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The Statistical Mechanics of Quantum Lattice Systems

A Path Integral Approach

European Mathematical Society

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To our families

Preface

It would not be an exaggeration to say that since the 1940s Feynman's path integrals substantially changed quantum physics. This concept led to the implementation of many path integration methods of different levels of mathematical sophistication, see [6], [84] and the numerous references therein. Spectacular progress in rigorous quantum field theory (at least for low-dimensional space-time models) was achieved through the Euclidean strategy in which the Minkowski space was converted into a Euclidean space by passing to imaginary values of time. The corresponding quantum field was constructed and studied in this Euclidean domain and then transferred to real time by a certain procedure. Due to this development, Feynman–Wiener path integrals and hence the theory of Markov processes as well as methods of classical statistical mechanics were widely applied. The state of the art in this domain up to the time of their respective publication was presented in the monographs by B. Simon [273] and J. Glimm and A. Jaffe [135]. The introduction to the former book gives a profound survey of ideas and historical facts behind the Euclidean strategy.

Quantum statistical mechanics is close, both conceptually and technically, to quantum field theory. Its rigorous version has been developed on the basis of the theory of operator algebras, whose fundamentals can be found in the monographs by O. Bratteli and D. W. Robinson [76], [77], G. G. Emch [114], and by M. Takesaki [300], [301], [302]. However, for a big class of important quantum models, especially those described by unbounded operators, these methods encountered considerable difficulties; see the discussion on page 241 of [77] and also in [160], [161].

The present book is dedicated to the rigorous statistical mechanics of infinite systems of interacting quantum anharmonic oscillators. It can be considered as a natural continuation of B. Simon's book "The Statistical Mechanics of Lattice Gases: I", where both classical and quantum models of this kind are on the list of *'models not to be discussed',* see pp. 19–26 in [277]. In addition, our book is connected with quantum field theory by the fact that the free quantum field can be interpreted as an infinite system of interacting quantum harmonic oscillators.

There is, however, one more important reason to develop the theory presented in this book. Since the 1960s, systems of quantum oscillators have been widely used in models of quantum solid state physics where they describe vibrations of light particles localized near sites of crystal lattices and their interaction with other particles and fields. In this context, we mention the book by A. A. Maradudin *et al.* [212], see also [154], [155], and the series of articles by A. Verbeure and his collaborators [310], [316]. The theory of quantum harmonic oscillators is relatively simple and therefore is quite well elaborated. The case of anharmonic oscillators is more complex. However, systems of anharmonic oscillators possess much richer properties and hence have much wider applications, which strongly stimulates the development of their theory, including its mathematically rigorous versions. Clearly, the properties of such systems depend on the geometry of interactions and hence on the configuration of the equilibrium positions of the oscillators. In the model studied in this book, they constitute a countable set $\mathbb{L} \subset \mathbb{R}^d$, equipped with the Euclidean distance and obeying a certain regularity condition. A particular case is the model where $\mathbb{L}$ is a crystal lattice. More general cases of $\mathbb{L}$ correspond to quantum particles irregularly distributed in $\mathbb{R}^d$. Most of our results apply to this general case, however, a number of them are valid for crystal lattices only.

In classical (i.e., non-quantum) statistical mechanics, a complete description of the equilibrium thermodynamic properties of an infinite-particle system can be given by constructing its Gibbs states. For some quantum models with bounded Hamiltonians, equilibrium thermodynamic states are defined as functionals on algebras of observables satisfying the Kubo–Martin–Schwinger (KMS) condition, see [145] and [77], which is an equilibrium condition reflecting a consistency between the dynamics and thermodynamics of the model. However, for an infinite system of interacting quantum anharmonic oscillators, the KMS condition cannot be formulated and hence the KMS states cannot even be defined. In this situation, a natural alternative is given by a version of the Euclidean strategy based on path integral techniques, which was successful in low dimensional quantum field theory. Because of their intuitive appeal, methods employing integration in function spaces on the 'physical' level of strictness enjoy great popularity among theoretical physicists working in quantum physics. This is assured by numerous monographs and textbooks in this field having appeared or having been reprinted recently, see e.g., [165], [178], [213], [220], [322]. Thus, the second goal of this book is to provide a firm mathematical background of path integral methods used in quantum statistical mechanics, based on the latest achievements in stochastic and functional analysis. We also believe that the mathematical problems arising here will stimulate development of the corresponding fields of mathematics.

In accordance with these goals, we address the book to both communities – physicists and mathematicians. Theoretical physicists, especially those who are concerned with the rigorous mathematical background of their results, can find here a concise collection of facts, concepts, and tools relevant for the application of path integrals and other methods based on measure and integration theory to problems of quantum physics. They can also find the latest results in the mathematical theory of quantum anharmonic crystals, which can be used as a basis for the study of equilibrium and non-equilibrium statistical mechanical properties of models employing quantum anharmonic oscillators. Mathematicians are given an opportunity to learn what kind of problems arise in quantum statistical mechanics and how to attack them. We believe that our methods are also applicable to other problems involving infinitely many variables, for example, in biology and economics.

In view of its interdisciplinary nature, this book consists of 'mathematical' and 'physical' parts, preceded by an introduction, where we outline the ideas on which our approach rests, formulate its main aspects, and briefly describe physical consequences of the theory developed on the basis of this approach.

The first part, comprising three chapters, starts with a description of the model considered throughout the book. For this model, we define local Gibbs states as functionals on the corresponding algebras of local observables and give the mathematical background of the theory of such states, which includes elements of the theory of linear operators in Hilbert spaces. Then we present a detailed description of the properties of Schrödinger operators of single quantum oscillators, both harmonic and anharmonic. Afterwards, we prepare the description of the local Gibbs states of our model in terms of stochastic processes and associated path measures. Here we present a number of facts from the theory of probability measures on topological spaces coming from various sources. Most of the statements are proven here, addressing those readers who would like to get into the details without using additional sources. Thereby, we develop a description of local Gibbs states in terms of path space measures, which in this book are called local Euclidean Gibbs measures. They have the same structure as the local Gibbs measures of the corresponding classical models of unbounded spins. Here, however, the 'spins' are not only unbounded, but also belong to an infinite-dimensional Banach space. In the next chapter, we develop tools for studying local Euclidean Gibbs measures, based on their approximation by the Gibbs measures of classical models with unbounded finite-dimensional spins. With the help of this approximation, we derive a number of correlation inequalities, which are then crucially used throughout the book. In Chapter 3, which is the main point of the first part and perhaps of the whole book, we introduce and study the (global) Euclidean Gibbs measures of our model. These measures contain all information about equilibrium thermodynamic properties of the model and play the same role as the KMS states do in the algebraic formulation of quantum statistical mechanics.

The second part of the book is dedicated to a description of some physical properties of our model which is based on the Euclidean Gibbs measures constructed in the first part. Here we present a complete theory of phase transitions and quantum effects. This theory is mainly based on various correlation inequalities, on regularity properties of the paths of the underlying stochastic processes, and on the spectral properties of the corresponding Schrödinger operators. It explains a large number of relevant experimental data, confirming our approach. In this context, one has to mention powerful methods of studying Gibbs states and phase transitions in classical lattice systems based on cluster, polymer, and other expansions and estimates. Some of them, like cluster expansions, have also found applications to quantum anharmonic crystals, see the works by R. A. Minlos, e.g. [217], and the bibliographic notes below. We expect that the general framework developed in this book will lead to a more effective use of these methods in the future. At the same time, a number of such methods, for instance, the Pirogov–Sinai theory of phase transitions [245], [321], are applicable to classical models only. In our approach, quantum anharmonic crystals are described as systems of classical albeit infinite-dimensional 'spins'. Thus, we hope that by means of our techniques the development of a version of the Pirogov–Sinai theory, applicable to our and similar models, will be possible. A relevant problem which we also leave for the future is the interpretation of our results in terms of states on von Neumann algebras in the spirit of works by the groups of J. Fröhlich [66], R. Gielerak [131], [132], and A. Verbeure [79], [313].

Although we did our best to make the book self-contained, the reader is supposed to have certain preliminary knowledge at a graduate level, both in mathematics and physics. As a book of great impact on this area we strongly recommend B. Simon's monograph [274], and also the monographs [76], [77], [114], [145], [300], [301], [302] as sources on the algebraic methods of quantum statistical mechanics. Since the construction of our Euclidean Gibbs measures is carried out in the framework of the Dobrushin–Lanford–Ruelle approach, we recommend learning its fundamentals from the books [129], [249], [281].

The line of research described in this book has its roots in original work by the late Raphael Høegh-Krohn, who discovered a fundamental duality in relativistic quantum statistical mechanics by representing the basic correlation functions in terms of a certain stochastic process, see [1]. For this, it seems more than appropriate to call this stochastic process the Høegh-Krohn process, as we do in this book. This should be understood as an expression of our admiration for a great mathematician, who departed much too early. Our work on the book was completed on the eve of the 20-th anniversary of Raphael's death to confirm that his spirit is still among us.

This book has mostly been written at BiBoS (Bielefeld–Bonn Stochastics) Research Centre at Bielefeld University. Some of its parts have been presented in lecture series at the International Graduate College (IGK) 'Stochastics and Real World Models', Bielefeld University. The underlying research, as well as the actual work on the book, were financially supported by the Deutsche Forschungsgemeinschaft through the projects No 436 POL 113/98/0-1 'Methods of stochastic analysis in the theory of collective phenomena: Gibbs states and statistical hydrodynamics' and No 436 POL 113/115/0-1 'Quantum infinite particle systems in a functional integral approach', as well as through the SFB 701 'Spektrale Strukturen und topologische Methoden in der Mathematik'. Yuri Kozitsky was also supported by the Komitet Badań Naukowych through the Grant 2P03A 02025. We sincerely appreciate this support for which we express our deep gratitude to the corresponding institutions. Our research on the subject of the book was strongly influenced by the works of our colleagues Ph. Blanchard, R. A. Minlos, and L. Streit. A substantial part of our results was obtained in collaboration with T. Pasurek (Tsikalenko). We want to express our deep respect for their role in the development of this field, as well as our acknowledgment of their collaboration. We also thank our colleagues A. Daletskii, K. Goebel, Y. Holovatch, T. Kuczumow, T. Kuna, O. Kutovyy, E. Lytvynov, R. Olkiewicz, R. P. Streater, Z. Rychlik, and E. Zhizhina for interesting discussions and continuous encouragement. Finally, we are grateful to two anonymous referees for their constructive criticism which helped to improve the quality of the book.

Contents

Introduction

A one-dimensional quantum harmonic oscillator is a quantized version of the model of a particle performing sinusoidal oscillations in a parabolic potential field, corresponding to Hooke's law. A ν-dimensional harmonic oscillator, $\nu \in \mathbb{N}$, performs independent simultaneous oscillations in all ν dimensions; its momentum and displacement are ν-dimensional vectors. For anharmonic oscillators, the corresponding potential fields are usually super-quadratic and may have multiple minima. The latter peculiarity entails essential changes in the particle dynamics as compared with the case of convex potentials. A typical example from physics is a hydrogen bound $\mathrm{O} - \mathrm{H} - \mathrm{O}$, consisting of two negative oxygen ions and a positive hydrogen ion (proton), which here stands for a quantum particle. Such a bound is the key structure element of many inorganic and organic compounds. The potential field created by the oxygen ions has two minima (wells), close to each of the ions. Then such a bound with the proton localized in one of the wells is an electric dipole. These dipoles interact with each other, which can force the protons to stay in one of the corresponding two wells. At the same time, the protons oscillate between the wells, even in low-energy states. This motion through a potential barrier, forbidden for classical particles, is called *quantum mechanical tunneling*, see Subsection 1.1.3 below. It produces a strong delocalizing effect, especially at low temperature.

Along with modeling localized quantum particles, quantum anharmonic oscillators are also involved in models describing the interaction of vibrating quantum particles with a radiation (photon) field or strong electron-electron correlations caused by the interaction of electrons with vibrating light ions. Infinite systems of interacting quantum particles of this kind possess interesting physical properties connected with ordering (phase transitions) and quantum effects. Most of them are related to solids, such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, like the hydrogen bounds mentioned above, or quantum crystals consisting entirely of such particles, e.g., crystalline helium. In the Born–Oppenheimer (called also adiabatic) approximation, the motion of heavy ions is neglected and the oscillators are attached to the sites of a regular crystal lattice – one oscillator per site. Other important physical objects of this kind are systems of localized light particles irregularly distributed (admixed) in a certain medium. In the corresponding model, the sites the oscillators are attached to constitute an irregular set and the localization potentials may vary from site to site. This can also include the case where $\mathbb{L}$ is a lattice but V_ℓ, as well as $J_{\ell\ell'}$, are random. Often, as in the case of hydrogen bounds, the described particles carry electric charges and their displacements from equilibrium positions produce dipole moments. Then the interaction between the particles is of dipole–dipole type and thereby has slow spatial decay. In what follows, infinite systems of interacting quantum anharmonic oscillators with possibly irregular spatial distribution of their equilibrium positions and with long-range interactions can be used in modeling a wide variety of physical objects. A rigorous mathematical description of such systems is still a challenging task, and one of the aims of the present book is to provide a framework for such a construct.

In the sequel, by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ we denote the sets of complex, real, integer, positive integer, and nonnegative integer numbers, respectively. The main object of our study is a system of interacting quantum anharmonic oscillators attached to the elements of a countable set $\mathbb{L} \subset \mathbb{R}^d$ equipped with the Euclidean distance $|\cdot|$ inherited from $\mathbb{R}^d$. We suppose that

$$\sup_{\ell \in \mathbb{L}} \sum_{\ell' \in \mathbb{L}} \frac{1}{(1 + |\ell - \ell'|)^{d+\epsilon}} < \infty, \tag{1}$$

for every $\epsilon > 0$, which in particular means that $\mathbb{L}$ has no accumulation points. The condition (1) implies that subsets of $\mathbb{R}^d$ of small volume cannot contain a large number of elements of $\mathbb{L}$. In general, this will be the only condition imposed on the set $\mathbb{L}$. However, some of our results have been obtained in the case where $\mathbb{L}$ is a crystal lattice, which is clearly indicated in the text. For simplicity, in such cases we always assume that $\mathbb{L} = \mathbb{Z}^d$. With a slight abuse of terminology we call our model a *quantum anharmonic crystal,* even if $\mathbb{L}$ is not a lattice. The heuristic Hamiltonian of our model is

$$H = -\frac{1}{2} \sum_{\ell, \ell'} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell} H_\ell, \tag{2}$$

where the interaction term is harmonic – the simplest possible choice, which, however, has a physical motivation (it is of dipole–dipole type). The indices in the sums run through the set $\mathbb{L}$, the displacement q_ℓ of the oscillator attached to a given $\ell \in \mathbb{L}$ is a ν-dimensional vector, whose components $q_\ell^{(j)}$, $j = 1, \ldots, \nu$, are position operators. By $(\cdot, \cdot)$ and $|\cdot|$ we denote the scalar product and norm in $\mathbb{R}^\nu$. The one-site Hamiltonian

$$H_\ell = H_\ell^{\mathrm{har}} + V_\ell(q_\ell) \stackrel{\mathrm{def}}{=} \frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell), \quad a > 0, \tag{3}$$

describes an isolated quantum anharmonic oscillator of mass m and momentum $p_\ell = (p_\ell^{(1)}, \ldots, p_\ell^{(\nu)})$. It is also called the *Schrödinger operator* of the oscillator. H_ℓ^{har} is the Schrödinger operator of a ν-dimensional harmonic oscillator of rigidity a. The components of p_ℓ and q_ℓ, which are operators in $L^2(\mathbb{R}^\nu)$, obey the canonical commutation relation

$$p_\ell^{(j)} q_{\ell'}^{(j')} - q_{\ell'}^{(j')} p_\ell^{(j)} = -\mathrm{i}\delta_{\ell\ell'}\delta_{jj'}, \quad j, j' = 1, \ldots, \nu, \quad \mathrm{i} = \sqrt{-1}.$$

In our presentation of this relation, Planck's constant $\hbar$ is included into the mass parameter

$$m = m_{\mathrm{ph}}/\hbar^2, \tag{4}$$

where m_{ph} is the physical mass of the particle. The anharmonic potentials $V_\ell \colon \mathbb{R}^\nu \to \mathbb{R}$, which may vary from site to site, are continuous functions obeying

$$b_V |x|^{2r} - c_V \le V_\ell(x) \le V(x),$$

with constants $b_V > 0$, $c_V \geq 0$, $r \in \mathbb{N} \setminus \{1\}$, and a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$. These bounds are responsible for the system stability. As for the interaction intensities, the only general restriction is

$$\hat{J}_0 \stackrel{\text{def}}{=} \sup_{\ell} \sum_{\ell'} |J_{\ell\ell'}| < \infty,$$

which is a stability condition as well. By imposing this condition we shall avoid problems with infinite forces acting on a given oscillator. In general, we do not assume that the model has special properties like translation invariance or that the interaction has finite range. Therefore, our model can describe also systems with long-range interactions and with spatial irregularities like impurities or the ones with random components.

The Hamiltonian (2) has no direct mathematical meaning and usually is 'represented' by local Hamiltonians H_Λ corresponding to finite $\Lambda \subset \mathbb{L}$. Here and in the sequel, the adjective 'local' characterizes a property, related to a finite $\Lambda \subset \mathbb{L}$, whereas 'global' will always refer to the whole 'lattice' $\mathbb{L}$. Cases of infinite $\Lambda \subsetneq \mathbb{L}$ are indicated explicitly. Each H_Λ describes the subsystem of oscillators attached to the lattice points $\ell \in \Lambda$, and hence is obtained from (2) by restricting the corresponding sums to Λ. It is a self-adjoint lower bounded operator in the physical Hilbert space $L^2(\mathbb{R}^{\nu|\Lambda|})$, the elements of which are called *wave functions*. The operator H_Λ has discrete spectrum and is such that

$$\text{trace}[\exp(-\tau H_\Lambda)] < \infty, \quad \text{for all } \tau > 0. \tag{5}$$

The quantum-mechanical states of the subsystem in Λ are defined by the wave functions $\psi \in L^2(\mathbb{R}^{\nu|\Lambda|})$ of unit norm in the following sense. Let $\mathfrak{C}_\Lambda$ be the algebra of all bounded linear operators in $L^2(\mathbb{R}^{\nu|\Lambda|})$. Its elements are called local observables. For the mentioned ψ, the state ω_ψ is defined on $\mathfrak{C}_\Lambda$ as the linear functional

$$\mathfrak{C}_\Lambda \ni A \mapsto \omega_\psi(A) = (\psi, A\psi)_{L^2(\mathbb{R}^{\nu|\Lambda|})},$$

where $(\cdot,\cdot)_{L^2(\mathbb{R}^{\nu|\Lambda|})}$ is the scalar product in $L^2(\mathbb{R}^{\nu|\Lambda|})$. Such a state can be extended to unbounded operators, which contain ψ in their domains. The state ω_ψ is *pure* (also called *extreme*), which means that it cannot be expressed as a nontrivial convex combination of other states. If ψ is the eigenfunction of H_Λ corresponding to the eigenvalue E, then the energy of the subsystem in the state ω_ψ is $\omega_\psi(H_\Lambda) = E$. By (5) and the Hilbert–Schmidt theorem it follows that there exists an orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}_0}$ of $L^2(\mathbb{R}^{\nu|\Lambda|})$, consisting of eigenvectors of H_Λ. Let $\{E_n\}_{n\in\mathbb{N}_0}$ be the set of the corresponding eigenvalues of H_Λ. According to the fundamental law of statistical mechanics, the equilibrium state $\varrho_{\beta,\Lambda}$ at a given value of the parameter $\beta = 1/k_B T$, called *inverse temperature*, is a mixture of the pure states ω_{ψ_n} with coefficients proportional to $\exp(-\beta E_n)$. Here $k_B > 0$ and $T > 0$ are Boltzmann's constant and absolute temperature, respectively. By these arguments we are immediately led to the formula

$$\varrho_{\beta,\Lambda}(A) = \frac{\text{trace}(Ae^{-\beta H_\Lambda})}{\text{trace}(e^{-\beta H_\Lambda})}. \tag{6}$$

This is the Gibbs state corresponding to the *canonical ensemble*. In the *grand canonical ensemble*, one includes also states with different numbers of particles. As in our case this number is constant and equal to $|\Lambda|$, we shall consider canonical ensembles only. Along with the thermodynamics of the considered system of oscillators, the Hamiltonian H_Λ determines its dynamics as well. There exist two equivalent approaches to the description of the dynamics of a quantum system. In the Schrödinger approach, the states ω_ψ evolve according to the Schrödinger equation, whereas the observables remain constant in time. In the Heisenberg picture, the states are constant but the observables evolve according to the following rule[1]

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}_t^\Lambda(A) \stackrel{\text{def}}{=} e^{\mathrm{i}tH_\Lambda} A e^{-\mathrm{i}tH_\Lambda}, \quad t \in \mathbb{R}. \tag{7}$$

As H_Λ is self-adjoint, the operators $e^{\mathrm{i}tH_\Lambda}$ are unitary; hence, the mappings $\mathfrak{a}_t^\Lambda$, $t \in \mathbb{R}$, constitute a one-parameter group of automorphisms of $\mathfrak{C}_\Lambda$. In our context, it is more appropriate to adopt the Heisenberg approach, at least because in both cases (6) and (7), one deals with mappings defined on one and the same set $\mathfrak{C}_\Lambda$. Note that the fact, that they are defined by the same operator H_Λ, is crucial. One might observe, however, that the picture just drawn has some deficiency since therein the subsystem in Λ is described separately from the rest of the system, the influence of which is thereby ignored. In the path integral approach developed below, this problem is settled by considering conditional Gibbs measures, in which the interaction of the subsystem in Λ with the remaining part of the system is taken into account.

We have come to the point where we can start to build up our Euclidean approach. In its first stage, we realize the state $\varrho_{\beta,\Lambda}$ with the help of a path measure. Here multiplication operators play a significant role. For a bounded Borel function, $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, the corresponding multiplication operator F acts according to

$$(F\psi)(x) = F(x)\psi(x), \quad \psi \in L^2(\mathbb{R}^{\nu|\Lambda|}).$$

In this case, we can write

$$\varrho_{\beta,\Lambda}(F) = \frac{\int_{\mathbb{R}^{\nu|\Lambda|}} F(x) K_\beta(x,x)\mathrm{d}x}{\int_{\mathbb{R}^{\nu|\Lambda|}} K_\beta(x,x)\mathrm{d}x}, \tag{8}$$

where $K_\tau(x,y)$ is the integral kernel of $\exp(-\tau H_\Lambda)$. This defines the restriction of the state (6) to the abelian subalgebra consisting of all multiplication operators by bounded Borel functions. Of course, such a result is not sufficient. To extend this kind of representation to the remaining elements of $\mathfrak{C}_\Lambda$ we proceed as follows. First we prove that the linear span of the products

$$\mathfrak{a}_{t_1}^\Lambda(F_1)\dots\mathfrak{a}_{t_n}^\Lambda(F_n)$$

with all possible choices of $n \in \mathbb{N}$, $t_1,\dots,t_n \in \mathbb{R}$, $F_1,\dots,F_n \in C_\mathrm{b}(\mathbb{R}^{\nu|\Lambda|})$, is dense in $\mathfrak{C}_\Lambda$ in a certain (σ-weak) topology, in which the state (6) is continuous. Here $C_\mathrm{b}(\mathbb{R}^{\nu|\Lambda|})$

[1] For convenience, we set $t = \text{time}/\hbar$, where $\hbar$ is Planck's constant.

is the set of all bounded continuous functions $F\colon \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$. Thereby, this state is fully determined by its values on such products, that is, by the Green functions

$$G^{\Lambda}_{F_1,\dots,F_n}(t_1,\dots,t_n) \stackrel{\text{def}}{=} \varrho_{\beta,\Lambda}[\mathfrak{a}^{\Lambda}_{t_1}(F_1)\dots\mathfrak{a}^{\Lambda}_{t_n}(F_n)], \quad F_1,\dots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|}). \quad (9)$$

Let us formally set here $t_k = \mathrm{i}\tau_k$, $\tau_k \in \mathbb{R}$, $k = 1,\dots,n$, and consider

$$\begin{aligned} \Xi(\tau_1,\dots,\tau_n) \stackrel{\text{def}}{=} \ & \mathrm{trace}\big\{e^{-\tau_1 H_\Lambda} F_1 e^{-(\tau_2-\tau_1)H_\Lambda} \\ & \times\dots\times F_{n-1}e^{-(\tau_n-\tau_{n-1})H_\Lambda} F_n e^{-(\beta-\tau_n)H_\Lambda}\big\}, \end{aligned} \quad (10)$$

which can be written in the form (8) provided $0 \le \tau_1 \le \dots \le \tau_n \le \beta$. Now the problem of relating (10) to (9) can be settled by means of an analytic continuation from the real to imaginary values of time. This is done by proving that each Green function is the restriction of a function $G^{\Lambda}_{F_1,\dots,F_n}$, which is analytic in the following complex tubular domain[2]

$$\mathcal{D}^n_\beta = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid 0 < \Im(z_1) < \dots < \Im(z_n) < \beta\}, \quad (11)$$

and continuous on its closure $\bar{\mathcal{D}}^n_\beta \subset \mathbb{C}^n$. Thereby, one shows that for any $n \in \mathbb{N}$, the 'imaginary time' subset

$$\{(z_1,\dots,z_n) \in \mathcal{D}^n_\beta \mid \Re(z_1) = \dots = \Re(z_n) = 0\}$$

is a set of uniqueness for functions analytic in $\mathcal{D}^n_\beta$. This means that if two such functions take equal values on this set, then they are equal everywhere and thus equal as functions. Therefore, the Green functions (9), and hence the state (6), are determined by the so-called Matsubara functions

$$\begin{aligned} \Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) &\stackrel{\text{def}}{=} G^{\Lambda}_{F_1,\dots,F_n}(\mathrm{i}\tau_1,\dots,\mathrm{i}\tau_n) \\ &= \Xi(\tau_1,\dots,\tau_n)/\mathrm{trace}[e^{-\beta H_\Lambda}] \\ &= \mathrm{trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda}\dots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}], \end{aligned} \quad (12)$$

taken at ordered arguments $0 \le \tau_1 \le \dots \le \tau_n \le \tau_1 + \beta \stackrel{\text{def}}{=} \tau_{n+1}$, with all possible choices of $n \in \mathbb{N}$ and $F_1,\dots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$. Their extensions to $[0,\beta]^n$ are defined as

$$\Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \Gamma^{\Lambda}_{F_{\sigma(1)},\dots,F_{\sigma(n)}}(\tau_{\sigma(1)},\dots,\tau_{\sigma(n)}),$$

where σ is the permutation of $\{1,2,\dots,n\}$ such that $\tau_{\sigma(1)} \le \tau_{\sigma(2)} \le \dots \le \tau_{\sigma(n)}$. This multiple-time analyticity can be thought of as a consequence of the above mentioned fact that the dynamics and thermodynamics of the subsystem are determined by the same local Hamiltonian and hence are in equilibrium. As follows from the representation (12), the Matsubara function $\Gamma^{\Lambda}_{F_1,\dots,F_n}$ can be written in the form

$$\Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \int_{\Omega_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1))\dots F_n(x_\Lambda(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}x_{\beta,\Lambda}), \quad (13)$$

[2] For a $z = x + \mathrm{i}y \in \mathbb{C}$, $x, y \in \mathbb{R}$, we write $x = \Re(z)$, $y = \Im(z)$.

where $\nu_{\beta,\Lambda}$ is a certain probability measure on the local path space $\Omega_{\beta,\Lambda}$. It is canonically associated with a β-periodic Markov process[3], for which the transition probabilities are defined by the kernels $K_\tau(x, y)$ mentioned above. This is the main point of the first stage of our approach. As was mentioned above, this approach is called Euclidean in view of the passage from the real to imaginary values of time. Correspondingly, the measure $\nu_{\beta,\Lambda}$ is called a local Euclidean Gibbs measure. By standard arguments, it is uniquely determined by the integrals (13); hence, since the Matsubara functions $\Gamma^{\Lambda}_{F_1,\ldots,F_n}$ uniquely determine the state $\varrho_{\beta,\Lambda}$, the representation (13) establishes a one-to-one correspondence between the local Gibbs states and local Euclidean Gibbs measures.

Now suppose that we are given an algebra of observables $\mathfrak{C}$ and a one-parameter group of time automorphisms $\alpha_t : \mathfrak{C} \to \mathfrak{C}$, $t \in \mathbb{R}$, which determines the dynamics of the underlying system. How can one find a σ-weakly continuous state ω on $\mathfrak{C}$ such that the continuation of the Green functions of this state to imaginary values of time is possible and thereby a kind of equilibrium can be established? Here we note that the family of such states need not be a singleton. The answer to the above question is related to the Kubo–Martin–Schwinger (KMS) property of ω. For $A, B \in \mathfrak{C}$, let us set

$$F^{\omega}_{A,B}(t) = \omega(B\alpha_t(A)), \qquad G^{\omega}_{A,B}(t) = \omega(\alpha_t(A)B), \quad t \in \mathbb{R}. \tag{14}$$

Then ω is called a β-KMS state if there exists a function F, analytic in $\mathcal{D}^1_\beta$ and continuous on its closure $\bar{\mathcal{D}}^1_\beta$, such that

$$F^{\omega}_{A,B}(t) = F(t), \quad G^{\omega}_{A,B}(t) = F(t + \mathrm{i}\beta), \quad \text{for all } t \in \mathbb{R}.$$

In [144], see also page 202 in [145], it was suggested to use the KMS property of ω as the defining property of an equilibrium state at a given value of β. It turns out that if ω is a KMS state, then each Green function

$$G^{\omega}_{A_1,\ldots,A_n}(t_1, \ldots, t_n) = \omega(\alpha_{t_1}(A_1) \ldots \alpha_{t_n}(A_n)),$$

for any $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathfrak{C}$, has a multiple-time analyticity property, the same as the Green functions (9). This fact was proven in [176]. Thus, a σ-weakly continuous KMS state is uniquely determined by the Matsubara functions corresponding to the operators from a maximal abelian subalgebra of $\mathfrak{C}$. Let us now analyze the possibility of using the idea just outlined in constructing global equilibrium states. As we have seen, the crucial elements of this construction are the algebra of observables and the group of time automorphisms. A candidate for such an algebra could be the norm-completion of the algebra of local observables

$$\mathfrak{C}^{\mathrm{loc}} = \bigcup_{\Lambda} \mathfrak{C}_\Lambda, \tag{15}$$

where the union is taken over all finite Λ. It is a C^*-algebra, but need not be a von Neumann algebra. The group of time automorphisms could be obtained in the

[3]The periodic Markov property was introduced and studied in [177].

infinite-volume limit $\Lambda \nearrow \mathbb{L}$ from the automorphisms (7). For some systems with bounded local Hamiltonians H_Λ, e.g., quantum spin models or the ideal Fermi gas, this 'algebraic' way of constructing equilibrium states can be realized, see [77]. However, in the case of the model (2), (3), it does not work since the construction of corresponding infinite-volume time automorphisms is beyond the technical possibilities existing at this time. As a consequence, the global KMS condition for this model cannot be formulated and hence the KMS states cannot even be defined[4].

What we have also learned from the above consideration is that the Matsubara functions can determine equilibrium states. For our model, this can be done for the local states by (12) and (13), where these functions are obtained as integrals with respect to local Euclidean Gibbs measures. On the global level, general abstract techniques of constructing equilibrium states from given (complete) sets of Matsubara functions were elaborated in [66], [131], [132], [133]. As follows from these works, the number of equilibrium states existing for the same values of the model parameters and temperature is in correspondence with the number of sets of Matsubara functions, which can be constructed for these values. Therefore, all the information about the thermodynamics of the considered model is contained in these functions. Our approach gives a way how to obtain them. Here we exploit the fact that the local states are represented by probability measures and hence can be interpreted as local Gibbs measures of classical lattice systems of unbounded spins. For such systems, a complete description of the equilibrium thermodynamic properties is achieved by constructing their Gibbs states as probability measures on appropriate configuration spaces. Here the use of the distributions of configurations in a finite $\Lambda \subset \mathbb{L}$ conditioned by configurations outside Λ is standard. The corresponding techniques constitute the Dobrushin–Lanford–Ruelle (DLR) theory, which now is well-elaborated and widely used. By virtue of the Feynman–Kac formula employed in its construction, each of the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ has the same structure as the local Gibbs measure of a classical lattice model. The only, but essential, difference is that here even the single-site spaces are infinite-dimensional (spaces of continuous paths). Therefore, the reference measure employed in the construction of $\nu_{\beta,\Lambda}$ cannot be Lebesgue measure, which does not exist for such spaces. Instead, we use a Gaussian measure, which serves as a local Euclidean Gibbs measure of a single harmonic oscillator. In spite of the mentioned difficulty, the local Euclidean Gibbs measures, as well as the corresponding local conditional Gibbs measures, possess properties which allow for employing most of the DLR techniques adapted, however, to infinite-dimensional single-site spaces. This is realized in the first part of the book.

As was mentioned above, the model (2) has various physical applications and the corresponding physical objects are well studied, both experimentally and theoretically, e.g., by means of numerical methods and computer simulations. At the same time, the rigorous mathematical description of its equilibrium thermodynamic properties based on a widely recognized method has not been given yet. In the first part of the present book we develop a version of such a description. Therefore, it would be quite natural to

[4] A more detailed analysis of similar problems, which appear in the theory of interacting Bose gases, can be found on page 349 of [77].

obtain in its framework a qualitative explanation of the basic known facts concerning the thermodynamic properties of these physical objects. This is done in the second part of the book.

Thus, the main points of the Euclidean approach developed below are

(a) Constructing the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ as measures on path spaces (spaces of continuous paths).

(b) Constructing and studying the conditional local Euclidean Gibbs measures and hence the local Gibbs specifications.

(c) Constructing the set $\mathscr{G}^{\mathrm{t}}_{\beta}$ of tempered Euclidean Gibbs measures, describing the whole infinite model as the set of probability measures which solve the DLR (equilibrium) equations defined by the local Gibbs specification.

(d) Studying the properties of $\mathscr{G}^{\mathrm{t}}_{\beta}$ and thereby describing phase transitions and quantum effects in the model considered.

This program is realized in the book as follows. Part I, as said above, is dedicated to the mathematical background. It consists of Chapters 1–3. In Chapter 1, we start by introducing the model and making natural stability assumptions regarding $J_{\ell\ell'}$ and V_ℓ. Then we introduce the state (6) and provide the essential facts concerning linear operators in Hilbert spaces, the Schrödinger operators of single harmonic and anharmonic oscillators, normal states, and von Neumann algebras. Afterwards, we prove the density theorem which allows for describing local Gibbs states by the Green functions corresponding to multiplication operators. Next, we give a complete proof of the multiple-time analyticity of the Green functions, which leads us to the Matsubara functions and then to the representation (13). In passing from the states (6) to the measures $\nu_{\beta,\Lambda}$, as a reference system we use the subsystem of noninteracting harmonic oscillators. Its Green and Matsubara functions are obtained explicitly. Thereby, we present and interpret a collection of concepts and tools from stochastic analysis, which will be used subsequently. This includes a number of facts from the theory of probability measures on complete separable metric spaces (called Polish spaces), in particular on separable Hilbert spaces. As we show, the Euclidean Gibbs measure of a single harmonic oscillator is the measure corresponding to the periodic Ornstein–Uhlenbeck velocity process (periodic oscillator process), which for the first time appeared in R. Høegh-Krohn's paper [156]. We call them Høegh-Krohn process and Høegh-Krohn measure respectively. The properties of the Høegh-Krohn measure play a significant role in our construction and are, therefore, analyzed in detail. We construct and study the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ by using a version of the Feynman–Kac formula. Chapter 1, as all subsequent chapters, is concluded with comments and bibliographic notes.

In Chapter 2, for the local Euclidean Gibbs measures, we prove a number of correlation inequalities and similar useful facts. The proof is based on the 'lattice approximation' of the measures $\nu_{\beta,\Lambda}$, in which the approximating measures are local Gibbs measures of classical models with 'unbounded spins'. The main point here is the approximation of the Høegh-Krohn process (which is a periodic Markov process) by a Markov chain. A similar approach is known in Euclidean quantum field theory. By means of this approximation we rederive the basic correlation inequalities known for classical spin models. Among the new results obtained here we mention the Lee–Yang property for a certain type of anharmonic potentials V_ℓ, scalar domination inequalities which allow for comparing scalar and vector versions of our model, and some new inequalities for Matsubara and Ursell functions.

Chapter 3 is dedicated to the construction and description of the Euclidean Gibbs states of the model (2) in complete generality. We start by discussing the thermodynamic limit and limiting Gibbs states. Then we introduce the spaces of all configurations Ω_β and tempered configurations $\Omega^{\mathrm{t}}_\beta$. The space Ω_β is constructed from the spaces of local configurations in a natural way. We equip Ω_β with the product topology that turns it into a Polish space. This fact is essential in view of the DLR techniques which we are going to use. The reason to introduce the space of tempered configurations $\Omega^{\mathrm{t}}_\beta$ is twofold. First, since the interaction intensities $J_{\ell\ell'}$ may have infinite range, we must impose some a priori restrictions on the L^2-norms $\|\xi_\ell\|_{L^2_\beta}$ of the components of configurations $\xi \in \Omega_\beta$. Otherwise, the local conditional Euclidean Gibbs measures $\nu_{\beta,\Lambda}(\cdot|\xi)$ cannot be defined. Second, even if $J_{\ell\ell'}$ had finite range, restrictions should be imposed to exclude measures which in a sense are 'improper'. By definition, *tempered* Euclidean Gibbs measures are to be supported by $\Omega^{\mathrm{t}}_\beta$. This is a usual procedure in the DLR theory of Gibbs measures of systems of 'unbounded spins'. However, as we show afterwards, the real support of the tempered Euclidean Gibbs measures is much smaller than $\Omega^{\mathrm{t}}_\beta$ and is independent of the way the latter set has been introduced. As to this way, the restrictions are imposed by means of weights, $\{w_\alpha\}_{\alpha\in\mathcal{I}}$, that among other properties have the one by which each function $-\log w_\alpha$, $\alpha \in \mathcal{I}$, is a metric on $\mathbb{L}$. We equip $\Omega^{\mathrm{t}}_\beta$ with a projective limit topology, so that it becomes a Polish space as well. In Section 3.2, we prove that the kernels $\pi_{\beta,\Lambda}$ obtained from the local conditional Gibbs measures obey certain exponential moment estimates, which play a key role in constructing and studying the tempered Euclidean Gibbs measures. In Section 3.3, we prove that the set of such measures $\mathcal{G}^{\mathrm{t}}_\beta$ is non-void and weakly compact. We also prove a number of statements characterizing $\mathcal{G}^{\mathrm{t}}_\beta$, among them the support property mentioned above. Next we develop an alternative approach to the construction of Euclidean Gibbs measures based on the Radon–Nikodym characterization. In this approach, $\mathcal{G}^{\mathrm{t}}_\beta$ is defined as the set of measures obeying an integration-by-parts formula. Subsequently, we present a more detailed study of the case of local interactions, where the intensities $J_{\ell\ell'}$ have finite range, and of the translation-invariant case, where $\mathbb{L} = \mathbb{Z}^d$, $V_\ell = V$, and $J_{\ell\ell'}$ are invariant with respect to the translations of $\mathbb{L}$. In the latter case, the set $\mathcal{G}^{\mathrm{t}}_\beta$ among others contains the so-called periodic elements. Finally, for $J_{\ell\ell'} \geq 0$ and $\nu = 1$, we introduce a stochastic order on $\mathcal{G}^{\mathrm{t}}_\beta$, with respect to which it has a minimal element, μ_-, and a maximal element, μ_+. This fact is then employed in further studying $\mathcal{G}^{\mathrm{t}}_\beta$. In particular, by means of these elements a uniqueness criterion is obtained.

Part II, comprising Chapters 4–7, is dedicated to the description of some physical properties of the model defined by (2) and (3). Here we concentrate on those related to phase transitions and critical points, as well as on quantum effects. In Chapter 4, we discuss in more detail which physical systems can be modeled by the Hamiltonians (2) and (3). Then we study the classical limit $m \to +\infty$, cf. (4), of the local Euclidean Gibbs measures and show that they coincide with those of the corresponding system of classical anharmonic oscillators. Next, we prove that $\mathscr{G}^{\mathrm{t}}_{\beta}$ is a singleton at high temperatures and/or weak interactions. Chapter 5 is dedicated to the study of the thermodynamic pressure, which up to a factor coincides with the free energy density. Here we suppose that $\mathbb{L}$ is a lattice and the model is translation-invariant. We begin by proving that the pressure exists and is the same for each state $\mu \in \mathscr{G}^{\mathrm{t}}_{\beta}$. Then we describe its dependence on the external field and formulate a uniqueness criterion in terms of differentiability of the pressure. Next, in Chapter 6, we study phase transitions. According to our definition, a phase transition occurs if the set of tempered Euclidean Gibbs measures contains more than one element that corresponds to the non-uniqueness of equilibrium phases. We also analyze the connection of this definition with the one based on an order parameter and with the definition of L. Landau. Then we prove that a number of versions of our model, including those with irregular $\mathbb{L}$, have a phase transition under certain conditions. The proof is based on the reflection positivity method, adapted here to the Euclidean approach, on correlation inequalities, and on appropriate analytic methods developed in the first part of the book. Next, we consider a hierarchical model of quantum anharmonic oscillators, which is a special case of the model (2), (3). For this model, we prove a statement describing its critical point.

The final Chapter 7 is dedicated to the theory of quantum effects in our model. Since the 1970s, understanding the influence of quantum effects on phase transitions is one of the main tasks in the theory of systems of this kind. As is commonly accepted, a ferroelectric phase transition in the KDP-type compounds is triggered by the ordering of protons on the hydrogen bounds and, therefore, the model (2) is quite appropriate to describe this class of physical objects. These ferroelectrics become less stable with respect to a structural phase transition if one replaces protons by deuterons[5]. On the other hand, high hydrostatic pressure, which increases tunneling of the particles by bringing minima of the wells closer to one another, decreases the transition temperature. We propose a theory, which qualitatively explains all these facts. It naturally comes from the results obtained above and is based on the following arguments. The key parameter here is $m\varDelta^2$, where m is the mass parameter (4) and $\varDelta$ is the least difference between the eigenvalues of the single-particle Hamiltonian H_ℓ (which depends on m). In the harmonic case, $m\varDelta^2$ is merely the oscillator rigidity and the stability of the crystal corresponds to large values of this parameter. That is why we call $m\varDelta^2$ *quantum rigidity*. If the tunneling between the wells gets more intensive (closer minima), or if the mass diminishes, $m\varDelta^2$ gets bigger and the particle 'forgets' the details of the potential energy in the vicinity of the origin (including instability) and oscillates as if its equilibrium at zero were stable, as in the harmonic case. We provide a complete

[5] This amounts to altering the particle mass in (3) without changing any other parameters.

mathematical background for these arguments. First, we prove that the quantum rigidity is a continuous function of m and that $m\Delta^2 \to +\infty$ as $m \to 0$. Then we prove that the model has no phase transitions, at any temperature, if $m\Delta^2 < \hat{J}_0$, where $\hat{J}_0$ is the total energy of the interaction of the particle with the rest of the system. This can be considered as a further confirmation that our Euclidean approach is adequate to describe a large class of phenomena arising in solid state physics.

Part I

Mathematical Background

Chapter 1
Quantum Mechanics and Stochastic Analysis

1.1 The Model and Preliminaries

In Subsection1.1.1, we introduce our main model, the description of which is performed in terms of linear operators on the corresponding Hilbert spaces. Then in Subsection 1.1.2, we present a concise collection of notions and facts about linear operators and Hilbert spaces, which are employed in the sequel. As the main element of our model is a quantum oscillator, in Subsection 1.1.3 we present its detailed theory. First, we consider a harmonic oscillator described by the Schrödinger operator H^{har}. We establish its domain, eigenfunctions, and the corresponding eigenvalues. Afterwards, we develop the theory of the Schrödinger operator $H = H^{\text{har}} + V$, where V is an anharmonic potential. Here, along with known facts, we also use some recent results obtained in the theory of such operators. Thereafter, we study the dependence of the gap parameter of the spectrum of H on the particle mass, which is used in the theory of physical properties of our model developed in Part II.

Throughout the book by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ we denote the sets of complex, real, positive real, integer, positive integer, and nonnegative integer numbers, respectively. For a complex number $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, we write $x = \Re(z)$, $y = \Im(z)$, and $\bar{z} = x - \mathrm{i}y$.

1.1.1 The Model

The model we consider in this book is a system of interacting quantum anharmonic oscillators indexed by the elements of a countable set $\mathbb{L} \subset \mathbb{R}^d$, $d \in \mathbb{N}$ (one oscillator per each $\ell \in \mathbb{L}$). The set $\mathbb{L}$ is equipped with the Euclidean distance

$$|\ell - \ell'| = [|\ell_1 - \ell'_1|^2 + \cdots + |\ell_d - \ell'_d|^2]^{1/2}, \quad \ell, \ell' \in \mathbb{L}, \tag{1.1.1}$$

inherited from the metric space $\mathbb{R}^d$. We impose a regularity condition,

$$\sup_{\ell \in \mathbb{L}} \sum_{\ell' \in \mathbb{L}} \frac{1}{(1 + |\ell - \ell'|)^{d+\epsilon}} < \infty, \tag{1.1.2}$$

which has to hold for every $\epsilon > 0$. This means that $\mathbb{L}$ has no accumulation points. In particular, $\mathbb{L}$ can be a crystalline lattice. We shall always assume that $\mathbb{L} = \mathbb{Z}^d$ if $\mathbb{L}$ is a lattice. For the sake of simplicity, we call our model *quantum anharmonic crystal*, even if $\mathbb{L}$ is not a crystal.

The *quantum anharmonic oscillator* is a mathematical model of a localized quantum particle moving in a potential field with possibly multiple stable equilibrium positions and sufficiently fast growth at infinity. We suppose that the space in which such a particle moves is $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$, that is, our particle performs ν-dimensional oscillations around a certain point, specific for each particle. Its displacement from this point is described by the operator $q_\ell = (q_\ell^{(j)})_{j=1}^\nu$, $\ell \in \mathbb{L}$. Correspondingly, the momentum of the particle is a vector, $p_\ell = (p_\ell^{(j)})_{j=1}^\nu$. The Schrödinger operator (also called Hamiltonian) of a single anharmonic oscillator of mass $m > 0$ is

$$H_\ell = H_\ell^{\rm har} + V_\ell(q_\ell) \stackrel{\rm def}{=} \frac{1}{2m}|p_\ell|^2 + \frac{a}{2}|q_\ell|^2 + V_\ell(q_\ell), \quad a > 0. \tag{1.1.3}$$

Here $|\cdot|$ stands also for the norm in $\mathbb{R}^\nu$. The first two terms constitute the Hamiltonian $H_\ell^{\rm har}$ of *a quantum harmonic oscillator* of mass m and *rigidity* a, whereas V_ℓ is an anharmonic correction, which is a function, $V_\ell : \mathbb{R}^\nu \to \mathbb{R}$. We shall also call it *anharmonic potential*. The total potential energy is the sum of the harmonic part $a|q_\ell|^2/2$ and the anharmonic potential; its minima correspond to the stable equilibrium positions of the oscillator. The Hamiltonian (1.1.3), as well as the momentum p_ℓ and displacement q_ℓ, are operators acting in a *single-site physical Hilbert space* $\mathcal{H}_\ell$, which in our case is the space $L^2(\mathbb{R}^\nu)$ of square-integrable *wave functions* $\psi : \mathbb{R}^\nu \to \mathbb{C}$. As usual, two such functions which differ on a subset of $\mathbb{R}^\nu$ of zero Lebesgue measure define the same element of $L^2(\mathbb{R}^\nu)$. The scalar product and norm in this space are

$$(\phi, \psi)_{\mathcal{H}_\ell} = \int_{\mathbb{R}^\nu} \bar{\phi}(x)\psi(x){\rm d}x, \quad \|\psi\|_{\mathcal{H}_\ell} = \sqrt{(\psi,\psi)_{\mathcal{H}_\ell}}, \tag{1.1.4}$$

where $\bar{\phi}$ is the complex conjugate of ϕ. Wave functions of unit norm determine states of the particle in the following sense. By definition, the Borel σ-algebra $\mathcal{B}(\mathbb{R}^\nu)$ of subsets of $\mathbb{R}^\nu$ is the smallest σ-algebra which contains all open sets. For $B \in \mathcal{B}(\mathbb{R}^\nu)$, the quantity

$$\mathsf{P}_\psi(B) = \int_B |\psi(x)|^2 {\rm d}x \tag{1.1.5}$$

is chosen to be the probability that the particle in state ψ is contained in set B. A brief presentation of the main notions of the theory of linear operators acting in Hilbert spaces is given in the next subsection, where we also return to the interpretation of wave functions as states, see Definition 1.1.12 and the discussion following it.

The components of the displacement and momentum operators $q_\ell^{(j)}$, $p_\ell^{(j)}$, $j = 1, \ldots, \nu$, satisfy on their common domain *the canonical commutation relation*

$$[q_\ell^{(j)}, p_{\ell'}^{(j')}] = q_\ell^{(j)} p_{\ell'}^{(j')} - p_{\ell'}^{(j')} q_\ell^{(j)} = {\rm i}\delta_{\ell\ell'}\delta_{jj'}, \quad {\rm i} = \sqrt{-1}, \tag{1.1.6}$$

where δ is the Kronecker symbol. The Planck constant $\hbar$, which usually appears in (1.1.6), has been included into the particle mass, i.e., the mass parameter m in (1.1.3) is

$$m = m_{\rm ph}/\hbar^2, \tag{1.1.7}$$

where $m_{\rm ph}$ is the physical mass of the particle.

The quantum anharmonic crystal we study throughout this book is described by a 'heuristic Hamiltonian'

$$H = -\frac{1}{2}\sum_{\ell,\ell'} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell} H_\ell, \tag{1.1.8}$$

where the indices of the sums run through the set $\mathbb{L}$, the brackets $(\cdot,\cdot)$ stand for the scalar product in $\mathbb{R}^\nu$, and H_ℓ is the Hamiltonian (1.1.3). The set $\mathbb{L}$ obeys the condition (1.1.2) only. We reiterate that the model is called a quantum anharmonic crystal, even in the general case where $\mathbb{L}$ is not a crystal at all. The first term in (1.1.8) describes the interaction between the particles. The interaction intensities

$$J_{\ell\ell} = 0, \quad J_{\ell\ell'} = J_{\ell'\ell} \in \mathbb{R}, \quad \ell, \ell' \in \mathbb{L}, \tag{1.1.9}$$

constitute the so-called dynamical matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$. Our choice of interaction term is motivated by physical applications and will be discussed in the second part of the book. We shall refer to this 'heuristic Hamiltonian' as the model we consider. The anharmonic potentials V_ℓ and the dynamical matrix $(J_{\ell\ell'})_{\mathbb{L}\times\mathbb{L}}$ are subject to the following

Assumption 1.1.1. All $V_\ell \colon \mathbb{R}^\nu \to \mathbb{R}$ are continuous and such that $V_\ell(0) = 0$. Furthermore, there exist an integer $r \geq 2$, constants $b_V > 0$, $c_V \geq 0$, and a continuous function $V \colon \mathbb{R}^\nu \to \mathbb{R}$, $V(0) = 0$, such that for all ℓ and $x \in \mathbb{R}^\nu$,

$$b_V |x|^{2r} - c_V \leq V_\ell(x) \leq V(x). \tag{1.1.10}$$

Finally, we assume that

$$\hat{J}_0 \stackrel{\text{def}}{=} \sup_{\ell} \sum_{\ell'} |J_{\ell\ell'}| < \infty. \tag{1.1.11}$$

By imposing (1.1.11) we shall avoid problems with infinite forces acting on a given oscillator. The lower bound in (1.1.10) is responsible for confining the particle to the vicinity of the point $q_\ell = 0$. According to this bound the anharmonic potentials have a *super-quadratic growth*, due to which they dominate the first term in (1.1.8). The upper bound in (1.1.10) guarantees that the oscillations of the particles located far from the origin are not suppressed. An example of V_ℓ to bear in mind is an even polynomial

$$V_\ell(x) = \sum_{s=1}^{r} b_\ell^{(s)} |x|^{2s} - (h, x), \quad b_\ell^{(s)} \in \mathbb{R}, \quad r \geq 2, \tag{1.1.12}$$

in which $h \in \mathbb{R}^\nu$ is an external field and the coefficients $b_\ell^{(s)}$ vary in certain intervals, such that both estimates in (1.1.10) hold. Thereby, the main object of our study is the model described by the Hamiltonians (1.1.3), (1.1.8). Its particular cases are specified in the following

Definition 1.1.2. The model (1.1.3), (1.1.8) is called rotation-invariant if $V_\ell(x) = V_\ell(Ux)$ for all ℓ and all orthogonal transformations $U \colon \mathbb{R}^\nu \to \mathbb{R}^\nu$. In the case $\nu = 1$, it merely means that $V_\ell(x) = V_\ell(-x)$ for all ℓ. The model is ferromagnetic if $J_{\ell\ell'} \geq 0$ for all ℓ, ℓ'. The interaction has finite range if there exists $R > 0$ such that $J_{\ell\ell'} = 0$ whenever $|\ell - \ell'| > R$.

