar{\psi}\,\partial_j\psi + \tfrac{1}{2}\,\mathrm{i}\,A_0^i\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi}) \\
&\qquad + A_0^i\,A_i\,|\psi|^2 + \tfrac{1}{4}\,(\bar{\psi}\,\Delta_0\psi + \psi\,\overline{\Delta_0\psi})\Big)\,\eta \\
&\quad = \tfrac{1}{2}\,f^0\,\Big(|\psi|\,\Delta_0|\psi| + G_0^{ij}\,\partial_i|\psi|\,\partial_j|\psi|\Big)\,\eta \\
&\quad = \tfrac{1}{2}\,f^0\left(|\psi|\,(G_0^{ij}\,\partial_{ij}|\psi| + \frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}}\,\partial_j|\psi|) + G_0^{ij}\,\partial_i|\psi|\,\partial_j|\psi|\right)\eta. \quad \square
\end{aligned}$$

Example 21.3.6 We have the following distinguished forms (see Example 12.1.4):

$$\mathfrak{d}_\eta[x^\lambda](\Psi) = 0, \qquad \mathfrak{d}_\eta[\mathcal{P}_j](\Psi) = 0, \qquad \mathfrak{d}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{4}\,\Delta_0|\psi|^2\,\eta. \quad \square$$

Eventually, we show the following result, which compares the vertical quantum current and the quantum expectation form associated with a time preserving special phase function.

Theorem 21.3.7 *For each* $f \in \operatorname{timspe}(J_1\boldsymbol{E}, I\!R)$ *and* $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, *with compact spacelike support, we obtain the vanishing fibrewise integral*

$$\int \mathfrak{d}_\eta[f](\Psi) = 0.$$

Proof. The proof follows from the expression of $\mathfrak{d}_\eta[f]$ provided by Definition 21.3.5, the definition of the differential operator Δ_0 and the Stokes theorem. □

Remark 21.3.8 [358] The above result shows the "physical equivalence" of vertical currents forms and expectation forms, which arise from quite different routes (on one side, quantum lifts of special phase functions and quantum Poincaré–Cartan form, on the other side, quantum operators).

This result might suggest an alternative approach to quantum expectation values of physical observables, based quantum currents, which generalise the probability current. As we have shown before, any quantum current is associated with a special phase function by means of a covariant procedure, where special phase functions play the role of physical observables of classical phase space. □

Chapter 22
Sectional Quantum Bundle

So far, we have been dealing with the geometric differential stuff of our covariant approach to Quantum Mechanics. Now, in conclusion of the present setting, we sketch an introduction to the Hilbert stuff, leaving to the reader further developments on this topic.

Standard Quantum Mechanics usually deals with a flat spacetime $\boldsymbol{E}$ and a given global inertial observer o, which yields a global splitting $\boldsymbol{E} = \boldsymbol{T} \times \boldsymbol{P}[o]$. Accordingly, one deals with the Hilbert space $\mathrm{H}\,[\boldsymbol{P}[o]]$ consisting of suitable maps $\Psi : \boldsymbol{P}[o] \to \mathbb{C}$. In this context, time essentially behaves as a parameter.

In our covariant approach to Quantum Mechanics, even in the flat case, we do not deal with a fixed observer o, hence we cannot avail of a given observed splitting of spacetime; moreover, in our approach, time is not a pure parameter. Therefore, we are led to replace the *Hilbert space* $\mathrm{H}\,[\boldsymbol{P}[o]]$ with an observer independent *Hilbert bundle* $\widehat{t} : \widehat{\boldsymbol{H}} \to \boldsymbol{T}$ based over time. Indeed, this bundle has no distinguished trivialisation, but every suitable global observer o yields such a trivialisation, as it is implicitly assumed in standard Quantum Mechanics.

So, by taking into account the above observation, we introduce the notion of "*sectional quantum bundle*" $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ based over time, whose fibres $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ consist of fibrewise smooth quantum sections with spacelike compact support $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ (Sect. 22.3). Then, a completion procedure yields a *"Hilbert quantum bundle"* $\widehat{t} : \widehat{\boldsymbol{H}} \to \boldsymbol{T}$ (Sect. 22.4).

Further, we translate the quantum differential operators to the Hilbert quantum bundle (Sect. 22.5). It is remarkable that in this context we can regard the Schrödinger operator as a connection of the sectional quantum bundle (Sect. 22.6).

A detailed study of such a bundle would require, besides the usual techniques of mathematical analysis concerning Hilbert spaces, hard geometric techniques concerning infinite dimensional bundles (see, for instance, [268, 303]). In order to manage such double methods at the same time, we would need to specify suitable hypotheses of our framework, by referring to specific examples.

The basic aim of the present book is an introduction to Covariant Quantum Mechanics, with main emphasis on the covariant geometric achievement of the

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Schrödinger equation, quantum operators and conservation laws. Here, we are not explicitly involved with the analytic study of the solutions of the Schrödinger equation; therefore, delicate problems of functional analysis and infinite dimensional geometry are unnecessary for our introductory goal.

So, in the introductory discussion of the present book, we are able to keep our general setting and formulate preliminary results on the sectional quantum bundle by means of a ploy. Namely, we relinquish the hard methods of true infinite dimensional geometry and resort to the more general and feeble technique of "*F-smooth spaces*" (see, for instance, [36, 130, 225, 303, 370]). In fact, this quite geometric structure is sufficient for our purposes. For convenience of the reader, we start the present Chapter with a concise introduction to F-smooth spaces (Sect. 22.1).

However, a full understanding of the Hilbert quantum bundle could not skip the use of more powerful methods of mathematical analysis and infinite dimensional geometry. We leave to the reader the task to proceed toward a full analytic development of our setting by a Hilbert completion procedure and subsequent steps. In practice, it would be reasonable to perform such a completion in the framework of a specific quantum system and not in our general setting.

Thus, we have pointed out that the covariance of the theory requires a Hilbert space for each time. This fact suggests the need of a possible extension of the postulates of standard Quantum Mechanics concerning quantum operators in order to include "quantum operators" like the "arrival time" (see, for instance, [26, 192, 235–237, 320] and literature therein).

For most topics analysed in the present book it is sufficient to assume that spacetime be a generic fibred manifold over time and to deal with local quantum sections. However, in the context of the sectional quantum bundle and related developments, for the sake of simplicity, we assume a more specific hypothesis on spacetime, namely that $t : \boldsymbol{E} \to \boldsymbol{T}$ be a bundle with contractible type fibre.

22.1 Concise Introduction to F-smooth Spaces

We start with preliminary recalls concerning *F-smooth spaces* and *F-smooth systems* (see [225]). More details and a comprehensive discussion can be found in [225]; the original source of this subject in the literature can be found, for instance, in [130, 253, 303, 370].

Indeed, the present concise review provided below is sufficient for most purposes of our application to Covariant Quantum Mechanics. Actually, the concept of F-smooth space is weaker than that of smooth manifold. Nevertheless, for our purposes, it is suitable to achieve, by simple means, some geometric results on "infinite dimensional" spaces consisting of smooth maps between smooth manifolds.

In this Section, we deal, at the same time, with smooth manifolds, F-smooth spaces, smooth maps between smooth manifolds, F-smooth maps between F-smooth spaces, and sets and maps with no smoothness property, as well. Therefore, for the sake of clarity, we specify every time which kind of smoothness we are dealing with;

when no specification of smoothness are emphasised, it means that no smoothness is required. All standard smooth manifolds are supposed to be finite dimensional.

In a few words, a standard *smooth manifold* $\boldsymbol{M}$ is defined via the choice of a suitable family of distinguished local functions $\boldsymbol{M} \to \mathbb{R}$; conversely, an *F-smooth space* $\boldsymbol{S}$ is defined via the choice of a suitable family of distinguished local curves $\mathbb{R} \to \boldsymbol{S}$.

Indeed, the concept of F-smooth space turns out to be more general and more feeble that of smooth manifold. However, the concept of F-smooth space can be easily used in an infinite dimensional context.

(1) An *F-smooth space* is defined to be a non empty set $\boldsymbol{S}$, along with a family $\mathcal{C}$ of "*basic curves*" $c : \boldsymbol{I}_c \to \boldsymbol{S}$, which fulfills the following simple properties: the constant curves belong to $\mathcal{C}$; if a curve belongs to $\mathcal{C}$, then its smooth reparametrisations belong to $\mathcal{C}$ as well (see [225]).

A map $f : \boldsymbol{M} \to \boldsymbol{N}$ between F-smooth spaces is said to be *F-smooth* if it maps the basic curves of $\boldsymbol{M}$ into the basic curves of $\boldsymbol{N}$ (see [225]).

In particular, all smooth manifolds are equipped with a natural F-smooth structure by assuming their standard smooth curves as basic curves (see [225]).

(2) Now, let us consider two smooth manifolds $\boldsymbol{M}$ and $\boldsymbol{N}$.

An *F-smooth system of smooth maps* is defined to be a family $\{f\}$ of global smooth maps $f : \boldsymbol{M} \to \boldsymbol{N}$, which is parametrised by the elements of a set $\boldsymbol{S}$ (see [225]).

Thus, an F-smooth system of smooth maps is characterised by a pair $(\boldsymbol{S}, \epsilon)$, where

$$\epsilon : \boldsymbol{S} \times \boldsymbol{M} \to \boldsymbol{N}$$

is the "*evaluation map*", which, for each $s \in \boldsymbol{S}$, provides the selected global smooth map

$$\check{s} := \epsilon_s : \boldsymbol{M} \to \boldsymbol{N}.$$

We denote the subset of selected global smooth maps selected by

$$\mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) \subset \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}).$$

The system is said to be "injective" if the induced map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}(\boldsymbol{M}, \boldsymbol{N}) : s \mapsto \check{s}$$

is injective, that is if the induced map

$$\epsilon_{\boldsymbol{S}} : \boldsymbol{S} \to \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) : s \mapsto \check{s}$$

is bijective. In this case, we denote the inverse map by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \mathrm{Map}_{\boldsymbol{S}}(\boldsymbol{M}, \boldsymbol{N}) \to \boldsymbol{S} : f \mapsto \widehat{f}\,.$$

We can prove that the set of parameters $\boldsymbol{S}$ turns out to be an F-smooth space in a natural way and that the evaluation map ϵ turns out to be F-smooth as well; actually, the basic curves of $\boldsymbol{S}$ are the curves $c : \boldsymbol{I}_c \to \boldsymbol{S}$, such that the induced map between smooth manifolds $c^*(\epsilon) : \boldsymbol{I}_c \times \boldsymbol{M} \to \boldsymbol{N}$ is smooth (see [225]).

(3) Next, let us consider a smooth double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$.

We recall (see [225]) that an open subset $\boldsymbol{U} \subset \boldsymbol{F}$ is said to be "tubelike" if it is of the type $\boldsymbol{U} = p^{-1}(\underline{\boldsymbol{U}})$, where $\underline{\boldsymbol{U}} \subset \boldsymbol{B}$ is an open submanifold.

We recall (see [225]) that a *fibrewise smooth tubelike section* is defined to be a local section $\phi : \boldsymbol{F} \to \boldsymbol{G}$, whose domain $\boldsymbol{D}[\phi] \subset \boldsymbol{F}$ is a tubelike subset and whose restriction to each fibre $\boldsymbol{F}_b$, with $b \in \boldsymbol{D}[\phi]$, is (global and) smooth, without any further smoothness requirement.

We denote the sheaf of *fibrewise smooth tubelike sections* and the subsheaf of *fully smooth sections* by

$$\underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G}) \qquad \text{and} \qquad \text{tub}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G}).$$

We recall (see [225]) that an *F-smooth system of smooth sections* is defined to be a family $\{\phi\}$ of fibrewise smooth tubelike sections $\phi : \boldsymbol{F} \to \boldsymbol{G}$, which is parametrised by the sections of a fibred set $\zeta : \boldsymbol{S} \to \boldsymbol{B}$.

Thus, an F-smooth system of smooth sections is characterised by a 3-plet

$$(\boldsymbol{S}, \zeta, \epsilon), \quad \text{with } \zeta : \boldsymbol{S} \to \boldsymbol{B}, \ \epsilon : \boldsymbol{S} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{G},$$

where the "*evaluation map*" ϵ provides, for each local section $\sigma : \boldsymbol{B} \to \boldsymbol{S}$, the selected fibrewise smooth tubelike section

$$\breve{\sigma} := \epsilon_\sigma : \boldsymbol{F} \to \boldsymbol{G}.$$

We denote by $\underline{\text{sec}}(\boldsymbol{B}, \boldsymbol{S})$ the sheaf of local sections $\widehat{\sigma} : \boldsymbol{B} \to \boldsymbol{S}$ without any smoothness requirement. Moreover, we denote by $\underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G})$ the subsheaf of fibrewise smooth tubelike sections selected by the system.

The system $(\boldsymbol{S}, \zeta, \epsilon)$ is said to be "injective" if the induced sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \underline{\text{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\text{tub}}(\boldsymbol{F}, \boldsymbol{G}) : s \mapsto \breve{s}$$

is injective, that is if the induced sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \underline{\text{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) : s \mapsto \breve{s}$$

is bijective. In this case, we denote the inverse map by

$$(\epsilon_{\boldsymbol{S}})^{-1} : \underline{\text{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \to \underline{\text{sec}}(\boldsymbol{B}, \boldsymbol{S}) : \phi \mapsto \widehat{\phi}.$$

We can prove that the set of parameters $\boldsymbol{S}$ turns out to be an F-smooth space in a natural way, and that the projection ζ and the evaluation map ϵ turn out to be F-smooth as well; actually, the basic curves of $\boldsymbol{S}$ are the curves $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{S}$, such that their base projection $c : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{B}$ is smooth and the induced section between smooth manifolds $c^*(\epsilon) : c^*(\boldsymbol{F}) \to c^*(\boldsymbol{G})$ is smooth (see [225]).

We denote the subsheaf of F-smooth local sections $\widehat{\sigma} : \boldsymbol{B} \to \boldsymbol{S}$ and the subsheaf of fully smooth tubelike sections $\sigma : \boldsymbol{B} \to \boldsymbol{S}$ selected by the system by

$$\mathrm{sec}(\boldsymbol{B}, \boldsymbol{S}) \subset \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \qquad \text{and} \qquad \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) \subset \mathrm{tub}(\boldsymbol{F}, \boldsymbol{G}).$$

We have seen that, by definition, the system provides a sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}) : \widehat{\sigma} \mapsto \breve{\sigma}.$$

Indeed, we can prove that $\breve{\sigma}$ is fully smooth if and only if $\widehat{\sigma}$ is F-smooth (see [225]).

Hence, the above sheaf morphism $\epsilon_{\boldsymbol{S}} : \underline{\mathrm{sec}}(\boldsymbol{B}, \boldsymbol{S}) \to \underline{\mathrm{tub}}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G})$ restricts to a sheaf morphism

$$\epsilon_{\boldsymbol{S}} : \mathrm{sec}(\boldsymbol{B}, \boldsymbol{S}) \to \mathrm{tub}_{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{G}).$$

22.2 The F-smooth Sectional Quantum Space

In this Section, we introduce the "*sectional quantum space*" in terms of the notions of "F-smooth system of smooth sections" of a double fibred manifold.

We regard the quantum bundle $\boldsymbol{Q}$ as a smooth double fibred manifold $\boldsymbol{Q} \to \boldsymbol{E} \to \boldsymbol{T}$ (see Sect. 14). Then, for each $\mathrm{t} \in \boldsymbol{T}$, we define the *sectional quantum space* $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$, which consists of all global smooth sections with compact support $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$. Clearly, this space inherits in a natural way the complex vector structure of $\boldsymbol{Q}_{\mathrm{t}}$.

Moreover, we define the *sectional quantum space* $\widehat{\boldsymbol{Q}}$ to be the disjoint union $\widehat{\boldsymbol{Q}} := \bigsqcup_{\mathrm{t} \in \boldsymbol{T}} \widehat{\boldsymbol{Q}}_{\mathrm{t}}$.

By definition, $\widehat{\boldsymbol{Q}}$ is equipped with a surjective fibring $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$. Hence, $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be a complex vector fibred set.

We can regard $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ as an injective F-smooth system of smooth maps $\boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ and $\widehat{\boldsymbol{Q}}$ as an injective F-smooth system of smooth sections $\boldsymbol{E} \to \boldsymbol{Q}$. This system yields a natural sheaf isomorphism $\mathrm{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi$ between the sheaf of local F-smooth sections $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$ and the sheaf of smooth tubelike sections with fibrewise compact support $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$.

Let us consider the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ (see Postulate Q.1 and Note 14.4.5).

We start by introducing the "regular quantum sections" $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$.

Definition 22.2.1 We define the following objects.

(1) An *almost regular quantum section* is defined to be a local quantum section (see [225])

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q},$$

(a) whose domain is a "tubelike open submanifold" $\boldsymbol{D}[\Psi] \subset \boldsymbol{E}$,

(b) for each $\mathrm{t} \in \underline{\boldsymbol{D}}[\Psi] \subset \boldsymbol{T}$, its restriction $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ is smooth (no further "transversal" smoothness assumption is required),

(c) for each $\mathrm{t} \in \underline{\boldsymbol{D}}[\Psi] \subset \boldsymbol{T}$, its restriction $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ has compact support.

(2) A *regular quantum section* is defined to be an almost regular quantum section

$$\Psi : \boldsymbol{E} \to \boldsymbol{Q},$$

which is fully smooth.

We denote the subsheaf of almost regular quantum sections and the subsheaf of regular quantum sections, respectively, by

$$\underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \qquad \text{and} \qquad \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \subset \underline{\mathrm{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}). \quad \square$$

Then, we define the "sectional quantum space" $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ at a given time $\mathrm{t} \in \boldsymbol{T}$.

Definition 22.2.2 For each $\mathrm{t} \in \boldsymbol{T}$, we define the *sectional quantum space at* t, to be the subset

$$\widehat{\boldsymbol{Q}}_{\mathrm{t}} := \mathrm{cpt}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}) \subset \mathrm{Sec}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}})$$

consisting of all *global (fibrewise) smooth sections* $\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}$ *with compact support*. □

Note 22.2.3 By definition, every element $\widehat{\Psi}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$ is a global smooth section with compact support

$$\widehat{\Psi}_{\mathrm{t}} = \big(\Psi_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}}\big)\,, \quad \text{for a certain } \mathrm{t} \in \boldsymbol{T}.$$

Thus, for each $\mathrm{t} \in \boldsymbol{T}$, we have the natural evaluation map

$$\epsilon_{\mathrm{t}} : \widehat{\boldsymbol{Q}}_{\mathrm{t}} \times \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}} : (\widehat{\Psi}_{\mathrm{t}}, \mathrm{e}_{\mathrm{t}}) \mapsto \Psi_{\mathrm{t}}(\mathrm{e}_{\mathrm{t}}). \quad \square$$

Note 22.2.4 For each $\mathrm{t} \in \boldsymbol{T}$, the set $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ turns out to be a complex vector space, by setting (see the above Note 22.2.3), for each $\widehat{\Psi}_{\mathrm{t}},\ \widehat{\acute{\Psi}}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$ and $k \in \mathbb{C}$,

$$\widehat{\Psi}_{\mathrm{t}} + \widehat{\acute{\Psi}}_{\mathrm{t}} := \widehat{\Psi_{\mathrm{t}} + \acute{\Psi}_{\mathrm{t}}} \qquad \text{and} \qquad k\,\widehat{\Psi}_{\mathrm{t}} := \widehat{k\,\Psi_{\mathrm{t}}}\,.$$

Proof. The sum of two smooth sections with compact support is smooth and with compact support. The product of a complex number times a smooth section with compact support is smooth and with compact support. □

The sectional quantum space $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ at $\mathrm{t} \in \boldsymbol{T}$ can be regarded as an *F-smooth system of smooth maps*, hence it turns out to be naturally equipped with an *F-smooth structure*.

In order to analyse this feature, the reader can refer to [225].

Proposition 22.2.5 *For each* $\mathrm{t} \in \boldsymbol{T}$, *the pair* $(\widehat{\boldsymbol{Q}}_{\mathrm{t}}, \epsilon_{\mathrm{t}})$ *turns out to be an F-smooth system of smooth maps (see [225]).*

Then, for each $\mathrm{t} \in \boldsymbol{T}$, *the set* $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ *turns out to be an F-smooth space (see [225]).*

The basic F-smooth curves of $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ *are all curves* $c : \boldsymbol{I}_c \to \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, *such that the induced curves*

$$c_{\boldsymbol{E}_{\mathrm{t}}} : \boldsymbol{I}_c \times \boldsymbol{E}_{\mathrm{t}} \to \boldsymbol{Q}_{\mathrm{t}} : (\lambda, \mathrm{e}) \mapsto (c(\lambda))\,(\mathrm{e})$$

be smooth. □

Further, we analyse the "sectional quantum fibred space" $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$.

Definition 22.2.6 We define the *sectional quantum space* to be the disjoint union

$$\widehat{\boldsymbol{Q}} := \bigsqcup_{\mathrm{t} \in \boldsymbol{T}} \widehat{\boldsymbol{Q}}_{\mathrm{t}}. \quad \square$$

Note 22.2.7 By definition, the sectional quantum space $\widehat{\boldsymbol{Q}}$ is naturally equipped with the natural surjective fibring

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T} : \widehat{\Psi}_{\mathrm{t}} \to \mathrm{t},$$

which makes $\widehat{\boldsymbol{Q}}$ a fibred set over $\boldsymbol{T}$.

We have the natural evaluation map

$$\epsilon : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q} : (\widehat{\Psi}, \mathrm{e}) \mapsto \Psi(\mathrm{e}). \quad \square$$

The sectional quantum space can be regarded as an *F-smooth system*, hence it turns out to be naturally equipped with an *F-smooth structure* (see [225]).

Proposition 22.2.8 *The 3-plet* $(\widehat{\boldsymbol{Q}}, \widehat{t}, \epsilon)$ *turns out to be an F-smooth system of sections (see [225]).*

Then, the set $\widehat{\boldsymbol{Q}}$ *turns out to be an F-smooth space.*

The basic F-smooth curves of $\widehat{\boldsymbol{Q}}$ *are all curves* $\widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \widehat{\boldsymbol{Q}}$, *such that the induced curves*

$$c := \widehat{t} \circ \widehat{c} : \boldsymbol{I}_{\widehat{c}} \to \boldsymbol{T} \qquad \text{and} \qquad \widehat{c}^{\,*}(\epsilon) : c^*(\boldsymbol{E}) \to c^*(\boldsymbol{Q})$$

be smooth.

Moreover, the maps

$$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T} \qquad \text{and} \qquad \epsilon : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q}$$

turn out to be F-smooth (see [225]).

Thus, $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an F-smooth fibred space. □

Note 22.2.9 We denote the sheaves of local sections $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$, *without any smoothness requirement*, and the subsheaf of local *F-smooth* sections $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$, respectively, by

$$\underline{\text{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \qquad \text{and} \qquad \text{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \subset \underline{\text{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}).$$

By definition of sectional quantum space (see Definition 22.2.6), we have natural mutually inverse sheaf isomorphism

$$\underline{\text{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \underline{\text{reg}}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi \quad \text{and} \quad \underline{\text{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \to \underline{\text{sec}}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi}\,.$$

Moreover, an almost regular quantum section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ is smooth if an only if $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$ is F-smooth.

Hence, the above sheaf isomorphism restricts to mutually inverse sheaf isomorphisms

$$\text{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \text{reg}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi \quad \text{and} \quad \text{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \text{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi}\,. \quad \square$$

Remark 22.2.10 In the present context of F-smooth systems of smooth sections, where the base space is a 1 dimensional affine space $\boldsymbol{B} = \boldsymbol{T}$, checking that a (local) section $\widehat{\Psi} : \boldsymbol{T} \to \widehat{\boldsymbol{Q}}$ be F-smooth reduces just to check that the induced fibrewise smooth section $\Psi : \boldsymbol{F} \to \boldsymbol{G}$ be fully smooth (see [225]).

So, in the present case, the Theorem which states the sheaf isomorphism

$$\text{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \text{tub}(\boldsymbol{F}, \boldsymbol{Q}),$$

reduces just to the definition of $\text{sec}(\boldsymbol{T}, \widehat{\boldsymbol{Q}})$ without the need to perform further independent check (see [225]). □

22.3 The F-smooth Sectional Quantum Bundle

Here, for the sake of simplicity, we suppose that the spacetime fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ be a bundle with contractible fibre; as a consequence, both the spacetime bundle $t : \boldsymbol{E} \to \boldsymbol{T}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ turn out to be trivial (see Remark 14.2.3).

For each $\mathrm{t} \in \boldsymbol{T}$, the choice of a quantum basis b_{t} on the spacetime fibre $\boldsymbol{E}_{\mathrm{t}}$ makes the sectional quantum fibre $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$ a (scaled) functional space in a rather usual way. Moreover, the sectional quantum fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an F-smooth bundle, whose type fibre is a rather usual (scaled) functional space.

But, if the spacetime fibring is not a bundle, then we cannot achieve a local F-smooth trivialisation of the sectional quantum fibred space. Moreover, we stress that, even in the case when the sectional quantum fibred set is an F-smooth bundle, there is no distinguished trivialisation.

These considerations are important for a comparison with standard Quantum Mechanics, which is essentially developed with reference to a flat spacetime and a given inertial observer.

For the sake of simplicity, *in the present context of the sectional quantum bundle*, we make the following simplified assumption.

Assumption Q.4 We assume that the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ be a bundle with contractible type fibre. □

Under the above hypothesis, in virtue of Remark 14.2.3, both the spacetime bundle $t : \boldsymbol{E} \to \boldsymbol{T}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ turn out to be trivial.

For each $\mathrm{t} \in \boldsymbol{T}$, the choice of a quantum basis b_{t} allows us to regard the set $\widehat{\boldsymbol{Q}}_{\mathrm{t}}$, with $\mathrm{t} \in \boldsymbol{T}$, as a scaled functional space in a rather usual way.

Note 22.3.1 For each $\mathrm{t} \in \boldsymbol{T}$, the choice of a global smooth quantum basis $\mathrm{b}_{\mathrm{t}} : \boldsymbol{E}_{\mathrm{t}} \to \mathbb{L}^{-3/2} \otimes \boldsymbol{Q}_{\mathrm{t}}$ yields a bijection (see Definition 22.2.2)

$$\widehat{\boldsymbol{Q}}_{\mathrm{t}} \to \mathrm{cpt}(\boldsymbol{E}_t, \mathbb{L}^{3/2} \otimes \mathbb{C}) : \widehat{\Psi}_{\mathrm{t}} \mapsto \psi_{\mathrm{t}},$$

where $\mathrm{cpt}(\boldsymbol{E}_t, \mathbb{L}^{3/2} \otimes \mathbb{C})$ denotes the space of global (fibrewise) smooth scaled functions with compact support. □

Next, we discuss the local and global trivialisation of the sectional quantum space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$.

Proposition 22.3.2 *Being $t : \boldsymbol{E} \to \boldsymbol{T}$ a bundle, the sectional quantum fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an* F-smooth bundle. *More precisely, we can achieve a local F-smooth bundle trivialisation of $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ in the following way.*

Let us consider two bundle charts of the bundles $t : \boldsymbol{E} \to \boldsymbol{T}$ and $\pi : \boldsymbol{Q} \to \boldsymbol{E}$

$$\Phi_{\boldsymbol{E}} : t^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathbb{E} \qquad \text{and} \qquad \Phi_{\boldsymbol{Q}} : \pi^{-1}\big(t^{-1}(\boldsymbol{U})\big) \to t^{-1}(\boldsymbol{U}) \times \mathbb{Q},$$

where $\boldsymbol{U} \subset \boldsymbol{T}$ is a (sufficiently small) open subset and $\mathbb{E}$ and $\mathbb{Q}$ are type fibres.

Then, for each $\mathrm{t} \in \boldsymbol{U}$, we obtain the smooth bijections

$$\Phi_{\boldsymbol{E}_{\mathrm{t}}} : \boldsymbol{E}_{\mathrm{t}} \to \mathbb{E} \qquad \text{and} \qquad \Phi_{\boldsymbol{Q}_{\mathrm{t}}} : \boldsymbol{Q}_{\mathrm{t}} \to \mathbb{E} \times \mathbb{Q}$$

and, for each $\widehat{\Psi}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, the map

$$\Psi_{\mathrm{t}} : \mathbb{E} \to \mathbb{Q}$$

given by the composition of maps

$$\mathbb{E} \xrightarrow{\Phi^{-1}_{\boldsymbol{E}_{\mathrm{t}}}} \boldsymbol{E}_{\mathrm{t}} \xrightarrow{\Psi_{\mathrm{t}}} \boldsymbol{Q}_{\mathrm{t}} \xrightarrow{\Phi_{\boldsymbol{Q}_{\mathrm{t}}}} \mathbb{E} \times \mathbb{Q}.$$

Indeed, being $\Phi_{\boldsymbol{E}_{\mathrm{t}}} : \boldsymbol{E}_{\mathrm{t}} \to \mathbb{E}$ *and* $\Phi_{\boldsymbol{Q}_{\mathrm{t}}} : \boldsymbol{Q}_{\mathrm{t}} \to \mathbb{E} \times \mathbb{Q}$ *diffeomorphisms, the section* $\Psi_{\mathrm{t}} : \mathbb{E} \to \mathbb{E} \times \mathbb{Q}$ *turns out to have compact support.*

Hence, we obtain the F-smooth bundle chart of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$

$$\widehat{\Phi} : \widehat{t}^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \widehat{\mathbb{Q}},$$

where, the type fibre

$$\widehat{\mathbb{Q}} := \mathrm{cpt}(\mathbb{E}, \mathbb{Q})$$

is the space of all smooth maps $\mathbb{E} \to \mathbb{Q}$ *with compact support.*

For each quantum basis $\mathrm{b} : t^{-1}(\boldsymbol{U}) \subset \boldsymbol{E} \to \boldsymbol{Q}$, *we obtain the bundle chart of the bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$

$$\Phi_{\boldsymbol{Q}} : \pi^{-1}\left(t^{-1}(\boldsymbol{U})\right) \to t^{-1}(\boldsymbol{U}) \times (\mathbb{L}^{3/2} \otimes \mathbb{C}).$$

Hence, we obtain the F-smooth bundle chart of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$

$$\widehat{\Phi} : \widehat{t}^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \widehat{\mathbb{Q}},$$

where, the type fibre

$$\widehat{\mathbb{Q}} := \mathrm{cpt}(\mathbb{E}, \mathbb{L}^{3/2} \otimes \mathbb{C})$$

is the space of all smooth scaled functions $\mathbb{E} \to \mathbb{L}^{3/2} \otimes \mathbb{C}$ *with compact support.* □

Corollary 22.3.3 *Being the quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ *globally trivialisable, the F-smooth sectional quantum bundle* $\widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ *turns out to be globally trivialisable by a global F-smooth bundle isomorphism*

$$\widehat{\Phi} : \widehat{\boldsymbol{Q}} \to \boldsymbol{E} \times \widehat{\mathbb{Q}},$$

where, the type fibre

$$\widehat{\mathbb{Q}} := \mathrm{cpt}(\mathbb{E}, \mathbb{L}^{3/2} \otimes \mathbb{C})$$

is the space of all smooth scaled functions $\mathbb{E} \to \mathbb{L}^{3/2} \otimes \mathbb{C}$ *with compact support.* □

Remark 22.3.4 According to Proposition 22.3.2, the sectional quantum F-smooth fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ turns out to be an F-smooth bundle only if the fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ is a bundle.

We stress that, even in this case, the sectional quantum F-smooth fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ has no distinguished local splittings into time and a functional type fibre. Such an F-smooth trivialisation depends on a bundle trivialisation of the spacetime fibred space and on a local quantum basis.

If the spacetime fibred space is a bundle, then suitable classical observers might provide a smooth splitting of the spacetime fibred space, hence an F-smooth splitting of the sectional quantum fibred space. But, we stress that such a F-smooth bundle splitting would be observer dependent. □

Remark 22.3.5 We cannot apply the Theorem on globally trivialisable bundles with contractible base space to the F-smooth sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$, because it is not smooth and finite dimensional. □

22.4 The Pre-Hilbert Sectional Quantum Bundle

The hermitian quantum metric h_η (see Proposition 14.3.1) equips the fibres of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ with a *pre-Hilbert metric* $\langle \, | \, \rangle : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \widehat{\boldsymbol{Q}} \to \mathbb{C}$, via integration on the fibres of spacetime.

Then, the F-smooth sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ can be made into a true *Hilbert bundle* by a completion procedure; however, a detailed analysis of such a completion is out of the scope of this book.

Let us recall the mutually inverse sheaf isomorphisms (see Note 22.2.9)

$$\sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi \quad \text{and} \quad \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi} \,.$$

Proposition 22.4.1 *For each* $\mathrm{t} \in \boldsymbol{T}$, *the* η*-hermitian quantum metric (see Definition 14.5.1)*

$$\mathrm{h}_\eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{C} \otimes \Lambda^3 V^* \boldsymbol{E} : (\Psi, \acute{\Psi}) \mapsto \bar{\psi} \, \acute{\psi} \, \eta$$

equips the complex vector fibres $\widehat{\boldsymbol{Q}}_\mathrm{t}$ *of the sectional quantum bundle* $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ *with the scalar product*

$$\langle \, | \, \rangle_\mathrm{t} : \widehat{\boldsymbol{Q}}_\mathrm{t} \times \widehat{\boldsymbol{Q}}_\mathrm{t} \to \mathbb{C} : (\widehat{\Psi}_\mathrm{t}, \widehat{\acute{\Psi}}_\mathrm{t}) \mapsto \langle \widehat{\Psi}_\mathrm{t} \mid \widehat{\acute{\Psi}}_\mathrm{t} \rangle_\mathrm{t} := \int_{\boldsymbol{E}_\mathrm{t}} \mathrm{h}_\mathrm{t}(\Psi_\mathrm{t}, \acute{\Psi}_\mathrm{t}) \, \eta_\mathrm{t} .$$

With reference to a spacetime chart (x^λ) *and to a quantum basis* b, *we have the coordinate expression*

$$\langle \widehat{\Psi}_\mathrm{t} \mid \widehat{\acute{\Psi}}_\mathrm{t} \rangle_\mathrm{t} = \int_{\boldsymbol{E}_\mathrm{t}} \bar{\psi}_\mathrm{t} \, \acute{\psi}_\mathrm{t} \sqrt{|g|} \, \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3 .$$

Moreover, for each $\widehat{\Psi}_{t\,1}$, $\widehat{\Psi}_{t\,2}$, $\widehat{\acute{\Psi}}_{t} \in \widehat{\boldsymbol{Q}}_{t}$ *and* $k \in \mathbb{C}$, *the following equalities hold*

$$\langle \widehat{\Psi}_{t\,1} + \widehat{\Psi}_{t\,2} \mid \widehat{\acute{\Psi}} \rangle = \langle \widehat{\Psi}_{t\,1} \mid \widehat{\acute{\Psi}}_{t} \rangle + \langle \widehat{\Psi}_{t\,2} \mid \widehat{\acute{\Psi}}_{t} \rangle,$$
$$\langle \widehat{\acute{\Psi}}_{t} \mid \widehat{\Psi}_{t\,1} + \widehat{\Psi}_{t\,2} \rangle = \langle \widehat{\acute{\Psi}}_{t} \mid \widehat{\acute{\Psi}}_{t\,1} \rangle + \langle \widehat{\acute{\Psi}}_{t} \mid \widehat{\acute{\Psi}}_{t\,2} \rangle,$$
$$\langle k\,\widehat{\Psi}_{t} \mid \widehat{\acute{\Psi}}_{t} \rangle = \bar{k}\,\langle \widehat{\Psi}_{t} \mid \widehat{\acute{\Psi}}_{t} \rangle, \qquad \langle \widehat{\Psi}_{t} \mid k\,\widehat{\acute{\Psi}}_{t} \rangle = k\,\langle \widehat{\Psi}_{t} \mid \widehat{\acute{\Psi}}_{t} \rangle,$$
$$\langle \widehat{\Psi}_{t} \mid \widehat{\acute{\Psi}}_{t} \rangle = \overline{\langle \widehat{\acute{\Psi}}_{t} \mid \widehat{\Psi}_{t} \rangle},$$
$$\widehat{\Psi}_{t} \neq 0 \qquad \Rightarrow \qquad \langle \widehat{\Psi}_{t} \mid \widehat{\Psi}_{t} \rangle \in I\!R^{+}.$$

Thus, the above scale product $\langle\,|\,\rangle_{t}$ *makes the complex vector space* $\widehat{\boldsymbol{Q}}_{t}$ *a* pre-Hilbert vector space.

Indeed, the scalar product $\langle\,|\,\rangle_{t} : \widehat{\boldsymbol{Q}}_{t} \times \widehat{\boldsymbol{Q}}_{t} \to \mathbb{C}$ *turns out to be an F-smooth map.* □

22.5 Quantum Operators

For each projectable special phase function (defined on a tubelike domain over spacetime), the associated quantum operator on quantum sections (see Sect. 22.5) can be regarded as a complex linear operator on the pre-Hilbert fibres of the sectional quantum bundle (Chap. 22).

Indeed, if a special phase function has constant time component, then the above associated pre-Hilbert complex linear operator turns out to be symmetric with respect to the pre-Hilbert scalar product. After a completion procedure of the pre-Hilbert sectional quantum bundle, we could prove the hermitianity of this operator; but such an analysis is out of the scope of the present book.

Thus, the special phase functions with constant time component turn out to be the *quantisable functions* in our approach.

We start by observing that each spacelike quantum differential operator O acting on the sections of the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ yields in a natural way an operator $\widehat{\mathrm{O}}$ acting on the F-smooth sections of the sectional quantum bundle $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ and an F-smooth fibred morphism over $\boldsymbol{T}$ of the sectional quantum bundle itself.

Lemma 22.5.1 *Each spacelike quantum differential operator (see Definition 20.1.1)*

$$\mathrm{O} : \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})$$

maps quantum sections with fibrewise compact support into quantum sections with fibrewise compact support, hence it factorises through a spacelike differential operator, denoted by the same symbol (see Definition 22.2.1)

$$\mathrm{O} : \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}),$$

according to the following commutative diagram

$$\begin{array}{ccc} \sec(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathrm{O}} & \sec(\boldsymbol{E}, \boldsymbol{Q}) \\ \cup \uparrow & & \uparrow \cup \\ \operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) & \xrightarrow{\mathrm{O}} & \operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) \end{array} \quad . \qquad \square$$

Lemma 22.5.2 *For each spacelike quantum differential operator*

$$\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}),$$

in virtue of the above Lemma 22.5.1 and of Note 22.2.9, we obtain the differential operator $\widehat{\mathrm{O}}$ given by the sheaf morphism

$$\widehat{\mathrm{O}} : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \widehat{\Psi} \mapsto \widehat{\mathrm{O}}(\widehat{\Psi}) := \widehat{\mathrm{O}(\Psi)}\,.$$

Therefore, the spacelike quantum differential operator $\mathrm{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$ *yields the F-smooth fibred morphism over $\boldsymbol{T}$, denoted by the same symbol (see [225])*

$$\widehat{\mathrm{O}} : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}},$$

uniquely defined, for each $\widehat{\Psi}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, with $\mathrm{t} \in \boldsymbol{T}$, by the equality

$$\widehat{\mathrm{O}}\left(\widehat{\Psi}_{\mathrm{t}}\right) = \widehat{\mathrm{O}_{\mathrm{t}}(\Psi_{\mathrm{t}})}\,, \qquad \text{where } \ \Psi_t \in \operatorname{reg}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}).$$

Proof. The proof follows, in virtue of Note 22.2.9, from the bijection $\operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}})$. □

Next, we recall our "*covariant correspondence principle*" expressed by Theorem 20.1.9, which provides the spacelike differential operator

$$\mathrm{O}[f] : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}) : \Psi \mapsto \mathfrak{i}\,\left(Y_\eta[f].\Psi - \mathrm{S}[f].\Psi\right)$$

associated with every projectable special phase function $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$.

Then, we discuss the induced differential operator

$$\widehat{\mathrm{O}}[f] : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}).$$

Proposition 22.5.3 *According to Lemma 22.5.2, for each projectable special phase function $f \in \operatorname{pro}\operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R})$, we obtain the $\mathbb{C}$-linear fibred morphism over $\boldsymbol{T}$*

$$\widehat{\mathrm{O}}[f] : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}},$$

uniquely defined, for each $\widehat{\Psi}_{\mathrm{t}} \in \widehat{\boldsymbol{Q}}_{\mathrm{t}}$, with $\mathrm{t} \in \boldsymbol{T}$, by the equality

$$\widehat{\mathrm{O}}[f]\left(\widehat{\Psi}_{\mathrm{t}}\right) = \widehat{\mathrm{O}_{\mathrm{t}}[f](\Psi_{\mathrm{t}})}, \quad \textit{where} \;\; \Psi_t \in \mathrm{reg}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}).$$

Thus, for each $\mathrm{t} \in \boldsymbol{T}$, *the fibred morphism* $\widehat{\mathrm{O}}[f] : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}}$ *restricts to a* $\mathbb{C}$*-linear* operator

$$\widehat{\mathrm{O}}[f]_{\mathrm{t}} : \widehat{\boldsymbol{Q}}_{\mathrm{t}} \to \widehat{\boldsymbol{Q}}_{\mathrm{t}}$$

acting on the pre-Hilbert fibres of the sectional quantum bundle. □

Next, we prove that, under the hypothesis that f be time preserving, the quantum operator $\widehat{\mathrm{O}}[f]$ turns out to be symmetric.

Lemma 22.5.4 *Given an observer* o, *let us consider the observed laplacian* $\Delta[o]$ *(see Sect. 16.3) and the induced operator* $\widehat{\Delta[o]} : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}}$.

Then, $\widehat{\Delta[o]}$ *is* symmetric, *with respect to the fibred scalar product* $\langle \mid \rangle$ *(Chap. 22), i.e.*

$$\langle \widehat{\Delta[o]}\,\widehat{\Phi} \mid \widehat{\Psi}\,\rangle = \langle \widehat{\Phi} \mid \widehat{\Delta[o]}\,\widehat{\Psi}\,\rangle, \quad \textit{for each} \;\; \Phi,\ \Psi \in \mathrm{reg}(\boldsymbol{E}, \boldsymbol{Q}).$$

Proof. We have locally the equality

$$\begin{aligned}
&\mathrm{h}(\Delta[o]\Phi, \Psi)_0 - \mathrm{h}(\Phi, \Delta[o]\Psi)_0 \\
&\quad = \left(G_0^{ij}\,(\bar{\nabla}_i\bar{\nabla}_j + K_i{}^h{}_j\,\bar{\nabla}_h)\,\bar{\phi}\right)\,\psi - \bar{\phi}\,\left(G_0^{ij}\,(\nabla_i\nabla_j + K_i{}^h{}_j\,\nabla_h)\,\psi\right) \\
&\quad = G_0^{ij}\,\left(\partial_i(\bar{\nabla}_j\bar{\phi}\,\psi) + \mathrm{i}\,A_i\bar{\nabla}_j\bar{\phi}\,\psi + K_i{}^h{}_j\,\bar{\nabla}_h\bar{\phi}\,\psi - \bar{\nabla}_j\bar{\phi}\,\partial_i\psi\right) \\
&\qquad - G_0^{ij}\,\left(\partial_i(\bar{\phi}\,\nabla_j\psi) - \mathrm{i}\,A_i\,\bar{\phi}\,\nabla_j\psi + K_i{}^h{}_j\,\bar{\phi}\,\nabla_h\psi - \partial_i\bar{\phi}\,\nabla_j\psi\right) \\
&\quad = G_0^{ij}\,\left(\partial_i(\bar{\nabla}_j\bar{\phi}\,\psi) + K_i{}^h{}_j\,\bar{\nabla}_h\bar{\phi}\,\psi - \partial_i(\bar{\phi}\,\nabla_j\psi) - K_i{}^h{}_j\,\bar{\phi}\,\nabla_h\psi\right) \\
&\quad = \frac{\partial_i(G_0^{ij}\,\bar{\nabla}_j\bar{\phi}\,\psi\,\sqrt{|g|})}{\sqrt{|g|}} - \frac{\partial_i(G_0^{ij}\,\bar{\phi}\,\nabla_j\psi\,\sqrt{|g|})}{\sqrt{|g|}},
\end{aligned}$$

which is expressed by two divergence terms.

Then, for each $\mathrm{t} \in \boldsymbol{T}$, by taking into account a partition of unity and the divergence theorem, we obtain

$$\int_{\boldsymbol{E}_{\mathrm{t}}} \left(\mathrm{h}_\eta(\Delta[o]\Phi, \Psi)_0 - \mathrm{h}_\eta(\Phi, \Delta[o]\Psi)_0\right) = 0. \quad \square$$

Theorem 22.5.5 *For each* $\mathrm{t} \in \boldsymbol{T}$ *and each* $f \in \mathrm{tim\,spe}(J_1\boldsymbol{E}, I\!R)$, *defined on a tube-like domain over* $\boldsymbol{E}$ *containing the fibre* $(J_1\boldsymbol{E})_{\mathrm{t}}$, *the operator*

$$\widehat{\mathrm{O}}[f]_{\mathrm{t}} : \widehat{\boldsymbol{Q}}_{\mathrm{t}} \to \widehat{\boldsymbol{Q}}_{\mathrm{t}}$$

turns out to be symmetric, *with respect to the fibred scalar product* $\langle \mid \rangle$ *(see Proposition 22), i.e.*

$$\langle \widehat{\mathrm{O}}[f]_{\mathrm{t}}(\widehat{\Phi}) \mid \widehat{\Psi} \rangle = \langle \widehat{\Phi} \mid \widehat{\mathrm{O}}[f]_{\mathrm{t}}(\widehat{\Psi}) \rangle, \quad \textit{for each } \Phi, \Psi \in \operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}).$$

Proof. We have locally

$$\begin{aligned}
&\mathrm{h}\big(\widehat{\mathrm{O}}[f](\Phi), \Psi\big) - \mathrm{h}\big(\Phi, \widehat{\mathrm{O}}[f](\Psi)\big) \\
&= \psi \left(\breve{f} + \tfrac{1}{2}\,\mathrm{i}\, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} + \mathrm{i} f^i \,\bar{\nabla}_i - \tfrac{1}{2} f^0 \,\bar{\Delta}_0 \right) \bar{\phi} - \bar{\phi} \left(\breve{f} - \tfrac{1}{2}\,\mathrm{i}\, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i} f^i \,\nabla_i - \tfrac{1}{2} f^0 \,\Delta_0 \right) \psi \\
&= \psi \,(\tfrac{1}{2}\,\mathrm{i}\, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} + \mathrm{i} f^i \,\partial_i - \tfrac{1}{2} f^0 \,\bar{\Delta}_0)\, \bar{\phi} - \bar{\phi}\, (-\tfrac{1}{2}\,\mathrm{i}\, \frac{\partial_i (f^i \sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i} f^i \,\partial_i - \tfrac{1}{2} f^0 \,\Delta_0)\, \psi \\
&= \mathrm{i}\, \frac{\partial_i (f^i \,\bar{\phi}\, \psi \sqrt{|g|})}{\sqrt{|g|}} - \psi\, \tfrac{1}{2} f^0 \,\bar{\Delta}_0 \,\bar{\phi} + \bar{\phi}\, \tfrac{1}{2} f^0 \,\Delta_0 \,\psi .
\end{aligned}$$

Then, for each $\mathrm{t} \in \boldsymbol{T}$, by taking into account a partition of unity, the above Lemma 22.5.4 and the divergence theorem, we obtain

$$\int_{\boldsymbol{E}_{\mathrm{t}}} \big(\mathrm{h}_\eta \,(\mathrm{O}[f](\Phi), \Psi) - \mathrm{h}_\eta \,(\Phi, \mathrm{O}[f](\Psi)) \big) = 0. \quad \square$$

22.6 Schrödinger Connection

The Schrödinger operator can be regarded in a natural way as the covariant differential operator with respect to an F-smooth connection of the F-smooth fibred space $\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$.

Indeed, the spacelike component of the Schrödinger operators behave as the symbol of this "infinite dimensional connection".

Let us recall the Schrödinger operator (see Theorem 17.6.5)

$$\mathrm{S} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$$

with coordinate expression

$$\begin{aligned}
\mathrm{S}(\Psi) = \big(&\partial_0 \psi - \tfrac{1}{2}\,\mathrm{i}\, G_0^{ij} \,\partial_{ij} \psi - (A_0^j + \tfrac{1}{2}\,\mathrm{i}\, \frac{\partial_i (G_0^{ij} \sqrt{|g|})}{\sqrt{|g|}})\, \partial_j \psi \\
&+ \tfrac{1}{2} \left(\frac{\partial_0 \sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i (A_0^i \sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\, \alpha_0 \right) \mathrm{b}.
\end{aligned}$$

Lemma 22.6.1 *The Schrödinger operator can be regarded as an F-smooth operator*

$$\widehat{\mathrm{S}} : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}}). \quad \square$$

Then, we can easily prove the following result.

Proposition 22.6.2 *There exists a unique F-smooth connection (see [225])*

$$Ш : \widehat{\boldsymbol{Q}} \to \mathbb{T}^* \otimes T\widehat{\boldsymbol{Q}},$$

whose associated covariant differential

$$\nabla[Ш] : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T},\ \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}})$$

is just the Schrödinger operator.

Proof. For the proof, see [225]. □

Remark 22.6.3 The spacelike differential operator

$$\psi \mapsto -\tfrac{1}{2}\,\mathrm{i}\,G_0^{ij}\,\partial_{ij}\psi - (A_0^j + \tfrac{1}{2}\,\mathrm{i}\,\frac{\partial_i(G_0^{ij}\sqrt{|g|})}{\sqrt{|g|}})\,\partial_j\psi \\ + \tfrac{1}{2}\left(\frac{\partial_0\sqrt{|g|}}{\sqrt{|g|}} - \frac{\partial_i(A_0^i\sqrt{|g|})}{\sqrt{|g|}} - \mathrm{i}\,\alpha_0\right)\psi$$

plays the role of the components of the F-smooth connection Ш, in the present infinite dimensional framework. □

Chapter 23
Feynman Path Integral

We have already seen that the upper quantum connection $\text{ч}^{\uparrow}$ of the upper quantum bundle $\boldsymbol{Q} \to J_1\boldsymbol{E}$ and the dynamical phase connection γ of the fibred manifold $J_1\boldsymbol{E} \to \boldsymbol{T}$ yield, in a covariant way, an *upper quantum connection over time* $\underline{\text{ч}}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^* \otimes \boldsymbol{Q}$ (see Theorem 15.3.1).

Now, we show that the above upper quantum connection over time naturally yields the "*Feynman amplitudes*" $\exp\big(\mathfrak{i} \int_{\boldsymbol{T}} (\mathcal{L} \circ j_1 s)\big)$ by integrating the equation $\nabla^{\uparrow}{}_s \Psi = 0$ of parallel transport of a quantum section Ψ along a motion s.

Thus, it is remarkable that the classical Lagrangian $\mathcal{L}$ appearing in the Feynman amplitudes arises from the upper quantum connection $\text{ч}^{\uparrow}$.

Here, we just discuss this covariant approach to the Feynman amplitudes; further developments of the path integral are out of the introductory scope of the present book.

23.1 Upper Quantum Covariant Differential Over Time

With reference to a motion s and a quantum section Ψ, we compute the upper quantum covariant differential $\nabla^{\uparrow}{}_s \Psi := ds \lrcorner (\nabla^{\uparrow} \Psi^{\uparrow}) \circ j_1 s : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{Q}$ of Ψ along s induced by the upper quantum connection (see Definition 15.2.1 and Postulate Q.2).

It is remarkable that this covariant differential involves the classical Lagrangian $\mathcal{L}$ evaluated along the motion s.

Actually, the above quantum derivative depends on the restriction of Ψ to the image of s. This fact suggests to achieve the same result by considering directly a quantum section over time $\underline{\Psi} \in \sec(\boldsymbol{T}, \boldsymbol{Q})$ (see Remark 23.1.2).

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We recall that each quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ can be regarded, by pullback, as a section of the upper quantum bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$

$$\Psi^{\uparrow} : J_1\boldsymbol{E} \to J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} : \mathrm{e}_1 \mapsto \big(\mathrm{e}_1,\ \Psi\big(t_0^1(\mathrm{e}_1)\big)\big)\,.$$

Moreover, let us recall the upper quantum connection (see Definition 15.2.1, Theorem 15.2.4 and Postulate Q.2)

$$\mathrm{Y}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$$

of the upper quantum bundle $\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to J_1\boldsymbol{E}$.

Its coordinate expression is

$$\mathrm{Y}^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^{i} \otimes \partial_{i}^{0} + A^{\uparrow}{}_{\lambda}\, d^{\lambda} \otimes \mathbb{I}^{\uparrow},$$

where

$$A^{\uparrow}[\mathrm{b}] = -\mathcal{H}[\mathrm{b}, o] + \mathcal{P}[\mathrm{b}, o] : J_1\boldsymbol{E} \to T^*\boldsymbol{E}$$

is the horizontal potential of the classical cosymplectic 2-form Ω associated with the quantum basis b and the observer o (see Theorem 10.1.4).

Furthermore, we recall that, for each quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, the upper covariant differential turns out to be a horizontal vertical valued form

$$\nabla^{\uparrow}\Psi : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes \boldsymbol{Q},$$

with coordinate expression

$$\nabla^{\uparrow}\Psi = (\partial_{\lambda}\psi - \mathrm{i}\, A^{\uparrow}{}_{\lambda}\, \psi)\, d^{\lambda} \otimes \mathrm{b}.$$

Indeed, $\nabla^{\uparrow}\Psi$ turns out to be valued in the horizontal subbundle $T^*\boldsymbol{E} \subset T^*J_1\boldsymbol{E}$ because the term of the type $(\nabla^{\uparrow}\Psi)^0_i\, d^i_0 \otimes \mathrm{b}$ disappears in virtue of the equalities $\partial^0_i\psi = 0$ and $A^0_i = 0$.

Now, let us consider a motion and a quantum section

$$s \in \sec(\boldsymbol{T}, \boldsymbol{E}) \qquad \text{and} \qquad \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}).$$

Proposition 23.1.1 *The* upper quantum covariant derivative *of* Ψ along the motion *s is the section*

$$\nabla^{\uparrow}{}_{s}\Psi := ds \,\lrcorner\, (\nabla^{\uparrow}\Psi^{\uparrow}) \circ j_1 s : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{Q},$$

with coordinate expression

$$(x^0, x^i;\ u_0 \otimes z) \circ \nabla^{\uparrow}{}_{s}\Psi = \big(x^0, s^i;\ \partial_0(\psi \circ s) - \mathrm{i}\,(\mathcal{L}_0 \circ j_1 s)\,(\psi \circ s)\big)\,,$$

where the classical lagrangian $\mathcal{L} : J_1\boldsymbol{E} \to H^*\boldsymbol{E}$ evaluated along the motion s has coordinate expression

$$\mathcal{L} \circ j_1 s = \big(\tfrac{1}{2} (G^0_{ij} \circ s)\, \partial_0 s^i\, \partial_0 s^j + (A_i \circ s)\, \partial_0 s^i + (A_0 \circ s)\big)\; u^0.$$

Proof. We have the sections

$$\nabla^{\uparrow}\Psi : J_1\boldsymbol{E} \to T^* J_1\boldsymbol{E} \otimes \boldsymbol{Q} \qquad \text{and} \qquad ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

with coordinate expressions

$$\nabla^{\uparrow}\Psi^{\uparrow} = (\partial_\lambda \psi - \mathrm{i}\, A^{\uparrow}{}_\lambda\, \psi)\, d^\lambda \otimes \mathsf{b} \qquad \text{and} \qquad ds = u^0 \otimes (\partial_0 + \partial_0 s^i\, \partial_i),$$

where

$$A^{\uparrow} = -(\tfrac{1}{2}\, G^0_{ij}\, x^i_0\, x^j_0 - A_0)\, d^0 + (G^0_{ij}\, x^j_0 + A_i)\, d^i.$$

Hence, we obtain

$$ds \lrcorner \nabla^{\uparrow}\Psi^{\uparrow} = \big((\partial_0 \psi) \circ s + (\partial_i \psi) \circ s\, \partial_0 s^i - \mathrm{i}\, \mathcal{L}_0 \circ j_1 s\big)\; u^0 \otimes \mathsf{b}. \quad \square$$

Remark 23.1.2 In Proposition 23.1.1, the section $\nabla^{\uparrow}{}_s\Psi : \boldsymbol{T} \to \mathbb{T}^* \otimes \boldsymbol{Q}$ depends on the section $\Psi : \boldsymbol{E} \to \boldsymbol{Q}$ through its restriction $\Psi_{|s(\boldsymbol{T})} : s(\boldsymbol{T}) \subset \boldsymbol{E} \to \boldsymbol{Q}$.

Indeed, this fact suggests a possible alternative approach. In fact, we might consider a section $\underline{\Psi} \in \sec(\boldsymbol{T}, \boldsymbol{Q})$ of the quantum fibred manifold over time $t \circ \pi : \boldsymbol{Q} \to \boldsymbol{T}$ and compute its covariant differential with respect to the connection $\underline{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^* \otimes T\boldsymbol{Q}^{\uparrow}$ (see Theorem 15.3.1).

Actually, the 2nd component of the above differential provides the above result (see Proposition 15.3.6). □

23.2 Feynman Amplitudes

We achieve the Feynman amplitudes $\exp\big(\mathrm{i} \int_{\boldsymbol{T}} (\mathcal{L} \circ j_1 s)\big) \in \operatorname{map}(\boldsymbol{T}, \mathbb{C})$ by integrating the equation $\nabla^{\uparrow}{}_s\Psi = 0$, in the unknown section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, for a given classical motion $s \in \sec(\boldsymbol{T}, \boldsymbol{E})$.

It is remarkable the fact that the classical Lagrangian $\mathcal{L}$ in the above formula is not postulated, but it arises naturally from the upper quantum connection $Ч^{\uparrow}$.

The original source of this approach goes back to [196].

In view of the comparison of the above formula with the standard literature, we recall that our Lagrangian $\mathcal{L}$ is rescaled through the Planck constant $\hbar$.

Lemma 23.2.1 *Let us consider a motion and a quantum section*

$$s \in \sec(\boldsymbol{T}, \boldsymbol{E}) \qquad \text{and} \qquad \Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}).$$

Moreover, let us consider a quantum basis $\mathrm{b} : \boldsymbol{U} \subset \boldsymbol{E} \to \boldsymbol{Q}$ *and suppose that* s *is defined in an interval* $\boldsymbol{I} := [\mathrm{t}_0, \mathrm{t}] \subset \bar{\boldsymbol{T}}$, *such that* $s(\boldsymbol{I}) \subset \boldsymbol{U}$.

Then, Ψ *fulfills the equation (see Proposition 23.1.1)*

$$\nabla^{\uparrow}{}_{s}\Psi = 0$$

if and only if

$$(\Psi \circ s)(\mathrm{t}) = (\Psi \circ s)(\mathrm{t}_0) \exp\left(\mathrm{i} \int_{[\mathrm{t}_0,\mathrm{t}]} (\mathcal{L} \circ j_1 s)\right).$$

Proof. In virtue of Proposition 23.1.1, we have $\nabla^{\uparrow}{}_{s}\underline{\Psi} = 0$ if and only if

$$\partial_0(\psi \circ s) - \mathrm{i}\,(\mathcal{L}_0 \circ j_1 s)\,(\psi \circ s) = 0.$$

Hence, by integration we obtain

$$(\Psi \circ s)(\mathrm{t}) = (\Psi \circ s)(\mathrm{t}_0) \exp\left(\mathrm{i} \int_{[\mathrm{t}_0,\mathrm{t}]} (\mathcal{L} \circ j_1 s)\right). \quad \square$$

Theorem 23.2.2 *Let us consider a motion* $s : \boldsymbol{T} \to \boldsymbol{E}$.

Then, according to the above Lemma 23.2.1, the solutions $\Psi \in \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q})$ *of the equation*

$$\nabla^{\uparrow}{}_{s}\Psi = 0,$$

yield, for each $\mathrm{t}_0, \mathrm{t} \in \boldsymbol{T}$, *a complex linear* h*-isometry*

$$\Pi_s[\mathrm{t}_0, \mathrm{t}] : \boldsymbol{Q}_{s(\mathrm{t}_0)} \to \boldsymbol{Q}_{s(\mathrm{t})} : (\psi \circ s)(\mathrm{t}_0)\,\mathrm{b}\,(s(\mathrm{t}_0)) \mapsto (\psi \circ s)(\mathrm{t})\,\mathrm{b}\,(s(\mathrm{t}))\,.$$

Proof. The map $\Pi_s[\mathrm{t}_0, \mathrm{t}]$ is obtained by gluing the gauged contributions related to different quantum bases. $\square$

The above map can be easily extended to all "broken" motions (i.e. to motions which are continuous and almost everywhere smooth).

Indeed, the above time function

$$\exp\left(\mathrm{i} \int_{\boldsymbol{T}} (\mathcal{L} \circ j_1 s)\right) \in \mathrm{map}(\boldsymbol{T}, \mathbb{C})$$

is just the "Feynman amplitude", which appears in the path integral formulation of Quantum Mechanics (see, for instance, [48, 49, 124, 125, 299]).

Part III
Examples

Standard Quantum Mechanics is basically formulated in a flat spacetime.

So, we conclude the present exposition with simple examples dealing with a flat spacetime, or a quasi-flat spacetime, in order to test our formalism in these simple cases and to make a comparison with standard Quantum Mechanics.

We start by discussing two examples of galilean spacetimes:

(1) the *flat newtonian spacetime*, which deals with a geometric formulation, in our geometric covariant language, of the flat spacetime of standard Classical Mechanics,

(2) the *curved newtonian spacetime*, which deals with a curved spacetime equipped with an additional background flat spacetime connection, fulfilling a suitable condition; this model of spacetime allows us to recover, in our covariant geometric language, the inertial observers and the standard Newton law of gravitation, via the Galilei–Einstein equation.

Then, with reference to an inertial observer, a uniformly accelerated observer and a uniformly rotating observer, we provide the expression of the basic classical and quantum objects discussed throughout the book, in three cases:

(a) a flat newtonian spacetime with vanishing electromagnetic field,

(b) a flat newtonian spacetime with a radial electric field,

(c) a flat newtonian spacetime with a constant magnetic field.

Chapter 24
Flat Newtonian Spacetime

One can think of generic curved models of spacetime, which fulfill our general postulates on the time fibring, the galilean spacelike metric, the galilean gravitational field and the galilean electromagnetic field (see Postulates C.1–C.4). Indeed, most notions and results established in the book hold for such generic curved models of spacetime.

Standard Quantum Mechanics is formulated in a flat spacetime and usually referred to inertial observers. So, in view of a comparison between Covariant Quantum Mechanics and standard Quantum Mechanics, it is useful to discuss in details the concept of flat spacetime and inertial observers, within our general geometric framework.

So, in this chapter, we analyse the standard "*flat newtonian spacetime*", which is a topologically trivial bundle $t : \boldsymbol{E} \to \boldsymbol{T}$ equipped with a gravitational connection $K^\natural$, whose curvature tensor $R[K^\natural]$ is *vanishing* (Sect. 24.1). In this model, the spacetime manifold $\boldsymbol{E}$ turns out to be an *affine space* and the spacetime fibred manifold $t : \boldsymbol{E} \to \boldsymbol{T}$ turns out to be an *abelian principal bundle* associated with a 3-dimensional vector space $\boldsymbol{S}$.

Moreover, in this model, we can achieve the concepts of *inertial motions*, *inertial observers* and *cartesian spacetime charts* (Sect. 24.2). Besides the inertial observer, in order to test our formalism, we consider also uniformly accelerated observers and uniformly rotating observers (Sects. 24.4 and 24.5).

Thus, this model turns out to be a geometric formulation, in our language, of the standard flat spacetime of Classical Mechanics.

Later, we shall consider another model of spacetime which is equipped with a more rich geometric structure (see Chap. 28).

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24.1 Flat Newtonian Spacetime

We define, the *flat newtonian spacetime* of standard Classical Mechanics in terms of our geometric language.

Namely, we postulate a topologically trivial bundle $t : \boldsymbol{E} \to \boldsymbol{T}$, along with its *metric* and *gravitational fields* g and $K^\natural$. Moreover, we suppose that the curvature $R[K^\natural]$ of the gravitational field vanishes.

These hypotheses yield an *affine structure* on the spacetime manifold $\boldsymbol{E}$ and an *abelian principal bundle structure*, associated with a 3-dimensional vector space $\boldsymbol{S}$, on the spacetime bundle $t : \boldsymbol{E} \to \boldsymbol{T}$. Accordingly, the time projection $t : \boldsymbol{E} \to \boldsymbol{T}$ turns out to be an affine map and the metric field g turns out to be a *euclidean metric* of the vector space $\boldsymbol{S}$.

We stress that it is important to make a clear distinction between the spacetime $\boldsymbol{E}$, regarded as an affine space, and the associated vector space $\bar{\boldsymbol{E}}$. Also in einsteinian Special Relativity, the Minkowski space of events and its associated vector space should be considered as distinct objects. The standard habit in physical literature to introduce the concepts in coordinates might yield a confusion between such objects.

Definition 24.1.1 We define a *flat newtonian spacetime* to be an oriented fibred manifold (see Postulate C.1)

$$t : \boldsymbol{E} \to \boldsymbol{T},$$

which

(1) is a *bundle* topologically homeomorphic to the trivial bundle (see, Appendix: Definition A.2.1 and Remark A.2.2)

$$\text{pro}_1 : \mathbb{R} \times R^3 \to \mathbb{R},$$

(2) is equipped with

(a) a *galilean metric* (according to Postulate C.2)

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}),$$

(b) a *galilean gravitational connection* $K^\natural$ (according to Postulate C.3)

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

which fulfills the additional condition

$$R[K^\natural] = 0.$$

Hence, by definition, $K^\natural$ is a linear, torsion free spacetime connection, such that

$$\nabla^\natural dt = 0, \qquad \overset{''}{\nabla}{}^\natural g = 0, \qquad R[K^\natural] = 0. \quad \square$$

Thus, let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ and analyse its main features.

We start by discussing the induced affine structure of spacetime.

Proposition 24.1.2 *The flat gravitational connection* $K^\natural$ *yields an* affine structure $(\boldsymbol{E}, \bar{\boldsymbol{E}}, +)$ *on the topologically trivial spacetime manifold, where* $\bar{\boldsymbol{E}}$ *is an oriented 4-dimensional vector space and*

$$+ : \boldsymbol{E} \times \bar{\boldsymbol{E}} \to \boldsymbol{E} : (e, \bar{v}) \mapsto e + \bar{v}$$

is a free and transitive action.

Moreover, the following properties hold.

(1) In virtue of the condition $\nabla^\natural dt = 0$, *the time projection* $t : \boldsymbol{E} \to \boldsymbol{T}$ *turns out to be an* affine map.
Thus, by definition of the above affine structure, for each $e \in \boldsymbol{E}$ *and* $\bar{v} \in \bar{\boldsymbol{E}}$, *we have*

$$t(e + \bar{v}) = t(e) + Dt(\bar{v}), \quad \text{with } Dt \in \mathbb{T} \otimes \bar{\boldsymbol{E}}^*.$$

(2) For all $\mathrm{t} \in \boldsymbol{T}$, *the fibres* $\boldsymbol{E}_{\mathrm{t}}$ *of the spacetime bundle turn out to be* affine spaces *associated with the* same 3*-dimensional vector subspace*

$$\boldsymbol{S} := \ker Dt \subset \bar{\boldsymbol{E}}.$$

In other words, $t : \boldsymbol{E} \to \boldsymbol{T}$ *turns out to be an* abelian principal bundle *associated with the vector space* $\boldsymbol{S}$. *Thus, we have a trivial vector bundle*

$$\bar{t} : \bar{\boldsymbol{E}} = \boldsymbol{T} \times \boldsymbol{S} \to \boldsymbol{T}.$$

(3) In virtue of the condition $\nabla^\natural g = 0$, *the galilean metric* g *can be regarded as a scaled* euclidean metric

$$g \in \mathbb{L}^2 \otimes (\boldsymbol{S}^* \otimes \boldsymbol{S}^*). \quad \square$$

Remark 24.1.3 The affine spacetime bundle $t : \boldsymbol{E} \to \boldsymbol{T}$ is trivialisable, but we do not postulate any distinguished splitting.

Conversely, the associated vector bundle $\bar{t} : \bar{\boldsymbol{E}} \to \boldsymbol{T}$ is equipped with the natural distinguished splitting $\bar{\boldsymbol{E}} = \boldsymbol{T} \times \boldsymbol{S}$. □

Then, we discuss the tangent bundle, the vertical tangent bundle, the iterated tangent bundle and the 1st-jet bundle associated with the affine spacetime.

Definition 24.1.4 With reference to the linear map

$$\tau := \mathrm{id}_{\mathbb{T}^*} \otimes Dt : \mathbb{T}^* \otimes \bar{\boldsymbol{E}} \to \mathbb{T}^* \otimes (\mathbb{T} \otimes \mathbb{R}) \simeq \mathbb{R} : u^0 \otimes \bar{v} \mapsto v^0 \, (u^0 \otimes u_0) \simeq v^0,$$

we define the 3-dimensional affine subspace

$$\boldsymbol{U} := \mathrm{T}^{-1}(1_{\mathbb{T}}) \subset \mathbb{T}^* \otimes \bar{\boldsymbol{E}},$$

which is associated with the vector space $\mathbb{T}^* \otimes \boldsymbol{S}$. □

In virtue of the above affine structure of spacetime, we have the following distinguished splittings.

Note 24.1.5 The tangent space $T\boldsymbol{E}$, the vertical space $V\boldsymbol{E}$ and the phase space $J_1\boldsymbol{E}$ can be naturally regarded as the trivial products (see Definition 2.2.1, and Proposition 2.5.1)

$$T\boldsymbol{E} = \boldsymbol{E} \times \bar{\boldsymbol{E}}, \qquad V\boldsymbol{E} = \boldsymbol{E} \times \boldsymbol{S}, \qquad J_1\boldsymbol{E} = \boldsymbol{E} \times \boldsymbol{U}.$$

Accordingly, a vector field X, a vertical vector field Y and an observer o can be written, respectively, as

$$X : \boldsymbol{E} \to T\boldsymbol{E} : e \mapsto \big(e, \tilde{X}(e)\big), \qquad Y : \boldsymbol{E} \to V\boldsymbol{E} : e \mapsto \big(e, \tilde{Y}(e)\big),$$
$$o : \boldsymbol{E} \to J_1\boldsymbol{E} : e \mapsto \big(e, \tilde{o}(e)\big),$$

where the components

$$\tilde{X} : \boldsymbol{E} \to \bar{\boldsymbol{E}}, \qquad \tilde{Y} : \boldsymbol{E} \to \boldsymbol{S}, \qquad \tilde{o} : \boldsymbol{E} \to \boldsymbol{U}$$

characterise X, Y, o, respectively. □

Note 24.1.6 In virtue of the affine structure of spacetime, the iterated tangent bundle of spacetime can be naturally regarded as the trivial fibred product (see Definition 2.2.1)

$$\tau_{T\boldsymbol{E}} = \mathrm{pro}_1 : TT\boldsymbol{E} = T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (T\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E}) \to T\boldsymbol{E}. \quad □$$

24.2 Inertial Observers

In a flat newtonian spacetime, we define the *inertial motions* and the *inertial observers* by taking into account the affine structure of $\boldsymbol{E}$. Indeed, these are just the standard observers used in standard Classical Mechanics and standard Quantum Mechanics.

Then, we choose an inertial observer $o_{\mathtt{in}}$ and define an adapted *cartesian spacetime chart* (x^0, x^i) to be used as basic reference in forthcoming examples. Further, we define an adapted *cylindrical spacetime chart* (x^0, ρ, ϕ, z). and an adapted *sherical spacetime chart* (x^0, r, θ, ϕ). Most notions are elementary and standard, but still some details deserver an additional care.

We stress that, in the general theory, it is convenient to use unscaled coordinates (see Definition 2.1.4); but in specific examples, we shall use scaled coordinates, in order to be close to the standard usage. Of course, this choice might change the dimensional scaling of the components of some tensors, but does not effect the dimensional scaling of intrinsic objects.

Definition 24.2.1 We define an *inertial motion* to be a global affine section

$$s : \boldsymbol{T} \to \boldsymbol{E},$$

according to the following equality, for each $\mathrm{t} \in \boldsymbol{T}$ and $u \in \mathbb{T} \otimes \mathbb{R}$,

$$s(\mathrm{t} + u) = s(\mathrm{t}) + Ds(u) \quad \text{with} \quad Ds \in \boldsymbol{U} \subset \mathbb{T}^* \otimes \bar{\boldsymbol{E}}.$$

We define an *inertial observer* to be a global section (see Definition 2.7.1 and Proposition 2.7.3)

$$o : \boldsymbol{E} \to J_1\boldsymbol{E}, \quad \text{such that} \;\; \nabla^{\natural} д[o] = 0. \quad \square$$

Note 24.2.2 An observer o is inertial if and only if $\tilde{o}$ is constant (see Note 24.1.5).

Thus, an observer o turns out to be inertial if and only if its flow

$$\mathscr{F}[o] : (\mathbb{T} \otimes \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E}$$

consists of inertial particle motions s with the same (constant) velocity $\tilde{o} = Ds$.

Hence, an inertial observer o is determined by each of the inertial particle motions of its flow $\mathscr{F}[o]$.

Thus, the flow $\mathscr{F}[o]$ of an inertial observer is of the type (see Note 24.1.5)

$$\mathscr{F}[o](\tau, e) = e + \tau\, \tilde{o}.$$

Indeed, the distances between the particles of the flow do not change along its time evolution; hence, an inertial observer is rigid (see Definition 3.2.22). □

Note 24.2.3 If o is an inertial observer, then another observer $\acute{o}$ is inertial if and only if $\vec{v} := \acute{o} - o$ is a constant scaled vector field $\vec{v} \in \mathbb{T}^* \otimes \boldsymbol{S}$. □

Next, we discuss distinguished spacetime charts adapted to an inertial observer.

From now on, in this section, we shall refer to an *inertial motion* and to the associated *inertial observer* (see Definition 24.2.1)

$$s_{\mathtt{in}} : \boldsymbol{T} \to \boldsymbol{E} \qquad \text{and} \qquad o_{\mathtt{in}} : \boldsymbol{E} \to J_1\boldsymbol{E}.$$

Definition 24.2.4 We define a scaled *cartesian spacetime chart*, adapted to the inertial observer $o_{\mathtt{in}}$,

$$(x^0, x^i) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$$

in the following way.

We choose an origin of time $t_0 \in \boldsymbol{T}$ and a scaled basis $(u_0, \mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ of $\bar{\boldsymbol{E}}$ consisting of a time scale $u_0 \in \mathbb{T}$ and a spacelike orthonormal scaled basis of $\boldsymbol{S}$ (see Proposition 2.7.3 and Definition 24.2.1)

$$(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3) \subset (\mathbb{L}^{-1} \otimes \boldsymbol{S})^3, \quad \text{with } g(\mathrm{e}_i, \mathrm{e}_j) = \delta_{ij}.$$

Thus, the dual basis $(u^0, \boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2, \boldsymbol{\epsilon}^3)$ of $\bar{\boldsymbol{E}}^*$ turns out to be given by the scaled forms

$$u^0 \in \mathbb{T}^*, \qquad (\boldsymbol{\epsilon}^1, \boldsymbol{\epsilon}^2, \boldsymbol{\epsilon}^3) = \big(g^\flat(\mathrm{e}_1),\ g^\flat(\mathrm{e}_2),\ g^\flat(\mathrm{e}_3)\big) \subset (\mathbb{L} \otimes \boldsymbol{S}^*)^3.$$

Then, we define the spacetime functions

$$x^0 \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}) : e \mapsto x^0(e) := u^0\big(t(e) - t_0\big),$$
$$x^i \in \operatorname{map}(\boldsymbol{E}, \mathbb{L} \otimes \mathbb{R}) : e \mapsto x^i(e) := \boldsymbol{\epsilon}^i\big(e - s_{\mathtt{in}}(t(e))\big).$$

Indeed, we obtain the equalities

$$Tt(\partial_0) = u_0 \in \mathbb{T}, \qquad d^0 = u^0 \in \mathbb{T}^*,$$
$$\partial_i = \mathrm{e}_i \in \mathbb{L}^{-1} \otimes \boldsymbol{S}, \qquad \check{d}^i = \boldsymbol{\epsilon}^i \in \mathbb{L} \otimes \boldsymbol{S}^*.$$

Thus, the scaled vector fields ∂_λ and the scaled forms d^λ are constant with respect to the affine structure of spacetime (see Definition 24.1.1 and Note 24.1.5). □

Definition 24.2.5 We define a scaled *cylindrical spacetime chart*, adapted to the observer $o_{\mathtt{in}}$, to be a spacetime chart

$$(x^0, \rho, \phi, z) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/2\pi \times (\mathbb{L} \otimes \mathbb{R}),$$

defined by the following transition law, with reference to the cartesian spacetime chart (x^0, x^i) (see Definition 24.2.4),

$$x^1 = \rho \cos\phi, \qquad x^2 = \rho \sin\phi, \qquad x^3 = z.$$

Thus, we have $\rho = \sqrt{(x^1)^2 + (x^2)^2}$.

We stress that this chart is defined in an open subset of spacetime; accordingly, we should implicitly take into account the due restrictions, even when they are not explicitly emphasised. □

Definition 24.2.6 We define a *spherical spacetime chart*, adapted to the observer $o_{\mathtt{in}}$, to be a spacetime chart

$$(x^0, r, \theta, \phi) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/\pi \times \mathbb{R}/2\pi,$$

defined by the following transition law, with reference to the cartesian spacetime chart (x^0, x^i) adapted to the inertial observer $o_{\mathtt{in}}$ (see Definition 24.2.4),

$$x^1 = r \sin\theta \cos\phi, \qquad x^2 = r \sin\theta \sin\phi, \qquad x^3 = r \cos\theta.$$

Thus, we have $r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

We stress that this chart is defined in an open subset of spacetime; accordingly, we should implicitly take into account the due restrictions, even when they are not explicitly emphasised. □

24.3 Inertial Observers Versus Affine Spacetime

This section is devoted to an excursus aimed at answering a possible question concerning the mutual relation between the affine structure of spacetime and inertial observers.

In the above section (see Sect. 24.2) we have seen that an affine spacetime is naturally equipped with the distinguished family of inertial observers; conversely, this distinguished family of inertial observers characterises the affine structure of spacetime.

Thus, this fact can be used for recovering a possible affine structure of spacetime by physical operative procedures, via inertial observers, according to Proposition 24.3.5.

Let us consider a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Definition 24.1.1).

Let us start with a standard elementary construction.

We can naturally recover the vector space $\bar{\boldsymbol{E}}$ and the associated affine structure of the spacetime $\boldsymbol{E}$ via a natural equivalence relation $\sim$ in the space of pairs $\boldsymbol{E} \times \boldsymbol{E}$.

Lemma 24.3.1 *We have a natural equivalence relation in the set $\boldsymbol{E} \times \boldsymbol{E}$ given by*

$$(e_1, e_2) \sim (\acute{e}_1, \acute{e}_2) \qquad \Leftrightarrow \qquad e_2 - e_1 = \acute{e}_2 - \acute{e}_1 .$$

Clearly, if $(e_1, e_2) \sim (\acute{e}_1, \acute{e}_2)$, then we have

$$\begin{aligned} e_2 - e_1 = \acute{e}_2 - \acute{e}_1 \qquad &\text{and} \qquad \acute{e}_1 - e_1 = \acute{e}_2 - e_2 , \\ t(e_2) - t(e_1) = t(\acute{e}_2) - t(\acute{e}_1) \qquad &\text{and} \qquad t(\acute{e}_1) - t(e_1) = t(\acute{e}_2) - t(e_2). \end{aligned}$$

The difference map

$$- : \boldsymbol{E} \times \boldsymbol{E} \to \bar{\boldsymbol{E}} : (e_1, e_2) \mapsto e_2 - e_1$$

passes to the quotient of the equivalence relation, hence yielding a natural bijection

$$\imath : (\boldsymbol{E} \times \boldsymbol{E})_{|\sim} \to \bar{\boldsymbol{E}} : [(e_1, e_2)]_\sim \mapsto e_2 - e_1 ,$$

which makes the quotient space $\widetilde{\boldsymbol{E}} := (\boldsymbol{E} \times \boldsymbol{E})_{|\sim}$ *a vector space naturally isomorphic to* $\bar{\boldsymbol{E}}$*. Moreover, the map*

$$\widetilde{+} : \boldsymbol{E} \times \widetilde{\boldsymbol{E}} \to \boldsymbol{E} : \big(e,\ [(e_1, e_2)]_\sim\big) \mapsto e + (e_2 - e_1)$$

is well defined and turns out to be a free and transitive action of the vector space $\widetilde{\boldsymbol{E}}$ *on the set* $\boldsymbol{E}$*.*

Hence, the spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$*, along with the vector space* $\widetilde{\boldsymbol{E}}$ *and the above free and transitive action* $\widetilde{+} : \boldsymbol{E} \times \widetilde{\boldsymbol{E}} \to \boldsymbol{E}$ *turns out to be an affine spacetime.*

Indeed, this affine spacetime coincides with the original one, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{E} \times \widetilde{\boldsymbol{E}} & \xrightarrow{\ \widetilde{+}\ } & \boldsymbol{E} \\ {\scriptstyle \mathrm{id} \times \imath}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\ \boldsymbol{E} \times \bar{\boldsymbol{E}} & \xrightarrow{\ +\ } & \boldsymbol{E} \end{array} . \quad \square$$

Next, let us consider the subspace

$$\boldsymbol{E} \overset{\shortparallel}{\times} \boldsymbol{E} := \{(e, \acute{e}) \in \boldsymbol{E} \times \boldsymbol{E} \ \mid\ t(e) \neq t(\acute{e})\} \subset \boldsymbol{E} \times \boldsymbol{E}.$$

Lemma 24.3.2 *For each* $(e,\ \acute{e}) \in \boldsymbol{E} \overset{\shortparallel}{\times} \boldsymbol{E}$*, there exists a unique observer* o*, whose flow* $\mathscr{F}[o]$ *shifts* e *onto* $\acute{e}$*, i.e. such that*

$$\mathscr{F}_{\tau_{(e,\acute{e})}}[o](e) = \acute{e}, \quad \textit{where} \ \ \tau_{(e,\acute{e})} := t(\acute{e}) - t(e) \in \mathbb{T} \otimes I\!R.$$

Actually, we have

$$\mathscr{F}[o] : (\mathbb{T} \otimes I\!R) \times \boldsymbol{E} : (\tau, e) \mapsto e + \frac{\tau}{\tau_{(e,\acute{e})}}\,(\acute{e} - e). \quad \square$$

We can avail of the following method, via inertial observers, to check the equivalence of pairs of events.

Lemma 24.3.3 *Two pairs* $(e_1,\ e_2),\ (\acute{e}_1,\ \acute{e}_2) \in \boldsymbol{E} \overset{\shortparallel}{\times} \boldsymbol{E}$ *are equivalent if and only if the unique observer* o*, whose flow* $\mathscr{F}[o]$ *shifts* e_1 *onto* $\acute{e}_1$*, shift also* e_2 *onto* $\acute{e}_2$*, according to the equalities*

$$\mathscr{F}_{\tau_{(e_1,\acute{e}_1)}}[o](e_1) = \acute{e}_1 \qquad \text{and} \qquad \mathscr{F}_{\tau_{(e_2,\acute{e}_2)}}[o](e_2) = \acute{e}_2. \quad \square$$

More generally, we can avail of the following method, via inertial observers, to check the equivalence of any pairs.

Lemma 24.3.4 *Two pairs* (e'_1, e'_2), $(e''_1, e''_2) \in \boldsymbol{E} \times \boldsymbol{E}$ *are equivalent if and only if there is a third pair* $(e_1, e_2) \in \boldsymbol{E} \times \boldsymbol{E}$, *with* $t(e_1) \neq t(e'_1)$ *and* $t(e_1) \neq t(e''_1)$, *such that*

(1) the unique observer o', *whose flow* $\mathscr{F}[o']$ *shifts* e'_1 *onto* e_1, *shifts also* e'_2 *onto* e_2,
(2) the unique observer o'', *whose flow* $\mathscr{F}[o'']$ *shifts* e''_1 *onto* e_1, *shifts also* e''_2 *onto* e_2, *according to the equalities*

$$\mathscr{F}_{\tau_{(e'_1,e_1)}}[o'](e'_1) = e_1 \quad \text{and} \quad \mathscr{F}_{\tau_{(e'_2,e_2)}}[o'](e'_2) = e_2,$$
$$\mathscr{F}_{\tau_{(e''_1,e_1)}}[o''](e''_1) = e_1 \quad \text{and} \quad \mathscr{F}_{\tau_{(e''_2,e_2)}}[o''](e''_2) = e_2. \quad \square$$

Proposition 24.3.5 *According to the above Lemma 24.3.4, the knowledge of the family of inertial observers associated with the affine spacetime* $t : \boldsymbol{E} \to \boldsymbol{T}$ *allows us to recover the equivalence relation in* $\boldsymbol{E} \times \boldsymbol{E}$ *and the quotient vector space* $\widetilde{\boldsymbol{E}} \simeq \bar{\boldsymbol{E}}$, *hence it allows us to recover the affine structure of spacetime, according to Lemma 24.3.1.*

Proof. It follows from Lemmas 24.3.1–24.3.4. □

24.4 Uniformly Accelerated Observer

Besides the inertial observer $o_{\mathtt{in}}$, we define a *uniformly accelerated observer* $o_{\mathtt{ac}}$, along with an adapted *cartesian spacetime chart* $(x^0, x^i{}_{\mathtt{ac}})$. This is a non inertial observer suitable to test our covariant formulation of Classical Mechanics and Quantum Mechanics.

Most notions are elementary and standard, but still some details deserver an additional care.

Definition 24.4.1 A *uniformly accelerated observer* is defined to be an observer $o_{\mathtt{ac}}$, whose flow

$$F[o_{\mathtt{ac}}] : (\mathbb{T} \otimes \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E},$$

with reference to the inertial observer $o_{\mathtt{in}}$, is of the type (see Note 2.7.1 and Definition 24.2.1)

$$F[o_{\mathtt{ac}}](\tau, e) = e + \tau\, д[o_{\mathtt{in}}] + \Big(\tau\, \nu\big(t(e)\big) + \tfrac{1}{2}\, \tau^2\, a\Big)\, \boldsymbol{e},$$

where

(1) $\boldsymbol{e} \in \mathbb{L}^{-1} \otimes \boldsymbol{S}$, with $g(\boldsymbol{e}, \boldsymbol{e}) = 1$,
(2) $\nu : \boldsymbol{T} \to (\mathbb{T}^{-1} \otimes \mathbb{L}) \otimes \mathbb{R}$ is an affine map,
(3) $a \in (\mathbb{T}^{-2} \otimes \mathbb{L}) \otimes \mathbb{R}$.

Thus, by definition, for each $e \in \boldsymbol{E}$, the particle motion s_e of the flow $F[o_{\mathtt{ac}}]$ passing through e is a uniformly accelerated motion with

(a) constant observed acceleration $\nabla^{\natural}{}_{ds_e} ds_e = a\, \boldsymbol{e} \in \mathbb{T}^{-2} \otimes \boldsymbol{S}$,
(b) affine observed velocity $ds_e : \mathbb{T} \otimes R \to \mathbb{T}^* \otimes \boldsymbol{S} : \tau \mapsto \Big(\tau \otimes a + \nu\big(t(e)\big)\Big)\, \boldsymbol{e}$.

Moreover, all particle motions s_e, where e belongs to the same spacetime fibre $\boldsymbol{E}_{\mathrm{t}}$, with $\mathrm{t} \in \boldsymbol{T}$, have the same velocity. Hence, the world line of all particle motions of the flow $F[o_{\mathtt{ac}}]$ are parallel with respect to the affine structure of spacetime.

There is a time $\mathrm{t}_0 \in \boldsymbol{T}$ such that, for all $e \in \boldsymbol{E}_{\mathrm{t}_0}$, the observed velocities $ds_e(\mathrm{t}_0)$ vanish.

Indeed, the distances between the particles of the flow do not change along the time evolution; hence, the uniformly accelerated observer $o_{\mathtt{ac}}$ is rigid (see Definition 3.2.22).

We have the equality (see Note 2.7.6 and Definition 24.2.4)

$$o_{\mathtt{ac}} = o_{\mathtt{in}} + \vec{v}_{\mathtt{ac}}\,, \qquad \text{with } \vec{v}_{\mathtt{ac}} = (\nu \circ t)\, \boldsymbol{e}.$$

From now on, just to fix the ideas, we shall suppose that (see Note 24.2.4)

$$x^0(\mathrm{t}_0) = 0 \qquad \text{and} \qquad \boldsymbol{e} = \boldsymbol{e}_1 = \partial_1.$$

Thus, in a cartesian spacetime chart (x^0, x^i) adapted to $o_{\mathtt{in}}$, we have the coordinate expression

$$\vec{v}_{\mathtt{ac}} = \big(x^0\, a_{00} + (\nu_0 \circ t)\big)\, u^0 \otimes \partial_1. \quad \square$$

Definition 24.4.2 We define a *cartesian uniformly accelerated chart*, adapted to the uniformly accelerated observer $o_{\mathtt{ac}}$,

$$(x^0, x^i{}_{\mathtt{ac}}) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$$

to be the scaled spacetime chart defined by the following transition rule, with reference to the cartesian spacetime chart (x^0, x^i), (see Definition 24.2.4)

$$x^1 = x_{\mathtt{ac}}{}^1 + \tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0, \qquad x^2 = x_{\mathtt{ac}}{}^2, \qquad x^3 = x_{\mathtt{ac}}{}^3.$$

Then, we obtain the transition rule

$$x^1_0 = x_{\mathtt{ac}}{}^1_0 + a_{00}\, x^0 + \nu_0. \quad \square$$

24.5 Uniformly Rotating Observer

Besides the inertial observer $o_{\mathtt{in}}$, we define a *uniformly rotating observer* $o_{\mathtt{ro}}$, along with an adapted *cartesian spacetime chart* $(x^0, x^i{}_{\mathtt{ro}})$ and an adapted *cylindrical spacetime chart* $(x^0, \rho_{\mathtt{ro}}, \phi_{\mathtt{ro}}, z_{\mathtt{ro}}) \equiv (x^0, \rho, \phi_{\mathtt{ro}}, z)$. This is a non inertial observer suitable to test our covariant formulation of Classical Mechanics and Quantum Mechanics.

Most notions are elementary and standard, but still some details deserver an additional care.

Definition 24.5.1 A *uniformly rotating observer* is defined to be an observer $o_{\mathtt{ro}}$, whose flow

$$F[o_{\mathtt{ro}}] : (\mathbb{T} \otimes \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E},$$

with reference to the inertial observer $o_{\mathtt{in}}$, is of the type (see Note 2.7.1 and Definition 24.2.1)

$$F[o_{\mathtt{ro}}](\tau, e) = s_{\mathtt{in}}\big(\tau + t(e)\big) + R_{\tau\omega}\Big(e - s_{\mathtt{in}}\big(t(e)\big)\Big),$$

where

(1) $\omega \in \mathbb{T}^* \otimes \mathbb{R}$,
(2) $R_{\tau\omega} : \boldsymbol{S} \to \boldsymbol{S}$ is the rotation operator of an angle $\tau\,\omega \in \mathbb{R}$, with axis $\boldsymbol{e} \in \mathbb{L}^{-1} \otimes \boldsymbol{S}$, normalised by the condition $g(\boldsymbol{e}, \boldsymbol{e}) = 1$.

Thus, for each $e \in \boldsymbol{E}$, the particle motion s_e of the flow $F[o_{\mathtt{ro}}]$ is, with reference to the observer $o_{\mathtt{in}}$, a uniform rotation with center $s_{\mathtt{in}}\big(t(e)\big)$, angular velocity ω and axis $\boldsymbol{e}$.

We have the equality (see Note 2.7.6 and Definition 24.2.4)

$$o_{\mathtt{ro}} = o_{\mathtt{in}} + \vec{v}_{\mathtt{ro}},$$

where, for each $e \in \boldsymbol{E}$,

$$\vec{v}_{\mathtt{ro}}(e) = \vec{\omega} \times \big(e - s_{\mathtt{in}}(t(e)\big), \quad \text{with } \vec{\omega} = \omega\, \mathrm{e} \in (\mathbb{T}^* \otimes \mathbb{L}^{-1}) \otimes \boldsymbol{S}.$$

Indeed, the distances between the particles of the flow do not change along the time evolution; hence, a uniformly rotating observer is rigid (see Definition 3.2.22).

From now on, just to fix the ideas, we shall suppose that (see Note 24.2.4)

$$\boldsymbol{e} = \boldsymbol{e}_3 = \partial_3.$$

In a cartesian spacetime chart adapted to $o_{\mathtt{in}}$, we have the coordinate expression

$$\vec{v}_{\mathtt{ro}} = \omega_0\, u^0 \otimes (x^1\, \partial_2 - x^2\, \partial_1). \quad \square$$

Definition 24.5.2 We define a *cartesian uniformly rotating spacetime chart*, adapted to the uniformly rotating observer $o_{\texttt{ro}}$,

$$(x^0, x^i{}_{\texttt{ro}}) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$$

to be the scaled spacetime chart defined by the following transition law, with reference to the cartesian spacetime chart (x^0, x^i) (see Definition 24.2.4)

$$\begin{aligned} x^1 &= x^1{}_{\texttt{ro}} \; \cos(\omega_0 \, x^0) - x^2{}_{\texttt{ro}} \; \sin(\omega_0 \, x^0), \\ x^2 &= x^1{}_{\texttt{ro}} \; \sin(\omega_0 \, x^0) + x^2{}_{\texttt{ro}} \; \cos(\omega_0 \, x^0), \\ x^3 &= x^3{}_{\texttt{ro}} . \quad \square \end{aligned}$$

Definition 24.5.3 We define a *cylindrical uniformly rotating spacetime chart*, adapted to the uniformly rotating observer $o_{\texttt{ro}}$,

$$(x^0, \rho_{\texttt{ro}}, \phi_{\texttt{ro}}, z_{\texttt{ro}}) : \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/2\pi \times (\mathbb{L} \otimes \mathbb{R})$$

to be the scaled spacetime chart defined by the following transition law, with reference to the cylindrical spacetime chart (x^0, ρ, ϕ, z) (see Definition 24.2.5)

$$x^0 = x_{\texttt{ro}}{}^0, \qquad \rho = \rho_{\texttt{ro}}, \qquad \phi = \phi_{\texttt{ro}} + \omega_0 \, x^0, \qquad z = z_{\texttt{ro}} .$$

Then, we have the following transition rules, which will be frequently used in the following,

$$\begin{aligned} & x_{\texttt{ro}}{}^0 = x^0, \qquad \rho_{\texttt{ro}} = \rho, \qquad \phi_{\texttt{ro}} = \phi - \omega_0 \, x^0, \qquad z_{\texttt{ro}} = z. \\ & d_{\texttt{ro}}{}^0 = d^0, \qquad d_{\texttt{ro}}{}^\rho = d^\rho, \qquad d_{\texttt{ro}}{}^\phi = d^\phi - \omega_0 \, d^0, \qquad d_{\texttt{ro}}{}^z = d^z, \\ & \partial_{\texttt{ro}\,0} = \partial_0 + \omega_0 \, \partial_\phi, \qquad \partial_{\texttt{ro}\,\rho} = \partial_\rho, \qquad \partial_{\texttt{ro}\,\phi} = \partial_\phi, \qquad \partial_{\texttt{ro}\,z} = \partial_z, \\ & \rho_{\texttt{ro}\,0} = \rho_0, \qquad \phi_{\texttt{ro}\,0} = \phi_0 - \omega_0, \qquad z_{\texttt{ro}\,0} = z_0, \\ & d_{\texttt{ro}}{}^\rho_0 = d^\rho_0, \qquad d_{\texttt{ro}}{}^\phi_0 = d^\phi_0, \qquad d_{\texttt{ro}}{}^z_0 = d^z_0, \\ & \partial_{\texttt{ro}}{}^0_\rho = \partial^0_\rho, \qquad \partial_{\texttt{ro}}{}^0_\phi = \partial^0_\phi, \qquad \partial_{\texttt{ro}}{}^0_z = \partial^0_z. \end{aligned}$$

Moreover, we have the coordinate expression

$$\vec{v}_{\texttt{ro}} = \omega_0 \, u^0 \otimes \partial_\phi .$$

Indeed, this chart is defined locally, hence we have to remove from its domain the points where the coordinates are not continuous. $\square$

Chapter 25
Dynamical Example 1: No Electromagnetic Field

In view of a comparison with standard Quantum Mechanics, we compute, in the framework of the standard flat spacetime, the basic classical and quantum objects discussed throughout the body of the book (Sects. 25.1 and 25.2).

In order to show how our machinery works with different observers, we consider, not only a standard inertial observer, but also a uniformly accelerated observer and a uniformly rotating observer. Moreover, we consider, not only a cartesian spacetime chart, but also curvilinear spacetime charts.

Special attention is devoted to the game of the upper potential $A^{\uparrow}[\mathfrak{b}]$ and of the observed potentials $A[\mathfrak{b}, o]$, also with relation to the upper quantum connection $Ч^{\uparrow}$.

Indeed, we hope that our covariant approach can be useful to better understand some features of Quantum Mechanics, even in the flat case.

In this 1st example, we deal only with the gravitational field, by supposing that the electromagnetic field vanishes. In the subsequent examples, we shall also consider, respectively, a radial electric field and a constant magnetic field.

25.1 Classical Objects

We consider a flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ (with vanishing electromagnetic field) and compute the main *classical objects* discussed throughout the book, with reference to an inertial observer $o_{\mathtt{in}}$, a uniformly accelerated observer $o_{\mathtt{ac}}$ and a uniformly rotating observer $o_{\mathtt{ro}}$ (see Definitions 24.2.1, 24.4.1 and 24.5.1).

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In particular, we select the distinguished classical gauge $\mathrm{b}^{\square}$, defined by some equivalent distinguished conditions.

Later, this classical gauge will be regarded as a distinguished quantum basis $\mathrm{b}^{\square}$ and will be used for postulating the upper quantum connection $\mathrm{Ч}^{\uparrow}$ in the present framework (see Hypothesis Q.1).

25.1.1 Starting Hypothesis of the Classical Theory

In this 1st example, the classical background spacetime (see Hypothesis C.1) is the standard flat spacetime discussed in the above Chap. 24.

Later, we shall complete the hypothesis by postulating the quantum bundle and the upper quantum connection of this 1st example (see Hypothesis Q.1).

Hypothesis C.1 In this 1st Example, we consider the standard *flat newtonian spacetime* $(\boldsymbol{E}, g, K^{\natural})$ defined in Definition 24.1.1. Moreover, we consider a particle of mass m. □

Thus, in this example we assume a vanishing electromagnetic field $F = 0$.

25.1.2 Inertial Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics and, with reference to an *inertial observer* $o_{\mathtt{in}}$, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.1, Definitions 24.1.1 and 24.2.1).

In order to show how our automatic formalism works, we perform these computations in a cartesian spacetime chart, in a cylindrical spacetime chart and in a spherical spacetime chart adapted to the inertial observer $o_{\mathtt{in}}$. In particular, we discuss the observed kinetic angular momentum $\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}]$ (see Definition 3.2.12).

Moreover, we select the distinguished classical gauge $\mathrm{b}^{\square}$ and the associated upper potential $A^{\uparrow}[\mathrm{b}^{\square}]$ and observed potential $A[\mathrm{b}^{\square}, o_{\mathtt{in}}]$.

25.1.2.1 Adapted Cartesian Spacetime Chart

We consider first a cartesian spacetime chart (x^0, x^i) adapted to the inertial observer $o_{\mathtt{in}}$ (see Definitions 24.2.1 and 24.2.4).

Proposition 25.1.1 *We have the following coordinate expressions (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):*

$$g_{ij} = \delta_{ij} \in I\!R, \qquad g^{ij} = \delta^{ij} \in I\!R,$$

$$G^0_{ij} = \tfrac{m}{\hbar_0}\,\delta_{ij} \in \mathbb{L}^{-2} \otimes I\!R, \qquad G^{ij}_0 = \tfrac{\hbar_0}{m}\,\delta^{ij} \in \mathbb{L}^2 \otimes I\!R,$$
$$K^\natural{}_\lambda{}^i{}_\mu = 0.$$

Hence, we obtain the equalities (see Proposition 3.2.4, Definitions 3.2.9 and 3.2.12)

$$\upsilon = u_0 \otimes (d^0 \wedge d^1 \wedge d^2 \wedge d^3), \qquad \eta = \check{d}^1 \wedge \check{d}^2 \wedge \check{d}^3,$$
$$\mathcal{K}[o_{\mathtt{in}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x^i_0\,x^j_0\,d^0, \qquad \mathcal{Q}[o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\,\delta_{ij}\,x^i_0\,d^j,$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$K^\natural = d^\lambda \otimes \partial_\lambda, \qquad \Phi^\natural[o_{\mathtt{in}}] = 0,$$
$$\Gamma^\natural = d^\lambda \otimes \partial_\lambda, \qquad \gamma^\natural = u^0 \otimes \big(\partial_0 + x^i_0\,\partial_i\big),$$
$$\Omega^\natural = \tfrac{m}{\hbar_0}\,\delta_{ij}\,d^i_0 \wedge \theta^j, \qquad \Lambda^\natural = \tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_i \wedge \partial^0_j. \quad \square$$

Further, we discuss observed kinetic angular momentum and the standard Poisson bracket for its cartesian components.

Proposition 25.1.2 *Let us consider the observed kinetic angular momentum (see Definition 3.2.12)*

$$\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}] : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E},$$

where $c_{\mathtt{in}} : \boldsymbol{T} \to \boldsymbol{E}$ is the motion of the centre of the chosen cartesian coordinate chart adapted to $o_{\mathtt{in}}$.

We have the following coordinate expressions, in the cartesian spacetime chart,

$$\begin{aligned}
\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}] &= \tfrac{m}{\hbar_0}\,\epsilon_{ijh}\,r^i\,x^j_0\,\check{d}^h,\\
\mathfrak{L}^2[c_{\mathtt{in}}, o_{\mathtt{in}}] &= (\tfrac{m}{\hbar_0})^2\Big((r^1)^2\big((x^2_0)^2 + (x^3_0)^2\big) + (r^2)^2\big((x^1_0)^2 + (x^3_0)^2\big)\\
&\quad + (r^3)^2\big((x^1_0)^2 + (x^2_0)^2\big)\Big)\\
&\quad - (\tfrac{m}{\hbar_0})^2\big(2\,r^1\,r^2\,x^1_0\,x^2_0 + 2\,r^2\,r^3\,x^2_0\,x^3_0 + 2\,r^1\,r^3\,x^1_0\,x^3_0\big).
\end{aligned}$$

Moreover, let us define the components of $\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}]$ in the cartesian spacetime chart as

$$\mathfrak{L}_h := \partial_h \lrcorner\, \mathfrak{L} = \tfrac{m}{\hbar_0}\,\epsilon_{ijh}\,r^i\,x^j_0.$$

In the cartesian spacetime chart, the components $\mathfrak{L}_h$ turn out to be affine special phase functions (see Definitions 3.2.12 and 12.1.3)

$$\mathfrak{L}_h = \tfrac{m}{\hbar_0}\,\epsilon_{ijh}\,r^i\,x^j_0 \in \mathrm{aff}(J_1\boldsymbol{E}, I\!R).$$

In the cartesian spacetime chart, the Poisson bracket of the above components fulfills the standard identities (see Definition 11.4.1)

$$\{\mathfrak{L}_h, \mathfrak{L}_k\} = \epsilon_{hkr}\, \mathfrak{L}^r, \quad \text{where } \mathfrak{L}^r := \delta^{rs}\, \mathfrak{L}_s. \quad \square$$

Remark 25.1.3 We stress that the standard identity $\{\mathfrak{L}_h, \mathfrak{L}_k\} = \epsilon_{hkr}\, \mathfrak{L}^r$ concerning the Poisson bracket between the components of the observed kinetic angular momentum $\mathfrak{L}$ (see, for instance, [302]), as in the above Proposition 25.1.2, holds if we define the components $\mathfrak{L}_h := \partial_h \lrcorner \mathfrak{L}$ by means of a cartesian spacetime chart (see Definition 24.2.4).

Clearly, once the above functions $\mathfrak{L}_h := \partial_h \lrcorner \mathfrak{L}$ have been defined, the above Poisson identities are true independently of the chart we use to express the given functions, because the Poisson bracket $\{\mathfrak{L}_h, \mathfrak{L}_k\}$ of two given functions $\mathfrak{L}_h$ and $\mathfrak{L}_k$ is an intrinsic object.

We might define the components $\acute{\mathfrak{L}}_h := \acute{\partial}_h \lrcorner \mathfrak{L}$ by means of another generic spacetime chart $(x^0, \acute{x}^i)$, so obtaining another triplet of phase function $(\acute{\mathfrak{L}}_h)$.

For instance, we might define the cylindrical or spherical components

$$\begin{aligned} &\mathfrak{L}_\rho := \partial_\rho \lrcorner \mathfrak{L}, &&\mathfrak{L}_\phi := \partial_\phi \lrcorner \mathfrak{L}, &&\mathfrak{L}_z := \partial_z \lrcorner \mathfrak{L}, \\ &\mathfrak{L}_r := \partial_r \lrcorner \mathfrak{L}, &&\mathfrak{L}_\theta := \partial_\theta \lrcorner \mathfrak{L}, &&\mathfrak{L}_\phi := \partial_\phi \lrcorner \mathfrak{L}. \end{aligned}$$

We shall show (see Remarks 25.1.6 and 25.1.8) that these components do not fulfill the standard Poisson identities (even if would express these given functions by means of a cartesian spacetime chart).

By the way, we stress also that we should not confuse the spacetime chart used to define the phase functions $\mathfrak{L}_h := \partial_h \lrcorner \mathfrak{L}$ with the spacetime chart used to express these given functions (actually, these charts need not to be the same).

Indeed, the failure of the standard Poisson identities for the components defined via a generic non cartesian spacetime chart shows, once more, that the observed kinetic angular momentum is deeply associated with a possible affine structure of the fibres of spacetime. □

Remark 25.1.4 We stress that the square of the observed angular momentum $\mathfrak{L}^2$ is not a special phase function because the coefficient of the quadratic term is not the metric G (see Definition 12.1.1).

Hence, we cannot apply to this phase function $\mathfrak{L}^2$ our covariant machinery. □

25.1.2.2 Adapted Cylindrical Spacetime Chart

Then, we consider a cylindrical spacetime chart (x^0, ρ, ϕ, z) adapted to the inertial observer $o_{\mathtt{in}}$ (see Definition 24.2.5).

Proposition 25.1.5 *The non vanishing components of the metric tensor and of the gravitational spacetime connection are (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):*

$$\begin{gathered}
g_{\rho\rho} = 1 \in I\!R, \qquad g_{\phi\phi} = \rho^2 \in \mathbb{L}^2 \otimes I\!R, \qquad g_{zz} = 1 \in I\!R, \\
g^{\rho\rho} = 1 \in I\!R, \qquad g^{\phi\phi} = \tfrac{1}{\rho^2} \in \mathbb{L}^{-2} \otimes I\!R, \qquad g^{zz} = 1 \in I\!R, \\
G^0_{\rho\rho} = \tfrac{m}{\hbar_0} \in \mathbb{L}^{-2} \otimes I\!R, \qquad G^0_{\phi\phi} = \tfrac{m}{\hbar_0}\,\rho^2 \in I\!R, \qquad G^0_{zz} = \tfrac{m}{\hbar_0} \in \mathbb{L}^{-2} \otimes I\!R, \\
G_0^{\rho\rho} = \tfrac{\hbar_0}{m} \in \mathbb{L}^2 \otimes I\!R, \qquad G_0^{\phi\phi} = \tfrac{\hbar_0}{m}\,\tfrac{1}{\rho^2} \in I\!R, \qquad G_0^{zz} = \tfrac{\hbar_0}{m} \in \mathbb{L}^2 \otimes I\!R, \\
K^{\natural}{}_{\phi}{}^{\rho}{}_{\phi} = \rho, \qquad K^{\natural}{}_{\rho}{}^{\phi}{}_{\phi} = K^{\natural}{}_{\phi}{}^{\phi}{}_{\rho} = -\tfrac{1}{\rho}\,.
\end{gathered}$$

Hence, we obtain the equalities (see Proposition 3.2.4, Note 3.2.9 and Definition 3.2.12)

$$\begin{gathered}
\upsilon = \rho\, u_0 \otimes (d^0 \wedge d^\rho \wedge d^\phi \wedge d^z), \qquad \eta = \rho\, \check{d}^\rho \wedge \check{d}^\phi \wedge \check{d}^z, \\
\mathcal{K}[o_{\mathtt{in}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_0)^2 + \rho^2\,(\phi_0)^2 + (z_0)^2\big)\, d^0, \\
\mathcal{Q}[o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\,(\rho_0\, d^\rho + \rho^2\,\phi_0\, d^\phi + z_0\, d^z),
\end{gathered}$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$\begin{aligned}
K^{\natural} &= d^0 \otimes \partial_0 + d^\rho \otimes \partial_\rho + d^\phi \otimes \partial_\phi + d^z \otimes \partial_z + \rho\,\dot{\phi}\, d^\phi \otimes \dot{\partial}_\rho \\
&\quad - \tfrac{1}{\rho}\,(\dot{\phi}\, d^\rho + \dot{\rho}\, d^\phi) \otimes \dot{\partial}_\phi, \\
\Gamma^{\natural} &= d^0 \otimes \partial_0 + d^\rho \otimes \partial_\rho + d^\phi \otimes \partial_\phi + d^z \otimes \partial_z + \rho\,\phi_0\, d^\phi \otimes \partial^0_\rho \\
&\quad - \tfrac{1}{\rho}\,(\rho_0\, d^\phi + \phi_0\, d^\rho) \otimes \partial^0_\phi, \\
\gamma^{\natural} &= u^0 \otimes \big(\partial_0 + \rho_0\,\partial_\rho + \phi_0\,\partial_\phi + z_0\,\partial_z + \rho\,(\phi_0)^2\,\partial^0_\rho - 2\,\tfrac{1}{\rho}\,\rho_0\,\phi_0\,\partial^0_\phi\big), \\
\Omega^{\natural} &= \tfrac{m}{\hbar_0}\,\big(d^\rho_0 \wedge (d^\rho - \rho_0\, d^0) + \rho^2\, d^\phi_0 \wedge (d^\phi - \phi_0\, d^0) + d^z_0 \wedge (d^z - z_0\, d^0)\big) \\
&\quad + \tfrac{m}{\hbar_0}\,\rho\,\phi_0\, d^\rho \wedge (d^\phi - \phi_0\, d^0) + \tfrac{m}{\hbar_0}\,\rho\,\phi_0\, d^\rho \wedge d^\phi, \\
\Lambda^{\natural} &= \tfrac{\hbar_0}{m}\,(\partial_\rho \wedge \partial^0_\rho + \tfrac{1}{\rho^2}\,\partial_\phi \wedge \partial^0_\phi + \partial_z \wedge \partial^0_z) + 2\,\tfrac{\hbar_0}{m}\,\tfrac{1}{\rho}\,\phi_0\,\partial^0_\rho \wedge \partial^0_\phi. \quad \square
\end{aligned}$$

Next, we show that the phase functions defined as the cylindrical components of the observed kinetic angular momentum do not fulfill the standard Poisson identities.

Remark 25.1.6 Let us consider the observed kinetic angular momentum (see Definition 3.2.12)

$$\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}] : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E},$$

where $c_{\mathtt{in}} : \boldsymbol{T} \to \boldsymbol{E}$ is the motion of the centre of the chosen cartesian coordinate system adapted to $o_{\mathtt{in}}$. Moreover, let us define the components of $\mathfrak{L}[c_{\mathtt{in}}, o_{\mathtt{in}}]$ in the cylindrical spacetime chart as

$$\begin{aligned}
&\mathfrak{L}_\rho := \partial_\rho \lrcorner\, \mathfrak{L} = -\tfrac{m}{\hbar_0}\,\rho\, z\,\phi_0, \quad \mathfrak{L}_\phi := \partial_\phi \lrcorner\, \mathfrak{L} = \tfrac{m}{\hbar_0}\,\rho\,(z\,\rho_0 - \rho\, z_0), \\
&\mathfrak{L}_z := \partial_z \lrcorner\, \mathfrak{L} = \tfrac{m}{\hbar_0}\,\rho^2\,\phi_0.
\end{aligned}$$

Then, we have the following equalities

$$\{\mathfrak{L}_\rho, \mathfrak{L}_z\} = 0, \qquad \{\mathfrak{L}_\phi, \mathfrak{L}_z\} = \tfrac{m}{\hbar_0}\, \rho\, \phi_0 \big(\rho^2 + z^2\big), \qquad \{\mathfrak{L}_\rho, \mathfrak{L}_\phi\} = 0.$$

Hence, the cylindrical components of the observed kinetic angular momentum do not fulfill the standard Poisson identities. □

25.1.2.3 Adapted Spherical Spacetime Chart

Eventually, we consider a spherical spacetime chart (x^0, r, θ, ϕ) adapted to the inertial observer $o_{\mathtt{in}}$ (aee Definition 24.2.6).

Proposition 25.1.7 *The non vanishing components of the metric tensor and of the gravitational spacetime connection are (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):*

$$g_{rr} = 1 \in I\!R, \qquad g_{\theta\theta} = r^2 \in \mathbb{L}^2 \otimes I\!R, \qquad g_{\phi\phi} = r^2 \sin^2\theta \in \mathbb{L}^2 \otimes I\!R,$$
$$g^{rr} = 1 \in I\!R, \qquad g^{\theta\theta} = 1/(r^2) \in \mathbb{L}^{-2} \otimes I\!R, \qquad g^{\phi\phi} = 1/(r^2\, \sin^2\theta) \in \mathbb{L}^{-2} \otimes I\!R,$$
$$G^0_{rr} = \tfrac{m}{\hbar_0} \in \mathbb{L}^{-2} \otimes I\!R, \qquad G^0_{\theta\theta} = \tfrac{m}{\hbar_0}\, r^2 \in I\!R, \qquad G^0_{\phi\phi} = \tfrac{m}{\hbar_0}\, r^2\, \sin^2\theta \in I\!R,$$
$$G^{rr}_0 = \tfrac{\hbar_0}{m} \in \mathbb{L}^2 \otimes I\!R, \qquad G^{\theta\theta}_0 = \tfrac{\hbar_0}{m}\, 1/(r^2) \in I\!R, \qquad G^{\phi\phi}_0 = \tfrac{\hbar_0}{m}\, 1/(r^2\, \sin^2\theta) \in I\!R,$$
$$K^\natural{}_\theta{}^r{}_\theta = r, \qquad K^\natural{}_\phi{}^r{}_\phi = r\, \sin^2\theta,$$
$$K^\natural{}_\theta{}^\theta{}_r = K^\natural{}_r{}^\theta{}_\theta = -1/r, \qquad K^\natural{}_\phi{}^\theta{}_\phi = \sin\theta\, \cos\theta$$
$$K^\natural{}_\phi{}^\phi{}_r = K^\natural{}_r{}^\phi{}_\phi = -1/r, \qquad K^\natural{}_\theta{}^\phi{}_\phi = K^\natural{}_\phi{}^\phi{}_\theta = -\cos\theta/\sin\theta.$$

Hence, we obtain the equalities (see Proposition 3.2.4, Note 3.2.9 and Definition 3.2.12)

$$\upsilon = r^2\, \sin\theta\, u_0 \otimes (d^0 \wedge d^r \wedge d^\theta \wedge d^\phi), \qquad \eta = r^2\, \sin\theta\, \check{d}^r \wedge \check{d}^\theta \wedge \check{d}^\phi,$$
$$\mathcal{K}[o_{\mathtt{in}}] = \tfrac{1}{2}\, \tfrac{m}{\hbar_0} \big((r_0)^2 + r^2\, (\theta_0)^2 + r^2\, (\phi_0)^2\, \sin^2\theta\big)\, d^0,$$
$$\mathcal{Q}[o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\, (r_0\, d^r + r^2\, \theta_0\, d^\theta + r^2\, \phi_0\, \sin^2\theta\, d^\phi),$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$\begin{aligned} K^\natural = {} & d^0 \otimes \partial_0 + d^r \otimes \partial_r + d^\theta \otimes \partial_\theta + d^\phi \otimes \partial_\phi \\ & + r\, \big(\dot\theta\, d^\theta + \dot\phi\, \sin^2\theta\, d^\phi\big) \otimes \dot\partial_r \\ & + \big(-\tfrac{1}{r}\, (\dot\theta\, d^r + \dot r\, d^\theta) + \dot\phi\, \sin\theta\, \cos\theta\, d^\phi\big) \otimes \dot\partial_\theta \\ & + \big(-\tfrac{1}{r}\, (\dot\phi\, d^r + \dot r\, d^\phi) - \tfrac{\cos\theta}{\sin\theta}\, (\dot\phi\, d^\theta + \dot\theta\, d^\phi)\big) \otimes \dot\partial_\phi, \end{aligned}$$

$$\begin{aligned} \Gamma^\natural = {} & d^0 \otimes \partial_0 + d^r \otimes \partial_r + d^\theta \otimes \partial_\theta + d^\phi \otimes \partial_\phi \\ & + r\, \big(\theta_0\, d^\theta + \phi_0\, \sin^2\theta\, d^\phi\big) \otimes \partial^0_r \end{aligned}$$

$$+ \big(-\tfrac{1}{r}\ (\theta_0\, d^r + r_0\, d^\theta) + \phi_0\ \sin\theta\ \cos\theta\, d^\phi \big) \otimes \partial^0_\theta$$
$$+ \big(-\tfrac{1}{r}\ (\phi_0\, d^r + r_0\, d^\phi) - \tfrac{\cos\theta}{\sin\theta}\ (\phi_0\, d^\theta + \theta_0\, d^\phi) \big) \otimes \partial^0_\phi,$$

$$\Omega^\natural = \tfrac{m}{\hbar_0} \Big(d^r_0 \wedge (d^r - r_0\, d^0) + r^2\, d^\theta_0 \wedge (d^\theta - \theta_0\, d^0) + r^2\ \sin^2\theta\, d^\phi_0 \wedge (d^\phi - \phi_0\, d^0)$$
$$+ 2\, r\, \theta_0\, d^r \wedge d^\theta + 2\, r\, \phi_0 \sin^2\theta\, d^r \wedge d^\phi + 2\, r^2\, \phi_0\ \sin\theta\ \cos\theta\, d^\theta \wedge d^\phi$$
$$+ d^0 \wedge \Big(r\, \big((\theta_0)^2 + (\phi_0)^2\ \sin^2\theta \big)\, d^r + r^2\, (\phi_0)^2\ \sin\theta\ \cos\theta\, d^\theta \Big) \Big),$$

$$\gamma^\natural = u^0 \otimes \Big(\partial_0 + r_0\, \partial_r + \theta_0\, \partial_\theta + \phi_0\, \partial_\phi + r\, \big((\theta_0)^2 + (\phi_0)^2\ \sin^2\theta \big)\, \partial^0_r$$
$$+ \big(-2\, \tfrac{1}{r}\, r_0\, \theta_0 + (\phi_0)^2 \sin\theta\ \cos\theta \big)\, \partial^0_\theta - \phi_0\, \big(2\, \tfrac{1}{r}\, r_0 + 2\, \tfrac{\cos\theta}{\sin\theta}\, \theta_0 \big)\, \partial^0_\phi \Big),$$

$$\Lambda^\natural = \tfrac{\hbar_0}{m} \Big(\partial_r \wedge \partial^0_r + \tfrac{1}{r^2}\, \partial_\theta \wedge \partial^0_\theta + \tfrac{1}{r^2\, \sin^2\theta}\, \partial_\phi \wedge \partial^0_\phi$$
$$+ 2\, \tfrac{1}{r}\, \theta_0\, \partial^0_r \wedge \partial^0_\theta + 2\, \tfrac{1}{r}\, \phi_0\, \partial^0_r \wedge \partial^0_\phi + 2\, \tfrac{1}{r^2}\, \phi_0\, \tfrac{\cos\theta}{\sin\theta}\ \partial^0_\theta \wedge \partial^0_\phi \Big). \quad \square$$

Next, we show that the phase functions defined as the spherical components of the observed kinetic angular momentum do not fulfill the standard Poisson identities.

Remark 25.1.8 Let us consider the observed kinetic angular momentum (see Definition 3.2.12)

$$\mathfrak{L}[c_{\texttt{in}}, o_{\texttt{in}}] : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E},$$

where $c_{\texttt{in}} : \boldsymbol{T} \to \boldsymbol{E}$ is the motion of the centre of the chosen cartesian coordinate system adapted to $o_{\texttt{in}}$.

Moreover, let us define the components of $\mathfrak{L}[c_{\texttt{in}}, o_{\texttt{in}}]$ in the spherical spacetime chart as

$$\mathfrak{L}_r := \partial_r \lrcorner \mathfrak{L} = 0, \qquad \mathfrak{L}_\theta := \partial_\theta \lrcorner \mathfrak{L} = -\tfrac{m}{\hbar_0}\, r^3\, \phi_0\ \sin\theta, \qquad \mathfrak{L}_\phi := \partial_\phi \lrcorner \mathfrak{L} = \tfrac{m}{\hbar_0}\, r^3\, \theta_0\ \sin\theta.$$

Then, we have the following equalities

$$\{\mathfrak{L}_r, \mathfrak{L}_\theta\} = 0, \qquad \{\mathfrak{L}_r, \mathfrak{L}_\phi\} = 0, \qquad \{\mathfrak{L}_\theta, \mathfrak{L}_\phi\} = \tfrac{m}{\hbar_0}\, r^4\, \phi_0\ \sin\theta\ \cos\theta.$$

Hence, the spherical components of the observed kinetic angular momentum do not fulfill the standard Poisson identities. □

25.1.2.4 Joined Objects

By taking into account that, in the present case, the electromagnetic field F is vanishing, the joined spacetime objects coincide with the corresponding gravitational objects.

Proposition 25.1.9 *The equality $F = 0$ implies the following equalities (see Theorems 6.3.1, 9.2.1, 9.2.6, 9.2.8, 9.2.11, Corollary 6.3.3, and Definition 10.1.3)*

$$K = K^{\natural}, \qquad \Phi[o_{\mathtt{in}}] = \Phi^{\natural}[o_{\mathtt{in}}] = 0,$$
$$\Gamma = \Gamma^{\natural}, \qquad \gamma = \gamma^{\natural}, \qquad \Omega = \Omega^{\natural}, \qquad \Lambda = \Lambda^{\natural}. \quad \square$$

25.1.2.5 Distinguished Gauge

We select a *distinguished global classical gauge* $\mathrm{b}^{\square}$ by choosing a distinguished potential $A^{\uparrow}[\mathrm{b}^{\square}]$ of Ω.

Actually, this potential $A^{\uparrow}[\mathrm{b}^{\square}]$ yields a vanishing observed potential $A[\mathrm{b}^{\square}, o_{\mathtt{in}}]$ of the vanishing observed spacetime 2–form $\Phi[o_{\mathtt{in}}] = 0$.

This natural classical gauge $\mathrm{b}^{\square}$ plays a fundamental role in the subsequent classical and quantum developments. Indeed, we shall refer to this classical gauge $\mathrm{b}^{\square}$ for the choice of the upper quantum connection $\mathrm{Ч}^{\uparrow}$ (see Hypothesis Q.1).

Actually, other observers will suggest further less natural and less simple distinguished classical gauges. We shall discuss and compare these gauges, in the framework of the classical theory and of the quantum theory.

Proposition 25.1.10 *The coordinate expression of Ω in a cartesian spacetime chart adapted to $o_{\mathtt{in}}$ (see Proposition 25.1.1)*

$$\Omega = \tfrac{m}{\hbar_0}\, \delta_{ij}\, d^i_0 \wedge \theta^j$$

shows that there exists a distinguished global potential of Ω (see Definitions 3.2.9, 10.1.3, Remark 10.1.6 and Notation 10.1.7)

$$A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{\mathtt{in}}],$$

where $\mathrm{b}^{\square}$ denotes the associated classical gauge.

In cartesian, cylindrical and spherical spacetime charts, respectively, we have the following coordinate expressions

$$\begin{aligned} A^{\uparrow}[\mathrm{b}^{\square}] &= \mathcal{C}[o_{\mathtt{in}}] \\ &= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x^i_0\, x^j_0\, d^0 + \tfrac{m}{\hbar_0}\, \delta_{ij}\, x^j_0\, d^i \\ &= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0} \big((\rho_0)^2 + \rho^2\, (\phi_0)^2 + (z_0)^2\big)\, d^0 \\ &\quad + \tfrac{m}{\hbar_0}\, (\rho_0\, d^{\rho} + \rho^2\, \phi_0\, d^{\phi} + z_0\, d^z) \\ &= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0} \big((r_0)^2 + r^2\, (\theta_0)^2 + r^2\, (\phi_0)^2\, \sin^2\theta\big)\, d^0 \\ &\quad + \tfrac{m}{\hbar_0} \big(r_0\, d^r + r^2\, \theta_0\, d^{\theta} + r^2\, \phi_0\, \sin^2\theta\, d^{\phi}\big). \end{aligned}$$

Then, the above potential $A^{\uparrow}[\mathrm{b}^{\square}]$ *yields the following observed potential* $A[\mathrm{b}^{\square}, o_{\mathtt{in}}]$ *of the spacetime 2–form* $\Phi[o_{\mathtt{in}}] = 0$ *(see Theorem 10.1.4, Remark 10.1.6 and Notation 10.1.7)*

$$A[\mathrm{b}^{\square}, o_{\mathtt{in}}] := o^*{}_{\mathtt{in}}\, A^{\uparrow}[\mathrm{b}^{\square}] = 0. \quad \square$$

25.1.2.6 Distinguished Phase 1–Forms

With reference to the above distinguished classical gauge $\mathrm{b}^{\square}$, hence to the vanishing observed potential $A[\mathrm{b}^{\square}, o_{\mathtt{in}}] = 0$, we obtain the observed hamiltonian $\mathcal{H}[\mathrm{b}^{\square}, o_{\mathtt{in}}]$ and the observed momentum $\mathcal{P}[\mathrm{b}^{\square}, o_{\mathtt{in}}]$.

Proposition 25.1.11 *We have the following distinguished phase 1–forms, expressed in a cartesian, cylindrical and spherical spacetime chart adapted to* $o_{\mathtt{in}}$*, respectively, (see Definition 3.2.9, Theorem 10.1.8 and Example 12.1.4)*

$$\begin{aligned}
\mathcal{H}[\mathrm{b}^{\square}, o_{\mathtt{in}}] = \mathcal{K}[o_{\mathtt{in}}] &= \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x^i_0\,x^j_0\,d^0 \\
&= \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_0)^2 + \rho^2\,(\phi_0)^2 + (z_0)^2\big)\,d^0 \\
&= \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\Big((r_0)^2 + r^2\,(\theta_0)^2 + r^2\,(\phi_0)^2\,\sin^2\theta\Big)\,d^0, \\
\mathcal{P}[\mathrm{b}^{\square}, o_{\mathtt{in}}] = \mathcal{Q}[o_{\mathtt{in}}] &= \tfrac{m}{\hbar_0}\,\delta_{ij}\,x^j_0\,d^i \\
&= \tfrac{m}{\hbar_0}\,(\rho_0\,d^{\rho} + \rho^2\,\phi_0\,d^{\phi} + z_0\,d^z) \\
&= \tfrac{m}{\hbar_0}\,\Big(r_0\,d^r + r^2\,\theta_0\,d^{\theta} + r^2\,\phi_0\,\sin^2\theta\,d^{\phi}\Big). \quad \square
\end{aligned}$$

25.1.3 Uniformly Accelerated Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics and, with reference to the uniformly accelerated observer $o_{\mathtt{ac}}$, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.1, Definitions 24.1.1 and 24.4.1).