Note that if we discuss the scalar case $\nu = 1$, we say that the model (1.1.3), (1.1.8) is *symmetric* if $V_\ell(x) = V_\ell(-x)$ for all ℓ.

1.1.2 Linear Operators in Hilbert Spaces

Basic notions

A Hilbert space $\mathcal{H}$ is a linear space over the complex field $\mathbb{C}$ endowed with a scalar product $(\cdot,\cdot)_{\mathcal{H}}$ and the corresponding norm $\|\cdot\|_{\mathcal{H}}$, cf. (1.1.4), which is complete as a normed space. The latter means that every Cauchy sequence of its elements $\{\psi_n\}_{n\in\mathbb{N}}$ has a limit in $\mathcal{H}$, that is, there exists $\psi \in \mathcal{H}$ such that $\psi_n \to \psi$, as $n \to +\infty$. The latter means that

$$\lim_{n\to+\infty} \|\psi_n - \psi\|_{\mathcal{H}} = 0.$$

By definition, a *Cauchy sequence* has the property that for any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\|\psi_n - \psi_m\|_{\mathcal{H}} < \varepsilon$ whenever $n, m > n_\varepsilon$. We always assume that $\dim \mathcal{H} > 0$, which just means that each of our Hilbert spaces contains nonzero vectors. We also assume that the map $(\phi,\psi) \mapsto (\phi,\psi)_{\mathcal{H}}$ is linear with respect to the second argument and is anti-linear with respect to the first one, cf. (1.1.4); that is, for all ϕ, ϕ_1, ϕ_2, ψ, $\psi_1, \psi_2 \in \mathcal{H}$ and all $\alpha_1, \alpha_2 \in \mathbb{C}$,

$$\begin{aligned} (\phi, \alpha_1\psi_1 + \alpha_2\psi_2)_{\mathcal{H}} &= \alpha_1(\phi,\psi_1)_{\mathcal{H}} + \alpha_2(\phi,\psi_2)_{\mathcal{H}}, \\ (\alpha_1\phi_1 + \alpha_2\phi_2, \psi)_{\mathcal{H}} &= \bar{\alpha}_1(\phi_1,\psi)_{\mathcal{H}} + \bar{\alpha}_2(\phi_2,\psi)_{\mathcal{H}}. \end{aligned} \tag{1.1.13}$$

As $(\phi,\psi)_{\mathcal{H}} = \overline{(\psi,\phi)_{\mathcal{H}}}$, the second line follows from the first one. The scalar product and norm obey the *polarization identity,*

$$(\phi,\psi)_{\mathcal{H}} = \frac{1}{4}\{\|\psi+\phi\|^2_{\mathcal{H}} - \|\psi-\phi\|^2_{\mathcal{H}} - \mathrm{i}\|\psi+\mathrm{i}\phi\|^2_{\mathcal{H}} + \mathrm{i}\|\psi-\mathrm{i}\phi\|^2_{\mathcal{H}}\}, \tag{1.1.14}$$

and the *Cauchy–Schwarz inequality*,

$$|(\phi,\psi)_{\mathcal{H}}| \le \|\phi\|_{\mathcal{H}} \cdot \|\psi\|_{\mathcal{H}}. \tag{1.1.15}$$

By the latter inequality one obtains that $(\phi_m,\psi_n)_{\mathcal{H}} \to (\phi,\psi)_{\mathcal{H}}$ whenever $\psi_n \to \psi$ and $\phi_m \to \phi$, as $n, m \to +\infty$. Thus, the map $(\phi,\psi) \mapsto (\phi,\psi)_{\mathcal{H}}$ is continuous. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-void subsets of $\mathcal{H}$. By definition, they are mutually orthogonal if $(\psi,\phi)_{\mathcal{H}} = 0$ for each $\psi \in \mathcal{A}$ and each $\phi \in \mathcal{B}$. For a non-void $\mathcal{A} \subset \mathcal{H}$, its *annihilator* is set to be

$$\mathcal{A}^\perp = \{\phi \in \mathcal{H} \mid \forall \psi \in \mathcal{A} : \ (\phi,\psi)_{\mathcal{H}} = 0\}. \tag{1.1.16}$$

A subset of a Hilbert space is called linear if it is closed under the linear operations. Such subsets are also called algebraic subspaces. By definition, $\mathcal{A}$ is a *dense subset* of $\mathcal{H}$ if each $\phi \in \mathcal{H}$ is the limit of a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$. If a linear subset is closed in norm (contains the limits of all its Cauchy sequences), we say that it is a *subspace* of the Hilbert space $\mathcal{H}$. The set (1.1.16) is a subspace of $\mathcal{H}$ for any $\mathcal{A}$. Below subsets closed in norm are called merely closed.

Remark 1.1.3. If $\mathcal{A} \subset \mathcal{H}$ is dense, then $\mathcal{A}^\perp = \{0\}$, that is, the only vector which can be orthogonal to all elements of a dense subset is the zero vector.

Indeed, for a given $\phi \in \mathcal{A}^\perp$, let $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{A}$ be such that $\phi_n \to \phi$. Then by the continuity of the scalar product one gets $(\phi, \phi)_{\mathcal{H}} = 0$.

A subset $\mathcal{B} \subset \mathcal{H}$ is called an *orthonormal family* if: (a) each $\phi \in \mathcal{B}$ is normalized, i.e., $\|\phi\|_{\mathcal{H}} = 1$; (b) any two distinct $\phi, \psi \in \mathcal{B}$ are orthogonal, i.e., $(\phi, \psi)_{\mathcal{H}} = 0$. Such a family is called *complete* if $\mathcal{B}^\perp = \{0\}$. Every Hilbert space contains complete orthonormal families. If $\mathcal{B} \subset \mathcal{H}$ is such a family, then for every $\psi \in \mathcal{H}$, there exists a subset $\{\phi_k\}_{k\in\mathsf{K}} \subset \mathcal{B}$, K being at most countable, such that

$$\psi = \sum_{k\in\mathsf{K}} (\phi_k, \psi)_{\mathcal{H}} \phi_k. \tag{1.1.17}$$

A Hilbert space $\mathcal{H}$ is called *separable* if it contains a countable dense subset. $\mathcal{H}$ is separable if and only if it contains a complete orthonormal family $\mathcal{B}$ which is at most countable. If $\mathcal{B}$ is finite, i.e., $|\mathcal{B}| = N$, then $\dim \mathcal{H} = N$. Otherwise, $\mathcal{H}$ is infinite-dimensional. If $\mathcal{H}$ is separable, every $\psi \in \mathcal{H}$ can be written as in (1.1.17) with one and the same orthonormal family $\{\phi_k\}_{k\in\mathsf{K}}$, which is then called an *orthonormal basis* of $\mathcal{H}$. Sometimes, the sum on the right-hand side of (1.1.17) is said to be the *Fourier series* of ψ. In the sequel, if not explicitly stated otherwise, we consider separable Hilbert spaces only.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{D}$ be a linear subset of $\mathcal{H}_1$. A linear operator[1] from $\mathcal{H}_1$ to $\mathcal{H}_2$ with domain $\mathcal{D}$ is a map, $T : \mathcal{D} \subset \mathcal{H}_1 \to \mathcal{H}_2$, such that

$$\forall \psi, \phi \in \mathcal{D}, \ \forall \alpha, \beta \in \mathbb{C}: \quad T(\alpha\psi + \beta\phi) = \alpha T\psi + \beta T\phi,$$

which in particular means that T maps the zero vector of $\mathcal{H}_1$ into the zero vector of $\mathcal{H}_2$. To indicate that $\mathcal{D}$ is the domain of T we write $\mathcal{D} = \mathrm{Dom}(T)$. By $\mathrm{Ran}(T)$ we denote the *range* of T, that is, the set of all $T\psi$, $\psi \in \mathrm{Dom}(T)$. Finally, the set $\mathrm{Ker}(T) = \{\psi \in \mathcal{H}_1 \mid T\psi = 0\}$ is called the *kernel* of T, it always contains the zero vector. Note that it may happen that $\mathrm{Ker}(T) = \mathrm{Dom}(T) = \{0\}$. The operator O, for which $\mathrm{Dom}(O) = \mathrm{Ker}(O) = \mathcal{H}$, is called the *zero operator*. For $\mathcal{A} \subset \mathrm{Ran}(T)$, by $T^{-1}\mathcal{A}$ we denote the pre-image of $\mathcal{A}$, that is, $T^{-1}\mathcal{A} = \{\phi \in \mathrm{Dom}(T) \mid T\phi \in \mathcal{A}\}$. One observes that $\mathrm{Ker}(T)$ is a linear subset of $\mathcal{H}_1$, whereas $\mathrm{Ran}(T)$ is a linear subset of $\mathcal{H}_2$.

A linear operator $T : \mathrm{Dom}(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is called *invertible* if there exists a linear operator $Q : \mathrm{Ran}(T) \subset \mathcal{H}_2 \to \mathcal{H}_1$, that is $\mathrm{Dom}(Q) = \mathrm{Ran}(T)$, such that $Q\phi = \psi$ whenever $T\psi = \phi$. Then Q is called the *inverse* of T and is denoted by T^{-1}. It is uniquely determined by T. Furthermore, T is invertible if and only if $\mathrm{Ker}(T) = \{0\}$. In this case, $\mathrm{Ker}(Q) = \{0\}$ and $Q^{-1} = T$.

Suppose that for given Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, there exists a linear operator, $T : \mathcal{H}_1 \to \mathcal{H}_2$, with $\mathrm{Dom}(T) = \mathcal{H}_1$, $\mathrm{Ker}(T) = \{0\}$, and $\mathrm{Ran}(T) = \mathcal{H}_2$. Then T is

[1]As throughout the whole book we consider only linear operators, by saying *operator* we shall always mean *linear operator*.

invertible and T^{-1}, which maps $\mathcal{H}_2$ *onto* $\mathcal{H}_1$, has the same properties as T. In this case, both T and T^{-1} are called *isomorphisms,* whereas the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are called *isomorphic*. An isomorphism $T\colon \mathcal{H} \to \mathcal{H}$ is called an *automorphism*. For $N \in \mathbb{N}$, the linear space $\mathbb{C}^N = \{x = (x_1, \dots, x_N) \mid x_j \in \mathbb{C},\, j = 1, \dots, N\}$ is endowed with the scalar product $(x, y) \mapsto (x, y)_{\mathbb{C}^N} = \bar{x}_1 y_1 + \cdots + \bar{x}_N y_N$. Thereby, it becomes an N-dimensional complex Hilbert space. Every complex Hilbert space $\mathcal{H}$, such that $\dim \mathcal{H} = N$, is isomorphic to $\mathbb{C}^N$. Linear operators $\theta\colon \mathcal{H} \to \mathbb{C}$ are called *linear functionals*.

Bounded operators

A linear operator $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ is called *bounded* if $\mathrm{Dom}(T) = \mathcal{H}_1$ and there exists $C > 0$ such that

$$\|T\phi\|_{\mathcal{H}_2} \le C\|\phi\|_{\mathcal{H}_1}, \tag{1.1.18}$$

for all $\phi \in \mathcal{H}_1$. The infimum of C obeying (1.1.18) is called the *norm* of T. It can be calculated from the formulas

$$\|T\| = \sup_{\phi\in\mathcal{H}_1,\ \phi\neq 0} \frac{\|T\phi\|_{\mathcal{H}_2}}{\|\phi\|_{\mathcal{H}_1}} = \sup_{\phi\in\mathcal{H}_1,\ \|\phi\|_{\mathcal{H}_1}=1} \|T\phi\|_{\mathcal{H}_2}. \tag{1.1.19}$$

Therefore,

$$\forall \phi \in \mathcal{H}_1: \quad \|T\phi\|_{\mathcal{H}_2} \le \|T\| \cdot \|\phi\|_{\mathcal{H}_1}. \tag{1.1.20}$$

Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of bounded operators. We say that $\{T_n\}_{n\in\mathbb{N}}$ converges to a bounded operator T *in norm* if $\|T - T_n\| \to 0$ as $n \to +\infty$. We say that $\{T_n\}_{n\in\mathbb{N}}$ converges to T *strongly* if $\|T\psi - T_n\psi\|_{\mathcal{H}_2} \to 0$ for all $\psi \in \mathcal{H}_1$. By (1.1.20) the convergence in norm implies strong convergence. For infinite-dimensional spaces, the converse is not true.

As in the general case of mappings between normed spaces, a linear operator $T\colon \mathrm{Dom}(T) \subset \mathcal{H}_1 \to \mathcal{H}_2$ is called *continuous at a given point* (*vector*) $\psi \in \mathrm{Dom}(T)$ if for every sequence $\{\psi_n\}_{n\in\mathbb{N}}$ such that $\psi_n \to \psi$ in $\mathcal{H}_1$, one has $T\psi_n \to T\psi$ in $\mathcal{H}_2$. Such an operator is called *continuous* (on its domain) if it is continuous at every $\psi \in \mathrm{Dom}(T)$. Suppose that T is continuous at ψ and take any other $\phi \in \mathrm{Dom}(T)$. For a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(T)$, such that $\phi_n \to \phi$, we set $\psi_n = \phi_n + (\psi - \phi)$. By linearity, $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(T)$. At the same time, $\psi_n \to \psi$ and hence $T\psi_n \to \psi$. By the linearity of T this yields $T\phi_n \to \phi$; thus, T is continuous also at ϕ and thereby everywhere on its domain. In summary, if T fails to be continuous at a single point, it is nowhere continuous, and to check the continuity of T on $\mathrm{Dom}(T)$, one only has to check it at $\psi = 0$. Bounded operators are continuous. Indeed, if a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ converges to $0 \in \mathcal{H}_1$, then by (1.1.20) one has $\|T\psi_n\|_{\mathcal{H}_2} \to 0$. The converse is also true – any continuous linear operator $T\colon \mathcal{H}_1 \to \mathcal{H}_2$ is bounded.

Let $\mathfrak{L}(\mathcal{H}_1, \mathcal{H}_2)$ be the set of all bounded linear operators $T\colon \mathcal{H}_1 \to \mathcal{H}_2$. For short, we write $\mathfrak{L}(\mathcal{H}) = \mathfrak{L}(\mathcal{H}, \mathcal{H})$. Each $T \in \mathfrak{L}(\mathcal{H})$ defines the mapping $\mathcal{H} \ni \psi \mapsto (\psi, T\psi)_{\mathcal{H}} \in \mathbb{C}$. The set

$$\mathrm{Num}(T) = \{(\psi, T\psi)_{\mathcal{H}} \mid \psi \in \mathcal{H},\ \|\psi\|_{\mathcal{H}} = 1\} \tag{1.1.21}$$

is called the *numerical range* of T. The mapping $\psi \mapsto (\psi, T\psi)_{\mathscr{H}}$ uniquely determines T in the following sense.

Proposition 1.1.4. *If $T, Q \in \mathfrak{C}(\mathscr{H})$ are such that $(\psi, T\psi)_{\mathscr{H}} = (\psi, Q\psi)_{\mathscr{H}}$ for all $\psi \in \mathscr{H}$, then $T = Q$.*

Proof. One can easily check that for any $T \in \mathfrak{C}(\mathscr{H})$,

$$\begin{aligned}(\phi, T\psi)_{\mathscr{H}} = \frac{1}{4}\{&(\phi+\psi, T(\phi+\psi))_{\mathscr{H}} - (\phi-\psi, T(\phi-\psi))_{\mathscr{H}} \\ &- \mathrm{i}(\phi+\mathrm{i}\psi, T(\phi+\mathrm{i}\psi))_{\mathscr{H}} + \mathrm{i}(\phi-\mathrm{i}\psi, T(\phi-\mathrm{i}\psi))_{\mathscr{H}}\}.\end{aligned} \tag{1.1.22}$$

Then the assumed property of T and Q is equivalent to $(\phi, T\psi)_{\mathscr{H}} = (\phi, Q\psi)_{\mathscr{H}}$, holding for all $\psi, \phi \in \mathscr{H}$. Therefore, $(T - Q)\psi$ is orthogonal to every ϕ and hence $Q = T$. □

For $T \in \mathfrak{C}(\mathscr{H})$, its adjoint operator $T^*\colon \mathscr{H} \to \mathscr{H}$ is defined by the condition

$$\forall\, \phi, \psi \in \mathscr{H}: \quad (\phi, T\psi)_{\mathscr{H}} = (T^*\phi, \psi)_{\mathscr{H}}. \tag{1.1.23}$$

T^* is hereby uniquely determined and bounded. T is called *self-adjoint* if $T = T^*$. By definition, the identity operator I is such that $I\psi = \psi$ for all $\psi \in \mathscr{H}$. Both I and O are apparently self-adjoint.

Let $\mathcal{K}$ be a subspace of $\mathscr{H}$. Then $\mathcal{K}^\perp$ is also a subspace, $\mathcal{K}^{\perp\perp} \stackrel{\text{def}}{=} (\mathcal{K}^\perp)^\perp = \mathcal{K}$, and each $\psi \in \mathscr{H}$ can uniquely be decomposed into the sum of $\phi \in \mathcal{K}$ and $\phi' \in \mathcal{K}^\perp$, that is, $\mathscr{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. By $P_{\mathcal{K}}$ and $P_{\mathcal{K}^\perp}$ we denote the linear operators which act on $\psi = \phi + \phi'$ according to

$$P_{\mathcal{K}}(\phi + \phi') = \phi, \quad P_{\mathcal{K}^\perp}(\phi + \phi') = \phi'. \tag{1.1.24}$$

These are the *orthogonal projections* onto the subspaces $\mathcal{K}$ and $\mathcal{K}^\perp$ respectively. They are bounded and self-adjoint.

The set $\mathfrak{C}(\mathscr{H})$ can be endowed with the following *point-wise* operations:

$$\forall \psi \in \mathscr{H}: \qquad \begin{aligned}(T+Q)\psi &\stackrel{\text{def}}{=} T\psi + Q\psi, \\ (TQ)\psi &\stackrel{\text{def}}{=} T(Q\psi), \\ (\alpha T)\psi &\stackrel{\text{def}}{=} \alpha(T\psi), \quad \alpha \in \mathbb{C}.\end{aligned} \tag{1.1.25}$$

Then, for all $T \in \mathfrak{C}(\mathscr{H})$, it follows that

$$IT = TI = T, \quad O + T = T, \quad OT = TO = O.$$

For the projections introduced above, we have

$$P^2_{\mathcal{K}} = P_{\mathcal{K}}, \quad P^2_{\mathcal{K}^\perp} = P_{\mathcal{K}^\perp}, \quad I = P_{\mathcal{K}} + P_{\mathcal{K}^\perp}, \quad P_{\mathcal{K}} P_{\mathcal{K}^\perp} = P_{\mathcal{K}^\perp} P_{\mathcal{K}} = O.$$

With the operations (1.1.25) the set $\mathfrak{C}(\mathcal{H})$ becomes a complex associative algebra with the zero and unit elements being O and T, respectively. This means that $\mathfrak{C}(\mathcal{H})$ is a linear space over the field $\mathbb{C}$ equipped with an associative product, which is distributive with respect to the addition. It can also be viewed as an associative ring with respect to the addition and multiplication introduced in (1.1.25), equipped with the multiplication by complex numbers (scalars). We note, however, that in general, complex associative algebras need not have unit elements.

The linear operations in (1.1.25) can also be defined for bounded operators $T : \mathcal{H}_1 \to \mathcal{H}_2$ with $\mathcal{H}_1 \neq \mathcal{H}_2$. With these operations the set $\mathfrak{C}(\mathcal{H}_1, \mathcal{H}_2)$ becomes a complex linear space. It is complete in the norm (1.1.19) and hence is a complex Banach space. A particular case is the space of all bounded linear functionals $\mathfrak{C}(\mathcal{H}, \mathbb{C})$. It is called the *dual space* of $\mathcal{H}$. By (1.1.13) and (1.1.15) for each $\phi \in \mathcal{H}$, the map $\mathcal{H} \ni \psi \mapsto (\phi, \psi)_{\mathcal{H}}$ is an element of $\mathfrak{C}(\mathcal{H}, \mathbb{C})$. It turns out that *every* element of $\mathfrak{C}(\mathcal{H}, \mathbb{C})$ can be realized in this way. This means that for every $\theta \in \mathfrak{C}(\mathcal{H}, \mathbb{C})$, there exists $\phi \in \mathcal{H}$, such that $\theta(\psi) = (\phi, \psi)_{\mathcal{H}}$ for all $\psi \in \mathcal{H}$. The corresponding statement is known as the Riesz lemma, see e.g., Theorem II.4 on page 43 in [255], according to which $\mathfrak{C}(\mathcal{H}, \mathbb{C})$ is isomorphic to $\mathcal{H}$.

A complex associative algebra $\mathfrak{C}$ is called *a normed algebra* if it is equipped with a norm that has the property

$$\|AB\| \leq \|A\| \, \|B\| \quad \text{for all } A, B \in \mathfrak{C}. \tag{1.1.26}$$

If $\mathfrak{C}$ is complete in this norm (that is, $\mathfrak{C}$ is a Banach space), then it is called *a Banach algebra*. A mapping $A \mapsto A^*$ of $\mathfrak{C}$ into itself is called *an involution* if the following conditions are satisfied:

$$\begin{aligned} &\text{(i)} && (A^*)^* = A, \\ &\text{(ii)} && (A+B)^* = A^* + B^*, \\ &\text{(iii)} && (AB)^* = B^* A^*, \\ &\text{(iv)} && (\alpha A)^* = \bar{\alpha} A^*. \end{aligned} \tag{1.1.27}$$

Definition 1.1.5. A complex associative algebra with an involution is called a $*$-algebra. A Banach $*$-algebra is called a C^*-algebra if for all its elements,

$$\|A^* A\| = \|A\|^2. \tag{1.1.28}$$

One observes that by (1.1.26) $\|A^* A\| \leq \|A^*\| \, \|A\|$; hence, (1.1.28) and claim (i) in (1.1.27) imply $\|A\| = \|A^*\|$.

As we already know, $\mathfrak{C}(\mathcal{H})$ is a complex associative algebra. Endowed with the operator norm (1.1.19) and with the involution $T \mapsto T^*$ defined by (1.1.23) it becomes a C^*-algebra with the unit element I. We say that $T \in \mathfrak{C}(\mathcal{H})$ is invertible in $\mathfrak{C}(\mathcal{H})$ if $T^{-1} \in \mathfrak{C}(\mathcal{H})$. Clearly, in this case $\mathrm{Ran}(T) = \mathcal{H}$ and $TT^{-1} = T^{-1}T = I$. Furthermore, $U \in \mathfrak{C}(\mathcal{H})$ is called *unitary* if $U^* = U^{-1}$. By $\mathfrak{C}_U(\mathcal{H})$ we denote the set of all unitary operators. It consists of all those U with $\mathrm{Ran}(U) = \mathcal{H}$, for which $(U\phi, U\psi)_{\mathcal{H}} = (\phi, \psi)_{\mathcal{H}}$ for all $\psi, \phi \in \mathcal{H}$. Clearly, $\mathfrak{C}_U(\mathcal{H})$ contains I and is a group. For each $U \in \mathfrak{C}_U(\mathcal{H})$, the map $\mathfrak{C}(\mathcal{H}) \ni T \mapsto U^*TU$ is a norm preserving automorphism of $\mathfrak{C}(\mathcal{H})$ since $\|T\| = \|U^*TU\|$. We say that Q and T are *unitary equivalent* if $Q = U^*TU$ for some $U \in \mathfrak{C}_U(\mathcal{H})$.

A linear operator $T : \mathcal{H} \to \mathcal{H}$ with $\mathrm{Dom}(T) = \mathcal{H}$ is called *compact*, if for any bounded sequence $\{\psi_n\}_{n\in\mathbb{N}}$ (i.e., $\|\psi_n\|_{\mathcal{H}} \leq C$ for all n and some $C > 0$) the sequence $\{T\psi_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. Clearly, each compact operator is continuous and hence belongs to $\mathfrak{C}(\mathcal{H})$.

Definition 1.1.6. An operator $T \in \mathfrak{C}(\mathcal{H})$ is said to be of finite rank if $\mathrm{Ran}(T)$ is a finite-dimensional subspace of $\mathcal{H}$.

For a finite rank operator T, each bounded sequence in $\mathrm{Ran}(T)$ contains a convergent subsequence. Hence, a finite rank operator is compact.

Proposition 1.1.7. *If T is compact and $Q \in \mathfrak{C}(\mathcal{H})$, then both TQ and QT are compact.*

Proof. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Then $\{Q\psi_n\}_{n\in\mathbb{N}}$ is also bounded and hence $\{TQ\psi_n\}_{n\in\mathbb{N}}$ contains a convergent subsequence. Thus, TQ is compact. Furthermore, $\{\phi_n\}_{n\in\mathbb{N}}$, where $\phi_n \stackrel{\mathrm{def}}{=} T\psi_n$, contains a convergent subsequence, say $\{\phi_{n_k}\}_{k\in\mathbb{N}}$, for which $\{Q\phi_{n_k}\}_{k\in\mathbb{N}} = \{QT\psi_{n_k}\}_{k\in\mathbb{N}}$ is also a convergent sequence as Q is continuous. $\square$

Definition 1.1.8. An operator $A \in \mathfrak{C}(\mathcal{H})$ is said to be positive if $(\psi, A\psi)_{\mathcal{H}} \geq 0$ for all $\psi \in \mathcal{H}$.

By $\mathfrak{C}^+(\mathcal{H})$ we denote the set of all positive elements of $\mathfrak{C}(\mathcal{H})$. For $A \in \mathfrak{C}^+(\mathcal{H})$, one has

$$(\psi, A\psi)_{\mathcal{H}} = \overline{(\psi, A\psi)_{\mathcal{H}}} = (A\psi, \psi)_{\mathcal{H}} = (\psi, A^*\psi)_{\mathcal{H}};$$

hence, $A^* = A$ in view of Proposition 1.1.4. If both $\pm A$ are positive, then $(\psi, A\psi)_{\mathcal{H}} = 0$ for all $\psi \in \mathcal{H}$. This immediately yields $A = O$, see Proposition 1.1.4. Then the relation

$$A \geq B \quad \text{if } A - B \in \mathfrak{C}^+(\mathcal{H}) \tag{1.1.29}$$

is an order on $\mathfrak{C}(\mathcal{H})$. A is positive if and only if $A = B^2$ for some self-adjoint $B \in \mathfrak{C}(\mathcal{H})$. In this case, there exists a unique positive B, such that $A = B^2$. This B is called the *square root* of A and is denoted by $\sqrt{A}$; it can be calculated from the series

$$\sqrt{A} = \|A\| \Big[I - \sum_{n=1}^{\infty} c_n \left(I - \|A\|^{-1} A \right)^n \Big], \tag{1.1.30}$$

where the numbers c_n are the coefficients of the Taylor expansion of the function $[0, 1] \ni x \mapsto \sqrt{1-x}$, see e.g., pages 33, 34 in [76] or pages 195, 196 in [255]. The series in (1.1.30) converges in norm, which in particular means that A and $\sqrt{A}$

commute. For a self-adjoint $B \in \mathfrak{C}(\mathscr{H})$, we set $|B| = \sqrt{B^2}$. Then each such B can be written in the form

$$B = B_+ - B_-, \quad \text{where } B_\pm = \frac{1}{2}\,(|B| \pm B), \tag{1.1.31}$$

both $B_\pm$ being positive. Since $|B|$ and B commute, one has

$$B_+B_- = \frac{1}{4}\left(|B|^2 - B^2\right) = O.$$

In view of this property, the representation $B = B_+ - B_-$ is often referred to as the orthogonal decomposition of B. We also note that $|B| = B_+ + B_-$.

An arbitrary $A \in \mathfrak{C}(\mathscr{H})$ can be decomposed according to

$$A = A_r + \mathrm{i}A_i, \quad A^* = A_r - \mathrm{i}A_i, \tag{1.1.32}$$

where both

$$A_r = \frac{1}{2}\left(A + A^*\right) \quad \text{and} \quad A_i = \frac{1}{2\mathrm{i}}\left(A - A^*\right)$$

are self-adjoint. Combining (1.1.32) with (1.1.31) we conclude that each $A \in \mathfrak{C}(\mathscr{H})$ has the representation

$$A = \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 + \alpha_4 B_4, \tag{1.1.33}$$

for some $\alpha_i \in \mathbb{C}$ and $B_i \in \mathfrak{C}^+(\mathscr{H})$, $i = 1, \ldots, 4$. For any $A \in \mathfrak{C}(\mathscr{H})$, one has $A^*A \in \mathfrak{C}^+(\mathscr{H})$ since $(\phi, A^*A\phi)_{\mathscr{H}} = \|A\phi\|^2_{\mathscr{H}} \geq 0$. Thus, for every $A \in \mathfrak{C}(\mathscr{H})$, there exists a positive operator

$$|A| \stackrel{\text{def}}{=} \sqrt{A^*A}. \tag{1.1.34}$$

It turns out that each $B \in \mathfrak{C}^+(\mathscr{H})$ can be written as $B = A^*A$ for some $A \in \mathfrak{C}(\mathscr{H})$, see Theorem 2.2.12 on page 36 in [76].

Since $\mathfrak{C}(\mathscr{H})$ is a linear space, one can define linear functionals $\theta\colon \mathfrak{D}_\theta \subset \mathfrak{C}(\mathscr{H}) \to \mathbb{C}$, where $\mathfrak{D}_\theta$ is a linear subset – the domain of θ. Let $\mathfrak{C}^*(\mathscr{H})$ be the set of such functionals obeying the following two conditions: (a) $\mathfrak{D}_\theta = \mathfrak{C}(\mathscr{H})$; (b) θ is norm-continuous. The latter property means that for every sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}(\mathscr{H})$, such that $\|A_n - A\| \to 0$, $n \to +\infty$, for some $A \in \mathfrak{C}(\mathscr{H})$, one has $\theta(A_n) \to \theta(A)$, as $n \to +\infty$.

Definition 1.1.9. A linear functional $\theta\colon \mathfrak{C}(\mathscr{H}) \to \mathbb{C}$ is said to be positive if $\theta(A) \geq 0$ whenever $A \in \mathfrak{C}^+(\mathscr{H})$.

Every positive linear functional is automatically norm-continuous, see Proposition 2.3.11 in [76]. However, if $\dim \mathscr{H} = +\infty$, a positive linear functional $\theta\colon \mathfrak{C}(\mathscr{H}) \to \mathbb{C}$ might not be strongly continuous. That is, for the convergence $\theta(A_n) \to \theta(A)$ it is not enough that A_n be strongly convergent to A, see Subsection 1.2.2 below. We recall that a sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}(\mathscr{H})$ strongly converges to $A \in \mathfrak{C}(\mathscr{H})$ if $\|A_n\psi - A\psi\|_{\mathscr{H}} \to 0$ for every $\psi \in \mathscr{H}$.

By standard arguments one proves that every positive $\theta \in \mathfrak{C}^*(\mathcal{H})$ obeys the Cauchy–Schwarz inequality, cf. (1.1.15),

$$|\theta(A^*B)|^2 \leq \theta(A^*A)\theta(B^*B), \quad A, B \in \mathfrak{C}(\mathcal{H}). \tag{1.1.35}$$

Let $\theta \colon \mathfrak{C}(\mathcal{H}) \to \mathbb{C}$ be a positive linear functional, such that both θ and $-\theta$ are positive. Then $\theta(A) = 0$ for all $A \in \mathfrak{C}(\mathcal{H})$, i.e., it is the zero functional. Indeed, as $\pm\theta(B) \geq 0$ for all $B \in \mathfrak{C}^+(\mathcal{H})$, θ takes zero values on $\mathfrak{C}^+(\mathcal{H})$, which by the representation (1.1.33) yields $\theta(A) = 0$.

For $\theta, \tilde{\theta} \in \mathfrak{C}^*(\mathcal{H})$, we say that θ *dominates* $\tilde{\theta}$ if $\theta - \tilde{\theta}$ is a positive linear functional. In view of the property of positive linear functional just shown, $\theta = \tilde{\theta}$ if these functionals dominate each other, and thereby domination is an order.

Proposition 1.1.10. *Let $\theta \in \mathfrak{C}^*(\mathcal{H})$ be positive and such that $\theta(I) = 0$. Then θ is the zero functional.*

Proof. For any $A \in \mathfrak{C}(\mathcal{H})$, by (1.1.35) one has

$$|\theta(A)|^2 = |\theta(IA)|^2 \leq \theta(I)\theta(A^*A),$$

which yields $\theta(A) = 0$. □

For any $\psi \in \mathcal{H}$, the map

$$\mathfrak{C}(\mathcal{H}) \ni A \mapsto \theta_\psi(A) = (\psi, A\psi)_\mathcal{H} \in \mathbb{C} \tag{1.1.36}$$

is a positive linear functional. It is nonzero if and only if $\psi \neq 0$. Such a functional is continuous in the strong topology. Indeed, by (1.1.15) one has

$$|\theta_\psi(A_n) - \theta_\psi(A)| = |(\psi, (A_n - A)\psi)_\mathcal{H}| \leq \|\psi\|_\mathcal{H} \, \|A_n\psi - A\psi\|_\mathcal{H}, \tag{1.1.37}$$

which yields $\theta_\psi(A_n) \to \theta_\psi(A)$ whenever $A_n \to A$ strongly.

Proposition 1.1.11. *Let $\theta \in \mathfrak{C}^*(\mathcal{H})$ be positive and dominated by the functional (1.1.36). Then θ is strongly continuous.*

Proof. Since θ is linear, it is enough to prove its continuity at zero, cf. (1.1.37). Let $A_n \to O$, strongly as $n \to +\infty$. Then $\theta(|A_n|^2) \to 0$ since

$$0 \leq \theta(|A_n|^2) \leq \theta_\psi(|A_n|^2) = \|A_n\psi\|^2_\mathcal{H}.$$

At the same time, by (1.1.35)

$$|\theta(A_n)|^2 = |\theta(IA_n)|^2 \leq \theta(I)\theta(|A_n|^2),$$

which yields the property stated. □

Definition 1.1.12. A positive linear functional $\omega \colon \mathfrak{C}(\mathcal{H}) \to \mathbb{C}$ is called *a state* on $\mathfrak{C}(\mathcal{H})$ if it is normalized, i.e., $\omega(I) = 1$.

If $\theta \in \mathfrak{C}^*(\mathcal{H})$ is positive and nonzero, then $\omega = \theta/\theta(I)$ is a state. Let ω_1, ω_2 be states on $\mathfrak{C}(\mathcal{H})$ and α be in $[0, 1]$. Then the *convex combination* $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$ is also a state on $\mathfrak{C}(\mathcal{H})$. Such a combination is called trivial if $\omega = \omega_1$ or $\omega = \omega_2$; otherwise, it is called nontrivial.

Definition 1.1.13. A state ω on $\mathfrak{C}(\mathcal{H})$ is called pure (or extreme) if $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$ for some states ω_1, ω_2 and $\alpha \in [0, 1]$, implies $\omega = \omega_1$ or $\omega = \omega_2$. That is, such ω cannot be a nontrivial convex combination of other states.

Proposition 1.1.14. *A state ω on $\mathfrak{C}(\mathcal{H})$ is pure if and only if the fact that a positive nonzero $\theta \in \mathfrak{C}^*(\mathcal{H})$ is dominated by ω implies that $\theta = \alpha\omega$ for some $\alpha > 0$.*

Proof. Let ω be a pure state, which dominates a given nonzero positive $\theta \in \mathfrak{C}^*(\mathcal{H})$. Then $\theta_2 = \omega - \theta$ is positive, ω dominates also θ_2, and $\theta(I) + \theta_2(I) = 1$. Then $\alpha \stackrel{\text{def}}{=} \theta(I) > 0$, see Proposition 1.1.10. If $\alpha = 1$, then $\omega = \theta$. If $\alpha < 1$, we set $\omega_1 = \theta/\alpha$ and $\omega_2 = \theta/(1-\alpha)$. Then $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$. By the purity of ω it follows that $\alpha\omega = \theta$ in each of the cases $\omega = \omega_i$, $i = 1, 2$. Now let ω possess the assumed property and $\omega = \alpha\omega_1 + (1-\alpha)\omega_2$ for some $\alpha \in (0, 1]$. Then ω dominates $\alpha\omega_1$; hence, $\omega = \omega_1$, which immediately yields also $\omega = \omega_2$ if $\alpha \in (0, 1)$. $\square$

Let us return to the functional (1.1.36). It is a state if $\|\psi\|_{\mathcal{H}} = 1$. In this case, we shall denote it by ω_ψ and call it a *vector state*. As is easily seen, for ψ of unit norm, the vector $\phi = \alpha\psi$ with $\alpha \in \mathbb{C}$, such that $|\alpha| = 1$, defines the same vector state, that is, $\omega_\phi = \omega_\psi$. If $\dim \mathcal{H} = 1$, then $\mathcal{H}$ is isomorphic to $\mathbb{C}$ and every linear operator $T : \mathcal{H} \to \mathcal{H}$ is a multiplication operator by a certain $\vartheta \in \mathbb{C}$. Such T is positive if $\vartheta \geq 0$. The only state here is the map $T \mapsto \omega_0(T) = \vartheta$. As a unique state, ω_0 is pure. It is a vector state.

Theorem 1.1.15. *Let $\mathcal{H}$ be an arbitrary separable complex Hilbert space and $\psi \in \mathcal{H}$ be any vector of unit norm. Then the vector state ω_ψ is a pure state on $\mathfrak{C}(\mathcal{H})$.*

Proof. Suppose that a positive $\theta \in \mathfrak{C}^*(\mathcal{H})$ is dominated by ω_ψ. To prove the theorem we have to show that $\theta = \alpha\omega_\psi$ for some $\alpha \geq 0$. Let P_ψ be the orthogonal projection on the one-dimensional space $\mathcal{H}_\psi$ spanned by ψ. Then the algebra

$$P_\psi\mathfrak{C}(\mathcal{H})P_\psi \stackrel{\text{def}}{=} \{P_\psi A P_\psi \mid A \in \mathcal{H}\}$$

is isomorphic to the algebra of all linear operators on the linear space $\mathbb{C}$ in the following sense. For $B \in P_\psi\mathfrak{C}(\mathcal{H})P_\psi$, there exists $\vartheta \in \mathbb{C}$, such that $B\psi = \vartheta\psi$. Then the isomorphism is $B \mapsto \vartheta$. Thus, the only state on $P_\psi\mathfrak{C}(\mathcal{H})P_\psi$ is the map $B \mapsto \vartheta$. Since every positive $B \in P_\psi\mathfrak{C}(\mathcal{H})P_\psi$ belongs to $\mathfrak{C}^+(\mathcal{H})$, the restriction of θ to $P_\psi\mathfrak{C}(\mathcal{H})P_\psi$ is dominated by the corresponding restriction of ω_ψ. Therefore, one finds $\alpha \geq 0$, such that $\theta(B) = \alpha\omega_\psi(B)$ for all $B \in P_\psi\mathfrak{C}(\mathcal{H})P_\psi$. Let us show now that for any $A \in \mathfrak{C}(\mathcal{H})$,

$$\theta(A) = \theta(AP_\psi) = \theta(P_\psi A) = \theta(P_\psi A P_\psi). \tag{1.1.38}$$

Set $Q_\psi = I - P_\psi$. Then $Q_\psi \in \mathfrak{C}^+(\mathcal{H})$ as Q_ψ is self-adjoint and $Q_\psi^2 = Q_\psi$. This yields $0 \le \theta(Q_\psi) \le \omega_\psi(Q_\psi) = 0$. Therefore, by (1.1.35)

$$|\theta(AQ_\psi)|^2 \le \theta(A^*A)\theta(Q_\psi) = 0,$$

which in view of $\theta(A) = \theta(AP_\psi + AQ_\psi) = \theta(AP_\psi) + \theta(AQ_\psi)$, yields the first equality in (1.1.38). The second equality can be obtained in the same way. The third equality is just a consequence of the first two. Clearly, the equalities (1.1.38) are valid also for ω_ψ. Thus, for any $A \in \mathfrak{C}(\mathcal{H})$, one has

$$\theta(A) = \theta(P_\psi A P_\psi) = \alpha\omega_\psi(P_\psi A P_\psi) = \alpha\omega_\psi(A),$$

which completes the proof. □

Let us now get back to the probability of the event defined in (1.1.5), where $\mathcal{H} = L^2(\mathbb{R}^\nu)$. By definition, the indicator function of a Borel set $B \subset \mathbb{R}^\nu$ is

$$\mathbb{I}_B(x) = \begin{cases} 1 & \text{if } \ x \in B; \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.39}$$

It determines a linear operator $P_B\colon L^2(\mathbb{R}^\nu) \to L^2(\mathbb{R}^\nu)$ by

$$(P_B\psi)(x) = \mathbb{I}_B(x)\psi(x), \quad x \in \mathbb{R}^\nu. \tag{1.1.40}$$

If B' is such that the set $(B \setminus B') \cup (B' \setminus B)$ has zero Lebesgue measure, the operator $P_{B'}$ coincides with P_B since $\mathbb{I}_B(x)\psi(x)$ and $\mathbb{I}_{B'}(x)\psi(x)$ coincide as elements of $L^2(\mathbb{R}^\nu)$. In the Hilbert spaces $L^2(\mathbb{R}^\nu)$, $\nu \in \mathbb{N}$, operators of this type are called *multiplication* operators. Then

$$(\psi, P_B\psi)_{\mathcal{H}} = \|P_B\psi\|^2_{L^2(\mathbb{R}^\nu)} = \int_B |\psi(x)|^2 \mathrm{d}x;$$

thus, P_B is bounded and positive. Its range is a subspace of $L^2(\mathbb{R}^\nu)$ and $P_B^2 = P_B$; hence, it is an orthogonal projection. Let $\psi \in L^2(\mathbb{R}^\nu)$ be of unit norm. Then the left-hand side of (1.1.5) is the value of the corresponding vector state at P_B, that is,

$$\mathsf{P}_\psi(B) = \omega_\psi(P_B) = (\psi, P_B\psi)_{L^2(\mathbb{R}^\nu)},$$

and $\mathcal{H}_B = \mathrm{Ran}(P_B)$ is the subspace of $L^2(\mathbb{R}^\nu)$ consisting of states *localized* in B. Two such subspaces $\mathcal{H}_B$ and $\mathcal{H}_{B'}$ are mutually orthogonal if the set $B \cap B'$ has zero Lebesgue measure.

Unbounded operators

Clearly, if $\dim\mathcal{H} > 1$, the set of all linear operators $T\colon \mathcal{H} \to \mathcal{H}$ with all possible linear domains contained in $\mathcal{H}$ is bigger than $\mathfrak{C}(\mathcal{H})$. The algebraic operations (1.1.25) can be extended to these operators as well. However, here one has to be careful with the domains of the resulting operators. For two such operators T and Q, their linear combination can act only on those $\psi \in \mathcal{H}$, which belong to the domains of both operators. Thus, we define

$$\mathrm{Dom}(\alpha T + \beta Q) = \mathrm{Dom}(T) \cap \mathrm{Dom}(Q). \tag{1.1.41}$$

It may happen that this intersection consists of the zero vector alone. In such a case, $\alpha T + \beta Q$ is a trivial operator, which sends the zero vector into itself. Similarly, the domain of the product TQ consists of those $\psi \in \mathrm{Dom}(Q)$ for which $Q\psi \in \mathrm{Dom}(T)$. Therefore, in this case,

$$\mathrm{Dom}(TQ) = Q^{-1}\mathrm{Dom}(T), \tag{1.1.42}$$

which allows, however, for $\mathrm{Dom}(TQ) = \{0\}$.

A sequence $\{\psi_n\}_{n\in\mathbb{N}} \subset \mathrm{Dom}(T)$ is called T-*convergent* if both $\{\psi_n\}_{n\in\mathbb{N}}$ and $\{T\psi_n\}_{n\in\mathbb{N}}$ are convergent sequences. Note that if T is continuous on its domain and $\{\psi_n\}_{n\in\mathbb{N}}$ is a convergent sequence, then $\{T\psi_n\}_{n\in\mathbb{N}}$ is a convergent sequence automatically.

Definition 1.1.16. An operator T is called closed if for every T-convergent sequence $\{\psi_n\}_{n\in\mathbb{N}}$, its limit ψ is in $\mathrm{Dom}(T)$ and

$$T\psi = \lim_{n\to+\infty} T\psi_n.$$

Obviously, if T is invertible and closed, its inverse is closed as well. If T is continuous and closed, then $\mathrm{Dom}(T)$ is also closed and hence is a subspace of $\mathcal{H}$. Endowed with the Hilbert space structure inherited from $\mathcal{H}$, it becomes a Hilbert space, say $\mathcal{H}_1$, and T can be redefined as an element of $\mathfrak{L}(\mathcal{H}_1, \mathcal{H})$. The definition of the numerical range can be generalized as follows, cf. (1.1.21),

$$\mathrm{Num}(T) = \{(\psi, T\psi)_{\mathcal{H}} \mid \psi \in \mathrm{Dom}(T),\ \|\psi\|_{\mathcal{H}} = 1\}. \tag{1.1.43}$$

Definition 1.1.17. An unbounded operator T is called positive if

$$\mathrm{Num}(T) \subset [0, +\infty).$$

If for two linear operators T and Q, $\mathrm{Dom}(T) \subset \mathrm{Dom}(Q)$, and $\forall \psi \in \mathrm{Dom}(T)$: $T\psi = Q\psi$, then Q is said to be an *extension* of T, and T – the *restriction* of Q to $\mathrm{Dom}(T)$. In this case, we write $Q \supset T$.

An operator T is called *closable* if it has a closed extension. T is closable if and only if for every T-convergent sequence $\{\psi_n\}_{n\in\mathbb{N}}$, such that $\psi_n \to 0$ and $T\psi_n \to \phi$, it follows that $\phi = 0$. If T is closable, the set of its closed extensions has a (unique) minimal element $\tilde{T}$. This means that $\mathrm{Dom}(\tilde{T})$ is contained in the domains of all other closed extensions. $\tilde{T}$ is called the *closure* of T. For a closed operator Q, a subset of its domain is called a *core* of Q, if Q is the closure of its restriction to this subset.

It turns out that T belongs to $\mathfrak{L}(\mathcal{H})$ already when it is closed and $\mathrm{Dom}(T) = \mathcal{H}$, without assuming continuity. This fact is known as the *closed graph theorem*.

Therefore, the set of all linear operators $T\colon \mathcal{H} \to \mathcal{H}$ falls into the following three groups: (a) $\mathfrak{C}(\mathcal{H})$; (b) continuous operators defined on non-dense linear subsets of $\mathcal{H}$; (c) nowhere continuous operators. Group (c) contains *unbounded operators*; it is non-void if and only if $\dim \mathcal{H} = +\infty$.

By the closed graph theorem mentioned above, the domain of a closed unbounded linear operator is a proper subset of $\mathcal{H}$. Among such operators, one can distinguish those defined on dense subsets of the corresponding Hilbert spaces. Two operators T and Q, which satisfy the condition

$$\forall \phi \in \mathrm{Dom}(Q),\ \forall \psi \in \mathrm{Dom}(T)\colon \quad (Q\phi, \psi)_{\mathcal{H}} = (\phi, T\psi)_{\mathcal{H}}, \tag{1.1.44}$$

are called formally adjoint to each other. For T being a densely defined linear operator, let us show that there exists a unique linear operator Q, with maximal domain, such that (1.1.44) holds. Set

$$\mathcal{D} = \{\psi \in \mathcal{H} \mid \exists C_\psi > 0\ \ \forall \phi \in \mathrm{Dom}(T)\colon\ |(\psi, T\phi)_{\mathcal{H}}| \le C_\psi \|\phi\|_{\mathcal{H}}\}. \tag{1.1.45}$$

Then, for each $\psi \in \mathcal{D}$, the map $\mathrm{Dom}(T) \ni \phi \mapsto (\psi, T\phi)_{\mathcal{H}}$ defines a bounded linear functional θ on a dense subset of $\mathcal{H}$, which clearly can be extended to an element of $\mathfrak{C}(\mathcal{H}, \mathbb{C})$. By the Riesz lemma, there exists $\varphi \in \mathcal{H}$, such that

$$\theta(\phi) = (\varphi, \phi)_{\mathcal{H}} = (\psi, T\phi)_{\mathcal{H}},$$

which holds for all $\phi \in \mathrm{Dom}(T)$. The composition

$$\mathcal{D} \ni \psi \mapsto \theta \mapsto Q(\psi) \stackrel{\mathrm{def}}{=} \varphi \in \mathcal{H}$$

defines a linear map $Q\colon \mathcal{D} \to \mathcal{H}$. In this case, we use the notation $Q = T^*$, which makes sense for densely defined operators T only.

Definition 1.1.18. For a densely defined linear operator T, the operator T^* is called the adjoint of T.

T^* is always closed, even if T is not closed or closable. However, it may happen that $\mathrm{Dom}(T^*) = \{0\}$. If T is closable, T^* is densely defined and hence there exists $T^{**} = (T^*)^*$. In this case, from (1.1.44) one readily concludes that $T \subset T^{**}$, that is, T^{**} is a closed extension of T.

A densely defined linear operator T is called *symmetric* if T^* is an extension of T, i.e.,

$$T^* \supset T. \tag{1.1.46}$$

Thus, a symmetric operator is closable and T^* is densely defined. If

$$T = T^*, \tag{1.1.47}$$

T is called *self-adjoint*. From (1.1.46) we readily derive that a densely defined operator T is symmetric if and only if

$$\forall \psi, \phi \in \mathrm{Dom}(T)\colon \quad (T\phi, \psi)_{\mathcal{H}} = (\phi, T\psi)_{\mathcal{H}}. \tag{1.1.48}$$

This holds if and only if $\mathrm{Num}(T) \subset \mathbb{R}$. One can show that $T^{**} = \widetilde{T}$ if T is closable. Hence, if T is symmetric, then $T^* \supset T^{**}$, and, therefore, $T^{***} \supset T^{**}$ (see the proof of Proposition 1.1.4). Thus, for a symmetric linear operator T, its closure T^{**} is also symmetric. A symmetric operator T is called *essentially self-adjoint* if its closure $\widetilde{T}$ is self-adjoint. In this case, it is easy to check that a symmetric linear operator T is essentially self-adjoint if and only if T^* is symmetric. Furthermore, for an essentially self-adjoint T, we have $\widetilde{T} = T^*$ and $\mathrm{Num}(T^*)$ is the closure of $\mathrm{Num}(T)$.

Usually, one defines a linear operator by its action on certain vectors and then tries to extend this action to a maximal set, which could be a core of some self-adjoint operator. As an example let us consider the components of the displacement and momentum operators $q_\ell^{(j)}$, $p_\ell^{(j)}$, $j = 1, \ldots, \nu$, which obey (1.1.6). For a suitable function $\psi \in L^2(\mathbb{R}^\nu)$, their action is canonically defined as

$$(q_\ell^{(j)}\psi)(x) = x_j\psi(x), \quad (p_\ell^{(j)}\psi)(x) = -\mathrm{i}\frac{\partial}{\partial x_j}\psi(x), \quad x \in \mathbb{R}^\nu, \tag{1.1.49}$$

by which $q_\ell^{(j)}$ is a multiplication operator. Clearly, such 'suitable' functions should be differentiable and square-integrable, even after being multiplied by x_j. Thus, it is impossible to extend these operators to the whole space $L^2(\mathbb{R}^\nu)$. For a function $\varphi\colon \mathbb{R}^\nu \to \mathbb{C}$ and $n \in \mathbb{N}_0^\nu$, that is $n = (n_1, \ldots, n_\nu)$, $n_j \in \mathbb{N}_0$, $j = 1, \ldots, \nu$, we set $|n| = n_1 + \cdots + n_\nu$ and

$$x^n = x_1^{n_1} \ldots x_\nu^{n_\nu}, \quad \varphi^{(n)}(x) = \frac{\partial^{|n|}\varphi}{\partial x_1^{n_1} \ldots \partial x_\nu^{n_\nu}}(x). \tag{1.1.50}$$

As usual, see page 133 in [255], let $\mathcal{S}(\mathbb{R}^\nu)$ be the complex linear space of the functions $\varphi\colon \mathbb{R}^\nu \to \mathbb{C}$ such that

$$\forall n, m \in \mathbb{N}_0^\nu: \quad \|\varphi\|_{n,m} \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}^\nu} \left\{\left|(1 + x^n)\varphi^{(m)}(x)\right|\right\} < \infty. \tag{1.1.51}$$

Clearly, $\mathcal{S}(\mathbb{R}^\nu) \subset L^2(\mathbb{R}^\nu)$ and the action in (1.1.49) of both operators is well defined on $\psi \in \mathcal{S}(\mathbb{R}^\nu)$, with values again in $\mathcal{S}(\mathbb{R}^\nu)$. Furthermore, $\mathcal{S}(\mathbb{R}^\nu)$ is dense in $L^2(\mathbb{R}^\nu)$ and the operators $q_\ell^{(j)}$, $p_\ell^{(j)}$, $j = 1, \ldots, \nu$, obey (1.1.6) on $\mathcal{S}(\mathbb{R}^\nu)$. Thus, for the time being, one can set

$$\mathrm{Dom}(q_\ell^{(j)}) = \mathrm{Dom}(p_\ell^{(j)}) = \mathcal{S}(\mathbb{R}^\nu).$$

According to (1.1.48), with this domain both operators are symmetric. For $q_\ell^{(j)}$, it is obvious; for $p_\ell^{(j)}$, this can be shown by integrating by parts. In the next subsection, we construct self-adjoint extensions of such $q_\ell^{(j)}$ and $p_\ell^{(j)}$. It is worth noting that the linear space $\mathcal{S}(\mathbb{R}^\nu)$ equipped with the topology defined by the family $\{\|\cdot\|_{n,m} \mid n, m \in \mathbb{N}_0^\nu\}$ becomes a complete locally convex space. This is the *Schwartz space of test functions*.

Spectra

If T is a closed linear operator, $T - \zeta I$ is also closed for any $\zeta \in \mathbb{C}$. For such an operator, the subset

$$\mathsf{R}(T) = \{\zeta \in \mathbb{C} \mid (T - \zeta I)^{-1} \in \mathfrak{C}(\mathcal{H})\} \tag{1.1.52}$$

is called the *resolvent set* of T. Note that by this definition $\mathrm{Ran}(T - \zeta I) = \mathcal{H}$ if $\zeta \in \mathsf{R}(T)$. The resolvent set is always open, and may be void or equal to $\mathbb{C}$. Its complement $\mathsf{S}(T) = \mathbb{C} \setminus \mathsf{R}(T)$ is called the *spectrum* of T. If $T \in \mathfrak{C}(\mathcal{H})$, then $\mathsf{S}(T)$ is a non-void bounded subset of $\mathbb{C}$. For $\zeta \in \mathsf{R}(T)$, the operator

$$R_\zeta(T) = (T - \zeta I)^{-1} \tag{1.1.53}$$

is called the *resolvent* of T. One can easily show that for any $\zeta, \zeta' \in \mathsf{R}(T)$, the resolvents $R_\zeta(T)$ and $R_{\zeta'}(T)$ commute and obey the *resolvent identity*

$$R_\zeta(T) - R_{\zeta'}(T) = (\zeta - \zeta')R_\zeta(T)R_{\zeta'}(T). \tag{1.1.54}$$

For $\mathsf{C} \subset \mathbb{C}$, its complex conjugate is the set $\overline{\mathsf{C}} = \{\zeta \mid \bar{\zeta} \in \mathsf{C}\}$. Clearly, $\mathsf{R}(T^*) = \overline{\mathsf{R}(T)}$ and hence $\mathsf{S}(T^*) = \overline{\mathsf{S}(T)}$. Thus,

$$\begin{array}{ll} \text{if } T \in \mathfrak{C}(\mathcal{H}) \text{ and } T^* = T, & \mathsf{S}(T) \subset [a, b], \\ \text{if } T \in \mathfrak{C}^+(\mathcal{H}), & \mathsf{S}(T) \subset [0, c], \end{array} \tag{1.1.55}$$

for certain real $a, b, a < b$, and positive c.

For closed unbounded operators, we have the following fact, see Theorem X.1, page 136 in [256], or Theorem 3.16, page 271 in [172].

Proposition 1.1.19. *Let T be closed and symmetric. Then the spectrum of T is one of the following subsets of the complex plane:* (a) *the closed upper half-plane;* (b) *the closed lower half-plane;* (c) *the entire plane $\mathbb{C}$;* (d) *a subset of the real line. T is self-adjoint if and only if $\mathsf{S}(T) \subset \mathbb{R}$. In this case, the resolvent $R_\zeta(T)$ exists for all $\zeta \in \mathbb{C} \setminus \mathbb{R}$, and obeys the estimates*

$$\|R_\zeta(T)\| \leq |\Im(\zeta)|^{-1}, \quad \|(T - \Re(\zeta)I)R_\zeta(T)\| \leq 1. \tag{1.1.56}$$

Corollary 1.1.20. *If T is closed and symmetric and $\mathsf{R}(T) \cap \mathbb{R} \neq \emptyset$, then T is self-adjoint.*

Proof. Since $\mathsf{R}(T)$ is open and contains a point $x \in \mathbb{R}$, its intersections with both upper and lower half-planes is non-void. Hence, by Proposition 1.1.19 we have $\mathsf{S}(T) \subset \mathbb{R}$ and thereby T is self-adjoint. □

Any non-zero vector ψ, such that $T\psi = \lambda\psi$ for a certain complex λ, is called an *eigenvector* (or an *eigenfunction*) of T, whereas λ is called the *eigenvalue* which corresponds to the eigenvector ψ. Clearly, $\lambda \in \mathsf{S}(T)$; it is called an *isolated eigenvalue* if there exists $\varepsilon > 0$ such that the disc $\{\zeta \in \mathbb{C} \mid |\zeta - \lambda| < \varepsilon\}$ contains no other points of $\mathsf{S}(T)$. Let ψ_1 and ψ_2 be eigenfunctions, corresponding to the same eigenvalue λ. Their linear combinations are also eigenfunctions, corresponding to this λ. By definition, an eigenvalue λ has *finite multiplicity* if the set of all corresponding eigenfunctions $\mathcal{K}_\lambda$ is a finite-dimensional subspace of $\mathcal{H}$. Then $\dim \mathcal{K}_\lambda$ is called the *multiplicity* of λ. Such an eigenvalue is called *simple* if $\dim \mathcal{K}_\lambda = 1$.

Spectra of compact operators have a special structure. Namely, by the Riesz–Schauder theorem[2], for each such T, $\mathsf{S}(T)$ is a countable set with no accumulation points other than zero. Each nonzero $\lambda \in \mathsf{S}(T)$ is an eigenvalue of finite multiplicity and $\bar{\lambda}$ is an eigenvalue of T^* with the same multiplicity. The following known statement gives further information on the spectra of compact operators

Proposition 1.1.21 (Hilbert–Schmidt Theorem). *Suppose that $\mathcal{H}$ is infinite-dimensional and $T \in \mathfrak{C}(\mathcal{H})$ is self-adjoint and compact. Then there exists an orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ such that $T\psi_n = \lambda_n\psi_n$ and $\lambda_n \to 0$ as $n \to +\infty$.*

For positive compact operators, a corollary of this theorem yields the following property.

Proposition 1.1.22. *Let $T \neq O$ be positive and compact. Then it can be written as*

$$T = \sum_{k\in\mathsf{K}} \lambda_k P_k, \quad \lambda_k > 0, \text{ for all } k \in \mathsf{K}, \tag{1.1.57}$$

where K may be finite or countable, λ_k and P_k, $k \in \mathsf{K}$, are eigenvalues and orthogonal projections onto finite-dimensional subspaces, respectively.

Definition 1.1.23. A positive compact operator T is said to be a trace-class operator if the sequence of its eigenvalues is summable. Then its trace is set to be

$$\operatorname{trace}(T) = \sum_{k\in\mathsf{K}} \lambda_k. \tag{1.1.58}$$

$Q \in \mathfrak{C}(\mathcal{H})$ is called a Hilbert–Schmidt operator if Q^*Q is a trace-class operator.

Note that for $Q \in \mathfrak{C}(\mathcal{H})$, the operator Q^*Q is positive since $(\psi, Q^*Q\psi)_{\mathcal{H}} = \|Q\psi\|^2_{\mathcal{H}} \geq 0$. Let $\{\psi_n\}_{n\in\mathbb{N}}$ be a bounded sequence. Then for any n, m, by the Cauchy–Schwarz inequality (1.1.15) one gets

$$\|Q\psi_n - Q\psi_m\|^2_{\mathcal{H}} \leq \|Q^*Q(\psi_n - \psi_m)\|_{\mathcal{H}} \cdot \|\psi_n - \psi_m\|_{\mathcal{H}}.$$

Hence, if Q^*Q is compact, then so is Q. Therefore, every Hilbert–Schmidt operator is compact. One observes that (1.1.58) can be rewritten as

$$\operatorname{trace}(T) = \sum_{n\in\mathbb{N}} (\psi_n, T\psi_n)_{\mathcal{H}}, \tag{1.1.59}$$

[2]The details on this theorem, as well as on the Hilbert–Schmidt theorem given below, can be found on page 203 of [255].

where $\{\psi_n\}_{n\in\mathbb{N}}$ is any orthonormal basis of $\mathcal{H}$. Vice versa, if for a $T \in \mathfrak{C}^+(\mathcal{H})$ and a given orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$, the series in (1.1.59) converges, then T is trace-class and possesses the properties (1.1.57) and (1.1.58). As T is positive, the summands in (1.1.59) are positive and the convergence of the series and its absolute convergence are equivalent.

Definition 1.1.24. $T \in \mathfrak{C}(\mathcal{H})$ is called a trace-class operator if the series in (1.1.59) absolutely converges for every orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$.

Note that here the condition of the absolute convergence *for every basis* is crucial. One can show that for each such an operator, the sum in (1.1.59) is independent of the choice of the basis; hence, trace(T) is well defined. Directly from (1.1.59) we deduce the following important fact, cf. Proposition 1.1.7.

Proposition 1.1.25. *For a trace-class operator T and any $Q \in \mathfrak{C}(\mathcal{H})$, the operators TQ and QT are trace-class and* trace(QT) = trace(TQ). *If T is positive, then*

$$|\mathrm{trace}(QT)| \leq \|Q\| \cdot \mathrm{trace}(T). \tag{1.1.60}$$

Another class of operators with relatively simple spectra is the class of closed operators with compact resolvent. Their spectra are described by the following statement, see Theorem 6.29 on page 187 of [172].

Proposition 1.1.26. *Let T be a closed operator, such that $R_{\zeta_0}(T)$ is compact at least for some $\zeta_0 \in \mathsf{R}(T)$. Then $R_\zeta(T)$ is compact for all $\zeta \in \mathsf{R}(T)$ and its spectrum $\mathsf{S}(T)$ consists entirely of isolated eigenvalues of finite multiplicity.*

The spectrum of an operator is called *discrete* if it is as in the above statement. Such a spectrum is called *non-degenerate* if each eigenvalue is simple.

We recall that $\mathcal{B}(\mathbb{R})$ denotes the Borel σ-algebra of subsets of $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is called a *Borel function* if for every $B \in \mathcal{B}(\mathbb{R})$, the f-pre-image $f^{-1}(B)$ is in $\mathcal{B}(\mathbb{R})$. Let $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{B}(\mathbb{R})$ be such that $B_n \cap B_m = \emptyset$ for $n \neq m$. For this sequence, we define

$$C = \bigcup_{n=1}^{\infty} B_n, \quad C_m = \bigcup_{n=1}^{m} B_n, \quad m \in \mathbb{N}, \tag{1.1.61}$$

and say that $\{B_n\}_{n\in\mathbb{N}}$ is a partition of C. Note that $C \in \mathcal{B}(\mathbb{R})$. Let $\mathcal{P}$ be the family of all projections P_B, $B \in \mathcal{B}(\mathbb{R})$, defined by (1.1.40).

Proposition 1.1.27. *The family $\mathcal{P}$ has the following properties:*

(a) *each element of $\mathcal{P}$ is an orthogonal projection;*

(b) *$P_{\mathbb{R}} = I$ and $P_\emptyset = O$;*

(c) *$P_{B_1} P_{B_2} = P_{B_1 \cap B_2}$ for all $B_1, B_2 \in \mathcal{B}(\mathbb{R})$;*

(d) *if* $\{B_n\}_{n\in\mathbb{N}}$ *is a partition of* $C \in \mathcal{B}(\mathbb{R})$ *and* C_m, $m \in \mathbb{N}$, *are as in* (1.1.61), *then* $P_{C_m} = \sum_{n=1}^{m} P_{B_n}$ *for all* $m \in \mathbb{N}$, *and the sequence* $\{P_{C_m}\}_{m\in\mathbb{N}}$ *strongly converges to* P_C.

Proof. By (1.1.39) it follows that $\mathbb{I}_{\mathbb{R}}(x) = 1$ and $\mathbb{I}_{\emptyset}(x) = 0$ for all $x \in \mathbb{R}$, which by (1.1.40) yields (a) and (b). Since $\mathbb{I}_{B_1}\mathbb{I}_{B_2} = \mathbb{I}_{B_1\cap B_2}$, also (c) follows. As each $x \in C_m$ belongs exactly to one B_n, $n = 1, \ldots, m$, one has $\mathbb{I}_{C_m}(x) = \sum_{n=1}^{m} \mathbb{I}_{B_n}(x)$. Furthermore, each $x \in C$ belongs to a certain C_m; hence, $\mathbb{I}_{C_m}(x) \to \mathbb{I}_C(x)$ for all $x \in \mathbb{R}$ (i.e., point-wise), which readily implies

$$\int_{\mathbb{R}} [\mathbb{I}_C(x) - \mathbb{I}_{C_m}(x)]|\psi(x)|^2 \mathrm{d}x \to 0,$$

that is, $\|P_{C_m}\psi - P_C\psi\|_{L^2(\mathbb{R})} \to 0$ for all $\psi \in \mathcal{H}$. □

In Proposition 1.1.27, the family $\mathcal{P}$ was constructed in a concrete Hilbert space, $L^2(\mathbb{R})$.

Definition 1.1.28. For a Hilbert space $\mathcal{H}$, let $\mathcal{P} = \{P_B \mid B \in \mathcal{B}(\mathbb{R})\} \subset \mathfrak{L}(\mathcal{H})$ be a family that possesses the properties described by Proposition 1.1.27. Then $\mathcal{P}$ is called a projection-valued measure.

Let $\mathcal{P} = \{P_B \mid B \in \mathcal{B}(\mathbb{R})\}$ be a projection-valued measure. Given $\psi \in \mathcal{H}$, the map $\mathcal{B}(\mathbb{R}) \ni B \mapsto (\psi, P_B\psi)_{\mathcal{H}}$ determines a probability measure on $\mathbb{R}$, which we denote by μ_ψ. For a bounded Borel function $f : \mathbb{R} \to \mathbb{R}$, a self-adjoint operator $T_f \in \mathfrak{L}(\mathcal{H})$ is defined as follows, see Proposition 1.1.4 and (1.1.22),

$$(\psi, T_f\psi)_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda)\mu_\psi(\mathrm{d}\lambda), \quad \psi \in \mathcal{H}. \tag{1.1.62}$$

Then for $\psi \in \mathcal{H}$, μ_ψ is called a *spectral measure* for T_f. Now let $f : \mathbb{R} \to \mathbb{R}$ be a Borel function, not necessarily bounded. Set

$$\mathcal{D}_f = \left\{\psi \in \mathcal{H} \mid \textstyle\int_{\mathbb{R}} [f(\lambda)]^2 \mu_\psi(\mathrm{d}\lambda) < \infty\right\}. \tag{1.1.63}$$

It turns out that for any such f, $\mathcal{D}_f$ is dense in $\mathcal{H}$ and the family of integrals as in (1.1.62) but with $\psi \in \mathcal{D}_f$ determines a self-adjoint operator with domain $\mathcal{D}_f$. Let T be the operator corresponding to $f(\lambda) = \lambda$. Then

$$\begin{aligned} (\psi, T\psi)_{\mathcal{H}} &= \int_{\mathbb{R}} \lambda\mu_\psi(\mathrm{d}\lambda), \quad \psi \in \mathrm{Dom}(T), \\ \mathrm{Dom}(T) &= \left\{\phi \in \mathcal{H} \mid \textstyle\int_{\mathbb{R}} \lambda^2 \mu_\phi(\mathrm{d}\lambda) < \infty\right\}. \end{aligned} \tag{1.1.64}$$

This representation suggests a way of defining functions of self-adjoint operators. If T is as in (1.1.64) and $f : \mathbb{R} \to \mathbb{R}$ is a Borel function, then the representation

$$(\psi, f(T)\psi)_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda)\mu_\psi(\mathrm{d}\lambda) \tag{1.1.65}$$

defines a self-adjoint operator with domain (1.1.63). The following statement, the proof of which can be found in [255], see Theorem VIII.6 on page 263, establishes a correspondence between projection-valued measures and self-adjoint operators.

Proposition 1.1.29 (Spectral Theorem). *Let $\mathcal{H}$ be a separable complex Hilbert space. There exists a one-to-one correspondence between projection-valued measures and self-adjoint operators in $\mathcal{H}$ established by* (1.1.64)*. If $f : \mathbb{R} \to \mathbb{R}$ is a Borel function, then the operator $f(T)$ defined by* (1.1.65) *with domain* (1.1.63) *is self-adjoint. If f is bounded, then $f(T) \in \mathfrak{L}(\mathcal{H})$.*

By (1.1.22) the spectral measures μ_ψ, $\psi \in \mathrm{Dom}(T)$, see (1.1.64), determine a complex-valued Borel measure $\mu_{\phi,\psi}$ such that

$$(\phi, T\psi)_{\mathcal{H}} = \int_{\mathbb{R}} \lambda \mu_{\phi,\psi}(\mathrm{d}\lambda), \quad \phi, \psi \in \mathrm{Dom}(T). \tag{1.1.66}$$

And similarly as in (1.1.65),

$$(\phi, f(T)\psi)_{\mathcal{H}} = \int_{\mathbb{R}} f(\lambda) \mu_{\phi,\psi}(\mathrm{d}\lambda), \quad \phi, \psi \in \mathrm{Dom}(T), \tag{1.1.67}$$

which holds for any bounded Borel function $f : \mathbb{R} \to \mathbb{R}$. An important application of the spectral theorem is the following statement, see Theorem VIII.7 on page 265 of [255].

Proposition 1.1.30. *For a self-adjoint operator T, let* $\exp(\mathrm{i}tT)$*, $t \in \mathbb{R}$, be defined by* (1.1.65)*. Then*

(a) *for any $t \in \mathbb{R}$, $U(t) = \exp(\mathrm{i}tT)$ is unitary;*

(b) *the family $\mathfrak{U}_T = \{U(t) \mid t \in \mathbb{R}\}$ is a one-parameter group under composition such that $U(t+s) = U(t)U(s)$;*

(c) *for any $s \in \mathbb{R}$ and $\psi \in \mathcal{H}$, $U(t)\psi \to U(s)\psi$ in $\mathcal{H}$ as $t \to s$;*

(d) *$(U(t)\psi - \psi)/t \to \mathrm{i}T\psi$ in $\mathcal{H}$ as $t \to 0$ if and only if $\psi \in \mathrm{Dom}(T)$.*

Claim (c) says that the group $\mathfrak{U}_T$ is strongly continuous. At the same time, $\mathfrak{U}_T$ is a subgroup of the group of all unitary operators $\mathfrak{L}_U(\mathcal{H})$. T is called the *generator* of $\mathfrak{U}_T$. It turns out that every strongly continuous one-parameter unitary group is generated by a self-adjoint operator, see Theorem VIII.8 on page 266 of [255].

Proposition 1.1.31 (Stone Theorem). *For every strongly continuous one-parameter unitary group $\mathfrak{U} = \{U(t) \mid t \in \mathbb{R}\}$ on a Hilbert space $\mathcal{H}$, there exists a self-adjoint operator T on $\mathcal{H}$ such that $U(t) = \exp(\mathrm{i}tT)$ for all $t \in \mathbb{R}$.*

1.1.3 Quantum Oscillators

Harmonic oscillator

The Hamiltonian of a one-dimensional quantum harmonic oscillator is, cf. (1.1.3),

$$H^{\mathrm{har}} = \frac{1}{2m} p^2 + \frac{a}{2} q^2, \quad a > 0. \tag{1.1.68}$$

The operators q and p act on the elements of $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ according to (1.1.49). Thus, for the time being, we assume that $\mathrm{Dom}(H^{\mathrm{har}}) = \mathcal{S}(\mathbb{R})$, cf. (1.1.41). In the description of the harmonic oscillator, we use the *Hermite functions* $\varphi_n : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}_0$, which can be defined as

$$\varphi_n(x) = (2^n n!)^{-1/2} (-1)^n \pi^{-1/4} e^{x^2/2} D^n e^{-x^2}, \quad D = \frac{d}{dx}.$$

It is easy to check that each φ_n solves the equation

$$\left(-D^2 + x^2\right) \varphi_n(x) = (2n+1)\varphi_n(x)$$

and can also be written in the form

$$\varphi_n(x) = h_n(x) e^{-x^2/2}, \tag{1.1.69}$$

where h_n is a *Hermite polynomial*. In order to relate the functions φ_n to the operator (1.1.68) we introduce

$$\psi_n(x) = \kappa^{1/2} \varphi_n(\kappa x), \quad \kappa = (ma)^{1/4}, \tag{1.1.70}$$

where m and a are as in (1.1.68). In view of (1.1.69), each ψ_n belongs to $\mathcal{S}(\mathbb{R})$, see (1.1.51), and solves the equation

$$\left(-\frac{1}{2m} D^2 + \frac{a}{2} x^2\right) \psi_n(x) = \delta \left(n + \frac{1}{2}\right) \psi_n(x), \tag{1.1.71}$$

where

$$\delta \stackrel{\mathrm{def}}{=} \sqrt{a/m}. \tag{1.1.72}$$

Therefore, we have

$$H^{\mathrm{har}} \psi_n = E_n^{\mathrm{har}} \psi_n, \quad E_n^{\mathrm{har}} = (n + 1/2)\delta, \quad n \in \mathbb{N}_0. \tag{1.1.73}$$

The family $\{\psi_n\}_{n \in \mathbb{N}_0}$ is an orthonormal basis for $L^2(\mathbb{R})$, see Lemma 3 on page 142 in [255]. In the description of the harmonic oscillator, it is convenient to use *creation* $A^\dagger$ *and annihilation* A operators defined as

$$A^\dagger = \frac{1}{\sqrt{2}} \left(\kappa q - \frac{\mathrm{i}}{\kappa} p\right), \quad A = \frac{1}{\sqrt{2}} \left(\kappa q + \frac{\mathrm{i}}{\kappa} p\right). \tag{1.1.74}$$

By (1.1.49) and (1.1.70) one gets

$$
\begin{aligned}
A^\dagger \psi_n &= \sqrt{n+1}\,\psi_{n+1}, && n \in \mathbb{N}_0, \\
A\psi_n &= \sqrt{n}\,\psi_{n-1}, && n \in \mathbb{N}, \\
A\psi_0 &= 0, && \\
N\psi_n &\stackrel{\text{def}}{=} A^\dagger A \psi_n = n\psi_n, && n \in \mathbb{N}_0.
\end{aligned}
\tag{1.1.75}
$$

As linear combinations of p and q, the operators A and $A^\dagger$ map $\mathcal{S}(\mathbb{R})$ into itself. For $\psi \in \mathcal{S}(\mathbb{R})$, by (1.1.75) one obtains

$$
\begin{aligned}
\|A\psi\|^2_{L^2(\mathbb{R})} &= \sum_{n\in\mathbb{N}} n|(\psi,\psi_n)_{L^2(\mathbb{R})}|^2, \\
\|A^\dagger\psi\|^2_{L^2(\mathbb{R})} &= \sum_{n\in\mathbb{N}_0} (n+1)|(\psi,\psi_n)_{L^2(\mathbb{R})}|^2.
\end{aligned}
\tag{1.1.76}
$$

Thus, the operators A and $A^\dagger$ can be extended to all those $\psi \in L^2(\mathbb{R})$ for which the series in (1.1.76) converge. In view of this, we set

$$
\mathrm{Dom}(A) = \mathrm{Dom}(A^\dagger) = \mathcal{D}^1, \tag{1.1.77}
$$

where for $\varkappa \in \mathbb{N}$,

$$
\mathcal{D}^\varkappa \stackrel{\text{def}}{=} \left\{\psi \in \mathcal{H} \mid \textstyle\sum_{n=1}^\infty n^\varkappa |(\psi,\psi_n)_{L^2(\mathbb{R})}|^2 < \infty\right\}. \tag{1.1.78}
$$

Note that each $\mathcal{D}^\varkappa$ is a linear subset of $L^2(\mathbb{R})$ and

$$
\mathcal{S}(\mathbb{R}) = \bigcap_{\varkappa\in\mathbb{N}} \mathcal{D}^\varkappa, \tag{1.1.79}
$$

see Theorem V.13 on page 143 in [255].

Theorem 1.1.32. *With the domains (1.1.77) both A and $A^\dagger$ are closed.*

Proof. Let $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{D}^1$ be A-convergent. Then there exist $\phi, \Phi \in L^2(\mathbb{R})$ such that $\phi_k \to \phi$ and $A\phi_k \to \Phi$, as $k \to +\infty$. We have to show that $\phi \in \mathcal{D}^1$ and $A\phi = \Phi$. As a convergent sequence, $\{A\phi_k\}_{k\in\mathbb{N}}$ is bounded in $L^2(\mathbb{R})$. Set

$$
\sup_{k\in\mathbb{N}} \|A\phi_k\|^2_{L^2(\mathbb{R})} = \sup_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} n|(\phi_k,\psi_n)_{L^2(\mathbb{R})}|^2 = C. \tag{1.1.80}
$$

Given $N \in \mathbb{N}$, we pick $l \in \mathbb{N}$, such that $\|\phi - \phi_l\|^2_{L^2(\mathbb{R})} < 1/N$, and estimate

$$\begin{aligned}\sum_{n=1}^{N} n|(\phi, \psi_n)_{L^2(\mathbb{R})}|^2 &= \sum_{n=1}^{N} n|(\phi - \phi_l + \phi_l, \psi_n)_{L^2(\mathbb{R})}|^2 \\ &\leq 2\sum_{n=1}^{N} n|(\phi - \phi_l, \psi_n)_{L^2(\mathbb{R})}|^2 + 2\sum_{n=1}^{\infty} n|(\phi_l, \psi_n)_{L^2(\mathbb{R})}|^2 \\ &\leq 2N\sum_{n=1}^{N} |(\phi - \phi_l, \psi_n)_{L^2(\mathbb{R})}|^2 + 2C \leq 2(1 + C).\end{aligned} \tag{1.1.81}$$

Passing here to the limit $N \to +\infty$, we obtain $\phi \in \mathcal{D}^1$. To prove that $A\phi = \Phi$ let us estimate

$$\begin{aligned}\|A\phi - \Phi\|^2_{L^2(\mathbb{R})} &= \|A(\phi - \phi_k) + A\phi_k - \Phi\|^2_{L^2(\mathbb{R})} \\ &\leq 2\|A(\phi - \phi_k)\|^2_{L^2(\mathbb{R})} + 2\|A\phi_k - \Phi\|^2_{L^2(\mathbb{R})},\end{aligned} \tag{1.1.82}$$

which holds for any $k \in \mathbb{N}$. Furthermore, for any $l, N \in \mathbb{N}$,

$$\begin{aligned}\|A(\phi - \phi_k)\|^2_{L^2(\mathbb{R})} &= \sum_{n=1}^{N} n\left|(\psi_n, \phi - \phi_k)_{L^2(\mathbb{R})}\right|^2 \\ &\quad + \sum_{n=N+1}^{\infty} n\left|(\psi_n, \phi - \phi_l + \phi_l - \phi_k)_{L^2(\mathbb{R})}\right|^2 \\ &\leq N\|\phi - \phi_k\|^2_{L^2(\mathbb{R})} + 4\sum_{n=N+1}^{\infty} n\left|(\psi_n, \phi)_{L^2(\mathbb{R})}\right|^2 \\ &\quad + 4\sum_{n=N+1}^{\infty} n\left|(\psi_n, \phi_l)_{L^2(\mathbb{R})}\right|^2 + 2\|A(\phi_l - \phi_k)\|^2_{L^2(\mathbb{R})}.\end{aligned} \tag{1.1.83}$$

Let $\varepsilon > 0$ be fixed. Then there exists $l_\varepsilon \in \mathbb{N}$, such that $\|A(\phi_l - \phi_k)\|^2_{L^2(\mathbb{R})} < \varepsilon/20$, whenever $k, l \geq l_\varepsilon$, which we assume to hold in the sequel. Then we fix l and pick N such that both following estimates hold, cf. (1.1.81),

$$\sum_{n=N+1}^{\infty} n|(\psi_n, \phi)_{L^2(\mathbb{R})}|^2 < \varepsilon/40, \qquad \sum_{n=N+1}^{\infty} n|(\psi_n, \phi_l)_{L^2(\mathbb{R})}|^2 < \varepsilon/40.$$

Finally, we pick k such that

$$\|A\phi_k - \Phi\|^2_{L^2(\mathbb{R})} < \varepsilon/10 \quad \text{and} \quad \|(\phi - \phi_k)\|^2_{L^2(\mathbb{R})} < \varepsilon/10N.$$

Employing all these estimates in (1.1.83) and then in (1.1.82) we arrive at

$$\|A\phi - \Phi\|^2_{L^2(\mathbb{R})} < \varepsilon.$$

As $\varepsilon > 0$ is an arbitrary number, the latter yields $A\phi = \Phi$ and thereby the closedness of A with domain $\mathcal{D}^1$ follows. In the same way, one proves the closedness of $A^\dagger$. $\square$

By (1.1.75) one easily gets that for all $\phi, \psi \in \mathcal{D}^1$,

$$(A^\dagger \phi, \psi)_{L^2(\mathbb{R})} = (\phi, A\psi)_{L^2(\mathbb{R})}, \quad (A\phi, \psi)_{L^2(\mathbb{R})} = (\phi, A^\dagger \psi)_{L^2(\mathbb{R})}.$$

As $\mathcal{D}^1$ is the maximal set where both A and $A^\dagger$ can be defined, see (1.1.76), this yields that

$$A^* = A^\dagger, \quad (A^\dagger)^* = A, \quad \text{and} \quad [A, A^\dagger] = AA^\dagger - A^\dagger A = I. \tag{1.1.84}$$

Clearly, the latter property holds on $\mathcal{D}^2$ – the domain of the products $AA^\dagger$ and $A^\dagger A$, see (1.1.42). By (1.1.74) we have

$$q = (A + A^\dagger)/\sqrt{2}\kappa, \quad p = -\mathrm{i}\kappa(A - A^\dagger)/\sqrt{2}; \tag{1.1.85}$$

hence,

$$H^{\mathrm{har}} = (A^\dagger A + I/2)\delta = (N + I/2)\delta. \tag{1.1.86}$$

As we shall see below, both p and q are not closed on $\mathcal{D}^1$.

Theorem 1.1.33. *The operator H^{har} with the domain*

$$\mathrm{Dom}(H^{\mathrm{har}}) = \mathcal{D}^2 \tag{1.1.87}$$

is self-adjoint. The resolvent of H^{har} is compact and the spectrum of H^{har} consists of simple isolated eigenvalues.

Proof. Clearly, H^{har} is symmetric. Exactly as in the proof of Theorem 1.1.32 one shows that H^{har} is closed on $\mathcal{D}^2$. Furthermore, by (1.1.73) one obtains that for all $\phi \in \mathcal{D}^2$,

$$(\phi, H^{\mathrm{har}}\phi)_{L^2(\mathbb{R})} \geq E_0 \|\phi\|^2_{L^2(\mathbb{R})},$$

which yields

$$\mathrm{Num}(H^{\mathrm{har}}) \subset [\delta/2, +\infty). \tag{1.1.88}$$

From the latter fact and (1.1.73) one can deduce that H^{har} is invertible and $(H^{\mathrm{har}})^{-1}$ is compact. Therefore, the resolvent set of H^{har} contains zero, that is, the intersection of $\mathsf{R}(H^{\mathrm{har}})$ with the real line is non-void. By Corollary 1.1.20 this yields that H^{har} is self-adjoint. The compactness of $R_\zeta(H^{\mathrm{har}})$ follows by Proposition 1.1.26, by which one also obtains that the spectrum of H^{har} consists of isolated eigenvalues of finite multiplicity. For $\nu = 1$, they are simple according to Proposition 3.3 on page 65 in [63]. $\square$

From now on, we assume that H^{har} has domain (1.1.87).

Let us turn to the vector case where $q = (q^{(1)}, \ldots, q^{(\nu)})$, $p = (p^{(1)}, \ldots, p^{(\nu)})$, $\nu > 1$. By $\mathcal{H}^{(j)}$ and H_j, $j = 1, \ldots, \nu$, we denote the copies of $L^2(\mathbb{R})$ and H^{har} given by (1.1.68), respectively. Set $H_j^{\mathrm{har}} = I \otimes \cdots \otimes H_j \otimes \cdots \otimes I$, where H_j is on the j-th position, and

$$L^2(\mathbb{R}^\nu) = \bigotimes_{j=1}^{\nu} \mathcal{H}^{(j)}, \quad H^{\mathrm{har}} = \sum_{j=1}^{\nu} H_j^{\mathrm{har}}. \tag{1.1.89}$$

Furthermore, for $n = (n_1, \ldots, n_\nu) \in \mathbb{N}_0^\nu$ and $x = (x_1, \ldots x_\nu) \in \mathbb{R}^\nu$, we set

$$\psi_n(x) = (\psi_{n_1} \otimes \cdots \otimes \psi_{n_\nu})(x_1, \ldots, x_\nu) = \psi_{n_1}(x_1) \ldots \psi_{n_\nu}(x_\nu). \tag{1.1.90}$$

Then $\{\psi_n\}_{n \in \mathbb{N}_0^\nu}$ is an orthonormal basis of $L^2(\mathbb{R}^\nu)$ and

$$H^{\mathrm{har}} \psi_n = E_n^{\mathrm{har}} \psi_n = \delta(|n| + \nu/2)\psi_n, \quad |n| = n_1 + \cdots + n_\nu. \tag{1.1.91}$$

By Theorem 1.1.33 we have the following

Theorem 1.1.34. *The operator H^{har} with the domain*

$$\mathrm{Dom}(H^{\mathrm{har}}) = \left\{\psi \in L^2(\mathbb{R}^\nu) \mid \sum_{n \in \mathbb{N}_0^\nu}^{\infty} |n|^2 \, |(\psi, \psi_n)_{\mathcal{H}}|^2 < \infty\right\} \tag{1.1.92}$$

is self-adjoint. Its spectrum consists of isolated eigenvalues of finite multiplicity.

Now let us look at the domain and spectrum of H^{har} having in mind the fact that H^{har} is a differential operator. By $C_0^\infty(\mathbb{R}^\nu)$ we denote the set of infinitely differentiable functions $\varphi \colon \mathbb{R}^\nu \to \mathbb{C}$, for each of which the support, i.e., the closure of the set $\{x \in \mathbb{R}^\nu \mid \varphi(x) \neq 0\}$, is compact. For $\phi \in L^2(\mathbb{R}^\nu)$ and $j = 1, \ldots, \nu$, we define the map

$$C_0^\infty(\mathbb{R}^\nu) \ni \psi \mapsto \theta_\phi^j(\psi) = -\int_{\mathbb{R}^\nu} \bar{\phi}(x) \frac{\partial \psi}{\partial x_j}(x) \mathrm{d}x, \tag{1.1.93}$$

where $\partial\psi(x)/\partial x_j$ is the usual partial derivative of $\psi \in C_0^\infty(\mathbb{R}^\nu)$. This map is a linear functional on $L^2(\mathbb{R}^\nu)$ with domain $C_0^\infty(\mathbb{R}^\nu)$. Clearly, if θ_ϕ^j and θ_ψ^j coincide as functionals, then ϕ and ψ coincide as elements of $L^2(\mathbb{R}^\nu)$. Moreover, if ϕ is differentiable in the usual sense, e.g., $\phi \in \mathcal{S}(\mathbb{R}^\nu)$, then

$$\theta_\phi^j(\psi) = \int_{\mathbb{R}^\nu} \frac{\partial \bar{\phi}}{\partial x_j}(x)\psi(x)\mathrm{d}x.$$

In view of this, the linear functional defined by (1.1.93) is referred to as the *weak derivative* of ϕ. It exists for all $\phi \in L^2(\mathbb{R}^\nu)$ and will be denoted by $\partial\phi(x)/\partial x_j$. For $n \in \mathbb{N}_0^\nu$, the weak derivative of order n, cf. (1.1.50), that is,

$$\phi^{(n)}(x) = \frac{\partial^{|n|}\phi}{\partial x_1^{n_1} \ldots \partial x_\nu^{n_\nu}}(x)$$

is defined as the linear functional

$$C_0^\infty(\mathbb{R}^\nu) \ni \psi \mapsto \theta_\phi^{(n)}(\psi) = (-1)^{|n|} \int_{\mathbb{R}^\nu} \bar{\phi}(x)\psi^{(n)}(x)\mathrm{d}x. \tag{1.1.94}$$

In the general case of $\phi \in L^2(\mathbb{R}^\nu)$, the functional (1.1.94) may be unbounded. If $\phi \in \mathcal{S}(\mathbb{R}^\nu)$, then integrating by parts we obtain

$$\theta_\phi^{(n)}(\psi) = \int_{\mathbb{R}^\nu} \bar{\phi}^{(n)}(x)\psi(x)\mathrm{d}x. \tag{1.1.95}$$

In this case, $\theta_\phi^{(n)}$ can be extended to a bounded linear functional $\theta_\phi^{(n)}: L^2(\mathbb{R}^\nu) \to \mathbb{C}$. Then, by the Riesz lemma, there exists an element of $L^2(\mathbb{R}^\nu)$, which we denote by $\phi^{(n)}$, such that (1.1.95) holds for all $\psi \in L^2(\mathbb{R}^\nu)$.

The equation (1.1.71) is an ordinary differential equation, the *strict solution* of which is the function ψ_n. Here *strict* means *usual* or *classical* since ψ_n, as an element of $\mathcal{S}(\mathbb{R}^\nu)$, is a differentiable function. Given $k \in \mathbb{N} \cup \{\infty\}$, by $C^k(\mathbb{R}^\nu)$ we denote the set of all functions $\varphi: \mathbb{R}^\nu \to \mathbb{C}$ which have continuous partial derivatives $\varphi^{(n)}$ of order $|n| \leq k$. Let λ be a complex number and $W: \mathbb{R}^\nu \to \mathbb{R}$ be a measurable function.

Definition 1.1.35. A function $\phi \in L^2(\mathbb{R}^\nu)$ is said to be a strict solution (respectively, a weak solution) of the differential equation

$$-\frac{1}{2m}\sum_{j=1}^{\nu}\frac{\partial^2\phi}{\partial x_j^2}(x) + W(x)\phi(x) = \lambda\phi(x), \tag{1.1.96}$$

if it is in $C^2(\mathbb{R}^\nu)$ and satisfies this equation (respectively, satisfies (1.1.96) with $\partial^2\phi/\partial x_j^2$, $j = 1, \ldots, \nu$ being weak derivatives).

It is known, see e.g., page 149 in [255], that if W is continuous and $\phi \in C^2(\mathbb{R}^\nu)$, then ϕ is a weak solution of the equation (1.1.96) if and only if it is its strict solution. A corollary of Weyl's lemma, see Theorem IX.26, page 54 in [256], yields the following

Proposition 1.1.36. *Let ϕ be a weak solution of the equation (1.1.96), where W is in $C^\infty(\mathbb{R}^\nu)$. Then also ϕ is in $C^\infty(\mathbb{R}^\nu)$, that is, each weak solution of the equation (1.1.96) is also its strict solution if $W \in C^\infty(\mathbb{R}^\nu)$.*

We note here that the above weak solution ϕ will be in $C^2(\mathbb{R}^\nu)$ already if $W \in C^\varkappa(\mathbb{R}^\nu)$ with certain $\varkappa > 0$, which depends on ν.

Let us proceed considering the harmonic oscillator described by the Hamiltonian H^{har}. Clearly, a given $\phi \in L^2(\mathbb{R}^\nu)$ is a weak solution of (1.1.96) with $W(x) = a|x|^2/2$ if and only if $(\psi, H^{\text{har}}\phi)_{L^2(\mathbb{R}^\nu)} = \lambda(\psi,\phi)_{L^2(\mathbb{R}^\nu)}$ holds for all $\psi \in C_0^\infty(\mathbb{R}^\nu)$. As $C_0^\infty(\mathbb{R}^\nu)$ is dense in $L^2(\mathbb{R}^\nu)$, the latter holds if and only if ϕ is an eigenfunction corresponding to the eigenvalue λ. Therefore, by Proposition 1.1.36 we obtain the following

Proposition 1.1.37. *The spectrum of H^{har} consists of the eigenvalues*

$$E_n^{\text{har}} = (|n| + \nu/2)\delta, \quad n \in \mathbb{N}_0^\nu,$$

only. If $\nu = 1$, these eigenvalues are simple.

Proof. By Theorem 1.1.34, the spectrum of H^{har} is discrete. For each eigenvalue λ, the corresponding function is a weak solution and hence a strict solution of (1.1.96) with $W(x) = a|x|^2/2$. Then it should be one of ψ_n and hence λ is the corresponding eigenvalue E_n^{har}, see (1.1.91). □

As a consequence of the result just proven we obtain that the set of the eigenfunctions (1.1.90) is a basis of $L^2(\mathbb{R}^\nu)$, cf. Lemma 3 on page 142 in [255].

Now we return to the weak derivatives introduced above and set

$$\mathcal{W}^{2,2}(\mathbb{R}^\nu) = \left\{\phi \in L^2(\mathbb{R}^\nu) \mid \phi^{(n)} \in L^2(\mathbb{R}^\nu) \text{ for all } |n| \le 2\right\}. \tag{1.1.97}$$

This is the classical *Sobolev space* of order 2 in $L^2(\mathbb{R}^\nu)$. Clearly, the linear operations preserve $\mathcal{W}^{2,2}(\mathbb{R}^\nu)$. For $\phi, \psi \in \mathcal{W}^{2,2}(\mathbb{R}^\nu)$, we set

$$(\phi, \psi)_{\mathcal{W}^{2,2}(\mathbb{R}^\nu)} = \sum_{0 \le |n| \le 2} (\phi^{(n)}, \psi^{(n)})_{L^2(\mathbb{R}^\nu)}, \tag{1.1.98}$$

where $\phi^{(n)}$ with $n = (0, \ldots, 0)$ stands for ϕ. One can show that (1.1.98) is a scalar product, which turns $\mathcal{W}^{2,2}(\mathbb{R}^\nu)$ into a separable Hilbert space, see e.g., pages 172, 173 in [209]. The following fact is well-known, see e.g., Theorem IX.27 on page 54 in [256].

Proposition 1.1.38. *For every $\nu \in \mathbb{N}$, the operator $|p|^2 = (p^{(1)})^2 + \cdots + (p^{(\nu)})^2$ with* $\mathrm{Dom}(|p|^2) = \mathcal{W}^{2,2}(\mathbb{R}^\nu)$ *is self-adjoint.*

Let us now introduce multiplication operators by Borel functions, cf. (1.1.40).

Definition 1.1.39. Let $F : \mathbb{R}^\nu \to \mathbb{C}$ be a Borel function and

$$\mathcal{A}^F \stackrel{\mathrm{def}}{=} \left\{\psi \in L^2(\mathbb{R}^\nu) \mid \int_{\mathbb{R}^\nu} |F(x)|^2 |\psi(x)|^2 \mathrm{d}x < \infty\right\}. \tag{1.1.99}$$

Then the operator $F(q)$ with $\mathrm{Dom}(F(q)) = \mathcal{A}^F$ which acts as

$$(F(q)\psi)(x) = F(x)\psi(x)$$

is called the multiplication operator by the function F.

In particular, the components of $q = (q^{(1)}, \ldots, q^{(\nu)})$ are multiplication operators. Given $j = 1, \ldots, \nu$, let $\mathcal{A}^j$ be the set (1.1.99) with $F(x) = x_j$ and let $\phi \in \mathcal{A}^j$ be of unit norm. Furthermore, for these j and ϕ, let μ_ϕ^j be the projection of the measure $|\phi(x)|^2 \mathrm{d}x$ onto the j-th axis of $\mathbb{R}^\nu$. By Proposition 1.1.29 the family $\{\mu_\phi^j\}_{\phi \in \mathcal{A}^j}$ defines a self-adjoint operator with domain $\mathcal{A}^j$. At the same time, for $\phi \in \mathcal{A}^j$, one has

$$(\phi, q^{(j)}\phi)_{L^2(\mathbb{R}^\nu)} = \int_{\mathbb{R}^\nu} x_j |\phi(x)|^2 \mathrm{d}x = \int_{\mathbb{R}} \lambda \mu_\phi^j(\mathrm{d}\lambda),$$

which means that the operator mentioned above is $q^{(j)}$, see Proposition 1.1.4 which can obviously be extended to unbounded densely defined operators. Thus, $q^{(j)}$ is self-adjoint on $\mathcal{A}^j$. Clearly, $\mathcal{A}^j$ can contain elements whose weak derivatives are not in

$L^2(\mathbb{R}^\nu)$. From this we see that the sum $A + A^\dagger$, see (1.1.85), cannot be closed on $\mathcal{D}^1$ as all elements of the latter set are in the domain of p. Similarly, one can show that p is not closed on $\mathcal{D}^1$ as well.

By Proposition 1.1.29 we have the following

Proposition 1.1.40. *If F is a real-valued Borel function, then the multiplication operator $F(q)$ with* $\mathrm{Dom}(F(q)) = \mathcal{A}^F$ *is self-adjoint.*

In the sequel, every multiplication operator $F(q)$ is assumed to be defined on the corresponding set $\mathcal{A}^F$, that is, by writing, e.g., $\mathrm{Dom}(|q|^{2r})$, we shall always mean $\mathcal{A}^F$ with $F(q) = |q|^{2r}$. Clearly, for all $r \in \mathbb{N}$,

$$\mathcal{S}(\mathbb{R}^\nu) \subset \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathrm{Dom}(|q|^{2r}). \tag{1.1.100}$$

Proposition 1.1.41. *The operator H^{har} defined as the sum*

$$H^{\mathrm{har}} = \frac{1}{2m}|p|^2 + \frac{a}{2}|q|^2, \tag{1.1.101}$$

with

$$\mathrm{Dom}(H^{\mathrm{har}}) = \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathrm{Dom}(|q|^2) \tag{1.1.102}$$

is self-adjoint.

Proof. We recall that by Theorem 1.1.34, H^{har} is self-adjoint on (1.1.92). Thus, we have to show that

$$\mathrm{RHS}(1.1.92) = \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathrm{Dom}(|q|^2). \tag{1.1.103}$$

Let us do this for $\nu = 1$. As the sum of $p^2/2m$ and $aq^2/2$, the operator (1.1.101) has the property

$$\mathcal{W}^{2,2}(\mathbb{R}) \cap \mathrm{Dom}(q^2) \ni \psi \mapsto H^{\mathrm{har}}\psi \in L^2(\mathbb{R}),$$

which in view of (1.1.86) yields that such ψ is in $\mathcal{D}^2$. To prove the opposite inclusion we take any $\phi \in \mathcal{D}^2$ and obtain, see (1.1.87) and (1.1.49),

$$\phi^{(2)} = (-p^2)\phi = \frac{\kappa^2}{2}\left[A^2 + \left(A^\dagger\right)^2 - (2N+1)\right]\sum_{n=0}^{\infty}(\psi_n, \phi)_{L^2(\mathbb{R})}\psi_n = \sum_{n=0}^{\infty}\alpha_n\psi_n,$$

where

$$\begin{aligned}\alpha_n = \frac{\kappa^2}{2}\Big[&\sqrt{(n+1)(n+2)}(\psi_{n+2}, \phi)_{L^2(\mathbb{R})} \\ &+ \sqrt{(n-1)n}(\psi_{n-2}, \phi)_{L^2(\mathbb{R})} - (2n+1)(\psi_n, \phi)_{L^2(\mathbb{R})}\Big].\end{aligned}$$

As $\phi \in \mathcal{D}^2$, the sequence $\{\alpha_n\}$ is square summable; hence, $\phi^{(2)} \in L^2(\mathbb{R})$. Exactly in the same way one shows that $q^2\phi \in L^2(\mathbb{R})$ if $\phi \in \mathcal{D}^2$. Thus, (1.1.103) holds for $\nu = 1$. The extension of this result to arbitrary ν is obvious. □

We conclude considering the harmonic case by proving the following statement where the ψ_n's are the eigenfunctions of H^{har} with $\nu = 1$.

Lemma 1.1.42. *For every integer $r \geq 2$, it follows that*

$$q^{2r}\psi_n = \frac{1}{2^r\kappa^{2r}} \sum_{k=-r}^{r} \pi_r(k;n)\psi_{n+2k}, \tag{1.1.104}$$

where the coefficients are such that $\pi_r(k;n) = 0$ if $n + 2k < 0$ and $\pi_r(k;n) > 0$ otherwise. Furthermore,

$$\pi_r(k;n) \leq 3^{r-1}(2n + 4r + 1)^r, \tag{1.1.105}$$

which holds for all $k, n \in \mathbb{N}_0$.

Proof. By (1.1.85)

$$q^{2r} = \frac{1}{2^r\kappa^{2r}}\left[A^2 + (2N+1) + \left(A^\dagger\right)^2\right]^r.$$

Since ψ_n, $n \in \mathbb{N}_0$ are as in (1.1.70) and (1.1.75), by direct calculation we obtain (1.1.104). Here for $n + 2k < 0$, as well as for $|k| > r$, we set $\pi_r(k;n) = 0$. Otherwise, $[\pi_r(k;n)]^2$ is a polynomial in n of degree $2r$. It can be calculated from

$$\begin{aligned} \pi_r(k;n) = {} & \pi_{r-1}(k-1;n)\sqrt{(n+2k)(n+2k-1)} \\ & + \pi_{r-1}(k;n)(2n+4k+1) \\ & + \pi_{r-1}(k+1;n)\sqrt{(n+2k+2)(n+2k+1)}, \end{aligned} \tag{1.1.106}$$

with the initial elements

$$\begin{aligned} \pi_1(-1;n) &= \sqrt{n(n-1)}, \\ \pi_1(0;n) &= 2n+1, \\ \pi_1(1;n) &= \sqrt{(n+2)(n+1)}. \end{aligned} \tag{1.1.107}$$

We note that

$$\forall k = 1, \ldots, r: \quad \pi_r(k;n) = \pi_r(-k;n+2k). \tag{1.1.108}$$

The estimate (1.1.105) is obtained by induction over r in (1.1.106) and then by (1.1.107). $\square$

Anharmonic oscillator: domain and spectrum

The Hamiltonian of the ν-dimensional quantum anharmonic oscillator is

$$H = H^{\text{har}} + V(q) = \frac{1}{2m}|p|^2 + \frac{a}{2}|q|^2 + V(q), \tag{1.1.109}$$

where the potential V obeys Assumption 1.1.1. Here and in the sequel, the operator H^{har} is supposed to be defined on the set given by (1.1.102) and (1.1.92). According to the definition of the sum of operators we set

$$\mathrm{Dom}(H) = \mathrm{Dom}(H^{\text{har}}) \cap \mathcal{A}^V = \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathcal{A}^V. \tag{1.1.110}$$

Obviously, $C_0^\infty(\mathbb{R}^\nu)$ is contained in (1.1.110). As V has the lower bound (1.1.10), by Theorem X.28, page 184 in [256], we obtain the following

Proposition 1.1.43. *The sum $H^{\text{har}} + V(q)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$. More precisely, the restriction of* (1.1.109) *to $C_0^\infty(\mathbb{R}^\nu)$ is an essentially self-adjoint operator.*

The operator $H_{\min}$ as in the above proposition is called the *minimal operator* defined by the sum $H^{\text{har}} + V(q)$, whereas the operator (1.1.109) with domain (1.1.110) is called the *maximal operator*, see e.g., page 274 in [172]. Since the closure $\widetilde{H}$ is an extension of H, one has

$$\mathrm{Dom}(\widetilde{H}) \supset \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathcal{A}^V. \tag{1.1.111}$$

It turns out that the opposite inclusion can also be proven under a certain additional condition imposed on the potential V. For $\rho > 0$, we set $B_\rho = \{x \in \mathbb{R}^\nu \mid |x| \leq \rho\}$ and let $|B_\rho|$ stand for the volume of B_ρ.

Definition 1.1.44. A continuous function $F\colon \mathbb{R}^\nu \to [0, +\infty)$ is said to belong to the class $\mathfrak{B}_s(\mathbb{R}^\nu)$ with $s \in (0, +\infty)$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B_\rho|} \int_{B_\rho} [F(x+y)]^s \mathrm{d}x \right)^{1/s} \leq \frac{C}{|B_\rho|} \int_{B_\rho} F(x+y)\mathrm{d}x \tag{1.1.112}$$

holds for all $y \in \mathbb{R}^\nu$ and $\rho > 0$. The class $\mathfrak{B}_\infty(\mathbb{R}^\nu)$ consists of those $F\colon \mathbb{R}^\nu \to [0, +\infty)$, for which (1.1.112) holds with $\sup_{x \in B_\rho} F(x+y)$ on the left-hand side. The smallest C in (1.1.112) is called the $\mathfrak{B}_s$-constant of F.