In order to show how our automatic formalism works, we perform these computations in an adapted cartesian spacetime chart. Moreover, as an exercise, we emphasise two further distinguished classical gauges $\mathrm{b}^{\triangleright}$ and $\acute{\mathrm{b}}^{\triangleright}$ and compare them with $\mathrm{b}^{\square}$.

For the analysis of an accelerated quantum particle in an einsteinian framework, see also [140, 300, 412].

Let us consider the uniformly accelerated observer $o_{\mathtt{ac}}$ defined in Definition 24.4.1.

25.1.3.1 Adapted Cartesian Spacetime Chart

We consider the cartesian spacetime chart $(x^0, x^i{}_{\mathtt{ac}})$ adapted to the uniformly accelerated observer $o_{\mathtt{ac}}$ (see Definition 24.4.2).

Proposition 25.1.12 *Let us refer to the uniformly accelerated observer $o_{\mathtt{ac}}$ and to the adapted cartesian spacetime chart $(x^0, x_{\mathtt{ac}}{}^i)$ (see Definitions 24.4.1 and 24.4.2).*

We have the following coordinate expressions of the metric tensor and of the non vanishing component of the gravitational connection (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):

$$g_{\mathtt{ac}\,ij} = \delta_{ij}, \quad g_{\mathtt{ac}}{}^{ij} = \delta^{ij}, \qquad G_{\mathtt{ac}}{}^{0}_{ij} = \tfrac{m}{\hbar_0}\,\delta_{ij}, \quad G_{\mathtt{ac}}{}^{ij}_{0} = \tfrac{\hbar_0}{m}\,\delta^{ij},$$
$$K^{\natural}{}_{\mathtt{ac}\,0}{}^{1}{}_{0} = -a_{00}.$$

Hence, we obtain the equalities (see Proposition 3.2.4 and Definition 3.2.9)

$$\upsilon = u_0 \otimes (d^0 \wedge d_{\mathtt{ac}}{}^1 \wedge d_{\mathtt{ac}}{}^2 \wedge d_{\mathtt{ac}}{}^3), \qquad \eta = \check{d}_{\mathtt{ac}}{}^1 \wedge \check{d}_{\mathtt{ac}}{}^2 \wedge \check{d}_{\mathtt{ac}}{}^3,$$
$$\mathcal{K}[o_{\mathtt{ac}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(\delta_{ij}\, x_{\mathtt{ac}}{}^i_0\, x_{\mathtt{ac}}{}^j_0)\, d^0, \qquad \mathcal{Q}[o_{\mathtt{ac}}] = \tfrac{m}{\hbar_0}\,\delta_{ij}\, x_{\mathtt{ac}}{}^i_0\, d_{\mathtt{ac}}{}^j,$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$K^{\natural} = d_{\mathtt{ac}}{}^{\lambda} \otimes \partial_{\mathtt{ac}\,\lambda} - a_{00}\,\dot{x}^0\, d^0 \otimes \dot{\partial}_{\mathtt{ac}\,1},$$
$$\Gamma^{\natural} = d_{\mathtt{ac}}{}^{\lambda} \otimes \partial_{\mathtt{ac}\,\lambda} - a_{00}\, d^0 \otimes \partial_{\mathtt{ac}}{}^0_1,$$
$$\gamma^{\natural} = u^0 \otimes (\partial_{\mathtt{ac}\,0} + x_{\mathtt{ac}}{}^i_0\,\partial_{\mathtt{ac}\,i} - a_{00}\,\partial_{\mathtt{ac}}{}^0_1),$$
$$\Omega^{\natural} = \tfrac{m}{\hbar_0}\,\delta_{ij}\, d_{\mathtt{ac}}{}^i_0 \wedge (d_{\mathtt{ac}}{}^j - x_{\mathtt{ac}}{}^j_0\, d^0) + \tfrac{m}{\hbar_0}\, a_{00}\, d^0 \wedge d_{\mathtt{ac}}{}^1,$$
$$\Lambda^{\natural} = \tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathtt{ac}\,i} \wedge \partial_{\mathtt{ac}}{}^0_j. \quad \square$$

25.1.3.2 Distinguished 1–Forms

We start by discussing the observed potential $A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] = 0$, with respect to the classical gauge $\mathfrak{b}^{\square}$ (see Proposition 25.1.10).

Then, we analyse the associated observed hamiltonian $\mathcal{H}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}]$, observed momentum $\mathcal{P}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}]$ and the upper quantum potential $A^{\uparrow}[\mathfrak{b}^{\square}]$.

Proposition 25.1.13 *The distinguished classical gauge $\mathfrak{b}^{\square}$ suggested by the inertial observer $o_{\mathtt{in}}$ yields the following observed potential of the observed spacetime 2–form $\Phi[o_{\mathtt{ac}}]$, which is associated with the same classical gauge $\mathfrak{b}^{\square}$,*

$$A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] := o_{\mathtt{ac}}{}^{*} A^{\uparrow}[\mathfrak{b}^{\square}].$$

Then, in virtue of the transition rule (see Remark 10.1.6 Notation 10.1.7 and also Theorem 15.2.26)

$$A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] = A[\mathfrak{b}^{\square}, o_{\mathtt{in}}] + \theta[o_{\mathtt{in}}] \lrcorner\, G^{\flat}(\vec{v}_{\mathtt{ac}}) - \tfrac{1}{2}\, G(\vec{v}_{\mathtt{ac}}, \vec{v}_{\mathtt{ac}}),$$

we obtain the following equality, in a cartesian spacetime chart adapted to $o_{\mathtt{in}}$, (see Definition 24.4.1)

$$A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] = \frac{m}{\hbar_0}\big(-\tfrac{1}{2}(a_{00}\,x^0 + \nu_0)^2\,d^0 + (a_{00}\,x^0 + \nu_0)\,d^1\big).$$

i.e., in a cartesian spacetime chart adapted to $o_{\mathtt{ac}}$,

$$A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] = \frac{m}{\hbar_0}\big(\tfrac{1}{2}(a_{00}\,x^0 + \nu_0)^2\,d^0 + (a_{00}\,x^0 + \nu_0)\,d_{\mathtt{ac}}{}^1\big). \quad \square$$

Corollary 25.1.14 *We have the following phase 1–forms (see Example 12.1.4)*

$$\begin{aligned}\mathcal{H}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] &= \mathcal{K}[o_{\mathtt{ac}}] - \partial[o_{\mathtt{ac}}] \lrcorner A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] \\ &= \frac{m}{\hbar_0}\big(\tfrac{1}{2}(\delta_{ij}\,x_{\mathtt{ac}}{}^i_0\,x_{\mathtt{ac}}{}^j_0) - \tfrac{1}{2}(a_{00}\,x^0 + \nu_0)^2\big)\,d^0, \\ \mathcal{P}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] &= \mathcal{Q}[o_{\mathtt{ac}}] + \theta[o_{\mathtt{ac}}] \lrcorner A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] \\ &= \frac{m}{\hbar_0}\big(\delta_{ij}\,x_{\mathtt{ac}}{}^i_0\,d_{\mathtt{ac}}{}^j + (a_{00}\,x^0 + \nu_0)\,d_{\mathtt{ac}}{}^1\big).\end{aligned}$$

Exercise 25.1.15 Let us consider the observer independent upper potential $A^{\uparrow}[\mathfrak{b}^{\square}]$, which has been defined, with reference to the distinguished observer $o_{\mathtt{in}}$, as (see Proposition 25.1.10)

$$A^{\uparrow}[\mathfrak{b}^{\square}] := \mathcal{C}[o_{\mathtt{in}}].$$

Then, this upper potential can be also written, with reference to the observer $o_{\mathtt{ac}}$, in the following way

$$\begin{aligned}A^{\uparrow}[\mathfrak{b}^{\square}] &= \mathcal{C}[o_{\mathtt{ac}}] + A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] = -\mathcal{H}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] + \mathcal{P}[\mathfrak{b}^{\square}, o_{\mathtt{ac}}] \\ &= -\frac{m}{\hbar_0}\big(\tfrac{1}{2}(\delta_{ij}\,x_{\mathtt{ac}}{}^i_0\,x_{\mathtt{ac}}{}^j_0) - \tfrac{1}{2}(a_{00}\,x^0 + \nu_0)^2\big)\,d^0 \\ &\quad + \frac{m}{\hbar_0}\big(\delta_{ij}\,x_{\mathtt{ac}}{}^i_0\,d_{\mathtt{ac}}{}^j + (a_{00}\,x^0 + \nu_0)\,d_{\mathtt{ac}}{}^1\big). \quad \square\end{aligned}$$

Exercise 25.1.16 Check the equality (see Proposition 25.1.12, Theorem 10.1.4 and Corollary 9.2.4)

$$dA^{\uparrow}[\mathfrak{b}^{\square}] = \Omega,$$

by using the expression

$$A^{\uparrow}[\mathfrak{b}^{\square}] = \mathcal{C}[o_{\mathtt{ac}}] + A[\mathfrak{b}^{\square}, o_{\mathtt{ac}}]. \quad \square$$

Exercise 25.1.17 With reference to the classical gauge $\mathfrak{b}^{\square}$ and the observers $o_{\mathtt{in}}$, $o_{\mathtt{ac}}$, let us check that the timelike 1–form (see Note 10.1.11 and Proposition 15.2.29)

$$\alpha[\mathfrak{b}^{\square}] = д[o] \lrcorner A[\mathfrak{b}^{\square}, o] - \tfrac{1}{2}\,\check{A}[\mathfrak{b}^{\square}, o] \lrcorner \vec{A}[\mathfrak{b}^{\square}, o]$$

is observer equivariant, even more, in the present case, that it is vanishing, according to Corollary 15.2.28.

Indeed, in virtue of Proposition 25.1.13, we obtain the equalities

$$д[o_{\mathtt{in}}] \lrcorner A[\mathrm{b}^{\square}, o_{\mathtt{in}}] - \tfrac{1}{2}\check{A}[\mathrm{b}^{\square}, o_{\mathtt{in}}] \lrcorner \vec{A}[\mathrm{b}^{\square}, o_{\mathtt{in}}] = 0,$$
$$д[o_{\mathtt{ac}}] \lrcorner A[\mathrm{b}^{\square}, o_{\mathtt{ac}}] - \tfrac{1}{2}\check{A}[\mathrm{b}^{\square}, o_{\mathtt{ac}}] \lrcorner \vec{A}[\mathrm{b}^{\square}, o_{\mathtt{ac}}] = 0. \quad \square$$

25.1.3.3 Further Distinguished Gauges Suggested by the Cartesian Chart

Besides the distinguished classical gauge $\mathrm{b}^{\square}$, that we have been already selected with reference to the observer $o_{\mathtt{in}}$ (see Proposition 25.1.10), the uniformly accelerated observer $o_{\mathtt{ac}}$ suggests, for instance, two further distinguished classical gauges $\mathrm{b}^{\triangleright}$ and $\acute{\mathrm{b}}^{\triangleright}$, which provide two distinguished observed potentials $A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}]$ and $A[\acute{\mathrm{b}}^{\triangleright}, o_{\mathtt{ac}}]$ of the observed spacetime 2–form $\Phi[o_{\mathtt{ac}}]$. Then, we compare the distinguished gauges $\mathrm{b}^{\square}$, $\mathrm{b}^{\triangleright}$ and $\acute{\mathrm{b}}^{\triangleright}$.

This comparison offers an opportunity to check our general machinery on the transition rules for observed potentials and upper potentials.

Note 25.1.18 The expression of K yields the equality (see Definition 4.2.11)

$$\Phi[o_{\mathtt{ac}}] = 2\,\tfrac{m}{\hbar_0}\,a_{00}\,d^0 \wedge d_{\mathtt{ac}}{}^1.$$

Therefore, there exists, for instance,

(i) a global classical gauge $\mathrm{b}^{\triangleright}$ such that the observed potential of $\Phi[o_{\mathtt{ac}}]$ is (see Theorem 4.3.3)

$$A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}] = \tfrac{m}{\hbar_0}\,a_{00}\,x^0\,d_{\mathtt{ac}}{}^1$$
$$= \tfrac{m}{\hbar_0}\,a_{00}\,x^0\,(d^1 - (a_{00}\,x^0 + \nu_0)\,d^0),$$

(ii) a global classical gauge $\acute{\mathrm{b}}^{\triangleright}$, such that the observed potential of $\Phi[o_{\mathtt{ac}}]$ is (see Theorem 4.3.3)

$$A[\acute{\mathrm{b}}^{\triangleright}, o_{\mathtt{ac}}] = -\tfrac{m}{\hbar_0}\,a_{00}\,x_{\mathtt{ac}}{}^1\,d^0$$
$$= -\tfrac{m}{\hbar_0}\,a_{00}\,(x^1 - \tfrac{1}{2}\,a_{00}\,(x^0)^2 - \nu_0\,x^0)\,d^0. \quad \square$$

Remark 25.1.19 The difference of the two observed potentials above is given by the differential of a spacetime function as follows

$$A[\acute{\mathrm{b}}^{\triangleright}, o_{\mathtt{ac}}] - A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}] = \tfrac{m}{\hbar_0}\,a_{00}\,d(x^0\,x_{\mathtt{ac}}{}^1).$$

Hence, in the classical context, we might remove the exact terms from these observed potential and write, up to an exact term,

$$A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}] \simeq A[\acute{\mathrm{b}}^{\triangleright}, o_{\mathtt{ac}}].$$

But, in the quantum context, once we have chosen the upper quantum connection $Ч^{\uparrow}$, the above removal and identification are forbidden by the consistency of the expressions of the gauge independent and observer independent objects derived from upper quantum connection $Ч^{\uparrow}$. □

In order to compare the classical gauges $\mathrm{b}^{\Box}$ and $\mathrm{b}^{\triangleright}$, which have been defined by means of different observers, we need to pass through the observer independent upper potentials $A^{\uparrow}[\mathrm{b}^{\Box}]$ and $A^{\uparrow}[\mathrm{b}^{\triangleright}]$, in the following way.

Corollary 25.1.20 *The upper potentials $A^{\uparrow}[\mathrm{b}^{\Box}]$ and $A^{\uparrow}[\mathrm{b}^{\triangleright}]$ are defined by the equalities (see Theorem 10.1.4, Proposition 25.1.10 and Note 25.1.18)*

$$A^{\uparrow}[\mathrm{b}^{\Box}] := \mathcal{C}[o_{\mathtt{in}}] = \mathcal{C}[o_{\mathtt{ac}}] + A[\mathrm{b}^{\Box}, o_{\mathtt{ac}}], \qquad A^{\uparrow}[\mathrm{b}^{\triangleright}] := \mathcal{C}[o_{\mathtt{ac}}] + A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}].$$

Then, a comparison of the above equalities yields

$$A^{\uparrow}[\mathrm{b}^{\Box}] = A^{\uparrow}[\mathrm{b}^{\triangleright}] + d\vartheta, \qquad A[\mathrm{b}^{\Box}, o_{\mathtt{ac}}] = A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}] + d\vartheta,$$

where

$$\vartheta := \tfrac{m}{\hbar_0}\, d\big(v_0\, x_{\mathtt{ac}}{}^1 + \tfrac{1}{6}\, a_{00}^2\, (x^0)^3 + \tfrac{1}{2}\, v_0^2\, x^0 + \tfrac{1}{2}\, a_{00}\, v_0\, (x^0)^2\big). \quad \Box$$

25.1.4 Uniformly Rotating Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics and, with reference to the uniformly rotating observer $o_{\mathtt{ro}}$, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.1, Definitions 24.1.1 and 24.5.1).

In order to show how our automatic formalism works, we perform these computations in adapted cartesian and cylindrical spacetime charts.

Moreover, as an exercise, we emphasise four further distinguished classical gauges $\mathrm{b}^{\Diamond}$ and $\acute{\mathrm{b}}^{\Diamond}$, $\mathrm{b}^{\odot}$ and $\acute{\mathrm{b}}^{\odot}$.

25.1.4.1 Adapted Cartesian Spacetime Chart

We consider first a cartesian spacetime chart $(x^0, x^i{}_{\mathtt{ro}})$ adapted to the uniformly rotating observer $o_{\mathtt{ro}}$ (see Definition 24.5.2).

Proposition 25.1.21 *Let us refer to the uniformly rotating observer $o_{\mathtt{ro}}$ and to the adapted spacetime chart $(x^0, x_{\mathtt{ro}}{}^i)$ (see Definitions 24.5.1 and 24.5.2).*

We have the following components of the metric tensor and non vanishing components of the gravitational connection (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):

$$g_{\mathrm{ro}\,ij} = \delta_{ij}, \qquad g_{\mathrm{ro}}{}^{ij} = \delta^{ij}, \qquad G_{\mathrm{ro}}{}^{0}_{ij} = \tfrac{m}{\hbar_0}\,\delta_{ij}, \qquad G_{\mathrm{ro}}{}^{ij}_{0} = \tfrac{\hbar_0}{m}\,\delta^{ij},$$
$$K^{\natural}{}_{\mathrm{ro}\,0}{}^{1}{}_{0} = (\omega_0)^2\,x_{\mathrm{ro}}{}^{1}, \qquad K^{\natural}{}_{\mathrm{ro}\,0}{}^{2}{}_{0} = (\omega_0)^2\,x_{\mathrm{ro}}{}^{2},$$
$$K^{\natural}{}_{\mathrm{ro}\,2}{}^{1}{}_{0} = K^{\natural}{}_{\mathrm{ro}\,0}{}^{1}{}_{2} = \omega_0, \qquad K^{\natural}{}_{\mathrm{ro}\,1}{}^{2}{}_{0} = K^{\natural}{}_{\mathrm{ro}\,0}{}^{2}{}_{1} = -\omega_0.$$

Hence, we obtain the equalities (see Proposition 3.2.4 and Definition 3.2.9)

$$\upsilon = u_0 \otimes (d^0 \wedge d_{\mathrm{ro}}{}^{1} \wedge d_{\mathrm{ro}}{}^{2} \wedge d_{\mathrm{ac}}{}^{3}), \qquad \eta = \check{d}_{\mathrm{ro}}{}^{1} \wedge \check{d}_{\mathrm{ro}}{}^{2} \wedge \check{d}_{\mathrm{ro}}{}^{3},$$
$$\mathcal{K}[o_{\mathrm{ro}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(\delta_{ij}\,x_{\mathrm{ro}}{}^{i}_{0}\,x_{\mathrm{ro}}{}^{j}_{0})\,d^0, \qquad \mathcal{Q}[o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ro}}{}^{i}_{0}\,d_{\mathrm{ro}}{}^{j},$$
$$\mathfrak{L}[c_{\mathrm{in}}, o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\epsilon_{ijh}\,x_{\mathrm{ro}}{}^{i}\,x_{\mathrm{ro}}{}^{j}_{0}\,\check{d}_{\mathrm{ro}}{}^{h},$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$\begin{aligned} K^{\natural} &= d_{\mathrm{ro}}{}^{\lambda} \otimes \partial_{\mathrm{ro}\,\lambda} + (\omega_0)^2\,x_{\mathrm{ro}}{}^{1}\,\dot{x}^0\,d^0 \otimes \dot{\partial}_{\mathrm{ro}\,1} + (\omega_0)^2\,x_{\mathrm{ro}}{}^{2}\,\dot{x}^0\,d^0 \otimes \dot{\partial}_{\mathrm{ro}\,2} \\ &\quad + \omega_0\,(\dot{x}^0\,d_{\mathrm{ro}}{}^{2} + \dot{x}_{\mathrm{ro}}{}^{2}\,d^0) \otimes \dot{\partial}_{\mathrm{ro}\,1} - \omega_0\,(\dot{x}^0\,d_{\mathrm{ro}}{}^{1} + \dot{x}_{\mathrm{ro}}{}^{1}\,d^0) \otimes \dot{\partial}_{\mathrm{ro}\,2}, \end{aligned}$$

$$\begin{aligned} \Gamma^{\natural} &= d_{\mathrm{ro}}{}^{\lambda} \otimes \partial_{\mathrm{ro}\,\lambda} + \omega_0\,(x_{\mathrm{ro}}{}^{2}_{0} + \omega_0\,x_{\mathrm{ro}}{}^{1})\,d^0 \otimes \partial_{\mathrm{ro}}{}^{0}_{1} \\ &\quad - \omega_0\,(x_{\mathrm{ro}}{}^{1}_{0} - \omega_0\,x_{\mathrm{ro}}{}^{2})\,d^0 \otimes \partial_{\mathrm{ro}}{}^{0}_{2} + \omega_0\,(d_{\mathrm{ro}}{}^{2} \otimes \partial_{\mathrm{ro}}{}^{0}_{1} - d_{\mathrm{ro}}{}^{1} \otimes \partial_{\mathrm{ro}}{}^{0}_{2}), \end{aligned}$$

$$\begin{aligned} \gamma^{\natural} &= u^0 \otimes (\partial_{\mathrm{ro}\,0} + x_{\mathrm{ro}}{}^{i}_{0}\,\partial_{\mathrm{ro}\,i}) \\ &\quad + u^0 \otimes \big(\omega_0\,(\omega_0\,x_{\mathrm{ro}}{}^{1} + 2\,x_{\mathrm{ro}}{}^{2}_{0})\,\partial_{\mathrm{ro}}{}^{0}_{1} + \omega_0\,(\omega_0\,x_{\mathrm{ro}}{}^{2} - 2\,x_{\mathrm{ro}}{}^{1}_{0})\,\partial_{\mathrm{ro}}{}^{0}_{2}\big), \end{aligned}$$

$$\begin{aligned} \Omega^{\natural} &= \tfrac{m}{\hbar_0}\,\delta_{ij}\,d_{\mathrm{ro}}{}^{i}_{0} \wedge (d_{\mathrm{ro}}{}^{j} - x_{\mathrm{ro}}{}^{j}_{0}\,d^0) \\ &\quad - \tfrac{m}{\hbar_0}\,(\omega_0)^2\,d_{\mathrm{ro}}{}^{0} \wedge (x_{\mathrm{ro}}{}^{1}\,d_{\mathrm{ro}}{}^{1} + x_{\mathrm{ro}}{}^{2}\,d_{\mathrm{ro}}{}^{2}) + 2\,\tfrac{m}{\hbar_0}\,\omega_0\,d_{\mathrm{ro}}{}^{1} \wedge d_{\mathrm{ro}}{}^{2}, \end{aligned}$$

$$\Lambda^{\natural} = \tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ro}\,i} \wedge \partial_{\mathrm{ro}}{}^{0}_{j} - 2\,\tfrac{\hbar_0}{m}\,\omega_0\,\partial_{\mathrm{ro}}{}^{0}_{1} \wedge \partial_{\mathrm{ro}}{}^{0}_{2}. \quad \square$$

25.1.4.2 Adapted Cylindrical Spacetime Chart

Then, we consider the cylindrical spacetime chart $(x^0, \rho_{\mathrm{ro}}, \phi_{\mathrm{ro}}, z_{\mathrm{ro}})$ adapted to the uniformly rotating observer o_{ro} (see Definition 24.5.3).

Proposition 25.1.22 *Let us refer to the uniformly rotating observer o_{ro} and to the adapted spacetime chart $(x^0,\ \rho_{\mathrm{ro}},\ \phi_{\mathrm{ro}}, z_{\mathrm{ro}})$ (see Definition 24.5.3).*

We have the following non vanishing components of the metric tensor and of the gravitational spacetime connection (see Definition 3.2.1, Theorem 4.2.13 and Corollary 9.2.4):

$$g_{\mathrm{ro}\,\rho\rho} = 1, \qquad g_{\mathrm{ro}\,\phi\phi} = {\rho_{\mathrm{ro}}}^2, \qquad g_{\mathrm{ro}\,zz} = 1,$$
$$g_{\mathrm{ro}}{}^{\rho\rho} = 1, \qquad g_{\mathrm{ro}}{}^{\phi\phi} = \tfrac{1}{{\rho_{\mathrm{ro}}}^2}\,, \qquad g_{\mathrm{ro}}{}^{zz} = 1,$$
$$G_{\mathrm{ro}}{}^{0}_{\rho\rho} = \tfrac{m}{\hbar_0}\,, \qquad G_{\mathrm{ro}}{}^{0}_{\phi\phi} = \tfrac{m}{\hbar_0}\,{\rho_{\mathrm{ro}}}^2, \qquad G_{\mathrm{ro}}{}^{0}_{zz} = \tfrac{m}{\hbar_0}\,,$$
$$G_{\mathrm{ro}}{}^{\rho\rho}_{0} = \tfrac{\hbar_0}{m}\,, \qquad G_{\mathrm{ro}}{}^{\phi\phi}_{0} = \tfrac{\hbar_0}{m}\,\tfrac{1}{{\rho_{\mathrm{ro}}}^2}\,, \qquad G_{\mathrm{ro}}{}^{zz}_{0} = \tfrac{\hbar_0}{m}\,,$$
$$K^{\natural}{}_{\mathrm{ro}\,0}{}^{\rho}{}_{0} = (\omega_0)^2\,\rho_{\mathrm{ro}}\,,$$
$$K^{\natural}{}_{\mathrm{ro}\,0}{}^{\rho}{}_{\phi} = K^{\natural}{}_{\mathrm{ro}\,\phi}{}^{\rho}{}_{0} = \omega_0\,\rho_{\mathrm{ro}}\,, \qquad K^{\natural}{}_{\mathrm{ro}\,\phi}{}^{\rho}{}_{\phi} = \rho_{\mathrm{ro}}\,,$$
$$K^{\natural}{}_{\mathrm{ro}\,0}{}^{\phi}{}_{\rho} = K^{\natural}{}_{\mathrm{ro}\,\rho}{}^{\phi}{}_{0} = -\omega_0\,\tfrac{1}{\rho_{\mathrm{ro}}}\,, \qquad K^{\natural}{}_{\mathrm{ro}\,\rho}{}^{\phi}{}_{\phi} = K^{\natural}{}_{\mathrm{ro}\,\phi}{}^{\phi}{}_{\rho} = -\tfrac{1}{\rho_{\mathrm{ro}}}\,.$$

Hence, we obtain the equalities (see Proposition 3.2.4 and Definition 3.2.9)

$$\upsilon = \rho_{\mathrm{ro}}\;u_0 \otimes d^0 \wedge d_{\mathrm{ro}}{}^{\rho} \wedge d_{\mathrm{ro}}{}^{\phi} \wedge d_{\mathrm{ro}}{}^{z}, \qquad \eta = \rho_{\mathrm{ro}}\;\check{d}_{\mathrm{ro}}{}^{\rho} \wedge \check{d}_{\mathrm{ro}}{}^{\phi} \wedge \check{d}_{\mathrm{ro}}{}^{z},$$

$$\mathcal{K}[o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,\big((\rho_{\mathrm{ro}\,0})^2 + (\rho_{\mathrm{ro}})^2\,(\phi_{\mathrm{ro}\,0})^2 + (z_{\mathrm{ro}\,0})^2\big)\,d^0,$$
$$\mathcal{Q}[o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\big(\rho_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\rho} + (\rho_{\mathrm{ro}})^2\,\phi_{\mathrm{ro}\,0}\,d^{\phi}{}_{\mathrm{ro}} + z_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{z}),$$

and (see Theorem 4.2.13 and Corollary 9.2.4)

$$\begin{aligned}
K^{\natural} &= d^0 \otimes \partial_{\mathrm{ro}\,0} + d_{\mathrm{ro}}{}^{\rho} \otimes \partial_{\mathrm{ro}\,\rho} + d_{\mathrm{ro}}{}^{\phi} \otimes \partial_{\mathrm{ro}\,\phi} + d_{\mathrm{ro}}{}^{z} \otimes \partial_{\mathrm{ro}\,z} \\
&\quad + (\omega_0)^2\,\rho\,\dot{x}^0\,d^0 \otimes \dot{\partial}_{\mathrm{ro}\,\rho} \\
&\quad + \omega_0\,\rho\,(\dot{\phi}_{\mathrm{ro}}\;d^0 + \dot{x}^0\,d_{\mathrm{ro}}{}^{\phi}) \otimes \dot{\partial}_{\mathrm{ro}\,\rho} \\
&\quad - \omega_0\,\tfrac{1}{\rho_{\mathrm{ro}}}\;(\dot{\rho}_{\mathrm{ro}}\;d^0 + \dot{x}^0\,d_{\mathrm{ro}}{}^{\rho}{}_{\mathrm{ro}}) \otimes \dot{\partial}_{\mathrm{ro}\,\phi} \\
&\quad + \rho\,\dot{\phi}_{\mathrm{ro}}\;d_{\mathrm{ro}}{}^{\phi} \otimes \dot{\partial}_{\mathrm{ro}\,\rho} - \tfrac{1}{\rho}\,(\dot{\rho}_{\mathrm{ro}}\;d_{\mathrm{ro}}{}^{\phi} + \dot{\phi}_{\mathrm{ro}}\;d_{\mathrm{ro}}{}^{\rho}) \otimes \dot{\partial}_{\mathrm{ro}\,\phi},
\end{aligned}$$

$$\begin{aligned}
\Gamma^{\natural} &= d^0 \otimes \partial_{\mathrm{ro}\,0} + d_{\mathrm{ro}}{}^{\rho} \otimes \partial_{\mathrm{ro}\,\rho} + d_{\mathrm{ro}}{}^{\phi} \otimes \partial_{\mathrm{ro}\,\phi} + d_{\mathrm{ro}}{}^{z} \otimes \partial_{\mathrm{ro}\,z} \\
&\quad + (\omega_0)^2\,d^0 \otimes \partial_{\mathrm{ro}}{}^{0}_{\rho} - \omega_0\,\rho_{\mathrm{ro}}\;(\phi_{\mathrm{ro}\,0}\,d^0 + d_{\mathrm{ro}}{}^{\phi}) \otimes \partial_{\mathrm{ro}}{}^{0}_{\rho\,\mathrm{ro}} \\
&\quad - \omega_0\,\tfrac{1}{\rho_{\mathrm{ro}}}\;(\rho_0\,d^0 + d_{\mathrm{ro}}{}^{\rho}) \otimes \partial_{\mathrm{ro}}{}^{0}_{\phi} \\
&\quad + \rho_{\mathrm{ro}}\;\phi_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\phi} \otimes \partial_{\mathrm{ro}}{}^{0}_{\rho\,\mathrm{ro}} - \tfrac{1}{\rho_{\mathrm{ro}}}\,(\rho_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\phi} + \phi_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\rho}) \otimes \partial_{\mathrm{ro}}{}^{0}_{\phi},
\end{aligned}$$

$$\begin{aligned}
\gamma^{\natural} &= u^0 \otimes (\partial_{\mathrm{ro}\,0} + \rho_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,\rho} + \phi_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,\phi} + z_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,z}) \\
&\quad + \rho_{\mathrm{ro}}\;u^0 \otimes (\omega_0 + \phi_{\mathrm{ro}\,0})^2\,\partial_{\mathrm{ro}}{}^{0}_{\rho} - 2\,\tfrac{1}{\rho_{\mathrm{ro}}}\,\rho_{\mathrm{ro}\,0}\,u^0 \otimes (\omega_0 + \phi_{\mathrm{ro}\,0})\,\partial_{\mathrm{ro}}{}^{0}_{\phi},
\end{aligned}$$

$$\begin{aligned}
\Omega^{\natural} &= \tfrac{m}{\hbar_0}\,d_{\mathrm{ro}}{}^{\rho}_{0} \wedge (d_{\mathrm{ro}}{}^{\rho} - \rho_{\mathrm{ro}\,0}\,d^0) \\
&\quad + \tfrac{m}{\hbar_0}\,\Big({\rho_{\mathrm{ro}}}^2\,d_{\mathrm{ro}}{}^{\phi}_{0} \wedge (d_{\mathrm{ro}}{}^{\phi} - \phi_{\mathrm{ro}\,0}\,d^0) + d_{\mathrm{ro}}{}^{z}_{0} \wedge (d_{\mathrm{ro}}{}^{z} - z_{\mathrm{ro}\,0}\,d^0)\Big) \\
&\quad + \tfrac{m}{\hbar_0}\,\Big(\rho_{\mathrm{ro}}\;\big((\phi_{\mathrm{ro}\,0})^2 - (\omega_0)^2\big)\,d^0 \wedge d_{\mathrm{ro}}{}^{\rho} \\
&\quad + 2\,\rho_{\mathrm{ro}}\;(\omega_0 + \phi_{\mathrm{ro}\,0})\,d_{\mathrm{ro}}{}^{\rho} \wedge d_{\mathrm{ro}}{}^{\phi}\Big),
\end{aligned}$$

$$\Lambda^{\natural} = \frac{\hbar_0}{m}\, \partial_{\mathtt{ro}\,\rho} \wedge \partial_{\mathtt{ro}}{}^{0}_{\rho}$$
$$+ \frac{\hbar_0}{m} \Big(\frac{1}{(\rho_{\mathtt{ro}})^2}\, \partial_{\mathtt{ro}\,\phi} \wedge \partial_{\mathtt{ro}}{}^{0}_{\phi} + \partial_{\mathtt{ro}\,z} \wedge \partial_{\mathtt{ro}}{}^{0}_{z}$$
$$+ 2\, \frac{1}{\rho_{\mathtt{ro}}}\, (\omega_0 + \phi_{\mathtt{ro}\,0})\, \partial_{\mathtt{ro}}{}^{0}_{\rho} \wedge \partial_{\mathtt{ro}}{}^{0}_{\phi} \Big). \quad \square$$

25.1.4.3 Distinguished 1–Forms

We start by discussing the observed potential $A[\mathtt{b}^{\square}, o_{\mathtt{ro}}] = 0$, with respect to the classical gauge $\mathtt{b}^{\square}$ (see Proposition 25.1.10).

Then, we analyse the associated observed hamiltonian $\mathcal{H}[\mathtt{b}^{\square}, o_{\mathtt{ro}}]$, observed momentum $\mathcal{P}[\mathtt{b}^{\square}, o_{\mathtt{ro}}]$ and the upper quantum potential $A^{\uparrow}[\mathtt{b}^{\square}]$.

Proposition 25.1.23 *The distinguished classical gauge $\mathtt{b}^{\square}$ suggested by the inertial observer $o_{\mathtt{in}}$ yields the following observed potential of the observed spacetime 2–form $\Phi[o_{\mathtt{ro}}]$, which is associated with the same classical gauge $\mathtt{b}^{\square}$,*

$$A[\mathtt{b}^{\square}, o_{\mathtt{ro}}] := o_{\mathtt{ro}}{}^{*} A^{\uparrow}[\mathtt{b}^{\square}].$$

Then, in virtue of the transition rule (see Remark 10.1.6 and also Theorem 15.2.26)

$$A[\mathtt{b}^{\square}, o_{\mathtt{ro}}] = A[\mathtt{b}^{\square}, o_{\mathtt{in}}] + \theta[o_{\mathtt{in}}] \lrcorner\, G^{\flat}(\vec{v}_{\mathtt{ro}}) - \tfrac{1}{2}\, G(\vec{v}_{\mathtt{ro}}, \vec{v}_{\mathtt{ro}}),$$

we obtain the following equality, in cartesian and cylindrical spacetime charts adapted to $o_{\mathtt{in}}$ (see Definitions 24.2.4 and 24.2.5),

$$A[\mathtt{b}^{\square}, o_{\mathtt{ro}}] = -\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, (\omega_0)^2 \big((x^1)^2 + (x^2)^2\big)\, d^0 + \tfrac{m}{\hbar_0}\, \omega_0\, (x^1\, d^2 - x^2\, d^1)$$
$$= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \rho^2\, (\omega_0)^2\, d^0 + \tfrac{m}{\hbar_0}\, \omega_0\, \rho^2\, d^{\phi},$$

i.e., in cartesian and cylindrical spacetime charts adapted to $o_{\mathtt{ro}}$ (see Definitions 24.5.2 and 24.5.3),

$$A[\mathtt{b}^{\square}, o_{\mathtt{ro}}] = \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, (\omega_0)^2 \big((x_{\mathtt{ro}}{}^1)^2 + (x_{\mathtt{ro}}{}^2)^2\big)\, d^0 + \tfrac{m}{\hbar_0}\, \omega_0\, (x_{\mathtt{ro}}{}^1\, d_{\mathtt{ro}}{}^2 - x_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^1),$$
$$= \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \rho_{\mathtt{ro}}{}^2\, (\omega_0)^2\, d^0 + \tfrac{m}{\hbar_0}\, \omega_0\, \rho_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^{\phi}. \quad \square$$

Corollary 25.1.24 *We have the following phase 1–forms (see Example 12.1.4)*

$$\mathcal{H}[\mathtt{b}^{\square}, o_{\mathtt{ro}}] = \mathcal{K}[o_{\mathtt{ro}}] - \partial[o_{\mathtt{ro}}] \lrcorner\, A[\mathtt{b}^{\square}, o_{\mathtt{ro}}]$$
$$= \tfrac{1}{2}\, \tfrac{m}{\hbar_0} \Big(\big(\delta_{ij}\, x_{\mathtt{ro}}{}^{i}_{0}\, x_{\mathtt{ro}}{}^{j}_{0}\big) - (\omega_0)^2 \big((x_{\mathtt{ro}}{}^1)^2 + (x_{\mathtt{ro}}{}^2)^2\big)\Big)\, d^0$$
$$= \tfrac{1}{2}\, \tfrac{m}{\hbar_0} \big((\rho_{\mathtt{ro}\,0})^2 + \rho_{\mathtt{ro}}{}^2\, (\phi_{\mathtt{ro}\,0})^2 + (z_{\mathtt{ro}\,0})^2 - (\omega_0)^2\, \rho_{\mathtt{ro}}{}^2\big)\, d^0,$$

$$\begin{aligned}\mathcal{P}[\mathrm{b}^{\square}, o_{\mathtt{ro}}] &= \mathcal{Q}[o_{\mathtt{ro}}] + \theta[o_{\mathtt{ro}}] \lrcorner A[\mathrm{b}^{\square}, o_{\mathtt{ro}}] \\ &= \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_{\mathtt{ro}}{}^{i}_{0}\, d_{\mathtt{ro}}{}^{j} + \tfrac{m}{\hbar_0}\, \omega_0\, (x_{\mathtt{ro}}{}^{1}\, d_{\mathtt{ro}}{}^{2} - x_{\mathtt{ro}}{}^{2}\, d_{\mathtt{ro}}{}^{1}) \\ &= \tfrac{m}{\hbar_0}\, \big(\rho_{\mathtt{ro}\,0}\, d_{\mathtt{ro}}{}^{\rho} + \rho_{\mathtt{ro}}{}^{2}\, (\phi_{\mathtt{ro}\,0} + \omega_0)\, d_{\mathtt{ro}}{}^{\phi} + z_{\mathtt{ro}\,0}\, d_{\mathtt{ro}}{}^{z}\big). \quad \square\end{aligned}$$

Exercise 25.1.25 Let us consider the observer independent upper potential $A^{\uparrow}[\mathrm{b}^{\square}]$, which has been defined, with reference to the distinguished observer $o_{\mathtt{in}}$, as (see Proposition 25.1.10)

$$A^{\uparrow}[\mathrm{b}^{\square}] := \mathcal{C}[o_{\mathtt{in}}].$$

Then, this upper potential can be also written, with reference to the observer $o_{\mathtt{ro}}$, in the following way

$$\begin{aligned}A^{\uparrow}[\mathrm{b}^{\square}] &= \mathcal{C}[o_{\mathtt{ro}}] + A[\mathrm{b}^{\square}, o_{\mathtt{ro}}] = -\mathcal{H}[\mathrm{b}^{\square}, o_{\mathtt{ro}}] + \mathcal{P}[\mathrm{b}^{\square}, o_{\mathtt{ro}}] \\ &= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0} \Big(\delta_{ij}\, x_{\mathtt{ro}}{}^{i}_{0}\, x_{\mathtt{ro}}{}^{j}_{0}) - (\omega_0)^2\, \big((x_{\mathtt{ro}}{}^{1})^2 + (x_{\mathtt{ro}}{}^{2})^2\big)\Big)\, d^0 \\ &\quad + \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_{\mathtt{ro}}{}^{i}_{0}\, d_{\mathtt{ro}}{}^{j} + \tfrac{m}{\hbar_0}\, \omega_0\, (x_{\mathtt{ro}}{}^{1}\, d_{\mathtt{ro}}{}^{2} - x_{\mathtt{ro}}{}^{2}\, d_{\mathtt{ro}}{}^{1}) \\ &= -\tfrac{1}{2}\, \tfrac{m}{\hbar_0} \big((\rho_{\mathtt{ro}\,0})^2 + (\rho_{\mathtt{ro}})^2\, (\phi_{\mathtt{ro}\,0})^2 + (z_{\mathtt{ro}\,0})^2 - (\omega_0)^2\, (\rho_{\mathtt{ro}})^2\big)\, d^0 \\ &\quad + \tfrac{m}{\hbar_0} \big(\rho_{\mathtt{ro}\,0}\, d_{\mathtt{ro}}{}^{\rho} + (\rho_{\mathtt{ro}})^2\, (\phi_{\mathtt{ro}\,0} + \omega_0)\, d_{\mathtt{ro}}{}^{\phi} + z_{\mathtt{ro}\,0}\, d_{\mathtt{ro}}{}^{z}\big). \quad \square\end{aligned}$$

Exercise 25.1.26 Check the equality (see Proposition 25.1.12, Theorem 10.1.4 and Corollary 9.2.4)

$$dA^{\uparrow}[\mathrm{b}^{\square}] = \Omega,$$

by using the expression

$$A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{\mathtt{ro}}] + A[\mathrm{b}^{\square}, o_{\mathtt{ro}}]. \quad \square$$

Exercise 25.1.27 With reference to the classical gauge $\mathrm{b}^{\square}$ and the observers $o_{\mathtt{in}}$, $o_{\mathtt{ro}}$, let us check that the timelike 1–form (see Note 10.1.11 and Proposition 15.2.29)

$$\alpha[\mathrm{b}^{\square}] = д[o] \lrcorner A[\mathrm{b}^{\square}, o] - \tfrac{1}{2}\, \check{A}[\mathrm{b}^{\square}, o] \lrcorner \vec{A}[\mathrm{b}^{\square}, o]$$

is observer equivariant, even more, in the present case, that it is vanishing, according to Corollary 15.2.28.

Indeed, in virtue of Proposition 25.1.23, we obtain the equalities

$$\begin{aligned}&д[o_{\mathtt{in}}] \lrcorner A[\mathrm{b}^{\square}, o_{\mathtt{in}}] - \tfrac{1}{2}\, \check{A}[\mathrm{b}^{\square}, o_{\mathtt{in}}] \lrcorner \vec{A}[\mathrm{b}^{\square}, o_{\mathtt{in}}] = 0, \\ &д[o_{\mathtt{ro}}] \lrcorner A[\mathrm{b}^{\square}, o_{\mathtt{ro}}] - \tfrac{1}{2}\, \check{A}[\mathrm{b}^{\square}, o_{\mathtt{ro}}] \lrcorner \vec{A}[\mathrm{b}^{\square}, o_{\mathtt{ro}}] = 0. \quad \square\end{aligned}$$

25.1.4.4 Further Distinguished Gauges Suggested by the Cartesian Chart

Besides the distinguished classical gauge $\mathrm{b}^{\square}$, that we have already selected with reference to the observer $o_{\mathtt{in}}$ (see Proposition 25.1.10), the uniformly rotating observer $o_{\mathtt{ro}}$ and the adapted cartesian chart suggest, for instance, two further distinguished classical gauges $\mathrm{b}^{\diamondsuit}$ and $\acute{\mathrm{b}}^{\diamondsuit}$, which provide two distinguished observed potentials $A[\mathrm{b}^{\diamondsuit}, o_{\mathtt{ro}}]$ and $A[\acute{\mathrm{b}}^{\diamondsuit}, o_{\mathtt{ro}}]$ of the observed spacetime 2–form $\Phi[o_{\mathtt{ro}}]$.

The comparison of the above classical gauges offers an opportunity to check our general machinery on the transition rules for observed potentials and upper potentials.

Note 25.1.28 The expression of K yields the equality (see Definition 4.2.11)

$$\Phi[o_{\mathtt{ro}}] = -2\,\tfrac{m}{\hbar_0}\,(\omega_0)^2\, d^0 \wedge (x_{\mathtt{ro}}{}^1\, d_{\mathtt{ro}}{}^1 + x_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^2) + 4\,\tfrac{m}{\hbar_0}\,\omega_0\, d_{\mathtt{ro}}{}^1 \wedge d_{\mathtt{ro}}{}^2.$$

Therefore, there exists, for instance,

(i) a global classical gauge $\mathrm{b}^{\diamondsuit}$ such that the observed potential of $\Phi[o_{\mathtt{ro}}]$ is (see Theorem 4.3.3)

$$\begin{aligned} A[\mathrm{b}^{\diamondsuit}, o_{\mathtt{ro}}] &= \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(\omega_0)^2 \big((x_{\mathtt{ro}}{}^1)^2 + (x_{\mathtt{ro}}{}^2)^2\big)\, d^0 \\ &\quad + \tfrac{m}{\hbar_0}\,\omega_0\,(x_{\mathtt{ro}}{}^1\, d_{\mathtt{ro}}{}^2 - x_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^1), \end{aligned}$$

(ii) a global classical gauge $\acute{\mathrm{b}}^{\diamondsuit}$, such that the observed potential of $\Phi[o_{\mathtt{ro}}]$ is (see Theorem 4.3.3)

$$\begin{aligned} A[\acute{\mathrm{b}}^{\diamondsuit}, o_{\mathtt{ro}}] &= -\tfrac{m}{\hbar_0}\,(\omega_0)^2\, x^0\,(x_{\mathtt{ro}}{}^1\, d_{\mathtt{ro}}{}^1 + x_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^2) \\ &\quad + \tfrac{m}{\hbar_0}\,\omega_0\,(x_{\mathtt{ro}}{}^1\, d_{\mathtt{ro}}{}^2 - x_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^1). \end{aligned}$$

Indeed, we have the equality (see Proposition 25.1.23)

$$A[\mathrm{b}^{\diamondsuit}, o_{\mathtt{ro}}] = A[\mathrm{b}^{\square}, o_{\mathtt{ro}}]. \quad \square$$

Remark 25.1.29 The difference of the two observed potentials above is given by the differential of a spacetime function as follows

$$\begin{aligned} A[\acute{\mathrm{b}}^{\diamondsuit}, o_{\mathtt{ro}}] - A[\mathrm{b}^{\diamondsuit}, o_{\mathtt{ro}}] &= -\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(\omega_0)^2\, d\Big(x^0\,\big((x_{\mathtt{ro}}{}^1)^2 + (x_{\mathtt{ro}}{}^1)^2\big)\Big) \\ &= -\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(\omega_0)^2\, d(x^0\,\rho_{\mathtt{ro}}{}^2). \end{aligned}$$

Hence, in the classical context, we might remove the exact terms from these observed potential and write, up to an exact term,

$$A[\mathrm{b}^{\diamondsuit}, o_{\mathtt{ro}}] \;\simeq\; A[\acute{\mathrm{b}}^{\diamondsuit}, o_{\mathtt{ro}}].$$

But, in the quantum context, once we have chosen the upper quantum connection $Ч^{\uparrow}$, the above removal and identification are forbidden by the consistency of the expressions of the gauge independent and observer independent objects derived from upper quantum connection $Ч^{\uparrow}$. □

25.1.4.5 Further Distinguished Gauges Suggested by the Cylindrical Chart

Besides the distinguished classical gauge $\mathrm{b}^{\Box}$, that we have already selected with reference to the observer $o_{\mathtt{in}}$ (see Proposition 25.1.10), the uniformly rotating observer $o_{\mathtt{ro}}$ and the adapted cylindrical chart suggest, for instance, two further distinguished classical gauges $\mathrm{b}^{\odot}$ and $\acute{\mathrm{b}}^{\odot}$, which provide two distinguished observed potentials $A[\mathrm{b}^{\odot}, o_{\mathtt{ro}}]$ and $A[\acute{\mathrm{b}}^{\odot}, o_{\mathtt{ro}}]$ of the observed spacetime 2–form $\Phi[o_{\mathtt{ro}}]$.

The comparison of the above classical gauges offers an opportunity to check our general machinery on the transition rules for observed potentials and upper potentials.

Note 25.1.30 The expression of K yields the equality (see Definition 4.2.11)

$$\Phi[o_{\mathtt{ro}}] = -2\, \tfrac{m}{\hbar_0} \big((\omega_0)^2\, \rho_{\mathtt{ro}}\, d^0 \wedge d_{\mathtt{ro}}{}^{\rho} - 2\, \omega_0\, \rho_{\mathtt{ro}}\, d_{\mathtt{ro}}{}^{\rho} \wedge d_{\mathtt{ro}}{}^{\phi}\big).$$

Therefore, there exists, for instance,

(i) a global classical gauge $\mathrm{b}^{\odot}$ such that the observed potential of $\Phi[o_{\mathtt{ro}}]$ is (see Theorem 4.3.3)

$$A[\mathrm{b}^{\odot}, o_{\mathtt{ro}}] = \tfrac{m}{\hbar_0} \big(\tfrac{1}{2}\, (\omega_0)^2\, \rho_{\mathtt{ro}}{}^2\, d^0 + \omega_0\, \rho_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^{\phi}\big),$$

(ii) a global classical gauge $\acute{\mathrm{b}}^{\odot}$, such that the observed potential of $\Phi[o_{\mathtt{ro}}]$ is (see Theorem 4.3.3)

$$A[\acute{\mathrm{b}}^{\odot}, o_{\mathtt{ro}}] = \tfrac{m}{\hbar_0} \big(-(\omega_0)^2\, x^0\, \rho_{\mathtt{ro}}\, d_{\mathtt{ro}}{}^{\rho} + \omega_0\, \rho_{\mathtt{ro}}{}^2\, d_{\mathtt{ro}}{}^{\phi}\big).$$

Indeed, we have the equality (see Proposition 25.1.23)

$$A[\mathrm{b}^{\odot}, o_{\mathtt{ro}}] = A[\mathrm{b}^{\Box}, o_{\mathtt{ro}}]. \quad \Box$$

Remark 25.1.31 The difference of these two observed potentials is given by the differential of a spacetime function as follows

$$A[\acute{\mathrm{b}}^{\odot}, o_{\mathtt{ro}}] - A[\mathrm{b}^{\odot}, o_{\mathtt{ro}}] = -\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, (\omega_0)^2\, d\big(x^0\, \rho_{\mathtt{ro}}{}^2\big).$$

Hence, in the classical context, we might remove the exact terms from the observed potential and write, up to an exact term,

$$A[\mathrm{b}^{\odot}, o_{\mathtt{ro}}] \simeq A[\acute{\mathrm{b}}^{\odot}, o_{\mathtt{ro}}].$$

But, in the quantum context, once we have chosen the upper quantum connection $\text{Ч}^{\uparrow}$, the above removal and identification are forbidden by the consistency of the expressions of the gauge independent and observer independent objects derived from upper quantum connection $\text{Ч}^{\uparrow}$. □

25.2 Quantum Objects

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ (with vanishing electromagnetic field) introduced by Hypothesis C.1, postulate a suitable quantum structure $(\boldsymbol{Q}, \mathsf{h}_{\eta}, \text{Ч}^{\uparrow})$ and compute the main quantum objects discussed throughout the book, with reference to an inertial observer, a uniformly accelerated observer and a uniformly rotating observer.

25.2.1 Starting Hypothesis of the Quantum Theory

In this 1st example, we consider the standard flat spacetime as classical background (see Hypothesis C.1).

Then, we postulate a trivial quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ and an upper quantum connection $\text{Ч}^{\uparrow}$, as source of all further quantum developments (see Hypothesis Q.1).

Indeed, the upper quantum connection $\text{Ч}^{\uparrow}$ is defined by means of the distinguished classical gauge $\mathsf{b}^{\square}$ and the associated upper potential $A^{\uparrow}[\mathsf{b}^{\square}]$, that have been emphasised in the classical theory (see Proposition 25.1.10).

Note 25.2.1 Preliminarily, we observe that, in virtue of Hypothesis C.1, the cohomology class of the phase 2–form Ω turns out to be integer (see [410] and recall Theorem 15.2.20).

Moreover, we recall the global classical gauge $\mathsf{b}^{\square}$, for which we have (see Proposition 25.1.10)

$$A^{\uparrow}[\mathsf{b}^{\square}] = \mathcal{C}[o_{\mathtt{in}}] \qquad \text{and} \qquad A[\mathsf{b}^{\square}, o_{\mathtt{in}}] = 0. \quad \square$$

Hypothesis Q.1 In this 1st Example, in the framework of the classical background postulated in Hypothesis C.1, we consider a *trivialisable* quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, along with an η–hermitian quantum metric $\mathsf{h}_{\eta} := \mathsf{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}$, according to Postulate Q.1, Proposition 14.3.1 and Definition 14.5.1.

Thus, in virtue of the triviality of the quantum bundle, there exist global quantum bases b of $\pi : \boldsymbol{Q} \to \boldsymbol{E}$.

Then, we choose a global quantum basis $\mathsf{b}^{\square}$ (denoted by the same symbol of the distinguished classical gauge $\mathsf{b}^{\square}$) and consider the upper quantum connection (see Theorem 15.2.4, Note 15.2.12, Postulate Q.2 and Definition 3.2.9)

$$Ч^{\uparrow} = \chi^{\uparrow}[\mathrm{b}^{\square}] + \mathrm{i}\, A^{\uparrow}[\mathrm{b}^{\square}]\, \mathbb{I}^{\uparrow}, \quad \text{such that } A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{\mathtt{in}}],$$

with coordinate expression, in the cartesian spacetime chart (x^0, x^i) (see Definition 24.2.4 and Proposition 25.1.1),

$$Ч^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + \mathrm{i}\big(- \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j\, d^0 + \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^j\, d^i\big) \otimes \mathbb{I}^{\uparrow}. \quad \square$$

Note 25.2.2 The distinguished global quantum basis $\mathrm{b}^{\square}$ yields the hermitian fibred isomorphism over $\boldsymbol{E}$

$$\boldsymbol{Q} \simeq \boldsymbol{E} \times \mathbb{C},$$

which allows us to use the language of standard Quantum Mechanics dealing with wave functions.

Thus, with reference to the above distinguished quantum basis $\mathrm{b}^{\square}$, we can identify the *quantum sections* Ψ with the associated *wave functions*:

$$\Psi \simeq \psi := \Psi/\mathrm{b}^{\square}. \quad \square$$

The above upper quantum connection $Ч^{\uparrow}$ yields, in the present specific example, all quantum objects that we have analysed in the general theory. Now, we shall provide their expressions with reference to the inertial observer $o_{\mathtt{in}}$, the uniformly accelerated observer $o_{\mathtt{ac}}$ and the uniformly rotating observer $o_{\mathtt{ro}}$.

25.2.2 Discussion on the Chosen Distinguished Gauge

In the classical theory, besides the distinguished classical gauge $\mathrm{b}^{\square}$, we have emphasised other distinguished classical gauges $\mathrm{b}^{\triangleright}$, $\acute{\mathrm{b}}^{\triangleright}$, $\mathrm{b}^{\diamond}$, $\acute{\mathrm{b}}^{\diamond}$, $\mathrm{b}^{\odot}$, $\acute{\mathrm{b}}^{\odot}$, which are suggested by different observed potentials $A[\mathrm{b}, o]$ of the spacetime 2–form $\Phi[o]$ referred to different classical observers o (see Proposition 25.1.10, Notes 25.1.18, 25.1.28, 25.1.30).

Now, in the quantum theory, in view of our starting hypothesis on the upper quantum connection $Ч^{\uparrow}$, we have chosen the classical gauge $\mathrm{b}^{\square}$, for the clear reason that it yields the simplest expressions of the derived quantum objects (see Hypothesis Q.1).

A natural question arises: what would happen if we had chosen another distinguished global gauge for our starting hypothesis on the quantum theory?

Indeed, we discuss an example, which shows that another choice of the distinguished global classical gauge would yield a global change of the quantum gauge, hence an equivalent formulation of the quantum theory.

We compare the quantum bases $\mathrm{b}^{\square}$ and $\mathrm{b}^{\triangleright}$. Analogous comparison can be made for the other distinguished quantum bases.

Remark 25.2.3 In the classical theory, with reference to the classical gauges, we have found the equality (see Corollary 25.1.20)

$$A^{\uparrow}[\mathrm{b}^{\triangleright}] = A^{\uparrow}[\mathrm{b}^{\square}] + d\vartheta,$$

where

$$\vartheta := -\tfrac{m}{\hbar_0}\, d\big(\nu_0\, x_{\mathtt{ac}}{}^1 - \tfrac{1}{6}\, a_{00}^2\, (x^0)^3 - \tfrac{1}{2}\, \nu_0^2\, x^0 - \tfrac{1}{2}\, a_{00}\, \nu_0\, (x^0)^2\big).$$

Now, in the quantum theory, in terms of quantum bases, we shall write

$$\mathrm{b}^{\triangleright} = \exp(\mathrm{i}\,\vartheta)\,\mathrm{b}^{\square}.$$

Indeed, the above equality turns out to be consistent with Hypothesis Q.1 and its consequences, in particular with the transition rule (see Theorem 15.2.26)

$$A[\mathrm{b}^{\triangleright}, o_{\mathtt{ac}}] = A[\mathrm{b}^{\square}, o_{\mathtt{in}}] - d\vartheta + \theta[o_{\mathtt{in}}] \lrcorner\, G^{\flat}(\vec{v}_{\mathtt{ac}}) - \tfrac{1}{2}\, G(\vec{v}_{\mathtt{ac}}, \vec{v}_{\mathtt{ac}}). \quad \square$$

25.2.3 Inertial Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics, the above quantum structure $(\boldsymbol{Q}, \mathrm{h}_{\eta}, \mathrm{Y}^{\uparrow})$ and, with reference to an inertial observer $o_{\mathtt{in}}$, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.1, Q.1 and Definition 24.2.1).

With reference to the quantum basis $\mathrm{b}^{\square}$, the observer $o_{\mathtt{in}}$ and the adapted cartesian chart (x^{λ}), we obtain the following coordinate expressions.

Proposition 25.2.4 *We obtain the quantum laplacian, quantum kinetic tensor, the probability current, the quantum lagrangian and the Schrödinger operator, whose coordinate expressions are (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2 and 17.6.5)*

$$\begin{aligned}
\Delta_0(\Psi) &= \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi\, \mathrm{b}^{\square},\\
\mathrm{Q}_0(\Psi) &= \big(\psi\, \partial_0 - \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_j\psi\, \partial_i\big) \otimes \mathrm{b}^{\square},\\
\mathrm{J}_0(\Psi) &= |\psi|^2\, \partial_0 + \mathrm{i}\, \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, (\psi\, \partial_j\bar{\psi} - \bar{\psi}\, \partial_j\psi)\, \partial_i,\\
\mathrm{L}_0(\Psi) &= \tfrac{1}{2}\, \big(-\tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_i\bar{\psi}\, \partial_j\psi + \mathrm{i}\, (\bar{\psi}\, \partial_0\psi - \psi\, \partial_0\bar{\psi})\big),\\
\mathrm{S}_0(\Psi) &= \big(\partial_0\psi - \tfrac{1}{2}\, \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi\big)\, \mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Next, let us consider the distinguished special phase functions (see Propositions 25.1.1 and 25.1.2)

$$\begin{aligned}
\mathcal{H}_0 &\equiv \mathcal{H}_0[\mathrm{b}^\square, o_{\mathtt{in}}] = \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_i &\equiv \mathcal{P}_i[\mathrm{b}^\square, o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^j \in \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
&1 \in \mathbb{R} \subset \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathfrak{L}_h &\equiv \mathfrak{L}_i[c_{\mathtt{in}}, o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\, \epsilon_{ijh}\, r^i\, x_0^j \in \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}).
\end{aligned}$$

Proposition 25.2.5 *We have the following coordinate expressions of distinguished quantum operators (see Theorem 20.1.9 and Example 20.1.12)*

$$\mathrm{O}[\mathcal{H}_0](\Psi) = -\tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk} \partial_{hk}\psi\, \mathrm{b}^\square, \qquad \mathrm{O}[\mathcal{P}_j](\Psi) = -\mathrm{i}\, \partial_j \psi\, \mathrm{b}^\square, \qquad \mathrm{O}[1](\Psi) = \psi\, \mathrm{b}^\square,$$
$$\mathrm{O}[\mathfrak{L}_h](\Psi) = -\mathrm{i}\, \epsilon_{ijh}\, \delta^{jk}\, x^i\, \partial_k \psi\, \mathrm{b}^\square. \quad \square$$

The coordinate expressions of the above special phase functions and the coordinate expression of γ (see Proposition 25.1.1) show immediately that these special phase functions are conserved.

Hence, they generate conserved classical and quantum currents.

Proposition 25.2.6 *We have the following coordinate expressions of distinguished quantum current forms (see Proposition 21.2.2 and Example 21.2.3)*

$$\begin{aligned}
\mathrm{j}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_h\bar\psi\, \partial_k\psi\, \upsilon_0^0 - \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\partial_h\bar\psi\, \partial_0\psi + \partial_h\psi\, \partial_0\bar\psi)\, \upsilon_k^0,\\
\mathrm{j}_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_h\bar\psi\, \partial_k\psi\, \upsilon_j^0 + \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\partial_j\bar\psi\, \partial_h\psi + \partial_j\psi\, \partial_h\bar\psi)\, \upsilon_k^0\\
&\quad - \tfrac{1}{2}\, \mathrm{i}\, \big((\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \upsilon_0^0 - (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\, \upsilon_j^0\big),\\
\mathrm{j}_\eta[1](\Psi) &= -\tfrac{1}{2}\, \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\bar\psi\, \partial_h\psi - \psi\, \partial_h\bar\psi)\, \upsilon_k^0 + |\psi|^2\, \upsilon_0^0,\\
\mathrm{j}_\eta[\mathfrak{L}_r](\Psi) &= -\epsilon_{pqr}\delta^{qj}\, x^p\, \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk} \partial_h\bar\psi\, \partial_k\psi\, \upsilon_j^0\\
&\quad + \epsilon_{pqr}\delta^{qj}\, x^p\, \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\partial_j\bar\psi\, \partial_h\psi + \partial_j\psi\, \partial_h\bar\psi)\, \upsilon_k^0\\
&\quad - \epsilon_{pqr}\delta^{qj}\, x^p\, \tfrac{1}{2}\, \mathrm{i}\, \big((\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \upsilon_0^0 - (\bar\psi\, \partial_0\psi - \psi\, \partial_0\bar\psi)\, \upsilon_j^0\big). \quad \square
\end{aligned}$$

Corollary 25.2.7 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6 and Example 21.2.7)*

$$\begin{aligned}
\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_h\bar\psi\, \partial_k\psi\, \eta,\\
\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2}\, \mathrm{i}\, (\bar\psi\, \partial_j\psi - \psi\, \partial_j\bar\psi)\, \eta,\\
\check{\mathrm{j}}_\eta[1](\Psi) &= |\psi|^2\, \eta,\\
\check{\mathrm{j}}_\eta[\mathfrak{L}_r](\Psi) &= -\epsilon_{ihr}\, \delta^{hk}\, x^i\, \tfrac{1}{2}\, \mathrm{i}\, (\bar\psi\, \partial_k\psi - \psi\, \partial_k\bar\psi)\, \eta. \quad \square
\end{aligned}$$

Proposition 25.2.8 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Proposition 21.3.2 and Example 21.3.3)*

$$\epsilon_\eta[\mathcal{H}_0](\Psi) = -\tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\psi\,\partial_{hk}\bar{\psi} + \bar{\psi}\,\partial_{hk}\psi)\,\eta,$$
$$\epsilon_\eta[\mathcal{P}_j](\Psi) = \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_i\bar{\psi} - \bar{\psi}\,\partial_i\psi)\,\eta,$$
$$\epsilon_\eta[1](\Psi) = |\psi|^2\,\eta,$$
$$\epsilon_\eta[\mathfrak{L}_r](\Psi) = -\tfrac{1}{2}\,\mathrm{i}\,\epsilon_{pqr}\delta^{qj}\,x^p\,(\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi})\,\eta. \quad \square$$

25.2.4 Uniformly Accelerated Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^\natural)$ of standard Classical Mechanics, the quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \Upsilon^\uparrow)$ and, with reference to the uniformly accelerated observer o_{ac}, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.1, Q.1 and Definition 24.4.1).

We stress that, in the present Example, we have postulated the upper quantum connection $\Upsilon^\uparrow$ by taking into account the inertial observer o_{in}, the quantum basis $\mathrm{b}^\square$, the upper quantum potential $A^\uparrow[\mathrm{b}^\square]$ and the observed potential $A[\mathrm{b}^\square, o_{\mathrm{in}}] = o_{\mathrm{in}}{}^* A^\uparrow[\mathrm{b}^\square]$ (see Hypothesis Q.1).

Now, we express the quantum objects with reference to the quantum basis $\mathrm{b}^\square$ and the uniformly accelerated observer o_{ac} (see Definition 24.4.1).

Lemma 25.2.9 *By recalling the coordinate expression of the upper quantum potential in a spacetime chart adapted to o_{ac} (see Exercise 25.1.15)*

$$A^\uparrow[\mathrm{b}^\square] = -\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big(\delta_{ij}\,x_{\mathrm{ac}}{}^i_0\,x_{\mathrm{ac}}{}^j_0 - (a_{00}\,x^0 + v_0)^2\big)\,d^0 + \tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ac}}{}^j_0\,d_{\mathrm{ac}}{}^i + \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + v_0)\,d_{\mathrm{ac}}{}^1,$$

the upper quantum connection can be written as

$$\begin{aligned}\Upsilon^\uparrow &= \chi^\uparrow[\mathrm{b}^\square] + \mathrm{i}\,A^\uparrow[\mathrm{b}^\square]\,\mathbb{I}^\uparrow \\ &= d^0 \otimes \partial_{\mathrm{ac}\,0} + d_{\mathrm{ac}}{}^i \otimes \partial_{\mathrm{ac}\,i} + d_{\mathrm{ac}}{}^i_0 \otimes \partial_{\mathrm{ac}}{}^0_i \\ &\quad - \mathrm{i}\,\tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,\big(\delta_{ij}\,x_{\mathrm{ac}}{}^i_0\,x_{\mathrm{ac}}{}^j_0 - (a_{00}\,x^0 + v_0)^2\big)\,d^0 \otimes \mathbb{I}^\uparrow \\ &\quad + \mathrm{i}\,\tfrac{m}{\hbar_0}\,\big(\delta_{ij}\,x_{\mathrm{ac}}{}^j_0\,d_{\mathrm{ac}}{}^i + (a_{00}\,x^0 + v_0)\,d_{\mathrm{ac}}{}^1\big) \otimes \mathbb{I}^\uparrow. \quad \square\end{aligned}$$

Then, we obtain the following coordinate expressions of dynamical quantum objects, by recalling the equality (see Proposition 25.1.13)

$$A[\mathrm{b}^\square, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,\big((a_{00}\,x^0 + v_0)\,d_{\mathrm{ac}}{}^1 + \tfrac{1}{2}\,(a_{00}\,x^0 + v_0)^2\,d^0\big).$$

Proposition 25.2.10 *We have the following coordinate expressions of dynamical quantum objects (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2 and 17.6.5)*

$$\begin{aligned}
\Delta_0(\Psi) &= \big(\tfrac{\hbar_0}{m}\,\delta^{ij}\partial_{\mathrm{ac}\,ij}\psi - 2\,\mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}\,1}\psi - \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + \nu_0)^2\,\psi\big)\,\mathrm{b}^{\square},\\
\mathrm{Q}_0(\Psi) &= \big(\psi\,\partial_{\mathrm{ac}\,0} - \mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}\,j}\psi\,\partial_{\mathrm{ac}\,i} - (a_{00}\,x^0 + \nu_0)\,\psi\,\partial_{\mathrm{ac}\,1}\big)\otimes\mathrm{b}^{\square},\\
\mathrm{J}_0(\Psi) &= |\psi|^2\,\partial_{\mathrm{ac}\,0} + \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\psi\,\partial_{\mathrm{ac}\,j}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ac}\,j}\psi)\,\partial_i - (a_{00}\,x^0 + \nu_0)\,|\psi|^2\partial_{\mathrm{ac}\,1},\\
\mathrm{L}_0(\Psi) &= \tfrac{1}{2}\,\big(-\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}\,i}\bar{\psi}\,\partial_{\mathrm{ac}\,j}\psi + \mathrm{i}\,(\bar{\psi}\,\partial_{\mathrm{ac}\,0}\psi - \psi\,\partial_{\mathrm{ac}\,0}\bar{\psi})\\
&\qquad - \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}\,1}\psi - \psi\,\partial_{\mathrm{ac}\,1}\bar{\psi})\big),\\
\mathrm{S}_0(\Psi) &= \big(\partial_{\mathrm{ac}\,0}\psi - (a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}\,1}\psi - \tfrac{1}{2}\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}\,ij}\psi\big)\,\mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Note 25.2.11 Let us consider the distinguished special phase functions (see Proposition 25.1.12)

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,\big((\delta_{ij}\,x_{\mathrm{ac}}{}^{i}_{0}\,x_{\mathrm{ac}}{}^{j}_{0}) - (a_{00}\,x^0 + \nu_0)^2\big) \in \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_1 &:= \mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,(x_{\mathrm{ac}}{}^{1}_{0} + a_{00}\,x^0 + \nu_0) \in \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_2 &:= \mathcal{P}_2[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,x_{\mathrm{ac}}{}^{2}_{0} \in \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_3 &:= \mathcal{P}_3[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,x_{\mathrm{ac}}{}^{3}_{0} \in \operatorname{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
&\qquad 1 \in \mathbb{R} \subset \operatorname{map}(\boldsymbol{E}, \mathbb{R}) \subset \operatorname{spe}(J_1\boldsymbol{E}, \mathbb{R}).
\end{aligned}$$

Moreover, let us recall the coordinate expression (see Proposition 25.1.12)

$$\gamma = u^0 \otimes (\partial_{\mathrm{ac}\,0} + x_{\mathrm{ac}}{}^{i}_{0}\,\partial_{\mathrm{ac}\,i} - a_{00}\,\partial_{\mathrm{ac}}{}^{0}_{1}).$$

Then, the coordinate expressions of $\mathcal{P}_i$ show immediately that these special phase functions are conserved. Hence, they generate conserved classical and quantum currents.

Conversely, $\mathcal{H}_0$ is not conserved. More explicitly, we have

$$\gamma_0 \cdot \mathcal{H}_0 = -\tfrac{m}{\hbar_0}\,a_{00}\,\big(x_{\mathrm{ac}}{}^{1}_{0} + a_{00}\,x^0 + \nu_0\big). \quad \square$$

Remark 25.2.12 In Hypothesis Q.1, we have assumed the upper quantum connection $Ч^{\uparrow}$ with reference to the quantum gauge $\mathrm{b}^{\square}$.

Hence, all quantum objects should be derived with reference to the same quantum gauge $\mathrm{b}^{\square}$.

It might be interesting to observe that, by changing the quantum gauge, we obtain different conserved and non conserved distinguished phase functions.

In particular, we have the following results.

(1) The phase function $\mathcal{H}_0[\acute{\mathrm{b}}^{\triangleright}, o_{\mathrm{ac}}]$ is conserved, but the phase functions $\mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathrm{ac}}]$ and $\mathcal{H}_0[\mathrm{b}^{\triangleright}, o_{\mathrm{ac}}]$ are not conserved.
(2) The phase functions $\mathcal{P}_i[\mathrm{b}^{\triangleright}, o_{\mathrm{ac}}]$ and $\mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathrm{ac}}]$ are conserved, but the phase function $\mathcal{P}_i[\acute{\mathrm{b}}^{\triangleright}, o_{\mathrm{ac}}]$ is not conserved. $\square$

From now on, we essentially refer to the classical gauge $\mathrm{b}^{\Box}$, according to Hypothesis Q.1. Possible reference to other classical gauges will be considered as an exercise and explicitly mentioned.

Proposition 25.2.13 *We have the following coordinate expressions of distinguished quantum operators (see Theorem 20.1.9 and Example 20.1.12)*

$$\begin{aligned}
\mathrm{O}[\mathcal{H}_0](\Psi) &= \big(-\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\partial_{\mathrm{ac}\;hk}\psi + \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}\;1}\psi\big)\,\mathrm{b}^{\Box},\\
\mathrm{O}[\mathcal{P}_j](\Psi) &= -\mathrm{i}\,\partial_{\mathrm{ac}\;j}\psi\,\mathrm{b}^{\Box},\\
\mathrm{O}[1](\Psi) &= \psi\,\mathrm{b}^{\Box}. \quad \Box
\end{aligned}$$

Proposition 25.2.14 *We have the following coordinate expressions of distinguished quantum current forms (see Proposition 21.2.2 and Example 21.2.3)*

$$\begin{aligned}
\mathrm{j}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\,\big(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\;h}\bar{\psi}\,\partial_{\mathrm{ac}\;k}\psi + \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}\;1}\psi - \psi\,\partial_{\mathrm{ac}\;1}\bar{\psi})\big)\,\upsilon^0_0\\
&\quad - \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\partial_{\mathrm{ac}\;h}\bar{\psi}\,\partial_{\mathrm{ac}\;0}\psi + \partial_{\mathrm{ac}\;h}\psi\,\partial_{\mathrm{ac}\;0}\bar{\psi})\,\upsilon^0_k\\
&\quad - \tfrac{1}{2}\,\mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}\;0}\psi - \psi\,\partial_{\mathrm{ac}\;0}\bar{\psi})\,\upsilon^0_1,\\
\mathrm{j}_\eta[\mathcal{P}_i](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\,\upsilon^0_0 + \tfrac{1}{2}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_i\psi - \psi\,\partial_i\bar{\psi})\,\upsilon^0_1\\
&\quad - \tfrac{1}{2}\,\Big(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\;h}\bar{\psi}\,\partial_{\mathrm{ac}\;k}\psi + \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}\;1}\psi - \psi\,\partial_{\mathrm{ac}\;1}\bar{\psi})\\
&\quad - \mathrm{i}\,(\bar{\psi}\,\partial_{\mathrm{ac}\;0}\psi - \psi\,\partial_{\mathrm{ac}\;0}\bar{\psi})\Big)\,\upsilon^0_i\\
&\quad + \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hj}\,\big(\partial_{\mathrm{ac}\;i}\bar{\psi}\,\partial_{\mathrm{ac}\;h}\psi + \partial_{\mathrm{ac}\;i}\psi\,\partial_{\mathrm{ac}\;h}\bar{\psi}\big)\,\upsilon^0_j,\\
\mathrm{j}_\eta[1](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\bar{\psi}\,\partial_{\mathrm{ac}\;h}\psi - \psi\,\partial_{\mathrm{ac}\;h}\bar{\psi})\,\upsilon^0_k + |\psi|^2\,\upsilon^0_0 - (a_{00}\,x^0 + \nu_0)\,|\psi|^2\,\upsilon^0_1. \quad \Box
\end{aligned}$$

Corollary 25.2.15 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6 and Example 21.2.7)*

$$\begin{aligned}
\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\,\big(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\;h}\bar{\psi}\,\partial_{\mathrm{ac}\;k}\psi + \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}\;1}\psi - \psi\,\partial_{\mathrm{ac}\;1}\bar{\psi})\big)\,\eta,\\
\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big((\bar{\psi}\,\partial_{\mathrm{ac}\;j}\psi - \psi\,\partial_{\mathrm{ac}\;j}\bar{\psi})\,\eta,\\
\check{\mathrm{j}}_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \Box
\end{aligned}$$

Proposition 25.2.16 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Proposition 21.3.2 and Example 21.3.3)*

$$\begin{aligned}
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\big(\tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\psi\,\partial_{\mathrm{ac}\;hk}\bar{\psi} + \bar{\psi}\,\partial_{\mathrm{ac}\;hk}\psi)\\
&\qquad + \tfrac{1}{2}\,\mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\psi\,\partial_{\mathrm{ac}\;1}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ac}\;1}\psi)\big)\,\eta,\\
\epsilon_\eta[\mathcal{P}_j](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{\mathrm{ac}\;j}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ac}\;j}\psi)\,\eta,\\
\epsilon_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \Box
\end{aligned}$$

Corollary 25.2.17 *We have (see Example 21.3.6)*

$$\mathfrak{d}_\eta\big[\mathcal{H}_0\big](\Psi) = \tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\,hk}|\psi|^2\eta, \qquad \mathfrak{d}_\eta\big[\mathcal{P}_i\big](\Psi) = 0, \qquad \mathfrak{d}_\eta\big[1\big](\Psi) = 0. \quad \square$$

Remark 25.2.18 The special phase function $\mathcal{H}_0[\mathrm{b}^\square, o_{\mathrm{ac}}]$ is not conserved, hence the corresponding classical and quantum currents are not conserved.

If we refer to the gauge $\acute{\mathrm{b}}^\triangleright$ and to the corresponding observed potential $A[\acute{\mathrm{b}}^\triangleright, o_{\mathrm{ac}}]$ (see Note 25.1.18), then we get the conserved special phase function

$$\mathcal{H}_0[\acute{\mathrm{b}}^\triangleright, o_{\mathrm{ac}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ac}}{}^i_0\,x_{\mathrm{ac}}{}^j_0 + \tfrac{m}{\hbar_0}\,a_{00}\,x_{\mathrm{ac}}{}^1$$

which yields the conserved quantum current

$$\begin{aligned}\mathrm{j}_\eta[\mathcal{H}_0[\acute{\mathrm{b}}^\triangleright, o_{\mathrm{ac}}]](\Psi) = \big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\,h}\bar{\psi}\,\partial_{\mathrm{ac}\,k}\psi + \tfrac{m}{\hbar_0}\,a_{00}\,x_{\mathrm{ac}}{}^1\,|\psi|^2\big)\,\upsilon^0_0 \\ - \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,\big(\partial_{\mathrm{ac}\,h}\bar{\psi}\,\partial_{\mathrm{ac}\,0}\psi + \partial_{\mathrm{ac}\,h}\psi\,\partial_{\mathrm{ac}\,0}\bar{\psi}\big)\,\upsilon^0_k,\end{aligned}$$

the vertical quantum current form

$$\check{\mathrm{j}}_\eta[\mathcal{H}_0[\acute{\mathrm{b}}^\triangleright{}_{\mathrm{ac}}, o_{\mathrm{ac}}]](\Psi) = \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\,h}\bar{\psi}\,\partial_{\mathrm{ac}\,k}\psi + \tfrac{m}{\hbar_0}\,a_{00}\,x_{\mathrm{ac}}{}^1\,|\psi|^2)\,\eta$$

and the expectation value form

$$\epsilon_\eta[\mathcal{H}_0[\acute{\mathrm{b}}^\triangleright{}_{\mathrm{ac}}, o_{\mathrm{ac}}]](\Psi) = \big(-\tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\psi\,\partial_{\mathrm{ac}\,hk}\bar{\psi} + \bar{\psi}\,\partial_{\mathrm{ac}\,hk}\psi) + \tfrac{m}{\hbar_0}\,a_{00}\,x_{\mathrm{ac}}{}^1\,|\psi|^2\big)\,\eta.$$

Moreover, we have

$$\mathfrak{d}_\eta\big[\mathcal{H}_0[\acute{\mathrm{b}}^\triangleright, o_{\mathrm{ac}}]\big](\Psi) = \tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_{\mathrm{ac}\,hk}|\psi|^2\eta. \quad \square$$

25.2.5 Uniformly Rotating Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^\natural)$ of standard Classical Mechanics, the quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \mathrm{Ч}^\uparrow)$ and, with reference to the uniformly rotating observer o_{ro}, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.1, Q.1 and Definition 24.5.1).

We stress that, in the present Example, we have postulated the upper quantum connection $\mathrm{Ч}^\uparrow$ by taking into account the inertial observer o_{in}, the quantum basis $\mathrm{b}^\square$, the upper quantum potential $A^\uparrow[\mathrm{b}^\square]$ and the observed potential $A[\mathrm{b}^\square, o_{\mathrm{in}}] = o_{\mathrm{in}}{}^* A^\uparrow[\mathrm{b}^\square]$ (see Hypothesis Q.1).

Now, we express the quantum objects with reference to the quantum basis $\mathrm{b}^\square$ and the uniformly rotating observer o_{ro} (see Definition 24.5.1).

Lemma 25.2.19 *By recalling the coordinate expression of the upper potential in a spacetime chart adapted to o_{ro} (see Exercise 25.1.25 and Definition 24.5.3)*

$$A^{\uparrow}[\mathrm{b}^{\square}] = -\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_{\mathrm{ro}\,0})^2 + (\rho_{\mathrm{ro}})^2\,(\phi_{\mathrm{ro}\,0})^2 + (z_{\mathrm{ro}\,0})^2 - (\omega_0)^2\,(\rho_{\mathrm{ro}})^2\big)\,d^0 + \tfrac{m}{\hbar_0}\,\big(\rho_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\rho} + (\rho_{\mathrm{ro}})^2\,(\phi_{\mathrm{ro}\,0} + \omega_0)\,d_{\mathrm{ro}}{}^{\phi} + z_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{z}\big),$$

the upper quantum connection can be written as

$$\begin{aligned}
\Psi^{\uparrow} &= \chi^{\uparrow}[\mathrm{b}^{\square}] + \mathrm{i}\,A^{\uparrow}[\mathrm{b}^{\square}]\,\mathbb{I}^{\uparrow}\\
&= d^0 \otimes \partial_{\mathrm{ro}\,0} + d_{\mathrm{ro}}{}^{\rho} \otimes \partial_{\mathrm{ro}\,\rho} + d_{\mathrm{ro}}{}^{\phi} \otimes \partial_{\mathrm{ro}\,\phi} + d_{\mathrm{ro}}{}^{z} \otimes \partial_{\mathrm{ro}\,z}\\
&\quad + d_{\mathrm{ro}}{}^{\rho}_{0} \otimes \partial_{\mathrm{ro}}{}^{0}_{\rho} + d_{\mathrm{ro}}{}^{\phi}_{0} \otimes \partial_{\mathrm{ro}}{}^{0}_{\phi} + d_{\mathrm{ro}}{}^{z}_{0} \otimes \partial_{\mathrm{ro}}{}^{0}_{z}\\
&\quad - \mathrm{i}\,\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_0)^2 + (\rho_{\mathrm{ro}})^2\,(\phi_{\mathrm{ro}\,0})^2 + (z_{\mathrm{ro}\,0})^2 - (\omega_0)^2\,(\rho_{\mathrm{ro}})^2\big)\,d^0 \otimes \mathbb{I}^{\uparrow}\\
&\quad + \mathrm{i}\,\tfrac{m}{\hbar_0}\,\big(\rho_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{\rho} + (\rho_{\mathrm{ro}})^2\,(\phi_{\mathrm{ro}\,0} + \omega_0)\,d_{\mathrm{ro}}{}^{\phi} + z_{\mathrm{ro}\,0}\,d_{\mathrm{ro}}{}^{z}\big) \otimes \mathbb{I}^{\uparrow}. \quad \square
\end{aligned}$$

Then, we obtain the following coordinate expressions of dynamical quantum objects, by recalling the equality (see Proposition 25.1.23)

$$A[\mathrm{b}^{\square}, o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\omega_0\,\rho_{\mathrm{ro}}{}^2\,(d_{\mathrm{ro}}{}^{\phi} - \tfrac{1}{2}\,\omega_0\,d^0).$$

Proposition 25.2.20 *We obtain the quantum kinetic tensor, the probability current, the quantum lagrangian and the Schrödinger operator, whose coordinate expressions are (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2, 17.6.5 and Proposition 25.1.22)*

$$\begin{aligned}
\Delta_0(\Psi) &= \Big(\tfrac{\hbar_0}{m}\,\big(\partial_{\mathrm{ro}\,\rho\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi\phi}\psi + \partial_{\mathrm{ro}\,zz}\psi + \tfrac{1}{\rho_{\mathrm{ro}}}\,\partial_{\mathrm{ro}\,\rho}\psi\big)\,\mathrm{b}^{\square}\\
&\qquad - 2\,\mathrm{i}\,\omega_0\,\partial_{\mathrm{ro}\,\phi}\psi - \tfrac{m}{\hbar_0}\,(\omega_0)^2\,\rho_{\mathrm{ro}}{}^2\,\psi\Big)\,\mathrm{b}^{\square},\\
\mathrm{Q}_0(\Psi) &= \psi\,\partial_{\mathrm{ro}\,0} \otimes \mathrm{b}^{\square}\\
&\qquad - \tfrac{\hbar_0}{m}\,\mathrm{i}\,\big(\partial_{\mathrm{ro}\,\rho}\psi\,\partial_{\mathrm{ro}\,\rho} + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}\psi\,\partial_{\mathrm{ro}\,\phi} + \partial_{\mathrm{ro}\,z}\psi\,\partial_{\mathrm{ro}\,z}\big)\\
&\qquad - \omega_0\,\psi\,\partial_{\mathrm{ro}\,\phi}\big) \otimes \mathrm{b}^{\square},\\
\mathrm{J}_0(\Psi) &= |\psi|^2\,\partial_{\mathrm{ro}\,0} - |\psi|^2\,\omega_0\,\partial_{\mathrm{ro}\,\phi}\\
&\qquad + \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\big((\psi\,\partial_{\mathrm{ro}\,\rho}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ro}\,\rho}\psi)\,\partial_{\mathrm{ro}\,\rho}\\
&\qquad + \tfrac{1}{\rho^2}\,(\psi\,\partial_{\mathrm{ro}\,\phi}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ro}\,\phi}\psi)\,\partial_{\mathrm{ro}\,\phi}\\
&\qquad + (\psi\,\partial_{\mathrm{ro}\,z}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ro}\,z}\psi)\,\partial_{\mathrm{ro}\,z}\big),\\
\mathrm{L}_0(\Psi) &= \tfrac{1}{2}\,\Big(-\tfrac{\hbar_0}{m}\,\big(\partial_{\mathrm{ro}\,\rho}\bar{\psi}\,\partial_{\mathrm{ro}\,\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}\bar{\psi}\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,z}\bar{\psi}\,\partial_{\mathrm{ro}\,z}\psi\big)\\
&\qquad + \mathrm{i}\,(\bar{\psi}\,\partial_{\mathrm{ro}\,0}\psi - \psi\,\partial_{\mathrm{ro}\,0}\bar{\psi}) + \mathrm{i}\,\omega_0\,(\bar{\psi}\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar{\psi})\Big),\\
\mathrm{S}_0(\Psi) &= (\partial_{\mathrm{ro}\,0}\psi)\,\mathrm{b}^{\square}\\
&\qquad - \Big(\omega_0\,\partial_{\mathrm{ro}\,\phi}\psi - \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\big(\partial_{\mathrm{ro}\,\rho\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi\phi}\psi\\
&\qquad + \partial_{\mathrm{ro}\,zz}\psi + \tfrac{1}{\rho_{\mathrm{ro}}}\,\partial_{\mathrm{ro}\,\rho}\psi\big)\Big)\,\mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Note 25.2.21 Let us consider the distinguished special phase functions (see Corollary 25.1.24)

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H}_0[\mathrm{b}^\square, o_{\mathrm{ro}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_{\mathrm{ro}\,0})^2 + \rho_{\mathrm{ro}}{}^2\,(\phi_{\mathrm{ro}\,0})^2 + (z_{\mathrm{ro}\,0})^2 - (\omega_0)^2\,\rho_{\mathrm{ro}}{}^2\big) \\
&\qquad\qquad \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_\rho &:= \mathcal{P}_\rho[\mathrm{b}^\square, o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\rho_{\mathrm{ro}\,0} \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_\phi &:= \mathcal{P}_\phi[\mathrm{b}^\square, o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,\rho_{\mathrm{ro}}{}^2\,(\phi_{\mathrm{ro}\,0} + \omega_0) \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_z &:= \mathcal{P}_i[\mathrm{b}^\square, o_{\mathrm{ro}}] = \tfrac{m}{\hbar_0}\,z_{\mathrm{ro}\,0} \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
&\qquad 1 \in \mathbb{R} \subset \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}).
\end{aligned}$$

Moreover, let us recall the coordinate expression (see Proposition 25.1.22)

$$\begin{aligned}
\gamma &= u^0 \otimes (\partial_{\mathrm{ro}\,0} + \rho_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,\rho} + \phi_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,\phi} + z_{\mathrm{ro}\,0}\,\partial_{\mathrm{ro}\,z}) \\
&\quad + \rho_{\mathrm{ro}}\,u^0 \otimes (\omega_0 + \phi_{\mathrm{ro}\,0})^2\,\partial_{\mathrm{ro}}{}^0_\rho - 2\,\tfrac{1}{\rho_{\mathrm{ro}}}\,\rho_{\mathrm{ro}\,0}\,u^0 \otimes (\omega_0 + \phi_{\mathrm{ro}\,0})\,\partial_{\mathrm{ro}}{}^0_\phi.
\end{aligned}$$

Then, the above coordinate expressions show immediately that the special phase functions $\mathcal{H}_0$, $\mathcal{P}_\phi$ and $\mathcal{P}_z$ are conserved.

Hence, they generate conserved classical and quantum currents.

Conversely, the phase function $\mathcal{P}_\rho$ is not conserved, hence the corresponding classical and quantum currents are not conserved. More explicitly, we have

$$\gamma_0.\mathcal{P}_\rho = \tfrac{m}{\hbar_0}\,\rho_{\mathrm{ro}}\,(\omega_0 + \phi_{\mathrm{ro}\,0})^2. \quad \square$$

Remark 25.2.22 With reference to the observer o_{ac}, the phase function $\mathcal{P}_i$ is conserved, while the phase function $\mathcal{H}_0$ is not conserved.

With reference to the observer o_{ro}, the phase function $\mathcal{H}_0$ is conserved, while the phase function $\mathcal{P}_\rho$ is not conserved. □

Note 25.2.23 We have the following coordinate expressions of distinguished quantum operators (see Theorem 20.1.9 and Example 20.1.12)

$$\begin{aligned}
\mathrm{O}[\mathcal{H}_0](\Psi) &= \big(-\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\partial_{\mathrm{ro}\,\rho\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi\phi}\psi + \partial_{\mathrm{ro}\,zz}\psi + \tfrac{1}{\rho_{\mathrm{ro}}}\,\partial_{\mathrm{ro}\,\rho}\psi) \\
&\qquad + \mathrm{i}\,\omega_0\,\partial_{\mathrm{ro}\,\phi}\psi\big)\,\mathrm{b}^\square, \\
\mathrm{O}[\mathcal{P}_\rho](\Psi) &= -\mathrm{i}\,\big(\partial_{\mathrm{ro}\,\rho}\psi + \tfrac{1}{2}\,\tfrac{1}{\rho_{\mathrm{ro}}}\,\psi\big)\,\mathrm{b}^\square, \\
\mathrm{O}[\mathcal{P}_\phi](\Psi) &= -\mathrm{i}\,\partial_{\mathrm{ro}\,\phi}\psi\,\mathrm{b}^\square, \\
\mathrm{O}[\mathcal{P}_z](\psi) &= -\mathrm{i}\,\partial_{\mathrm{ro}\,z}\psi\,\mathrm{b}^\square, \\
\mathrm{O}[1](\Psi) &= \psi\,\mathrm{b}^\square. \quad \square
\end{aligned}$$

Proposition 25.2.24 *We have the following coordinate expressions of distinguished quantum current forms (see Proposition 21.2.2 and Example 21.2.3)*

$$
\begin{aligned}
\mathrm{j}_\eta[\mathcal{H}_0](\Psi) = {}& \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big(\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi\Big)\,\upsilon_{\mathrm{ro}}{}^0_0 \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,(\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_0 \\
&- \tfrac{1}{2}\,\mathrm{i}\;\omega_0\,\big(\bar\psi\,\partial_{\mathrm{ro}\,0}\psi - \psi\,\partial_{\mathrm{ro}\,0}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big((\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,0}\psi + \partial_{\mathrm{ro}\,\rho}\psi\,\partial_{\mathrm{ro}\,0}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,(\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,0}\psi + \partial_{\mathrm{ro}\,\phi}\psi\,\partial_{\mathrm{ro}\,0}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&+ (\partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,0}\psi + \partial_{\mathrm{ro}\,z}\psi\,\partial_{\mathrm{ro}\,0}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_z\Big), \\
\mathrm{j}_\eta[\mathcal{P}_\rho](\Psi) = {}& -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi - \psi\,\partial_{\mathrm{ro}\,\rho}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_0 \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big(-\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi + \tfrac{1}{\rho^2}\,\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi\Big)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&- \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,(\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big(\tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,(\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,\rho}\psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big((\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi + \partial_{\mathrm{ro}\,\rho}\psi\,\partial_{\mathrm{ro}\,z}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_z\Big) \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,\big(\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi - \psi\,\partial_{\mathrm{ro}\,\rho}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,0}\psi - \psi\,\partial_{\mathrm{ro}\,0}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\tfrac{1}{\rho_{\mathrm{ro}}}\,\Big(\partial_{\mathrm{ro}\,\rho}|\psi|^2\,\upsilon_{\mathrm{ro}}{}^0_\rho + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}|\psi|^2\,\upsilon_{\mathrm{ro}}{}^0_\phi + \partial_{\mathrm{ro}\,z}|\psi|^2\,\upsilon_{\mathrm{ro}}{}^0_z\Big), \\
\mathrm{j}_\eta[\mathcal{P}_\phi](\psi) = {}& -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_0 \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big(\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi - \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi\Big)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big((\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi + \partial_{\mathrm{ro}\,\phi}\psi\,\partial_{\mathrm{ro}\,\rho}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi + \partial_{\mathrm{ro}\,\phi}\psi\,\partial_{\mathrm{ro}\,z}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_z\Big) \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,0}\psi - \psi\,\partial_{\mathrm{ro}\,0}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_\phi, \\
\mathrm{j}_\eta[\mathcal{P}_z](\Psi) = {}& -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,z}\psi - \psi\,\partial_{\mathrm{ro}\,z}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_0 \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big(\partial_{\mathrm{ro}\,\rho}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi + \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,\partial_{\mathrm{ro}\,\phi}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,z}\psi\Big)\,\upsilon_{\mathrm{ro}}{}^0_z \\
&- \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,(\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_z \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\Big((\partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi + \partial_{\mathrm{ro}\,z}\psi\,\partial_{\mathrm{ro}\,\rho}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,(\partial_{\mathrm{ro}\,z}\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi + \partial_{\mathrm{ro}\,z}\psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\phi\Big) \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,\big(\bar\psi\,\partial_{\mathrm{ro}\,z}\psi - \psi\,\partial_{\mathrm{ro}\,z}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&+ \tfrac{1}{2}\,\mathrm{i}\,\big(\bar\psi\,\partial_{\mathrm{ro}\,0}\psi - \psi\,\partial_{\mathrm{ro}\,0}\bar\psi\big)\,\upsilon_{\mathrm{ro}}{}^0_z, \\
\mathrm{j}_\eta[1](\Psi) = {}& |\psi|^2\,\upsilon_{\mathrm{ro}}{}^0_0 - \omega_0\,|\psi|^2\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&- \tfrac{1}{2}\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\Big((\bar\psi\,\partial_{\mathrm{ro}\,\rho}\psi - \psi\,\partial_{\mathrm{ro}\,\rho}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\rho \\
&+ \tfrac{1}{\rho_{\mathrm{ro}}{}^2}\,(\bar\psi\,\partial_{\mathrm{ro}\,\phi}\psi - \psi\,\partial_{\mathrm{ro}\,\phi}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_\phi \\
&+ (\bar\psi\,\partial_{\mathrm{ro}\,z}\psi - \psi\,\partial_{\mathrm{ro}\,z}\bar\psi)\,\upsilon_{\mathrm{ro}}{}^0_z\Big). \quad \square
\end{aligned}
$$

Corollary 25.2.25 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6 and Example 21.2.7)*

$$\begin{aligned}
\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\Big(\tfrac{\hbar_0}{m}\ \big(\partial_{\mathtt{ro}\,\rho}\bar{\psi}\,\partial_{\mathtt{ro}\,\rho}\psi + \tfrac{1}{\rho_{\mathtt{ro}}{}^2}\ \partial_{\mathtt{ro}\,\phi}\bar{\psi}\,\partial_{\mathtt{ro}\,\phi}\psi + \partial_{\mathtt{ro}\,z}\bar{\psi}\,\partial_{\mathtt{ro}\,z}\psi\big) \\
&\qquad + \mathrm{i}\,\omega_0\,(\bar{\psi}\,\partial_{\mathtt{ro}\,\phi}\psi - \psi\,\partial_{\mathtt{ro}\,\phi}\bar{\psi})\Big)\,\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_\rho](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{\mathtt{ro}\,\rho}\psi - \psi\,\partial_{\mathtt{ro}\,\rho}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_\phi](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{\mathtt{ro}\,\phi}\psi - \psi\,\partial_{\mathtt{ro}\,\phi}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_z](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{\mathtt{ro}\,z}\psi - \psi\,\partial_{\mathtt{ro}\,z}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}$$

Proposition 25.2.26 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Proposition 21.3.2 and Example 21.3.3)*

$$\begin{aligned}
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\tfrac{1}{2}\Big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\ \big((\psi\,\partial_{\mathtt{ro}\,\rho\rho}\bar{\psi} + \bar{\psi}\,\partial_{\mathtt{ro}\,\rho\rho}\psi) \\
&\qquad + \tfrac{1}{\rho_{\mathtt{ro}}{}^2}\ (\psi\,\partial_{\mathtt{ro}\,\phi\phi}\bar{\psi} + \bar{\psi}\,\partial_{\mathtt{ro}\,\phi\phi}\psi) \\
&\qquad + (\psi\,\partial_{\mathtt{ro}\,zz}\bar{\psi} + \bar{\psi}\,\partial_{\mathtt{ro}\,zz}\psi) \\
&\qquad + \tfrac{1}{\rho_{\mathtt{ro}}}\ (\psi\,\partial_{\mathtt{ro}\,\rho}\bar{\psi} + \bar{\psi}\,\partial_{\mathtt{ro}\,\rho}\psi)\big) \\
&\qquad + \mathrm{i}\,\omega_0\,(\psi\,\partial_{\mathtt{ro}\,\phi}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ro}\,\phi}\psi)\Big)\,\eta, \\
\epsilon_\eta[\mathcal{P}_\rho](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{\mathtt{ro}\,\rho}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ro}\,\rho}\psi)\,\eta, \\
\epsilon_\eta[\mathcal{P}_\phi](\psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{\mathtt{ro}\,\phi}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ro}\,\phi}\psi)\,\eta, \\
\epsilon_\eta[\mathcal{P}_z](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{\mathtt{ro}\,z}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ro}\,z}\psi)\,\eta, \\
\epsilon_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}$$

Corollary 25.2.27 *We have (see 21.3.6)*

$$\begin{aligned}
\mathfrak{d}_\eta\big[\mathcal{H}_0\big](\Psi) &= \tfrac{1}{4}\,\tfrac{\hbar_0}{m}\ \Big(\partial_{\mathtt{ro}\,\rho\rho}|\psi|^2 + \tfrac{1}{\rho_{\mathtt{ro}}{}^2}\,\partial_{\mathtt{ro}\,\phi\phi}|\psi|^2 + \partial_{\mathtt{ro}\,zz}|\psi|^2 + \tfrac{1}{\rho_{\mathtt{ro}}}\ \partial_{\mathtt{ro}\,\rho}|\psi|^2\Big)\,\eta \\
&= \tfrac{1}{4}\ \Delta_0|\psi|^2\,\eta. \quad \square
\end{aligned}$$

Chapter 26
Dynamical Example 2: Radial Electric Field

In this 2nd example, we extend the survey provided by the 1st example (see the above Chap. 25) by computing, in the framework of the *standard flat spacetime* equipped with a *given radial electric field*, the basic classical and quantum objects discussed throughout the body of the book (Sects. 26.1 and 26.2).

In order to show how our machinery works with different observers, we consider, not only a standard inertial observer, but also a *uniformly accelerated observer*; we leave to the reader the case of a uniformly rotating observer as an exercise.

26.1 Classical Objects

We consider a flat newtonian spacetime equipped with a *given radial electric field* and compute the main classical objects discussed throughout the book, with reference to an inertial observer and a uniformly accelerated observer.

26.1.1 Starting Hypothesis of the Classical Theory

In this 2nd example, the classical background spacetime (see Hypothesis C.2) is the standard flat spacetime discussed in Sect. 24.1 (after deleting a singular world line), along with a radial electric field.

Later, we shall complete the hypothesis on the classical theory by postulating the quantum bundle and the upper quantum connection of this 2nd example (see Hypothesis Q.2).

Hypothesis C.2 *In this 2nd Example, with reference to the standard* flat newtonian spacetime $(\boldsymbol{E}, g, K^\natural)$ *defined in Definition 24.1.1, we consider the open submanifold*

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Fundamental Theories of Physics 205,
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$$\acute{\boldsymbol{E}} := \boldsymbol{E}/s_{\mathtt{in}}(\boldsymbol{T}) \subset \boldsymbol{E},$$

where $s_{\mathtt{in}} : \boldsymbol{T} \to \boldsymbol{E}$ *is the inertial motion chosen in Definition 24.2.4.*

Thus, by a suitable restriction, we obtain the "flat newtonian spacetime" $(\acute{\boldsymbol{E}}, g, K^{\natural})$.

Then, we consider the "radial" electric field *(see Definitions 5.2.1 and 5.1.1)*

$$\vec{E} = k \, \tfrac{1}{r^3} \, \vec{r} \in \sec\Big(\acute{\boldsymbol{E}}, \; (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\acute{\boldsymbol{E}}\Big),$$

where (see Definitions 24.2.1 and 24.2.6)

$$k \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2}, \qquad \vec{r}(e) = e - s_{\mathtt{in}}\,(t(e))\,, \qquad r := \sqrt{g(\vec{r}, \vec{r})} \in \mathbb{L}.$$

In other words, in the cartesian spacetime chart adapted to $o_{\mathtt{in}}$ *(see Definition 24.2.4), we have the coordinate expression*

$$\vec{E} = k \, \tfrac{1}{r^3} \, x^i \, \partial_i \,, \quad \text{with } r := \sqrt{\delta_{ij}\, x^i \, x^j}.$$

Moreover, we consider a particle of mass m *and charge* q. □

Remark 26.1.1 We stress that, in the present example, according to Proposition 5.5.1, the electric field $\vec{E}$ turns out to be observer independent, because the magnetic field B is supposed to be vanishing. □

26.1.2 Inertial Observer

We consider the flat newtonian spacetime $(\acute{\boldsymbol{E}}, g, K^{\natural})$ equipped with the given *radial electric field* $\vec{E}$, and, with reference to the inertial observer $o_{\mathtt{in}}$, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.2 and Definition 24.2.1).

Lemma 26.1.2 *The electromagnetic field turns out to be the scaled spacetime 2–form (see Proposition 5.4.1)*

$$F = -2\, dt \wedge E \in \sec\Big(\acute{\boldsymbol{E}}, \; (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\acute{\boldsymbol{E}}\Big),$$

where (see Definition 5.3.1)

$$E := g^{\flat}(\vec{E}) \in \sec\Big(\acute{\boldsymbol{E}}, \; (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\acute{\boldsymbol{E}}\Big).$$

In other words, in coordinates, we have

$$F = -2\, k_0 \tfrac{1}{r^3}\; \delta_{ij}\, x^i\, d^0 \wedge d^j, \qquad F_{0j} = -k_0 \tfrac{1}{r^3}\; \delta_{ij}\, x^i, \qquad F_{ij} = 0.$$

Moreover, the above electromagnetic field can be regarded as an unscaled spacetime 2–form $\frac{q}{\hbar} F \in \sec(\boldsymbol{E}, \Lambda^2 T^* \boldsymbol{E})$. *Then, according to the equality* $\frac{q}{\hbar} F = 2\, d A^{\mathfrak{e}}$, *we have the distinguished, gauge dependent and observer independent, timelike electromagnetic potential*

$$A^{\mathfrak{e}} = A^{\mathfrak{e}}{}_0\, d^0, \quad \text{with } A^{\mathfrak{e}}{}_0 = -\tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r} \;\in \operatorname{map}(\boldsymbol{E}, I\!R).$$

Indeed, the above equalities yield $E = d\,(-k\,\frac{1}{r})$. □

Proposition 26.1.3 *The joined spacetime connection associated with the postulated gravitational and electromagnetic fields turns out to be (see Theorem 6.3.1 and Proposition 25.1.1)*

$$K \;\equiv\; K^{\natural} + K^{\mathfrak{e}} = K^{\natural} - \tfrac{1}{2}\, \tfrac{q}{m}\; (dt \otimes \hat{F} + \hat{F} \otimes dt),$$

with coordinate expression, in the cartesian spacetime chart adapted to the observer $o_{\mathtt{in}}$,

$$K = d^{\lambda} \otimes \partial_{\lambda} + \tfrac{q_0}{m}\, k_0\, \tfrac{1}{r^3}\; x^i\, \dot{x}^0\, d^0 \otimes \dot{\partial}_i.$$

Thus, we have

$$K_0{}^i{}_0 = K^{\mathfrak{e}}{}_0{}^i{}_0 = \tfrac{q_0}{m}\; k_0\, \tfrac{1}{r^3}\; x^i,$$
$$K_0{}^i{}_j = K_j{}^i{}_0 = K^{\natural}{}_0{}^i{}_j = K^{\natural}{}_j{}^i{}_0 = 0, \qquad K_h{}^i{}_k = K^{\natural}{}_h{}^i{}_k = 0. \quad \square$$

Corollary 26.1.4 *The joined observed spacetime 2–form turns out to be (see Corollary 6.3.3 and Proposition 25.1.1)*

$$\Phi[o_{\mathtt{in}}] = \Phi^{\natural}[o_{\mathtt{in}}] + \tfrac{q}{\hbar}\, F = -2\, \tfrac{q}{\hbar}\, dt \wedge E,$$

i.e., in coordinates,

$$\Phi[o_{\mathtt{in}}] = -2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\; \delta_{ij}\, x^i\, d^0 \wedge d^j.$$

Thus, we have

$$\Phi_{0j} = \Phi^{\mathfrak{e}}{}_{0j} = -\tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\; \delta_{ij}\, x^i, \qquad \Phi^{\natural}{}_{0j} = 0, \qquad \Phi_{ij} = \Phi^{\natural}{}_{ij} = 0.$$

Then, there exists a global classical gauge $\mathtt{b}^{\square}$, *such that the observed potential is (see Theorem 4.3.3 and Lemma 26.1.2)*

$$A[\mathtt{b}^{\square}, o_{\mathtt{in}}] = -\tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\; d^0. \quad \square$$

Indeed, in the present example, we shall be involved with the above distinguished gauge $\mathrm{b}^{\square}$ throughout the classical and the quantum frameworks.

Corollary 26.1.5 *The joined spacetime connection K yields the following joined phase objects, whose coordinate expressions in a cartesian spacetime chart adapted to the observer $o_{\mathtt{in}}$ are (see Theorem 6.3.1 and Corollary 9.2.4 and Proposition 25.1.21)*

$$\begin{aligned}
\Gamma &= d^{\lambda} \otimes \partial_{\lambda} + \tfrac{q_0}{m}\, k_0\, \tfrac{1}{r^3}\; x^i\, d^0 \otimes \partial_i^0, \\
\gamma_0 &= \partial_0 + x_0^i\, \partial_i + \tfrac{q_0}{m}\, k_0\, \tfrac{1}{r^3}\; x^i\, \partial_i^0, \\
\Omega &= \tfrac{q}{\hbar}\, \delta_{ij}\, d_0^i \wedge \theta^j - \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\; \delta_{ij}\, x^i\, d^0 \wedge d^j, \\
\Lambda &= \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_i \wedge \partial_j^0. \quad \square
\end{aligned}$$

Thus, the observed electric field effects Γ, γ, and Ω, but does not effect Λ.

Corollary 26.1.6 *According to the above results, we have the distinguished horizontal potential of Ω (see Theorem 10.1.4, Corollaries 26.1.4 and 26.1.5)*

$$A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{\mathtt{in}}] + A[\mathrm{b}^{\square}, o_{\mathtt{in}}] = -\mathcal{K}[o_{\mathtt{in}}] + \mathcal{Q}[o_{\mathtt{in}}] + A[\mathrm{b}^{\square}, o_{\mathtt{in}}],$$

with coordinate expression, in a cartesian spacetime chart adapted to the observer $o_{\mathtt{in}}$,

$$A^{\uparrow}[\mathrm{b}^{\square}] = -(\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j + \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\,)\, d^0 + \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, d^j. \quad \square$$

Indeed, this distinguished upper potential $A^{\uparrow}[\mathrm{b}^{\square}]$ will be used to postulate the upper quantum connection $\mathrm{y}^{\uparrow}$ (see Hypothesis Q.2).

26.1.3 Uniformly Accelerated Observer

We consider the flat newtonian spacetime $(\acute{\boldsymbol{E}}, g, K^{\natural})$ equipped with the given *radial electric field* $\vec{E}$, and, with reference to the *uniformly accelerated observer* $o_{\mathtt{ac}}$, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.2 and Definition 24.4.1).

Lemma 26.1.7 *In the spacetime chart $(x^0, x_{\mathtt{ac}}{}^i)$ adapted to the uniformly accelerated observer $o_{\mathtt{ac}}$, we have the coordinate expression (see Lemma 26.1.2 and Definition 24.4.1)*

$$\tfrac{q}{\hbar}\, F = -2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\, d^0 \wedge \big(\delta_{ij}\, x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)\, \delta_{1j}\big)\; d_{\mathtt{ac}}{}^j,$$

where

$$r = \sqrt{\delta_{ij}\, x_{\mathtt{ac}}{}^i\, x_{\mathtt{ac}}{}^j + 2\, x_{\mathtt{ac}}{}^1\, (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0) + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)^2}.$$

Moreover, the electromagnetic potential $A^{\mathfrak{e}}[\mathtt{b}]$ *selected in Lemma 26.1.2 reads as*

$$A^{\mathfrak{e}} = A^{\mathfrak{e}}{}_0\, d^0, \quad \text{with } A^{\mathfrak{e}}{}_0 = -\tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r} \in \operatorname{map}(\boldsymbol{E}, I\!R). \quad \square$$

Proposition 26.1.8 *In the spacetime chart* $(x^0, x_{\mathtt{ac}}{}^i)$ *adapted to the uniformly accelerated observer* $o_{\mathtt{ac}}$*, we have the coordinate expression of the joined spacetime connection* K *(see Theorem 6.3.1, Propositions 26.1.3 and 25.1.12)*

$$K = d_{\mathtt{ac}}{}^{\lambda} \otimes \partial_{\mathtt{ac}\lambda} + \left(-a_{00}\, \delta^{i1} + \tfrac{q_0}{m}\, k_0\, \tfrac{1}{r^3}\ \left(x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)\, \delta^{i1}\right)\right) \\ \dot{x}^0\, d^0 \otimes \dot{\partial}_{\mathtt{ac}i}.$$

Thus, we have

$$K_0{}^i{}_0 = K^{\natural}{}_0{}^i{}_0 + K^{\mathfrak{e}}{}_0{}^i{}_0 = -a_{00}\, \delta^{i1} + \tfrac{q_0}{m}\, k_0\, \tfrac{1}{r^3}\ \left(x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)\, \delta^{i1}\right),$$
$$K_0{}^i{}_j = K_0{}^i{}_j = K^{\natural}{}_0{}^i{}_j = K^{\natural}{}_j{}^i{}_0 = 0, \qquad K_h{}^i{}_k = K^{\natural}{}_h{}^i{}_k = 0. \quad \square$$

Corollary 26.1.9 *The joined observed spacetime 2–form associated with the uniformly accelerated observer* $o_{\mathtt{ac}}$ *turns out to be (see Definition 4.2.11 and Note 25.1.18)*

$$\Phi[o_{\mathtt{ac}}] = \Phi^{\natural}[o_{\mathtt{ac}}] + \tfrac{q}{\hbar}\, F = \Phi^{\natural}[o_{\mathtt{ac}}] - 2\, \tfrac{q}{\hbar}\, dt \wedge E,$$

i.e., in a cartesian coordinate chart adapted to $o_{\mathtt{ac}}$*,*

$$\Phi[o_{\mathtt{ac}}] = \left(2\, \tfrac{m}{\hbar_0}\, a_{00}\, \delta_{1j} - 2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\ \left(\delta_{ij}\, x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)\, \delta_{1j}\right)\right) \\ d^0 \wedge d_{\mathtt{ac}}{}^j.$$

Thus, we have

$$\Phi_{0j} = \Phi^{\natural}{}_{0j} + \Phi^{\mathfrak{e}}{}_{0j} = \tfrac{m}{\hbar_0}\, a_{00}\, \delta_{1j} - \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\ \left(\delta_{ij}\, x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\, a_{00}\, (x^0)^2 + \nu_0\, x^0)\, \delta_{1j}\right),$$
$$\Phi_{ij} = \Phi^{\natural}{}_{ij} = 0. \quad \square$$

Proposition 26.1.10 *The distinguished classical gauge* $\mathtt{b}^{\square}$ *suggested by the inertial observer* $o_{\mathtt{in}}$ *yields the following observed potential of the observed spacetime 2–form* $\Phi[o_{\mathtt{ac}}]$*, which is associated with the same classical gauge* $\mathtt{b}^{\square}$*, (see Corollary 26.1.6)*

$$A[\mathtt{b}^{\square}, o_{\mathtt{ac}}] := o_{\mathtt{ac}}{}^* A^{\uparrow}[\mathtt{b}^{\square}].$$

Then, in virtue of the transition rule (see Remark 10.1.6 and also Theorem 15.2.26)

$$A[\mathtt{b}^{\square}, o_{\mathtt{ac}}] = A[\mathtt{b}^{\square}, o_{\mathtt{in}}] + \theta[o_{\mathtt{in}}] \lrcorner\, G^{\flat}(\vec{v}_{\mathtt{ac}}) - \tfrac{1}{2}\, G(\vec{v}_{\mathtt{ac}}, \vec{v}_{\mathtt{ac}}),$$

we obtain the following equality, in a cartesian spacetime chart adapted to $o_{\mathtt{in}}$ *, (see Definition 24.4.1, Proposition 25.1.13 and Lemma 26.1.7)*

$$A[\mathrm{b}^{\square}, o_{\mathtt{ac}}] = \tfrac{m}{\hbar_0}\,\Big(-\big(\tfrac{1}{2}\,(a_{00}\,x^0 + \nu_0)^2 + \tfrac{q_0}{m}\,k_0\,\tfrac{1}{r}\big)\,d^0 + (a_{00}\,x^0 + \nu_0)\,d^1\Big)\,,$$

i.e., in a cartesian spacetime chart adapted to $o_{\mathtt{ac}}$,

$$A[\mathrm{b}^{\square}, o_{\mathtt{ac}}] = \tfrac{m}{\hbar_0}\,\Big(\big(\tfrac{1}{2}\,(a_{00}\,x^0 + \nu_0)^2 - \tfrac{q_0}{m}\,k_0\,\tfrac{1}{r}\big)\,d^0 + (a_{00}\,x^0 + \nu_0)\,d_{\mathtt{ac}}{}^1\Big)\,. \quad \square$$

Proposition 26.1.11 *The joined spacetime connection* K *yields the following joined phase objects, whose coordinate expressions in a cartesian spacetime chart adapted to the observer* $o_{\mathtt{ac}}$ *are (see Theorem 6.3.1, Corollary 9.2.4 and Proposition 25.1.12)*

$$\begin{aligned}
\Gamma &= d_{\mathtt{ac}}{}^{\lambda} \otimes \partial_{\mathtt{ac}\lambda} \\
&\quad + \Big(\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r^3}\,\,\big(x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\,a_{00}\,(x^0)^2 + \nu_0\,x^0)\,\delta^{i1}\big) - a_{00}\,\delta^{i1}\Big)\,d^0 \otimes \partial_{\mathtt{ac}}{}^0_i\,, \\
\gamma_0 &= \partial_{\mathtt{ac}0} + x_{\mathtt{ac}}{}^i_0\,\partial_{\mathtt{ac}i} \\
&\quad + \Big(\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r^3}\,\,\big(x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\,a_{00}\,(x^0)^2 + \nu_0\,x^0)\,\delta^{i1}\big) - a_{00}\,\delta^{i1}\Big)\,\partial_{\mathtt{ac}}{}^0_i\,, \\
\Omega &= \tfrac{m}{\hbar_0}\,\delta_{ij}\,d_{\mathtt{ac}}{}^i_0 \wedge \theta_{\mathtt{ac}}{}^j \\
&\quad - \tfrac{m}{\hbar_0}\,\Big(\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r^3}\,\,\big(\delta_{ij}x_{\mathtt{ac}}{}^i + (\tfrac{1}{2}\,a_{00}\,(x_{\mathtt{ac}}{}^0)^2 + \nu_0\,x_{\mathtt{ac}}{}^0)\,\delta_{j1}\big) - a_{00}\,\delta_{j1}\Big) \\
&\qquad d^0 \wedge d_{\mathtt{ac}}{}^j\,, \\
\Lambda &= \tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathtt{ac}i} \wedge \partial_{\mathtt{ac}}{}^0_j\,. \quad \square
\end{aligned}$$

Thus, the observed electric field does not effect Λ.

Corollary 26.1.12 *According to the above results, the coordinate expression of the distinguished horizontal potential of* Ω, *in a cartesian spacetime chart adapted to* $o_{\mathtt{ac}}$, *is (see Theorem 10.1.4, Proposition 25.1.12, Corollaries 26.1.9, and 26.1.5)*

$$\begin{aligned}
A^{\uparrow}[\mathrm{b}^{\square}] &= -\,\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\Big(\delta_{ij}\,x_{\mathtt{ac}}{}^i_0\,x_{\mathtt{ac}}{}^j_0 - (a_{00}\,x^0 + \nu_0)^2 + 2\,\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r}\Big)\,d^0 \\
&\quad + \tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathtt{ac}}{}^j_0\,d_{\mathtt{ac}}{}^i + \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + \nu_0)\,d_{\mathtt{ac}}{}^1\,. \quad \square
\end{aligned}$$

26.2 Quantum Objects

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$, equipped with the *radial electric field* $\vec{E}$, introduced in Hypothesis C.2, postulate a suitable quantum structure $(\boldsymbol{Q}, \mathrm{h}_{\eta}, \mathrm{ч}^{\uparrow})$ and compute the main quantum objects discussed throughout the book, with reference to an inertial observer and a uniformly accelerated observer.

26.2.1 Starting Hypothesis of the Quantum Theory

In this 2nd example, we consider a flat spacetime equipped with a given radial electric field $\vec{E}$ as classical background (see Hypothesis C.2).

Then, we postulate a trivial quantum bundle $\pi : \boldsymbol{Q} \to \acute{\boldsymbol{E}}$ and an upper quantum connection $\Psi^{\uparrow}$, as source of all further quantum developments (see Hypothesis Q.2).

Indeed, the upper quantum connection $\Psi^{\uparrow}$ is defined by means of the distinguished classical gauge $\mathrm{b}^{\square}$ and the associated upper potential $A^{\uparrow}[\mathrm{b}^{\square}]$, that have been emphasised in the classical theory (see Corollary 26.1.6).

Note 26.2.1 Preliminarily, we observe that, in virtue of Hypothesis C.2, the cohomology class of the phase 2–form Ω turns out to be integer (see [410] and recall Theorem 15.2.20).

Moreover, we recall the global classical gauge $\mathrm{b}^{\square}$, for which the classical upper potential of the phase 2–form Ω is (see Corollary 26.1.6):

$$A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}^{\natural}[o_{\mathtt{in}}] + A[\mathrm{b}^{\square}, o_{\mathtt{in}}] = -(\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_0^i\,x_0^j + \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r})\,d^0 + \tfrac{m}{\hbar_0}\,\delta_{ij}\,x_0^i\,d^j\,. \quad \square$$

Hypothesis Q.2 *In this 2nd Example, in the framework of the classical background postulated in Hypothesis C.2, we consider a* trivialisable *quantum bundle* $\pi : \boldsymbol{Q} \to \acute{\boldsymbol{E}}$*, along with an* η*–hermitian quantum metric* $\mathrm{h}_{\eta} := \mathrm{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \acute{\boldsymbol{E}} \otimes \mathbb{C}$*, according to Postulate Q.1, Proposition 14.3.1 and Definition 14.5.1.*

Thus, in virtue of the triviality of the quantum bundle, there exist global quantum bases b *of* $\pi : \boldsymbol{Q} \to \acute{\boldsymbol{E}}$*.*

Then, we choose a global quantum basis $\mathrm{b}^{\square}$ *(denoted by the same symbol of the distinguished classical gauge* $\mathrm{b}^{\square}$*) and consider the upper quantum connection (see Definition 3.2.9, Theorem 15.2.4, Note 15.2.12, Postulate Q.2 and Corollary 26.1.9)*

$$\Psi^{\uparrow} = \chi^{\uparrow}[\mathrm{b}^{\square}] + \mathrm{i}\,A^{\uparrow}[\mathrm{b}^{\square}]\,\mathbb{I}^{\uparrow}, \ \textit{such that} \ A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{\mathtt{in}}] + A[\mathrm{b}^{\square}, o_{\mathtt{in}}],$$

with coordinate expression, in a cartesian spacetime chart adapted to the inertial observer $o_{\mathtt{in}}$*,*

$$\Psi^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 + \mathrm{i}\,\Big(-(\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_0^i\,x_0^j + \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r})\,d^0 + (\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_0^i)\,d^j\Big) \otimes \mathbb{I}^{\uparrow}. \quad \square$$

The above upper quantum connection $\Psi^{\uparrow}$ yields, in the present specific example, all quantum objects that we have analysed in the general theory. Now, we shall provide their expressions with reference to the inertial observer $o_{\mathtt{in}}$ and the uniformly accelerated observer $o_{\mathtt{ac}}$.

26.2.2 Inertial Observer

We consider the flat newtonian spacetime $(\acute{\boldsymbol{E}}, g, K^{\natural})$ equipped with the given radial electric field $\vec{E}$, the above quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \Upsilon^\uparrow)$ and, with reference to the inertial observer $o_{\mathtt{in}}$, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.3, Q.2 and Definition 24.2.1).

With reference to the quantum basis $\mathrm{b}^{\square}$, the observer $o_{\mathtt{in}}$ and the adapted cartesian chart (x^λ), we obtain the following coordinate expressions.

Proposition 26.2.2 *With reference to a cartesian spacetime chart adapted to the observer $o_{\mathtt{in}}$, we obtain the following coordinate expressions of the quantum laplacian, the quantum kinetic tensor, the quantum probability current, the quantum lagrangian and the Schrödinger operator (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2 and 17.6.5)*

$$\begin{aligned}
\Delta_0(\Psi) &= \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi\, \mathrm{b}^{\square}, \\
\mathrm{Q}_0(\Psi) &= (\psi\, \partial_0 - \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_j\psi\, \partial_i) \otimes \mathrm{b}^{\square}, \\
\mathrm{J}_0(\Psi) &= |\psi|^2\, \partial_0 + \mathrm{i}\, \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, (\psi\, \partial_j\bar{\psi} - \bar{\psi}\, \partial_j\psi)\, \partial_i, \\
\mathrm{L}_0(\Psi) &= \tfrac{1}{2}\, \big(-\tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_i\bar{\psi}\, \partial_j\psi + \mathrm{i}\, (\bar{\psi}\, \partial_0\psi - \psi\, \partial_0\bar{\psi}) - 2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\, |\psi|^2\big), \\
\mathrm{S}_0(\Psi) &= (\partial_0\psi - \tfrac{1}{2}\, \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi + \mathrm{i}\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\, \psi)\, \mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the electric field does not effect $\Delta_0(\Psi)$, $\mathrm{Q}_0(\Psi)$ and $\mathrm{J}_0(\Psi)$.

Note 26.2.3 With reference to the inertial observer $o_{\mathtt{in}}$ and the distinguished classical gauge $\mathrm{b}^{\square}$, we obtain the distinguished special phase functions (see Definition 3.2.9, Theorem 10.1.8 and Example 12.1.4)

$$\begin{aligned}
\mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathtt{in}}] &:= \mathcal{K}_0[o_{\mathtt{in}}] - A_0[\mathrm{b}^{\square}, o_{\mathtt{in}}] &&\in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathtt{in}}] &:= \mathcal{Q}_i[o_{\mathtt{in}}] - A_i[\mathrm{b}^{\square}, o_{\mathtt{in}}] &&\in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
&1 &&\in \mathbb{R} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square
\end{aligned}$$

Proposition 26.2.4 *We have the following coordinate expressions, in a cartesian spacetime chart adapted to $o_{\mathtt{in}}$ (see Example 12.1.4, Propositions 26.1.3 and 25.1.2)*

$$\mathcal{H}_0 := \mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathtt{in}}] = \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j + \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}, \qquad \mathcal{P}_i := \mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathtt{in}}] = \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^j. \quad \square$$

Thus, the electric field does not effect $\mathcal{P}_i$.

Proposition 26.2.5 *We obtain the following coordinate expressions of distinguished quantum operators (see Theorem 20.1.9, Example 20.1.12 and Proposition 25.2.5)*

$$\mathrm{O}[\mathcal{H}_0](\Psi) = -\left(\tfrac{1}{2}\,\Delta_0\,\psi - \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,\psi\right)\mathfrak{b}^{\square}\,, \quad \mathrm{O}[\mathcal{P}_j](\Psi) = -\mathfrak{i}\,\partial_j\psi\,\mathfrak{b}^{\square}\,,$$
$$\mathrm{O}[1](\Psi) = \psi\,\mathfrak{b}^{\square}. \quad \square$$

Thus, the electric field does not effect $\mathrm{O}[\mathcal{P}_j](\Psi)$ and $\mathrm{O}[1](\Psi)$.

Eventually, we discuss the possible conservation of the distinguished phase functions and study the associated quantum current forms and expectation forms.

Lemma 26.2.6 *The special phase function $\mathcal{H}_0$ is conserved, while $\mathcal{P}_i$ is not conserved (see Definition 11.6.2 and Proposition 11.6.3).*

Proof. We have the following equalities (see Corollary 26.1.5 and Note 26.2.3)

$$\gamma_0.\mathcal{H}_0 = \tfrac{q}{\hbar}\,k_0\,x_0^i\,\partial_i\tfrac{1}{r} + \tfrac{1}{2}\,\tfrac{q_0}{m}\,k_0\,\tfrac{m}{\hbar_0}\,\tfrac{1}{r^3}\,x^i\,\partial_i^0(\delta_{hk}\,x_0^h\,x_0^k) = -\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r^3}\,\delta_{ij}\,x_0^i\,x^j$$
$$+ \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r^3}\,\delta_{ij}\,x_0^j\,x^i = 0,$$
$$\gamma_0.\mathcal{P}_i = \tfrac{q_0}{m}\,k_0\,\tfrac{1}{r^3}\,x^j\,\partial_j^0(\tfrac{m}{\hbar_0}\,\delta_{ik}\,x_0^k) = \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r^3}\,\delta_{ik}\,x^k \neq 0. \quad \square$$

Remark 26.2.7 We stress that $\mathcal{H}_0$ is conserved with respect to the joined dynamical phase connection γ, but it is not conserved with respect to the gravitational dynamical phase connection $\gamma^{\natural}$, as we have $\gamma^{\natural}.\mathcal{H}_0 \neq 0$. □

Proposition 26.2.8 *We have the following coordinate expressions of distinguished quantum current forms (see Propositions 21.2.2, 25.2.6, and Example 21.2.3)*

$$\mathrm{j}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{2}\,\left(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar{\psi}\,\partial_k\psi + 2\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,|\psi|^2\right)\upsilon_0^0 - \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}$$
$$(\partial_h\bar{\psi}\,\partial_0\psi + \partial_h\psi\,\partial_0\bar{\psi})\,\upsilon_k^0,$$
$$\mathrm{j}_\eta[\mathcal{P}_j](\Psi) = -\tfrac{1}{2}\,\left(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar{\psi}\,\partial_k\psi + 2\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,|\psi|^2\right)\upsilon_j^0 + \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}$$
$$(\partial_j\bar{\psi}\,\partial_h\psi + \partial_j\psi\,\partial_h\bar{\psi})\,\upsilon_k^0$$
$$- \tfrac{1}{2}\,\mathfrak{i}\,\left((\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi})\,\upsilon_0^0 - (\bar{\psi}\,\partial_0\psi - \psi\,\partial_0\bar{\psi})\,\upsilon_j^0\right),$$
$$\mathrm{j}_\eta[1](\Psi) = -\tfrac{1}{2}\,\mathfrak{i}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\bar{\psi}\,\partial_h\psi - \psi\,\partial_h\bar{\psi})\,\upsilon_k^0 + |\psi|^2\,\upsilon_0^0. \quad \square$$

Thus, the electric field does not effect $\mathrm{j}_\eta[1](\Psi)$.