Proposition 1.1.45. *If $s' > s$, then $\mathfrak{B}_{s'}(\mathbb{R}^\nu) \subset \mathfrak{B}_s(\mathbb{R}^\nu)$. All positive polynomials belong to $\mathfrak{B}_\infty(\mathbb{R}^\nu)$. Moreover, for $F \in \mathfrak{B}_\infty(\mathbb{R}^\nu)$, also $g \circ F \in \mathfrak{B}_\infty(\mathbb{R}^\nu)$, where $g\colon [0, +\infty) \to [0, +\infty)$ is increasing, convex, and such that for any $\vartheta \geq 1$, $g(\vartheta t)/g(t)$ is bounded on $[0, +\infty)$ (as for example, $g(t) = t^\eta$ with $\eta \geq 1$).*

Proof. We prove only the final part of the statement – see [48] for the rest. For given $\vartheta \geq 1$, we set $\gamma(\vartheta) = \sup_{t \geq 0} g(\vartheta t)/g(t)$. As F belongs to $\mathfrak{B}_\infty(\mathbb{R}^\nu)$, for any $y \in \mathbb{R}^\nu$ and $\rho > 0$,

$$\sup_{x \in B_\rho} F(x+y) \leq \frac{C}{|B_\rho|} \int_{B_\rho} F(x+y)\mathrm{d}x.$$

Therefrom, by monotonicity and Jensen's inequality one gets

$$\begin{aligned} \sup_{x \in B_\rho} (g \circ F)(x+y) &\leq \left(\frac{C}{|B_\rho|} \int_{B_\rho} F(x+y)\mathrm{d}x \right) \\ &\leq \frac{\gamma(C)}{|B_\rho|} \int_{B_\rho} (g \circ F)(x+y)\,\mathrm{d}x. \end{aligned}$$

$\square$

Employing Theorem 1.1 of [48] we obtain the following

Proposition 1.1.46. *Let V obey (1.1.10) and be such that*

$$\frac{a}{2}|x|^2 + V(x) + c_V \in \mathfrak{B}_2(\mathbb{R}^\nu). \tag{1.1.113}$$

Then there exist positive constants C_1 and C_2, such that for all $\psi \in C_0^\infty(\mathbb{R}^\nu)$,

$$\begin{aligned} \|H^{\text{har}}\psi\|_{L^2(\mathbb{R}^\nu)} &\le C_1\|(H + c_V I)\psi\|_{L^2(\mathbb{R}^\nu)}, \\ \|(V(q) + c_V I)\psi\|_{L^2(\mathbb{R}^\nu)} &\le C_2\|(H + c_V I)\psi\|_{L^2(\mathbb{R}^\nu)}. \end{aligned} \tag{1.1.114}$$

Theorem 1.1.47. *If V is as in Proposition 1.1.46, then the operator (1.1.109) with domain (1.1.110) is self-adjoint.*

The proof of this theorem is based on a simple fact, which we are going to use also in other situations. In view of this, we formulate it in a more general form in the next statement.

Lemma 1.1.48. *Let T and Q be closed linear operators on a Hilbert space $\mathcal{H}$, and let $\mathcal{D} \subset \mathcal{H}$ be a core of the two. If for all $\phi \in \mathcal{D}$, one has*

$$\|Q\phi\|_{\mathcal{H}} \le \|T\phi\|_{\mathcal{H}}, \tag{1.1.115}$$

then $\text{Dom}(T) \subset \text{Dom}(Q)$.

Proof. As $\mathcal{D}$ is a core of T, for arbitrary $\phi \in \text{Dom}(T)$, one finds a sequence $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$, which is T-convergent to ϕ. For any $n, m \in \mathbb{N}$, by (1.1.115) we have

$$\|Q\phi_n - Q\phi_m\|_{\mathcal{H}} \le \|T\phi_n - T\phi_m\|_{\mathcal{H}}.$$

Thus, $\{\phi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}$ is also Q-convergent to ϕ and thereby $\phi \in \text{Dom}(Q)$. □

Proof of Theorem 1.1.47. In view of (1.1.111) and (1.1.110), the proof will be done if we show that

$$\text{Dom}(\widetilde{H}) \subset \text{Dom}(H^{\text{har}}) \cap \mathcal{A}^V = \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathcal{A}^V,$$

which readily follows by Lemma 1.1.48 and the estimates (1.1.114). Here we have used the fact that $C_0^\infty(\mathbb{R}^\nu)$ is a core for H, H^{har}, and $V(q)$. □

For a positive self-adjoint operator $T : \mathcal{H} \to \mathcal{H}$, an orthonormal set $\{\varphi_n\}_{n\in\mathbb{N}_0} \subset \mathcal{H}$ is called a *trial system* if $\varphi_n \in \text{Dom}(\sqrt{T})$ for all $n \in \mathbb{N}_0$. The next statement gives a way of estimating eigenvalues of operators, which are either compact or have compact resolvents. It is an adaptation of a known result, see e.g., pages 300 and 301 in [209] or pages 75–79 in [257].

Proposition 1.1.49. *Let $T : \mathcal{H} \to \mathcal{H}$ be a positive and self-adjoint linear operator. Suppose also that $\mathsf{S}(T)$ consists entirely of eigenvalues λ_n, $n \in \mathbb{N}_0$, of finite multiplicity, and there exists an orthonormal basis of $\mathcal{H}$ consisting of the corresponding eigenfunctions ψ_n. Then for every $n \in \mathbb{N}_0$, one has*

$$\lambda_{n+1} = \max_{\varphi_0,\ldots,\varphi_n} \min \left\{ (\phi, T\phi)_{\mathcal{H}} \mid \phi \in \mathcal{K}_n^{\perp},\ \|\phi\|_{\mathcal{H}} = 1 \right\}, \tag{1.1.116}$$

$$\lambda_n = \min_{\varphi_0,\ldots,\varphi_n} \max \left\{ (\phi, T\phi)_{\mathcal{H}} \mid \phi \in \mathcal{K}_n,\ \|\phi\|_{\mathcal{H}} = 1 \right\}, \tag{1.1.117}$$

where the first max *and* min *are taken over all trial systems for T, and $\mathcal{K}_n$ is the linear span of $\varphi_0, \ldots, \varphi_n$.*

Proof. For $l \in \mathbb{N}_0$, by $\mathcal{H}_l$ we denote the linear span of $\psi_0, \ldots, \psi_l$. Let us show that for any trial system, there exists $\psi \in \mathcal{H}_{n+1}$ such that $\|\psi\|_{\mathcal{H}} = 1$ and $\psi \in \mathcal{K}_n^{\perp}$. Fix $\{\varphi_k\}$ and let $\mathcal{T}$ be any finite-dimensional subspace of $\mathcal{H}$, which contains both $\mathcal{H}_{n+1}$ and $\mathcal{K}_n$. If $\dim \mathcal{T} = m$, then $\dim \mathcal{T}^c = m - n$, where $\mathcal{T}^c \stackrel{\text{def}}{=} \mathcal{T} \cap \mathcal{K}_n^{\perp}$. The vector in question can be taken from $\mathcal{T}^c \cap \mathcal{H}_{n+1}$, which contains nonzero vectors, since one would have $\dim \mathcal{T} > m$ otherwise. In the same way, one shows the existence of $\phi \in \mathcal{K}_n$ such that $\phi \in \mathcal{H}_{n-1}^{\perp}$.

Let γ_{n+1} be the right-hand side of (1.1.116). Then

$$\gamma_{n+1} \geq \min_{\psi \in \mathcal{H}_n^{\perp}} (\psi, T\psi)_{\mathcal{H}} = (\psi_{n+1}, T\psi_{n+1})_{\mathcal{H}} = \lambda_{n+1}.$$

On the other hand, for each $\varphi_0, \ldots, \varphi_n$, one finds $\psi \in \mathcal{H}_{n+1}$, $\|\psi\|_{\mathcal{H}} = 1$, such that $\psi \in \mathcal{K}_n^{\perp}$. Then $\gamma_{n+1} \leq (\psi, T\psi)_{\mathcal{H}} \leq \lambda_{n+1}$ which proves (1.1.116). Now let γ_n be the right-hand side of (1.1.117). Then

$$\gamma_n \leq \max_{\psi \in \mathcal{H}_n} (\psi, T\psi)_{\mathcal{H}} = (\psi_n, T\psi_n)_{\mathcal{H}} = \lambda_n.$$

At the same time, for each $\varphi_0, \ldots, \varphi_n$, one finds $\phi \in \mathcal{K}_n$, $\|\phi\|_{\mathcal{H}} = 1$, such that $\phi \in \mathcal{H}_{n-1}^{\perp}$. Then $\gamma_n \geq (\phi, T\phi)_{\mathcal{H}} \geq \lambda_n$, which completes the proof. □

Now let V not necessarily belong to $\mathfrak{B}_s(\mathbb{R}^\nu)$. For c_V being as in (1.1.10) and $\phi \in \mathrm{Dom}(H)$ given by (1.1.110), we have

$$\begin{aligned} (\phi, [H + c_V I]\phi)_{L^2(\mathbb{R}^\nu)} &= (\phi, H^{\mathrm{har}}\phi)_{L^2(\mathbb{R}^\nu)} \\ &\quad + (\phi, [V(q) + c_V I]\phi)_{L^2(\mathbb{R}^\nu)} \\ &\geq E_0^{\mathrm{har}} = \delta\nu/2. \end{aligned} \tag{1.1.118}$$

In the next statement, $\widetilde{H}$ stands for the closure of the operator defined in (1.1.109), (1.1.110).

Theorem 1.1.50. *Suppose that V obeys (1.1.10) only. Then the spectrum of $\widetilde{H}$ consists entirely of eigenvalues of finite multiplicity E_k, $k \in \mathbb{N}_0$, which obey the estimate*

$$\forall k \in \mathbb{N}_0 : \quad E_k \geq (k + \nu/2)\delta - c_V, \tag{1.1.119}$$

where c_V and δ are the same as in (1.1.10) and in (1.1.91), respectively. If $\nu = 1$, the eigenvalues are simple.

Proof. Both H^{har} and $\widetilde{H} + c_V I$ are self-adjoint and positive, see (1.1.118). Then there exist $(H^{\text{har}})^{\pm 1/2}$ and $(\widetilde{H} + c_V I)^{\pm 1/2}$, see Proposition 1.1.29. This, in particular, means that $\text{Ran}(\widetilde{H} + c_V I)^{1/2} = L^2(\mathbb{R}^\nu)$. Also by (1.1.118) it follows that

$$\|Q\| \leq 1, \quad \text{where } Q \stackrel{\text{def}}{=} (H^{\text{har}})^{1/2}(\widetilde{H} + c_V I)^{-1/2}. \tag{1.1.120}$$

In view of (1.1.91), the operator $(H^{\text{har}})^{-1/2}$ is compact. By Proposition 1.1.7 this yields that $(\widetilde{H} + c_V I)^{-1/2}$ is also compact and hence $\widetilde{H}$ has compact resolvent. Then the general properties of the spectrum of $\widetilde{H}$ follow by Proposition 1.1.26, by which, and by Proposition 1.1.21, we also have that there exists an orthonormal basis $\{\psi_k\}_{k\in\mathbb{N}_0}$ of $L^2(\mathbb{R}^\nu)$ consisting of the eigenfunctions of $\widetilde{H}$. Let us show that $\{\psi_k\}_{k\in\mathbb{N}_0}$ is a trial system for H^{har}. Clearly, each ψ_k belongs to $\text{Dom}(\sqrt{H + c_V I})$. By (1.1.118) one obtains that for all ϕ in $C_0^\infty(\mathbb{R}^\nu)$, which is a core for both H and H^{har}, it follows that

$$\|\sqrt{H^{\text{har}}}\,\phi\|_{L^2(\mathbb{R}^\nu)} \leq \|\sqrt{H + c_V I}\,\phi\|_{L^2(\mathbb{R}^\nu)}.$$

Therefrom by Lemma 1.1.48 one obtains

$$\text{Dom}(\sqrt{H + c_V I}) \subset \text{Dom}(\sqrt{H^{\text{har}}}),$$

which yields $\psi_k \in \text{Dom}(\sqrt{H^{\text{har}}})$ for all $k \in \mathbb{N}_0$. Now we prove (1.1.119). For $k \in \mathbb{N}_0$, by $\mathcal{H}_k$ we denote the linear span of ψ_l with $l \leq k$. Then for $n \in \mathbb{N}_0^\nu$, such that $|n| \leq k$, by (1.1.117) and (1.1.120) we get

$$\begin{aligned}
E_n^{\text{har}} &\leq \max\left\{\|(H^{\text{har}})^{1/2}\phi\|^2_{L^2(\mathbb{R}^\nu)} \mid \phi \in \mathcal{H}_k,\ \|\phi\|_{L^2(\mathbb{R}^\nu)} = 1\right\}\\
&= \max\left\{\|Q(\widetilde{H} + c_V I)^{1/2}\phi\|^2_{L^2(\mathbb{R}^\nu)} \mid \phi \in \mathcal{H}_k,\ \|\phi\|_{L^2(\mathbb{R}^\nu)} = 1\right\}\\
&\leq \max\left\{\|(\widetilde{H} + c_V I)^{1/2}\phi\|^2_{L^2(\mathbb{R}^\nu)} \mid \phi \in \mathcal{H}_k,\ \|\phi\|_{L^2(\mathbb{R}^\nu)} = 1\right\},
\end{aligned}$$

which by (1.1.91) yields (1.1.119). The claimed simplicity of E_k follows by Proposition 3.3 on page 65 of [63]. □

For particular cases of V, one can get a more precise bound than (1.1.119). The corresponding technique comes from the theory of differential equations. We recall that the notion of a strict and a weak solution was introduced in Definition 1.1.35. Let us begin by establishing the following fact.

Theorem 1.1.51. *Let V be in $C^\infty(\mathbb{R}^\nu)$ and obey (1.1.113). Then*

$$\text{Dom}(\widetilde{H}) = \mathcal{W}^{2,2}(\mathbb{R}^\nu) \cap \mathcal{A}^V, \tag{1.1.121}$$

and every eigenfunction ψ_k, $k \in \mathbb{N}_0$, of $\widetilde{H}$ is a strict solution of the equation

$$-\frac{1}{2m}\Delta\psi(x) + \left(\frac{a}{2}|x|^2 + V(x)\right)\psi(x) = E_k\psi(x), \tag{1.1.122}$$

where E_k is the corresponding eigenvalue and Δ is the Laplacian in $\mathbb{R}^\nu$.

Proof. The first part of the statement, i.e., (1.1.121) follows by Theorem 1.1.47. Furthermore, as in the proof of Proposition 1.1.37 we obtain that every eigenfunction of $\widetilde{H}$ is a weak solution of the corresponding equation (1.1.122). Thus, by Proposition 1.1.36 it is also a strict solution of this equation. □

Now let us consider the simplest case of (1.1.109), where $\nu = 1$ and

$$H = T \stackrel{\text{def}}{=} H^{\text{har}} + bq^{2r} = \frac{1}{2m}p^2 + \frac{a}{2}q^2 + bq^{2r}, \qquad r \in \mathbb{N},\ r \geq 2,\ b > 0, \tag{1.1.123}$$

which by Theorem 1.1.46 is self-adjoint on

$$\mathrm{Dom}(T) = \mathrm{Dom}(H^{\text{har}}) \cap \mathrm{Dom}(q^{2r}). \tag{1.1.124}$$

The spectrum of T is described in Theorem 1.1.50. Let $\{\phi_n\}_{n\in\mathbb{N}_0}$ be an orthonormal basis of $L^2(\mathbb{R})$ consisting of the eigenfunctions of T. Set

$$T\phi_n = \Theta_n\phi_n, \qquad n \in \mathbb{N}_0. \tag{1.1.125}$$

We recall that each Θ_n is simple. Since both $\phi_n^{\pm}$, where $\phi_n^{\pm}(x) \stackrel{\text{def}}{=} \phi_n(\pm x)$, correspond to the same eigenvalue, each ϕ_n is either even or odd. Exactly as in the proof of the closedness of A, see Theorem 1.1.32 and also (1.1.87), one shows that

$$\mathrm{Dom}(T) = \left\{\phi \in L^2(\mathbb{R}) \mid \textstyle\sum_{k\in\mathbb{N}_0} \Theta_k^2 |(\phi, \phi_k)_{L^2(\mathbb{R})}|^2 < \infty\right\}. \tag{1.1.126}$$

By Theorem 1.1.50 $\Theta_k \geq (k + 1/2)\delta$, which can also be proven directly with the help of (1.1.116). More precise information on the growth of Θ_k can be obtained by studying the corresponding differential equations. Each ϕ_k is a strict solution of the corresponding one-dimensional version of (1.1.122) subject to the condition $\phi(\pm\infty) = 0$, see Theorem 1.1.51. This allows us to employ a number of classical results, e.g., following the line of arguments developed in [305]. The next statement will also be used in obtaining a more accurate bound for Θ_n.

Proposition 1.1.52 (Sturm Theorem). *Let g and h be continuous real-valued functions, defined on $(c, d) \subset [0, +\infty)$, such that $g(x) < h(x)$ on a certain interval $(a, b) \subset (c, d)$. Let u (respectively, v) be a solution of the equation (1.1.127) (respectively, (1.1.128)), where*

$$u''(x) + g(x)u(x) = 0, \tag{1.1.127}$$

$$v''(x) + h(x)v(x) = 0. \tag{1.1.128}$$

Assume also that there exist $\alpha, \beta \in (a, b)$, $\alpha < \beta$, such that $u(\alpha) = u(\beta) = 0$ and $u(x) > 0$ for all $x \in (\alpha, \beta)$. Then there exists $\gamma \in (\alpha, \beta)$, such that $v(\gamma) = 0$.

Proof. The assumed properties of u and v imply that $u'(\alpha) \geq 0$, $u'(\beta) \leq 0$. Suppose v is positive on (α, β), and hence nonnegative at the endpoints. Then subtracting the equations and integrating by parts we get

$$v(\beta)u'(\beta) - v(\alpha)u'(\alpha) = \int_\alpha^\beta [h(x) - g(x)]u(x)v(x)\mathrm{d}x > 0,$$

which is impossible. □

Now we are in a position to get a more accurate bound for Θ_n. Recall that in (1.1.123), as well as in the lemma below, m stands for the particle mass.

Lemma 1.1.53. *The eigenvalues Θ_n, $n \in \mathbb{N}_0$, of the operator (1.1.123) have the following property. There exist $\Theta_* > 0$ and $n_* \in \mathbb{N}$, which depend on the parameters a, b, m, and r only, such that for all $n \geq n_*$,*

$$\Theta_n \geq \Theta_* n^{2r/(r+1)}. \tag{1.1.129}$$

Proof. First, we point out that the equation

$$-v''(x) + q(x)v(x) = \lambda v(x), \quad x \in \mathbb{R},$$

with a continuous function q, such that $q(x) \to +\infty$ as $x \to \pm\infty$, has solutions $v \in L^2(\mathbb{R})$ only for λ belonging to a discrete set $\{\lambda_n\}_{n\in\mathbb{N}_0}$, such that $\lambda_n \to +\infty$ as $n \to +\infty$. For each $n \in \mathbb{N}_0$, the corresponding solution v_n has exactly n zeros in $\mathbb{R}$. Furthermore, if x_n is such that $q(x) - \lambda_n > 0$ for all $x > x_n$, then the number of zeros of v_n which exceed x_n is at most 1. If q is even, then the solutions v_{2n}, $n \in \mathbb{N}$, are even (respectively, v_{2n-1}, $n \in \mathbb{N}$, are odd), and the number of zeros of v_{2n} on $(0, +\infty)$ is n (respectively, the number of zeros of v_{2n-1} on $(0, +\infty)$ is $n - 1$), see Theorem 3.5 on page 66 of [63].

For $\theta > 0$, let $w(\theta)$ be the positive solution of the equation

$$\frac{1}{2}aw + bw^r = \theta.$$

Then $w(\theta)$, and hence $\theta w(\theta)$, are increasing functions. Let $\theta_* > 0$ be the solution of

$$\theta w(\theta) = \frac{9r^2}{8m}. \tag{1.1.130}$$

As $\Theta_n \geq (n + 1/2)\delta$, one can pick an integer $n_* \geq 3$, such that for $n \geq n_*$,

$$\Theta_n \geq \theta_*. \tag{1.1.131}$$

We fix such an n and set

$$x_n = \sqrt{w(\Theta_n)}. \tag{1.1.132}$$

Clearly,

$$x_n < \left(\frac{\Theta_n}{b}\right)^{1/2r}. \tag{1.1.133}$$

Now for $\theta > \Theta_n$, let us consider the function $\Psi(x,\theta)$ defined on $x \in (0, x_n)$ by

$$\begin{aligned}\Psi(x,\theta) &= \left(2\theta m - amx^2 - 2bmx^{2r}\right)^{-1/4}\cos\rho(x,\theta),\\ \rho(x,\theta) &= \int_0^x \left(2\theta m - amt^2 - 2bmt^{2r}\right)^{1/2} \mathrm{d}t.\end{aligned} \tag{1.1.134}$$

Set $\Psi_n(x) = \Psi(x, 2\Theta_n)$ and $\rho_n(x) = \rho(x, 2\Theta_n)$. By a direct calculation,

$$-\frac{1}{2m}\Psi_n''(x) + \left[\frac{a}{2}x^2 + bx^{2r} - \varphi(x, 2\Theta_n)\right]\Psi_n(x) = \Theta_n\Psi_n(x), \tag{1.1.135}$$

where

$$\begin{aligned}\varphi(x,\theta) = \theta - \Theta_n &- \frac{a/2 + br(2r-1)x^{2r-2}}{4m(\theta - ax^2/2 - bx^{2r})}\\ &- \frac{5}{8m}\left(\frac{ax/2 + brx^{2r-1}}{\theta - ax^2/2 - bx^{2r}}\right)^2.\end{aligned}$$

Clearly, $\varphi(x,\theta) \geq \varphi(x_n,\theta)$; thus,

$$\begin{aligned}\varphi(x, 2\Theta_n) &\geq \varphi(x_n, 2\Theta_n)\\ &\geq \Theta_n\left(1 - \frac{1}{x_n^2\Theta_n}\cdot\frac{9r^2 - 2r}{8m}\right) > 0.\end{aligned} \tag{1.1.136}$$

The latter estimate holds since $x_n^2\Theta_n \geq 9r^2/8m$, see (1.1.130), (1.1.131), and (1.1.132). Along with (1.1.135) we consider the problem

$$(T\psi)(x) = -\frac{1}{2m}\psi''(x) + [ax^2/2 + bx^{2r}]\psi(x) = \Theta_n\psi(x), \tag{1.1.137}$$

$x \in \mathbb{R}$ with $\psi \in L^2(\mathbb{R})$. Its solution is ϕ_n. In view of the facts mentioned at the very beginning of this proof, ϕ_{2n} has at least $n-1$ zeros on $(0, x_{2n})$. Given $n \in \mathbb{N}$, let l be the number of zeros of Ψ_{2n}. Comparing the equations (1.1.135) and (1.1.137) and taking into account the estimate (1.1.136) we conclude that $l \geq n-2$, see Proposition 1.1.52. These zeros can be found from the equation $\cos\rho_{2n}(x) = 0$, cf. (1.1.134). The function $\rho_{2n}(x)$ is increasing on $(0, x_{2n})$. The m-th zero of Ψ_{2n} is defined by the equation $\rho_{2n}(\xi_m) = (m + 1/2)\pi$, $m = 0, \ldots, l-1$, where, see (1.1.134),

$$\rho_n(x) = 2\sqrt{m\Theta_n}\int_0^x\left[1 - \frac{at^2/2 + bt^{2r}}{2(ax_n^2/2 + bx_n^{2r})}\right]^{1/2}\mathrm{d}t \leq 2x\sqrt{m\Theta_n}.$$

Therefore, for $n \geq 3$, we have

$$\rho_{2n}(x_{2n}) \geq \rho_{2n}(\xi_{l-1}) = (l - 1/2)\pi \geq (n - 5/2)\pi \geq n\pi/6,$$

which together with the estimate

$$\rho_{2n}(x_{2n}) \leq 2x_{2n}\Theta_{2n}^{1/2}\sqrt{m} < 2\sqrt{m}\left(\Theta_{2n}^{1+r}/b\right)^{1/2r},$$

see (1.1.133), yields

$$\Theta_{2n} > n^{2r/(r+1)} \left(\frac{\pi b^{1/2r}}{12\sqrt{m}} \right)^{\frac{2r}{1+r}}.$$

Therefore, taking into account that $\Theta_{2n+1} > \Theta_{2n}$, we obtain (1.1.129) with

$$\Theta_* = \left(\frac{\pi}{24\sqrt{m}} \right)^{2r/(r+1)} b^{1/(1+r)}. \tag{1.1.138}$$

For $r = 1$ and $b = a/2$, the estimate (1.1.129) takes the form $\Theta_n \geq (\pi/24\sqrt{2})\delta n$, which can be compared with the exact value $\Theta_n = (n + 1/2)\delta$. □

To extend the result just obtained to more general operators than that given by (1.1.123) we employ a comparison technique, based on the following notion.

Definition 1.1.54. Let Q and T be densely defined linear operators, such that $\mathrm{Dom}(Q) \supset \mathrm{Dom}(T)$. Then Q is T-bounded if there exist positive a and b, such that for all $\psi \in \mathrm{Dom}(T)$,

$$\|Q\psi\|_{\mathcal{H}} \leq a\|T\psi\|_{\mathcal{H}} + b\|\psi\|_{\mathcal{H}}. \tag{1.1.139}$$

The infimum of a in (1.1.139) is called the T-bound for Q. If this infimum is zero, one writes $Q \ll T$.

The following statement is very useful in establishing domains of sums of operators, see Section X.2 of [256].

Proposition 1.1.55 (Kato–Rellich Theorem). *Let T be self-adjoint and Q be symmetric and T-bounded with the T-bound $a < 1$. Then the operator $T + Q$ is self-adjoint on* $\mathrm{Dom}(T)$.

We use this statement to extend the result of Lemma 1.1.53 to more general types of H.

Theorem 1.1.56. *Let T be as in (1.1.123) and $W : \mathbb{R} \to \mathbb{R}$ be continuous and have the following property. For every $\alpha > 0$, there exists $\beta > 0$ such that, for all $x \in \mathbb{R}$,*

$$[W(x)]^2 \leq \alpha b^2 x^{4r} + \beta, \tag{1.1.140}$$

where r and b are as in (1.1.123). Then the operator $T + W(q)$ is self-adjoint on $\mathrm{Dom}(T)$ *given by* (1.1.124).

Proof. By (1.1.140) the multiplication operator $W(q)$ is symmetric on $\mathrm{Dom}(q^{2r})$, see (1.1.99) and (1.1.100), and, therefore, on $\mathrm{Dom}(T)$. By (1.1.114) it follows that

$$\forall \psi \in \mathrm{Dom}(T): \quad \begin{aligned} \|H^{\mathrm{har}}\psi\|_{L^2(\mathbb{R}^\nu)} &\leq C_1\|T\psi\|_{L^2(\mathbb{R}^\nu)}, \\ \|bq^{2r}\psi\|_{L^2(\mathbb{R}^\nu)} &\leq C_2\|T\psi\|_{L^2(\mathbb{R}^\nu)}, \end{aligned}$$

for some positive constants C_1 and C_2. Then by (1.1.140) we obtain

$$\begin{aligned}\forall \psi \in \mathrm{Dom}(T): \quad \|W(q)\psi\|^2_{L^2(\mathbb{R})} &\leq \alpha\|bq^{2r}\psi\|^2_{L^2(\mathbb{R})} + \beta\|\psi\|^2_{L^2(\mathbb{R})} \\ &< \alpha C_2\|T\psi\|^2_{L^2(\mathbb{R})} + \beta\|\psi\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

For positive a, b, x, y, z, the estimate $x^2 \leq a^2y^2 + b^2z^2$ implies $x \leq ay + bz$. Therefore, $W \ll T$ and the proof follows by the Kato–Rellich theorem. □

A direct generalization of the latter statement is the following

Theorem 1.1.57. *Let ν be an arbitrary positive integer and V be as in Theorem 1.1.46. Let also $W : \mathbb{R}^\nu \to \mathbb{R}$ be continuous and such that*

$$[W(x)]^2 \leq \alpha[V(x) + c_V]^2 + \beta,$$

with $\alpha < 1/C_2$ and a certain $\beta > 0$, C_2 being the same as in (1.1.114). Then the operator $H = H^{\mathrm{har}} + W(q) + V(q)$ is self-adjoint on

$$\mathrm{Dom}(H) = \mathrm{Dom}(H^{\mathrm{har}}) \cap \mathcal{A}^V.$$

Now we obtain an extension of Lemma 1.1.53.

Theorem 1.1.58. *Let V and r be as in (1.1.10) and V obey the conditions of Theorem 1.1.51. Then for every $\nu \geq 1$, the eigenvalues E_k, $k \in \mathbb{N}_0$, of the operator (1.1.109) have the following property. There exist $E_* > 0$ and $k_* \in \mathbb{N} \setminus \{1\}$ such that for all $k \geq k_*$,*

$$E_k \geq E_* k^{2r/(r+1)}. \tag{1.1.141}$$

Proof. We recall that the space $L^2(\mathbb{R}^\nu)$ is the tensor product of ν copies of $L^2(\mathbb{R})$. Let T be as in (1.1.123) with $b = b_V$, b_V being the same as in (1.1.10). For $j = 1, \ldots, \nu$, we set

$$T_j = I \otimes \cdots \otimes I \otimes T \otimes I \otimes \cdots \otimes I,$$

where a copy of the operator (1.1.123) is on the j-th position. Consider

$$\begin{aligned} S &\stackrel{\mathrm{def}}{=} \sum_{j=1}^{\nu} T_j = H^{\mathrm{har}} + P_r(q; b), \\ P_r(x; b) &\stackrel{\mathrm{def}}{=} b\sum_{j=1}^{\nu}(x^{(j)})^{2r}, \quad x \in \mathbb{R}^\nu, \end{aligned} \tag{1.1.142}$$

where $b \in (0, b_V]$ and H^{har} is as in (1.1.91). In view of Proposition 1.1.43, S is essentially self-adjoint on $C_0^\infty(\mathbb{R}^\nu)$ and self-adjoint on

$$\mathrm{Dom}(S) = \mathrm{Dom}(H^{\mathrm{har}}) \cap \mathcal{A}^{P_r}.$$

Let ϕ_k be the (unique) normalized eigenfunction of T corresponding to the eigenvalue Θ_k, $k \in \mathbb{N}_0$. Then

$$\phi_n = \phi_{n_1} \otimes \phi_{n_2} \otimes \cdots \otimes \phi_{n_\nu}, \quad n \in \mathbb{N}_0^\nu$$

is the eigenfunction of S corresponding to the eigenvalue

$$\Sigma_n = \Theta_{n_1} + \cdots + \Theta_{n_\nu}.$$

The growth of Σ_n, as $|n| \to +\infty$, is described by Lemma 1.1.53. By Theorem 1.1.51 we have that each eigenfunction ψ_k of $\widetilde{H}$ belongs to $\mathrm{Dom}(S)$. In view of (1.1.10), we have that for all $x \in \mathbb{R}^\nu$,

$$G(x) \stackrel{\mathrm{def}}{=} V(x) - P_r(x; b) + c_V \geq 0. \tag{1.1.143}$$

Let $\{\psi_k\}_{k \in \mathbb{N}_0}$ be an orthonormal basis of $L^2(\mathbb{R}^\nu)$ consisting of eigenfunctions of $\widetilde{H}$. Given $k \in \mathbb{N}_0$, let $\mathcal{H}_k$ be the linear span of $\psi_0, \dots, \psi_k$ and $\mathcal{I}_k = \{\phi \in \mathcal{H}_k \mid \|\phi\|_{L^2(\mathbb{R}^\nu)} = 1\}$. Then by (1.1.117) one obtains that for any $n \in \mathbb{N}_0^\nu$, such that $|n| \leq k$,

$$\Sigma_n \leq \max_{\phi \in \mathcal{I}_k} (\phi, S\phi)_{L^2(\mathbb{R}^\nu)}. \tag{1.1.144}$$

But by (1.1.143)

$$(\phi, S\phi)_{L^2(\mathbb{R}^\nu)} = (\phi, H\phi)_{L^2(\mathbb{R}^\nu)} + c_V - (\phi, G(q)\phi)_{L^2(\mathbb{R}^\nu)} \leq (\phi, H\phi)_{L^2(\mathbb{R}^\nu)} + c_V,$$

which immediately yields in (1.1.144)

$$\Sigma_n \leq E_k + c_V.$$

Thereafter, the stated property follows by Lemma 1.1.53. □

Anharmonic oscillator: tunneling and gap estimates

Now let us consider the case where $\nu = 1$ and $V(x)$ is even. Figure 1.1 presents an example of the corresponding potential $(a/2)x^2 + V(x)$. Such double well potentials are typical for hydrogen bounds (see Introduction), e.g., in KDP-type ferroelectrics.

The dynamics of oscillators with convex potentials and with potentials like the one depicted in Figure 1.1 are qualitatively different. For quantum oscillators, however, this difference is not so essential due to a purely quantum effect, called tunneling. If the energy of the classical oscillator is lower than the height of the barrier separating the wells, the oscillator is forced to oscillate within the well. By symmetry, such oscillations in the opposite well correspond to the same energy. Thus, the classical oscillator can be in two different states with the same energy, which for its quantum counterpart is impossible as each of the corresponding eigenvalues is simple, see Theorem 1.1.50. Therefore, the quantum oscillator leaves the wells, even if its energy is lower than the barrier. This phenomenon can be interpreted in the following way. As we know,

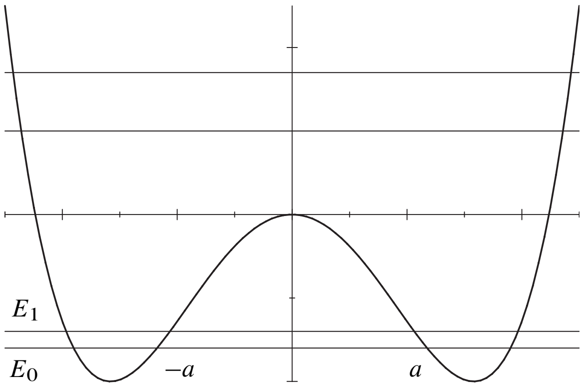

Figure 1.1. Double well potential.

the localization of a quantum particle is a random event, see (1.1.5), whose probability distribution is expressed in terms of the corresponding wave function. Thus, if the state (the eigenfunction of H) is, say, ψ_0, and the corresponding energy (the eigenvalue of H) is E_0, the probability to find the particle in a Borel subset B is given by (1.1.5) with this ψ_0. If one takes $B = [-a, a]$, see Figure 1.1, this probability could be positive if $|\psi_0(x)|^2$ were essentially positive in this interval. The point is that *it is positive*; thus, the particle can enter the area which is forbidden for its classical counterpart. In fact, $\psi_0(x)$ has an oscillatory character for x such that $E_0 - [(a/2)x^2 + V(x)] \geq 0$, and decreases otherwise – the faster, the bigger the difference $[(a/2)x^2 + V(x)] - E_0$ is, see page 165 in [305]. But if this difference is not too big and the interval $[-a, a]$ is not too long, the probability expressed by the integral in (1.1.5) is positive. This motion between the wells in low energy states is called *tunneling*. Due to it the degeneracy occurring in the classical case is lifted, which manifests itself in the *tunneling splitting* of the degenerate eigenvalue E_0 into two simple eigenvalues E_0 and E_1, which are close to each other. The gap parameter $\Delta = E_1 - E_0$ is called the *tunneling frequency*. It is small if the parameter m is big. This means that the tunneling disappears in the macro-scale limit $m_{\mathrm{ph}} \to +\infty$, as well as in the classical limit $\hbar \to 0$, see (1.1.7). In quantum anharmonic crystals, tunneling plays a very important role in the quantum effects studied in Chapter 7, where we also describe how Δ tends to zero in the limit $m \to +\infty$. As we show there, the gap parameter is responsible for the *quantum stabilization*, the theory of which is developed in Section 7.1.

Until the end of this subsection, for $\nu = 1$ and

$$H = H^{\mathrm{har}} + W(q) + b_r q^{2r} = \frac{1}{2m} p^2 + \frac{a}{2} q^2 + W(q) + b_r q^{2r}, \qquad (1.1.145)$$

we study the dependence of the gap parameter on the mass m. Here $b_r = b_V$, b_V and r

are as in (1.1.10), and

$$W(q) = \sum_{s=1}^{r-1} b_s q^{2s}, \quad b_s \in \mathbb{R}, \text{ for all } s = 2, \ldots, 2r-1. \tag{1.1.146}$$

As we know, H is self-adjoint on $\mathrm{Dom}(T)$ given by (1.1.124), see Theorem 1.1.56. Its eigenvalues E_n, $n \in \mathbb{N}_0$, are described in Theorem 1.1.58. More precise information about them is given in the following

Theorem 1.1.59. *The eigenvalues of the operator* (1.1.145) *have the property*

$$\lim_{n\to+\infty}(E_{n+1} - E_n) = +\infty.$$

Proof. For a certain positive $c < +\infty$, let

$$g : \mathbb{R} \to [0, c], \quad g(x) = g(-x), \tag{1.1.147}$$

be an infinitely differentiable function, obeying the conditions:

$$\begin{aligned} &\text{(a)} \quad g(0) = 0, \\ &\text{(b)} \quad \sup_{x\in\mathbb{R}} |g'(x)| < \infty, \\ &\text{(c)} \quad \forall x \in \mathbb{R}: \quad g''(x) \geq -a - W''(x) - 2r(2r-1)b_r x^{2(r-1)}. \end{aligned} \tag{1.1.148}$$

The latter condition guarantees that

$$U''(x) \geq 0, \tag{1.1.149}$$

where

$$U(x) \stackrel{\text{def}}{=} g(x) + \frac{1}{2}ax^2 + W(x) + b_r x^{2r}.$$

Now we set

$$T = \frac{1}{2m}p^2 + U(q), \tag{1.1.150}$$

and let ϕ_n and Θ_n, $n \in \mathbb{N}_0$, be the eigenfunctions and the corresponding eigenvalues of T. By ψ_n, $n \in \mathbb{N}_0$, we denote the eigenfunctions of (1.1.145). In view of Theorem 1.1.57,

$$\mathrm{Dom}(H) = \mathrm{Dom}(T),$$

which means that both sets $\{\phi_n\}_{n\in\mathbb{N}_0}$ and $\{\psi_n\}_{n\in\mathbb{N}_0}$ can be used as trial functions for T and $H + cI$, see Proposition 1.1.49 and Theorem 1.1.47. As g is positive,

$$\forall \phi \in \mathrm{Dom}(H): \quad (\phi, H\phi)_{L^2(\mathbb{R})} \leq (\phi, T\phi)_{L^2(\mathbb{R})}.$$

Hence, by (1.1.117) one obtains, cf. (1.1.144),

$$E_n \leq \Theta_n, \quad n \in \mathbb{N}_0. \tag{1.1.151}$$

On the other hand, by the same bound (1.1.117)

$$\begin{aligned}\Theta_n &\le \max\left\{(\phi, T\phi)_{L^2(\mathbb{R})} \mid \phi \in \mathcal{K}_n,\ \ \|\phi\|_{L^2(\mathbb{R})} = 1\right\}\\ &\le c + \max\left\{(\phi, H\phi)_{L^2(\mathbb{R})} \mid \phi \in \mathcal{K}_n,\ \ \|\phi\|_{L^2(\mathbb{R})} = 1\right\}\\ &= c + E_n,\end{aligned} \tag{1.1.152}$$

where $\mathcal{K}_n$ is the linear span of $\{\psi_0, \dots, \psi_n\}$. Then the stated property will follow if we show that

$$\lim_{n\to+\infty} (\Theta_{n+1} - \Theta_n) = +\infty. \tag{1.1.153}$$

As the function U is convex, see (1.1.149), for any $\theta > 0$ the equation

$$\theta = U(x) \tag{1.1.154}$$

has a unique solution on $\mathbb{R}_+$, which we denote by $x(\theta)$. Clearly, $x(\theta)$ is increasing, concave, and differentiable. Furthermore, $b_r[x(\theta)]^{2r}/\theta \to 1$, that is,

$$x(\theta) = O\big(\theta^{1/2r}\big), \quad \text{as } \theta \to +\infty, \tag{1.1.155}$$

and hence

$$x'(\theta) = \frac{1}{U'(x(\theta))} = O\big(\theta^{-1+1/2r}\big). \tag{1.1.156}$$

Now for $n \in \mathbb{N}$, we set $x_n = x(\Theta_n)$. By Theorem 1.1.58 $\Theta_n \ge \Theta_* n^{2r/(r+1)}$ for all $n \in \mathbb{N}_0$ and a certain $\Theta_* > 0$. In addition to the convexity, the function U is three times differentiable, such that for $k = 1, 2, 3$,

$$\frac{U^{(k)}(x)}{U^{(k-1)}(x)} = O\left(\frac{1}{x}\right), \quad x \to \infty, \tag{1.1.157}$$

cf. (7.8.1) in [305], page 151. Therefore, by a modification of the method used in the proof of Lemma 1.1.53 one can prove that

$$\frac{\sqrt{8m}}{\pi} \int_0^{x_n} [\Theta_n - U(x)]^{1/2}\, \mathrm{d}x = n + 1/2 + O\left(\frac{1}{n}\right), \tag{1.1.158}$$

see (7.7.4) in [305], page 151. It is worthwhile to note that the error term here tends to zero as $n \to +\infty$. Thereby,

$$\begin{aligned}I_n &\stackrel{\text{def}}{=} \int_0^{x_{n+1}} [\Theta_{n+1} - U(x)]^{1/2}\, \mathrm{d}x - \int_0^{x_n} [\Theta_n - U(x)]^{1/2}\, \mathrm{d}x\\ &= \frac{\pi}{\sqrt{8m}} + O\left(\frac{1}{n}\right).\end{aligned} \tag{1.1.159}$$

It can also be written as $I_n = J_n + K_n$, where

$$\begin{aligned}J_n &= \int_0^{x_n} \frac{[\Theta_{n+1} - \Theta_n]\, \mathrm{d}x}{[\Theta_{n+1} - U(x)]^{1/2} + [\Theta_n - U(x)]^{1/2}},\\ K_n &= \int_{x_n}^{x_{n+1}} [\Theta_{n+1} - U(x)]^{1/2}\, \mathrm{d}x.\end{aligned} \tag{1.1.160}$$

Now for $n \in \mathbb{N}$, we set $\varphi_n(t) = U(tx_n)/\Theta_n$. This function is convex. Furthermore, $\varphi_n(0) = 0$, $\varphi_n(1) = 1$, and $\varphi_n : [0, 1] \to [0, 1]$. Therefore, $\varphi_n(t) \leq t$, which implies

$$\begin{aligned} J_n &= \frac{x_n\left[\Theta_{n+1} - \Theta_n\right]}{\sqrt{\Theta_n}} \int_0^1 \frac{\mathrm{d}t}{\left[\Theta_{n+1}/\Theta_n - \varphi_n(t)\right]^{1/2} + \left[1 - \varphi_n(t)\right]^{1/2}} \\ &\leq \frac{x_n\left[\Theta_{n+1} - \Theta_n\right]}{2\sqrt{\Theta_n}} \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t}} = \frac{x_n\left[\Theta_{n+1} - \Theta_n\right]}{\sqrt{\Theta_n}}. \end{aligned}$$

At the same time, as $x(\theta)$ is concave, $x'(\theta) \leq x'(\Theta_n)$ for $\theta \geq \Theta_n$. Hence,

$$\begin{aligned} K_n &= \int_{\Theta_n}^{\Theta_{n+1}} \left[\Theta_{n+1} - \theta\right]^{1/2} x'(\theta)\mathrm{d}\theta \\ &\leq x'(\Theta_n) \int_{\Theta_n}^{\Theta_{n+1}} \left[\Theta_{n+1} - \theta\right]^{1/2} \mathrm{d}\theta \\ &= \frac{2}{3} x'(\Theta_n)\left[\Theta_{n+1} - \Theta_n\right]^{3/2}. \end{aligned}$$

Now suppose that the sequence $\Theta_{n+1} - \Theta_n$, $n \in \mathbb{N}_0$, contains a bounded subsequence. That is, there exists a sequence of integers $\{n_k\}_{k\in\mathbb{N}_0}$, tending to $+\infty$, such that

$$\Theta_{n_k+1} - \Theta_{n_k} \leq \theta_* \tag{1.1.161}$$

for all $k \in \mathbb{N}_0$ and some finite $\theta_* > 0$. Then the above estimates and (1.1.161) yield $J_{n_k} \to 0$, see (1.1.155), and $K_{n_k} \to 0$, see (1.1.156), which contradicts (1.1.159). □

It is easy to see that in Theorem 1.1.59 the concrete form of the anharmonic potential in (1.1.145) was not especially important. The only properties we needed were its three-times differentiability, the asymptotic properties (1.1.157), and the concavity of $x(\theta)$. The latter is readily guaranteed by the estimate (1.1.10), according to which the potential energy of (1.1.145) is 'asymptotically convex'. Concerning the choice of the function (1.1.147), it has to be convex enough in the vicinity of the origin in order to compensate the eventual non-convexity of the potential energy. Far from the origin it should be slightly concave and increasing to c as $x \to \pm\infty$. An example here can be $g(x) = c[1 - \exp(-\alpha^2 x^2)]$ with sufficiently small $\alpha > 0$ and big enough c.

Now let us turn to the study of the dependence of the eigenvalues of (1.1.145), and hence of the gap parameter, on the mass m. To visualize this dependence the corresponding quantities will be supplied with the subscript m. In particular, the Hamiltonian (1.1.145) is now written as H_m. Its gap parameter is

$$\Delta_m = \min_{n\in\mathbb{N}}(E_n - E_{n-1}). \tag{1.1.162}$$

In the next statement, by writing $f \sim g$ we mean that $\lim(f/g) = 1$.

Theorem 1.1.60. *For every integer $r \geq 2$, the gap parameter* (1.1.162) *of the Hamiltonian* (1.1.145) *is a continuous function of $m \in (0, +\infty)$, such that*

$$\Delta_m \sim \Delta_0 m^{-r/(r+1)}, \quad \textit{as } m \to 0, \tag{1.1.163}$$

with a certain $\Delta_0 > 0$.

The proof of this theorem is based on the analytic perturbation theory for self-adjoint operators and will be preceded by the introduction of the corresponding notions and facts. But before let us discuss some consequences of Theorem 1.1.60. In the harmonic case, we have, see (1.1.73) and (1.1.72),

$$\Delta_m^{\mathrm{har}} = \delta = \sqrt{a/m}; \quad \text{hence,} \;\; a = m[\Delta_m^{\mathrm{har}}]^2, \tag{1.1.164}$$

which agrees with (1.1.163). As we shall see below in Subsection 7.1.1, the parameter

$$\mathfrak{r}_m = m\Delta_m^2 \tag{1.1.165}$$

plays a crucial role in the description of quantum effects in quantum anharmonic crystals. Since in the harmonic case $\mathfrak{r}_m$ coincides with the oscillator rigidity, we will call it the *quantum rigidity* of the oscillator. By Theorem 1.1.60, $\mathfrak{r}_m$ is a continuous function of m and

$$\mathfrak{r}_m \sim \Delta_0^2 m^{-(r-1)/(r+1)}, \quad \text{as } m \to 0. \tag{1.1.166}$$

Hence, $\mathfrak{r}_m \to +\infty$, as $m \to 0$, since $r \geq 2$.

In the sequel, a *domain* will also mean a connected open subset of $\mathbb{C}^n$, with $n \in \mathbb{N}$ depending on the context. A typical example is the disc

$$\mathsf{D}_\varepsilon(\zeta) \stackrel{\mathrm{def}}{=} \{z \mid |z - \zeta| < \varepsilon\}.$$

Definition 1.1.61. Let $\mathsf{C} \subset \mathbb{C}$ be a domain. A map $\mathsf{C} \ni z \mapsto \psi_z \in \mathcal{H}$ is called analytic at $\zeta \in \mathsf{C}$ if there exist $\varepsilon > 0$ and a dense set $\mathcal{A} \subset \mathcal{H}$ such that for every $\phi \in \mathcal{A}$, the map $z \mapsto (\phi, \psi_z)_{\mathcal{H}} \in \mathbb{C}$ is analytic and bounded in $\mathsf{D}_\varepsilon(\zeta) \subset \mathsf{C}$. The map $\mathsf{C} \ni z \mapsto \psi_z \in \mathcal{H}$ is called analytic in C if it is analytic at every $\zeta \in \mathsf{C}$.

Definition 1.1.62. Let $\mathsf{C} \subset \mathbb{C}$ be a domain and for every $z \in \mathsf{C}$, let $T(z)$ be a closed operator with a non-void resolvent set. Then $\{T(z) \mid z \in \mathsf{C}\}$ is called an analytic family of type (A) in C if the following conditions are satisfied:

(a) all $T(z)$, $z \in \mathsf{C}$, have the same domain $\mathrm{Dom}(T(z)) = \mathcal{A}$;

(b) for each $\phi \in \mathcal{A}$, the map $\mathsf{C} \ni z \mapsto T(z)\phi \in \mathcal{H}$ is analytic in C.

If $\mathsf{C} = \mathbb{C}$, then $\{T(z) \mid z \in \mathsf{C}\}$ is called an entire family of type (A).

Remark 1.1.63. An analytic family of type (A) has a compact resolvent for all $z \in \mathsf{C}$ or for no $z \in \mathsf{C}$, see Theorem 2.4 on page 377 of [172].

For the families defined by the series $T(z) = T + zQ_1 + z^2 Q_2 + \cdots$, a criterion of analyticity is presented by another Kato–Rellich theorem, see Theorem 2.6 on page 377 of [172] and also Lemma on page 16 of [257]. We recall that the relative boundedness of operators was introduced in Definition 1.1.54.

Proposition 1.1.64. *Let T be a closed linear operator with a non-void resolvent set, and $Q_n : \mathcal{H} \to \mathcal{H}$, $n \in \mathbb{N}$, be linear operators, such that* $\mathrm{Dom}(Q_n) \supset \mathrm{Dom}(T)$. *Suppose also that there exist positive a, b, and c, such that*

$$\forall \psi \in \mathrm{Dom}(T),\ n \in \mathbb{N}: \quad \|Q_n \psi\|_{\mathcal{H}} \le c^{n-1} \left(a\|T\psi\|_{\mathcal{H}} + b\|\psi\|_{\mathcal{H}}\right). \tag{1.1.167}$$

Then for $|z| < 1/c$, the series

$$T(z) = T + zQ_1 + z^2 Q_2 + \cdots \tag{1.1.168}$$

defines an operator with $\mathrm{Dom}(T(z)) = \mathrm{Dom}(T)$. *For every $z \in \mathsf{D}_{1/(a+c)}(0)$, the operator $T(z)$ is closable and the closures $\widetilde{T}(z)$ constitute an analytic family of type* (A) *in* $\mathsf{D}_{1/(a+c)}(0)$.

Definition 1.1.65. An analytic family of type (A) is called self-adjoint if: (a) its domain C is symmetric with respect to the real axis; (b) $T(z)$ is densely defined for each $z \in \mathsf{C}$; (c) $[T(z)]^* = T(\bar{z})$, which in particular means that $T(z)$ is self-adjoint for real z.

Yet another Kato–Rellich theorem describes the eigenvalues of such families, see Theorem 3.9 on page 392 in [172].

Proposition 1.1.66. *Let $T(z)$, $z \in \mathsf{C}$, be a self-adjoint analytic family of type* (A). *Suppose that C contains an interval $\mathsf{I} \subset \mathbb{R}$, and let $T(z)$, $z \in \mathsf{C}$, have compact resolvent. Then there exists a sequence of scalar-valued functions $\lambda_n(z)$ and a sequence of vector-valued functions $\psi_n(z)$, all analytic in the vicinity of I, such that for $z \in \mathsf{I}$, the $\lambda_n(z)$'s represent all the repeated eigenvalues of $T(z)$ and the $\psi_n(z)$'s form a complete orthonormal family of the associated eigenvectors of $T(z)$.*

Remark 1.1.67. In Proposition 1.1.66, each $\lambda_n(z)$ is analytic in a domain, $\mathsf{C}_n \supset \mathsf{I}$, which may depend on n. Thus, for sure, only finite families of $\lambda_n(z)$ have common domains of analyticity.

Proof of Theorem 1.1.60. Given $n \in \mathbb{N}_0$, we set

$$\mathfrak{d}(m; n) = E_{n+1} - E_n,$$

where m is the mass. Then according to Theorem 1.1.59,

$$\Delta_m = \mathfrak{d}(m; k_1) = \cdots = \mathfrak{d}(m; k_s), \tag{1.1.169}$$

for a certain $s \in \mathbb{N}$ and $k_1, \ldots, k_s \in \mathbb{N}_0$. If each E_n is a continuous function of m, then so is each $\mathfrak{d}(m; n)$. Therefore, if m' is close to m, then

$$\Delta_{m'} = \min\{\mathfrak{d}(m'; k_1), \ldots, \mathfrak{d}(m'; k_s)\},$$

and hence is close to Δ_m. Thus, to complete the proof of the continuity of Δ_m we have to establish the continuity of the eigenvalues, which will be done by means of Proposition 1.1.66.