Corollary 26.2.9 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Propositions 21.2.6, Example 21.2.7 and Corollary 25.2.7)*

$$\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{2}\,\left(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar{\psi}\,\partial_k\psi + 2\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,|\psi|^2\right)\eta,$$
$$\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) = -\tfrac{1}{2}\,\mathfrak{i}\,(\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi})\,\eta,$$
$$\check{\mathrm{j}}_\eta[1](\Psi) = |\psi|^2\,\eta. \quad \square$$

Thus, the electric field does not effect $\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi)$ and $\check{\mathrm{j}}_\eta[1](\Psi)$.

Corollary 26.2.10 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Propositions 21.3.2, 25.2.8, and Example 21.3.3)*

$$\epsilon_\eta[\mathcal{H}_0](\Psi) = -\tfrac{1}{2}\,\Big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}(\psi\,\partial_{hk}\bar{\psi} + \bar{\psi}\,\partial_{hk}\psi) - 2\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,|\psi|^2\Big)\,\eta,$$
$$\epsilon_\eta[\mathcal{P}_j](\Psi) = -\tfrac{1}{2}\,\mathrm{i}\,(\bar{\psi}\,\partial_j\psi - \psi\,\partial_j\bar{\psi})\,\eta\,,$$
$$\epsilon_\eta[1](\Psi) = |\psi|^2\,\eta. \quad \square$$

Thus, the electric field does not effect $\epsilon_\eta[\mathcal{P}_j](\Psi)$ and $\epsilon_\eta[1](\Psi)$.

26.2.3 Uniformly Accelerated Observer

We consider the flat newtonian spacetime $(\acute{\boldsymbol{E}}, g, K^\natural)$ of standard Classical Mechanics equipped with the given radial electric field $\vec{E}$, the above quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \mathrm{Ч}^\uparrow)$ and, with reference to the uniformly accelerated observer o_{ac}, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.2, Q.2 and Definition 24.4.1).

Let us recall the expression of the upper quantum potential in a cartesian spacetime chart adapted to o_{ac} (see Corollary 26.1.12)

$$A^\uparrow[\mathrm{b}^\square] = -\,\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\Big(\delta_{ij}\,x_{\mathrm{ac}}{}^i_0\,x_{\mathrm{ac}}{}^j_0 - (a_{00}\,x^0 + v_0)^2 + 2\,\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r}\Big)\,d^0$$
$$+\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ac}}{}^j_0\,d_{\mathrm{ac}}{}^i + \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + v_0)\,d_{\mathrm{ac}}{}^1.$$

Lemma 26.2.11 *The coordinate expression of the upper quantum connection in a cartesian spacetime chart adapted to o_{ac} is*

$$\mathrm{Ч}^\uparrow = \chi^\uparrow[\mathrm{b}^\square] + \mathrm{i}\,A^\uparrow[\mathrm{b}_{\mathrm{ac}}]\,\mathbb{I}^\uparrow$$
$$= d_{\mathrm{ac}}{}^\lambda \otimes \partial_{\mathrm{ac}\lambda} + d_{\mathrm{ac}}{}^i_0 \otimes \partial_{\mathrm{ac}}{}^0_i$$
$$-\,\mathrm{i}\,\tfrac{m}{\hbar_0}\,\tfrac{1}{2}\,\Big(\delta_{ij}\,x_{\mathrm{ac}}{}^i_0\,x_{\mathrm{ac}}{}^j_0 - (a_{00}\,x^0 + v_0)^2 + 2\,\tfrac{q_0}{m}\,k_0\,\tfrac{1}{r}\Big)\,d^0 \otimes \mathbb{I}^\uparrow$$
$$+\,\mathrm{i}\,\tfrac{m}{\hbar_0}\,\Big(\delta_{ij}\,x_{\mathrm{ac}}{}^j_0\,d_{\mathrm{ac}}{}^i + (a_{00}\,x^0 + v_0)\,d_{\mathrm{ac}}{}^1\Big) \otimes \mathbb{I}^\uparrow. \quad \square$$

Then, we obtain the following coordinate expressions of dynamical quantum objects.

Proposition 26.2.12 *With reference to a cartesian spacetime chart adapted to the observer o_{ac}, we obtain the following coordinate expressions of the quantum laplacian, the quantum kinetic tensor, the quantum probability current, the quantum lagrangian and the Schrödinger operator (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2, 17.6.5 and Proposition 25.2.10)*

$$\Delta_0(\Psi) = \Big(\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}ij}\psi - 2\,\mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}1}\psi - \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + \nu_0)^2\,\psi\Big)\,\mathrm{b}^{\square},$$

$$\mathrm{Q}_0(\Psi) = \big(\psi\,\partial_{\mathrm{ac}0} - \mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}j}\psi\,\partial_{\mathrm{ac}i} - (a_{00}\,x^0 + \nu_0)\,\psi\,\partial_{\mathrm{ac}1}\big)\otimes\mathrm{b}^{\square},$$

$$\mathrm{J}_0(\Psi) = |\psi|^2\,\partial_{\mathrm{ac}0} + \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\psi\,\partial_{\mathrm{ac}j}\bar{\psi} - \bar{\psi}\,\partial_{\mathrm{ac}j}\psi)\,\partial_{\mathrm{ac}i} - (a_{00}\,x^0 + \nu_0)\,|\psi|^2\,\partial_{\mathrm{ac}1}\big)\otimes\mathrm{b}^{\square},$$

$$\mathrm{L}_0(\Psi) = \tfrac{1}{2}\,\big(-\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}i}\bar{\psi}\,\partial_{\mathrm{ac}j}\psi + \mathrm{i}\,(\bar{\psi}\,\partial_{\mathrm{ac}0}\psi - \psi\,\partial_{\mathrm{ac}0}\bar{\psi}) - \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,(\bar{\psi}\,\partial_{\mathrm{ac}1}\psi - \psi\,\partial_{\mathrm{ac}1}\bar{\psi}) - 2\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,|\psi|^2\big)\,,$$

$$\mathrm{S}_0(\Psi) = \big(\partial_{\mathrm{ac}0}\psi + \mathrm{i}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,\psi - (a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}1}\psi - \tfrac{1}{2}\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_{\mathrm{ac}ij}\psi\big)\,\mathrm{b}^{\square}. \quad \square$$

Thus, the electric field does not effect $\Delta(\Psi)$, $\mathrm{Q}(\Psi)$ and $\mathrm{J}(\Psi)$.

Next, let us consider the distinguished special phase functions (see Definition 3.2.9, Theorem 10.1.8 and Example 12.1.4)

$$\begin{aligned}
\mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathrm{ac}}] &:= \mathcal{K}_0[o_{\mathrm{ac}}] - A_0[\mathrm{b}^{\square}, o_{\mathrm{ac}}] && \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathrm{ac}}] &:= \mathcal{Q}_i[o_{\mathrm{ac}}] - A_i[\mathrm{b}^{\square}, o_{\mathrm{ac}}] && \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
& 1 && \in \mathbb{R} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}).
\end{aligned}$$

Proposition 26.2.13 *We have the following coordinate expressions, in a cartesian spacetime chart adapted to o_{ac} (see Example 12.1.4 and Proposition 25.1.12)*

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ac}0}^{i}\,x_{\mathrm{ac}0}^{j} - \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + \nu_0)^2 + \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,,\\
\mathcal{P}_i &:= \mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathrm{ac}}] = \tfrac{m}{\hbar_0}\,\delta_{ij}\,x_{\mathrm{ac}0}^{j} + \tfrac{m}{\hbar_0}\,(a_{00}\,x^0 + \nu_0)\,\delta_{i1}. \quad \square
\end{aligned}$$

Thus, the electric field does not effect $\mathcal{P}_i$.

Proposition 26.2.14 *We have the following coordinate expression of the distinguished quantum operators (see Theorem 20.1.9, Example 20.1.12 and Proposition 25.2.13)*

$$\begin{aligned}
\mathrm{O}[\mathcal{H}_0](\Psi) &= \big(-\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\partial_{\mathrm{ac}hk}\psi + \mathrm{i}\,(a_{00}\,x^0 + \nu_0)\,\partial_{\mathrm{ac}1}\psi + \tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,\psi\big)\,\mathrm{b}^{\square},\\
\mathrm{O}[\mathcal{P}_j](\Psi) &= -\mathrm{i}\,\partial_{\mathrm{ac}j}\psi\,\mathrm{b}^{\square},\\
\mathrm{O}[1](\Psi) &= \psi\,\mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the electric field does not effect $\mathrm{O}[\mathcal{P}_j](\Psi)$ and $\mathrm{O}[1](\Psi)$.

Eventually, we discuss the possible conservation of the distinguished phase functions and study the associated quantum current forms and expectation forms.

Lemma 26.2.15 *The special phase functions $\mathcal{H}_0$ and $\mathcal{P}_i$ are not conserved (see Definition 11.6.2 and Proposition 11.6.3).*

Proof. We have the following equalities (see Corollary 26.1.5 and Proposition 26.2.13)

$$\gamma_0.\mathcal{H}_0 = \tfrac{m}{\hbar_0}\, a_{00}\, \big(a_{00}\, x^0 + v_0 + x_{\mathrm{ac}}{}^1_0\big)$$
$$- \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\, \Big(\big(x_{\mathrm{ac}}{}^1 + a_{00}\,(x^0)^2 + v_0\, x^0\big)\, (a_{00}\, x^0 + v_0)$$
$$- x_{\mathrm{ac}}{}^1_0\, \big(\tfrac{1}{2}\, a_{00}\,(x^0)^2 + v_0\, x^0\big)\Big)\,,$$
$$\gamma_0.\mathcal{P}_i = \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r^3}\, \big(\delta_{ij}\, x_{\mathrm{ac}}{}^j + (\tfrac{1}{2}\, a_{00}\,(x^0)^2 + v_0\, x^0)\, \delta_{i1}\big)\,.$$

Hence, $\mathcal{H}_0$ and $\mathcal{P}_i$ are not conserved. □

Note 26.2.16 We have the following coordinate expressions of distinguished quantum current forms (see Propositions 21.2.2, 25.2.14, and Example 21.2.3)

$$\mathrm{j}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{2}\, \big(\tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_{\mathrm{ac}h}\bar{\psi}\, \partial_{\mathrm{ac}k}\psi + \mathrm{i}\,(a_{00}\, x^0 + v_0)\, (\bar{\psi}\, \partial_{\mathrm{ac}1}\psi - \psi\, \partial_{\mathrm{ac}1}\bar{\psi})$$
$$+ 2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\, |\psi|^2\big)\, \upsilon^0_0$$
$$- \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\partial_{\mathrm{ac}h}\bar{\psi}\, \partial_{\mathrm{ac}0}\psi + \partial_{\mathrm{ac}h}\psi\, \partial_{\mathrm{ac}0}\bar{\psi})\, \upsilon^0_k$$
$$- \tfrac{1}{2}\, \mathrm{i}\,(a_{00}\, x^0 + v_0)\, (\bar{\psi}\, \partial_{\mathrm{ac}0}\psi - \psi\, \partial_{\mathrm{ac}0}\bar{\psi})\, \upsilon^0_1,$$
$$\mathrm{j}_\eta[\mathcal{P}_i](\Psi) = -\tfrac{1}{2}\, \mathrm{i}\,(\bar{\psi}\, \partial_i\psi - \psi\, \partial_i\bar{\psi})\, \upsilon^0_0 + \tfrac{1}{2}\, (a_{00}\, x^0 + v_0)\, (\bar{\psi}\, \partial_i\psi - \psi\, \partial_i\bar{\psi})\, \upsilon^0_1$$
$$- \tfrac{1}{2}\, \big(\tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_{\mathrm{ac}h}\bar{\psi}\, \partial_{\mathrm{ac}k}\psi + \mathrm{i}\,(a_{00}\, x^0 + v_0)\, (\bar{\psi}\, \partial_{\mathrm{ac}1}\psi - \psi\, \partial_{\mathrm{ac}1}\bar{\psi})$$
$$-\mathrm{i}\,(\bar{\psi}\, \partial_{\mathrm{ac}0}\psi - \psi\, \partial_{\mathrm{ac}0}\bar{\psi}) + 2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\, |\psi|^2\big)\, \upsilon^0_i$$
$$+ \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{hj}\, \big(\partial_{\mathrm{ac}i}\bar{\psi}\, \partial_{\mathrm{ac}h}\psi + \partial_{\mathrm{ac}i}\psi\, \partial_{\mathrm{ac}h}\bar{\psi}\big)\, \upsilon^0_j,$$
$$\mathrm{j}_\eta[1](\Psi) = -\tfrac{1}{2}\, \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{hk}\, (\bar{\psi}\, \partial_{\mathrm{ac}h}\psi - \psi\, \partial_{\mathrm{ac}h}\bar{\psi})\, \upsilon^0_k + |\psi|^2\, \upsilon^0_0$$
$$- (a_{00}\, x^0 + v_0)\, |\psi|^2\, \upsilon^0_1. \quad \square$$

Thus, the electric field does not effect $\mathrm{j}_\eta[1](\Psi)$.

Note 26.2.17 We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6, Example 21.2.7 and Corollary 25.2.15)

$$\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{2}\, \big(\tfrac{\hbar_0}{m}\, \delta^{hk}\, \partial_{\mathrm{ac}h}\bar{\psi}\, \partial_{\mathrm{ac}k}\psi + \mathrm{i}\,(a_{00}\, x^0 + v_0)\, (\bar{\psi}\, \partial_{\mathrm{ac}1}\psi - \psi\, \partial_{\mathrm{ac}1}\bar{\psi})$$
$$+ 2\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{r}\, |\psi|^2\big)\, \eta,$$
$$\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) = -\tfrac{1}{2}\, \mathrm{i}\, \big((\bar{\psi}\, \partial_{\mathrm{ac}j}\psi - \psi\, \partial_{\mathrm{ac}j}\bar{\psi})\, \eta,$$
$$\check{\mathrm{j}}_\eta[1](\Psi) = |\psi|^2\, \eta. \quad \square$$

Thus, the electric field does not effect $\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi)$ and $\check{\mathrm{j}}_\eta[1](\Psi)$.

Note 26.2.18 We have the following coordinate expressions of distinguished quantum expectation value forms (see Propositions 21.3.2, 25.2.16, and Example 21.3.3)

$$\begin{aligned}
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\Big(\tfrac{1}{4}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\psi\,\partial_{\mathtt{ac}hk}\bar{\psi} + \bar{\psi}\,\partial_{\mathtt{ac}hk}\psi) + \tfrac{1}{2}\,\mathfrak{i}\,(a_{00}\,x^0 + \nu_0) \\
&\qquad (\psi\,\partial_{\mathtt{ac}1}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ac}1}\psi) \\
&\quad + \tfrac{1}{4}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\;|\psi|^2\Big)\;\eta, \\
\epsilon_\eta[\mathcal{P}_j](\Psi) &= \tfrac{1}{2}\,\mathfrak{i}\,(\psi\,\partial_{\mathtt{ac}j}\bar{\psi} - \bar{\psi}\,\partial_{\mathtt{ac}j}\psi)\,\eta, \\
\epsilon_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}$$

Thus, the electric field does not effect $\epsilon_\eta[\mathcal{P}_j](\Psi)$ and $\epsilon_\eta[1](\Psi)$.

Chapter 27
Dynamical Example 3: Constant Magnetic Field

In this 3rd example, we extend the survey provided by the 1st example (see Chap. 25) by computing, in the framework of the *standard flat spacetime* equipped with a *given constant magnetic field*, the basic classical and quantum objects discussed throughout the body of the book (Sects. 27.1 and 27.2).

In order to show how our machinery works with different observers, we consider, not only a standard inertial observer, but also a *uniformly rotating observer*, whose rotation axis is orthogonal to the magnetic field; we leave to the reader the case of a uniformly accelerating observer as an exercise.

27.1 Classical Objects

We consider a flat newtonian spacetime equipped with a given *constant magnetic field* and compute the main classical objects discussed throughout the book, with reference to an inertial observer and a uniformly rotating observer.

27.1.1 Starting Hypothesis of the Classical Theory

In this 3rd example, the classical background spacetime (see Hypothesis C.3) is the standard flat spacetime discussed in Chap. 24, along with a constant magnetic field.

Later, we shall complete the hypothesis on the classical theory by postulating the quantum bundle and the upper quantum connection of this 3rd example (see Hypothesis Q.3).

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Hypothesis C.3 *In this third Example, we consider the standard flat spacetime* $(\boldsymbol{E}, g, K^{\natural})$ *defined in Definition 24.1.1, along with the constant* magnetic field *(see Definitions 5.1.1 and 5.2.1)*

$$\vec{B} \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}\big),$$

whose expression is, in a cartesian spacetime chart adapted to the observer o_{in}, *(see Definition 24.2.4)*

$$\vec{B} = k\, \mathrm{e}_3, \quad \text{with } k \in \mathbb{T}^{-1} \otimes \mathbb{L}^{-1/2} \otimes \mathbb{M}^{1/2}.$$

Moreover, we consider a particle of mass m and charge q. □

27.1.2 Inertial Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ equipped with the given *constant magnetic field* $\vec{B}$, and, with reference to an inertial observer o_{in}, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.3 and Definition 24.2.1).

Lemma 27.1.1 *The electromagnetic field turns out to be the scaled spacetime 2-form (see Proposition 5.4.1)*

$$F = 2\, \tfrac{1}{c}\, \theta[o_{in}]^{*}\big(i_{\vec{B}}\, \eta\big) \in \sec\big(\boldsymbol{E},\ (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^{*}\boldsymbol{E}\big),$$

with coordinate expression, in the cartesian spacetime chart adapted to o_{in},

$$F = 2\, k\, \tfrac{1}{c}\, d^1 \wedge d^2.$$

Moreover, the above electromagnetic field can be regarded as the unscaled spacetime 2-form $\frac{q}{\hbar} F \in \sec(\boldsymbol{E}, \Lambda^2 T^{*}\boldsymbol{E})$. *Then, according to the equality* $\frac{q}{\hbar} F = 2\, dA^{\mathfrak{e}}$, *we have the, gauge dependent and observer independent, spacelike electromagnetic potential*

$$A^{\mathfrak{e}} = \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k\, \tfrac{1}{c}\, (x^1\, d^2 - x^2\, d^1) = \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k\, \tfrac{1}{c}\, \epsilon_{ij3}\, x^i\, d^j. \quad \square$$

Proposition 27.1.2 *The joined spacetime connection associated with the postulated gravitational and electromagnetic fields turns out to be (see Theorem 6.3.1 and Proposition 25.1.1)*

$$K \equiv K^{\natural} + K^{\mathfrak{e}} = K^{\natural} - \tfrac{1}{2}\, \tfrac{q}{m}\, (dt \otimes \hat{F} + \hat{F} \otimes dt),$$

with coordinate expression, in the cartesian spacetime chart adapted to the observer o_{in},

$$K = d^\lambda \otimes \partial_\lambda + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big((\dot{x}^2\,d^0 + \dot{x}^0\,d^2) \otimes \dot{\partial}_1 - (\dot{x}^0\,d^1 + \dot{x}^1\,d^0) \otimes \dot{\partial}_2\big).$$

Thus, we have

$$\begin{aligned}
K_0{}^i{}_0 &= K^\natural{}_0{}^i{}_0 = 0,\\
K_0{}^2{}_1 = K_1{}^2{}_0 = -K_0{}^1{}_2 = -K_2{}^1{}_0 &= -\tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,,\\
K_0{}^3{}_i = K_i{}^3{}_0 = K^\natural{}_0{}^3{}_i &= K^\natural{}_i{}^3{}_0 = 0,\\
K_h{}^i{}_k &= K^\natural{}_h{}^i{}_k = 0. \quad \square
\end{aligned}$$

Corollary 27.1.3 *The joined observed spacetime 2-form turns out to be (see Corollary 6.3.3 and Proposition 25.1.1)*

$$\Phi[o_{\mathtt{in}}] = \Phi^\natural[o_{\mathtt{in}}] + \tfrac{q}{\hbar}\,F = 2\,\tfrac{q}{\hbar}\,\tfrac{1}{c}\,\theta[o_{\mathtt{in}}]^*\big(i_{\vec{B}}\,\eta\big),$$

i.e., in coordinates,

$$\Phi[o_{\mathtt{in}}] = 2\,\tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,d^1 \wedge d^2.$$

Then, there exists a global gauge $\mathsf{b}^\square$, *such that (see Theorem 4.3.3)*

$$A[\mathsf{b}^\square, o_{\mathtt{in}}] = \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,(x^1\,d^2 - x^2\,d^1) = \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,\epsilon_{ij3}\,x^i\,d^j. \quad \square$$

Corollary 27.1.4 *The joined spacetime connection K yields the following joined phase objects, whose coordinate expressions in a cartesian spacetime chart adapted to the observer* $o_{\mathtt{in}}$ *are (see Theorem 6.3.1 and Corollary 9.2.4 and Proposition 25.1.21)*

$$\begin{aligned}
\Gamma &= d^\lambda \otimes \partial_\lambda + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\epsilon_{hj3}\,\delta^{hi}\,(d^j + x^j_0\,d^0) \otimes \partial^0_i,\\
\gamma_0 &= \partial_0 + x^i_0\,\partial_i + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\epsilon_{kh3}\,x^h_0\,\delta^{ik}\,\partial^0_i,\\
\Omega &= \tfrac{m}{\hbar_0}\,\delta_{ij}\,d^i_0 \wedge \theta^j + \tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,d^1 \wedge d^2,\\
\Lambda &= \tfrac{\hbar_0}{m}\,\delta^{ij}\,\partial_i \wedge \partial^0_j + \tfrac{q}{m}\,\tfrac{\hbar_0}{m}\,k_0\,\tfrac{1}{c}\,\partial^0_1 \wedge \partial^0_2. \quad \square
\end{aligned}$$

Thus, the magnetic field $\vec{B}$ effects all above objects Γ, γ, Ω and Λ.

Corollary 27.1.5 *According to the above results, we consider the distinguished horizontal potential of* Ω *(see Theorem 10.1.4, Corollaries 27.1.3 and 27.1.4)*

$$A^\uparrow[\mathsf{b}^\square] = \mathcal{C}[o_{\mathtt{in}}] + A[\mathsf{b}^\square, o_{\mathtt{in}}] = -\mathcal{K}[o_{\mathtt{in}}] + \mathcal{Q}[o_{\mathtt{in}}] + A[\mathsf{b}^\square, o_{\mathtt{in}}],$$

with coordinate expression, in a cartesian spacetime chart adapted to the observer $o_{\mathtt{in}}$,

$$A^\uparrow[\mathsf{b}^\square] = -(\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\delta_{ij}\,x^i_0\,x^j_0)\,d^0 + \big(\tfrac{m}{\hbar_0}\,\delta_{ij}\,x^i_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,\epsilon_{ij3}\,x^i\big)\,d^j. \quad \square$$

27.1.3 Uniformly Rotating Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, \underset{\to}{g}, K^\natural)$ of standard Classical Mechanics, equipped with the *constant magnetic field* $\vec{B}$, and, with reference to the *uniformly rotating observer* o_{ro}, we compute the expressions of basic objects of Classical Mechanics discussed in the body of the book (see Hypothesis C.3 and Definition 24.5.1).

Lemma 27.1.6 *In the cylindrical spacetime chart* $(x^0, \rho_{ro}, \phi_{ro}, z_{ro})$ *adapted to the uniformly rotating observer* o_{ro}*, we have (see Definition 24.5.3)*

$$F = 2\,k\,\tfrac{1}{c}\;\rho_{ro}\,\big(-\,\omega_0\,d^0 \wedge d_{ro}{}^{\rho} + d_{ro}{}^{\rho} \wedge d_{ro}{}^{\phi}\big).$$

Then, according to the equality $\frac{q}{\hbar}\,F = 2\,dA^{\mathfrak{e}}$, *we have the gauge dependent and observer independent spacelike electromagnetic potential selected in Lemma 27.1.1*

$$A^{\mathfrak{e}} = \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,\rho_{ro}{}^2\,\big(\omega_0\,d^0 + d_{ro}{}^{\phi}\big). \quad \square$$

Proposition 27.1.7 *The joined spacetime galilean connection associated with the postulated gravitational and electromagnetic fields turns out to be (see Theorem 6.3.1)*

$$K \;\equiv\; K^\natural + K^{\mathfrak{e}} = K^\natural - \tfrac{1}{2}\,\tfrac{q}{m}\,(dt \otimes \hat{F} + \hat{F} \otimes dt),$$

with coordinate expression (see Proposition 25.1.22)

$$\begin{aligned}
K = {} & d^0 \otimes \partial_{ro\,0} + d_{ro}{}^{\rho} \otimes \partial_{ro\,\rho} + d_{ro}{}^{\phi} \otimes \partial_{ro\,\phi} + d_{ro}{}^{z} \otimes \partial_{ro\,z} \\
& + \big((\omega_0)^2 + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0\big)\,\rho_{ro}\,\dot{x}^0\,d^0 \otimes \dot{\partial}_{ro\,\rho} \\
& + \big(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big)\,\rho_{ro}\,(\dot{\phi}_{ro}\,d^0 + \dot{x}^0\,d_{ro}{}^{\phi}) \otimes \dot{\partial}_{ro\,\rho} \\
& - \big(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big)\,\tfrac{1}{\rho_{ro}}\,(\dot{\rho}_{ro}\,d^0 + \dot{x}^0\,d_{ro}{}^{\rho}) \otimes \dot{\partial}_{ro\,\phi} \\
& + \rho_{ro}\,\dot{\phi}_{ro}\,d_{ro}{}^{\phi} \otimes \dot{\partial}_{ro\,\rho} - \tfrac{1}{\rho_{ro}}\,(\dot{\rho}_{ro}\,d_{ro}{}^{\phi} + \dot{\phi}_{ro}\,d_{ro}{}^{\rho}) \otimes \dot{\partial}_{ro\,\phi}.
\end{aligned}$$

So, the non vanishing coefficients of the joined galilean connection are

$$\begin{aligned}
& K_{ro\,0}{}^{\rho}{}_0 = K^\natural{}_{ro\,0}{}^{\rho}{}_0 + K^{\mathfrak{e}}{}_{ro\,0}{}^{\rho}{}_0 = (\omega_0)^2\,\rho_{ro} + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0\,\rho_{ro}, \\
& K_{ro\,0}{}^{\rho}{}_{\phi} = K_{ro\,\phi}{}^{\rho}{}_0 = K^\natural{}_{ro\,0}{}^{\rho}{}_{\phi} + K^{\mathfrak{e}}{}_{ro\,0}{}^{\rho}{}_{\phi} \\
& \qquad = K^\natural{}_{ro\,\phi}{}^{\rho}{}_0 + K^{\mathfrak{e}}{}_{ro\,\phi}{}^{\rho}{}_0 = \rho_{ro}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,), \\
& K_{ro\,\phi}{}^{\rho}{}_{\phi} = K^\natural{}_{ro\,\phi}{}^{\rho}{}_{\phi} = \rho_{ro}\,, K_{ro\,0}{}^{\phi}{}_{\rho} + K_{ro\,\rho}{}^{\phi}{}_0 = K^\natural{}_{ro\,0}{}^{\phi}{}_{\rho} + K^{\mathfrak{e}}{}_{ro\,0}{}^{\phi}{}_{\rho} \\
& \qquad = K^\natural{}_{ro\,\rho}{}^{\phi}{}_0 + K^{\mathfrak{e}}{}_{ro\,\rho}{}^{\phi}{}_0 = -\tfrac{1}{\rho_{ro}}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,), \\
& K_{ro\,\rho}{}^{\phi}{}_{\phi} = K_{ro\,\phi}{}^{\phi}{}_{\rho} = K^\natural{}_{ro\,\rho}{}^{\phi}{}_{\phi} = K^\natural{}_{ro\,\phi}{}^{\phi}{}_{\rho} = -\tfrac{1}{\rho_{ro}}\,. \quad \square
\end{aligned}$$

Corollary 27.1.8 *The joined observed spacetime 2-form associated with the uniformly rotating observer o_{ro} turns out to be (see Definition 4.2.11, Proposition 25.1.22 and Lemma 27.1.6)*

$$\Phi[o_{ro}] = \Phi^{\natural}[o_{ro}] + \tfrac{q}{\hbar} F,$$

i.e., in a cylindrical coordinate chart adapted to o_{ro},

$$\Phi[o_{ro}] = -2 \tfrac{m}{\hbar_0} \rho_{ro} \Big(\big((\omega_0)^2 + \tfrac{q}{m} k_0 \tfrac{1}{c} \omega_0\big) d^0 \wedge d_{ro}{}^{\rho} - 2 \big(\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}\big) d_{ro}{}^{\rho} \wedge d_{ro}{}^{\phi}\Big).$$

Thus, we have the non vanishing coefficients

$$\Phi_{0\rho} = \Phi^{\natural}{}_{0\rho} + \Phi^{\mathfrak{e}}{}_{0\rho} = -\tfrac{m}{\hbar_0} \big((\omega_0)^2 + \tfrac{q}{m} k_0 \tfrac{1}{c} \omega_0\big) \rho_{ro},$$
$$\Phi_{\rho\phi} = \Phi^{\natural}{}_{\rho\phi} + \Phi^{\mathfrak{e}}{}_{\rho\phi} = 2 \tfrac{m}{\hbar_0} (\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}) \rho_{ro}. \quad \square$$

Proposition 27.1.9 *The distinguished classical gauge $\mathfrak{b}^{\square}$ suggested by the inertial observer o_{in} yields the following observed potential of the observed spacetime 2-form $\Phi[o_{ac}]$, which is associated with the same classical gauge $\mathfrak{b}^{\square}$, (see Corollary 26.1.6)*

$$A[\mathfrak{b}^{\square}, o_{ro}] := o_{ro}{}^{*} A^{\uparrow}[\mathfrak{b}^{\square}].$$

Then, in virtue of the transition rule (see Remark 10.1.6 and also Theorem 15.2.26)

$$A[\mathfrak{b}^{\square}, o_{ro}] = A[\mathfrak{b}^{\square}, o_{in}] + \theta[o_{in}] \lrcorner G^{\flat}(\vec{v}_{ro}) - \tfrac{1}{2} G(\vec{v}_{ro}, \vec{v}_{ro}),$$

we obtain the following equality, in cartesian and cylindrical spacetime charts adapted to o_{in}, (see Definitions 24.2.4 and 24.2.5),

$$\begin{aligned} A[\mathfrak{b}^{\square}, o_{ro}] &= -\tfrac{1}{2} \tfrac{m}{\hbar_0} (\omega_0)^2 \big((x^1)^2 + (x^2)^2\big) d^0 + \tfrac{m}{\hbar_0} \big(\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}\big) (x^1 d^2 - x^2 d^1) \\ &= -\tfrac{1}{2} \tfrac{m}{\hbar_0} (\omega_0)^2 \rho^2 d^0 + \tfrac{m}{\hbar_0} \big(\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}\big) \rho^2 d^{\phi}, \end{aligned}$$

i.e., in cartesian and cylindrical spacetime charts adapted to o_{ro},

$$\begin{aligned} A[\mathfrak{b}^{\square}, o_{ro}] &= \tfrac{1}{2} \tfrac{m}{\hbar_0} \big((\omega_0)^2 + \tfrac{q}{m} k_0 \tfrac{1}{c} \omega_0\big) \big((x_{ro}{}^1)^2 + (x_{ro}{}^2)^2\big) d^0 \\ &\quad + \tfrac{m}{\hbar_0} \big(\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}\big) (x_{ro}{}^1 d_{ro}{}^2 - x_{ro}{}^2 d_{ro}{}^1) \\ &= \tfrac{1}{2} \tfrac{m}{\hbar_0} \big((\omega_0)^2 + \tfrac{q}{m} k_0 \tfrac{1}{c} \omega_0\big) \rho_{ro}{}^2 d^0 + \tfrac{m}{\hbar_0} \big(\omega_0 + \tfrac{1}{2} \tfrac{q}{m} k_0 \tfrac{1}{c}\big) \rho_{ro}{}^2 d_{ro}{}^{\phi}. \quad \square \end{aligned}$$

Corollary 27.1.10 *The joined spacetime connection K yields the following joined phase objects (see Proposition 25.1.22, Theorem 6.3.1 and Corollary 9.2.4)*

$$\begin{aligned}
\Gamma &= d^0 \otimes \partial_{ro\,0} + d_{ro}{}^{\rho} \otimes \partial_{ro\,\rho} + d_{ro}{}^{\phi} \otimes \partial_{ro\,\phi} + d_{ro}{}^{z} \otimes \partial_{ro\,z} \\
&\quad + \big((\omega_0)^2 + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0\big)\,\rho_{ro}\;d^0 \otimes \partial_{ro}{}^{0}_{\rho} \\
&\quad + \big(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big)\,\rho_{ro}\;(\phi_{ro\,0}\,d^0 + d_{ro}{}^{\phi}) \otimes \partial_{ro}{}^{0}_{\rho} \\
&\quad - \big(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big)\,\tfrac{1}{\rho_{ro}}\;(\rho_{ro\,0}\,d^0 + d_{ro}{}^{\rho}) \otimes \partial_{ro}{}^{0}_{\phi} \\
&\quad + \rho_{ro}\;\phi_{ro\,0}\,d_{ro}{}^{\phi} \otimes \partial_{ro}{}^{0}_{\rho} - \tfrac{1}{\rho_{ro}}\,(\rho_{ro\,0}\,d_{ro}{}^{\phi} + \phi_{ro\,0}\,d_{ro}{}^{\rho}) \otimes \partial_{ro}{}^{0}_{\phi},
\end{aligned}$$

$$\begin{aligned}
\gamma_0 &= \partial_{ro\,0} + \rho_{ro\,0}\,\partial_{ro\,\rho} + \phi_{ro\,0}\,\partial_{ro\,\phi} + z_{ro\,0}\,\partial_{ro\,z} \\
&\quad + \rho_{ro}\;(\omega_0 + \phi_{ro\,0})\,(\omega_0 + \phi_{ro\,0} + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,)\,\partial_{ro}{}^{0}_{\rho} \\
&\quad - 2\,\tfrac{1}{\rho_{ro}}\;\rho_{ro\,0}\,(\omega_0 + \phi_{ro\,0} + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,)\,\partial_{ro}{}^{0}_{\phi},
\end{aligned}$$

$$\begin{aligned}
\Omega &= \tfrac{m}{\hbar_0}\,\Big(d_{ro}{}^{\rho}_{0} \wedge (d_{ro}{}^{\rho} - \rho_{ro\,0}\,d^0) \\
&\qquad + \rho_{ro}{}^{2}\,d_{ro}{}^{\phi}_{0} \wedge (d_{ro}{}^{ro\,\phi} - \phi_{ro\,0}\,d^0) + d_{ro}{}^{z}_{0} \wedge (d_{ro}{}^{z} - z_{ro\,0}\,d^0) \\
&\qquad + \rho_{ro}\;\big((\phi_{ro\,0})^2 - (\omega_0)^2 - \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\;\omega_0\big)\,d^0 \wedge d_{ro}{}^{\rho} \\
&\qquad + 2\,\rho_{ro}\;(\omega_0 + \phi_{ro\,0} + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,)\,d_{ro}{}^{\rho} \wedge d_{ro}{}^{\phi}\Big),
\end{aligned}$$

$$\begin{aligned}
\Lambda &= \tfrac{\hbar_0}{m}\,\big(\partial_{ro\,\rho} \wedge \partial_{ro}{}^{0}_{\rho} + \tfrac{1}{(\rho_{ro})^2}\,\partial_{ro\,\phi} \wedge \partial_{ro}{}^{0}_{\phi} + \partial_{ro\,z} \wedge \partial_{ro}{}^{0}_{z} \\
&\quad + 2\,\tfrac{1}{\rho_{ro}}\,(\omega_0 + \phi_{ro\,0} + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,)\,\partial_{ro}{}^{0}_{\rho} \wedge \partial_{ro}{}^{0}_{\phi}\big). \quad \square
\end{aligned}$$

Thus, the magnetic field effects Γ, γ, Ω and Λ.

Corollary 27.1.11 *Accordingly, we consider the distinguished horizontal potential of* Ω *(see Theorem 10.1.4 and Corollary 27.1.8)*

$$A^{\uparrow}[\mathrm{b}^{\square}] = \mathcal{C}[o_{ro}] + A[\mathrm{b}^{\square}, o_{ro}] = -\mathcal{K}^{\natural}[o_{ro}] + \mathcal{Q}^{\natural}[o_{ro}] + A[\mathrm{b}^{\square}, o_{ro}],$$

with coordinate expression

$$\begin{aligned}
A^{\uparrow}[\mathrm{b}^{\square}] &= -\,\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\Big((\rho_{ro\,0})^2 + \rho_{ro}{}^{2}\,(\phi_{ro\,0})^2 + (z_{ro\,0})^2 - \big((\omega_0)^2 + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0\big)\,\rho_{ro}{}^{2}\Big)\,d^0 \\
&\quad + \tfrac{m}{\hbar_0}\,\Big(\rho_{ro\,0}\,d_{ro}{}^{\rho} + \rho_{ro}{}^{2}\,\big(\phi_{ro\,0} + \omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\big)\,d_{ro}{}^{\phi} + z_{ro\,0}\,d_{ro}{}^{z}\Big). \quad \square
\end{aligned}$$

27.2 Quantum Objects

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$, equipped with the constant magnetic field $\vec{B}$, introduced in Hypothesis C.3, postulate a suitable quantum structure $(\boldsymbol{Q}, \mathrm{h}_{\eta}, \text{ч}^{\uparrow})$ and compute the main quantum objects discussed throughout the book, with reference to an inertial observer and a uniformly rotating observer.

Note 27.2.1 Preliminarily, we observe that, in virtue of Hypothesis C.3, the cohomology class of the phase 2-form Ω turns out to be integer (see [410] and recall Theorem 15.2.20).

Moreover, we recall the global classical gauge $\mathrm{b}^{\square}$, for which the classical upper potential of the phase 2-form Ω is (see Corollary 27.1.5):

$$\begin{aligned} A^{\uparrow}[\mathrm{b}^{\square}] &= -\mathcal{K}^{\natural}[o_{\mathtt{in}}] + \mathcal{Q}^{\natural}[o_{\mathtt{in}}] + A[\mathrm{b}^{\square}, o_{\mathtt{in}}] \\ &= -(\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j)\, d^0 + \big(\tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i + \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k\, \tfrac{1}{c}\, \epsilon_{ij3}\, x^i\big)\, d^j. \quad \square \end{aligned}$$

Hypothesis Q.3 *In this 3rd Example, in the framework of the classical background postulated in Hypothesis C.3, we consider a* trivialisable *quantum bundle* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, *along with an* η*-hermitian quantum metric* $\mathrm{h}_\eta := \mathrm{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}$, *according to Postulate Q.1, Proposition 14.3.1 and Definition 14.5.1.*

Thus, there exist global quantum bases b *of* $\pi : \boldsymbol{Q} \to \boldsymbol{E}$.

After having observed that the cohomology class of Ω *turns out to be integer (see Theorem 15.2.20 and [410]), we choose the global quantum basis* $\mathrm{b}^{\square}$ *(denoted by the same symbol of the distinguished classical gauge* $\mathrm{b}^{\square}$*) and consider the upper quantum connection (see Theorem 15.2.4, Note 15.2.12 and Postulate Q.2)*

$$\mathrm{Ч}^{\uparrow} = \chi^{\uparrow}[\mathrm{b}^{\square}] + \mathrm{i}\, A^{\uparrow}[\mathrm{b}^{\square}]\, \mathbb{I}^{\uparrow},$$

with coordinate expression, in a cartesian spacetime chart,

$$\begin{aligned} \mathrm{Ч}^{\uparrow} &= d^{\lambda} \otimes \partial_{\lambda} + d_0^i \otimes \partial_i^0 \\ &\quad + \mathrm{i}\, \Big(- (\tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j)\, d^0 + \big(\tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i + \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k\, \tfrac{1}{c}\, \epsilon_{ij3}\, x^i\big)\, d^j \Big) \otimes \mathbb{I}^{\uparrow}. \quad \square \end{aligned}$$

The above upper quantum connection $\mathrm{Ч}^{\uparrow}$ yields, in the present specific example, all quantum objects that we have analysed in the general theory. Now, we shall provide their expressions with reference to the inertial observer $o_{\mathtt{in}}$ and the uniformly rotating observer $o_{\mathtt{ro}}$.

27.2.1 Inertial Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics, equipped with the given constant magnetic field $\vec{B}$, the above quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \mathrm{Ч}^{\uparrow})$ and, with reference to an inertial observer $o_{\mathtt{in}}$, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.3, Q.3 and Definition 24.2.1).

Proposition 27.2.2 *With reference to a cartesian spacetime chart adapted to the observer* $o_{\mathtt{in}}$, *we obtain the following coordinate expressions of the quantum laplacian, the quantum kinetic tensor, the quantum probability current, the quantum lagrangian and the Schrödinger operator (see Proposition 16.3.2, Theorems 17.3.2, 17.4.2, 17.5.2 and 17.6.5).*

$$\begin{aligned}
\Delta_0(\Psi) =& \big(\tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi - \mathrm{i}\, \tfrac{q}{m}\, k_0\, \tfrac{1}{c}\, \delta^{ij} \epsilon_{pj3}\, x^p\, \partial_i\psi - \tfrac{1}{4}\, \tfrac{q\, q_0}{m\, \hbar}\, k^2\, \tfrac{1}{c^2}\, \big((x^1)^2 + (x^2)^2\big)\psi\big)\, \mathrm{b}^{\square},\\
\mathrm{Q}_0(\Psi) =& \big(\psi\, \partial_0 - \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, (\partial_j\psi - \mathrm{i}\, \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k_0\, \tfrac{1}{c}\, \epsilon_{pj3}\, x^p\, \psi)\, \partial_i\big) \otimes \mathrm{b}^{\square},\\
\mathrm{J}_0(\Psi) =& |\psi|^2\, \partial_0 + \mathrm{i}\, \tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, (\psi\, \partial_j\bar{\psi} - \bar{\psi}\, \partial_j\psi)\, \partial_i - \tfrac{1}{2}\, \tfrac{q}{m}\, k_0\, \tfrac{1}{c}\, \delta^{ij} \epsilon_{pj3}\, x^p\, |\psi|^2\, \partial_i,\\
\mathrm{L}_0(\Psi) =& \tfrac{1}{2}\, \big(- \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_i\bar{\psi}\, \partial_j\psi + \mathrm{i}\, (\bar{\psi}\, \partial_0\psi - \psi\, \partial_0\bar{\psi})\\
&+ \tfrac{1}{2}\, \tfrac{q}{m}\, k_0\, \tfrac{1}{c}\, \delta^{ij} \epsilon_{pj3}\, x^p\, (\bar{\psi}\, \partial_i\psi - \psi\, \partial_i\bar{\psi}) - \tfrac{1}{4}\, \tfrac{q\, q_0}{m\, \hbar}\, k^2\, \tfrac{1}{c^2}\, \big((x^1)^2 + (x^2)^2\big)|\psi|^2\big),\\
\mathrm{S}_0(\Psi) =& \big(\partial_0\psi - \tfrac{1}{2}\, \mathrm{i}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi - \tfrac{1}{2}\, \tfrac{q}{m}\, k_0\, \tfrac{1}{c}\, \delta^{ij} \epsilon_{pj3}\, x^p\, \partial_i\psi + \mathrm{i}\, \tfrac{1}{8}\, \tfrac{q\, q_0}{m\, \hbar}\, k^2\, \tfrac{1}{c^2}\, \big((x^1)^2 + (x^2)^2\big)\psi\big)\, \mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the magnetic field effects all above quantum objects.

Note 27.2.3 With reference to the inertial observer $o_{\mathtt{in}}$ and the distinguished classical gauge $\mathrm{b}^{\square}$, we obtain the distinguished special phase functions (see Definition 3.2.9, Theorem 10.1.8 and Example 12.1.4)

$$\begin{aligned}
\mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathtt{in}}] &:= \mathcal{K}_0[o_{\mathtt{in}}] - A_0[\mathrm{b}^{\square}, o_{\mathtt{in}}] \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
\mathcal{P}_i[\mathrm{b}^{\square}, o_{\mathtt{in}}] &:= \mathcal{Q}_i[o_{\mathtt{in}}] - A_i[\mathrm{b}^{\square}, o_{\mathtt{in}}] \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),\\
&\quad 1 \in \mathbb{R} \in \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}). \quad \square
\end{aligned}$$

Proposition 27.2.4 *We have the following coordinate expressions, in a cartesian spacetime chart adapted to o_{in} (see Example 12.1.4 and Proposition 27.1.2)*

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H}_0[\mathrm{b}^{\square}, o_{in}] = \tfrac{1}{2}\, \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^i\, x_0^j,\\
\mathcal{P}_i &:= \mathcal{P}_i[\mathrm{b}^{\square}, o_{in}] = \tfrac{m}{\hbar_0}\, \delta_{ij}\, x_0^j + \tfrac{1}{2}\, \tfrac{q}{\hbar}\, k\, \tfrac{1}{c}\, \epsilon_{pi3}\, x^p. \quad \square
\end{aligned}$$

Thus, the constant magnetic field effects $\mathcal{P}_i$ but not $\mathcal{H}_0$.

Proposition 27.2.5 *We have the following coordinate expression of the distinguished quantum operators (see Theorem 20.1.9, Example 20.1.12 and Proposition 25.2.5)*

$$\begin{aligned}
\mathrm{O}[\mathcal{H}_0](\Psi) &= \big(-\tfrac{1}{2}\, \tfrac{\hbar_0}{m}\, \delta^{ij}\, \partial_{ij}\psi + \tfrac{1}{2}\, \mathrm{i}\, \tfrac{q}{m}\, k_0\, \tfrac{1}{c}\, \delta^{ij}\, \epsilon_{pj3}\, x^p\, \partial_i\psi\\
&\quad + \tfrac{1}{8}\, \tfrac{q\, q_0}{m\, \hbar}\, k^2\, \tfrac{1}{c^2}\, \big((x^1)^2 + (x^2)^2\big)\, \psi\big)\, \mathrm{b}^{\square},\\
\mathrm{O}[\mathcal{P}_j](\Psi) &= -\mathrm{i}\, \partial_j\psi\, \mathrm{b}^{\square},\\
\mathrm{O}[1](\Psi) &= \psi\, \mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the constant magnetic field effects the quantum operator $\mathrm{O}[\mathcal{H}_0]$ but not the quantum operator $\mathrm{O}[\mathcal{P}_i]$.

Proposition 27.2.6 *We have the following coordinate expressions of distinguished quantum current forms (see Propositions 21.2.2, 25.2.6, and Example 21.2.3)*

$$
\begin{aligned}
\mathrm{j}_\eta[\mathcal{H}_0](\Psi) = \tfrac{1}{2}\big(&\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar\psi\,\partial_k\psi + \tfrac{1}{2}\,\mathrm{i}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\delta^{ij}\,\epsilon_{pj3}\,x^p\,(\bar\psi\,\partial_i\psi - \psi\,\partial_i\bar\psi)\\
&+ \tfrac{1}{4}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\big((x^1)^2 + (x^2)^2\big)\,|\psi|^2\big)\,\upsilon^0_0\\
&- \big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\partial_h\bar\psi\,\partial_0\psi + \partial_h\psi\,\partial_0\bar\psi)\\
&+ \tfrac{1}{4}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\delta^{kh}\,\epsilon_{ph3}\,x^p\,(\bar\psi\,\partial_0\psi - \psi\,\partial_0\bar\psi)\big)\,\upsilon^0_k,\\
\mathrm{j}_\eta[\mathcal{P}_j](\Psi) = -\,\tfrac{1}{2}&\Big(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar\psi\,\partial_k\psi + \tfrac{1}{4}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\big((x^1)^2 + (x^2)^2\big)\,|\psi|^2\Big)\,\upsilon^0_j\\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\partial_j\bar\psi\,\partial_h\psi + \partial_j\psi\,\partial_h\bar\psi)\,\upsilon^0_k\\
&- \tfrac{1}{2}\,\mathrm{i}\,\big((\bar\psi\,\partial_j\psi - \psi\,\partial_j\bar\psi)\,\upsilon^0_0 - (\bar\psi\,\partial_0\psi - \psi\,\partial_0\bar\psi)\,\upsilon^0_j\big)\\
&+ \tfrac{1}{2}\,\mathrm{i}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\delta^{kh}\,\epsilon_{ph3}\,x^p\,\big((\bar\psi\,\partial_j\psi - \psi\,\partial_j\bar\psi)\,\upsilon^0_k - (\bar\psi\,\partial_k\psi - \psi\,\partial_k\bar\psi)\,\upsilon^0_j\big),\\
\mathrm{j}_\eta[1](\Psi) = -\,\tfrac{1}{2}&\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\bar\psi\,\partial_h\psi - \psi\,\partial_h\bar\psi)\,\upsilon^0_k + |\psi|^2\,\upsilon^0_0. \quad \square
\end{aligned}
$$

Thus, the magnetic field effects the quantum current forms $\mathrm{j}_\eta[\mathcal{H}_0]$ and $\mathrm{j}_\eta[\mathcal{P}_i]$, but not $\mathrm{j}_\eta[1]$.

Corollary 27.2.7 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6, Example 21.2.7 and Corollary 25.2.7)*

$$
\begin{aligned}
\check{\mathrm{j}}_\eta[\mathcal{H}_0](\Psi) &= \tfrac{1}{2}\big(\tfrac{\hbar_0}{m}\,\delta^{hk}\,\partial_h\bar\psi\,\partial_k\psi + \tfrac{1}{2}\,\mathrm{i}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\delta^{ij}\,\epsilon_{pj3}\,x^p\,(\bar\psi\,\partial_i\psi - \psi\,\partial_i\bar\psi)\\
&\qquad + \tfrac{1}{4}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\big((x^1)^2 + (x^2)^2\big)\,|\psi|^2\big)\,\eta,\\
\check{\mathrm{j}}_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,(\bar\psi\,\partial_j\psi - \psi\,\partial_j\bar\psi)\,\eta,\\
\check{\mathrm{j}}_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}
$$

Thus, the magnetic field effects the vertical quantum current form $\check{\mathrm{j}}_\eta[\mathcal{H}_0]$, but not the quantum current forms $\check{\mathrm{j}}_\eta[\mathcal{P}_j]$ and $\check{\mathrm{j}}_\eta[1]$.

Proposition 27.2.8 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Propositions 21.3.2, 27.2.2, 25.2.8, and Example 21.3.3)*

$$
\begin{aligned}
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\tfrac{1}{4}\big(\psi\,\overline{\Delta_0(\Psi)} + \bar\psi\,\Delta_0(\Psi)\big)\\
&= -\tfrac{1}{4}\,\Big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\delta^{hk}\,(\bar\psi\,\partial_{hk}\psi + \psi\,\partial_{hk}\bar\psi)\\
&\qquad - \mathrm{i}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\delta^{ij}\,\epsilon_{pj3}\,x^p\,(\bar\psi\,\partial_i\psi - \psi\,\partial_i\bar\psi)\\
&\qquad - \tfrac{1}{2}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\big((x^1)^2 + (x^2)^2\big)\,|\psi|^2\Big)\,\eta,\\
\epsilon_\eta[\mathcal{P}_j](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,(\bar\psi\,\partial_j\psi - \psi\,\partial_j\bar\psi)\,\eta,\\
\epsilon_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}
$$

Thus, the magnetic field effects $\epsilon_\eta[\mathcal{H}_0]$, but not $\epsilon_\eta[\mathcal{P}_j]$ and $\epsilon_\eta[1]$.

27.2.2 Uniformly Rotating Observer

We consider the flat newtonian spacetime $(\boldsymbol{E}, g, K^{\natural})$ of standard Classical Mechanics, equipped with the given constant magnetic field $\vec{B}$, the above quantum structure $(\boldsymbol{Q}, \mathrm{h}_\eta, \mathrm{ч}^\uparrow)$ and, with reference to the uniformly rotating observer o_{ro}, we compute the expressions of basic objects of Quantum Mechanics discussed in the body of the book (see Hypotheses C.3, Q.3 and Definition 24.5.1).

Note 27.2.9 By recalling the expression of the upper potential (see Corollary 27.1.11)

$$\begin{aligned} A^\uparrow[\mathrm{b}^\square] &= -\mathcal{K}^\natural[o_{\text{ro}}] + \mathcal{Q}^\natural[o_{\text{ro}}] + A[\mathrm{b}^\square, o_{\text{ro}}] \\ &= -\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_{\text{ro}\,0})^2 + \rho_{\text{ro}}{}^2\,(\phi_{\text{ro}\,0})^2 + (z_{\text{ro}\,0})^2 - ((\omega_0)^2 + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0)\,\rho_{\text{ro}}{}^2\big)d^0 \\ &\quad + \tfrac{m}{\hbar_0}\,\big(\rho_{\text{ro}\,0}\,d_{\text{ro}}{}^\rho + \rho_{\text{ro}}{}^2\,(\phi_{\text{ro}\,0} + \omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c})\,d_{\text{ro}}{}^\phi + z_{\text{ro}\,0}\,d_{\text{ro}}{}^z\big), \end{aligned}$$

the upper quantum connection can be written as

$$\begin{aligned} \mathrm{ч}^\uparrow &= \chi^\uparrow[\mathrm{b}^\square] + \mathrm{i}\,A^\uparrow[\mathrm{b}^\square]\,\mathbb{I}^\uparrow \\ &= d^0 \otimes \partial_{\text{ro}\,0} + d_{\text{ro}}{}^\rho \otimes \partial_{\text{ro}\,\rho} + d_{\text{ro}}{}^\phi \otimes \partial_{\text{ro}\,\phi} + d_{\text{ro}}{}^z \otimes \partial_{\text{ro}\,z} \\ &\quad + d_{\text{ro}}{}^\rho_0 \otimes \partial_{\text{ro}}{}^0_\rho + d_{\text{ro}}{}^\phi_0 \otimes \partial_{\text{ro}}{}^0_\phi + d_{\text{ro}}{}^z_0 \otimes \partial_{\text{ro}}{}^0_z \\ &\quad - \mathrm{i}\,\tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_{\text{ro}\,0})^2 + \rho_{\text{ro}}{}^2\,(\phi_{\text{ro}\,0})^2 + (z_{\text{ro}\,0})^2 - ((\omega_0)^2 + \tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,\omega_0)\,\rho_{\text{ro}}{}^2\big)\,d^0 \otimes \mathbb{I}^\uparrow \\ &\quad + \mathrm{i}\,\tfrac{m}{\hbar_0}\,\big(\rho_{\text{ro}\,0}\,d_{\text{ro}}{}^\rho + \rho_{\text{ro}}{}^2\,(\phi_{\text{ro}\,0} + \omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c})\,d_{\text{ro}}{}^\phi + z_{\text{ro}\,0}\,d_{\text{ro}}{}^z\big) \otimes \mathbb{I}^\uparrow. \quad \square \end{aligned}$$

Then, we obtain the following coordinate expressions of dynamical quantum objects.

Proposition 27.2.10 *We obtain the quantum laplacian, the quantum kinetic tensor, the probability current, the quantum lagrangian and the Schrödinger operator, whose coordinate expressions are (see Propositions 16.3.2, 25.1.22, Theorems 17.3.2, 17.4.2, 17.5.2, 17.6.5, and Definition 24.1.1)*

$$\begin{aligned} \Delta_0(\Psi) &= \Big(\tfrac{\hbar_0}{m}\,(\partial_{ro\,\rho\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi\phi}\psi + \partial_{ro\,zz}\psi + \tfrac{1}{\rho_{ro}}\,\partial_{ro\,\rho}\psi) \\ &\qquad - 2\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c})\,\partial_{ro\,\phi}\psi \\ &\qquad - \big(\tfrac{m}{\hbar_0}\,(\omega_0)^2 + \tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,\omega_0 + \tfrac{1}{4}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\big)\,\rho_{ro}{}^2\,\psi\Big)\,\mathrm{b}^\square, \\ \mathrm{Q}_0(\Psi) &= \big(\psi\,\partial_{ro\,0} - \mathrm{i}\,\tfrac{\hbar_0}{m}\,(\partial_{ro\,\rho}\psi\,\partial_{ro\,\rho} + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi}\psi\,\partial_{ro\,\phi} + \partial_{ro\,z}\psi\,\partial_{ro\,z}) \\ &\qquad - (\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\tfrac{1}{c})\,\psi\,\partial_{ro\,\phi}\big) \otimes \mathrm{b}^\square, \\ \mathrm{J}_0(\Psi) &= |\psi|^2\,\partial_{ro\,0} - (\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c})\,|\psi|^2\,\partial_{ro\,\phi} \\ &\qquad + \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\big((\psi\,\partial_{ro\,\rho}\bar\psi - \bar\psi\,\partial_{ro\,\rho}\psi)\,\partial_{ro\,\rho} + \tfrac{1}{\rho_{ro}{}^2}\,(\psi\,\partial_{ro\,\phi}\bar\psi - \bar\psi\,\partial_{ro\,\phi}\psi)\,\partial_{ro\,\phi} \\ &\qquad + (\psi\,\partial_{ro\,z}\bar\psi - \bar\psi\,\partial_{ro\,z}\psi)\,\partial_{ro\,z}\big), \end{aligned}$$

$$\begin{aligned}
\mathrm{L}_0(\Psi) = &-\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi) \\
&+ \tfrac{1}{2}\,\mathrm{i}\,(\bar{\psi}\,\partial_{ro\,0}\psi - \psi\,\partial_{ro\,0}\bar{\psi}) + \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,)\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}) \\
&- \tfrac{1}{8}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\,\rho_{ro}{}^2\,|\psi|^2, \\
\mathrm{S}_0(\Psi) = &\big(\partial_{ro\,0}\psi - (\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{r}\,)\,\partial_{ro\,\phi}\psi \\
&- \mathrm{i}\,\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,(\partial_{ro\,\rho\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi\phi}\psi + \partial_{ro\,zz}\psi + \tfrac{1}{\rho_{ro}}\,\partial_{ro\,\rho}\psi) \\
&+ \tfrac{1}{8}\,\mathrm{i}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\,\rho_{ro}{}^2\,\psi\big)\,\mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the magnetic field effects all above quantum dynamical objects.

Note 27.2.11 Let us remark that the constant magnetic field does not effect the distinguished special phase functions (see Corollaries 25.1.24 and 27.1.8)

$$\begin{aligned}
1 &\in \mathbb{R} \subset \mathrm{map}(\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_\rho &:= \mathcal{P}_\rho[\mathrm{b}^{\square}, o_{\mathrm{ro}}\,] \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_z &:= \mathcal{P}_z[\mathrm{b}^{\square}, o_{\mathrm{ro}}\,] \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),
\end{aligned}$$

with coordinate expressions

$$\mathcal{P}_\rho = \tfrac{m}{\hbar_0}\,\rho_{\mathrm{ro}\,0}, \qquad \mathcal{P}_z = \tfrac{m}{\hbar_0}\,z_{\mathrm{ro}\,0}.$$

The special phase functions

$$\begin{aligned}
\mathcal{H}_0 &:= \mathcal{H}_0[\mathrm{b}^{\square}, o_{\mathrm{ro}}\,] \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}), \\
\mathcal{P}_\phi &:= \mathcal{P}_z[\mathrm{b}^{\square}, o_{\mathrm{ro}}\,] \in \mathrm{aff}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})
\end{aligned}$$

split as $\mathcal{H}_0 = \mathcal{H}^{\natural}{}_0 + \mathcal{H}^{\mathfrak{e}}{}_0$ and $\mathcal{P}_\phi = \mathcal{P}^{\natural}{}_\phi + \mathcal{P}^{\mathfrak{e}}{}_\phi$ and from Corollary 25.1.24 and Proposition 27.1.9 we get the coordinate expressions

$$\begin{aligned}
\mathcal{H}_0 &= \tfrac{1}{2}\,\tfrac{m}{\hbar_0}\,\big((\rho_{\mathrm{ro}\,0})^2 + \rho_{\mathrm{ro}}{}^2\,(\phi_{\mathrm{ro}\,0})^2 + (z_{\mathrm{ro}\,0})^2 - \rho_{\mathrm{ro}}{}^2\,((\omega_0)^2 + \tfrac{q}{m}\,k_0\tfrac{1}{c}\,\omega_0)\big), \\
\mathcal{P}_\phi &= \tfrac{m}{\hbar_0}\,\rho_{\mathrm{ro}}{}^2(\phi_{\mathrm{ro}\,0} + \omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\tfrac{1}{c}\,). \quad \square
\end{aligned}$$

Proposition 27.2.12 *We have the following coordinate expressions of distinguished quantum operators (see Theorem 20.1.9 and Example 20.1.12)*

$$\begin{aligned}
\mathrm{O}[\mathcal{H}_0](\Psi) &= -\tfrac{1}{2}\Big(\tfrac{\hbar_0}{m}\,(\partial_{ro\,\rho\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\;\partial_{ro\,\phi\phi}\psi + \partial_{ro\,zz}\psi + \tfrac{1}{\rho_{ro}}\,\partial_{ro\,\rho}\psi) \\
&\qquad - 2\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{m}\,k_0\,\tfrac{1}{c}\,)\,\partial_{ro\,\phi}\psi - \tfrac{1}{4}\;\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\;\rho_{ro}{}^2\,\psi\Big)\,\mathrm{b}^{\square}, \\
\mathrm{O}[\mathcal{P}_\rho](\Psi) &= -\mathrm{i}\,\big(\partial_{ro\,\rho}\psi + \tfrac{1}{2}\,\tfrac{1}{\rho_{ro}}\;\psi\big)\,\mathrm{b}^{\square}, \\
\mathrm{O}[\mathcal{P}_\phi](\Psi) &= -\mathrm{i}\,\partial_{ro\,\phi}\psi\,\mathrm{b}^{\square}, \\
\mathrm{O}[\mathcal{P}_z](\psi) &= -\mathrm{i}\,\partial_{ro\,z}\psi\,\mathrm{b}^{\square}, \\
\mathrm{O}[1](\Psi) &= \psi\mathrm{b}^{\square}. \quad \square
\end{aligned}$$

Thus, the magnetic field effects $\mathrm{O}[\mathcal{H}_0]$ but not the other operators.

Proposition 27.2.13 *We have the following coordinate expressions of distinguished quantum current forms (see Proposition 21.2.2 and Example 21.2.3)*

$$\begin{aligned}
&\mathrm{j}_\eta[\mathcal{H}_0](\psi) \\
&= \Big(\tfrac{1}{2}\;\tfrac{\hbar_0}{m}\;(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi) \\
&\quad + \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\;\tfrac{q}{\hbar}\;k_0\,\tfrac{1}{r}\,)\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}) + \tfrac{1}{2}\,\mathrm{i}\,\omega_0\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}) \\
&\quad + \tfrac{1}{8}\;\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\;\rho_{ro}{}^2\,|\psi|^2\Big)\,\upsilon_{ro}{}^0_0 \\
&\quad - \tfrac{1}{2}\;\tfrac{\hbar_0}{m}\,\big((\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,0}\psi + \partial_{ro\,\rho}\psi\,\partial_{ro\,0}\bar{\psi})\,\upsilon_{ro}{}^0_\rho \\
&\quad + \Big(\tfrac{1}{\rho_{ro}{}^2}\;(\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,0}\psi + \partial_{ro\,\phi}\psi\,\partial_{ro\,0}\bar{\psi}) \\
&\quad - \tfrac{1}{2}\,\mathrm{i}\;(\omega_0 + \tfrac{1}{2}\;\tfrac{q}{\hbar}\;k_0\,\tfrac{1}{c}\,)\,\big(\bar{\psi}\,\partial_{ro\,0}\psi - h\psi\,\partial_{ro\,0}\bar{\psi})\Big)\,\upsilon_{ro}{}^0_\phi \\
&\quad + (\partial_{ro\,z}\bar{\psi}\,\partial_{ro\,0}\psi + \partial_{ro\,z}\psi\,\partial_{ro\,0}\bar{\psi})\,\upsilon_{ro}{}^0_z,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{j}_\eta[\mathcal{P}_\rho](\Psi) \\
&= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{ro\,\rho}\psi - \psi\,\partial_{ro\,\rho}\bar{\psi}\big)\,\upsilon_{ro}{}^0_0 \\
&\quad - \tfrac{1}{2}\;\tfrac{\hbar_0}{m}\;\big(-\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\;\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi\big)\,\upsilon_{ro}{}^0_\rho \\
&\quad - \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\;\tfrac{q}{\hbar}\;k_0\,\tfrac{1}{c}\,)\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi})\,\upsilon_{ro}{}^0_\rho \\
&\quad - \tfrac{1}{8}\;\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\;\rho_{ro}{}^2\,|\psi|^2\,\upsilon_{ro}{}^0_\rho \\
&\quad + \tfrac{1}{2}\;\tfrac{\hbar_0}{m}\,\big(\tfrac{1}{\rho_{ro}{}^2}\;(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,\rho}\psi\,\partial_{ro\,\phi}\bar{\psi})\,\upsilon_{ro}{}^0_\phi \\
&\quad + (\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,z}\psi + \partial_{ro\,\rho}\psi\,\partial_{ro\,z}\bar{\psi})\,\upsilon_{ro}{}^0_z\big) \\
&\quad + \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\;\tfrac{q}{\hbar}\;k_0\,\tfrac{1}{c}\,)\,\big(\bar{\psi}\,\partial_{ro\,\rho}\psi - \psi\,\partial_{ro\,\rho}\bar{\psi}\big)\,\upsilon_{ro}{}^0_\phi \\
&\quad + \tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{ro\,0}\psi - \psi\,\partial_{ro\,0}\bar{\psi}\big)\,\upsilon_{ro}{}^0_\rho \\
&\quad + \tfrac{1}{4}\;\tfrac{\hbar_0}{m}\;\tfrac{1}{\rho_{ro}}\;\big(\partial_{ro\,\rho}|\psi|^2\,\upsilon_{ro}{}^0_\rho + \tfrac{1}{\rho_{ro}{}^2}\;\partial_{ro\,\phi}|\psi|^2\,\upsilon_{ro}{}^0_\phi + \partial_{ro\,z}|\psi|^2\,\upsilon_{ro}{}^0_z\big),
\end{aligned}$$

$$\begin{aligned}
&\mathrm{j}_\eta[\mathcal{P}_\phi](\psi) \\
&= -\tfrac{1}{2}\,\mathrm{i}\left(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}\right) \upsilon_{ro}{}^0_0 \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\left(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi - \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi\right) \upsilon_{ro}{}^0_\phi \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\left((\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\rho}\psi + \partial_{ro\,\phi}\psi\,\partial_{ro\,\rho}\bar{\psi})\,\upsilon_{ro}{}^0_\rho \right.\\
&\quad \left. + (\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,z}\psi + \partial_{ro\,\phi}\psi\,\partial_{ro\,z}\bar{\psi})\,\upsilon_{ro}{}^0_z\right) \\
&+ \tfrac{1}{2}\,\mathrm{i}\left(\bar{\psi}\,\partial_{ro\,0}\psi - \psi\,\partial_{ro\,0}\bar{\psi}\right) \upsilon_{ro}{}^0_\phi \\
&- \tfrac{1}{8}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\,\rho_{ro}{}^2\,|\psi|^2\,\upsilon_{ro}{}^0_\phi,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{j}_\eta[\mathcal{P}_z](\Psi) \\
&= -\tfrac{1}{2}\,\mathrm{i}\left(\bar{\psi}\,\partial_{ro\,z}\psi - \psi\,\partial_{ro\,z}\bar{\psi}\right) \upsilon_{ro}{}^0_0 \\
&- \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\left(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi + \tfrac{1}{\rho_{ro}{}^2}\,\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi - \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi\right) \upsilon_{ro}{}^0_z \\
&- \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{c}\,)\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi})\,\upsilon_{ro}{}^0_z \\
&- \tfrac{1}{8}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\,\rho_{ro}{}^2\,|\psi|^2\,\upsilon_{ro}{}^0_z \\
&+ \tfrac{1}{2}\,\tfrac{\hbar_0}{m}\left((\partial_{ro\,z}\bar{\psi}\,\partial_{ro\,\rho}\psi + \partial_{ro\,z}\psi\,\partial_{ro\,\rho}\bar{\psi})\,\upsilon_{ro}{}^0_\rho \right.\\
&\quad \left. + \tfrac{1}{\rho_{ro}{}^2}\,(\partial_{ro\,z}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\psi\,\partial_{ro\,\phi}\bar{\psi})\,\upsilon_{ro}{}^0_\phi\right) \\
&+ \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{c}\,)\left(\bar{\psi}\,\partial_{ro\,z}\psi - \psi\,\partial_{ro\,z}\bar{\psi}\right) \upsilon_{ro}{}^0_\phi \\
&+ \tfrac{1}{2}\,\mathrm{i}\left(\bar{\psi}\,\partial_{ro\,0}\psi - \psi\,\partial_{ro\,0}\bar{\psi}\right) \upsilon_{ro}{}^0_z,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{j}_\eta[1](\Psi) \\
&= |\psi|^2\,\upsilon_{ro}{}^0_0 - (\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\,k_0\,\tfrac{1}{c}\,)\,|\psi|^2\,\upsilon_{ro}{}^0_\phi \\
&- \tfrac{1}{2}\,\mathrm{i}\,\tfrac{\hbar_0}{m}\,\left((\bar{\psi}\,\partial_{ro\,\rho}\psi - \psi\,\partial_{ro\,\rho}\bar{\psi})\,\upsilon_{ro}{}^0_\rho + \tfrac{1}{\rho_{ro}{}^2}\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi})\,\upsilon_{ro}{}^0_\phi \right.\\
&\left. + (\bar{\psi}\,\partial_{ro\,z}\psi - \psi\,\partial_{ro\,z}\bar{\psi})\,\upsilon_{ro}{}^0_z\right). \quad \square
\end{aligned}$$

Thus, the magnetic field effects all distinguished current forms.

Corollary 27.2.14 *We have the following coordinate expressions of distinguished vertical quantum current forms (see Proposition 21.2.6 and Example 21.2.7)*

$$\begin{aligned}
\check{\mathrm{j}}_\eta[\mathcal{H}_0](\psi) &= \Big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\;(\partial_{ro\,\rho}\bar{\psi}\,\partial_{ro\,\rho}\psi + \tfrac{1}{{\rho_{ro}}^2}\,\partial_{ro\,\phi}\bar{\psi}\,\partial_{ro\,\phi}\psi + \partial_{ro\,z}\bar{\psi}\,\partial_{ro\,z}\psi) \\
&\quad + \tfrac{1}{2}\,\mathrm{i}\,(\omega_0 + \tfrac{1}{2}\,\tfrac{q}{\hbar}\;k_0\,\tfrac{1}{r}\,)\,(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}) \\
&\quad + \tfrac{1}{8}\,\tfrac{q\,q_0}{m\,\hbar}\,k^2\,\tfrac{1}{c^2}\;{\rho_{ro}}^2\,|\psi|^2\Big)\;\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_\rho](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{ro\,\rho}\psi - \psi\,\partial_{ro\,\rho}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_\phi](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{ro\,\phi}\psi - \psi\,\partial_{ro\,\phi}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[\mathcal{P}_z](\Psi) &= -\tfrac{1}{2}\,\mathrm{i}\,\big(\bar{\psi}\,\partial_{ro\,z}\psi - \psi\,\partial_{ro\,z}\bar{\psi}\big)\,\eta, \\
\check{\mathrm{j}}_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}$$

Thus, the magnetic field effects the current form $\check{\mathrm{j}}_\eta[\mathcal{H}_0]$ but not the others distinguished current forms.

Proposition 27.2.15 *We have the following coordinate expressions of distinguished quantum expectation value forms (see Proposition 21.3.2 and Example 21.3.3)*

$$\begin{aligned}
\epsilon_\eta[\mathcal{H}_0](\Psi) &= -\tfrac{1}{2}\,\Big(\tfrac{1}{2}\,\tfrac{\hbar_0}{m}\,\big((\psi\,\partial_{ro\,\rho\rho}\bar{\psi} + \bar{\psi}\,\partial_{ro\,\rho\rho}\psi) + \tfrac{1}{{\rho_{ro}}^2}\;(\psi\,\partial_{ro\,\phi\phi}\bar{\psi} + \bar{\psi}\,\partial_{ro\,\phi\phi}\psi) \\
&\quad + (\psi\,\partial_{ro\,zz}\bar{\psi} + \bar{\psi}\,\partial_{ro\,zz}\psi) + \tfrac{1}{\rho_{ro}}\;(\psi\,\partial_{ro\,\rho}\bar{\psi} + \bar{\psi}\,\partial_{ro\,\rho}\psi)\big) \\
&\quad + \mathrm{i}\,\omega_0\,(\psi\,\partial_{ro\,\phi}\bar{\psi} - \bar{\psi}\,\partial_{ro\,\phi}\psi) + (\tfrac{m}{\hbar_0}\,(\omega_0)^2 + \tfrac{q}{\hbar}\,k\,\tfrac{1}{c}\,\omega_0)\,{\rho_{ro}}^2|\psi|^2\Big)\,\eta, \\
\epsilon_\eta[\mathcal{P}_\rho](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{ro\,\rho}\bar{\psi} - \bar{\psi}\,\partial_{ro\,\rho}\psi)\,\eta, \\
\epsilon_\eta[\mathcal{P}_\phi](\psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{ro\,\phi}\bar{\psi} - \bar{\psi}\,\partial_{ro\,\phi}\psi)\,\eta, \\
\epsilon_\eta[\mathcal{P}_z](\Psi) &= \tfrac{1}{2}\,\mathrm{i}\,(\psi\,\partial_{ro\,z}\bar{\psi} - \bar{\psi}\,\partial_{ro\,z}\psi)\,\eta, \\
\epsilon_\eta[1](\Psi) &= |\psi|^2\,\eta. \quad \square
\end{aligned}$$

Thus, the magnetic field effects $\epsilon_\eta[\mathcal{H}_0]$ but not the others distinguished quantum expectation valued forms.

Chapter 28
Curved Newtonian Spacetime

The examples discussed in the above chapters of the present 3rd Part deal with a *newtonian flat spacetime*, which is nothing but the standard spacetime of standard Classical Mechanics and standard Quantum Mechanics. Thus, such a model of spacetime has given us the opportunity to test the main classical and quantum objects discussed throughout the body of the book in the arena of standard Quantum Mechanics.

Of course, we might consider several other models of spacetime standing in between the above fully flat model and a fully curved model.

Here, we focus our attention to an intermediate model of spacetime, which gives us the opportunity to recover the Newton law of gravitation within our general formalism (Sect. 28.1).

Thus, we discuss the "*curved newtonian spacetime*", which is equipped with

- a gravitational connection $K^{\natural}$, whose Ricci tensor $r[K^{\natural}]$ is *timelike*,
- an additional background flat connection $K^{\parallel}$, fulfilling the compatibility condition $\overset{\shortparallel}{K}{}^{\natural} = \overset{\shortparallel}{K}{}^{\parallel}$.

In this model, the Galilei–Einstein equation yields just the *Newton law of gravitation* (Sect. 28.4).

We leave to the reader the possible task to evaluate in the present model of spacetime all further main objects discussed throughout the body of the book.

28.1 Curved Newtonian Spacetime

We define *curved newtonian spacetime* to be a topologically trivial spacetime bundle $t : \boldsymbol{E} \to \boldsymbol{T}$ equipped with a galilean metric g, a galilean gravitational connection $K^{\natural}$, whose Ricci tensor $r^{\natural}$ is timelike, and a "background" flat spacetime connection $K^{\parallel}$, which fulfill the condition $\overset{\shortparallel}{K}{}^{\natural} = \overset{\shortparallel}{K}{}^{\parallel}$.

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Indeed, this additional background flat connection $K^{\parallel}$ does not fully comply with the true spirit of general relativity, but it provides a geometric way to introduce the distinguished family of inertial observers, which is typical in standard newtonian Classical Mechanics.

Thus, in the present model, spacetime inherits an affine structure from the background connection $K^{\parallel}$ and the fibres of spacetime turn out to be affine in virtue of the gravitational connection $K^{\natural}$. Indeed, these affine structures turn out to be compatible.

Definition 28.1.1 We define a *curved newtonian spacetime* to be an oriented fibred manifold (according to Postulate C.1) $t : \boldsymbol{E} \to \boldsymbol{T}$, which

(1) is a *bundle* topologically homeomorphic to the trivial bundle (see Appendix: Definition A.2.1 and Remark A.2.2)

$$\mathrm{pro}_1 : \mathrm{I\!R} \times R^3 \to \mathrm{I\!R},$$

(2) is equipped with:

(a) a *galilean metric* (according to Postulate C.2)

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E}),$$

(b) a *galilean gravitational connection* $K^{\natural}$ (according to Postulate C.3)

$$K^{\natural} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

which fulfills the additional condition

$$r^{\natural} = \mathrm{r}^{\natural}\, dt \otimes dt, \quad \text{with } \mathrm{r}^{\natural} \in \mathrm{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathrm{I\!R}),$$

(c) a "*background*" flat spacetime connection

$$K^{\parallel} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

such that the restrictions to the vertical subbundle $V\boldsymbol{E} \subset T\boldsymbol{E}$ of $K^{\natural}$ and $K^{\parallel}$ coincide (see Proposition 4.1.2)

$$\overset{''}{K}{}^{\natural} = \overset{''}{K}{}^{\parallel} : V\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TV\boldsymbol{E}.$$

Thus, by definition, $K^{\natural}$ is a linear, torsion free spacetime connection, such that

$$\overset{''}{K}{}^{\natural} = \overset{''}{K}{}^{\parallel}, \qquad \nabla^{\natural} dt = 0, \qquad \overset{''}{\nabla}{}^{\natural} g = 0, \qquad A\underline{R}^{\natural} = 0, \qquad r^{\natural} = \mathrm{r}^{\natural}\, dt \otimes dt. \quad \square$$

Thus, let us consider a curved newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ and analyse preliminary features that arise immediately from the background connection $K^{\parallel}$.

Indeed, the curved spacetime inherits, via the background spacetime connection $K^{\|}$, an affine structure, along with the associated objects and properties of a flat newtonian spacetime, that have been discussed in the above Sect. 24.1.

In particular, we can define the following notions.

Definition 28.1.2 By availing of the background affine structure, we say that:

(1) an observer o is *inertial* if it is inertial with respect to the background affine spacetime structure induced by $K^{\|}$ (see Definition 24.2.1),
(2) a spacetime chart is *cartesian* if it is affine with respect to the background affine structure induced by $K^{\|}$ (see Definition 24.2.4). □

Proposition 28.1.3 *We have* $\overset{\shortparallel}{\nabla}{}^{\|} g = 0$, *i.e., in any cartesian spacetime chart,* $\partial_\lambda g = 0$.

Hence, the background affine connection $K^{\|}$ *turns out to be metric preserving.*

Moreover, being g *constant with respect to the background affine structure of spacetime, it can be regarded as a constant scaled tensor (see Proposition 24.1.2)*

$$g \in \mathbb{L}^2 \otimes (\bar{\boldsymbol{S}}^* \otimes \bar{\boldsymbol{S}}^*),$$

hence a scaled euclidean metric on the fibres of spacetime.

Proof. The proof follows immediately from the hypotheses $\overset{\shortparallel}{K}{}^{\|} = \overset{\shortparallel}{K}{}^{\natural}$ and $\overset{\shortparallel}{\nabla}{}^{\natural} g = 0$. □

Proposition 28.1.4 *The full vertical restrictions of* $K^{\natural}$ *and* $K^{\|}$ *coincide (see Proposition 4.1.2):*

$$\overset{\vee}{K}{}^{\natural} = \overset{\vee}{K}{}^{\|} : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{E}} V\boldsymbol{E}.$$

Thus, the fibres of spacetime turn out to be flat with respect to both connections $K^{\natural}$ *and* $K^{\|}$.

In particular, with reference to the gravitational connection $K^{\natural}$, *the spacetime bundle* $t : \boldsymbol{E} \to \boldsymbol{T}$ *turns out to be an affine bundle (see Appendix: Definition A.3.4).*

Proof. The result follows directly from the condition $\overset{\shortparallel}{K}{}^{\natural} = \overset{\shortparallel}{K}{}^{\|}$. □

Note 28.1.5 The vertical restriction of the gravitational Ricci tensor and the gravitational scalar curvature vanish:

$$\check{r}^{\natural} = 0 \qquad \text{and} \qquad C^{\natural} = 0.$$

Indeed, this fact can be regarded both as a consequence of the above Proposition 28.1.4 and also as a consequence of the hypothesis that $r^{\natural}$ be timelike. □

Proposition 28.1.6 *We stress that the condition* $\check{r}^{\|} = \check{r}^{\natural} = 0$ *implies that the spacelike connections* $\overset{\vee}{K}{}^{\|}$ *and* $\overset{\vee}{K}{}^{\natural}$ *induced on the* 3–dimensional *fibres of spacetime be flat, according to a general result of riemannian geometry (see, for instance, [138]).* □

28.2 Gravitational Connection

Now, we discuss an explicit expression of the gravitational connection $K^\natural$, by further comparing $K^\natural$ with $K^\|$. Actually, we find the equality $K^\natural = K^\| + dt \otimes N^\natural \otimes dt$, where $N = \vec{d} U$, with $U \in \mathrm{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R})$.

The above expression tuns out later to play a role in the formulation of Newton law of gravitation in the present context (see Theorem 28.4.2).

Given an inertial observer o, we express the gravitational connection $K^\natural$ in terms of the background connection $K^\|$ and the observed spacetime 2–form $\Phi^\natural[o]$.

Lemma 28.2.1 *If o and $\acute{o} = o + v$ are inertial observers, then we obtain* $\Phi^\natural[\acute{o}] = \Phi^\natural[o]$.

Proof. The proof follows immediately from the definition of $\Phi^\natural[\acute{o}]$ and $\Phi^\natural[o]$ (see Definition 4.2.11) and from the condition $\nabla^\natural v = \nabla^\| v = 0$ (see Definition 28.1.2). □

Lemma 28.2.2 *For each inertial observer o, we have the equality (see Definition 4.2.11 and Notation 4.2.4)*

$$K^\natural = K^\| - \tfrac{1}{2}\big(dt \otimes \widehat{\Phi}^\natural[o] + \widehat{\Phi}^\natural[o] \otimes dt\big),$$

where

$$\Phi^\natural[o] : \boldsymbol{E} \to H^*\boldsymbol{E} \wedge V^*\boldsymbol{E} \subset \Lambda^2 T^*\boldsymbol{E} \qquad \text{and} \qquad \widehat{\Phi}^\natural[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E},$$

i.e., in any spacetime chart,

$$\Phi^\natural[o] = 2\,\Phi^\natural{}_{0j}\, d^0 \wedge d^j \qquad \text{and} \qquad \widehat{\Phi}^\natural[o] = G^{ij}_0\, \Phi^\natural{}_{0j}\, u^0 \otimes u^0 \otimes \partial_i .$$

Hence, we have $\check{\Phi}^\natural[o] = 0$, i.e., in any spacetime chart, $\Phi^\natural{}_{ij} = 0$.

Indeed, $\widehat{\Phi}^\natural[o]$ turns out to be observer independent *(with respect to inertial observers).*

In any cartesian spacetime chart, we have the coordinate expressions

$$K^\natural{}_0{}^i{}_0 = -G^{ij}_0\, \Phi^\natural{}_{0j} \qquad \text{and} \qquad K^\natural{}_0{}^i{}_k = K^\natural{}_k{}^i{}_0 = K^\natural{}_h{}^i{}_k = K^\natural{}_k{}^i{}_h = 0.$$

Proof. We proceed by several steps.

(1) By definition of the background connection $K^\|$ and of a cartesian spacetime chart, in any cartesian spacetime chart, we have the equality $K^\|{}_\lambda{}^i{}_\mu = 0$.

Hence, the condition $\ddot{K}^\natural = \ddot{K}^\|$ can be expressed, in any cartesian spacetime chart, by the equality (see Proposition 4.1.2 and Definition 4.1.4)

$$K^\natural{}_0{}^i{}_k = K^\natural{}_h{}^i{}_0 = K^\natural{}_h{}^i{}_k = 0.$$

(2) In a cartesian spacetime chart, we have the equality $\partial_\lambda G^0_{ij} = 0$ and in any spacetime chart, we have the equalities (see Theorem 4.2.13)

$$K^\natural{}_0{}^i{}_0 = -G_0^{ij}\, \Phi^\natural{}_{0j},$$
$$K^\natural{}_0{}^i{}_h = K^\natural{}_h{}^i{}_0 = -\tfrac{1}{2}\, G_0^{ij}\, (\partial_0 G^0_{hj} + \Phi^\natural{}_{hj}),$$
$$K^\natural{}_k{}^i{}_h = K^\natural{}_h{}^i{}_k = -\tfrac{1}{2}\, G_0^{ij}\, (\partial_h G^0_{jk} + \partial_k G^0_{jh} - \partial_j G^0_{hk}).$$

Hence, in virtue of step (1), in any cartesian spacetime chart, we obtain the equality $\Phi^\natural{}_{ij} = 0$.

(3) Therefore, for each inertial observer o, we obtain the equalities

$$K^\natural = K^\| - \tfrac{1}{2}\big(dt \otimes \widehat{\Phi}{}^\natural[o] + \widehat{\Phi}{}^\natural[o] \otimes dt\big), \qquad \check{\Phi}^\natural[o] = 0,$$

where

$$\Phi^\natural[o] : \boldsymbol{E} \to H^*\boldsymbol{E} \wedge V^*\boldsymbol{E} \subset \Lambda^2 T^*\boldsymbol{E} \qquad \text{and} \qquad \widehat{\Phi}{}^\natural[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}.$$

In other words, in any cartesian spacetime chart, we have the coordinate expressions

$$K^\natural{}_0{}^i{}_0 = -G_0^{ij}\, \Phi^\natural{}_{0j}, \qquad K^\natural{}_0{}^i{}_k = K^\natural{}_k{}^i{}_0 = K^\natural{}_h{}^i{}_k = 0,$$
$$\Phi^\natural[o] = 2\, \Phi_{0j}\, d^0 \wedge d^j, \qquad \widehat{\Phi}{}^\natural[o] = G_0^{ij}\, \Phi_{0j}\, u^0 \otimes u^0 \otimes \partial_i .$$

(4) The above equality

$$K^\natural = K^\| - \tfrac{1}{2}\big(dt \otimes \widehat{\Phi}{}^\natural[o] + \widehat{\Phi}{}^\natural[o] \otimes dt\big)$$

and the fact that $K^\|$ is observer independent yield, for any inertial observers o and $\acute{o}$, the equalities

$$\widehat{\Phi}{}^\natural[o] = \widehat{\Phi}{}^\natural[\acute{o}] \qquad \text{and} \qquad \Phi^\natural[o] = \Phi^\natural[\acute{o}]. \quad \square$$

Next, we prove in two steps that the observed spacetime 2–form $\Phi^\natural[o]$ can be expressed through a suitable potential $U^\natural[o]$.

Lemma 28.2.3 *For each inertial observer o, the condition $\check{\Phi}[o] = 0$ yields*

$$\check{d}\check{A}^\natural[o] = 0,$$

hence, locally, in any spacetime chart,

$$A^\natural[o] = A^\natural{}_0\, d^0 + \partial_i V^\natural\, d^i,$$

where $V^\natural[o] \in \mathrm{map}(\boldsymbol{E}, I\!R)$. $\square$

Lemma 28.2.4 *For each inertial observer o, we have*

$$\Phi^\natural[o] = -2\, dt \wedge \check{d}\, U^\natural[o],$$

where (see the above Lemma 28.2.3)

$$U^\natural[o] := \partial[o] \lrcorner A^\natural[o] - \partial[o].V^\natural[o] \in \mathrm{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}).$$

In other words, in a spacetime chart adapted to o, we have

$$\Phi^\natural[o] = -2\, \partial_j U^\natural{}_0\, d^0 \wedge d^j, \quad \text{where } U^\natural[o] = u^0 \otimes (A^\natural{}_0 - \partial_0 V^\natural).$$

Indeed, for any inertial observers o and $\acute{o}$, we have

$$U^\natural[\acute{o}] = U^\natural[o] + \varphi, \quad \text{where } \varphi \in \mathrm{map}(\boldsymbol{T}, \mathbb{T}^* \otimes \mathbb{R}).$$

Proof. In virtue of the above Lemma 28.2.3, in any spacetime chart,

$$\Phi^\natural[o] = 2\, (\partial_0 A^\natural{}_j - \partial_j A^\natural{}_0)\, d^0 \wedge d^j = 2\, (\partial_{0j} V^\natural - \partial_j A^\natural{}_0)\, d^0 \wedge d^j = -2\, \partial_j U^\natural{}_0\, d^0 \wedge d^j.$$

For any inertial observers o and $\acute{o} = o + v$, we have $\Phi^\natural[\acute{o}] = \Phi^\natural[o]$, hence we obtain

$$U^\natural[\acute{o}] = U^\natural[o] + \varphi, \quad \text{where } \varphi \in \mathrm{map}(\boldsymbol{T}, \mathbb{T}^* \otimes \mathbb{R}). \quad \square$$

Eventually, we obtain an intrinsic expression of $K^\natural$ through $K^\parallel$.

Theorem 28.2.5 *The gravitational galilean connection $K^\natural$ can be uniquely written as*

$$K^\natural = K^\parallel + dt \otimes N^\natural \otimes dt,$$

where $N^\natural$ is an observer independent *scaled vertical vector field*

$$N^\natural : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

which is locally the gradient of the observer independent *scaled function defined (up to a scaled function of time) in the above Lemma 28.2.4*

$$N^\uparrow = \vec{d}\, U^\uparrow, \quad \text{with } U^\uparrow \in \mathrm{map}(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R}).$$

We have the coordinate expression, in any spacetime chart,

$$N^\natural = N^\natural{}_0{}^i{}_0\, (u^0 \otimes u^0) \otimes \partial_i = G^{ij}_0\, \partial_j U_0\, (u^0 \otimes u^0) \otimes \partial_i.$$

For each inertial observer o, we have

$$N^\natural = -\widehat{\Phi}^\natural[o].$$

In any cartesian spacetime chart, we have the equalities

$$K^\natural{}_0{}^i{}_k = K^\natural{}_k{}^i{}_0 = K^\natural{}_h{}^i{}_k = K^\natural{}_k{}^i{}_h = 0, \qquad N^\natural{}_0{}^i{}_0 = K^\natural{}_0{}^i{}_0.$$

Proof. The above Lemma 28.2.2 yields

$$\begin{aligned}
K^\natural &= K^\Vert - \tfrac{1}{2}\big(dt \otimes \widehat{\Phi}^\natural[o] + \widehat{\Phi}^\natural[o] \otimes dt\big) \\
&= K^\Vert - \tfrac{1}{2}\, G_0^{ij}\, \Phi_{0j}\,\big(u_0 \otimes d^0 \otimes u^0 \otimes u^0 \otimes \partial_i + u^0 \otimes u^0 \otimes \partial_i \otimes u_0 \otimes d^0\big) \\
&= K^\Vert - \tfrac{1}{2}\, G_0^{ij}\, \Phi_{0j}\,\big(d^0 \otimes u^0 \otimes \partial_i + u^0 \otimes \partial_i \otimes d^0\big) \\
&= K^\Vert - \tfrac{1}{2}\, G_0^{ij}\, \Phi_{0j}\,\big(u^0 \otimes u^0 \otimes \partial_i + u^0 \otimes u^0 \otimes \partial_i\big) \\
&= K^\Vert - G_0^{ij}\, \Phi_{0j}\, u^0 \otimes u^0 \otimes \partial_i \\
&= K^\Vert + N^\natural{}_0{}^i{}_0\, u^0 \otimes u^0 \otimes \partial_i.
\end{aligned}$$

Moreover, Lemma 28.2.4 yields $N = \vec{d}\, U$, with $U \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R})$. □

Remark 28.2.6 So far, in this Section, we have been involved with no specific mass m. Nevertheless, we have used the unscaled spacetime 2–form $\Phi[o]$ and the contravariant metric $\bar{G}$, which have been normalised, respectively, through the scales $m/\hbar$ and $\hbar/m$ (see Definitions 4.2.11 and 3.2.1). Indeed, in this context, the use of these objects is due only to a convenience of notation and the normalisation is obtained by referring to an arbitrary mass m.

We stress that, eventually, the arbitrary mass m and the Plank constant $\hbar$ disappear in the expression of the main object $N^\natural = \vec{d}\, U^\natural$, which does not involve any reference to a mass and to $\hbar$. □

28.3 Gravitational Curvature

Next, we discuss an explicit expression of the curvature $R^\natural$ and of the Ricci tensor $r^\natural$ of the gravitational connection $K^\natural$.

Lemma 28.3.1 *By considering the natural inclusion* $\boldsymbol{E} \times \mathbb{T}^* \simeq H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$, *the scaled vertical vector field* $N^\natural : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ *can be, equivalently, regarded as a scaled vertical valued 1–form*

$$\tilde{N}^\natural : \boldsymbol{E} \to \mathbb{T}^* \otimes (H^*\boldsymbol{E} \otimes V\boldsymbol{E}) \subset \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E}),$$

with coordinate expression

$$\tilde{N}^\natural = N^\natural{}_{00}{}^i\, u^0 \otimes (d^0 \otimes \partial_i), \quad \text{with } N^\natural{}_{00}{}^i = N^\natural{}_0{}^i{}_0 = K^\natural{}_0{}^i{}_0. \quad □$$

Proposition 28.3.2 *The curvature tensor $R^\natural$ of the gravitational connection $K^\natural$ fulfills the equality*

$$R^\natural = -2\, d^\natural \tilde{N} : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E},$$

where $d^\natural$ denotes the exterior covariant differential with respect to the gravitational connection $K^\natural$ (see Appendix: Definition F.1.6).

In any affine spacetime chart, we have the following coordinate expression

$$R^\natural = 2\, \partial_j N^\natural{}_{00}{}^i\, d^0 \wedge d^j \otimes \partial_i \otimes d^0. \quad \square$$

Corollary 28.3.3 *The Ricci tensor $r^\natural$ of the gravitational connection $K^\natural$ is given by the tensor (see the above Proposition 28.3.2)*

$$r^\natural = \mathrm{div}_\eta \tilde{N}^\natural = \Delta[G] U^\natural : \boldsymbol{E} \to H^*\boldsymbol{E} \otimes H^*\boldsymbol{E},$$

which is locally expressed by the equality (see Definition 3.2.20)

$$r^\natural = (\Delta[G]\, U^\natural)\, dt \otimes dt = (\mathrm{div}_\eta \vec{d}\, U^\natural)\, dt \otimes dt.$$

In any affine spacetime chart, we have the coordinate expression

$$r^\natural = \partial_j N^\natural{}_{00}{}^j\, d^0 \otimes d^0 = G_0^{ij}\, \partial_{ij} U^\natural{}_0\, d^0 \otimes d^0. \quad \square$$

Next, in view of the Newton law of gravitation, we study a further property of the vector field $N^\natural$.

Lemma 28.3.4 *The metric g yields the scaled vertical 1–form*

$$\underline{N}^\natural := g^\flat(N^\natural) : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^* \otimes \mathbb{L}^2) \otimes V^*\boldsymbol{E},$$

with coordinate expression

$$\underline{N}^\natural = N^\natural{}_{00i}\, (u^0 \otimes u^0) \otimes \check{d}^i, \quad \text{with } N^\natural{}_{00i} = K^\natural{}_{0i0}. \quad \square$$

Further, we analyse the condition $A\,\underline{R}^\natural = 0$, by which the gravitational connection $K^\natural$ is galilean (see Definition 4.3.1). In fact, we show that this condition is equivalent to the closure of the scaled spacetime 1–form $\underline{N}^\natural$, hence to the local exactness of $\underline{N}^\natural$.

In this way we find again that the scaled vector field N is the gradient of a scaled spacetime function.

Corollary 28.3.5 *The following equivalence holds*

$$A\,\underline{R}^\natural = 0 \quad \Leftrightarrow \quad \check{d}\underline{N} = 0.$$

In any spacetime chart, the above equalities read as

$$\partial_j N^{\natural}{}_{00\,i}\,(u^0 \otimes u^0) \otimes d^j \wedge d^i = 0.$$

Hence, the following equivalence holds locally

$$A\underline{R}^{\natural} = 0 \qquad \Leftrightarrow \qquad N^{\natural} = \vec{d}\, U^{\natural} := G^{\sharp}(dU^{\natural}),$$

where the potential $U^{\natural}$ *is a local scaled function*

$$U^{\natural} : \boldsymbol{E} \to \mathbb{T}^* \otimes I\!R. \quad \square$$

28.4 Newton Law of Gravitation

Further, we show that the Galilei–Einstein equation $r^{\natural} = \mathrm{r}\,\mu\, dt \otimes dt$ for a curved newtonian spacetime, whose source is a classical continuum with mass density μ (see Definition 8.3.2), turns out to be just the standard Newton law of gravitation $\Delta[G]\, U = \mathrm{r}\,\mu$.

We can interpret the gradient $\vec{d}\, U$ as the gravitational force, via the background flat newtonian structure of spacetime, according to the Newton law of motion $m\, \nabla^{\|}{}_{ds} ds = m\,(\vec{d}\, U) \circ s$.

These results can be easily extended to the case when spacetime is equipped also with a given electromagnetic field F.

Let us consider a curved newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Definition 28.1.1).

Corollary 28.4.1 *Let us consider the Galilei–Einstein equation (see Definition 8.3.2 and Proposition 8.3.3)*

$$r^{\natural} = \mathrm{r}\, \underline{\mathscr{T}}^{\natural}$$

and suppose that the source of the gravitational connection $K^{\natural}$ *be a stress energy tensor (see Definition 7.3.3)*

$$\underline{\mathscr{T}}^{\natural} : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}).$$

Then, we obtain

$$\underline{\mathscr{T}}^{\natural} : \boldsymbol{E} \to (\mathbb{T}^2 \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (H^*\boldsymbol{E} \otimes H^*\boldsymbol{E}).$$

In other words, the coordinate expression of $\underline{\mathscr{T}}^{\natural}$ *is of the type*

$$\underline{\mathscr{T}}^{\natural} = \underline{\mathscr{T}}_{00}\, d^0 \otimes d^0. \quad \square$$

In particular, let us suppose that the source of the gravitational field be the timelike scaled tensor

$$\underline{\mathscr{T}}^{\natural} := \text{r}\, \mu\, dt \otimes dt : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$$

associated with a mass density $\mu : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \mathbb{M} \otimes \mathbb{R}$.

Theorem 28.4.2 *The gravitational field $K^{\natural}$ is a solution of the Galilei–Einstein equation (see Definition 8.3.2 and Proposition 8.3.3)*

$$r^{\natural} = \text{r}\, \mu\, dt \otimes dt$$

if and only if the potential U fulfills the Newton equation of gravitation (see Definition 3.2.20)

$$\Delta[G]\, U = \text{r}\, \mu.$$

Proof. It follows from Corollary 28.3.3. □

Corollary 28.4.3 *The Newton law of motion for a particle of mass $m \in \mathbb{M}$ effected by the gravitational connection $K^{\natural}$ (see Assumption C.2)*

$$m\, \nabla^{\natural}{}_{ds} ds = 0$$

can be written, by referring to the background spacetime connection $K^{\|}$, as

$$m\, \nabla^{\|}{}_{ds} ds = m\, N^{\natural} \circ s = \vec{d}\, U \circ s,$$

i.e., in an affine spacetime chart

$$m\, \partial_{00} s^i = m\, N^{\natural}{}_{00}{}^i \circ s = m\, G^{ij}_0\, \partial_j U_0 \circ s.$$

Thus, the scaled vector field $\bar{N}^{\natural}$ turns out to be the gravitational force *per unit mass, with respect to the background affine connection $K^{\|}$.*

Proof. Theorem 28.2.5 yields $\nabla^{\natural}{}_{ds} ds = m\, \nabla^{\|}{}_{ds} ds - N^{\natural} \circ s$.

Hence, the result follows from the Newton law of motion $\nabla^{\natural}{}_{ds} ds = 0$ for a particle affected by the gravitational field (see Assumption C.2). □

We can easily extend the above results to the case when spacetime is equipped also with a given electromagnetic field F. In this case, the gravitational field is generated by a mass density and an electromagnetic field (see Postulate C.6).

In this way, the Galilei–Einstein equation yields a gravitational effect of the "galilean electromagnetic energy" $\check{F}^2 = \frac{4}{c^2}\, g(\vec{B}, \vec{B})$ (see Theorem 8.3.4 and Definition 5.6.5).

This fact, agrees with the principle of General Relativity, by which energy has a gravitational effect. In the present galilean model of curved spacetime, the gravitational effect of the "galilean electromagnetic energy" can be regarded as a gravitational "action at distance force" acting on massive test particles.

Moreover, in presence of an electromagnetic field, the Newton law of motion for a charged particle includes also the Lorentz force.

Note 28.4.4 Let us suppose that spacetime be equipped with both a given mass density μ and a given electromagnetic field F.

Then, the above Theorem 28.4.2 and Corollary 28.4.3 can be extended as follows.

(1) The source of the gravitational field turns out to be the timelike scaled tensor (see Definition 8.3.2 and Proposition 8.3.3)

$$\underline{\mathscr{T}}^{\natural} := \mathrm{r}\,\big(\mu + \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\big)\, dt \otimes dt : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}.$$

Then, according to the Galilei–Einstein equation, the Newton law of gravitation becomes

$$\Delta[G]\, U = \mathrm{r}\,\big(\mu + \tfrac{1}{c^2}\, g(\vec{B}, \vec{B})\big).$$

(2) The Newton law of motion for a charged particle becomes (see Assumption C.2)

$$m\, \nabla^{\natural}{}_{ds} ds = m\, N^{\natural} \circ s + \mathsf{f}. \quad \square$$

28.5 Further Properties

Eventually, we analyse further minor properties of a curved newtonian spacetime. In particular, we discuss a characterisation of inertial observers, a characterisation of curved newtonian spacetimes and uniqueness features of the background affine structure.

We mention that R. Vitolo has proved a uniqueness property of curved newtonian spacetime with spherical symmetry [410].

Proposition 28.5.1 *Let us consider a curved newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$. An observer o is inertial with respect to the background affine structure if and only if*

$$\nabla^{\natural}{}_{\partial[o]}\partial[o] = -N^{\natural},$$

i.e., if and only if, in a cartesian spacetime chart,

$$\nabla^{\natural}{}_{\partial[o]}\partial[o] = -N^{\natural}{}_{00}{}^{i}\, u^0 \otimes (d^0 \otimes \partial_i).$$

Thus, the scaled vector field $N^{\natural}$ expresses (up to sign) the acceleration of inertial observers with respect to the gravitational connection $K^{\natural}$. □

Proposition 28.5.2 *Given a flat newtonian spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$ equipped with a flat affine connection $K^{\Vert}$ (see Definition 24.1.1) and a scaled vertical vector field*

$$N^{\natural} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E},$$

the spacetime connection

$$K^{\natural} := K^{\parallel} + dt \otimes N^{\natural} \otimes dt$$

turns out to be galilean if and only if $\check{d}\, \underline{N}^{\natural} = 0$.

Proof. We easily see that the above connection $K^{\natural}$ is metric preserving, linear, torsion free and time preserving. Then, Corollary 28.3.5 implies that $K^{\natural}$ is galilean if and only if $\check{d}\, \underline{N}^{\natural} = 0$. □

Corollary 28.5.3 *Given a flat newtonian spacetime* $t : \boldsymbol{E} \to \boldsymbol{T}$, *the map*

$$N^{\natural} \mapsto K^{\natural} = K^{\parallel} + dt \otimes N^{\natural} \otimes dt$$

yields a bijection between scaled spacelike vector fields

$$N^{\natural} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}, \quad \text{such that } \check{d}\, \underline{N}^{\natural} = 0,$$

and gravitational connections $K^{\natural}$ *of curved newtonian spacetimes.* □

We have the following result, which, in a sense, analyses the "non uniqueness" of the background affine structure of a curved newtonian spacetime.

Proposition 28.5.4 *Let us consider a curved newtonian spacetime*

$$(t : \boldsymbol{E} \to \boldsymbol{T},\ g,\ K^{\natural},\ K^{\parallel\prime}),$$

with a certain background affine structure associated with the affine connection $K^{\parallel\prime}$.

Then, another affine background spacetime connection $K^{\parallel\prime\prime}$ *is compatible with the above curved newtonian spacetime if and only if*

$$K^{\parallel\prime\prime} = K^{\parallel\prime} + dt \otimes X \otimes dt,$$

where $X : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ *is a scaled vertical vector field, which is spacelikely constant with respect to the 1st affine structure, i.e. if and only if in an cartesian spacetime chart adapted to the 1st affine structure, we have*

$$\partial_j X_0{}^i{}_0 = 0\,. \quad \square$$

Proof. According to Theorem 28.2.5, the gravitational connection $K^{\natural}$ can be written as

$$K^{\natural} = K^{\parallel\prime} + dt \otimes N' \otimes dt, \quad \text{with } \check{d}\underline{N}' = 0.$$

(1) Let us consider a scaled vector field $X : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$, which is spacelikely constant with respect to the 1st affine structure, and define the spacetime connection $K^{\parallel\prime\prime} := K^{\parallel\prime} + dt \otimes X \otimes dt$.

Then, the connection $K^{\|'}$ has the following properties:

(a) it is linear, because $K^{\|'}$ is linear and $dt \otimes X \otimes dt$ is a linear vertical valued 1—form;
(b) it is torsion free, because $dt \otimes X \otimes dt$ is a symmetric tensor;
(c) it is metric preserving, because $K^{\|'}$ is metric preserving and $\overset{\shortparallel}{K}{}^{\|''} = \overset{\shortparallel}{K}{}^{\natural} = \overset{\shortparallel}{K}{}^{\|'}$.

Moreover, the curvature $R^{\|''}$ of $K^{\|''4}$ vanishes, because its coordinate expression is

$$R^{\|''} = -\partial_j X_0{}^h{}_0 \, d^0 \wedge d^j \otimes \partial_h \otimes d^0 = 0\,.$$

Then, the flat spacetime connection $K^{\|''}$ yields an affine structure of spacetime. Moreover, being $K^{\|''}$ time preserving, the map dt turns out to be affine.
Eventually, the gravitational connection $K^{\natural}$ turns out to fulfill the equality $K^{\natural} = K^{\|''} + dt \otimes N'' \otimes dt$, where $N'' := N' + X$.
Therefore, the connection $K^{\|''}$ induces a background affine structure of spacetime, according to the definition of flat newtonian spacetime (see Definition 24.1.1).

(2) Let $K^{\|''}$ be a 2nd flat spacetime connection, which is compatible with the 1st curved newtonian spacetime.

Then, we have $K^{\natural} = K^{\|'} + dt \otimes N' \otimes dt = K^{\|''} + dt \otimes N'' \otimes dt$.

Indeed, the two affine structures induced on the fibres of spacetime coincide. Then, the scaled vector field $X := N'' - N' : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ is spacelikely constant with respect to both affine structures, because N'' and N' are spacelikely constant with respect to both affine structures. □

The above non uniqueness of the background affine structure can be illustrated in terms of inertial observers, according to the following results.

Corollary 28.5.5 *Let us consider two curved newtonian spacetimes*

$$(\boldsymbol{E}, g, K^{\natural}, K^{\|'}) \qquad \text{and} \qquad (\boldsymbol{E}, g, K^{\natural}, K^{\|''}),$$

such that

$$K^{\|''} = K^{\|'} + dt \otimes X \otimes dt\,.$$

Let o' be a 1st inertial observer with respect to the 1st background affine structure and $o'' = o' + \vec{v}$ a 2nd observer. Then, the observer o'' is inertial with respect to the 2nd background affine structure if and only if the scaled vector field $\vec{v} : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ fulfills the condition

$$\nabla^{\natural} \vec{v} = X\,. \quad □$$

Note 28.5.6 The condition $\nabla^{\natural}\vec{v} = \nabla^{\|'}\vec{v} = \nabla^{\|''}\vec{v} = X$ analysed in the above Corollary 28.5.5 implies that the vertical derivative of $\vec{v}$, with respect to the affine structures of the fibres, is spacelikely constant. This means that the relative motion of one observer with respect to the other one is spacelikely affine.

This occurs when one observer performs a "translation motion", with respect to the other one. □

Part IV
Conclusions and Further Developments

We devote the present concluding part to

• the conclusions concerning our approach to the covariant formulation of classical and quantum theory of a scalar particle in a curved galilean framework discussed so far,

• a sketch of the developments of our approach to the quantum theory of a spin particle in a curved galilean framework (Chap. 29),

• a sketch of some results achieved in the einsteinian framework along with possible hints (Chap. 30).

Chapter 29
Conclusions

29.1 Main Features of Our Approach

- We do not propose an alternative theory with respect to standard Quantum Mechanics; conversely, this theory is the touchstone of our discussion and developments.

Our main aim is to show that implementing the requirement of covariance into Quantum Mechanics suggests essentially new approaches and methods, which yield unusual viewpoints.

- Our true main goal deals with Quantum Mechanics in a flat spacetime; the curved spacetime is not our primary aim. Actually, we deal with a curved framework for three reasons:

(1) once we have developed a covariant theory of Quantum Mechanics in a flat galilean spacetime, its generalisation to a curved spacetime is rather simple and natural,
(2) the curved case suggests several hints also for the flat case,
(3) a discussion of Covariant Quantum Mechanics in a curved spacetime fosters an interesting comparison with the einsteinian General Relativity.

- Surprisingly, our Covariant Classical Mechanics inherits several ideas and methods of einsteinian General Relativity (including the geometric description of gravity and general observers), but it is quite far from the lorentzian geometry of einsteinian Special Relativity.
- We deal with a *broad covariance requirement*, which includes not only equivariance with respect to observers and coordinates, but also with respect to units of measurement. For this reason, we have systematically adopted a rigorous formal mathematical language which allows us to include an explicit mention of the possible scale factor of every object (see Sect. 1.3.5 and Appendix: Sect. K).

In particular, the equivariance with respect to units of measurement plays an essential role in the hamiltonian lift of phase functions and in the linearity of Schrödinger operator (see Definition 11.3.6, Remarks 11.3.9, and 17.7.13).

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- Most objects of the classical and quantum theories are systematically expressed in an *intrinsic way*, in *coordinates* and in an intermediate "*observed way*" (see Sect. 1.2.6). Each of these expressions emphasise different features of the object.
- We systematically deal with *general observers* (see Definition 2.7.1). The corresponding expressions are achieved by a simple general geometric procedure, not by "ad hoc" approaches.
- In particular, in the quantum theory, we introduce the concept of "*rest observer*" o_Ψ for every proper quantum section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q}_{/0})$, by means of several independent approaches. This concept can be hardly achieved in standard Quantum Mechanics, because, in general, the rest observer is non inertial. Indeed, in a sense, the concept of rest observer has some analogies with the concept of rest mass in General Relativity.

In particular, the rest observer o_Ψ is characterised as the observer for which the observed potential $A[\mathrm{b}_\Psi, o_\Psi]$ is timelike and is regarded as the observer at rest with respect to the hydrodynamical fluid associated with the proper quantum section Ψ (see Theorem 15.2.31 and Definition 18.1.3).

So, we systematically provide the observed expressions of several quantum objects, not only with respect to a generic observer o, but also with respect to the rest observer o_Ψ. In this second case, we emphasise relevant non standard meanings of related formulas (see, for instance, Corollaries 17.3.3, 17.3.5, 17.5.3, 17.6.9, 17.6.12 and 18.1.10, Theorems 17.4.2, 18.2.1, and 18.2.2).

- We deal with a self-consistent version of the *galilean gravitational and electromagnetic fields* (see Postulates C.2 and C.3)

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \quad \text{and} \quad F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}.$$

  All developments of Covariant Classical and Quantum Mechanics are derived from the above fields. Actually, we deal only with these classical fields, which deserve the right to be considered as fundamental in Classical Mechanics. Other more phenomenological fields would hardly yield a covariant theory.
- We show that the gravitational and electromagnetic fields can be naturally encoded into an original "*joined galilean spacetime connection*", by a *natural minimal coupling*, (see Theorem 6.3.1)

$$K \equiv K^\natural + K^\mathfrak{e} := K^\natural - \tfrac{1}{2}\,\mathrm{k}\,(dt \otimes \hat{F} + \hat{F} \otimes dt) : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

  So, all classical and quantum developments are derived from this joined connection in a synthetic way. Of course, we emphasise the splitting of all joined formulas into their gravitational and electromagnetic components (see, for instance, Propositions 7.2.2 and 8.3.3, Theorems 9.2.5, 9.2.6, 9.2.8, 9.2.11 and 10.1.8, and Definition 15.2.1).
- In Covariant Classical Mechanics and Covariant Quantum Mechanics *time is not just a parameter*, but plays an essential role in several respects.
  In particular, in Covariant Classical Mechanics, spacetime is supposed to be

equipped with an observer independent projection over time, but not with a splitting into space and time (see Postulate C.1 and Note 2.7.5). This fact has several relevant consequences.

- In order to fulfill the manifest covariance of the theory, a criterion of minimality and the essential role of time, we choose the odd dimensional affine 1-jet bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ as *classical phase space* (see Proposition 2.5.1). This choice plays a strategic role in classical and quantum theories.
- For the same reason as above, the *classical cosymplectic geometry* replaces the more standard *classical symplectic geometry* (see Theorems 9.2.8 and 10.1.1). So, a fundamental role is played by the cosymplectic pair (dt, Ω), where

$$dt : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E} \quad \text{and} \quad \Omega : J_1\boldsymbol{E} \to \Lambda^2 T^* J_1\boldsymbol{E}.$$

- The *hamiltonian methods* associated with the cosymplectic structure require relevant modifications with respect to the more standard methods associated with a symplectic structure (see Proposition 11.1.4, Theorems 11.2.11, 11.3.1, 11.3.4 and 11.3.8, Definition 11.4.1).

For instance, we deal with the following non standard *hamiltonian lift* of a phase function $f \in \text{map}(J_1\boldsymbol{E}, \mathbb{R})$ (see Definition 11.3.6)

$$X^{\uparrow}{}_{\text{ham}}[f] := X^{\uparrow}_{\text{ham}}[f'', f] = \gamma(f'') + \Lambda^{\sharp}(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E}).$$

We do not deal with a given global gauge independent and observer independent classical *hamiltonian function*. Conversely, the cosymplectic 2-form encodes such a phase function, which can be extracted by means of a geometric procedure, by choosing a gauge and an observer, (see Theorem 10.1.8)

$$\mathcal{H}[\mathrm{b}, o] := -\,д[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}).$$

Analogously, we do not deal with a given global gauge independent classical *lagrangian function*. Conversely, the cosymplectic 2-form encodes such a phase function, which can be extracted by means of a geometric procedure, by choosing a gauge, (see Theorem 10.1.8)

$$\mathcal{L}[\mathrm{b}] := -\,д \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E}).$$

- We introduce the original *Lie algebra of special phase functions*

$$\text{spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{map}(J_1\boldsymbol{E}, \mathbb{R}),$$

which includes, on the same footing, the spacetime functions x^λ, the components $\mathcal{P}_i$ of momentum and the component $\mathcal{H}_0$ of hamiltonian (see Example 12.1.4).

Indeed, the Lie algebra of special phase functions plays a relevant role in our theory and allows us to unify several items of the classical and quantum theories.

- The original *phase Lie bracket*

$$[\![f, \acute{f}]\!] := \Lambda(df, d\acute{f}) + \gamma(f'').\acute{f} - \gamma(\acute{f}'').f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$$

  arises in a natural way from a natural modification of the standard Poisson bracket (see Definition 12.1.1 and Theorem 12.5.3). Indeed, several facts exhibit the above bracket (see, for instance, Remark 19.1.11).
- We achieve, in a covariant way, several distinguished phase and quantum lifts of special phase functions, which yield several interplays and consequences.

The special phase functions $f \in spe(J_1\boldsymbol{E}, \mathbb{R})$ admit the non standard *tangent phase lift* (see Theorem 12.2.1)

$$X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E}),$$

which plays an essential role in several developments.

The tangent lift of special phase functions $f \in \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$ naturally yields the original *holonomic lift* (see Definition 12.3.2)

$$X^{\uparrow}{}_{\mathrm{hol}}[f] := r^1 \circ J_1 X[f] \in \sec(J_1\boldsymbol{E}, T J_1\boldsymbol{E}).$$

The original tangent *hamiltonian lift*

$$X^{\uparrow}{}_{\mathrm{ham}}[f] := \gamma(f'') + \Lambda^{\sharp}(df) \in \sec(J_1\boldsymbol{E}, T J_1\boldsymbol{E})$$

of special phase functions $f \in spe(J_1\boldsymbol{E}, \mathbb{R})$ arises in a natural way from a natural modification of the standard hamiltonian lift (see Definition 12.4.1).

A comparison of the hamiltonian and holonomic lifts of special phase functions allows us to introduce the original *holonomic Lie subalgebra* (see Definition 12.6.8)

$$\mathrm{hol\,spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R}),$$

which plays a role in the study of classical and quantum infinitesimal symmetries.

The tangent lift of projectable special phase functions $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ naturally yields the original *η-quantum lift* (see Definition 19.1.3)

$$\begin{aligned} Y_\eta[f] &:= X[f] \lrcorner \text{Ч}[o] + (\mathrm{i}\,\check{f}[o] - \tfrac{1}{2}\,\mathrm{div}_\eta f)\,\mathbb{I} \\ &= X[f] \lrcorner \chi[\mathfrak{b}] + (\mathrm{i}\,\hat{f}[\mathfrak{b}] - \tfrac{1}{2}\,\mathrm{div}_\eta f)\,\mathbb{I} \in \sec(\boldsymbol{Q}, T\boldsymbol{Q}). \end{aligned}$$

The tangent lift of projectable special phase functions $f \in \mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ naturally yields the original *η-upper quantum lift* (see Theorem 19.2.2)

$$Y^{\uparrow}{}_\eta[f] = \text{Ч}^{\uparrow}\big(X^{\uparrow}[f]\big) + \mathrm{i}\, f\, \mathbb{I}^{\uparrow} \in \sec(\boldsymbol{Q}^{\uparrow}, T\boldsymbol{Q}^{\uparrow}).$$

- The above lifts of special phase functions naturally yield a unified classification of the *infinitesimal symmetries* of classical and quantum theories as follows.

The Lie subalgebra $\text{pro spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ of *projectable special phase functions* generates the *infinitesimal symmetries of the hermitian quantum structure* $\mathfrak{h}_\eta$ (see Theorem 19.1.7)

$$Y_\eta[f] \in \sec(\boldsymbol{Q}, T\boldsymbol{Q}).$$

It is remarkable that the above classification theorem yields a *natural Lie algebra isomorphism* between classical and quantum Lie algebras

$$Y_\eta : \text{pro spe}(J_1\boldsymbol{E}, \mathbb{R}) \to \text{her}_\eta(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y_\eta[f].$$

The Lie subalgebra $\text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ of *conserved time preserving special phase functions* generates the *infinitesimal symmetries of the quantum structure* $(\mathfrak{h}_\eta, \mathfrak{Y}^\uparrow)$ (see Theorem 19.2.2)

$$Y^\uparrow{}_\eta[f] \in \sec(\boldsymbol{Q}^\uparrow, T\boldsymbol{Q}^\uparrow).$$

A further Lie subalgebra of $\text{cns tim spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$ generates the *infinitesimal symmetries of the dynamical quantum structure* (dt, L) (see Theorem 19.3.2)

$$Y^1_\eta[f] \in \sec(J_1\boldsymbol{Q}, TJ_1\boldsymbol{Q}).$$

- The Schrödinger operator and the Lie subalgebra of time preserving projectable special phase functions

$$\mathrm{S} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}) \quad \text{and} \quad \text{tim pro spe}(J_1\boldsymbol{E}, \mathbb{R}) \subset \text{spe}(J_1\boldsymbol{E}, \mathbb{R})$$

  generate, in a covariant way, *differential quantum operators*

$$\mathrm{O}[f] : \Psi \mapsto \mathrm{O}[f].\Psi,$$

  associated with special phase functions, including, on the same footing, the quantum operators associated with the spacetime functions x^λ, the components $\mathcal{P}_i$ of momentum and the component $\mathcal{H}_0$ of hamiltonian (see Theorem 20.1.9).
- The *quantum bundle*

$$\pi : \boldsymbol{Q} \to \boldsymbol{E}$$

  can be equivalently defined as a 2-dimensional real vector bundle equipped with a suitable real metric, or as a 1-dimensional complex vector bundle equipped with a hermitian metric (see Postulate Q.1, Propositions 14.2.1, 14.3.1, 14.4.1 and 14.4.2 and Note 14.4.5).

We use both languages. The complex language is suitable for discussing standard features of the quantum theory, while the real language is more suitable for discussing essentially real features. Moreover, the real language emphasises *two gauge dependent real degrees of freedom* of the scalar quantum particle.

- We take into due consideration the *proper quantum bundle*, defined by dropping the zero section, (see Definition 14.6.1)

$$\pi_{/0} : \boldsymbol{Q}_{/0} \subset \boldsymbol{Q} \to \boldsymbol{E}.$$

Accordingly, we discuss a gauge free *polar splitting of the proper quantum bundle*

$$\boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{(\!(\,)\!)}_{/0} : \Psi_e \mapsto \big(\|\Psi\|_e, (\!(\Psi)\!)_e\big),$$

where the type fibre $\boldsymbol{S}_1$ replaces the more standard gauge dependent type fibre $U(1)$ (see Proposition 14.7.2). This polar splitting emphasises *two gauge independent real degrees of freedom* of the scalar quantum particle.

We frequently use this polar splitting in several contexts of the quantum theory, including the dynamical quantum objects and the hydrodynamical picture (see, for instance, Corollaries 17.3.5, 17.5.3, 17.6.12, 17.6.18 and 18.1.4, Theorems 17.6.11, 18.1.1, 18.2.1 and 18.2.2).

- The original base space of the quantum bundle $\boldsymbol{Q}$ is the spacetime $\boldsymbol{E}$. However, later we define the *upper quantum bundle*

$$\pi^{\uparrow} : \boldsymbol{Q}^{\uparrow} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$$

by enlarging the base space to the classical phase space $J_1\boldsymbol{E}$, via a pullback (see Definition 14.11.1). This is a strategic minimal choice of our approach. In particular, in this natural way the bundle $\boldsymbol{Q}^{\uparrow}$ has *no redundant dimension* and we have no need to search for polarisations.
- We are involved with a *game of the joined potential* (gravitational and electromagnetic) throughout the classical and quantum theory.

First of all, in the classical theory we deal with the gauge dependent and observer dependent *potential*

$$A[\mathfrak{b}, o] \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$$

of the observed spacetime 2-form

$$\Phi[o] \in \sec(\boldsymbol{E}, \Lambda^2 T^*\boldsymbol{E})$$

associated with the joined spacetime connection K (see Definition 4.2.11, Theorems 4.3.3 and 6.3.1, Corollary 6.3.3, Remark 6.3.4).

Then, we deal with the gauge dependent and observer independent *horizontal upper potential*

$$A^{\uparrow}[\mathfrak{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$$

of the gauge independent and observer independent joined cosymplectic 2-form Ω (see Definition 10.1.3).

The above potentials are linked by the equality (see Theorem 10.1.4)

$$A[\mathrm{b}, o] := o^* A^{\uparrow}[\mathrm{b}].$$

Further, in the quantum theory, the observed potential $A[\mathrm{b}, o]$ and the upper potential $A^{\uparrow}[\mathrm{b}]$ appear in the gauge dependent and observer dependent expressions of the upper quantum connection $Ч^{\uparrow}$, according to the equalities (see Theorem 15.2.4)

$$\begin{aligned} Ч^{\uparrow} &= \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\, A^{\uparrow}[\mathrm{b}] \otimes \mathbb{I}^{\uparrow} \\ &= \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\big(- \mathcal{K}[o] + \mathcal{Q}[o] + A[\mathrm{b}, o]\big) \otimes \mathbb{I}^{\uparrow}. \end{aligned}$$

So, the above classical potentials will play a role in all quantum objects derived from the upper quantum connection $Ч^{\uparrow}$.

- We propose an original *differential geometric procedure* for deriving several quantum objects from the upper quantum connection $Ч^{\uparrow}$, including

(a) the *quantum velocity* (see Theorem 17.2.2)

$$\mathrm{V}[\Psi] := д + \vec{\nabla}^{\uparrow (\!(0)\!)}(\!(\Psi)\!) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E}),$$

(b) the *kinetic quantum tensor* (see Theorem 17.3.2)

$$\mathrm{Q}[\Psi] := д \otimes \Psi - \mathrm{i}\, \vec{\nabla}^{\uparrow} \Psi \in \sec\big(\boldsymbol{E}, \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big),$$

(c) the *quantum probability current* (see [219] and Theorem 17.4.2)

$$\mathrm{J}[\Psi] := д \otimes \|\Psi\|^2 - \mathrm{re}\, \mathrm{h}(\Psi, \mathrm{i}\, \vec{\nabla}^{\uparrow} \Psi) \in \sec\big(\boldsymbol{E}, \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big),$$

(d) the *quantum lagrangian* (see Theorem 17.5.2)

$$\begin{aligned} \mathrm{L}[\Psi] \;:=\; -dt \wedge \Big(&\mathrm{im}\, \mathrm{h}_\eta(\Psi, д \lrcorner \nabla^{\uparrow} \Psi) \\ &+ \tfrac{1}{2}\, (\bar{G} \otimes \mathrm{h}_\eta)(\check{\nabla}^{\uparrow} \Psi, \check{\nabla}^{\uparrow} \Psi)\Big) : \boldsymbol{E} \to \Lambda^4 T^* \boldsymbol{E}, \end{aligned}$$

(e) the *Schrödinger operator* (see [219] and Theorem 17.6.5)

$$\mathrm{S}[\Psi] := \tfrac{1}{2}\big(д \lrcorner \nabla^{\uparrow} \Psi + \delta^{\uparrow}(\mathrm{Q}[\Psi])\big) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q}),$$

(f) the *quantum operators* associated with special phase functions (see [222] and Theorem 20.1.9)

$$\mathrm{O}[f] = \mathrm{i}\big(Y_\eta[f] - \mathrm{S}[f]\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q}),$$

(g) the *quantum currents* associated with special phase functions (see [358] and Definition 21.1.3)

$$\mathrm{j}_\eta[f] := -\, i_{Y^1_\eta[f]}\, \mathrm{C} = -i_{Y_\eta[f]}\, \mathrm{C} \in \sec(J_1\boldsymbol{Q}, \Lambda^3 T^*\boldsymbol{Q}),$$

(h) the *quantum expectations forms* associated with special phase functions (see [358] and Definition 21.3.1)

$$\epsilon_\eta[f](\Psi) := \mathrm{re}\, \mathrm{h}_\eta\big(\Psi,\ \mathrm{O}[f](\Psi)\big) \in \sec(\boldsymbol{E}, \Lambda^3 V^*\boldsymbol{E}).$$

It is remarkable the fact that the *probability current* turns out to be just the current associated with the distinguished special phase function $f = 1$, but the above procedure associates a quantum current with every special phase function (see Example 21.1.5).

Moreover, for every *conserved special phase function* we obtain a *conserved quantum current form* (see Theorem 21.2.4).

It is remarkable that all dynamical quantum objects above, including the Schrödinger operator, are covariant. Even more, not only we show that the Schrödinger operator is covariant, but that this operator can be achieved by means of the only requirement of covariance (see Theorem 17.7.12). Further, in this context, we prove that the equivariance with respect to units of measurement is a necessary condition for achieving a linear operator (see Remark 17.7.13).

- As we have already mentioned, the observed potential $A[\mathrm{b}, o]$ and the upper potential $A^\uparrow[\mathrm{b}]$ depend on an arbitrary gauge b.

Moreover, every proper quantum section Ψ yields in a natural way a distinguished gauge b_Ψ (see Note 14.6.3).

Then, every proper quantum section Ψ yields the distinguished *timelike joined potential*

$$A[\Psi] := A[o_\psi, \mathrm{b}_\Psi] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E}),$$

which can be regarded as the potential intrinsically "*seen*" by the quantum particle, regardless of any arbitrary observers and gauges (see Theorem 15.2.31).

In a sense, this distinguished potential has some analogies with the rest mass of General Relativity.

The distinguished potential $A[\Psi]$ appears in the gauge independent expression of several quantum objects, so emphasising relevant non standard meanings, (see, for instance, Corollaries 17.5.3, 17.6.9 and 17.6.12, Theorems 18.2.1 and 18.2.2).

In particular, we show an intrinsic *polar splitting of the Schrödinger equation* into a system involving $|\Psi|$ and the joined timelike potential $A[\Psi]$ "seen" by the proper quantum section Ψ (see Corollary 17.6.18)

$$0 = д[o_\Psi].\|\Psi\| + \tfrac{1}{2}\, \|\Psi\|\, \mathrm{div}_\eta д[o_\Psi] \quad \text{and} \quad 0 = \Delta[G]\|\Psi\| + 2\, \|\Psi\|\, A[\Psi].$$

29.2 Open Problems

Among the open problems, we mention two items.

- We have shown that a manifestly covariant formulation of Quantum Mechanics referred to a generic observer cannot deal with a unique time independent *Hilbert space*. Conversely, such a theory is based on a Hilbert bundle over time. So, we have a Hilbert space for every time. Indeed, there is no natural trivialisation of such a bundle, i.e. no natural isomorphism between the Hilbert spaces associated with different times (see Definitions 22.2.2 and 22.2.6, Propositions 22.3.2 and 22.4.1, Remark 22.3.4, Theorem 22.5.5).

So, this fact suggests the opportunity to possibly enlarge the standard axioms of standard Quantum Mechanics in order to include quantum operators acting between different Hilbert spaces at different times. Such operators would possibly describe measurements involving different times (see, for instance [13, 15, 54, 192, 235–237, 320]).

However, here we just suggest such a hint; a definite proposal is out of the scope of the present book.

It would be quite interesting to investigate the possible relation of our approach with the "relational interpretation of quantum mechanics" (see [291, 351, 371]).

- We have shown that several serious problems concerning *angular momentum* arise even in the flat case (see Proposition 25.1.2, Remarks 25.1.3, 25.1.4, 25.1.6 and 25.1.8). For instance, the standard bracket for angular momentum needs not to work in curvilinear coordinates.

The deep reason for that is related to the essentially affine nature of the standard angular momentum (see Sect. 1.5.19). The same essential problem holds also in einsteinian General Relativity (see, for instance [378]).

Indeed, one might possibly find a much more general concept holding in a curvilinear framework which in the particular case of a flat spacetime yields the standard concept of angular momentum. However, here we just suggest such a hint; a definite proposal is out of the scope of the present book.

Chapter 30
Developments in Galilean Spin Particle

In the present book we explicitly deal with the approach to Covariant Quantum Mechanics of a *scalar quantum particle*. However, this approach can be easily extended also to a *quantum particle with spin* in a curved galilean framework (see [40, 42]). In other words, we can extend the covariant approach of Schrödinger equation to a covariant approach of *Pauli equation*. Here, we just sketch the basic ideas of this approach.

30.1 Classical Spinning Particle

30.1.1 Classical Sphere Bundle

Let us consider the classical galilean spacetime $t : \boldsymbol{E} \to \boldsymbol{T}$, equipped with the galilean metric field g, the galilean gravitational connection $K^{\natural}$ and the electromagnetic field F.

According to [42], we focus our attention on the scaled spacetime vertical vector bundle $\mathbb{L}^* \otimes V\boldsymbol{E} \to \boldsymbol{E}$ and on its subbundle $S\boldsymbol{E} \subset \mathbb{L}^* \otimes V\boldsymbol{E}$ consisting of *spheres of radius* 1.

We consider a *mass*, a *charge* and two *coupling scales*

$$m \in \mathbb{M}, \qquad q \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R},$$
$$q/m \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}, \qquad \mu \in \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}.$$

The coupling scales yield two *joined galilean spacetime connection* (see Theorem 6.3.1)

$$K_{q/m} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E} \qquad \text{and} \qquad K_{2\mu} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}.$$

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Moreover, the spacetime connection $K_{2\mu}$ naturally yields a linear connection

$$\check{K}_{2\mu} : V\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TV\boldsymbol{E}.$$

The history of a *classical spinning particle* can be described by a section

$$\varsigma : \boldsymbol{T} \to S\boldsymbol{E},$$

which projects over a classical motion $s := \tau_{\boldsymbol{E}} \circ \varsigma : \boldsymbol{T} \to \boldsymbol{E}$.

The *intrinsic angular momentum* is represented by the section

$$\tfrac{1}{2}\,\hbar\,\varsigma : \boldsymbol{T} \to (\mathbb{T}^{-1} \otimes \mathbb{L} \otimes \mathbb{M}) \otimes V\boldsymbol{E}.$$

The equation of motion of a charged spinning particle can be written by means of the connections $K_{q/m}$ and $\check{K}_{2\mu}$.

30.1.2 Lie Algebra of Spin Special Phase Functions

As *spin phase space* for the classical spinning particle we consider the bundle (see [40, 42])

$$U_1\boldsymbol{E} := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{L}^* \otimes V\boldsymbol{E}).$$

Then, we define the *spin special phase functions* to be the local functions of the type

$$(f, \phi) : U_1\boldsymbol{E} \to \mathbb{R}, \quad \text{with } f \in \text{spe}(J_1\boldsymbol{E}, \mathbb{R}),\ \phi \in \text{lin}(\mathbb{L}^* \otimes V\boldsymbol{E},\ \mathbb{R}).$$

Next, let us consider the natural linear fibred isomorphism

$$\mathrm{j} : \mathbb{L} \otimes V^*\boldsymbol{E} \to \text{End}(V\boldsymbol{E}) : \alpha \mapsto \mathrm{j}(\alpha), \quad \text{where } \mathrm{j}(\alpha) : V\boldsymbol{E} \to V\boldsymbol{E} : X \mapsto i_\alpha i_{g^\flat(X)}\bar{\eta},$$

and the linear fibred morphism

$$\rho \equiv \check{R} \lrcorner (\mathrm{j})^{-1} : \boldsymbol{E} \to \mathbb{L} \otimes \Lambda^2 T^*\boldsymbol{E} \otimes V^*\boldsymbol{E},$$

where

$$\check{R} : V\boldsymbol{E} \to \Lambda^2\, T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes V^*\boldsymbol{E}$$

is the curvature tensor of the connection

$$\check{K} : V\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TV\boldsymbol{E}.$$

Then, we define the *spin special Lie bracket* given by the equality

$$[\![(f + \phi), (\acute{f} + \acute{\phi})]\!] := \Big([\![f, \acute{f}]\!], \;\; \acute{\phi} \times \phi + \check{\nabla}_{X[f]} \acute{\phi} - \check{\nabla}_{X[\acute{f}]} \phi - \rho\big(X[f], X[\acute{f}]\big) \Big),$$

where $\acute{\phi} \times \phi := i_{g^\sharp(\phi)} i_{g^\sharp(\acute{\phi})} \eta$ is the cross product of scaled forms.

30.1.3 Classical Spin Bundle

Following [42], we start by considering as *classical spin bundle* a 2-dimensional complex vector bundle over the galilean spacetime $\boldsymbol{E}$

$$\pi_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{E}$$

equipped with a *hermitian metric* and a *normalised non singular 2-form*

$$\mathrm{h}_{\boldsymbol{S}} : \boldsymbol{E} \to \overline{\boldsymbol{S}^\star} \otimes \boldsymbol{S}^\star \qquad \text{and} \qquad \epsilon_{\boldsymbol{S}} : \boldsymbol{E} \to \Lambda^2 \boldsymbol{S}^\star,$$

where $\boldsymbol{S}^\star$ and $\overline{\boldsymbol{S}^\star}$ denote the complex dual and anti-dual bundles. Hence, $\pi_{\boldsymbol{S}} : \boldsymbol{S} \to \boldsymbol{E}$ can be regarded as a bundle associated with a principal bundle with structure group $SU(2)$.

We consider an $\mathrm{h}_{\boldsymbol{S}}$-*orthonormal frame* (ζ_{A}) of $\boldsymbol{S}$ and its complex dual frame (z^{A}), with $\mathrm{A} = 1, 2$. Accordingly, we deal with a linear fibred chart $(x^\lambda, z^{\mathrm{A}})$ of $\boldsymbol{S}$ and the conjugate chart $(x^\lambda, \bar{z}^{\mathrm{A}^\star})$ of $\boldsymbol{S}^\star$. We define a *normal spin frame* to be an ordered $\mathrm{h}_{\boldsymbol{S}}$-orthonormal frame such that $\epsilon_{\boldsymbol{S}} = z^1 \wedge z^2$.

Next, we focus our attention to the vector bundle $\operatorname{End} \boldsymbol{S} = \boldsymbol{S} \otimes \boldsymbol{S}^\star$ of complex linear endomorphisms, which is equipped with the standard fibred associative algebra and Lie algebra. This bundle splits into the direct sum of real bundles consisting of hermitian and anti-hermitian endomorphisms

$$\operatorname{End} \boldsymbol{S} = \boldsymbol{H} \underset{\boldsymbol{E}}{\oplus} \mathrm{i}\, \boldsymbol{H}.$$

Further, $\boldsymbol{H}$ splits into the sum of the vector subbundle $\boldsymbol{H}_1$ generated by the identity and the vector subbundle $\boldsymbol{H}_0$ of traceless endomorphisms. So, we obtain the splitting

$$\operatorname{End} \boldsymbol{S} = (\boldsymbol{H}_1 \underset{\boldsymbol{E}}{\oplus} \boldsymbol{H}_0) \underset{\boldsymbol{E}}{\oplus} (\mathrm{i}\, \boldsymbol{H}_1 \underset{\boldsymbol{E}}{\oplus} \mathrm{i}\, \boldsymbol{H}_0).$$

The 3-dimensional real vector bundle $\boldsymbol{H}_0$ is constituted by all endomorphisms ϕ whose matrix is of the type

$$(\phi^A{}_B) = \begin{pmatrix} r & c \\ \bar{c} & -r \end{pmatrix}, \quad \text{with} \;\; r \in \mathbb{R}, \; c \in \mathbb{C}.$$

We obtain the fibred *euclidean metric*

$$k : \boldsymbol{H}_0 \underset{\boldsymbol{E}}{\times} \boldsymbol{H}_0 \to \mathrm{I\!R} : (\phi, \theta) \mapsto \tfrac{1}{2}\, \mathrm{tr}(\phi \circ \theta),$$

which makes $\boldsymbol{H}_0$ a bundle associated with a principal bundle with structure group $O(3)$.

We can show that the bundle $\boldsymbol{H}_0$ turns out to be equipped with a distinguished orientation of its fibres. Then, the metric k and this orientation yield a global 3-form

$$\tilde{\eta} : \boldsymbol{E} \to \Lambda^3 \boldsymbol{H}_0.$$

Hence, the bundle $\boldsymbol{H}_0$ can be regarded as associated with a principal bundle with structure group $SO(3)$.

We consider an orthonormal frame of $\boldsymbol{H}$

$$(\sigma_A), \quad \text{with } A = 0, 1, 2, 3, \ \sigma_0 := 1_{\boldsymbol{S}}.$$

In particular, for any $\mathrm{h}_{\boldsymbol{S}}$-orthonormal frame (ζ_A), we consider the endomorphisms

$$(\sigma_r) := \sigma_r{}^A{}_B\, \zeta_A \otimes z^B,$$

associated with the *Pauli matrices*

$$(\sigma_r{}^A{}_B) := \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right), \quad \text{with } r = 1, 2, 3.$$

Then, we can write $\tilde{\eta} = \sigma_1 \wedge \sigma_2 \wedge \sigma_3$.

30.1.4 Spin Connection

Next, we define a *spin connection* to be a complex linear connection (see [42])

$$\text{Б} : \boldsymbol{S} \to T^*\boldsymbol{E} \otimes T\boldsymbol{S},$$

such that $\nabla[\text{Б}]\mathrm{h}_{\boldsymbol{S}} = 0$ and $\nabla[\text{Б}]\epsilon_{\boldsymbol{S}} = 0$. Its coordinate expression is of the type

$$\text{Б} = d^\lambda \otimes (\partial_\lambda + \mathrm{i}\, \text{Б}_\lambda{}^A{}_B\, z^B\, \partial_A),$$

where

$$\text{Б}_\lambda{}^A{}_B = \text{Б}_\lambda{}^r\, \sigma_r{}^A{}_B, \quad \text{with } \text{Б}_\lambda{}^\mu \in \mathrm{map}(\boldsymbol{E}, \mathrm{I\!R}), \ r = 1, 2, 3.$$

We obtain the further identity $\nabla[\text{Б}]\,k = 0$. Even more, this condition determines the spin connection Б.

30.1.5 Pauli Map

We postulate a *Pauli map*, i.e. an orientation preserving linear fibred isometry over $\boldsymbol{E}$ (see [42])

$$\Sigma : \mathbb{L}^* \otimes V\boldsymbol{E} \to \boldsymbol{H}_0.$$

Then, we obtain a Lie algebra isomorphism $-\frac{\mathfrak{i}}{2}\,\Sigma : \mathbb{L}^* \otimes V\boldsymbol{E} \to \mathfrak{i}\,\boldsymbol{H}_0$.

One can prove that for every metric linear connection C of $V\boldsymbol{E} \to \boldsymbol{E}$, there is a unique spin connection Б of $\boldsymbol{S} \to \boldsymbol{E}$, such that

$$\Sigma(\nabla[C]X) = \nabla[\text{Б}](\Sigma(X)), \quad \text{for each } X \in \sec(\boldsymbol{E}, \mathbb{L}^* \otimes V\boldsymbol{E}).$$

In coordinates, we have $\text{Б}_\lambda{}^A{}_B = \frac{1}{4}\,\epsilon_r{}^{sp}\,C_\lambda{}^r{}_s\,\sigma_p{}^A{}_B$.

Then, the curvature tensor of Б and the covariant curvature tensor of C fulfill the equality

$$R[\text{Б}] = -\tfrac{1}{4}\,\Sigma(*\underline{R}[C]).$$

The connection $\check{K}_{2\mu}$ of the vector bundle $V\boldsymbol{E} \to \boldsymbol{E}$, determines the spin connection $\text{Б} \equiv \text{Б}_{2\mu}$ of $\boldsymbol{S} \to \boldsymbol{E}$, by means of the condition

$$\Sigma\big(\nabla[\overset{''}{K}]X\big) = \nabla[\text{Б}]\big(\Sigma(X)\big), \qquad \forall X \in \sec(\boldsymbol{E},\, \mathbb{L}^* \otimes V\boldsymbol{E}).$$

In coordinates, we have $\text{Б}_\lambda{}^A{}_B = \frac{1}{4}\,\epsilon_r{}^{sp}\,K_\lambda{}^r{}_s\,\sigma_p{}^A{}_B$.

30.2 Quantum Spin

30.2.1 Quantum Spin Bundle

Let us recall the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ of the scalar quantum particle and the local normal quantum bases $\mathsf{b} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$, along their dual functions $z \in \text{map}(\boldsymbol{Q}, \mathbb{C})$.

Then, we define the *spin quantum bundle* to be the tensor product bundle (see [40, 42])

$$\pi_U : \boldsymbol{U} := \boldsymbol{Q} \otimes \boldsymbol{S} \to \boldsymbol{E}.$$

A quantum basis b and a spin frame ζ_A yield the normal spin frame of $\boldsymbol{U}$ and its dual

$$\mathsf{b}_A := \mathsf{b} \otimes \zeta_A \qquad \text{and} \qquad w^A := z \otimes z^A .$$

The *quantum history of a spin particle* will be described by a section

$$\Psi : \boldsymbol{E} \to \boldsymbol{U},$$

with coordinate expression

$$\Psi = \Psi^A \otimes \zeta_A = \psi^{\mathrm{A}} \, \mathsf{b}_A, \quad \text{with } A = 1, 2,$$

where $\Psi^A = \psi^A \, \mathsf{b} \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ and $\psi^A \in \operatorname{map}(\boldsymbol{E}, \mathbb{C})$.

Clearly, we obtain the hermitian metric $\mathfrak{h}_{\boldsymbol{U}} := \mathfrak{h}_{\boldsymbol{Q}} \otimes \mathfrak{h}_{\boldsymbol{S}}$.

Further, we define, by pullback, the *upper spin quantum bundle*

$$\pi^{\uparrow}{}_{\boldsymbol{U}} : \boldsymbol{U}^{\uparrow} := \boldsymbol{Q}^{\uparrow} \otimes \boldsymbol{S} \to J_1\boldsymbol{E}.$$

30.2.2 Quantum Spin Connection

The galilean upper quantum connection $Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^{\uparrow}$ and the spin connection $Б : \boldsymbol{S} \to T^*\boldsymbol{E} \otimes T\boldsymbol{S}$ yield the *upper quantum spin connection*

$$Ч^{\uparrow}{}_{\boldsymbol{U}} := Ч^{\uparrow} \otimes Б : \boldsymbol{U}^{\uparrow} \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{U}^{\uparrow}.$$

Then, for each section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{U})$, we obtain the covariant differential with coordinate expression

$$\nabla^{\uparrow}\Psi = (\partial_\lambda \, \psi^A - \mathrm{i}\, A^{\uparrow}{}_\lambda \, \psi^A - \mathrm{i}\, Б_\lambda{}^A{}_B \, \psi^B) \, d^\lambda \otimes \mathsf{b}_A .$$

30.2.3 Quantum Spin Lagrangian

For each section $\Psi \in \sec(\boldsymbol{E}, \boldsymbol{U})$, we consider the sections (see [42])

$$\overset{o}{\nabla}\Psi := д \lrcorner \nabla^{\uparrow}\Psi : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \boldsymbol{U} \qquad \text{and} \qquad \check{\nabla}\Psi := \nabla^{\uparrow}\Psi_{V\boldsymbol{E}} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes \boldsymbol{U}.$$

We have the distinguished phase 4-forms

$$\overset{o}{\mathrm{L}}[\Psi] := \tfrac{1}{2}\big(\mathrm{h}(\Psi, \mathrm{i}\,\overset{o}{\nabla}\Psi) + \mathrm{h}(\mathrm{i}\,\overset{o}{\nabla}\Psi, \Psi)\big)\,\upsilon : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^4 \boldsymbol{T}^*\boldsymbol{E},$$
$$\check{\mathrm{L}}[\Psi] := \tfrac{\hbar}{2m}\,(g \otimes \mathrm{h})(\check{\nabla}\Psi, \check{\nabla}\Psi)\,\upsilon : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^4 \boldsymbol{T}^*\boldsymbol{E},$$

which yield the distinguished *quantum lagrangian* given by the (observer independent) spacetime 4-form

$$\mathrm{L}[\Psi] := \overset{o}{\mathrm{L}}[\Psi] - \check{\mathrm{L}}[\Psi].$$

Then, we obtain the *quantum momentum*

$$\mathrm{p} := (\mathrm{re}\,\mathrm{h})^\sharp(*V_U \mathrm{L}) : J_1\boldsymbol{U} \to \mathbb{T}^* \otimes \boldsymbol{TE} \otimes \boldsymbol{U}.$$

30.2.4 Pauli Equation on the Curved Galilean Spacetime

The Euler–Lagrange equation associated with the above lagrangian L yields the *Pauli equation* on our curved galilean spacetime, which can be written as (see [42])

$$\mathrm{P}[\Psi] := *\,\mathrm{E}^\star[\Psi] = 2\left(\mathrm{i}\,D[o]\Psi + \tfrac{\hbar}{2m}\,\check{\Delta}[o]\Psi\right) = 0,$$

where, in coordinates,

$$D[o]\Psi = (\partial_0\psi^A + \frac{\partial_0\sqrt{|g|}}{2\sqrt{|g|}}\,\psi^A - \mathrm{i}\,A_0\,\psi^A - \mathrm{i}\,\mathrm{B}_{0\,A}B\,\psi^B)\,u^0 \otimes \mathrm{b}_A,$$
$$(\check{\Delta}[o]\Psi)^A = g^{hk}\left(\delta^A_B\,(\partial_h - \mathrm{i}\,A_h) - \mathrm{i}\,\mathrm{B}_h{}^A{}_B\right)\left(\delta^B_C\,(\partial_k - \mathrm{i}\,A_k) - \mathrm{i}\,\mathrm{B}_k{}^B{}_C\right)\psi^C.$$

So, we have obtained the *Pauli operator*

$$\mathrm{P} : J_2\boldsymbol{U} \to \mathbb{T}^* \otimes \boldsymbol{U},$$

which, in the case of spin particles, replaces the Schrödinger operator.

Following [42], in virtue of the Noether theorem, the action of the group $U(1)$

$$\mathbb{R} \times \boldsymbol{U} \to \boldsymbol{U} : (\phi, \zeta) \mapsto e^{-\mathrm{i}\,\phi}\zeta$$

yields the conserved probability current.

Moreover, in the case of a flat spacetime and vanishing electromagnetic field, the lagrangian is invariant with respect to the action group $SU(2)$ and we obtain the conservation of the expectation value of the spin.

30.2.5 Quantum Spin Operators

Let us consider the hermitian metric $\mathfrak{h}_\eta := \mathfrak{h} \otimes \eta$. Analogously to the scalar case, we have a natural (gauge independent and observer independent) *Lie algebra isomorphism* (see [40])

$$\operatorname{spe}(J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \mathbb{L}^* \otimes V\boldsymbol{E},\ \mathbb{R})) \to \operatorname{her}_\eta(\boldsymbol{U}, T\boldsymbol{U}) : (f, \phi) \mapsto Y_\eta[(f, \phi)]$$

between the Lie algebra of spin special phase functions and the Lie algebra of projectable η-hermitian spin quantum vector fields. In coordinates, we have the equality

$$\begin{aligned} Y_\eta[(f, \phi)] = {} & f^0\, \partial_0 - f^j\, \partial_j \\ & + \big(\mathfrak{i}\,(f^0 A_0 - f^j A_j + \breve{f})\, \delta^A_B - \tfrac{1}{2}\, \mathfrak{i}\,(f^0\, C^i_0 - f^j\, C^i_j + \phi^i)\, \sigma_i{}^A{}_B)) \\ & - \tfrac{1}{2}\, \operatorname{div}_\eta X[f]\, \delta^A_B\big)\, z^B\, \partial_A . \end{aligned}$$

Then, analogously to the scalar case, for each spin special phase function (f, ϕ), we obtain the *quantum operator*

$$Z_\eta[(f, \phi)] := \mathfrak{i}\,(Y_\eta[f, \phi] - f'' \lrcorner\, \mathrm{P}) : \sec(\boldsymbol{E}, \boldsymbol{U}) \to \mathbb{T}^* \otimes \boldsymbol{U} .$$

In particular, we have the following quantum operators associated with the spacetime coordinates x^λ, the components of the classical momentum $\mathcal{P}_i$, the classical hamiltonian $\mathcal{H}_0$ and the classical spin direction $n^\flat$ (see [40])

$$\begin{aligned} Z[x^\lambda, 0](\Psi) &= x^\lambda\, \Psi, \\ Z[\mathcal{P}_i, 0](\Psi) &= -\mathfrak{i}\,(\partial_i \psi - C_i{}^A{}_B\, \psi^B + \tfrac{1}{2}\, \frac{\partial_i \sqrt{|g|}}{\sqrt{|g|}}\, \psi^A)\, \zeta_A, \\ Z[\mathcal{H}_0, 0](\Psi) &= (-\tfrac{1}{2}\, \Delta_0[o]\psi^A - A_0\, \psi^A + \tfrac{1}{2}\, u_0\, \mu\, B^i\, \sigma_i{}^A{}_B\, \psi^B)\, \zeta_A, \\ Z[0, n^\flat](\Psi) &= \tfrac{1}{2}\, n^i\, \sigma_I{}^A{}_N\, \psi^B\, \zeta_A . \end{aligned}$$

Further, analogously to the scalar case, we can define a quantum Hilbert bundle over time and obtain self-adjoint quantum operators.

Chapter 31
Developments in Einsteinian General Relativity

This book is devoted to an introductory discussion on a covariant approach to Quantum Mechanics in the framework of a curved galilean spacetime. For this purpose we have developed some original geometric techniques and have applied them to the classical and quantum theories.

Then, a natural question arises whether these methods can be rephrased in the einsteinian general relativistic framework; the answer is positive. Here, we just summarise such developments and add a few comments or hints. Further details can be found in the original papers (see, for instance, [199, 201, 205–211, 213–215, 217, 218, 220, 222, 223, 229, 286, 409]; see, also, [187, 188]).

31.1 Einsteinian Spacetime

31.1.1 The Einsteinian Spacetime

As einsteinian general relativistic *spacetime* (see [213, 222]) we consider a 4-dimensional manifold $\boldsymbol{E}$ equipped with a *scaled lorentzian metric* with signature $(-+++)$

$$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}),$$

where $\mathbb{L}$ is the positive space of lengths (see Introduction: Sect. 1.3.5 and Appendix: Sect. K).

Analogously to the galilean case, we consider the *rescaled lorentzian metrics*

$$G := \tfrac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \qquad \text{and} \qquad \tilde{G} := \tfrac{1}{c^2}\, g : \boldsymbol{E} \to \mathbb{T}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}).$$

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
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Moreover, we define the *unscaled lorentzian metric*

$$\mathrm{G} := \tfrac{m^2 c^2}{\hbar^2}\, g : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}.$$

The corresponding contravariant metrics are denoted by $\bar{g}$, $\bar{G}$, $\bar{\tilde{G}}$, $\bar{\mathrm{G}}$.

We suppose spacetime to be oriented and time-oriented. We deal with *local unscaled spacetime charts* $(x^0, x^i) : \boldsymbol{E} \to \mathbb{R} \times \mathbb{R}^3$ adapted to the lorentzian structure.

The lorentzian metric g and the orientation of spacetime yield the dual scaled *volume form* and *volume vector*

$$\upsilon : \boldsymbol{E} \to \mathbb{L}^4 \otimes \Lambda^4 T^*\boldsymbol{E} \qquad \text{and} \qquad \bar{\upsilon} : \boldsymbol{E} \to \mathbb{L}^{-4} \otimes \Lambda^4 T\boldsymbol{E},$$

with coordinate expressions

$$\upsilon = \sqrt{|g|}\, d^0 \wedge d^1 \wedge d^2 \wedge d^3 \qquad \text{and} \qquad \bar{\upsilon} = \frac{1}{\sqrt{|g|}}\, \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3.$$

By the way, the rescaled spacetime volumes

$$\tfrac{1}{c}\, \upsilon : \boldsymbol{E} \to (\mathbb{T} \otimes \mathbb{L}^3) \otimes \Lambda^4 T^*\boldsymbol{E} \qquad \text{and} \qquad c\, \bar{\upsilon} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{-3}) \otimes \Lambda^4 T\boldsymbol{E}$$

have the same scaling of the spacetime volumes in the galilean case.

31.1.2 Gravitational Connection

The lorentzian metric g yields the *Levi–Civita connection*

$$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E},$$

with coordinate expression

$$K^\natural = d^\mu \otimes (\partial_\mu + K^\natural{}_\mu{}^\lambda{}_\nu \dot{x}^\nu \dot{\partial}_\lambda), \text{ where } K^\natural{}_\mu{}^\lambda{}_\nu = -\tfrac{1}{2}\, g^{\mu\alpha}\, (\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu}).$$

We regard $K^\natural$ as the *spacetime gravitational connection*.

In the galilean case, the metric g is only spacelike and for this reason we are dealing with the gauge dependent and observer dependent gravitational potential $A^\natural$. In the einsteinian case, where the metric g is 4-dimensional, we might regard $(g_{00,} g_{0i})$ as the analogue of the above galilean potential $(A^\natural{}_0, A^\natural{}_i)$.

31.1.3 Motions

A *motion* is defined to be a 1-dimensional timelike submanifold (see [222])

$$s : \boldsymbol{T}_s \hookrightarrow \boldsymbol{E}.$$

Accordingly, the tangent space of every motion s can be *locally* regarded as a product

$$Ts = \boldsymbol{T}_s \times (\mathbb{T} \otimes \mathbb{R}),$$

where $\mathbb{T}$ is the *positive space* which represents the intervals of *proper time* (see Introduction: Sect. 1.3.5 and Appendix: Sect. K).

Indeed, the space $\mathbb{T} \times \mathbb{R}$ is the same for all motions (at least locally); in fact, the lorentzian metric naturally yields such an identity. We stress that this fact (which deals only with time intervals) has nothing to do with synchronisation of clocks, hence it does not conflict with relativity of time.

Thus, in the galilean case, motions are represented by sections of the spacetime fibred manifold, while, in the einsteinian case, they are represented by suitable 1-dimensional submanifolds of spacetime.

31.2 Einsteinian Phase Space

31.2.1 The Einsteinian Phase Space

We consider the *k-jet bundles* of 1-dimensional submanifolds $\boldsymbol{N} \subset \boldsymbol{E}$ (see, for instance, [6, 315])

$$J_k(\boldsymbol{E}, 1) \to \boldsymbol{E}$$

and the *k-jet bundles* of *timelike* 1-dimensional submanifolds $s \subset \boldsymbol{E}$

$$\mathrm{J}_k(\boldsymbol{E}, 1) \subset J_k(\boldsymbol{E}, 1) \to \boldsymbol{E}.$$

In particular, we define the *phase space* to be the 1-jet subbundle of timelike 1-dimensional submanifolds (i.e. the 1-jet bundle of motions) (see [213, 222])

$$U_1\boldsymbol{E} \equiv \mathrm{J}_1(\boldsymbol{E}, 1) \to \boldsymbol{E}.$$

Thus, the manifold $U_1\boldsymbol{E}$ is 7-dimensional analogously to the galilean case.

We stress that, unlike the galilean case, this bundle is not affine; this fact yields a great difference between the galilean and the einsteinian cases.

Each spacetime chart (x^0, x^i) yields naturally a fibred chart (x^0, x^i, x^i_0) of $U_1\boldsymbol{E}$.

31.2.2 Contact Map and Contact Form

The lorentzian metric g yields naturally the *contact map* (see [222])

$$\text{д} : U_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E},$$

characterised by the following identity, for each motion s,

$$\text{д} \circ j_1 s = ds.$$

We have the coordinate expression

$$\text{д} = c\,\alpha^0\,\text{д}_0 = c\,\alpha^0\,(\partial_0 + x_0^i\,\partial_i),$$

where

$$\alpha^0 := 1/\|\text{д}_0\| = 1/\sqrt{|g_{00} + 2\,g_{0j}\,x_0^j + g_{ij}\,x_0^i\,x_0^j|} \in \operatorname{map}(U_1\boldsymbol{E},\ \mathbb{L}^*).$$

We have the identity

$$g \circ (\text{д}, \text{д}) = -c^2.$$

This map is the analogue of the contact map $\text{д} : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ in the galilean case. We stress that in the galilean case the contact map is induced by the fibring over spacetime, while in the einsteinian case it is induced by the lorentzian metric.

We define the *contact form* to be the scaled form

$$\tau := -\,\tfrac{1}{c^2}\,g^\flat \circ \text{д} : U_1\boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E},$$

with coordinate expression

$$\tau = \tau_\lambda\,d^\lambda = -\tfrac{1}{c}\,\alpha^0\,(g_{0\lambda} + g_{i\lambda}\,x_0^i)\,d^\lambda.$$

We have the identity

$$\text{д} \lrcorner \tau = c\,\alpha^0\,(\tau_0 + \tau_i\,x_0^i) = 1.$$

The form τ is the analogue of the galilean form dt and the identity $\text{д} \lrcorner \tau = 1$ resembles the galilean identity $\text{д} \lrcorner dt = 1$. However, we stress that dt is generated by the time fibring of spacetime and lives on spacetime, while τ is generated by the lorentzian metric and lives on the phase space. Moreover, dt is closed and globally exact, while τ is not closed.

We define the *complementary contact map* to be the linear fibred morphism over $U_1\boldsymbol{E}$

$$\theta := 1 - \text{д} \otimes \tau : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{E},$$

with coordinate expression

$$\theta = d^\lambda \otimes \partial_\lambda + (\alpha^0)^2 \, (g_{0\lambda} + g_{i\lambda} \, x_0^i) \, d^\lambda \otimes (\partial_0 + x_0^j \, \partial_j).$$

We have the identities

$$д \lrcorner \theta = 0 \qquad \text{and} \qquad \tau \lrcorner \theta = 0.$$

With reference to a particle of mass m and by taking into account the Planck constant $\hbar$, we obtain the *unscaled phase 1-form*

$$\hat{\tau} := \tfrac{m}{\hbar} \, c^2 \, \tau : U_1\boldsymbol{E} \to T^*\boldsymbol{E},$$

with coordinate expression

$$\hat{\tau} = -c_0 \, \alpha^0 \, (G^0_{\lambda 0} + G^0_{\lambda j} \, x_0^j) \, d^\lambda.$$

31.2.3 Orthogonal Projection

The contact map д and the contact form τ naturally yield a linear splitting of the tangent space of spacetime into timelike and spacelike components *over the phase space* (see [222])

$$U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} = H_д\boldsymbol{E} \underset{U_1\boldsymbol{E}}{\oplus} V_\tau\boldsymbol{E} \qquad \text{and} \qquad U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} = H^*_\tau\boldsymbol{E} \underset{U_1\boldsymbol{E}}{\oplus} V^*_д\boldsymbol{E},$$

where

$$\begin{aligned} H_д\boldsymbol{E} &\subset T\boldsymbol{E} && \text{is the subspace generated by } д, \\ V_\tau\boldsymbol{E} &\subset T\boldsymbol{E} && \text{is the subspace annihilated by } \tau, \\ H^*_\tau\boldsymbol{E} &\subset T^*\boldsymbol{E} && \text{is the subspace generated by } \tau, \\ V^*_д\boldsymbol{E} &\subset T^*\boldsymbol{E} && \text{is the subspace annihilated by } д. \end{aligned}$$

The corresponding projections are

$$\begin{aligned} \pi^{\parallel} &= \tau \otimes д && : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to H_д\boldsymbol{E} && \text{and} && \pi_{\parallel} = д \otimes \tau && : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \to H^*_\tau\boldsymbol{E}, \\ \pi^{\perp} &= \theta && : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to V_\tau\boldsymbol{E} && \text{and} && \pi_{\perp} = \theta^* && : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T^*\boldsymbol{E} \to V^*_д\boldsymbol{E}. \end{aligned}$$

The dual bases adapted to the above splittings are $(b_0, \; b_i)$ and $(\beta^0, \; \beta^i)$, where

$$b_0 := \partial_0 + x_0^i\,\partial_i \quad \in \sec(U_1\boldsymbol{E},\ H_{\text{д}}\boldsymbol{E}),$$
$$b_i := \partial_i - c\,\alpha^0\,\tau_i\,(\partial_0 + x_0^j\,\partial_j) \quad \in \sec(U_1\boldsymbol{E},\ V_\tau\boldsymbol{E}),$$
$$\beta^0 := d^0 + c\,\alpha^0\,\tau_i\,(d^i - x_0^i\,d^0) \quad \in \sec(U_1\boldsymbol{E},\ H^*_\tau\boldsymbol{E}),$$
$$\beta^i := d^i - x_0^i\,d^0 \quad \in \sec(U_1\boldsymbol{E},\ V^*_{\text{д}}\boldsymbol{E}).$$

This splitting resembles the galilean splitting into horizontal and vertical components of the tangent space over spacetime.

The restriction of the lorentzian metric to the spacelike component yields a scaled metric with coordinate expression

$$g^{\perp}{}_{ij} = g_{ij} + c^2\,\tau_i\,\tau_j \qquad \text{and} \qquad g_{\perp}^{ij} = g^{ij} - g^{i0}\,x_0^j - g^{j0}\,x_0^i + g^{00}\,x_0^i\,x_0^j.$$

We have the equalities

$$\breve{g}_{0\lambda} := g\,(b_0, \partial_\lambda) = g_{\rho\lambda}\,\breve{\delta}_0^\rho, \qquad \breve{g}^{0\lambda} := \bar{g}\,(\beta^0, d^\lambda) = -(\alpha^0)^2\,\breve{\delta}_0^\lambda,$$
$$\breve{g}_{i\lambda} := g(b_i,\ \partial_\lambda) = g_{i\lambda} + (\alpha^0)^2\,\breve{g}_{0i}\,\breve{g}_{0\lambda}, \qquad \breve{g}^{i\lambda} := \bar{g}(\beta^i,\ d^\lambda) = \breve{\delta}^i_\rho\,g^{\rho\lambda},$$

where we have set

$$\breve{\delta}_0^\lambda = \delta_0^\lambda + \delta_i^\lambda\,x_0^i, \qquad \breve{\delta}^i_\mu = \delta^i_\mu - \delta^0_\mu\,x_0^i.$$

The contact map and the contact form yield the dual scaled spacelike volume form and spacelike volume vector

$$\eta := i_{\text{д}}\upsilon : U_1\boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3\,V^*_{\text{д}}\boldsymbol{E} \qquad \text{and} \qquad \bar{\eta} := i_\tau\bar{\upsilon} : U_1\boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3\,V_\tau\boldsymbol{E}.$$

31.2.4 Vertical Space of the Phase Space

The vertical prolongation of the contact map yields the mutually inverse linear fibred isomorphisms over $U_1\boldsymbol{E}$ (see [222])

$$\nu_\tau : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V_\tau\boldsymbol{E}) \to VU_1\boldsymbol{E} \qquad \text{and} \qquad \nu_\tau^{-1} : U_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V_{\text{д}}\boldsymbol{E}) \to V^*U_1\boldsymbol{E},$$

with coordinate expressions

$$\nu_\tau = \frac{1}{c_0\,\alpha^0}\,u_0 \otimes \beta^i \otimes \partial_i^0 \qquad \text{and} \qquad \nu_\tau^{-1} = c_0\,\alpha^0\,u^0 \otimes d_0^i \otimes b_i.$$

The above representation of the vertical tangent space of the phase space in the einsteinian case is analogue to the identity $V J_1\boldsymbol{E} = J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} (\mathbb{T}^* \otimes V\boldsymbol{E})$ in the galilean case.

31.2.5 Observers

An *observer* is defined to be a section $o : \boldsymbol{E} \to U_1\boldsymbol{E}$ (see [222]).

The definitions of observer are formally identical in the galilean and einsteinian cases; however, great differences arise from the different structure of the phase space in the two cases.

Every observer o yields the observed *contact map*, *contact form* and *complementary contact map*

$$д[o] := д \circ o : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}, \quad \tau[o] := \tau \circ o : \boldsymbol{E} \to \mathbb{T} \otimes T^*\boldsymbol{E}, \quad \theta[o] := \theta \circ o : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T\boldsymbol{E},$$

with coordinate expressions

$$\begin{aligned} д[o] &= c\,\alpha^0[o]\,(\partial_0 + o_0^i\,\partial_i), \\ \tau[o] &= -\tfrac{1}{c}\,\alpha^0[o]\,(g_{0\lambda} + g_{i\lambda}\,o_0^i)\,d^\lambda, \\ \theta[o] &= d^\lambda \otimes \partial_\lambda + (\alpha^0)^2[o]\,(g_{0\lambda} + g_{i\lambda}\,o_0^i)\,d^\lambda \otimes (\partial_0 + o_0^j\,\partial_j), \end{aligned}$$

where

$$\alpha^0[o] := 1/\sqrt{|g_{00} + 2\,g_{0j}\,o_0^j + g_{ij}\,o_0^i\,o_0^j|} \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^*).$$

Accordingly, we obtain a linear splitting of the tangent space of spacetime into timelike and spacelike components

$$T\boldsymbol{E} = H_{д[o]}\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} V_{\tau[o]}\boldsymbol{E} \qquad \text{and} \qquad T^*\boldsymbol{E} = H^*_{\tau[o]}\boldsymbol{E} \underset{\boldsymbol{E}}{\oplus} V^*_{д[o]}\boldsymbol{E},$$

where

$$\begin{aligned} H_{д[o]}\boldsymbol{E} &\subset T\boldsymbol{E} && \text{is the subspace generated by} \quad д[o], \\ V_{\tau[o]}\boldsymbol{E} &\subset T\boldsymbol{E} && \text{is the subspace annihilated by} \quad \tau[o], \\ H^*_{\tau[o]}\boldsymbol{E} &\subset T^*\boldsymbol{E} && \text{is the subspace generated by} \quad \tau[o], \\ V^*_{д[o]}\boldsymbol{E} &\subset T^*\boldsymbol{E} && \text{is the subspace annihilated by} \quad д[o]. \end{aligned}$$

The corresponding projections are

$$\begin{aligned} &\pi^{\parallel}[o] = \tau[o] \otimes д[o] \quad : T\boldsymbol{E} \to H_{д[o]}\boldsymbol{E} \quad \text{and} \qquad \pi_{\parallel}[o] = д[o] \otimes \tau[o] \quad : T^*\boldsymbol{E} \to H^*_{\tau[o]}\boldsymbol{E}, \\ &\pi^{\perp}[o] = \theta[o] \quad : T\boldsymbol{E} \to V_{\tau[o]}\boldsymbol{E} \qquad \text{and} \qquad \pi_{\perp}[o] = \theta[o]^* \quad : T^*\boldsymbol{E} \to V^*_{д[o]}\boldsymbol{E}. \end{aligned}$$

The dual bases adapted to the above splittings are $(b_0[o], b_i[o])$ and $(\beta^0[o], \ \beta^i[o])$, where

$$b_0[o] := \partial_0 + o_0^i\, \partial_i \quad \in \sec(\boldsymbol{E},\ H_{\text{д}[o]}\boldsymbol{E}),$$
$$b_i[o] := \partial_i - c\,\alpha^0\, \tau[o]_i\, (\partial_0 + o_0^j\, \partial_j) \quad \in \sec(\boldsymbol{E},\ V_{\tau[o]}\boldsymbol{E}),$$
$$\beta^0[o] := d^0 + c\,\alpha^0[o]\, \tau_i[o]\, (d^i - o_0^i\, d^0) \quad \in \sec(\boldsymbol{E},\ H^*_{\tau[o]}\boldsymbol{E}),$$
$$\beta^i[o] := d^i - o_0^i\, d^0 \quad \in \sec(\boldsymbol{E},\ V^*_{\text{д}[o]}\boldsymbol{E}).$$

We define the spacetime charts (x^0, x^i) *adapted* to an observer o by the condition

$$\tfrac{1}{c\,\alpha^0[o]}\, \text{д}[o] = \partial_0.$$

In other words, a spacetime chart is adapted to o if and only if $o_0^i = 0$.

Hence, all formulas above can be simplified by referring to adapted chart.

For each motion s, we have obtain the splitting into the observed timelike and spacelike components (see [213])

$$\text{д} \circ j_1 s = \delta\, (\text{д} \circ o_{|s} + \vec{\beta}),$$

where

$$\delta = \frac{c}{\sqrt{c^2 - \|\vec{\beta}\|^2}} : s \to \mathbb{R} \quad \text{and} \quad \vec{\beta} : s \to \mathbb{T}^* \otimes T^{\perp}\boldsymbol{E}.$$

Indeed, we have

$$\delta > 1 \quad \text{and} \quad \|\vec{\beta}\| < c.$$

31.2.6 Observed Spacelike Volume

Every observer o yields the dual observed scaled *spacelike volume form* and *spacelike volume vector*

$$\eta[o] := \tfrac{1}{c}\, i_{\text{д}[o]}\upsilon : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3\, V^*_{\text{д}[o]}\boldsymbol{E}, \quad \text{and} \quad \bar{\eta}[o] := c\, i_{\tau[o]}\bar{\upsilon} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3\, V_{\tau[o]}\boldsymbol{E}.$$

31.3 Phase Objects

31.3.1 Dynamical Phase Objects

(1) We define a *phase connection* to be a connection of the phase space (see [213])

$$\Gamma : U_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TU_1\boldsymbol{E},$$

with coordinate expression

$$\Gamma = d^\lambda \otimes (\partial_\lambda + \Gamma_\lambda{}^i{}_0\, \partial_i^0), \quad \text{where } \Gamma_\lambda{}^i{}_0 \in \text{map}(U_1\boldsymbol{E}, \mathbb{R}).$$

The associate projection is the fibred morphism over $U_1\boldsymbol{E}$

$$\nu_\tau[\Gamma] : U_1\boldsymbol{E} \to T^*U_1\boldsymbol{E} \otimes (\mathbb{T}^* \otimes V_\tau\boldsymbol{E}),$$

with coordinate expression

$$\nu_\tau[\Gamma] = c\,\alpha^0\,(d_0^i - \Gamma_\lambda{}^i{}_0\, d^\lambda) \otimes b_i, \quad \text{with } \Gamma_\lambda{}^i{}_0 \in \text{map}(U_1\boldsymbol{E},\ \mathbb{R}).$$

(2) We define a *dynamical phase connection* to be a scaled vector field

$$\gamma : U_1\boldsymbol{E} \to \mathbb{T}^* \otimes TU_1\boldsymbol{E},$$

which projects on д. It can be regarded as a *2nd order connection*, i.e. as a section

$$\gamma : U_1\boldsymbol{E} \to U_2\boldsymbol{E}.$$

We have the coordinate expression

$$\gamma = c\,\alpha^0\,(\partial_0 + x_0^i\,\partial_i + \gamma_0{}^i{}_0\,\partial_i^0), \quad \text{where } \gamma_0{}^i{}_0 \in \text{map}(U_1\boldsymbol{E}, \mathbb{R}).$$

With reference to a particle of mass m and by taking into account the Planck constant $\hbar$, we obtain the unscaled dynamical phase connection

$$\hat{\gamma} := \tfrac{\hbar}{m\,c^2}\,\gamma : U_1\boldsymbol{E} \to TU_1\boldsymbol{E},$$

with coordinate expression

$$\hat{\gamma} = \tfrac{\hbar}{m\,c}\,\alpha^0\,(\partial_0 + x_0^i\,\partial_i + \gamma_0{}^i{}_0\,\partial_i^0).$$

We still have the identities

$$\gamma \lrcorner \tau = \hat{\gamma} \lrcorner \hat{\tau} = 1.$$

(3) Each phase connection Γ yields the associated

dynamical phase connection	$\gamma \equiv \gamma[\Gamma] \quad := д \lrcorner \Gamma : U_1\boldsymbol{E} \to \mathbb{T}^* \otimes TU_1\boldsymbol{E},$
dynamical phase 2-form	$\Omega \equiv \Omega[G,\Gamma] \quad := G \lrcorner \big(\nu_\tau[\Gamma] \wedge \theta\big) : U_1 \to \Lambda^2 T^*U_1\boldsymbol{E},$
dynamical phase 2-vector	$\Lambda \equiv \Lambda[G,\Gamma] \quad := \bar{G} \lrcorner (\Gamma \wedge \nu_\tau) : U_1\boldsymbol{E} \to \Lambda^2 TU_1\boldsymbol{E},$

with coordinate expressions

$$\gamma = c\,\alpha^0 \left(\partial_0 + x_0^i\,\partial_i + (\Gamma_{0}{}^{i}_{0} + \Gamma_{j}{}^{i}_{0}\,x_0^j)\,\partial_i^0\right),$$
$$\Omega = c_0\,\alpha^0\,\breve{G}^0_{i\mu}\,(d_0^i - \Gamma_{\lambda}{}^{i}_{0}\,d^\lambda) \wedge d^\mu,$$
$$\Lambda = 1/(c_0\,\alpha^0)\,\breve{G}_0^{j\lambda}\,(\partial_\lambda + \Gamma_{\lambda}{}^{i}_{0}\,\partial_i^0) \wedge \partial_j^0.$$

The sections

$$-\hat{\tau} \wedge \Omega \wedge \Omega \wedge \Omega : U_1\boldsymbol{E} \to \Lambda^7\,T^*U_1\boldsymbol{E} \quad \text{and} \quad -\hat{\gamma} \wedge \Lambda \wedge \Lambda \wedge \Lambda : U_1\boldsymbol{E} \to \Lambda^7\,TU_1\boldsymbol{E}$$

with coordinate expressions

$$-\hat{\tau} \wedge \Omega \wedge \Omega \wedge \Omega = 3!\,(c_0\,\alpha^0)^4\,|G^0_{\lambda\mu}|\,d_0^1 \wedge d_0^2 \wedge d_0^3 \wedge d^0 \wedge d^1 \wedge d^2 \wedge d^3,$$
$$-\hat{\gamma} \wedge \Lambda \wedge \Lambda \wedge \Lambda = -3!\,\frac{1}{(c_0\,\alpha^0)^4}\,|G_0^{\lambda\mu}|\,\partial_1^0 \wedge \partial_2^0 \wedge \partial_3^0 \wedge \partial_0 \wedge \partial_1 \wedge \partial_2 \wedge \partial_3$$

turn out to be an *unscaled volume form* and an *unscaled volume vector* of the phase space.

Hence, $(-\hat{\tau}, \Omega)$ and $(-\hat{\gamma}, \Lambda)$ turn out to be a *"regular covariant pair"* and a *"regular contravariant pair"* of the phase space (see [222]).

The above relations resemble analogue relations in the galilean case, but the degrees of polynomial expressions are different in the two cases.

31.3.2 Gravitational Phase Objects

There is a natural map (see [220])

$$\chi : K \mapsto \Gamma,$$

which maps linear spacetime connections K into phase connections Γ. Its coordinate expression is

$$\Gamma_{\lambda}{}^{i}_{0} = K_{\lambda}{}^{i}{}_{0} + K_{\lambda}{}^{i}{}_{j}\,x_0^j - x_0^i\,(K_{\lambda}{}^{0}{}_{0} + K_{\lambda}{}^{0}{}_{j}\,x_0^j).$$

We recall that an analogous natural map χ holds in the galilean case. However, we stress that in the galilean case this map is invertible, while in the einsteinian case it is not.

Then, the lorentzian metric g naturally yields the *gravitational phase connection* (see [220])

$$\Gamma^\natural := \chi(K^\natural],$$

with coordinate expression

$$\Gamma^\natural{}_{\lambda}{}^{i}_{0} = K^\natural{}_{\lambda}{}^{i}{}_{0} + K^\natural{}_{\lambda}{}^{i}{}_{j}\,x_0^j - x_0^i\,(K^\natural{}_{\lambda}{}^{0}{}_{0} + K^\natural{}_{\lambda}{}^{0}{}_{j}\,x_0^j).$$

Further, the above gravitational phase connection $\Gamma^{\natural}$ naturally yields the *associated gravitational phase objects*

$$\begin{aligned} &gravitational\,dynamical\,phase\,connection && \gamma^{\natural} \equiv \gamma[\Gamma^{\natural}], \\ &gravitational\,dynamical\,phase\,2\text{-}form && \Omega^{\natural} \equiv \Omega[G, \Gamma^{\natural}], \\ &gravitational\,dynamical\,phase\,2\text{-}vector && \Lambda^{\natural} \equiv \Lambda[G, \Gamma^{\natural}], \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \gamma^{\natural} &= c\,\alpha^0 \left(\partial_0 + x_0^i\,\partial_i + (\Gamma^{\natural}{}_{0}{}^{i}_{0} + \Gamma^{\natural}{}_{j}{}^{i}_{0}\,x_0^j)\,\partial_i^0\right), \\ \Omega^{\natural} &= c\,\alpha^0\,\breve{G}^0_{i\mu}\,(d_0^i - \Gamma^{\natural}{}_{\lambda}{}^{i}_{0}\,d^{\lambda}) \wedge d^{\mu}, \\ \Lambda^{\natural} &= \tfrac{1}{c_0\,\alpha^0}\,\breve{G}_0^{j\lambda}\,(\partial_{\lambda} + \Gamma^{\natural}{}_{\lambda}{}^{i}_{0}\,\partial_i^0) \wedge \partial_j^0. \end{aligned}$$

The above gravitational phase objects resemble the corresponding objects in the galilean case, but the polynomial character of these formulas is quite different in the two cases.

We have the identities

$$\begin{aligned} \Omega^{\natural} = -d\hat{\tau}, \qquad [\Lambda^{\natural}, \Lambda^{\natural}] = -2\,\hat{\gamma}^{\natural} \wedge \Lambda^{\natural}, \qquad [\Lambda^{\natural}, \Lambda^{\natural}] = 0, \\ i_{\hat{\gamma}^{\natural}}\hat{\tau} = 1, \qquad i_{\hat{\gamma}^{\natural}}\Omega^{\natural} = 0, \qquad i_{\hat{\tau}}\Lambda^{\natural} = 0, \\ (\Omega^{\natural\flat}{}_{|\mathrm{im}\,\Lambda^{\natural\sharp}})^{-1} = \Lambda^{\natural\sharp}{}_{|\mathrm{im}\,\Omega^{\natural\flat}}, \qquad (\Lambda^{\natural\sharp}{}_{|\mathrm{im}\,\Omega^{\natural\flat}})^{-1} = \Omega^{\natural\flat}{}_{|\mathrm{im}\,\Lambda^{\natural\sharp}}, \end{aligned}$$

where [,] denotes the Schouten bracket.

Hence, $(-\hat{\tau}, \Omega^{\natural})$ and $(-\hat{\gamma}^{\natural}, \Lambda^{\natural})$ turn out to be dual *"contact structure"* and *"Jacobi structure"* of the phase space (see [222]).

We stress that, in comparison with the galilean case, the above contact structure replaces the cosymplectic structure and the above Jacobi structure replaces the co-Poisson structure.

31.4 Electromagnetic Field

Next, let us consider the scaled *electromagnetic 2-form* (see [213])

$$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2\,T^*\boldsymbol{E}.$$

With reference to an observer o, we define the *observed magnetic field* and the *observed electric field*,

$$\begin{aligned} \vec{B}[o] &:= \tfrac{1}{2}\,c\,(\theta^*[o]F) \lrcorner\, \bar{\eta}[o] && : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V_{\tau}\boldsymbol{E}, \\ \vec{E}[o] &:= -\,g^{\sharp}(д[o] \lrcorner\, F) && : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V_{\tau}\boldsymbol{E}. \end{aligned}$$

Accordingly, we obtain the observed splitting,

$$F = -2\,\tau[o] \wedge g^{\flat}(\vec{E}[o]) + \tfrac{2}{c}\; i_{\vec{B}[o]}\eta[o].$$

The *Lorentz force*, is defined the scaled 1-form

$$\mathsf{f} := -\tfrac{1}{2}\,q\,i_{д}F : U_1\boldsymbol{E} \to (\mathbb{T}^{-2}\otimes\mathbb{L}^{2}\otimes\mathbb{M})\otimes T^{*}\boldsymbol{E},$$

whose expression in terms of the observed magnetic and observed electric fields is

$$\mathsf{f} = q\,\big(-g\,(д,\,\vec{E}[o])\,\tau[o] + g^{\flat}(\vec{E}[o])\,i_{д}\tau[o] + \tfrac{1}{c}\,g^{\flat}(д \times_o \vec{B}[o])\big),$$

where $\times_o$ denotes the observed "cross product".

Then, the observed Lorentz force effecting a charged particle is obtained by composition with the velocity $j_1 s : \boldsymbol{T} \to U_1\boldsymbol{E}$ of its motion.

The 2-form F is essentially the same of the galilean case. However, the observed magnetic field and electric field are defined in an analogous way in the galilean and einsteinian cases, but differences arise from the different features of observers in the two cases.

We suppose that F fulfills the *1st Maxwell equation* and the *2nd Maxwell equation*

$$dF = 0 \qquad \text{and} \qquad \bar{g} \lrcorner \nabla^{\natural}F = J,$$

where

$$J := \tfrac{1}{c^2}\,\rho\,g^{\flat}\,(д \circ \mathscr{V}) : \boldsymbol{E} \to (\mathbb{L}^{-3/2}\otimes\mathbb{M}^{1/2})\otimes T^{*}\boldsymbol{E}$$

is the charge scaled current 1-form associated with the charge density

$$\rho : \boldsymbol{E} \to (\mathbb{T}^{-1}\otimes\mathbb{L}^{-3/2}\otimes\mathbb{M}^{1/2})\otimes\mathbb{R}$$

and the velocity field of the source charged fluid

$$\mathscr{V} : \boldsymbol{E} \to U_1\boldsymbol{E}.$$

Here, the scale factors of the above objects have been chosen in such a way to yield scales analogue to the galilean case.

31.5 Joined Phase Objects

Analogously to the galilean case, a minimal coupling of the gravitational objects and the electromagnetic field yields the "*joined phase objects*" (see [220])

$$\begin{aligned}
&\textit{joined phase connection} && \Gamma := \Gamma^{\natural} + \Gamma^{\mathfrak{e}}, \\
&\textit{joined dynamical phase connection} && \gamma := \gamma^{\natural} + \gamma^{\mathfrak{e}}, \\
&\textit{joined dynamical phase 2-form} && \Omega := \Omega^{\natural} + \Omega^{\mathfrak{e}}, \\
&\textit{joined dynamical phase 2-vector} && \Lambda := \Lambda^{\natural} + \Lambda^{\mathfrak{e}},
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^{\mathfrak{e}} &= -\tfrac{1}{2}\,\tfrac{q}{m}\,(\nu_\tau \circ g^{\sharp 2})\,\big(F + 2\,\tau \wedge (\text{д} \lrcorner F)\big), \\
\gamma^{\mathfrak{e}} &= -\tfrac{q}{m}\,(\nu_\tau \circ g^{\sharp 2}) \circ (\text{д} \lrcorner F), \\
\Omega^{\mathfrak{e}} &= \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F, \\
\Lambda^{\mathfrak{e}} &= \tfrac{1}{2}\,\text{Alt}\,\big((\nu_\tau \lrcorner G^{\sharp}) \otimes (\nu_\tau \lrcorner G^{\sharp})\big)(F),
\end{aligned}$$

with coordinate expressions

$$\begin{aligned}
\Gamma^{\mathfrak{e}} &= -\tfrac{1}{2}\,\tfrac{q}{m}\,\tfrac{1}{c\,\alpha^0}\,\breve{g}^{i\rho}\,\Big(F_{\lambda\rho} - (\alpha^0)^2\,\breve{g}_{0\lambda}\,F_{\sigma\rho}\,\breve{\delta}^{\sigma}_0\Big)\,d^{\lambda} \otimes \partial^0_i, \\
\gamma^{\mathfrak{e}} &= -\tfrac{q}{m}\,\breve{g}^{i\mu}\,(F_{0\mu} + F_{j\mu}\,x^j_0)\,u^0 \otimes \partial^0_i, \\
\Omega^{\mathfrak{e}} &= \tfrac{1}{2}\,\tfrac{q}{\hbar}\,F_{\lambda\mu}\,d^{\lambda} \wedge d^{\mu}, \\
\Lambda^{\mathfrak{e}} &= \tfrac{1}{2}\,\tfrac{q}{\hbar}\,\tfrac{1}{(c\,\alpha^0)^2}\,\breve{G}^{j\lambda}\,\breve{G}^{i\rho}\,F_{\lambda\rho}\,\partial^0_i \wedge \partial^0_j.
\end{aligned}$$

We have the identities

$$-d\hat{\tau} = \Omega^{\natural}, \qquad i_{\hat{\gamma}}\hat{\tau} = 1, \qquad i_{\hat{\gamma}}\Omega = 0, \qquad d\Omega = 0,$$
$$\hat{\tau} \wedge \Omega \wedge \Omega \wedge \Omega = \hat{\tau} \wedge \Omega^{\natural} \wedge \Omega^{\natural} \wedge \Omega^{\natural} \not\equiv 0, \qquad \hat{\gamma} \wedge \Lambda \wedge \Lambda \wedge \Lambda = \hat{\gamma} \wedge \Lambda^{\natural} \wedge \Lambda^{\natural} \wedge \Lambda^{\natural} \not\equiv 0.$$

Hence, the pair $(-\hat{\tau}, \Omega)$ is an almost-cosymplectic contact structure of $U_1\boldsymbol{E}$ (see [223]). Moreover, $\hat{\gamma}$ turns out to be the unique 2nd order connection which fulfills the conditions $i_{\hat{\gamma}}\hat{\tau} = 1$ and $i_{\hat{\gamma}}\Omega = 0$.

The standard law of motion for a charged particle affected by the electromagnetic field can be written in joined form as

$$\nabla[\gamma]j_1 s = 0, \quad \text{i.e.} \quad \nabla[\gamma^{\natural}]j_1 s = -\gamma^{\mathfrak{e}} \circ j_1 s.$$

In other words, $\gamma^{\mathfrak{e}}$ turns out to be just the Lorentz force.

The cosymplectic phase 2-form Ω admits a *horizontal potential*

$$A^{\uparrow} \in \sec(U_1\boldsymbol{E}, T^*\boldsymbol{E}), \quad \text{such that } dA^{\uparrow} = \Omega,$$

which is locally defined up to a gauge of the type $\phi \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$.

We have the equality

$$A^{\uparrow} = -\hat{\tau} + A^{\mathfrak{e}},$$

where $A^{\mathfrak{e}}$ is the potential of $\frac{q}{\hbar} F$.

We remark that in the galilean case both gravitational and electromagnetic potentials are gauge dependent, while in the einsteinian case the gravitational potential $\hat{\tau}^{\natural}$ is gauge independent and observer independent.

31.6 Dynamical 1-Forms

Analogously to the galilean case, we define the following distinguished dynamical 1-forms, by deriving them from the upper potential $A^{\uparrow}$, via the contact structure of the phase space (see [222, 286]). So, we define the

lagrangian	$\mathcal{L} := (д \lrcorner A^{\uparrow})\, \tau : U_1\boldsymbol{E} \to H^*_\tau \boldsymbol{E}$,
momentum	$\mathcal{M} := A^{\uparrow} - (д \lrcorner A^{\uparrow})\, \tau : U_1\boldsymbol{E} \to V^*_д \boldsymbol{E}$,
observed hamiltonian	$\mathcal{H}[o] := -\,(д[o] \lrcorner A^{\uparrow})\, \tau[o] : U_1\boldsymbol{E} \to H^*_{\tau[o]} \boldsymbol{E}$,
observed momentum	$\mathcal{P}[o] := A^{\uparrow} - (д[o] \lrcorner A^{\uparrow})\, \tau[o] : U_1\boldsymbol{E} \to V^*_{д[o]} \boldsymbol{E}$,

with coordinate expressions

$$\mathcal{L} = -\hat{\tau}_\lambda\, d^\lambda + c\,\alpha^0\, ({A^{\mathfrak{e}}}_0 + {A^{\mathfrak{e}}}_i\, x^i_0)\, \tau_\lambda\, d^\lambda,$$
$$\mathcal{M} = \big({A^{\mathfrak{e}}}_\lambda - c\,\alpha^0\, ({A^{\mathfrak{e}}}_0 + {A^{\mathfrak{e}}}_i\, x^i_0)\, \tau_\lambda\big)\, d^\lambda,$$

and, in an adapted spacetime chart,

$$\mathcal{H}[o] = -(c_0\,\alpha^0\, \breve{G}^0_{00} + {A^{\mathfrak{e}}}_0)\, d^0,$$
$$\mathcal{P}[o] = (c_0\,\alpha^0\, \breve{G}^0_{0i} + {A^{\mathfrak{e}}}_i)\, d^i.$$

31.7 Hamiltonian Lift

For every time scale $\sigma \in \mathrm{map}(U_1\boldsymbol{E}, \bar{\mathbb{T}})$, we define the σ-hamiltonian lift

$$X^{\uparrow}{}_{\mathrm{ham}}[\sigma] : \mathrm{map}(U_1\boldsymbol{E}, \mathbb{R}) \to \sec(U_1\boldsymbol{E}, TU_1\boldsymbol{E}) : f \mapsto X^{\uparrow}[\sigma, f] := \sigma \lrcorner \gamma + \Lambda^{\sharp}{}_0(df),$$

with coordinate expression

$$X^{\uparrow}{}_{\mathrm{ham}}[\sigma, f] = \sigma^0\, c_0\, \alpha^0\, (\partial_0 + x^i_0\, \partial_i + \gamma_0{}^i_0\, \partial^0_i) \\ - \frac{1}{c_0\,\alpha^0}\, \big(\breve{G}^{j\lambda}_0\, \partial^0_j f\, \partial_\lambda - (\breve{G}^{i\lambda}_0\, \partial_\lambda f + \breve{\Xi}^{ij}_{00}\, \partial^0_j f)\, \partial^0_i\big),$$

where $\breve{\Xi}^{ij}_{00} = \breve{G}^{ih}_0\, \Gamma_h{}^j_0 - \breve{G}^{jh}_0\, \Gamma_h{}^i_0$.

31.8 Phase Lie Brackets

31.8.1 Poisson Lie Bracket

We define the *Poisson Lie bracket* of all phase functions

$$\{,\} : \operatorname{map}(U_1\boldsymbol{E}, \mathbb{R}) \to \operatorname{map}(U_1\boldsymbol{E}, \mathbb{R}) : (f, \acute{f}) \mapsto \{f, \acute{f}\} := i_{df \wedge d\acute{f}} \Lambda,$$

whose coordinate expression is

$$\{f, \acute{f}\} = \tfrac{1}{c_0\, \alpha^0} \left(\breve{G}_0^{i\lambda} \, (\partial_\lambda f \, \partial_i^0 \acute{f} - \partial_\lambda \acute{f} \, \partial_i^0 f) - \breve{\Xi}_{00}^{ij} \, \partial_i^0 f \, \partial_j^0 \acute{f} \right).$$

This bracket is analogous to the corresponding bracket in the galilean case.

31.8.2 Special Phase Lie Bracket

We define a *special phase function* to be a function $f \in \operatorname{map}(U_1\boldsymbol{E}, \mathbb{R})$ of the type (see [215])

$$f = -G(\text{д}, X[f]) + \breve{f}, \quad \text{where } X[f] \in \sec(\boldsymbol{E}, T\boldsymbol{E}), \;\; \breve{f} \in \operatorname{map}(\boldsymbol{E}, \mathbb{R}).$$

We say that

$$\begin{aligned} &X[f] \in \sec(\boldsymbol{E},\, T\boldsymbol{E}) && \textit{is the tangent lift},\\ &\phi[f] := G^\flat\,(X[f]) \in \sec(\boldsymbol{E},\, \mathbb{T} \otimes T^*\boldsymbol{E}) && \textit{is the cotangent lift},\\ &\sigma[f] := \tau\,(X[f]) \in \operatorname{map}(J_1\boldsymbol{E},\, \bar{\mathbb{T}}) && \textit{is the time scale},\\ &\breve{f} \in \operatorname{map}(\boldsymbol{E},\, \mathbb{R}) && \textit{is the spacetime component}. \end{aligned}$$

Hence, for every special phase function f, we have the following equivalent expressions

$$f = -G\big(\text{д}, X[f]\big) + \breve{f} = -\text{д} \lrcorner \phi[f] + \breve{f} = X[f] \lrcorner \hat{\tau} + \breve{f} = \tfrac{m\,c^2}{\hbar}\, \sigma[f] + \breve{f}.$$

In coordinates we can write

$$f = -\frac{c_0\,(G_{\lambda 0}^0 + G_{\lambda j}^0\, x_0^j)\, f^\lambda}{\sqrt{|g_{00} + 2\, g_{0k}\, x_0^k + g_{hk}\, x_0^h\, x_0^k|}} + \breve{f},$$

i.e.,

$$
\begin{aligned}
f &= -c_0\, \alpha^0\, (f^0\, G^0_{00} + f^0\, G^0_{0j}\, x^j_0 + f^i\, G^0_{i0} + f^i\, G^0_{ij}\, x^j_0) + \breve{f}, \\
&\quad \text{with } f^\lambda := X^\lambda = G^{\lambda\mu}_0\, \phi^0_\mu, \\
&= -c_0\, \alpha^0\, (\phi^0_0 + \phi^0_i\, x^i_0) + \breve{f}, \quad \text{with } \phi^0_\lambda = f^0_\lambda := G^0_{\lambda\mu}\, X^\mu, \\
&= -c_0\, \alpha^0\, (f^0_0 + f^0_i\, x^i_0) + \breve{f}, \quad \text{with } f^0_\lambda = \phi^0_\lambda := G^0_{\lambda\mu}\, X^\mu.
\end{aligned}
$$

We denote the subsheaf of special phase functions by

$$\mathrm{spe}(U_1\boldsymbol{E}, \mathbb{R}) \subset \mathrm{map}(U_1\boldsymbol{E}, \mathbb{R}).$$

We have the gauge independent and observer independent maps

$$
\begin{aligned}
X &: \mathrm{spe}(U_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{map}(\boldsymbol{E}, T\boldsymbol{E}) : f \mapsto X[f], \\
\breve{} &: \mathrm{spe}(U_1\boldsymbol{E}, \mathbb{R}) \to \mathrm{map}(\boldsymbol{E}, \mathbb{R}) : f \mapsto \breve{f}.
\end{aligned}
$$

Indeed, the pair $\big(X[f],\ \breve{f}\big)$ characterises f.

The above expressions have evident analogies with the galilean case, but also relevant differences. Also in the einsteinian case, the definition of special phase functions arises naturally in several contexts.

We define the *special phase Lie bracket* of special phase functions by means of the equality (see [215])

$$[\![f,\ \acute{f}]\!] := \{f, \acute{f}\} + (\sigma[f])\, (\gamma.\acute{f}) - \big(\sigma[\acute{f}]\big)\, (\gamma.f).$$

We have the following further expression

$$[\![f,\ \acute{f}]\!] = -\text{д} \lrcorner G^\flat\big[X[f],\ X[\acute{f}]\big] + X[f].\breve{\acute{f}} - X[\acute{f}].\breve{f} + \tfrac{q}{\hbar}\, F\big(X[f], X[\breve{f}]\big).$$

For instance, the following phase functions are special:

$$
\begin{aligned}
& x^\lambda \quad \text{with } X[x^\lambda] = 0,\ \breve{x}^\lambda = x^\lambda, \\
& \hat{\tau}_0 = -c_0\, \alpha^0\, \breve{G}^0_{00}, \quad \text{with } X[\hat{\tau}_0] = \partial_0,\ \breve{\hat{\tau}}_0 = 0, \\
& \hat{\tau}_i = -c_0\, \alpha^0\, \breve{G}^0_{i0}, \quad \text{with } X[\hat{\tau}_i] = \partial_i,\ \breve{\hat{\tau}}_i = 0, \\
& \mathcal{H}_0 = -c_0\, \alpha^0\, \breve{G}^0_{00} - A^{\mathfrak{e}}{}_0, \quad \text{with } X[\mathcal{H}_0] = \partial_0,\ \breve{\mathcal{H}}_0 = -A^{\mathfrak{e}}{}_0, \\
& \mathcal{P}_j = c_0\, \alpha^0\, \breve{G}^0_{0j} + A^{\mathfrak{e}}{}_j, \quad \text{with } X[\mathcal{P}_j] = -\partial_i,\ \breve{\mathcal{P}}_j = A^{\mathfrak{e}}{}_j.
\end{aligned}
$$

Accordingly, we obtain, for instance, the equalities

$$[\![x^\lambda, \mathcal{H}_0]\!] = -\delta^\lambda_0 \,, \qquad [\![x^\lambda, \mathcal{P}_j]\!] = \delta^\lambda_j \,,$$
$$[\![x^\lambda, \hat{\tau}_0]\!] = -\delta^\lambda_0 \,, \qquad [\![x^\lambda, \hat{\tau}_j]\!] = -\delta^\lambda_j \,,$$
$$[\![\hat{\tau}_0, \hat{\tau}_i]\!] = \tfrac{q}{\hbar} \, F_{0i} \,, \qquad [\![\mathcal{H}_0, \mathcal{P}_j]\!] = 0.$$

For the correct interpretation of the above formulas, we recall that the above functions $\mathcal{H}_0, \mathcal{P}_j, \ldots$ incorporate the Planck constant $\hbar$ through their normalisation.

Analogously to the galilean case, we define the *conserved phase functions* f by means of the condition $\gamma.f = 0$.

The sheaf of conserved special phase functions is a Lie subalgebra of the Lie algebra of special phase functions with respect to the special Lie bracket.

31.9 Classical Symmetries

In einsteinian case we regard the almost-cosymplectic-contact pair $(-\hat{\tau}, \Omega)$ as *classical structure*. Then, we define an *infinitesimal symmetry of the classical structure* to be a vector field $Y : U_1\boldsymbol{E} \to TU_1\boldsymbol{E}$, such that

$$L_Y \hat{\tau} = 0 \qquad \text{and} \qquad L_Y \Omega = 0.$$

Analogously to the galilean case, the Lie algebra of infinitesimal symmetries of the classical structure is generated by the Lie algebra of conserved special phase functions (see [229]).

Namely, a phase vector field Y is a classical infinitesimal symmetry if and only if it is the hamiltonian lift of a conserved special phase function $f = -G(\text{д}, X) + \breve{f}$ i.e. a special phase function such that

$$\hat{\gamma}.f = 0, \qquad L_X g = 0, \qquad d\breve{f} = \tfrac{q}{\hbar} X \lrcorner F.$$

31.10 Quantum Stuff

31.10.1 Quantum Bundle

For the quantum theory of a scalar particle, we proceed in close analogy with the galilean case (see [222]).

We consider a 1-dimensional complex *quantum bundle* over spacetime and the associated *upper quantum bundle*, defined as its pullback over the phase space

$$\pi : \boldsymbol{Q} \to \boldsymbol{E} \qquad \text{and} \qquad \pi^\uparrow : \boldsymbol{Q}^\uparrow \to U_1\boldsymbol{E}.$$

Moreover, we consider a scaled hermitian product

$$\mathrm{h} : \boldsymbol{E} \to (\mathbb{L}^{-4} \otimes \mathbb{C}) \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*).$$

We refer to normalised local *quantum bases*

$$\mathrm{b} \in \sec(\boldsymbol{E}, \mathbb{L}^2 \otimes \boldsymbol{Q}), \quad \text{with } \mathrm{h}(\mathrm{b}, \mathrm{b}) = 1.$$

Further, we suppose the cohomology class of F to be integer and postulate an *upper quantum connection*

$$\mathrm{Ч}^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to T^* U_1 \boldsymbol{E} \otimes T \boldsymbol{Q}^{\uparrow},$$

which is hermitian, reducible and whose curvature fulfills the condition

$$R[\mathrm{Ч}^{\uparrow}] = -2\,\mathrm{i}\,\Omega \otimes \mathbb{I}.$$

We have a natural splitting

$$\mathrm{Ч}^{\uparrow} = \mathrm{Ч}^{\uparrow \mathfrak{e}} - \mathrm{i}\,\hat{\tau} \otimes \mathbb{I}^{\uparrow},$$

where

$$\mathrm{Ч}^{\uparrow \mathfrak{e}} : \boldsymbol{Q}^{\uparrow} \to T^* U_1 \boldsymbol{E} \otimes T \boldsymbol{Q}^{\uparrow},$$

is the pullback of a hermitian connection

$$\mathrm{Ч}^{\mathfrak{e}} : \boldsymbol{Q} \to T^* \boldsymbol{E} \otimes T \boldsymbol{Q},$$

whose curvature fulfills the condition

$$R[\mathrm{Ч}^{\mathfrak{e}}] = -\mathrm{i}\,\tfrac{q}{\hbar}\,F \otimes \mathbb{I}.$$

In coordinates, we have expressions of the type

$$\mathrm{Ч}^{\uparrow} = d^{\lambda} \otimes \partial_{\lambda} + d^i_0 \otimes \partial^0_i + \mathrm{i}\,(c_0\,\alpha^0\,\breve{G}^0_{0\lambda} + A^{\mathfrak{e}}{}_{\lambda})\,d^{\lambda} \otimes \mathbb{I}^{\uparrow},$$
$$\mathrm{Ч}^{\mathfrak{e}} = d^{\lambda} \otimes (\partial_{\lambda} + \mathrm{i}\,A^{\mathfrak{e}}{}_{\lambda}\,\mathbb{I}).$$

In comparison with the galilean case, here the gravitational component of the potential is the natural form $\hat{\tau}$. So, the gauge of the joined potential depends only on the electromagnetic component.

31.10.2 Hermitian Vector Fields

Analogously to the galilean case, we consider the subsheaf of *hermitian quantum vector fields* (see [215, 220])

$$\mathrm{her}(\boldsymbol{E}, \boldsymbol{Q}) \subset \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}).$$

Then, we have a gauge independent and observer independent Lie algebra isomorphism

$$\mathrm{spe}(U_1 \boldsymbol{E}, \mathbb{R}) \to \mathrm{her}(\boldsymbol{Q}, T\boldsymbol{Q}) : f \mapsto Y[f],$$

with coordinate expression

$$Y[f] = f^\lambda \, \partial_\lambda + \mathrm{i}\, (f^\lambda \, A^{\mathfrak{e}}{}_\lambda + \breve{f}) \, \mathbb{I}.$$

For instance, we obtain the distinguished hermitian quantum vector fields

$$Y[x^\lambda] = \mathrm{i}\, x^\lambda \, \mathbb{I}, \qquad Y[\hat{\tau}_\lambda] = \partial_\lambda + \mathrm{i}\, A^{\mathfrak{e}}{}_\lambda \, \mathbb{I}, \qquad Y[\mathcal{H}_0[o]] = \partial_0, \qquad Y[\mathcal{P}_i[o]] = \partial_i .$$

31.10.3 Quantum Dynamics

Analogously to the galilean case, we can prove that the natural scaled *quantum lagrangian* (see [201])

$$\mathrm{L} \equiv \mathrm{l}\, \upsilon : \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathrm{sec}(\boldsymbol{E}, \Lambda^4 T^* \boldsymbol{E}) : \Psi \mapsto \mathrm{L}[\Psi]$$

in the einsteinian framework is generated by the two functions

$$\begin{aligned} \mathrm{l}_1 &: \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathbb{L}^{-4} \otimes \mathbb{R} \qquad : \Psi \mapsto \tfrac{1}{2}\, \mathrm{h}(\Psi, \Psi), \\ \mathrm{l}_2 &: \mathrm{sec}(\boldsymbol{E}, \boldsymbol{Q}) \to \mathbb{L}^{-4} \otimes \mathbb{R} \qquad : \Psi \mapsto \tfrac{\hbar^2}{2\, m^2 c^2}\, (\bar{g} \otimes \mathrm{h})\, (\nabla^{\mathfrak{e}} \Psi, \nabla^{\mathfrak{e}} \Psi), \end{aligned}$$

with coordinate expressions

$$\begin{aligned} \mathrm{l}_1 &= \tfrac{1}{2}\, \bar{z}\, z, \\ \mathrm{l}_2 &= \tfrac{\hbar^2}{2\, m^2 c^2}\, g^{\lambda\mu}\, \bar{z}_\lambda\, z_\mu + \mathrm{i}\, \tfrac{q\hbar}{2\, m^2 c^3}\, g^{\lambda\mu}\, A^{\mathfrak{e}}{}_\lambda\, (z\, \bar{z}_\mu - z_\mu\, \bar{z}) + \tfrac{q^2}{2\, m^2 c^4}\, g^{\lambda\mu}\, A^{\mathfrak{e}}{}_\lambda\, A^{\mathfrak{e}}{}_\mu\, \bar{z}\, z. \end{aligned}$$

Among the possible natural lagrangians above, the most physically reasonable appears to be

$$\mathrm{L} := \mathrm{l}_2\, \upsilon - \mathrm{l}_1\, \upsilon.$$

Indeed, the associated Euler–Lagrange equation turns out to be just the *Klein–Gordon equation*

$$\langle \bar{g}, \nabla^{\mathfrak{e}} \nabla^{\mathfrak{e}} \Psi \rangle + \tfrac{m^2 c^2}{\hbar^2}\, \Psi = 0,$$

whose coordinate expression is

$$\begin{aligned} g^{\lambda\mu} \big(& -\partial_{\lambda\mu}\psi + 2\,\mathrm{i}\, \tfrac{q}{\hbar c}\, A^{\mathfrak{e}}{}_{\lambda}\, \partial_{\mu}\psi + \mathrm{i}\, \tfrac{q}{\hbar c}\, \psi\, \partial_{\lambda} A^{\mathfrak{e}}{}_{\mu} \big) \\ & + \tfrac{q^2}{\hbar^2 c^2}\, \psi\, A^{\mathfrak{e}}{}_{\lambda}\, A^{\mathfrak{e}}{}_{\mu} - K^{\natural}{}_{\lambda}{}^{\nu}{}_{\mu}\, \partial_{\nu}\psi \\ & + \mathrm{i}\, \tfrac{q}{\hbar c}\, K^{\natural}{}_{\lambda}{}^{\nu}{}_{\mu}\, \psi\, A^{\mathfrak{e}}{}_{\nu}) + \tfrac{m^2 c^2}{\hbar^2}\, \psi = 0. \end{aligned}$$

31.11 Further Hints

A fully intrinsic approach to Quantum Mechanics of a scalar particle in the einsteinian framework might perhaps suggest to normalise the quantum section Ψ on the whole spacetime, regardless of any choice of arbitrary spacelike slicing of spacetime.

The meaning of such a possible choice would be that there is probability 1 to detect the particle in one event. But, in such a case, for consistency reasons, once detected, the particle would be absorbed and there would be no place for probability current. If one would try to follow this hint, then further developments would require a deep revision and would lead us directly to the scheme of quantum field theory along with creation and destruction operators.

But such possible developments are out of the scope of the present book.

Appendix on Geometric Methods

We assume the reader to be familiar with elementary notions of modern Differential Geometry.

This Appendix is devoted to rather non standard topics of Differential Geometry, which are involved in the present book. We just sketch the main notions and results, without any pretension of completion. For further details, the reader can refer to well established text (see, for instance, [51, 94–96, 138, 146, 241, 242, 246, 268]).

As usual, in the present book, all manifolds are supposed to be smooth and finite dimensional and all maps between manifolds are supposed to be smooth, unless a different hypothesis is explicitly mentioned.

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1

Appendix A
Fibred Manifolds and Bundles

The geometric language of the present book is largely based on the concepts of *fibred manifolds*, *bundles*, *structured bundles* and their *tangent prolongations*.

In particular, in the present book, just as an example, spacetime is a fibred manifold over time, the classical phase space is an affine bundle over spacetime, the quantum bundle is a vector bundle over spacetime.

Here we recall the basic notions, in order to establish our language and our notation.

By the way, our approach presents a few minor differences with respect to the current geometric literature. For instance, we define a bundle as a particular case of fibred manifold and we stress the conceptual difference between bundle and structured bundle. Moreover, we deal with the unusual definition of Lie affine bundle and regard the principal bundles as a particular case of Lie affine bundles.

For further details the reader can refer to well established texts (see, for instance, [146, 241, 242, 246, 341, 374]).

A.1 Fibred Manifolds

We discuss the concepts of *fibred manifold*, along with the concepts of (local) sections, (local) fibred morphisms, sheaf of sections, sheaf morphisms, local trivialisations and pullbacks.

In the present book, the main example of fibred manifold is provided by the classical spacetime fibring $t : \boldsymbol{E} \to \boldsymbol{T}$ (see Postulate C.1).

Definition A.1.1 A *fibred manifold* is defined to be a manifold $\boldsymbol{F}$ along with a surjective map $p : \boldsymbol{F} \to \boldsymbol{B}$, whose rank (see, for instance, [46]) is equal to the dimension of $\boldsymbol{B}$. The manifold $\boldsymbol{F}$ is called the *total space*, the manifold $\boldsymbol{B}$ the *base space* and the map p the *base projection*, or the *fibring*. For each $b \in \boldsymbol{B}$, the submanifold $\boldsymbol{F}_b := p^{-1}(b) \subset \boldsymbol{F}$ is called the *fibre* over b.

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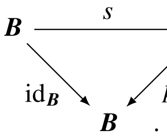
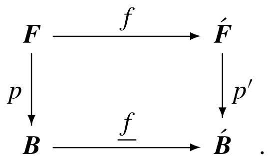
A *section* is defined to be a (local) map $s : \boldsymbol{B} \to \boldsymbol{F}$, such that $p \circ s = \mathrm{id}_{\boldsymbol{B}}$, according to the following commutative diagram



We define a *fibred morphism* between two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \acute{\boldsymbol{B}}$ to be a map

$$f : \boldsymbol{F} \to \acute{\boldsymbol{F}},$$

which *preserves the fibres*, i.e. which projects on a map $\underline{f} : \boldsymbol{B} \to \acute{\boldsymbol{B}}$, according to the following commutative diagram



In the particular case when $\boldsymbol{B} = \acute{\boldsymbol{B}}$ and $\underline{f} = \mathrm{id}_{\boldsymbol{B}}$, we also say that the fibred morphism is *over* $\boldsymbol{B}$.

A *local section* and a *local fibred morphism* are defined to be, respectively, a section and a fibred morphism defined in an open subset

$$s : \boldsymbol{U} \to \boldsymbol{F} \quad \text{and} \quad f : \boldsymbol{V} \to \acute{\boldsymbol{F}}, \quad \text{where } \boldsymbol{U} \subset \boldsymbol{B} \text{ and } \boldsymbol{V} \subset \boldsymbol{F}.$$

We denote the set of local sections $s : \boldsymbol{B} \to \boldsymbol{F}$ and the set of local fibred morphisms $f : \boldsymbol{F} \to \acute{\boldsymbol{F}}$ over $\boldsymbol{B}$, respectively, by

$$\mathrm{sec}(\boldsymbol{B}, \boldsymbol{F}) \qquad \text{and} \qquad \mathrm{fib}(\boldsymbol{F}, \acute{\boldsymbol{F}}). \quad \square$$

Note A.1.2 Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and a local section $s : \boldsymbol{U} \subset \boldsymbol{B} \to \boldsymbol{E}$. Then, the image of the section $s(\boldsymbol{U}) \subset \boldsymbol{F}$ turns out to be submanifold of $\boldsymbol{F}$. □

In the present book, we often use the basic elementary terminology concerning *sheaves* and *sheaf morphisms*. This is nothing but a quick way to express simple properties, as explained by the following Note.

Note A.1.3 We say that the set $\mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$ of local sections of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ is a *sheaf*, as it behaves well with respect to restrictions and glueing, according to the following properties:

(1) Let $\boldsymbol{U}, \acute{\boldsymbol{U}} \subset \boldsymbol{B}$ be open subsets and $\acute{\boldsymbol{U}} \subset \boldsymbol{U}$; then

$$s_{\boldsymbol{U}} \in \sec(\boldsymbol{B}, \boldsymbol{F}) \qquad \Rightarrow \qquad s_{\boldsymbol{U}}|_{\acute{\boldsymbol{U}}} \in \sec(\boldsymbol{B}, \boldsymbol{F}).$$

(2) Let $\boldsymbol{U} \subset \boldsymbol{B}$ be an open subset, $\{\boldsymbol{U}_i\}$ an open covering of $\boldsymbol{U}$ and let us consider a family $\{s_i : \boldsymbol{U}_i \to \boldsymbol{F}\}|_{i \in I} \in \sec(\boldsymbol{B}, \boldsymbol{F})$; then

$$\forall i, j \in I, s_i|_{\boldsymbol{U}_i \cap \boldsymbol{U}_j} = s_j|_{\boldsymbol{U}_i \cap \boldsymbol{U}_j} \qquad \Rightarrow \qquad \exists!\, s_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{F} \in \sec(\boldsymbol{B}, \boldsymbol{F}) \mid s_{\boldsymbol{U}}|_{\boldsymbol{U}_i} = s_i.$$

In a similar way, we say that the set of local fibred morphisms $\mathrm{fib}(\boldsymbol{F}, \acute{\boldsymbol{F}})$ over $\boldsymbol{B}$ of two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \boldsymbol{B}$ is a *sheaf*.

Analogously, given two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \boldsymbol{B}$, we define a *sheaf morphism*

$$f : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, \acute{\boldsymbol{F}}) : s_{\boldsymbol{U}} : \boldsymbol{U} \to \boldsymbol{B} \quad \mapsto \quad f(s_{\boldsymbol{U}}) : \boldsymbol{U} \to \acute{\boldsymbol{F}}$$

to be a map which behaves well with respect to restrictions, according to the following property:

Let $\boldsymbol{U}, \acute{\boldsymbol{U}} \subset \boldsymbol{B}$ be open subsets and $\acute{\boldsymbol{U}} \subset \boldsymbol{U}$; then

$$s_{\boldsymbol{U}} \in \sec(\boldsymbol{B}, \boldsymbol{F}) \qquad \Rightarrow \qquad f(s_{\boldsymbol{U}})|_{\acute{\boldsymbol{U}}} = f(s_{\boldsymbol{U}}|_{\acute{\boldsymbol{U}}}). \quad \square$$

For instance, the exterior differential and the covariant differential

$$\begin{aligned} d &: \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M}) \to \sec(\boldsymbol{M}, \Lambda^{r+1} T^*\boldsymbol{M}), \\ \nabla &: \sec(\boldsymbol{M}, \otimes^r T\boldsymbol{M}) \ \to \sec(\boldsymbol{M}, T^*\boldsymbol{M} \otimes (\otimes^r T\boldsymbol{M})) \end{aligned}$$

are sheaf morphisms, as they behave well with respect to local restrictions.

Definition A.1.4 A *fibred submanifold* of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ is defined to be a fibred manifold $\acute{p} : \acute{\boldsymbol{F}} \to \acute{\boldsymbol{B}}$, such that $\acute{\boldsymbol{F}} \subset \boldsymbol{F}$ and $\acute{\boldsymbol{B}} \subset \boldsymbol{B}$ are open submanifolds and $\acute{\boldsymbol{F}} \subset \boldsymbol{F}$ is a fibred morphism over $\acute{\boldsymbol{B}} \subset \boldsymbol{B}$. $\square$

Note A.1.5 Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.
Then, in virtue of the rank Theorem (see, for instance, [47]), for each $\mathsf{f} \in \boldsymbol{F}$, there exists a *local fibred trivialisation* (called also *fibred manifold chart*) in a neighbourhood $\boldsymbol{V} \subset \boldsymbol{F}$ of f.

In other words, for each $\mathsf{f} \in \boldsymbol{F}$, there exist an open neighbourhood $\boldsymbol{V} \subset \boldsymbol{F}$ of f, which projects on an open subset $\boldsymbol{U} := p(\boldsymbol{V}) \subset \boldsymbol{B}$, a manifold F (called *local type fibre*) and a fibred isomorphism of the type

$$\Phi : \boldsymbol{V} \to p(\boldsymbol{V}) \times \mathsf{F},$$

which makes the following diagram commutative

$$\begin{array}{ccc} \boldsymbol{V} & \xrightarrow{\ \Phi\ } & p(\boldsymbol{V}) \times \mathsf{F} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pro}_1} \\ p(\boldsymbol{V}) & \xrightarrow{\ \underline{\Phi}\ } & p(\boldsymbol{V}) \end{array} \quad .$$

A *fibred manifold atlas* is defined to be a family of fibred manifold charts

$$\mathcal{A} := \left\{\Phi_i : \boldsymbol{V}_i \to p(\boldsymbol{V}_i) \times \mathsf{F}_i\right\}_{i \in I},$$

such that $\left\{\boldsymbol{V}_i\right\}_{i \in I}$ is an open covering of $\boldsymbol{F}$. □

Note A.1.6 Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

A *fibred (coordinate) chart* is defined to be a local chart (x^λ, y^i) of $\boldsymbol{F}$, which is adapted to a local trivialisation, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{V} & \xrightarrow{\ \Phi\ } & p(\boldsymbol{V}) \times \mathsf{F} \\ {\scriptstyle (x^\lambda, y^i)}\big\downarrow & & \big\downarrow{\scriptstyle (x^\lambda) \times (y^i)} \\ \mathbb{R}^{m+l} & \xrightarrow{\qquad} & \mathbb{R}^m \times \mathbb{R}^l \end{array} \quad . □$$

Then, by definition, a fibred (coordinate) chart (x^λ, y^i) yields a chart (x^λ) of $\boldsymbol{U} := p(\boldsymbol{V}) \subset \boldsymbol{B}$ and a chart (y^i) of F.

Conversely, a local fibred trivialisation $\Phi : \boldsymbol{V} \to p(\boldsymbol{V}) \times \mathsf{F}$, a chart (x^λ) of $\boldsymbol{U} := p(\boldsymbol{V})$ and a chart (y^i) of F yield a fibred (coordinate) chart (x^λ, y^i) of $\boldsymbol{F}$. □

Note A.1.7 If a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ is globally trivialisable through the global fibred isomorphism $\Phi : \boldsymbol{F} \to \boldsymbol{B} \times \mathsf{F}$, then, for each $\mathsf{f} \in \mathsf{F}$, it admits the global section

$$s : \boldsymbol{B} \to \boldsymbol{F} : b \mapsto \Phi^{-1}(b, \mathsf{f}), \quad \text{with } \mathsf{f} \in \mathsf{F}. \quad □$$

Note A.1.8 The *fibred product* of two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \boldsymbol{B}$ over the same base $\boldsymbol{B}$ is defined to be the fibred manifold

$$\boldsymbol{F} \underset{\boldsymbol{B}}{\times} \acute{\boldsymbol{F}} := \bigsqcup_{b \in \boldsymbol{B}} \boldsymbol{F}_b \times \acute{\boldsymbol{F}}_b \to \boldsymbol{B}. \quad □$$

In the present book, we often use the concept of pullback.

Note A.1.9 Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and a map $f : \boldsymbol{M} \to \boldsymbol{B}$.

Then, the *pullback of* $p : \boldsymbol{F} \to \boldsymbol{B}$ *with respect to* f is defined to be the fibred manifold

$$f^*(p) : f^*(\boldsymbol{F}) := \bigsqcup_{x \in \boldsymbol{M}} \boldsymbol{F}_{f(x)} \to \boldsymbol{M}.$$

Thus, by definition, every element $y \in \big(f^*(\boldsymbol{F})\big)_x$ is equal to an element $y \in \boldsymbol{F}_{f(x)}$.

Therefore, we have a natural fibred morphism $\widehat{f} : f^*(\boldsymbol{F}) \to \boldsymbol{F}$ over $f : \boldsymbol{M} \to \boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc} f^*(\boldsymbol{F}) & \xrightarrow{\;\widehat{f}\;} & \boldsymbol{F} \\ {\scriptstyle f^*(p)}\downarrow & & \downarrow{\scriptstyle p} \\ \boldsymbol{M} & \xrightarrow{\;f\;} & \boldsymbol{B} \end{array} \; .$$

Moreover, we define the *pullback of a section* $s : \boldsymbol{B} \to \boldsymbol{F}$ to be the section

$$f^*(s) : \boldsymbol{M} \to f^*(\boldsymbol{F}) : x \mapsto s\big(f(x)\big),$$

according to the following commutative diagram

$$\begin{array}{ccc} f^*(\boldsymbol{F}) & \xrightarrow{\;\widehat{f}\;} & \boldsymbol{F} \\ {\scriptstyle f^*(s)}\uparrow & & \uparrow{\scriptstyle s} \\ \boldsymbol{M} & \xrightarrow{\;f\;} & \boldsymbol{B} \end{array} \; . \square$$

Example A.1.10 The *fibred product* of two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \boldsymbol{B}$ over the same base $\boldsymbol{B}$ turns out to be the pullback, with respect to the diagonal map $d : \boldsymbol{B} \to \boldsymbol{B} \times \boldsymbol{B} : x \mapsto (x, x)$,

$$\boldsymbol{F} \underset{\boldsymbol{B}}{\times} \acute{\boldsymbol{F}} = d^*\big(\boldsymbol{F} \times \acute{\boldsymbol{F}}\big)$$

of the cartesian product of fibred manifolds $(p \times \acute{p}) : \boldsymbol{F} \times \acute{\boldsymbol{F}} \to \boldsymbol{B} \times \boldsymbol{B}$. □

A.2 Bundles

We define the concept of *bundle* as a distinguished type of fibred manifold which admits "tubelike fibred trivialisations". We stress that our definition of bundle is simplified, with respect to the original one due to N. Steenrode (see [374]).

In the present book, we deal with a large number of bundles; for instance, just as an example, we deal with the classical phase space bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ (see Assumption C.1 and Postulate Q.1).

Definition A.2.1 A *bundle* is defined to be a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, which admits, for each $b \in \boldsymbol{B}$, a local *tubelike fibred trivialisation* (called also *bundle trivialisation*) in a neighbourhood $\boldsymbol{U} \subset \boldsymbol{B}$ of b. In other words, by definition, a bundle admits, for each $b \in \boldsymbol{B}$, an open neighbourhood $\boldsymbol{U} \subset \boldsymbol{B}$ of b, a manifold F (called a *type fibre*) and a fibred isomorphism (called *bundle trivialisation*, or *bundle chart*) of the type

$$\Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F},$$

which makes the following diagram commutative

$$\begin{array}{ccc} p^{-1}(\boldsymbol{U}) & \xrightarrow{\ \Phi\ } & \boldsymbol{U} \times \mathsf{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_1} \\ \boldsymbol{U} & \xrightarrow{\ \underline{\Phi}\ } & \boldsymbol{U} \end{array} \quad .$$

A bundle is said to be *trivial* if it admits a global bundle trivialisation of the type

$$\Phi : \boldsymbol{F} \to \boldsymbol{B} \times \mathsf{F}.$$

A *bundle atlas* is defined to be a family of bundle charts

$$\mathcal{A} := \left\{\Phi_i : p^{-1}(\boldsymbol{U}_i) \to \boldsymbol{U}_i \times \mathsf{F}_i\right\},$$

such that $\left\{\boldsymbol{U}_i\right\}$ is an open covering of $\boldsymbol{B}$.

A *fibred chart* (x^λ, y^i) is said to be *adapted* to the bundle if it is adapted to a bundle chart $\Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F}$, i.e. if it factorises through manifold charts (denoted by the same symbol)

$$(x^\lambda) : \boldsymbol{U} \to \mathbb{R}^{\dim \boldsymbol{B}} \qquad \text{and} \qquad (y^i) : \mathsf{F} \to \mathbb{R}^{\dim \mathsf{F}},$$

according to the following commutative diagrams

$$\begin{array}{ccc} p^{-1}(\boldsymbol{U}) & \xrightarrow{\ \mathrm{pro}_1 \circ \Phi\ } & \boldsymbol{U} \\ {\scriptstyle (x^\lambda)}\downarrow & & \downarrow{\scriptstyle (x^\lambda)} \\ \mathbb{R}^{\dim \boldsymbol{B}} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{R}^{\dim \boldsymbol{B}} \end{array} \qquad \begin{array}{ccc} p^{-1}(\boldsymbol{U}) & \xrightarrow{\ \mathrm{pro}_2 \circ \Phi\ } & \mathsf{F} \\ {\scriptstyle (y^i)}\downarrow & & \downarrow{\scriptstyle (y^i)} \\ \mathbb{R}^{\dim \mathsf{F}} & \xrightarrow{\ \mathrm{id}\ } & \mathbb{R}^{\dim \mathsf{F}} \end{array} \ . \square$$

Remark A.2.2 A fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ either is a bundle, or is not a bundle. In other words, being a bundle is a possible property of a fibred manifold, not an additional structure!

A different case is that of structured bundles (for instance, vector bundles, affine bundles, principal bundles, and so on), which are bundles *along with* an additional geometric structure. □

Remark A.2.3 According to our definition, a trivial bundle needs not to be a product $\boldsymbol{F} = \boldsymbol{B} \times \mathsf{F}$ in a distinguished way. In fact, if a global bundle trivialisation exists, then many such global trivialisations exist. □

Note A.2.4 A fibred morphism $f : \boldsymbol{F} \to \acute{\boldsymbol{F}}$ between two bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \boldsymbol{F} \to \boldsymbol{B}$ preserves locally the bundle trivialisations.

Accordingly, we define the *bundle morphisms* to be just the fibred morphisms between bundles, without any additional condition.

Proof. The proof can be achieved by taking into account that the map $\underline{f} : \boldsymbol{B} \to \acute{\boldsymbol{B}}$ is continuous. □

Remark A.2.5 Let us consider a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and a point $\mathsf{f} \in \boldsymbol{F}$.

Then, the open subset

$$p : \boldsymbol{F} - \{\mathsf{f}\} \to \boldsymbol{B}$$

is no longer a bundle, but it is still a fibred manifold. □

In the classical and quantum theories, we use frequently the result below.

Theorem A.2.6 *[374] Each bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ with a "contractible" base space $\boldsymbol{B}$ is globally trivialisable by a global bundle isomorphism of the type $\Phi : \boldsymbol{F} \to \boldsymbol{B} \times \mathsf{F}$, where the manifold F is the type fibre.* □

A.3 Structured Bundles

Next, we define the main types of structured bundles that we are concerned with. Thus, we define the *vector bundles*, *affine bundles*, *Lie group bundles* and *Lie affine bundles* in an analogous way, by assuming a "smoothly endowed" algebraic structure of their fibres.

Then, the *principal bundles* can be defined just as Lie affine bundles associated with a trivial Lie group bundle.

A.3.1 Vector Bundles

Vector bundles are the simplest and largely used type of structured bundles that we deal with.

In the present book, we deal with a large number of vector bundles; for instance, just as an example, we deal with the tangent bundle of spacetime $\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$ and the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$ (see Definition 2.2.1 and Postulate Q.1).

Definition A.3.1 A *vector bundle* is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ along with a "smoothly endowed" vector structure of its fibres.

A fibred morphism $f : \boldsymbol{F} \to \acute{\boldsymbol{F}}$ over $\underline{f} : \boldsymbol{B} \to \acute{\boldsymbol{B}}$ between two vector bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \acute{\boldsymbol{B}}$ is said *linear* if it restricts to linear maps between their fibres. □

Note A.3.2 In other words, a vector bundle is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, along with a vector structure of every fibre $\boldsymbol{F}_b$, with $b \in \boldsymbol{B}$, such that there exists a bundle atlas (called *vector bundle atlas*) consisting of bundle charts $\Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F}$, where the type fibre F is a vector space and, for each $b \in \boldsymbol{U}$, the fibred morphism Φ restricts to a linear isomorphism $\Phi_b : \boldsymbol{F}_b \to \mathsf{F}$.

Thus, every vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ turns out to be equipped with the "algebraic" fibred morphisms over $\boldsymbol{B}$ and the zero section

$$\cdot : \mathbb{R} \times \boldsymbol{F} \to \boldsymbol{F}, \qquad + : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}, \qquad - : \boldsymbol{F} \to \boldsymbol{F} \qquad \text{and} \qquad 0 : \boldsymbol{B} \to \boldsymbol{F}.$$

A fibred chart (x^λ, y^i) adapted to the bundle is said to be *linear* if, for each $b \in \boldsymbol{B}$, the restricted map

$$(y^i)_b : \boldsymbol{F}_b \to \mathbb{R}^{\dim \boldsymbol{F}_b}$$

is a linear isomorphism. □

We can define the dual of a vector bundle by taking the dual vector spaces of its fibres.

Analogously, we can define the tensor product of two vector bundles by taking the tensor product of their fibres.

A.3.2 Affine Bundles

The structure of affine bundle is achieved by means of an additional associated vector bundle and a fibred free and transitive action.

In the present book we are concerned with several affine bundles. The most important one is the classical phase space bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ (see Proposition 2.5.1).

We recall the following standard elementary definition.

Definition A.3.3 An *affine space* is defined to be a set $\boldsymbol{A}$ along with a vector space $\bar{\boldsymbol{A}}$ (called the *associated vector space*), which acts freely and transitively on $\boldsymbol{A}$ through a map $\tau : \boldsymbol{A} \times \bar{\boldsymbol{A}} \to \boldsymbol{A} : (a, h) \mapsto a + h$ (called *translation*). Thus, by definition, we have

$$\begin{aligned}
&(1) \qquad \forall a \in \boldsymbol{A},\ v, v' \in \bar{\boldsymbol{A}}, \qquad (a + v) + v' = a + (v + v'),\\
&(2) \qquad \forall a \in \boldsymbol{A}, \qquad a + 0 = a,\\
&(3) \qquad \forall a \in \boldsymbol{A},\ v \in \bar{\boldsymbol{A}}, \qquad a + v = a \qquad \Rightarrow \qquad v = 0,\\
&(4) \qquad \forall a, a' \in A, \qquad \exists v \in \bar{\boldsymbol{A}}, \quad \text{such that} \quad a' = a + v.
\end{aligned}$$

Given $a, a' \in \boldsymbol{A}$, we can easily see that the element $v \in \bar{\boldsymbol{A}}$ arising in (4) is unique. Hence, according to (4), we obtain a well defined *difference map* denote by

$$\boldsymbol{A} \times \boldsymbol{A} \to \bar{\boldsymbol{A}} : (a, a') \mapsto a' - a := v.$$

Moreover, a map $f : \boldsymbol{A} \to \boldsymbol{A}'$ between two affine spaces $\boldsymbol{A}$ and $\boldsymbol{A}'$ is said to be *affine* if, for each $a \in \boldsymbol{A}$ and $h \in \bar{\boldsymbol{A}}$, we have $f(a+h) = f(a) + Df(h)$, where $Df : \bar{\boldsymbol{A}} \to \bar{\boldsymbol{A}}'$ is a linear map. □

Definition A.3.4 An *affine bundle* is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ along with a "smoothly endowed" affine structure of its fibres.

A fibred morphism $f : \boldsymbol{F} \to \boldsymbol{F}'$ over $\underline{f} : \boldsymbol{B} \to \boldsymbol{B}'$ between two affine bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}'$ is said to be *affine* if it restricts to affine maps between their fibres. □

Note A.3.5 In other words, an affine bundle is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, along with a vector bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and a fibred morphism over $\boldsymbol{B}$ (called *fibred action*)

$$\tau : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F},$$

which, for each $b \in \boldsymbol{B}$, restricts to a free and transitive action $\tau_b : \boldsymbol{F}_b \times \bar{\boldsymbol{F}}_b \to \boldsymbol{F}_b$.

Clearly, the fibred action τ equips every fibre $\boldsymbol{F}_b$ with an affine structure associated with the vector space $\bar{\boldsymbol{F}}_b$.

Moreover, there exist a vector bundle atlas of $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and a bundle atlas of $p : \boldsymbol{F} \to \boldsymbol{B}$ (called *affine bundle atlas*) consisting of bundle charts

$$\bar{\Phi} : \bar{p}^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \bar{\mathsf{F}} \qquad \text{and} \qquad \Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F},$$

where F is an affine space associated with the vector space $\bar{\mathsf{F}}$ and, for each $b \in \boldsymbol{B}$, the fibred morphism Φ restricts to an affine isomorphism $\Phi_b : \boldsymbol{F}_b \to \mathsf{F}$.

A fibred chart (x^λ, y^i) adapted to the bundle is said to be *affine* if, for each $b \in \boldsymbol{B}$, the restricted map $(y^i)_b : \boldsymbol{F}_b \to \mathbb{R}^{\dim \boldsymbol{F}_b}$ is an affine isomorphism. □

A.3.3 Lie Group Bundles

The definition of *Lie group bundle* rephrases that of vector bundle by replacing the vector structure with a Lie group structure.

In the present book, we deal with some Lie affine bundles, mainly with relation to principal bundles. The most important one is the phase quantum bundle $\pi^{(0)}_{/0} : \boldsymbol{Q}^{(0)}_{/0} \to \boldsymbol{E}$ (see Proposition 14.7.1).

Definition A.3.6 A *Lie group bundle* is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ along with a "smoothly endowed" Lie group structure of its fibres.

A fibred morphism $f : \boldsymbol{F} \to \acute{\boldsymbol{F}}$ over $\underline{f} : \boldsymbol{B} \to \acute{\boldsymbol{B}}$ between two Lie group bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $\acute{p} : \acute{\boldsymbol{F}} \to \acute{\boldsymbol{B}}$ is said to be a *Lie group fibred morphism* if it restricts to group morphisms between their fibres. □

Note A.3.7 In other words, a Lie group bundle is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, along with a Lie group structure of every fibre $\boldsymbol{F}_b$, with $b \in \boldsymbol{B}$, such that there exists a bundle atlas consisting of bundle charts

$$\Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F},$$

where the type fibre F is a Lie group and, for each $b \in \boldsymbol{U}$, the fibred morphism Φ restricts to a Lie group isomorphism $\Phi_b : \boldsymbol{F}_b \to \mathsf{F}$.

Thus, every Lie group bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ turns out to be equipped with the "algebraic" fibred morphisms over $\boldsymbol{B}$ and the unit section

$$m : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F} : (\mathsf{f}, \acute{\mathsf{f}}) \mapsto \mathsf{f}\,\acute{\mathsf{f}}, \qquad i : \boldsymbol{F} \to \boldsymbol{F} : \mathsf{f} \mapsto \mathsf{f}^{-1}, \qquad e : \boldsymbol{B} \to \boldsymbol{F}. \quad \square$$

A.3.4 Lie Affine Bundles

The definition of *Lie affine bundle* rephrases that of affine bundle by replacing the associated vector bundle with a Lie group bundle. We stress the importance to make a distinction between Lie affine bundles and principal bundles.

In the present book, we deal with some principal bundles. In particular, we mention the phase quantum bundle $\pi^{(0)}_{/0} : \boldsymbol{Q}^{(0)}_{/0} \to \boldsymbol{E}$ (see Proposition 14.7.1).

The following rather unusual, but useful, definition resembles the standard definition of affine space.

Note A.3.8 A *right Lie affine space* is defined to be a set $\boldsymbol{A}$ along with a Lie group $\bar{\boldsymbol{A}}$ (called the *associated Lie group*) which acts on the right freely and transitively on $\boldsymbol{A}$ through a map $\tau : \boldsymbol{A} \times \bar{\boldsymbol{A}} \to \boldsymbol{A} : (a, h) \mapsto a\,h$. Thus, by definition, we have

$$\begin{aligned}
&(1) && \forall a \in \boldsymbol{A},\ h, h' \in \bar{\boldsymbol{A}}, && (a\,h)\,h' = a\,(h\,h'),\\
&(2) && \forall a \in \boldsymbol{A}, && a\,e = a,\\
&(3) && \forall a \in \boldsymbol{A},\ h \in \bar{\boldsymbol{A}}, && a\,h = a \quad \Rightarrow \quad h = e,\\
&(4) && \forall a, a' \in A, && \exists h \in \bar{\boldsymbol{A}}, \quad \text{such that} \quad a' = a\,h.
\end{aligned}$$

Given $a, a' \in \boldsymbol{A}$, we can easily see that the element $h \in \bar{\boldsymbol{A}}$ arising in (4) is unique. Hence, according to (4), we obtain a well defined *ratio map* denote by

$$\boldsymbol{A} \times \boldsymbol{A} \to \bar{\boldsymbol{A}} : (a, a') \mapsto a^{-1}a' := h.$$

Moreover, a map $f : \boldsymbol{A} \to \boldsymbol{A}'$ between two Lie affine spaces $\boldsymbol{A}$ and $\boldsymbol{A}'$ is said to be a *Lie affine morphism* if, for each $a \in \boldsymbol{A}$ and $h \in \bar{\boldsymbol{A}}$, we have $f(a\,h) = f(a)\,Df(h)$, where $Df : \bar{\boldsymbol{A}} \to \bar{\boldsymbol{A}}'$ is a Lie group morphism. $\square$

Definition A.3.9 A *Lie affine bundle* is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ along with a "smoothly endowed" Lie affine structure of its fibres.

A fibred morphism $f : \boldsymbol{F} \to \boldsymbol{F}'$ over $\underline{f} : \boldsymbol{B} \to \boldsymbol{B}'$ between two Lie affine bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}'$ is said to be *Lie affine* if its restricts to Lie affine maps between their fibres. □

Note A.3.10 In other words, a Lie affine bundle is defined to be a bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, along with a Lie group bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and a fibred morphism (called *fibred action*) over $\boldsymbol{B}$

$$\tau : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F},$$

which, for each $b \in \boldsymbol{B}$, restricts to a right free and transitive action $\tau_b : \boldsymbol{F}_b \times \bar{\boldsymbol{F}}_b \to \boldsymbol{F}_b$.

Clearly, the fibred action τ equips every fibre $\boldsymbol{F}_b$ with a Lie affine structure associated with the Lie group $\bar{\boldsymbol{F}}_b$.

Moreover, there exist a Lie group bundle atlas of $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and a bundle atlas of $p : \boldsymbol{F} \to \boldsymbol{B}$ (called *Lie affine bundle atlas*) consisting of bundle charts

$$\Phi : p^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \mathsf{F} \qquad \text{and} \qquad \bar{\Phi} : \bar{p}^{-1}(\boldsymbol{U}) \to \boldsymbol{U} \times \bar{\mathsf{F}},$$

where F is a Lie affine space associated with the Lie group $\bar{\mathsf{F}}$ and, for each $b \in \boldsymbol{B}$, the fibred morphism Φ restricts to a Lie affine isomorphism $\Phi_b : \boldsymbol{F}_b \to \mathsf{F}$. □

Remark A.3.11 A Lie affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ needs not to admits a global section $s : \boldsymbol{B} \to \boldsymbol{F}$. □

A.3.5 Principal Bundles

A *principal bundle* can be defined, in a straightforward way, just as as a Lie affine bundle associated with a trivial Lie group bundle. We stress that the present definition of principal bundle (involving the notion of Lie affine space) is essentially equivalent to the standard one in current literature, but, in our opinion, it is more direct and suitable for comparison with other structured bundles.

In the present book, we deal with some principal bundles. In particular, we mention the flat spacetime of standard Classical Mechanics $t : \boldsymbol{E} \to \boldsymbol{T}$ and the phase quantum bundle $\pi^{\circledcirc}_{/0} : \boldsymbol{Q}^{\circledcirc}_{/0} \to \boldsymbol{E}$ (see Propositions 24.1.2 and 14.7.1).

Definition A.3.12 A G-principal bundle is defined to be a Lie affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ associated with the trivial Lie group bundle

$$\bar{p} : \bar{\boldsymbol{F}} := \boldsymbol{B} \times G \to \boldsymbol{B}. \quad \square$$

Note A.3.13 A principal bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ is globally trivialisable if and only if it admits a global section.

Proof. (1) If $s : \boldsymbol{B} \to \boldsymbol{F}$ is a global section, then we obtain the global fibred affine isomorphism

$$\Phi : \boldsymbol{F} \to \boldsymbol{B} \times G : \mathfrak{f}_b \mapsto \big(b,\ (s(b))^{-1} f_b\big).$$

(2) If $f : \boldsymbol{F} \to \boldsymbol{B} \times G$ is a global affine fibred isomorphism, then we obtain the global section

$$s : \boldsymbol{B} \to \boldsymbol{F} : b \mapsto f^{-1}(b, e). \quad \square$$

Appendix B
Tangent Bundle

We recall a few notions concerning the *tangent prolongation* of manifolds and maps; in particular, we deal also with the tangent prolongation of fibred manifolds and fibred morphisms. In this context, we discuss the tangent prolongation of structured bundles.

Eventually, we recall a few notions concerning the iterated tangent bundle of a manifold.

B.1 Tangent Prolongation of Manifolds

We recall the standard tangent prolongation of manifolds and maps between manifolds, in order to establish our notation (for further details, see, for instance, [51, 146, 241, 242, 246]).

Let us consider a manifold $\boldsymbol{M}$ of dimension n and denote its typical chart by (x^λ).

Note B.1.1 For every point $p \in \boldsymbol{M}$, if two curves $c : \boldsymbol{I} \to \boldsymbol{M}$ and $c' : \boldsymbol{I}' \to \boldsymbol{M}$ fulfill in a chart the property

$$c^\mu(\lambda) = x^\mu(p) = c'^\mu(\lambda') \qquad \text{and} \qquad \partial_0 c^\mu(\lambda) = \partial_0 c'^\mu(\lambda'),$$

then they fulfill the same property in any other chart. In such a case we say that the two curves have a *1st order contact* in p.

The 1st order contact in p yields an equivalence relation between pairs (c, λ). We denote an equivalence class by $dc(\lambda) \equiv [(c, \lambda)]_\sim$. □

Definition B.1.2 We define:

- the *tangent space* at $p \in \boldsymbol{M}$ to be the set of equivalence classes of pairs which have a 1st order contact in p

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1

$$T_p\boldsymbol{M} := \{[(c, \lambda)]_\sim\},$$

- the *tangent space* of $\boldsymbol{M}$ to be the set

$$T\boldsymbol{M} := \bigsqcup_{p \in \boldsymbol{M}} T_p\boldsymbol{M}.$$

Accordingly, for each curve $c : \boldsymbol{I} \to \boldsymbol{M}$ we obtain the curve (called *tangent prolongation* of c)

$$dc : \boldsymbol{I} \to T\boldsymbol{M} : \lambda \mapsto dc(\lambda). \quad \square$$

Note B.1.3 The tangent space $T\boldsymbol{M}$ turns out to be, in a natural way, a smooth manifold and a vector bundle $\tau_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ with type fibre $\mathbb{R}^n$.

Moreover, for each curve $c : \boldsymbol{I} \to \boldsymbol{M}$, the curve $dc : \boldsymbol{I} \to T\boldsymbol{M}$ turns out to be smooth.

Each chart (x^λ) of $\boldsymbol{M}$ yields, in a natural way, the fibred chart $(x^\lambda, \dot{x}^\lambda)$ of $T\boldsymbol{M}$, which is characterised, for each curve $c : \boldsymbol{I} \to \boldsymbol{M}$, by the equality $\dot{x}^\lambda \circ dc = \partial_0 c^\lambda$.

Moreover, each chart (x^λ) of $\boldsymbol{M}$ yields, in a natural way, for each $x \in \boldsymbol{M}$, the basis $(\partial_\lambda(x)) \subset T_x\boldsymbol{M}$, consisting of the tangent prolongations

$$\partial_\lambda(x) := \partial x_\lambda(x) \in T_x\boldsymbol{M}$$

of the coordinate curves $x_\lambda : \mathbb{R} \times \boldsymbol{M} \to \boldsymbol{M}$, where ∂ denotes the tangent map, with respect to the parameter, evaluated at $0 \in \mathbb{R}$.

Then, the coordinate expression of a *tangent vector field* $X : \boldsymbol{M} \to T\boldsymbol{M}$ is of the type

$$X = X^\lambda\, \partial_\lambda, \quad \text{with } X^\lambda \in \operatorname{map}(\boldsymbol{M}, \mathbb{R}). \quad \square$$

Note B.1.4 For each map $f : \boldsymbol{M} \to \boldsymbol{N}$ there is a unique fibred morphism over f (called the *tangent prolongation of* f)

$$Tf : T\boldsymbol{M} \to T\boldsymbol{N},$$

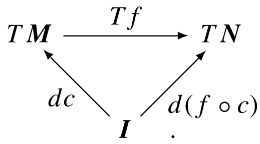
according to the following commutative diagram

$$\begin{array}{ccc} T\boldsymbol{M} & \xrightarrow{\;Tf\;} & T\boldsymbol{N} \\ {\scriptstyle \tau_{\boldsymbol{M}}}\downarrow & & \downarrow{\scriptstyle \tau_{\boldsymbol{N}}} \\ \boldsymbol{M} & \xrightarrow{\;f\;} & \boldsymbol{N} \end{array},$$

which, for each curve $c : \boldsymbol{I} \to \boldsymbol{M}$, fulfills the equality

$$Tf \circ dc = d(f \circ c),$$

according to the following commutative diagram

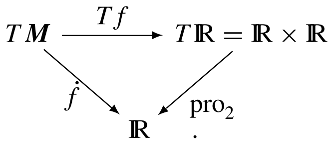


Indeed, the fibred morphism Tf turns out to be linear and has coordinate expression

$$(y^\lambda, \dot{y}^\lambda) \circ Tf = (f^\lambda, \partial_\mu f^\lambda \, \dot{x}^\mu).$$

Moreover, for each manifold $\boldsymbol{M}$, we have the equality $T\,\mathrm{id}_{\boldsymbol{M}} = \mathrm{id}_{T\boldsymbol{M}}$ and, for each pair of maps $f : \boldsymbol{M} \to \boldsymbol{M}'$ and $f' : \boldsymbol{M}' \to \boldsymbol{M}''$, we have the equality $T(f' \circ f) = Tf' \circ Tf$. □

Note B.1.5 For each function $f : \boldsymbol{M} \to \mathbb{R}$, we obtain the function $\dot{f} := \mathrm{pro}_2 \circ Tf$, according to the following commutative diagram



Indeed, the above notation $\dot{f}$ is consistent with the notation $\dot{x}^\lambda$ used in Note B.1.3. □

Note B.1.6 The tangent vector fields constitute an $\mathbb{R}$-Lie algebra with respect to the natural bracket

$$[\,,\,] : \mathrm{sec}(\boldsymbol{M}, T\boldsymbol{M}) \times \mathrm{sec}(\boldsymbol{M}, T\boldsymbol{M}) \to \mathrm{sec}(\boldsymbol{M}, T\boldsymbol{M}) : (X, Y) \mapsto [X, Y],$$

with coordinate expression

$$[X, Y] = (X^\mu \, \partial_\mu Y^\lambda - Y^\mu \, \partial_\mu X^\lambda) \, \partial_\lambda. \quad \square$$

Note B.1.7 We define the *cotangent bundle* of the manifold $\boldsymbol{M}$ to be the dual bundle $\tau^{\boldsymbol{M}} : T^*\boldsymbol{M} \to \boldsymbol{M}$ of the tangent bundle $\tau_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$.

We denote the fibred charts naturally induced on the cotangent bundle by $(x^\lambda, \dot{x}_\lambda)$.

For each function $f : \boldsymbol{M} \to \mathbb{R}$, the map $\dot{f} : T\boldsymbol{M} \to \mathbb{R}$ can be naturally identified with the section (called *differential* of f) $df : \boldsymbol{M} \to T^*\boldsymbol{M}$.

Moreover, each chart (x^λ) of $\boldsymbol{M}$ yields, in a natural way, for each $x \in \boldsymbol{M}$, the basis $(d^\lambda(x)) \subset T^*\boldsymbol{M}_x$, consisting of the differentials $d^\lambda(x) := dx^\lambda(x) \in T^*_x\boldsymbol{M}$ of the coordinate functions $x^\lambda : \boldsymbol{M} \to \mathbb{R}$.

Then, the coordinate expression of a *1-form* $\alpha : \boldsymbol{M} \to T^*\boldsymbol{M}$ is of the type

$$\alpha = \alpha_\lambda \, d^\lambda, \qquad \text{with } \alpha_\lambda \in \operatorname{map}(\boldsymbol{M}, \mathbb{R}). \quad \square$$

The tangent spaces of vector spaces, affine spaces, Lie groups and Lie affine spaces turn out to be trivial bundles, which inherit the algebraic structure of their source spaces.

Example B.1.8 We have the following distinguished examples of tangent spaces of manifolds equipped with algebraic structure.

The tangent space of a vector space $\boldsymbol{V}$ turns out to be the trivial bundle $T\boldsymbol{V} = \boldsymbol{V} \times \boldsymbol{V}$.

The tangent space of an affine space $\boldsymbol{A}$ turns out to be the trivial bundle $T\boldsymbol{A} = \boldsymbol{A} \times \bar{\boldsymbol{A}}$.

The tangent space of a Lie group G turns out to be the trivial bundle $TG = G \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of the group.

The tangent space of a Lie affine space $\boldsymbol{A}$ associated with the Lie group G turns out to be the trivial bundle $T\boldsymbol{A} = \boldsymbol{A} \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. □

B.2 Tangent Prolongation of Fibred Manifolds

We discuss the essential notions concerning the tangent prolongation of fibred manifolds. In particular, we recall the vertical prolongation and discuss the *projectable vector fields*.

In the present book we deal with tangent prolongations of several fibred manifolds. In particular we mention the tangent prolongation $T\boldsymbol{t} : T\boldsymbol{E} \to T\boldsymbol{T}$ of the spacetime fibring (see Definition 2.2.1).

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical fibred charts by (x^λ, y^i) (see Sect. A.3.1).

Note B.2.1 The tangent prolongation of the manifold $\boldsymbol{F}$ makes it a vector bundle over $\boldsymbol{F}$ and a fibred manifold over $T\boldsymbol{B}$

$$\tau_{\boldsymbol{F}} : T\boldsymbol{F} \to \boldsymbol{F} \qquad \text{and} \qquad Tp : T\boldsymbol{F} \to T\boldsymbol{B},$$

according to the following commutative diagram

$$\begin{array}{ccc} T\boldsymbol{F} & \xrightarrow{\tau_{\boldsymbol{F}}} & \boldsymbol{F} \\ {\scriptstyle Tp}\downarrow & & \downarrow{\scriptstyle p} \\ T\boldsymbol{B} & \xrightarrow{\tau_{\boldsymbol{B}}} & \boldsymbol{B} \end{array} \quad .$$

The induced fibred charts are, respectively,

$$(x^\lambda, y^i;\ \dot{x}^\lambda, \dot{y}^i) \qquad \text{and} \qquad (x^\lambda, \dot{x}^\lambda;\ y^i, \dot{y}^i).$$

Moreover, the induced basis of vector fields of $\boldsymbol{F}$ is $(\partial_\lambda, \partial_i)$.

Hence, the coordinate expression of a vector field $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ is of the type

$$X = X^\lambda \, \partial_\lambda + X^i \, \partial_i, \quad \text{with } X^\lambda \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}), \quad X^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}).$$

With reference two the fibred charts (x^λ, y^i) and (x'^μ, y'^j), we have the transition rules

$$\dot{x}'^\mu = \partial_\lambda x'^\mu \, \dot{x}^\lambda \qquad \text{and} \qquad \dot{y}'^j = \partial_\lambda y'^j \, \dot{x}^\lambda + \partial_i y'^j \, \dot{y}^i. \quad \square$$

Note B.2.2 We define the *vertical (tangent) bundle* of $p : \boldsymbol{F} \to \boldsymbol{B}$ to be the vector subbundle over $\boldsymbol{F}$

$$V\boldsymbol{F} := \ker(Tp) \subset T\boldsymbol{F}.$$

Indeed, the vertical subbundle $(V\boldsymbol{F})_b$, over $b \in \boldsymbol{B}$, consists of the tangent vectors dc of vertical curves c, i.e. of curves $c : \boldsymbol{I} \to \boldsymbol{F}_b \subset \boldsymbol{F}$, valued in the fibre $\boldsymbol{F}_b$.

Thus, $V\boldsymbol{F} \subset T\boldsymbol{F}$ turns out to be the vector subbundle over $\boldsymbol{F}$ characterised by the constraint $\dot{x}^\lambda = 0$, according to the following commutative diagram

$$\begin{array}{ccc} V\boldsymbol{F} & \xrightarrow{\tau_{\boldsymbol{F}}} & \boldsymbol{F} \\ {\scriptstyle Tp}\downarrow & & \downarrow{\scriptstyle p} \\ T\boldsymbol{B} & \xleftarrow{0} & \boldsymbol{B} \end{array} .$$

Hence, the induced fibred chart of $V\boldsymbol{F}$ is $(x^\lambda, y^i;\ \dot{y}^i)$ and the induced basis of vertical vector fields is (∂_i).

Therefore, the coordinate expression of a vertical vector field $X \in \sec(\boldsymbol{F}, V\boldsymbol{F})$ is of the type

$$X = X^i \, \partial_i, \quad \text{with } X^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}). \quad \square$$

Note B.2.3 We have the natural linear injective and the linear surjective fibred morphism over $\boldsymbol{F}$

$$j_{\boldsymbol{F}} : V\boldsymbol{F} \to T\boldsymbol{F} \qquad \text{and} \qquad \pi_{\boldsymbol{F}} := (\tau_{\boldsymbol{F}}, Tp) : T\boldsymbol{F} \to \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B},$$

such that $\operatorname{im}(j_{\boldsymbol{F}}) = \ker(\pi_{\boldsymbol{F}})$.