Given $\alpha > 0$, let $U_\alpha \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unitary operator

$$(U_\alpha \psi)(x) = \sqrt{\alpha}\psi(\alpha x). \tag{1.1.170}$$

By (1.1.49)

$$U_\alpha^{-1} p U_\alpha = \alpha p, \quad U_\alpha^{-1} q U_\alpha = \alpha^{-1} q.$$

For a fixed m_0 and a given m, we set

$$\alpha = \rho^{1/2} = (m/m_0)^{1/2(r+1)}.$$

Then

$$U_\alpha^{-1} H_m U_\alpha \stackrel{\text{def}}{=} \widehat{H}_m = \rho^{-r} T(\rho - 1), \tag{1.1.171}$$

where

$$\begin{aligned} T(\rho-1) &= H_{m_0} + Q(\rho-1, q) \\ &= \frac{1}{2m_0} p^2 + \rho^{r-1}(b_1 + a/2)q^2 + \rho^{r-2} b_2 q^4 + \cdots + b_r q^{2r}, \end{aligned} \tag{1.1.172}$$

$$\begin{aligned} Q(z,q) = z\big[&p_{r-1}(z)(b_1 + a/2)q^2 \\ &+ p_{r-2}(z) b_2 q^4 + \cdots + p_{r-s}(z) b_s q^{2s} + \cdots + b_{r-1} q^{2(r-1)}\big], \end{aligned}$$

and

$$p_k(z) = \sum_{l=1}^{k-1} \frac{k!}{l!(k-l)!} z^{l-1}.$$

Therefore,

$$Q(z,q) = z Q_1(q) + \cdots + z^{r-1} Q_{r-1}(q),$$

where every $Q_s(q)$ is an even real polynomial in q, such that $\deg Q_s = 2(r-s)$, $s = 1, \ldots, r-1$. By Theorem 1.1.56 H_{m_0} is self-adjoint on

$$\mathrm{Dom}(H_{m_0}) = \mathcal{W}^{2,2}(\mathbb{R}) \cap \mathrm{Dom}(q^{2r}).$$

Since each $Q_s(x)$ obeys (1.1.140) with arbitrarily small α and big enough β, as in the proof of Theorem 1.1.56 one shows that for any $\varepsilon > 0$, there exists a $\delta > 0$, big enough, such that for all $s = 1, 2, \ldots, r-1$ and $\phi \in \mathrm{Dom}(H_{m_0})$,

$$\|Q_s(q)\phi\|_{L^2(\mathbb{R})} \le \left(\frac{\varepsilon}{2}\right)^{s-1} \left[\left(\frac{\varepsilon}{2}\right) \|H_{m_0}\phi\|_{L^2(\mathbb{R})} + \delta \|\phi\|_{L^2(\mathbb{R})}\right].$$

Then by Proposition 1.1.64, $T(z)$, $z \in \mathsf{D}_{1/\varepsilon}(0)$, with $T(z)$ given by (1.1.172), is an analytic family of type (A) since each $T(z)$ is already closed. This in particular means that the resolvent set of each $T(z)$ is non-void, see Definition 1.1.62. As $T(0) = H_{m_0}$

has compact resolvent, then so does each $T(z)$, $z \in \mathsf{D}_{1/\varepsilon}(0)$, see Remark 1.1.63. In what follows, the family $T(z)$, $z \in \mathsf{D}_{1/\varepsilon}(0)$, is self-adjoint, see Definition 1.1.65, and hence satisfies the conditions of Proposition 1.1.66. Therefore, for any $k \in \mathbb{N}$, one finds a domain $\mathsf{C}_k \subset \mathsf{D}_{1/\varepsilon}(0)$, such that the eigenvalues $\lambda_0(z), \ldots, \lambda_k(z)$ of $T(z)$ are analytic in C_k, and hence real and continuous at zero as functions of $\rho - 1$. In view of (1.1.171), for $\rho \in (1-\epsilon, 1+\epsilon)$, each $\rho^{-r}\lambda_n(\rho - 1)$ is an eigenvalue of $\widehat{H}_m$, and hence of H_m, as these operators are unitary equivalent. This yields the continuity in question.

To prove (1.1.163) we rewrite (1.1.171) as

$$\widehat{H}_m = \rho^{-r} S(\rho), \quad S(\rho) = H_{m_0}^{(0)} + Q(\rho, q), \tag{1.1.173}$$

where as above $\rho = (m/m_0)^{1/(r+1)}$. But

$$\begin{aligned} Q(\rho, q) &= \rho\big(\rho^{r-2}(b_1 + a/2)q^2 + \rho^{r-3} b_2 q^4 + \cdots + b_{r-1} q^{2(r-1)}\big), \\ H_{m_0}^{(0)} &= \frac{1}{2m_0} p^2 + b_r q^{2r}. \end{aligned}$$

Repeating the above perturbation arguments one concludes that the family $S(z)$ is self-adjoint and analytic near zero. Hence, the gap parameter (1.1.169) tends, as $\rho \to 0$, to that of the operator $H_{m_0}^{(0)}$, which we denote by $\varDelta_0$. Thereby, the asymptotics (1.1.163) follows by (1.1.173) and by the unitary equivalence of $\widehat{H}_m$ and H_m. □

1.2 Local Gibbs States

In this section, we study the properties of finite collections of interacting anharmonic oscillators, which includes also their thermodynamic states. In Subsection 1.2.1, we introduce and study local Hamiltonians H_Λ, which are the Schrödinger operators of sets of interacting quantum anharmonic oscillators, indexed by finite $\Lambda \subset \mathbb{L}$. This study is based on the theory of single quantum oscillators developed above. By means of H_Λ we introduce local Gibbs states $\varrho_{\beta,\Lambda}$. As these states are normal, in Subsection 1.2.2 we present elements of the theory of such states on von Neumann algebras. Here we introduce the notion of a *complete family of multiplication operators*, which is important for representing the states $\varrho_{\beta,\Lambda}$ via path integrals. In Subsection 1.2.3, we prove Høegh-Krohn's theorem according to which the states $\varrho_{\beta,\Lambda}$ are determined by their values on the operators evolving from complete families. Such values, considered as functions of time, are called Green functions. They, and their extensions to complex values of time variables, are studied in Subsection 1.2.4. The main result here is Theorem 1.2.32, by which the states $\varrho_{\beta,\Lambda}$ are determined by Matsubara functions, constructed for multiplication operators belonging to a complete family. Such functions can also be defined by path integrals. In Subsection 1.2.5, this is done for noninteracting systems of harmonic oscillators, which in the subsequent sections is extended to finite systems of interacting anharmonic oscillators.

1.2.1 Local Hamiltonians and Gibbs States

In the sequel, the adjective ‘local’ characterizes a property, related to a finite $\Lambda \subset \mathbb{L}$, whereas ‘global’ will always refer to the whole model, i.e., to $\Lambda = \mathbb{L}$. Cases of infinite $\Lambda \subsetneq \mathbb{L}$ are indicated explicitly. The family of all non-void finite subsets $\Lambda \subset \mathbb{L}$ (respectively, all subsets) will be denoted by $\mathfrak{L}_{\text{fin}}$ (respectively, by $\mathfrak{L}$). For $\Lambda \in \mathfrak{L}_{\text{fin}}$, the *local physical Hilbert space* is

$$\mathcal{H}_\Lambda = L^2(\mathbb{R}^{\nu|\Lambda|}) = \bigotimes_{\ell\in\Lambda} \mathcal{H}_\ell. \tag{1.2.1}$$

The definitions of the spaces $C_0^\infty(\mathbb{R}^{\nu|\Lambda|})$ and $\mathcal{W}^{2,2}(\mathbb{R}^{\nu|\Lambda|})$ are obvious extensions of those given in Subsection 1.1.3. In $\mathcal{H}_\Lambda$, the operators $q_\ell^{(j)}$ and $p_\ell^{(j)}$ are defined as the corresponding tensor products of the operators given by (1.1.49) and the identity operators. Likewise,

$$H_\Lambda^{(0)} \stackrel{\text{def}}{=} \sum_{\ell\in\Lambda} H_\ell, \tag{1.2.2}$$

with

$$\mathrm{Dom}(H_\Lambda^{(0)}) = \mathcal{W}^{2,2}(\mathbb{R}^{|\nu|\Lambda|}) \cap \mathcal{A}_\Lambda. \tag{1.2.3}$$

Here, cf, (1.1.99),

$$\mathcal{A}_\Lambda \stackrel{\text{def}}{=} \left\{\psi \in \mathcal{H}_\Lambda \mid \textstyle\int_{\mathbb{R}^{\nu|\Lambda|}} \prod_{\ell\in\Lambda} [V_\ell(x_\ell)]^2 \, |\psi(x)|^2 \mathrm{d}x < \infty\right\}. \tag{1.2.4}$$

As was already mentioned, the Hamiltonian (1.1.8) has no direct mathematical meaning and is usually ‘represented’ by *local Hamiltonians*. For $\Lambda \in \mathfrak{L}_{\text{fin}}$, we set

$$\begin{aligned} H_\Lambda &= -\frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda} H_\ell \\ &= \frac{1}{2m} \sum_{\ell\in\Lambda} |p_\ell|^2 + \Big(\sum_{\ell\in\Lambda} \frac{a}{2} |q_\ell|^2 + V_\ell(q_\ell) - \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) \Big). \end{aligned} \tag{1.2.5}$$

In the latter line, the multiplication operator in $(\ldots)$ is the potential energy of the system of oscillators located in Λ. Since the corresponding function is below bounded, continuous, and tending to $+\infty$ as $\sum_{\ell\in\Lambda} |x_\ell|^2 \to +\infty$, H_Λ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^{\nu|\Lambda|})$, see Proposition 1.1.43. From now on, by the Hamiltonian H_Λ we mean the self-adjoint operator being the closure of (1.2.5) as defined on $C_0^\infty(\mathbb{R}^{\nu|\Lambda|})$. Exactly as in Theorem 1.1.50, one can prove that the spectrum of the Hamiltonian H_Λ is discrete with no accumulation points other than infinity. Its more detailed description is given in the statement below, where r and c_V are as in (1.1.10).

Theorem 1.2.1. *The eigenvalues E_s, $s \in \mathbb{N}_0$, of H_Λ obey the estimate*

$$\forall s \geq s_\Lambda : \quad E_s \geq \epsilon_\Lambda s, \tag{1.2.6}$$

with certain $\epsilon_\Lambda > 0$ *and* $s_\Lambda \in \mathbb{N}$. *If all* V_ℓ *simultaneously are infinitely differentiable or obey* (1.1.113), *then*

$$\forall s \geq s_\Lambda : \quad E_s \geq \epsilon_\Lambda s^{2r/(r+1)}, \tag{1.2.7}$$

with appropriate $\epsilon_\Lambda > 0$ *and* $s_\Lambda \in \mathbb{N}$, *not necessarily the same as in* (1.2.6). *If* V_ℓ *are as in* (1.1.113), *then also*

$$\mathrm{Dom}(H_\Lambda) = \mathcal{W}^{2,2}(\mathbb{R}^{|\nu|\Lambda|}) \cap \mathcal{A}_\Lambda. \tag{1.2.8}$$

Proof. We prove only (1.2.7) as the proof of (1.2.6) is based on the same arguments and Theorem 1.1.50. Given $\ell \in \Lambda$, by S_ℓ we denote the tensor product of the identity operators and a copy of the operator (1.1.142) with $b = b_V/2$, standing at ℓ-th position. For $n_\ell \in \mathbb{N}_0^\nu$, let Σ_{n_ℓ} be its eigenvalue. Set

$$S_\Lambda = \sum_{\ell \in \Lambda} S_\ell.$$

Then the eigenvalues of S_Λ are

$$\Sigma_{n_\Lambda} = \sum_{\ell \in \Lambda} \Sigma_{n_\ell}, \quad n_\Lambda = (n_\ell)_{\ell \in \Lambda} \in \mathbb{N}_0^{\nu|\Lambda|}. \tag{1.2.9}$$

For each $\ell \in \Lambda$, we pick $c_\ell \geq c_V$ in such a way that

$$\begin{aligned} \forall x_\Lambda \in \mathbb{R}^{\nu|\Lambda|} : \quad W_\Lambda(x_\Lambda) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} \frac{1}{2} b_V \left(x_\ell^{(j)}\right)^{2r} + \sum_{\ell \in \Lambda} (c_\ell - c_V) \\ - \frac{1}{2} \sum_{\ell, \ell' \in \Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'}) \geq 0. \end{aligned} \tag{1.2.10}$$

Let $G_\ell(x_\ell)$ be as in (1.1.143) with $V = V_\ell$ and $b = b_V$. Then

$$\forall x_\Lambda \in \mathbb{R}^{\nu|\Lambda|} : \quad F_\Lambda(x_\Lambda) \stackrel{\text{def}}{=} \sum_{\ell \in \Lambda} G_\ell(x_\ell) + W_\Lambda(x_\Lambda) \geq 0. \tag{1.2.11}$$

Let ψ_s, $s \in \mathbb{N}_0$, be the eigenfunctions of H_Λ. As each V_ℓ obeys the conditions of Theorem 1.1.51, these eigenfunctions belong to the domain of the sum on the right-hand side of (1.2.5). Hence, $\psi_s \in \mathrm{Dom}(S_\Lambda)$ for all $s \in \mathbb{N}_0$. Let $\mathcal{H}_s$ be the linear span of $\psi_0, \ldots, \psi_s$ and $\mathcal{I}_s = \{\phi \in \mathcal{H}_s \mid \|\phi\|_{\mathcal{H}_\Lambda} = 1\}$. Then by (1.1.117) one gets that, for any n_Λ, such that $|n_\Lambda| \leq s$,

$$\Sigma_{n_\Lambda} \leq \max_{\phi \in \mathcal{I}_s} (\phi, S_\Lambda \phi)_{\mathcal{H}_\Lambda}.$$

At the same time, for $\phi \in \mathcal{I}_s$, one has

$$\begin{aligned} (\phi, S_\Lambda \phi)_{\mathcal{H}_\Lambda} &= (\phi, H_\Lambda \phi)_{\mathcal{H}_\Lambda} + \sum_{\ell \in \Lambda} c_\ell - (\phi, F_\Lambda(q_\Lambda)\phi)_{\mathcal{H}_\Lambda} \\ &\leq (\phi, H_\Lambda \phi)_{\mathcal{H}_\Lambda} + \sum_{\ell \in \Lambda} c_\ell \\ &\leq E_s + \sum_{\ell \in \Lambda} c_\ell. \end{aligned}$$

Then the proof follows from (1.2.9) and Theorem 1.1.58. □

Directly from (1.2.6) one gets the following

Corollary 1.2.2. *For any $\beta > 0$, the operator* $\exp(-\beta H_\Lambda)$ *is of trace-class.*

Set $\mathfrak{C}_\Lambda = \mathfrak{C}(\mathcal{H}_\Lambda)$. The elements of $\mathfrak{C}_\Lambda$ are called *observables*. In the definition below, $T > 0$ is absolute temperature and k_B is Boltzmann's constant. A temperature state on $\mathfrak{C}_\Lambda$ is defined as a mixture of (pure) vector states $\mathfrak{C}_\Lambda \ni A \mapsto (\psi_s, A\psi_s)_{\mathcal{H}_\Lambda}$ as follows, cf. Definitions 1.1.12 and 1.1.13.

Definition 1.2.3. For $\Lambda \in \mathfrak{L}_{\rm fin}$ and a given inverse temperature $\beta = 1/k_{\rm B}T$, the local Gibbs state $\varrho_{\beta,\Lambda}$ on $\mathfrak{C}_\Lambda$ is

$$\varrho_{\beta,\Lambda}(A) = \frac{\mathrm{trace}\,[A\exp(-\beta H_\Lambda)]}{\mathrm{trace}\,[\exp(-\beta H_\Lambda)]}, \quad A \in \mathfrak{C}_\Lambda. \tag{1.2.12}$$

In the Heisenberg approach to quantum mechanics, the dynamics of the subsystem of oscillators located in Λ is described by the maps

$$\mathfrak{C}_\Lambda \ni A \mapsto \mathfrak{a}^t_\Lambda(A) = U^t_\Lambda A [U^t_\Lambda]^{-1}, \tag{1.2.13}$$

where $t \in \mathbb{R}$ is time and

$$U^t_\Lambda = \exp\left(\mathrm{i}tH_\Lambda\right). \tag{1.2.14}$$

As H_Λ is self-adjoint, by Proposition 1.1.30 we have easily

Proposition 1.2.4. *For each $t \in \mathbb{R}$, U^t_Λ is a unitary operator on $\mathcal{H}_\Lambda$ and $U^{t+s}_\Lambda = U^t_\Lambda U^s_\Lambda$ for all $t, s \in \mathbb{R}$. For any $\phi \in \mathcal{H}_\Lambda$ and $s \in \mathbb{R}$,*

$$U^t_\Lambda \phi \to U^s_\Lambda \phi, \quad \textit{as } t \to s.$$

By this statement, the maps (1.2.13) are norm-preserving, i.e., $\|\mathfrak{a}^t_\Lambda(A)\| = \|A\|$ for all $A \in \mathfrak{C}_\Lambda$. They are linear, bijective, and such that for all $A, B \in \mathfrak{C}_\Lambda$,

$$\mathfrak{a}^t_\Lambda(AB) = \mathfrak{a}^t_\Lambda(A)\mathfrak{a}^t_\Lambda(B), \quad \left(\mathfrak{a}^t_\Lambda(A)\right)^* = \mathfrak{a}^t_\Lambda(A^*). \tag{1.2.15}$$

The latter property means that each $\mathfrak{a}^t_\Lambda$ is a $*$-automorphism of the corresponding C^*-algebra. Moreover, for any $s, t \in \mathbb{R}$,

$$\mathfrak{a}^s_\Lambda\left(\mathfrak{a}^t_\Lambda(A)\right) = \mathfrak{a}^{s+t}_\Lambda(A), \tag{1.2.16}$$

and for $s = 0$, $\mathfrak{a}^s_\Lambda$ is the identity map. Therefore, the maps $A \mapsto \mathfrak{a}^t_\Lambda(A) \in \mathfrak{C}_\Lambda$, $t \in \mathbb{R}$, form a one-parameter group. This group, denoted by $\mathfrak{A}_\Lambda$, is called the *group of time automorphisms* which determine the *time evolution* of the elements of $\mathfrak{C}_\Lambda$.

Remark 1.2.5. The state (1.2.12) can be extended to some unbounded operators. Indeed, given a trace-class operator T, the operators AT and TA may be trace-class not only for bounded operators A. For example, if $T = \exp(-\beta H_\Lambda)$, $\beta > 0$ and $A = \exp(\lambda H_\Lambda)$, $\lambda \in \mathbb{C}$, the operator AT is trace-class provided $\Re(\lambda) < \beta$.

The set of all operators to which the above state $\varrho_{\beta,\Lambda}$ can be extended is denoted by $\overline{\mathfrak{C}}_{\beta,\Lambda}$. Since the trace is cyclic, cf. Proposition 1.1.25, that is,

$$\mathrm{trace}(ABC) = \mathrm{trace}(BCA) = \mathrm{trace}(CAB), \tag{1.2.17}$$

the state $\varrho_{\beta,\Lambda}$ has the property

$$\varrho_{\beta,\Lambda}(\mathfrak{a}^t_\Lambda(A)B) = \varrho_{\beta,\Lambda}(A\mathfrak{a}^{-t}_\Lambda(B)), \quad A, B \in \mathfrak{C}_\Lambda, \ t \in \mathbb{R}. \tag{1.2.18}$$

Taking the identity operator I as B one obtains

$$\varrho_{\beta,\Lambda}(\mathfrak{a}^t_\Lambda(A)) = \varrho_{\beta,\Lambda}(A). \tag{1.2.19}$$

Therefore, the local Gibbs states are invariant with respect to the time evolution; moreover, the time automorphisms can be extended to $\overline{\mathfrak{C}}_{\beta,\Lambda}$ and $\mathfrak{a}^t_\Lambda : \overline{\mathfrak{C}}_{\beta,\Lambda} \to \overline{\mathfrak{C}}_{\beta,\Lambda}$.

The fact that the local Gibbs state may be extended to the operators $\exp(\lambda H_\Lambda)$ with complex λ gives us the possibility to define the operators (1.2.14) and hence the time automorphisms also for complex $t = \theta + \mathrm{i}\tau$. For example, the operator

$$\mathfrak{a}^{\theta+\mathrm{i}\tau}_\Lambda(B) = \exp(-\tau H_\Lambda)\mathfrak{a}^\theta_\Lambda(B)\exp(\tau H_\Lambda),$$

with $B \in \mathfrak{C}_\Lambda$, $\theta \in \mathbb{R}$, and $\tau \in (0,\beta)$ is an element of $\overline{\mathfrak{C}}_{\beta,\Lambda}$. In what follows, the function

$$F_{A,B}(\theta + \mathrm{i}\tau) = \varrho_{\beta,\Lambda}(A\mathfrak{a}^{\theta+\mathrm{i}\tau}_\Lambda(B)), \quad A, B \in \mathfrak{C}_\Lambda, \tag{1.2.20}$$

can be defined on the strip $\{\theta + \mathrm{i}\tau \in \mathbb{C} \mid \theta \in \mathbb{R}, \ \tau \in [0,\beta)\}$.

Definition 1.2.6. Let $\mathfrak{A} = \{\mathfrak{a}^\theta \mid \theta \in \mathbb{R}\}$ be a one-parameter group of automorphisms and ω be a state on the algebra of observables $\mathfrak{C}_\Lambda$. Then ω is called *a β-Kubo–Martin–Schwinger (β-KMS) state* relative to the group $\mathfrak{A}$ if for any $A, B \in \mathfrak{C}_\Lambda$, there exists a function $F^\omega_{A,B}(z)$, analytic in the open strip $\{t = \theta + \mathrm{i}\tau \mid \theta \in \mathbb{R}, \ \tau \in (0,\beta)\}$ and continuous on its closure, which satisfies *the Kubo–Martin–Schwinger condition*

$$F^\omega_{A,B}(\theta) = \omega(A\mathfrak{a}^\theta(B)), \quad F^\omega_{A,B}(\theta + \mathrm{i}\beta) = \omega(\mathfrak{a}^\theta(B)A), \tag{1.2.21}$$

for all $\theta \in \mathbb{R}$.

Although the function (1.2.20) is defined on the strip $\{\theta + \mathrm{i}\tau \in \mathbb{C} \mid \theta \in \mathbb{R}, \ \tau \in [0,\beta)\}$, so far we know nothing about its analytic properties; hence, we do not know if the state (1.2.12) is a β-KMS state relative to the group $\mathfrak{A}_\Lambda$ of the automorphisms (1.2.13). If we knew this, by means of *the Gelfand–Naimark–Segal construction* (see e.g., page 46 of [244]), we would get a very important property of this state. It is *the multiple-time analyticity* (see Theorem 2.1 in [176]), which plays a key role in our theory. Below, in Theorem 1.2.32, we establish this analyticity directly and obtain the KMS property of $\varrho_{\beta,\Lambda}$ as a consequence.

1.2.2 Von Neumann Algebras and Normal States

Given two sets, X and Y, let $\mathcal{F}$ be a family of maps $f : X \to Y$. Often, it is useful to find a subset $X_0 \subset X$, such that every $f \in \mathcal{F}$ can be identified by its values on X_0. For example, a continuous function $f : \mathbb{R} \to \mathbb{R}$ can be identified by its values on a dense subset of $\mathbb{R}$, e.g., on the set of rational numbers. Another example is the set of functions of a single complex variable, holomorphic in a domain $D \subset \mathbb{C}$. Here as such a set one may take a subset of D with an accumulation point other than infinity. In the present subsection, we find subsets of $\mathfrak{C}_\Lambda$, which can be used for identifying local Gibbs states.

On the algebra $\mathfrak{C}_\Lambda$ along with the norm topology we will use the *strong, σ-weak, and σ-strong topologies*. Given $A \in \mathfrak{C}_\Lambda$ and $\psi \in \mathcal{H}_\Lambda$, we set $\|A\|_\psi = \|A\psi\|_{\mathcal{H}_\Lambda}$. Then $\|\cdot\|_\psi$ is a seminorm. The strong topology is defined as a locally convex topology on $\mathfrak{C}_\Lambda$, generated by the family of seminorms $\{\|\cdot\|_\psi \mid \psi \in \mathcal{H}_\Lambda\}$. A sequence of operators, $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}_\Lambda$, converges in this topology (*strongly* converges) to a certain $A \in \mathfrak{C}_\Lambda$ if for every $\psi \in \mathcal{H}_\Lambda$, the sequence $\{A_n\psi\}_{n\in\mathbb{N}} \subset \mathcal{H}_\Lambda$ converges in $\mathcal{H}_\Lambda$ to $A\psi$. This agrees with the definition of the strong convergence given above.

Let $\mathcal{F}_\Lambda$ be the set of sequences $\Psi = \{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{H}_\Lambda$, such that

$$\|\Psi\|^2_{\mathcal{F}_\Lambda} \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \|\psi_n\|^2_{\mathcal{H}_\Lambda} < \infty. \tag{1.2.22}$$

For $\Psi, \Phi \in \mathcal{F}_\Lambda$ and $A \in \mathfrak{C}_\Lambda$, we set

$$\|A\|_\Psi = \Big[\sum_{n=1}^{\infty} \|A\psi_n\|^2_{\mathcal{H}_\Lambda}\Big]^{1/2}, \quad \|A\|_{\Psi,\Phi} = \sum_{n=1}^{\infty} \big|(\phi_n, A\psi_n)_{\mathcal{H}_\Lambda}\big|. \tag{1.2.23}$$

Obviously, for any $A \in \mathfrak{C}_\Lambda$,

$$\|A\|_\Psi \le \|A\| \cdot \|\Psi\|_{\mathcal{F}_\Lambda}, \tag{1.2.24}$$

that is, $A\Psi \stackrel{\text{def}}{=} \{A\psi_n\}_{n\in\mathbb{N}} \in \mathcal{F}_\Lambda$ if $\Psi \in \mathcal{F}_\Lambda$. Both $\|\cdot\|_\Psi$ and $\|\cdot\|_{\Psi,\Phi}$ are seminorms on $\mathfrak{C}_\Lambda$. The locally convex topology on $\mathfrak{C}_\Lambda$ generated by the seminorms $\{\|\cdot\|_\Psi\}$ with all possible choices of $\Psi \in \mathcal{F}_\Lambda$ is called *the σ-strong topology* . Respectively, the locally convex topology on $\mathfrak{C}_\Lambda$ generated by the seminorms $\{\|\cdot\|_{\Psi,\Phi}\}$ with all possible choices of $\Psi, \Phi \in \mathcal{F}_\Lambda$ is called *the σ-weak topology*. A sequence $\{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}_\Lambda$ converges to a certain $A \in \mathfrak{C}_\Lambda$ in the σ-strong topology (respectively, in the σ-weak topology) if for any $\Psi \in \mathcal{F}_\Lambda$ (respectively, for any $\Psi, \Phi \in \mathcal{F}_\Lambda$), one has $\|A_n - A\|_\Psi \to 0$ (respectively, $\|A_n - A\|_{\Psi,\Phi} \to 0$), as $n \to +\infty$. The strong, σ-strong, and σ-weak topologies are weaker than the norm topology. Furthermore, the σ-strong topology is stronger than the strong and σ-weak topologies. At the same time, the latter two topologies are incomparable in general (see page 70 of [76]).

In our case, where the Hilbert space $\mathcal{H}_\Lambda$ is infinite-dimensional, multiplication $\mathfrak{C}_\Lambda \times \mathfrak{C}_\Lambda \ni (A, B) \mapsto AB \in \mathfrak{C}_\Lambda$ is not σ-strongly continuous. However, its restriction to bounded subsets of $\mathfrak{C}_\Lambda$ is continuous in the following sense, see Proposition 2.4.1 on page 67 of [76]. For $r > 0$, we set

$$\mathfrak{K}^r_\Lambda = \{A \in \mathfrak{C}_\Lambda \mid \|A\| \leq r\}. \tag{1.2.25}$$

Proposition 1.2.7. *For every $r > 0$, the topologies induced on $\mathfrak{K}^r_\Lambda$ by the strong and σ-strong topologies, respectively, coincide. The map $\mathfrak{K}^r_\Lambda \times \mathfrak{C}_\Lambda \ni (A, B) \mapsto AB \in \mathfrak{C}_\Lambda$ is σ-strongly continuous.*

In the sequel, we use the following corollaries of this statement. Recall that the automorphisms $\mathfrak{a}^t_\Lambda$ were introduced in (1.2.13) and (1.2.14).

Proposition 1.2.8. *For every $A \in \mathfrak{C}_\Lambda$, the map $\mathbb{R} \ni t \mapsto \mathfrak{a}^t_\Lambda(A) \in \mathfrak{C}_\Lambda$ is strongly and hence σ-strongly continuous.*

Proof. In view of the property (1.2.16), it is enough to prove the stated continuity at $t = 0$. For $\phi \in \mathcal{H}_\Lambda$, we set $\psi = A\phi$. Then

$$\|A - \mathfrak{a}^t_\Lambda(A)\|_\phi \leq \|(U^t_\Lambda - I)\psi\|_{\mathcal{H}_\Lambda} + \|A\| \cdot \|(U^{-t}_\Lambda - I)\phi\|_{\mathcal{H}_\Lambda}. \tag{1.2.26}$$

By Proposition 1.2.4, both terms can be made arbitrarily small by taking small enough $|t|$, which yields the strong continuity. The σ-strong continuity follows from this fact by Proposition 1.2.7. □

Proposition 1.2.9. *Given $k \in \mathbb{N}$ and $r > 0$, let the operators $A_j \in \mathfrak{C}_\Lambda$ and the sequences $\{A^{(m)}_j\}_{m=1}^\infty \subset \mathfrak{K}^r_\Lambda$, $j = 1, \ldots, k$, be such that for each j, $A^{(m)}_j \to A_j$, σ-strongly. Then*

$$A^{(m)}_1 \ldots A^{(m)}_k \to A_1 \ldots A_k, \quad \text{as } m \to +\infty, \tag{1.2.27}$$

also σ-strongly.

Proof. Given $\Phi \in \mathcal{F}_\Lambda$, by (1.1.26) and (1.2.24) it follows that

$$\begin{aligned}
&\|A^{(m)}_1 \ldots A^{(m)}_k - A_1 \ldots A_k\|_\Phi \\
&\quad \leq \|A^{(m)}_1 \ldots A^{(m)}_{k-1}\| \cdot \|A^{(m)}_k - A_k\|_\Phi + \|A^{(m)}_1 \ldots A^{(m)}_{k-1} - A_1 \ldots A_{k-1}\|_{A_k\Phi} \\
&\quad \leq \|A^{(m)}_1 \ldots A^{(m)}_{k-1}\| \cdot \|A^{(m)}_k - A_k\|_\Phi + \|A^{(m)}_1 \ldots A^{(m)}_{k-2}\| \cdot \|A^{(m)}_{k-1} - A_{k-1}\|_{A_k\Phi} \\
&\qquad + \cdots + \|A^{(m)}_1\| \cdot \|A^{(m)}_2 - A_2\|_{A_3 \ldots A_k\Phi} + \|A^{(m)}_1 - A_1\|_{A_2 \ldots A_k\Phi},
\end{aligned}$$

which yields the convergence to be proven. □

We recall that the set of all positive elements $\mathfrak{C}^+_\Lambda \subset \mathfrak{C}_\Lambda$ (see Definition 1.1.8) can be used to define an order on $\mathfrak{C}_\Lambda$. For $A, B \in \mathfrak{C}_\Lambda$, one sets $A \geq B$ if $A - B \in \mathfrak{C}^+_\Lambda$. It is an order since if both $A - B$ and $B - A$ belong to $\mathfrak{C}^+_\Lambda$, then $(\phi, A\phi)_{\mathcal{H}_\Lambda} = (\phi, B\phi)_{\mathcal{H}_\Lambda}$ for all $\phi \in \mathcal{H}_\Lambda$. By Proposition 3.1.14 this yields $A = B$. A net $\{A_\alpha\} \subset \mathfrak{C}_\Lambda$ is called *increasing* if $A_\alpha \geq A_{\alpha'}$ for any $\alpha \geq \alpha'$. Let $\{A_\alpha\} \subset \mathfrak{C}_\Lambda^+$ be increasing and bounded. The latter means that there exists $A \in \mathfrak{C}_\Lambda^+$, such that $A \geq A_\alpha$ for all α. An upper bound $\tilde{A}$ of this net is called the *least upper bound* if for every upper bound A, one has $A \geq \tilde{A}$. It turns out that, see Lemma 2.4.19 in [76], the following holds.

Proposition 1.2.10. *Every bounded increasing net $\{A_\alpha\} \subset \mathfrak{C}_\Lambda$ has a least upper bound $\tilde{A}$, and $A_\alpha \to \tilde{A}$ in the σ-strong topology.*

In the theory of quantum Gibbs states, an important role is played by *von Neumann algebras*. Let $\mathfrak{M}$ be a subset of the algebra of all bounded linear operators $\mathfrak{C}_\Lambda$ on the complex Hilbert space $\mathscr{H}_\Lambda$ defined by (1.2.1). By $\mathfrak{M}'$ we denote its *commutant* – the subset of $\mathfrak{C}_\Lambda$ consisting of those operators which commute with each element of $\mathfrak{M}$. This set is clearly nonempty as it contains $\mathbb{C}I \stackrel{\text{def}}{=} \{\lambda I \mid \lambda \in \mathbb{C}\}$. For any $A, B \in \mathfrak{M}'$ and $\alpha, \beta \in \mathbb{C}$, one has $\alpha A + \beta B \in \mathfrak{M}'$ and $AB \in \mathfrak{M}'$, $BA \in \mathfrak{M}'$, which means that $\mathfrak{M}'$ is an algebra. If $\mathfrak{M}$ is self-adjoint (i.e., $A^* \in \mathfrak{M}$ for every $A \in \mathfrak{M}$), the commutant $\mathfrak{M}'$ is also self-adjoint; hence, it is a $*$ -algebra. Moreover, $\mathfrak{M}'$ is closed in the strong and σ-weak topologies and therefore is closed also in the σ-strong and norm topologies. Since $\mathfrak{M}'$ is again a subset of $\mathfrak{C}_\Lambda$, one can define its commutant, which in turn is the *bicommutant* $\mathfrak{M}''$ of the initial set $\mathfrak{M}$. Successively, we define commutants of higher orders

$$\mathfrak{M}^{(\mathrm{n})} = \left(\mathfrak{M}^{(\mathrm{n}-1)}\right)',$$

for which,

$$\mathfrak{M} \subseteq \mathfrak{M}'' = \mathfrak{M}^{(\mathrm{iv})} = \mathfrak{M}^{(\mathrm{vi})} = \cdots,$$

$$\mathfrak{M}' = \mathfrak{M}''' = \mathfrak{M}^{(\mathrm{v})} = \mathfrak{M}^{(\mathrm{vii})} = \cdots.$$

Definition 1.2.11. A $*$-subalgebra $\mathfrak{M} \subset \mathfrak{C}_\Lambda$ is called a von Neumann algebra if

$$\mathfrak{M} = \mathfrak{M}''.$$

Since the commutant of the whole algebra $\mathfrak{C}_\Lambda$ is exactly $\mathbb{C}I$, its bicommutant is again $\mathfrak{C}_\Lambda$; hence, both $\mathfrak{C}_\Lambda$ and $\mathbb{C}I$ are von Neumann algebras. Every von Neumann algebra we consider is a subalgebra of the algebra of bounded linear operators on a certain Hilbert space. Thus, the definition of positive elements and of the corresponding ordering on $\mathfrak{M}$ is the same as above, see Definition 1.1.8 and (1.1.29), respectively.

As in the case of C^*-algebras, on every von Neumann algebra $\mathfrak{M}$ one can define a state.

Definition 1.2.12. A state ω on a von Neumann algebra is called normal if for any bounded increasing net of positive elements $\{A_\alpha\}$ with the least upper bound A, the least upper bound of the net $\{\omega(A_\alpha)\}$ is $\omega(A)$.

As was mentioned above, see Remark 1.2.5, the local Gibbs states can be extended to some unbounded operators defined on dense subsets of the Hilbert space $\mathscr{H}_\Lambda$. Such operators should satisfy certain conditions, which are summarized in the following notion (see page 87 of [76] or page 164 of [244]).

Definition 1.2.13. Let $\mathfrak{M}$ be a von Neumann algebra of bounded linear operators on $\mathcal{H}_\Lambda$. A closed operator A on $\mathcal{H}_\Lambda$ is said to be affiliated with $\mathfrak{M}$ if for every $T \in \mathfrak{M}'$: (a) T maps the domain of A into itself; (b) the operator AT is an extension of TA.

In the sequel, we use the following fact, see page 164 of [244].

Proposition 1.2.14. *A necessary and sufficient condition for a self-adjoint operator A to be affiliated with $\mathfrak{M}$ is that $f(A) \in \mathfrak{M}$ for every bounded Borel function f defined on the spectrum of A.*

By this statement the displacement and momentum operators (1.1.49) are affiliated with the algebra $\mathfrak{C}_\Lambda$.

By Proposition 1.1.25 one can show that for any $A \in \mathfrak{C}_\Lambda$,

$$|\varrho_{\beta,\Lambda}(A)| \leq \|A\|. \tag{1.2.28}$$

Thus, as each state, the local Gibbs state $\varrho_{\beta,\Lambda}$ is continuous in the norm topology. Its continuity in a weaker topology would be a stronger property. The following statement (see Theorem 2.4.21 in [76]) establishes such a property for the states $\varrho_{\beta,\Lambda}$.

Proposition 1.2.15. *Let ω be a state on a von Neumann algebra $\mathfrak{M}$ of linear bounded operators acting on a Hilbert space $\mathcal{H}$. The following properties of ω are equivalent:*

(a) *ω is normal;*

(b) *ω is σ-weakly continuous;*

(c) *there exists a density matrix, i.e., a positive trace-class operator $T : \mathcal{H} \to \mathcal{H}$ with $\mathrm{trace}(T) = 1$, such that*

$$\omega(A) = \mathrm{trace}(AT) \quad \textit{for all } A \in \mathfrak{M}. \tag{1.2.29}$$

Hence, we have the following

Proposition 1.2.16. *The local Gibbs states $\varrho_{\beta,\Lambda}$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, are normal and thereby σ-weakly continuous.*

An example of a von Neumann algebra, which plays an important role in the sequel, is given by the algebra of all bounded multiplication operators. Let a Borel function, $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, be essentially bounded, which means that there exists a subset of $\mathbb{R}^{\nu|\Lambda|}$ of zero Lebesgue measure, such that F is bounded on its complement. As usual, we define

$$\|F\|_{L^\infty} = \inf\{C > 0 \mid |F| \leq C \text{ outside of a Lebesgue measure zero set}\}.$$

The set of all such functions is denoted by $L^\infty(\mathbb{R}^{\nu|\Lambda|})$. The set $C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ of all bounded continuous functions $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$ is a subset of $L^\infty(\mathbb{R}^{\nu|\Lambda|})$. Both $C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ and $L^\infty(\mathbb{R}^{\nu|\Lambda|})$ are commutative $*$-algebras with respect to the point-wise linear operations, multiplication, and involution $F \mapsto \bar{F}$, where as above, $\bar{F}$ means complex conjugation. Every $F \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$ defines a linear multiplication operator, also denoted by F, see Definition 1.1.39. Obviously, two such functions $F, F' \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$ which differ on a set of zero Lebesgue measure define the same multiplication operator. On the other hand, if two functions from $C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ define the same multiplication operator, then they coincide. Furthermore,

$$\|F\psi\|^2_{\mathcal{H}_\Lambda} = \int_{\mathbb{R}^{\nu|\Lambda|}} |F(x)|^2 |\psi(x)|^2 \mathrm{d}x \le \|F\|^2_{L^\infty} \|\psi\|^2_{\mathcal{H}_\Lambda},$$

which means that the multiplication operator F is bounded and its norm $\|F\|$ does not exceed $\|F\|_{L^\infty}$. One can show, however, that these two norms coincide (see e.g., page 88 of [104]). Let $\mathfrak{M}_\Lambda$ be the set of all multiplication operators by functions from $L^\infty(\mathbb{R}^{\nu|\Lambda|})$. The following statement can be proven e.g., by means of Proposition 4.22, page 89 of [104].

Proposition 1.2.17. *The set $\mathfrak{M}_\Lambda$ of all multiplication operators by functions from $L^\infty(\mathbb{R}^{\nu|\Lambda|})$ satisfies*

$$\mathfrak{M}_\Lambda = \mathfrak{M}'_\Lambda = \mathfrak{M}''_\Lambda,$$

in particular, $\mathfrak{M}_\Lambda$ is a maximal commutative von Neumann algebra contained in $\mathfrak{C}_\Lambda$.

Definition 1.2.18. A $*$-subalgebra $\mathfrak{D} \subset \mathfrak{C}_\Lambda$ is called non-degenerate if the closure of the set

$$\{A\psi \mid A \in \mathfrak{D}, \quad \psi \in \mathcal{H}_\Lambda\} \subset \mathcal{H}_\Lambda$$

is $\mathcal{H}_\Lambda$.

Note that if $\mathfrak{D}$ contains the identity operator, it is automatically non-degenerate.

Important information about von Neumann algebras is given by the following statement – the von Neumann density theorem (see Corollary 2.4.15 in [76]).

Proposition 1.2.19. *Given a Hilbert space $\mathcal{H}$, let $\mathfrak{D}$ be a non-degenerate $*$-subalgebra of $\mathfrak{C}(\mathcal{H})$. Then it is dense in its bicommutant $\mathfrak{D}''$ in the strong, σ-strong, and σ-weak topologies. Therefore, every von Neumann algebra is closed in these topologies.*

The set $\mathbb{C}I$ is a non-degenerate von Neumann algebra of multiplication operators; hence, it is closed in the mentioned topologies. Let $\mathfrak{D}_\Lambda \subset \mathfrak{C}_\Lambda$ be a non-degenerate $*$-algebra of bounded multiplication operators. As its commutant $\mathfrak{D}'_\Lambda$ contains all bounded multiplication operators, and hence $\mathfrak{D}''_\Lambda \subset \mathfrak{M}_\Lambda$, the algebra $\mathfrak{D}_\Lambda$ is σ-weakly, σ-strongly and strongly dense in the algebra $\mathfrak{M}_\Lambda$ if and only if $\mathfrak{D}''_\Lambda = \mathfrak{M}_\Lambda$. Then one can say that such an algebra has 'enough' elements. Let $\mathfrak{F}_\Lambda$ be a self-adjoint family of bounded multiplication operators on $\mathcal{H}_\Lambda$. By definition, the algebra $\mathfrak{C}(\mathfrak{F}_\Lambda)$ *generated by this family* is the minimal algebra which contains $\mathfrak{F}_\Lambda$. It is the linear span of all finite products of the elements of $\mathfrak{F}_\Lambda$. If $\mathfrak{F}_\Lambda$ is closed also with respect to the multiplication, then $\mathfrak{C}(\mathfrak{F}_\Lambda)$ is just the linear span of $\mathfrak{F}_\Lambda$. As a corollary of Proposition 1.2.17 we have the following

Proposition 1.2.20. *Let $\mathfrak{F}_\Lambda$ be a self-adjoint family of bounded multiplication operators and $\mathfrak{C}(\mathfrak{F}_\Lambda)$ be the algebra generated by this family. Then $\mathfrak{C}(\mathfrak{F}_\Lambda)'' = \mathfrak{M}_\Lambda$ if and only if $\mathfrak{F}'_\Lambda = \mathfrak{M}_\Lambda$. In this case, $\mathfrak{C}(\mathfrak{F}_\Lambda)$ is dense in $\mathfrak{M}_\Lambda$ in the σ-strong, σ-weak, and strong topologies.*

Definition 1.2.21. A family, $\mathfrak{F}_\Lambda$, of multiplication operators is said to be complete if: (a) it is self-adjoint; (b) it contains the identity operator; (c) $\mathfrak{F}'_\Lambda = \mathfrak{M}_\Lambda$.

According to Proposition 1.2.20, if $\mathfrak{F}_\Lambda$ is complete, the algebra $\mathfrak{C}(\mathfrak{F}_\Lambda)$ is dense in $\mathfrak{M}_\Lambda$ in the topologies mentioned there.

We recall that $\mathcal{B}(\mathbb{R}^{\nu|\Lambda|})\}$ stands for the Borel σ-algebra of subsets of $\mathbb{R}^{\nu|\Lambda|}$. Given $B \in \mathcal{B}(\mathbb{R}^{\nu|\Lambda|})$, let $\mathcal{H}_\Lambda(B)$ be the range of the projection (1.1.40), which is a subspace of $\mathcal{H}_\Lambda$. Such projections have the properties $P_B = P_B^*$ and $P_B P_{B'} = P_{B\cap B'}$, i.e., the family $\{P_B \mid B \in \mathcal{B}(\mathbb{R}^{\nu|\Lambda|})\}$ is closed under involution and multiplication. In the case of multiplication operators by bounded continuous functions, we have the following result. By definition, a family of functions from $C_b(\mathbb{R}^{\nu|\Lambda|})$ separates points if for any distinct $x, y \in \mathbb{R}^{\nu|\Lambda|}$, one can find a member of this family, F, such that $F(x) \neq F(y)$. In the sequel, if we say that a family of multiplication operators by bounded continuous functions separates points, we mean that the corresponding family of functions does so.

Theorem 1.2.22. *Let a family, $\mathfrak{F}_\Lambda$, of multiplication operators by bounded continuous functions possess the following properties:* (a) *it is self-adjoint;* (b) *it separates points;* (c) $I \in \mathfrak{F}_\Lambda$. *Then $\mathfrak{F}'_\Lambda = \mathfrak{M}_\Lambda$ and thereby $\mathfrak{F}_\Lambda$ is complete.*

Remark 1.2.23. Obviously, the algebra $C_b(\mathbb{R}^{\nu|\Lambda|})$ is complete.

The proof of Theorem 1.2.22 is given in Subsection 1.3.4 below where we develop the corresponding tools based on measure theory.

1.2.3 Høegh-Krohn's Theorem

We recall that the dynamics of the subsystem of oscillators attached to the points of Λ is described by the group of time automorphisms $\mathfrak{a}^t_\Lambda$, see (1.2.13). For a self-adjoint set of multiplication operators $\mathfrak{F}_\Lambda \subset \mathfrak{C}_\Lambda$, let $\mathfrak{A}(\mathfrak{F}_\Lambda)$ be the algebra generated by the operators $\mathfrak{a}^t_\Lambda(F)$, $F \in \mathfrak{F}_\Lambda$, $t \in \mathbb{R}$. In view of the property (1.2.15), $\mathfrak{A}(\mathfrak{F}_\Lambda)$ is a $*$-algebra. It is the linear span of the set of operators

$$A = \mathfrak{a}^{t_1}_\Lambda(F_1)\ldots\mathfrak{a}^{t_n}_\Lambda(F_n), \quad n \in \mathbb{N},\ t_1,\ldots,t_n \in \mathbb{R},\ F_1,\ldots F_n \in \mathfrak{F}_\Lambda. \tag{1.2.30}$$

The main result of this subsection is the following theorem of Høegh-Krohn [156], see also [195].

Theorem 1.2.24. *Let $\mathfrak{F}_\Lambda \subset \mathfrak{M}_\Lambda$ be complete. Then the local Gibbs state $\varrho_{\beta,\Lambda}$ can be identified by its values on the algebra $\mathfrak{A}(\mathfrak{F}_\Lambda)$. This means that if ω is a normal state on $\mathfrak{C}_\Lambda$ and*

$$\omega(A) = \varrho_{\beta,\Lambda}(A), \tag{1.2.31}$$

for all A having the form (1.2.30), then $\omega = \varrho_{\beta,\Lambda}$.

The proof of this theorem will be done in several steps. First we prove the following fact. Set

$$D_\Lambda = -\frac{1}{2m}\sum_{\ell\in\Lambda}\sum_{j=1}^{\nu}\left(\frac{\partial}{\partial x_\ell^{(j)}}\right)^2 = \frac{1}{2m}\sum_{\ell\in\Lambda}|p_\ell|^2, \qquad \mathrm{Dom}(D_\Lambda) = \mathcal{W}^{2,2}\left(\mathbb{R}^{\nu|\Lambda|}\right), \tag{1.2.32}$$

where m and p_ℓ, $\ell \in \Lambda$, are the same as in (1.1.3). By Proposition 1.1.38 D_Λ is self-adjoint. Comparing (1.1.3) and (1.2.5) with (1.2.32) one obtains

$$H_\Lambda = D_\Lambda - W_\Lambda, \qquad W_\Lambda \stackrel{\mathrm{def}}{=} \frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(q_\ell, q_{\ell'}) - \sum_{\ell\in\Lambda}\left[(a/2)|q_\ell|^2 + V_\ell(q_\ell)\right]. \tag{1.2.33}$$

For any $t \in \mathbb{R}$, $\exp(\mathrm{i}tD_\Lambda)$ is a unitary operator on $\mathcal{H}_\Lambda$, see Proposition 1.1.30. For $t = 0$, one has $\exp(\mathrm{i}tD_\Lambda) = I$. For $t \neq 0$, this operator can be defined by its integral kernel (see e.g., page 6 of [274])

$$K(x, y; t) = \left(\frac{\mathrm{i}m}{2\pi t}\right)^{\nu|\Lambda|/2}\exp\left(-\frac{\mathrm{i}m}{2t}\sum_{\ell\in\Lambda}|x_\ell - y_\ell|^2\right), \quad x, y \in \mathbb{R}^{\nu|\Lambda|}. \tag{1.2.34}$$

For $t = 0$, the kernel is

$$K(x, y; 0) = \delta(x - y), \tag{1.2.35}$$

where δ is the Dirac delta-function, which is a distribution satisfying

$$\int_{\mathbb{R}^{\nu|\Lambda|}} \delta(x - y)\psi(y)\mathrm{d}y = \psi(x). \tag{1.2.36}$$

For any $t \in \mathbb{R}$,

$$\exp(\mathrm{i}tD_\Lambda)\exp(-\mathrm{i}tD_\Lambda) = I.$$

Thus, for $t \neq 0$, the kernel of this product may be written as

$$\begin{aligned}
[\exp(\mathrm{i}tD_\Lambda)\exp(-\mathrm{i}tD_\Lambda)]\,(x, y) &= \int_{\mathbb{R}^{\nu|\Lambda|}} K(x, z; t)K(z, y; -t)\mathrm{d}z \\
&= \left(\frac{m}{2\pi t}\right)^{\nu|\Lambda|}\int_{\mathbb{R}^{\nu|\Lambda|}}\exp\left(-\frac{\mathrm{i}m}{2t}\sum_{\ell\in\Lambda}|x_\ell - z_\ell|^2 + +\frac{\mathrm{i}m}{2t}\sum_{l\in\Lambda}|z_\ell - y_\ell|^2\right)\mathrm{d}z \\
&= \exp\left(-\frac{\mathrm{i}m}{2t}\sum_{\ell\in\Lambda}(|x_\ell|^2 - |y_\ell|^2)\right) \\
&\quad\times\left(\frac{m}{2\pi t}\right)^{\nu|\Lambda|}\int_{\mathbb{R}^{\nu|\Lambda|}}\exp\left(\frac{\mathrm{i}m}{t}\sum_{\ell\in\Lambda}(z_\ell, x_\ell - y_\ell)\right)\mathrm{d}z = \delta(x - y).
\end{aligned}$$

Given $\tau = (\tau_\ell^{(j)}) \in \mathbb{R}^{\nu|\Lambda|}$, we set

$$\begin{aligned} Q(\tau) &= \prod_{\ell\in\Lambda}\prod_{j=1}^{\nu} Q_\ell^{(j)}(\tau_\ell^{(j)}), \quad Q_\ell^{(j)}(\tau_\ell^{(j)}) = \exp\left(\mathrm{i}\tau_\ell^{(j)} q_\ell^{(j)}\right), \\ P(\tau) &= \prod_{\ell\in\Lambda}\prod_{j=1}^{\nu} P_\ell^{(j)}(\tau_\ell^{(j)}), \quad P_\ell^{(j)}(\tau_\ell^{(j)}) = \exp\left(\mathrm{i}\tau_\ell^{(j)} p_\ell^{(j)}\right). \end{aligned} \tag{1.2.37}$$

Both $Q(\tau)$, $P(\tau)$ are unitary operators on $\mathcal{H}_\Lambda$. By (1.1.49) for $\psi \in \mathcal{S}(\mathbb{R}^{\nu|\Lambda|})$,

$$\left(P_\ell^{(j)}(\tau_\ell^{(j)})\psi\right)(x) = \sum_{n=0}^{\infty} \frac{[\tau_\ell^{(j)}]^n}{n!}\left(\frac{\partial}{\partial x_\ell^{(j)}}\right)^n \psi(x) = \psi\left(x + \tau_\ell^{(j)} e_\ell^{(j)}\right),$$

where $e_\ell^{(j)} \in \mathbb{R}^{\nu|\Lambda|}$ has the component with the indices ℓ and j equal to 1 and the zero remaining components. By continuity this action can be extended to all elements of $\mathcal{H}_\Lambda$. Similarly, for any $\psi \in \mathcal{H}_\Lambda$,

$$Q_\ell^{(j)}(\tau_\ell^{(j)})\psi(x) = \exp\left(\mathrm{i}\tau_\ell^{(j)} x_\ell^{(j)}\right)\psi(x).$$

Thereby, the operators (1.2.37) act as follows:

$$\begin{aligned} (Q(\tau)\psi)(x) &= \exp(\mathrm{i}\tau\cdot x)\,\psi(x), \\ (P(\tau)\psi)(x) &= \psi(x+\tau), \\ \tau\cdot x &\stackrel{\text{def}}{=} \sum_{\ell\in\Lambda}\sum_{j=1}^{\nu} \tau_\ell^{(j)} x_\ell^{(j)}. \end{aligned} \tag{1.2.38}$$

These formulas immediately yield Weyl's commutation rule

$$P(\tau)Q(\theta) = \exp(\mathrm{i}\tau\cdot\theta)\, Q(\theta)P(\tau). \tag{1.2.39}$$

By (1.2.32) the operators D_Λ and $p_\ell^{(j)}$, $\ell \in \Lambda$, $j = 1, \ldots, \nu$, commute. Thus, by Theorem VIII.13 of [255] one has

Proposition 1.2.25. *For any $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^{\nu|\Lambda|}$, the operators $\exp(\mathrm{i}tD_\Lambda)$ and $P(\tau)$ commute.*

Proposition 1.2.26. *If a given $B \in \mathfrak{C}_\Lambda$ commutes with all $P(\tau)$ and all $Q(\tau')$, $\tau, \tau' \in \mathbb{R}^{\nu|\Lambda|}$, then $B \in \mathbb{C}I$.*

Proof. Set $\mathfrak{Q} = \{Q(\tau)\}_{\tau\in\mathbb{R}^{\nu|\Lambda|}}$. It is a complete family, see Definition 1.2.21 and Theorem 1.2.22. Then $\mathfrak{Q}' = \mathfrak{M}_\Lambda$; hence, each $B \in \mathfrak{Q}'$ is a multiplication operator. That is,

$$(B\psi)(x) = B(x)\psi(x), \quad \psi \in \mathcal{H}_\Lambda,$$

where $B(\cdot)$ is a Borel function. Thus, by (1.2.38) for any $\tau \in \mathbb{R}^{\nu|\Lambda|}$, one has

$$\begin{aligned}(P(\tau)B\psi)(x) &= B(x+\tau)\psi(x+\tau),\\ (BP(\tau)\psi)(x) &= B(x)\psi(x+\tau),\end{aligned}$$

which yields that $B(\cdot)$ is constant. □

Given $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^{\nu|\Lambda|}$, we set

$$R(\tau, t) = \exp(-\mathrm{i}tD_\Lambda)Q(\tau)\exp(\mathrm{i}tD_\Lambda). \tag{1.2.40}$$

Lemma 1.2.27. *For any $t \in \mathbb{R}$ and $\tau \in \mathbb{R}^{\nu|\Lambda|}$, it follows that*

$$\begin{aligned}R(\tau, t) &= \exp\left(\frac{\mathrm{i}t}{2m}\tau\cdot\tau\right)P(-(t/m)\tau)Q(\tau),\\ R(-\tau, t) &= \exp\left(-\frac{\mathrm{i}t}{2m}\tau\cdot\tau\right)Q(-\tau)P((t/m)\tau).\end{aligned} \tag{1.2.41}$$

Proof. One observes that each line in (1.2.41) follows from the other one by the commutation rule (1.2.39). By (1.2.34) and (1.2.38) for $t \neq 0$ and $\psi \in \mathcal{S}(\mathbb{R}^{\nu|\Lambda|})$, one has

$$\begin{aligned}[R(\tau,t)\psi](x) &= \left(\frac{m}{2\pi t}\right)^{\nu|\Lambda|}\int_{\mathbb{R}^{\nu|\Lambda|}}\left\{\int_{\mathbb{R}^{\nu|\Lambda|}}\exp\left\{\frac{\mathrm{i}m}{2t}(x-y)\cdot(x-y)\right.\right.\\ &\qquad\left.\left. +\,\mathrm{i}\tau\cdot y - \frac{\mathrm{i}m}{2t}(y-z)\cdot(y-z)\right\}\mathrm{d}y\right\}\psi(z)\mathrm{d}z\\ &= \int_{\mathbb{R}^{\nu|\Lambda|}}\left[\left(\frac{m}{2\pi t}\right)^{\nu|\Lambda|}\int_{\mathbb{R}^{\nu|\Lambda|}}\exp\left\{\frac{\mathrm{i}m}{t}y\cdot(z+(t/m)\tau - x)\right\}\mathrm{d}y\right]\\ &\qquad\times\exp\left\{\frac{\mathrm{i}m}{2t}\left[x\cdot x - z\cdot z\right]\right\}\psi(z)\mathrm{d}z\\ &= \int_{\mathbb{R}^{\nu|\Lambda|}}\prod_{\ell\in\Lambda}\prod_{j=1}^{\nu}\delta\left(z_\ell^{(j)} + (t/m)\tau_\ell^{(j)} - x_\ell^{(j)}\right)\\ &\qquad\times\exp\left\{\frac{\mathrm{i}m}{2t}\left[x\cdot x - z\cdot z\right]\right\}\psi(z)\mathrm{d}z\\ &= \exp\left(-\frac{\mathrm{i}t}{2m}\tau\cdot\tau + \mathrm{i}\tau\cdot x\right)\psi(x-(t/m)\tau)\\ &\qquad\times\left[\exp\left(-\frac{\mathrm{i}t}{2m}\tau\cdot\tau\right)Q(\tau)P(-(t/m)\tau)\psi\right](x)\\ &= \left[\exp\left(\frac{\mathrm{i}t}{2m}\tau\cdot\tau\right)P(-(t/m)\tau)Q(\tau)\psi\right](x),\end{aligned}$$

which can obviously be extended to the whole space $\mathcal{H}_\Lambda$. □

Lemma 1.2.28. *Given* $F \in \mathfrak{M}_\Lambda$, *if*

$$A(F;t) = \exp(\mathrm{i}tD_\Lambda)F\exp(-\mathrm{i}tD_\Lambda) \tag{1.2.42}$$

is a multiplication operator for all $t \in \mathbb{R}$, *then* $F \in \mathbb{C}I$.