We have the coordinate expressions

$$(x^\lambda, y^i; \dot{x}^\lambda, \dot{y}^i) \circ j_{\boldsymbol{F}} = (x^\lambda, y^i; 0, \dot{y}^i) \qquad \text{and} \qquad (x^\lambda, y^i; \dot{x}^\lambda) \circ \pi_{\boldsymbol{F}} = (x^\lambda, y^i; \dot{x}^\lambda).$$

Moreover, the fibred manifold

$$\pi_{\boldsymbol{F}} : T\boldsymbol{F} \to \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}$$

turns out to be an affine bundle associated with the vector bundle $\tau_{\boldsymbol{F}} : V\boldsymbol{F} \to \boldsymbol{F}$. □

Note B.2.4 A vector field $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ is said to be *projectable* if it factorises through a vector field $\underline{X} \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ (called the *base projection* of X), according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;\;X\;\;} & T\boldsymbol{F} \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle Tp} \\ \boldsymbol{B} & \xrightarrow{\;\;\underline{X}\;\;} & T\boldsymbol{B} \end{array} .$$

A vector field $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ turns out to be projectable if and only if its coordinate expression is of the type

$$X = X^\lambda \,\partial_\lambda + X^i \,\partial_i, \quad \text{with } X^\lambda \in \operatorname{map}(\boldsymbol{B}, \mathbb{R}), \quad X^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}).$$

We denote the subsheaf of (local) projectable vector fields by

$$\operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F}) \subset \sec(\boldsymbol{F}, T\boldsymbol{F}).$$

In particular, the vertical vector fields $X \in \sec(\boldsymbol{F}, V\boldsymbol{F})$ turn out to be just the vector fields $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, which are projectable on $\underline{X} = 0$. □

Remark B.2.5 Clearly, the vertical vector fields $X : \boldsymbol{F} \to V\boldsymbol{F}$ are vector fields $X : \boldsymbol{F} \to T\boldsymbol{F}$ valued in the subspace $V\boldsymbol{F} \subset T\boldsymbol{F}$. However, the projectable vector fields cannot be defined by selecting a subbundle of $T\boldsymbol{F}$. □

Note B.2.6 The subsheaf $\operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F}) \subset \sec(\boldsymbol{F}, T\boldsymbol{F})$ of projectable vector fields is closed with respect to the Lie bracket $[,]$.

Even more, the Lie bracket commutes with respect to the base projection, i.e. we have the equality

$$\underline{[X, Y]} = [\underline{X}, \underline{Y}], \quad \text{for each } X,\ Y \in \operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F}).$$

For each $X, Y \in \operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F})$, we have the coordinate expression

$$[X, Y] = (X^\mu \,\partial_\mu Y^\lambda - Y^\mu \,\partial_\mu X^\lambda)\,\partial_\lambda + (X^\mu \,\partial_\mu Y^i + X^j \,\partial_j Y^i - Y^\mu \,\partial_\mu X^i - Y^j \,\partial_j X^i)\,\partial_i.$$

In particular, the subsheaf $\sec(\boldsymbol{F}, V\boldsymbol{F})$ of vertical vector fields is closed with respect to the Lie bracket $[,]$. For each $X, Y \in \sec(\boldsymbol{F}, V\boldsymbol{F})$, we have the coordinate expression

$$[X, Y] = (X^j \,\partial_j Y^i - Y^j \,\partial_j X^i)\,\partial_i. \quad \square$$

Note B.2.7 If $X \in \operatorname{pro}\sec(\boldsymbol{F},\ T\boldsymbol{F})$ and $Y \in \sec(\boldsymbol{F},\ V\boldsymbol{F})$, then their Lie bracket turns out to be a vertical vector field $[X, Y] \in \sec(\boldsymbol{F},\ V\boldsymbol{F})$, with coordinate expression

$$[X, Y] = (X^\mu \,\partial_\mu Y^i + X^j \,\partial_j Y^i - Y^j \,\partial_j X^i)\,\partial_i.$$

Hence, $\sec(\boldsymbol{F}, V\boldsymbol{F})$ is an ideal of $\operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F})$, with respect to the Lie bracket $[,]$. □

B.3 Tangent Prolongation of Structured Bundles

We discuss the tangent prolongations of vector bundles, affine bundles, Lie group bundles, Lie affine bundles and principal bundles.

Actually, the tangent functor T is a covariant functor, hence it preserves the algebraic properties of the algebraic operations assumed on the fibres of the structured bundles.

Note B.3.1 If $p : \boldsymbol{F} \to \boldsymbol{B}$ is a bundle with type fibre F, then its tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be a bundle with type fibre $T\mathsf{F}$, according to the commutative diagram

$$\begin{array}{ccc} T\boldsymbol{F} & \xrightarrow{T\Phi} & T\boldsymbol{B} \times T\mathsf{F} \\ \tau_{\boldsymbol{F}} \downarrow & & \downarrow \tau_{\boldsymbol{B}} \times \tau_{\mathsf{F}} \\ \boldsymbol{F} & \xrightarrow{\Phi} & \boldsymbol{B} \times \mathsf{F} \end{array} \; . \square$$

B.3.1 Tangent Prolongation of Vector Bundles

We discuss the vector structure of the tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ of a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$.

In this case, we obtain some relevant facts, which are often used throughout the present book. In fact, we have a natural linear splitting of the vertical tangent bundle $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$, which yields the concept of Liouville vector field $\mathbb{I} : \boldsymbol{F} \to V\boldsymbol{F}$ and the natural identification between sections and "basic" vertical vector fields $s \simeq Y$.

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical linear fibred charts by (x^λ, y^i).

Proposition B.3.2 *The tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be a vector bundle. Indeed, this prolonged vector structure is provided by the algebraic fibred morphisms and by the zero section over $T\boldsymbol{B}$*

$$T+ : T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{F}, \qquad T\cdot : \mathbb{R} \times T\boldsymbol{F} \to T\boldsymbol{F} \qquad \textit{and} \qquad T0 : T\boldsymbol{B} \to T\boldsymbol{F},$$

which are given by the tangent prolongations of the fibred morphisms and of the zero section over $\boldsymbol{B}$

$$+ : F \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}, \qquad \cdot : \mathbb{R} \times \boldsymbol{F} \to \boldsymbol{F} \qquad \textit{and} \qquad 0 : \boldsymbol{B} \to \boldsymbol{F}.$$

The linear fibred charts induced on the vector bundle $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ *are* $(x^\lambda, \dot{x}^\lambda;\ y^i, \dot{y}^i)$.

The transition rule between two such linear fibred charts of $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ *is*

$$\acute{y}^i = \acute{S}^i_j\, y^j \qquad \textit{and} \qquad \dot{\acute{y}}^i = (\partial_\lambda \acute{S}^i_j)\, \dot{x}^\lambda\, y^j + \acute{S}^i_j\, \dot{y}^j,$$

where $\acute{S}^i_j \in \mathrm{map}(\boldsymbol{B}, I\!R)$. □

We have a natural linear splitting of the vertical tangent bundle $V\boldsymbol{F}$.

Proposition B.3.3 *We have a natural linear fibred isomorphism over* $\boldsymbol{F}$

$$V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F},$$

according to the following commutative diagram

$$\begin{array}{ccc} V\boldsymbol{F} & \xrightarrow{\ \simeq\ } & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_1} \\ V\boldsymbol{F} & \xrightarrow{\ \tau_{\boldsymbol{F}}\ } & \boldsymbol{F} \end{array} \quad .$$

The linear fibred charts induced on the vector bundle $Vp : V\boldsymbol{F} \to \boldsymbol{B}$ *are* $(x^\lambda;\ y^i, \dot{y}^i)$. □

Corollary B.3.4 *We have a natural linear fibred inclusion* $\jmath : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \subset T\boldsymbol{F}$ *over* $\boldsymbol{F}$, *with coordinate expression* $(x^\lambda, y^i;\ \dot{x}^\lambda, \dot{y}^i) \circ \jmath = (x^\lambda, y^i_1;\ 0, y^i_2)$. □

On a vector bundle, we have distinguished types of vector fields: the *linear projectable vector fields*, the *basic vector fields* and the *Liouville vector field*.

Note B.3.5 By recalling Proposition B.3.2, a vector field $Y : \boldsymbol{F} \to T\boldsymbol{F}$, which is projectable on the base vector field $X : \boldsymbol{B} \to T\boldsymbol{B}$, is said to be *linear* if it is a linear fibred morphism over $X : \boldsymbol{B} \to T\boldsymbol{B}$, according to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\ Y\ } & T\boldsymbol{F} \\ \downarrow & & \downarrow \\ \boldsymbol{B} & \xrightarrow{\ X\ } & T\boldsymbol{B} \end{array} \quad .$$

We denote the subsheaf of linear projectable vector fields by

$$\mathrm{lin\,pro\,sec}(\boldsymbol{F}, T\boldsymbol{F}) \subset \mathrm{pro\,sec}(\boldsymbol{F}, T\boldsymbol{F}) \subset \mathrm{sec}(\boldsymbol{F}, T\boldsymbol{F}).$$

The coordinate expression of linear projectable vector fields is of the type

$$Y = X^\lambda\, \partial_\lambda + Y^i_j\, y^j\, \partial_i, \quad \text{with } X^\lambda,\ Y^i_j \in \operatorname{map}(\boldsymbol{B}, \mathbb{R}). \quad \square$$

Let us take recall again the natural linear isomorphism $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$ over $\boldsymbol{F}$ (see Proposition B.3.3).

Note B.3.6 The following natural mutually inverse identifications hold.

(1) If $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ is a section, then we obtain the vertical vector field $\widetilde{s} \in \sec(\boldsymbol{F}, V\boldsymbol{F})$ given by the map $\widetilde{s} : \boldsymbol{F} \to V\boldsymbol{F} : \mathsf{f} \mapsto \big(\mathsf{f},\, s\big(p(\mathsf{f})\big)\big)$.
Thus, by definition, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\ \widetilde{s}\ } & V\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{B} & \xrightarrow{\ s\ } & \boldsymbol{F} \end{array}$$

and

$$V\widetilde{s} = 0 : V\boldsymbol{F} \to VV\boldsymbol{F}.$$

In other words, in coordinates, we have $\widetilde{s} = s^i\, \partial_i$ and $\partial_j \widetilde{s}^i = 0$.

(2) If $Y \in \sec(\boldsymbol{F},\ V\boldsymbol{F})$ is a vertical vector field such that $VY = 0 : V\boldsymbol{F} \to VV\boldsymbol{F}$, i.e., in coordinates, such that $Y = Y^i\, \partial_i$ with $\partial_j Y^i = 0$, then Y projects over a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$, through the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\ Y\ } & V\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{B} & \xrightarrow{\ s\ } & \boldsymbol{F} \end{array}$$

and we have $Y^i = s^i$.

(3) Indeed, the above maps $s \mapsto \widetilde{s}$ and $Y \mapsto s$ are mutually inverse bijections.

The vertical vector fields as above $Y \in \sec(\boldsymbol{F},\ V\boldsymbol{F})$ are called *basic*.

According to the above observations, we shall often write $s \simeq Y \simeq \widetilde{s}$. $\square$

Later, we shall extend the above result to vector valued forms and vertical tangent valued forms (see Note C.2.16).

Note B.3.7 We define the *Liouville vector field* of the vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ to be the distinguished vertical vector field

$$\mathbb{I} : \boldsymbol{F} \to V\boldsymbol{F} \subset T\boldsymbol{F} : \mathsf{f} \mapsto (\mathsf{f}, \mathsf{f}),$$

whose coordinate expression is $\mathbb{I} = y^i\, \partial_i$. $\square$

B.3.2 Tangent Prolongation of Affine Bundles

We discuss the affine structure of the tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ of affine bundles $p : \boldsymbol{F} \to \boldsymbol{B}$. In this case, we obtain the linear fibred splitting $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}}$, which is often used throughout the present book.

Let us consider an affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ associated with the vector bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and denote its typical affine fibred charts by (x^λ, y^i) (see Sect. A.3.2).

Proposition B.3.8 *The tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be an affine bundle associated with the vector bundle $T\bar{p} : T\bar{\boldsymbol{F}} \to T\boldsymbol{B}$, Indeed, this prolonged affine structure is provided by the algebraic fibred morphism over $T\boldsymbol{B}$*

$$T\tau : T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\bar{\boldsymbol{F}} \to T\boldsymbol{F},$$

which is given by the tangent prolongation of the fibred morphism over $\boldsymbol{B}$

$$\tau : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F}.$$

The affine fibred charts induced on the affine bundle $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ are $(x^\lambda, \dot{x}^\lambda;\; y^i, \dot{y}^i)$.

The transition rule between two such affine fibred charts of $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ is

$$\acute{y}^i = \acute{S}^i_j\, y^j + \acute{U}^i \qquad \text{and} \qquad \dot{\acute{y}}^i = (\partial_\lambda \acute{S}^i_j)\, \dot{x}^\lambda\, y^j + (\partial_\lambda \acute{U}^i)\, \dot{x}^\lambda + \acute{S}^i_j\, \dot{y}^j,$$

where $\acute{S}^i_j,\; \acute{U}^i \in \mathrm{map}(\boldsymbol{B}, I\!R)$. □

We have a natural linear splitting of the vertical tangent bundle $V\boldsymbol{F}$.

Proposition B.3.9 *We have a natural linear fibred isomorphism over $\boldsymbol{F}$*

$$V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}},$$

according to the following commutative diagram

$$\begin{array}{ccc} V\boldsymbol{F} & \xrightarrow{\;\simeq\;} & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_1} \\ V\boldsymbol{F} & \xrightarrow{\;\tau_{\boldsymbol{F}}\;} & \boldsymbol{F} \end{array} .$$

The linear fibred charts induced on the vector bundle $Vp : V\boldsymbol{F} \to \boldsymbol{B}$ are $(x^\lambda;\; y^i, \dot{y}^i)$. □

B.3.3 Tangent Prolongation of Lie Group Bundles

We discuss the Lie group structure of the tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ of Lie group bundles $p : \boldsymbol{F} \to \boldsymbol{B}$.

Let us consider a Lie group bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, along with the associated Lie algebra bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ (see Sect. A.3.3).

Proposition B.3.10 *The tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be a Lie group bundle. Indeed, this prolonged Lie group structure is provided by the algebraic fibred morphism and by the unit section over $T\boldsymbol{B}$*

$$T\cdot : T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\boldsymbol{F} \to T\boldsymbol{F} \qquad \textit{and} \qquad T\mathrm{e} : T\boldsymbol{B} \to T\boldsymbol{F},$$

which are given by the tangent prolongations of the fibred morphism and of the unit section over $\boldsymbol{B}$

$$\cdot : F \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}, \qquad \textit{and} \qquad \mathrm{e} : \boldsymbol{B} \to \boldsymbol{F}.$$

In particular, if $p : \boldsymbol{F} := \boldsymbol{B} \times \boldsymbol{G} \to \boldsymbol{B}$ is a trivial Lie group bundle, then we obtain

$$T\boldsymbol{F} = T\boldsymbol{B} \times (G \times \mathfrak{g}),$$

where $\mathfrak{g}$ is the Lie algebra of $\boldsymbol{g}$. □

We have a natural linear splitting of the vertical tangent bundle $V\boldsymbol{F}$.

Proposition B.3.11 *We have a natural linear fibred isomorphism $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}}$ over $\boldsymbol{F}$ according to the following commutative diagram*

$$\begin{array}{ccc} V\boldsymbol{F} & \xrightarrow{\ \simeq\ } & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_1} \\ V\boldsymbol{F} & \xrightarrow{\ \tau_{\boldsymbol{F}}\ } & \boldsymbol{F} \end{array} .$$

Hence, the vertical tangent prolongation of $p : \boldsymbol{F} \to \boldsymbol{B}$ turns out to be the fibred product $Vp : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{B}$ over $\boldsymbol{B}$. □

B.3.4 Tangent Prolongation of Lie Affine Bundles

We discuss the Lie affine structure of the tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ of Lie affine bundles $p : \boldsymbol{F} \to \boldsymbol{B}$. Then, the result found for Lie affine bundles is easily translated for principal bundles, by taking into account the triviality of their associated Lie group bundles.

Let us consider a Lie affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ associated with the Lie group bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$; moreover, let us consider the Lie algebra bundle $\tilde{p} : \mathfrak{F} \to \boldsymbol{B}$ associated with the Lie group bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$.

Proposition B.3.12 *The tangent prolongation* $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ *of* $p : \boldsymbol{F} \to \boldsymbol{B}$ *turns out to be a Lie affine bundle associated with the Lie group bundle* $T\bar{p} : T\bar{\boldsymbol{F}} \to T\boldsymbol{B}$.

Indeed, this prolonged vector structure is provided by the algebraic fibred morphism over $T\boldsymbol{B}$

$$T\tau : T\boldsymbol{F} \underset{T\boldsymbol{B}}{\times} T\bar{\boldsymbol{F}} \to T\boldsymbol{F}$$

which is given by the tangent prolongation of the fibred morphism over $\boldsymbol{B}$

$$\tau : F \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F}. \quad \square$$

We have a natural linear splitting of the vertical tangent bundle $V\boldsymbol{F}$.

Proposition B.3.13 *We have a natural linear fibred isomorphism over* $\boldsymbol{F}$

$$V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \mathfrak{F},$$

according to the following commutative diagram

$$\begin{array}{ccc} V\boldsymbol{F} & \xrightarrow{\ \simeq\ } & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \mathfrak{F} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_1} \\ V\boldsymbol{F} & \xrightarrow{\ \tau_{\boldsymbol{F}}\ } & \boldsymbol{F} \end{array} \quad . \square$$

The above results can be applied to principal bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ in a straightforward way, by taking into account the associated Lie group bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ is trivial.

Corollary B.3.14 *Let* $p : \boldsymbol{F} \to \boldsymbol{B}$ *be a principal bundle associated with the trivial Lie group bundle* $\bar{p} : (\boldsymbol{B} \times G) \to \boldsymbol{B}$.

Then, its tangent prolongation $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ *turns out to be a principal bundle associated with the trivial Lie group bundle* $T\bar{p} : \big(T\boldsymbol{B} \times (G \times \mathfrak{g})\big) \to T\boldsymbol{B}$, *where* $\mathfrak{g}$ *is the Lie algebra of* G. $\square$

B.4 Iterated Tangent Bundle

In Sect. B.2 we have discussed the tangent prolongation $\tau_{\boldsymbol{F}} : T\boldsymbol{F} \to T\boldsymbol{B}$ of any fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$. Now, we analyse a distinguished particular case.

Namely, we study the tangent prolongation $\tau_{TM} : TTM \to TM$ of the tangent bundle $\tau_M : TM \to M$ of a manifold M.

In this case, we have $F = TB$, hence several interesting facts occur (see, for instance, [146]). In particular, we obtain two distinct fibrings of TTM, a natural inclusion and a natural involution, which yields a prolongation of vector fields of M to vector fields of TM.

The iterated tangent bundle of a manifold is used in several contexts of the present book. In particular, it is used for the discussion on the galilean spacetime connections (see Definition 4.1.1).

Let us consider a manifold M and denote its typical charts by (x^λ).

Note B.4.1 We define the *iterated tangent space* of M to be the iterated tangent prolongation TTM of M, which turns out to be both a vector bundle

$$\tau_{TM} : TTM \to TM$$

and an affine bundle

$$T\tau_M : TTM \to TM$$

associated with the vector bundle

$$\mathrm{pro}_1 : VTM = TM \underset{M}{\times} TM \to TM.$$

Thus, the following diagram commutes

$$\begin{array}{ccc} TTM & \xrightarrow{\ \tau_{TM}\ } & TM \\ {\scriptstyle T\tau_M}\downarrow & & \downarrow{\scriptstyle \tau_M} \\ TM & \xrightarrow{\ \tau_M\ } & M \end{array} \quad .$$

The fibred charts naturally induced on the above bundles

$$\tau_{TM} : TTM \to TM, \qquad T\tau_M : TTM \to TM, \qquad TM \underset{M}{\times} TM \to M$$

will be denoted, respectively, by

$$(x^\lambda, \dot{x}^\lambda; \overset{\shortparallel}{x}{}^\lambda, \ddot{x}^\lambda), \qquad (x^\lambda, \overset{\shortparallel}{x}{}^\lambda; \dot{x}^\lambda, \ddot{x}^\lambda), \qquad (x^\lambda, \dot{x}^\lambda; \overset{\shortparallel}{x}{}^\lambda).$$

Moreover, the coordinate expressions of the above projections τ_{TM} and $T\tau_M$ are

$$(x^\lambda, \dot{x}^\lambda) \circ \tau_{TM} = (x^\lambda, \dot{x}^\lambda) \qquad \text{and} \qquad (x^\lambda, \dot{x}^\lambda) \circ T\tau_M = (x^\lambda, \overset{\shortparallel}{x}{}^\lambda).$$

With reference to the charts (x^μ) and (x'^λ), we have the transition rules

$$\dot{x}'^{\lambda} = \partial_{\mu} x'^{\lambda}\, \dot{x}^{\mu}, \qquad \overset{\shortparallel}{x}'^{\lambda} = \partial_{\mu} x'^{\lambda}\, \overset{\shortparallel}{x}^{\mu}, \qquad \ddot{x}'^{\lambda} = \partial_{\mu\nu} x'^{\lambda}\, \dot{x}^{\nu}\, \dot{x}^{\nu} + \partial_{\mu} x'^{\lambda}\, \ddot{x}^{\mu}. \quad \square$$

Next, we recall a distinguished natural injective linear fibred morphism.

Note B.4.2 By taking into account the equality $VT\boldsymbol{M} = T\boldsymbol{M} \underset{\boldsymbol{M}}{\times} T\boldsymbol{M}$, the natural linear fibred inclusion over $T\boldsymbol{M}$

$$j_{\boldsymbol{M}} : VT\boldsymbol{M} \to TT\boldsymbol{M}$$

can be regarded as a natural fibred inclusion over $T\boldsymbol{M}$

$$\nu_{T\boldsymbol{M}} : T\boldsymbol{M} \underset{\boldsymbol{M}}{\times} T\boldsymbol{M} \to TT\boldsymbol{M},$$

with coordinate expression

$$(x^{\lambda}, \dot{x}^{\lambda}; \overset{\shortparallel}{x}^{\lambda}, \ddot{x}^{\lambda}) \circ \nu_{T\boldsymbol{M}} = (x^{\lambda}, \dot{x}^{\lambda}_{1}; 0, \dot{x}^{\lambda}_{2}). \quad \square$$

Eventually, we recall the natural involution of the iterated tangent bundle.

Lemma B.4.3 *Given a map* $f : \mathbb{R} \times \mathbb{R} \to \boldsymbol{M}$, *we obtain the four maps*

$$d_1 f : \mathbb{R} \times \mathbb{R} \to T\boldsymbol{M}, \qquad d_2 d_1 f : \mathbb{R} \times \mathbb{R} \to TT\boldsymbol{M},$$
$$d_2 f : \mathbb{R} \times \mathbb{R} \to T\boldsymbol{M}, \qquad d_1 d_2 f : \mathbb{R} \times \mathbb{R} \to TT\boldsymbol{M},$$

with coordinate expressions

$$\begin{aligned}(x^{\lambda}, \dot{x}^{\lambda}) \circ d_1 f &= (f^{\lambda},\ \partial_1 f^{\lambda}) \quad (x^{\lambda}, \dot{x}^{\lambda};\ \overset{\shortparallel}{x}^{\lambda}, \ddot{x}^{\lambda}) \circ d_2 d_1 f \\ &= (f^{\lambda},\ \partial_1 f^{\lambda};\ \partial_2 f^{\lambda},\ \partial_2 \partial_1 f^{\lambda}) \\ (x^{\lambda}, \dot{x}^{\lambda}) \circ d_2 f &= (f^{\lambda},\ \partial_2 f^{\lambda}) \quad (x^{\lambda}, \dot{x}^{\lambda};\ \overset{\shortparallel}{x}^{\lambda}, \ddot{x}^{\lambda}) \circ d_1 d_2 f \\ &= (f^{\lambda},\ \partial_2 f^{\lambda};\ \partial_1 f^{\lambda},\ \partial_1 \partial_2 f^{\lambda}).\end{aligned}$$

Indeed, we have

$$\partial_1 \partial_2 f^{\lambda} = \partial_2 \partial_1 f^{\lambda}$$

$$\tau_{T\boldsymbol{M}} \circ d_2 d_1 f = d_1 f = T\tau_{\boldsymbol{M}} \circ d_1 d_2 f,$$
$$\tau_{T\boldsymbol{M}} \circ d_1 d_2 f = d_2 f = T\tau_{\boldsymbol{M}} \circ d_2 d_1 f. \quad \square$$

Lemma B.4.4 *For each* $Y \in TT\boldsymbol{M}$, *there exists a map* $f : \mathbb{R} \times \mathbb{R} \to \boldsymbol{M}$, *such that* $(d_2 d_1 f)(0, 0) = Y$. $\square$

Proposition B.4.5 *There exists a unique map*

$$\sigma : TT\boldsymbol{M} \to TT\boldsymbol{M},$$

such that, for each map $f : I\!R \times I\!R \to \boldsymbol{M}$, *the following diagram commutes*

$$\begin{array}{ccc} T T\boldsymbol{M} & \xrightarrow{\ \sigma\ } & T T\boldsymbol{M} \\ {\scriptstyle d_2 d_1 f}\ \nwarrow & & \nearrow\ {\scriptstyle d_1 d_2 f} \\ & I\!R \times I\!R & . \end{array}$$

Moreover, the map σ *makes the following diagram commutative*

$$\begin{array}{ccc} T T\boldsymbol{M} & \xrightarrow{\ \sigma\ } & T T\boldsymbol{M} \\ {\scriptstyle \tau_{T\boldsymbol{M}} \times T\tau_{\boldsymbol{M}}}\ \searrow & & \swarrow\ {\scriptstyle T\tau_{\boldsymbol{M}} \times \tau_{T\boldsymbol{M}}} \\ & T\boldsymbol{M} \underset{\boldsymbol{M}}{\times} T\boldsymbol{M} & \end{array}$$

and is an involution, i.e. $\sigma \circ \sigma = \mathrm{id}_{T T\boldsymbol{M}}$.

We have the following coordinate expression

$$(x^\lambda, \dot{x}^\lambda;\ \overset{\shortparallel}{x}{}^\lambda, \ddddot{x}^\lambda) \circ \sigma = (x^\lambda, \overset{\shortparallel}{x}{}^\lambda;\ \dot{x}^\lambda, \ddddot{x}^\lambda).$$

Proof. *Uniqueness.* In virtue of Lemma B.4.4, for each $Y \in T T\boldsymbol{M}$, there exists a map
$f : \mathbb{R} \times \mathbb{R} \to \boldsymbol{M}$ such that $(d_2 d_1 f)(0, 0) = Y$. Hence, if σ exists, then we have $\sigma(Y) = d_1 d_2 f(0, 0)$.

Existence. If $\acute{f} : \mathbb{R} \times \mathbb{R} \to \boldsymbol{M}$ is another map such that $(d_2 d_1 f)(0, 0) = Y$, then it yields the same map $T T\boldsymbol{M} \to T T\boldsymbol{M}$. Moreover, the above diagram commutes for any value $(t_1, t_2) \in \mathbb{R} \times \mathbb{R}$. Hence, the map σ exists. Lemma B.4.3 yields the coordinate expression and the other properties of σ. □

We have the following natural prolongation of vector fields.

Note B.4.6 For each vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$, we obtain its tangent prolongation

$$\widetilde{X} := \sigma \circ T X : T\boldsymbol{M} \to T T\boldsymbol{M},$$

which makes the following diagram commutative

$$\begin{array}{ccc} T\boldsymbol{M} & \xrightarrow{\ \widetilde{X}\ } & T T\boldsymbol{M} \\ {\scriptstyle \tau_{\boldsymbol{M}}} \downarrow & & \downarrow {\scriptstyle T\tau_{\boldsymbol{M}}} \\ \boldsymbol{M} & \xrightarrow{\ X\ } & T\boldsymbol{M} \ . \end{array}$$

We have the coordinate expression $\widetilde{X} = X^\lambda\, \partial_\lambda + \partial_\mu X^\lambda\, \dot{x}^\mu\, \dot{\partial}_\lambda$. □

Eventually, we discuss the vertical differential on the iterated tangent bundle, which we use for the definition of the Liouville form (see Proposition 3.2.14).

Note B.4.7 Let us consider the vertical valued 1-form on $T\boldsymbol{M}$ defined in Note B.4.2

$$\nu : T\boldsymbol{M} \to T^*\boldsymbol{M} \otimes VT\boldsymbol{M}$$

with coordinate expression $\nu = d^\lambda \otimes \dot{\partial}_\lambda$. Then, we define the vertical differential (see [146, p. 163]) to be the differential operator

$$d_\nu = i_\nu \circ d.$$

For instance, for each function $f : T\boldsymbol{M} \to \mathbb{R}$, we have coordinate expression

$$d_\nu f = \dot{\partial}_\lambda f \, d^\lambda. \quad \square$$

Appendix C
Tangent Valued Forms

We introduce the notion of "*tangent valued form*" $\phi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M})$ of a manifold $\boldsymbol{M}$. In particular, we deal with the "*tangent valued forms*" and "*projectable tangent valued forms*" of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

In the further specific cases of a vector bundle, or an affine bundle, we can define the *linear, or affine, projectable tangent valued forms*.

In particular, the *vector valued forms* $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes \boldsymbol{F})$ of a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ can be naturally regarded as vertical valued forms $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$.

Indeed, the projectable tangent valued forms of a fibred manifold turn out to be an important tool of our covariant theory. In particular, we use the projectable tangent valued forms of a fibred manifold for the definition of curvature and torsion of general connections (see, for instance, Sects. F.1.3 and F.1.6 and applications to spacetime connections and phase connections).

C.1 Conventions on Exterior Forms

Preliminarily, we state our conventions on the notation concerning exterior forms.

Anti-symmetric tensors require normalising conventions; actually, in current literature one can find different conventions. So, we need to state our choice; we do it in such a way to get simple formulas as far as possible.

Thus, let us consider a vector space $\boldsymbol{V}$ of dimension n and refer to a basis (b_i) and to its dual basis (β^j).

(1) For each $v \in \boldsymbol{V}$, $\alpha \in \boldsymbol{V}^*$, $t \in \otimes^r \boldsymbol{V}$, $\tau \in \otimes^r \boldsymbol{V}^*$, we write

$$v = v^i\, b_i, \quad \alpha = \alpha_j\, \beta^j, \quad t = t^{i_1 \dots i_r}\, b_{i_1} \otimes \dots \otimes b_{i_r} v,$$
$$\tau = \tau_{j_1 \dots j_r}\, \beta^{j_1} \otimes \dots \otimes \beta^{j_r}.$$

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1

(2) The *duality map* is denoted by

$$\langle\ ,\ \rangle : \boldsymbol{V}^* \times \boldsymbol{V} \to \mathbb{R} : (\alpha, v) \mapsto \alpha(v) = \alpha_i\, v^i, \qquad \langle \beta_j,\ b^i \rangle = \delta^i_j.$$

(3) We have the standard *natural isomorphism*

$$\otimes^r \boldsymbol{V}^* \hookrightarrow L^r(\boldsymbol{V}, \mathbb{R}), \qquad (\alpha_1 \otimes \ldots \otimes \alpha_r)(v_1, \ldots, v_r) = \alpha_1(v_1) \ldots \alpha_r(v_r),$$

where L^r denotes the space of r-linear maps.

(4) For, $1 \leq r \leq s \leq n$, we define the *standard contraction*

$$\lrcorner \equiv C^{1\ldots r}_{1\ldots r} : \otimes^r \boldsymbol{V} \times \otimes^s \boldsymbol{V}^* \to \otimes^{s-r} \boldsymbol{V}^* : (t,\ \tau) \mapsto t \lrcorner \tau,$$
$$(v_1 \otimes \ldots \otimes v_r) \lrcorner \tau := v_r \lrcorner \ldots \lrcorner v_1 \lrcorner \tau, \qquad t \lrcorner \tau = t^{i_1\ldots i_r}\, \tau_{i_1\ldots i_r\, i_{r+1}\ldots i_s}.$$

As a particular case, when $r = s$, we obtain the standard extended *duality map*

$$\langle\ ,\ \rangle : \otimes^r \boldsymbol{V} \times \otimes^r \boldsymbol{V}^* \to \mathbb{R} : (t,\ \tau) \mapsto \langle t,\ \tau \rangle := t \lrcorner \tau.$$

(5) The *anti-symmetrisation* operator is defined by

$$A : \otimes^r \boldsymbol{V} \to \Lambda^r \boldsymbol{V} \subset \otimes^r \boldsymbol{V} : v_1 \otimes \ldots \otimes v_r \mapsto \tfrac{1}{r!} \sum_\sigma |\sigma|\, v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)}.$$

Actually, the normalising factor $\frac{1}{r!}$ has been chosen in such a way that $A^2 = A$.

(6) The *exterior product* is defined by the anti-symmetric r-linear map

$$\wedge^r : \boldsymbol{V} \times \ldots \times \boldsymbol{V} \to \Lambda^r \boldsymbol{V} : (v_1, \ldots, v_r)$$
$$\mapsto v_1 \wedge \ldots \wedge v_r \equiv \tfrac{1}{r!} \sum_\sigma |\sigma|\, v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)}.$$

(7) For $1 \leq r, s \leq n$, we define the extended *exterior product* given by the bilinear map

$$\wedge : \Lambda^r \boldsymbol{V} \times \Lambda^s \boldsymbol{V} \to \Lambda^{r+s} \boldsymbol{V} : (x, y) \mapsto x \wedge y \equiv A(x \otimes y),$$
$$(v_1 \wedge \ldots \wedge v_r) \wedge (v_{r+1} \wedge \ldots \wedge v_{r+s}) = v_1 \wedge \ldots \wedge v_{r+s}.$$

(8) We obtain the bases

$$(b_{i_1} \wedge \ldots \wedge b_{i_r})_{1<i_1<i_r<n} \subset \Lambda^r \boldsymbol{V} \quad \textit{and} \quad (\beta^{i_1} \wedge \ldots \wedge \beta^{i_r})_{1<i_1<i_r<n} \subset \Lambda^r \boldsymbol{V}^*.$$

Accordingly, if $t \in \Lambda^r \boldsymbol{V}$ and $\tau \in \Lambda^r \boldsymbol{V}^*$, then we can write

$$t = t^{i_1 \dots i_r} \, b_{i_1} \otimes \dots \otimes b_{i_r} = r! \sum_{1 \le i_1 \le \dots \le i_r \le n} t^{i_1 \dots i_r} \, b_{i_1} \wedge \dots \wedge b_{i_r}$$

$$\tau = \tau_{j_1 \dots j_r} \, \beta^{j_1} \otimes \dots \otimes \beta^{j_r} = r! \sum_{1 \le j_1 \le \dots \le j_r \le n} \tau_{j_1 \dots j_r} \, \beta^{j_1} \wedge \dots \wedge \beta^{j_r}.$$

(9) By considering $t \in \Lambda^r \boldsymbol{V} \subset \otimes^r \boldsymbol{V}$ and $\tau \in \Lambda^r \boldsymbol{V}^* \subset \otimes^r \boldsymbol{V}^*$ as standard tensors, we can also write

$$t = t^{i_1 \dots i_r} \, b_{i_1} \otimes \dots \otimes b_{i_r} = t^{i_1 \dots i_r} \, b_{i_1} \wedge \dots \wedge b_{i_r} = \sum_{i_1 \dots i_r} t^{i_1 \dots i_r} \, b_{i_1} \wedge \dots \wedge b_{i_r},$$

$$\tau = \tau_{j_1 \dots j_r} \, \beta^{j_1} \otimes \dots \otimes \beta^{j_r} = \tau_{j_1 \dots j_r} \, \beta^{j_1} \wedge \dots \wedge \beta^{j_r} = \sum_{i_1 \dots i_r} \tau_{j_1 \dots j_r} \, \beta^{j_1} \wedge \dots \wedge \beta^{j_r}.$$

(10) We stress that, in the formulas in items 8) and 9), t and τ have the same "components" if regarded as tensors or as anti-symmetric tensors.
But we must take care with these expressions. In fact, the sets $(b_{i_1} \otimes \dots \otimes b_{i_r})$ and $(\beta^{j_1} \otimes \dots \otimes \beta^{j_r})$ constitute bases of the tensor powers but not of the anti-symmetric tensor powers. So, the components $t^{i_1 \dots i_r}$ and $\tau_{j_1 \dots j_r}$ and the sets $(b_{i_1} \wedge \dots \wedge b_{i_r})$ and $(\beta^{j_1} \wedge \dots \wedge \beta^{j_r})$ are not independent.
The comparison of the expressions in item 8) and 9) is provided by the equalities

$$t = \sum_{i_1 \dots i_r} t^{i_1 \dots i_r} \, b_{i_1} \wedge \dots \wedge b_{i_r} = r! \sum_{1 \le i_1 \le \dots \le i_r \le n} t^{i_1 \dots i_r} \, b_{i_1} \wedge \dots \wedge b_{i_r}$$

$$\tau = \sum_{j_1 \dots j_r} \tau_{j_1 \dots j_r} \, \beta^{j_1} \wedge \dots \wedge \beta^{j_r} = r! \sum_{1 \le j_1 \le \dots \le j_r \le n} \tau_{j_1 \dots j_r} \, \beta^{j_1} \wedge \dots \wedge \beta^{j_r}.$$

(11) For $1 \le s \le n$, and each $\tau \in \Lambda^s \boldsymbol{V}^*$, we define the *interior product*

$$i : \boldsymbol{V} \times \Lambda^s \boldsymbol{V}^* \to \Lambda^{s-1} \boldsymbol{V}^* : (v, \ \tau) \mapsto i_v \tau,$$

$$i_v(\alpha_1 \wedge \dots \wedge \alpha_s) := \sum_i (-1)^{i-1} \alpha_i(v) \, \alpha_1 \wedge \dots \wedge \widehat{\alpha_i} \wedge \dots \wedge \alpha_s.$$

For $\tau \in \Lambda^s \boldsymbol{V}^*$ and $\sigma \in \Lambda^t \boldsymbol{V}^*$, we obtain the *derivation rule*

$$i_v(\tau \wedge \sigma) = i_v \tau \wedge \sigma + (-1)^s \tau \wedge i_v \sigma.$$

(12) The interior product yields the natural *isomorphism*

$$\Lambda^r \boldsymbol{V}^* \hookrightarrow A^r(\boldsymbol{V}, \mathbb{R}) \qquad \text{given by} \qquad \alpha(v_1, \dots, v_r)_\wedge := i_{v_r} \dots i_{v_1} \alpha,$$

where A^r denotes the space of anti-symmetric r-multilinear maps.

(13) Each anti-symmetric tensor can be regarded as a generic tensor; hence, we can use the standard contraction $\lrcorner$. Moreover, for anti-symmetric tensors we

have the natural "interior product" i, which fulfills the useful properties of derivations.

We can compare the two contractions $\lrcorner$ and i of a vector v with an antisymmetric r-form. In fact, if $v \in \boldsymbol{V}$ and $\tau \in \Lambda^s \boldsymbol{V}^*$, then we obtain

$$v \lrcorner \tau = \tfrac{1}{s}\, i_v \tau.$$

Thus, we obtain two natural ways of operating anti-symmetric forms τ of degree s on vectors $(v_1, \dots, v_s)$, according to the equality

$$v_s \lrcorner \dots v_1 \lrcorner \tau := \tau(v_1, \dots, v_s) = \tfrac{1}{s!}\, \tau(v_1, \dots, v_s)_\wedge := i_{v_r} \dots i_{v_1} \tau.$$

In the 1st case we regard τ as a generic tensor, in the 2nd case we emphasise the fact that τ is an anti-symmetric tensor.

(14) For $1 \le r \le s \le n$, we extend the *interior product* in the following way

$$i : \Lambda^r \boldsymbol{V} \times \Lambda^s \boldsymbol{V}^* \to \Lambda^{s-r} \boldsymbol{V}^* : (v_1 \wedge \dots \wedge v_r,\ \tau) \mapsto i_{v_r} \dots i_{v_1} \tau.$$

(15) As a particular case, when $r = s$, we obtain the *duality map*

$$\langle\ ,\ \rangle_\wedge : \Lambda^r \boldsymbol{V} \times \Lambda^r \boldsymbol{V}^* \to \mathbb{R} : (t,\ \alpha) \mapsto \langle t,\ \alpha\rangle_\wedge.$$

C.2 Tangent Valued Forms on a Manifold

We start with the notion of *tangent valued form* $\phi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M})$ of a manifold $\boldsymbol{M}$, in view of the next extension of this concept to a fibred manifold (see Sect. C.2.1).

Let us consider a manifold $\boldsymbol{M}$ and denote its typical charts by (x^λ).

Preliminarily, we recall that we can regard each form $\alpha \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M})$ as antisymmetric r-linear form in two ways, by regarding, respectively, the form as a generic tensor and by taking into account its antisymmetry (see Appendix: Sect. C.1).

Note C.2.1 For each exterior form $\alpha \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M})$ we obtain, in a natural way, two r-linear maps, with respect to the functions $f \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})$,

$$\alpha : \times^r \sec(\boldsymbol{M}, T\boldsymbol{M}) \to \operatorname{map}(\boldsymbol{M}, \mathbb{R}) : (X_1, \dots X_r) \mapsto \alpha\,(X_1, \dots X_r) \ := X_r \lrcorner \dots X_1 \lrcorner \alpha,$$
$$\alpha : \times^r \sec(\boldsymbol{M}, T\boldsymbol{M}) \to \operatorname{map}(\boldsymbol{M}, \mathbb{R}) : (X_1, \dots X_r) \mapsto \alpha\,(X_1, \dots X_r)_\wedge := i_{X_r} \dots i_{X_1} \alpha,$$

where $X \lrcorner$ denotes the standard contraction of tensors and i_X denotes the interior product.

We have the expressions

$$\alpha\,(X_1,\ldots X_r) = \tfrac{1}{r!}\,\alpha\,(X_1,\ldots X_r)_\wedge \text{ and } \alpha\,(X_1,\ldots X_r)_\wedge = r!\,\alpha\,(X_1,\ldots X_r)$$

and, in coordinates,

$$\alpha\,(X_1,\ldots X_r) = \alpha_{\lambda_1\ldots\lambda_r}\,X_1^{\lambda_1}\ldots X_r^{\lambda_r} \qquad \text{and} \qquad \alpha\,(X_1,\ldots X_r)_\wedge = r!\,\alpha_{\lambda_1\ldots\lambda_r}\,X_1^{\lambda_1}\ldots X_r^{\lambda_r}. \quad \square$$

Definition C.2.2 A *tangent valued form* of degree r is defined to be a section

$$\phi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M}),$$

with coordinate expression

$$\phi = \phi^\mu_{\lambda_1\ldots\lambda_r}\,d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_r}\otimes\partial_\mu, \quad \text{with } \phi^\mu_{\lambda_1\ldots\lambda_r} \in \text{map}(\boldsymbol{M}, \mathbb{R}).$$

A tangent valued form ϕ of degree r is said to be *decomposable* if it is of the type

$$\phi = \alpha\otimes X, \quad \text{with } \alpha \in \sec(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M}),\ \ X \in \sec(\boldsymbol{M},\ T\boldsymbol{M}),$$

i.e., in coordinates,

$$\phi{=}(\alpha_{\lambda_1\ldots\lambda_r}\,d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_r})\otimes(X^\mu\,\partial_\mu), \quad \textit{with} \quad \alpha_{\lambda_1\ldots\lambda_r},\ X^\mu \in \text{map}(\boldsymbol{M}, \mathbb{R}). \quad \square$$

In particular, the vector fields $X \in \sec(\boldsymbol{M},\ T\boldsymbol{M})$ are the tangent valued forms of degree 0.

Note C.2.3 Each tangent valued form $\phi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M}\otimes T\boldsymbol{M})$ can be regarded as an r-linear map, with respect to the functions $f \in \text{map}(\boldsymbol{M}, \mathbb{R})$,

$$\phi : \times^r \sec(\boldsymbol{M}, T\boldsymbol{M}) \to \sec(\boldsymbol{M}, T\boldsymbol{M}) : (X_1,\ldots,X_r) \mapsto \phi(X_1,\ldots,X_r)_\wedge := i_{X_r}\ldots i_{X_1}\phi.$$

We have the coordinate expression $\phi(X_1,\ldots,X_r)_\wedge = r!\,X_1^{\lambda_1}\ \ldots\ X_r^{\lambda_1}\,\phi^\mu_{\lambda_1\ldots\lambda_r}\,\partial_\mu$. $\square$

We have a useful local decomposition of tangent valued forms.

Note C.2.4 Each tangent valued form $\phi \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M}\otimes T\boldsymbol{M})$ can be written locally, by referring to a chart, as a finite sum of decomposable tangent valued forms of the type

$$\alpha\otimes X, \quad \text{with } \alpha \in \sec(\boldsymbol{M}, \Lambda^r T^*\boldsymbol{M}) \textit{ and } X \in \sec(\boldsymbol{M}, T\boldsymbol{M}).$$

Even more, we can always choose the above decomposable tangent valued forms with the condition $d\alpha = 0$.

Note C.2.5 The tangent valued forms constitute a sheaf of modules over the sheaf of functions $\text{map}(\boldsymbol{M}, \mathbb{R})$. $\square$

C.2.1 Tangent Valued Forms on a Fibred Manifold

In the above Section (see Sect. C.2), we have introduced the notion of tangent valued form on a manifold. In the particular case when the manifold is a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ the tangent valued forms inherit further features from the fibring.

Then, we can consider the more specific notions of "*projectable tangent valued form*" $\phi \in \operatorname{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ and of "*vertical valued form*" $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$.

In the present book, we are involved with several projectable tangent valued forms, such as observers and several kinds of connections.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical fibred chart by (x^λ, y^i) (see Sect. A.1).

Note C.2.6 A tangent valued form $\phi \in \sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ is said to be *projectable on* $\boldsymbol{B}$ if there exists a tangent valued form of the base space (which turns out to be unique) $\underline{\phi} \in \sec(\boldsymbol{B},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B})$, which makes the following diagram commutative

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\ \phi\ } & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\ \underline{\phi}\ } & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} .$$

In particular, a tangent valued form ϕ is said to be a *vertical valued form* if it is projectable on $\underline{\phi} = 0$, i.e. if $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$.

We denote the subsheaf of projectable tangent valued forms by

$$\operatorname{pro}\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$$

and the subsheaf of vertical valued forms by

$$\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F}) \subset \operatorname{pro}\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}).$$

A tangent valued form $\phi \in \sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ is projectable on $\boldsymbol{B}$ if and only if its coordinate expression is of type

$$\phi = \phi^\mu_{\lambda_1\ldots\lambda_r}\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_\mu + \phi^i_{\lambda_1\ldots\lambda_r}\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_i ,$$

with $\phi^\mu_{\lambda_1\ldots\lambda_r} \in \operatorname{map}(\boldsymbol{B}, \mathbb{R})$ and $\phi^i_{\lambda_1\ldots\lambda_r} \in \operatorname{map}(\boldsymbol{F}, \mathbb{R})$.

Indeed, if ϕ is projectable, then we obtain

$$\underline{\phi} = \phi^\mu_{\lambda_1\ldots\lambda_r}\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_\mu .$$

In particular, ϕ is vertical valued if and only if $\phi^\mu_{\lambda_1\ldots\lambda_r} = 0$. □

Note C.2.7 Each projectable tangent valued form $\phi \in \text{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ can be regarded as an r-linear map, with respect to the functions $f \in \text{map}(\boldsymbol{B}, \mathbb{R})$,

$$\phi : \times^r \sec(\boldsymbol{B}, T\boldsymbol{B}) \to \sec(\boldsymbol{F}, T\boldsymbol{F}) : (X_1, \ldots, X_r) \mapsto \phi(X_1, \ldots, X_r)_\wedge := i_{X_r} \ldots i_{X_1}\phi.$$

We have the coordinate expression

$$\phi(X_1, \ldots, X_r) = X_1^{\lambda_1} \ldots X_r^{\lambda_1} (\phi^{\mu}_{\lambda_1 \ldots \lambda_r} \partial_\mu + \phi^{i}_{\lambda_1 \ldots \lambda_r} \partial_i). \quad \square$$

We have a useful local decomposition of projectable tangent valued forms (see Note C.2.4).

Note C.2.8 Each $\phi \in \text{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ can be written locally, by referring to a chart, as a finite sum of decomposable tangent valued forms of the type (see Note C.2.4)

$$\alpha \otimes X, \quad \text{with } \alpha \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B}) \textit{ and } X \in \text{pro}(\boldsymbol{F}, T\boldsymbol{F}).$$

Even more, we can always choose the above decomposable tangent valued forms with the condition $\alpha \in \sec(\boldsymbol{B}, \Lambda^r T^*\boldsymbol{B})$ and $d\alpha = 0$.

Note C.2.9 The projectable tangent valued forms constitute a sheaf of modules over $\text{map}(\boldsymbol{B}, \mathbb{R})$. We stress that, because of the condition of projectability, this is not a sheaf of modules over $\text{map}(\boldsymbol{F}, \mathbb{R})$. $\square$

As a particular case, let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a linear fibred chart (x^λ, y^i) (see Definition A.3.1). We recall that $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be a vector bundle (see Proposition B.3.2).

Note C.2.10 A projectable tangent valued form ϕ is said to be *linear* if it is a linear fibred morphism over its projection $\underline{\phi}$, according to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\ \phi\ } & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \text{id} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\ \underline{\phi}\ } & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} .$$

We denote the subsheaf of linear projectable tangent valued forms by

$$\text{lin}\,\text{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \text{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}).$$

A projectable tangent valued form ϕ is linear if and only if its coordinate expression is of the type

$$\phi = \phi^{\mu}_{\lambda_1 \ldots \lambda_r} \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_\mu + \phi^{i}_{\lambda_1 \ldots \lambda_r \, j} \, y^j \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_i ,$$

with $\phi^{\mu}_{\lambda_1 \ldots \lambda_r}, \; \phi^{i}_{\lambda_1 \ldots \lambda_r \, j} \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$. □

Note C.2.11 The linear projectable tangent valued forms constitute a sheaf of modules over $\mathrm{map}(\boldsymbol{B}, \mathbb{R})$. We stress that, because of the conditions of projectability and linearity, this is not a sheaf of modules over $\mathrm{map}(\boldsymbol{F}, \mathbb{R})$. □

As a further particular case, let us consider an affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ associated with the vector bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$ and refer to an affine fibred chart (x^λ, y^i) (see Definition A.3.4). We recall that $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be an affine bundle associated with the vector bundle $T\bar{p} : T\bar{\boldsymbol{F}} \to T\boldsymbol{B}$ (see Proposition B.3.2).

Note C.2.12 A projectable tangent valued form ϕ is said to be *affine* if it is an affine fibred morphism over its projection $\underline{\phi}$, according to the commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;\phi\;} & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\;\underline{\phi}\;} & \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} \; .$$

We denote the subsheaf of affine projectable tangent valued forms by

$$\mathrm{aff\,pro\,sec}(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \mathrm{pro\,sec}(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}).$$

A projectable tangent valued form ϕ is affine if and only if its coordinate expression is of the type

$$\phi = \phi^{\mu}_{\lambda_1 \ldots \lambda_r} \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_\mu + (\phi^{i}_{\lambda_1 \ldots \lambda_r \, j} \, y^j + \phi^{i}_{\lambda_1 \ldots \lambda_r \, \circ}) \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_i ,$$

with $\phi^{\mu}_{\lambda_1 \ldots \lambda_r}, \; \phi^{i}_{\lambda_1 \ldots \lambda_r \, \circ}, \; \phi^{i}_{\lambda_1 \ldots \lambda_r \, j} \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$. □

By taking into account the fact that

$$\boldsymbol{F} \to \boldsymbol{B} \qquad \text{and} \qquad \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F} \to \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B}$$

are affine bundles associated with the vector bundles and

$$\bar{\boldsymbol{F}} \to \boldsymbol{B} \qquad \text{and} \qquad \Lambda^r T^*\boldsymbol{B} \otimes T\bar{\boldsymbol{F}} \to \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{B},$$

we can compute the fibre derivative of affine projectable tangent valued forms.

Note C.2.13 The "*fibre derivative*" of an affine projectable tangent valued form $\phi \in \mathrm{aff\,pro\,sec}(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ is a linear projectable tangent valued form

$$\bar{\phi} := D\phi \in \operatorname{lin}\operatorname{pro}\sec(\bar{\boldsymbol{F}}, \Lambda^r T^*\boldsymbol{B} \otimes T\bar{\boldsymbol{F}}),$$

with coordinate expression

$$\bar{\phi} = \phi^{\mu}_{\lambda_1 \ldots \lambda_r} \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_\mu + \phi^{i}_{\lambda_1 \ldots \lambda_r \, j} \, y^j \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \partial_i . \quad \square$$

Note C.2.14 The affine projectable tangent valued forms constitute a sheaf of modules over $\operatorname{map}(\boldsymbol{B}, \mathbb{R})$. We stress that, because of the condition of projectability and affinity, this is not a sheaf of modules over $\operatorname{map}(\boldsymbol{F}, \mathbb{R})$. □

C.2.2 Vector Valued Forms on a Vector Bundle

We introduce the notions of *vector valued form* $\psi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes \boldsymbol{F})$ and of *basic vertical valued form* $\widetilde{\psi} \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$ on a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ (see, also, Note B.3.6). Then, we show that they can be naturally identified with vertical valued forms $\psi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$.

In this way, it will be possible to extend the geometric techniques developed for tangent valued forms, including the FN-bracket (see Sect. E.2), to vector valued forms.

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a linear fibred chart (x^λ, y^i) and to the associated basis (b_i) (see Definition A.3.1).

Note C.2.15 A *vector valued form* is defined to be a section

$$\psi \in \sec(\boldsymbol{B}, \Lambda^r T^*\boldsymbol{B} \otimes \boldsymbol{F}).$$

The coordinate expression of a vector valued form ϕ is of the type

$$\psi = \psi^i_{\lambda_1 \ldots \lambda_r} \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_r} \otimes \mathrm{b}_i , \quad \text{with } \psi^i_{\lambda_1 \ldots \lambda_r} \in \operatorname{map}(\boldsymbol{B}, \mathbb{R}).$$

We can naturally identify vector valued forms with vertical valued forms, by extending Note B.3.6, which deals with an analogous result on vector fields.

Note C.2.16 The following natural mutually inverse identifications hold.

(1) If $\psi \in \sec(\boldsymbol{B}, \Lambda^s T^*\boldsymbol{B} \otimes \boldsymbol{F})$ is a vector valued form, then we obtain the vertical valued form $\tilde{\psi} \in \sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F})$ given by the map

$$\widetilde{\psi} : \boldsymbol{F} \to \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F} : f \mapsto \Big(f, \, \psi\big(p(f)\big)\Big).$$

Thus, by definition, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\widetilde{\psi}} & \Lambda^s T^* \boldsymbol{B} \otimes V \boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{B} & \xrightarrow{\psi} & \Lambda^s T^* \boldsymbol{B} \otimes \boldsymbol{F} \end{array}$$

and

$$V\widetilde{\psi} = 0 : V\boldsymbol{F} \to \Lambda^s T^* \boldsymbol{B} \otimes V_{\boldsymbol{B}} V \boldsymbol{F}.$$

I.e., in coordinates, we have $\widetilde{\psi} = \psi^i_{\lambda_1 \dots \lambda_r}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_i$ and $\partial_j \widetilde{\psi}^i_{\lambda_1 \dots \lambda_r} = 0$.

(2) If $\widetilde{\psi} \in \sec(\boldsymbol{F},\ \Lambda^s T^* \boldsymbol{B} \otimes V \boldsymbol{F})$ is a vertical valued form such that

$$V\widetilde{\psi} = 0 : V\boldsymbol{F} \to \Lambda^s T^* \boldsymbol{B} \otimes V_{\boldsymbol{B}} V \boldsymbol{F},$$

i.e., in coordinates, such that

$$\widetilde{\psi} = \widetilde{\psi}^i_{\lambda_1 \dots \lambda_r}\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_r} \otimes \partial_i \qquad \text{and} \qquad \partial_j \widetilde{\psi}^i_{\lambda_1 \dots \lambda_r} = 0,$$

then $\widetilde{\psi}$ projects over a vector valued form $\psi \in \sec(\boldsymbol{B},\ \Lambda^s T^* \boldsymbol{B} \otimes \boldsymbol{F})$, through the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\widetilde{\psi}} & \Lambda^s T^* \boldsymbol{B} \otimes V \boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{B} & \xrightarrow{\psi} & \Lambda^s T^* \boldsymbol{B} \otimes \boldsymbol{F} \end{array}$$

and we have $\widetilde{\psi}^i_{\lambda_1 \dots \lambda_r} = \psi^i_{\lambda_1 \dots \lambda_r}$.

(3) Indeed, the above maps $\psi \mapsto \widetilde{\psi}$ and $\widetilde{\psi} \mapsto \psi$ are mutually inverse bijections.

The vertical valued forms

$$\widetilde{\psi} \in \sec(\boldsymbol{F},\ \Lambda^s T^* \boldsymbol{B} \otimes V \boldsymbol{F})$$

as above are called *basic*.

According to the above observations, we shall often write $\psi \simeq \widetilde{\psi}$. □

Appendix D
Lie Derivatives

We discuss a few rather unusual topics dealing with Lie derivatives, such as *Lie derivatives of generic sections of a vector bundle* and *Lie derivatives of vertical covariant tensors* (Sects. D.1 and D.2). Then, we discuss the *infinitesimal symmetries* of tensors (Sect. D.3).

D.1 Lie Derivatives of Sections

We analyse the Lie derivative of a section with respect to a vector field. This concept is used in Sect. 20.1.2.

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote the linear fibred charts by (x^λ, y^i) and the associated basis by $b_i \in \sec(\boldsymbol{B}, \boldsymbol{F})$.

Lemma D.1.1 *We have a natural bijection between sections* $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ *and vertical valued vector fields* $\tilde{s} \in \sec(\boldsymbol{F}, V_{\boldsymbol{B}}\boldsymbol{F})$, *whose 2nd component is projectable on* $\boldsymbol{B}$, *according to the following commutative diagram*

$$\begin{array}{ccc} V_{\boldsymbol{B}}\boldsymbol{F} & \xrightarrow{\ \simeq\ } & \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \\ {\scriptstyle \tilde{s}}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{pro}_2} \\ \boldsymbol{B} & \xrightarrow{\ s\ } & \boldsymbol{F} \end{array} .$$

We have the coordinate expressions

$$s = s^i\, b_i \qquad \textit{and} \qquad \tilde{s} = s^i\, \partial_i\,, \quad \text{with } s^i \in \operatorname{map}(\boldsymbol{B}, I\!R). \quad \square$$

Proposition D.1.2 *If* $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ *and* $Y \in \operatorname{pro}_{\boldsymbol{B}}(\boldsymbol{F}, T\boldsymbol{F})$, *then the Lie derivative* $L_Y\tilde{s} \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ *turns out to be a* $\boldsymbol{B}$*-vertical vector field, whose 2nd component is projectable on* $\boldsymbol{B}$, *in the sense of the above Lemma.*

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Hence, the vector field $L_Y \tilde{s} \in \sec(\boldsymbol{F}, V_{\boldsymbol{B}}\boldsymbol{F})$ *can be regarded as a section (called the* Lie derivative *of s) denoted by*

$$Y_{\bullet}s \simeq L_Y \tilde{s} \in \sec(\boldsymbol{B}, \boldsymbol{F}).$$

We have the coordinate expression

$$Y_{\bullet}s = (X^{\mu} \, \partial_{\mu} s^i - s^j \, \partial_j Y^i) \, b_i. \quad \square$$

Remark D.1.3 We stress the minus sign appearing in the above formula. □

Proposition D.1.4 *If* $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ *and* $Y, \acute{Y} \in \mathrm{pro}_{\boldsymbol{B}}(\boldsymbol{F}, T\boldsymbol{F})$, *then*

$$[Y, \acute{Y}]_{\bullet}s = (Y_{\bullet} \circ \acute{Y}_{\bullet} - \acute{Y}_{\bullet} \circ Y_{\bullet})\, s. \quad \square$$

We can extend the above result as follows.

Let us assume a further bundle fibring $\boldsymbol{C} \to \boldsymbol{B}$ and denote its typical fibred charts by (x^{λ}, z^a).

D.2 Lie Derivatives of Vertical Covariant Tensors

Given a fibred manifold, the Lie derivative of a vertical covariant tensor, with respect to a vector field of the fibred manifold, does not make sense, because we have no natural inclusion of the vertical cotangent space into the cotangent space.

However, the Lie derivative of a vertical covariant tensor with respect to a *projectable* vector field of the fibred manifold is well defined as a special case of "general Lie derivative" (see [246, p. 376]).

This result is used several times throughout the book (see, for instance, Sects. 3.2.8, 4.3, 12.2.1, 20.1.2, 13.1 and 19.2).

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote the fibred charts by (x^{λ}, y^i).

We stress that, if $\alpha \in \sec(\boldsymbol{F}, \otimes^r V^*_{\boldsymbol{B}}\boldsymbol{F})$ and $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, then the Lie derivative $L_X \alpha$ is not well-defined.

If $X \in \mathrm{pro}_{\boldsymbol{B}}(\boldsymbol{F}, T\boldsymbol{F})$, then the Lie derivative $L_X \alpha$ is well-defined, according to the following result.

Proposition D.2.1 *If* $\alpha \in \sec(\boldsymbol{F}, \otimes^r V^*_{\boldsymbol{B}}\boldsymbol{F})$ *and* $X \in \mathrm{pro}_{\boldsymbol{B}}(\boldsymbol{F}, T\boldsymbol{F})$, *then the vertical restriction*

$$L_X \alpha := (L_X \widetilde{\alpha})^{\vee} \in \sec(\boldsymbol{F}, V^*_{\boldsymbol{B}}\boldsymbol{F})$$

of the standard Lie derivative of any extension $\widetilde{\alpha} \in \sec(\boldsymbol{F}, \otimes^r T^*_{\boldsymbol{B}}\boldsymbol{F})$ *of* α *(called the* Lie derivative *of* α*) does not depend on the extension* $\widetilde{\alpha}$.

Actually, we have the following equality, for each $Y_1, \ldots, Y_r \in \sec(\boldsymbol{F}, V\boldsymbol{F})$,

$$L_X\alpha\,(Y_1,\dots,Y_r) = X.\big(\alpha(Y_1,\dots,Y_r)\big) - \sum_i \alpha\big([X,Y_i],Y_1,\dots,\widehat{Y}_i,\dots,Y_r\big)$$

and the coordinate expression

$$L_X\,\alpha = (X^\lambda\,\partial_\lambda\alpha_{i_1\dots i_r} + X^j\,\partial_j\alpha_{i_1\dots i_r} + \sum_{1\le k\le r}\alpha_{i_1\dots i_{k-1}j i_{k+1}\dots i_r}\,\partial_{i_k}X^j)\,\check{d}^{i_1}\otimes\dots\otimes\check{d}^{i_r}. \quad \square$$

Remark D.2.2 We stress that, in the above coordinate expression, the terms appearing in the standard Lie derivative and containing the derivatives $\partial_i X^\lambda$ are missing. □

D.3 Infinitesimal Symmetries of Tensors

We study the Lie algebra of infinitesimal symmetries of a tensor field. These concepts are used in our discussion on classical and quantum symmetries (see Chaps. 13 and 19).

Definition D.3.1 Let us consider a manifold $\boldsymbol{M}$ and a tensor field

$$\mathsf{T} \in \sec\big(\boldsymbol{M},\ (\otimes^r T^*\boldsymbol{M})\otimes(\otimes^s T\boldsymbol{M})\big).$$

We define an *infinitesimal symmetry* of T to be a vector field

$$X \in \sec(\boldsymbol{M}, T\boldsymbol{M}), \qquad \text{such that} \qquad L_X\mathsf{T} = 0.$$

We denote the subsheaf of infinitesimal symmetries of T by

$$\mathrm{isy}_{\mathsf{T}}(\boldsymbol{M}, T\boldsymbol{M}) \subset \sec(\boldsymbol{M}, T\boldsymbol{M}). \quad \square$$

Proposition D.3.2 *For each* $X, \acute{X} \in \sec(\boldsymbol{M}, T\boldsymbol{M})$, *we have*

$$L_{[X,\acute{X}]}\mathsf{T} = (L_X\,L_{\acute{X}} - L_{\acute{X}}\,L_X)\,\mathsf{T}.$$

Then, the subsheaf

$$\mathrm{isy}_{\mathsf{T}}(\boldsymbol{M}, T\boldsymbol{M}) \subset \sec(\boldsymbol{M}, T\boldsymbol{M})$$

of infinitesimal symmetries of T *turns out to be closed with respect to the Lie bracket, hence it is an* $\mathbb{R}$*-Lie subalgebra.* □

These concepts and results can be easily extended to the case of vertical valued tensor fields $\mathsf{T} \in \sec(\boldsymbol{F}, \otimes^r V^*_{\boldsymbol{B}}\boldsymbol{F})$ and projectable vector fields $Y \in \mathrm{pro}_{\boldsymbol{B}}(\boldsymbol{F}, T\boldsymbol{F})$.

Appendix E
The Frölicher–Nijenhuis Bracket

The "*Frölicher–Nijenhuis bracket*" (FN-bracket) is a natural extension of the Lie bracket of vector fields to tangent valued forms (see, for instance, [131, 132, 197, 245, 246, 284, 305, 311, 326]). Actually, the FN-bracket has been originally achieved by studying the Lie algebra of derivations of exterior forms (see [131, 246]); here we follow a direct approach proposed in [284].

We study the FN-bracket by subsequent steps: on a manifold, on a fibred manifold, on a structured bundle, on vector valued forms (Sects. E.1, E.2, E.3 and E.4).

In the present book, we use tangent valued forms and their FN-bracket as an essential tool for the definition of the curvature and the torsion of general connections (Sects. F.1.3, F.1.6, 4.1.2, 4.1.3, 4.1.4, 4.2.5 and 9.1.1).

E.1 The FN-Bracket on a Manifold

We introduce the "Frölicher–Nijenhuis bracket" $[\phi, \psi] \in \sec(\boldsymbol{M}, \ \Lambda^{r+s} T^*\boldsymbol{M} \otimes T\boldsymbol{M})$ of tangent valued forms ϕ and ψ of degrees r and s on a manifold $\boldsymbol{M}$. The FN-bracket is a "graded Lie bracket", as it fulfills distinguished commutation and Jacobi rules. Originally, the FN-bracket has been introduced as a byproduct of the study of derivations of exterior forms [131, 246]. Here, we provide a direct approach to this subject [284].

In the present book, we mainly deal with the FN-bracket of projectable tangent valued forms on a fibred manifold, in the context of general connections (see Sects. E.2 and F.1).

Here, this section, which is devoted to the FN-bracket on a manifold, is essentially intended as an introduction to the following Sect. E.2.

Theorem E.1.1 *[284] For each tangent valued forms*

$$\phi \in \sec(\boldsymbol{M}, \ \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \qquad \textit{and} \qquad \psi \in \sec(\boldsymbol{M}, \ \Lambda^s T^*\boldsymbol{M} \otimes T\boldsymbol{M}),$$

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let us define the map

$$[\phi,\ \psi] : \times^{r+s} \operatorname{sec}(\boldsymbol{M},\ T\boldsymbol{M}) \to \operatorname{sec}(\boldsymbol{M},\ T\boldsymbol{M})$$

given, for each $X_1, \ldots, X_{r+s} \in \operatorname{sec}(\boldsymbol{M},\ T\boldsymbol{M})$, *by the following equality, which is expressed in terms of the standard Lie bracket of vector fields,* [1]

$$\begin{aligned}
&[\phi,\ \psi]\big(X_1, \ldots, X_{r+s}\big)_\wedge \\
:= \tfrac{1}{r!\,s!} &\sum_\sigma |\sigma| \big[\phi(X_{\sigma(1)}, \ldots, X_{\sigma(r)})_\wedge,\ \psi(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big] \\
&- \tfrac{r}{r!\,s!} \sum_\sigma |\sigma|\, \phi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(r-1)},\ \big[X_{\sigma(r)},\ \psi(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big]\Big)_\wedge \\
&+ \tfrac{(-1)^{rs}\, s}{r!\,s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ \phi(X_{\sigma(s+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big]\Big)_\wedge \\
&+ \tfrac{rs}{2\,r!\,s!} \sum_\sigma |\sigma|\, \phi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(r-1)},\ \psi\big([X_{\sigma(r)},\ X_{\sigma(r+1)}], X_{\sigma(r+2)}, \ldots, X_{\sigma(r+s)}\big)_\wedge\Big)_\wedge \\
&- \tfrac{(-1)^{rs}\, rs}{2\,r!\,s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)},\ \phi\big([X_{\sigma(s)},\ X_{\sigma(s+1)}], X_{\sigma(s+2)}, \ldots, X_{\sigma(r+s)}\big)_\wedge\Big)_\wedge,
\end{aligned}$$

where the summation is extended to all the permutations σ *of* $(1, \ldots, r+s)$ *and* $|\sigma|$ *denotes the sign of the permutation* σ.

Indeed, the map $[\phi, \psi]$ *turns out to be antisymmetric and* $(r+s)$*-linear with respect to the functions* $f \in \operatorname{map}(\boldsymbol{M}, I\!R)$.

Hence, $[\phi, \psi]$ *turns out to be an element*

$$[\phi,\ \psi] \in \operatorname{sec}(\boldsymbol{M},\ \Lambda^{r+s} T^*\boldsymbol{M} \otimes T\boldsymbol{M}),$$

which is called Frölicher–Nijenhuis bracket *(FN-bracket) of* ϕ *and* ψ.

Thus, the previous rule provides a natural map

$$\begin{aligned}
[\ ,\ ] : \operatorname{sec}(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \times \operatorname{sec}(\boldsymbol{M},\ \Lambda^s T^*\boldsymbol{M} \otimes T\boldsymbol{M}) \\
\to \operatorname{sec}(\boldsymbol{M},\ \Lambda^{r+s} T^*\boldsymbol{M} \otimes T\boldsymbol{M}) : (\phi, \psi) \mapsto [\phi, \psi].
\end{aligned}$$

The FN-bracket turns out to be a graded Lie bracket. In other words, for each

$$\begin{gathered}
\phi, \phi' \in \operatorname{sec}(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M} \otimes T\boldsymbol{M}), \qquad \psi, \psi' \in \operatorname{sec}(\boldsymbol{M},\ \Lambda^s T^*\boldsymbol{M} \otimes T\boldsymbol{M}), \\
\theta \in \operatorname{sec}(\boldsymbol{M},\ \Lambda^t T^*\boldsymbol{M} \otimes T\boldsymbol{M}), \qquad k \in I\!R,
\end{gathered}$$

we have the following linearity, antisymmetry and Jacobi properties

[1] The following formula should be read with this condition. According to the possible values of the integers r and s, if in a term of the expression of $[\phi, \psi]$ the possible variables $X_{\sigma(i)}$ do not exist, then this term should be omitted.

$$[\phi + \phi',\ \psi] = [\phi, \psi] + [\phi', \psi], \qquad [\phi,\ \psi + \psi'] = [\phi, \psi] + [\phi, \psi'],$$
$$[k\phi,\ \psi] = k\,[\phi,\ \psi] = [\phi, k\psi],$$
$$[\phi,\ \psi] = -(-1)^{rs}[\psi,\ \phi],$$
$$[\theta,\ [\phi,\ \psi]] = [[\theta,\ \phi],\ \psi] + (-1)^{rt}[\phi,\ [\theta,\ \psi]].$$

We have the coordinate expression

$$\begin{aligned}[\phi,\ \psi] =&(\phi^{\rho}_{\lambda_1\dots\lambda_r}\,\partial_\rho\psi^{\mu}_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\,\psi^{\rho}_{\lambda_1\dots\lambda_s}\,\partial_\rho\phi^{\mu}_{\lambda_{s+1}\dots\lambda_{r+s}}\\ &- r\,\phi^{\mu}_{\lambda_1\dots\lambda_{r-1}\rho}\,\partial_{\lambda_r}\psi^{\rho}_{\lambda_{r+1}\dots\lambda_{r+s}} + (-1)^{rs}\,s\,\psi^{\mu}_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\phi^{\rho}_{\lambda_{s+1}\dots\lambda_{r+s}})\\ &d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{r+s}}\otimes\partial_\mu. \quad \square\end{aligned}$$

Note E.1.2 The Jacobi property can be equivalently written as

$$(-1)^{|\theta||\phi|}\left[\theta, [\phi, \psi]\right] + (-1)^{|\psi||\phi|}\left[\psi, [\theta, \phi]\right] + (-1)^{|\phi||\theta|}\left[\phi, [\psi, \theta]\right] = 0,$$

where $|\alpha|$ denotes the degree of the tangent valued form α. □

The following Proposition E.1.3 provides a simple useful expression of the FN-bracket in terms of decomposable tangent valued forms (see Note C.2.4. For the reference, see, for instance, [246].

Proposition E.1.3 *The FN-bracket is the unique $\mathbb{R}$-bilinear sheaf morphism*

$$\begin{aligned}[\ ,\ ] : \mathrm{sec}(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M}\otimes T\boldsymbol{M}) \times \mathrm{sec}(\boldsymbol{M},\ \Lambda^s T^*\boldsymbol{M}\otimes T\boldsymbol{M})\\ \to \mathrm{sec}(\boldsymbol{M},\ \Lambda^{r+s} T^*\boldsymbol{M}\otimes T\boldsymbol{M}) : (\phi, \psi) \mapsto [\phi, \psi],\end{aligned}$$

which is given, for decomposable forms, by the equality

$$\begin{aligned}[\alpha\otimes X,\ \beta\otimes Y] = \alpha\wedge\beta\otimes[X, Y]\\ +\alpha\wedge L_X\beta\otimes Y - (-1)^{rs}\,\beta\wedge L_Y\alpha\otimes X + (-1)^r\,i_Y\alpha\wedge d\beta\otimes X - (-1)^{rs+s}\,i_X\beta\wedge d\alpha\otimes Y,\end{aligned}$$

where $\alpha \in \mathrm{sec}(\boldsymbol{M},\ \Lambda^r T^*\boldsymbol{M})$, $\beta \in \mathrm{sec}(\boldsymbol{M},\ \Lambda^s T^*\boldsymbol{M})$, $X, Y \in \mathrm{sec}(\boldsymbol{M},\ T\boldsymbol{M})$.

In particular, if α and β are closed, the above equality becomes

$$[\alpha\otimes X,\ \beta\otimes Y] = \alpha\wedge\beta\otimes[X, Y] + \alpha\wedge di_X\beta\otimes Y - (-1)^{rs}\,\beta\wedge di_Y\alpha\otimes X. \quad \square$$

Next, we analyse particular cases related to the degrees r and s of the tangent valued forms.

We use the FN-bracket mainly for the covariant differential of tangent valued forms, which will discussed later (see Sect. F.1.2). In view of this subject, we provide the example of the FN-bracket in the case $r = 1$.

Example E.1.4 In the particular case when $r = 1$, the expression of the FN-bracket becomes

$$
\begin{aligned}
[\phi,\ \psi]\big(X_1,\dots,X_{r+s}\big)_\wedge = {} & \tfrac{1}{s!}\sum_\sigma |\sigma|\,\big[\phi(X_{\sigma(1)}),\ \psi(X_{\sigma(2)},\dots,X_{\sigma(1+s)})_\wedge\big] \\
& - \tfrac{1}{s!}\sum_\sigma |\sigma|\,\phi\Big(\big[X_{\sigma(1)},\ \psi(X_{\sigma(2)},\dots,X_{\sigma(1+s)})_\wedge\big]\Big) \\
& + \tfrac{(-1)^s s}{s!}\sum_\sigma |\sigma|\,\psi\Big(X_{\sigma(1)},\dots,X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ \phi(X_{\sigma(s+1)})\big]\Big)_\wedge \\
& + \tfrac{s}{2\,s!}\sum_\sigma |\sigma|\,\phi\Big(\psi\big([X_{\sigma(1)},\ X_{\sigma(2)}], X_{\sigma(3)},\dots,X_{\sigma(1+s)}\big)_\wedge\Big) \\
& - \tfrac{(-1)^s s}{2\,s!}\sum_\sigma |\sigma|\,\psi\Big(X_{\sigma(1)},\dots,X_{\sigma(s-1)},\ \phi\big([X_{\sigma(s)},\ X_{\sigma(s+1)}]\big)\Big)_\wedge,
\end{aligned}
$$

and the coordinate expression becomes

$$
\begin{aligned}
[\phi,\ \psi] = {} & (\phi^\rho_{\lambda_1}\,\partial_\rho\psi^\mu_{\lambda_2\dots\lambda_{1+s}} - (-1)^s\,\psi^\rho_{\lambda_1\dots\lambda_s}\,\partial_\rho\phi^\mu_{\lambda_{s+1}} \\
& - \phi^\mu_\rho\,\partial_{\lambda_1}\psi^\rho_{\lambda_2\dots\lambda_{1+s}} + (-1)^s\,s\,\psi^\mu_{\lambda_1\dots\lambda_{s-1}\rho}\,\partial_{\lambda_s}\phi^\rho_{\lambda_{s+1}}) \\
& d^{\lambda_1}\wedge\dots\wedge d^{\lambda_{1+s}}\otimes\partial_\mu. \quad \square
\end{aligned}
$$

Example E.1.5 In the further particular case when $r = s = 1$, the expression of the FN-bracket becomes

$$
\begin{aligned}
[\phi,\ \psi]\big(X_1,\dots,X_{r+s}\big)_\wedge = {} & \sum_\sigma |\sigma|\,\big[\phi(X_{\sigma(1)}),\ \psi(X_{\sigma(2)})\big] \\
& - \sum_\sigma |\sigma|\,\phi\Big(\big[X_{\sigma(1)},\ \psi(X_{\sigma(2)})\big]\Big) - \sum_\sigma |\sigma|\,\psi\Big(\big[X_{\sigma(1)},\ \phi(X_{\sigma(2)})\big]\Big) \\
& + \tfrac{1}{2}\sum_\sigma |\sigma|\,\phi\Big(\psi\big([X_{\sigma(1)},\ X_{\sigma(2)}]\big)\Big) + \tfrac{1}{2}\sum_\sigma |\sigma|\,\psi\Big(\phi\big([X_{\sigma(1)},\ X_{\sigma(2)}]\big)\Big),
\end{aligned}
$$

and the coordinate expression becomes

$$
[\phi,\ \psi] = (\phi^\rho_{\lambda_1}\,\partial_\rho\psi^\mu_{\lambda_2} + \psi^\rho_{\lambda_1}\,\partial_\rho\phi^\mu_{\lambda_2} - \phi^\mu_\rho\,\partial_{\lambda_1}\psi^\rho_{\lambda_2} - \psi^\mu_\rho\,\partial_{\lambda_1}\phi^\rho_{\lambda_2})\,d^{\lambda_1}\wedge d^{\lambda_2}\otimes\partial_\mu. \quad \square
$$

We stress that, in the case $r = s = 1$, we have $[\phi, \psi] = [\psi, \phi]$.

In the particular case when $\psi = \phi$, the FN-bracket $[\phi, \phi] \in \sec(\boldsymbol{F}, \Lambda^2 T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ turns out to be just the "Nijenhuis tensor", "torsion" of ϕ, which is introduced in the literature by an ad hoc definition (see, for instance, [242, p. 123]).

In the particular case when $r = 0$, i.e. when the 1st tangent valued form is a vector field, the FN-bracket becomes the standard Lie derivative.

Example E.1.6 If

$$
\phi := X \in \sec(\boldsymbol{M}, T\boldsymbol{M}) \qquad \text{and} \qquad \psi \in \sec(\boldsymbol{M},\ \Lambda^s T^*\boldsymbol{M} \otimes T\boldsymbol{M}),
$$

then the FN-bracket turns out to be the Lie derivative

$$
[\phi, \psi] = [X,\ \psi] = L_X\psi.
$$

Moreover, for each $X_1, \dots, X_s \in \sec(\boldsymbol{M}, T\boldsymbol{M})$, we obtain the equality

$$[X,\ \psi]\big(X_1, \dots, X_s\big) = \big[X,\ \psi(X_1, \dots, X_s)\big] - \sum_{1 \le i \le s} \psi\big(X_1, \dots, [X,\ X_i], \dots X_s\big).$$

We have the coordinate expression

$$[X,\ \psi] = (X^\rho\, \partial_\rho \psi^\mu_{\lambda_1 \dots \lambda_s} - \psi^\rho_{\lambda_1 \dots \lambda_s}\, \partial_\rho X^\mu + s\, \psi^\mu_{\lambda_1 \dots \lambda_{s-1}\rho}\, \partial_{\lambda_s} X^\rho)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu.$$

In the particular case of a decomposable tangent valued form $\psi = \beta \otimes Y$, we obtain the equality (see Proposition E.1.3)

$$[X, \psi] = [X,\ \beta \otimes Y] = \beta \otimes [X, Y] + L_X \beta \otimes Y.$$

Indeed, the above results fit the standard Leibniz rule for Lie derivatives. □

Even more, in the particular case when the two tangent valued forms are vector fields, the FN-bracket reduced to the standard Lie bracket.

Example E.1.7 If

$$\phi := X \in \sec(\boldsymbol{M}, T\boldsymbol{M}) \qquad \text{and} \qquad \psi := Y \in \sec(\boldsymbol{M}, T\boldsymbol{M}),$$

then the FN-bracket turns out to be the standard Lie bracket

$$[\phi,\ \psi] = [X, Y] = L_X Y,$$

with coordinate expression

$$[X,\ Y] = (X^\rho\, \partial_\rho Y^\mu - Y^\rho\, \partial_\rho X^\mu)\, \partial_\mu. \quad \square$$

E.2 The FN-Bracket on a Fibred Manifold

In the above Sect. E.1, we have introduced notion of FN-bracket for tangent valued forms on a manifold $\boldsymbol{M}$ (see Sect. C.2). Clearly, if a manifold $\boldsymbol{F}$ is equipped with a fibring $p : \boldsymbol{F} \to \boldsymbol{B}$, then the FN-bracket can be still achieved. Even more, the FN-bracket inherits further properties from the fibring. In particular, we can see how the FN-bracket restricts to projectable tangent valued forms and vertical valued forms.

In the present book, we use several times these results. In fact, we are involved with several fibred manifolds, such as spacetime, the tangent space of spacetime, the phase space, the quantum bundle, the upper quantum bundle.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical fibred chart by (x^λ, y^i).

We recall that if $X_1, X_2 \in \text{pro}(\boldsymbol{F}, T\boldsymbol{F})$ are projectable on $\underline{X}_1, \underline{X}_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then the Lie bracket $[X_1, X_2] \in \text{pro}(\boldsymbol{F}, T\boldsymbol{F})$ is projectable on $[\underline{X}_1, \underline{X}_2] \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ (see Note B.2.6). Indeed, we can extend this property to projectable tangent valued forms in the following way.

Lemma E.2.1 *If $\phi \in \text{pro}\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ and $X_1, \ldots, X_r \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then the vector field*

$$\phi(X_1, \ldots, X_r) = \tfrac{1}{r!}\ \phi(X_1, \ldots, X_r)_\wedge \in \text{pro}\sec(\boldsymbol{F}, T\boldsymbol{F})$$

is projectable on

$$\underline{\phi}(X_1, \ldots, X_r) = \tfrac{1}{r!}\ \underline{\phi}(X_1, \ldots, X_r)_\wedge \in \text{pro}\sec(\boldsymbol{B}, T\boldsymbol{B}). \square$$

Lemma E.2.2 *A projectable tangent valued form $\phi \in \text{pro}\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ on the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ can be naturally regarded, by pullback with respect to the fibring $p : \boldsymbol{F} \to \boldsymbol{B}$, as a tangent valued form of the manifold $\boldsymbol{F}$*

$$\bar{\phi} \in \sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{F} \otimes T\boldsymbol{F}).$$

Accordingly, if $\bar{X}_1, \ldots, \bar{X}_r \in \text{pro}(\boldsymbol{F}, T\boldsymbol{F})$ are projectable on $X_1, \ldots, X_r \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then we have

$$\bar{\phi}(\bar{X}_1, \ldots, \bar{X}_r) = \tfrac{1}{r!}\ \bar{\phi}(\bar{X}_1, \ldots, \bar{X}_r)_\wedge = \phi(X_1, \ldots, X_r) = \tfrac{1}{r!}\ \phi(X_1, \ldots, X_r)_\wedge. \quad \square$$

Theorem E.2.3 *The FN-bracket of two projectable tangent valued forms is a projectable tangent valued form. Moreover, for each projectable tangent valued form ϕ and ψ, we obtain*

$$\underline{[\phi, \psi]} = [\underline{\phi}, \underline{\psi}].$$

If $\phi \in \text{pro}\sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ and $\psi \in \text{pro}\sec(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F})$, then, the expression of the FN-bracket (see Theorem E.1.1) becomes, for each $X_1, \ldots, X_{r+s} \in \sec(\boldsymbol{B},\ T\boldsymbol{B})$,

$$\begin{aligned}
&[\phi,\ \psi]\big(X_1, \ldots, X_{r+s}\big)_\wedge \\
&:= \tfrac{1}{r!s!} \sum_\sigma |\sigma| \big[\phi(X_{\sigma(1)}, \ldots, X_{\sigma(r)})_\wedge,\ \psi(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big] \\
&\quad - \tfrac{r}{r!s!} \sum_\sigma |\sigma|\, \phi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(r-1)},\ \big[X_{\sigma(r)},\ \underline{\psi}(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big]\Big)_\wedge \\
&\quad + (-1)^{rs}\, \tfrac{s}{r!s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ \underline{\phi}(X_{\sigma(s+1)}, \ldots, X_{\sigma(r+s)})_\wedge\big]\Big)_\wedge \\
&\quad + \tfrac{rs}{2r!s!} \sum_\sigma |\sigma|\, \phi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(r-1)},\ \underline{\psi}\big([X_{\sigma(r)},\ X_{\sigma(r+1)}], X_{\sigma(r+2)}, \ldots, X_{\sigma(r+s)}\big)_\wedge\Big)_\wedge \\
&\quad - (-1)^{rs}\, \tfrac{rs}{2r!s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)},\ \underline{\phi}\big([X_{\sigma(s)},\ X_{\sigma(s+1)}], X_{\sigma(s+2)}, \ldots, X_{\sigma(r+s)}\big)_\wedge\Big)_\wedge .
\end{aligned}$$

Accordingly, the coordinate expression becomes

$$
\begin{aligned}
[\phi, \psi] = \big(&\phi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho \psi^{\mu}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \psi^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho \phi^{\mu}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\
&- r\, \phi^{\mu}_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \psi^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}\, s\, \psi^{\mu}_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \phi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu \\
&+ \big(\phi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \psi^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho \phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\
&+ \phi^{j}_{\lambda_1\ldots\lambda_r}\, \partial_j \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \psi^{j}_{\lambda_1\ldots\lambda_s}\, \partial_j \phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\
&- r\, \phi^{i}_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \psi^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}\, s\, \psi^{i}_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \phi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_i \,. \square
\end{aligned}
$$

Then, the following Proposition E.2.4 provides a simple useful expression of the FN-bracket in terms of decomposable projectable tangent valued forms (see Note C.2.8 and Proposition E.1.3).

Proposition E.2.4 *The FN-bracket for projectable tangent valued forms is the unique* $\mathbb{R}$*-bilinear sheaf morphism*

$$
\begin{aligned}
[\,,\,] : \mathrm{pro}(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \times \mathrm{pro}(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \to \\
\to \mathrm{pro}(\boldsymbol{F},\ \Lambda^{r+s} T^*\boldsymbol{B} \otimes T\boldsymbol{F}) : (\phi, \psi) \mapsto [\phi, \psi],
\end{aligned}
$$

which is given, for decomposable projectable forms, by

$$
\begin{aligned}
[\alpha \otimes X,\ \beta \otimes Y] = \alpha \wedge \beta \otimes [X, Y] \\
+\alpha \wedge L_X \beta \otimes Y - (-1)^{rs}\, \beta \wedge L_Y \alpha \otimes X + (-1)^r\, i_Y \alpha \wedge d\beta \otimes X - (-1)^{rs+s}\, i_X \beta \wedge d\alpha \otimes Y,
\end{aligned}
$$

where $\alpha \in \mathrm{sec}(\boldsymbol{B},\ \Lambda^r T^*\boldsymbol{B})$, $\beta \in \mathrm{sec}(\boldsymbol{B},\ \Lambda^s T^*\boldsymbol{B})$, $X, Y \in \mathrm{sec}(\boldsymbol{F},\ T\boldsymbol{F})$.

In particular, if $\alpha \in \mathrm{sec}(\boldsymbol{B}, \Lambda^r T^*\boldsymbol{B})$ *and* $\beta \in \mathrm{sec}(\boldsymbol{B}, \Lambda^s T^*\boldsymbol{B})$ *are closed, then the above equality becomes*

$$
\begin{aligned}
[\alpha \otimes X,\ \beta \otimes Y] = \alpha \wedge \beta \otimes [X, Y] + \alpha \wedge d i_{\underline{X}} \beta \otimes Y \\
- (-1)^{rs}\, \beta \wedge d i_{\underline{Y}} \alpha \otimes X. \quad \square
\end{aligned}
$$

Next, we discuss the case when one or both projectable tangent forms are vertical valued.

The FN-bracket of a projectable tangent valued form with a vertical valued form turns out to be a vertical valued form.

Corollary E.2.5 *If* $\phi \in \mathrm{pro\,sec}(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F})$ *and* $\psi \in \mathrm{sec}(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F})$, *then we obtain*

$$
[\phi, \psi] \in \mathrm{sec}(\boldsymbol{F},\ \Lambda^{r+s} T^*\boldsymbol{B} \otimes V\boldsymbol{F}).
$$

Moreover, the expression of the FN-bracket becomes

$$[\phi,\ \psi]\big(X_1, \dots, X_{r+s}\big)_\wedge$$
$$:= \tfrac{1}{r!s!} \sum_\sigma |\sigma| \big[\phi(X_{\sigma(1)}, \dots, X_{\sigma(r)})_\wedge,\ \psi(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)})_\wedge\big]$$
$$+ (-1)^{rs} \tfrac{s}{r!s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \dots, X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ \underline{\phi}(X_{\sigma(s+1)}, \dots, X_{\sigma(r+s)})_\wedge\big]\Big)_\wedge$$
$$-(-1)^{rs} \tfrac{rs}{2r!s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \dots, X_{\sigma(s-1)},\ \underline{\phi}\big([X_{\sigma(s)},\ X_{\sigma(s+1)}], X_{\sigma(s+2)}, \dots, X_{\sigma(r+s)}\big)_\wedge\Big)_\wedge,$$

Accordingly, the coordinate expression becomes

$$[\phi, \psi] = (\phi^\rho_{\lambda_1\dots\lambda_r}\, \partial_\rho \psi^i_{\lambda_{r+1}\dots\lambda_{r+s}} + \phi^j_{\lambda_1\dots\lambda_r}\, \partial_j \psi^i_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\, \psi^j_{\lambda_1\dots\lambda_s}\, \partial_j \phi^i_{\lambda_{s+1}\dots\lambda_{r+s}}$$
$$+ (-1)^{rs}\, s\, \psi^i_{\lambda_1\dots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \phi^\rho_{\lambda_{s+1}\dots\lambda_{r+s}})\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \partial_i.$$

Hence, the subsheaf

$$\mathrm{sec}(\boldsymbol{F},\ \Lambda T^*\boldsymbol{B} \otimes V\boldsymbol{F}) \subset \mathrm{pro}(\boldsymbol{F},\ \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F})$$

turns out to be an ideal. □

In the further particular case when the tangent valued forms are vertical, we obtain a further simplification of the the FN-bracket.

Corollary E.2.6 *If* $\phi \in \mathrm{sec}(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$ *and* $\psi \in \mathrm{sec}(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F})$, *then we obtain*

$$[\phi, \psi] \in \mathrm{sec}(\boldsymbol{F},\ \Lambda^{r+s} T^*\boldsymbol{B} \otimes V\boldsymbol{F}),$$

with the expression

$$[\phi,\ \psi]\big(X_1, \dots, X_{r+s}\big)_\wedge = \tfrac{1}{r!s!} \sum_\sigma |\sigma| \big[\phi(X_{\sigma(1)}, \dots, X_{\sigma(r)})_\wedge,\ \psi(X_{\sigma(r+1)}, \dots, X_{\sigma(r+s)})_\wedge\big]$$

and the coordinate expression

$$[\phi, \psi] = (\phi^j_{\lambda_1\dots\lambda_r}\, \partial_j \psi^i_{\lambda_{r+1}\dots\lambda_{r+s}} - (-1)^{rs}\, \psi^j_{\lambda_1\dots\lambda_s}\, \partial_j \phi^i_{\lambda_{s+1}\dots\lambda_{r+s}})\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \partial_i.$$

Hence, the subsheaf

$$\mathrm{sec}(\boldsymbol{F},\ \Lambda T^*\boldsymbol{B} \otimes V\boldsymbol{F}) \subset \mathrm{pro\,sec}(\boldsymbol{F},\ \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F})$$

turns out to be a subalgebra. □

In the particular case when the 1st projectable tangent valued form is a vector field $\phi := Y$, the FN-bracket $[Y, \psi]$ reduces to the standard Lie derivative $L_Y \psi$.

Proposition E.2.7 *If* $\phi := Y \in \mathrm{pro\,sec}(\boldsymbol{F}, T\boldsymbol{F})$ *and* $\psi \in \mathrm{pro\,sec}(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F})$, *then the F-N bracket turns out to be the Lie derivative*

$$[\phi, \psi] := [Y, \psi] = L_Y \psi,$$

with coordinate expression

$$\begin{aligned}[Y, \psi] = {} & (Y^\rho \, \partial_\rho \psi^\mu_{\lambda_1 \ldots \lambda_s} - \psi^\rho_{\lambda_1 \ldots \lambda_s} \, \partial_\rho Y^\mu + s \, \psi^\mu_{\lambda_1 \ldots \lambda_{s-1} \rho} \, \partial_{\lambda_s} Y^\rho) \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_s} \otimes \partial_\mu \\ & + (Y^\rho \, \partial_\rho \psi^i_{\lambda_1 \ldots \lambda_s} - \psi^\rho_{\lambda_1 \ldots \lambda_s} \, \partial_\rho Y^i \\ & + Y^j \, \partial_j \psi^i_{\lambda_1 \ldots \lambda_s} - \psi^j_{\lambda_1 \ldots \lambda_s} \, \partial_j Y^i + s \, \psi^i_{\lambda_1 \ldots \lambda_{s-1} \rho} \, \partial_{\lambda_s} Y^\rho) \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_s} \otimes \partial_i . \quad \square\end{aligned}$$

Next, we analyse particular cases related to the degrees r and s of the projectable tangent valued forms.

We use the FN-bracket mainly for the covariant differential of projectable tangent valued forms, which will discussed later (see Sect. F.1.2). In view of this subject, we provide the example of the FN-bracket for projectable tangent valued forms in the case $r = 1$.

Example E.2.8 In the particular case when $r = 1$, the expression of the FN-bracket becomes

$$\begin{aligned}[\phi, \psi] \big(X_1, \ldots, X_{r+s}\big)_\wedge = {} & \tfrac{1}{s!} \sum_\sigma |\sigma| \big[\phi(X_{\sigma(1)}), \; \psi(X_{\sigma(2)}, \ldots, X_{\sigma(1+s)})_\wedge\big] \\ & - \tfrac{1}{s!} \sum_\sigma |\sigma| \, \phi\Big(\big[X_{\sigma(1)}, \; \underline{\psi}(X_{\sigma(2)}, \ldots, X_{\sigma(1+s)})_\wedge\big]\Big) \\ & + \tfrac{(-1)^s \, s}{s!} \sum_\sigma |\sigma| \, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)}, \; \big[X_{\sigma(s)}, \; \underline{\phi}(X_{\sigma(s+1)})\big]\Big)_\wedge \\ & + \tfrac{s}{2\,s!} \sum_\sigma |\sigma| \, \phi\Big(\underline{\psi}\big([X_{\sigma(1)}, \; X_{\sigma(2)}], X_{\sigma(3)}, \ldots, X_{\sigma(1+s)}\big)_\wedge\Big) \\ & - \tfrac{(-1)^s \, s}{2\,s!} \sum_\sigma |\sigma| \, \psi\Big(X_{\sigma(1)}, \ldots, X_{\sigma(s-1)}, \; \underline{\phi}\big([X_{\sigma(s)}, \; X_{\sigma(s+1)}]\big)\Big)_\wedge ,\end{aligned}$$

and the coordinate expression becomes

$$\begin{aligned}[\phi, \psi] = {} & \big(\phi^\rho_{\lambda_1} \, \partial_\rho \psi^\mu_{\lambda_2 \ldots \lambda_{s+1}} - (-1)^s \, \psi^\rho_{\lambda_1 \ldots \lambda_s} \, \partial_\rho \phi^\mu_{\lambda_{s+1}} \\ & - \phi^\mu_\rho \, \partial_{\lambda_1} \psi^\rho_{\lambda_2 \ldots \lambda_{s+1}} + (-1)^s \, s \, \psi^\mu_{\lambda_1 \ldots \lambda_{s-1} \rho} \, \partial_{\lambda_s} \phi^\rho_{\lambda_{s+1}}\big) \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{s+1}} \otimes \partial_\mu \\ & + \big(\phi^\rho_{\lambda_1} \, \partial_\rho \psi^i_{\lambda_2 \ldots \lambda_{s+1}} - (-1)^s \, \psi^\rho_{\lambda_1 \ldots \lambda_s} \, \partial_\rho \phi^i_{\lambda_{s+1}} \\ & + \phi^j_{\lambda_1} \, \partial_j \psi^i_{\lambda_2 \ldots \lambda_{s+1}} - (-1)^s \, \psi^j_{\lambda_1 \ldots \lambda_s} \, \partial_j \phi^i_{\lambda_{s+1}} \\ & - \phi^i_\rho \, \partial_{\lambda_1} \psi^\rho_{\lambda_2 \ldots \lambda_{s+1}} + (-1)^s \, s \, \psi^i_{\lambda_1 \ldots \lambda_{s-1} \rho} \, \partial_{\lambda_s} \phi^\rho_{\lambda_{s+1}}\big) \, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{s+1}} \otimes \partial_i . \quad \square\end{aligned}$$

Example E.2.9 In the further particular case when $r = s = 1$, the expression of the FN-bracket becomes

$$
\begin{aligned}
[\phi,\ \psi]\big(X_1,\dots,X_{r+s}\big)_\wedge = {} & \sum_\sigma |\sigma|\, \big[\phi(X_{\sigma(1)}),\ \psi(X_{\sigma(2)})\big] \\
& - \sum_\sigma |\sigma|\, \phi\Big(\big[X_{\sigma(1)},\ \underline{\psi}(X_{\sigma(2)})\big]\Big) - \sum_\sigma |\sigma|\, \psi\Big(\big[X_{\sigma(1)},\ \underline{\phi}(X_{\sigma(2)})\big]\Big) \\
& + \tfrac{1}{2} \sum_\sigma |\sigma|\, \phi\Big(\underline{\psi}\big([X_{\sigma(1)},\ X_{\sigma(2)}]\big)\Big) + \tfrac{1}{2} \sum_\sigma |\sigma|\, \psi\Big(\underline{\phi}\big([X_{\sigma(1)},\ X_{\sigma(2)}]\big)\Big),
\end{aligned}
$$

and the coordinate expression becomes

$$
\begin{aligned}
& [\phi, \psi] = \\
& = \big(\phi^\rho_{\lambda_1}\, \partial_\rho \psi^\mu_{\lambda_2} + \psi^\rho_{\lambda_1}\, \partial_\rho \phi^\mu_{\lambda_2} - \phi^\mu_\rho\, \partial_{\lambda_1} \psi^\rho_{\lambda_2} - \psi^\mu_\rho\, \partial_{\lambda_1} \phi^\rho_{\lambda_2}\big)\, d^{\lambda_1} \wedge d^{\lambda_2} \otimes \partial_\mu \\
& + \big(\phi^\rho_{\lambda_1}\, \partial_\rho \psi^i_{\lambda_2} + \psi^\rho_{\lambda_1}\, \partial_\rho \phi^i_{\lambda_2} + \phi^j_{\lambda_1}\, \partial_j \psi^i_{\lambda_2} + \psi^j_{\lambda_1}\, \partial_j \phi^i_{\lambda_2} \\
& \quad - \phi^i_\rho\, \partial_{\lambda_1} \psi^\rho_{\lambda_2} - \psi^i_\rho\, \partial_{\lambda_1} \phi^\rho_{\lambda_1}\big)\, d^{\lambda_1} \wedge d^{\lambda_2} \otimes \partial_i .
\end{aligned}
$$

We stress that, in the case $r = s = 1$, we have

$$[\phi, \psi] = [\psi, \phi]. \quad \square$$

Even more, in the particular case when the two projectable tangent valued forms are projectable vector fields, the FN-bracket reduced to the standard Lie bracket.

Example E.2.10 If $\phi := X \in \mathrm{pro}(\boldsymbol{F}, T\boldsymbol{F})$ and $\psi := Y \in \mathrm{pro}(\boldsymbol{F}, T\boldsymbol{F})$, then the FN-bracket turns out to be the standard Lie derivative

$$[\phi,\ \psi] = [X, Y] = L_X Y,$$

with coordinate expression

$$
\begin{aligned}
[X,\ Y] = {} & (X^\rho\, \partial_\rho Y^\mu - Y^\rho\, \partial_\rho X^\mu)\, \partial_\mu \\
& + (X^\rho\, \partial_\rho Y^i - Y^\rho\, \partial_\rho X^i + X^j\, \partial_j Y^i - Y^j\, \partial_j X^i)\, \partial_i . \quad \square
\end{aligned}
$$

E.3 The FN-Bracket on a Structured Bundle

In the above Sect. E.2, we have discussed the FN-bracket of tangent valued forms on any fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$. Clearly, if the fibred manifold $\boldsymbol{F}$ is a bundle equipped with an algebraic structure, then the FN-bracket can be still achieved. Even more, the FN-bracket inherits further properties from the algebraic structure of the bundle.

Here, we study the cases of vector and affine bundles. Indeed, the sheaves of linear and affine projectable tangent valued forms turn out to be closed with respect to the FN-bracket (see Notes C.2.10 and C.2.12).

E.3.1 The FN-Bracket on a Vector Bundle

The sheaf of linear projectable tangent valued forms on a vector bundle turns out to be closed with respect to the FN-bracket (see Note C.2.10).

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a linear fibred chart (x^λ, y^i).

Proposition E.3.1 *For each*

$$\phi \in \operatorname{lin}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \quad \textit{and} \quad \psi \in \operatorname{lin}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F}),$$

we obtain

$$[\phi, \psi] \in \operatorname{lin}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^{r+s} T^*\boldsymbol{B} \otimes T\boldsymbol{F}).$$

Thus, the subsheaf of linear projectable tangent valued forms

$$\operatorname{lin}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \operatorname{pro}\sec(\boldsymbol{F}, \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F})$$

is closed with respect to the FN-bracket. We have the coordinate expression

$$\begin{aligned}
[\phi, \psi] = \big(&\phi^\rho_{\lambda_1\ldots\lambda_r}\, \partial_\rho \psi^\mu_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \psi^\rho_{\lambda_1\ldots\lambda_s}\, \partial_\rho \phi^\mu_{\lambda_{s+1}\ldots\lambda_{r+s}} \\
&- r\, \phi^\mu_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \psi^\rho_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, s\, \psi^\mu_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \phi^\rho_{\lambda_{s+1}\ldots\lambda_{r+s}}\big) \\
&\qquad\qquad d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu \\
&+ \big(\phi^\rho_{\lambda_1\ldots\lambda_r}\, \partial_\rho \psi^i_{\lambda_{r+1}\ldots\lambda_{r+s}\, h} - (-1)^{rs}\, \psi^\rho_{\lambda_1\ldots\lambda_s}\, \partial_\rho \phi^i_{\lambda_{s+1}\ldots\lambda_{r+s}\, h} \\
&+ \phi^j_{\lambda_1\ldots\lambda_r\, h}\, \psi^i_{\lambda_{r+1}\ldots\lambda_{r+s}\, j} - (-1)^{rs}\, \psi^j_{\lambda_1\ldots\lambda_s h}\, \phi^i_{\lambda_{s+1}\ldots\lambda_{r+s}\, j} \\
&- r\, \phi^i_{\lambda_1\ldots\lambda_{r-1}\rho\, h}\, \partial_{\lambda_r} \psi^\rho_{\lambda_{r+1}\ldots\lambda_{r+s}} + (-1)^{rs}\, s\, \psi^i_{\lambda_1\ldots\lambda_{s-1}\rho\, h}\, \partial_{\lambda_s} \phi^\rho_{\lambda_{s+1}\ldots\lambda_{r+s}}\big) \\
&\qquad\qquad y^h\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_i .
\end{aligned}$$

In particular, the FN-bracket of linear vertical valued forms becomes a "commutator", which is a purely algebraic operation, *with the following coordinate expression*

$$[\phi, \psi] = (\phi^j_{\lambda_1\ldots\lambda_r\, h}\, \psi^i_{\lambda_{r+1}\ldots\lambda_{r+s}\, j} - (-1)^{rs}\, \psi^j_{\lambda_1\ldots\lambda_s\, h}\, \phi^i_{\lambda_{s+1}\ldots\lambda_{r+s}\, j})\, y^h\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_i . \quad \square$$

E.3.2 The FN-Bracket on an Affine Bundle

The sheaf of affine projectable tangent valued forms on an affine bundle turns out to be closed with respect to the FN-bracket (see Note C.2.12).

Let us consider an affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to an affine fibred chart (x^λ, y^i).

Proposition E.3.2 *For each*

$$\phi \in \operatorname{aff}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \quad \textit{and} \quad \psi \in \operatorname{aff}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F}),$$

we obtain

$$[\phi, \psi] \in \operatorname{aff}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda^{r+s} T^*\boldsymbol{B} \otimes T\boldsymbol{F}).$$

Thus, the subsheaf of affine projectable tangent valued forms

$$\operatorname{aff}\operatorname{pro}\sec(\boldsymbol{F}, \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F}) \subset \operatorname{pro}\sec(\boldsymbol{F}, \Lambda T^*\boldsymbol{B} \otimes T\boldsymbol{F})$$

is closed with respect to the FN-bracket. We have the coordinate expression

$$\begin{aligned}
&[\phi, \psi] \\
&= \big(\phi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho \psi^{\mu}_{\lambda_{r+1}\ldots\lambda_{r+s}} - (-1)^{rs}\, \psi^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho \phi^{\mu}_{\lambda_{s+1}\ldots\lambda_{r+s}} \\
&\quad - r\, \phi^{\mu}_{\lambda_1\ldots\lambda_{r-1}\rho}\, \partial_{\lambda_r} \psi^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} \\
&\qquad + (-1)^{rs}\, s\, \psi^{\mu}_{\lambda_1\ldots\lambda_{s-1}\rho}\, \partial_{\lambda_s} \phi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}\big)\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_\mu \\
&\quad + \big(\phi^{\rho}_{\lambda_1\ldots\lambda_r}\, \partial_\rho(\psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}\, h}\, y^h + \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}\, o}) \\
&\quad - (-1)^{rs}\, \psi^{\rho}_{\lambda_1\ldots\lambda_s}\, \partial_\rho(\phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}\, h}\, y^h + \phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}\, o}) \\
&\quad + (\phi^{j}_{\lambda_1\ldots\lambda_r\, h}\, y^h + \phi^{j}_{\lambda_1\ldots\lambda_r\, o})\, \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}\, j} \\
&\quad - (-1)^{rs}\, (\psi^{j}_{\lambda_1\ldots\lambda_s\, h}\, y^h + \psi^{j}_{\lambda_1\ldots\lambda_s\, o})\, \phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}\, j} \\
&\quad - r\, (\phi^{i}_{\lambda_1\ldots\lambda_{r-1}\rho\, h}\, y^h + \phi^{i}_{\lambda_1\ldots\lambda_{r-1}\rho\, o})\, \partial_{\lambda_r} \psi^{\rho}_{\lambda_{r+1}\ldots\lambda_{r+s}} \\
&\quad + (-1)^{rs}\, s\, (\psi^{i}_{\lambda_1\ldots\lambda_{s-1}\rho\, h}\, y^h + \psi^{i}_{\lambda_1\ldots\lambda_{s-1}\rho\, o})\, \partial_{\lambda_s} \phi^{\rho}_{\lambda_{s+1}\ldots\lambda_{r+s}}\big)\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_i .
\end{aligned}$$

In particular, the FN-bracket of affine vertical valued forms becomes a "commutator", which is a purely algebraic operation*, with the following coordinate expression*

$$\begin{aligned}
[\phi, \psi] = \Big(&(\phi^{j}_{\lambda_1\ldots\lambda_r\, h}\, y^h + \phi^{j}_{\lambda_1\ldots\lambda_r\, o})\, \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}\, j} \\
&- (-1)^{rs}\, (\psi^{j}_{\lambda_1\ldots\lambda_s\, h}\, y^h + \psi^{j}_{\lambda_1\ldots\lambda_s\, o})\, \phi^{i}_{\lambda_{s+1}\ldots\lambda_{r+s}\, j}\Big)\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{r+s}} \otimes \partial_i . \quad \square
\end{aligned}$$

E.4 The FN-Bracket of Vector Valued Forms

We have seen that the vector valued forms $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes \boldsymbol{F})$ on a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ can be naturally regarded as vertical valued forms $\phi \in \sec(\boldsymbol{F}, \Lambda^r T^*\boldsymbol{B} \otimes V\boldsymbol{F})$ (see Note C.2.16). Accordingly, the FN-bracket can be easily extended to vector valued forms.

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and refer to a linear fibred chart (x^λ, y^i) and to the associated basis (b_i).

Note E.4.1 By taking into account the natural identification between vector valued forms and tangent valued forms (see Note C.2.16), for each vector valued forms

$$\phi \in \sec(\boldsymbol{F}, \Lambda^r T^* \boldsymbol{B} \otimes \boldsymbol{F}) \qquad \text{and} \qquad \psi \in \sec(\boldsymbol{F}, \Lambda^s T^* \boldsymbol{B} \otimes \boldsymbol{F})$$

we obtain the FN-bracket

$$[\phi, \psi] \in \sec(\boldsymbol{F}, \Lambda^{r+s} T^* \boldsymbol{B} \otimes \boldsymbol{F}),$$

with coordinate expression

$$[\phi, \psi] = (\phi^j_{\lambda_1 \dots \lambda_r} \, \partial_j \psi^i_{\lambda_{r+1} \dots \lambda_{r+s}} \\ - (-1)^{rs} \, \psi^j_{\lambda_1 \dots \lambda_s} \, \partial_j \phi^i_{\lambda_{s+1} \dots \lambda_{r+s}}) \, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes \mathrm{b}_i . \quad \square$$

In the context of Quantum Mechanics, we deal also with vector valued forms on a complex vector bundle. Actually, the FN-bracket which has been discussed for real vector bundles can be easily extended to complex vector bundles. In fact, a complex vector bundle is also a real vector bundle; hence the FN-bracket is defined according to this real vector structure. We can emphasise the additional complex structure by exhibiting the behaviour of the FN-bracket with respect to the complex unit $\mathfrak{i}$.

Let us consider a complex vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and recall that the vertical bundle $V\boldsymbol{F} = \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{B}$ turns out to be a complex vector bundle.

Note E.4.2 We can easily prove that the FN-bracket of vertical tangent valued forms turns out to be linear with respect to the complex constants.

In other words, for each

$$\phi \in \sec(\boldsymbol{F}, \Lambda^r T^* \boldsymbol{B} \otimes V\boldsymbol{F}) \qquad \text{and} \qquad \psi \in \sec(\boldsymbol{F}, \Lambda^s T^* \boldsymbol{B} \otimes V\boldsymbol{F}),$$

we have

$$[\mathfrak{i}\, \phi, \psi] = \mathfrak{i}\, [\phi, \psi] = [\phi, \mathfrak{i}\, \psi]. \quad \square$$

Appendix F
Connections

In the standard literature of Differential Geometry and Mathematical Physics, the most popular approaches to connections deal with linear connections on a vector bundle (hence, in particular, linear connections of a manifold), Levi–Civita connections on a pseudo-riemannian bundle and principal connections on a principal bundle (see, for instance, [51, 94–96, 138, 241, 242, 246, 251]).

Usually, the linear connections are defined via the associated covariant differential, which fulfills certain linearity properties; accordingly, the curvature, the torsion and the Bianchi identities of a linear connection are usually defined via the associated covariant differential. The principal connections on a principal bundle turn out to be a more general and mathematically sophisticated approach to connections on bundles equipped with a symmetry structure group.

In the present book, we deal with several kinds of connections, which are non standard in some respects. So, it is convenient to avail of a more general and unifying approach to connections on a fibred manifold.

Thus, following the scheme proposed in [284, 285, 287, 311], we sketch the general theory of connections on a fibred manifold based on the language of the *Frölicher–Nijenhuis graded Lie algebra of tangent valued forms* (see, also, [97, 246]).

We present our approach step by step. We start with *general connections on a fibred manifold*; then we specialise the above theory to *linear connections of vector bundles*, and, in particular, we consider *affine connections of affine bundles* and *linear connections of a manifold* (Sects. F.2, F.3, and F.4).

F.1 General Connections

A very general and straightforward approach to connections of a fibred manifold can be conveniently achieved via tangent valued forms and their Frölicher–Nijenhuis

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graded Lie algebra. Indeed, this approach is able to account for standard and non standard connections in a simple unified scheme.

In this Section we refer to a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical fibred charts by (x^λ, y^i).

F.1.1 Connections as Tangent Valued Forms

We start by defining a *connection* of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ as a projectable tangent valued form $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ (see Definition C.2.2 and Note C.2.6).

Moreover, we discuss the associated *splitting* $T\boldsymbol{F} = H_c\boldsymbol{F} \underset{\boldsymbol{B}}{\oplus} V\boldsymbol{F}$ of the tangent bundle $T\boldsymbol{F}$ into the horizontal and vertical subbundles and the associated *covariant differential* $\nabla s : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ of sections $s : \boldsymbol{B} \to \boldsymbol{F}$.

Definition F.1.1 A *connection* of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ is defined to be a section $c : \boldsymbol{F} \to J_1\boldsymbol{F}$, i.e. a tangent valued form

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F},$$

which is projectable on $\mathbf{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;c\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\;\mathbf{1}_{\boldsymbol{B}}\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} .$$

The coordinate expression of a connection c is of the type

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i \, \partial_i), \quad \text{with } c_\lambda{}^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}).$$

Thus, a connection c yields the complementary *horizontal prolongation* and the *vertical projection*

$$c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F} \qquad \text{and} \qquad \nu[c] : T\boldsymbol{F} \to V\boldsymbol{F},$$

with coordinate expressions, for each $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ and $Y \in \sec(\boldsymbol{F}, T\boldsymbol{F})$,

$$c(X) = X^\lambda \, (\partial_\lambda + c_\lambda{}^i \, \partial_i) \qquad \text{and} \qquad \nu[c](Y) = (Y^i - c_\lambda{}^i \, Y^\lambda) \, \partial_i .$$

Indeed, we have

$$Tp\big(c(X)\big) = X \circ p \qquad \text{and} \qquad \nu[c](Y) = Y, \quad \text{for each } Y : \boldsymbol{F} \to V\boldsymbol{F}.$$

The subbundle $H_c\boldsymbol{F} := \operatorname{im}(c) \subset T\boldsymbol{F}$ is said to be the *horizontal subbundle*.

Then, c yields a linear splitting over $\boldsymbol{F}$

$$T\boldsymbol{F} = H_c\boldsymbol{F} \underset{\boldsymbol{F}}{\oplus} V\boldsymbol{F}.$$

Actually, the vertical projection $\nu[c]$ characterises the connection c.

We define the *covariant differential* of a section $s : \boldsymbol{B} \to \boldsymbol{F}$, to be the linear fibred morphism over s

$$\nabla[c]s := \nu[c] \circ Ts : T\boldsymbol{B} \to V\boldsymbol{F},$$

with coordinate expression

$$\nabla[c]s = (\partial_\lambda s^i - c_\lambda{}^i \circ s)\,(\partial_i \circ s). \quad \square$$

Note F.1.2 A connection and a soldering form

$$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F} \qquad \text{and} \qquad \sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$$

yield the new connection

$$\acute{c} := c + \sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F},$$

with coordinate expression

$$\acute{c} = d^\lambda \otimes \big(\partial_\lambda + (c_\lambda{}^i + \sigma_\lambda{}^i)\,\partial_i\big).$$

Conversely, the difference of any two connections c and $\acute{c}$ is a soldering form σ. □

Remark F.1.3 We stress that our components $c_\lambda{}^i$ have different sign with respect to the standard notation. Our unusual convention reflects the fact that we have introduced the notion of connection via the horizontal prolongation and not via the covariant differential. □

Remark F.1.4 We stress that, in our general context, assuming algebraic properties for a general covariant differential (as it is usually done in the context of standard linear connections of a manifold) would not make any sense. □

Remark F.1.5 Let us consider a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$. Then, in virtue of the definition

$$\nabla[c]s := ds - c \circ s,$$

the covariant differential $\nabla[c]s$ vanishes if and only if

$$Ts(T\boldsymbol{B}) \subset H_c\boldsymbol{F} \subset T\boldsymbol{F}$$

is contained in the horizontal subbundle.

Thus, the covariant differential "measures" the discrepancy of the tangent map Ts to be "horizontal" in the above sense. □

F.1.2 Covariant Differential of Tangent Valued Forms

Given a connection c, besides the covariant differential of sections, we introduce, through the FN-bracket, the *covariant differential* $d_c\psi := [c, \psi] \in \sec(\boldsymbol{F}, \Lambda^{s+1}T^*\boldsymbol{F} \otimes T\boldsymbol{F})$ of a tangent valued form $\psi \in \sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{F} \otimes T\boldsymbol{F})$ (see Definition C.2.2).

Later, we use the covariant differential of projectable tangent valued forms to achieve the concepts of curvature and torsion of a connection (see Sects. F.1.3 and F.1.6).

This concept of covariant differential of tangent valued forms turns out to be a generalisation of the standard Koszul exterior covariant differential of vector valued forms, in the case of vector bundles (see Sect. F.2.1 and see, for instance, [51, 251]).

Let us consider a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Definition F.1.6 We define the *(exterior) covariant differential* of a tangent valued form $\psi \in \sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{F} \otimes T\boldsymbol{F})$ to be the tangent valued form

$$d_c\psi := [c, \psi] \in \sec(\boldsymbol{F}, \Lambda^{s+1}T^*\boldsymbol{F} \otimes T\boldsymbol{F}).$$

In particular, if $\psi \in \operatorname{pro}\sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{B} \otimes T\boldsymbol{F})$, then its covariant differential turns out to be a vertical valued form

$$d_c\psi \in \sec(\boldsymbol{F}, \Lambda^{s+1}T^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

For each $X_1, \dots, X_{s+1} \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we obtain

$$\begin{aligned} d_c\psi(X_1, \dots, X_{s+1})_\wedge = {} & \tfrac{1}{s!} \sum_\sigma |\sigma| \big[c(X_{\sigma(1)}),\ \psi(X_{\sigma(2)}, \dots, X_{\sigma(s+1)})_\wedge\big] \\ & - \tfrac{1}{s!} \sum_\sigma |\sigma|\, c\Big(\big[X_{\sigma(1)},\ \underline{\psi}(X_{\sigma(2)}, \dots, X_{\sigma(s+1)})_\wedge\big]\Big) \\ & + (-1)^s\, \tfrac{s}{2\,s!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \dots, X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ X_{\sigma(s+1)}\big]\Big)_\wedge \\ & + \tfrac{s}{2\,s!} \sum_\sigma |\sigma|\, c\Big(\underline{\psi}\big([X_{\sigma(1)},\ X_{\sigma(2)}], X_{\sigma(3)}, \dots, X_{\sigma(s+1)}\big)_\wedge\Big). \end{aligned}$$

Moreover, we have the coordinate expression

$$\begin{aligned} d_c\psi = \big(& - \partial_\rho c_{\lambda_1}{}^i\, \psi^\rho_{\lambda_2 \dots \lambda_{s+1}} - c_\rho{}^i\, \partial_{\lambda_1} \psi^\rho_{\lambda_2 \dots \lambda_{s+1}} \\ & + \partial_{\lambda_1} \psi^i_{\lambda_2 \dots \lambda_{s+1}} + c_{\lambda_1}{}^j\, \partial_j \psi^i_{\lambda_2 \dots \lambda_{s+1}} \\ & - \partial_j c_{\lambda_1}{}^i\, \psi^j_{\lambda_2 \dots \lambda_{s+1}} \big)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{s+1}} \otimes \partial_i. \end{aligned}$$

In particular, if $\psi \in \sec(\boldsymbol{F},\ \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F})$, then its covariant differential turns out to be a vertical valued form

$$d_c\psi \in \sec(\boldsymbol{F},\ \Lambda^{s+1} T^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

For each $X_1, \dots, X_{s+1} \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we obtain

$$d_c\psi\,\big(X_1, \dots, X_{s+1}\big)_\wedge = \tfrac{1}{s!} \sum_\sigma |\sigma|\, \big[c(X_{\sigma(1)}),\ \psi(X_{\sigma(2)}, \dots, X_{\sigma(s+1)})_\wedge\big]$$
$$+ (-1)^s\, \tfrac{1}{2\,(s-1)!} \sum_\sigma |\sigma|\, \psi\Big(X_{\sigma(1)}, \dots, X_{\sigma(s-1)},\ \big[X_{\sigma(s)},\ X_{\sigma(s+1)}\big]\Big)_\wedge.$$

Moreover, we have the coordinate expression

$$d_c\psi = \big(\partial_{\lambda_1}\psi^i_{\lambda_2\dots\lambda_{s+1}} + c_{\lambda_1}{}^j\,\partial_j\psi^i_{\lambda_2\dots\lambda_{s+1}} - \partial_j c_{\lambda_1}{}^i\,\psi^j_{\lambda_2\dots\lambda_{s+1}}\big)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{s+1}} \otimes \partial_i. \quad \square$$

We have the following further particular cases.

Example F.1.7 If $\psi \equiv Y \in \operatorname{pro}\sec(\boldsymbol{F}, T\boldsymbol{F})$, then its covariant differential turns out to be a vertical valued 1-form

$$d_c Y \in \sec(\boldsymbol{F},\ T^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

For each $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we obtain

$$(d_c Y)\,(X) = \big[c(X),\ Y\big] - c\big([X,\ \underline{Y}]\big),$$

where $\underline{Y}$ $\sec(\boldsymbol{B}, T\boldsymbol{B})$ is the projection of Y.

Moreover, we have the coordinate expression

$$d_c Y = \big(-\partial_\rho c_\lambda{}^i\, Y^\rho - c_\rho{}^i\,\partial_\lambda Y^\rho + \partial_\lambda Y^i + c_\lambda{}^j\,\partial_j Y^i - \partial_j c_\lambda{}^i\, Y^j\big)\, d^\lambda \otimes \partial_i.$$

In particular, if $\psi \equiv Y \in \sec(\boldsymbol{F}, V\boldsymbol{F})$, then we obtain the equality

$$(d_c Y)(X) = [c(X), Y] = L_{c(X)}\, Y = -L_Y\big(c(X)\big)$$

and the coordinate expression

$$(d_c Y)(X) = X^\lambda\,\big(\partial_\lambda Y^i + c_\lambda{}^j\,\partial_j Y^i - \partial_j c_\lambda{}^i\, Y^j\big)\,\partial_i. \quad \square$$

In the particular case of vertical vector field $Y \in \sec(\boldsymbol{F}, V\boldsymbol{F})$, we can recover the standard notion of covariant differential ∇, as a particular case of our general approach.

Note F.1.8 For each $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ and $Y \in \sec(\boldsymbol{F}, V\boldsymbol{F})$, the vertical vector field

$$(d_c Y)(X) = [c(X), Y] \in \sec(\boldsymbol{F}, V\boldsymbol{F})$$

turns out to be a map$(\boldsymbol{B}, \mathbb{R})$-linear with respect to X.

For this reason, we can adopt the standard notation

$$\nabla_X Y := (d_c Y)(X) = \big[c(X), Y\big].$$

Thus, for each

$$X,\ \acute{X} \in \sec(\boldsymbol{B}, T\boldsymbol{B}), \quad Y,\ \acute{Y} \in \sec(\boldsymbol{B}, V\boldsymbol{F}) \quad \text{and} \quad \underline{f} \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}), \quad f \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),$$

we have

$$\nabla_{X+\acute{X}} Y = \nabla_X Y + \nabla_{\acute{X}} Y, \qquad \nabla_{\underline{f} X} Y = \underline{f}\, \nabla_X Y,$$
$$\nabla_X (Y + \acute{Y}) = \nabla_X Y + \nabla_X \acute{Y}, \qquad \nabla_X (f\, Y) = f\, \nabla_X Y + df(X) \otimes Y. \quad \square$$

F.1.3 Curvature

We define the *curvature* of a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ as the covariant differential $-d_c c : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ of the connection itself (up to a normalising sign).

We stress that, in the present general context, the curvature lives on the total space $\boldsymbol{F}$. Later, we shall see that it is possible to project the curvature on the base space $\boldsymbol{B}$ if we add some suitable symmetry hypothesis, for instance, that the fibred manifold be a vector (or affine) bundle and the connection be linear (or affine). Correspondingly, in our general context, the coordinate expression of the curvature involves some partial derivatives, which replace algebraic contractions in the case of standard connections.

Let us consider a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Definition F.1.9 The *curvature* of c is defined to be the vertical valued 2-form

$$R[c] := -\, d_c c := -\, [c,\ c] \in \sec(\boldsymbol{F}, \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

For each base vector fields $X_1, X_2 \in \sec(\boldsymbol{B},\ T\boldsymbol{B})$, we obtain

$$R[c](X_1, X_2) = c\big([X_1, X_2]\big) - \big[c(X_1), c(X_2)\big].$$

Moreover, we have the coordinate expression

$$R[c] = -2\, (\partial_\lambda c_\mu{}^i + c_\lambda{}^j\, \partial_j c_\mu{}^i)\, d^\lambda \wedge d^\mu \otimes \partial_i.$$

In the particular case when $R[c] = 0$, we say that the connection c is *flat*. $\square$

Note F.1.10 If we consider a soldering form $\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ and the connection $\acute{c} = c + \sigma$, then we have the following transition rule

$$R[\acute{c}] = R[c] - 2\,[c, \sigma] - [\sigma, \sigma],$$

with coordinate expression

$$\begin{aligned} R[\acute{c}] = &- 2\,(\partial_\lambda c_\mu{}^i + c_\lambda{}^j\,\partial_j c_\mu{}^i)\,d^\lambda \wedge d^\mu \otimes \partial_i \\ &- 2\,(\sigma_\lambda{}^j\,c_\mu{}^i + c_\lambda{}^j\,\sigma_\mu{}^i)\,d^\lambda \wedge d^\mu \otimes \partial_i - 2\,\sigma_\lambda{}^j\,\partial_j \sigma_\mu{}^i\,d^\lambda \wedge d^\mu \otimes \partial_i. \quad \square \end{aligned}$$

Then, we have the following geometric interpretations of the curvature.

Lemma F.1.11 *[Frobenius] Let us consider a manifold $\boldsymbol{M}$. Then, a vector subbundle $U\boldsymbol{M} \subset T\boldsymbol{M}$ over $\boldsymbol{M}$ is said to be* integrable *if, for each $x \in \boldsymbol{M}$, there is locally a submanifold $\boldsymbol{N}$ passing through x, such that $T\boldsymbol{N} \subset U\boldsymbol{M}$.*

Indeed, the vector subbundle $U\boldsymbol{M} \subset T\boldsymbol{M}$ is integrable if and only if the following property holds (see, for instance, [146, 241])

$$X, Y \in \mathrm{sec}(\boldsymbol{M}, U\boldsymbol{M}) \qquad \Rightarrow \qquad [X, Y] \in \mathrm{sec}(\boldsymbol{M}, U\boldsymbol{M}). \quad \square$$

Proposition F.1.12 *We have the following geometric interpretations of the curvature.*

(1) The equality

$$R[c](X_1, X_2) = c\big([X_1, X_2]\big) - \big[c(X_1), c(X_2)\big]$$

says that the curvature "measures" the discrepancy of the horizontal prolongation

$$c : \mathrm{sec}(\boldsymbol{B}, T\boldsymbol{B}) \to \mathrm{sec}(\boldsymbol{F}, T\boldsymbol{F})$$

from being a morphism of Lie algebras.

(2) The horizontal vector subbundle

$$H_c\boldsymbol{F} := c(\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}) \subset T\boldsymbol{F}$$

is integrable if and only if

$$R[c] = 0.$$

(3) The following conditions are equivalent:

(a) for each $f \in \boldsymbol{F}_b$, there exists a local section $s \in \mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$ passing through f, such that $\nabla[c]s = 0$,

(b) $$R[c] = 0.$$

Proof. (1) Follows immediately from the expression of the curvature.

(2) In virtue of the above Frobenius Lemma, the horizontal vector subbundle is integrable if and only if, for each $X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, the vector field $\big[c(X_1), c(X_2)\big]$ is horizontal, i.e. if and only if there exists $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ such that

$$(*) \qquad \big[c(X_1), c(X_2)\big] = c(X).$$

The projectability properties of the Lie bracket $\big[c(X_1), c(X_2)\big]$ (see Note B.2.4) and of the connection c imply

$$[X_1, X_2] = X.$$

Then, equality (*) is equivalent to

$$\big[c(X_1), c(X_2)\big] = c\big([X_1, X_2]\big).$$

Therefore, the above equality and the expression of the curvature prove our claim.

(3) Preliminary, we make the following observation. If s is a local section such that $\nabla[c]s = 0$, then we have $Ts(T\boldsymbol{B}) \subset H_c\boldsymbol{F} \subset T\boldsymbol{F}$.
(a) $\Rightarrow$ (b). (a) means that the vector subbundle $H_c\boldsymbol{F} \subset T\boldsymbol{F}$ is integrable.
Then, the above item (2) implies (b).
(b) $\Rightarrow$ (a). If $R[c] = 0$, then, in virtue of the above item (2), the vector subbundle $H_c\boldsymbol{F} \subset T\boldsymbol{F}$ is integrable. This means that condition (a) holds. □

Note F.1.13 Let $p : \boldsymbol{F} \to \boldsymbol{B}$ be a bundle.

(1) Let us consider a (local) bundle splitting $\varsigma : \boldsymbol{F} \to \boldsymbol{B} \times \boldsymbol{S}$ and the induced tangent splitting $T\varsigma : T\boldsymbol{F} \to T\boldsymbol{B} \times T\boldsymbol{S}$. Then, we obtain (locally) the connection

$$c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F} : (f_b, X_b) \mapsto (T\varsigma)^{-1}(X_b, 0).$$

Clearly, in any fibred chart adapted to the bundle splitting the symbols of the connection c vanish, hence we have $R[c] = 0$.

(2) Let us consider a flat connection c. Then, we obtain (locally) the bundle splitting

$$\varsigma : \boldsymbol{F} \to \boldsymbol{B} \times \boldsymbol{S} : f_b \mapsto \big(b,\ [f_b]\big),$$

where $\boldsymbol{S}$ is the quotient space, with respect to the (local) equivalence relation in $\boldsymbol{F}$ induced by the sections with vanishing covariant differential, and $[f_b]$ is the equivalence class of the element $f_b \in \boldsymbol{F}$.

(3) The above points (1) and (2) yield locally a bijection between bundle splittings and flat connections.

Proof. (1) The connection c is flat, because, by construction, we have

$$c\big([X_1, X_2]\big) = \big[c(X_1), c(X_2)\big], \qquad \text{for each} \qquad X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B}).$$

(2) The sections with vanishing covariant differential yield an equivalence relation in $\boldsymbol{F}$ in virtue of items (2) and (3) of the above Proposition F.1.12. □

F.1.4 Identities for Curvature

The curvature of riemannian connections fulfills the well known Bianchi identities, which usually are proved by an ad hoc procedure (see, for instance, [51, 241, 242]).

However, the curvature of general connections fulfills such properties in the most general context of fibred manifolds, just as an immediate consequence of the Jacobi property of the FN-bracket.

Here, we start by proving a first set of these identities. Later, we shall exhibit further identities, in the context of the torsion of a connection, after having chosen an additional soldering form (see Sect. F.1.7).

Indeed, the standard Bianchi identities of riemannian geometry turn out to be a particular case of the present results (see Sect. F.4).

In particular, these identities show that the curvature is the only tangent valued form that can be derived from a connection by means of the FN-bracket (unless we consider other objects).

We stress that, in our general context, the coordinate expression of these identities involve some partial derivatives, which replace algebraic contractions in the case of standard connections.

Let us consider a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Theorem F.1.14 *We have the (generalised)* 2nd Bianchi identity

$$d_c\, R[c] := \big[c, R[c]\big] = 0,$$

with coordinate expression

$$(\partial_\lambda R_{\mu\nu}{}^i + c_\lambda{}^j\, \partial_j R_{\mu\nu}{}^i - \partial_j c_\lambda{}^i\, R_{\mu\nu}{}^j)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0.$$

Proof. The Jacobi and commutativity properties of the FN-bracket yield $\big[c, [c, c]\big] = -2\big[c, [c, c]\big]$, hence $\big[c, [c, c]\big] = 0$.

Therefore, we obtain $-d_c\, R[c] := -\big[c, R[c]\big] = \big[c, [c, c]\big] = 0$.

Moreover, in virtue of Definition F.1.6, we have the following coordinate expression

$$\big[c, R[c]\big] = \big(\partial_\lambda R_{\mu\nu}{}^i + c_\lambda{}^j\, \partial_j R_{\mu\nu}{}^i - \partial_j c_\lambda{}^i\, R_{\mu\nu}{}^j\big)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i. \quad □$$

Proposition F.1.15 *We have the identity*

$$\big[R[c],\, R[c]\big] = 0.$$

Proof. The commutativity of the FN-bracket yields $\big[R[c],\ R[c]\big] = -(-1)^4 \big[R[c],\ R[c]\big]$, hence $2\,\big[R[c],\ R[c]\big] = 0$. □

Proposition F.1.16 *If* $\psi \in \operatorname{pro\,sec}(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{F} \otimes T\boldsymbol{F})$, *then we have the following equality*

$$d_c^2 \psi = -\tfrac{1}{2}\,\big[R[c], \psi\big],$$

with coordinate expression

$$d_c^2 \psi = -\tfrac{1}{2}\,\big(-\psi^{\rho}_{\lambda_1 \ldots \lambda_s}\, \partial_\rho R_{\lambda_{s+1}\lambda_{s+2}}{}^{i} + R_{\lambda_1\lambda_2}{}^{i}\, \partial_j \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}} - \psi^{j}_{\lambda_1\ldots\lambda_s}\, \partial_j R_{\lambda_{s+1}\lambda_{r+2}}{}^{i} - 2\, R_{\lambda_1\rho}{}^{i}\, \partial_{\lambda_2} \psi^{\rho}_{\lambda_3\ldots\lambda_{s+2}} \big)\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{s+2}} \otimes \partial_i .$$

In particular, if $\psi \in \sec(\boldsymbol{F},\ \Lambda^r T^*\boldsymbol{F} \otimes V\boldsymbol{F})$, *then the above coordinate expression becomes*

$$d_c^2 \psi = -\tfrac{1}{2}\,\big(R_{\lambda_1\lambda_2}{}^{j}\, \partial_j \psi^{i}_{\lambda_{r+1}\ldots\lambda_{r+s}} - \psi^{j}_{\lambda_1\ldots\lambda_s}\, \partial_j R_{\lambda_{s+1}\lambda_{r+2}}{}^{i} \big)\, d^{\lambda_1} \wedge \ldots \wedge d^{\lambda_{s+2}} \otimes \partial_i .$$

Proof. The Jacobi property of the FN-bracket yields $\big[c,\ [c, \psi]\big] = \big[[c, c],\ \psi\big] - \big[c,\ [c, \psi]\big]$, hence $2\,\big[c,\ [c, \psi]\big] = \big[[c, c],\ \psi\big]$, which yields $\big[c,\ [c, \psi]\big] = -\frac{1}{2}\,\big[R[c], \psi\big]$. □

Remark F.1.17 The above Proposition F.1.16 shows that the 2nd covariant differential $d_c^2 \psi$ depends, at most, on the 1st jet of ψ and on the 2nd jet of c.

This result reflects the fact that, in the coordinate expression of $d_c^2\psi$, the 2nd partial derivatives of ψ are cancelled by antisymetrisation, in virtue of the Schwarz Theorem. □

F.1.5 Lie Derivatives of the Connection

Now, we collect some useful results concerning connections and Lie derivatives.

Let us consider a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Note F.1.18 We have the following Lie derivatives.

(1) For each $Y \in \operatorname{pro\,sec}(\boldsymbol{F}, T\boldsymbol{F})$, we have

$$L_Y c = -d_c Y = [Y, c].$$

(2) For each $Y \in \text{pro}\sec(\boldsymbol{F}, V\boldsymbol{F})$, we have

$$L_Y c = -\nabla[c]Y.$$

(3) For each $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we have

$$L_{c(X)}\, c = -\tfrac{1}{2}\, i_X R[c].$$

Proof. (1) In virtue of the Leibniz rule of Lie derivatives, the Lie bracket of vector fields $\big[Y, c(X)\big]$ is given by $\big[Y, c(X)\big] = L_Y \big(c(X)\big) = (L_Y\, c)(X) + c\,(\underline{L_Y\, X}) = (L_Y\, c)(X) + c\,\big([\underline{Y}, X]\big)$, where the underline denotes the projection on the base space $\boldsymbol{B}$. Hence, we obtain

$$(L_Y\, c)(X) = \big[Y, c(X)\big] - c\,\big([\underline{Y}, X]\big) = -\big[c(X), Y\big] + c\,\big([X, \underline{Y}]\big).$$

In virtue of the above Example F.1.7, we have $(d_c Y)\,(X) = \big[c(X),\ Y\big] - c\big([X, \underline{Y}]\big)$.

(2) In virtue of the above item 1), we have $L_Y c = -d_c Y$.
In the case of a vertical vector field Y, in virtue of Note F.1.8, we can write $d_c Y = \nabla[c]Y$.

(3) In virtue of Note F.1.18, Example F.1.7 and Definition F.1.9, we have the equalities

$$\begin{aligned}\big(L_{c(X)}\, c\big)(Z) &= -\Big(\big[c, c(X)\big]\Big)(Z) = -\big[c(Z), c(X)\big] + c\big([Z, X]\big) = \tfrac{1}{2}\, R[c](Z, X)_\wedge \\ &= -\tfrac{1}{2}\,\big(i_X R[c]\big)(Z). \quad \square\end{aligned}$$

Note F.1.19 If $X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then we have

$$L_{c(X_1)}\, c(X_2) = c\big([X_1, X_2]\big) - \tfrac{1}{2}\, R[c](X_1, X_2).$$

Proof. Definition F.1.9 yields $L_{c(X_1)}\, c(X_2) = \big[c(X_1),\ c(X_2)\big] = c\big([X_1, X_2]\big) - R[c](X_1, X_2)$. □

An infinitesimal symmetry of the connection c turns out to be also an infinitesimal symmetry of the curvature $R[c]$.

Note F.1.20 If $Y \in \text{pro}(\boldsymbol{F}, T\boldsymbol{F})$, then the following implication holds

$$L_Y\, c = 0 \qquad \Rightarrow \qquad L_Y\, R[c] = 0.$$

Proof. In virtue of Proposition F.1.16, we have $L_Y\, R[c] = \big[Y, R[c]\big] = \big[R[c], Y\big] = -2\, d_c^2 Y$.

In virtue of Note F.1.18, we have $L_Y\, c = -d_c Y$.

Hence, if $L_Y\, c = 0$, then, we have $L_Y\, R[c] = -2\, d_c^2 Y = -2\, d_c(d_c Y) = 0$. □

F.1.6 Torsion

We define the *torsion* of a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ as the covariant differential (up to a suitable normalising factor) $d_c\sigma : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ of an auxiliary vertical valued 1-form $\sigma : T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ (see, for instance, [311]).

In general, this soldering form depends on an arbitrary choice. However, in all cases of our interest, the actual structure of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ provides a natural soldering form, which is a suitable candidate to be chosen for the definition of torsion. Clearly, the geometric meaning of the torsion depends, case by case, on the geometric meaning of the soldering form σ.

We stress that, in the present general context, the torsion lives on the total space $\boldsymbol{F}$. Later, we shall see that it is possible to project the torsion on the base space $\boldsymbol{B}$ if we add some suitable symmetry hypothesis, for instance, that the fibred manifold be a vector (or affine) bundle and the connection be linear (or affine) (see Sects. F.2 and F.3).

We stress that our general definition of torsion is not based on a trivial algebraic operation, but involves an important differential operation. Correspondingly, in our general context, the coordinate expression of the torsion involves some partial derivatives, which replace algebraic contractions in the case of standard connections.

Let us consider a connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ and a soldering form

$$\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}.$$

Note F.1.21 The *torsion* of c (with respect to the *soldering form* σ) is defined to be the vertical valued 2-form

$$T_\sigma[c] := 2\, d_c\sigma := 2\,[c,\ \sigma] \in \sec(\boldsymbol{F}, \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

For each $X_1, X_2 \in \sec(\boldsymbol{B},\ T\boldsymbol{B})$, we obtain the equality

$$\begin{aligned} T_\sigma[c](X_1, X_2) &= \big[c(X_1),\ \sigma(X_2)\big] - \big[c(X_2),\ \sigma(X_1)\big] - \sigma\big([X_1,\ X_2]\big) \\ &= \nabla_{X_1}\sigma(X_2) - \nabla_{X_2}\sigma(X_1) - \sigma\big([X_1,\ X_2]\big). \end{aligned}$$

We have the coordinate expression

$$\begin{aligned} T_\sigma[c] &= 2\,(\partial_\lambda\sigma_\mu{}^i + c_\lambda{}^j\,\partial_j\sigma_\mu{}^i - \partial_j c_\lambda{}^i\,\sigma_\mu{}^j)\, d^\lambda \wedge d^\mu \otimes \partial_i \\ &= \big(\partial_\lambda\sigma_\mu{}^i + c_\lambda{}^j\,\partial_j\sigma_\mu{}^i - \partial_j c_\lambda{}^i\,\sigma_\mu{}^j - \partial_\mu\sigma_\lambda{}^i \\ &\quad - c_\mu{}^j\,\partial_j\sigma_\lambda{}^i + \partial_j c_i{}^\mu\,\sigma_\lambda{}^j)\big)\, d^\lambda \otimes d^\mu \otimes \partial_i. \quad \square \end{aligned}$$

Proof. In virtue of Definition F.1.6, for each $X_1, X_2 \in \sec(\boldsymbol{B},\ T\boldsymbol{B})$, we obtain the equality

$$
\begin{aligned}
d_c\sigma\,\big(X_1, X_2\big) &= \big[c(X_1),\ \sigma(X_2)\big] - \big[c(X_2),\ \sigma(X_1)\big] \\
&\quad - \sigma\big([X_1,\ X_2]\big) - \sigma\big([X_2,\ X_1]\big) \\
&\quad + \tfrac{1}{2}\,\sigma\big([X_1,\ X_2]\big) - \tfrac{1}{2}\,\sigma\big([X_2,\ X_1]\big) \\
&= \big[c(X_1),\ \sigma(X_2)\big] - \big[c(X_2),\ \sigma(X_1)\big] - \sigma\big([X_1,\ X_2]\big),
\end{aligned}
$$

hence

$$2\,d_c\sigma\,\big(X_1, X_2\big) = \big[c(X_1),\ \sigma(X_2)\big] - \big[c(X_2),\ \sigma(X_1)\big] - \sigma\big([X_1,\ X_2]\big).$$

Therefore, in virtue of the the equalities (see Note F.1.8)

$$\big[c(X_1), \sigma(X_2)\big] = \nabla_{X_1}\sigma(X_2) \qquad \text{and} \qquad \big[c(X_2), \sigma(X_1)\big] = \nabla_{X_2}\sigma(X_1),$$

we obtain

$$T_\sigma[c](X_1, X_2) = \big[c(X_1),\ \sigma(X_2)\big] - \big[c(X_2),\ \sigma(X_1)\big] - \sigma\big([X_1,\ X_2]\big).$$

Furthermore, again in virtue of Note F.1.8, by setting $\phi = c$ and $\psi = \sigma$, we obtain the coordinate expression

$$2\,d_c\sigma = 2\left(\partial_\lambda\sigma_\mu{}^i + c_\lambda{}^j\,\partial_j\sigma_\mu{}^i - \partial_j c_\lambda{}^i\,\sigma_\mu{}^j\right) d^\lambda \wedge d^\mu \otimes \partial_i. \quad \square$$

Remark F.1.22 The equality

$$T_\sigma[c](X_1, X_2) = \nabla_{X_1}\sigma(X_2) - \nabla_{X_2}\sigma(X_1) - \sigma\big([X_1,\ X_2]\big)$$

exhibited in the above Note F.1.21 is clearly, a generalisation of the standard definition of torsion in the case of linear connections of a vector bundle (see Sect. F.4). □

F.1.7 Identities for Torsion

The curvature and the torsion of riemannian connections fulfill the well known Bianchi identities, which usually are proved by an ad hoc procedure (see, for instance, [51, 241, 242]).

In Sect. F.1.4, we have proved a 1st set of identities involving the curvature $R[c]$ of a general connection c, just as an immediate consequence of the Jacobi property of the FN-bracket.

Now, after having introduced the torsion $T[c]$, we prove a 2nd set of identities involving the curvature $R[c]$ and the torsion $T[c]$, again as an immediate consequence of the Jacobi property of the FN-bracket.

We stress that the standard 1st Bianchi identity of a linear connection of a manifold is derived from the curvature $R[c]$ through an algebraic operation, while, in the present general context, the 1st Bianchi identity is derived from the curvature $R[c]$ through a 1st order differential operator (see Sect. F.4).

Theorem F.1.23 *We have the (generalised)* 1st Bianchi identity

$$d_c\, T_\sigma[c] = -\big[R[c], \sigma\big],$$

with coordinate expression

$$\begin{aligned}&\big(\partial_\lambda T_{\mu\nu}{}^i + c_\lambda{}^j\, \partial_j T_{\mu\nu}{}^i - \partial_j c_\lambda{}^i\, T_{\mu\nu}{}^j\big)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ &\quad = (R_{\lambda\mu}{}^j\, \partial_j \sigma_\nu{}^i - \sigma_\lambda{}^j\, \partial_j R_{\mu\nu}{}^i)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i\,.\end{aligned}$$

In particular, if the torsion $T[c]$ of the connection c vanishes, then the 1st Bianchi identity reduces to the equality

$$\big[R[c],\ \sigma\big] = 0,$$

with coordinate expression

$$(R_{\lambda\mu}{}^j\, \partial_j \sigma_\nu{}^i - \sigma_\lambda{}^j\, \partial_j R_{\mu\nu}{}^i)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0.$$

Proof. In virtue of Proposition F.1.16, we obtain

$$d_c\, T_\sigma[c] = 2\, d_c^2 \sigma = -\big[R[c], \sigma\big].$$

Moreover, the coordinate expressions follow from Definition F.1.6 and Corollary E.2.6. □

We have the further identity, by taking the FN-bracket of the torsion with itself.

Proposition F.1.24 *We have the identity*

$$\big[T_\sigma[c],\ T_\sigma[c]\big] = 0.$$

Proof. The commutativity property of the FN-bracket yields $\big[T_\sigma[c],\ T_\sigma[c]\big] = -(-1)^4 \big[T_\sigma[c],\ T_\sigma[c]\big]$, hence $2\,\big[T_\sigma[c],\ T_\sigma[c]\big] = 0$. □

Further, by considering the FN-bracket $N[\sigma] := [\sigma, \sigma]$, we obtain a further identity, which here will be "conventionally" called *3rd Bianchi identity*.

Thus, let us consider the FN-bracket

$$N[\sigma] := [\sigma, \sigma] \in \sec(\boldsymbol{F}, \Lambda^2 T^* \boldsymbol{B} \otimes V \boldsymbol{F}),$$

with coordinate expression

$$N[\sigma] = 2\,\sigma_\lambda{}^j\,\partial_j\sigma_\mu{}^i\,d^\lambda \wedge d^\mu \otimes \partial_i\,.$$

Proposition F.1.25 [3rd Bianchi identity]
We have the identity

$$d_c N[\sigma] = -\tfrac{1}{4}\,\big[\sigma, T_\sigma[c]\big],$$

with coordinate expressions

$$\big(\partial_\lambda N_{\mu\nu}{}^i + c_\lambda{}^j\,\partial_j N_{\mu\nu}{}^i - \partial_j c_\lambda{}^i\,N_{\mu\nu}{}^j\big)\,d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i$$
$$= -\tfrac{1}{4}\,(\sigma_\lambda{}^j\,\partial_j T_{\mu\nu}{}^i - T_{\lambda\mu}{}^j\,\partial_j\sigma_\nu{}^i)\,d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i\,.$$

In particular, iff $N[\sigma] := [\sigma, \sigma] = 0$, *then we have the identity*

$$\big[\sigma, T_\sigma[c]\big] = 0,$$

with coordinate expression

$$\sigma_\lambda{}^j\,\partial_j T_{\mu\nu}{}^i - T_{\lambda\mu}{}^j\,\partial_j\sigma_\nu{}^i = 0.$$

Proof. In virtue of the Jacobi property of the FN-bracket, we obtain the equalities (see Theorem E.2.3)

$$d_c N[\sigma] = \big[c, [\sigma, \sigma]\big] = \big[[\sigma, \sigma], c\big] - \big[\sigma, [c, \sigma]\big] = -\big[c, [\sigma, \sigma]\big] - \tfrac{1}{2}\,\big[\sigma, T_\sigma[c]\big]$$
$$= -d_c N[\sigma] - \tfrac{1}{2}\,\big[\sigma, T_\sigma[c]\big],$$

hence $d_c[\sigma, \sigma] = -\frac{1}{4}\,\big[\sigma, T_\sigma[c]\big]$.
Moreover, the coordinate expressions follows from Corollary E.2.6. □

F.2 Linear Connections of Vector Bundles

Next, we consider a vector bundle and analyse the particular case of linear connections. In this context we recover more standard notions and results.

In particular, the standard linear connections of a manifold are the linear connections of its tangent bundle.

For linear connections on vector bundles, we can achieve, in the usual way, the standard notion of dual connection and tensor product of connections.

Let us consider a vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical linear fibred charts by (x^λ, y^i) and the associated basis by (b_i).

We recall that the fibred manifold $Tp : T\boldsymbol{F} \to T\boldsymbol{B}$ turns out to be a vector bundle, as well (see Proposition B.3.2).

Moreover, we recall the natural linear fibred isomorphism $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F}$ over $\boldsymbol{B}$, by which we shall often identify the above spaces (see Proposition B.3.2).

Accordingly, $V\boldsymbol{F} \simeq \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{B}$ turns out to be naturally a vector bundle.

Definition F.2.1 A connection $c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F}$ is said to be *linear* if it is a linear fibred morphism over $\mathrm{id} : T\boldsymbol{B} \to T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\quad c \quad} & T\boldsymbol{F} \\ {\scriptstyle \mathrm{pro}_2}\downarrow & & \downarrow{\scriptstyle Tp} \\ T\boldsymbol{B} & \xrightarrow{\quad \mathrm{id} \quad} & T\boldsymbol{B} \end{array} .$$

The coordinate expression of a linear connection is

$$c = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i{}_j \, y^j \, \partial_i), \qquad \text{with } c_\lambda{}^i{}_j \in \mathrm{map}(\boldsymbol{B}, \mathrm{I\!R}). \quad \square$$

In the linear case, the covariant differentials of sections turn out to be vector valued forms, which fulfill classical algebraic properties.

Let us consider a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Note F.2.2 We can naturally regard the covariant differential $\nabla[c]s$ of a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ (see Definition F.1.1) as a vector valued 1-form

$$\nabla[c]s \in \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \boldsymbol{F}),$$

with coordinate expression

$$\nabla[c]s = (\partial_\lambda s^i - c_\lambda{}^i{}_j \, s^j) \, d^\lambda \otimes b_i .$$

Moreover, for each $s, s' \in \sec(\boldsymbol{B}, \boldsymbol{F})$ and $f \in \mathrm{map}(\boldsymbol{B}, \mathrm{I\!R})$, we have the equalities

(*)
$$\nabla[c](s + s') = \nabla[c]s + \nabla[c]s' \qquad \text{and} \qquad \nabla[c](f\,s) = f\,\nabla[c]s + df \otimes s .$$

Indeed, the above covariant differential $\nabla[c]s$ coincides with the covariant differential $\nabla[c]$ (see Note F.1.8) of the associated basic vertical vector field $\widetilde{s}$, i.e. we can write

$$\nabla[c]s = \nabla[c]\widetilde{s}.$$

Thus, each linear connection c yields a sheaf morphism

$$\nabla[c] : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \boldsymbol{F}),$$

which fulfills the above property $(*)$. Conversely, for each sheaf morphism

$$\nabla : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \boldsymbol{F}),$$

which fulfills the above property $(*)$, there is a unique linear connection c, such that

$$\nabla = \nabla[c]. \quad \square$$

Remark F.2.3 We stress that general connections on a generic fibred manifold cannot be characterised, through their covariant differential of sections, by means of algebraic properties as above. □

F.2.1 Covariant Differential of Vector Calued Forms

In the linear case, we obtain the Koszul covariant differential of vector valued forms [251] as a particular case of our covariant differential of vertical valued forms.

Moreover, the covariant differential of sections, which has been discussed in Note F.1.8, can be recovered as a particular case of covariant differential of tangent valued forms.

Furthermore, the standard exterior differential of forms of a manifold can be regarded as a particular case of the covariant differential of vector valued forms.

Let us consider a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Let us recall the natural identification between vector valued forms and "basic" vertical valued forms (see Note C.2.16).

Then, we compute the *exterior covariant differential of vector valued forms*, through the exterior covariant differential of tangent valued forms (see Sect. F.1.2).

Note F.2.4 If $\psi \in \sec(\boldsymbol{B}, \Lambda^s T^*\boldsymbol{B} \otimes \boldsymbol{F})$, then the form

$$d_c\tilde{\psi} \in \sec(\boldsymbol{F}, \Lambda^s T^*\boldsymbol{B} \otimes V\boldsymbol{F})$$

turns out to be basic, hence it can be regarded as a section

$$d_c\psi \in \sec(\boldsymbol{B}, \Lambda^s T^*\boldsymbol{B} \otimes \boldsymbol{F}).$$

Indeed, for each $X_1, \dots, X_{s+1} \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we obtain

$$\begin{aligned} d_c\psi\,(X_1, \dots, X_{s+1})_\wedge = & \sum_{1\le i\le s+1} (-1)^{i-1}\, \nabla[c]_{X_i}\, \psi(X_1, \dots, \widehat{X_i}, \dots, X_{s+1})_\wedge \\ & + \sum_{1\le i<j\le s+1} (-1)^{i+j}\, \psi\big([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{\sigma(s-1)}\big)_\wedge . \end{aligned}$$

Moreover, we have the coordinate expression

$$d_c\psi = \big(\partial_{\lambda_1}\psi^i_{\lambda_2\dots\lambda_{s+1}} - c_{\lambda_1}{}^i{}_j\, \psi^j_{\lambda_2\dots\lambda_{s+1}}\big)\, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{s+1}} \otimes \mathfrak{b}_i .$$

Proof. The Proposition follows immediately from Definition F.1.6. □

We have a distinguished example of vector valued forms.

Example F.2.5 Let us consider a linear vertical valued form of the type $\psi = \alpha \otimes \mathbb{I}$, with coordinate expression

$$\psi = \alpha_{\lambda_1 \dots \lambda_r}\, y^i\, \partial_i, \qquad \text{where } \alpha \in \sec(\boldsymbol{B}, \Lambda^r T^* \boldsymbol{B})$$

is a form of the base space and $\mathbb{I} : \boldsymbol{F} \to V\boldsymbol{F}$ is the Liouville vertical vector field of the vector bundle (see Note B.3.7).

Then, we have the equality

$$d_c \psi = d\alpha \otimes \mathbb{I}.$$

Proof. The Example can be deduced directly from the coordinate expression provided by Definition F.1.6. □

By regarding sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$, as basic vector valued 0-forms $\tilde{s} \in \sec(\boldsymbol{F}, V\boldsymbol{F})$, we can compare the covariant differential of sections with the exterior covariant differential of vector valued forms.

Example F.2.6 If $s \in \sec(\boldsymbol{B},\ \boldsymbol{F})$, then the vertical valued 1-form

$$d_c \tilde{s} \in \sec(\boldsymbol{F},\ T^*\boldsymbol{B} \otimes V\boldsymbol{F})$$

turns out to be basic, hence it can be regarded as a section

$$\nabla[c]\, s \ \equiv\ d_c s \in \sec(\boldsymbol{B},\ T^*\boldsymbol{B} \otimes \boldsymbol{F}).$$

Indeed, we obtain

$$\nabla[c]\, s \ \equiv\ d_c s.$$

Moreover, for each $X \in \sec(\boldsymbol{B},\ T\boldsymbol{B})$, we obtain

$$\nabla[c]_X\, s := \nabla[c]s\,(X) = \big[c(X),\ \tilde{s}\big]. \quad \square$$

F.2.2 Curvature

We analyse the curvature $R[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes \boldsymbol{F}$ of a linear connection c as a particular case of the general approach to curvature (see Sect. F.1.3).

Indeed, in this way, we recover classical formulas and interpretations. In particular, we recover the standard definition of curvature of linear connections and the standard expression of the 2nd Bianchi identity (see, for instance, [51, 241]).

Moreover, we discuss the relation between local bases and local flat linear connections.

Let us consider a linear connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Proposition F.2.7 *The curvature* $R[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ *can be naturally regarded as a linear fibred morphism over* $\boldsymbol{B}$

$$R[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F},$$

i.e., equivalently, as a vector valued 2-form

$$R[c] : \boldsymbol{B} \to \Lambda^2 T^*\boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*),$$

whose coordinate expression is

$$R[c] = -(\partial_\lambda c_\mu{}^i{}_j + c_\lambda{}^h{}_j\, c_\mu{}^i{}_h - \partial_\mu c_\lambda{}^i{}_j - c_\mu{}^h{}_j\, c_\lambda{}^i{}_h)\, y^j\, d^\lambda \wedge d^\mu \otimes b_i. \quad \square$$

We recover the standard definition of curvature for linear connections on a vector bundle (see, for instance, [51, 241]), as a particular case of our general approach.

Note F.2.8 For each $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ and $X, Y \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we obtain (see Note F.2.2)

$$\big(R[c](X, Y)\big)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s.$$

Proof. In virtue of Note F.1.8, we have $\nabla_X s = \big[c(X), \widetilde{s}\big]$.

Then, by recalling the Jacobi property of the bracket (see Theorem E.1.1) and Note F.2.4, we obtain

$$\begin{aligned}
\nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s &= \Big[c(X),\ \big[c(Y), \widetilde{s}\big]\Big] \\
&- \Big[c(Y),\ \big[c(X), \widetilde{s}\big]\Big] - \Big[c\big([X, Y]\big), \widetilde{s}\Big] \\
&= \Big[\big[c(X),\ c(Y)\big], \widetilde{s}\Big] + \Big[c(Y),\ \big[c(X), \widetilde{s}\big]\Big] - \Big[c(Y),\ \big[c(X), \widetilde{s}\big]\Big] \\
&- \Big[c\big([X, Y]\big), \widetilde{s}\Big] \\
&= -\Big[\widetilde{s},\ \big[c(X), c(Y)\big]\Big] + \Big[\widetilde{s}, c\big([X, Y]\big)\Big] = \big[\widetilde{s},\ R(X, Y)\big] = R(X, Y)\,(s). \quad \square
\end{aligned}$$

Remark F.2.9 In the linear case, the coordinate expression of the curvature coincides essentially with the standard one. The apparent difference in the sign is just due to our non standard convention on the sign of the symbols of the connection, for the reasons explained before (see Remark F.1.3). $\square$

Next, we show how the 2nd Bianchi identity looks in the case of a linear connection (see Theorem F.1.23). Later, we shall show an analogous translation for the 1st Bianchi identity (see Note F.2.18).

Proposition F.2.10 *The* 2nd Bianchi identity *can be expressed, in terms of the exterior covariant differential of the vector valued 2-form* $R[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)$, *with respect to the linear connection* c, *by means of the equality*

$$d_c R[c] = 0,$$

with coordinate expression

$$(\partial_\lambda R_{\mu\nu}{}^i{}_h + R_{\nu\lambda}{}^i{}_j \, c_\mu{}^j{}_h - R_{\mu\nu}{}^j{}_h \, c_\lambda{}^i{}_j) \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \otimes d^h = 0.$$

Proof. The Proposition follows easily from Theorem F.1.14, Note F.2.4 and Proposition F.2.7. □

In the linear case, the expressions of the 2nd covariant differential of linear and of basic vector valued forms through the curvature of the connection turn out to be purely algebraic operations (see Notes F.1.16 and F.2.4).

Preliminary, we introduce the following algebraic operation.

Lemma F.2.11 *For each decomposable linear vector valued forms*

$$\phi := \alpha \otimes \Sigma \in \sec\big(\boldsymbol{B},\ \Lambda^r T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big) \quad \textit{and} \quad \psi := \beta \otimes \Xi \in \sec\big(\boldsymbol{B},\ \Lambda^s T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big),$$

we define the decomposable linear vector valued form

$$\phi \mathbin{\bar\wedge} \psi := \alpha \wedge \beta \otimes C^2_1(\Sigma \otimes \Xi) \in \sec\big(\boldsymbol{B},\ \Lambda^{r+2} T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big),$$

with coordinate expression

$$\phi \mathbin{\bar\wedge} \psi = \alpha_{\lambda_1 \dots \lambda_r} \, \beta_{\lambda_{r+1} \dots \lambda_{r+s}} \, \Sigma^i_j \, \Xi^j_h \, d^{\lambda_1} \wedge \dots \wedge d^{\lambda_{r+s}} \otimes b_i \otimes y^h.$$

The above product $\bar\wedge$ *can be easily extended by linearity to all linear vector valued forms*

$$\phi \in \sec\big(\boldsymbol{B},\ \Lambda^r T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big) \qquad \textit{and} \qquad \psi \in \sec\big(\boldsymbol{B},\ \Lambda^s T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big). \quad \square$$

Then, we have the following result.

Note F.2.12 For each linear vector valued form

$$\psi \in \sec\big(\boldsymbol{B},\ \Lambda^s T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big),$$

the 2nd covariant differential turns out to be a linear vector valued form

$$d_c^2 \, \psi = -\tfrac{1}{2} \big[R[c],\ \psi\big] \in \sec\big(\boldsymbol{F},\ \Lambda^{r+2} T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)\big),$$

given by the purely algebraic equality

$$d_c^2\,\psi = -\tfrac{1}{2}\,\big[R[c],\ \psi\big] = -\tfrac{1}{2}\,\big(R[c]\,\bar{\wedge}\,\psi - \psi\,\bar{\wedge}\,R[c]\big),$$

with coordinate expression

$$d_c^2\psi = -\tfrac{1}{2}\,\big(R_{\lambda_1\lambda_2}{}^i{}_j\,\psi_{\lambda_3\ldots\lambda_{s+2}}{}^j{}_h - \psi_{\lambda_1\ldots\lambda_s}{}^i{}_j\,R_{\lambda_{s+1}\lambda_{s+2}}{}^j{}_h\big)\,d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_{s+2}}\otimes\partial_i\otimes y^h.$$

Proof. It follows immediately from Proposition F.1.16 and Lemma F.2.11. □

Then, we discuss the case of basic vector valued forms.
Preliminary, we introduce the following algebraic operation.

Lemma F.2.13 *For each decomposable linear and basic vector valued forms*

$$\phi := \alpha\otimes\Sigma \in \sec\big(\boldsymbol{B},\ \Lambda^r T^*\boldsymbol{B}\otimes(\boldsymbol{F}\otimes\boldsymbol{F}^*)\big) \quad \textit{and} \quad \psi := \beta\otimes\Xi \in \sec\big(\boldsymbol{B},\ \Lambda^s T^*\boldsymbol{B}\otimes\boldsymbol{F}\big),$$

we define the decomposable basic vector valued form

$$\phi\,\bar{\wedge}\,\psi := \alpha\wedge\beta\otimes C_1^2(\Sigma\otimes\Xi) \in \sec(\boldsymbol{B},\ \Lambda^{r+2}T^*\boldsymbol{B}\otimes\boldsymbol{F}),$$

with coordinate expression

$$\phi\,\bar{\wedge}\,\psi = \alpha_{\lambda_1\ldots\lambda_r}\,\beta_{\lambda_{r+1}\ldots\lambda_{r+s}}\,\Sigma^i_j\,\Xi^j\,d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_{r+s}}\otimes b_i.$$

The above product $\bar{\wedge}$ *can be easily extended by linearity to all linear and basic vector valued forms*

$$\phi\in\sec\big(\boldsymbol{B},\ \Lambda^r T^*\boldsymbol{B}\otimes(\boldsymbol{F}\otimes\boldsymbol{F}^*)\big) \qquad \textit{and} \qquad \psi\in\sec\big(\boldsymbol{B},\ \Lambda^s T^*\boldsymbol{B}\otimes\boldsymbol{F}\big).$$ □

Then, we have the following result.

Note F.2.14 For each basic vector valued form

$$\psi\in\sec(\boldsymbol{B},\ \Lambda^r T^*\boldsymbol{B}\otimes\boldsymbol{F}),$$

the 2nd covariant differential turns out to be the basic vector valued form

$$d_c^2\,\psi = -\tfrac{1}{2}\,\big[R[c],\ \psi\big] \in \sec(\boldsymbol{F},\ \Lambda^{r+2}T^*\boldsymbol{B}\otimes\boldsymbol{F}),$$

given by the purely algebraic equality

$$d_c^2\,\psi = -\tfrac{1}{2}\,\big[R[c],\ \psi\big] = -\tfrac{1}{2}\,R[c]\,\bar{\wedge}\,\psi,$$

with coordinate expression

$$d_c^2\psi = -\tfrac{1}{2}\,\big(R_{\lambda_1\lambda_2}{}^i{}_j\,\psi_{\lambda_3\ldots\lambda_{s+2}}{}^j\big)\,d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_{s+2}}\otimes\partial_i.$$

Proof. It follows immediately from Proposition F.1.16 and Lemma F.2.13. □

Next, let us consider the Liouville vector field $\mathbb{I} : \boldsymbol{F} \to V\boldsymbol{F}$, with coordinate expression $\mathbb{I} = y^i \, \partial_i$.

Note F.2.15 We have the following distinguished Lie derivatives:

(1) If $\phi \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$, then we have $L_{\phi \mathbb{I}} \, c = -d\phi \otimes \mathbb{I}$.
(2) If $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then we have $[c(X), \, \mathbb{I}] = 0$.
(3) If $\phi \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$ and $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, then we have $[c(X), \, \phi\, \mathbb{I}] = (X.\phi)\, \mathbb{I}$.
(4) If $\phi, \acute{\phi} \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$, then we have $[\phi\, \mathbb{I}, \, \acute{\phi}\, \mathbb{I}] = 0$. □

F.2.3 Torsion

We analyse the torsion $T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F}$ of a linear connection c, with respect to a linear soldering 1-form σ and with respect to a basic soldering form σ, as a particular case of the general approach to torsion (see Sect. F.1.6).

Indeed, in this way, we recover classical formulas and interpretations. In particular, we recover the standard definition of torsion of linear connections and the standard expression of the 1st Bianchi identity (see, for instance, [51, 241]).

We start by considering a generic *soldering 1-form*

$$\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F},$$

with coordinate expression

$$\sigma = \sigma_\lambda{}^i \, d^\lambda \otimes \partial_i, \quad \text{with } \sigma_\lambda{}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}).$$

Proposition F.2.16 *The torsion (see Note F.1.21)*

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$$

is a vertical valued form, which can be naturally regarded as a fibred morphism over $\boldsymbol{B}$

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F},$$

with coordinate expression

$$T_\sigma[c] = \big(\partial_\lambda \sigma_\mu{}^i + c_\lambda{}^j{}_h \, y^h \, \partial_j \sigma_\mu{}^i - c_\lambda{}^i{}_j \, \sigma_\mu{}^j - \partial_\mu \sigma_\lambda{}^i - c_\mu{}^j{}_h \, y^h \, \partial_j \sigma_\lambda{}^i + c_\mu{}^i{}_j \, \sigma_\lambda{}^j\big) \, d^\lambda \wedge d^\mu \otimes \partial_i .$$

Indeed, according to the general theory of torsion, for each $X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, *we have the equality*

$$T_\sigma[c](X_1, X_2) = \nabla_{X_1}\sigma(X_2) - \nabla_{X_2}\sigma(X_1) - \sigma\big([X_1,\ X_2]\big).$$

Proof. The coordinate expression follows immediately from Note F.1.21. □

Next, we discuss the case of a linear soldering form σ. Thus, let us consider a linear soldering 1-form

$$\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F},$$

with coordinate expression

$$\sigma = \sigma_\lambda{}^i{}_j\, y^j\, d^\lambda \otimes \partial_i, \qquad \text{with} \qquad \sigma_\lambda{}^i{}_j \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}).$$

Note F.2.17 The torsion

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$$

turns out to be a linear vertical valued form, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \Lambda^2 T^*\boldsymbol{B} & \xrightarrow{\quad T[c] \quad} & V\boldsymbol{F} \\ \big\downarrow & & \big\downarrow \\ \boldsymbol{B} & \xrightarrow{\quad \mathrm{id} \quad} & \boldsymbol{B} \end{array} \quad .$$

Indeed, the torsion can be naturally regarded as a linear fibred morphism over $\boldsymbol{B}$

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F},$$

i.e., equivalently, as a vector valued 2-form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^*\boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*),$$

whose coordinate expression is

$$\begin{aligned} T_\sigma[c] = (&\partial_\lambda \sigma_\mu{}^i{}_j + c_\lambda{}^h{}_j\, \sigma_\mu{}^i{}_h - c_\lambda{}^i{}_h\, \sigma_\mu{}^h{}_j - \partial_\mu \sigma_\lambda{}^i{}_j \\ &- c_\mu{}^h{}_j\, \sigma_\lambda{}^i{}_h + c_\mu{}^i{}_h\, \sigma_\lambda{}^h{}_j)\, y^h\, d^\lambda \wedge d^\mu \otimes b_i. \end{aligned}$$

Proof. The coordinate expression follows easily from Proposition F.2.16. □

Next, we show how the 1st Bianchi identity looks like in the case of a linear connection c and a linear soldering form σ (see Theorem F.1.14).

Note F.2.18 The *1st Bianchi identity* can be expressed, in terms of the exterior covariant differential of the vector valued 2-form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^*\boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)$$

with respect to the linear connection c, by means of the equality (see Lemma F.2.11)

$$d_c\, T_\sigma[c] = -\big[R[c], \sigma\big] = \big(R[c] \,\bar{\wedge}\, \sigma - \sigma \,\bar{\wedge}\, R[c]\big),$$

with coordinate expression

$$\begin{aligned}&\big(\partial_\lambda T_{\mu\nu}{}^i{}_h + c_\lambda{}^j{}_h\, T_{\mu\nu}{}^i{}_j - c_\lambda{}^i{}_j\, T_{\mu\nu}{}^j{}_h\big) y^h\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ &\quad = -(R_{\lambda\mu}{}^j{}_h\, \sigma_\nu{}^i{}_j - \sigma_\lambda{}^j{}_h\, R_{\mu\nu}{}^i{}_j)\, y^h\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i .\end{aligned}$$

In particular, if the torsion $T_\sigma[c]$ vanishes, then the 1st Bianchi identity reduces to the equality

$$\big[R[c],\ \sigma\big] = \big(R[c] \,\bar{\wedge}\, \sigma - \sigma \,\bar{\wedge}\, R[c]\big) = 0,$$

with coordinate expression

$$\big[R[c],\ \sigma\big] = (R_{\lambda\mu}{}^j{}_h\, \sigma_\nu{}^i{}_j - \sigma_\lambda{}^j{}_h\, R_{\mu\nu}{}^i{}_j)\, y^h\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0.$$

Proof. The Proposition follows easily from Theorem F.1.23 and Lemma F.2.11. □

Further, we show how the 3rd Bianchi identity looks like in the case of a linear connection c and a linear soldering form σ (see Proposition F.1.25).

Lemma F.2.19 *The FN-bracket $N[\sigma] := [\sigma, \sigma]$ turns out to be a linear vector valued form*

$$N[\sigma] = -2\, \sigma \,\bar{\wedge}\, \sigma \, : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*),$$

with coordinate expression

$$N[\sigma] = -2\, \sigma_\lambda{}^i{}_j\, \sigma_\mu{}^j{}_h\, y^h\, d^\lambda \wedge d^\mu \otimes \partial_i . \quad \square$$

Note F.2.20 The 3rd Bianchi identity

$$d_c N[\sigma] = -\tfrac{1}{4}\, \big[\sigma, T_\sigma[c]\big]$$

has coordinate expressions

$$\begin{aligned}&\big(\partial_\lambda N_{\mu\nu}{}^i{}_h + c_\lambda{}^j{}_h\, N_{\mu\nu}{}^i{}_j - c_\lambda{}^i{}_j\, N_{\mu\nu}{}^j{}_h\big)\ y^h\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ &\quad = -\tfrac{1}{4}\, (\sigma_\lambda{}^j{}_h\, T_{\mu\nu}{}^i{}_j - T_{\lambda\mu}{}^j{}_h\, \sigma_\nu{}^i{}_j)\, y^h\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i . \quad \square\end{aligned}$$

Eventually, we discuss the case of a basic soldering form σ. Indeed, a basic soldering form plays a role in the case of linear connections of a manifold (see Sect. F.4). Thus, let us consider a basic soldering 1-form (see Note C.2.16)

$$\sigma : \boldsymbol{F} \to T^* \boldsymbol{B} \otimes V \boldsymbol{F},$$

with coordinate expression

$$\sigma = \sigma_\lambda{}^i \, d^\lambda \otimes \partial_i, \quad \text{with } \sigma_\lambda{}^i \in \text{map}(\boldsymbol{B}, \mathbb{R}).$$

Thus, by hypothesis, we have $\partial_j \sigma_\lambda{}^i = 0$.

Note F.2.21 The torsion

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes V \boldsymbol{F}$$

turns out to be a basic vertical valued form which can be naturally regarded as a vector valued form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \boldsymbol{F},$$

whose coordinate expression is

$$T_\sigma[c] = (\partial_\lambda \sigma_\mu{}^i - c_\lambda{}^i{}_h \, \sigma_\mu{}^h - \partial_\mu \sigma_\lambda{}^i) \, d^\lambda \wedge d^\mu \otimes b_i \, .$$

Proof. The proof follows easily from Proposition F.2.16. □

Further, we show how the 1st Bianchi identity looks like in the case of a linear connection c and a basic soldering form σ (see Theorem F.1.23).

Note F.2.22 The *1st Bianchi identity* can be expressed, in terms of the exterior covariant differential of the vector valued 2-form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \boldsymbol{F}$$

by means of the equality (see Lemma F.2.13)

$$d_c \, T_\sigma[c] = -\big[R[c], \sigma\big] = R[c] \, \bar{\wedge} \, \sigma,$$

with coordinate expression

$$\begin{aligned} d_c T_\sigma[c] &= (\partial_\lambda T_{\mu\nu}{}^i - c_\lambda{}^i{}_j \, T_{\mu\nu}{}^j) \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ &= -\sigma_\lambda{}^j \, R_{\mu\nu}{}^i{}_j \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \, . \end{aligned}$$

Proof. The proof follows easily from Theorem F.1.14. □

Eventually, we show how the 3rd Bianchi identity looks like in the case of a linear connection c and a basic soldering form σ (see Proposition F.1.25).

Lemma F.2.23 *The FN-bracket* $N[\sigma] := [\sigma, \sigma]$ *vanishes.* □

Note F.2.24 The 3rd Bianchi identity becomes

$$\tfrac{1}{4}\big[\sigma, T_\sigma[c]\big] = 0.$$

Indeed, the FN-bracket of two vertical valued basic forms vanishes identically. □

F.3 Affine Connections of Affine Bundles

Further, we consider an affine bundle and analyse the particular case of affine connections. In the present book, we are involved with the affine *phase connection* Γ of the affine phase bundle $t^1_0 : J_1\boldsymbol{E} \to \boldsymbol{E}$ (see Definition 9.1.1).

Let us consider an affine bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ associated with the vector bundle $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$.

We denote the typical affine fibred charts and linear fibred charts of the affine bundle and of the vector bundle, respectively, by (x^λ, y^i) and $(x^\lambda, \bar{y}^i)$.

Definition F.3.1 A connection $c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F}$ is said to be *affine* if it is an affine fibred morphism over $\text{id} : T\boldsymbol{B} \to T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\;\;c\;\;} & T\boldsymbol{F} \\ {\scriptstyle \text{pro}_2}\big\downarrow & & \big\downarrow{\scriptstyle Tp} \\ T\boldsymbol{B} & \xrightarrow{\;\;\text{id}\;\;} & T\boldsymbol{B} \end{array} .$$

The coordinate expression of an affine connection is

$$c = d^\lambda \otimes \big(\partial_\lambda + (c_\lambda{}^i{}_j\, y^j + c_\lambda{}^i{}_o)\, \partial_j\big), \quad \text{with } c_\lambda{}^i{}_j,\; c_\lambda{}^i{}_o \in \text{map}(\boldsymbol{B}, \mathbb{R}). \quad \square$$

We can take the fibred derivative of affine connections in the following way.

Note F.3.2 By taking into account the two affine bundles

$$\text{pro}_2 : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{B} \qquad \text{and} \qquad Tp : T\boldsymbol{F} \to T\boldsymbol{B},$$

we can take the fibred derivative over $\text{id} : T\boldsymbol{B} \to T\boldsymbol{B}$ of an affine connection $c : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\boldsymbol{F}$.

Indeed, the fibre derivative of c turns out to be the linear connection

$$\bar{c} : \bar{\boldsymbol{F}} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T\bar{\boldsymbol{F}},$$

with coordinate expression

$$\bar{c} = d^\lambda \otimes (\partial_\lambda + c_\lambda{}^i{}_j \, \bar{y}^j \, \partial_j), \quad \text{with } c_\lambda{}^i{}_j \in \text{map}(\boldsymbol{B}, \mathbb{R}). \quad \square$$

Now, let us consider an affine connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

In the affine case, the covariant differentials of sections turn out to be a vector valued forms, which fulfill an algebraic property.

Note F.3.3 We can regard the covariant differential of a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ as a vector valued 1-form

$$\nabla[c]s \in \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \bar{\boldsymbol{F}}),$$

with coordinate expression

$$\nabla[c]s = \big(\partial_\lambda s^i - (c_\lambda{}^i{}_j \, s^j + c_\lambda{}^i{}_o)\big) \, b_i .$$

Moreover, for each $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ and $\bar{s} \in \sec(\boldsymbol{B}, \bar{\boldsymbol{F}})$, we have the equality

$$(*) \qquad \nabla[c](s + \bar{s}) = \nabla[c]s + \nabla[\bar{c}]\bar{s}.$$

Indeed, each affine connection c yields a sheaf morphism

$$\nabla[c] : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \boldsymbol{F}),$$

which fulfills the above property $(*)$.

Conversely, for each linear connection $\bar{c} : \bar{\boldsymbol{F}} \to T^*\boldsymbol{B} \otimes T\bar{\boldsymbol{F}}$ and for each sheaf morphism

$$\nabla : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, T^*\boldsymbol{B} \otimes \boldsymbol{F}),$$

which fulfills the above property $(*)$, there is a unique affine connection c, such that

$$\nabla = \nabla[c] \qquad \text{and} \qquad Dc = \bar{c}. \quad \square$$

F.3.1 Curvature

We analyse the curvature $R[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \bar{\boldsymbol{F}}$ of an affine connection c as a particular case of the general approach to curvature (see Sect. F.1.3).

Let us consider an affine connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

Proposition F.3.4 *The curvature*

$$R[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$$

turns out to be an affine vertical valued form, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \Lambda^2 T^* \boldsymbol{B} & \xrightarrow{R[c]} & V\boldsymbol{F} \\ \downarrow & & \downarrow \\ \boldsymbol{B} & \xrightarrow{\text{id}} & \boldsymbol{B} \end{array} .$$

Indeed, the curvature $R[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes V\boldsymbol{F}$ *can be naturally regarded as an affine fibred morphism over* $\boldsymbol{B}$

$$R[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes \bar{\boldsymbol{F}},$$

i.e., equivalently, as a vector valued 2-form

$$R[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \operatorname{aff}(\boldsymbol{F}, \bar{\boldsymbol{F}}),$$

whose coordinate expression is

$$R[c] = (R_{\lambda\mu}{}^{i}{}_{j}\, y^j + R_{\lambda\mu}{}^{i}{}_{o})\, d^\lambda \wedge d^\mu \otimes b_i,$$

where

$$\begin{aligned} R_{\lambda\mu}{}^{i}{}_{j} &= -(\partial_\lambda c_\mu{}^{i}{}_{j} + c_\lambda{}^{h}{}_{j}\, c_\mu{}^{i}{}_{h} - \partial_\mu c_\lambda{}^{i}{}_{j} - c_\mu{}^{h}{}_{j}\, c_\lambda{}^{i}{}_{h}), \\ R_{\lambda\mu}{}^{i}{}_{o} &= -(\partial_\lambda c_\mu{}^{i}{}_{o} + c_\lambda{}^{h}{}_{o}\, c_\mu{}^{i}{}_{h} - \partial_\mu c_\lambda{}^{i}{}_{o} - c_\mu{}^{h}{}_{o}\, c_\lambda{}^{i}{}_{h}). \end{aligned}$$

Proof. It follows immediately from Definition F.1.9. □

We recover an "almost standard" definition of curvature for affine connections on an affine bundle, as a particular case of our general approach.

Proposition F.3.5 *For each* $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ *and* $X, Y \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, *we obtain (see Notes F.3.3 and F.2.2)*

$$\big(R[c](X, Y)\big)(s) = \overline{\nabla}_X \nabla_Y s - \overline{\nabla}_Y \nabla_X s - \nabla_{[X,Y]} s,$$

where $\overline{\nabla}$ *is the covariant differential with respect to the linear connection* $\bar{c}$ *(see Note F.3.2)*

Remark F.3.6 We stress that, according to the above equality

$$\big(R[c](X, Y)\big)(s) = \overline{\nabla}_X \nabla_Y s - \overline{\nabla}_Y \nabla_X s - \nabla_{[X,Y]} s,$$

the fibred morphism

$$R[c](X, Y) \in \sec(\boldsymbol{F}, \bar{\boldsymbol{F}})$$

is affine. Indeed, it is not linear as in the case of linear connections on linear bundles. Actually, in the present affine case, linearity with respect to s would not make sense. □

F.3.2 Torsion

We analyse the torsion $T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F}$ of an affine connection c, with respect to an affine soldering 1-form σ and with respect to a basic soldering form σ, as a particular case of the general approach to torsion (see Sect. F.1.6).

Let us consider an affine connection $c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$.

We start by considering a generic soldering 1-form $\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$, with coordinate expression $\sigma = \sigma_\lambda{}^i \, d^\lambda \otimes \partial_i$, with $\sigma_\lambda{}^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R})$.

Note F.3.7 The torsion

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$$

is a vertical valued form, which can be naturally regarded as a fibred morphism over $\boldsymbol{B}$

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes \bar{\boldsymbol{F}},$$

with coordinate expression

$$\begin{aligned} T_\sigma[c] = \big(&\partial_\lambda \sigma_\mu{}^i + (c_\lambda{}^j{}_h \, y^h + c_\lambda{}^j{}_o) \, \partial_j \sigma_\mu{}^i \; - c_\lambda{}^i{}_j \, \sigma_\mu{}^j - \partial_\mu \sigma_\lambda{}^i \\ &- (c_\mu{}^j{}_h \, y^h + c_\mu{}^j{}_o) \, \partial_j \sigma_\lambda{}^i + c_\mu{}^i{}_j \, \sigma_\lambda{}^j\big) \, d^\lambda \wedge d^\mu \otimes \partial_i , \end{aligned}$$

Indeed, according to the general theory of torsion, for each $X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, we have the equality

$$T_\sigma[c](X_1, X_2) = \nabla_{X_1} \sigma(X_2) - \nabla_{X_2} \sigma(X_1) - \sigma\big([X_1, \, X_2]\big).$$

Proof. The coordinate expression follows immediately from Note F.1.21. □

Then, we consider an affine soldering 1-form $\sigma : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes V\boldsymbol{F}$, with coordinate expression $\sigma = (\sigma_\lambda{}^i{}_j \, y^j + \sigma_\lambda{}^i{}_o) \, d^\lambda \otimes \partial_i$, where $\sigma_\lambda{}^i{}_j, \; \sigma_\lambda{}^i{}_o \in \operatorname{map}(\boldsymbol{B}, \mathbb{R})$.

Note F.3.8 The torsion $T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ turns out to be a vertical valued form, according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \Lambda^2 T^*\boldsymbol{B} & \xrightarrow{\;T[c]\;} & V\boldsymbol{F} \\ \big\downarrow & & \big\downarrow \\ \boldsymbol{B} & \xrightarrow{\;\mathrm{id}\;} & \boldsymbol{B} \end{array} .$$

Indeed, the torsion can be naturally regarded as an affine fibred morphism over $\boldsymbol{B}$

$$T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes \boldsymbol{F},$$

i.e., equivalently, as a vector valued 2-form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \mathrm{Aff}(\boldsymbol{F}, \bar{\boldsymbol{F}}),$$

whose coordinate expression is

$$\begin{aligned} T_\sigma[c] = \big(\partial_\lambda \sigma_\mu{}^i{}_h + c_\lambda{}^j{}_h\, \sigma_\mu{}^i{}_j - \partial_\mu \sigma_\lambda{}^i{}_h - c_\mu{}^j{}_h\, \sigma_\lambda{}^i{}_j\big)\, y^h\, d^\lambda \wedge d^\mu \otimes \partial_i \\ + \big(\partial_\lambda \sigma_\mu{}^i{}_o + c_\lambda{}^j{}_o\, \sigma_\mu{}^i{}_j - c_\mu{}^i{}_j\, \sigma_\lambda{}^j{}_o - \partial_\mu \sigma_\lambda{}^i{}_o \\ - c_\mu{}^j{}_o\, \sigma_\lambda{}^i{}_j + c_\lambda{}^i{}_j\, \sigma_\mu{}^j{}_o\big)\, d^\lambda \wedge d^\mu \otimes \partial_i . \end{aligned}$$

Proof. The coordinate expression follows immediately from Note F.3.7. □

Next, we show how the 1st Bianchi identity looks like in the case of an affine connection c and an affine soldering form σ (see Theorem F.1.23).

Note F.3.9 The *1st Bianchi identity* can be expressed, in terms of the exterior covariant differential of the vector valued 2-form $T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes (\boldsymbol{F} \otimes \boldsymbol{F}^*)$ with respect to the affine connection c by means of the equality (see Lemma F.2.11)

$$d_c\, T_\sigma[c] = -\big[R[c], \sigma\big],$$

with coordinate expression

$$\begin{aligned} \big(\partial_\lambda T_{\mu\nu}{}^i{}_h\, y^h + \partial_\lambda T_{\mu\nu}{}^i{}_o + (c_\lambda{}^j{}_h\, y^h + c_\lambda{}^j{}_o)\, T_{\mu\nu}{}^i{}_j \\ - c_\lambda{}^i{}_j\, (T_{\mu\nu}{}^j{}_h\, y^h + T_{\mu\nu}{}^j{}_o)\big) d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ = -\big(R_{\lambda\mu}{}^j{}_h\, y^h + R_{\lambda\mu}{}^j{}_o)\, \sigma_\nu{}^i{}_j \\ - (\sigma_\lambda{}^j{}_h\, y^h + \sigma_\lambda{}^j{}_o)\, R_{\mu\nu}{}^i{}_j\big)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i . \end{aligned}$$

Proof. The Proposition follows easily from Theorem F.1.23. □

Corollary F.3.10 *If the torsion $T_\sigma[c]$ of the connection c vanishes, then the 1st Bianchi identity reduces to the equality*

$$\big[R[c],\, \sigma\big] = 0,$$

with coordinate expression

$$\big[R[c],\, \sigma\big] = \big(R_{\lambda\mu}{}^j{}_h\, y^h + R_{\lambda\mu}{}^j{}_o)\, \sigma_\nu{}^i{}_j - (\sigma_\lambda{}^j{}_h\, y^h + \sigma_\lambda{}^j{}_o)\, R_{\mu\nu}{}^i{}_j\big)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0. \quad \square$$

Further, we show how the 3rd Bianchi identity looks like in the case of an affine connection c and a linear soldering form σ (see Proposition F.1.25).

Lemma F.3.11 *The FN-bracket* $N[\sigma] := [\sigma, \sigma]$ *turns out to be an affine vector valued form*

$$N[\sigma] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \mathrm{Aff}(\boldsymbol{F}, \bar{\boldsymbol{F}}),$$

with coordinate expression

$$N[\sigma] = -2\, \sigma_\lambda{}^i{}_j \, (\sigma_\mu{}^j{}_h \, y^h + \sigma_\mu{}^j{}_o) \, d^\lambda \wedge d^\mu \otimes \partial_i . \quad \square$$

Note F.3.12 The 3rd Bianchi identity

$$d_c N[\sigma] = -\tfrac{1}{4} \big[\sigma, T_\sigma[c]\big]$$

has coordinate expressions given by the equality

$$\begin{aligned}&\big(\partial_\lambda N_{\mu\nu}{}^i{}_h \, y^h + \partial_\lambda N_{\mu\nu}{}^i{}_o + (c_\lambda{}^j{}_h \, y^h + c_\lambda{}^j{}_o) \, N_{\mu\nu}{}^i{}_j \\ &\quad - c_\lambda{}^i{}_j \, (N_{\mu\nu}{}^j{}_h \, y^h + N_{\mu\nu}{}^j{}_o)\big) d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i \\ &= \tfrac{1}{4} \big((\sigma_\lambda{}^j{}_h \, y^h + \sigma_\lambda{}^j{}_o) \, T_{\mu\nu}{}^i{}_j - (T_{\lambda\mu}{}^j{}_h \, y^h + T_{\lambda\mu}{}^j{}_o) \, \sigma_\nu{}^i{}_j\big) \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i . \quad \square\end{aligned}$$

Eventually, we discuss the case of a basic soldering form σ.

Thus, let us consider a basic soldering 1-form $\sigma : \boldsymbol{F} \to T^* \boldsymbol{B} \otimes V \boldsymbol{F}$, with coordinate expression $\sigma = \sigma_\lambda{}^i \, d^\lambda \otimes \partial_i$, where $\sigma_\lambda{}^i \in \mathrm{map}(\boldsymbol{B}, \mathbb{R})$.

Hence, by hypothesis, we have $\partial_j \sigma_\lambda{}^i = 0$.

Note F.3.13 The torsion $T_\sigma[c] : \boldsymbol{F} \to \Lambda^2 T^* \boldsymbol{B} \otimes V \boldsymbol{F}$ turns out to be a basic vertical valued form which can be naturally regarded as a vector valued form

$$T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \bar{\boldsymbol{F}},$$

whose coordinate expression is

$$T_\sigma[c] = (\partial_\lambda \sigma_\mu{}^i - c_\lambda{}^i{}_h \, \sigma_\mu{}^h - \partial_\mu \sigma_\lambda{}^i + c_\mu{}^i{}_h \, \sigma_\lambda{}^h) \, d^\lambda \wedge d^\mu \otimes \partial_i .$$

Proof. It follows easily from Note F.1.21. $\square$

Further, we show how the 1st Bianchi identity looks like in the case of an affine connection c and a basic soldering form σ (see Theorem F.1.23).

Proposition F.3.14 *The* 1st Bianchi identity *can be expressed, in terms of the exterior covariant differential of the vector valued 2-form* $T_\sigma[c] : \boldsymbol{B} \to \Lambda^2 T^* \boldsymbol{B} \otimes \boldsymbol{F}$ *by means of the equality (see Lemma F.2.13)*

$$d_c \, T_\sigma[c] = -\big[R[c], \sigma\big] = R[c] \, \bar{\wedge} \, \sigma,$$

with coordinate expression

$$d_c T_\sigma[c] = (\partial_\lambda T_{\mu\nu}{}^i - c_\lambda{}^i{}_j \, T_{\mu\nu}{}^j) \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = R_{\lambda\mu}{}^i{}_j \, \sigma_\nu{}^j \, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_i = 0. \quad \square$$

Proof. The Proposition follows easily from Theorem F.1.23 and Lemma F.2.11. □

Corollary F.3.15 *If the torsion $T_\sigma[c]$ of the connection c vanishes, then the 2nd Bianchi identity reduces to the equality*

$$\big[R[c],\, \sigma\big] = R[c]\,\bar{\wedge}\,\sigma = 0,$$

with coordinate expression

$$\big[R[c],\, \sigma\big] = R_{\lambda\mu}{}^{i}{}_{j}\,\sigma_{\lambda}{}^{j}\, d^{\lambda} \wedge d^{\mu} \wedge d^{\nu} \otimes \partial_i = 0. \quad \square$$

Eventually, we show how the 3rd Bianchi identity looks like in the case of aan connection c and a basic soldering form σ (see Proposition F.1.25).

Note F.3.16 The FN-bracket $N[\sigma] := [\sigma, \sigma]$ vanishes. Hence, the 3rd Bianchi identity becomes

$$\tfrac{1}{4}\big[\sigma,\, T_\sigma[c]\big] = 0.$$

Indeed, the FN-bracket of two basic vertical valued forms vanishes identically. □

F.4 Linear Connections of a Manifold

We recover the standard notion of linear connection $c : T\boldsymbol{M} \to T^*\boldsymbol{M} \otimes TT\boldsymbol{M}$ of a manifold $\boldsymbol{M}$, as a particular case of our general approach to connections.

In this way, we recover standard formulas for the curvature $R[c]$.

In particular, we avail of the distinguished natural soldering form $\sigma = \mathbf{1}_{\boldsymbol{M}}$ for the definition of torsion $T[c]$. Hence, we recover the standard purely algebraic expression of the torsion as a particular case of our general approach (see [248, 311]).

Now, let us consider a manifold $\boldsymbol{M}$ and refer to the tangent vector bundle

$$\boldsymbol{F} := T\boldsymbol{M} \to \boldsymbol{B} := \boldsymbol{M}.$$

Note F.4.1 We define a *linear connection of the manifold* $\boldsymbol{M}$ to be a linear connection of the vector bundle $T\boldsymbol{M} \to \boldsymbol{M}$

$$c : T\boldsymbol{M} \to T^*\boldsymbol{M} \otimes TT\boldsymbol{M}.$$

The coordinate expression of a linear connection of the manifold $\boldsymbol{M}$ is

$$c = d^{\lambda} \otimes (\partial_{\lambda} + c_{\lambda}{}^{\nu}{}_{\mu}\, \dot{x}^{\mu}\, \dot{\partial}_{\nu}), \quad \text{with } c_{\lambda}{}^{\nu}{}_{\mu} \in \text{map}(\boldsymbol{M}, \mathbb{R}). \quad \square$$

Let us consider a linear connection $c : T\boldsymbol{M} \to T^*\boldsymbol{M} \otimes TT\boldsymbol{M}$.

We start by recovering the standard curvature $R[c]$ of c in the framework of our general approach to connections.

Note F.4.2 The curvature $R[c] : T\boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes VT\boldsymbol{M}$ can be naturally regarded as a linear fibred morphism over $\boldsymbol{M}$

$$R[c] : T\boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes T\boldsymbol{M},$$

i.e., equivalently, as a vector valued 2-form

$$R[c] : \boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes (T\boldsymbol{M} \otimes T^*\boldsymbol{M}),$$

whose coordinate expression is

$$R[c] = -(\partial_\lambda c_\mu{}^\alpha{}_\beta + c_\lambda{}^\rho{}_\beta\, c_\mu{}^\alpha{}_\rho - \partial_\mu c_\lambda{}^\alpha{}_\beta - c_\mu{}^\rho{}_\beta\, c_\lambda{}^\alpha{}_\rho)\, \dot{x}^\beta\, d^\lambda \wedge d^\mu \otimes \partial_\alpha.$$

For each $X, Y, Z \in \sec(\boldsymbol{M}, T\boldsymbol{M})$, we obtain the standard equality

$$\big(R[c](X, Y)\big)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Proof. The proof follows easily from Proposition F.2.7 and Note F.2.8. □

Further, we show how the 2nd Bianchi identity looks in the case of a linear connection of a manifold (see Theorem F.1.14).

Note F.4.3 The *2nd Bianchi identity* can be expressed, in terms of the exterior covariant differential of the vector valued 2-form

$$R[c] : \boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes (T\boldsymbol{M} \otimes T^*\boldsymbol{M})$$

with respect to the linear connection c, by means of the equality

$$d_c R[c] = 0,$$

with coordinate expression

$$(\partial_\lambda R_{\mu\nu}{}^\alpha{}_\rho + R_{\nu\lambda}{}^\alpha{}_\beta\, c_\mu{}^\beta{}_\rho - R_{\mu\nu}{}^\beta{}_\rho\, c_\lambda{}^\alpha{}_\beta)\, d^\lambda \wedge d^\mu \wedge d^\nu \otimes \partial_\alpha \otimes d^\rho = 0.$$

Proof. It follows easily from Proposition F.2.10. □

Next, we recover the standard torsion $T[c]$ of c in the framework of our general approach to connections.

Lemma F.4.4 *The equality* $VT\boldsymbol{M} = T\boldsymbol{M} \underset{\boldsymbol{M}}{\times} T\boldsymbol{M}$ *yields the natural basic soldering form*

$$\sigma := \mathbf{1}_M : T\boldsymbol{M} \to T^*\boldsymbol{M} \otimes VT\boldsymbol{M},$$

which can be naturally regarded as a vector valued form

$$\sigma := \mathbf{1}_{\boldsymbol{M}} : \boldsymbol{M} \to T^*\boldsymbol{M} \otimes T\boldsymbol{M},$$

with coordinate expression

$$\sigma = \delta^{\mu}_{\lambda}\, d^{\lambda} \otimes \partial_{\mu}. \quad \square$$

Note F.4.5 We define the *torsion* of the linear connection c of the manifold $\boldsymbol{M}$ to be the basic 2-form

$$T[c] := 2\,[c, \sigma] : T\boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes VT\boldsymbol{M},$$

which can be naturally regarded as a vector valued 2-form

$$T[c] := 2\,[c, \sigma] : T\boldsymbol{M} \to \Lambda^2 T^*\boldsymbol{M} \otimes T\boldsymbol{M}.$$

For each $X_1, X_2 \in \sec(\boldsymbol{B}, T\boldsymbol{B})$ we have the equality

$$T_{\sigma}[c](X_1, X_2) = \nabla_{X_1} X_2 - \nabla_{X_2} X_1 - [X_1,\ X_2].$$

Moreover, we have the coordinate expression

$$T_{\sigma}[c] = -c_{\lambda}{}^{\nu}{}_{\mu}\, d^{\lambda} \wedge d^{\mu} \otimes \partial_{\nu}.$$

Proof. The proof follows easily from Proposition F.2.16. □

Remark F.4.6 In the case of linear connections of a manifold, the coordinate expression of the torsion coincides essentially with the standard one. The apparent difference in the sign is just due to our non standard convention on the sign of the symbols of the connection, for the reasons explained before (see Remark F.1.3). □

Further, we show how the 1st Bianchi identity looks in the case of a linear connection of a manifold (see Theorem F.1.23).

Note F.4.7 The *1st Bianchi identity* can be expressed, in terms of the exterior covariant differential of the vector valued 2-form $T_{\sigma}[c] : \boldsymbol{B} \to \Lambda^2 T^*\boldsymbol{B} \otimes \boldsymbol{F}$ by means of the equality

$$d_c\, T_{\sigma}[c] = \big[R[c], \sigma\big] = -R[c]\, \bar{\wedge}\, \sigma,$$

with coordinate expression

$$\big(\partial_{\lambda} T_{\mu\nu}{}^{\alpha} - c_{\lambda}{}^{\alpha}{}_{\beta}\, T_{\mu\nu}{}^{\beta}\big)\, d^{\lambda} \wedge d^{\mu} \wedge d^{\nu} \otimes \partial_{\alpha} = R_{\lambda\mu}{}^{\alpha}{}_{\nu}\, d^{\lambda} \wedge d^{\mu} \wedge d^{\nu} \otimes \partial_{\alpha}.$$

In particular, if the torsion $T_{\sigma}[c]$ of the connection c vanishes, then the 1st Bianchi identity reduces to the equality

$$\big[R[c],\ \sigma\big] = R[c]\,\bar{\wedge}\,\sigma = 0,$$

with coordinate expression

$$\big[R[c],\ \sigma\big] = R_{\lambda\mu}{}^{\alpha}{}_{\nu}\, d^{\lambda} \wedge d^{\mu} \wedge d^{\nu} \otimes \partial_{\alpha} = 0. \quad \square$$

Proof. The proof follows easily from Theorem F.1.23 and Lemma F.2.13. □

Appendix G
Jets

The geometric language of the present book is largely based on techniques related to jets. Here we recall the basic notions on these topics. In the book, we essentially deal with 1st and 2nd order jets; however, for a general exposition it is quite natural to deal with jets of any order.

Our exposition develops step by step: *jets of fibred manifolds*, *jets of double fibred manifolds*, *contact structure*, *jet functor*, *exchange map*, *holonomic prolongation of vector fields* (Sects. G.1, G.2, G.3, G.4, G.5, and G.6).

The reader who is interested in further knowledge on the subject can consult standard texts on the subject (see, for instance, [246, 360]).

We stress that in this Chapter we discuss jets of sections of fibred manifolds because they are largely used throughout the book. However, in Chap. J we deal also with jets of maps between manifolds and in Chap. 31 we deal with jets of submanifolds. The reader can define and analyse these types of jets analogously to jets of sections of fibred manifolds (see also, for instance, [213, 222, 360] and literature therein).

G.1 Jet Spaces of Fibred Manifolds

We introduce the concept of *k-jet space* $J_k\boldsymbol{F}$ of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$.

G.1.1 Multi-indices

In order to discuss higher order jet spaces, it is convenient to introduce a suitable notation for the indices of higher order partial derivatives of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$. For this reason, we introduce the preliminary notion of *multi-index*.

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1

Let us consider a manifold $\boldsymbol{M}$ of dimension m and denote its typical chart by (x^λ).

Note G.1.1 We define a *multi-index* of *range m* to be a row with m entries

$$\underline{\lambda} := (\lambda_1, \ldots, \lambda_m), \qquad \lambda_i \in \mathbb{N}.$$

The *length* of a multi-index $\underline{\lambda}$ is defined to be the integer

$$|\underline{\lambda}| := \lambda_1 + \cdots + \lambda_m \in \mathbb{N}.$$

The zero multi-index of range m and the sum of two multi-indices of range m are defined by

$$\underline{0} := (0, \ldots, 0) \qquad \text{and} \qquad \underline{\lambda} + \underline{\mu} := (\lambda_1 + \mu_1, \ldots, \lambda_m + \mu_m).$$

Each index μ, with $1 \leq \mu \leq m$, can be naturally identified with a multi-index as follows

$$\mu \simeq (\underset{1}{0}, \ldots, \underset{\mu}{1}, \ldots, \underset{m}{0}).$$

Accordingly, the sum of a multi-index $\underline{\lambda}$ and an index μ turns out to be the multi-index

$$\underline{\lambda} + \mu = (\lambda_1, \ldots, \lambda_\mu + 1, \ldots, \lambda_m). \quad \square$$

Note G.1.2 For each multi-index $\underline{\lambda}$ such that $|\underline{\lambda}| \geq 1$, we define the *multi-partial derivatives* of a function $f \in \mathrm{map}(\boldsymbol{B}, \mathrm{I\!R})$, as

$$\partial_{\underline{\lambda}} f := (\partial_1)^{\lambda_1} \ldots (\partial_m)^{\lambda_m} f.$$

Thus, the integer $|\underline{\lambda}|$ turns out to be the order of the above multi-partial derivative.

It is convenient to extend the above definition to the zero multi-index. Namely, for the zero multi-index $\underline{0}$, we define the multi-differential of order zero as the identity operator

$$\partial_{\underline{0}} f := (\partial_1)^0 \ldots (\partial_m)^0 f \equiv f. \quad \square$$

G.1.2 Jet Spaces

For ahort, the k-jet space of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ of a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ is defined to be the fibred manifold $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ consisting, for each base point $b \in \boldsymbol{B}$, of equivalence classes $j_k s(b) := [s]_{(b,k)}$ of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$, whose partial derivatives in b are equal up to order k.

Then, we discuss the natural smooth structure of $J_k\boldsymbol{F}$ and analyse the natural fibring $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ over the base space $\boldsymbol{B}$ and the bundle $p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F}$ over the h-jet spaces of lower degrees.

Moreover, we show that the bundle $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{B}$ turns out to be equipped with an affine structure associated with the vector bundle $S^k T^*\boldsymbol{B} \otimes V\boldsymbol{F}$.

We exhibit explicitly the transition rules for jet spaces of degree 1 and 2.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, with $\dim \boldsymbol{B} = m$ and $\dim \boldsymbol{F} = m + n$.

We denote the typical fibred chart of $\boldsymbol{F}$ by (x^λ, y^i), with $1 \le \lambda \le m, \quad 1 \le i \le n$.

Lemma G.1.3 *Let us consider two sections* $s, \acute{s} \in \sec(\boldsymbol{B}, \boldsymbol{F})$ *and an integer* k, *with* $0 \le k$. *If in a fibred chart we have* $\partial_{\underline{\lambda}} s^i = \partial_{\underline{\lambda}} \acute{s}^i$, *for each multi-index* $\underline{\lambda}$, *with* $0 \le |\underline{\lambda}| \le k$, *then the same equality holds in any other fibred chart.* □

Definition G.1.4 Let $b \in \boldsymbol{B}$. Then, two sections $s, \acute{s} \in \sec(\boldsymbol{B}, \boldsymbol{F})$, defined in a neighbourhood of b, are said to be *equivalent* at *order* k if

$$\partial_{\underline{\lambda}} s^i(b) = \partial_{\underline{\lambda}} \acute{s}^i(b), \quad \text{for each} \quad \underline{\lambda}, \quad \text{with } 0 \le |\underline{\lambda}| \le k.$$

The equivalence class of order k at b of the section s is said to be the *k-jet* of s and we use the notation

$$j_k s : \boldsymbol{B} \to J_k\boldsymbol{F} : b \mapsto j_k s(b) := [s]_{(k,b)}.$$

The *k-jet space* of sections of the fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ is defined to be the set

$$J_k\boldsymbol{F} := \coprod_{b \in \boldsymbol{B}} J_{k\,b}\boldsymbol{F},$$

where

$$J_{k\,b}\boldsymbol{F} := \left\{ j_k s(b) \right\}_{s \in \sec(\boldsymbol{B}, \boldsymbol{F})}$$

is the set of equivalence classes, at order k, of sections at b. □

Note G.1.5 In particular, for $k = 0$, we obtain, by definition,

$$J_0\boldsymbol{F} = \boldsymbol{F}. \quad \square$$

Proposition G.1.6 *The k-jet space* $J_k\boldsymbol{F}$ *is equipped with the following natural maps.*

- *We have the natural surjective map*

$$p^k : J_k\boldsymbol{F} \to \boldsymbol{B} : j_k s(b) \mapsto b.$$

- *For each integer* h *such that* $0 \le h < k$, *we have the natural surjective map*

$$p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F} : j_k s(b) \mapsto j_h s(b).$$

Moreover, we have

$$p^h \circ p^k_h = p^k, \qquad 0 \le h < k \qquad \textit{and} \qquad p^h_l \circ p^k_h = p^k_l, \qquad 0 \le l < h < k. \quad \square$$

Remark G.1.7 We stress that, in general, there is NO natural map

$$J_h \boldsymbol{F} \to J_k \boldsymbol{F}, \quad \text{for } 0 \le h < k. \quad \square$$

Indeed, the above geometric objects turn out to be smooth according to the following Theorem G.1.9.

Lemma G.1.8 *For each $b \in \boldsymbol{B}$, given any set of numbers*

$$r^i_{\underline{\lambda}} \in I\!R, \quad \text{with } 0 \le |\underline{\lambda}| \le k, \quad 1 \le i \le n,$$

there exists a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ in a neighbourhood of b such that

$$\partial_{\underline{\lambda}} s^i(b) = r^i_{\underline{\lambda}}.$$

Proof. It suffices to take the section with coordinate expression

$$s^i = \frac{1}{\lambda_1! \dots \lambda_m!} \sum_{|\underline{\lambda}|=k} r^i_{\underline{\lambda}} (x^1)^{\lambda_1} \dots (x^m)^{\lambda_m}. \quad \square$$

Theorem G.1.9 *Each fibred chart (x^λ, y^i) defined in an open subset $\boldsymbol{U} \subset \boldsymbol{B}$ induces naturally the chart $(x^\lambda, y^i_{\underline{\lambda}})$, with $0 \le |\underline{\lambda}| \le k$, on the subset $(p^k)^{-1}(\boldsymbol{U}) \subset J_k \boldsymbol{F}$, according to the defining equality*

$$y^i_{\underline{\lambda}}(j_k s) := \partial_{\underline{\lambda}} s^i, \quad \text{with } 0 \le |\underline{\lambda}| \le k, \quad 1 \le i \le n.$$

Moreover, the atlas of $J_k \boldsymbol{F}$ consisting of these natural charts induces a smooth structure on $J_k \boldsymbol{F}$. $\square$

Corollary G.1.10 *We have the coordinate expressions*

$$(x^\lambda) \circ p^k = (x^\lambda),$$
$$(x^\lambda, y^i_{\underline{\lambda}}) \circ p^k_h = (x^\lambda, y^i_{\underline{\lambda}}), \qquad 0 \le |\underline{\lambda}| \le h, \quad 0 \le h < k.$$

Hence, the maps $p^k : J_k \boldsymbol{F}$ and $p^k_h : J_k \boldsymbol{F} \to J_h \boldsymbol{F}$ are smooth.
Furthermore,

(1)
$$p^k : J_k \boldsymbol{F} \to \boldsymbol{B}, \quad \text{with } 0 \le k$$

turns out to be a fibred manifold;

(2) $$p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F}, \quad \text{with } 0 \le h < k,$$

turns out to be a bundle, with type fibre $\mathbb{R}^L$, *where* L *is an integer which depends on* k *and* h *by means of a combinatorial formula.* □

Even more, $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$ turns out to be an affine bundle associated with the vector bundle $S^kT^*\boldsymbol{B} \otimes V\boldsymbol{F}$, according to the following Theorem.

Theorem G.1.11 *The natural bundle atlas of* $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ *yields on the bundle*

$$p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$$

an affine structure associated with the vector bundle

$$\overline{J_k\boldsymbol{F}} = S^kT^*\boldsymbol{B} \otimes V\boldsymbol{F}.$$

Hence, the vertical tangent space $V_{k-1}J_k\boldsymbol{F}$ *of the bundle* $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$ *is given by*

(*) $$V_{k-1}J_k\boldsymbol{F} = J_{k-1}\boldsymbol{F} \underset{\boldsymbol{F}}{\times} (S^kT^*\boldsymbol{B} \otimes V\boldsymbol{F}).$$

In fact, given two fibred charts of $p^k : J_k\boldsymbol{E} \to \boldsymbol{B}$

$$(x^\lambda, y^i_{\underline{\lambda}}) \quad \textit{and} \quad (\acute{x}^\lambda, \acute{y}^i_{\underline{\lambda}}), \quad \text{with } 0 \le |\underline{\lambda}| \le k,$$

we obtain transition rules of the type

$$\acute{y}^i_{\underline{\lambda}} = \acute{\partial}_{\lambda_1}x^{\mu_1} \dots \acute{\partial}_{\lambda_k}x^{\mu_k}\, \partial_j\acute{y}^i\, y^j_{\underline{\mu}} + f^i_{\underline{\lambda}}, \quad \text{with } f^i_{\underline{\lambda}} \in \operatorname{map}(J_{k-1}\boldsymbol{F}, \mathbb{R}). \quad \square$$

For $h < k-1$, the bundle $p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F}$ has a more complicated algebraic structure (see [247]).

Eventually, via jets, we can define the differential operators of a certain order with respect to two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $p' : \boldsymbol{F}' \to \boldsymbol{B}$.

Definition G.1.12 A sheaf morphism

$$\Phi : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, \boldsymbol{F}'),$$

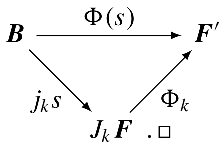
is said to be a *differential operator* of order k, with $0 \le k$, if it depends on the partial derivatives of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ up to order k, i.e., more precisely, if it factorises through a fibred morphism over $\boldsymbol{B}$

$$\Phi_k : J_k\boldsymbol{F} \to \boldsymbol{F}',$$

according to the equality

$$\Phi(s) = \Phi_k \circ j_k s,$$

i.e. to the commutative diagram



□

The explicit expressions of general full transition rules for the fibred charts of the natural smooth atlas of $J_k \boldsymbol{F} \to \boldsymbol{B}$ can be obtained by induction via a combinatorial procedure; indeed, they are rather complicated for high values of k.

Here, we provide the explicit formulas for $J_1 \boldsymbol{F}$ and $J_2 \boldsymbol{F}$.

Given two fibred charts (x^λ, y^i) and $(\acute{x}^\lambda, \acute{y}^i)$ of $\boldsymbol{F}$, we set

$$\begin{aligned}
&\acute{S}^\lambda_\mu := \partial_\mu \acute{x}^\lambda \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}), \quad && \acute{S}^i_\mu := \partial_\mu \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),\\
&\acute{S}^i_j := \partial_j \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && S^\lambda_\mu := \acute{\partial}_\mu x^\lambda \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}),\\
&S^i_\mu := \acute{\partial}_\mu y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && S^i_j := \acute{\partial}_j y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),\\
&\acute{S}^\lambda_{\mu\nu} := \partial_{\mu\nu} \acute{x}^\lambda \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}), \quad && \acute{S}^i_{\mu\nu} := \partial_{\mu\nu} \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),\\
&\acute{S}^i_{\mu k} := \partial_{\mu k} \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && \acute{S}^i_{h\nu} := \partial_{h\nu} \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),\\
&\acute{S}^i_{hk} := \partial_{hk} \acute{y}^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && S^\lambda_{\mu\nu} := \acute{\partial}_{\mu\nu} x^\lambda \in \mathrm{map}(\boldsymbol{B}, \mathbb{R}),\\
&S^i_{\mu\nu} := \acute{\partial}_{\mu\nu} y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && S^i_{\mu k} := \acute{\partial}_{\mu k} y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}),\\
&S^i_{h\nu} := \acute{\partial}_{h\nu} y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}), \quad && S^i_{hk} := \acute{\partial}_{hk} y^i \in \mathrm{map}(\boldsymbol{F}, \mathbb{R}).
\end{aligned}$$

Example G.1.13 Let us consider a section $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$.

Then, we have the following transition rules for the 1st and 2nd order partial derivatives of the section s.

For $k = 1$, we have

$$\acute{\partial}_\lambda \acute{s}^i = S^\mu_\lambda \, (\acute{S}^i_j \, \partial_\mu s^j + \acute{S}^i_\mu).$$

For $k = 2$, we have

$$\begin{aligned}
\acute{\partial}_\lambda \acute{s}^i &= S^\mu_\lambda \, (\acute{S}^i_j \, \partial_\mu s^j + \acute{S}^i_\mu),\\
\acute{\partial}_{\lambda_1\lambda_2} \acute{s}^i &= S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_j \, \partial_{\mu_1\mu_2} s^j + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{hk} \, \partial_{\mu_1} s^h \, \partial_{\mu_2} s^k\\
&\quad + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_1 j} \, \partial_{\mu_2} s^j + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_2 j} \, \partial_{\mu_1} s^j + S^\mu_{\lambda_1\lambda_2} \, \acute{S}^i_j \, \partial_\mu s^j\\
&\quad + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_1\mu_2} + S^\mu_{\lambda_1\lambda_2} \, \acute{S}^i_\mu. \quad \square
\end{aligned}$$

Example G.1.14 Given two fibred charts (x^λ, y^i) and $(\acute{x}^\lambda, \acute{y}^i)$, we obtain the following transition rules, in virtue of the above Example G.1.13.

For $k = 1$, we obtain

$$\acute{y}^i_\lambda = S^\mu_\lambda \, (\acute{S}^i_j \, y^j_\mu + \acute{S}^i_\mu).$$

For $k = 2$, we obtain

$$\begin{aligned}
\acute{y}^i_\lambda &= S^\mu_\lambda \, (\acute{S}^i_j \, y^j_\mu + \acute{S}^i_\mu), \\
\acute{y}^i_{\lambda_1\lambda_2} &= S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_j \, y^j_{\mu_1\mu_2} + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{hk} \, y^h_{\mu_1} \, y^k_{\mu_2} \\
&\quad + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_1 j} \, y^j_{\mu_2} + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_2 j} \, y^j_{\mu_1} + S^\mu_{\lambda_1\lambda_2} \, \acute{S}^i_j \, y^j_\mu \\
&\quad + S^{\mu_1}_{\lambda_1} \, S^{\mu_2}_{\lambda_2} \, \acute{S}^i_{\mu_1\mu_2} + S^\mu_{\lambda_1\lambda_2} \, \acute{S}^i_\mu .
\end{aligned}$$

Thus,

- $p^1_0 : J_1\boldsymbol{F} \to \boldsymbol{F}$ turns out to be an affine bundle associated with the vector bundle $T^*\boldsymbol{B} \otimes V\boldsymbol{F}$,
- $p^2_1 : J_2\boldsymbol{F} \to J_1\boldsymbol{F}$ turns out to be an affine bundle associated with the vector bundle $S^2T^*\boldsymbol{B} \otimes V\boldsymbol{F}$. □

G.1.3 *Vertical Bundle of Jet Spaces*

We conclude this Section by discussing the vertical spaces of the k-jet spaces.

Note G.1.15 For $0 \le h < k$, we define the *h-vertical bundle* of $J_k\boldsymbol{F}$ to be the vertical subbundle $V_h J_k\boldsymbol{F} \subset T J_k\boldsymbol{F}$ of the fibred manifold $p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F}$ (see Note B.2.2).

For $0 < h \le k-1$, the subbundle $V_h J_k\boldsymbol{F} \subset T J_k\boldsymbol{F}$ is locally characterised by the constraints $\dot{x}^\lambda = 0, \; \dot{y}^i_{\underline{\mu}} = 0$, with $0 \le |\underline{\mu}| \le h$.

Then, we have the following sequence of natural linear inclusions

$$V_{k-1}J_k\boldsymbol{F} \xrightarrow{\subset} V_h J_k\boldsymbol{F} \xrightarrow{\subset} V_0 J_k\boldsymbol{F} \xrightarrow{\subset} V J_k\boldsymbol{F} \xrightarrow{\subset} T J_k\boldsymbol{F}.$$

In particular, in virtue of Theorem G.1.11, we have

$$V_{k-1}J_k\boldsymbol{F} = J_{k-1}\boldsymbol{F} \underset{\boldsymbol{F}}{\times} S^kT^*\boldsymbol{B} \otimes V\boldsymbol{F}. \quad \square$$

Throughout the book, we frequently use the following particular case.

Example G.1.16 We have

$$V_0 J_1\boldsymbol{E} = T^*\boldsymbol{B} \otimes V\boldsymbol{F}. \quad \square$$

G.2 Jet Spaces of Double Fibred Manifolds

In the previous Sect. G.1, we have discussed the jet spaces of sections $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ of a fibred manifold $\boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$.

Now, we discuss the jet spaces of sections $s \in \sec(\boldsymbol{F}, \boldsymbol{G})$ of a double fibred manifold $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$. This concept can be regarded as a particular case of the concept of jet space of sections of a fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$. We just replace the target fibred manifold $\boldsymbol{F}$ with the double fibred manifold $\boldsymbol{G}$ and the source base manifold $\boldsymbol{B}$ with the fibred manifold $\boldsymbol{F}$. In practice, it means that we replace the family of base coordinates (x^λ) with the double family of fibred coordinates (x^λ, y^i).

Thus, this new framework turns out to be richer, but more complicated with respect to that of fibred manifolds.

In the context of double fibred manifolds we can define, besides the order k of a differential operator, also its horizontal order h.

Indeed, the transition rules for higher order partial derivatives are rather complicated, but, we do not need their explicit general expression; for our purposes some simple properties of these transition rules are sufficient.

Let us consider a double fibred manifold

$$\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}, \quad \text{with } \dim \boldsymbol{B} = m, \ \dim \boldsymbol{F} = m + n, \ \dim \boldsymbol{G} = m + n + l.$$

We denote the typical double fibred chart of $\boldsymbol{G}$ by

$$(x^\lambda, y^i, z^a), \quad \text{with } 1 \le \lambda \le m, \ 1 \le i \le n, \ 1 \le a \le l.$$

We denote the indices and the multi-indices referred to the base space $\boldsymbol{F}$ of $q : \boldsymbol{G} \to \boldsymbol{F}$ by the capital characters $\Lambda, \ldots$ and $\underline{\Lambda}, \ldots$.

Definition G.2.1 The *k-jet space* of sections $s \in \sec(\boldsymbol{F}, \boldsymbol{G})$ of the double fibred manifold $q : \boldsymbol{G} \to \boldsymbol{F}$ is defined to be just the k-jet space $J_k\boldsymbol{G}$ of sections $s \in \sec(\boldsymbol{F}, \boldsymbol{G})$ according to Definition G.1.4. □

Note G.2.2 According to Theorem G.1.9, each double fibred chart (x^λ, y^i, z^a) of $\boldsymbol{G}$ yields the fibred chart

$$(x^\lambda, y^i; z^a_{\underline{\Lambda}}), \quad \text{with } 0 \le |\underline{\Lambda}| \le k, \ 1 \le a \le l,$$

of $J_k\boldsymbol{G}$ defined by the equality

$$z^a_{\underline{\Lambda}} \circ j_k s := \partial_{\underline{\Lambda}} s^a. \quad \square$$

In the context of fibred manifolds, we have introduced the concept of differential operator of order k, via jets, according to Definition G.1.12.

This concept can be applied to the context of double fibred manifolds as well, as a particular case.

Let us consider two double fibred manifolds $\boldsymbol{G} \xrightarrow{q} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$ and $\boldsymbol{G}' \xrightarrow{q'} \boldsymbol{F} \xrightarrow{p} \boldsymbol{B}$. According to Definition G.1.12, a sheaf morphism $\boldsymbol{F}$

$$\Phi : \sec(\boldsymbol{F}, \boldsymbol{G}) \to \sec(\boldsymbol{F}, \boldsymbol{G}')$$

is said to be a *differential operator* of order k, with $0 \leq k$, if it depends on the partial derivatives of sections $s \in \sec(\boldsymbol{F}, \boldsymbol{G})$ up to order k, i.e., more precisely, if it factorises through a fibred morphism over $\boldsymbol{F}$

$$\Phi_k : J_k\boldsymbol{G} \to \boldsymbol{G}', \quad \text{according to the equality } \Phi(s) = \Phi_k \circ j_k s.$$

Besides the order k of a differential operator, we can define also the "horizontal" order $k_h \leq k$ and the "vertical" order $k_v \leq k$.

Lemma G.2.3 *With reference to two fibred charts* (x^λ, y^i, z^a) *and* $(\acute{x}^\mu, \acute{y}^j, \acute{z}^b)$ *of the double fibred manifold* $\boldsymbol{G}$*, the transition rule of the fibred charts* $(x^\lambda, y^i, z^a_{\underline{\Lambda}})$ *and* $(\acute{x}^\mu, \acute{y}^j, \acute{z}^b_{\underline{\Lambda}'})$ *of* $J_k\boldsymbol{G}$*, preserve the order, respectively, of the "horizontal" partial derivatives* $\partial_{\underline{\mu}}\acute{x}^\lambda$ *and of the "vertical" partial derivatives* $\partial_{\underline{j}}\acute{x}^\lambda$ *(see, for instance, Example G.2.5.* □

Definition G.2.4 A differential operator $\Phi : \sec(\boldsymbol{F}, \boldsymbol{G}) \to \sec(\boldsymbol{F}, \boldsymbol{G}')$ of order k is said to be, respectively, of *horizontal order* $k_h \leq k$ and of *vertical order* $k_v \leq k$ if it involves horizontal partial derivatives $\partial_{\underline{\lambda}} s^a$ up to order k_h and vertical partial derivatives $\partial_{\underline{j}} s^a$ up to order k_v. □

The explicit expressions of general transition rules for the fibred charts of the above smooth atlas of $J_k\boldsymbol{G} \to \boldsymbol{B}$ can be obtained by induction via a combinatorial procedure; indeed, they are rather complicated even for low values of k.

Here, we provide the explicit formulas for $J_1\boldsymbol{G}$ and $J_2\boldsymbol{G}$.

Given two fibred charts (x^λ, y^i, z^a) and $(\acute{x}^\lambda, \acute{y}^i, \acute{z}^a)$ of $\boldsymbol{G}$, we set

$$\begin{aligned}
\acute{S}^\lambda_\mu &:= \partial_\mu \acute{x}^\lambda \in \operatorname{map}(\boldsymbol{B}, \mathbb{R}), & \acute{S}^i_\mu &:= \partial_\mu \acute{y}^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}),\\
\acute{S}^i_j &:= \partial_j \acute{y}^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}), & \acute{S}^a_\mu &:= \partial_\mu \acute{z}^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}),\\
\acute{S}^a_j &:= \partial_j \acute{z}^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}), & \acute{S}^a_b &:= \partial_b \acute{z}^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}),\\
S^\lambda_\mu &:= \acute{\partial}_\mu x^\lambda \in \operatorname{map}(\boldsymbol{B}, \mathbb{R}), & S^i_\mu &:= \acute{\partial}_\mu y^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}),\\
S^i_j &:= \acute{\partial}_j y^i \in \operatorname{map}(\boldsymbol{F}, \mathbb{R}), & S^a_\mu &:= \acute{\partial}_\mu z^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}),\\
S^a_j &:= \acute{\partial}_j z^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}), & S^a_b &:= \acute{\partial}_b z^a \in \operatorname{map}(\boldsymbol{G}, \mathbb{R}),
\end{aligned}$$

and analogously for the 2nd order transition functions.

Example G.2.5 Let us consider a section $s \in \sec(\boldsymbol{F}, \boldsymbol{G})$.

Then, we have the following transition rules for the 1st and 2nd order partial derivatives of the section s.

For $k = 1$, we have

$$\acute{\partial}_\lambda \acute{s}^a = S^\mu_\lambda \, (\acute{S}^a_b \, \partial_\mu s^b + \acute{S}^a_\mu) + S^h_\lambda \, (\acute{S}^a_b \, \partial_h s^b + \acute{S}^a_h), \qquad \acute{\partial}_i \acute{s}^a = S^j_i \, (\acute{S}^a_b \, \partial_j s^b + \acute{S}^a_j). \quad \square$$

Example G.2.6 Given two double fibred charts (x^λ, y^i, z^a) and $(\acute{x}^\lambda, \acute{y}^i, \acute{z}^a)$, we obtain the following transition rules, in virtue of Example G.2.5.

For $k = 1$, we obtain

$$\acute{z}^a_\lambda = S^\mu_\lambda \, (\acute{S}^a_b \, z^b_\mu + \acute{S}^a_\mu) + S^j_\lambda \, (\acute{S}^a_b \, z^b_j + \acute{S}^a_j), \qquad \acute{z}^a_i = S^j_i \, (\acute{S}^a_b \, z^b_j + \acute{S}^a_j). \quad \square$$

Example G.2.7 The Schrödinger operator $\mathrm{S} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$ is of order $k = 2$ (see Theorem 17.6.5). Moreover, it is of horizontal order (with respect to time) $k_h = 1$ and of vertical order (with respect to space) $k_v = 2$. $\square$

G.3 Contact Structure

The k-jet space $J_k\boldsymbol{F}$ is equipped with a "*contact map*" and its "*complementary contact map*"

$$\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F} \qquad \text{and} \qquad \theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} T J_{k-1}\boldsymbol{F} \to V J_{k-1}\boldsymbol{F}.$$

These maps yield, via pullback with respect to $J_k\boldsymbol{F}$, a splitting of the tangent space

$$T J_{k-1}\boldsymbol{F} = T\boldsymbol{B} \underset{\boldsymbol{B}}{\oplus} V J_{k-1}\boldsymbol{F}.$$

In the present book, we make large use of the contact map and of the complementary contact map, mostly at 1st order.

G.3.1 Contact Maps

We introduce the k-contact map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ and analyse its basic properties. Indeed, the k-contact map is a fundamental objects related to jet spaces.

In simple words, the concept of k-contact map can be explained as follows.

Clearly, there is no natural prolongation $T\boldsymbol{B} \to T J_h\boldsymbol{F}$, which maps vectors of the base space $\boldsymbol{B}$ to vectors of the tangent space of the jet space $J_h\boldsymbol{F}$. However, let us consider a section $s : \boldsymbol{B} \to \boldsymbol{F}$ and its h-jet prolongation $j_h s : \boldsymbol{B} \to J_h\boldsymbol{F}$; moreover, let us consider a point of the base space $b \in \boldsymbol{B}$, and the $(h+1)$-jet $j_{h+1}s(b) \in$

$J_{h+1}\boldsymbol{F}$. Then, for each vector $X_b \in T_b\boldsymbol{B}$, the vector $T j_h s(X_b) \in T_{j_h s(b)} J_h\boldsymbol{F}$ is determined just by $j_{h+1}s(b) \in J_{h+1}\boldsymbol{F}$.

Thus, the knowledge of a point in $J_{h+1}\boldsymbol{F}$ yields the desired prolongation $T\boldsymbol{B} \to T J_h\boldsymbol{F}$, without reference to a specific section s.

Indeed, there is a close relation between the concepts of contact map and general connection (see, Sect. F.1).

The k-contact map д^k can be naturally regarded as a fibred morphism in different ways.

An important feature of the k-contact map д^k is that it provides an affine inclusion over $J_{k-1}\boldsymbol{F}$ of the affine bundle $J_k\boldsymbol{F}$ into a distinguished affine subbundle of the vector bundle $T^*\boldsymbol{B} \otimes T J_{k-1}\boldsymbol{F}$.

In the particular case $k = 1$, the above map provides a natural identification of $J_1\boldsymbol{F}$ with the affine subbundle of $T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ which projects onto the section $\mathbf{1} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and an integer $k \geq 1$.

We start by introducing the k-contact map through the following Theorem.

Theorem G.3.1 *There is a unique map, called the k-*contact map*,*

$$\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F},$$

such that, for each $s \in \sec(\boldsymbol{B}, \boldsymbol{F})$ and $X \in \sec(\boldsymbol{B}, T\boldsymbol{B})$, the following diagram commutes

$$\begin{array}{ccc}
J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\ \text{д}^k\ } & T J_{k-1}\boldsymbol{F} \\
{\scriptstyle (j_k s \times \mathrm{id}_{T\boldsymbol{B}})} \nwarrow & & \nearrow {\scriptstyle T j_{k-1} s} \\
& T\boldsymbol{B} & .
\end{array}$$

The coordinate expression of д^k is

$$(x^\lambda,\ y^i_{\underline{\mu}},\ \dot{x}^\lambda,\ \dot{y}^i_{\underline{\mu}}) \circ \text{д}^k = (x^\lambda,\ y^i_{\underline{\mu}},\ \dot{x}^\lambda,\ y^i_{\underline{\mu}+\lambda}\dot{x}^\lambda), \quad \text{with } 0 \leq |\underline{\mu}| \leq k-1. \square$$

We analyse the k-contact map as fibred morphism in two ways.

Proposition G.3.2 *The k-contact map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ turns out to be a fibred morphism in the following ways.*

(1) The k-contact map is an injective linear fibred morphism over $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$, according to the following commutative diagram

$$\begin{array}{ccc}
J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\ \text{д}^k\ } & T J_{k-1}\boldsymbol{F} \\
{\scriptstyle \mathrm{pro}_1} \downarrow & & \downarrow {\scriptstyle \tau_{J_{k-1}\boldsymbol{F}}} \\
J_k\boldsymbol{F} & \xrightarrow{\ p^k_{k-1}\ } & J_{k-1}\boldsymbol{F} \quad .
\end{array}$$

(2) *The k-contact map is a fibred morphism over $T\boldsymbol{B}$, according to the following commutative diagram*

$$\begin{array}{ccc} J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\ \text{д}^k\ } & T J_{k-1}\boldsymbol{F} \\ & \searrow^{\mathrm{pro}_2} \quad \swarrow_{Tp^{k-1}} & \\ & T\boldsymbol{B} & . \square \end{array}$$

Further, let us recall that $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$ is an affine bundle associated with the vector bundle $S_k T^*\boldsymbol{B} \otimes V\boldsymbol{F}$ (see Theorem G.1.11).

Then, the k-contact map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ can be easily regarded as a natural affine inclusion over $J_{k-1}\boldsymbol{F}$

$$J_k\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes T J_{k-1}\boldsymbol{F}.$$

Proposition G.3.3 *The k-contact map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ can be naturally regarded as an injective affine fibred morphism over $J_{k-1}\boldsymbol{F}$*

$$\text{д}^k : J_k\boldsymbol{F} \to T^*\boldsymbol{B} \otimes T J_{k-1}\boldsymbol{F},$$

according to the following commutative diagram

$$\begin{array}{ccc} J_k\boldsymbol{F} & \xrightarrow{\ \text{д}^k\ } & T^*\boldsymbol{B} \otimes T J_{k-1}\boldsymbol{F} \\ & \searrow^{p^k_{k-1}} \quad \swarrow_{\tau_{J_{k-1}\boldsymbol{F}}} & \\ & J_{k-1}\boldsymbol{F} & . \end{array}$$

The coordinate expression of the map $\text{д}^k : J_k\boldsymbol{F} \to T^*\boldsymbol{B} \otimes T J_{k-1}\boldsymbol{F}$ is

$$(x^\lambda,\ y^i_{\underline{\mu}}) \circ \text{д}^k = (x^\lambda,\ y^i_{\underline{\mu}}), \qquad \text{with } 0 \le |\underline{\mu}| \le k-1,$$

$$(\dot{x}_\lambda \otimes \dot{x}^\mu,\ \dot{x}_\lambda \otimes \dot{y}^i_{\underline{\mu}}) \circ \text{д}^k = (\delta^\mu_\lambda,\ y^i_{\underline{\mu}+\lambda}), \qquad \text{with } 0 \le |\underline{\mu}| \le k-1,$$

i.e.

$$\text{д}^k = d^\lambda \otimes (\partial_\lambda + y^i_{\underline{\mu}+\lambda}\, \partial^{\underline{\mu}}_i), \quad \text{with } 0 \le |\underline{\mu}| \le k-1. \square$$

We can analyse the image of the above fibred morphism д^k as follows. For the sake of clarity it is convenient to distinguish the cases $k = 1$ and $k \ge 2$.

Theorem G.3.4 *For $k = 1$, the image of the map $\text{д}^1 : J_1\boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ turns out to be the affine subbundle over $\boldsymbol{F}$*

$$J_1\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes T\boldsymbol{F},$$

which projects on $\mathbf{1}_{\boldsymbol{B}} : \boldsymbol{B} \to T^*\boldsymbol{B} \otimes T\boldsymbol{B}$, *according to the following commutative diagram*

$$\begin{array}{ccc} J_1\boldsymbol{F} & \xrightarrow{\;\text{д}^1\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{F} \\ {\scriptstyle p^1}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{T^*\boldsymbol{B}} \otimes Tp} \\ \boldsymbol{B} & \xrightarrow{\;\mathbf{1}_{\boldsymbol{B}}\;} & T^*\boldsymbol{B} \otimes T\boldsymbol{B} \end{array} .$$

The vector bundle associated with the affine subbundle $J_1\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ *turns out to be the vector subbundle (see Theorem G.1.11)*

$$\overline{J}_1\boldsymbol{F} = T^*\boldsymbol{B} \otimes V\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes T\boldsymbol{F}. \quad \square$$

Theorem G.3.5 *For* $k \geq 2$, *the image of the map* $\text{д}^k : J_k\boldsymbol{F} \to T^*\boldsymbol{B} \otimes TJ_{k-1}\boldsymbol{F}$ *turns out to be the affine subbundle over* $J_{k-1}\boldsymbol{F}$

$$J_k\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes TJ_{k-1}\boldsymbol{F},$$

characterised, in coordinates, by the constraint

$$(\dot{x}_\lambda \otimes \dot{x}^\mu) \circ \text{д}^k = \delta^\mu_\lambda \qquad \text{and} \qquad (\dot{x}_\lambda \otimes \dot{y}^i_{\underline{\mu}}) \circ \text{д}^k = y^i_{\underline{\mu}+\lambda}.$$

The vector bundle associated with the affine subbundle $J_k\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes TJ_{k-1}\boldsymbol{F}$ *turns out to be the ("symmetrised") vector subbundle (see Theorem G.1.11)*

$$\overline{J}_k\boldsymbol{F} = S^kT^*\boldsymbol{B} \otimes V\boldsymbol{F} \subset T^*\boldsymbol{B} \otimes (V_{k-2}J_{k-1}\boldsymbol{F}) = T^*\boldsymbol{B} \otimes (S^{k-1}T^*\boldsymbol{B} \otimes V\boldsymbol{F}),$$

where $V_{J_{k-2}}J_{k-1}\boldsymbol{F}$ *denotes the vertical bundle of the affine bundle* $p^{k-1}_{k-2} : J_{k-1}\boldsymbol{F} \to J_{k-2}\boldsymbol{F}$ *(see Theorem G.1.11).*

The coordinate expression of the fibred derivative of the affine fibred morphism $\text{д}^k : J_k\boldsymbol{F} \to T^*\boldsymbol{B} \otimes TJ_{k-1}\boldsymbol{F}$ *over* $J_{k-1}\boldsymbol{F}$ *can be written as*

$$\begin{aligned} (x^\lambda, \; y^i_{\underline{\mu}}) \circ D\text{д}^k &= (x^\lambda, \; y^i_{\underline{\mu}}), && \text{with } 0 \leq |\underline{\mu}| \leq k-1, \\ (\dot{x}_\lambda, \; \dot{x}_\lambda \otimes \dot{y}^i_{\underline{\mu}}) \circ D\text{д}^k &= 0, && \text{with } 0 \leq |\underline{\mu}| \leq k-2, \\ (\dot{x}_\lambda \otimes \dot{y}^i_{\underline{\mu}}) \circ D\text{д}^k &= y^i_{\underline{\mu}+\lambda}, && \text{with } |\underline{\mu}| = k-1. \square \end{aligned}$$

Eventually, we can compare the contact maps of different degrees in the following way.

Note G.3.6 For each integer $h < k$, the following diagram commutes

$$\begin{array}{ccc} J_k \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\text{д}^k} & T J_{k-1}\boldsymbol{F} \\ {\scriptstyle (p^k_h \times \mathrm{id}_{T\boldsymbol{B}})} \downarrow & & \downarrow {\scriptstyle T p^{k-1}_{h-1}} \\ J_h \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} & \xrightarrow{\text{д}^h} & T J_{h-1}\boldsymbol{F}. \end{array} \quad \square$$

Example G.3.7 For $k = 1, 2$, we have the coordinate expressions

$$\text{д}^1 = d^\lambda \otimes \text{д}^1_\lambda \qquad \text{and} \qquad \text{д}^2 = d^\lambda \otimes \text{д}^2_\lambda,$$

where

$$\begin{aligned} \text{д}^1_\lambda &= \partial_\lambda + y^i_\lambda\, \partial_i \in \mathrm{fib}(J_1\boldsymbol{F},\ T\boldsymbol{F}), \\ \text{д}^2_\lambda &= \partial_\lambda + y^i_\lambda\, \partial_i + y^i_{\lambda\mu}\, \partial^\mu_i \in \mathrm{fib}(J_2\boldsymbol{F},\ T J_1\boldsymbol{F}). \square \end{aligned}$$

In other words, for $k = 1, 2$, we have the coordinate expressions

$$\begin{aligned} (x^\lambda,\ y^i,\ \dot{x}^\lambda,\ \dot{y}^i) \circ \text{д}^1 &= (x^\lambda,\ y^i,\ \dot{x}^\lambda,\ y^i_\mu\, \dot{x}^\mu) \\ (x^\lambda,\ y^i,\ \dot{x}^\lambda,\ \dot{y}^i,\ \dot{y}^i_\lambda) \circ \text{д}^2 &= (x^\lambda,\ y^i,\ \dot{x}^\lambda,\ y^i_\mu\, \dot{x}^\mu,\ y^i_{\lambda\mu}\, \dot{x}^\mu). \square \end{aligned}$$

G.3.2 Complementary Contact Maps

The contact map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ yields in a natural way a "complementary contact map" $\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} T J_{k-1}\boldsymbol{F} \to V J_{k-1}\boldsymbol{F}$.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and an integer $k \geq 1$.

Proposition G.3.8 *The map $\text{д}^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}$ yields the fibred morphism over $J_{k-1}\boldsymbol{F}$, called k-*complementary contact map*,*

$$\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} T J_{k-1}\boldsymbol{F} \to V J_{k-1}\boldsymbol{F},$$

given by

$$\theta_k(f_k,\ X_{k-1}) = X_{k-1} - \text{д}^k\big(f_k,\ T p^{k-1}(X_{k-1})\big).$$

The coordinate expression of θ_k is

$$(x^\lambda,\ y^i_{\underline{\mu}};\ \dot{y}^i_{\underline{\mu}}) \circ \theta_k = (x^\lambda,\ y^i_{\underline{\mu}};\ \dot{y}^i_{\underline{\mu}} - y^i_{\underline{\mu}+\lambda}\, \dot{x}^\lambda), \quad \text{with } 0 \leq |\underline{\mu}| \leq k-1. \quad \square$$

The complementary contact map θ_k has the following properties.

Proposition G.3.9 *The map* $\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F} \to VJ_{k-1}\boldsymbol{F}$ *is a surjective linear morphism over* $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$, *according to the commutative diagram*

$$\begin{array}{ccc}
J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F} & \xrightarrow{\quad \theta_k \quad} & VJ_{k-1}\boldsymbol{F} \\
\mathrm{pro}_1 \big\downarrow & & \big\downarrow \\
J_k\boldsymbol{F} & \xrightarrow{\quad p^k_{k-1} \quad} & J_{k-1}\boldsymbol{F} \quad .
\end{array}$$

Indeed, we have

$$\theta_k(f_k, X_{k-1}) = X_{k-1}, \quad \textit{for each } \forall (f_k, X_{k-1}) \in J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} VJ_{k-1}\boldsymbol{F}.$$

Moreover, the kernel of the linear fibred morphism $\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F} \to VJ_{k-1}\boldsymbol{F}$ *over* $p^k_{k-1} : J_k\boldsymbol{F} \to J_{k-1}\boldsymbol{F}$ *coincides with the image of the map* ∂^k

$$\ker(\theta_k) = \mathrm{im}(\text{д}^k) \subset J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F}. \square$$

Note G.3.10 The map $\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F} \to VJ_{k-1}\boldsymbol{F}$ can be equivalently written as

$$\theta_k : J_k\boldsymbol{F} \to T^*J_{k-1}\boldsymbol{F} \otimes VJ_{k-1}\boldsymbol{F}.$$

Accordingly, we have the coordinate expression

$$\theta_k = \theta^i_{\underline{\mu}} \otimes \partial^{\underline{\mu}}_i, \quad \text{with } 0 \le |\underline{\mu}| \le k-1,$$

where the maps $\theta^i_{\underline{\mu}} \in \mathrm{fib}(J_k\boldsymbol{F},\ T^*J_{k-1}\boldsymbol{F})$ are defined by

$$\theta^i_{\underline{\mu}} := \theta_k \lrcorner\, d^i_{\underline{\mu}} = d^i_{\underline{\mu}} - y^i_{\underline{\mu}+\lambda}\, d^\lambda. \quad \square$$

The complementary contact map θ_k fulfills the following interesting property.

Proposition G.3.11 *Let us consider a section* $s_k \in \sec(\boldsymbol{B}, J_k\boldsymbol{F})$ *and its projections*

$$s := p^k_0 \circ s_k \in \sec(\boldsymbol{B}, \boldsymbol{F}) \qquad \textit{and} \qquad s_{k-1} := p^k_{k-1} \circ s_k \in \sec(\boldsymbol{B}, J_{k-1}\boldsymbol{F}).$$

Then, the following conditions are equivalent:

(1)
$$s_k = j_k s,$$

(2)
$$(Ts_{k-1}) \lrcorner\, (\theta_k \circ s_{k-1}) = 0. \quad \square$$

Example G.3.12 For $k = 1, 2$, we have the coordinate expressions

$$(x^\lambda,\ y^i;\ \dot{y}^i) \circ \theta_1 = (x^\lambda,\ y^i;\ \dot{y}^i - y^i_\mu\, \dot{x}^\mu)$$
$$(x^\lambda,\ y^i;\ \dot{y}^i,\ \dot{y}^i_\lambda) \circ \theta_2 = (x^\lambda,\ y^i;\ \dot{y}^i - y^i_\mu\, \dot{x}^\mu,\ \dot{y}^i_\lambda - y^i_{\lambda\mu}\, \dot{x}^\mu).$$

Equivalently, for $k = 1, 2$, we can write

$$\theta_1 = \theta^i \otimes \partial_i, \qquad \text{and} \qquad \theta_2 = \theta^i \otimes \partial_i + \theta^i_\mu \otimes \partial^\mu_i,$$

where

$$\theta^i = d^i - y^i_\lambda\, d^\lambda \qquad \text{and} \qquad \theta^i_\mu = d^i_\mu - y^i_{\mu\lambda}\, d^\lambda. \quad \square$$

G.3.3 Contact Splitting of the Tangent Space

The contact map $Д^k$ and the complementary contact map θ_k yield in a natural way a splitting of the tangent space $T J_{k-1}\boldsymbol{F}$, via pullback over $J_k\boldsymbol{F}$.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and an integer $k \geq 1$.

We start by considering the a natural inclusion and projection related to the tangent space $T J_{k-1}\boldsymbol{F}$.

Note G.3.13 We have the natural *injective linear fibred morphism* over $J_{k-1}\boldsymbol{F}$

$$i_{k-1} := i_{J_{k-1}\boldsymbol{F}} : V J_{k-1}\boldsymbol{F} \to T J_{k-1}\boldsymbol{F}$$

and the natural *surjective linear fibred morphism* over $J_{k-1}\boldsymbol{F}$

$$\pi_{k-1} := \tau_{J_{k-1}\boldsymbol{F}} \times T p^{k-1} : T J_{k-1}\boldsymbol{F} \to J_{k-1}\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B},$$

which fulfill the equality

$$\operatorname{im}(i_{k-1}) = \ker(\pi_{k-1}). \quad \square$$

Remark G.3.14 There is no natural injective linear fibred morphism over $J_{k-1}\boldsymbol{F}$

$$c_{k-1} : J_{k-1}\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to T J_{k-1}\boldsymbol{F}.$$

There is no natural surjective linear fibred morphism over $J_{k-1}\boldsymbol{F}$

$$\nu_{k-1} : T J_{k-1}\boldsymbol{F} \to V J_{k-1}\boldsymbol{F}.$$

Hence, there is no natural splitting over $J_{k-1}\boldsymbol{F}$

$$T J_{k-1}\boldsymbol{F} = \left(J_{k-1}\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}\right) \underset{J_{k-1}\boldsymbol{F}}{\oplus} V J_{k-1}\boldsymbol{F}. \quad \square$$

Proposition G.3.15 *The contact map* $д^k$ *and the complementary contact map* θ_k *yield a natural splitting over* $J_k\boldsymbol{F}$

$$J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} T J_{k-1}\boldsymbol{F} = \left(J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}\right) \underset{J_k\boldsymbol{F}}{\oplus} \left(J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} V J_{k-1}\boldsymbol{F}\right),$$

with coordinate expression

$$\begin{aligned} X^\lambda\, \partial_\lambda + X^i_{\underline{\mu}}\, \partial_i^{\underline{\mu}} &= X^\lambda\, (\partial_\lambda + y^i_{\underline{\mu}+\lambda}\, \partial_i^{\underline{\mu}}) \\ &\quad + (X^i_{\underline{\mu}} - y^i_{\underline{\mu}+\lambda}\, X^\lambda)\, \partial_i^{\underline{\mu}}, \quad \text{with } 0 \le |\underline{\mu}| \le k-1. \quad \square \end{aligned}$$

Example G.3.16 For $k = 1$, the contact map $д^1$ and the complementary contact map θ_1 yield a natural splitting over $J_1\boldsymbol{F}$

$$J_1\boldsymbol{F} \underset{\boldsymbol{F}}{\times} T\boldsymbol{F} = \left(J_1\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}\right) \underset{J_1\boldsymbol{F}}{\oplus} \left(J_1\boldsymbol{F} \underset{\boldsymbol{F}}{\times} V\boldsymbol{F}\right),$$

with coordinate expression

$$X^\lambda\, \partial_\lambda + X^i\, \partial_i = X^\lambda\, (\partial_\lambda + y^i_\lambda\, \partial_i) + (X^i - y^i_\lambda\, X^\lambda)\, \partial_i. \quad \square$$

G.4 Jet Functor

A fibred morphism $f : \boldsymbol{F} \to \boldsymbol{G}$ over the base space $\boldsymbol{B}$ can be naturally prolonged to a fibred morphism $J_k f : J_k\boldsymbol{F} \to J_k\boldsymbol{G}$ via the chain rule of derivatives.

The k-jet prolongation of bundles equipped with an algebraic structure inherit this algebraic structures by k-jet prolongation.

Thus, J_k turns out to be a functor from the category of fibred manifolds to the category of double fibred manifolds (see Definition J.1.4).

Let us consider two fibred manifolds $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ along with their typical fibred charts (x^λ, y^i) and (x^λ, z^a).

Theorem G.4.1 *Let us consider a fibred morphism* $f : \boldsymbol{F} \to \boldsymbol{G}$ *over* $\boldsymbol{B}$.

Then, for each integer $k \ge 1$, *there is a unique fibred morphism, called* k-jet prolongation *of* f,

$$J_k f : J_k\boldsymbol{F} \to J_k\boldsymbol{G},$$

such that, for each $s \in \mathrm{sec}(\boldsymbol{B}, \boldsymbol{F})$, *we have*

$$J_k(f \circ s) = J_k f \circ j_k s,$$

according to the following commutative diagram

$$\begin{array}{ccc} J_k \boldsymbol{F} & \xrightarrow{J_k f} & J_k \boldsymbol{G} \\ & {\scriptstyle j_k s} \nwarrow \quad \nearrow {\scriptstyle j_k(f \circ s)} & \\ & \boldsymbol{B} & \end{array} .$$

Indeed, the following diagram commutes

$$\begin{array}{ccc} J_k \boldsymbol{F} & \xrightarrow{J_k f} & J_k \boldsymbol{G} \\ {\scriptstyle p^k_0} \downarrow & & \downarrow {\scriptstyle q^k_0} \\ \boldsymbol{F} & \xrightarrow{f} & \boldsymbol{G} \end{array} .$$

We have the coordinate expression

$$x^\alpha \circ J_k f = x^\alpha ,$$

$$z^a_{\underline{\lambda}} \circ J_k f = \sum \partial_{\underline{\mu}} \partial_{j_1} \ldots \partial_{j_r} f^a \, y^{j_1}_{\underline{\nu}_1} \ldots y^{j_r}_{\underline{\nu}_r} , \quad \textit{with } 0 \le |\underline{\lambda}| \le k, \; 0 \le r, \qquad (*)$$

where the sum is extended to all indices and multi-indices such that

$$\underline{\mu} + \underline{\nu}_1 + \cdots + \underline{\nu}_r = \underline{\lambda} . \quad \square$$

Proposition G.4.2 *The following properties hold:*

(1) if $p : \boldsymbol{F} \to \boldsymbol{B}$ is a fibred manifold, then

$$J_k \, \mathrm{id}_{\boldsymbol{F}} = \mathrm{id}_{J_k \boldsymbol{F}} .$$

(2) if $f : \boldsymbol{F} \to \boldsymbol{G}$ and $g : \boldsymbol{G} \to \boldsymbol{H}$ are fibred morphisms over $\boldsymbol{B}$, then we obtain

$$J_k(g \circ f) = J_k g \circ J_k f . \quad \square$$

Proposition G.4.3 *If $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ are fibred manifolds and $f : \boldsymbol{F} \to \boldsymbol{G}$ is an injective fibred morphism over $\boldsymbol{B}$, then also the k-jet prolongation $J_k f : J_k \boldsymbol{F} \to J_k \boldsymbol{G}$ turns out to be an injective fibred morphism over $\boldsymbol{B}$.* $\square$

The k-jet prolongations of a vector bundle, an affine bundle, a Lie group bundle, a Lie affine bundle turn out to be in a natural way bundles with the same algebraic structure (see Sect. A.3). Actually, analogous results hold for several other bundles equipped with algebraic structures.

Proposition G.4.4 *The following facts hold (see Sect. A.3).*

(1) Let $p : \boldsymbol{F} \to \boldsymbol{B}$ *be a vector bundle, equipped with the algebraic fibred operations and the zero section*

$$\cdot : I\!R \times \boldsymbol{F} \to \boldsymbol{F}, \quad + : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}, \quad - : \boldsymbol{F} \to \boldsymbol{F}, \quad 0 : \boldsymbol{B} \to \boldsymbol{F}.$$

Then the k*-jet prolongations*

$$J_k\cdot : I\!R \times J_k\boldsymbol{F} \to J_k\boldsymbol{F}, \quad J_k+ : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\boldsymbol{F} \to J_k\boldsymbol{F},$$
$$J_k- : J_k\boldsymbol{F} \to J_k\boldsymbol{F}, \quad j_k0 : \boldsymbol{B} \to \boldsymbol{F}$$

make the bundle $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ *a vector bundle.*

(2) Let $p : \boldsymbol{F} \to \boldsymbol{B}$ *be an affine bundle associated with the vector bundle* $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$*, equipped with the algebraic fibred operation* $\tau : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F}$*.*
Then the k*-jet prolongation*

$$J_k\tau : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\bar{\boldsymbol{F}} \to J_k\boldsymbol{F}$$

makes the bundle $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ *an affine bundle associated with the vector bundle* $\bar{p}^k : J_k\bar{\boldsymbol{F}} \to \boldsymbol{B}$*.*

(3) Let $p : \boldsymbol{F} \to \boldsymbol{B}$ *be a Lie group bundle, equipped with the algebraic fibred operations and the unit section*

$$m : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \to \boldsymbol{F}, \quad i : \boldsymbol{F} \to \boldsymbol{F}, \quad e : \boldsymbol{B} \to \boldsymbol{F}.$$

Then the k*-jet prolongations*

$$J_km : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\boldsymbol{F} \to J_k\boldsymbol{F}, \qquad J_ki : J_k\boldsymbol{F} \to J_k\boldsymbol{F}, \qquad j_ke : \boldsymbol{B} \to J_k\boldsymbol{F}$$

make the bundle $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ *a Lie group bundle.*

(4) Let $p : \boldsymbol{F} \to \boldsymbol{B}$ *be a Lie affine bundle associated with the Lie group bundle* $\bar{p} : \bar{\boldsymbol{F}} \to \boldsymbol{B}$*, equipped with the algebraic fibred operation* $\tau : \boldsymbol{F} \underset{\boldsymbol{B}}{\times} \bar{\boldsymbol{F}} \to \boldsymbol{F}$*.*

Then the k*-jet prolongation* $J_k\tau : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k\bar{\boldsymbol{F}} \to J_k\boldsymbol{F}$ *makes the bundle* $p^k : J_k\boldsymbol{F} \to \boldsymbol{B}$ *a Lie affine bundle associated with the Lie group bundle* $\bar{p}^k : J_k\bar{\boldsymbol{F}} \to \boldsymbol{B}$*.* □

Example G.4.5 For $k = 1$, we have the equality

$$z^a_\lambda \circ (J_1 f) = \partial_\lambda f^a + \partial_j f^a \, y^j_\lambda. \quad \square$$

Example G.4.6 For $k = 2$, we have the equalities

$$z^a_\lambda \circ (J_2 f) = \partial_\lambda f^a + \partial_j f^a \, y^j_\lambda,$$
$$z^a_{\lambda\mu} \circ (J_2 f) = \partial_{\lambda\mu} f^a + \partial_{\lambda j} f^a \, y^j_\mu + \partial_{\mu j} f^a \, y^j_\lambda + \partial_{jh} f^a \, y^j_\lambda \, y^h_\mu + \partial_j f^a \, y^j_{\lambda\mu}. \quad \square$$

Example G.4.7 The k-jet prolongation $(\tau_{\boldsymbol{B}})^k : J_k T\boldsymbol{B} \to \boldsymbol{B}$ of the tangent vector bundle $\tau_{\boldsymbol{B}} : T\boldsymbol{B} \to \boldsymbol{B}$ turns out to be a vector bundle. $\square$

G.5 The Exchange Map

We discuss the natural *exchange* map $r_k : J_k T\boldsymbol{F} \to T J_k \boldsymbol{F}$, which exchanges the functors J_k and T (see [282, 285]). We stress that there is no natural map in the opposite direction.