Proof. For $\tau \in \mathbb{R}^{\nu|\Lambda|}$, we set

$$A_\tau(F;t) = Q(\tau)A(F;t)Q(-\tau). \tag{1.2.43}$$

By (1.2.40)

$$\begin{aligned} Q(\tau)\exp(\mathrm{i}tD_\Lambda) &= \exp(\mathrm{i}tD_\Lambda)R(\tau,t),\\ \exp(-\mathrm{i}tD_\Lambda)Q(-\tau) &= R(-\tau,t)\exp(-\mathrm{i}tD_\Lambda). \end{aligned} \tag{1.2.44}$$

Then

$$A_\tau(F;t) = \exp(\mathrm{i}tD_\Lambda)R(\tau,t)FR(-\tau,t)\exp(-\mathrm{i}tD_\Lambda),$$

and by (1.2.41)

$$A_\tau(F;t) = \exp(\mathrm{i}tD_\Lambda)P(-(t/m)\tau)Q(\tau)FQ(-\tau)P((t/m)\tau)\exp(-\mathrm{i}tD_\Lambda).$$

Taking into account that F (respectively, $\exp(\pm\mathrm{i}tD_\Lambda)$) commutes with $Q(\tau)$ (respectively, with $P(\pm(t/m)\tau)$, see Proposition 1.2.25), one gets

$$A_\tau(F;t) = P(-(t/m)\tau)A(F;t)P((t/m)\tau). \tag{1.2.45}$$

Thus, as $A(F;t)$ is a multiplication operator, it commutes with all $Q(\tau)$ and hence by (1.2.43) coincides with $A_\tau(F;t)$ for any $\tau \in \mathbb{R}^{\nu|\Lambda|}$. Then by (1.2.45) it commutes with all $P(\tau)$, $\tau \in \mathbb{R}^{\nu|\Lambda|}$, which yields by Proposition 1.2.26 $A(F;t) \in \mathbb{C}I$, and hence $F \in \mathbb{C}I$. $\square$

In the sequel, we shall use the Trotter–Kato product formula (see Theorem 1.1, page 4 of [274]).

Proposition 1.2.29. *Let* A *and* B *be essentially self-adjoint operators on a separable Hilbert space so that the operator* $A+B$ *defined on* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$, *is also essentially self-adjoint. Then for all* $t \in \mathbb{R}$,

$$\exp\left(\mathrm{i}t(A+B)\right) = \mathrm{lim}^{\mathrm{s}}_{n\to+\infty}\left[\exp\left(\mathrm{i}(t/n)A\right)\exp\left(\mathrm{i}(t/n)B\right)\right]^n. \tag{1.2.46}$$

If, moreover, A *and* B *are bounded from below, then for all* $t \geq 0$,

$$\exp\left(-t(A+B)\right) = \mathrm{lim}^{\mathrm{s}}_{n\to+\infty}\left[\exp\left(-(t/n)A\right)\exp\left(-(t/n)B\right)\right]^n, \tag{1.2.47}$$

where $\mathrm{lim}^{\mathrm{s}}$ *means strong limit.*

The following statement is the second key element, along with Lemma 1.2.28, in the proof of Theorem 1.2.24. We recall that the $*$-algebra $\mathfrak{A}(\mathfrak{D}_\Lambda)$ mentioned in this theorem is the linear span of the operators (1.2.30).

Lemma 1.2.30. *Let a family of multiplication operators* $\mathfrak{D}_\Lambda$ *be complete. Then*

$$\mathfrak{A}(\mathfrak{D}_\Lambda)'' = \mathfrak{C}_\Lambda.$$

Proof. The proof will be done by showing that $\mathfrak{A}(\mathfrak{D}_\Lambda)' = \mathbb{C}I$. By definition, an arbitrary $B \in \mathfrak{A}(\mathfrak{D}_\Lambda)'$ commutes with every $\mathfrak{a}^t_\Lambda(F)$, $t \in \mathbb{R}$, $F \in \mathfrak{D}_\Lambda$. Hence,

$$\exp(-\mathrm{i}tH_\Lambda)\, B \exp(\mathrm{i}tH_\Lambda)\, F = F \exp(-\mathrm{i}tH_\Lambda)\, B \exp(\mathrm{i}tH_\Lambda),$$

for any $t \in \mathbb{R}$ and $F \in \mathfrak{D}_\Lambda$. This immediately yields that $\mathfrak{a}^t_\Lambda(B)$ commutes with every $F \in \mathfrak{D}_\Lambda$; hence, $\mathfrak{a}^t_\Lambda(B) \in \mathfrak{M}_\Lambda$ for any $t \in \mathbb{R}$, see Definition 1.2.21. In particular, $B \in \mathfrak{M}_\Lambda$. If in (1.2.33) one substitutes q_ℓ by $x_\ell \in \mathbb{R}^\nu$, then the corresponding function W_Λ will be continuous on $\mathbb{R}^{\nu|\Lambda|}$, see Assumption 1.1.1. This means that the operator $\exp(\mathrm{i}sW_\Lambda)$ belongs to $\mathfrak{M}_\Lambda$ for any real s and hence commutes with $\mathfrak{a}^t_\Lambda(B)$ for any real t. Then for any $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\exp(\mathrm{i}(t/n)W_\Lambda)\, \mathfrak{a}^{t/n}_\Lambda(B) \exp(-\mathrm{i}(t/n)W_\Lambda) = \mathfrak{a}^{t/n}_\Lambda(B),$$

which yields

$$\begin{aligned}\exp(\mathrm{i}(t/n)H_\Lambda) \times \exp(\mathrm{i}(t/n)W_\Lambda)\, \mathfrak{a}^{t/n}_\Lambda(B) \exp(-\mathrm{i}(t/n)W_\Lambda) \times \exp(-\mathrm{i}(t/n)H_\Lambda) \\ = \mathfrak{a}^{2t/n}_\Lambda(B).\end{aligned}$$

By iteration,

$$\begin{aligned}&[\exp(\mathrm{i}(t/n)W_\Lambda)\exp(\mathrm{i}(t/n)H_\Lambda)]^n\, B \\ &\quad \times [\exp(-\mathrm{i}(t/n)W_\Lambda)\exp(-\mathrm{i}(t/n)H_\Lambda)]^n = \mathfrak{a}^t_\Lambda(B).\end{aligned} \tag{1.2.48}$$

The left-hand side of (1.2.48) is an element of $\mathfrak{M}_\Lambda$; hence, $\mathfrak{a}^t_\Lambda(B) \in \mathfrak{M}_\Lambda$. Passing here to the limit $n \to +\infty$, by Proposition 1.2.29 and (1.2.33) one gets

$$\forall t \in \mathbb{R}: \qquad \exp(\mathrm{i}tD_\Lambda)\, B \exp(-\mathrm{i}tD_\Lambda) = \mathfrak{a}^t_\Lambda(B). \tag{1.2.49}$$

This means that for any $t \in \mathbb{R}$, both B and $\mathfrak{a}^t_\Lambda(B)$ belong to $\mathfrak{M}_\Lambda$, which by Lemma 1.2.28 and (1.2.49) is possible if and only if $B \in \mathbb{C}I$. □

Proof of Theorem 1.2.24. In view of Proposition 1.2.16, the local Gibbs states are σ-weakly continuous; hence, they are uniquely determined by their values on a σ-weakly dense subset of the corresponding algebra of observables. But by Proposition 1.2.19 and Lemma 1.2.28, $\mathfrak{A}(\mathfrak{F}_\Lambda)$ is σ-weakly dense in $\mathfrak{C}_\Lambda$. By linearity, the values of the state $\varrho_{\beta,\Lambda}$ on the algebra $\mathfrak{A}(\mathfrak{F}_\Lambda)$ are linear combinations of its values on the operators (1.2.30). □

Corollary 1.2.31. *Let* $\mathfrak{Q}$ *be a family of multiplication operators by continuous functions, which has the following properties:* (a) *is closed under multiplication and involution* $\mathfrak{Q} \ni Q \mapsto Q^*$*;* (b) *separates points;* (c) $I \in \mathfrak{Q}$. *Then the state* $\varrho_{\beta,\Lambda}$ *is uniquely determined by its values on the operators* (1.2.30) *with* $F_1, \ldots, F_n \in \mathfrak{Q}$. *The family* $\{Q(\tau) \mid \tau \in \mathbb{R}\}$ *of the operators defined by* (1.2.37) *possesses these properties.*

Proof. By Proposition 1.2.22, the family $\mathfrak{Q}$ is complete. Then the property claimed follows by Theorem 1.2.24. The family $\{Q(\tau) \mid \tau \in \mathbb{R}\}$ consists of multiplication operators by continuous functions; it contains the identity operator $I = Q(0)$; it is closed under multiplication, $Q(\tau)Q(\tau') = Q(\tau + \tau')$, and involution $Q(\tau) \mapsto Q^*(\tau)Q(-\tau)$. Finally, it obviously separates points. □

1.2.4 Green and Matsubara Functions

For $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathfrak{C}_\Lambda$, and $t_1, \dots, t_n \in \mathbb{R}$, we set

$$G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n) = \varrho_{\beta,\Lambda}\left(\mathfrak{a}^{t_1}_\Lambda(A_1)\dots\mathfrak{a}^{t_n}_\Lambda(A_n)\right). \tag{1.2.50}$$

Considered as a mapping $\mathbb{R}^n \ni (t_1, \dots, t_n) \to \mathbb{C}$ this expression is called *a Green function* constructed on the operators $A_1, \dots, A_n$.

Given a domain $\mathcal{O} \subset \mathbb{C}^n$, let $\mathrm{Hol}(\mathcal{O})$ be the set of all holomorphic functions $f : \mathcal{O} \to \mathbb{C}$, and

$$\mathcal{D}^n_\beta \stackrel{\mathrm{def}}{=} \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid 0 < \Im(t_1) < \dots < \Im(t_n) < \beta\}. \tag{1.2.51}$$

$\mathcal{D}^n_\beta$ is a domain, its closure is denoted by $\bar{\mathcal{D}}^n_\beta$. Finally, we set

$$\mathcal{D}^n_\beta(0) \stackrel{\mathrm{def}}{=} \{(t_1, \dots, t_n) \in \mathcal{D}^n_\beta \mid \Re(t_i) = 0,\ i = 1, \dots, n\}. \tag{1.2.52}$$

In view of the linearity of the local Gibbs state $\varrho_{\beta,\Lambda}$, it can be identified by the values of the Green functions (1.2.50) constructed on operators, which constitute a family satisfying the conditions of Corollary 1.2.31. The next statement establishes the multiple-time analyticity and thereby the KMS property of the state $\varrho_{\beta,\Lambda}$.

Theorem 1.2.32. *For each collection $A_1, \dots, A_n \in \mathfrak{C}_\Lambda$,*

(a) *the Green function $G^{\beta,\Lambda}_{A_1,\dots,A_n}$ defined by* (1.2.50) *is the restriction to $\mathbb{R}^n$ of a member of* $\mathrm{Hol}(\mathcal{D}^n_\beta)$, *for which we will use the same notation;*

(b) *the function $G^{\beta,\Lambda}_{A_1,\dots,A_n} \in \mathrm{Hol}(\mathcal{D}^n_\beta)$ mentioned in* (a) *is continuous on the closure of $\mathcal{D}^n_\beta$; moreover, for all $(t_1, \dots, t_n) \in \bar{\mathcal{D}}^n_\beta$,*

$$|G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n)| \le \|A_1\| \dots \|A_n\|; \tag{1.2.53}$$

(c) *the set* (1.2.52) *is such that for arbitrary $f, g \in \mathrm{Hol}(\mathcal{D}^n_\beta)$, the equality $f = g$ on $\mathcal{D}^n_\beta(0)$ implies that $f = g$ on the whole $\mathcal{D}^n_\beta$.*

Proof. We recall that the spectrum of the Hamiltonian H_Λ (1.2.5) consists of the eigenvalues $E_s > 0$, see Theorem 1.2.1. Let $\{\psi_s\}_{s\in\mathbb{N}}$ be an orthonormal basis consisting of the corresponding eigenvectors. Set

$$A^{(i)}_{ss'} = (\psi_s, A_i\psi_{s'})_{\mathcal{H}_\Lambda}, \qquad A_i \in \mathfrak{C}_\Lambda,\ i = 1, \dots, n. \tag{1.2.54}$$

Then

$$
\begin{aligned}
G^{\beta,\Lambda}_{A_1,\ldots,A_n}(t_1,\ldots,t_n) = \frac{1}{Z_{\beta,\Lambda}} \sum_{s_1,\ldots,s_n\in\mathbb{N}} & A^{(1)}_{s_1s_2}\exp[\mathrm{i}(t_2-t_1)E_{s_2}] \\
& \times\cdots\times A^{(n-1)}_{s_{n-1}s_n}\exp[\mathrm{i}(t_n-t_{n-1})E_{s_n}] \\
& \times A^{(n)}_{s_ns_1}\exp[\mathrm{i}(t_1-t_n+\mathrm{i}\beta)E_{s_1}],
\end{aligned}
\tag{1.2.55}
$$

where

$$
Z_{\beta,\Lambda} = \mathrm{trace}\left(\exp\left[-\beta H_\Lambda\right]\right). \tag{1.2.56}
$$

For $i=1,\ldots,n$, we set

$$
t_i=\theta_i+\mathrm{i}\tau_i,\quad \theta_i\in\mathbb{R},\ \tau_i\in[0,\beta], \tag{1.2.57}
$$

and also

$$
\begin{aligned}
\mathcal{C}^n_\beta &= \{(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n \mid 0<\tau_1<\cdots<\tau_n<\beta\},\\
\bar{\mathcal{C}}^n_\beta &= \{(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n \mid 0\le\tau_1\le\cdots\le\tau_n<\beta\}.
\end{aligned}
\tag{1.2.58}
$$

Clearly, $\bar{\mathcal{C}}^n_\beta$ is the closure of $\mathcal{C}^n_\beta$ in $\mathbb{R}^n$. Then

$$
\begin{aligned}
\mathcal{D}^n_\beta &= \{(t_1,\ldots,t_n)\in\mathbb{C}^n \mid (\tau_1,\ldots,\tau_n)\in\mathcal{C}^n_\beta\},\\
\bar{\mathcal{D}}^n_\beta &= \{(t_1,\ldots,t_n)\in\mathbb{C}^n \mid (\tau_1,\ldots,\tau_n)\in\bar{\mathcal{C}}^n_\beta\}.
\end{aligned}
\tag{1.2.59}
$$

To prove the multiple-time analyticity stated in claim (a) one can use the fact that each summand of the series (1.2.55) is an entire function of $(t_1,\ldots,t_n)\in\mathbb{C}^n$. Thus, our goal will be achieved if we show that the series (1.2.55) converges uniformly on compact subsets of $\mathcal{D}^n_\beta$. Each such a subset can be embedded into the set

$$
\mathcal{D}^n_\beta(K) \stackrel{\mathrm{def}}{=} \{(t_1,\ldots,t_n)\in\mathcal{D}^n_\beta \mid (\tau_1,\ldots,\tau_n)\in K\}, \tag{1.2.60}
$$

with a closed $K\subset\mathcal{C}^n_\beta$. Each K in turn can be embedded into a finite union of the parallelepipeds

$$
K^n_{c,d}=\prod_{i=1}^n[c_i,d_i],\quad 0<c_1<d_1<c_2<d_2<\cdots<d_{n-1}<c_n<d_n<\beta. \tag{1.2.61}
$$

Thus, the multiple-time analyticity will follow from the uniform convergence of the series (1.2.55) on the sets $\mathcal{D}^n_\beta(K^n_{c,d})$ with all possible choices of c_i and d_i, $i=1,\ldots,n$. For each such a choice, one has the following estimate of the summands of (1.2.55):

$$
\begin{aligned}
&\Big|A^{(1)}_{s_1s_2}\exp[\mathrm{i}(t_2-t_1)E_{s_2}]\ldots A^{(n-1)}_{s_{n-1}s_n}\exp[\mathrm{i}(t_n-t_{n-1})E_{s_n}] \\
&\qquad\times A^{(n)}_{s_ns_1}\exp[\mathrm{i}(t_1-t_n+\mathrm{i}\beta)E_{s_1}]\Big| \\
&\quad\le \left|A^{(1)}_{s_1s_2}\right|\exp[-(c_2-d_1)E_{s_2}] \\
&\qquad\times\cdots\times\left|A^{(n-1)}_{s_{n-1}s_n}\right|\exp[-(c_n-d_{n-1})E_{s_n}] \\
&\qquad\times\left|A^{(n)}_{s_ns_1}\right|\exp[-(\beta-d_n+c_1)E_{s_1}].
\end{aligned}
\tag{1.2.62}
$$

Taking into account that the eigenfunctions ψ_s are normalized and that for a bounded operator,

$$\left|A^{(i)}_{s_i s_{i+1}}\right| = \left|(\psi_{s_i}, A_i \psi_{s_{i+1}})\right| \leq \|A_i\|, \tag{1.2.63}$$

we obtain the convergence to be proven, as well as the estimate (1.2.53) but so far for $(t_1, \ldots, t_n) \in \mathcal{D}^n_\beta$ only.

In the proof of claim (b), we use the boundary $\partial\mathcal{D}^n_\beta$ of the tubular domain (1.2.51). By (1.2.58), (1.2.59) the tuple $(t_1, \ldots t_n)$ belongs to $\partial\mathcal{D}^n_\beta$ if and only if $(\tau_1, \ldots, \tau_n) \in \bar{\mathcal{C}}^n_\beta \setminus \mathcal{C}^n_\beta$, that is, certain τ_i's should coincide with each other or with the endpoints 0, β. For $k = 1, \ldots, n-1$, let us consider

$$\mathfrak{n} = (n_0, n_1, \ldots, n_k, n_{k+1}), \quad n_0, n_{k+1} \in \mathbb{N}_0,\ n_1, \ldots, n_k \in \mathbb{N}, \tag{1.2.64}$$

subject to the condition

$$n_0 + n_1 + \cdots + n_k + n_{k+1} = n. \tag{1.2.65}$$

Given $\mathfrak{n}$ and $j = 1, \ldots, k$, we set

$$\mathcal{N}^{(j)}_\mathfrak{n} = \{m \in \mathbb{N} \mid n_0 + \cdots + n_{j-1} + 1 \leq m \leq n_0 + \cdots + n_j\}. \tag{1.2.66}$$

We also set

$$\begin{aligned} \mathcal{N}^{(0)}_\mathfrak{n} &= \{1, 2, \ldots, n_0\}, \\ \mathcal{N}^{(0)}_\mathfrak{n} &= \emptyset, \quad \text{for } n_0 = 0, \\ \mathcal{N}^{(k+1)}_\mathfrak{n} &= \{m \in \mathbb{N} \mid n_0 + \cdots + n_k + 1 \leq m \leq n\}, \\ \mathcal{N}^{(k+1)}_\mathfrak{n} &= \emptyset, \quad \text{for } n_{k+1} = 0. \end{aligned} \tag{1.2.67}$$

For $\vartheta_0, \vartheta_1, \ldots, \vartheta_k, \vartheta_{k+1} \in [0, \beta]$ obeying the condition

$$0 = \vartheta_0 < \vartheta_1 < \cdots < \vartheta_k < \vartheta_{k+1} = \beta, \tag{1.2.68}$$

we set

$$\begin{aligned} &\Xi^\beta_\mathfrak{n}(\vartheta_1, \ldots, \vartheta_k) \\ &\quad = \{(t_1, \ldots, t_n) \in \mathbb{C}^n \mid \tau_i = \vartheta_j \text{ for } i \in \mathcal{N}^{(j)}_\mathfrak{n}, i = 1, \ldots, n,\ j = 0, \ldots, k+1\}, \end{aligned} \tag{1.2.69}$$

and

$$\Xi^\beta_\mathfrak{n} = \bigcup \Xi^\beta_\mathfrak{n}(\vartheta_1, \ldots, \vartheta_k), \quad \Xi^\beta = \bigcup \Xi^\beta_\mathfrak{n}, \tag{1.2.70}$$

where the first union is taken over all $\vartheta_1, \ldots, \vartheta_k$ obeying (1.2.68), whereas the second one is taken over all $\mathfrak{n}$ (with all possible choices of k) obeying (1.2.65). Then

$$\partial\mathcal{D}^n_\beta = \Xi^\beta \cup \Sigma^\beta, \tag{1.2.71}$$

where Σ^β is the *skeleton* of the tubular domain $\mathcal{D}^n_\beta$, which is

$$\begin{aligned}
\Sigma^\beta &= \bigcup_{j=0}^{n} \Sigma^\beta_j, \\
\Sigma^\beta_0 &= \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid \tau_1 = \dots = \tau_n = \beta\}, \\
\Sigma^\beta_n &= \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid \tau_1 = \dots = \tau_n = 0\}, \\
\Sigma^\beta_j &= \{(t_1, \dots, t_n) \in \mathbb{C}^n \mid \tau_1 = \dots = \tau_j = 0,\ \tau_{j+1} \dots = \tau_n = \beta\}.
\end{aligned} \tag{1.2.72}$$

For $\Re(t_i) = \theta_i \in \mathbb{R}$, cf. (1.2.57), we set

$$B_i = \mathfrak{a}^{\theta_i}_\Lambda(A_i), \quad i = 1, \dots, n, \tag{1.2.73}$$

and for a given $\mathfrak{n}$,

$$\begin{aligned}
C_0 &= I, && \text{if } n_0 = n_{k+1} = 0, \\
C_0 &= B_{n_1+\dots+n_k+1} \dots B_n, && \text{if } n_0 = 0, n_{k+1} > 0, \\
C_0 &= B_1 \dots B_{n_0}, && \text{if } n_0 > 0, n_{k+1} = 0, \\
C_0 &= B_{n_0+\dots+n_k+1} \dots B_n \cdot B_1 \dots B_{n_0}, && \text{if } n_0, n_{k+1} > 0, \\
C_j &= B_{n_0+\dots+n_{j-1}+1} \dots B_{n_0+\dots+n_j}, && \text{for } j = 1, \dots, k,
\end{aligned} \tag{1.2.74}$$

which are bounded operators on $\mathcal{H}_\Lambda$. As in (1.2.54) we set

$$C^{(j)}_{ss'} = (\psi_s, C_j \psi_{s'})_{\mathcal{H}_\Lambda}, \quad j = 0, 1, \dots, k.$$

Then for $(t_1, \dots, t_n) \in \Xi^\beta_{\mathfrak{n}}(\vartheta_1, \dots, \vartheta_k)$, one may write the series (1.2.55) in the form

$$\begin{aligned}
G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n) = \left[Z_{\beta,\Lambda}\right]^{-1} &\sum_{s_0,\dots,s_k \in \mathbb{N}} C^{(0)}_{s_0 s_1} e^{-\vartheta_1 E_{s_1}} \\
&\times C^{(1)}_{s_1 s_2} e^{-(\vartheta_2 - \vartheta_1) E_{s_2}} \dots C^{(k)}_{s_k s_0} e^{-(\beta - \vartheta_k) E_{s_0}}.
\end{aligned} \tag{1.2.75}$$

As above, one can show that this series, as a function of $(\vartheta_1, \dots, \vartheta_k) \in \mathcal{C}^k_\beta$, converges uniformly on the parallelepipeds $K^k_{c,d}$, which yields the continuity of the Green function on the set $\Xi^\beta \cup \mathcal{D}^n_\beta$. Thus, in view of (1.2.71), to prove claim (b) one has to show that for any sequence $\{(t^{(m)}_1, \dots, t^{(m)}_n)\}_{m \in \mathbb{N}} \subset \Xi^\beta \cup \mathcal{D}^n_\beta$, which converges to a point $(t_1, \dots, t_n) \in \Sigma^\beta$, the corresponding sequence of values of the Green function converges to its value at such $(t_1, \dots, t_n)$. To this end we first prove that the series (1.2.55) converges at every $(t_1, \dots, t_n) \in \Sigma^\beta$. Take $(t_1, \dots, t_n) \in \Sigma^\beta_j$, $j = 0, \dots, n$, see (1.2.72). For this $(t_1, \dots, t_n)$, we rewrite (1.2.50) in the form

$$\begin{aligned}
&G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n) \\
&\quad = \left[Z_{\beta,\Lambda}\right]^{-1} \times \sum_{s=1}^{+\infty} (\psi_s, B_{j+1} \dots B_n B_1 \dots B_j \psi_s)_{\mathcal{H}_\Lambda} e^{-\beta E_s},
\end{aligned} \tag{1.2.76}$$

B_i being defined by (1.2.73). Then, as in (1.2.63), one gets

$$\begin{aligned}\left|(\psi_s, B_{j+1} \dots B_n B_1 \dots B_j \psi_s)_{\mathcal{H}_\Lambda}\right| &\le \|B_{j+1} \dots B_n B_1 \dots B_j\| \\ &\le \|A_1\| \dots \|A_n\|,\end{aligned}$$

which yields in (1.2.76)

$$\left|G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n)\right| \le \|A_1\| \dots \|A_n\|. \tag{1.2.77}$$

Now let us prove that

$$G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1^{(m)}, \dots, t_n^{(m)}) \to G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1, \dots, t_n), \quad m \to +\infty, \tag{1.2.78}$$

for

$$\Xi^\beta \cup \mathcal{D}^n_\beta \ni (t_1^{(m)}, \dots, t_n^{(m)}) \to (t_1, \dots, t_n) \in \Sigma^\beta.$$

Let $j = 0, \dots, n$ be such that $(t_1, \dots, t_n) \in \Sigma^\beta_j$, see (1.2.72). Then

$$(t_1^{(m)}, \dots, t_n^{(m)}) \to (t_1, \dots, t_n), \quad \text{as } m \to +\infty,$$

means that

$$\begin{aligned} \theta_k^{(m)} &\to \theta_k, && \text{for all } k = 1, \dots n, \\ \tau_k^{(m)} &\to 0, && \text{for } k \le j, \end{aligned}$$

and

$$\tau_k^{(m)} \to \beta, \quad \text{for } k > j.$$

In this case, one has

$$\begin{aligned} &Z_{\beta,\Lambda} \cdot G^{\beta,\Lambda}_{A_1,\dots,A_n}(t_1^{(m)}, \dots, t_n^{(m)}) \\ &\quad = \operatorname{trace}\bigl\{B^{(m)}_{j+1} \exp[-\varepsilon^{(m)}_{j+1} H_\Lambda] \\ &\qquad\quad \times B^{(m)}_{j+2} \exp[-\varepsilon^{(m)}_{j+2} H_\Lambda] \dots B^{(m)}_n \exp[-\varepsilon^{(m)}_n H_\Lambda] \\ &\qquad\quad \times B^{(m)}_1 \exp[-\varepsilon^{(m)}_1 H_\Lambda] \dots B^{(m)}_j \exp[-\varepsilon^{(m)}_j H_\Lambda] \cdot \exp[-(\beta/2) H_\Lambda]\bigr\}, \end{aligned} \tag{1.2.79}$$

where the operators $B_k^{(m)}$, $k = 1, \dots, n$, are given by (1.2.73) with $\theta_k = \theta_k^{(m)}$. Furthermore, $\varepsilon_k^{(m)} \downarrow 0$ as $m \to +\infty$, for all $k = 1, \dots, j-1, j+1, \dots, n$ and $\varepsilon_j^{(m)} \uparrow \beta/2$. The right-hand side of (1.2.79) can be considered as the value of the functional

$$\mathfrak{C}_\Lambda \ni A(\varepsilon_1^{(m)}, \dots, \varepsilon_n^{(m)}) \mapsto \operatorname{trace}\bigl\{A(\varepsilon_1^{(m)}, \dots, \varepsilon_n^{(m)}) \exp[-(\beta/2) H_\Lambda]\bigr\}. \tag{1.2.80}$$

By Proposition 1.2.15 this functional is continuous in the σ-weak and hence in the σ-strong topologies. Thus, the convergence (1.2.78) will hold if

$$A(\varepsilon_1^{(m)}, \dots, \varepsilon_n^{(m)}) \to B_{j+1} \dots B_n B_1 \dots B_j \exp[-(\beta/2) H_\Lambda], \tag{1.2.81}$$

σ-strongly as $m \to +\infty$. The operator $A(\varepsilon_1^{(m)}, \ldots, \varepsilon_n^{(m)})$ is the product of $B_k^{(m)}$ and the operators $\exp[-\varepsilon_k^{(m)} H_\Lambda]$, $k = 1, \ldots, n$ with $\varepsilon_j^{(m)} \uparrow \beta/2$ and $\varepsilon_k^{(m)} \downarrow 0$, for $k \neq j$. By Proposition 1.2.8, for each k, $B_k^{(m)} \to B_k$, σ-strongly. Thus, in view of Proposition 1.2.9, to prove (1.2.81) we have to show that

$$\begin{aligned} &\exp[-\varepsilon_j^{(m)} H_\Lambda] \to \exp[-(\beta/2) H_\Lambda], \\ &\exp[-\varepsilon_k^{(m)} H_\Lambda] \to I, \quad k = 1, \ldots, j-1, j+1, \ldots, n, \end{aligned} \tag{1.2.82}$$

σ-strongly as $m \to +\infty$. Note that one cannot expect that the same convergence holds in the norm topology. Let us take any $\Phi = \{\phi_n\}_{n \in \mathbb{N}} \in \mathcal{F}_\Lambda$, see (1.2.22), and let $\{\psi_s\}_{s \in \mathbb{N}}$ be as in (1.2.54). The condition (1.2.22) yields

$$\sum_{s \in \mathbb{N}} \alpha_s \stackrel{\text{def}}{=} \sum_{n,s \in \mathbb{N}} \left|(\psi_s, \phi_n)_{\mathcal{H}_\Lambda}\right|^2 < \infty. \tag{1.2.83}$$

Thus, see (1.2.23),

$$\begin{aligned} \left\|\left(I - \exp[-\varepsilon_k^{(m)} H_\Lambda]\right)\right\|_\Phi^2 &= \sum_{n,s \in \mathbb{N}} \left|(\psi_s, \phi_n)_{\mathcal{H}_\Lambda}\right|^2 \left(1 - \exp[-\varepsilon_k^{(m)} E_s]\right)^2 \\ &\leq \sum_{s=1}^{s_0} \left(1 - \exp[-\varepsilon_k^{(m)} E_s]\right)^2 \alpha_s + \sum_{s \geq s_0 + 1} \alpha_s, \end{aligned}$$

which holds for any $s_0 \in \mathbb{N}$. Given $\epsilon > 0$, one finds s_0 such that the second term gets smaller than $\epsilon/2$. Afterwards, one picks m_0 such that for all $m > m_0$, the first term also gets smaller than $\epsilon/2$. This yields that $\exp[-\varepsilon_k^{(m)} H_\Lambda]$ converges σ-strongly to the identity operator. The convergence (1.2.82) of $\exp[-\varepsilon_j^{(m)} H_\Lambda]$ can be proven in the same way.

To complete the proof of claim (b) we employ the maximum modulus principle (see e.g., [269], pages 21, 22), by which the module of a function, holomorphic in the domain $\mathcal{D}_\beta^n$ and continuous on its closure, achieves its maximal value on the Shilov boundary of $\mathcal{D}_\beta^n$, which is its skeleton (1.2.72). Therefore, the estimate (1.2.53) follows from (1.2.77).

To prove claim (c) let us show that a function $f \in \mathrm{Hol}(\mathcal{D}_\beta^n)$, which is zero on $\mathcal{D}_n^\beta(0)$, is identically zero on the whole domain $\mathcal{D}_\beta^n$. To this end we use the following known fact from complex analysis[3]: if a function φ, that is holomorphic in a domain $D \subset \mathbb{C}^n$, vanishes at some point $a \in D$ together with all its partial derivatives, then $\varphi \equiv 0$. For a point $t^0 = (t_1, \ldots, t_n) \in \mathcal{D}_\beta^n$ and a small enough $\delta > 0$, the *imaginary δ-neighborhood* of t^0 is set to be

$$U_\delta(t^0) = \{(t_1, \ldots, t_n) \in \mathcal{D}_\beta^n \mid |\Im(t_i) - \Im(t_i^0)| < \delta, \ \Re(t_i) = \Re(t_i^0), \ i = 1, \ldots, n\}.$$

[3] See Theorem 5, page 21 in [269].

Clearly, for every $t^0 \in \mathcal{D}^n_\beta(0)$, one finds small enough $\delta > 0$, such that $U_\delta(t^0) \subset \mathcal{D}^n_\beta$. Suppose now that a given $f \in \mathrm{Hol}(\mathcal{D}^n_\beta)$ vanishes in $U_\delta(t^0)$ for a certain $t^0 \in \mathcal{D}^n_\beta(0)$ and an appropriate $\delta > 0$. As a holomorphic function, f has the series expansion

$$f(t) = \sum_{k \in \mathbb{N}_0^n} c_k (t - t^0)^k,$$

where $t^k = t_1^{k_1} \dots t_n^{k_n}$, see (1.1.50). This series converges in some open subset of $\mathcal{D}^n_\beta$, which we suppose to contain $U_\delta(t^0)$. Set $t_i = \theta_i + \mathrm{i}\tau_i$, $i = 1, \dots, n$. Then restricting the above expansion to $U_\delta(t^0)$ we find that

$$\sum_{k \in \mathbb{N}_0^n} \mathrm{i}^{|k|} c_k (\tau - \tau^0)^k \equiv 0,$$

for all $|\tau_i - \tau_i^0| < \delta$. As the latter series is still absolutely convergent, we can differentiate it with respect to τ and then set $\tau = \tau^0$, which immediately will yield $c_k = 0$ for all $k \in \mathbb{N}_0^n$. Then by Theorem 5 of [269] $f \equiv 0$ in $\mathcal{D}^n_\beta$ □

Corollary 1.2.33. *For each $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, the local Gibbs state $\varrho_{\beta,\Lambda}$ is a β-KMS state relative to the groups of time automorphisms $\mathfrak{A}_\Lambda$.*

Proof. The function $F_{A,B}$ mentioned in Definition 1.2.6 is set to be

$$F_{A,B}(t + \mathrm{i}\tau) = G^{\beta,\Lambda}_{A,B}(0, t + \mathrm{i}\tau), \quad t \in \mathbb{R}, \ \tau \in (0, \beta).$$

By claims (a) and (b), this function is analytic in $\{t + \mathrm{i}\tau \mid t \in \mathbb{R}, \ \ \tau \in (0,\beta)\}$ and continuous on the closure of this set. By (1.2.55)

$$\begin{aligned} G^{\beta,\Lambda}_{B,A}(t,0) &= \frac{1}{Z_{\beta,\Lambda}(0)} \sum_{s_1,s_2 \in \mathbb{N}} B_{s_2 s_1} \exp(-\mathrm{i}tE_{s_1}) A_{s_1 s_2} \exp(\mathrm{i}(t + \mathrm{i}\beta)E_{s_2}) \\ &= \frac{1}{Z_{\beta,\Lambda}(0)} \sum_{s_1,s_2 \in \mathbb{N}} A_{s_1 s_2} \exp(\mathrm{i}(t + \mathrm{i}\beta)E_{s_2}) \\ &\qquad \times B_{s_2 s_1} \exp(-\mathrm{i}(t + \mathrm{i}\beta)E_{s_1}) \exp(-\beta E_{s_1}) \\ &= G^{\beta,\Lambda}_{A,B}(0, t + \mathrm{i}\beta). \end{aligned}$$

Therefore, the KMS condition (1.2.21) is satisfied. □

The restriction of the function $G^{\beta,\Lambda}_{A_1,\dots,A_n}$ to $\mathcal{D}^n_\beta(0)$, that is,

$$\Gamma^{\beta,\Lambda}_{A_1,\dots,A_n}(\tau_1, \dots, \tau_n) \stackrel{\mathrm{def}}{=} G^{\beta,\Lambda}_{A_1,\dots,A_n}(\mathrm{i}\tau_1, \dots, \mathrm{i}\tau_n), \tag{1.2.84}$$

is called *a Matsubara function*; it is defined on $\overline{\mathcal{C}}^n_\beta$, see (1.2.58). In the sequel, we use the extension of (1.2.84) to $[0,\beta]^n$ defined as follows. For given $(\tau_1, \dots, \tau_n) \in [0,\beta]^n$, let σ be the permutation of $\{1,\ldots,n\}$ such that $\tau_{\sigma(1)} \leq \tau_{\sigma(2)} \leq \cdots \leq \tau_{\sigma(n)}$. Then we set

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \Gamma^{\beta,\Lambda}_{A_{\sigma(1)},\ldots,A_{\sigma(n)}}(\tau_{\sigma(1)},\ldots,\tau_{\sigma(n)}), \tag{1.2.85}$$

with the right-hand side defined by (1.2.84).

In addition to the notion of a positive operator, see Definition 1.1.8, we introduce now another notion of this kind[4]. Recall that $\mathcal{H}_\Lambda = L^2(\mathbb{R}^{\nu|\Lambda|})$.

Definition 1.2.34. An operator $A \in \mathfrak{C}_\Lambda$ is said to be positivity preserving if $(A\psi)(x) \geq 0$ for almost all $x \in \mathbb{R}^{\nu|\Lambda|}$, whenever $\psi(x) \geq 0$ for almost all $x \in \mathbb{R}^{\nu|\Lambda|}$.

A multiplication operator F is positive if and only if the corresponding Borel function $F\colon \mathbb{R}^{\nu|\Lambda|} \to \mathbb{R}$ is almost everywhere nonnegative. In this case, F is positivity preserving. For a $\psi \in \mathcal{H}_\Lambda$, $\beta > 0$, and the operator D_Λ defined by (1.2.32), by (1.2.34) we have

$$\begin{aligned}
&(\exp(-\beta D_\Lambda)\,\psi)(x) \\
&\quad = \left(\frac{m}{2\pi\beta}\right)^{\nu|\Lambda|/2} \int_{\mathbb{R}^{\nu|\Lambda|}} \exp\left(-\frac{m}{2\beta}\sum_{\ell\in\Lambda}|x_\ell - y_\ell|^2\right)\psi(y)\mathrm{d}y.
\end{aligned} \tag{1.2.86}$$

Let the above wave function be nonnegative almost everywhere. Then by (1.2.86),

$$(\exp(-\beta D_\Lambda)\,\psi)\,(x) > 0, \tag{1.2.87}$$

for all x. Therefore, $\exp(-\beta D_\Lambda)$ is positivity preserving for any β. By (1.2.33) and (1.2.47) we have

$$\begin{aligned}
\exp(-\beta H_\Lambda) &= \exp(-\beta D_\Lambda + \beta W_\Lambda) \\
&= \mathrm{lim}^{\mathrm{s}}_{n\to+\infty}\left[\exp(-(\beta/n)D_\Lambda)\exp((\beta/n)W_\Lambda)\right]^n .
\end{aligned} \tag{1.2.88}$$

Note that $\exp(\gamma W_\Lambda)$ is bounded for any $\gamma \geq 0$, see (1.1.10), and hence positivity preserving. Since the product of finite number of positivity preserving operators, as well as the strong limit of a sequence of such operators, are positivity preserving, the operator $\exp(-\beta H_\Lambda)$ is also positivity preserving for any $\beta > 0$.

Theorem 1.2.35. *The Matsubara functions* (1.2.85) *are continuous on* $[0,\beta]^n$ *and have the following properties:*

(a) *positivity: for all positive bounded multiplication operators* $F_1,\ldots F_n$*,*

$$\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) \geq 0, \tag{1.2.89}$$

(b) *shift invariance: for every* $\theta \in [0,\beta]$*,*

$$\Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1,\ldots,\tau_n) = \Gamma^{\beta,\Lambda}_{A_1,\ldots,A_n}(\tau_1+\theta,\ldots,\tau_n+\theta), \tag{1.2.90}$$

where addition is modulo β.

[4]See Definition 18.3 in [176], where in Section 18 one can find detailed information about this notion.

Proof. The continuity follows directly from claim (b) of Theorem 1.2.32. Let us prove (1.2.89). By (1.2.56),

$$Z_{\beta,\Lambda} = \sum_{s=1}^{\infty} e^{-\beta E_s} > 0. \tag{1.2.91}$$

As the operator $\exp(-\gamma H_\Lambda)$ is positivity preserving for all $\gamma > 0$, by Theorem 18.4 of [176] it follows that

$$\begin{aligned}&\mathrm{trace}\{F_1 \exp[-(\tau_2 - \tau_1)H_\Lambda] F_{n-1} \exp[-(\tau_n - \tau_{n-1})H_\Lambda] \\ &\quad \times F_n \exp[-(\beta - \tau_n + \tau_1)H_\Lambda]\} \geq 0.\end{aligned}$$

Then by (1.2.91), one gets

$$\begin{aligned}\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) &= \frac{1}{Z_{\beta,\Lambda}} \mathrm{trace}\{F_1 \exp[-(\tau_2 - \tau_1)H_\Lambda] \\ &\quad \times \cdots \times F_{n-1} \exp[-(\tau_n - \tau_{n-1})H_\Lambda] \\ &\quad \times F_n \exp[-(\beta - \tau_n + \tau_1)H_\Lambda]\} \\ &\geq 0.\end{aligned} \tag{1.2.92}$$

The shift invariance (1.2.90), corresponding to the KMS periodicity (1.2.21), follows immediately from (1.2.92). □

In view of Theorem 1.2.24 and Corollary 1.2.31, the Green functions constructed according to (1.2.50) on multiplication operators, which form a family satisfying the conditions of Corollary 1.2.31, fully determine the local Gibbs states $\varrho_{\beta,\Lambda}$. Claim (c) of Theorem 1.2.32 yields in turn that these states are determined by the Matsubara functions (1.2.84) constructed on such operators. Therefore, the system of Matsubara functions constructed on multiplication operators, which form a family with the mentioned properties, contains the whole information about the local Gibbs states. The main point of the Euclidean approach is the representation of such Matsubara functions as integrals with respect to probability measures on certain infinite-dimensional spaces.

The positivity property (1.2.89) corresponds to the *stochastic positivity* of the states (1.2.12). According to Definition 3.1 in [176], the state is stochastically positive with respect to the abelian subalgebra $\mathfrak{M}_\Lambda$ of the algebra $\mathfrak{C}_\Lambda$ if for all positive elements of $\mathfrak{M}_\Lambda$, the Matsubara functions constructed on such operators have the property (1.2.89). Correspondingly (see Definition 3.2 in [176]), the tuple $(\mathfrak{C}_\Lambda, \mathfrak{M}_\Lambda, \mathfrak{A}_\Lambda, \varrho_{\beta,\Lambda})$ is a *stochastically positive KMS system*. It is worth noting, that here the first element of the tuple is the algebra of *all* bounded linear operators $T : \mathcal{H}_\Lambda \to \mathcal{H}_\Lambda$. In accordance with the general theory developed in [176], see also [175], the stochastic positivity mentioned above will allow us to construct probability measures (associated with certain stochastic processes) and to obtain the representations mentioned above. First we construct such probability measures for the system described by (1.1.3), (1.2.5) but with all $V_\ell = 0$ and all $J_{\ell\ell'} = 0$, that is, for the system of noninteracting quantum harmonic oscillators. Then the measures corresponding to the original model will be obtained as perturbations of such measures.

1.2.5 Gibbs States of Noninteracting Harmonic Oscillators

The Hamiltonian of a system of noninteracting harmonic oscillators located in Λ has the form, see (1.1.3) and (1.2.5),

$$H_{\Lambda}^{\mathrm{har}} = \sum_{\ell \in \Lambda} \sum_{j=1}^{\nu} H_{\ell,j}^{\mathrm{har}}, \tag{1.2.93}$$

where $H_{\ell,j}^{\mathrm{har}}$ is a copy of the operator (1.1.68). This decomposition yields

$$\exp(-\beta H_{\Lambda}^{\mathrm{har}}) = \prod_{\ell \in \Lambda} \prod_{j=1}^{\nu} \exp\big(-\beta H_{\ell,j}^{\mathrm{har}}\big). \tag{1.2.94}$$

Moreover, the Hilbert space $\mathcal{H}_{\Lambda}$, see (1.2.1), can be decomposed according to

$$\mathcal{H}_{\Lambda} = L^2\big(\mathbb{R}^{\nu|\Lambda|}\big) = \bigotimes_{\ell \in \Lambda} \bigotimes_{j=1}^{\nu} \mathcal{H}_{\ell}^{(j)}, \quad \mathcal{H}_{\ell}^{(j)} = L^2(\mathbb{R}), \tag{1.2.95}$$

thereby,

$$\varrho_{\beta,\Lambda}^{\mathrm{har}} = \bigotimes_{\ell \in \Lambda} \bigotimes_{j=1}^{\nu} \varrho_{\ell,j}, \tag{1.2.96}$$

where the state

$$\varrho_{\ell,j}(A) = \mathrm{trace}\big[A \exp(-\beta H_{\ell,j}^{\mathrm{har}})\big] / \mathrm{trace}\big[\exp(-\beta H_{\ell,j}^{\mathrm{har}})\big], \quad A \in \mathfrak{C}_{\ell,j}, \tag{1.2.97}$$

is defined on the algebra $\mathfrak{C}_{\ell,j}$ of all bounded linear operators on the Hilbert space $\mathcal{H}_{\ell}^{(j)}$. Until the end of this subsection we consider only the one-dimensional oscillator described by the Hamiltonian (1.1.68); hence, we drop the sub- and superscripts ℓ, j and write p, q, H^{har}, $\mathcal{H}$, $\mathfrak{C}$ and ϱ instead of $p_{\ell}^{(j)}$, $q_{\ell}^{(j)}$, $H_{\ell,j}^{\mathrm{har}}$, $\mathcal{H}_{\ell}^{(j)}$, $\mathfrak{C}_{\ell,j}$, and $\varrho_{\ell,j}$, respectively.