We shall use the exchange map for the definition of holonomic prolongation of vector fields (see Sect. G.6).

Indeed, in the present book we are involved with holonomic prolongation of vector fields in the contest of classical and quantum symmetries.

Actually, in the present book we deal only with the 1st order exchange map; here we discuss this map at any order, because such a generalisation does not imply extra effort.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and an integer $1 \leq k$.

This fibred manifold yields several further fibred manifolds and bundles, through the tangent bundle T and the k-jet functor J_k (taken with respect to the base space $\boldsymbol{B}$).

In the present section, we deal with the following fibred manifolds and bundles along with their typical fibred charts

$$\begin{array}{rll}
\tau_{\boldsymbol{B}} : T\boldsymbol{B} \to \boldsymbol{B}, & (x^\alpha, \dot{x}^\alpha), & \\
(\tau_{\boldsymbol{B}})^k_0 : J_k T\boldsymbol{B} \to T\boldsymbol{B}, & (x^\alpha, \dot{x}^\alpha;\ \dot{x}^\alpha_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k, \\
p : \boldsymbol{F} \to \boldsymbol{B}, & (x^\alpha, y^i) & \\
\tau_{\boldsymbol{F}} : T\boldsymbol{F} \to \boldsymbol{F}, & (x^\alpha, y^i;\ \dot{x}^\alpha, \dot{y}^i), & \\
\tau_{\boldsymbol{F}} : V\boldsymbol{F} \to \boldsymbol{F}, & (x^\alpha, y^i;\ \dot{y}^i), & \\
p^k : J_k\boldsymbol{F} \to \boldsymbol{B}, & (x^\alpha, y^i_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k, \\
\tau_{J_k\boldsymbol{F}} : T J_k\boldsymbol{F} \to J_k\boldsymbol{F}, & (x^\alpha, y^i_{\underline{\lambda}};\ \dot{x}^\alpha, \dot{\dot{y}}^i_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k, \\
\tau_{J_k\boldsymbol{F}} : V J_k\boldsymbol{F} \to J_k\boldsymbol{F}, & (x^\alpha, y^i_{\underline{\lambda}};\ \dot{\dot{y}}^i_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k, \\
(p \circ \tau_{\boldsymbol{F}})^k : J_k T\boldsymbol{F} \to \boldsymbol{B}, & (x^\alpha;\ y^i_{\underline{\lambda}}, \dot{x}^\alpha_{\underline{\lambda}}, \dot{y}^i_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k, \\
(p \circ \tau_{\boldsymbol{F}})^k : J_k V\boldsymbol{F} \to \boldsymbol{B}, & (x^\alpha;\ y^i_{\underline{\lambda}}, \dot{y}^i_{\underline{\lambda}}), & 0 \leq |\underline{\lambda}| \leq k.
\end{array}$$

We stress the difference between the coordinates $\dot{\dot{y}}^i_{\underline{\lambda}}$ of $T J_k \boldsymbol{F}$ and $\dot{y}^i_{\underline{\lambda}}$ of $J_k T\boldsymbol{F}$. In fact, in the 1st case, first we perform the k-jet prolongation of the coordinates y^i

and then their tangent prolongation; conversely, in the 2nd case, first we perform the tangent prolongation of the coordinates y^i and then their k-jet prolongation. Indeed, the above coordinates $\dot{\breve{y}}^i_{\underline{\lambda}}$ and $\dot{y}^i_{\underline{\lambda}}$ undergo different transition rules. So, we have introduced the notation $\dot{\breve{y}}^i_{\underline{\lambda}}$ just in order to mark this difference.

The coordinates $\dot{\breve{y}}^i_{\underline{\lambda}}$ of $V J_k \boldsymbol{F}$ and $\dot{y}^i_{\underline{\lambda}}$ of $J_k V \boldsymbol{F}$, which are the vertical restrictions of the above coordinates, undergo the same transition rules.

We recall (see Example G.4.7) that we have the following natural affine bundles and the respective associated vector bundles

$$J_k T \boldsymbol{F} \to J_k \boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k T \boldsymbol{B}, \quad \text{associated with } J_k V \boldsymbol{F} \to J_k \boldsymbol{F},$$
$$T J_k \boldsymbol{F} \to J_k \boldsymbol{F} \underset{\boldsymbol{B}}{\times} T \boldsymbol{B}, \quad \text{associated with } V J_k \boldsymbol{F} \to J_k \boldsymbol{F}.$$

For the sake of simplicity, we start by discussing the vertical exchange map.

Proposition G.5.1 *There is a unique fibred morphism over* $J_k \boldsymbol{F}$, *called* vertical exchange map*,*

$$i_k : J_k V \boldsymbol{F} \to V J_k \boldsymbol{F},$$

such that, for each 1-parameter family of local sections $\sigma : \mathbb{R} \times \boldsymbol{B} \to \boldsymbol{F}$, *we have*

$$\partial j_k \sigma = i_k \circ \partial \sigma,$$

where ∂ *denotes the partial differential with respect to the parameter, evaluated at* $0 \in \mathbb{R}$, *according to the following commutative diagram*

$$\begin{array}{ccc} J_k V \boldsymbol{F} & \xrightarrow{\;i_k\;} & V J_k \boldsymbol{F} \\ {\scriptstyle j_k \partial \sigma} \nwarrow & & \nearrow {\scriptstyle \partial j_k \sigma} \\ & \boldsymbol{B} & \end{array} .$$

Indeed, i_k *turns out to be a linear isomorphism over* $J_k \boldsymbol{F}$, *according to the following commutative diagram*

$$\begin{array}{ccc} J_k V \boldsymbol{F} & \xrightarrow{\;i_k\;} & V J_k \boldsymbol{F} \\ {\scriptstyle J_k \tau_{\boldsymbol{F}}} \big\downarrow & & \big\downarrow {\scriptstyle \tau_{J_k \boldsymbol{F}}} \\ J_k \boldsymbol{F} & \xrightarrow{\;\mathrm{id}\;} & J_k \boldsymbol{F} \end{array} .$$

We have the coordinate expression

$$(x^\lambda, y^i_{\underline{\lambda}};\ \dot{\breve{y}}^i_{\underline{\lambda}}) \circ i_k = (x^\lambda, y^i_{\underline{\lambda}};\ \dot{y}^i_{\underline{\lambda}}). \quad \square$$

Then, we discuss the general exchange map.

Theorem G.5.2 *There exists a unique fibred morphism over* $J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{F}$, *called* k-*exchange map*

$$r_k : J_k T\boldsymbol{F} \to T J_k \boldsymbol{F}$$

according to the following commutative diagram

$$\begin{array}{ccc} J_k T\boldsymbol{F} & \xrightarrow{\ r_k\ } & T J_k\boldsymbol{F} \\ & \searrow \quad \swarrow & \\ & J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{F} & \end{array} \quad ,$$

such that, for each local section $s : \boldsymbol{B} \to \boldsymbol{F}$,

$$T j_k s \circ (\tau_{\boldsymbol{B}})^k_0 = r_k \circ J_k T s : J_k T\boldsymbol{B} \to T J_k \boldsymbol{F},$$

according to the following commutative diagram

$$\begin{array}{ccc} J_k T\boldsymbol{F} & \xrightarrow{\ r_k\ } & T J_k\boldsymbol{F} \\ {\scriptstyle J_k T s}\uparrow & & \uparrow{\scriptstyle T j_k s} \\ J_k T\boldsymbol{B} & \xrightarrow{\ (\tau_{\boldsymbol{B}})^k_0\ } & T\boldsymbol{B} \end{array} \quad .$$

We have the coordinate expression

$$(x^\lambda, y^i_{\underline{\lambda}};\ \dot{x}^\lambda, \dot{y}^i_{\underline{\lambda}}) \circ r_k = (x^\lambda, y^i_{\underline{\lambda}};\ \dot{x}^\lambda, \dot{y}^i_{\underline{\lambda}} - y^i_{\underline{\mu}+\beta}\, \dot{x}^\beta_{\underline{\nu}}), \quad \textit{with } 0 \le |\underline{\lambda}| \le k,$$

where the sum is extended to all multi-indices $\underline{\mu}$ *and* $\underline{\nu}$ *such that*

$$\underline{\mu} + \underline{\nu} = \underline{\lambda} \qquad \textit{and} \qquad 0 < |\underline{\nu}|.$$

Indeed, the map r_k *turns out to be a surjective affine morphism over the map* $J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k T\boldsymbol{B} \to J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B}$, *according to the following commutative diagram*

$$\begin{array}{ccc} J_k T\boldsymbol{F} & \xrightarrow{\ r_k\ } & T J_k\boldsymbol{F} \\ \downarrow & & \downarrow \\ J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} J_k T\boldsymbol{B} & \longrightarrow & J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \end{array} \quad ,$$

and the fibred derivative of r_k *is*

$$Dr_k = i_k.$$

Moreover, the restriction of $r_k : J_kT\boldsymbol{F} \to TJ_k\boldsymbol{F}$ *to the subbundle* $J_kV\boldsymbol{F}$ *factorises through the map*

$$i_k : J_kV\boldsymbol{F} \to VJ_k\boldsymbol{F},$$

according to the following commutative diagram

$$\begin{array}{ccc} J_kT\boldsymbol{F} & \xrightarrow{r_k} & TJ_k\boldsymbol{F} \\ \cup\uparrow & & \uparrow\cup \\ J_kV\boldsymbol{F} & \xrightarrow{i_k} & VJ_k\boldsymbol{F} \end{array} \quad . \square$$

Thus, for each section $s : \boldsymbol{B} \to \boldsymbol{F}$, the following diagram commutes

$$\begin{array}{ccccccc} T\boldsymbol{F} & \xleftarrow{(Tp)^k_0} & J_kT\boldsymbol{F} & \xrightarrow{r_k} & TJ_k\boldsymbol{F} & \xrightarrow{\tau_{J_k\boldsymbol{F}}} & J_k\boldsymbol{F} \\ Ts\uparrow & & \uparrow J_kTs & & Tj_ks\uparrow & & \uparrow j_ks \\ T\boldsymbol{B} & \xleftarrow{(\tau_{\boldsymbol{B}})^k_0} & J_kT\boldsymbol{B} & \xrightarrow{(\tau_{\boldsymbol{B}})^k_0} & T\boldsymbol{B} & \xrightarrow{\tau_{\boldsymbol{B}}} & \boldsymbol{B} \end{array} \quad .$$

Note G.5.3 For each integer $1 \leq h < k$, the following diagram commutes

$$\begin{array}{ccc} J_kT\boldsymbol{F} & \xrightarrow{r_k} & TJ_k\boldsymbol{F} \\ \big(p \circ (\tau_{\boldsymbol{F}})\big)^k_h\downarrow & & \downarrow T(p^k_h) \\ J_hT\boldsymbol{F} & \xrightarrow{r_h} & TJ_h\boldsymbol{F} \end{array} \; . \square$$

Example G.5.4 For $k = 1$, we have the natural *exchange* fibred morphism over $J_1\boldsymbol{F} \underset{\boldsymbol{F}}{\times} T\boldsymbol{F}$

$$r_1 : J_1T\boldsymbol{F} \to TJ_1\boldsymbol{F}$$

with coordinate expression

$$(x^\alpha,\ y^i,\ y^i_\lambda;\ \dot{x}^\alpha,\ \dot{y}^i_\lambda) \circ r_1 = (x^\alpha,\ y^i,\ y^i_\lambda;\ \dot{x}^\alpha,\ \dot{y}^i_\lambda - y^i_\beta\,\dot{x}^\beta_\lambda). \quad \square$$

G.6 Holonomic Prolongation of Vector Fields

We discuss the natural k-holonomic prolongation of vector fields of a fibred manifold. Here, we introduce this concept through a direct approach via the k-jet functor J_k and the exchange map r^k ([282]).

This result might be obtained also via integration of the vector field and k-jet prolongation of its flow [246, 360]. Indeed, after such a usual procedure, in three steps (integration, jet prolongation and differentiation), we obtain as a result, just a differentiation that coincides with the above procedure.

In the present book, we use several times the 1-holonomic prolongation of vector fields in the context of special phase functions and classical and quantum symmetries.

Even if in the present book we are essentially involved with the 1st holonomic prolongation, we analyse more generally the k-holonomic prolongation, for the sake of completeness and because such a generality requires little additional effort.

Let us consider a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and denote its typical fibred charts by (x^λ, y^i).

Lemma G.6.1 *For each vector field $X \in \mathrm{sec}(\boldsymbol{F}, T\boldsymbol{F})$, we obtain the section*

$$J_k X : J_k \boldsymbol{F} \to J_k T\boldsymbol{F},$$

where the functor J_k is taken with respect to the base space $\boldsymbol{B}$.

Indeed, $J_k X$ turns out to be a fibred morphism over X, according to the commutative diagram

$$\begin{array}{ccc} J_k\boldsymbol{F} & \xrightarrow{J_k X} & J_k T\boldsymbol{F} \\ {\scriptstyle p^k_0}\downarrow & & \downarrow{\scriptstyle (p\circ\tau_{\boldsymbol{F}})^k_0} \\ \boldsymbol{F} & \xrightarrow{X} & T\boldsymbol{F} \end{array} .$$

We have the coordinate expression

$$\begin{aligned} (x^\alpha, y^i_{\underline{\lambda}}) \circ J_k X &= (x^\alpha, y^i_{\underline{\lambda}}), \\ \dot{x}^\alpha_{\underline{\lambda}} \circ J_k X &= \partial_{\underline{\mu}} \partial_{j_1} \dots \partial_{j_r} X^\alpha \, y^{j_1}_{\underline{\nu}_1} \dots y^{j_r}_{\underline{\nu}_r}, \\ \dot{x}^i_{\underline{\lambda}} \circ J_k X &= \partial_{\underline{\mu}} \partial_{j_1} \dots \partial_{j_r} X^i \, y^{j_1}_{\underline{\nu}_1} \dots y^{j_r}_{\underline{\nu}_r}, \end{aligned}$$

with

$$0 \le |\underline{\lambda}| \le k, \qquad \underline{\mu} + \underline{\nu}_1 + \dots + \underline{\nu}_r = \underline{\lambda}, \qquad 0 \le r. \quad \square$$

Note G.6.2 In particular, if $X : \boldsymbol{F} \to T\boldsymbol{F}$ is projectable on $\underline{X} : \boldsymbol{B} \to T\boldsymbol{B}$, then the fibred morphism $J_k X : J_k \boldsymbol{F} \to J_k T\boldsymbol{F}$ is projectable on the section $j_k \underline{X} : \boldsymbol{B} \to J_k T\boldsymbol{B}$, according to the following commutative diagram

$$\begin{array}{ccc} J_k\boldsymbol{F} & \xrightarrow{J_k X} & J_k T\boldsymbol{F} \\ {\scriptstyle p^k}\big\downarrow & & \big\downarrow{\scriptstyle J_k T p} \\ \boldsymbol{B} & \xrightarrow{j_k \underline{X}} & J_k T\boldsymbol{B} \end{array} ,$$

and the above coordinate expression becomes

$$\begin{aligned} (x^\alpha, y^i_{\underline{\lambda}}) \circ J_k X &= (x^\alpha, y^i_{\underline{\lambda}}), \\ \dot{x}^\alpha_{\underline{\lambda}} \circ J_k X &= \partial_{\underline{\lambda}} X^\alpha, \\ \dot{x}^i_{\underline{\lambda}} \circ J_k X &= \partial_{\underline{\mu}} \partial_{j_1} \dots \partial_{j_r} X^i \, y^{j_1}_{\underline{\nu}_1} \dots y^{j_r}_{\underline{\nu}_r}, \end{aligned}$$

with

$$0 \le |\underline{\lambda}| \le k, \qquad \underline{\mu} + \underline{\nu}_1 + \dots + \underline{\nu}_r = \underline{\lambda}, \qquad 0 \le r. \quad \square$$

Theorem G.6.3 *For each vector field* $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, *we obtain the vector field, called the* k-jet prolongation *of* X,

$$X^k := r^k \circ J_k X \in \sec(J_k\boldsymbol{F}, T J_k\boldsymbol{F}),$$

which is projectable on X, *according to the following commutative diagram*

$$\begin{array}{ccccc} J_k\boldsymbol{F} & \xrightarrow{J_k X} & J_k T\boldsymbol{F} & \xrightarrow{r^k} & T J_k\boldsymbol{F} \\ {\scriptstyle p^k_0}\big\downarrow & & & & \big\downarrow{\scriptstyle T p^k_0} \\ \boldsymbol{F} & & \xrightarrow{\quad X \quad} & & T\boldsymbol{F} \end{array} .$$

We have the following coordinate expression

$$\begin{aligned} X^k &= X^\alpha \, \partial_\alpha \\ &\quad + (\partial_{\underline{\mu}} \partial_{j_1} \dots \partial_{j_r} X^i \, y^{j_1}_{\underline{\nu}_1} \dots y^{j_r}_{\underline{\nu}_r} - y^i_{\underline{\rho}+\beta} \, \partial_{\underline{\sigma}} \partial_{h_1} \dots \partial_{h_s} X^\beta \, y^{h_1}_{\underline{\sigma}_s} \dots y^{h_s}_{\underline{\sigma}_s}) \, \partial^{\underline{\lambda}}_i, \end{aligned}$$

where the sum is extended to all multi-indices and indices such that

$$\begin{aligned} &0 \le |\underline{\lambda}| \le k, \quad \underline{\mu} + \underline{\nu}_1 + \dots + \underline{\nu}_r = \underline{\lambda}, \\ &\underline{\rho} + \underline{\sigma} + \underline{\sigma}_1 + \dots + \underline{\sigma}_r = \underline{\lambda}, \quad 0 \le r, \quad 0 < s. \quad \square \end{aligned}$$

Note G.6.4 In particular, if the vector field $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ is projectable on $\boldsymbol{B}$, then the above coordinate expression becomes

$$X^k = X^\alpha \, \partial_\alpha + (\partial_{\underline{\mu}} \partial_{j_1} \dots \partial_{j_r} X^i \, y^{j_1}_{\underline{\nu}_1} \dots y^{j_r}_{\underline{\nu}_r}) \, \partial^{\underline{\lambda}}_i,$$

where the sum is extended to all multi-indices and indices such that

$$0 \le |\underline{\lambda}| \le k, \qquad \underline{\mu} + \underline{\nu}_1 + \cdots + \underline{\nu}_r = \underline{\lambda}, \qquad 0 \le r. \quad \square$$

Example G.6.5 For $k = 1$, the coordinate expression of the 1-holonomic prolongation of a vector field $X \in \sec(\boldsymbol{F}, T\boldsymbol{F})$ is

$$X^1 = X^\lambda \,\partial_\lambda + X^i \,\partial_i + (\partial_\lambda X^i + \partial_j X^i \, y^j_\lambda - \partial_\lambda X^\mu \, y^i_\mu - \partial_j X^\mu \, y^i_\mu \, y^j_\lambda) \,\partial^\mu_i .$$

In the particular case when the vector field is projectable on $\boldsymbol{B}$, the above coordinate expression becomes

$$X^1 = X^\lambda \,\partial_\lambda + X^i \,\partial_i + (\partial_\lambda X^i + \partial_j X^i \, y^j_\lambda - \partial_\lambda X^\mu \, y^i_\mu) \,\partial^\mu_i . \quad \square$$

Theorem G.6.6 *The sheaf morphism*

$$^k : \sec(\boldsymbol{F},\ T\boldsymbol{F}) \to \sec(J_k\boldsymbol{F},\ T J_k\boldsymbol{F}) : X \mapsto X^k$$

is a sheaf morphism of Lie algebras, i.e., for each $X, Y \in \sec(\boldsymbol{F},\ T\boldsymbol{F})$, *we have*

$$[X^k, Y^k] = [X, Y]^k . \quad \square$$

Appendix H
Lagrangian Formalism

We present a synthetic geometric formulation of the 1st order *lagrangian formalism* on a generic fibred manifold (see, for instance, [1, 147, 244, 249, 255–257, 283, 331, 360, 411]).

Thus, we deal with a fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$, with $\dim \boldsymbol{F} = n$ and $\dim \boldsymbol{B} = m$, and denote its typical fibred chart by (c^λ, y^i).

We start by considering a 1st order *lagrangian form* $\mathcal{L}$.

Then, we define the associated *momentum form* $\mathcal{P}$ and *Poincaré–Cartan form* $\mathcal{C}$ (Sect. H.1).

Further, we define the *Euler–Lagrange form* $\mathcal{E}$, which can be naturally regarded as a 2nd order differential operator (Sect. H.2).

Eventually, for each vector field $Z \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, we define the associated *current* $\mathrm{j}[Z]$ (Sect. H.3)

So, we can formulate the *Noether theorem*, by saying that if the projectable vector field Z of $\boldsymbol{F}$ fulfills the equality $L_{Z^1}\mathcal{L} = 0$, then the current form $\mathrm{j}[Z]$ turns out to be conserved along the critical sections of the Euler–Lagrange equation.

In the present book, we are involved with the lagrangian formalism in the context of several topics, including the *classical kinetic objects*, the *classical dynamical phase 1 forms*, the *classical law of motion*, the *symmetries of classical dynamics*, the *quantum lagrangian*, the *quantum momentum form*, the *quantum Poincaré–Cartan form*, the lagrangian approach to the *Schrödinger equation*, the *symmetries of quantum dynamics*, the *quantum Noether theorem*, the *quantum currents* (see Definition 3.2.9, Theorems 10.1.8, 17.5.2, 17.5.10, 17.6.23, 17.6.26, 19.3.2, 21.1.4, Propositions 11.5.3, 13.2.1, 17.5.7).

H.1 Momentum and Poincaré–Cartan Form

We consider a 1st order *lagrangian form*

$$\mathcal{L} : J_1\boldsymbol{F} \to \Lambda^m T^*\boldsymbol{B}$$

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and define the associated *momentum form* and *Poincaré–Cartan form*

$$\mathcal{P} := \theta \mathbin{\bar\wedge} V_0\mathcal{L} : J_1\boldsymbol{F} \to \Lambda^m T^*\boldsymbol{F} \qquad \text{and} \qquad \mathcal{C} := \mathcal{L} + \mathcal{P} \to \Lambda^m T^*\boldsymbol{F},$$

with coordinate expressions

$$\begin{aligned}
\mathcal{L} &= L\, d^1 \wedge \ldots d^m, \\
\mathcal{P} &= \partial_i^\lambda L\ (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \wedge d^i, \\
\mathcal{C} &= L\, d^1 \wedge \ldots d^m + \partial_i^\lambda L\ (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \wedge d^i.
\end{aligned}$$

We start by considering a 1st order *lagrangian form*

$$\mathcal{L} : J_1\boldsymbol{F} \to \Lambda^m T^*\boldsymbol{B},$$

with coordinate expression

$$\mathcal{L} = L\, d^1 \wedge \ldots \wedge d^m, \quad \text{where } L \in \mathrm{map}(J_1\boldsymbol{F}, \mathbb{R}).$$

For each $1 \le r \le m$, let us recall the natural linear inclusion over $\boldsymbol{F}$

$$\Lambda^r T^*\boldsymbol{B} \underset{\boldsymbol{B}}{\times} \boldsymbol{F} \subset \Lambda^r T^*\boldsymbol{F}.$$

Lemma H.1.1 *The map*

$$T\boldsymbol{B} \otimes \Lambda^m T^*\boldsymbol{B} \to \Lambda^{m-1} T^*\boldsymbol{B} : X \otimes \alpha \mapsto i_X\alpha$$

is a natural linear fibred isomorphism over $\boldsymbol{B}$, *whose coordinate expression is given by the equalities*

$$\begin{aligned}
X^\lambda\, \partial_\lambda \otimes (d^1 \wedge \ldots \wedge d^m) &\mapsto (-1)^{\lambda-1}\, X^\lambda\, d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m, \\
(-1)^{\lambda-1}\, X^\lambda\, d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m &\mapsto X^\lambda\, \partial_\lambda \otimes (d^1 \wedge \ldots \wedge d^m). \quad \square
\end{aligned}$$

Lemma H.1.2 *The vertical map of* $\mathcal{L}$

$$V_0\mathcal{L} : V_0 J_1\boldsymbol{F} \simeq J_1\boldsymbol{F} \underset{\boldsymbol{F}}{\times} (T^*\boldsymbol{B} \otimes V\boldsymbol{F}) \to \Lambda^m T^*\boldsymbol{B}$$

can be naturally regarded as the fibred morphism over $\boldsymbol{F}$

$$V_0\mathcal{L} : J_1\boldsymbol{F} \to T\boldsymbol{B} \otimes (\Lambda^m T^*\boldsymbol{B} \otimes V^*\boldsymbol{F}) \simeq \Lambda^{m-1} T^*\boldsymbol{B} \otimes V^*\boldsymbol{F},$$

with coordinate expression

$$V_0\mathcal{L} = \partial_i^\lambda L\ (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \otimes \partial_i. \quad \square$$

Definition H.1.3 We define the *momentum form* associated with $\mathcal{L}$ to be the form

$$\mathcal{P} := \theta \bar{\wedge} V_0 \mathcal{L} : J_1 \boldsymbol{F} \to \Lambda^{m-1} T^* \boldsymbol{B} \wedge T^* \boldsymbol{F} \subset \Lambda^m T^* \boldsymbol{F},$$

with coordinate expression

$$\mathcal{P} = \partial_i^\lambda L \, (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \wedge d^i. \quad \square$$

Definition H.1.4 We define the *Poincaré–Cartan form* associated with $\mathcal{L}$ to be the form

$$\mathcal{C} := \mathcal{L} + \mathcal{P} : J_1 \boldsymbol{F} \to \Lambda^m T^* \boldsymbol{F},$$

with coordinate expression

$$\mathcal{C} = L \, d^1 \wedge \ldots d^m + \partial_i^\lambda L \, (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \wedge d^i. \quad \square$$

H.2 Euler–Lagrange Operator

According to the standard variational approach, the Euler–Lagrange operator is obtained by a procedure aimed at minimalising a certain integral. Eventually, this procedure yields a differential operator.

Nowadays, we can obtain the same differential operator (along with the Helmoltz operator and cohomological properties as well) by means of an intrinsic purely geometric procedure, which analyses a certain bicomplex (see, for instance, [147, 255–257, 360, 411]).

Here, we just show the intrinsic geometric expression of the Euler–Lagrange operator $\mathcal{E} := d_V \mathcal{P} - d\mathcal{C} : J_2 \boldsymbol{F} \to \Lambda^m T^* \boldsymbol{B} \otimes V^* \boldsymbol{F}$.

Definition H.2.1 The *vertical differential* of $\mathcal{P}$ is defined to be the form (see [411])

$$d_V \mathcal{P} := i_{\vartheta_2} d\mathcal{P} - d i_{\vartheta_1} \mathcal{P} : J_2 \boldsymbol{F} \to \Lambda^{m+1} T^* J_1 \boldsymbol{F},$$

where

$$\vartheta_1 : J_1 \boldsymbol{F} \to T^* \boldsymbol{F} \otimes V \boldsymbol{F} \qquad \text{and} \qquad \vartheta_2 : J_2 \boldsymbol{F} \to T^* J_1 \boldsymbol{E} \otimes (T^* \boldsymbol{B} \otimes V \boldsymbol{F})$$

are the 1st and 2nd order complementary contact maps (see Proposition G.3.8).

We have the coordinate expression

$$\begin{aligned} d_V \mathcal{P} = {} & \partial_j \, \partial_i^\lambda L \, \vartheta^j \wedge \vartheta^i \wedge i_{\partial_\lambda} (d^1 \wedge \ldots \wedge d^m) \\ & + \partial_j^\mu \, \partial_i^\lambda L \, \vartheta_\mu^j \wedge \vartheta^i \wedge i_{\partial_\lambda} (d^1 \wedge \ldots \wedge d^m). \quad \square \end{aligned}$$

Then, the standard Euler–Lagrange operator arising from a variational procedure can be expressed in the following way as an intrinsic purely geometric differential operator.

Definition H.2.2 We define the *Euler–Lagrange form* to be the fibred morphism over $\boldsymbol{F}$

$$\mathcal{E} = := d_V \mathcal{P} - d\mathcal{C} : J_2\boldsymbol{F} \to \Lambda^{m+1} T^* J_1 \boldsymbol{F},$$

with coordinate expression

$$\begin{aligned}\mathcal{E} &= \partial_j\, \partial_i^\lambda L\, \vartheta^j \wedge \vartheta^i \wedge i_{\partial_\lambda}(d^1 \wedge \ldots \wedge d^m) \\ &+ \partial_j^\mu\, \partial_i^\lambda L\, \vartheta_\mu^j \wedge \vartheta^i \wedge i_{\partial_\lambda}(d^1 \wedge \ldots \wedge d^m) \\ &+ L\, d^1 \wedge \ldots \wedge d^m + \partial_i^\lambda L\ (d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m) \wedge d^i . \quad \square\end{aligned}$$

Proposition H.2.3 *We can naturally regard $\mathcal{E}$ as a fibred morphism over $\boldsymbol{B}$, called* Euler–Lagrange operator,

$$\mathcal{E} : J_2\boldsymbol{F} \to \Lambda^m T^* \boldsymbol{B} \otimes V^* \boldsymbol{F},$$

with coordinate expression,

$$\mathcal{E} = \Big((\partial_\lambda + y_\lambda^j\, \partial_j + y_{\lambda\mu}^j\, \partial_j^\mu)\, \partial_i^\lambda L - \partial_i L\Big)\, \check{d}^i \otimes d^1 \wedge \ldots \wedge d^m . \quad \square$$

H.3 Currents

We define the current $\mathrm{j}[Z] := -\, i_{Z^1}\mathcal{C} = -i_Z\mathcal{C} \in \sec(J_1\boldsymbol{F}, \Lambda^{m-1}T^*\boldsymbol{F})$ associated with a vector field $Z \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, and reformulate the Noether theorem in terms of currents.

For each vector field $Z \in \mathrm{pro}_{\boldsymbol{B}} \sec(\boldsymbol{F}, T\boldsymbol{F})$, let us recall its *1-jet holonomic prolongation* $Z^1 := r^1 \circ J_1 Z \in \mathrm{pro}_{\boldsymbol{F},\boldsymbol{B}} \sec(J_1\boldsymbol{F}, T J_1\boldsymbol{F})$, with coordinate expression $Z^1 = Z^\lambda\, \partial_\lambda + Z^i\, \partial_i + (\partial_\lambda Z^i + \partial_j Z^i\, y_\lambda^j - \partial_\lambda Z^\mu\, y_\mu^i)\, \partial_i^\lambda$ (see Theorem G.6.3).

Definition H.3.1 For each vector field $Z \in \sec(\boldsymbol{F}, T\boldsymbol{F})$, we define the associated *current* to be the horizontal $(n-1)$-form

$$\mathrm{j}[Z] := -\, i_{Z^1}\mathcal{C} = -i_Z\mathcal{C} \in \sec(J_1\boldsymbol{F}, \Lambda^{m-1}T^*\boldsymbol{F}),$$

with coordinate expression

$$
\begin{aligned}
\mathrm{j}[Z] = &- L\, Z^{\lambda}\, i_{\partial_\lambda}(d^1 \wedge \ldots \wedge d^m) \\
&- \partial_i^{\lambda} L\, \big((Z^i - y^i_\mu\, Z^\mu)\, i_{\partial_\lambda}(d^1 \wedge \ldots \wedge d^m) \\
&- X^\nu\, (d^i - y^i_\mu\, d^\mu) \otimes i_{\partial_\nu} i_{\partial_\lambda}(d^1 \wedge \ldots \wedge d^m)\big). \quad \square
\end{aligned}
$$

Lemma H.3.2 *[283] For each vector field $Z \in \mathrm{pro}_{\boldsymbol{B}}\, \mathrm{sec}(\boldsymbol{F}, T\boldsymbol{F})$, the following equivalence holds*

$$L_{Z^1}\mathcal{L} = 0 \qquad \Leftrightarrow \qquad L_{Z^1}\mathcal{C} = 0. \quad \square$$

Then, the we can formulate well-known Noether theorem (see, for instance, [243, 283, 331, 411]).

Theorem H.3.3 [Noether] *If the vector field $Z \in \mathrm{pro}_{\boldsymbol{B}}\, \mathrm{sec}(\boldsymbol{F}, T\boldsymbol{F})$ generates an infinitesimal symmetry of $\mathcal{L}$, i.e. if $L_{Z^1}\mathcal{L} = 0$, then the current form*

$$\mathrm{j}[Z] := -\, i_{Z^1}\mathcal{C} = -i_Z\mathcal{C} \in \mathrm{sec}(J_1\boldsymbol{F}, \Lambda^{m-1}T^*\boldsymbol{F})$$

turns out to be conserved along the sections $s : \boldsymbol{B} \to \boldsymbol{F}$ which are solutions of the Euler–Lagrange equation ("critical sections"), i.e.

$$d\big((j_1 s)^*\mathrm{j}[Z]\big) = 0. \quad \square$$

Appendix I
Geometric Structures

According to the general scheme proposed in [223], we discuss the *cosymplectic* and *coPoisson* geometric structures of odd dimensional manifolds given by pairs of the type (ω, Ω) and (E, Λ), which fulfill certain properties (see, also, for instance, [75, 76, 275, 416]).

In the present book, our specific application deals with the scaled cosymplectic structure (dt, Ω) and the scaled coPoisson structure (γ, Λ) on the odd dimensional classical phase space $J_1\boldsymbol{E}$ (see Theorems 10.1.1 and 10.2.1).

We stress that considering an unscaled 1-form ω and an unscaled vector field E, as it is usually done in standard literature, reflects the representation of time by means of the real line $\mathbb{R}$. Indeed, this representation implicitly involves the choice of a time unit of measurement. Our analogue of the 1-form ω is the scaled spacetime 1-form dt and the analogue of the unscaled vector field E is the scaled dynamical phase vector field γ. Indeed, these scaled objects reflect the fact that we represent time by an affine space $\boldsymbol{T}$, without reference to any selected time unit of measurement (see, also, Remark 12.5.9). □

I.1 Schouten Bracket

The "Schouten bracket" plays a relevant role in the classical theory. Here, we recall a convenient characterisation of this bracket (the present exposition follows the paper [275]).

Let us consider a manifold $\boldsymbol{M}$.

Note I.1.1 For each sections $X \in \sec(\boldsymbol{M}, \Lambda^p T\boldsymbol{M})$ and $Y \in \sec(\boldsymbol{M}, \Lambda^q T\boldsymbol{M})$, their "*Schouten bracket*" can be characterised as the unique section

$$[X, Y] \in \sec(\boldsymbol{M}, \Lambda^{p+q-1} T\boldsymbol{M}),$$

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which fulfills the following equality, for each closed form $\beta \in \sec(\boldsymbol{M}, \Lambda^{p+q-1} T^*\boldsymbol{M})$,

$$i_{[X,Y]}\beta = (-1)^{pq+q}\, i_X d i_Y \beta + (-1)^p\, i_Y d i_X \beta.$$

In particular, if $X \in \sec(\boldsymbol{M}, T\boldsymbol{M})$, then we have $[X, Y] = L_X Y$. □

I.1.1 Regular Pairs

We start by defining the "*regular covariant and contravariant pairs*" on a odd dimensional manifold and analysing the associated "*musical morphisms*" (see, for instance, [223, 416]).

Let us consider a manifold $\boldsymbol{M}$ with odd dimension $2\,n + 1$.

Definition I.1.2 We define:

- a *regular covariant pair* to be a pair (ω, Ω) consisting of a 1-form ω and of a 2-form Ω of constant rank $2\,n$, such that $\omega \wedge \Omega^n \not\equiv 0$,
- a *regular contravariant pair* to be a pair (E, Λ) consisting of a vector field E and a 2-vector Λ, of constant rank $2\,n$, such that $E \wedge \Lambda^n \not\equiv 0$. □

Let us consider a regular covariant pair (ω, Ω) and a regular contravariant pair (E, Λ).

Note I.1.3 The ranks of Ω and Λ imply $\Omega^n \not\equiv 0$ and $\Lambda^n \not\equiv 0$. □

Note I.1.4 We define the following linear *musical morphisms* over $\boldsymbol{M}$

$$\Omega^\flat : T\boldsymbol{M} \to T^*\boldsymbol{M} : X \mapsto X^\flat := i_X \Omega, \qquad \Lambda^\sharp : T^*\boldsymbol{M} \to T\boldsymbol{M} : X \mapsto \alpha^\sharp := i_\alpha \Lambda,$$

and obtain the following vector subspaces

$$\langle\omega\rangle := \{\lambda\,\omega \mid \lambda \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})\} \subset T^*\boldsymbol{M}, \quad \langle E\rangle := \{\lambda\, E \mid \lambda \in \operatorname{map}(\boldsymbol{M}, \mathbb{R})\} \subset T\boldsymbol{M},$$
$$\ker\omega := \{X \in T\boldsymbol{M} \mid \omega(X) = 0\}, \quad \ker E := \{\alpha \in T^*\boldsymbol{M} \mid \alpha(E) = 0\}.$$

Then, the following equalities hold:

$$\dim(\operatorname{im}\Omega^\flat) = 2\,n, \quad \dim(\ker\Omega^\flat) = 1, \quad \dim(\ker\omega) = 2\,n,$$
$$\dim(\operatorname{im}\Lambda^\sharp) = 2\,n, \quad \dim(\ker\Lambda^\sharp) = 1, \quad \dim(\ker E) = 2\,n. \quad □$$

I.1.2 Dual Regular Pairs

Then, we introduce the concept of "*dual regular pairs*" and recall a uniqueness result along other useful properties (see [223]).

Definition I.1.5 A regular covariant pair (ω, Ω) and a regular contravariant pair (E, Λ) are said to be *mutually dual* if the following conditions hold (see, also, [277]):

(1) the following maps are isomorphisms

$$\Omega^{\flat}{}_{\mathrm{im}\,(\Lambda^{\sharp})} : \mathrm{im}\,(\Lambda^{\sharp}) \to \mathrm{im}\,(\Omega^{\flat}) \subset T^*\boldsymbol{M}, \quad \Lambda^{\sharp}{}_{\mathrm{im}\,(\Omega^{\flat})} : \mathrm{im}\,(\Omega^{\flat}) \to \mathrm{im}\,(\Lambda^{\sharp}) \subset T\boldsymbol{M},$$

(2) $$\big(\Omega^{\flat}{}_{\mathrm{im}\,(\Lambda^{\sharp})}\big)^{-1} = \Lambda^{\sharp}{}_{\mathrm{im}\,(\Omega^{\flat})}, \quad \big(\Lambda^{\sharp}{}_{\mathrm{im}\,(\Omega^{\flat})}\big)^{-1} = \Omega^{\flat}{}_{\mathrm{im}\,(\Lambda^{\sharp})},$$

(3) $$i_E\Omega = 0, \quad i_E\omega = 1, \quad i_\omega\Lambda = 0. \quad \square$$

Now, let us state some results on mutually dual pairs (ω, Ω) and (E, Λ).

Theorem I.1.6 *If* (ω, Ω) *is a regular covariant pair, then there is a unique dual regular contravariant pair* (E, Λ). □

In this context, E is called the *Reeb vector field*.

Proposition I.1.7 *For two dual regular pairs* (ω, Ω) *and* (E, Λ), *we have*

$$\langle E\rangle = \ker \Omega^{\flat}, \qquad \langle\omega\rangle = \ker \Lambda^{\sharp}, \qquad \mathrm{im}\,(\Lambda^{\sharp}) = \ker \omega, \qquad \mathrm{im}\,(\Omega^{\flat}) = \ker E. \quad \square$$

Proposition I.1.8 *Two dual regular pairs* (ω, Ω) *and* (E, Λ), *yield the following splittings*

$$T\boldsymbol{M} = \langle E\rangle \oplus \mathrm{im}\,(\Lambda^{\sharp}) \qquad \textit{and} \qquad T^*\boldsymbol{M} = \langle\omega\rangle \oplus \mathrm{im}\,(\Omega^{\flat}).$$

Accordingly, for each $X \in \sec(\boldsymbol{M}, T\boldsymbol{M})$ *and* $\alpha \in \sec(\boldsymbol{M}, T^*\boldsymbol{M})$, *we have the splittings*

$$X = \omega(X)\,E + \big(X - \omega(X)\,E\big) \qquad \textit{and} \qquad \alpha = \alpha(E)\,\omega + \big(\alpha - \alpha(E)\,\omega\big),$$

Thus, the maps

$$\Lambda^{\sharp} \circ \Omega^{\flat} : T\boldsymbol{M} \to \mathrm{im}\,(\Lambda^{\sharp}) \qquad \textit{and} \qquad \Omega^{\flat} \circ \Lambda^{\sharp} : T^*\boldsymbol{M} \to \mathrm{im}\,(\Omega^{\flat})$$

turn out to be the "orthogonal" projections of the above splittings of $T\boldsymbol{M}$ *and* $T^*\boldsymbol{M}$. □

Proposition I.1.9 *We have*

$$(\Lambda^{\sharp} \otimes \Lambda^{\sharp})(\Omega) = -\Lambda \qquad \textit{and} \qquad (\Omega^{\flat} \otimes \Omega^{\flat})(\Lambda) = -\Omega. \quad \square$$

I.1.3 Cosymplectic and coPoisson Structures

Eventually, we define the "*cosymplectic structure*" and the "*coPoisson structure*" associated with two distinguished regular covariant pairs (see [223] and [76]).

Indeed, these structures play an important role in the present book.

Definition I.1.10 We define:

- a *cosymplectic structure* to be a covariant pair (ω, Ω), such that

$$d\omega = 0, \qquad d\Omega = 0, \qquad \omega \wedge \Omega^n \not\equiv 0,$$

- a *coPoisson structure* is defined to be a contravariant pair (E, Λ), such that

$$[E, \Lambda] = 0, \qquad [\Lambda, \Lambda] = 0, \qquad E \wedge \Lambda^n \not\equiv 0. \quad \square$$

Theorem I.1.11 *The covariant pair* (ω, Ω) *is cosymplectic if and only if the dual contravariant pair* (E, Λ) *is coPoisson.* $\square$

Appendix J
Covariance

The standard concept of "covariance" widely used in the literature of physics has got a rigorous meaning and has been refined by mathematicians. Nowadays, it appears in literature of mathematics under the name of "naturality" or "gauge naturality". A modern comprehensive reference for the geometry of naturality is the book [246]; indeed, the present chapter is highly based on this book. Further information can be found, for instance, in [110, 120, 121, 203, 204, 244, 254, 258, 327, 332, 388, 397], as well.

Here, we still use the word "covariance" according to the above concept of "naturality".

In this context, the independence from the choice of coordinates and the choice of units of measurement appears on the same footing.

Indeed, in Covariant Classical Mechanics we mostly deal with natural bundle functors and natural differential operators, while in Covariant Quantum Mechanics we mostly deal with gauge natural bundle functors and gauge natural differential operators. This difference emphasises an important feature of Quantum Mechanics, where the quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$ is not achieved from the spacetime $\boldsymbol{E}$ by means of a natural procedure.

In the standard literature of physics, a largely used approach to different theories is based on the choice of some groups, as starting step, and the subsequent analysis of their representations. Accordingly, the covariance of different objects is expressed by their equivariance with respect to the action of these groups.

In the present book, we mostly start with manifolds, bundles, and so on, which are equipped with distinguished geometric structures (such as, fibrings, metrics, connections, and so on). Then, we are implicitly involved with the groups of automorphisms of these structured manifolds, bundles, and so on. In general, these groups are only present in the background of our development and do not play an explicit role. Actually, our procedure starts with the assumption of a few structured space and derives from them new objects by "intrinsic" methods. In this way, the covariance of the theory is guaranteed a priory.

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When we classify all objects which fulfill certain properties, it is convenient to follow the approach of groups and group representations (see, for instance, Sect. 17.7). Here, this approach is implemented via the concepts of “natural covariance” and “gauge natural covariance” (Sects. J.4.2 and J.4.3).

The present chapter of Appendix is intended as a comprehensive general mathematical exposition of concepts frequently used in the body of the book. In particular, several concepts and technical procedures concerning covariance are used in Sect. 17.7.

By considering the quite technical character of this chapter, the reader might just grasp the main ideas and read details if needed.

J.1 Categories and Functors

We start by recalling the concepts of categories and functors (see, for instance, [16]).

J.1.1 Categories

We recall the notion of category.

In simple words, a category is a family of objects and of morphisms between objects, which fulfill reasonable conditions.

We provide a few examples of categories that we shall be involved with (categories of manifolds, of fibred manifolds, of bundles, of vector bundles, of affine bundles, of principal bundles).

Definition J.1.1 A *category* is defined to be a pair $\mathscr{C} = \Big(\mathrm{Obj}(\mathscr{C}),\ \mathrm{Mor}(\mathscr{C})\Big)$ constituted by a family of *objects* and a family of *morphisms*

$$\mathrm{Obj}(\mathscr{C}) \equiv \{\boldsymbol{M}\} \qquad \text{and} \qquad \mathrm{Mor}(\mathscr{C}) := \coprod_{M,N \in \mathrm{Obj}(\mathscr{C})} \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N}),$$

where, for each $\boldsymbol{M}, \boldsymbol{N} \in \mathrm{Obj}(\mathscr{C})$, we set

$$\mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N}) \equiv \{f : \boldsymbol{M} \to \boldsymbol{N}\},$$

along with a map

$$\mathrm{Obj}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{C}) : \boldsymbol{M} \mapsto \mathrm{id}_{\boldsymbol{M}} : \boldsymbol{M} \to \boldsymbol{M},$$

and, for each $\boldsymbol{M}, \boldsymbol{N}, \boldsymbol{P} \in \mathrm{Obj}(\mathscr{C})$, a map

$$\mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N}) \times \mathrm{Mor}(\boldsymbol{N}, \boldsymbol{P}) \to \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{P}) : (f, g) \mapsto g \circ f,$$

such that

(1) for each $f \in \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N})$, $g \in \mathrm{Mor}(\boldsymbol{N}, \boldsymbol{P})$ and $h \in \mathrm{Mor}(\boldsymbol{P}, \boldsymbol{Q})$ we have

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

(2) for each $f \in \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N})$, we have

$$\mathrm{id}_{\boldsymbol{M}} \circ f = f = f \circ \mathrm{id}_{\boldsymbol{N}} . \quad \square$$

Example J.1.2 We are involved with the following categories of general character:

- the category $\mathscr{M}$, whose objects are the *manifolds* and whose morphisms are (local) *smooth maps*;
- the category $\mathscr{M}[n]$, whose objects are the *manifolds* with fixed dimension n and whose morphisms are (local) *diffeomorphisms*;
- the category $\mathscr{F}$, whose objects are the *fibred manifolds* and whose morphisms are *fibred morphisms* over smooth maps of base manifolds;
- the category $\mathscr{F}[n]$, whose objects are the *fibred manifolds* with base space of dimension n and whose morphisms are *fibred morphisms* over diffeomorphisms of base manifolds;
- the category $\mathscr{B}$, whose objects are the *bundles* and whose morphisms are *fibred morphisms* over smooth maps of base manifolds;
- the category $\mathscr{B}[n]$, whose objects are the *bundles* with base space of dimension n and whose morphisms are *fibred morphisms* over diffeomorphisms of base manifolds.

In particular, we consider:

- the category $\mathscr{V}$, whose objects are the *vector bundles* and whose morphisms are *linear fibred morphisms* over smooth maps of base manifolds;
- the category $\mathscr{V}[n]$, whose objects are the *vector bundles* with base space of dimension n and whose morphisms are *linear fibred morphisms* over diffeomorphisms of base manifolds;
- the category $\mathscr{A}$, whose objects are the *affine bundles* and whose morphisms are *affine fibred morphisms* over smooth maps of base manifolds;
- the category $\mathscr{A}[n]$, whose objects are the *affine bundles* with base space of dimension n and whose morphisms are *affine fibred morphisms* over diffeomorphisms of base manifolds;
- the category $\mathscr{P}[G, n]$, whose objects are the *principal bundles* with the structure group G and with base space of dimension n and whose morphisms are *principal fibred morphisms* over diffeomorphisms of base manifolds. $\square$

Example J.1.3 In the present book we are also involved with the following more specific categories:

- the category $\mathscr{E}$ of galilean spacetimes, whose morphisms are the fibred isomorphisms projectable over affine isomorphisms of the base space,

- the category $\mathscr{E}[g]$ of galilean spacetimes equipped with a galilean metric, whose morphisms are the metric preserving fibred isomorphisms projectable over affine isomorphisms of the base space,
- the category $\mathscr{Q}$ of quantum bundles over galilean spacetime, whose morphisms are the complex linear fibred isomorphisms over galilean fibred isomorphisms of the base space,
- the category $\mathscr{Q}[\mathrm{h}]$ of quantum bundles over galilean spacetime equipped with a hermitian metric, whose morphisms are the hermitian metric preserving complex linear fibred isomorphisms over galilean fibred isomorphisms of the base space.□

J.1.2 Functors

We recall the notion of functor.

In simple words, a functor maps objects and morphisms of a category to objects and morphisms of another category, fulfilling reasonable conditions.

We provide a few examples of functors that we shall be involved with (the identity functor, the base functor, the tensor product functor). Other examples of functors will be discussed later, in the context of natural bundle functors and gauge natural bundle functors.

Definition J.1.4 Let us consider two categories $\mathscr{C}$ and $\mathscr{D}$. A *(covariant) functor*

$$\mathcal{F} : \mathscr{C} \to \mathscr{D}$$

is defined to be a pair of maps

$$\mathcal{F} := \Big(\mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D}), \quad \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D}) \Big),$$

which fulfill the following properties:

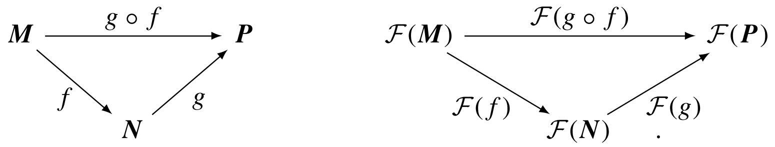
(1) If $(f : \boldsymbol{M} \to \boldsymbol{N}) \in \mathrm{Mor}(\mathscr{C})$, then

$$\Big(\mathcal{F}(f) : \mathcal{F}(\boldsymbol{M}) \to \mathcal{F}(\boldsymbol{N}) \Big) \in \mathrm{Mor}(\mathscr{D}).$$

(2) For each $(f : \boldsymbol{M} \to \boldsymbol{N}) \in \mathrm{Mor}(\mathscr{C})$ and $(g : \boldsymbol{N} \to \boldsymbol{P}) \in \mathrm{Mor}(\mathscr{C})$, we have the equality

$$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f),$$

according to the following commutative diagrams



(3) For each $\boldsymbol{M} \in \mathrm{Obj}(\mathscr{C})$, we have

$$\mathcal{F}(\mathrm{id}_{\boldsymbol{M}}) = \mathrm{id}_{\mathcal{F}(\boldsymbol{M})} \, . \quad \square$$

Example J.1.5 We are involved with the following functors:

- the *identity functor* $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ of a category $\mathscr{C}$, which associates the object $\boldsymbol{M}$ with every object $\boldsymbol{M}$ and the morphism $f : \boldsymbol{M} \to \boldsymbol{N}$ with every morphism $f : \boldsymbol{M} \to \boldsymbol{N}$;
- the *base functor* $\underline{\mathcal{B}} : \mathscr{F} \to \mathscr{M}$, which associates the base space $\boldsymbol{B}$ with each fibred manifold $p : \boldsymbol{F} \to \boldsymbol{B}$ and the base morphism $\underline{f} : \boldsymbol{B} \to \boldsymbol{B}'$ with each fibred morphism $f : \boldsymbol{F} \to \boldsymbol{F}'$;
- the r*-tensor power* $\mathscr{T}^r : \mathscr{V} \to \mathscr{V}$, which associates the r-tensor power bundle $p^r : \otimes^r \boldsymbol{F} \to \boldsymbol{B}$ with each vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$ and the r-tensor power $\otimes^r f : \otimes^r \boldsymbol{F} \to \otimes^r \boldsymbol{F}'$ with each linear fibred morphism $f : \boldsymbol{F} \to \boldsymbol{F}'$. □

In the next Sect. J.2.1, we shall be involved with other important functors.

J.2 Natural Bundle Functors and Operators

Next, we discuss the main properties of natural bundles and natural differential operators [212, 244, 246, 254, 258, 327, 332, 388].

J.2.1 Natural Bundle Functors

We discuss the notion of *natural bundle functor*.

In simple words, a natural bundle functor $\mathcal{B}$ maps objects and morphisms of a subcategory $\mathscr{C} \subset \mathscr{M}$ of manifolds into objects and morphisms of the category $\mathscr{B}$ of bundles, fulfilling the conditions of functors (see Definition J.1.4) and additional reasonable conditions of *prolongation* and *locality*.

All natural bundle functors $\mathcal{B}$ defined in the category $\mathscr{M}[n]$ have finite order (i.e. the natural bundle functors depend on finite order jets).

All natural bundles $\mathcal{B}(\boldsymbol{M})$ induced by a natural bundle functor $\mathcal{B}$ defined on the category $\mathscr{C} := \mathscr{M}[n]$ have the same type fibre $\boldsymbol{S}_{\mathcal{B}} := \mathcal{B}_0(\mathbb{R}^n)$.

Among the natural bundle functors, a distinguished role is played by the r-order frame bundle $\mathcal{F}^r$, which associates the r-order frame bundle $\mathcal{F}^r(\boldsymbol{M})$ with each n-dimensional manifold.

For each r-order natural bundle functor $\mathcal{B}$ defined on the category $\mathscr{M}[n]$, the bundle $\mathcal{B}(\boldsymbol{M})$ turns out to be a bundle associated with the frame bundle $\mathcal{F}^r(\boldsymbol{M})$ and the type fibre $\boldsymbol{S}_{\mathcal{B}}$.

We discuss the jet prolongation of natural bundle functors and the flow lift of vector fields under the continuity condition.

Eventually, we mention a few examples of natural bundle functors that we will be involved with (tangent functor, cotangent functor, tensor functor, metrics functor, linear connections functor).

Here, for the sake of simplicity, we define directly the natural bundle functors valued in the category of bundles. However, in the literature, the natural bundle functors are often defined, more generally, with values in the category of fibred manifolds; then, it can be proved that these fibred manifolds turn out always to be bundles (see, for instance, [246, 254, 388]).

A natural bundle functor defined on the subcategory $\mathscr{M}[n]$ of $\mathscr{M}$, for a certain n, was firstly defined by Nijenhuis [327] as a *natural lift functor*.

Recently, the concept of natural lift functor was generalised to the concept of natural bundle functor, [244, 246].

In literature natural bundle functors are also called *prolongation functors*.

Definition J.2.1 A *natural bundle functor*, on a subcategory $\mathscr{C} \subset \mathscr{M}$ of manifolds, is defined to be a covariant functor from the category $\mathscr{C}$ to the category $\mathscr{B}$ of bundles

$$\mathcal{B} : \mathscr{C} \to \mathscr{B},$$

i.e. a pair of maps

$$\mathcal{B} := \Big(\mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{B}), \quad \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{B}) \Big)$$

satisfying the properties (1), (2) and (3) of Definition J.1.4, which fulfill the following additional properties:

(a) (*Prolongation*) We have

$$\underline{\mathcal{B}} \circ \mathcal{B} = \mathrm{id}_{\mathscr{C}},$$

i.e.,

- for each $\boldsymbol{M} \in \mathrm{Obj}(\mathscr{C})$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\ \mathcal{B}\ } & \mathcal{B}(\boldsymbol{M}) \\ & {\scriptstyle \mathrm{id}_{\boldsymbol{M}}} \searrow \quad \swarrow {\scriptstyle p_{\boldsymbol{M}}} & \\ & \boldsymbol{M} & \end{array},$$

- for each $\underline{f} \in \mathrm{Mor}(\mathscr{C})$, the fibred morphism $\mathcal{B}(\underline{f}) \in \mathrm{Mor}(\mathscr{B})$ is over $\underline{f}$.

(b) (*Locality*) If $\imath_{\boldsymbol{U}} : \boldsymbol{U} \hookrightarrow \boldsymbol{M}$ is the inclusion of an open submanifold, then

$$\mathcal{B}(\boldsymbol{U}) = p_{\boldsymbol{M}}^{-1}(\boldsymbol{U}) \qquad \text{and} \qquad \mathcal{B}(\imath_{\boldsymbol{U}}) = \imath_{\mathcal{B}(\boldsymbol{U})} : \mathcal{B}(\boldsymbol{U}) \hookrightarrow \mathcal{B}(\boldsymbol{M}).$$

Given a natural bundle functor $\mathcal{B}$, the 3-plet

$$\big(\mathcal{B}(\boldsymbol{M}), p_{\boldsymbol{M}}, \boldsymbol{M}\big), \quad \text{for each } \boldsymbol{M} \in \mathrm{Obj}(\mathscr{C}),$$

is said to be the *natural bundle prolongation* of $\boldsymbol{M}$. □

Next, we discuss the order of natural bundle functors.

Definition J.2.2 A natural bundle functor $\mathcal{B} : \mathscr{C} \to \mathscr{B}$ is said to be of *finite order* r, with $0 \leq r < \infty$, if r is the minimal integer such that, for all morphisms $f, g : \boldsymbol{M} \to \boldsymbol{N}$ and every point $x \in \boldsymbol{M}$, the equality $j_{r\,x} f = j_{r\,x} g$ implies

$$\mathcal{B}(f)_{|\mathcal{B}_x(\boldsymbol{M})} = \mathcal{B}(g)_{|\mathcal{B}_x(\boldsymbol{M})}. \quad \square$$

Theorem J.2.3 *All natural bundle functors* $\mathcal{B} \equiv \mathcal{B}[n]$ *defined on the category* $\mathscr{C} := \mathscr{M}[n]$ *have finite order.*

Proof. See [246, 332]. □

Further, we discuss the fibre of natural bundle functors.

Theorem J.2.4 *Let us consider a natural bundle* $\mathcal{B} \equiv \mathcal{B}[n]$ *defined on the category* $\mathscr{C} := \mathscr{M}[n]$ *of manifolds with dimension* n.

Then, for each manifold $\boldsymbol{M}$ *of dimension* n, *the fibred manifold* $p_{\boldsymbol{M}} : \mathcal{B}(\boldsymbol{M}) \to \boldsymbol{M}$ *turns out to be a bundle with type fibre*

$$\boldsymbol{S}_{\mathcal{B}} = \mathcal{B}_0(\mathbb{R}^n), \quad \text{with } 0 \in \mathbb{R}.$$

Furthermore, local charts (x^λ) *on* $\boldsymbol{M}$ *and* (y^i) *on* $\boldsymbol{S}_{\mathcal{B}}$ *induce a fibred chart* (x^λ, y^i) *on* $\mathcal{B}(\boldsymbol{M})$, *which is said to be a* natural fibred chart*.*

Thus, the type fibre $\boldsymbol{S}_{\mathcal{B}}$ *of the bundle* $\mathcal{B}(\boldsymbol{M})$ *does not depend on the choice of* $\boldsymbol{M} \in \mathrm{Obj}(\mathscr{M}[n])$, *but depends only on* $\mathcal{B}$.

Proof. See [246, 332]. □

Note J.2.5 Let us note that there are ∞-order natural bundle functors defined on the category $\mathscr{M}$ of manifolds (see [306]). □

Furthermore, we study the distinguished r-order natural frame bundle functor $\mathcal{F}^r[n] : \mathscr{M}[n] \to \mathscr{P}[G_n^r, n]$.

Moreover, we show that, for each r-order natural bundle functor $\mathcal{B}$, the natural bundle $\mathcal{B}(\boldsymbol{M}) \to \boldsymbol{M}$ over an n-dimensional manifold $\boldsymbol{M}$ turns out to be the bundle with type fibre $\boldsymbol{S}_{\mathcal{B}}$ associated with the principal frame bundle $\mathcal{F}^r(\boldsymbol{M})$.

Definition J.2.6 We define the *r-order frame bundle* of an n-dimensional manifold $\boldsymbol{M}$ to be the bundle

$$\pi^r : \mathcal{F}^r(\boldsymbol{M}) := \text{inv}\, J_{r\,0}(\mathbb{R}^n, \boldsymbol{M}) \to \boldsymbol{M},$$

whose total space $\mathcal{F}^r(\boldsymbol{M})$ consists of r-jets, at $0 \in \mathbb{R}$, of diffeomorphisms $\mathbb{R}^n \to \boldsymbol{M}$.

Then, we define the *r-order frame bundle functor* to be the natural bundle functor of order r,

$$\mathcal{F}^r \equiv \mathcal{F}^r[n] : \mathscr{M}[n] \to \mathscr{F}[n],$$

given by

- for each $\boldsymbol{M} \in \text{Obj}(\mathscr{M}[n])$,

$$\mathcal{F}^r(\boldsymbol{M}) := \text{inv}\, J_{r\,0}(\mathbb{R}^n, \boldsymbol{M}),$$

- for each $f \in \text{Mor}(\mathscr{M}[n])$,

$$\mathcal{F}^r(f)(j_{r\,0}\alpha) = j_{r\,0}(f \circ \alpha), \quad \text{where } j_{r\,0}\alpha \in \mathcal{F}^r(\boldsymbol{M}). \quad \square$$

Definition J.2.7 We define the *r-order differential group* to be the Lie group

$$G_n^r := \text{inv}\, J_{r\,0}(\mathbb{R}^n, \mathbb{R}^n)_0$$

consisting of r-jets, at $0 \in \mathbb{R}^n$, of diffeomorphisms of $\mathbb{R}^n$ which preserve $0 \in \mathbb{R}^n$.

Here, the group multiplication is defined, for each $(j_{r0}\alpha),\ (j_{r0}\beta) \in G_n^r$, by

$$(j_{r0}\alpha)\,(j_{r0}\beta) := j_{r0}(\alpha \circ \beta). \quad \square$$

Proposition J.2.8 *Let us consider an n-dimensional manifold* $\boldsymbol{M}$.

Then, the standard fibre S_n^r *of the r-frame bundle* $p^r : \mathcal{F}^r(\boldsymbol{M}) \to \boldsymbol{M}$ *turns out to be the r-order differential group*

$$S_n^r = G_n^r := \text{inv}\, J_{r\,0}(\mathbb{R}^n, \mathbb{R}^n)_0.$$

Moreover, the action of G_n^r *on the standard fibre* S_n^r *is given, for each* $j_{r0}\alpha \in G_n^r$ *and each* $j_{r0}\beta \in S_n^r$ *by*

$$j_{r0}(\alpha)(j_{r0}\beta) := j_{r0}(\alpha \circ \beta). \quad \square$$

Corollary J.2.9 *The r-order frame functor* $\mathcal{F}^r[n]$ *turns out to be valued in the category* $\mathscr{P}[G_n^r, n]$ *of principal bundles with structure group* G_n^r *and with n-dimensional bases and principal fibred isomorphisms over local diffeomorphisms of base manifolds.* $\square$

Theorem J.2.10 *Let us consider an r-order natural bundle functor* $\mathcal{B} \equiv \mathcal{B}[n]$ *on the category* $\mathscr{M}[n]$.

Then, the following fact hold.

(1) The standard fibre $\boldsymbol{S}_{\mathcal{B}} = \mathcal{B}_0(\mathbb{R}^n)$ *turns out to be a left* G_n^r*-manifold and the left action of* G_n^r *on* $\boldsymbol{S}_{\mathcal{B}}$ *is given by*

$$(j_{r0}\alpha)(y) = \mathcal{B}(\alpha)(y), \quad \text{forall } j_{r0}\alpha \in \text{inv } J_{r\,0}(I\!R^n, I\!R^n)_0, \quad y \in \boldsymbol{S}_{\mathcal{B}}.$$

(2) For each n-dimensional manifold $\boldsymbol{M}$, *the natural bundle* $\mathcal{B}(\boldsymbol{M})$ *turns out to associated with the principal r-frame bundle* $\mathcal{F}^r(\boldsymbol{M})$.

In other words, we have

$$\mathcal{B}(\boldsymbol{M}) = (\mathcal{F}^r(\boldsymbol{M}) \times \boldsymbol{S}_{\mathcal{B}})/G_n^r, \quad \textit{for each } \boldsymbol{M} \in \text{Obj}(\mathscr{M}[n]),$$
$$\mathcal{B}(f) = (\mathcal{F}^r(f) \times \text{id}_{S_{\mathcal{B}}})/G_n^r, \quad \textit{for each } f \in \text{Mor}(\mathscr{M}[n]).$$

Proof. See [254, 388]. □

Corollary J.2.11 *There is a bijection between r-order natural bundle functors* $\mathcal{B}$ *defined on the category* $\mathscr{M}[n]$ *and left* G_n^r*-manifolds* $\boldsymbol{S}_{\mathcal{B}}$. □

Now, we discuss the jet prolongation of natural bundle functors and the flow lift of vector fields under the continuity condition.

Proposition J.2.12 *Let* $\mathcal{B}$ *be a natural bundle functor of order r on the category* $\mathscr{M}[n]$.

For each diffeomorphism $(f : \boldsymbol{M} \to \boldsymbol{N}) \in \text{Mor}(\mathscr{M}[n])$, *we get the following commutative diagram, by using the standard jet prolongation,*

$$\begin{array}{ccc}
J_s\mathcal{B}(\boldsymbol{M}) & \xrightarrow{\;J_s\mathcal{B}(f)\;} & J_s\mathcal{B}(\boldsymbol{N}) \\
{\scriptstyle (p_{\boldsymbol{M}})_0^s}\big\downarrow & & \big\downarrow{\scriptstyle (p_{\boldsymbol{N}})_0^s} \\
\mathcal{B}(\boldsymbol{M}) & \xrightarrow{\;\mathcal{B}(f)\;} & \mathcal{B}(\boldsymbol{N}) \\
{\scriptstyle p_{\boldsymbol{M}}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\boldsymbol{N}}} \\
\boldsymbol{M} & \xrightarrow{\;f\;} & \boldsymbol{N}
\end{array}\quad .$$

Hence, $J_s\mathcal{B} := J_s \circ \mathcal{B}$ *turns out to be a natural bundle functor of order* $(r + s)$. *If* $\boldsymbol{S}_{\mathcal{B}}$ *is the standard fibre of* $\mathcal{B}$, *then the standard fibre of* $J_s\mathcal{B}$ *is*

$$\boldsymbol{S}_{J_s\mathcal{B}} = J_s\mathcal{B}_0(I\!R^n) = T_n^s S_{\mathcal{B}} = J_{s\,0}(I\!R^n, S_{\mathcal{B}})$$

and the action of G_n^{r+s} *on* $\boldsymbol{S}_{J_s\mathcal{B}}$ *is obtained by the jet prolongation of the action of* G_n^r *on* $\boldsymbol{S}_{\mathcal{B}}$. □

Note J.2.13 Natural bundle functors $\mathcal{B}$ satisfy the *continuity condition* (see [114]), which says that a smoothly parametrised family of diffeomorphisms $\boldsymbol{M} \to \boldsymbol{M}'$ is prolonged into a smoothly parametrised family of fibred isomorphisms $\mathcal{B}(\boldsymbol{M}) \to \mathcal{B}(\boldsymbol{M}')$.

The continuity condition allows us to prolong a vector field X on $\boldsymbol{M}$ to the vector field $\mathcal{B}(X)$ on $\mathcal{B}(\boldsymbol{M})$ by the rule, [114, 212],

$$\exp(t\,\mathcal{B}(X)) := \mathcal{B}(\exp(t\,X)).$$

The vector field $\mathcal{B}(X)$ is projectable on X and is called the *flow lift* of X. □

Eventually, we present a few examples of natural bundle functors that we shall be involved with.

Example J.2.14 The tangent functor T, which associates the tangent bundle $\tau_{\boldsymbol{M}} : T\boldsymbol{M} \to \boldsymbol{M}$ with each manifold $\boldsymbol{M} \in \mathscr{M}$ and the tangent map $Tf : T\boldsymbol{M} \to T\boldsymbol{N}$ with each map $f \in \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{N})$, is a 1st order natural bundle functor from the category $\mathscr{M}$ to the category $\mathscr{V}$.

In dimension n, the type fibre of T is $\mathbb{R}^n$ with the associated tensor action of the differential group $G^1_n = \mathrm{Gl}(n, \mathbb{R})$ □

Example J.2.15 The cotangent functor T^*, which associates the cotangent bundle $\tau_{\boldsymbol{M}} : T^*\boldsymbol{M} \to \boldsymbol{M}$ with each manifold $\boldsymbol{M} \in \mathscr{M}[n]$ and the map $(T^* f)^{-1} : T^*\boldsymbol{M} \to T^*\boldsymbol{N}$ with each map $(f : \boldsymbol{M} \to \boldsymbol{N}) \in \mathrm{Mor}(\mathscr{M}[n])$, is a 1st order natural bundle functor from the category $\mathscr{M}[n]$ to the category $\mathscr{V}[n]$.

In dimension n, the type fibre of T^* is $\mathbb{R}^{n*}$ with the associated tensor action of the differential group $G^1_n = \mathrm{Gl}(n, \mathbb{R})$. □

Example J.2.16 The tensor functor $T^{(r,s)} := (\otimes^r T) \otimes (\otimes^s T^*)$ of tensors of type (r, s) is a 1st order natural bundle functor from the category $\mathscr{M}[n]$ to the category $\mathscr{V}[n]$.

In dimension n, the standard fibre of $T^{(r,s)}$ is $(\otimes^r \mathbb{R}^n) \otimes (\otimes^s \mathbb{R}^{*n})$ with the associated tensor action of the differential group $G^1_n = \mathrm{Gl}(n, \mathbb{R})$.

Analogously, we have the natural bundle functor $\wedge^p T^*$ of p-forms. □

Example J.2.17 The functor $\mathcal{G}$ of *metrics* is a 1st order natural vector bundle functor.

Its standard fibre $\boldsymbol{S}_{\mathcal{G}}$ is the subspace in $\odot^2 \mathbb{R}^{n*}$ of non-degenerate symmetric matrices and the associated differential group is $G^1_n = \mathrm{Gl}(n, \mathbb{R})$. □

Example J.2.18 The functor $\mathcal{C}$ of *linear connections* is a 2nd order natural bundle functor from the category $\mathscr{M}[n]$ to the category $\mathscr{A}[n]$.

Its standard fibre is $\mathbb{R}^n \otimes (\otimes^2 \mathbb{R}^{n*})$ with the associated action of the differential group G^2_n given by the transformation rules of the Christoffel symbols.

Analogously, we have the natural bundle functor of torsion free linear connections $\mathcal{C}_\tau$. □

J.2.2 Natural Differential Operators

We discuss the natural differential operators $D_{\boldsymbol{M}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{M})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{M})\big)$ between two natural bundle functors $\mathcal{B}[n]$ and $\mathcal{B}'[n]$.

In a few words, a natural differential operator is a differential operator which fulfills reasonable conditions of naturality, locality and regularity.

A theorem provides a bijection between natural differential operators and equivariant maps between standard fibres. According to this result, in order to classify natural differential operators between two natural bundle functors, it is sufficient to classify equivariant maps between their standard fibres. A very important tool for the classifications of equivariant maps is the *orbit reduction theorem* (see Theorem J.2.28 and [246, 258]).

For each natural bundle functor $\mathcal{B}$, we define the generalised Lie derivative $L_X\sigma$ of sections $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$ and show that this is a distinguished natural differential operator.

Eventually, we provide a few examples of natural differential operators that we shall be involved with (exterior differential, Levi–Civita connection, curvature operator).

Definition J.2.19 Let us consider a natural bundle functor $\mathcal{B} : \mathscr{M}[n] \to \mathscr{B}[n]$.

For each morphism $(f : \boldsymbol{M} \to \boldsymbol{M}') \in \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{M}')$ and each section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$, we define the *pullback section*

$$f^*\sigma := \mathcal{B}(f) \circ \sigma \circ f^{-1} : \boldsymbol{M}' \to \mathcal{B}(\boldsymbol{M}'),$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{M}' & \xrightarrow{\;f^*\sigma\;} & \mathcal{B}(\boldsymbol{M}') \\ {\scriptstyle f^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle \mathcal{B}(f)} \\ \boldsymbol{M} & \xrightarrow{\;\sigma\;} & \mathcal{B}(\boldsymbol{M}) \end{array} . \square$$

Definition J.2.20 Let us consider two natural bundle functors

$$\mathcal{B} : \mathscr{M}[n] \to \mathscr{B}[n] \qquad \text{and} \qquad \mathcal{B}' : \mathscr{M}[n] \to \mathscr{B}[n].$$

A *natural differential operator* D from $\mathcal{B}$ to $\mathcal{B}'$ is defined to be a family of differential operators

$$\Big\{D_{\boldsymbol{M}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{M})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{M})\big)\Big\}_{\boldsymbol{M}\in\mathrm{Obj}(\mathscr{M}[n])},$$

which fulfills the following conditions:

(i) (*Naturality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{M})\big)$ and each $f : \boldsymbol{M} \to \boldsymbol{M}' \in \mathrm{Mor}(\mathscr{M}[n])$,

$$D_{\boldsymbol{M}'}(f^*\sigma) = f^* D_{\boldsymbol{M}}(\sigma),$$

according to the following commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{B}(\boldsymbol{M}) & \xrightarrow{\quad \mathcal{B}(f) \quad} & \mathcal{B}(\boldsymbol{M}') \\
 & & \sigma \Big\uparrow & & \Big\uparrow f^*\sigma \\
\boldsymbol{M}' & \xrightarrow{\quad f^{-1} \quad} & \boldsymbol{M} & \xrightarrow{\quad f \quad} & \boldsymbol{M}' \\
f^* D_{\boldsymbol{M}}(\sigma) \Big\downarrow & & D_{\boldsymbol{M}}(\sigma) \Big\downarrow & & \Big\downarrow D_{\boldsymbol{M}'}(f^*\sigma) \\
\mathcal{B}'(\boldsymbol{M}') & \xleftarrow{\quad \mathcal{B}'(f) \quad} & \mathcal{B}'(\boldsymbol{M}) & \xrightarrow{\quad \mathcal{B}'(f) \quad} & \mathcal{B}'(\boldsymbol{M}') \\
\mathrm{id} \Big\downarrow & & & & \Big\uparrow \mathrm{id} \\
\mathcal{B}'(\boldsymbol{M}') & & \xrightarrow{\qquad\qquad \mathrm{id} \qquad\qquad} & & \mathcal{B}'(\boldsymbol{M}') \quad .
\end{array}$$

(ii) (*Locality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{M})\big)$ and each open subset $\boldsymbol{U} \subset \boldsymbol{M}$,

$$D_{\boldsymbol{U}}(\sigma_{|\boldsymbol{U}}) = (D_{\boldsymbol{M}}\sigma)_{|\boldsymbol{U}}.$$

(iii) (*Regularity*) Every smoothly parametrised family of sections of $\mathcal{B}(\boldsymbol{M})$ is transformed into a smoothly parametrised family of sections of $\mathcal{B}'(\boldsymbol{M})$. □

Remark J.2.21 From the definition of natural differential operators it follows that they are invariant with respect to change of local coordinates, which means that natural operators have the same coordinate expressions in all natural fibred coordinate charts. □

Note J.2.22 A natural differential operator D from $\mathcal{B}[n]$ to $\mathcal{B}'[n]$ is said to be of a *finite order k* if all $D_{\boldsymbol{M}}$, with $\boldsymbol{M} \in \mathrm{Obj}(\mathscr{M}[n])$, are of order k, i.e. if they depend only on k-order jets of sections of $\mathcal{B}(\boldsymbol{M})$.

Thus, a k-order natural differential operator D from $\mathcal{B}[n]$ to $\mathcal{B}'[n]$ is characterised by the family of fibred morphisms over the identity (denoted by the same symbol)

$$\Big\{D_{\boldsymbol{M}} : J_k\mathcal{B}(\boldsymbol{M}) \to \mathcal{B}'(\boldsymbol{M})\Big\}_{\boldsymbol{M} \in \mathrm{Obj}\,\mathscr{M}[n]},$$

such that, for each $f : \boldsymbol{M} \to \boldsymbol{M}' \in \mathrm{Mor}(\mathscr{M}[n])$, the following diagram commutes

$$\begin{array}{ccc}
J_k\mathcal{B}(\boldsymbol{M}) & \xrightarrow{\quad J_k\mathcal{B}(f) \quad} & J_k\mathcal{B}(\boldsymbol{M}') \\
D_{\boldsymbol{M}} \Big\downarrow & & \Big\downarrow D_{\boldsymbol{M}'} \\
\mathcal{B}'(\boldsymbol{M}) & \xrightarrow{\quad \mathcal{B}'(f) \quad} & \mathcal{B}'(\boldsymbol{M}') \quad .
\end{array}$$ □

Eventually, we provide a few examples of natural differential operators that we shall be involved with.

Example J.2.23 We are involved with the following natural differential operators:

- the *exterior differential* d is a first order natural differential operator from the natural bundle functor $\wedge^p T^*$, with $p \geq 0$, to the natural bundle functor $\wedge^{p+1} T^*$,
- the *Levi-Civita connection* is a first order natural differential operator from the natural bundle functor of metrics $\mathcal{G}$ to the natural bundle functor of linear connections $\mathcal{C}$,
- the *curvature operator* is a 1-order natural differential operator from the natural bundle functor of linear connections to the natural bundle functor $T \otimes (\otimes^3 T^*)$.□

The following Theorem J.2.24, which provides a bijection between natural differential operators and equivariant maps between standard fibres, plays an important role in the classification of natural differential operators.

Theorem J.2.24 [Classification of natural operators via standard fibres]
Let $\mathcal{B}$ and $\mathcal{B}'$ be natural bundle functors of order $\leq r$.
Then, we have a bijection

$$D_{\boldsymbol{M}} \mapsto (f : \boldsymbol{S}_{J_k \mathcal{B}} \to \boldsymbol{S}_{\mathcal{B}'})$$

between natural differential operators $D_{\boldsymbol{M}}$ of order k from $\mathcal{B}$ to $\mathcal{B}'$ and G_n^{r+k}-equivariant maps $f : \boldsymbol{S}_{J_k \mathcal{B}} \to \boldsymbol{S}_{\mathcal{B}'}$ from the standard fibre of $J_k \mathcal{B}$ to the standard fibre of $\mathcal{B}'$ (see Note J.3.21 and Definition J.2.7).

Proof. See [246, p. 145] . □

According to the above Theorem J.2.24, in order to classify natural differential operators it is sufficient to classify equivariant maps between their standard fibres.

A very important tool for the classification of equivariant maps is the following *orbit reduction theorem*, [246, 258].

Definition J.2.25 Let us consider a Lie group G. Then, we define:
- a *left G-space* to be a manifold $\boldsymbol{M}$ along with a left action $G \times \boldsymbol{M} \to \boldsymbol{M}$,
- a *G-map* to be a map $f : \boldsymbol{M} \to \boldsymbol{N}$ between two left G-manifolds, such that, for each $g \in G$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\ f\ } & \boldsymbol{N} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ \boldsymbol{M} & \xrightarrow{\ f\ } & \boldsymbol{N} \end{array} ,$$

- a *G-orbit* in the G-space $\boldsymbol{M}$ to be a subset of the type

$$\left\{ g(m) \mid m \in \boldsymbol{M} \right\}_{g \in G} . \quad \square$$

Definition J.2.26 Let us consider a Lie group morphism $p : G \to H$, a left G-space $\boldsymbol{M}$ and a left H-space $\boldsymbol{N}$.

Then, a map $f : \boldsymbol{M} \to \boldsymbol{N}$ is said to be *p-equivariant* if, for each $g \in G$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\ f\ } & \boldsymbol{N} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle p(g)} \\ \boldsymbol{M} & \xrightarrow{\ f\ } & \boldsymbol{N} \end{array} \; . \square$$

Note J.2.27 If $p : G \to H$ is a Lie group epimorphism, then we can consider every left H-space $\boldsymbol{N}$ as a left G-space by means of the equality

$$g\, y = p(g)\, y, \quad \text{for all } y \in \boldsymbol{N}, \quad g \in G. \quad \square$$

Theorem J.2.28 [Orbit reduction theorem] *Let $p : G \to H$ be a Lie group epimorphism with kernel K, $\boldsymbol{M}$ a left G-space, $\boldsymbol{Q}$ a left H-space and $\pi : \boldsymbol{M} \to \boldsymbol{Q}$ a p-equivariant fibring (see Appendix: Definition J.2.26 and Definition A.1.1).*

If, for each $q \in \boldsymbol{Q}$, the fibre $\pi^{-1}(q) \subset \boldsymbol{M}$ turns out to be a K-orbit in $\boldsymbol{M}$, then there is a bijection between the G-maps $f : \boldsymbol{M} \to \boldsymbol{N}$ and the H-maps $\varphi : \boldsymbol{Q} \to \boldsymbol{N}$ given by the equality

$$f = \varphi \circ \pi.$$

Proof. See [246, p. 233] . □

In several cases, application of the above "*orbit reduction theorem*" to natural differential operators allows us to reduce the order of the differential group acting on standard fibres of natural bundles and to factorise natural operators through simpler operators.

For instance, the following Example J.2.29 deals with typical natural operators occurring in our classical framework.

Example J.2.29 [*Higher order Utiyama like interaction for torsion free linear connections*] [204, 246, 397].

Let us consider an n-dimensional manifold $\boldsymbol{M}$ and a natural differential operator $D_{\boldsymbol{M}} : (\Gamma, \Phi) \mapsto \Psi$, of order $(r-1, r)$ which maps torsion free linear connections Γ and tensor fields Φ of $\boldsymbol{M}$ into tensor fields Ψ of $\boldsymbol{M}$.

Then the standard fibres of the corresponding source and target natural bundles are G_n^{r+1}-spaces and G_n^1-space, respectively (see Definition J.2.7).

We can apply the *orbit reduction theorem* with reference to the Lie group epimorphism

$$\pi_1^{r+1} : G_n^{r+1} \to G_n^1.$$

Then, the corresponding G_n^{r+1}-equivariant map can be factorised through the map on the so called "curvature" and "Ricci subspaces", which are the subspaces induced by the Bianchi and Ricci identities (see [246, pp. 235, 240]). As consequence, the operator $D_{\boldsymbol{M}}$ factorises through a zero order natural differential operator $\widetilde{D}_{\boldsymbol{M}}$ on the curvature tensor $R[\Gamma]$, its covariant derivatives up to the order $(r-2)$ and covariant derivatives of Φ up to the order r, i.e.

$$D_{\boldsymbol{M}}\,(j_{r-1}\Gamma,\ j_r\Phi) = \widetilde{D}_{\boldsymbol{M}}\,(\nabla^{(r-2)}R[\Gamma],\ \nabla^{(r)}\Phi), \quad \text{where } \nabla^{(r)} = (\mathrm{id}, \nabla, \ldots, \nabla^r). \quad \square$$

Next we discuss the generalised Lie derivative as a natural differential operator induced by natural bundle functors.

Definition J.2.30 Let $\mathcal{B}$ be a natural bundle functor of order r and $X \in \sec(\boldsymbol{M}, T\boldsymbol{M})$ be a vector field of a manifold $\boldsymbol{M}$.

Then, we define the *generalised Lie derivative* of a section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$ by the rule (see [246, p. 376])

$$L_X\sigma := T\sigma \circ X - \mathcal{B}(X) \circ \sigma : \boldsymbol{M} \to V\mathcal{B}(\boldsymbol{M}),$$

where $V\mathcal{B}(\boldsymbol{M})$ is the vertical tangent bundle of $\mathcal{B}(\boldsymbol{M})$. □

Proposition J.2.31 *Let $\mathcal{B}$ be a natural bundle functor of order r and*

$$X \in \sec(\boldsymbol{M}, T\boldsymbol{M})$$

be a vector field of a manifold $\boldsymbol{M}$.

Then, the generalised Lie derivative is a natural operator from $T \times \mathcal{B}$ to $V\mathcal{B}$, of order r with respect to T and of order 1 with respect to $\mathcal{B}$.

Proof. See [246, p. 376]. □

Proposition J.2.32 *Let $\mathcal{B}$ and $\mathcal{B}'$ be natural bundle functors and $D : \mathcal{B} \to \mathcal{B}'$ a natural differential operator.*

Then, for each vector field $X : \boldsymbol{M} \to T\boldsymbol{M}$ and each section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$, the following commutation rule holds

$$L_X\big(D_{\boldsymbol{M}}(\sigma)\big) = V D_{\boldsymbol{M}}(L_X\sigma).$$

Proof. See [212]. □

J.3 Gauge Natural Bundles and Operators

We introduce the notions of gauge natural bundle functor and gauge natural differential operator $\mathcal{B} : \mathscr{P} \to \mathscr{B}$ and $D_{\boldsymbol{P}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big) \to \sec\big((\boldsymbol{M}, \mathcal{B}'(\boldsymbol{P})\big)$ by following a scheme analogous to that of the above Sect. J.2.1, which deals with natural bundles $\mathcal{B} : \mathscr{C} \subset \mathscr{M} \to \mathscr{B}$ and natural differential operators $D_{\boldsymbol{M}} : \sec\big(\boldsymbol{C}, \mathcal{B}(\boldsymbol{M})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{M})\big)$.

Thus, we replace a generic manifold $\boldsymbol{M}$ with a principal bundle $\boldsymbol{P}$ (see, for instance, [110, 120, 244, 246]).

J.3.1 Gauge Natural Bundle Functors

We discuss the notion of *gauge natural bundle functor*.

In simple words, a gauge natural bundle functor $\mathcal{B} \equiv \mathcal{B}[G, n]$ maps objects and morphisms of the category of principal bundles $\mathscr{P} \equiv \mathscr{P}[G, n]$ into objects and morphisms of the category of bundles $\mathscr{B} \equiv \mathscr{B}[n]$, fulfilling the conditions of functors (see Definition J.1.4) and additional reasonable conditions of *base space preservation*, *base map preservation* and *locality*.

We discuss the distinguished gauge natural bundle functor $\mathcal{W}^r_n$, which maps each principal bundle $\boldsymbol{P}$, with structure group G and n-dimensional base manifold, to the principal bundle $\mathcal{W}^r_n(\boldsymbol{P}) := J_{r\,(0,e)}(\mathrm{I\!R}^n \times G,\ \boldsymbol{P})$, consisting of r-jets of principal isomorphisms, whose structure group is $\mathcal{W}^r_n[G] := J_{r\,(0,e)}(\mathrm{I\!R}^n \times G,\ \mathrm{I\!R}^n \times G)_0$ (see Definition J.3.2 and Proposition J.3.3).

All gauge natural bundles $\mathcal{B}(\boldsymbol{P})$ induced by a gauge natural bundle functor $\mathcal{B}[G, n]$ turn out to have the same type fibre $\boldsymbol{S}_{\mathcal{B}} := \mathcal{B}_0(\mathrm{I\!R}^n \times G)$, where $\mathrm{I\!R}^n \times G \to \mathrm{I\!R}^n$ is a canonical principal bundle (see Proposition J.3.4).

Moreover, all gauge natural bundles $\mathcal{B}(\boldsymbol{P})$ turn out to have finite order r, i.e. can be factorised through r-order jets and can be regarded as bundles associated with the principal bundle $\mathcal{W}^r_n(\boldsymbol{P})$ and the type fibre $\boldsymbol{S}_{\mathcal{B}}$ (see Theorem J.3.7 and Theorem J.3.8).

We define the gauge order (r, s) of gauge natural bundle functors and show that the natural bundle functors can be regarded as $(r, 0)$ gauge natural bundle functors (see Note J.3.10 and Note J.3.11).

Indeed, each group G and G-left space $\boldsymbol{S}$ yield the gauge natural bundle functor which maps every principal bundle $\boldsymbol{P}$ with structure group G to the associated bundle with type fibre $\boldsymbol{S}$ (see Proposition J.3.12).

Furthermore, we discuss the jet prolongation of natural bundle functors (see Note J.3.13).

Here, we present simple typical definitions; however, in practice, we might deal with more complex cases, which require more elaborate concepts. We leave to the reader the simple task to achieve such generalised concepts, as needed case by case.

Eventually, we mention the principal connections functor as a distinguished example of gauge natural bundle functors (see Example J.3.15).

Here, for the sake of simplicity, we define directly the gauge natural bundle functors valued in the category of bundles. However, in the literature the gauge natural bundle functors are often defined, more generally, with values in the category of fibred manifolds; then, it can be proved that these fibred manifolds turn out to be always bundles (see, for instance, [110, 246]).

Definition J.3.1 A *gauge natural bundle functor* is defined to be a covariant functor $\mathcal{B}$ from the category $\mathscr{P}[G]$ of principal bundles with structure group G to the category $\mathscr{B}$ of bundles $\mathcal{B} : \mathscr{P}[G] \to \mathscr{B}$, i.e. a pair of maps

$$\mathcal{B} := \Big(\text{Obj}(\mathscr{P}[G] \to \text{Obj}(\mathscr{B}), \quad \text{Mor}(\mathscr{P}[G]) \to \text{Mor}(\mathscr{B}) \Big)$$

satisfying the properties (1), (2) and (3) of Definition J.1.4, which fulfills the following additional properties:

(a) (*Base space preservation*) Each principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$, over a manifold $\boldsymbol{M}$, is mapped to a bundle *over* $\boldsymbol{M}$

$$p : \mathcal{B}(\boldsymbol{P}) \to \boldsymbol{M}.$$

(b) (*Base map preservation*) Each principal morphism $f : \boldsymbol{P} \to \boldsymbol{P}'$ over $\underline{f} : \boldsymbol{M} \to \boldsymbol{M}'$ is mapped to a fibred morphism *over* $\underline{f} : \boldsymbol{M} \to \boldsymbol{M}'$

$$\mathcal{B}(f) : \mathcal{B}(\boldsymbol{P}) \to \mathcal{B}(\boldsymbol{P}'),$$

according to the commutative diagrams

$$\begin{array}{ccc} \boldsymbol{P} & \xrightarrow{\;f\;} & \boldsymbol{P}' \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi'} \\ \boldsymbol{M} & \xrightarrow{\;\underline{f}\;} & \boldsymbol{M}' \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{B}(\boldsymbol{P}) & \xrightarrow{\;\mathcal{B}(f)\;} & \mathcal{B}(\boldsymbol{P}') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ \boldsymbol{M} & \xrightarrow{\;\underline{f}\;} & \boldsymbol{M}' \end{array} .$$

(c) (*Locality*) For each open subset $\boldsymbol{U} \subset \boldsymbol{M}$, the inclusion $\iota : \pi^{-1}(\boldsymbol{U}) \hookrightarrow \boldsymbol{P}$ is transformed into the inclusion $\mathcal{B}(\iota) : p^{-1}(\boldsymbol{U}) \hookrightarrow \mathcal{B}(\boldsymbol{P})$.

Given a gauge natural bundle functor $\mathcal{B}$, the 3-plet

$$\big(\mathcal{B}(\boldsymbol{P}),\ p,\ \boldsymbol{M}\big), \quad \text{for each } \boldsymbol{P}[G] \in \text{Obj}(\mathscr{P}[G]),$$

is said to be a *gauge natural bundle*. □

Next, we introduce the gauge natural bundle functor $\mathcal{W}^r_n$, which plays a fundamental role in the theory of gauge natural bundles (see, for instance, [110, 246]).

Definition J.3.2 Let us consider a principal bundle $(\pi : \boldsymbol{P} \to \boldsymbol{M}) \in \text{Obj}(\mathscr{P}[G, n])$ and the trivial principal bundle $\mathbb{R}^n \times G \to \mathbb{R}^n$, with the same structure group.

Then, we define the *r-jet lift of* $\boldsymbol{P}$ to be the bundle over $\boldsymbol{M}$

$$\mathcal{W}^r_n(\boldsymbol{P}) := J_{r\,(0,e)}(\mathbb{R}^n \times G,\ \boldsymbol{P})$$

consisting of r-jets, at $0 \in \mathbb{R}^n$, of principal isomorphisms $\mathbb{R}^n \times G \to \boldsymbol{P}$ over base diffeomorphisms.

Moreover, for each principal fibred morphism $(f : \boldsymbol{P} \to \boldsymbol{P}') \in \mathrm{Mor}(\mathscr{P}[G, n])$, we define the principal fibred morphism

$$\mathcal{W}^r_n(f) : \mathcal{W}^r_n(\boldsymbol{P}) \to \mathcal{W}^r_n(\boldsymbol{P}')$$

by means of the jet composition

$$\mathcal{W}^r_n(f)(j_{r\,(0,e)}\alpha) := j_{r\,(0,e)}(f \circ \alpha), \quad \text{where } j_{r\,(0,e)}\alpha \in \mathcal{W}^r_n(\boldsymbol{P}). \quad \square$$

Proposition J.3.3 *Let us consider a principal bundle* $(\pi : \boldsymbol{P} \to \boldsymbol{M}) \in \mathrm{Obj}(\mathscr{P}[G, n])$ *and the trivial principal bundle* $\mathbb{R}^n \times G \to \mathbb{R}^n$, *with the same structure group.*

Then, the bundle $\mathcal{W}^r_n(\boldsymbol{P})$ *turns out to be a principal bundle over* $\boldsymbol{M}$, *with the structure group*

$$\mathcal{W}^r_n[G] := J_{r\,(0,e)}(\mathbb{R}^n \times G,\ \mathbb{R}^n \times G)_0$$

consisting of r-jets, at $0 \in \mathbb{R}^n$, *of principal isomorphisms* $(\mathbb{R}^n \times G) \to (\mathbb{R}^n \times G)$ *over base diffeomorphisms, which preserve* $0 \in \mathbb{R}^n$.

Indeed, the above group $\mathcal{W}^r_n[G]$ *turns out to be the semi-direct product*

$$\mathcal{W}^r_n[G] = G^r_n \rtimes T^r_n G,$$

where (see Definition J.2.7 and Proposition J.2.12)

$$G^r_n := \mathrm{inv}\, J_{r\,0}(\mathbb{R}^n, \mathbb{R}^n)_0 \qquad \textit{and} \qquad T^r_n G := J_{r\,0}(\mathbb{R}^n, G)$$

and where the semi-direct product is taken with respect to the action of G^r_n *on* $T^r_n G$ *given by the jet composition.*

Indeed, the pair of maps

$$\Big(\mathcal{W}^r_n : \boldsymbol{P} \mapsto \mathcal{W}^r_n(\boldsymbol{P}),\ \ \mathcal{W}^r_n : f \mapsto \mathcal{W}^r_n(f)\Big)$$

turns out to be a gauge natural bundle functor from the category $\mathscr{P}[G, n]$ *to the category* $\mathscr{P}\big[\mathcal{W}^r_n[G], n\big]$. $\square$

Proposition J.3.4 *Let us consider a gauge natural bundle functor* $\mathcal{B}$ *from the category* $\mathscr{P}[G, n]$ *of principal bundles with structure group G and n-dimensional base manifold to the category* $\mathscr{B}[n]$ *of bundles with n-dimensional base manifold.*

Moreover, let us consider the trivial principal bundle $\mathbb{R}^n \times G \to \mathbb{R}^n$ *and the fibre* $\boldsymbol{S}_{\mathcal{B}} := \mathcal{B}_0(\mathbb{R}^n \times G)$, *over* $0 \in \mathbb{R}^n$, *of the induced bundle* $\mathcal{B}(\mathbb{R}^n \times G) \to \mathbb{R}^n$.

Then, for each principal bundle $\boldsymbol{P}$, *with structure group G and n-dimensional base space, the induced bundle* $p : \mathcal{B}(\boldsymbol{P}) \to \boldsymbol{M}$ *turns out to have the type fibre* $\boldsymbol{S}_{\mathcal{B}}$.

Proof. See [246, p. 396]. $\square$

Thus, all bundles $p : \mathcal{B}(\boldsymbol{P}) \to \boldsymbol{M}$ induced by the gauge natural bundle functor $\mathcal{B}$ have the same type fibre $\mathcal{S}_{\boldsymbol{B}}$.

Next, we discuss the order of natural bundle functors.

Lemma J.3.5 *Let us consider two principal bundles* $\boldsymbol{P}, \boldsymbol{P}' \in \mathrm{Obj}(\mathscr{P}[G, n])$ *and to principal fibred morphisms* $f, g : \boldsymbol{P} \to \boldsymbol{P}'$.

For each integer r, *if* $j_{ry} f = j_{ry} g$, *for a certain* $y \in \boldsymbol{P}_x$, *then we have* $j_{rz} f = j_{rz} g$, *for any* $z \in \boldsymbol{P}_x$.

Hence, for each $x \in \boldsymbol{M}$ *the* r*-jet* $j_{rx} f$ *at* $x \in \boldsymbol{M}$, *of a principal fibred morphism* $f : \boldsymbol{P} \to \boldsymbol{P}'$ *is well defined.*

Proof. The proof follows by taking into account the right translation of principal bundles. □

Definition J.3.6 A gauge natural bundle functor $\mathcal{B}$ is said to be of *finite order* r, with $0 \leq r < \infty$, if r is the minimal integer such that, for all principal fibred morphisms $f, g : \boldsymbol{P} \to \boldsymbol{P}'$ and every point $x \in \boldsymbol{M}$, the equality $j_{rx} f = j_{rx} g$ implies

$$\mathcal{B}(f)_{|\mathcal{B}_x(\boldsymbol{P})} = \mathcal{B}(g)_{|\mathcal{B}_x(\boldsymbol{P})}. \quad \square$$

Theorem J.3.7 *All gauge natural bundle functors* $\mathcal{B}$ *defined on the category* $\mathscr{P}[G, n]$ *have finite order.*

Proof. See [246, p. 396]. □

Theorem J.3.8 *Each* r*-order gauge natural bundle* $\mathcal{B}(\boldsymbol{P})$ *from* $\mathscr{P}[G, n]$ *to* $\mathscr{B}[n]$ *is a bundle associated with the gauge natural principal bundle* $\mathcal{W}^r_n(\boldsymbol{P})$, *with the type fibre* $\boldsymbol{S}_{\mathcal{B}}$. *More precisely, we have*

$$\mathcal{B}(\boldsymbol{P}) = (\mathcal{W}^r_n(\boldsymbol{P}) \times \boldsymbol{S}_{\mathcal{B}})/\mathcal{W}^r_n(G),$$

where the quotient is taken with respect to the left action of the group $\mathcal{W}^r_n[G]$ *on* $\boldsymbol{S}_{\mathcal{B}}$ *given by (see Definition J.3.2)*

$$(j_{r\,(0,e)}\alpha)(y) = \mathcal{B}(\alpha)(y), \quad \textit{for each } j_{r\,(0,e)}\alpha \in \mathcal{W}^r(\boldsymbol{P}), \;\; y \in \boldsymbol{S}_{\mathcal{B}}.$$

Moreover, for each $f \in \mathrm{Mor}(\mathscr{P}[G, n])$ *we have*

$$\mathcal{B}(f) = (\mathcal{W}^r_n(f) \times \mathrm{id}_{\boldsymbol{S}_{\mathcal{B}}})/\mathcal{W}^r_n(G).$$

Proof. See [110, 246]. □

Proposition J.3.9 *Let us consider a gauge natural bundle* $\mathcal{B}[G]$ *and a principal bundle* $\boldsymbol{P}$ *with structure group* G.

A local fibred chart (x^λ, z^A) *of* $\boldsymbol{P}$ *and a chart* (y^i) *of the type fibre* $\boldsymbol{S}_{\mathcal{B}}$ *yield a fibred chart* (x^λ, y^i) *of* $\mathcal{B}(\boldsymbol{P})$, *which is said to be the* (gauge) natural fibred chart. □

Note J.3.10 Let us consider a gauge natural bundle $\mathcal{B}[G]$ and a principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$ with structure group G and n-dimensional base space.

In the above Theorem J.3.8, we have seen that the bundle $\mathcal{B}(\boldsymbol{B})$ is associated with the principal bundle $\mathcal{W}^r_n(\boldsymbol{P})$, with structure group $\mathcal{W}^r_n[G] := G^r_n \rtimes T^r_n G$, for a certain integer r. But, it might happen that, for an integer $s \le r$, the action of the group $\mathcal{W}^r_n[G]$ on the standard fibre $\boldsymbol{S}_{\mathcal{B}}$ factorise through the canonical projection $\pi^r_s : T^r_n G \to T^s_n G$ via the commutative diagram

$$\begin{array}{ccc} (G^r_n \rtimes T^r_n G) \times \boldsymbol{S}_{\mathcal{B}} & \longrightarrow & \boldsymbol{S}_{\mathcal{B}} \\ \downarrow & & \downarrow \\ (G^r_n \rtimes T^s_n G) \times \boldsymbol{S}_{\mathcal{B}} & \longrightarrow & \boldsymbol{S}_{\mathcal{B}} \end{array} .$$

Then, we define the *gauge order* of $\mathcal{B}$ to be the pair (r, s), where r is the order of $\mathcal{W}^r_n(\boldsymbol{B})$ and s is the minimal integer $s \le r$, fulfilling the above condition. □

Note J.3.11 The gauge natural bundle functor associated with the trivial group $G = \{e\}$ reproduces the natural bundles on the subcategories $\mathscr{C}$ of manifolds $\mathscr{M}$.

Thus, each r-order natural bundle functor can be considered as an $(r, 0)$-order gauge natural bundle functor. □

Each G-manifold $\boldsymbol{S}$ yields the distinguished gauge natural bundle, which maps every principal bundle $\boldsymbol{P}$, with structure group G, to the associated bundle with type fibre $\boldsymbol{S}$.

Proposition J.3.12 *Let us consider a Lie group G and a left G-manifold $\boldsymbol{S}$.*

Then, we obtain the 0*-order* gauge natural associated bundle functor $\mathcal{B}[\boldsymbol{S}]$, *which maps*

- *each principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$, with structure group G, to the associated quotient bundle over $\boldsymbol{M}$ with type fibre $\boldsymbol{S}$*

$$\mathcal{B}[\boldsymbol{S}](\boldsymbol{P}) := (\boldsymbol{P} \times \boldsymbol{S})/G,$$

- *each principal fibred morphism $f : \boldsymbol{P} \to \boldsymbol{P}'$ to the quotient fibred morphism*

$$\mathcal{B}[\boldsymbol{S}](f) := (f \times \mathrm{id}_{\boldsymbol{S}})/G : \mathcal{B}(\boldsymbol{P}) \to \mathcal{B}(\boldsymbol{P}').$$

In particular, if $G = \mathrm{Gl}(n, \mathbb{R})$ and $\boldsymbol{S} = \mathbb{R}^n$, then the above gauge natural bundle $\mathcal{B}$ yields vector bundles $\mathcal{B}[\boldsymbol{S}](\boldsymbol{P})$ with n-dimensional fibres and linear fibred morphisms of vector bundles. □

Note J.3.13 If $\mathcal{B}$ is a gauge natural bundle functor of order (r, s), then $J_k\mathcal{B}$ turns out to be a gauge natural bundle functor of order at most

$$(r', s'), \quad \text{with } r' = r + k, \quad s' \le s + k.$$

For instance, if $\mathcal{B}$ is an r-order natural bundle functor, i.e. an $(r, 0)$-order gauge natural bundle functor, then $J_k\mathcal{B}$ turns out to be an $(r+k)$-order natural bundle functor, i.e. an $(r+k, 0)$-order gauge natural bundle functor. □

Note J.3.14 Gauge natural bundle functors $\mathcal{B}$ satisfy the *continuity condition* (see [246]), which says that a smoothly parametrised family of principal fibred isomorphisms $\boldsymbol{P} \to \boldsymbol{P}'$ is prolonged into a smoothly parametrised family of fibred isomorphisms $\mathcal{B}(\boldsymbol{P}) \to \mathcal{B}(\boldsymbol{P}')$.

The continuity condition allows us to transform a G-invariant vector field Ξ of a principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$ projectable on the vector field X of $\boldsymbol{M}$ to the vector field $\mathcal{B}(\Xi)$ of $\mathcal{B}(\boldsymbol{P})$ by the rule, [212],

$$\exp(t\,\mathcal{B}(\Xi)) := \mathcal{B}(\exp(t\,\Xi)).$$

The vector field $\mathcal{B}(\Xi)$ is projectable on X and is called the *flow transformation* of Ξ. □

The following example presents a distinguished gauge natural bundle.

Example J.3.15 The functor $\mathcal{C}$ of principal connections is a gauge natural bundle functor of order $(1, 1)$. □

Example J.3.16 Let us consider the subcategory $\mathscr{P}'[U(1,\mathbb{C})] \subset \mathscr{P}[G]$ of $U(1,\mathbb{C})$-principal bundles over galilean spacetimes and the category of quantum bundles $\mathscr{Q}$ over galilean spacetimes (see Example J.1.3).

Then, we have the "quantum" gauge natural bundle functor $\mathcal{Q}$, which maps every $U(1,\mathbb{C})$-principal bundle $\boldsymbol{P} \to \boldsymbol{E}$ to the associated quantum bundle $\boldsymbol{Q} \to \boldsymbol{E}$, by means of the quotient

$$\boldsymbol{Q} := (\boldsymbol{P} \times \mathbb{C})/U(1,\mathbb{C}). \quad \square$$

J.3.2 Natural Operators of Gauge Natural Bundles

We discuss the theory of *natural differential operators* and of *gauge natural differential operators* $D_{\boldsymbol{P}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{P})\big)$ between two natural bundle functors $\mathcal{B} : \mathscr{P} \to \mathscr{B}$ and $\mathcal{B}' : \mathscr{P} \to \mathscr{B}$.

In a few words, a natural and gauge natural differential operator are differential operators which fulfill reasonable conditions of naturality, locality and regularity; in the 1st case we are involved with generic diffeomorphisms of the base space, while in the 2nd case we are involved with the identity diffeomorphism of the base space.

A theorem provides a bijection between natural differential operators of order k and equivariant maps between the standard fibre of $J_k\mathcal{B}$ and the standard fibre of $\mathcal{B}'$. According to this result, in order to classify natural differential operators between two gauge natural bundle functors, it is sufficient to classify equivariant maps between the above standard fibres. Very important tool in classifications of equivariant maps is the *orbit reduction theorem* (see Theorem J.2.28 and [246, 258]) and the *homogeneous function theorem* (see Theorem J.3.26 and [246]).

For each gauge natural bundle functor $\mathcal{B}$, we define the generalised Lie derivative $L_\Xi \sigma$ of sections $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$, with respect to a G-invariant vector field Ξ of $\boldsymbol{P}$, and show that this is a distinguished natural differential operator. The natural differential operators commute with the generalised Lie derivative (see Proposition J.3.30).

Eventually, we present the curvature of principal connections as a distinguished example of gauge natural differential operators.

In the present book we deal with many "gauge independent" objects of Covariant Quantum Mechanics; by using the language of this section, it means that these objects are derived by means of a gauge natural differential operator.

Definition J.3.17 Let us consider a gauge natural bundle functor

$$\mathcal{B} \equiv \mathcal{B}[n] : \mathscr{P}[G, n] \to \mathscr{B}[n].$$

For each principal fibred isomorphism $(f : \boldsymbol{P} \to \boldsymbol{P}') \in \mathrm{Mor}(\mathscr{P}[G, n])$ over the base diffeomorphism $\underline{f} : \boldsymbol{M} \to \boldsymbol{M}'$, and each section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{P})$, we define the *pullback section*

$$f^*\sigma := \mathcal{B}(f) \circ \sigma \circ \underline{f}^{-1} : \boldsymbol{M}' \to \mathcal{B}(\boldsymbol{P}'),$$

according to the following commutative diagram

$$\begin{array}{ccc} \boldsymbol{M}' & \xrightarrow{\;f^*\sigma\;} & \mathcal{B}(\boldsymbol{P}') \\ {\scriptstyle \underline{f}^{-1}}\downarrow & & \uparrow{\scriptstyle \mathcal{B}(f)} \\ \boldsymbol{M} & \xrightarrow{\;\sigma\;} & \mathcal{B}(\boldsymbol{P}) \end{array} \;. \square$$

Let us consider two similar types of differential operators, according to the following Definitions.

Definition J.3.18 Let us consider two gauge natural bundle functors

$$\mathcal{B} \equiv \mathcal{B}[n] : \mathscr{P}[G, n] \to \mathscr{B}[n] \qquad \text{and} \qquad \mathcal{B}' \equiv \mathcal{B}'[n] : \mathscr{P}[G, n] \to \mathscr{B}[n].$$

A *natural differential operator* D from $\mathcal{B}$ to $\mathcal{B}'$ is defined to be a family of differential operators

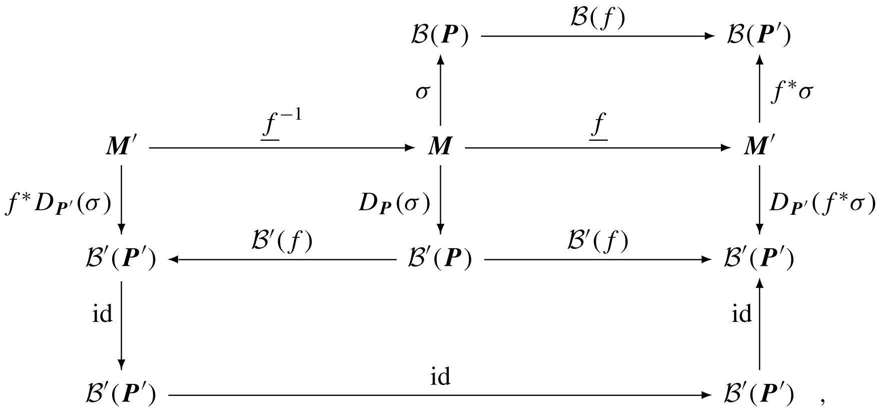
$$\Big\{ D_{\boldsymbol{P}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{P})\big) \Big\}_{(\pi : \boldsymbol{P} \to \boldsymbol{M}) \in \mathrm{Obj}(\mathscr{P}[G, n])},$$

which fulfill the following conditions:

(i) (*Naturality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big)$ and each principal fibred isomorphism $f : \boldsymbol{P} \to \boldsymbol{P}'$, over a base diffeomorphism $\underline{f} : \boldsymbol{M} \to \boldsymbol{M}'$,

$$D_{\boldsymbol{P}'}(f^*\sigma) = f^* D_{\boldsymbol{P}}(\sigma),$$

according to the following commutative diagram



(ii) (*Locality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big)$ and each open subset $\boldsymbol{U} \subset \boldsymbol{M}$,

$$D_{\pi^{-1}(\boldsymbol{U})}(\sigma_{|\boldsymbol{U}}) = (D_{\boldsymbol{P}}\sigma)_{|\boldsymbol{U}}.$$

(iii) (*Regularity*) Every smoothly parametrised family of sections of $\mathcal{B}(\boldsymbol{P})$ is transformed into a smoothly parametrised family of sections of $\mathcal{B}'(\boldsymbol{P})$. □

Definition J.3.19 Let us consider two gauge natural bundle functors

$$\mathcal{B} : \mathscr{P}[\boldsymbol{M}] \to \mathcal{P}[\boldsymbol{M}] \qquad \text{and} \qquad \mathcal{B}' : \mathscr{P}[\boldsymbol{M}] \to \mathcal{P}[\boldsymbol{M}],$$

where $\mathscr{P}[\boldsymbol{M}]$ is the category of principal bundles with a given base space $\boldsymbol{M}$.

A *gauge natural differential operator* D from $\mathcal{B}$ to $\mathcal{B}'$ is defined to be a family of differential operators

$$\Big\{D_{\boldsymbol{P}} : \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big) \to \sec\big(\boldsymbol{M}, \mathcal{B}'(\boldsymbol{P})\big)\Big\}_{(\pi:\boldsymbol{P}\to\boldsymbol{M})\in\mathrm{Obj}(\mathscr{P}[n])},$$

which fulfill the following conditions:

(i) (*Naturality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big)$ and each principal fibred isomorphism $f : \boldsymbol{P} \to \boldsymbol{P}'$, over the identity $\mathrm{id} : \boldsymbol{M} \to \boldsymbol{M}$,

$$D_{\boldsymbol{P}'}(f^*\sigma) = f^* D_{\boldsymbol{P}}(\sigma),$$

according to the following commutative diagram

$$
\begin{array}{ccccc}
 & & \mathcal{B}(\boldsymbol{P}) & \xrightarrow{\mathcal{B}(f)} & \mathcal{B}(\boldsymbol{P}') \\
 & & \uparrow{\scriptstyle \sigma} & & \uparrow{\scriptstyle f^*\sigma} \\
\boldsymbol{M} & \xrightarrow{\mathrm{id}_{\boldsymbol{M}}} & \boldsymbol{M} & \xrightarrow{\mathrm{id}_{\boldsymbol{M}}} & \boldsymbol{M} \\
{\scriptstyle f^*D_{\boldsymbol{P}}(\sigma)}\downarrow & & {\scriptstyle D_{\boldsymbol{P}}(\sigma)}\downarrow & & \downarrow{\scriptstyle D_{\boldsymbol{P}'}(f^*\sigma)} \\
\mathcal{B}'(\boldsymbol{P}') & \xleftarrow{\mathcal{B}'(f)} & \mathcal{B}'(\boldsymbol{P}) & \xrightarrow{\mathcal{B}'(f)} & \mathcal{B}'(\boldsymbol{P}') \\
{\scriptstyle \mathrm{id}}\downarrow & & & & \uparrow{\scriptstyle \mathrm{id}} \\
\mathcal{B}'(\boldsymbol{P}') & & \xrightarrow{\mathrm{id}} & & \mathcal{B}'(\boldsymbol{P}')
\end{array} \quad ,
$$

(ii) (*Locality*) For each $\sigma \in \sec\big(\boldsymbol{M}, \mathcal{B}(\boldsymbol{P})\big)$ and each open subset $\boldsymbol{U} \subset \boldsymbol{M}$,

$$D_{\pi^{-1}(\boldsymbol{U})}(\sigma_{|\boldsymbol{U}}) = (D_{\boldsymbol{P}}\sigma)_{|\boldsymbol{U}}.$$

(iii) (*Regularity*) Every smoothly parametrised family of sections of $\mathcal{B}(\boldsymbol{P})$ is transformed into a smoothly parametrised family of sections of $\mathcal{B}'(\boldsymbol{P})$. □

Remark J.3.20 A gauge natural differential operator, in the sense of the above Definition J.3.19, is not a natural differential operator, in the sense of Definition J.3.18.

In fact, in Definition J.3.19, we require that the base space $\boldsymbol{M}$ is fixed and the base diffeomorphism is the identity $\mathrm{id}_{\boldsymbol{M}}$. □

Note J.3.21 A natural differential operator D from $\mathcal{B}[n]$ to $\mathcal{B}'[n]$ is said to be of a *finite order* k if all $D_{\boldsymbol{P}}$, with $(\pi : \boldsymbol{P} \to \boldsymbol{M}) \in \mathrm{Obj}(\mathscr{P}[G, n])$, are of order k, i.e. if they depend only on k-order jets of sections of $\mathcal{B}(\boldsymbol{P})$.

Thus, a k-order natural differential operator from $\mathcal{B}$ to $\mathcal{B}'$ is characterised by the family of fibred morphisms over the identity

$$\Big\{D_{\boldsymbol{P}} : J_k\mathcal{B}(\boldsymbol{P}) \to \mathcal{B}'(\boldsymbol{P})\Big\}_{\boldsymbol{P}\in\mathrm{Obj}(\mathscr{P}[G,n])},$$

over the identity of $\boldsymbol{M}$, such that, for each principal fibred morphism $f : \boldsymbol{P} \to \boldsymbol{P}'$, the following diagram commutes

$$
\begin{array}{ccc}
J_k\mathcal{B}(\boldsymbol{P}) & \xrightarrow{J_k\mathcal{B}(f)} & J_k\mathcal{B}(\boldsymbol{P}') \\
{\scriptstyle D_{\boldsymbol{P}}}\downarrow & & \downarrow{\scriptstyle D_{\boldsymbol{P}'}} \\
\mathcal{B}'(\boldsymbol{P}) & \xrightarrow{\mathcal{B}'(f)} & \mathcal{B}'(\boldsymbol{P}')
\end{array} \quad .
$$

If the operator D is gauge natural, then all morphisms in the above diagram are over the identity of $\boldsymbol{M}$. □

Remark J.3.22 From the definition of natural differential operators it follows that they are invariant with respect to change of natural charts.

Gauge natural differential operators are invariant with respect to change of natural fibred charts with respect to the fibres.

So, natural operators of gauge natural bundles have the same coordinate expressions in all natural fibred charts. □

The curvature of principal connections provides a distinguished example of natural differential operators.

Example J.3.23 The curvature operator of principal connections of a principal bundle $\boldsymbol{P}[G]$ is a 1st-order natural differential operator from the gauge natural bundle functor of principal connections $\mathcal{C}$ to the gauge natural bundle functor of curvature type

$$\mathcal{R} : \mathcal{C} \to (\Lambda^2 T^*) \otimes \mathcal{B}[\mathfrak{g}],$$

where $\mathcal{B}[\mathfrak{g}]$ is the gauge natural bundle functor associated with the type fibre $\boldsymbol{S}$ given by the Lie algebra $\mathfrak{g}$, with the adjoint action (see Proposition J.3.12). □

The following fundamental Theorem J.3.24, shows that in order to classify natural or gauge natural differential operators it is sufficient to classify equivariant maps between their standard fibres.

Theorem J.3.24 [natural operators and standard fibres]

Let $\mathcal{B}$ and $\mathcal{B}'$ be gauge natural bundle functors of order $\leq r$.

Then, we have a bijection between natural differential operators of order k from $\mathcal{B}$ to $\mathcal{B}'$ and $\mathcal{W}_n^{r+k}[G]$-equivariant maps from the standard fibre of $J_k\mathcal{B}$ to the standard fibre of $\mathcal{B}'$.

Proof. See [110]. □

Moreover, a very important tool for the classification of equivariant maps is the "*orbit reduction*" Theorem J.2.28, which allows us to reduce the order of groups acting on standard fibres of gauge natural bundles and factorise natural differential operators of high order through natural differential operators of lower order (see [246, p. 233]).

For instance, the following Example J.3.25 deals with typical natural operators occurring in our quantum framework.

Example J.3.25 Let us consider as gauge group the general linear group $\mathrm{Gl}(n, \mathbb{R})$.

Any vector bundle $p : \boldsymbol{F} \to \boldsymbol{B}$, with n-dimensional fibres and m-dimensional base space, can be considered as a 0-order gauge natural bundle associated with a principal $\mathrm{Gl}(n, \mathbb{R})$-bundle $\pi : \boldsymbol{P} \to \boldsymbol{B}$ (see Proposition J.3.12).

Now, consider a natural differential operator $D_{\boldsymbol{P}}$ transforming torsion free linear connections Γ on the base manifold $\boldsymbol{B}$ (at order $(r-1)$), linear connections K on $\boldsymbol{F}$ (at order $(r-1)$) and sections $\Phi : \boldsymbol{B} \to \boldsymbol{F}$ (at order r) into sections of a gauge natural bundle $\boldsymbol{G} \to \boldsymbol{B}$ of order (1,0).

Then, we have the actions of the groups

$$\mathcal{W}_m^{r+1,r}[\mathrm{Gl}(n,\mathbb{R})] = G_m^{r+1} \rtimes T_m^r \mathrm{Gl}(n,\mathbb{R}) \qquad \text{and} \qquad G_m^1 \times \mathrm{Gl}(n,\mathbb{R})$$

on the source standard fibre and on the target standard fibre, respectively.

We can apply the "*orbit reduction*" Theorem J.2.28 for the Lie group epimorphism

$$\pi_{(1,0)}^{(r+1,r)} : \mathcal{W}_m^{r+1,r}[\mathrm{Gl}(n,\mathbb{R})] \to G_m^1 \times \mathrm{Gl}(n,\mathbb{R}).$$

Then, the corresponding $\mathcal{W}_m^{r+1,r}[\mathrm{Gl}(n,\mathbb{R})]$-equivariant map can be factorised through the map on the so called "curvature" and "Ricci subspaces" (see [203] and [246, pp. 235, 240]).

As a consequence, the natural differential operator $D_{\boldsymbol{P}}$ factorises through a 0-order natural differential operator $\widetilde{D}_{\boldsymbol{P}}$ on the curvature tensors $R[\Gamma]$ and $R[K]$, their covariant derivatives up to the order $(r-2)$ and covariant derivatives of Φ up to the order r.

In other words, we can write

$$D_{\boldsymbol{P}}(j_{r-1}\Gamma,\ j_{r-1}K,\ j_r\Phi) = \widetilde{D}_{\boldsymbol{P}}\,(\nabla^{(r-2)}R[\Gamma],\ \nabla^{(r-2)}R[K],\ \nabla^{(r)}\Phi),$$

where $\nabla^{(r)} := (\mathrm{id}, \nabla, \dots, \nabla^r)$ and where the covariant derivatives of $R[K]$ and Φ are taken with respect to the tensor product connection $\Gamma \otimes K$.

This construction is referred in the literature as the *higher order Utiyama like interaction for classical and general linear connections* [203, 397]. □

Furthermore, in order to classify natural differential operator $\widetilde{D}_{\boldsymbol{P}}$ as in the above Example J.3.25 it is very convenient to use the *homogeneous function theorem.*

Theorem J.3.26 [Homogeneous function theorem]

Let us consider a product $V := V_1 \times \ldots \times V_n$ *of finite dimensional vector spaces and denote their elements by* $x_i \in V_i$, *with* $i = 1, \dots, n$.

Let $f : V \to \mathbb{R}$ *be a smooth function. Suppose that* $a_i > 0$ *and* b *be real numbers such that, for every real number* $k > 0$,

$$(*) \qquad k^b f(x_1, \dots, x_n) = f(k^{a_1} x_1, \dots, k^{a_n} x_n),$$

Then, f *is a sum of the polynomials of degree* d_i *in* x_i *which satisfy the equality*

$$a_1 d_1 + \cdots + a_n d_n = b.$$

If there are no non-negative integers $d_1, \dots, d_n$ *with this property, then* f *is the zero function.*

Moreover, if the function f *satisfies the condition* $*)$ *with* $b < 0$, *then* $f = 0$.

Proof. See [246, p. 213]. □

Example J.3.27 With reference to Example J.3.25, let us consider a natural differential operator $D_{\boldsymbol{P}}(j_1\Gamma,\ j_1K,\ j_2\Phi)$ with values in the $(1,0)$-order gauge natural bundle $\boldsymbol{F} \otimes T^*\boldsymbol{B} \otimes T^*\boldsymbol{B} \to \boldsymbol{B}$.

According to Example J.3.25, this operator factorises through a 0-order operator according to the equality

$$D_{\boldsymbol{P}}(j_1\Gamma,\ j_1K,\ j_2\Phi) = \widetilde{D}_{\boldsymbol{P}}(R[\Gamma],\ R[K],\ \Phi,\ \nabla\Phi,\ \nabla^2\Phi).$$

Now, let us apply the homogeneous function theorem on the $\mathrm{Gl}(n,\mathbb{R}) \times G^1_m$-equivariant maps between standard fibres.

By applying the equivariance with respect to positive multiples of the unit in $\mathrm{Gl}(n,\mathbb{R})$, we get that the operator $\widetilde{D}_{\boldsymbol{P}}$ fulfills the equality

$$k\,\widetilde{D}_{\boldsymbol{P}}(R[\Gamma],\ R[K],\ \Phi,\ \nabla\Phi,\ \nabla^2\Phi) = \widetilde{D}_{\boldsymbol{P}}(R[\Gamma],\ R[K],\ k\,\Phi,\ k\,\nabla\Phi,\ k\,\nabla^2\Phi).$$

Thus, according to the homogeneous function Theorem J.3.26, the operator $\widetilde{D}_{\boldsymbol{P}}$ is a polynomial of degree c_1 in Φ, degree c_2 in $\nabla\Phi$ and degree c_3 in $\nabla^2\Phi$, such that

$$c_1 + c_2 + c_3 = 1.$$

So, the operator $\widetilde{D}_{\boldsymbol{P}}$ is of the form

$$\widetilde{D}_{\boldsymbol{P}}(R[\Gamma],\ R[K],\ \Phi,\ \nabla\Phi,\ \nabla^2\Phi) = A\,(\Phi) + B\,(\nabla\Phi) + C\,(\nabla^2\Phi),$$

where

- $A := A_{\boldsymbol{P}}(R[\Gamma],\ R[K])$ is a $\boldsymbol{F} \otimes \boldsymbol{F}^* \otimes T\boldsymbol{B} \otimes (\otimes^2 T^*\boldsymbol{B})$-valued natural operator,
- $B := B_{\boldsymbol{P}}(R[\Gamma],\ R[K])$ is a $\boldsymbol{F} \otimes \boldsymbol{F}^* \otimes (\otimes T^2\boldsymbol{B}) \otimes (\otimes^2 T^*\boldsymbol{B})$-valued natural operator,
- $C := C_{\boldsymbol{P}}(R[\Gamma],\ R[K])$ is a $\boldsymbol{F} \otimes \boldsymbol{F}^* \otimes (\otimes T^2\boldsymbol{B}) \otimes (\otimes^2 T^*\boldsymbol{B})$-valued natural operator.

Moreover, if we consider the equivariance with respect to positive multiples of the unit in G^1_m, then we get that the operators $A_{\boldsymbol{P}},\ B_{\boldsymbol{P}},\ C_{\boldsymbol{P}}$ satisfy the equalities

$$k^2\,A_{\boldsymbol{P}}(R[\Gamma],\ R[K]) = A_{\boldsymbol{P}}(k^2\,R[\Gamma],\ k^2\,R[K]),$$
$$k\,B_{\boldsymbol{P}}(R[\Gamma],\ R[K]) = B_{\boldsymbol{P}}(k^2\,R[\Gamma],\ k^2\,R[K]),$$
$$C_{\boldsymbol{P}}(R[\Gamma],\ R[K]) = C_{\boldsymbol{P}}(k^2\,R[\Gamma],\ k^2\,R[K]).$$

Furthermore, according to the homogeneous function Theorem J.3.26, the operator $A_{\boldsymbol{P}}$ is a polynomial of degree a in $R[\Gamma]$ and degree b in $R[K]$, with the condition

$$2\,a + 2\,b = 2,$$

and $A_{\boldsymbol{P}}$ is linear in $R[\Gamma]$ and $R[K]$.

Therefore, the corresponding component of the operator $\widetilde{D}_{\boldsymbol{P}}$ has coordinate expression of the form

$$y^i_{\lambda_1\lambda_2} \circ \widetilde{D}_{\boldsymbol{P}} = A^{i\mu_1\mu_2\mu_3}_{\lambda_1\lambda_2 j\lambda_3}\, \Phi^j\, R[\Gamma]_{\mu_1\mu_2}{}^{\lambda_3}{}_{\mu_3} + B^{il\mu_1\mu_2}_{\lambda_1\lambda_2 j_1 j_2}\, \Phi^{j_1}\, R[K]_{\mu_1\mu_2}{}^{j_2}{}_l,$$

where $A^{i\mu_1\mu_2\mu_3}_{\lambda_1\lambda_2 j\lambda_3}$, $B^{il\mu_1\mu_2}_{\lambda_1\lambda_2 j_1 j_2}$ are invariant tensors (see Examples J.4.4 and J.4.8).

Then, we have

$$A^{i\mu_1\mu_2\mu_3}_{\lambda_1\lambda_2 j\lambda_3} = \sum_\sigma a_\sigma\, \delta^i_j\, \delta^{\mu_1}_{\lambda_{\sigma(1)}}\, \delta^{\mu_2}_{\lambda_{\sigma(2)}}\, \delta^{\mu_3}_{\lambda_{\sigma(3)}},$$

where a_σ are real constants and σ runs all permutations of $(1, 2, 3)$.

Similarly, we have

$$B^{il\mu_1\mu_2}_{\lambda_1\lambda_2 j_1 j_2} = \sum_{\rho,\sigma} b_{\rho,\sigma}\, \delta^i_{j_{\rho(1)}}\, \delta^l_{j_{\rho(2)}}\, \delta^{\mu_1}_{\lambda_{\sigma(1)}}\, \delta^{\mu_2}_{\lambda_{\sigma(2)}},$$

where $b_{\rho,\sigma}$ are real constants and ρ, σ run all permutations of $(1, 2)$.