One observes that, cf. (1.1.73) and (1.1.86),

$$N\psi_n = n\psi_n, \quad H^{\mathrm{har}}\psi_n = (n + 1/2)\delta\psi_n; \tag{1.2.98}$$

thus, for any $\zeta \in \mathbb{C}$ and $n \in \mathbb{N}_0$, it follows that, see (1.1.65),

$$\begin{aligned} \exp(\zeta H^{\mathrm{har}}) A \exp(-\zeta H^{\mathrm{har}})\psi_n &= e^{-\zeta\delta}\sqrt{n}\psi_{n-1} = e^{-\zeta\delta} A\psi_n, \\ \exp(\zeta H^{\mathrm{har}}) A^{\dagger} \exp(-\zeta H^{\mathrm{har}})\psi_n &= e^{\zeta\delta}\sqrt{n+1}\psi_{n+1} = e^{\zeta\delta} A^{\dagger}\psi_n, \end{aligned}$$

which yields

$$\begin{aligned} \mathfrak{a}^t(A) &= \exp(\mathrm{i}tH^{\mathrm{har}}) A \exp(-\mathrm{i}tH^{\mathrm{har}}) = e^{-\mathrm{i}\delta t} A, \\ \mathfrak{a}^t(A^{\dagger}) &= \exp(\mathrm{i}tH^{\mathrm{har}}) A^{\dagger} \exp(-\mathrm{i}tH^{\mathrm{har}}) = e^{\mathrm{i}\delta t} A^{\dagger}, \end{aligned} \tag{1.2.99}$$

and

$$\alpha^t(q) = \exp(\mathrm{i}tH^{\mathrm{har}})q\exp(-\mathrm{i}tH^{\mathrm{har}}) = \frac{1}{\kappa\sqrt{2}}\big(e^{-\mathrm{i}\delta t}A + e^{\mathrm{i}\delta t}A^\dagger\big). \tag{1.2.100}$$

This allows for calculating explicitly the Green function

$$G_2(t_1,t_2) \stackrel{\mathrm{def}}{=} \varrho\big(\alpha^{t_1}(q)\alpha^{t_2}(q)\big), \quad t_1,t_2 \in \mathcal{D}_\beta^2, \tag{1.2.101}$$

and the corresponding Matsubara function $\Gamma_2(\tau_1,\tau_2) = G_2(\mathrm{i}\tau_1,\mathrm{i}\tau_2)$. Taking into account that $\varrho(AA) = \varrho(A^\dagger A^\dagger) = 0$, we get

$$G_2(t_1,t_2) = \frac{1}{2\kappa^2}\big[e^{\mathrm{i}(t_1-t_2)\delta}\varrho(A^\dagger A) + e^{\mathrm{i}(t_2-t_1)\delta}\varrho(AA^\dagger)\big]. \tag{1.2.102}$$

By (1.1.75) and (1.1.86)

$$\begin{aligned}\operatorname{trace}[A^\dagger A\exp(-\beta H^{\mathrm{har}})] &= \sum_{n=0}^{\infty}\big(\psi_n, A^\dagger A\exp(-\beta H^{\mathrm{har}})\psi_n\big)_{\mathcal{H}}\\ &= \sum_{n=0}^{\infty} n e^{-\beta\delta(n+1/2)}\\ &= e^{-\beta\delta/2}e^{-\beta\delta}\big[1-e^{-\beta\delta}\big]^{-2}.\end{aligned}$$

Similarly,

$$\begin{aligned}\operatorname{trace}[AA^\dagger\exp(-\beta H^{\mathrm{har}})] &= e^{-\beta\delta/2}\big[1-e^{-\beta\delta}\big]^{-2},\\ \operatorname{trace}[\exp(-\beta H^{\mathrm{har}})] &= e^{-\beta\delta/2}\big[1-e^{-\beta\delta}\big]^{-1}.\end{aligned}$$

Employing this in (1.2.102), we obtain

$$\begin{aligned}G_2(t_1,t_2) &= \frac{1}{2\kappa^2}\left(\frac{e^{-\beta\delta}}{1-e^{-\beta\delta}}e^{\mathrm{i}(t_1-t_2)\delta} + \frac{1}{1-e^{-\beta\delta}}e^{\mathrm{i}(t_2-t_1)\delta}\right),\\ \Gamma_2(\tau_1,\tau_2) &= \frac{1}{2\kappa^2[1-e^{-\beta\delta}]}\big(e^{-(\tau_2-\tau_1)\delta} + e^{-(\beta-\tau_2+\tau_1)\delta}\big), \quad \tau_1 \le \tau_2.\end{aligned} \tag{1.2.103}$$

For $(t_1,\dots,t_{2n}) \in \mathcal{D}_\beta^{2n}$ and $0 \le \tau_1 \le \cdots \le \tau_{2n} \le \beta$, we set

$$\begin{aligned}G_{2n}(t_1,\dots t_{2n}) &= \varrho(\alpha^{t_1}(q)\dots\alpha^{t_{2n}}(q)),\\ \Gamma_{2n}(\tau_1,\dots\tau_{2n}) &= G_{2n}(\mathrm{i}\tau_1,\dots\mathrm{i}\tau_{2n}).\end{aligned} \tag{1.2.104}$$

In the sequel, we use the symmetric extensions (1.2.85) of the Matsubara functions (1.2.104). In particular,

$$\Gamma_2(\tau_1,\tau_2) = \frac{1}{2\kappa^2[1-e^{-\beta\delta}]}\big(e^{-|\tau_2-\tau_1|\delta} + e^{-(\beta-|\tau_2-\tau_1|)\delta}\big). \tag{1.2.105}$$

Theorem 1.2.36. *The symmetric extensions* (1.2.85) *of the Matsubara functions* (1.2.104) *have the property*

$$\Gamma_{2n}(\tau_1,\dots\tau_{2n}) = \sum \Gamma_2(\tau_{i_1},\tau_{i_2})\dots\Gamma_2(\tau_{i_{2n-1}},\tau_{i_{2n}}), \tag{1.2.106}$$

where the sum is taken over all possible partitions of the set $\{1,\dots,2n\}$ *onto the unordered pairs* $(i_1,i_2),\dots,(i_{2n-1},i_{2n})$.

Proof. Clearly, it is enough to prove the statement for $0 \le \tau_1 \le \dots \le \tau_n < \beta$. In this case, one can write

$$\Gamma_{2n}(\tau_1,\dots\tau_{2n}) = \varrho\left(\mathfrak{a}^{\mathrm{i}\tau_1}(q)\mathfrak{a}^{\mathrm{i}\tau_2}(q)\dots\mathfrak{a}^{\mathrm{i}\tau_{2n}}(q)\right). \tag{1.2.107}$$

Set

$$\Gamma_{2n}^{+} = \varrho(A\mathfrak{a}^{\mathrm{i}\tau_2}(q)\dots\mathfrak{a}^{\mathrm{i}\tau_{2n}}(q)), \quad \Gamma_{2n}^{-} \ = \ \varrho(A^{\dagger}\mathfrak{a}^{\mathrm{i}\tau_2}(q)\dots\mathfrak{a}^{\mathrm{i}\tau_{2n}}(q)).$$

Then by (1.2.99), (1.2.100), and (1.2.104)

$$\Gamma_{2n}(\tau_1,\dots\tau_{2n}) = \frac{e^{\tau_1\delta}}{\kappa\sqrt{2}}\Gamma_{2n}^{+} + \frac{e^{-\tau_1\delta}}{\kappa\sqrt{2}}\Gamma_{2n}^{-}. \tag{1.2.108}$$

By (1.1.84), (1.2.100), for $j = 2,\dots,2n$, one has

$$\begin{aligned} A\mathfrak{a}^{\mathrm{i}\tau_j}(q) &= \mathfrak{a}^{\mathrm{i}\tau_j}(q)A + [A,\mathfrak{a}^{\mathrm{i}\tau_j}(q)] \\ &= \mathfrak{a}^{\mathrm{i}\tau_j}(q)A + e^{-\tau_j\delta}/\kappa\sqrt{2}, \\ A^{\dagger}\mathfrak{a}^{\mathrm{i}\tau_j}(q) &= \mathfrak{a}^{\mathrm{i}\tau_j}(q)A^{\dagger} + [A^{\dagger},\mathfrak{a}^{\mathrm{i}\tau_j}(q)] \\ &= \mathfrak{a}^{\mathrm{i}\tau_j}(q)A^{\dagger} - e^{\tau_j\delta}/\kappa\sqrt{2}, \end{aligned} \tag{1.2.109}$$

and

$$\begin{aligned} A\exp(-\beta H^{\mathrm{har}}) &= \exp(-\beta H^{\mathrm{har}})Ae^{-\beta\delta}, \\ A^{\dagger}\exp(-\beta H^{\mathrm{har}}) &= \exp(-\beta H^{\mathrm{har}})A^{\dagger}e^{\beta\delta}. \end{aligned} \tag{1.2.110}$$

Then by (1.2.109),

$$\begin{aligned} \Gamma_{2n}^{+} &= \frac{e^{-\tau_2\delta}}{\kappa\sqrt{2}}\Gamma_{2n-2}(\tau_3,\dots,\tau_{2n}) + \varrho(\mathfrak{a}^{\mathrm{i}\tau_2}(q)A\mathfrak{a}^{\mathrm{i}\tau_3}(q)\dots\mathfrak{a}^{\mathrm{i}\tau_{2n}}(q)) \\ &= \frac{e^{-\tau_2\delta}}{\kappa\sqrt{2}}\Gamma_{2n-2}(\tau_3,\tau_4,\dots,\tau_{2n}) + \frac{e^{-\tau_3\delta}}{\kappa\sqrt{2}}\Gamma_{2n-2}(\tau_2,\tau_4,\dots,\tau_{2n}) \\ &\quad + \dots + \frac{e^{-\tau_{2n}\delta}}{\kappa\sqrt{2}}\Gamma_{2n-2}(\tau_2,\tau_3,\dots,\tau_{2n-1}) + \varrho(\mathfrak{a}^{\mathrm{i}\tau_2}(q)\mathfrak{a}^{\mathrm{i}\tau_3}(q)\dots\mathfrak{a}^{\mathrm{i}\tau_{2n}}(q)A). \end{aligned} \tag{1.2.111}$$

Making use of the cyclicity of the trace and of the commutation rules (1.2.110) we get

$$\begin{aligned}\varrho(\alpha^{i\tau_2}(q)\alpha^{i\tau_3}(q)\dots\alpha^{i\tau_{2n}}(q)A) &= e^{-\beta\delta}\varrho(A\alpha^{i\tau_2}(q)\alpha^{i\tau_3}(q)\dots\alpha^{i\tau_{2n}}(q))\\ &= e^{-\beta\delta}\Gamma^+_{2n},\end{aligned}$$

which yields by (1.2.111),

$$\Gamma^+_{2n} = \frac{1}{\kappa\sqrt{2}}\sum_{i=2}^{2n}\frac{e^{-\tau_i\delta}}{1-e^{-\beta\delta}}\Gamma_{2n-2}(\tau_2,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_{2n}).$$

Similarly,

$$\Gamma^-_{2n} = \frac{1}{\kappa\sqrt{2}}\sum_{i=2}^{2n}\frac{e^{\tau_i\delta}}{e^{\beta\delta}-1}\Gamma_{2n-2}(\tau_2,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_{2n}).$$

Inserting these two expressions into (1.2.108) we arrive at

$$\Gamma_{2n}(\tau_1,\dots,\tau_{2n}) = \sum_{i=2}^{2n}\Gamma_2(\tau_1,\tau_i)\Gamma_{2n-2}(\tau_2,\dots,\tau_{i-1},\tau_{i+1},\dots,\tau_{2n}).$$

Applying the same procedure to Γ_{2n-2} we finally get (1.2.106). □

With the help of the result just proven we express the Matsubara functions of a complete family of multiplication operators[5] in terms of the integral kernel of the operator S, that is, by means of $\Gamma_2(\tau,\tau')$. To this end, for $\lambda\in\mathbb{R}$, we set $Q(\lambda) = \exp(i\lambda q)$ (cf. (1.2.37)). Then every $Q(\lambda)$ is the multiplication operator by the function $\exp(i\lambda x)$. By Corollary 1.2.31 the state ϱ (1.2.97) is uniquely determined by its values on the operators,

$$\alpha^{t_1}(Q(\lambda_1))\dots\alpha^{t_n}(Q(\lambda_n)),\quad n\in\mathbb{N},\ \lambda_1,\dots,\lambda_n\in\mathbb{R}.$$

Lemma 1.2.37. *For every $\lambda, t\in\mathbb{R}$, it follows that*

$$\alpha^t(Q(\lambda)) = \exp\left(i\lambda\alpha^t(q)\right). \tag{1.2.112}$$

Proof. By (1.1.99) for all $t\in\mathbb{R}$, we have

$$\mathrm{Dom}(\alpha^t(q)) = \mathrm{Dom}(q) = \left\{\psi\in L^2(\mathbb{R}) \mid \int_{\mathbb{R}} x^2|\psi(x)|^2 dx < \infty\right\}.$$

Now we take $\phi\in\mathrm{Dom}(q)$ and, for a fixed $t\in\mathbb{R}$, set $\psi = \exp(-itH^{\mathrm{har}})\phi$, cf. (1.2.13) and (1.2.14). Let μ_ϕ and μ_ψ be the spectral measures of q corresponding to ϕ and ψ, respectively, see (1.1.64). Then

$$(\phi,\alpha^t(q)\phi)_{L^2(\mathbb{R})} = \int_{\mathbb{R}} x\mu_\psi(dx),$$

[5] See Definition 1.2.21 and Corollary 1.2.31.

that is, μ_ψ is the spectral measure of $\mathfrak{a}^t(q)$ corresponding to ϕ. Thus,

$$\begin{aligned}(\phi, \mathfrak{a}(Q(\lambda))\phi)_{L^2(\mathbb{R})} &= (\psi, Q(\lambda)\psi)_{L^2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \exp(\mathrm{i}\lambda x)\mu_\psi(\mathrm{d}x \\ &= \left(\phi, \exp\left(\mathrm{i}\lambda \mathfrak{a}^t(q)\right)\phi\right)_{L^2(\mathbb{R})},\end{aligned}$$

which by Proposition 1.1.4 completes the proof, taking into account that $\mathrm{Dom}(q)$ is dense in $L^2(\mathbb{R})$. □

According to Theorems 1.2.24 and 1.2.32 the family of the Matsubara functions

$$\Gamma_{\lambda_1,\ldots,\lambda_n}(\tau_1,\ldots,\tau_n) = G_{\lambda_1,\ldots,\lambda_n}(\mathrm{i}\tau_1,\ldots,\mathrm{i}\tau_n), \quad n\in\mathbb{N}, \tag{1.2.113}$$

where $0 \le \tau_1 \le \cdots \le \tau_n \le \beta$ and

$$G_{\lambda_1,\ldots,\lambda_n}(t_1,\ldots,t_n) \stackrel{\mathrm{def}}{=} \varrho\left(\mathfrak{a}^{t_1}(Q(\lambda_1))\ldots\mathfrak{a}^{t_n}(Q(\lambda_n))\right),$$

completely determines the state ϱ. For the harmonic oscillators considered here, these functions may be computed explicitly.

Theorem 1.2.38. *For every $n \in \mathbb{N}$, one has*

$$\Gamma_{\lambda_1,\ldots,\lambda_n}(\tau_1,\ldots,\tau_n) = \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n}\Gamma_2(\tau_i,\tau_j)\lambda_i\lambda_j\right\}, \tag{1.2.114}$$

where the Matsubara function Γ_2 is taken in its symmetrized version (1.2.105).

Proof. As in the case of Theorem 1.2.36, we prove the statement for ordered $0 \le \tau_1 \le \cdots \le \tau_n < \beta$. To this end we consider the dependence of the function (1.2.114) on the real parameters $\lambda_1,\ldots,\lambda_n$. Set

$$\Psi_n(\lambda_1,\ldots,\lambda_n) = \Gamma_{\lambda_1,\ldots,\lambda_n}(\tau_1,\ldots,\tau_n). \tag{1.2.115}$$

Clearly, this function is differentiable. By (1.2.112)

$$\begin{aligned}\frac{\partial}{\partial\lambda_1}\Psi_n(\lambda_1,\ldots,\lambda_n) &= \mathrm{i}\theta^+\varrho\left(A\exp\left(\mathrm{i}\lambda_1\mathfrak{a}^{\mathrm{i}\tau_1}(q)\right)\ldots\exp\left(\mathrm{i}\lambda_n\mathfrak{a}^{\mathrm{i}\tau_n}(q)\right)\right) \\ &\quad + \mathrm{i}\theta^-\varrho\left(A^\dagger\exp\left(\mathrm{i}\lambda_1\mathfrak{a}^{\mathrm{i}\tau_1}(q)\right)\ldots\exp\left(\mathrm{i}\lambda_n\mathfrak{a}^{\mathrm{i}\tau_n}(q)\right)\right),\end{aligned} \tag{1.2.116}$$

where

$$\theta^\pm = e^{\pm\mathrm{i}\tau_1}/\kappa\sqrt{2}.$$

This can be considered as a differential equation for the function (1.2.115) subject to the initial condition

$$\Psi_n(0,\lambda_2,\ldots,\lambda_n) = \Psi_{n-1}(\lambda_2,\ldots,\lambda_n). \tag{1.2.117}$$

By means of the commutation relations (1.2.109), (1.2.110), similarly as it was done in obtaining (1.2.111), we transform (1.2.116) into the following equation

$$\frac{\partial}{\partial \lambda_1}\Psi_n(\lambda_1,\ldots,\lambda_n) = -\Psi_n(\lambda_1,\ldots,\lambda_n)\Big(\sum_{i=1}^{n}\Gamma_2(\tau_1,\tau_i)\lambda_i\Big), \tag{1.2.118}$$

the solution of which satisfying (1.2.117) is

$$\Psi_n(\lambda_1,\ldots,\lambda_n) = \exp\Big\{-\frac{1}{2}\Gamma_2(\tau_1,\tau_1)\lambda_1^2 - \sum_{i=2}^{n}\Gamma_2(\tau_1,\tau_i)\lambda_i\Big\}\Psi_{n-1}(\lambda_2,\ldots,\lambda_n).$$

Applying this procedure to the function $\Psi_{n-1}(\lambda_2,\ldots,\lambda_n)$ we finally arrive at (1.2.114). $\square$

In what follows, the state ϱ is completely determined by the two-point Matsubara function $\Gamma_2(\tau,\tau')$ defined by (1.2.105). The same situation occurs also for the ν-dimensional harmonic oscillator. Let us return to the decomposition (1.2.93), (1.2.96), which we rewrite here as

$$\begin{aligned} H_\Lambda^{\mathrm{har}} &= \sum_{\ell\in\Lambda} H_\ell^{\mathrm{har}}, & H_\ell^{\mathrm{har}} &= \sum_{j=1}^{\nu} H_{\ell,j}^{\mathrm{har}}, \\ \varrho_{\beta,\Lambda}^{\mathrm{har}} &= \bigotimes_{\ell\in\Lambda}\varrho_\ell, & \varrho_\ell &= \bigotimes_{j=1}^{\nu}\varrho_{\ell,j}, \end{aligned} \tag{1.2.119}$$

where the Hamiltonian H_ℓ^{har} and the state ϱ_ℓ describe the ν-dimensional rotational invariant harmonic oscillator. The latter decomposition allows us to determine the state ϱ_ℓ by its values on the operators

$$\mathfrak{a}_\ell^{t_1}(Q_\ell(\lambda_1))\ldots\mathfrak{a}_\ell^{t_n}(Q_\ell(\lambda_n)), \tag{1.2.120}$$

where, cf. (1.2.37),

$$Q_\ell(\lambda) = \exp\Big(\mathrm{i}\sum_{j=1}^{\nu}\lambda^{(j)}q_\ell^{(j)}\Big), \quad \lambda = (\lambda^{(1)},\ldots,\lambda^{(\nu)}) \in \mathbb{R}^\nu, \tag{1.2.121}$$

and

$$\begin{aligned} \mathfrak{a}_\ell^{t}(Q_\ell(\lambda)) &\stackrel{\mathrm{def}}{=} \exp\big(\mathrm{i}tH_\ell^{\mathrm{har}}\big)\,Q_\ell(\lambda)\exp\big(-\mathrm{i}tH_\ell^{\mathrm{har}}\big) \\ &= \exp\Big(\mathrm{i}\sum_{j=1}^{\nu}\lambda^{(j)}\mathfrak{a}_\ell^{t}(q_\ell^{(j)})\Big), \end{aligned} \tag{1.2.122}$$

where we have also used Lemma 1.2.37. Here $\mathfrak{a}^t$ is the same as in (1.2.112). Set

$$\begin{aligned} G^{\beta,\ell}_{Q(\lambda_1),\ldots,Q(\lambda_n)}(t_1,\ldots,t_n) &= \varrho_\ell\left(\mathfrak{a}_\ell^{t_1}(Q_\ell(\lambda_1))\ldots\mathfrak{a}_\ell^{t_n}(Q_\ell(\lambda_n))\right), \\ \Gamma^{\beta,\ell}_{Q(\lambda_1),\ldots,Q(\lambda_n)}(\tau_1,\ldots,\tau_n) &= G^{\beta,\ell}_{Q(\lambda_1),\ldots,Q(\lambda_n)}(\mathrm{i}\tau_1,\ldots,\mathrm{i}\tau_n). \end{aligned} \tag{1.2.123}$$

Then as a consequence of Theorem 1.2.38 we have the following

Corollary 1.2.39. *The Matsubara functions* (1.2.123) *satisfy*

$$\begin{aligned}\Gamma^{\beta,\ell}_{Q(\lambda_1),\ldots,Q(\lambda_n)}(\tau_1,\ldots,\tau_n) &= \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n}\sum_{k=1}^{\nu}\Gamma_2(\tau_i,\tau_j)\lambda_i^{(k)}\lambda_j^{(k)}\right\}\\ &= \exp\left\{-\frac{1}{2}\sum_{i,j=1}^{n}\big(\Theta(\tau_i,\tau_j)\lambda_i,\lambda_j\big)\right\},\end{aligned} \tag{1.2.124}$$

where the $\nu \times \nu$*-matrix* $\Theta(\tau,\tau')$ *is defined by its elements*

$$\Theta(\tau,\tau')_{jk} = \delta_{jk}\Gamma_2(\tau,\tau') = \frac{\delta_{jk}}{2\kappa^2}\cdot\frac{e^{-|\tau-\tau'|\delta}+e^{-(\beta-|\tau-\tau'|)\delta}}{1-e^{-\beta\delta}}. \tag{1.2.125}$$

As we shall see in the next section, the latter formula can be related to the Fourier transform of a Gaussian measure on the Hilbert space $L^2_\beta = L^2([0,\beta]\to\mathbb{R}^\nu)$. This fact is crucial for our theory – it will allow us to construct the local Gibbs states described in this section by means of probability measures. In the next section we study such measures and related stochastic processes in more detail.

1.3 Stochastic Analysis

In this section, we present basic facts and notions from stochastic analysis which are then used in the description of the Gibbs states of the model (1.1.3), (1.1.8) in terms of path measures. In Subsection 1.3.2, we introduce Gaussian processes. Among them is the Høegh-Krohn process which we employ to describe the Gibbs state of a single harmonic oscillator. Then, in Subsection 1.3.3, we present elements of the theory of stochastic processes with emphasis on their Hölder continuity and related properties. As we employ various realizations of our basic processes, we present here the Kuratowski theorem by which we relate these realizations with each other. Afterwards, in Subsections 1.3.4 and 1.3.5, we present a number of facts from the theory of probability measures on Polish and Hilbert spaces, respectively. In Subsection 1.3.6, we study in detail properties of the Høegh-Krohn process, as well as construct Markov chains, approximating this process. Finally, in Subsection 1.3.7 we present a description of the Gibbs states of harmonic oscillators by means of the Høegh-Krohn process.

1.3.1 Beginnings

Let $\mathbb{X}$ be a nonempty set. A family of its subsets $\mathcal{X}$ is called a σ-algebra (or σ-field), if

(a) $\mathbb{X} \in \mathcal{X}$;

(b) for any $A \in \mathcal{X}$, $A^c \stackrel{\text{def}}{=} \mathbb{X} \setminus A$ also belongs to $\mathcal{X}$;

(c) if for every $n \in \mathbb{N}$, $A_n \in \mathcal{X}$, then

$$\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{X}.$$

The pair $(\mathbb{X}, \mathcal{X})$ will be called a *measurable space*. The map $\mu\colon \mathcal{X} \to [0, +\infty]$ is called a *measure* on $(\mathbb{X}, \mathcal{X})$ if it is σ-additive and $\mu(\emptyset) = 0$. The σ-additivity means that for all $\{B_n\}_{n\in\mathbb{N}} \subset \mathcal{X}$, such that $B_n \cap B_m = \emptyset$ whenever $n \neq m$, one has

$$\mu\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mu(B_n). \tag{1.3.1}$$

Sometimes, we say that μ is a measure on $\mathbb{X}$ if it is clear which σ-algebra is meant. The measure μ is called *finite* if $\mu(\mathbb{X}) < \infty$; it is a *probability measure* if $\mu(\mathbb{X}) = 1$. In this case, the triple $(\mathbb{X}, \mathcal{X}, \mu)$ is called a *probability space*. If

$$\mathbb{X} = \bigcup_{n-1}^{\infty} A_n, \quad \text{such that } \mu(A_n) < \infty, \text{ for all } n \in \mathbb{N},$$

then μ is called σ*-finite*.

Let $(\mathbb{X}, \mathcal{X})$ and $(\mathbb{Y}, \mathcal{Y})$ be measurable spaces and $f : \mathbb{X} \to \mathbb{Y}$ be a map. Then f is called *measurable* if for every $B \in \mathcal{Y}$, its pre-image $f^{-1}(B)$ belongs to $\mathcal{X}$. Sometimes, in order to indicate which σ-algebras are meant we say that f is $\mathcal{X}/\mathcal{Y}$-measurable. Let μ be a measure on $(\mathbb{X}, \mathcal{X})$ and $f : \mathbb{X} \to \mathbb{Y}$ be measurable. For $B \in \mathcal{Y}$, we set

$$\nu_f(B) \stackrel{\text{def}}{=} (\mu \circ f^{-1})(B) = \mu\left[f^{-1}(B)\right]. \tag{1.3.2}$$

Then ν_f is a measure on $(\mathbb{Y}, \mathcal{Y})$. It is also called the *distribution* of f.

If $\mathbb{X}$ is a topological space, its Borel σ-algebra is the smallest σ-algebra containing all open subsets. Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces, $\mathcal{X}$ and $\mathcal{Y}$ being their Borel σ-algebras. If $f : \mathbb{X} \to \mathbb{Y}$ is continuous, then it is $\mathcal{X}/\mathcal{Y}$-measurable. We say that a sequence of maps $f_n : \mathbb{X} \to \mathbb{Y}$, $n \in \mathbb{N}$, converges point-wise to a map $f : \mathbb{X} \to \mathbb{Y}$, if for every $x \in \mathbb{X}$, $f_n(x) \to f(x)$ as $n \to +\infty$. The point-wise limit of a sequence of measurable maps is measurable.

If $\mathbb{Y} = \mathbb{R}$ and $(\mathbb{X}, \mathcal{X}, \mu)$ is a probability space, a measurable function $f : \mathbb{X} \to \mathbb{Y}$ is called a *random variable*. Let μ be a measure on $(\mathbb{X}, \mathcal{X})$. A measurable function $f : \mathbb{X} \to \mathbb{R}$ is *integrable* with respect to μ (μ-integrable), if

$$\int_{\mathbb{X}} |f(x)|\mu(\mathrm{d}x) < \infty.$$

In the sequel, we often use the following known fact, see e.g., Theorem D, page 110 of [147],

Proposition 1.3.1 (Lebesgue dominated convergence theorem). *Let μ be a measure on $(\mathbb{X}, \mathcal{X})$ and $\{f_n\}_{n\in\mathbb{N}}$, $f_n : \mathbb{X} \to \mathbb{R}$, be a sequence of μ-integrable functions, convergent* *point-wise to a function* $f : \mathbb{X} \to \mathbb{R}$. *Suppose also that there exists a* μ*-integrable function* $g : \mathbb{X} \to [0, +\infty)$, *such that*

$$\forall x \in \mathbb{X}, \ \forall n \in \mathbb{N}: \quad |f_n(x)| \le g(x). \tag{1.3.3}$$

Then f *is also* μ*-integrable and*

$$\lim_{n \to +\infty} \int_{\mathbb{X}} f_n(x)\mu(\mathrm{d}x) = \int_{\mathbb{X}} f(x)\mu(\mathrm{d}x).$$

The main object of our study in this section will be certain families of random variables called *stochastic processes.* We begin by introducing a particular kind of such processes.

1.3.2 Gaussian Processes

Kolmogorov extension theorem

Let I be any index set. For each $\alpha \in I$, let $\mathbb{X}_\alpha$ be a copy of the Euclidean space $\mathbb{X} = \mathbb{R}^\nu$, $\nu \in \mathbb{N}$, and $\mathcal{B}_\alpha$ be its Borel σ-algebra. For $I' \subseteq I$, we set

$$\mathbb{X}^{I'} = \prod_{\alpha \in I'} \mathbb{X}_\alpha. \tag{1.3.4}$$

The elements of $\mathbb{X}^{I'}$ are $x^{I'} = (x_\alpha)_{\alpha \in I'}$. In this case, $x_\alpha \in \mathbb{X}_\alpha$ are called the *components* of $x^{I'}$. Equipped with the component-wise linear operations each $\mathbb{X}^{I'}$ becomes a real linear space. For $I'' \subset I'$, let $\pi_{I'',I'} : \mathbb{X}^{I'} \to \mathbb{X}^{I''}$ be the projection $x^{I'} \mapsto x^{I''}$ such that the components of $x^{I'}$ and $x^{I''}$ indexed by the same $\alpha \in I''$ coincide. Given $x^{I'}$ and $I'' \subset I'$, $J = I' \setminus I''$, we write

$$x^{I'} = x^{I''} \times x^J, \quad \text{where } x^{I''} = \pi_{I'',I'}\big(x^{I'}\big) \text{ and } x^J = \pi_{J,I'}\big(x^{I'}\big). \tag{1.3.5}$$

The pre-image of $x^{I''}$ in $\mathbb{X}^{I'}$ is the set

$$\pi^{-1}_{I'',I'}\big(x^{I''}\big) = \{x^{I'} = x^{I''} \times x^J \mid x^J \in \mathbb{X}^J\}.$$

Similarly, for a subset $A^{I''} \subset \mathbb{X}^{I''}$, we write

$$\pi^{-1}_{I'',I'}\big(A^{I''}\big) = \{x^{I'} = x^{I''} \times x^J \mid x^{I''} \in A^{I''}, \ x^J \in \mathbb{X}^J\} \subset \mathbb{X}^{I'}.$$

Let $0^{I'}$ stand for the zero element of $\mathbb{X}^{I'}$, that is, all the components of $0^{I'}$ are equal to the zero element of $\mathbb{X} = \mathbb{R}^\nu$. For $I'' \subset I'$, the set $\{x^{I''} \times 0^J \mid x^{I''} \in \mathbb{X}^{I''}\}$ is a subspace of $\mathbb{X}^{I'}$, which is isomorphic to $\mathbb{X}^{I''}$; hence, it can and will be identified with $\mathbb{X}^{I''}$.

Let $J \subset I$ be finite. We denote by $|J|$ its cardinality and set $J^c = I \setminus J$. The family of all finite subsets of I will be denoted by $\mathcal{I}(I)$. For $J \in \mathcal{I}(I)$, $\mathbb{X}^J$ is a Euclidean space; by $\mathcal{B}^J$ we denote its Borel σ-algebra. For $A^J \in \mathcal{B}^J$, the set $\{x^I = x^J \times x^{J^c} \mid x^J \in A^J, \ x^{J^c} \in X^{J^c}\} \subset \mathbb{X}^I$, is called a *cylinder* subset of $\mathbb{X}^I$ with base A^J.

Definition 1.3.2. $\mathcal{B}^I$ is the smallest σ-algebra of subsets of $\mathbb{X}^I$ which contains all cylinder subsets with all possible choices of finite $J \subset I$ and $A^J \in \mathcal{B}^J$. It is called the σ-algebra generated by the cylinder subsets.

Assume that for every finite $J \subset I$, a probability measure μ^J on $\mathcal{B}^J$ is given. The family of such measures $\{\mu^J\}_{J\in\mathcal{I}(I)}$ is said to be *consistent* if for any $J \in \mathcal{I}(I)$ and $J' \subset J$, we have

$$\mu^{J'} = \mu^J \circ \pi^{-1}_{J',J}.$$

The family $\{\mu^J\}_{J\in\mathcal{I}(I)}$ determines a probability measure on $\mathcal{B}^I$ according to the following known theorem the proof of which can be found e.g., in [274].

Proposition 1.3.3 (Kolmogorov's extension theorem). *If the family of probability measures $\{\mu^J\}_{J\in\mathcal{I}(I)}$ described above is consistent, there exists a unique probability measure μ^I on $(\mathbb{X}^I, \mathcal{B}^I)$ such that*

$$\mu^J = \mu^I \circ \pi^{-1}_{J,I}.$$

Given $J \in \mathcal{I}(I)$, we set

$$\begin{aligned}\varphi_{\mu^J}(y^J) &= \Big\langle \exp\Big(\mathrm{i}\sum_{\alpha\in J}(y_\alpha,\cdot)\Big)\Big\rangle_{\mu^J} \\ &= \int \exp\Big(\mathrm{i}\sum_{\alpha\in J}(y_\alpha,x_\alpha)\Big)\mu^J(\mathrm{d}x^J), \quad y^J \in \mathbb{X}^J,\end{aligned} \tag{1.3.6}$$

where $(\cdot,\cdot)$ is the scalar product in $\mathbb{R}^\nu$. Here and in the sequel, for a probability measure μ and a μ-integrable function f, we write

$$\langle f\rangle_\mu = \int f\,\mathrm{d}\mu. \tag{1.3.7}$$

The above φ_{μ^J} is the Fourier transform of the measure μ^J, called also the characteristic function of μ^J. By Bochner's theorem it is continuous on $\mathbb{X}^J$, positive definite, and $\varphi_{\mu^J}(0) = 1$. The positive definiteness of φ_{μ^J} is the following property: for every $n \in \mathbb{N}$, $y_1^J, \dots y_n^J \in \mathbb{X}^J$, and for all $(\xi_1, \dots \xi_n) \in \mathbb{C}^n \setminus \{0\}$,

$$\sum_{k,l=1}^n \varphi_{\mu^J}(y_k^J - y_l^J)\xi_k\bar{\xi}_l > 0. \tag{1.3.8}$$

Conversely, each function with the above three properties is a characteristic function of a certain probability measure on $\mathbb{X}^J$ and this measure is uniquely determined by its characteristic function. Here it is important that $\mathbb{X}^J$ is finite-dimensional. The consistency of the family of measures $\{\mu^J\}_{J\in\mathcal{I}(I)}$ can be formulated in terms of their characteristic functions. Namely, this family is consistent if and only if for every $J' \subset J \in \mathcal{I}(I)$ and for any $y^{J'} \in \mathbb{X}^{J'}$, the corresponding characteristic functions satisfy the condition

$$\varphi_{\mu^J}(y^{J'} \times 0^{J\setminus J'}) = \varphi_{\mu^{J'}}(y^{J'}). \tag{1.3.9}$$

The notion of a Gaussian process

Let $X : I \times \mathbb{X}^I \to \mathbb{R}^\nu$ be such that $X_\alpha(x) \stackrel{\text{def}}{=} X(\alpha, x) = x_\alpha$. For every fixed $\alpha \in I$, X_α is measurable. Let $\mathbb{M}$ be the set of mappings $\sigma : I \times I \to \mathbb{R}^\nu \times \mathbb{R}^\nu$, the values of which may naturally be associated with real $\nu \times \nu$ matrices, possessing the following properties. First, each $\sigma \in \mathbb{M}$ is symmetric, that is, $\sigma(\alpha, \alpha') = \sigma^T(\alpha, \alpha')$ and $\sigma(\alpha, \alpha') = \sigma(\alpha', \alpha)$ for all $(\alpha, \alpha') \in I \times I$. Here $\sigma^T(\alpha, \alpha')$ stands for the transposed $\sigma(\alpha, \alpha')$ (as a $\nu \times \nu$-matrix). Second, for any $J \in \mathcal{I}(I)$ and every nonzero $\xi^J \in \mathbb{X}^J$,

$$\sum_{\alpha, \alpha' \in J} (\sigma(\alpha, \alpha')\xi_\alpha, \xi_{\alpha'}) > 0. \tag{1.3.10}$$

Let $a \in \mathbb{X}^I$ and $\sigma \in \mathbb{M}$ be given. Then for a given $J \in \mathcal{I}(I)$, we set

$$\varphi_{a,\sigma}(y^J) = \exp\Big(\mathrm{i} \sum_{\alpha \in J} (a_\alpha, y_\alpha) - \frac{1}{2} \sum_{\alpha, \alpha' \in J} (\sigma(\alpha, \alpha') y_\alpha, y_{\alpha'})\Big). \tag{1.3.11}$$

It is the characteristic function of a Gaussian measure, say γ^J, on the Euclidean space $\mathbb{X}^J$, for which

$$\begin{aligned} \langle X_\alpha^{(j)} \rangle_{\gamma^J} &= a_\alpha^{(j)}, \quad \alpha \in J, \ j = 1, \ldots, \nu; \\ \mathrm{cov}_{\gamma^J}(X_\alpha^{(j)}, X_{\alpha'}^{(k)}) &\stackrel{\text{def}}{=} \langle X_\alpha^{(j)} X_{\alpha'}^{(k)} \rangle_{\gamma^J} - \langle X_\alpha^{(j)} \rangle_{\gamma^J} \cdot \langle X_{\alpha'}^{(k)} \rangle_{\gamma^J} \\ &= \sigma_{jk}(\alpha, \alpha'). \end{aligned} \tag{1.3.12}$$

The family $\{\gamma^J\}_{J \in \mathcal{I}(I)}$ is consistent since the condition (1.3.9) is readily satisfied. Therefore, it defines a measure on the whole space $\mathbb{X}^I$. This measure γ will be called a Gaussian measure on $(\mathbb{X}^I, \mathcal{B}^I)$, its first moments and covariance are given by (1.3.12), which now holds for all $\alpha \in I$. In case $I \subseteq \mathbb{R}$, the mapping X is called a Gaussian random process. Now we give some examples, for which the corresponding measures are symmetric, that is $a = 0^I$.

Basic processes

1. Brownian motion. Here $I = [0, +\infty)$ and for $s, t \in [0, +\infty)$,

$$\sigma_{jk}(s, t) = \delta_{jk} \min(s, t), \quad j, k = 1, 2, \ldots, \nu, \tag{1.3.13}$$

where δ_{jk} is Kronecker's symbol. Let us check whether this σ obeys (1.3.10). To this end we use the representation

$$\min(s, t) = \int_0^{+\infty} \mathbb{I}_{[0,s]}(\tau) \mathbb{I}_{[0,t]}(\tau) \mathrm{d}\tau, \tag{1.3.14}$$

where $\mathbb{I}_B$ is the indicator function (1.1.39). Then we take $\{t_1, t_2, \dots t_n\} \subset [0, +\infty)$, $n \in \mathbb{N}$, such that $t_1 < t_2 < \cdots < t_n$, and compute

$$\begin{aligned}
&\sum_{j,k=1}^{\nu} \sum_{l_1,l_2=1}^{n} \delta_{jk} \min(t_{l_1}, t_{l_2}) \xi_{l_1}^{(j)} \bar{\xi}_{l_2}^{(k)} \\
&\quad = \sum_{j=1}^{\nu} \sum_{l_1,l_2=1}^{n} \xi_{l_1}^{(j)} \bar{\xi}_{l_2}^{(j)} \int_0^{+\infty} \mathbb{I}_{[0,t_{l_1}]}(\tau) \mathbb{I}_{[0,t_{l_2}]}(\tau) \mathrm{d}\tau \\
&\quad = \sum_{j=1}^{\nu} \int_0^{+\infty} \left| \sum_{l=1}^{n} \xi_l^{(j)} \mathbb{I}_{[0,t_l]}(\tau) \tau \right|^2 \mathrm{d}\tau \\
&\quad \geq 0.
\end{aligned}$$

The Gaussian process with the covariance (1.3.13) is called a ν-dimensional *Brownian motion* or *Wiener process*. Usually, in the definition of the Wiener process, as well as of the Gaussian processes introduced below, one includes the requirement that their paths are continuous. This property will be discussed in detail in the next subsection.

2. Brownian bridge. For $\beta > 0$, we set $I = [0, \beta]$ and

$$\sigma_{jk}(s,t) = \delta_{jk} \left[\beta \min(s,t) - st \right], \quad s,t \in [0,\beta], \; j,k = 1,2,\dots,\nu. \tag{1.3.15}$$

Let us prove that this σ obeys (1.3.10). Here again we use the representation (1.3.14) but with the integral taken over $[0, \beta]$. Then we choose $\{t_1, \dots, t_n\} \subset [0, \beta]$ as above and write

$$\begin{aligned}
&\sum_{j,k=1}^{\nu} \sum_{l_1,l_2=1}^{n} \sigma_{jk}(t_{l_1}, t_{l_2}) \xi_{l_1}^{(j)} \xi_{l_2}^{(k)} \\
&\quad = \beta^2 \sum_{j=1}^{\nu} \left\{ \beta^{-1} \int_0^{\beta} \left| \sum_{l=1}^{n} \xi_l^{(j)} \mathbb{I}_{[0,t_l]}(\tau) \right|^2 \mathrm{d}\tau - \left| \sum_{l=1}^{n} \xi_l^{(j)} \beta^{-1} \int_0^{\beta} \mathbb{I}_{[0,t_l]}(\tau) \mathrm{d}\tau \right|^2 \right\} \\
&\quad = \beta^2 \sum_{j=1}^{\nu} \beta^{-1} \int_0^{\beta} \left| \sum_{l=1}^{\beta} \xi_l^{(j)} \mathbb{I}_{[0,t_l]}(\tau) - \sum_{l=1}^{\beta} \xi_l^{(j)} \beta^{-1} \int_0^{\beta} \mathbb{I}_{[0,t_l]}(\tau) \mathrm{d}\tau \right|^2 \mathrm{d}\tau \\
&\quad \geq 0.
\end{aligned}$$

The corresponding Gaussian random process is called the ν-dimensional Brownian bridge which starts at $t = 0$ and ends at $t = \beta$.

3. Oscillator (Ornstein–Uhlenbeck) process. Here $I = [0, +\infty)$ and

$$\sigma_{jk}(s,t) = \delta_{jk} \frac{1}{2a} \exp\left(-a|s-t|\right), \quad a > 0. \tag{1.3.16}$$

To check that this σ obeys (1.3.10) we rewrite it as

$$\sigma_{jk}(s,t) = \delta_{jk}\frac{1}{2a}\exp(-as)\exp(-at)\max\{\exp(2as), \exp(2at)\},$$

and use the calculations from the above example. Another proof is based on the representation

$$\frac{1}{2a}\exp(-a|s-t|) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp[\mathrm{i}x(t-s)]}{x^2+a^2}\mathrm{d}x, \tag{1.3.17}$$

by which

$$\sum_{j,k=1}^{\nu}\sum_{l_1,l_2=1}^{n}\sigma_{jk}(t_{l_1},t_{l_2})\xi_{l_1}^{(j)}\xi_{l_2}^{(k)} = \frac{1}{2\pi}\sum_{j=1}^{\nu}\int_{-\infty}^{+\infty}\frac{1}{x^2+a^2}|F^{(j)}(\xi,x)|^2\mathrm{d}x \geq 0,$$

where

$$F^{(j)}(\xi,x) = \sum_{l=1}^{n}\exp(\mathrm{i}t_l x)\xi_l^{(j)}, \quad j = 1,\dots,\nu.$$

By definition, the corresponding random process is the ν-dimensional oscillator process. It is also called a ν-dimensional Ornstein–Uhlenbeck velocity process.

4. Høegh-Krohn's process. This process is also known as a periodic Ornstein–Uhlenbeck velocity process (see e.g., [94] and [177]). We call it Høegh-Krohn's process since it first appeared in Høegh-Krohn's paper [156]. For this process, one sets $I = [0,\beta]$, $\beta > 0$ and

$$\begin{aligned}\sigma_{jk}(s,t) = &\frac{\delta_{jk}}{2\kappa^2[1-\exp(-\beta\delta)]}\\ &\times\{\exp(-\delta|s-t|) + \exp(-\delta(\beta-|s-t|))\},\end{aligned} \tag{1.3.18}$$

where $\kappa > 0$ and $\delta > 0$ are parameters. For $\kappa = (ma)^{1/4}$ and $\delta = (a/m)^{1/2}$ (see (1.1.70), (1.1.72)), the above σ_{jk} coincides with the kernel (1.2.125). Set

$$|s-t|_\beta = \min\{|s-t|, \beta-|s-t|\}; \tag{1.3.19}$$

this can be considered as the distance on the circle of length β which one obtains by identifying the endpoints of the interval $[0,\beta]$. Then the above $\sigma_{jk}(s,t)$ as a function of $s,t \in [0,\beta]$ depends on $|s-t|$ and remains unchanged if one replaces $|s-t|$ by $|s-t|_\beta$. This means that it can be extended periodically to the whole $\mathbb{R}$. This enables us to prove that σ defined by (1.3.18) obeys the condition (1.3.10) by means of a Fourier transformation, similarly as it was done for the oscillator process. Set

$$\mathcal{K} = \{k = (2\pi/\beta)\varkappa \mid \varkappa \in \mathbb{Z}\}. \tag{1.3.20}$$

Then

$$\begin{aligned}&\frac{1}{2\kappa^2[1-\exp(-\beta\delta)]}\left\{\exp\left(-\delta|s-t|\right)+\exp\left(-\delta(\beta-|s-t|)\right)\right\}\\&\qquad=\sum_{k\in\mathcal{K}}\frac{\delta\kappa^{-2}}{\delta^2+k^2}\exp\left[\mathrm{i}k(t-s)\right],\end{aligned}\tag{1.3.21}$$

which yields

$$\sum_{j,k=1}^{\nu}\sum_{l_1,l_2=1}^{n}\sigma_{jk}(t_{l_1},t_{l_2})\xi_{l_1}^{(j)}\xi_{l_2}^{(k)}=\sum_{j=1}^{n}\sum_{k\in\mathcal{K}}\frac{\delta\kappa^{-2}}{\delta^2+k^2}|F^{(j)}(\xi,k)|^2\geq 0,$$

with

$$F^{(j)}(\xi,k)=\sum_{l=1}^{n}\exp(\mathrm{i}t_l k)\xi_l^{(j)},\quad j=1,\dots,\nu.$$

1.3.3 Stochastic Processes in General

The notion of a stochastic process

Let $(\Omega,\mathcal{F},\mathsf{P})$ be a complete probability space, which means that if $A\in\mathcal{F}$ and $\mathsf{P}(A)=0$, then any $B\subset A$ belongs to $\mathcal{F}$. To simplify notation we set $\langle\cdot\rangle=\langle\cdot\rangle_{\mathsf{P}}$ until the end of this subsection, see (1.3.7).

Let I be a subset of $\mathbb{R}$. A map $X: I\times\Omega\to\mathbb{R}^\nu$ is called a stochastic process with state space $\mathbb{R}^\nu$ if for every $t\in I$, the map $X(t,\cdot)\colon\Omega\to\mathbb{R}^\nu$ is measurable, i.e., for any Borel set $A\subset\mathbb{R}^\nu$, its pre-image $\{\omega\in\Omega\mid X(t,\omega)\in A\}$ is an element of $\mathcal{F}$. For every fixed $\omega\in\Omega$, the map $I\ni t\mapsto X(t,\omega)$ is an element of $\mathbb{X}^I$, with $\mathbb{X}=\mathbb{R}^\nu$. We shall denote this map by $X(\cdot,\omega)$ and call it a *path*. For such I, let the measurable space $(\mathbb{X}^I,\mathcal{B}^I)$ be as in the previous subsection. Then the map $\Omega\ni\omega\mapsto X(\cdot,\omega)\in\mathbb{X}^I$ is $\mathcal{F}/\mathcal{B}^I$-measurable. The distribution of this map defines a probability measure μ^I, on $(\mathbb{X}^I,\mathcal{B}^I)$. It is called the distribution of X. Analogously we obtain the measures $\mu^{I'}$ for any $I'\subset I$.

Definition 1.3.4. Two processes X and Y defined on the same probability space are said to be versions of each other if for all $t\in I$,

$$\mathsf{P}(X(t,\cdot)=Y(t,\cdot))=1.$$

Given $t\in I$ and two processes, X and Y, we set

$$A_t=\{\omega\mid X(t,\omega)=Y(t,\omega)\},\quad A=\bigcap_{t\in I}A_t.\tag{1.3.22}$$

One observes that if I is uncountable, the set A need not be measurable.

Definition 1.3.5. The processes X and Y are said to be indistinguishable, if the set A given by (1.3.22) is measurable and $\mathsf{P}(A) = 1$.

Of course, indistinguishable processes are versions of each other, but the converse is not true in general.

Continuity

Now we want to address the question whether for a given stochastic process there exists a version that possesses certain desirable properties. If the index set is of the form $I = \mathbb{R}$, $I = [0, +\infty)$, $I = [a, b]$ with $a, b \in \mathbb{R}$, $a < b$, such a property can be continuity. To exploit it we first present a number of facts about continuous functions. In view of the applications below, we consider the case of $I = [a, b]$ only.

Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^\nu$ and at the same time stand for the absolute value of a real number. By $C[a, b]$ we denote the set of all continuous functions $x : [a, b] \to \mathbb{R}^\nu$. For $x \in C[a, b]$, $\delta > 0$, and $D \subset [a, b]$, we set

$$w_D(x, \delta) = \sup\{|x(t) - x(s)| \mid |t - s| < \delta, \ \ s, t \in D\}. \tag{1.3.23}$$

Furthermore, we write $w(x, \delta) = w_{[a,b]}(x, \delta)$. For $D \subset [a, b]$ and $x : D \to \mathbb{R}^\nu$, a function $\tilde{x} : [a, b] \to \mathbb{R}^\nu$ is called an extension of x to $[a, b]$ if $x(t) = \tilde{x}(t)$ for all $t \in D$.

Proposition 1.3.6. *Let D be a dense subset of $[a, b]$. A function $x : D \to \mathbb{R}^\nu$ has a continuous extension to $[a, b]$ if and only if $w_D(x, \delta) \to 0$ as $\delta \downarrow 0$. This extension $\tilde{x}$ is unique and for all $\delta \in (0, b - a)$,*

$$w_D(x, \delta) = w(\tilde{x}, \delta). \tag{1.3.24}$$

The proof of this statement is immediate.

For an appropriate $x \in C[a, b]$ and $\sigma > 0$, we set

$$K_\sigma(x) = \sup\left\{\frac{|x(t) - x(s)|}{|t - s|^\sigma} \mid t \neq s, \ \ t, s \in [a, b]\right\}, \tag{1.3.25}$$

and

$$C_\sigma[a, b] = \{x \in C[a, b] \mid K_\sigma(x) < \infty\}. \tag{1.3.26}$$

The elements of the latter set are called Hölder continuous functions of order σ.

Proposition 1.3.7. *Let D be a dense subset of $[a, b]$. For a function $x : D \to \mathbb{R}^\nu$ and $\sigma > 0$, suppose that the sequence $\{2^{n\sigma} w_D(x, 2^{-n}(b - a))\}_{n \in \mathbb{N}}$ is bounded. Then x has a Hölder continuous extension to $[a, b]$ of order σ.*

Proof. Since $w_D(x, \delta) \leq w_D(x, \delta')$ for $\delta < \delta'$, the assumed boundedness implies that $w_D(x, \delta) \to 0$ as $\delta \downarrow 0$. Then by Proposition 1.3.6, x has a continuous extension $\tilde{x}$ for which (1.3.24) holds. Let us show that $\tilde{x} \in C_\sigma[a, b]$. Set

$$B_0 = \{(s, t) \in [a, b]^2 \mid (b - a)/2 \leq |t - s| \leq b - a\},$$
$$B_n = \{(s, t) \in [a, b]^2 \mid (b - a)/2^{n+1} \leq |t - s| < (b - a)/2^n\}, \quad n \in \mathbb{N}.$$

Then

$$\{(s,t) \in [a,b]^2 \mid s \neq t\} = \bigcup_{n=0}^{\infty} B_n,$$

and either $K_\sigma(\tilde{x}) = |b-a|^{-\sigma}|\tilde{x}(b) - \tilde{x}(a)|$, or else

$$\begin{aligned} K_\sigma(\tilde{x}) &= \sup_{n \in \mathbb{N}_0} \{\sup\{|\tilde{x}(t) - \tilde{x}(s)| \cdot |t-s|^{-\sigma} \mid (s,t) \in B_n\}\} \\ &\leq \left(\frac{2}{b-a}\right)^{\sigma} \sup_{n \in \mathbb{N}_0} \{2^{n\sigma} w(\tilde{x}, 2^{-n}(b-a))\}, \end{aligned} \tag{1.3.27}$$

which completes the proof. □

Two indistinguishable processes, say X and Y, have the same paths P-almost surely, that is, $X(\cdot, \omega) = Y(\cdot, \omega)$ for P-almost all ω. At the same time, the processes which are just versions of each other may have different paths.

Definition 1.3.8. A stochastic process $X : [a,b] \times \Omega \to \mathbb{R}^\nu$ is called continuous (respectively, Hölder continuous of order $\sigma > 0$) if $X(\cdot, \omega) \in C[a,b]$ (respectively, $X(\cdot, \omega) \in C_\sigma[a,b]$) for all $\omega \in \Omega$. A process X is said to have a continuous version (respectively, Hölder continuous version of order $\sigma > 0$) if there exists its version $\widetilde{X}$ which is continuous (respectively, Hölder continuous of order σ).

Clearly, a Hölder continuous process is continuous. The existence of Hölder continuous versions of a given process is established by the following known statement. Recall that we have set $\langle \cdot \rangle = \langle \cdot \rangle_{\mathsf{P}}$, see (1.3.7).

Theorem 1.3.9 (Kolmogorov Lemma). *Let a process X on $[a,b]$ be such that for all $t, s \in [a,b]$,*

$$\langle |X(t,\cdot) - X(s,\cdot)|^p \rangle \leq C|t-s|^{1+q}, \tag{1.3.28}$$

with certain positive C and p, q, such that $0 < q < p$. Then for every $\sigma \in (0, q/p)$, X has a Hölder continuous version of order σ.

In view of the importance of this statement, we present its complete proof here. It is based on the following two facts, which will also be extensively employed in the other parts of the book. The first one is Chebyshev's inequality

$$\mathsf{P}(\{\omega \mid f(\omega) \geq \eta\}) \leq \langle f \rangle / \eta, \tag{1.3.29}$$

which holds for any $\eta > 0$ and any measurable function $f : \Omega \to [0, +\infty)$. Indeed,

$$\begin{aligned} \langle f \rangle &\geq \int_{\{\omega | f(\omega) \geq \eta\}} f(\omega) \mathsf{P}(d\omega) \\ &\geq \eta \int_{\{\omega | f(\omega) \geq \eta\}} \mathsf{P}(d\omega) = \eta \cdot \mathsf{P}(\{\omega \mid f(\omega) \geq \eta\}). \end{aligned}$$

The second fact is established by the following statement, which is a part of the Borel–Cantelli lemma, see e.g., [75].