Further, the operator $B_{\boldsymbol{P}}$ is a polynomial of degree a in $R[\Gamma]$ and degree b in $R[K]$, with the condition

$$2\,a + 2\,b = 1.$$

Actually, there is no non negative solution in integers of above equation, hence $B_{\boldsymbol{P}}$ turns out to be the zero operator.

Eventually, the operator $C_{\boldsymbol{P}}$ is a polynomial of degree a in $R[\Gamma]$ and degree b in $R[K]$, with the condition

$$2\,a + 2\,b = 0.$$

There is only zero solution in non negative integers of above equation, hence $C_{\boldsymbol{P}}$ turns out to be a constant and invariant tensor of the type

$$C^{i\mu_1\mu_2}_{\lambda_1\lambda_2 j} = \sum_\sigma c_\sigma\, \delta^i_j\, \delta^{\mu_1}_{\lambda_{\sigma(1)}}\, \delta^{\mu_2}_{\lambda_{\sigma(2)}},$$

where c_σ are real constants and σ runs all permutations of $(1, 2)$.

Hence, in virtue of the properties of the curvature tensors and the 2nd order covariant derivatives of Φ, we get the equality

$$\begin{aligned} &y^i_{\lambda_1\lambda_2} \circ \widetilde{D}_{\boldsymbol{P}} \\ &= a\,\Phi^i\, R[\Gamma]_{\lambda_1\lambda_2}{}^\mu{}_\mu + b\,\Phi^i\, R[\Gamma]_{\mu\lambda_1}{}^\mu{}_{\lambda_2} + c\,\Phi^i\, R[K]_{\lambda_1\lambda_2}{}^j{}_j + d\,\Phi^j\, R[K]_{\lambda_1\lambda_2}{}^i{}_j + e\,\nabla_{\lambda_1\lambda_2}\Phi^i. \end{aligned}$$

Therefore, all natural operators $D_{\boldsymbol{P}}$ are linear combinations with real coefficients of five operators

$$\Phi \otimes C^1_3 R[\Gamma], \quad \Phi \otimes C^1_1 R[\Gamma], \quad \Phi \otimes C^1_3 R[K], \quad R[K](\Phi), \quad \nabla^2\Phi. \quad \square$$

J.3.3 Generalised Lie Derivatives

Next we discuss the generalised Lie derivative as a natural differential operator induced by gauge natural bundle functors.

Definition J.3.28 Let $\mathcal{B}$ be a gauge natural bundle functor of order r and Ξ a G-invariant vector field of a principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$ projectable on the vector field X of $\boldsymbol{M}$.

Then, we define the *generalised Lie derivative* of a section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{P})$ by the rule

$$L_\Xi \sigma := T\sigma \circ X - \mathcal{B}(\Xi) \circ \sigma : \boldsymbol{M} \to V\mathcal{B}(\boldsymbol{P}),$$

where $V\mathcal{B}(\boldsymbol{P})$ is the vertical tangent bundle of $\mathcal{B}(\boldsymbol{P})$. □

Proposition J.3.29 *Let $\mathcal{B}$ be a gauge natural bundle functor of order r and Ξ a G-invariant vector field of a principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$ projectable on the vector field X of $\boldsymbol{M}$.*

Then, the generalised Lie derivative is a natural differential operator from $T \times \mathcal{B}$ to $V\mathcal{B}$, of order r with respect to T and of order 1 with respect to $\mathcal{B}$. □

Proposition J.3.30 *Let $\mathcal{B}$ and $\mathcal{B}'$ be gauge natural bundle functors and $D : \mathcal{B} \to \mathcal{B}'$ be a natural differential operator.*

Then, for each G-invariant vector field Ξ of a principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$ projectable on the vector field X of $\boldsymbol{M}$ and each section $\sigma : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{P})$, the following commutation rule holds

$$L_\Xi(D_{\boldsymbol{P}}(\sigma)) = VD_{\boldsymbol{P}}(L_\Xi \sigma).$$

Proof. See [212]. □

J.4 Naturality and Covariance

We start by defining the concepts of *equivariance*.

Then, we define the concepts of *covariance* and *gauge covariance* in terms of natural and gauge natural bundle functors, via equivariance with respect to arbitrary fibred transformations.

Eventually, we provide simple examples taken from differential geometry and a fundamental example taken from Covariant Quantum Mechanics. Actually, in the present book we deal with a great number of covariant objects.

J.4.1 Equivariant Sections and Morphisms

We discuss the concepts of *equivariant sections* and *equivariant fibred morphism*.

In simple words, a section or a fibred morphism are equivariant if they are not effected by a certain family of fibred morphisms.

Indeed, the concept of equivariance will be an essential tool for the subsequent concept of covariance, which will be discussed in the next section (see Sect. J.4.2).

Definition J.4.1 Let us consider a subcategory $\mathscr{F}' \subset \mathscr{F}$ and a fibred manifold

$$(p : \boldsymbol{F} \to \boldsymbol{B}) \in \mathrm{Obj}(\mathscr{F}').$$

Then, a section $s : \boldsymbol{B} \to \boldsymbol{F}$ is said to be *equivariant* if, for each $\phi \in \mathrm{Mor}(\boldsymbol{F}, \boldsymbol{F})$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{B} & \xrightarrow{\;s\;} & \boldsymbol{F} \\ {\scriptstyle \underline{\phi}}\downarrow & & \downarrow{\scriptstyle \phi} \\ \boldsymbol{B} & \xrightarrow{\;s\;} & \boldsymbol{F} \end{array} \;. \square$$

Definition J.4.2 Let us consider a subcategory $\mathscr{B}' \subset \mathscr{B}$ and two bundles over the same base space

$$(p : \boldsymbol{F} \to \boldsymbol{B}) \in \mathrm{Obj}(\mathscr{B}') \qquad \text{and} \qquad q : \boldsymbol{G} \to \boldsymbol{B} \in \mathrm{Obj}(\mathscr{B}').$$

Then, a fibred morphism $f : \boldsymbol{F} \to \boldsymbol{G}$, over the base diffeomorphism $\underline{f} : \boldsymbol{B} \to \boldsymbol{B}$, is said to be *equivariant* if, for each $\phi \in \mathrm{Mor}(\boldsymbol{F}, \boldsymbol{F})$ and $\psi \in \mathrm{Mor}(\boldsymbol{G}, \boldsymbol{G})$, over the same base diffeomorphism $\underline{\phi} = \underline{\psi} : \boldsymbol{B} \to \boldsymbol{B}$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;f\;} & \boldsymbol{G} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \boldsymbol{F} & \xrightarrow{\;f\;} & \boldsymbol{G} \end{array} \;. \square$$

In practice, if the bundles $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ admit a type fibre (for instance this is true when the base space $\boldsymbol{B}$ is connected), then the classification of equivariant fibred morphisms f can be reduced to the classification of equivariant maps between the type fibres, according to the following Proposition.

Proposition J.4.3 *Let us consider two bundles $p : \boldsymbol{F} \to \boldsymbol{B}$ and $q : \boldsymbol{G} \to \boldsymbol{B}$ with type fibres $\boldsymbol{S_F}$ and $\boldsymbol{S_G}$. Moreover, let us consider two fibred morphisms $f,\ g : \boldsymbol{F} \to \boldsymbol{G}$ over the base diffeomorphism $\underline{f} = \underline{g} : \boldsymbol{B} \to \boldsymbol{B}$.*

Then, the following conditions are equivalent.

(1) For each $\phi \in \mathrm{fib}(\boldsymbol{F}, \boldsymbol{F})$ and $\psi \in \mathrm{fib}(\boldsymbol{G}, \boldsymbol{G})$, over the same base diffeomorphism $\underline{\phi} = \underline{\psi} : \boldsymbol{B} \to \boldsymbol{B}$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{F} & \xrightarrow{\;f\;} & \boldsymbol{G} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \psi} \\ \boldsymbol{F} & \xrightarrow{\;f\;} & \boldsymbol{G} \end{array} \,.$$

(2) For each $b \in \boldsymbol{B}$, the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{S_F} & \xrightarrow{\;f_b\;} & \boldsymbol{S_G} \\ {\scriptstyle \phi_b}\downarrow & & \downarrow{\scriptstyle \psi_b} \\ \boldsymbol{S_F} & \xrightarrow{\;f_b\;} & \boldsymbol{S_G} \end{array} \,,$$

where $f_b, g_b, \phi_b, \psi_b : \boldsymbol{S_F} \to \boldsymbol{S_G}$, are the maps between the type fibres, induced by a bundle trivialisation of $\boldsymbol{F}$ and $\boldsymbol{G}$ in tubelike neighbourhoods $\boldsymbol{F}_b$ and $\boldsymbol{G}_b$.

Proof. Clearly, (1) $\Rightarrow$ (2).

Conversely, (2) $\Rightarrow$ (1).

In fact, (2) implies (1), for each ϕ and ψ defined in trivial tubelike subbundles of $\boldsymbol{F}$ and $\boldsymbol{G}$.

Then, this local implication can be extended to a global implication by means of a glueing procedure. □

We present a few typical examples of equivariant sections and fibred morphisms.

Example J.4.4 We have the following distinguished examples of equivariant sections:

- Let us consider the subcategory of vector bundles $\mathscr{V} \subset \mathscr{B}$, with linear fibred morphisms.
  Then, for each vector bundle $\boldsymbol{F} \to \boldsymbol{B}$, the zero section $0 : \boldsymbol{B} \to \boldsymbol{F}$ turns out to be equivariant.
- Let us consider the subcategory $\mathscr{B}' \subset \mathscr{V}$ of vector bundles of the type $\boldsymbol{F}^* \otimes \boldsymbol{F} \to \boldsymbol{B}$, where $\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle, and fibred morphisms of the type $\phi^{-1*} \otimes \phi : \boldsymbol{F}^* \otimes \boldsymbol{F} \to \boldsymbol{F}^* \otimes \boldsymbol{F}$, where $\phi : \boldsymbol{F} \to \boldsymbol{F}$ is a linear fibred isomorphisms.
  Then, for each vector bundle $\boldsymbol{F} \to \boldsymbol{B}$, the section $\boldsymbol{1_F} : \boldsymbol{B} \to \boldsymbol{F}^* \otimes \boldsymbol{F}$ turns out to be equivariant.
- Let us consider any vector bundle $\boldsymbol{F} \to \boldsymbol{B}$.

The zero section $0 : \boldsymbol{B} \to (\otimes^p \boldsymbol{F}^*) \otimes (\otimes^q \boldsymbol{F})$ is equivariant for any p, q, with $p + q > 0$.

A non zero section $s : \boldsymbol{B} \to \otimes^p \boldsymbol{F}^* \otimes \otimes^q \boldsymbol{F}$ is equivariant if and only if $p = q > 0$ and s is a linear combination with constant coefficients of tensor products of p sections $\boldsymbol{1_F}$ applied to all possible choices of $\boldsymbol{F}^*$ and $\boldsymbol{F}$ in $(\otimes^p \boldsymbol{F}^*) \otimes (\otimes^p \boldsymbol{F})$.

With reference to a linear fibred linear chart (x^λ, y^i) of $\boldsymbol{F}$ and to the induced linear fibred chart (x^λ, y_i) of $\boldsymbol{F}^*$, the equivariant non zero section has coordinate expression

$$\boldsymbol{S}^{i_1 \ldots i_p}_{j_1 \ldots j_p} = \sum_{\sigma} a_\sigma \, \delta^{i_1}_{j_{\sigma(1)}} \ldots \delta^{i_p}_{j_{\sigma(p)}},$$

where a_σ are constants and σ runs all permutations of $(1, \ldots, p)$.

Such equivariant sections are also called *invariant tensors*, [246, p. 230]. □

Example J.4.5 We have the following distinguished examples of equivariant fibred morphisms:

- Let us refer to the category of fibred manifolds $\mathscr{F}$. Then, for each fibred manifold $\boldsymbol{F} \to \boldsymbol{B}$, the fibred morphism $\mathrm{id} : \boldsymbol{F} \to \boldsymbol{F}$ turns out to be equivariant.
- Let us refer to the category of complex vector bundles $\mathscr{V}_{\mathbb{C}} \subset \mathscr{V} \subset \mathscr{B}$. Then, for each complex vector bundle $\boldsymbol{F} \to \boldsymbol{B}$, the complex linear fibred morphism $\mathfrak{i} : \boldsymbol{F} \to \boldsymbol{F}$ turns out to be equivariant. □

J.4.2 Covariant Sections and Morphisms

With reference to natural bundle functors, we define the concepts of *covariant section* and *covariant* fibred morphism, in terms of suitable equivariance properties.

Here, we present simple typical definitions; however, in practice, we might deal with more complex cases, which require more elaborate concepts. We leave to the reader the simple task to achieve such generalised concepts, as needed case by case.

Eventually, we provide a few typical examples of covariant sections and fibred morphisms taken from standard Differential Geometry.

Definition J.4.6 Let us consider a natural bundle functor $\mathcal{B} : \mathscr{C} \to \mathscr{B}$ and a manifold $\boldsymbol{M}$. Then, a section $s : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{M})$ is said to be *covariant* if, for each $\phi \in \mathrm{Mor}(\boldsymbol{M}, \boldsymbol{M})$ the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{M} & \xrightarrow{\;s\;} & \mathcal{B}(\boldsymbol{M}) \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathcal{B}(\phi)} \\ \boldsymbol{M} & \xrightarrow{\;s\;} & \mathcal{B}'(\boldsymbol{M}) \end{array} \quad . \square$$

Definition J.4.7 Let us consider two natural bundle functors $\mathcal{B}, \mathcal{B}' : \mathscr{C} \to \mathscr{B}$ and a manifold $\boldsymbol{M}$. Then, a fibred morphism $f : \mathcal{B}(\boldsymbol{M}) \to \mathcal{B}'(\boldsymbol{M})$ over $\boldsymbol{M}$ is said to be *covariant* if, for each $\phi \in \mathrm{map}(\boldsymbol{M}, \boldsymbol{M})$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}(\boldsymbol{M}) & \xrightarrow{\;f\;} & \mathcal{B}'(\boldsymbol{M}) \\ {\scriptstyle \mathcal{B}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathcal{B}'(\phi)} \\ \mathcal{B}(\boldsymbol{M}) & \xrightarrow{\;f\;} & \mathcal{B}'(\boldsymbol{M}) \end{array} \quad . \square$$

We present a few typical examples of covariant sections and fibred morphisms.

Example J.4.8 We have the following distinguished examples of equivariant sections:

- Let us consider the natural tangent bundle functor $T : \mathscr{M} \to \mathscr{B}$.
  Then, for each section $s : \boldsymbol{B} \to \boldsymbol{F}$, its tangent prolongation $Ts : T\boldsymbol{B} \to T\boldsymbol{F}$ turns out to be covariant.
- Let us consider the tangent natural bundle functor $T : \mathscr{M}[n] \to \mathscr{B}[n]$.
  Then, for each manifold $\boldsymbol{M}$, , the section $\mathbf{1}_{T\boldsymbol{M}} : \boldsymbol{M} \to T^*\boldsymbol{M} \otimes T\boldsymbol{M}$ turns out to be covariant.
- The zero section $0 : \boldsymbol{M} \to (\otimes^p T\boldsymbol{M}) \otimes (\otimes^q T^*\boldsymbol{M})$ is equivariant for any p, q, with $p + q > 0$.

A non zero section $s : \boldsymbol{M} \to (\otimes^p T\boldsymbol{M}) \otimes (\otimes^q T^*\boldsymbol{M})$ is equivariant if and only if $p = q > 0$ and s is a linear combination with constant coefficients of tensor products of p sections $\mathbf{1}_{T\boldsymbol{M}}$ applied to all possible choices of $T\boldsymbol{M}$ and $T^*\boldsymbol{M}$ in $(\otimes^p T\boldsymbol{M}) \otimes (\otimes^p T^*\boldsymbol{M})$.

With reference to a natural linear fibred chart induced by a chart (x^λ) of $\boldsymbol{M}$, the equivariant non zero section has coordinate expression

$$s^{\lambda_1 \ldots \lambda_p}_{\mu_1 \ldots \mu_p} = \sum_\sigma a_\sigma\, \delta^{\lambda_1}_{\mu_{\sigma(1)}} \ldots \delta^{\lambda_p}_{\mu_{\sigma(p)}},$$

where a_σ are constants and σ spans all permutations of $(1, \ldots, p)$.

Such equivariant sections are also called *invariant tensors*, [246]. □

Example J.4.9 We have the following distinguished examples of equivariant fibred morphisms:

- Let us consider the natural tangent bundle functor $T : \mathscr{M} \to \mathscr{B}$.
  Then, for each map $f : \boldsymbol{M} \to \boldsymbol{N}$, its tangent prolongation $Tf : T\boldsymbol{M} \to T\boldsymbol{N}$ turns out to be covariant.
- Let us consider the natural cotangent bundle functor $T^* : \mathscr{M}[n] \to \mathscr{B}[n]$.

Then, for each diffeomorphism $f : \boldsymbol{M} \to \boldsymbol{N}$, its prolongation $T^* f^{-1} : T^*\boldsymbol{M} \to T^*\boldsymbol{N}$ turns out to be covariant. □

J.4.3 Gauge Covariant Sections and Morphisms

With reference to gauge natural bundle functors, we define the concepts of *gauge covariant section* and *gauge covariant* fibred morphism, in terms of suitable equivariance properties.

Here, we present simple typical definitions; however, in practice, we might deal with more complex cases, which require more elaborate concepts. We leave to the reader the simple task to achieve such generalised concepts, as needed case by case.

Eventually, we provide a few typical examples of gauge covariant sections and fibred morphisms taken from standard Differential Geometry and Covariant Quantum Mechanics.

Definition J.4.10 Let us consider a gauge natural bundle functor $\mathcal{B} : \mathscr{P} \to \mathscr{B}$ and a G-principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$, called the *structure bundle*. Then, a section $s : \boldsymbol{M} \to \mathcal{B}(\boldsymbol{P})$ is said to be *gauge covariant* if, for each $\phi \in \mathrm{Mor}(\boldsymbol{P}, \boldsymbol{P})$ the following diagram commutes

$$\begin{array}{ccc} \boldsymbol{B} & \xrightarrow{\ s\ } & \mathcal{B}(\boldsymbol{P}) \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\mathcal{B}(\phi)} \\ \boldsymbol{B} & \xrightarrow{\ s\ } & \mathcal{B}(\boldsymbol{P}) \end{array} \; . \square$$

Definition J.4.11 Let us consider two gauge natural bundle functors $\mathcal{B} : \mathscr{P} \to \mathscr{B}$ and $\mathcal{B}' : \mathscr{P} \to \mathscr{B}$ and a G-principal bundle $\pi : \boldsymbol{P} \to \boldsymbol{M}$, called the *structure bundle*.

Then, a fibred morphism $f : \mathcal{B}(\boldsymbol{P}) \to \mathcal{B}'(\boldsymbol{P})$ over $\boldsymbol{M}$ is said to be *gauge covariant* if, for each $\phi \in \mathrm{Mor}(\boldsymbol{P}, \boldsymbol{P})$ projectable over $\mathrm{id}_{\boldsymbol{M}}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}(\boldsymbol{P}) & \xrightarrow{\ f\ } & \mathcal{B}'(\boldsymbol{P}) \\ {\scriptstyle\mathcal{B}(\phi)}\downarrow & & \downarrow{\scriptstyle\mathcal{B}'(\phi)} \\ \mathcal{B}(\boldsymbol{P}) & \xrightarrow{\ f\ } & \mathcal{B}'(\boldsymbol{P}) \end{array} \; . \square$$

Eventually, we present typical examples of gauge covariant sections and fibred morphisms taken from Covariant Quantum Mechanics.

Indeed, the quantum bundle $\pi : \boldsymbol{Q} \to \boldsymbol{E}$, equipped with the hermitian quantum metric h, can be regarded as a bundle associated with the principal $U(1, \mathbb{C})$-bundle $\boldsymbol{P} \to \boldsymbol{E}$, where the Lie group $U(1, \mathbb{C})$ is regarded as subgroup of the group $\mathrm{Gl}(1, \mathbb{C})$.

Thus, the principal $U(1, \mathbb{C})$-bundle $\boldsymbol{P} \to \boldsymbol{E}$ turns out to be the fundamental structure group of Covariant Quantum Mechanics.

Example J.4.12 Each hermitian quantum vector field $Y : \boldsymbol{Q} \to T\boldsymbol{Q}$ (see Definition 19.1.5) turns out to be gauge covariant section (see Appendix: Definition J.4.10).

In particular, the gauge covariance implies the fact that the hermitian quantum vectors field are "gauge independent", i.e. invariant with respect to the change of a quantum basis. □

The rule that maps special phase functions $f : J_1\boldsymbol{E} \to \mathbb{R}$ to hermitian quantum vector fields $Y[f] : \boldsymbol{Q} \to T\boldsymbol{Q}$ (see Definition 19.1.3) provides an example of gauge covariant fibred morphism, according to the following Example.

Example J.4.13 Let us consider the category $\mathscr{P}$ of $U(1, \mathbb{C})$-principal bundles over galilean spacetimes and the gauge natural bundle functors $\mathcal{B}$ and $\mathcal{B}'$ given by the following maps

$$\mathcal{B}(\boldsymbol{P}) = G\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \qquad \text{and} \qquad \mathcal{B}'(\boldsymbol{P}) = T\,\boldsymbol{Q},$$

where $G\boldsymbol{E} \to \boldsymbol{E}$ is the natural bundle of galilean metrics of $\boldsymbol{E}$.

Then, the rule

$$f \in \sec\big(J_1\boldsymbol{E},\ J_1\boldsymbol{E} \times \mathbb{R}\big) \ \mapsto\ Y[f] \in \sec(\boldsymbol{Q}, T\,\boldsymbol{Q})$$

turns out to be a gauge covariant fibred morphism, according to the following commutative diagram induced by every special phase function $f : J_1\boldsymbol{E} \to \mathbb{R}$

$$\begin{array}{ccc} G\boldsymbol{E} \underset{\boldsymbol{E}}{\times} J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} & \xrightarrow{\ Y[f]\ } & T\,\boldsymbol{Q} \\ {\scriptstyle \mathrm{pro}_3}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{\boldsymbol{Q}}} \\ \boldsymbol{Q} & \xrightarrow{\ \mathrm{id}_{\boldsymbol{Q}}\ } & \boldsymbol{Q} \end{array} .$$

In particular, the gauge covariance implies the fact that the hermitian quantum vector field $Y[f]$ is "observer independent", i.e. invariant with respect to the change of observer. □

Example J.4.14 Suitable gauge natural bundles yield the associated gauge natural bundles

$$\boldsymbol{Q} \to \boldsymbol{E}, \qquad J_2\boldsymbol{Q} \to \boldsymbol{E}, \qquad \mathbb{T}^* \otimes \boldsymbol{Q} \to \boldsymbol{E}.$$

Indeed, the Schrödinger operator (see Theorem 17.7.12)

$$\mathrm{S} : J_2\boldsymbol{Q} \to \mathbb{T}^* \otimes \boldsymbol{Q}$$

turns out to be a gauge covariant fibred morphism (see Definition J.4.11).

In particular, the gauge covariance implies the fact that the Schrödinger operator is "gauge independent", i.e. invariant with respect to the change of a quantum basis. □

Appendix K
Scales

There is a very huge mathematical and physical literature concerning the "*dimensional analysis*" of units of measurements, with different goals and perspectives (see for instance, [32, 43, 175, 232, 239, 269, 297, 389, 424, 428]; actually, a quite larger literature can be found in the above paper [175]).

In the present book on Covariant Quantum Mechanics, we consider the equivariance of our theory with respect to units of measurement as one of the requirements of covariance, besides the covariance with respect to observers, gauges and coordinates. For this reason, we are led to include explicitly, in a formal mathematical way, the units of measurement involved in all formulas.

Thus, we need a rigorous mathematical formalism concerning the notion of units of measurement and their relations. Indeed, the paper [228] is just aimed at providing such a formalism, suitable for a systematic explicit use of units of measurement at any step of our classical and quantum theory.

Indeed, rigorous mathematical foundations of this formalism requires the formal notions and proofs established in the above paper [228]. But, eventually, once the reader has got that the formal procedures are well established, he needs not to reconsider the subtle definitions and proofs at any step. Luckily, the resulting language is rather intuitive and reflects closely the standard way of dealing with units of measurement in physics. In fact, in practice, they can be treated as scalars, via tensor product and rational powers.

In the literature, a standard name for such a formalism is "dimensional analysis". In our context this name might create confusion with the geometric concept of "dimension", which is also frequently used in the present book. So, conventionally, we use the term "scale" for "unit of measurement". Thus, a "scaled object" will denote an object referred to certain units of measurement, while, an "unscaled object" will denote an object without any reference to units of measurement.

The reader, who is possibly interested in checking all details and proofs of our formalism on scales car refer to the above paper [228]. In particular, in this paper is discussed an algebraic treatment of the *scale dimension* of scaled objects and a remark on the treatment of scale factors under *differential operators*.

J. Janyška and M. Modugno, *An Introduction to Covariant Quantum Mechanics*, Fundamental Theories of Physics 205,
https://doi.org/10.1007/978-3-030-89589-1

Here, for the convenience of the reader, we provide just a synthetic summary of the aspects that are explicitly involved in the present book.

K.1 Positive Spaces

We define a *positive space* $\mathbb{U}$ to be a semi-vector space on which $\mathbb{R}^+$ acts freely and transitively.

We stress that the reason why we deal with positive spaces modelled on $\mathbb{R}^+$ and not on vector spaces modelled on $\mathbb{R}$ is related to the definition of rational power.

K.1.1 Definition of Positive Spaces

We start by defining a *semi-vector space* $\boldsymbol{U}$ on $\mathbb{R}^+$ through axioms which are similar to those of a vector space over $\mathbb{R}$, but with the field $\mathbb{R}$ replaced by the semi-field $\mathbb{R}^+$ [173].

Then, we define a *positive space* $\mathbb{U}$ to be a semi-vector space on which $\mathbb{R}^+$ acts freely and transitively. Thus, a positive space $\mathbb{U}$ is just an algebraic model of a geometric semi-straight line.

Further, we define the *semi-linear* maps between positive spaces.

In particular, we define the *dual* $\mathbb{U}^{¥}$ of a positive space $\mathbb{U}$ and show the natural bijective map $\mathbb{U} \to \mathbb{U}^{¥} : u \mapsto 1/u$, which is largely used throughout the book.

Definition K.1.1 A *positive space* is defined to be a semi-vector space $\mathbb{U}$, whose scalar multiplication $\cdot : \mathbb{R}^+ \times \mathbb{U} \to \mathbb{U}$ yields a left free and transitive action of the group $(\mathbb{R}^+, \cdot)$ on $\mathbb{U}$. □

In simple words, a positive space is just the algebraic version of an oriented semi-straight line (without origin). The present algebraic language is aimed at introducing further algebraic operations.

By definition, a positive space $\mathbb{U}$ has no neutral element 0.

For each $u, b \in \mathbb{U}$, we can uniquely write

$$u = (u/b)\, b, \quad \text{with } u/b \in \mathbb{R}^+.$$

Thus, each element $b \in \mathbb{U}$ behaves as a “basis”. Indeed, the map $\mathbb{U} \to \mathbb{R}^+ : u \mapsto u/b$ turns out to be a semi-linear isomorphism. Hence, we can identify $\mathbb{U}$ with $\mathbb{R}^+$, *after having chosen a “basis” b*.

If $\mathbb{U}$ and $\mathbb{V}$ are positive spaces, then the set $\text{s-Lin}(\mathbb{U}, \mathbb{V})$ of semi-linear maps turns out to be a positive space.

In particular, we have the following distinguished cases.

(1) We set

$$\mathbb{U}^* := \text{s-Lin}(\mathbb{U}, \mathbb{R}^+).$$

We have a natural bijective map (*which is frequently used throughout the book*)

$$\text{inv} : \mathbb{U} \to \mathbb{U}^* : u \mapsto 1/u,$$

where $1/u \in \mathbb{U}^*$ is the unique element such that $(1/u)(u) = 1$.
Indeed, for each $r \in \mathbb{R}^+$ and $u \in \mathbb{U}$, we have $1/(ru) = (1/r)\,(1/u)$.

(2) Each semi-linear map $f : \mathbb{U} \to \mathbb{U}$ turns out to be a semi-linear isomorphism of the type

$$f : u \mapsto r\,u, \quad \text{with } r \in \mathbb{R}^+.$$

Thus, we have a natural semi-linear isomorphism

$$\text{s-Lin}(\mathbb{U}, \mathbb{U}) \;\simeq\; \mathbb{R}^+.$$

(3) We have the natural semi-linear isomorphism

$$\text{s-Lin}(\mathbb{R}^+, \mathbb{U}) \to \mathbb{U} : \phi \mapsto \phi(1).$$

K.1.2 Tensor Product of Positive Spaces

Then, we sketch the notion of *tensor product* in three steps: tensor product between a positive space and a vector space, tensor product between $\mathbb{R}$ and a positive space, tensor product between two positive spaces.

These algebraic constructions can be achieved via the universal property of the tensor product analogously to the case of vector spaces. However, an additional care is needed because the positive spaces have no zero element.

Here, we just sketch these concepts; explicit statements and proofs can be found in [228].

(1) One can start by defining the *tensor product* $\mathbb{U} \otimes \boldsymbol{V} \;\equiv\; \boldsymbol{V} \otimes \mathbb{U}$ between a *positive space* $\mathbb{U}$ and a *vector space* $\boldsymbol{V}$. This algebraic concept can be achieved analogously to that of the tensor product between vector spaces, but it needs a special additional care because the positive space has no zero element.
Thus the tensor product $\mathbb{U} \otimes \boldsymbol{V}$ is achieved independently of any choice of bases. Once a basis $b \in \mathbb{U}$ has been chosen, the above tensor product can be read as the tensor product of $\mathbb{R}^+ \otimes \boldsymbol{V}$; but, this identification depends on the choice of b.

(2) We define the *universal vector extension* of $\mathbb{U}$ to be the tensor product $\bar{\mathbb{U}} := \mathbb{R} \otimes\, \mathbb{U}$.

This vector space $\mathbb{R} \otimes \mathbb{U}$ has the following very intuitive properties.
For each $b \in \mathbb{U}$, the vector $1 \otimes b$ turns out to be a basis of $\mathbb{R} \otimes \mathbb{U}$.
The positive spaces

$$\mathbb{U}_{+} := \{1 \otimes u \mid u \in \mathbb{U}\} \subset \bar{\mathbb{U}} \qquad \text{and} \qquad \mathbb{U}_{-} := \{(-1) \otimes u \mid u \in \mathbb{U}\} \subset \bar{\mathbb{U}}$$

yield the disjoint union

$$\bar{\mathbb{U}} = \mathbb{U}_{-} \cup \{0\} \cup \mathbb{U}_{+}.$$

Moreover, we have the natural semi-linear inclusion

$$\mathbb{U} \to \mathbb{R} \otimes \mathbb{U} : u \mapsto 1 \otimes u.$$

Further, we have a natural linear isomorphism

$$(\bar{\mathbb{U}})^* := (\mathbb{R} \otimes \mathbb{U})^* \simeq \overline{\mathbb{U}^*} := \mathbb{R} \otimes (\mathbb{U}^*).$$

Additionally, for each vector space $\boldsymbol{V}$, there is a natural linear isomorphism (*which is frequently used throughout the book*)

$$j : \mathbb{U} \otimes \boldsymbol{V} \to \bar{\mathbb{U}} \otimes \boldsymbol{V},$$

by which we make the identification

$$\mathbb{U} \otimes \boldsymbol{V} \simeq \bar{\mathbb{U}} \otimes \boldsymbol{V}.$$

(3) Eventually, one can define the *tensor product* $\mathbb{U} \otimes \mathbb{V}$ between two positive spaces $\mathbb{U}$ and $\mathbb{V}$. Also this algebraic concept can be achieved analogously to that of the tensor product between vector spaces, but it needs a special additional care because the positive space has no zero element.

K.1.3 Rational Maps Between Positive Spaces

Next, we discuss the notion of *q-rational map* between positive spaces (for details, see [228]).

This formal mathematical concept involves rather subtle features, in order to achieve a result that is independent of the choice of any basis $b \in \mathbb{U}$. However, the final result is very intuitive; in practice, we can treat the rational maps between positive spaces quite analogously to rational maps of positive numbers.

Let us consider two positive space $\mathbb{U}$ and $\mathbb{V}$ and a rational number $q \in \mathbb{Q}$.

Definition K.1.2 A map $f : \mathbb{U} \to \mathbb{V}$ is said to be *rational of degree q* if

$$f(r\,u) = r^q\, f(u), \qquad \forall r \in \mathbb{R}^+, \quad u \in \mathbb{U}.$$

We denote the subspace of q-rational maps between $\mathbb{U}$ and $\mathbb{V}$ by

$$\mathrm{Rat}^q(\mathbb{U}, \mathbb{V}) \subset \mathrm{Map}(\mathbb{U}, \mathbb{V}). \quad \square$$

Then, we obtain the following natural semi-linear isomorphisms:

$$\begin{gathered}
\mathrm{Rat}^0(\mathbb{U}, \mathbb{V}) \simeq \mathbb{V}, \qquad \mathrm{Rat}^0(\mathbb{U}, \mathrm{I\!R}^+) \simeq \mathrm{I\!R}^+, \\
\mathrm{Rat}^1(\mathbb{U}, \mathbb{V}) = \text{s-Lin}(\mathbb{U}, \mathbb{V}), \qquad \mathrm{Rat}^1(\mathbb{U}, \mathrm{I\!R}^+) = \text{s-Lin}(\mathbb{U}, \mathrm{I\!R}^+) = \mathbb{U}^*, \\
\mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{V}) \simeq \text{s-Lin}(\mathbb{U}^*, \mathbb{V}), \qquad \mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{U}) \simeq \text{s-Lin}(\mathbb{U}^*, \mathbb{U}), \\
\mathbb{U}^{**} \simeq \mathbb{U}.
\end{gathered}$$

Moreover, we have (see Sect. K.1.1)

$$\mathrm{inv} \in \mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{U}^*), \quad \text{where } \mathrm{inv} : \mathbb{U} \to \mathbb{U}^* : u \mapsto 1/u.$$

K.1.4 Rational Powers of a Positive Space

Eventually, we introduce the *rational powers* of a positive space $\mathbb{U}$.

The basic idea is quite simple and could be achieved in an elementary way, by referring to a "semi-basis" $b \in \mathbb{U}$ and showing that the result is independent of the choice of b.

However, a full understanding of this concept is more subtle than it might appear at first insight and suggests a more sophisticated formal approach.

As soon as these formal notions and results have been rigorously established, the resulting practical rules turn out to be very simple and intuitive. Actually, at the end, in practice, the rational powers of a positive spaces $\mathbb{U}$ can be treated as the rational powers of $\mathrm{I\!R}^+$.

Let us consider a positive space $\mathbb{U}$ and a rational number $q \in \mathbb{Q}$.

Definition K.1.3 We define the *q-rational map*

$$\pi^q : \mathbb{U} \to \mathrm{Rat}^q(\mathbb{U}^*, \mathrm{I\!R}^+) : u \mapsto u^q,$$

where $u^q \in \mathrm{Rat}^q(\mathbb{U}^*, \mathrm{I\!R}^+)$ is the unique element such that $u^q(1/u) = 1$.

Then, we define the *q-power* of $\mathbb{U}$ is defined to be the pair $(\mathbb{U}^q, \pi^q)$ defined by

$$\mathbb{U}^q := \mathrm{Rat}^q(\mathbb{U}^*, \mathrm{I\!R}^+) \qquad \text{and} \qquad \pi^q : \mathbb{U} \to \mathbb{U}^q : u \mapsto u^q. \quad \square$$

We can re-interpret the above notion in a natural way in terms of tensor product as follows. We have the following distinguished cases.

(1) For $q = 0$, we have the natural semi-linear isomorphism

$$\mathbb{U}^0 := \mathrm{Rat}^0(\mathbb{U}^*, \mathrm{I\!R}^+) \simeq \mathrm{I\!R}^+$$

and the natural 0-rational map

$$\pi^0 : \mathbb{U} \to \mathbb{U}^0 : u \mapsto 1.$$

(2) For $q = n \in \mathbb{Z}^+$, we have the natural mutually inverse semi-linear isomorphisms

$$\otimes^n \mathbb{U} \to \mathbb{U}^n := \mathrm{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) : u \otimes \ldots \otimes u \to f_u,$$
$$\mathbb{U}^n := \mathrm{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) \to \otimes^n \mathbb{U} : f \mapsto u_f \otimes \ldots \otimes u_f,$$

where $f_u : \mathbb{U}^* \to \mathbb{R}^+ : \omega \mapsto \omega(u) \ldots \omega(u)$ and $u_f := 1/\omega_f \in \mathbb{U}$, being $\omega_f \in \mathbb{U}^*$ the unique element such that $f(\omega_f) = 1$.
Moreover, we have the natural n-rational map

$$\pi^n : \mathbb{U} \to \mathbb{U}^n := \mathrm{Rat}^n(\mathbb{U}^*, \mathbb{R}^+) : u \mapsto f_u.$$

In particular, if $q \equiv n = 1$ then we have the natural semi-linear isomorphism

$$\mathbb{U}^1 := \mathrm{Rat}^1(\mathbb{U}^*, \mathbb{R}^+) = \text{s-Lin}(\mathbb{U}^*, \mathbb{R}^+) := \mathbb{U}^{**} \simeq \mathbb{U}.$$

(3) For $q \equiv 1/n$, with $n \in \mathbb{Z}^+$, we have the natural mutually inverse semi-linear isomorphisms

$$\otimes^n \mathbb{U}^{1/n} := \otimes^n \mathrm{Rat}^{1/n}(\mathbb{U}, \mathbb{R}^+) \to \text{s-Lin}(\mathbb{U}, \mathbb{R}^+) := \mathbb{U} \simeq \mathbb{U} : f \otimes \ldots \otimes f \mapsto f^n,$$
$$\mathbb{U} \simeq \mathbb{U} := \text{s-Lin}(\mathbb{U}, \mathbb{R}^+) \to \otimes^n \mathbb{U}^{1/n} := \otimes^n \mathrm{Rat}^{1/n}(\mathbb{U}, \mathbb{R}^+) : f \mapsto f^{1/n} \otimes \ldots \otimes f^{1/n},$$

where $f^n : \mathbb{U} \to \mathbb{R}^+ : \omega \mapsto f(\omega) \ldots f(\omega)$ and $f^{1/n} : \mathbb{U} \to \mathbb{R}^+ : \omega \mapsto (f(\omega))^{1/n}$.
Moreover, we have the $1/n$-rational isomorphism

$$\pi^{1/n} : \mathbb{U} \simeq \mathbb{U} \to \mathbb{U}^{1/n} := \mathrm{Rat}^{1/n}(\mathbb{U}, \mathbb{R}^+) : f \mapsto f^{1/n}.$$

(4) For $q \equiv -n \in \mathbb{Z}^-$, we have the natural mutually inverse semi-linear isomorphisms

$$\otimes^n \mathbb{U} \to \mathbb{U}^{-n} := \mathrm{Rat}^{-n}(\mathbb{U}, \mathbb{R}^+) : \omega \otimes \ldots \otimes \omega \to f_\omega,$$
$$\mathbb{U}^{-n} := \mathrm{Rat}^{-n}(\mathbb{U}, \mathbb{R}^+) \to \otimes^n \mathbb{U} : f \mapsto \omega_f \otimes \ldots \otimes \omega_f,$$

where $f_\omega : \mathbb{U} \to \mathbb{R}^+ : \alpha \mapsto \omega(1/\alpha) \ldots \omega(1/\alpha)$ and $\omega_f \in \mathbb{U}, \quad f(\omega_f) = 1$.
Moreover, we have the natural $(-n)$-rational isomorphism

$$\pi^{-n} : \mathbb{U} \to \mathbb{U}^{-n} := \mathrm{Rat}^{-n}(\mathbb{U}, \mathbb{R}^+) : u \mapsto f_{1/u}.$$

In particular, if $q = -1$, then we have the natural semi-linear isomorphism

$$\mathbb{U}^{-1} := \mathrm{Rat}^{-1}(\mathbb{U}, \mathbb{R}^+) \simeq \text{s-Lin}(\mathbb{U}, \mathbb{R}^+) \simeq \text{s-Lin}(\mathbb{U}, \mathbb{R}^+) := \mathbb{U}.$$

(5) For $p, q \in \mathbb{Z}$, we have the natural semi-bilinear map

$$\mathbb{U}^p \times \mathbb{U}^q := \mathrm{Rat}^p(\mathbb{U}^*, \mathbb{R}^+) \times \mathrm{Rat}^q(\mathbb{U}^*, \mathbb{R}^+) \to \mathbb{U}^{p+q} := \mathrm{Rat}^{p+q}(\mathbb{U}, \mathbb{R}^+) : (f, g) \mapsto fg.$$

(6) For $p, q \in \mathbb{Z}$, we have the natural semi-linear isomorphism

$$c : (\mathbb{U}^p)^q := \mathrm{Rat}^q\left(\mathrm{Rat}^p(\mathbb{U}^*, \mathbb{R}^+), \mathbb{R}^+\right) \to \mathbb{U}^{pq} := \mathrm{Rat}^{pq}(\mathbb{U}^*, \mathbb{R}^+) : f \mapsto g_f,$$

where $g_f : \mathbb{U} \to \mathbb{R}^+ : 1/u \mapsto f(1/u^p)$.s
Moreover, we have $\pi^{pq} = c \circ (\pi^q \circ \pi^p)$.

(7) For $q \in \mathbb{Z}$, we have the natural semi-linear isomorphism

$$(\mathbb{U}^q)^* \simeq (\mathbb{U}^*)^q.$$

(8) For $p < q \in \mathbb{Z}$, we have the natural semi-linear isomorphisms

$$\otimes^q \mathbb{U} \otimes (\otimes^p \mathbb{U}^*) \simeq \mathbb{U}^q \otimes \mathbb{U}^{-p} = \mathbb{U}^{q-p} \simeq \otimes^{q-p}\mathbb{U},$$
$$\otimes^p \mathbb{U} \otimes (\otimes^q \mathbb{U}^*) \simeq \mathbb{U}^p \otimes \mathbb{U}^{-q} = \mathbb{U}^{p-q} \simeq \otimes^{|p-q|}\mathbb{U}^*.$$

K.2 Physical Scales

Eventually, we select our basic scales and define the scale dimension of scales. Moreover, we discuss the scaled bundles and related differential operators.

K.2.1 Units and Scales

We discuss the fundamental scale spaces used in the present book.

According to a possible standard usage, it is convenient to derive all scales from time, length and mass scales, via their rational powers. Of course, other choices might be possible.

We provide a formal definition of the *scale dimension*. Moreover, we define the notion of *scale basis* and show a condition by which a triplet of scales is a scale basis.

Hypothesis K.1 We consider as *basic spaces of scales* the following positive spaces:

(1) *the space* $\mathbb{T}$ *of* time scales,
(2) *the space* $\mathbb{L}$ *of* length scales,
(3) *the space* $\mathbb{M}$ *of* mass scales.

Accordingly,
(1) *each element* $u_0 \in \mathbb{T}$ *is said to be a* time scale,
(2) *each element* $\mathfrak{l} \in \mathbb{L}$ *is said to be a* length scale,
(3) *each element* $\mathfrak{m} \in \mathbb{M}$ *is said to be a* mass scale. □

For each time scale $u_0 \in \mathbb{T}$, we denote its dual by $u^0 := 1/u_0 \in \mathbb{T}^*$.

Definition K.2.1 We define a *scale space* to be a positive space of the type (see Definition K.1.1)

$$\mathbb{S} := \mathbb{T}^{d_1} \otimes \mathbb{L}^{d_2} \otimes \mathbb{M}^{d_2}, \quad \text{where } d_i \in \mathbb{Q}.$$

We define the *scale dimension* of $\mathbb{S}$ to be the above 3-plet (d_1, d_2, d_3) of rational numbers. We call *scale* each element $k \in \mathbb{S}$. For each scale $k \in \mathbb{S}$, we set

$$\lfloor k \rfloor \equiv \big(\lfloor k \rfloor_1, \lfloor k \rfloor_2, \lfloor k \rfloor_3\big) := (d_1, d_2, d_3). \quad \square$$

Definition K.2.2 A 3-plet of scales $(e_1, e_2, e_3) \in \mathbb{S}$ is said to be a *scale basis* if each scale $k \in \mathbb{S}$ can be written in a unique way as

$$k = r\,(e_1)^{c_1} \otimes (e_2)^{c_2} \otimes (e_3)^{c_3}, \quad \text{with } r \in \mathbb{R}^+, \quad c_i \in \mathbb{Q}. \quad \square$$

Proposition K.2.3 *A 3-plet of scales* (e_1, e_2, e_3) *is a scale basis if and only if*

$$\det(\lfloor e_j \rfloor_i) \neq 0.$$

Further, if (e_1, e_2, e_3) *is a scale basis and* k *a scale, then* $(c_1, c_2, c_3) \in \mathbb{Q}^3$ *is the unique solution of the linear system*

$$\lfloor k \rfloor_i = \sum_j \lfloor e_j \rfloor_i \, c_j. \quad \square$$

K.2.2 Scaled Objects

We provide a formal definition of scaled vector bundles via the tensor product of a positive space and a vector bundle.

Moreover, we show that the scale factors can be easily treated as numerical constants, with respect to differential operators.

Lemma K.2.4 *For each positive space* $\mathbb{U}$, *we can write* $T\mathbb{U} \simeq \mathbb{U} \times \bar{\mathbb{U}}$. □

Lemma K.2.5 *For each positive space* $\mathbb{U}$ *and vector bundle* $\boldsymbol{F} \to \boldsymbol{B}$, *we can naturally define the vector bundle* $\mathbb{U} \otimes \boldsymbol{F} \to \boldsymbol{B}$ *(see Sect. K.1.2, 1).* □

Note K.2.6 Let $\mathbb{S}$ be a positive space and $\boldsymbol{F} \to \boldsymbol{B}$ and $\boldsymbol{G} \to \boldsymbol{B}$ two vector bundles.

Then, for each linear differential operator

$$\mathcal{D} : \sec(\boldsymbol{B}, \boldsymbol{F}) \to \sec(\boldsymbol{B}, \boldsymbol{G}),$$

the following linear differential operator (denoted by the same symbol) is well defined

$$\mathcal{D} : \sec(\boldsymbol{B}, \mathbb{S} \otimes \boldsymbol{F}) \to \sec(\boldsymbol{B}, \mathbb{S} \otimes \boldsymbol{G}) : u \otimes s \mapsto \mathcal{D}(u \otimes s) := u \otimes \mathcal{D}(s),$$

for any $u \in \mathbb{S}$. □

The above result can be easily applied, for instance, to the exterior differential of scaled forms, to the Lie derivative of scaled tensors, to the covariant differential of scaled tensors, and so on.

Definition K.2.7 If $\boldsymbol{F} \to \boldsymbol{B}$ is a vector bundle and $\mathbb{S}$ a scale space, then we say that the tensor product bundle $(\mathbb{S} \otimes \boldsymbol{F}) \to \boldsymbol{B}$ is a *scaled bundle*, whose *scale dimension* is the scale dimension of $\mathbb{S}$. □

Remark K.2.8 In the book we are involved with the logarithm of a scaled function. Such a logarithm is defined by a convenient abuse of notation, that can be justified as follows (see Proposition 16.1.21).

Let us consider a scaled function $f \in \mathrm{map}(\boldsymbol{M}, \mathbb{S} \otimes \mathbb{R}^+)$.

Then, for each scale $u \in \mathbb{S}^*$, we obtain the unscaled function $F := u(f) \in \mathrm{map}(\boldsymbol{M}, \mathbb{R}^+)$.

We can define the logaritm of the unscaled function F, as usual, according to the equality $\exp(\log F) = F$.

We can also compute the differential of the unscaled function F, as usual, according to the equality $d \log F = dF/F$.

Now, let us choose another scale factor $\acute{u} := k\, u$, with $k \in \mathbb{R}^+$.

Then, we obtain the new functions $\acute{F} = k\, F$ and $\log \acute{F} = \log k + \log F$.

Indeed, being k a constant function, we obtain the equalities

$$d \log F = \frac{dF}{F} = \frac{k\, dF}{k\, F} = \frac{d\acute{F}}{\acute{F}} = d \log \acute{F}.$$

Thus, we have shown the following facts:

(1) in principle, the function $\log f$ is not well defined (unless we unscale f by means of a scale factor $u \in \mathbb{S}^*$),
(2) the 1-form $d \log F$ does not depend on the choice of the scale factor $u \in \mathbb{S}^*$.

For this reason, we shall write, by abuse of notation,

$$d \log f := \frac{df}{f} = \frac{d(u(f))}{u(f)} = d \log \big(u(f)\big),$$

regardless to the choice of a scale factor $u \in \mathbb{S}^*$. □

LIST OF SYMBOLS

Here we list the main symbols, with reference to the place where they have introduced.

Cyrillic characters used in the book

д cyrillic character "d" Pro 2.6.1
Ч cyrillic character "ch" Def 15.1.1
Щ cyrillic character "shch" Def 6.1.1
г cyrillic character "g" Pro 22.6.2

INTRODUCTION

Introduction: Scales

$\mathbb{T}$ positive space of time scales Sect. 1.3.5
$\mathbb{L}$ positive space of length scales Sect. 1.3.5
$\mathbb{M}$ positive space of mass scales Sect. 1.3.5
$\mathbb{Q} := \mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{1/2} \otimes \mathbb{R}$ space of charges Sect. 1.3.5
$c \in \mathbb{T}^{-1} \otimes \mathbb{L}$ speed of light Sect. 1.3.5
$\hbar \in \mathbb{T}^{-1} \otimes \mathbb{L}^2 \otimes \mathbb{M}$ Plank constant Sect. 1.3.5
$\text{г} \in \mathbb{T}^{-2} \otimes \mathbb{L}^3 \otimes \mathbb{M}^{-1}$ gravitational constant Sect. 1.3.5

PART I: COVARIANT CLASSICAL MECHANICS

Chapter 2: Spacetime

$t : \boldsymbol{E} \to \boldsymbol{T}$ spacetime fibred manifold Pos C.1
$\boldsymbol{T}$ and $\bar{\mathbb{T}}$ time and space of time intervals Pos C.1
$(x^\lambda) \equiv (x^0, x^i) : \boldsymbol{E} \to \mathbb{R} \times \mathbb{R}^3$ spacetime charts Def 2.1.4
$\tau_{\boldsymbol{E}} : T\boldsymbol{E} \to \boldsymbol{E}$ tangent bundle of spacetime Def 2.2.1
$\tau^{\boldsymbol{E}} : T^*\boldsymbol{E} \to \boldsymbol{E}$ cotangent bundle of spacetime Def 2.2.1
$(x^\lambda, \dot{x}^\lambda)$ fibred chart of $T\boldsymbol{E}$ Def 2.2.1
$(x^\lambda, \dot{x}_\lambda)$ fibred chart of $T^*\boldsymbol{E}$ Def 2.2.1
(∂_λ) basis of $T\boldsymbol{E}$ Def 2.2.1
(d^λ) basis of $T^*\boldsymbol{E}$ Def 2.2.1
$V\boldsymbol{E} \subset T\boldsymbol{E}$ vertical subspace of $T\boldsymbol{E}$ Pro 2.2.4
$H^*\boldsymbol{E} \subset T^*\boldsymbol{E}$ horizontal subspace of $T^*\boldsymbol{E}$ Pro 2.2.4
$\vee : T^*\boldsymbol{E} \to V^*\boldsymbol{E}$ vertical restriction of the spacetime forms Pro 2.2.4
$\check{\alpha}$ vertical restriction of the spacetime form α Pro 2.2.4
(∂_i) basis of $V\boldsymbol{E}$ Pro 2.2.4
$(\check{d}^i)$ basis of $V^*\boldsymbol{E}$ Pro 2.2.4
$\text{pro}\sec(\boldsymbol{E}, T\boldsymbol{E}) \subset \sec(\boldsymbol{E}, T\boldsymbol{E})$ subsheaf of projectable vector fields Def 2.2.6
$\text{tim}\sec(\boldsymbol{E}, T\boldsymbol{E}) \subset \sec(\boldsymbol{E}, T\boldsymbol{E})$ subsheaf of time preserving vector fields Def 2.2.6
$\tau_{T\boldsymbol{E}} : TT\boldsymbol{E} \to T\boldsymbol{E}$ iterated tangent bundle of spacetime Def 2.3.1
$(x^\lambda, \dot{x}^\lambda; \overset{\shortparallel}{x}{}^\lambda, \ddot{x}^\lambda)$ fibred chart of $TT\boldsymbol{E}$ Def 2.3.1
$(\partial_\lambda, \dot{\partial}_\lambda)$ basis of $TT\boldsymbol{E}$ Def 2.3.1
$(d^\lambda, \dot{d}^\lambda)$ basis of $T^*T\boldsymbol{E}$ Def 2.3.1
$s : \boldsymbol{T} \to \boldsymbol{E}$ particle motion Def 2.4.1
$ds : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ velocity of particle motion Def 2.4.1
$\mathscr{C} : \bar{\mathbb{T}} \times \boldsymbol{E} \to \boldsymbol{E}$ continuum motion Def 2.4.2
$\partial\mathscr{C} : \boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ velocity of continuum motion Def 2.4.2

$t_0^1 : J_1\boldsymbol{E} \to \boldsymbol{E}$ and $t^1 : J_1\boldsymbol{E} \to \boldsymbol{T}$ classical phase space Pro 2.5.1
(x^λ, x_0^i) fibred chart of $J_1\boldsymbol{E}$ Pro 2.6.1
$j_1 s : \boldsymbol{T} \to J_1\boldsymbol{E}$ velocity of the motion s Pro 2.5.1
$д = u^0 \otimes (\partial_0 + x_0^i\, \partial_i) : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes T\boldsymbol{E}$ contact map Pro 2.6.1
$\theta = (d^i - x_0^i\, d^0) : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E}$ contact map Cor 2.6.2
$o : \boldsymbol{E} \to J_1\boldsymbol{E} \subset \mathbb{T}^* \otimes T\boldsymbol{E}$ observer Def 2.7.1
$T\boldsymbol{E} = T_o\boldsymbol{E} \oplus_{\boldsymbol{E}} V\boldsymbol{E}$ observed splitting Pro 2.7.3
$T^*\boldsymbol{E} = H^*\boldsymbol{E} \oplus_{\boldsymbol{E}} V_o^*\boldsymbol{E}$ observed splitting Pro 2.7.3
$\nabla[o] = (x_0^i - o_0^i)\, d^0 \otimes \partial_i : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ observed covariant differential Pro 2.7.3
$\nabla[o]s \equiv d[o]s := \theta[o] \lrcorner ds : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ observed velocity of the motion s Def 2.7.7
$\partial[o]\mathscr{C} = (\delta_0\mathscr{C}^i - o_0^i)\, \partial_i : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ observed velocity of the continuum motion $\mathscr{C}$ Def 2.7.8

Chapter 3: Galilean metric field

$\mathbf{g} := dt \otimes dt : \boldsymbol{E} \to \mathbb{T}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ timelike galilean metric Def 3.1.1
$\mathfrak{g} := c^2\, \mathbf{g} = c^2\, dt \otimes dt : \boldsymbol{E} \to \mathbb{L}^2 \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ rescaled timelike galilean metric Not 3.1.2
$g : \boldsymbol{E} \to \mathbb{L}^2 \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$ galilean metric Pos C.2
$G := \frac{m}{\hbar}\, g : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes V^*\boldsymbol{E})$ rescaled galilean metric Def 3.2.1
$\bar{g} : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$ contravariant galilean metric Def 3.2.2
$\bar{G} : \boldsymbol{E} \to \mathbb{T}^* \otimes (V\boldsymbol{E} \otimes V\boldsymbol{E})$ contravariant rescaled galilean metric Def 3.2.2
$g^\flat : V\boldsymbol{E} \to \mathbb{L}^2 \otimes V^*\boldsymbol{E}$ metric musical morphism Def 3.2.2
$g^\sharp : V^*\boldsymbol{E} \to \mathbb{L}^{-2} \otimes V\boldsymbol{E}$ metric musical morphism Def 3.2.2
$G^\flat : V\boldsymbol{E} \to \mathbb{T} \otimes V^*\boldsymbol{E}$ metric musical morphism Def 3.2.2
$G^\sharp : V^*\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ metric musical morphism Def 3.2.2
$\eta : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^3 V^*\boldsymbol{E}$ spacelike volume form Pro 3.2.4
$\upsilon : \boldsymbol{E} \to \mathbb{T} \otimes \mathbb{L}^3 \otimes \Lambda^4 T^*\boldsymbol{E}$ spacetime volume form Pro 3.2.4
$\bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes \Lambda^3 V\boldsymbol{E}$ contravariant spacelike volume form Pro 3.2.4
$\bar{\upsilon} : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{L}^{-3} \otimes \Lambda^4 T\boldsymbol{E}$ contravariant spacetime volume form Pro 3.2.4
$\eta[o] := \theta^*[o]\eta : \boldsymbol{E} \to \Lambda^3 T^*\boldsymbol{E}$ observed spacelike volume form Def 3.2.6
$_\eta : \Lambda^r V^*\boldsymbol{E} \to \Lambda^{3-r} V^*\boldsymbol{E} : \phi \mapsto i_{g^\sharp(\phi)}\eta$ spacelike Hodge isomorphism Cor 3.2.7
$X \times Y := i_{g^\flat(X) \wedge g^\flat(Y)}\, \bar{\eta} \in \sec(\boldsymbol{E},\, \mathbb{L} \otimes V\boldsymbol{E})$ cross product Cor 3.2.8
$\mathcal{K}[o] := \frac{1}{2}\, G\,(\nabla[o], \nabla[o]) \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$ observed kinetic energy Def 3.2.9
$\mathcal{Q}[o] := \theta[o] \lrcorner (G^\flat\, \nabla[o]) \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ observed kinetic momentum Def 3.2.9
$\mathcal{C}[o] := -\mathcal{K}[o] + \mathcal{Q}[o] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ observed kinetic Poincaré–Cartan form Def 3.2.9
$\mathfrak{L} \equiv \mathfrak{L}[c, o] := \underline{r} \times \check{\mathfrak{Q}}[o] : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E}$ observed kinetic angular momentum Def 3.2.12
$\varkappa[g] = \varkappa[G] : V\boldsymbol{E} \to V^*\boldsymbol{E} \otimes V_{\boldsymbol{T}} V\boldsymbol{E}$ metric spacelike connection Pro 3.2.13
$C[g] := \bar{g} \lrcorner r : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes \mathbb{R}$ scaler curvature Pro 3.2.13
$C[G] := \bar{G} \lrcorner r : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$ rescaled scaler curvature Pro 3.2.13
$\omega[g] := \check{d}_v g^\square : V\boldsymbol{E} \to \mathbb{L}^2 \otimes \Lambda^2\, V_{\boldsymbol{E}}^* V\boldsymbol{E}$ spacelike symplectic form Pro 3.2.14
$\omega[g] = g \lrcorner \big(\nu[\varkappa] \wedge \nu\big) : V\boldsymbol{E} \to \mathbb{L}^2 \otimes \Lambda^2\, V_{\boldsymbol{E}}^* V\boldsymbol{E}$ spacelike symplectic form Pro 3.2.15
$\mathbf{d} f := g^\sharp(df) : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes V\boldsymbol{E}$ gradient Def 3.2.16
$\vec{d}\, f := G^\sharp(df) : \boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ rescaled gradient Def 3.2.16
$\operatorname{div}_\upsilon X := i_{\bar{\upsilon}}(L_X \upsilon) : \boldsymbol{E} \to \mathbb{R}$ spacetime divergence Def 3.2.17
$\operatorname{div}_\eta X := i_{\bar{\eta}}(L_X \eta) : \boldsymbol{E} \to \mathbb{R}$ spacelike divergence Def 3.2.17
$\operatorname{div}_\upsilon д[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$ spacetime divergence of an observer Def 3.2.17
$\operatorname{div}_\eta д[o] : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathbb{R}$ spacelike divergence of an observer Def 3.2.17
$\operatorname{curl} X := i_{\check{d}(g^\flat(X))}\, \bar{\eta} : \boldsymbol{E} \to \mathbb{L}^{-1} \otimes V\boldsymbol{E}$ curl of spacelike vector fields Def 3.2.19
$\Delta[g] f := \operatorname{div}_\eta \circ g^\sharp \circ \check{d} f : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes \mathbb{R}$ spacelike metric laplacian Def 3.2.20

$\Delta[G]f := \text{div}_\eta \circ G^\sharp \circ \check{d}f : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathrm{I\!R}$ rescaled spacelike metric laplacian Def 3.2.20
$L_{д[o]}G : \boldsymbol{E} \to V^*\boldsymbol{E} \otimes V^*\boldsymbol{E}$ observed Lie derivative of the metric Pro 3.2.21

Chapter 4: Galilean gravitational field

$K : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ spacetime connection Def 4.1.1
$\nu[K] : TT\boldsymbol{E} \to T\boldsymbol{E}$ projection of the connection K Pro 4.1.2
$\overset{\shortparallel}{K} : V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TV\boldsymbol{E}$ 1st vertical restriction Pro 4.1.3
$\check{K} : V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TT\boldsymbol{E}$ 2nd vertical restriction Pro 4.1.3
$\overset{\shortparallel}{K} : V\boldsymbol{E} \underset{\boldsymbol{E}}{\times} V\boldsymbol{E} \to V_{\boldsymbol{E}}V\boldsymbol{E}$ full vertical restriction Pro 4.1.3
$\nabla dt = 0$ condition for time preserving Pro 4.1.6
$R[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E}$ curvature of spacetime connection Pro 4.1.7
$r[K] := C^1_1 R[K] : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}$ Ricci tensor of spacetime connections Def 4.1.8
$C[g] := \bar{g} \lrcorner r[K] : \boldsymbol{E} \to \mathbb{L}^2 \otimes \mathrm{I\!R}$ scalar curvature of spacetime connections Def 4.1.8
$C[G] := \bar{G} \lrcorner r[K] : \boldsymbol{E} \to \mathbb{T}^* \otimes \mathrm{I\!R}$ scalar curvature of spacetime connections Def 4.1.8
$\nu_{T\boldsymbol{E}} : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes V_{\boldsymbol{E}}T\boldsymbol{E}$ natural soldering form of spacetime Lem 4.1.12
$\widehat{\Xi} := G^{\sharp 2}(\Xi) : \boldsymbol{E} \to \mathbb{T}^* \otimes T^*\boldsymbol{E} \otimes V\boldsymbol{E}$ mixed spacetime tensor Not 4.2.4
$T[K] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E} \otimes V\boldsymbol{E}$ torsion of spacetime connections Pro 4.1.13
$\widehat{\Sigma}[G, o] : \boldsymbol{E} \to \mathbb{T}^* \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ observed symmetric spacetime tensor Lem 4.2.9
$\Phi[G, o] : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ observed spacetime form Def 4.2.11
$\mathsf{k}[G, o] : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ distinguished observed spacetime connection Lem 4.2.7
$\mathsf{K}[G, o] : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ distinguished observed spacetime connection Lem 4.2.10
$\underline{R}[G, K] : \boldsymbol{E} \to \mathbb{T} \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes (V^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ covariant curvature Lem 4.2.25
$A[o] \equiv A[K, G, o] \in \text{sec}(\boldsymbol{E}, T^*\boldsymbol{E})$ observed potential The 4.3.3
$K^\natural : T\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TT\boldsymbol{E}$ galilean gravitational field Pos C.3
$\text{div}\, X : \boldsymbol{E} \to \otimes^{r-1} T\boldsymbol{E}$ divergence of contravariant tensors Def 4.4.5
$\text{div}\, \alpha : \boldsymbol{E} \to \mathbb{L}^{-2} \otimes (\otimes^{r-1} T^*\boldsymbol{E})$ divergence of covariant tensors Def 4.4.5
$\text{curl}\, X : \boldsymbol{E} \to \mathbb{L}^{-1} \otimes V\boldsymbol{E}$ curl of time preserving vector field Pro 4.4.7
$\text{curl}\, д[o] : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{L}^{-1}) \otimes V\boldsymbol{E}$ curl of an observer Exa 4.4.8

Chapter 5: Galilean electromagnetic field

$F : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ electromagnetic field Def 5.1.1
$\frac{q}{m} F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^2) \otimes \Lambda^2 T^*\boldsymbol{E}$ rescaled electromagnetic field Not 5.1.2
$\frac{q}{\hbar} F : \boldsymbol{E} \to \Lambda^2 T^*\boldsymbol{E}$ unscaled electromagnetic field Not 5.1.2
$\hat{F} : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$ mixed electromagnetic field Not 5.1.2
$\widehat{F} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes (T^*\boldsymbol{E} \otimes V\boldsymbol{E})$ mixed electromagnetic field Not 5.1.2
$\check{F} : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V^*\boldsymbol{E}$ magnetic 2-form Def 5.2.1
$\bar{F} : \boldsymbol{E} \to (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 V\boldsymbol{E}$ magnetic 2-vector Def 5.2.1
$B : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\boldsymbol{E}$ magnetic 1-form Def 5.2.1
$\mathbf{B} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}$ magnetic vector field Def 5.2.1
$\check{F}[o] : \boldsymbol{E} \to (\mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes \Lambda^2 T^*\boldsymbol{E}$ observed magnetic 1-form Not 5.2.2
$B[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$ observed magnetic 1-form Not 5.2.2
$E[o] := -д[o] \lrcorner F : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{1/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$ observed electric form Def 5.3.1
$\mathbf{E}[o] := g^\sharp(\check{E}[o]) : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}$ observed electric field Def 5.3.1
$\mathcal{F} : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-2} \otimes \mathbb{M}) \otimes \mathrm{I\!R}$ 1st electromagnetic algebraic invariant Def 5.6.1

$\check{F}^2 : \boldsymbol{E} \to (\mathbb{L}^{-3} \otimes \mathbb{M}) \otimes \mathbb{R}$ 2nd electromagnetic algebraic invariant Def 5.6.3
$\mathbb{M}^{\mathfrak{e}} : \boldsymbol{E} \to (\mathbb{L}^{-1} \otimes \mathbb{M}) \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ 3rd electromagnetic algebraic invariant Def 5.6.6
$\mathrm{C}^{\mathfrak{e}} : \boldsymbol{E} \to (\mathbb{L}^{-3} \otimes \mathbb{M}) \otimes V^*\boldsymbol{E} \otimes V\boldsymbol{E}$ 4th electromagnetic algebraic invariant Def 5.6.10
$\mathrm{f} \equiv \mathrm{f}[s, q] := -\frac{1}{2}\, q\, i_{ds} F : \boldsymbol{T} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^2 \otimes \mathbb{M}) \otimes T^*\boldsymbol{E}$ Lorentz form Def 5.7.1
$\mathrm{f} \equiv \mathrm{f}[s, q] := -\frac{1}{2}\, q\, g^\sharp(i_{ds} F) : \boldsymbol{T} \to (\mathbb{T}^{-2} \otimes \mathbb{M}) \otimes V\boldsymbol{E}$ Lorentz force Def 5.7.1
$\mathrm{f} \equiv \mathrm{f}[\mathscr{V}, \rho] := -\frac{1}{2}\, \rho\, i_{\mathscr{V}} F : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-1} \otimes \mathbb{M}) \otimes T^*\boldsymbol{E}$ Lorentz form density Def 5.7.3
$\mathrm{f} \equiv \mathrm{f}[\mathscr{V}, \rho] := -\frac{1}{2}\, \rho\, g^\sharp(i_{\mathscr{V}} F) : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes V\boldsymbol{E}$ Lorentz force density Def 5.7.3
$\mathrm{div}^{\natural} E[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}$ divergence of the electric field Lem 5.9.1
$\mathrm{div}_\eta\, \mathbf{E}[o] : \boldsymbol{E} \to (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}$ divergence of the electric field Lem 5.9.1
$\mathrm{curl}\, \mathbf{B} := i_{\check{d}B}\, \bar{\eta} : \boldsymbol{E} \to (\mathbb{L}^{-7/2} \otimes \mathbb{M}^{1/2}) \otimes V\boldsymbol{E}$ curl of the magnetic field Def 5.9.2
$\overset{\vee}{\mathrm{div}}{}^{\natural} \check{F} : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes V^*\boldsymbol{E}$ divergence of the electromagnetic field Lem 5.9.3
$\mathrm{div}^{\natural} \check{F}[o] : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$ divergence of the magnetic field Lem 5.9.4
$\mathrm{div}^{\natural} F : \boldsymbol{E} \to (\mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes T^*\boldsymbol{E}$ divergence of the magnetic field Lem 5.9.4

Chapter 6: Joined spacetime connection

$\mathrm{k} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}$ galilean joining constant scale Def 6.1.1
$\mathrm{k} := \frac{q}{m} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}$ electromagnetic joining constant scale Def 6.1.1
$\mathrm{k} := \sqrt{\mathrm{r}} \in (\mathbb{T}^{-1} \otimes \mathbb{L}^{3/2} \otimes \mathbb{M}^{-1/2}) \otimes \mathbb{R}$ gravitational joining constant scale Def 6.1.1
$\mathrm{k}\, dt \otimes \hat{F} : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E}$ joining electromagnetic term Lem 6.2.1
$\mathrm{k}\, \hat{F} \otimes dt : \boldsymbol{E} \to T^*\boldsymbol{E} \otimes V\boldsymbol{E} \otimes T^*\boldsymbol{E}$ joining electromagnetic term Lem 6.2.1
$K \equiv K^{\natural} + K^{\mathfrak{e}} := K^{\natural} - \frac{1}{2}\, \mathrm{k}\, (dt \otimes \hat{F} + \hat{F} \otimes dt)$ joined spacetime connection The 6.3.1
$\Phi[o] = \Phi^{\natural}[o] + \frac{m}{\hbar}\, \mathrm{k}\, F$ joined spacetime 2-form Cor 6.3.3
$A[o] = A^{\natural}[o] + A^{\mathfrak{e}}$ joined observed potential Cor 6.3.3
$R[K] = -[K^{\natural}, K^{\natural}] - 2\,[K^{\natural}, K^{\mathfrak{e}}] - [K^{\mathfrak{e}}, K^{\mathfrak{e}}]$ joined spacetime curvature The 6.4.3
$r[K] = r[K^{\natural}] - \frac{1}{2}\, \mathrm{k}\, dt \otimes \mathrm{div}^{\natural} F - \frac{1}{2}\, \mathrm{k}\, \mathrm{div}^{\natural} F \otimes dt - \frac{1}{4}\, \mathrm{k}^2\, \check{F}^2\, dt \otimes dt$ joined Ricci tensor The 6.5.2

Chapter 7: Classical dynamics

$d^2 s : \boldsymbol{T} \to \mathbb{T}^* \otimes T(\mathbb{T}^* \otimes T\boldsymbol{E})$ iterated velocity of particle motions Def 7.1.1
$\nabla^{\natural}{}_{ds} ds := \nu[K^{\natural}] \circ d^2 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ gravitational acceleration Def 7.1.3
$\nabla_{ds} ds := \nu[K] \circ d^2 s : \boldsymbol{T} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ joined acceleration Def 7.1.3
$d[o]s := \theta[o] \circ ds : \boldsymbol{T} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ observed velocity of particle motions Def 7.1.6
$s[o] := p[o] \circ s : \boldsymbol{T} \to \boldsymbol{P}[o]$ observed motion Def 7.1.7
$d\big(s[o]\big) : \boldsymbol{T} \to \mathbb{T}^* \otimes T\boldsymbol{P}[o]$ velocity of the observed motion Def 7.1.7
$ds = д[o] \circ s + d[o]s$ splitting of the observed velocity of particle motions Not 7.1.8
$\mathrm{A}_{\mathrm{obs}}[o] := \nabla_{ds}\, (d[o]s)$ observed acceleration Def 7.1.10
$\mathrm{A}_{\mathrm{rlt}}[o] := \nabla_{d[o]s}\, (д[o] \circ s)$ observed relative acceleration Def 7.1.10
$\mathrm{A}_{\mathrm{drg}}[o] := \nabla^{\natural}{}_{д[o]\circ s}\, (д[o] \circ s)$ observed dragging acceleration Def 7.1.10
$\mathrm{A}_{\mathrm{dfr}}[o] := -\frac{1}{2}\, \big(d[o]s \lrcorner \widehat{\Sigma}{}^{\natural}[o]\big)$ observed deformation acceleration Def 7.1.10
$\mathrm{A}_{\mathrm{crl}}[o] := -\frac{1}{2}\, \big(d[o]s \lrcorner \widehat{\Phi}{}^{\natural}[o]\big)$ observed Coriolis acceleration Def 7.1.10
$\nabla_{ds}\, ds = \mathrm{A}_{\mathrm{obs}}[o] + \mathrm{A}_{\mathrm{rlt}}[o] + \mathrm{A}_{\mathrm{drg}}[o]$ observed splitting of the acceleration Pro 7.1.11
$G^{\flat}(\nabla^{\natural}{}_{ds}\, ds) = -\frac{q}{\hbar}\, (ds \lrcorner F)$ classical law of motion Lem 7.2.1

$G^{\flat}(\nabla_{ds}\, ds) = 0$ classical law of motion Lem 7.2.1

$\mathscr{C} : (\mathbb{T} \times \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E}$ continuum motion Def 7.3.1

$\mu \in \mathrm{map}(\boldsymbol{E},\ \mathbb{L}^{-3} \otimes \mathbb{M} \otimes \mathbb{R})$ mass density Def 7.3.1

$\rho \in \mathrm{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \mathbb{R}\big)$ charge density Def 7.3.1

$\mathrm{p} \in \mathrm{map}\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{2}) \otimes \mathbb{R}\big)$ pressure density Def 7.3.1

$\mathscr{V} := \partial\mathscr{C} \in \sec(\boldsymbol{E},\ \mathbb{T}^{*} \otimes \boldsymbol{T}\boldsymbol{E})$ velocity of a continuum Def 7.3.2

$\mathscr{P} := \mu\, \mathscr{V} \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-1} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes \boldsymbol{T}\boldsymbol{E}\big)$ mass density current Def 7.3.2

$\mathscr{J} := \rho\, \mathscr{V} \in \sec\big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \boldsymbol{T}\boldsymbol{E}\big)$ charge density current Def 7.3.2

$\mathscr{T}^{\mathfrak{m}} := \mu\, \mathscr{V} \otimes \mathscr{V} - \mathrm{p}\, \bar{g} : \boldsymbol{E} \to (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (\boldsymbol{T}\boldsymbol{E} \otimes \boldsymbol{T}\boldsymbol{E})$ energy–momentum Def 7.3.2

$\underline{\mathscr{T}}^{\mathfrak{m}} := \mathbf{g}^{\flat}(\mathscr{T}^{\mathfrak{m}}) = \mu\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^{2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes H^{*}\boldsymbol{E} \otimes H^{*}\boldsymbol{E}$ energy tensor Def 7.3.2

$\mathscr{A}^{\natural} := \nabla^{\natural}{}_{\mathscr{V}}\, \mathscr{V} : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ gravitational acceleration of a continuum Def 7.3.4

$\mathscr{A} := \nabla_{\mathscr{V}}\, \mathscr{V} : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ joined acceleration of a continuum Def 7.3.4

$\mathscr{C}[o] : \bar{\mathbb{T}} \times \big(\boldsymbol{T} \times \boldsymbol{P}[o]\big) \to \boldsymbol{P}[o]$ observed continuum motion Def 7.3.7

$\mathscr{A}_{\mathrm{lag}}[o] := \nabla_{\mathscr{V}}\, \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ lagrangian acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{eul}}[o] := \nabla_{\text{д}[o]} \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ eulerian acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{spc}}[o] := \nabla_{\vec{\mathscr{V}}[o]} \vec{\mathscr{V}}[o] : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ spacelike acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{rlt}}[o] := \nabla_{\vec{\mathscr{V}}[o]} \text{д}[o] : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ relative acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{drg}}[o] := \nabla_{\text{д}[o]} \text{д}[o] : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ dragging acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{dfr}}[o] := -\frac{1}{2}\big(\vec{\mathscr{V}}[o] \lrcorner \widehat{\Sigma}^{\natural}[o]\big) : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ deformation acceleration Def 7.3.8

$\mathscr{A}_{\mathrm{crl}}[o] := -\frac{1}{2}\big(\vec{\mathscr{V}}[o] \lrcorner \widehat{\Phi}^{\natural}[o]\big) : \boldsymbol{E} \to (\mathbb{T}^{*} \otimes \mathbb{T}^{*}) \otimes V\boldsymbol{E}$ Coriolis acceleration Def 7.3.8

$\mathrm{div}_{\upsilon}\, \mathscr{P} = 0$ mass continuity equation Ass C.3

$\mathrm{div}_{\upsilon}\, \mathscr{J} = 0$ charge continuity equation Ass C.3

$\mu\, \mathscr{A}^{\natural} = -\rho\, g^{\sharp}(\mathscr{V} \lrcorner F) + \mu\, \mathbf{d}\,\mathrm{p}$ equation of motion Ass C.3

$\mu\, \mathscr{A} = \mu\, \mathbf{d}\,\mathrm{p}$ equation of motion Pro 7.4.2

Chapter 8: Sources of gravitational and electromagnetic fields

$\mathscr{J} := \rho\, \mathscr{V} \in \big(\boldsymbol{E},\ (\mathbb{T}^{-2} \otimes \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes \boldsymbol{T}\boldsymbol{E}\big)$ contravariant electric current Def 8.1.1

$\underline{\mathscr{J}} := \mathbf{g}^{\flat}(\mathscr{J}) = \rho\, dt \in \sec\big(\boldsymbol{E},\ \mathbb{L}^{-3/2} \otimes \mathbb{M}^{1/2}) \otimes H^{*}\boldsymbol{E}\big)$ covariant electric current Def 8.1.1

$\mathrm{div}^{\natural} F = \underline{\mathscr{J}}^{\mathfrak{q}} \equiv \rho\, dt$ Galilei–Maxwell equation Pos C.5

$r^{\natural} - \frac{1}{2}\, C^{\natural}\, \mathfrak{g} = \mathrm{r}\,(\mu + \frac{1}{4}\, \check{F}^{2})\, dt \otimes dt$ Galilei–Einstein equation Pro 8.2.1

$r^{\natural} = \mathrm{r}\,(\mu + \frac{1}{4}\, \check{F}^{2})\, dt \otimes dt$ Galilei–Einstein equation Pro 8.2.1

$\underline{\mathscr{T}}^{\mathfrak{m}} := \mu \otimes dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^{2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T^{*}\boldsymbol{E} \otimes T^{*}\boldsymbol{E})$ matter energy Def 7.3.3

$\underline{\mathscr{T}}^{\mathfrak{e}} := \frac{1}{4}\, \check{F}^{2}\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^{2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T^{*}\boldsymbol{E} \otimes T^{*}\boldsymbol{E})$ matter energy Def 5.6.3

$\underline{\mathscr{T}}^{\mathfrak{q}} := \mathrm{k}^{-1}\, \rho\, dt \otimes dt : \boldsymbol{E} \to (\mathbb{T}^{2} \otimes \mathbb{L}^{-3} \otimes \mathbb{M}) \otimes (T^{*}\boldsymbol{E} \otimes T^{*}\boldsymbol{E})$ joined energy density Def 8.3.1

$\underline{\mathscr{T}} := \underline{\mathscr{T}}^{\mathfrak{m}} + \underline{\mathscr{T}}^{\mathfrak{q}} = (\mu + \mathrm{k}^{-1}\, \rho)\, dt \otimes dt$ joined energy density Def 8.3.1

$r = r - \frac{1}{2}\, C\, \mathfrak{g} = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\, \rho)\, dt \otimes dt$ joined Galilei–Einstein equation Pro 8.3.3

$r = \mathrm{r}\,(\mu + \mathrm{r}^{-1/2}\,\rho)\,dt \otimes dt$ joined Galilei–Einstein equation Pro 8.3.3

Chapter 9: Fundamental fields of the phase space

$\Gamma : J_1\boldsymbol{E} \to T^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ phase connection Def 9.1.1
$T[\Gamma] := 2\,[\Gamma, \theta] : \boldsymbol{E} \to \Lambda^2\, T^*\boldsymbol{E} \otimes V\boldsymbol{E}$ torsion of a phase connection Def 9.1.1
$\nu[\Gamma] : TJ_1\boldsymbol{E} \to \mathbb{T}^* \otimes V\boldsymbol{E}$ vertical projection of a phase connection Def 9.1.1
$\check{\Gamma} : J_1\boldsymbol{E} \to V^*\boldsymbol{E} \otimes TJ_1\boldsymbol{E}$ vertical restriction of a phase connection Def 9.1.1
$\gamma : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ dynamical phase connection Def 9.1.2
$\nu[\gamma] : TJ_1\boldsymbol{E} \to V_{\boldsymbol{T}}J_1\boldsymbol{E}$ vertical projection of a dynamical phase connection Def 9.1.2
$\Omega \equiv \Omega[G, \Gamma] := G \lrcorner \big(\nu[\Gamma] \wedge \theta\big) : J_1\boldsymbol{E} \to \Lambda^2\, T^*J_1\boldsymbol{E}$ dynamical phase 2-form Def 9.1.3
$\Lambda \equiv \Lambda[G, \Gamma] := \bar{G} \lrcorner (\check{\Gamma} \wedge \nu) : J_1\boldsymbol{E} \to \Lambda^2 V_{\boldsymbol{T}}J_1\boldsymbol{E}$ dynamical phase 2-vector Def 9.1.4
$dt \wedge \Omega \wedge \Omega \wedge \Omega : J_1\boldsymbol{E} \to \mathbb{T} \otimes \Lambda^7\, T^*J_1\boldsymbol{E}$ phase volume form Pro 9.1.5
$\gamma \wedge \Lambda \wedge \Lambda \wedge \Lambda : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes \Lambda^7 TJ_1\boldsymbol{E}$ phase volume vector Pro 9.1.5
$\Gamma \equiv \Gamma[K] : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to TJ_1\boldsymbol{E}$ phase connection associated with K The 9.2.1
$\gamma[K] : J_1\boldsymbol{E} \to \mathbb{T}^* \otimes TJ_1\boldsymbol{E}$ dynamical phase connection associated with K Cor 9.2.4
$\Omega[K] : J_1\boldsymbol{E} \to \Lambda^2 T^*J_1\boldsymbol{E}$ dynamical phase 2-form associated with K Cor 9.2.4
$\Lambda[K] : J_1\boldsymbol{E} \to \Lambda^2 V_{\boldsymbol{T}}J_1\boldsymbol{E}$ dynamical phase connection associated with K Cor 9.2.4
$\Gamma^{\mathfrak{e}} = -\frac{1}{2}\,\frac{q}{m}\, g^{\sharp 2}\,(i_{д}F + \theta \lrcorner F)$ electromagnetic component of $\Gamma[K]$ The 9.2.5
$\gamma^{\mathfrak{e}} = д \lrcorner \Gamma^{\mathfrak{e}} = -\frac{q}{\hbar}\, G^{\sharp}\,(д \lrcorner F)$ electromagnetic component of $\gamma[K]$ The 9.2.6
$\Omega^{\mathfrak{e}} = \frac{1}{2}\,\frac{q}{\hbar}\,F$ electromagnetic component of $\Omega[K]$ The 9.2.8
$\Lambda^{\mathfrak{e}} = \frac{1}{2}\,\frac{q}{m}\,\frac{\hbar}{m}\,\bar{F}$ electromagnetic component of $\Lambda[K]$ The 9.2.11

Chapter 10: Geometric structures of phase space

(dt, Ω) cosymplectic structure of phase space The 10.1.1
$\Phi[o] = 2\,o^*\Omega$ Φ and Ω Pro 10.1.2
$\Omega = dA^{\uparrow}$ upper potential Def 10.1.3
$A^{\uparrow} \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E}) \subset \sec(J_1\boldsymbol{E}, T^*J_1\boldsymbol{E})$ horizontal upper potential The 10.1.4
$A[o] := o^*A^{\uparrow}$ observed potential and upper potential The 10.1.4
$\mathcal{L}[\mathrm{b}] := д \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$ classical lagrangian The 10.1.8
$\mathcal{M}[\mathrm{b}] := \theta \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ classical momentum The 10.1.8
$\mathcal{H}[\mathrm{b}, o] := -\,д[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, H^*\boldsymbol{E})$ observed hamiltonian The 10.1.8
$\mathcal{P}[\mathrm{b}, o] := \theta[o] \lrcorner A^{\uparrow}[\mathrm{b}] \in \sec(J_1\boldsymbol{E}, T^*\boldsymbol{E})$ observed momentum The 10.1.8
$\frac{1}{2}\,\mathcal{P}^2[\mathrm{b}] := \frac{1}{2}\,\bar{G}\,\big(\mathcal{P}[\mathrm{b}, o], \mathcal{P}[\mathrm{b}, o]\big) = \mathcal{L}[\mathrm{b}] - \alpha[\mathrm{b}]$ square of the observed momentum Cor 10.1.10
$\alpha[\mathrm{b}] = д[o] \lrcorner A[\mathrm{b}, o] - \frac{1}{2}\,\bar{G}\big(A[\mathrm{b}, o], A[\mathrm{b}, o]\big) \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$ timelike 1-form Not 10.1.11
$(\gamma, \Lambda) \equiv \big(\gamma[K], \Lambda[G, K]\big)$ coPoisson structure of phase space The 10.2.1

Chapter 11: Hamiltonian formalism

$TJ_1\boldsymbol{E} = H_{\gamma}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{T}}J_1\boldsymbol{E}$ phase splitting Pro 11.1.1
$T^*J_1\boldsymbol{E} = H^*_{\boldsymbol{T}}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V^*_{\gamma}J_1\boldsymbol{E}$ phase splitting Pro 11.1.1
$TJ_1\boldsymbol{E} = H_{\Gamma}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{E}}J_1\boldsymbol{E}$ phase splitting Pro 11.1.2
$T^*J_1\boldsymbol{E} = H^*_{\boldsymbol{E}}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V^*_{\Gamma}J_1\boldsymbol{E}$ phase splitting Pro 11.1.2
$TJ_1\boldsymbol{E} = H_{\gamma}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\Gamma}J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V_{\boldsymbol{E}}J_1\boldsymbol{E}$ phase splitting Pro 11.1.4

$T^*J_1\boldsymbol{E} = H^*_{\boldsymbol{T}} J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} H^*_\gamma J_1\boldsymbol{E} \underset{J_1\boldsymbol{E}}{\oplus} V^*_\Gamma J_1\boldsymbol{E}$ phase splitting Pro 11.1.4
$\Omega^\flat : TJ_1\boldsymbol{E} \to T^*J_1\boldsymbol{E} : X^\uparrow \mapsto i_{X^\uparrow}\,\Omega$ linear phase musical morphism Def 11.2.1
$\Lambda^\sharp : T^*J_1\boldsymbol{E} \to TJ_1\boldsymbol{E} : \alpha^\uparrow \mapsto i_{\alpha^\uparrow}\,\Lambda$ linear phase musical morphism Def 11.2.1
$H_\tau J_1\boldsymbol{E} := \{X^\uparrow \in TJ_1\boldsymbol{E}) \mid i_{X^\uparrow}\,dt = \tau\} \subset TJ_1\boldsymbol{E}$ τ-horizontal subbundle Def 11.2.9
${\Omega^\flat}_\tau : H_\tau J_1\boldsymbol{E} \to V^*_\gamma J_1\boldsymbol{E} : X^\uparrow \mapsto i_{X^\uparrow}\,\Omega$ affine phase musical morphism The 11.2.11
${\Lambda^\sharp}_\tau : V^*_\gamma J_1\boldsymbol{E} \to H_\tau J_1\boldsymbol{E} : \alpha^\uparrow \to \gamma(\tau) + \Lambda^\sharp(\alpha^\uparrow)$ affine phase musical morphism The 11.2.11
${X^\uparrow}_{\mathrm{ham}}[\tau, f] := \tau \lrcorner \gamma + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ scaled phase lift Def 11.3.1
${X^\uparrow}_{\mathrm{ham}}[f] = \gamma(f'') + \Lambda^\sharp(df) \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ hamiltonian phase lift Def 11.3.6
$\{f, \acute{f}\} = \big(\Lambda^\sharp(df)\big).\acute{f} = -\big(\Lambda^\sharp(d\acute{f})\big).f \in \mathrm{map}(J_1\boldsymbol{E}, \mathbb{R})$ Poisson bracket Def 11.4.1
$\mathcal{E}\big[\mathcal{L}[\mathfrak{b}]\big] : J_2\boldsymbol{E} \to V^*\boldsymbol{E}$ classical Euler–Lagrange operator Lem 11.5.2

Chapter 12: Lie algebra of special phase functions

$\mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$ sheaf of special phase functions Def 12.1.1
$\mathrm{pro\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ subsheaf of projectable special phase functions Def 12.1.3
$\mathrm{tim\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ subsheaf of time preserving special phase functions Def 12.1.3
$\mathrm{aff\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ subsheaf of affine special phase functions Def 12.1.3
$\mathrm{map\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ subsheaf of spacetime special phase functions Def 12.1.3
$\mathrm{hol\,sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ subsheaf of holonomic vector fields Def 12.3.2
$\mathrm{ham\,sec}(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ subsheaf of hamiltonian vector fields Def 12.4.1
$X[f] = f^0\,\partial_0 - f^i\,\partial_i$ tangent lift of special phase functions The 12.2.1
$\mathrm{div}_\eta f := \mathrm{div}_\eta X[f] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ divergence of special phase functions Def 12.2.7
$r^1 : J_1T\boldsymbol{F} \to TJ_1\boldsymbol{F}$ exchange map Pro 12.3.1
$\bar{f} \equiv \bar{f}[o] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ observed component of special phase function Pro 12.2.9
$\breve{f} \equiv \breve{f}[o] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ observed component of special phase function Pro 12.2.9
$\tilde{\iota} \equiv \tilde{f}[\mathfrak{b}] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ gauge component of special phase function Pro 12.2.9
$\hat{f} \equiv \hat{f}[\mathfrak{b}] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ gauge component of special phase function Pro 12.2.9
${X^\uparrow}_{\mathrm{hol}}[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ holonomic phase lift of special phase functions Pro 12.3.3
${X^\uparrow}_{\mathrm{ham}}[f] \in \sec(J_1\boldsymbol{E}, TJ_1\boldsymbol{E})$ hamiltonian phase lift of special phase functions Def 12.4.1
$[\![f, \acute{f}]\!]$ special phase Lie bracket Def 12.5.1
$\big[(X, \phi)\,,\ (\acute{X}, \acute{\phi})\big]_\Phi$ Lie bracket of pairs Not 12.5.6
$\mathrm{cns\,spe}(J_1\boldsymbol{E}, \mathbb{R})$ subsheaf of conserved special phase functions Def 12.6.10
$Z^1 := r^1 \circ J_1Z \in \mathrm{pro}_{\boldsymbol{F},\boldsymbol{B}} \sec(J_1\boldsymbol{F}, TJ_1\boldsymbol{F})$ 1-jet holonomic prolongation of Z Pro 12.3.1

Chapter 13: Classical symmetries

$\mathfrak{c}[f, \mathfrak{b}] := -\,i_{X[f]}A^\uparrow[\mathfrak{b}] \in \mathrm{srt}_{\mathfrak{b}}\,\mathrm{spe}(J_1\boldsymbol{E}, \mathbb{R})$ classical current Def 13.3.1

PART II: COVARIANT QUANTUM MECHANICS

Chapter 14: Quantum bundle

$\pi : \boldsymbol{Q} \to \boldsymbol{E}$ quantum bundle Pos Q.1
$\mathfrak{g}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^{-3} \otimes (\boldsymbol{Q}^* \otimes \boldsymbol{Q}^*)$ real quantum metric Pos Q.1
$\| \| : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{R} : \Psi_e \mapsto \sqrt{|\mathfrak{g}(\Psi_e, \Psi_e)|}$ quantum norm Pro 14.1.3
$\bar{\eta}_{\boldsymbol{Q}} : \boldsymbol{E} \to \mathbb{L}^3 \otimes \Lambda^2\boldsymbol{Q}$ quantum volume form Pro 14.1.3
$\Psi \wedge \acute{\Psi} : \boldsymbol{E} \to \Lambda^2\boldsymbol{Q}$ volume vector Pro 14.1.3
$\mathfrak{b}_{\mathfrak{a}} : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$ real quantum base Def 14.1.4

$w^{a} : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{R}$ real linear coordinates Def 14.1.4
$(\partial_\lambda, \partial w_a) = (\partial_\lambda, \partial w_1, \partial w_2)$ local basis of vector fields Def 14.1.4
$(d^\lambda, dw^a) = (d^\lambda, dw^1, dw^2)$ local basis of forms Def 14.1.4
$\bar{\eta}_{\boldsymbol{Q}} = \mathfrak{b}_1 \wedge \mathfrak{b}_2$ real quantum volume vector Pro 14.1.5
$\mathfrak{i} : \boldsymbol{Q} \to \boldsymbol{Q} : q \mapsto i_{g^\flat_{\boldsymbol{Q}}(q)} \bar{\eta}_{\boldsymbol{Q}}$ imaginary unit Pro 14.2.1
$\mathfrak{b} := \mathfrak{b}_1 : \boldsymbol{E} \to \mathbb{L}^{3/2} \otimes \boldsymbol{Q}$ complex quantum basis Pro 14.2.1
$z : \boldsymbol{Q} \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$ complex linear coordinate Pro 14.2.1
$\mathfrak{h} : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \mathbb{L}^{-3} \otimes \mathbb{C}$ hermitian quantum metric Pro 14.3.1
$\mathfrak{h}(\Psi_e, \acute{\Psi}_e) = g_{\boldsymbol{Q}}(\Psi_e, \acute{\Psi}_e) + \mathfrak{i}\,(\Psi_e \wedge \acute{\Psi}_e)/\bar{\eta}_{\boldsymbol{Q}})$ hermitian quantum metric Pro 14.3.1
$\mathfrak{h}_\eta := \mathfrak{h} \otimes \eta : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}$ η-hermitian quantum metric Def 14.5.1
$\pi_{/0} : \boldsymbol{Q}_{/0} \subset \boldsymbol{Q} \to \boldsymbol{E}$ proper quantum bundle Def 14.6.1
$\varrho : \boldsymbol{Q}_{/0} \to \mathbb{L}^{-3/2} \otimes \mathbb{R}^+, \quad \phi : \boldsymbol{Q}_{/0} \to \mathbb{R}/2\pi$ quantum polar coordinates Def 14.6.2
$\Psi = |\psi|\, e^{\mathfrak{i}\varphi}\, \mathfrak{b}$ polar expression of Ψ Def 14.6.2
$\mathfrak{b}_\Psi \equiv \mathfrak{b}[\Psi] := \Psi/\|\Psi\|\, \sec(\boldsymbol{E}, \mathbb{L}^{3/2} \otimes \boldsymbol{Q})$ distinguished quantum basis Not 14.6.3
$\pi^{\|\|}_{/0} : \boldsymbol{Q}^{\|\|}_{/0} = \boldsymbol{E} \times \mathbb{L}^{-3/2} \to \boldsymbol{E}$ norm quantum bundle Pro 14.7.1
$\pi^{(())}_{/0} : \boldsymbol{Q}^{(())}_{/0} \to \boldsymbol{E}$ phase quantum bundle Pro 14.7.1
$\boldsymbol{Q}^{(())}_{/0} \to \boldsymbol{E} \times \mathbb{R}/2\pi : e^{\mathfrak{i}\varphi}\, \mathfrak{b} \mapsto \varphi$ phase projection Pro 14.7.1
$(()) : \boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{(())}_{/0} : \Psi_e \mapsto ((\Psi))_e := \Psi_e/\|\Psi\|_e$ phase projection Pro 14.7.1
$\boldsymbol{Q}_{/0} \to \boldsymbol{Q}^{\|\|}_{/0} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q}^{(())}_{/0} : \Psi_e \mapsto (\|\Psi\|_e, ((\Psi))_e)$ polar splitting Pro 14.7.2
$\mathfrak{h}(\Psi, \acute{\Psi}) = \|\Psi\|\, \|\acute{\Psi}\| + \mathfrak{i}\,(((\acute{\Psi})) - ((\Psi)))$ hermitian quantum metric Pro 14.7.5
$\Psi \in \sec(\boldsymbol{E}, \boldsymbol{Q})$ quantum section Ass Q.1
$\psi := z \circ \Psi \in \mathrm{map}(\boldsymbol{E}, \mathbb{L}^{-3/2} \otimes \mathbb{C})$ quantum section Ass Q.1
$\mathbb{I} = w^a\, \partial w_a = z\, \partial_z$ Liouville quantum vector field Def 14.10.1
$\mathbb{I} = w^a\, \partial w_a = z\, \partial_z$ Liouville quantum vector field Def 14.10.1
$\pi^\uparrow : \boldsymbol{Q}^\uparrow := J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to J_1\boldsymbol{E}$ upper quantum bundle Def 14.11.1
$\mathfrak{h}^\uparrow : \boldsymbol{Q}^\uparrow \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^\uparrow \to \mathbb{L}^{-3/2} \otimes \mathbb{C}$ upper quantum hermitian metric Not 14.11.3
$\mathfrak{h}^\uparrow{}_\eta : \boldsymbol{Q}^\uparrow \underset{J_1\boldsymbol{E}}{\times} \boldsymbol{Q}^\uparrow \to \Lambda^3 V^* \boldsymbol{E} \otimes \mathbb{C}$ upper quantum η-hermitian metric Not 14.11.3
$\mathbb{I}^\uparrow : \boldsymbol{Q}^\uparrow \to V_{J_1\boldsymbol{E}}\, \boldsymbol{Q}^\uparrow$ upper quantum Liouville vector field Not 14.11.3

Chapter 15: Galilean upper quantum connection

$\text{Ч} : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ quantum connection Def 15.1.1
$\xi : J_1\boldsymbol{E} \underset{\boldsymbol{E}}{\times} \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ system of quantum connection Def 15.1.1
$\text{Ч}[o] := o^*\xi : \boldsymbol{Q} \underset{\boldsymbol{E}}{\times} T\boldsymbol{E} \to T\boldsymbol{Q}$ observed quantum connection Def 15.1.1
$\text{Ч}^\uparrow : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ upper quantum connection Def 15.1.5
$\chi[\mathfrak{b}] : \boldsymbol{Q} \to T^*\boldsymbol{E} \otimes T\boldsymbol{Q}$ quantum connection associated with $\mathfrak{b}$ Lem 15.1.14
$\chi^\uparrow[\mathfrak{b}] : \boldsymbol{Q}^\uparrow \to T^*J_1\boldsymbol{E} \otimes T\boldsymbol{Q}^\uparrow$ upper quantum connection associated with $\mathfrak{b}$ Lem 15.1.14
$\text{Ч} = \chi[\mathfrak{b}] + \check{\text{Ч}}[\mathfrak{b}]$ gauge splitting of a quantum connection Pro 15.1.15
$\text{Ч}^\uparrow = \chi^\uparrow[\mathfrak{b}] + \check{\text{Ч}}^\uparrow[\mathfrak{b}]$ gauge splitting of an upper quantum connection Pro 15.1.16
$R[\text{Ч}] : \boldsymbol{Q} \to \Lambda^2 T^*\boldsymbol{E} \otimes \boldsymbol{Q}$ curvature of a quantum connection Pro 15.1.17
$R[\text{Ч}] = -2\,\mathfrak{i}\, dA[\mathfrak{b}] \otimes \mathbb{I}$ curvature of a hermitian quantum connection Pro 15.1.17
$R[\text{Ч}^\uparrow] = -2\,\mathfrak{i}\, dA^\uparrow[\mathfrak{b}] \otimes \mathbb{I}^\uparrow$ curvature of a reduc. herm. upper quantum connection Pro 15.1.18
$\text{Ч}^\uparrow = \chi^\uparrow[\mathfrak{b}] + \mathfrak{i}\,(-\mathcal{H}[\mathfrak{b}, o] + \mathcal{P}[\mathfrak{b}, o]) \otimes \mathbb{I}^\uparrow$ galilean upper quantum connection The 15.2.4

$Ч^{\uparrow} = \chi^{\uparrow}[\mathrm{b}] + \mathrm{i}\big(-\mathcal{K}[o] + \mathcal{Q}[o] + A[\mathrm{b}, o]\big) \otimes \mathbb{I}^{\uparrow}$ galilean upper quantum connection The 15.2.4
$\nu[\mathrm{b}] := д[o] - G^{\sharp}(\check{A}[\mathrm{b}, o]) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$ invariant Cor 15.2.28
$\alpha[\mathrm{b}] := д[o] \lrcorner A[\mathrm{b}, o] - \frac{1}{2}\bar{G}\big(\check{A}[\mathrm{b}, o], \check{A}[\mathrm{b}, o]\big) \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$ invariant Cor 15.2.28
$o_{\Psi} \equiv o[\Psi] := o\big[\mathrm{b}_{\Psi}\big] \in \sec(\boldsymbol{E}, J_1\boldsymbol{E})$ distinguished observer The 15.2.31
$A[\Psi] := A\big[\mathrm{b}_{\Psi}\big] \in \sec(\boldsymbol{E}, H^*\boldsymbol{E})$ distinguished timelike potential The 15.2.31
$\underline{Ч}^{\uparrow} := \gamma \lrcorner Ч^{\uparrow} : \boldsymbol{Q}^{\uparrow} \to \mathbb{T}^* \otimes T\boldsymbol{Q}^{\uparrow}$ upper quantum connection over time Def 15.3.2

Chapter 16: Quantum differentials

$\nabla[o]\Psi := \nabla\big[Ч[o]\big]\Psi \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes \boldsymbol{Q})$ observed covariant differential Pro 16.1.1
$\vec{\nabla}[o]\Psi := G^{\sharp} \circ \check{\nabla}[o]\Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes V\boldsymbol{E} \otimes \boldsymbol{Q})$ spacelike observed covariant diff. Def 16.1.3
$\nabla^{(\!)}[o]((\Psi)) \in \sec(\boldsymbol{E}, T^*\boldsymbol{E})$ observed phase differential Pro 16.1.14
$\nabla[o]\Psi/\Psi \in \sec(\boldsymbol{E}, \mathbb{C} \otimes T^*\boldsymbol{E})$ complex form Lem 16.1.20
$\nabla^2[o]\Psi \in \sec\big(\boldsymbol{E}, (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E}) \otimes \boldsymbol{Q}\big)$ 2nd observed differential Pro 16.2.1
$\nabla^{(\!)2}[o]((\Psi)) \in \sec(\boldsymbol{E}, T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})$ 2nd observed phase differential Pro 16.2.5
$\nabla^2[o]\Psi/\Psi \in \sec\big(\boldsymbol{E}, \mathbb{C} \otimes (T^*\boldsymbol{E} \otimes T^*\boldsymbol{E})\big)$ complex tensor Lem 16.2.9
$\Delta[o]\Psi \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$ observed quantum laplacian Def 16.3.1
$\Delta^{(\!)}[G, o]((\Psi)) \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \mathbb{R})$ observed phase quantum laplacian Def 16.3.10

Chapter 17: Quantum dynamics

$\mathsf{V}[\Psi] \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes T\boldsymbol{E})$ quantum velocity The 17.2.2
$\mathsf{Q}[\Psi] \in \sec\big(\boldsymbol{E}, \mathbb{T}^* \otimes (T\boldsymbol{E} \otimes \boldsymbol{Q})\big)$ quantum kinetic tensor The 17.3.2
$\mathsf{Q}[\Psi]/\Psi \in \sec\big(\boldsymbol{E}, (\mathbb{T}^* \otimes T\boldsymbol{E} \otimes \mathbb{C})\big)$ quantum kinetic vector field Cor 17.3.5
$\mathsf{J}[\Psi] \in \sec\big(\boldsymbol{E}, \mathbb{L}^{-3} \otimes (\mathbb{T}^* \otimes T\boldsymbol{E})\big)$ probability curent The 17.4.2
$\underline{\mathsf{J}}[\Psi] \in \sec(\boldsymbol{E}, \Lambda^3 T\boldsymbol{E})$ quantum probability current form Pro 17.4.6
$\mathsf{L}[\Psi] : \boldsymbol{E} \to \Lambda^4 T^*\boldsymbol{E}$ quantum lagrangian The 17.5.2
$\mathsf{P} := \vartheta_{\boldsymbol{Q}} \bar{\wedge} V_{\boldsymbol{Q}}\mathsf{L} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{Q}$ quantum momentum form Pro 17.5.7
$\mathsf{C} : J_1\boldsymbol{Q} \to \Lambda^4 T^*\boldsymbol{Q}$ quantum Poincare–Cartan form The 17.5.10
$\mathsf{S}[\Psi] \in \sec(\boldsymbol{E}, \mathbb{T}^* \otimes \boldsymbol{Q})$ Schrödinger operator The 17.6.5
$\mathsf{E}[\mathsf{L}] : J_2\boldsymbol{Q} \to \Lambda^5 T^* J_1\boldsymbol{Q}$ quantum Euler–Lagrange form Pro 17.6.22

Chapter 18: Hydrodynamical picture of Quantum Mechanics

$\big(\mathscr{C}, \mu, \rho\big) \equiv \big(\mathscr{C}[\Psi], \mu[\Psi], \rho[\Psi]\big)$ associated charged fluid The 18.1.1
$\mathscr{P}[\Psi] := \mu[\Psi]\,\mathscr{V}[\Psi]$ associated mass current Def 18.1.14
$\mathscr{J}[\Psi] := \rho[\Psi]\,\mathscr{V}[\Psi]$ associated charge current Def 18.1.14
$\mathscr{A}^{\natural}[\Psi] := \nabla^{\natural}{}_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi]$ associated gravitational acceleration Def 18.1.16
$\mathscr{A}[\Psi] := \nabla_{\mathscr{V}[\Psi]}\mathscr{V}[\Psi]$ associated joined acceleration Def 18.1.16
$\mathscr{A}^{\natural}{}_{\mathrm{lag}}[\Psi, o] := \nabla^{\natural}{}_{\mathscr{V}[\Psi]}\vec{\mathscr{V}}[\Psi, o]$ lagrangian acceleration Pro 18.1.19
$\mathscr{A}^{\natural}{}_{\mathrm{eul}}[\Psi, o] := \nabla^{\natural}{}_{д[o]}\vec{\mathscr{V}}[\Psi, o]$ eulerian acceleration Pro 18.1.19
$\mathscr{A}^{\natural}{}_{\mathrm{spc}}[\Psi, o] := \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi, o]}\vec{\mathscr{V}}[\Psi, o]$ spacelike acceleration Pro 18.1.19
$\mathscr{A}^{\natural}{}_{\mathrm{rlt}}[\Psi, o] := \nabla^{\natural}{}_{\vec{\mathscr{V}}[\Psi, o]}д[o]$ relative acceleration Pro 18.1.19
$\mathscr{A}^{\natural}{}_{\mathrm{drg}}[\Psi, o] := \nabla^{\natural}{}_{д[o]}д[o]$ dragging acceleration Pro 18.1.19

$\boldsymbol{p}[\Psi] = -\frac{\hbar}{m} A[\Psi]$ quantum pressure The 18.2.1

Chapter 19: Quantum symmetries

$Y[f] := X[f] \lrcorner Ч[o] + \mathrm{i}\, \check{f}[o] \mathbb{I} = X[f] \lrcorner \chi[\mathrm{b}] + \mathrm{i}\, \hat{f}[\mathrm{b}] \mathbb{I}$ quantum lift of s.p.f. Def 19.1.3
$Y_\eta[f] := Y[f] - \frac{1}{2} \operatorname{div}_\eta f \mathbb{I}$ η-quantum lift of s.p.f. Def 19.1.3
$Y^{\uparrow}{}_\eta[f] = Ч^{\uparrow}\big(X^{\uparrow}[f]\big) + \mathrm{i}\, f\, \mathbb{I}^{\uparrow}$ upper η-quantum lift of s.p.f. The 19.2.2

Chapter 20: Quantum differential operators

$\mathsf{O} : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$ quantum differential operator Def 20.1.1
$\operatorname{ope}_k\big(\sec(\boldsymbol{E}, \boldsymbol{Q}), \sec(\boldsymbol{E}, \boldsymbol{Q})\big)$ sheaf of quantum differential operators Def 20.1.1
$\mathsf{O}[f] = \mathrm{i}\big(Y_\eta[f] - \mathsf{S}[f]\big) : \sec(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{E}, \boldsymbol{Q})$ special differential operator Def 20.1.10
$\big[\mathsf{O}[\acute{f}], \mathsf{O}[f]\big]$ commutator of special quantum differential operators Def 20.1.23

Chapter 21: Currents and expectation forms

$\mathrm{j}[Z] := -i_Z \Theta \in \sec(J_1 \boldsymbol{F}, \Lambda^{n-1} T^* \boldsymbol{F})$ current associated with the vector field Z Def H.3.1
$Y^1_\eta[f] := (r^1 \circ J_1)(Y_\eta[f]) \in \operatorname{pro}_{\boldsymbol{E},\boldsymbol{Q}}(J_1 \boldsymbol{Q}, T J_1 \boldsymbol{Q})$ holonomic prolongation of $Y_\eta[f]$ Lem 21.1.2
$\mathrm{j}_\eta[f] := -i_{Y_\eta[f]} \mathsf{C} \in \sec(J_1 \boldsymbol{Q}, \Lambda^3 T^* \boldsymbol{Q})$ quantum current Def 21.1.3
$\operatorname{cur}_\eta(J_1 \boldsymbol{Q}, \Lambda^3 T^* \boldsymbol{Q}) \subset \sec(J_1 \boldsymbol{Q}, \Lambda^3 T^* \boldsymbol{Q})$ subsheaf of quantum currents Def 21.1.3
$\check{\operatorname{cur}}_\eta(J_1 \boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}} \boldsymbol{Q}) \subset \sec(J_1 \boldsymbol{Q}, \Lambda^3 V^*_{\boldsymbol{T}} \boldsymbol{Q})$ vertical quantum current The 21.1.9
$\mathrm{j}_\eta[f](\Psi) := (j_1 \Psi)^* \mathrm{j}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3 T^* \boldsymbol{E})$ quantum current form Def 21.2.1
$(j_1 \Psi)^{*} \check{\mathrm{j}}_\eta[f] = (\check{j}_1 \Psi)^* \check{\mathrm{j}}_\eta[f] \in \sec(\boldsymbol{E}, \Lambda^3 V^* \boldsymbol{E})$ vertical quantum current forms Lem 21.2.5
$\epsilon_\eta[f](\Psi) := \operatorname{re} \mathrm{h}_\eta\big(\Psi, \mathsf{O}[f](\Psi)\big) \in \sec(\boldsymbol{E}, \Lambda^3 V^* \boldsymbol{E})$ quantum expectation form Def 21.3.1
$\mathfrak{d}_\eta[f](\Psi) := \check{\mathrm{j}}[f](\Psi) - \epsilon[f](\Psi) \in \sec(\boldsymbol{E}, \Lambda^3 V^* \boldsymbol{E})$ difference form Def 21.3.5

Chapter 22: Sectional quantum bundle

$\underline{\operatorname{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \sec(\boldsymbol{E}, \boldsymbol{Q})$ almost regular quantum sections Def 22.2.1
$\operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) \subset \underline{\operatorname{reg}}(\boldsymbol{E}, \boldsymbol{Q}) \subset \sec(\boldsymbol{E}, \boldsymbol{Q})$ almost regular quantum sections Def 22.2.1
$\widehat{\boldsymbol{Q}}_{\mathrm{t}} := \operatorname{cpt}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}}) \subset \operatorname{Sec}(\boldsymbol{E}_{\mathrm{t}}, \boldsymbol{Q}_{\mathrm{t}})$ sectional quantum space at the time t Def 22.2.6
$\widehat{\boldsymbol{Q}} := \bigsqcup_{\mathrm{t} \in \boldsymbol{T}} \widehat{\boldsymbol{Q}}_{\mathrm{t}}$ sectional quantum space Def 22.2.6
$\epsilon : \widehat{\boldsymbol{Q}} \underset{\boldsymbol{T}}{\times} \boldsymbol{E} \to \boldsymbol{Q} : (\widehat{\Psi}, \mathrm{e}) \mapsto \Psi(\mathrm{e})$ evaluation map Def 22.2.6
$\sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) : \widehat{\Psi} \mapsto \Psi$ isomorphism Not 22.2.9
$\operatorname{reg}(\boldsymbol{E}, \boldsymbol{Q}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \Psi \mapsto \widehat{\Psi}$ isomorphism Not 22.2.9
$\widehat{t} : \widehat{\boldsymbol{Q}} \to \boldsymbol{T}$ F-smooth sectional quantum bundle Pro 22.3.2
$\langle\,|\,\rangle_{\mathrm{t}} : (\widehat{\Psi}_{\mathrm{t}}, \widehat{\acute{\Psi}}_{\mathrm{t}}) \mapsto \langle \widehat{\Psi}_{\mathrm{t}} \mid \widehat{\acute{\Psi}}_{\mathrm{t}} \rangle_{\mathrm{t}} := \int_{\boldsymbol{E}_{\mathrm{t}}} \mathrm{h}_{\mathrm{t}}(\Psi_{\mathrm{t}}, \acute{\Psi}_{\mathrm{t}})\, \eta_{\mathrm{t}}$ scalar product Pro 22.4.1
$\widehat{\mathsf{O}} : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) : \widehat{\Psi} \mapsto \widehat{\mathsf{O}}(\widehat{\Psi}) := \widehat{\mathsf{O}(\Psi)}$ quantum operators Lem 22.5.2
$\widehat{\mathsf{O}}[f] : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}}$ quantum operators Pro 22.5.3
$\widehat{\Delta[o]} : \widehat{\boldsymbol{Q}} \to \widehat{\boldsymbol{Q}}$ quantum laplacian Lem 22.5.4
$\widehat{\mathsf{S}} : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}})$ Schrödinger operator Lem 22.6.1
$\text{Ш} : \widehat{\boldsymbol{Q}} \to \mathbb{T}^* \otimes T \widehat{\boldsymbol{Q}}$ Schrödinger operator Pro 22.6.2

$\nabla[\text{Ш}] : \sec(\boldsymbol{T}, \widehat{\boldsymbol{Q}}) \to \sec(\boldsymbol{T}, \mathbb{T}^* \otimes \widehat{\boldsymbol{Q}})$ Schrödinger operator Pro 22.6.2

Chapter 23: Feynman path integral

$(\Psi \circ s)(\mathrm{t}) = (\Psi \circ s)(\mathrm{t}_0) \exp\left(\mathrm{i} \int_{[\mathrm{t}_0,\mathrm{t}]} (\mathcal{L} \circ j_1 s)\right)$ Feynman amplitude Lem 23.2.1

PART III: EXAMPLES

Chapter 24: Flat newtonian spacetime

$\boldsymbol{S} := \ker Dt \subset \bar{\boldsymbol{E}}$ vertical space Pro 24.1.2
$(e_1, e_2, e_3) \subset (\mathbb{L}^{-1} \otimes \boldsymbol{S})^3)$ cartesian basis Def 24.2.4
$(x^0, x^i) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$ cartesian spacetime chart Def 24.2.4
$(x^0, \rho, \phi, z) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/2\pi \times (\mathbb{L} \otimes \mathbb{R})$ cylindrical chart Def 24.2.5
$(x^0, r, \theta, \phi) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/\pi \times \mathbb{R}/2\pi$ sherical chart Def 24.2.6
$F[o_{\mathsf{ac}}] : (\mathbb{T} \otimes \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E}$ uniformly accelerated observer Def 24.4.1
$(x^0, x^i{}_{\mathsf{ac}}) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$ cartesian uniformly accelerated chart Def 24.4.2
$F[o_{\mathsf{ro}}] : (\mathbb{T} \otimes \mathbb{R}) \times \boldsymbol{E} \to \boldsymbol{E}$ uniformly rotating observer Def 24.5.1
$(x^0, x^i{}_{\mathsf{ro}}) : \boldsymbol{E} \to \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R})^3$ cartesian uniformly rotating chart Def 24.5.2
$(x^0, \rho_{\mathsf{ro}}, \phi_{\mathsf{ro}}, z_{\mathsf{ro}}) : \mathbb{R} \times (\mathbb{L} \otimes \mathbb{R}^+) \times \mathbb{R}/2\pi \times (\mathbb{L} \otimes \mathbb{R})$ cylindrical u.r. chart Def 24.5.3

Chapter 25: Example 1: no electromagnetic field

$\mathfrak{L}[c,o] : J_1\boldsymbol{E} \to \mathbb{L} \otimes V^*\boldsymbol{E}$ observed kinetic angular momentum Pro 25.1.2
$A^{\uparrow}[\mathsf{b}^{\square}] = \mathcal{C}[o]$ distinguished gauge Pro 25.1.10
$A[\mathsf{b}^{\square}, o_{\mathsf{ac}}] := o^*_{\mathsf{ac}} A^{\uparrow}[\mathsf{b}^{\square}]$ distinguished observed potential Pro 25.1.13
$A[\mathsf{b}^{\square}, o_{\mathsf{ro}}] := o^*_{\mathsf{ro}} A^{\uparrow}[\mathsf{b}^{\square}]$ distinguished observed potential Pro 25.1.23

Chapter 26: Example 2: radial electric field

$\mathbf{E} = k \, \frac{1}{r^3} \, \mathbf{r}$ radial electric field Hyp C.2

Chapter 27: Example 3: constant magnetic field

$\mathbf{B} = k \, e_3$ constant magnetic field Hyp C.3

Chapter 28: Curved newtonian spacetime

$V^{\natural}[o] \in \mathrm{map}(\boldsymbol{E}, \mathbb{R})$ observed gravitational potential Lem 28.2.3
$U^{\natural}[o] := д[o] \lrcorner A^{\natural}[o] - д[o].V^{\natural}[o])$ observed gravitational potential Lem 28.2.4
$N^{\natural} : \boldsymbol{E} \to (\mathbb{T}^* \otimes \mathbb{T}^*) \otimes V\boldsymbol{E}$ gravitational vector field The 28.2.5

APPENDIX

Chapter A: Fibred manifolds and bundles

$p : \boldsymbol{F} \to \boldsymbol{B}$ fibred manifold Def A.1.1
$\sec(\boldsymbol{B}, \boldsymbol{F})$ sheaf of sections Def A.1.1
$\mathrm{fib}(\boldsymbol{F}, \acute{\boldsymbol{F}})$ sheaf of fibred morphisms Def A.1.1
$f^*(p) : f^*(\boldsymbol{F}) := \bigsqcup_{x \in \boldsymbol{M}} \boldsymbol{F}_{f(x)} \to \boldsymbol{M}$ pullback Not A.1.9

Chapter B: Tangent bundle

$T\boldsymbol{M} := \bigsqcup_{p \in \boldsymbol{M}} T_p\boldsymbol{M}$ tangent space Def B.1.2

$dc : \mathbb{R} \to T\boldsymbol{M}$ tangent map of a curve Def B.1.2
$Tf : T\boldsymbol{M} \to T\boldsymbol{N}$ tangent prolongation of a map Not B.1.4
$\tau_{T\boldsymbol{M}} : TT\boldsymbol{M} \to T\boldsymbol{M}$ iterated tangent space Not B.4.1
$T\tau_{\boldsymbol{M}} : TT\boldsymbol{M} \to T\boldsymbol{M}$ iterated tangent space Not B.4.1
$\nu : T\boldsymbol{M} \to T^*T\boldsymbol{M} \otimes VT\boldsymbol{M}$ distinguished vertical valued 1-form Not B.4.7
$d_\nu = i_\nu \circ d$ vertical differential Not B.4.7

Chapter C: Tangent valued forms

$\langle\ ,\ \rangle : \boldsymbol{V}^* \times \boldsymbol{V} \to \mathbb{R}$ duality map Sect. C.1
$\lrcorner \ \equiv\ C^{1\dots r}_{1\dots r} : \otimes^r \boldsymbol{V} \times \otimes^s \boldsymbol{V}^* \to \otimes^{s-r}\boldsymbol{V}^*$ standard contraction Sect. C.1
$A : \otimes^r \boldsymbol{V} \to \Lambda^r \boldsymbol{V} \subset \otimes^r \boldsymbol{V}$ anti-symmetrisation Sect. C.1
$\wedge^r : \boldsymbol{V} \times \ldots \times \boldsymbol{V} \to \Lambda^r \boldsymbol{V}$ extirior product Sect. C.1
$i : \boldsymbol{V} \times \Lambda^s \boldsymbol{V}^* \to \Lambda^{s-1}\boldsymbol{V}^*$ interior product Sect. C.1
$\alpha(v_1, \ldots, v_r)_\wedge := i_{v_r} \ldots i_{v_1} \alpha$ evaluation as exterior form Sect. C.1
$\tau(v_1, \ldots, v_s) := v_s \lrcorner \ldots v_1 \lrcorner \tau$ evaluation as tensor Sect. C.1

Chapter D: Lie derivatives

$Y.s \ \simeq\ L_Y \tilde{s} \in \sec(\boldsymbol{B}, \boldsymbol{F})$ vector field acting on a section Pro D.1.2

Chapter E: The Frölicher–Nijenhuis bracket

$[\phi,\ \psi] : \times^{r+s} \sec(\boldsymbol{M},\ T\boldsymbol{M}) \to \sec(\boldsymbol{M},\ T\boldsymbol{M})$ FN bracket on a manifold The E.1.1

Chapter F: Connections

$c : \boldsymbol{F} \to T^*\boldsymbol{B} \otimes T\boldsymbol{F}$ general connection Def F.1.1
$\nabla[c]s := \nu[c] \circ Ts : T\boldsymbol{B} \to V\boldsymbol{F}$ covariant differential of sections Def F.1.1
$d_c\psi \in \sec(\boldsymbol{F},\ \Lambda^{s+1}T^*\boldsymbol{B} \otimes V\boldsymbol{F})$ covariant differential of tangent valued forms Def F.1.6
$R[c] := -d_c c := -[c,\ c]$ curvature of connections Def F.1.9
$T_\sigma[c] := 2\, d_c\sigma := 2\,[c,\ \sigma]$ torsion of connections Not F.1.21

Chapter G: Jets

$\underline{\lambda} := (\lambda_1, \ldots, \lambda_m)$ multi-index Not G.1.1
$|\underline{\lambda}| := \lambda_1 + \cdots + \lambda_m \in \mathbb{N}$ length of a multi-index Not G.1.1
$\underline{\lambda} + \mu = (\lambda_1, \ldots, \lambda_\mu + 1, \ldots, \lambda_m)$ sum of a multi-index and an index Not G.1.1
$\partial_{\underline{\lambda}} f := (\partial_1)^{\lambda_1} \ldots (\partial_m)^{\lambda_m} f$ multi-partial derivative Not G.1.2
$j_k s(b) := [s]_{(k,b)}$ k-jet of a section Def G.1.4
$J_k \boldsymbol{F} := \coprod_{b \in \boldsymbol{B}} J_{k\,b}\boldsymbol{F}$ k-jet space Def G.1.4
$J_{k\,b}\boldsymbol{F} := \left\{ j_k s(b) \right\}_{s \in \sec(\boldsymbol{B}, \boldsymbol{F})}$ k-jet space at b Def G.1.4
$p^k : J_k\boldsymbol{F} \to \boldsymbol{B} : j_k s(b) \mapsto b$ projectjon Pro G.1.6
$p^k_h : J_k\boldsymbol{F} \to J_h\boldsymbol{F} : j_k s(b) \mapsto j_h s(b)$ projectjon Pro G.1.6
$(x^\lambda, y^i_{\underline{\lambda}})$ fibred chart of jet spaces The G.1.9
$\daleth^k : J_k\boldsymbol{F} \underset{\boldsymbol{B}}{\times} T\boldsymbol{B} \to TJ_{k-1}\boldsymbol{F}$ k-contact map Tht G.3.1
$\theta_k : J_k\boldsymbol{F} \underset{J_{k-1}\boldsymbol{F}}{\times} TJ_{k-1}\boldsymbol{F} \to VJ_{k-1}\boldsymbol{F}$ complementary k-contact map Pro G.3.8
$J_k f : J_k\boldsymbol{F} \to J_k\boldsymbol{G}$ k-jet functor The G.4.1
$i_k : J_kV\boldsymbol{F} \to VJ_k\boldsymbol{F}$ vertical exchange map Pro G.5.1
$r_k : J_kT\boldsymbol{F} \to TJ_k\boldsymbol{F}$ exchange map The G.5.2

$X^k := r^k \circ J_k X \in \sec(J_k \boldsymbol{F},\ T J_k \boldsymbol{F})$ k-holonomic prolongation of a vector field The G.6.3

Chapter H: Lagrangian formalism

$\mathcal{L} : J_1 \boldsymbol{F} \to \Lambda^m T^* \boldsymbol{B}$ lagrangian form Sect. H.1
$\mathcal{P} := \theta \bar{\wedge} V_0 \mathcal{L} : J_1 \boldsymbol{F} \to \Lambda^{m-1} T^* \boldsymbol{B} \wedge T^* \boldsymbol{F} \subset \Lambda^m T^* \boldsymbol{F}$ momentum form Def H.1.3
$\mathcal{C} := \mathcal{L} + \mathcal{P} : J_1 \boldsymbol{F} \to \Lambda^m T^* \boldsymbol{F}$ Poincaré–Cartan form Def H.1.4
$d_V \mathcal{P} := i_{\vartheta_2} d\mathcal{P} - d i_{\vartheta_1} \mathcal{P} : J_2 \boldsymbol{F} \to \Lambda^{m+1} T^* J_1 \boldsymbol{F}$ vertical differential of $\mathcal{P}$ Def H.2.1
$d_V \mathcal{P} - d\mathcal{C} : J_2 \boldsymbol{F} \to \Lambda^5 T^* J_1 \boldsymbol{F}$ Euler–Lagrange form Def H.2.2
$\mathcal{E} : J_2 \boldsymbol{F} \to \Lambda^m T^* \boldsymbol{B} \otimes V^* \boldsymbol{F}$ Euler–Lagrange operator Pro H.2.3
$\mathrm{j}[Z] := -i_{Z^1} \mathcal{C} = -i_Z \mathcal{C} \in \sec(J_1 \boldsymbol{F}, \Lambda^{m-1} T^* \boldsymbol{F})$ current Def H.3.1

Chapter I: Geometric structures

$[X, Y] \in \sec(\boldsymbol{M}, \Lambda^{p+q-1} T \boldsymbol{M})$ Schouten bracket Not I.1.1
(ω, Ω) regular covariant pair Def I.1.2
(E, Λ) regular contravariant pair Def I.1.2
$\Omega^\flat : T\boldsymbol{M} \to T^*\boldsymbol{M} : X \mapsto X^\flat := i_X \Omega$ musical morphism Not I.1.4
$\Lambda^\sharp : T^*\boldsymbol{M} \to T\boldsymbol{M} : X \mapsto \alpha^\sharp := i_\alpha \Lambda$ musical morphism Not I.1.4

Chapter J: Covariance

$\mathscr{C} = \Big(\mathrm{Obj}(\mathscr{C}),\ \mathrm{Mor}(\mathscr{C})\Big)$ category Def J.1.1
$\mathcal{F} := \Big(\mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{D}),\ \ \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{D})\Big)$ covariant functor Def J.1.4
$\mathcal{B} := \Big(\mathrm{Obj}(\mathscr{C}) \to \mathrm{Obj}(\mathscr{B}),\ \ \mathrm{Mor}(\mathscr{C}) \to \mathrm{Mor}(\mathscr{B})\Big)$ natural bundle functor Def J.2.1
$\pi^r : \mathcal{F}^r(\boldsymbol{M}) := \mathrm{inv}\, J_r{}_0(\mathbb{R}^n, \boldsymbol{M}) \to \boldsymbol{M}$ frame bundle Def J.2.6
$\mathcal{B} := \Big(\mathrm{Obj}(\mathscr{P}[G] \to \mathrm{Obj}(\mathscr{B}),\ \ \mathrm{Mor}(\mathscr{P}[G]) \to \mathrm{Mor}(\mathscr{B})\Big)$ gauge natural bundle functor Def J.3.1

Chapter K: Scales

$\mathbb{U}$ positive space Def K.1.1
$\mathbb{U}^q := \mathrm{Rat}^q(\mathbb{U}^{¥}, \mathbb{R}^+)$ q-rational power of $\mathbb{U}$ Def K.1.3

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