Proposition 1.3.10 (Borel–Cantelli Lemma). *Given a probability space* $(\Omega, \mathcal{F}, \mathsf{P})$ *and a sequence* $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$*, let*

$$C = \{\omega \mid \omega \in A_n \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m.$$

Then

$$\sum_{n=1}^{\infty} \mathsf{P}(A_n) < \infty$$

implies $\mathsf{P}(C) = 0$.

Proof. For any $\varepsilon > 0$, one finds $n \in \mathbb{N}$, such that

$$\mathsf{P}(C) \le \mathsf{P}\Big(\bigcup_{m=n}^{\infty} A_m\Big) \le \sum_{m=n}^{\infty} \mathsf{P}(A_m) < \varepsilon. \qquad \square$$

Proof of Theorem 1.3.9. For $n \in \mathbb{N}$, we set

$$D_n = \{t_k = a + (b-a)2^{-n}k \mid k = 0, 1, \dots, 2^n\}, \quad D = \bigcup_{n\in\mathbb{N}} D_n. \tag{1.3.30}$$

Clearly, D is countable and dense in $[a, b]$. Given $n \in \mathbb{N}$ and $k \in \{0, 1, \dots, 2^n\}$, let us consider

$$Z_{n,k}(\omega) \overset{\text{def}}{=} |X(t_{k+1}, \omega) - X(t_k, \omega)|. \tag{1.3.31}$$

By (1.3.28) it follows that

$$\langle Z_{n,k}^p \rangle \le C(b-a)^{1+q} 2^{-n(1+q)}.$$

Then for $\alpha \in (0, q/p)$, by (1.3.29) we get

$$\mathsf{P}(Z_{n,k} > 2^{-n\alpha}) = \mathsf{P}(Z_{n,k}^p > 2^{-n\alpha p}) \le C(b-a)^{1+q} 2^{-n-n(q-\alpha p)}. \tag{1.3.32}$$

Therefore,

$$\sum_{n=1}^{\infty} \sum_{k=0}^{2^n} \mathsf{P}(Z_{n,k} > 2^{-n\alpha}) < \infty,$$

which by Proposition 1.3.10 yields that there exist $A \in \mathcal{F}$, such that $\mathsf{P}(A) = 1$, and $n_A \in \mathbb{N}$, for which

$$\begin{aligned} Z_{n,k}(\omega) &< \infty, && \text{for all } \omega \in A; \\ Z_{n,k}(\omega) &\le 2^{-n\alpha}, && \text{for all } n > n_A,\ k = 0, \dots 2^n,\ \omega \in A. \end{aligned} \tag{1.3.33}$$

For $s, t \in D$, such that $0 \le t - s < (b-a)2^{-n}$, one can find $k \in \{0, \dots, 2^n - 1\}$, for which $\max\{|t - t_{k+1}|, |s - t_k|\} < (b-a)2^{-n}$. Then by the triangle inequality,

$$\begin{aligned} |X(s,\omega) - X(t,\omega)| \le{}& |X(t_{k+1},\omega) - X(t_k,\omega)| \\ &+ |X(s,\omega) - X(t_k,\omega)| + |X(t,\omega) - X(t_{k+1},\omega)|. \end{aligned} \tag{1.3.34}$$

Furthermore, one finds integers $l, m \geq n$, for which $s \in D_l$ and $t \in D_m$. Thus,

$$s = a + (b-a)2^{-n}k + \sum_{i=n+1}^{l} \gamma_i 2^{-j},$$
$$t = a + (b-a)2^{-n}(k+1) + \sum_{j=n+1}^{m} \gamma_j' 2^{-j}, \quad \gamma_i, \gamma_j' = 0, 1.$$

Thereby, one finds $k_i \in \{0, 1, \ldots, 2^i\}$ and $\kappa_j \in \{0, 1, \ldots, 2^j\}$, such that

$$|X(s,\omega) - X(t_k,\omega)| \leq \sum_{i=n+1}^{l} Z_{i,k_i}(\omega),$$
$$|X(t,\omega) - X(t_{k+1},\omega)| \leq \sum_{j=n+1}^{m} Z_{j,\kappa_j}(\omega).$$

Then by (1.3.33) and (1.3.34) one obtains

$$\begin{aligned} |X(s,\omega) - X(t,\omega)| &\leq Z_{n,k}(\omega) + \sum_{i=n+1}^{l} Z_{i,k_i}(\omega) + \sum_{j=n+1}^{m} Z_{j,\kappa_j}(\omega) \\ &\leq 2^{-n\alpha} + 2\sum_{j=n+1}^{\infty} 2^{-j\alpha} = 2^{-n\alpha}\left[1 + 2/(2^\alpha - 1)\right], \end{aligned} \tag{1.3.35}$$

which holds for all $\omega \in A$ and $t, s \in D$, such that $|t-s| < (b-a)2^{-(n+1)}$ for all $n > n_A$. Thereby, for every $\omega \in A$, one finds $C(\omega) > 0$, such that

$$2^{n\alpha} w_D\left(X(\cdot,\omega), 2^{-n}(b-a)\right) \leq C(\omega).$$

By Proposition 1.3.7 this yields that for every $\omega \in A$, the function $X(\cdot,\omega)$ has a Hölder continuous extension of order α, which we denote by $\widetilde{X}(\cdot,\omega)$, such that

$$w_D(X(\cdot,\omega),\delta) = w(\widetilde{X}(\cdot,\omega),\delta), \quad \text{for all } \delta \in (0,(b-a)). \tag{1.3.36}$$

For $\omega \in A^c$, we set $\widetilde{X}(t,\omega) = 0$ for all $t \in [a,b]$. Then $\widetilde{X}(\cdot,\omega)$ is Hölder continuous of order α for all $\omega \in \Omega$. Let us show now that $\widetilde{X}$ is a version of X. For $t \in D$, we have $\widetilde{X}(t,\omega) = X(t,\omega)$ for all $\omega \in A$. Since $\widetilde{X}$ is continuous and D is dense in $[a,b]$, for any $t \in [a,b] \setminus D$, one finds a sequence $\{s_n\}_{n\in\mathbb{N}} \subset D$, such that $s_n \to t$ and hence $X(s_n,\omega) \to \widetilde{X}(t,\omega)$ for all $\omega \in A$. On the other hand by (1.3.28) and the Chebyshev inequality (1.3.29) one obtains that for every positive ε and η, there exists $n(\varepsilon,\eta) \in \mathbb{N}$, such that for all $n > n(\varepsilon,\eta)$,

$$\mathsf{P}\left(\{\omega \mid |X(s_n,\omega) - X(t,\omega)| > \eta\}\right) < \varepsilon. \tag{1.3.37}$$

Now let us pick a sequence $\{n_k\}_{k\in\mathbb{N}}$, such that $n_k \geq n(2^{-k}, 2^{-k})$ for all $k \in \mathbb{N}$. Then

$$\mathsf{P}\big(\{\omega \mid |X(s_{n_k}, \omega) - X(t,\omega)| > 2^{-k}\}\big) \leq 2^{-k},$$

which for the event

$$B \stackrel{\text{def}}{=} \bigcap_{n\geq 1} \bigcup_{k\geq n} \{\omega \mid |X(s_{n_k}, \omega) - X(t,\omega)| > 2^{-k}\} \tag{1.3.38}$$

yields $\mathsf{P}(B) = 0$, see Proposition 1.3.10. Thus, for $\omega \in B^c$, the estimate $|X(s_{n_k}, \omega) - X(t,\omega)| \leq 2^{-k}$ fails to hold for finitely many k only; hence, $X(s_{n_k}, \omega) \to X(t,\omega)$ for such ω. Therefore, for $\omega \in B^c \cap A$, one has $\widetilde{X}(t,\omega) = X(t,\omega)$ for all $t \in [a,b]$. But $\mathsf{P}(B^c \cap A) = 1$; thereby, $\widetilde{X}$ is a version of X, which completes the proof. □

Along with Theorem 1.3.9 we are going to use a result known as the Garsia–Rodemich–Rumsey (GRR) lemma. Since this lemma has many applications, we present its complete proof here.

Theorem 1.3.11 (GRR lemma). *Let $p, \Psi\colon [0,+\infty) \to [0,+\infty)$ be continuous and strictly increasing functions, such that $p(0) = \Psi(0) = 0$ and $\lim_{t\to+\infty} \Psi(t) = +\infty$. Given $T > 0$ and $\phi \in C([0,T] \to \mathbb{R}^\nu)$, we set*

$$\Xi(t,s) = \Psi\left(\frac{|\phi(t) - \phi(s)|}{p(|t-s|)}\right), \quad t, s \in [0,T], \tag{1.3.39}$$

and

$$B = \int_0^T \int_0^T \Xi(t,s)\mathrm{d}t\mathrm{d}s. \tag{1.3.40}$$

If $B < \infty$, then for any $0 \leq s < t \leq T$,

$$|\phi(t) - \phi(s)| \leq 8 \int_0^{t-s} \Psi^{-1}\left(4B/u^2\right) p(\mathrm{d}u), \tag{1.3.41}$$

where Ψ^{-1} is the inverse of Ψ.

Proof. First let us prove the following fact. For given strictly positive a, b, and d, and for $x, y \in C([0,d] \to [0,+\infty))$, suppose that

$$\int_0^d x(t)\mathrm{d}t \leq a, \quad \int_0^d y(t)\mathrm{d}t \leq b.$$

Then there exists $\theta \in [0,d]$ such that $x(\theta) < 2a/d$ and $y(\theta) < 2b/d$. Indeed, otherwise

$$2a = \int_0^d (2a/d)\mathrm{d}t \leq \int_0^d x(t)\mathrm{d}t \leq a,$$

and likewise for y and b.

Now we set

$$I(t) = \int_0^T \Xi(t,s)\mathrm{d}s \tag{1.3.42}$$

and pick two non-increasing sequences, $\{t_n\}_{n\in\mathbb{N}_0} \subset [0,T]$ and $\{d_n\}_{n\in\mathbb{N}_0} \subset [0,T]$, satisfying

$$p(d_n) = p(t_n)/2, \tag{1.3.43}$$

which in particular means that $d_n < t_n$ for all n. Now, for given t_{n-1} and d_{n-1}, we have to show how to choose t_n. To this end we set $x = I$, $y = \Xi(t_{n-1}, \cdot)$, and $d = d_{n-1}$. By (1.3.40) and (1.3.42) we have $a = B$ and $b = I(t_{n-1})$. Then the above arguments yield the existence of $t_n \in [0, d_{n-1}]$ such that both estimates

$$I(t_n) \le 2B/d_{n-1}, \quad \Xi(t_n, t_{n-1}) \le 2I(t_{n-1})/d_{n-1} \tag{1.3.44}$$

hold true. By construction, $t_n \le d_{n-1}$; thus,

$$d_n < t_n \le d_{n-1} < t_{n-1}, \tag{1.3.45}$$

and $t_n \to 0$ as $n \to \infty$. Let us show now how to choose t_0. Set $d_{-1} = T$. By (1.3.40) and (1.3.42) there exists $t_0 \in [0,T]$ such that $I(t_0) \le B/T < 2B/T$.

By (1.3.45) it follows that

$$p(t_n - t_{n+1}) \le p(t_n) = 2p(d_n) = 4[p(d_n) - p(d_n)/2] \le 4[p(d_n) - p(d_{n+1})].$$

Combining the latter estimate with (1.3.44) and (1.3.45) we get

$$\begin{aligned}
|\phi(t_n) - \phi(t_{n+1})| &\le \Psi^{-1}\left(\frac{2I(t_n)}{d_n}\right) p(t_n - t_{n+1}) \\
&\le \Psi^{-1}\left(\frac{4B}{d_n d_{n-1}}\right) 4[p(d_n) - p(d_{n+1})] \\
&\le 4\Psi^{-1}\left(\frac{4B}{d_n^2}\right)[p(d_n) - p(d_{n+1})] \\
&\le 4\int_{d_{n+1}}^{d_n} \Psi^{-1}\left(4B/u^2\right) p(\mathrm{d}u).
\end{aligned}$$

Summing over $n \in \mathbb{N}_0$, we obtain

$$|\phi(t_0) - \phi(0)| \le 4\int_0^T \Psi^{-1}\left(4B)/u^2\right) p(\mathrm{d}u). \tag{1.3.46}$$

Thus, the variation of $\phi(t)$ on the interval $[0,t_0] \subset [0,T]$ is bounded by the integral over the whole $[0,T]$. To estimate the variation on $[t_0,T]$ we replace $\phi(t)$ by $\phi(T-t)$ and repeat the above construction, which yields

$$|\phi(T) - \phi(t_0)| \le 4\int_0^T \Psi^{-1}\left(4B)/u^2\right) p(\mathrm{d}u),$$

and thereby

$$|\phi(T) - \phi(0)| \leq 8 \int_0^T \Psi^{-1}\left(4B/u^2\right) p(\mathrm{d}u). \tag{1.3.47}$$

Now we fix $s, t \in [0, T]$, $s < t$, and introduce

$$\begin{aligned} \tilde{\phi}(u) &= \phi\Big(s + \frac{t-s}{T}u\Big), \quad \tilde{p}(u) = p\Big(s + \frac{t-s}{T}u\Big), \\ \widetilde{\Xi}(u, v) &= \Psi\Big(\frac{|\tilde{\phi}(u) - \tilde{\phi}(v)|}{\tilde{p}(|u-v|)}\Big), \quad u, v \in [0, T]. \end{aligned} \tag{1.3.48}$$

Then

$$\begin{aligned} \int_0^T \int_0^T \widetilde{\Xi}(u, v)\mathrm{d}u\mathrm{d}v &= \Big(\frac{T}{t-s}\Big)^2 \int_s^t \int_s^t \Xi(u, v)\mathrm{d}u\mathrm{d}v \\ &\leq (T/(t-s))^2\, B \stackrel{\text{def}}{=} \widetilde{B}. \end{aligned}$$

The functions (1.3.48) readily satisfy the conditions of the theorem; hence, they obey the estimate (1.3.47), that is,

$$|\tilde{\phi}(T) - \tilde{\phi}(0)| \leq 8 \int_0^T \Psi^{-1}\left(4\widetilde{B}/u^2\right) \tilde{p}(\mathrm{d}u),$$

which yields (1.3.41) after an obvious change of variables. □

In the sequel, the GRR lemma will be used mostly in situations where one needs to interchange integration with taking supremum. This will be done by means of the following corollary of Theorem 1.3.11.

Proposition 1.3.12 (GRR inequality). *Let X be the Hölder continuous version of the process which obeys* (1.3.28)*. For $\vartheta \in (0, b-a]$, $\alpha \in (0, q/p)$, we set*

$$L_{\alpha,\vartheta}(\omega) = \sup\left\{\tfrac{|X(t,\omega) - X(s,\omega)|}{|t-s|^\alpha} \;\Big|\; 0 < |t-s| < \vartheta\right\}. \tag{1.3.49}$$

Then

$$\langle L_{\alpha,\vartheta}^p \rangle \leq D(\alpha, p, q) C \vartheta^{1+q-\alpha p}, \tag{1.3.50}$$

where $D(\alpha, p, q) > 0$ is a constant, independent of ϑ, and C is the same as in (1.3.28).

Proof. For fixed ϑ and $\theta \in [a, b-\vartheta]$, we set

$$B_\theta(\omega) = \int_0^\vartheta \int_0^\vartheta \frac{|X(\theta + t, \omega) - X(\theta + s, \omega)|^p}{|t-s|^{2+\alpha p}} \mathrm{d}t\mathrm{d}s. \tag{1.3.51}$$

Then by Fubini's theorem and (1.3.28),

$$\langle B_\theta \rangle \leq \frac{2C\vartheta^{1+q-\alpha p}}{(q-\alpha p)(1+q-\alpha p)}. \tag{1.3.52}$$

On the other hand,

$$L^p_{\alpha,\vartheta}(\omega) = \sup_{\theta\in[a,b-\vartheta]} \sup_{t,s\in[0,\vartheta),\ t\neq s} \frac{|X(\theta+t,\omega)-X(\theta+s,\omega)|^p}{|t-s|^{\alpha p}},$$

and by (1.3.41) with $\Psi(t) = t^p$ and $p(t) = t^{\alpha+2/p}$ we get

$$\frac{|X(\theta+t,\omega)-X(\theta+s,\omega)|^p}{|t-s|^{\alpha p}} \leq 2^{3p+2}\left(1+\frac{2}{\alpha p}\right)^p B_\theta(\omega),$$

which along with (1.3.52) yields (1.3.50) with

$$D(\alpha, p, q) = \frac{2^{3(p+1)}}{(q-\alpha p)(1+q-\alpha p)}\left(1+\frac{2}{\alpha p}\right)^p. \tag{1.3.53}$$

□

It can be instructive to look at the result just proven as follows. By (1.3.28)

$$\sup\left\{\left\langle \tfrac{|X(t,\cdot)-X(s,\cdot)|^p}{|t-s|^{\alpha p}}\right\rangle \,\Big|\, 0<|t-s|\leq\vartheta\right\} \leq C\vartheta^{1+q-\alpha p}.$$

At the same time, by (1.3.50) we have

$$\left\langle \sup\left\{\tfrac{|X(t,\cdot)-X(s,\cdot)|^p}{|t-s|^{\alpha p}} \,\Big|\, 0<|t-s|\leq\vartheta\right\}\right\rangle \leq D(\alpha,p,q)C\vartheta^{1+q-\alpha p}.$$

We also note that for $\alpha' < \alpha$,

$$\{\omega \mid L_{\alpha,\vartheta}(\omega) < +\infty\} \subset \{\omega \mid L_{\alpha',\vartheta}(\omega) < +\infty\}. \tag{1.3.54}$$

A consequence of Theorem 1.3.9 is the following statement.

Proposition 1.3.13. *For any $\sigma \in (0, 1/2)$, the Brownian bridge and the Høegh-Krohn processes possess Hölder continuous versions of order σ.*

Proof. For any Gaussian ν-dimensional random vector X and any $p \in \mathbb{N}$, one has

$$\langle |X|^{2p}\rangle = \left[C(p,\nu)\langle |X|^2\rangle\right]^p, \tag{1.3.55}$$

with a certain constant $C(p,\nu) > 0$. We apply this identity to $X = X(t,\cdot) - X(s,\cdot)$, which is a Gaussian random vector, and obtain

$$\langle |X(t,\cdot)-X(s,\cdot)|^{2p}\rangle \leq [\nu C(p,\nu)\varkappa|s-t|]^p, \tag{1.3.56}$$

where $\varkappa = \beta$ for the Brownian bridge, and $\varkappa = \delta\kappa^{-2} = 1/m$ for the Høegh-Krohn process. For $p \geq 2$, we set $q = p-1$ and apply Theorem 1.3.9, which gives the property stated with any $\sigma < (p-1)/2p$; hence, for any $\sigma < 1/2$ since p in (1.3.56) can be taken arbitrarily big. □

From now on, by the Høegh-Krohn process we mean the Hölder continuous version as in Proposition 1.3.13. In the sequel, we will use the following spaces of continuous functions:

$$\begin{aligned} C_\beta &\overset{\text{def}}{=} \{x \in C([0,\beta] \to \mathbb{R}^\nu) \mid x(0) = x(\beta)\}, \\ C_\beta^\sigma &\overset{\text{def}}{=} \{x \in C_\sigma([0,\beta] \to \mathbb{R}^\nu) \mid x(0) = x(\beta)\}, \quad \sigma \in (0, 1/2). \end{aligned} \tag{1.3.57}$$

Remark 1.3.14. The Høegh-Krohn process satisfies (1.3.28) with $|t-s|$ replaced by the distance $|t-s|_\beta$ defined by (1.3.19).

Canonical realization

The results obtained above may be summarized as follows. The process $X : [a,b] \times \Omega \to \mathbb{R}^\nu$ considered as a map $\Omega \ni \omega \mapsto X(\cdot, \omega) \in (\mathbb{R}^\nu)^{[a,b]}$ may have a version, say $\widetilde{X}$, for which all values of this map belong to the subset $C[a,b] \overset{\text{def}}{=} C([a,b] \to \mathbb{R}^\nu)$ of $(\mathbb{R}^\nu)^{[a,b]}$, consisting of continuous functions. Note, however, that the set $C[a,b]$ need not belong to the cylindric σ-algebra $\mathcal{B}^{[a,b]}$, see Definition 1.3.2. On the other hand, $C[a,b]$ is a Banach space, and, therefore, can be endowed with the corresponding Borel σ-algebra $\mathcal{B}(C[a,b])$. So, if we can prove that the map $\Omega \ni \omega \mapsto \widetilde{X}(\cdot, \omega) \in C[a,b]$ is $\mathcal{F}/\mathcal{B}(C[a,b])$-measurable, we shall be able to deal with a 'concrete' probability space $(C[a,b], \mathcal{B}(C[a,b]), \mu)$ instead of $(\Omega, \mathcal{F}, \mathsf{P})$, where μ is the image measure of P under $\widetilde{X}$. Let us do this. The first step is to prove the measurability just mentioned.

Proposition 1.3.15. *Let the process $X : [a,b] \times \Omega \to \mathbb{R}^\nu$ be continuous. Then the map $\Omega \ni \omega \mapsto X(\cdot, \omega) \in C[a,b]$ is $\mathcal{F}/\mathcal{B}(C[a,b])$-measurable.*

Proof. Let $\widetilde{\mathcal{B}}$ be the σ-algebra of subsets of $C[a,b]$ generated by the maps $C[a,b] \ni x \mapsto x(t)$, with rational t, i.e., with $t \in Q_{[a,b]} \overset{\text{def}}{=} \mathbb{Q} \cap [a,b]$. Clearly, X is $\mathcal{F}/\widetilde{\mathcal{B}}$-measurable. So, it suffices to show that $\widetilde{\mathcal{B}} = \mathcal{B}(C[a,b])$. Since the maps $x \mapsto x(t)$ are continuous, we have $\widetilde{\mathcal{B}} \subset \mathcal{B}(C[a,b])$. As the Banach space $C[a,b]$ is separable, it is second countable, which means here that each of its open subsets is a countable union of the balls $B_r(y) = \{x \mid \sup_{t \in [a,b]} |x(t) - y(t)| < r\}$, $r > 0$. Then, to complete the proof it suffices to show that each $B_r(y) \in \widetilde{\mathcal{B}}$. Given $r > 0$, $y \in C[a,b]$, and $t \in Q_{[a,b]}$, the set $K_r^t(y) = \{x \| x(t) - y(t)| \leq r\}$ is in $\widetilde{\mathcal{B}}$. Let $\{r_n\}_{n \in \mathbb{N}}$ be any sequence of positive numbers such that $r_n \uparrow r$. Then

$$B_r(y) = \bigcup_{n \in \mathbb{N}} \bigcap_{t \in Q_{[a,b]}} K_{r_n}^t(y) \in \widetilde{\mathcal{B}},$$

which yields $\mathcal{B}(C[a,b]) \subset \widetilde{\mathcal{B}}$ and hence $\widetilde{\mathcal{B}} = \mathcal{B}(C[a,b])$. □

Corollary 1.3.16. *Let μ and ν be two probability measures on $(C[a,b], \mathcal{B}(C[a,b]))$, such that for every $k \in \mathbb{N}$, $A \in \mathcal{B}(\mathbb{R}^{k\nu})$, and $t_1, \ldots, t_k \in [a,b]$,*

$$\mu\left(\{x \mid (x(t_1), \ldots, x(t_k)) \in A\}\right) = \nu(\{x \mid (x(t_1), \ldots, x(t_k)) \in A\}).$$

Then $\mu = \nu$.

Proof. Clearly, $\widetilde{\mathcal{B}}$ is generated by the maps $x \mapsto x(t)$ with all $t \in [a,b]$. Then, $\mu(B) = \nu(B)$ for all $B \in \widetilde{\mathcal{B}}$, which by the above statement yields $\mu = \nu$. □

Definition 1.3.17. For a continuous process $X: [a,b] \times \Omega \to \mathbb{R}^\nu$, let the probability measure μ on $\mathcal{B}(C[a,b])$ be the X-image of P. Then the process $\xi: [a,b] \times C[a,b] \to \mathbb{R}^\nu$ such that

$$\xi(\cdot, x) = x, \quad \text{for all } x \in C[a,b],$$

is called the canonical realization of the process X on the space $C[a,b]$.

Kuratowski theorem

In the sequel, it will be convenient for us to deal with more than one realization of a given process. To relate such realizations with each other we will use the following known theorem, see [204], page 489, or Theorem 3.9, page 21 in [239].

Proposition 1.3.18 (Kuratowski theorem). *Let $\mathbb{Y}_1$, $\mathbb{Y}_2$ be Polish spaces and $\mathcal{B}_1$, $\mathcal{B}_2$ be their Borel σ-algebras respectively. If $\phi: \mathbb{Y}_1 \to \mathbb{Y}_2$ is a measurable injection, then $\phi(B) \in \mathcal{B}_2$ for any $B \in \mathcal{B}_1$.*

If the map ϕ in the above theorem is bijective then the σ-algebras $\mathcal{B}_1$ and $\mathcal{B}_2$ are *isomorphic* and ϕ is called *a measurable isomorphism.*

Now let $\mathbb{Y}$ be a Polish space and d be its metric. Given $E \in \mathcal{B}(\mathbb{Y})$, we set $\mathcal{B}(E) = \{A \cap E \mid A \in \mathcal{B}(\mathbb{Y})\}$. It is a subset of $\mathcal{B}(\mathbb{Y})$ and $\mathcal{B}(E) = \{B \subset E \mid B \in \mathcal{B}(\mathbb{Y})\}$. Suppose now that the set E is equipped with another metric, say $\tilde{d}$, such that for any $x, y \in E$, $d(x,y) \leq \tilde{d}(x,y)$. Suppose also that the metric space $(E, \tilde{d})$ is complete and separable. Let $\widetilde{\mathcal{B}}(E)$ be the corresponding Borel σ-algebra.

Lemma 1.3.19. *The above σ-algebras coincide, that is $\mathcal{B}(E) = \widetilde{\mathcal{B}}(E)$.*

Proof. Let $\phi: E \to \mathbb{Y}$ be the embedding map, that is, the map which sends the metric space $(E, \tilde{d})$ into the metric space $\mathbb{Y}$ such that $\phi(y) = y$ for all $y \in E$. By construction, this map is an injection; by the assumption regarding the metrics ϕ it is continuous, hence measurable. Then by the Kuratowski theorem $E \in \mathcal{B}(\mathbb{Y})$. Thus, $\mathcal{B}(E) \subset \widetilde{\mathcal{B}}(E)$. Since for any $A \subset E$, A and $\phi(A)$ consist of the same elements, every $B \in \widetilde{\mathcal{B}}(E)$ belongs to $\mathcal{B}(E)$; hence, $\widetilde{\mathcal{B}}(E) \subset \mathcal{B}(E)$. □

The Kuratowski theorem admits an extension, which does not use the topology of the state space. Let $(\mathbb{E}, \mathcal{E})$ be a measurable space. We say that a family, $\mathcal{D} \subset \mathcal{E}$, *generates* $\mathcal{E}$ if $\mathcal{E}$ is the smallest σ-algebra of subsets of X which contains $\mathcal{D}$.

Definition 1.3.20. A measurable space $(\mathbb{E}, \mathcal{E})$ is countably generated if there exists a countable $\mathcal{D} \subset \mathcal{E}$, which generates $\mathcal{E}$. It is called separable if $\mathcal{E}$ contains all single point subsets $\{x\}$, $x \in \mathbb{E}$. Finally, a countably generated space $(\mathbb{E}, \mathcal{E})$ is a standard Borel space if there exists a complete separable metric space $\mathbb{Y}$ and a measurable bijection $\phi: \mathbb{Y} \to \mathbb{E}$, such that the measure spaces $(\mathbb{Y}, \mathcal{B}(\mathbb{Y}))$ and $(\mathbb{E}, \mathcal{E})$ are isomorphic.

The following extended version of Kuratowski's theorem holds, see Theorem 2.4 on page 135 of [239].

Proposition 1.3.21. *Let* $(\mathbb{E}, \mathcal{E})$ *be a standard Borel space and* $(\mathbb{K}, \mathcal{K})$ *be separable and countably generated. Suppose also that* $\phi\colon \mathbb{E} \to \mathbb{K}$ *is injective and* $\mathcal{E}/\mathcal{K}$*-measurable. Then* $\mathbb{K}' = \phi(\mathbb{E})$ *belongs to* $\mathcal{K}$ *and the measurable spaces* $(\mathbb{E}, \mathcal{E})$ *and* $(\mathbb{K}', \mathcal{K}')$ *are isomorphic. Here* $\mathcal{K}' = \{B \cap \mathbb{K}' \mid B \in \mathcal{K}\}$.

Consider again the proof of Proposition 1.3.15. The space $(C[a,b], \widetilde{\mathcal{B}})$ is separable and countably generated whereas the space $(C[a,b], \mathcal{B}(C[a,b]))$ is standard as $C[a,b]$ is separable. Let $\phi\colon C[a,b] \to C[a,b]$ be the identity map. As $\widetilde{\mathcal{B}} \subset \mathcal{B}(C[a,b])$, this map is $\mathcal{B}(C[a,b])/\widetilde{\mathcal{B}}$-measurable, which by Proposition 1.3.21 yields $\widetilde{\mathcal{B}} = \mathcal{B}(C[a,b])$.

Now let us return to the sets of periodic continuous functions $x\colon [0,\beta] \to \mathbb{R}^\nu$ introduced in (1.3.57). They are equipped with the norms:

$$\|x\|_{C_\beta} = \sup_{t\in[0,\beta]} |x(t)|,$$

and, cf. (1.3.25),

$$\begin{aligned} \|x\|_{C^\sigma_\beta} &= |x(0)| + \beta^\sigma \cdot \sup_{s,t\in[0,\beta],\ s\neq t} \frac{|x(s)-x(t)|}{|s-t|^\sigma_\beta} \\ &= |x(0)| + \beta^\sigma K_\sigma(x), \quad \sigma > 0. \end{aligned} \tag{1.3.58}$$

With these norms they become real Banach spaces. Since for all $\tau, \tau' \in [0,\beta]$,

$$|x(\tau) - x(\tau')| \le \beta^\sigma K_\sigma(x),$$

one has $\|x\|_{C_\beta} \le \|x\|_{C^\sigma_\beta}$ for all $\sigma > 0$. Recall that L^2_β stands for the real Hilbert space $L^2([0,\beta] \to \mathbb{R}^\nu)$. Then for all $0 < \sigma < \sigma'$, it follows that

$$\beta^{-1/2}\|x\|_{L^2_\beta} \le \|x\|_{C_\beta} \le \|x\|_{C^\sigma_\beta} \le \|x\|_{C^{\sigma'}_\beta}. \tag{1.3.59}$$

Therefore, $C^{\sigma'}_\beta \subset C^\sigma_\beta \subset C_\beta \subset L^2_\beta$. As metric spaces, L^2_β, C_β, C^σ_β are complete. At the same time, the spaces L^2_β nd C_β are separable, but the Hölder spaces $C^\sigma_\beta, \sigma \in (0,1)$ are not separable (see e.g., page 37 of [203]). By (1.3.59) the embedding maps $C^{\sigma'}_\beta \hookrightarrow C^\sigma_\beta \hookrightarrow C_\beta \hookrightarrow L^2_\beta$ are continuous, hence measurable. Then by Proposition 1.3.18 the embedding $C_\beta \hookrightarrow L^2_\beta$ is a measurable injection. In what follows, by Lemma 1.3.19 the Borel σ-algebras of subsets of C_β generated by the topology induced on C_β from L^2_β and its own topology coincide. One can show that for any $\sigma \in (0,1/2)$, C^σ_β is a Borel subset of C_β or L^2_β (see e.g., page 278 of [256]), although some measurable subsets of C^σ_β may not be measurable in C_β and L^2_β.

Now let $X : [0,\beta] \times \Omega \to \mathbb{R}^\nu$ be the Høegh-Krohn process. By Proposition 1.3.13 it is Hölder-continuous. Hence, it has a canonical realization on C_β. Set

$$\chi_\beta = \mathsf{P} \circ X^{-1}. \tag{1.3.60}$$

Then $\chi_\beta(C_\beta^\sigma) = 1$ for any $\sigma \in (0, 1/2)$, and for any $A \in \mathcal{B}(C_\beta)$,

$$\chi_\beta(A) = \chi_\beta(A \cap C_\beta^\sigma). \tag{1.3.61}$$

Similarly, one can construct the realization of the Høegh-Krohn process on a wider space, provided it is still a separable metric space, as e.g., L^2_β.

1.3.4 Probability Measures on Polish Spaces

In this subsection, we collect a number of facts concerning probability measures on Polish spaces, which will be used in the subsequent parts of the book. Recall that a metric space is called Polish if it is complete and separable. Until the end of this subsection $\mathbb{E}$ will stand for such a space. By $\mathcal{P}(\mathbb{E})$ we denote the set of all probability measures defined on the Borel σ-algebra $\mathcal{B}(\mathbb{E})$. A function $f : \mathbb{E} \to \mathbb{R}$ is called *Borel* if it is $\mathcal{B}(\mathbb{E})/\mathcal{B}(\mathbb{R})$-measurable, i.e., $f^{-1}(B) \in \mathcal{B}(\mathbb{E})$ for each Borel subset $B \subset \mathbb{R}$. By $B_\mathrm{b}(\mathbb{E})$ we denote the set of all bounded real-valued Borel functions. The set of bounded continuous functions $C_\mathrm{b}(\mathbb{E})$ is a subset of $B_\mathrm{b}(\mathbb{E})$.

Proposition 1.3.22. *For every $\mu \in \mathcal{P}(\mathbb{E})$ and all $p \geq 1$, the set $C_\mathrm{b}(\mathbb{E})$ is dense in $L^p(\mathbb{E}, \mu)$.*

The proof of this statement will be given below.

Definition 1.3.23. A family of probability measures $\mathcal{M}$ on a topological space $\mathbb{Y}$ is called *tight* if for any $\varepsilon > 0$, there exists a compact subset $D_\varepsilon \subset \mathbb{Y}$, such that for all $\mu \in \mathcal{M}$, $\mu(D_\varepsilon) > 1 - \varepsilon$. A probability measure is tight if the one-element family $\{\mu\}$ is tight.

Proposition 1.3.24. *Every probability measure on a Polish space, $\mathbb{E}$, is tight. Moreover, for every such μ and any $B \in \mathcal{B}(\mathbb{E})$,*

$$\begin{aligned} \mu(B) &= \inf\{\mu(A) \mid A \supset B,\ A \textit{ is open}\} \\ &= \sup\{\mu(D) \mid D \subset B,\ D \textit{ is compact}\}. \end{aligned} \tag{1.3.62}$$

The proof of these facts can be found e.g., in [57], page 158 or in [222], pages 1–5.

Proposition 1.3.25. *Every probability measure on $\mathbb{E}$ may be uniquely determined by its values on compact subsets of $\mathbb{E}$.*

This property of compact sets means that if, for given $\mu, \nu \in \mathcal{P}(\mathbb{E})$, one has $\mu(D) = \nu(D)$ for all compact $D \subset \mathbb{E}$, then $\mu = \nu$. Let $\mathcal{D}(\mathbb{E})$ be the family of all compact subsets of $\mathbb{E}$. Then by Proposition 1.3.25 the family $\{\mathbb{I}_D \mid D \in \mathcal{D}(\mathbb{E})\}$ of indicator functions (1.1.39) is a *measure-defining class*. That is, a family of functions $\mathfrak{F} \subset B_\mathrm{b}(\mathbb{E})$ is a measure-defining class if for any $\mu, \nu \in \mathcal{P}(\mathbb{E})$, the equality $\langle f \rangle_\mu = \langle f \rangle_\nu$ for all $f \in \mathfrak{F}$, implies[6] $\mu = \nu$.

[6]Recall that $\langle f \rangle_\mu$ stands for the integral of f with respect to μ.

Now we want to obtain conditions under which families of bounded continuous functions $f : \mathbb{E} \to \mathbb{R}$ are measure-defining. The result we are going to get will be used in particular to make an alternative proof of Theorem 1.2.22. We recall that the notion of completeness of a family of multiplication operators was introduced in Definition 1.2.21.

Theorem 1.3.26. *Let $\mathfrak{F}$ be a family of bounded continuous functions $f : \mathbb{E} \to \mathbb{R}$, which possesses the following properties:* (a) *for any $x, x' \in \mathbb{E}$, $x \neq x'$, there exists $f \in \mathfrak{F}$ such that $f(x) \neq f(x')$;* (b) *for any $f, g \in \mathfrak{F}$, the point-wise product $f \cdot g$ also belongs to $\mathfrak{F}$. Then $\mathfrak{F}$ is a measure-defining class, that is, if for given probability measures μ, ν on $\mathbb{E}$, one has $\langle f \rangle_\mu = \langle f \rangle_\nu$ for all $f \in \mathfrak{F}$, then $\mu = \nu$.*

The proof of this theorem is given below and will be based on two facts formulated as Propositions 1.3.27 and 1.3.28, preceded by preparatory observations. For a family $\mathfrak{F}$ of bounded functions $f : \mathbb{E} \to \mathbb{R}$, let $\Sigma(\mathfrak{F})$ be the σ-algebra of subsets of $\mathbb{E}$ generated by $\mathfrak{F}$, i.e., the smallest σ-algebra of subsets of $\mathbb{E}$ with respect to which all the elements of $\mathfrak{F}$ are measurable. If $\mathfrak{F}$ consists of Borel functions, the algebra $\Sigma(\mathfrak{F})$ is a subalgebra of $\mathcal{B}(\mathbb{E})$. If the family $\mathfrak{F}$ is rich enough, then $\Sigma(\mathfrak{F}) = \mathcal{B}(\mathbb{E})$. The following statement, which we borrow from the book [308], page 6, Theorem 1.2, describes the families of continuous functions which generate the whole $\mathcal{B}(\mathbb{E})$.

Proposition 1.3.27. *Let $\mathbb{E}$ be a Polish space, and let $\mathfrak{F}$ be a family of bounded continuous functions $f : \mathbb{E} \to \mathbb{R}$, which separates points of $\mathbb{E}$. Then $\Sigma(\mathfrak{F}) = \mathcal{B}(\mathbb{E})$. In particular, $\Sigma(C_\mathrm{b}(\mathbb{E})) = \mathcal{B}(\mathbb{E})$.*

The second fact we use to prove Theorem 1.3.26 is taken from the book [215], page 28, Theorem 20.

Proposition 1.3.28. *Let $\mathbb{E}$ be a nonempty set and $\mathbb{V}$ be a real linear space of bounded functions $f : \mathbb{E} \to \mathbb{R}$, including the constant functions. Suppose that the space $\mathbb{V}$ possesses the following property: if a bounded increasing sequence of positive functions $\{f_n\}_{n\in\mathbb{N}} \subset \mathbb{V}$ converges point-wise in $\mathbb{E}$ to f, then $f \in \mathbb{V}$. Let $\mathfrak{V} \subset \mathbb{V}$ be closed with respect to point-wise multiplication, $\Sigma(\mathfrak{V})$ denote the σ-algebra generated by $\mathfrak{V}$, and $\Sigma(\mathfrak{V})_\mathrm{b}$ be the set of all bounded functions $f : \mathbb{E} \to \mathbb{R}$, measurable with respect to $\Sigma(\mathfrak{V})$. Then $\Sigma(\mathfrak{V})_\mathrm{b} \subset \mathbb{V}$.*

Proof of Theorem 1.3.26. Given μ and $\nu \in \mathcal{P}(\mathbb{E})$, we set

$$\mathbb{V} = \{f \in B_\mathrm{b}(\mathbb{E}) \mid \langle f \rangle_\mu = \langle f \rangle_\nu\}. \tag{1.3.63}$$

This set equipped with the linear operations becomes a real linear space, satisfying the conditions of Proposition 1.3.28 which follows from Levi's theorem, see page 305 of [180]. The subset $\mathfrak{F} \subset \mathbb{V}$ is closed under multiplication; by Proposition 1.3.27 it generates the whole σ-algebra $\mathcal{B}(\mathbb{E})$. Hence, by Proposition 1.3.28 the space $\mathbb{V}$ contains $B_\mathrm{b}(\mathbb{E})$, which means that these two sets coincide; hence, the condition $\langle f \rangle_\mu = \langle f \rangle_\nu$ is satisfied on the whole of $B_\mathrm{b}(\mathbb{E})$. □

Proof of Proposition 1.3.22. Given $p \geq 1$, we define

$$\mathbb{V} = \{f \in B_{\mathrm{b}}(\mathbb{E}) \mid \exists \{f_n\}_{n\in\mathbb{N}} \subset C_{\mathrm{b}}(\mathbb{E}),\ f_n \to f \text{ in } L^p(\mathbb{E}, \mu)\}. \tag{1.3.64}$$

The set $\mathbb{V}$ satisfies all the assumptions of Proposition 1.3.28. Thus, we apply it with $\mathfrak{V} = C_{\mathrm{b}}(\mathbb{E})$, see Proposition 1.3.27. Since $B_{\mathrm{b}}(\mathbb{E})$ is dense in $L^p(\mathbb{E}, \mu)$, the assertion follows. □

Proof of Theorem 1.2.22. The proof will be done if we show that the bicommutant of the family $\mathfrak{F}_\Lambda$ is $\mathfrak{M}_\Lambda$ – the algebra of all multiplication operators by bounded measurable functions. As $\mathfrak{F}_\Lambda$ consists of multiplication operators and $\mathfrak{M}_\Lambda$ is a maximal commutative subalgebra of $\mathfrak{C}_\Lambda$ consisting of multiplication operators, it follows that $\mathfrak{F}''_\Lambda \subset \mathfrak{M}_\Lambda$.

According to Definition 1.2.11, $\mathfrak{F}''_\Lambda$ is a von Neumann algebra. Then, as a linear space, it obeys the conditions of Proposition 1.3.28, see Propositions 1.2.10 and 1.2.19. On the other hand, by Proposition 1.3.27 $\Sigma(\mathfrak{F}_\Lambda)$ is the Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\nu|\Lambda|})$. Hence, by Proposition 1.3.28 $\mathfrak{M}_\Lambda \subset \mathfrak{F}''_\Lambda$, which completes the proof. □

Now we introduce a topology on the set of probability measures $\mathcal{P}(\mathbb{E})$. Given $\mu \in \mathcal{P}(\mathbb{E})$, $n \in \mathbb{N}$, positive $\varepsilon_1, \ldots \varepsilon_n$, and $f_1, \ldots, f_n \in C_{\mathrm{b}}(\mathbb{E})$, we set

$$U^{\varepsilon_1,\ldots,\varepsilon_n}_{f_1,\ldots,f_n}(\mu) = \left\{\nu \in \mathcal{P}(\mathbb{E}) \mid \left|\int_{\mathbb{E}} f_i \mathrm{d}\mu - \int_{\mathbb{E}} f_i \mathrm{d}\nu\right| < \varepsilon_i,\ \ i = 1, \ldots, n\right\}. \tag{1.3.65}$$

Definition 1.3.29. The topology on $\mathcal{P}(\mathbb{E})$ for which the subsets (1.3.65) define a base of neighborhoods is called the weak topology.

In this topology, $\mathcal{P}(\mathbb{E})$ is separable, since $\mathbb{E}$ is separable, and can be completely metrized, since $\mathbb{E}$ is complete. Therefore, it is a Polish space (see Theorem 2.1.1, page 19 of [75] or Theorem 6.2, page 43 and Theorem 6.5, page 46 of [239]). For more details on the weak topology see [65], Chapters V, VI of [64], Chapter 2 of [75], and Chapter II of [239].

We say that a net $\{\mu_\theta\}_{\theta\in\Theta} \subset \mathcal{P}(\mathbb{E})$ *weakly converges* to a measure $\mu \in \mathcal{P}(\mathbb{E})$, in writing $\mu_\theta \Rightarrow \mu$, if it converges in the weak topology. This holds if and only if

$$\forall f \in C_{\mathrm{b}}(\mathbb{E}): \qquad \int_{\mathbb{E}} f \mathrm{d}\mu_\theta \to \int_{\mathbb{E}} f \mathrm{d}\mu. \tag{1.3.66}$$

As a metrizable space is first countable, in the study of its topological properties, it is enough to consider sequences only. Thus, in the sequel we consider sequences of measures $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{E})$ rather than nets.

It turns out that the weak convergence holds if (1.3.66) occurs for functions belonging to certain proper subsets of $C_{\mathrm{b}}(\mathbb{E})$. Such subsets are called *weak convergence defining classes*. Let d be any metric on $\mathbb{E}$ consistent with its topology[7]. A function $f : \mathbb{E} \to \mathbb{R}$ is called Lipschitz if

$$\|f\|_L \stackrel{\text{def}}{=} \sup_{x,y\in\mathbb{E},\ x\neq y} |f(x) - f(y)|/d(x, y) < \infty. \tag{1.3.67}$$

[7]This means that the topology of $\mathbb{E}$ and the metric topology defined by d are equivalent.

Such a function is evidently continuous. If f is bounded, one can define

$$\|f\|_{BL} = \|f\|_L + \sup_{x\in\mathbb{E}} |f(x)|. \tag{1.3.68}$$

Clearly, $\|\cdot\|_{BL}$ is a norm. By $BL(\mathbb{E}, d)$ we denote the set of all bounded Lipschitz functions. It is a subset of $C_b^u(\mathbb{E}, d)$ – the set of all bounded functions, $f : \mathbb{E} \to \mathbb{R}$, uniformly continuous with respect to the metric d. By definition, each such a function has the property: for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$, whenever $d(x, y) < \delta$. The point here is that δ is the same for *all* x and y.

For $\mu, \nu \in \mathcal{P}(\mathbb{E})$, we set

$$D(\mu, \nu) = \sup_{f\in BL(\mathbb{E},d),\ \|f\|_{BL}\le 1} \left|\langle f\rangle_\mu - \langle f\rangle_\nu\right|. \tag{1.3.69}$$

The following statements are versions of Proposition 13.3.2 and Theorem 13.3.3, page 310 of [107], respectively.

Proposition 1.3.30. *For any metric space* $(\mathbb{E}, d)$*,* D *defined in* (1.3.69) *is a metric on* $\mathcal{P}(\mathbb{E})$.

Proposition 1.3.31. *Let* $(\mathbb{E}, d)$ *be a separable metric space. Then the following three kinds of the convergence of a sequence* $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{E})$ *to a measure* $\mu \in \mathcal{P}(\mathbb{E})$*, as* $n \to +\infty$*, are equivalent:*

(a) $\mu_n \Rightarrow \mu$;

(b) $\langle f\rangle_{\mu_n} \to \langle f\rangle_\mu$, *for all* $f \in BL(\mathbb{E}, d)$;

(c) $D(\mu_n, \mu) \to 0$.

Since $BL(\mathbb{E}, d) \subset C_b^u(\mathbb{E}, d)$, the latter statement yields

Proposition 1.3.32. *Let* $(\mathbb{E}, d)$ *be a separable metric space. A sequence,* $\{\mu_n\}_{n\in\mathbb{N}}$*, of probability measures on* $\mathbb{E}$ *weakly converges to a measure* $\mu \in \mathcal{P}(\mathbb{E})$*, if and only if*

$$\forall f \in C_b^u(\mathbb{E}, d): \quad \int_{\mathbb{E}} f\, d\mu_n \to \int_{\mathbb{E}} f\, d\mu. \tag{1.3.70}$$

There exists a connection between compactness in the weak topology and tightness, see Definition 1.3.23. The following fact is a direct corollary of Theorem 2, page 94 of [64] (see also Theorems 6.1, 6.2 page 37 of [65] and Theorem 2.3.1, page 25 of [75]).

Proposition 1.3.33 (Prokhorov's Theorem). *A family* $\mathcal{M} \subset \mathcal{P}(\mathbb{E})$ *is relatively compact in the weak topology if and only if it is tight.*

Let $(\mathbb{E}, d)$ be Polish. For $r > 0$ and $y \in \mathbb{E}$, we set $B_r(y) = \{x \in \mathbb{E} \mid d(x, y) \le r\}$. Let $\mathcal{M} \subset \mathcal{P}(\mathbb{E})$ be a family of measures such that for some $y \in \mathbb{E}$ (hence, for all such y),

$$\int_{\mathbb{E}} d(x, y)\mu(dx) < \infty, \quad \text{for all } \mu \in \mathcal{M}. \tag{1.3.71}$$

Definition 1.3.34. A continuous function $\phi \colon \mathbb{E} \to [0, +\infty)$ is said to be a compact function if for every $a > 0$, the set

$$A_a = \{x \in \mathbb{E} \mid \phi(x) \le a\}$$

is compact.

Proposition 1.3.35. *Let a family $\mathcal{M} \subset \mathcal{P}(\mathbb{E})$ be such that*

$$\int_{\mathbb{E}} \phi(x)\mu(\mathrm{d}x) \le M, \quad \textit{for all } \mu \in \mathcal{M}, \tag{1.3.72}$$

for certain positive M and compact ϕ. Then $\mathcal{M}$ is tight and hence relatively weakly compact.

Proof. For any $\mu \in \mathcal{M}$, by Chebyshev's inequality (1.3.29) one obtains $\mu(A_a) > 1-\varepsilon$ if $a > M/\varepsilon$, which yields the tightness of $\mathcal{M}$, see Definition 1.3.23. □

There exists yet another interesting property of measures on Polish spaces, formulated as Proposition 1.3.36 and Theorem 1.3.37 below. It occurs if $\mathbb{E}$ is a real separable Fréchet space. Recall that a linear topological space is called a Fréchet space if its zero element has a countable base of neighborhoods consisting of sets, which are balanced, absorbing, and convex. For such a space $\mathbb{E}$, we define $T_\pm \colon \mathbb{E} \times \mathbb{E} \to \mathbb{E}$, by

$$T_\pm(x_1, x_2) = (x_1 \pm x_2)/\sqrt{2}. \tag{1.3.73}$$

The following is known as Fernique's theorem, see [95], page 16, and also the original source [118].

Proposition 1.3.36 (Fernique's Theorem). *Let $\mathbb{E}$ be a real separable Fréchet space and $\phi \colon \mathbb{E} \to [0, +\infty]$ be a measurable sub-additive function with the property that $\phi(\alpha x) = |\alpha|\phi(x)$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{E}$. Next, let $\mu \in \mathcal{P}(\mathbb{E})$ be such that the measure $\mu \otimes \mu$ on $\mathbb{E}^2$ is invariant under the transformation*

$$\mathbb{E}^2 \ni (x_1, x_2) \mapsto (T_-(x_1, x_2), T_+(x_1, x_2)) \in \mathbb{E}^2. \tag{1.3.74}$$

If $\mu(\{x|\phi(x) < +\infty\}) = 1$, then there exists $\alpha > 0$ such that

$$\int_{\mathbb{E}} \exp\left[\alpha\phi^2(x)\right] \mu(\mathrm{d}x) < \infty. \tag{1.3.75}$$

We shall use a version of Fernique's theorem, which describes sequences of measures. Its proof, which we give below, is a slight modification of the proof of Theorem 1.3.24 in [95].

Theorem 1.3.37. *Let the space $\mathbb{E}$ and the functional ϕ be as in Proposition 1.3.36, and $\{\mu_N\}_{N\in\mathbb{N}} \subset \mathcal{P}(\mathbb{E})$ be a sequence, each element of which is such that $\mu_N \otimes \mu_N$ is invariant under the transformation (1.3.74). Assume also that for all $N \in \mathbb{N}$,*

$$\int_{\mathbb{E}} \phi^2(x)\mu_N(\mathrm{d}x) \le b, \tag{1.3.76}$$

for a certain $b > 0$. Then, for every

$$\alpha \in (0, \varsigma/b), \quad \textit{where } \varsigma = \left(\frac{\sqrt{2}-1}{4}\right)^2 \log 3, \tag{1.3.77}$$

and all $N \in \mathbb{N}$, it follows that

$$\int_{\mathbb{E}} \exp\left[\alpha\phi^2(x)\right] \mu_N(\mathrm{d}x) \leq C_\alpha \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 3^{-2^n(1-\alpha b/\varsigma)}. \tag{1.3.78}$$

Proof. Given $0 < s \leq t$, we set

$$B_s^- = \{x \in \mathbb{E} \mid \phi(x) \leq s\}, \quad B_t^+ = \{x \in \mathbb{E} \mid \phi(x) \geq t\}. \tag{1.3.79}$$

Then, for any μ_N,

$$\begin{aligned}
\mu_N(B_s^-) \cdot \mu_N(B_t^+) &= (\mu_N \otimes \mu_N)(B_s^- \times B_t^+) \\
&= (\mu_N \otimes \mu_N)\left[\{(x_1, x_2) \mid T_-(x_1, x_2) \in B_s^-,\ T_+(x_1, x_2) \in B_t^+\}\right] \\
&= (\mu_N \otimes \mu_N)\left[\{(x_1, x_2) \mid \phi(x_1 - x_2) \leq \sqrt{2}s;\ \phi(x_1 + x_2) \geq \sqrt{2}t\}\right] \\
&\leq (\mu_N \otimes \mu_N)\left[\{(x_1, x_2) \mid |\phi(x_1) - \phi(x_2)| \leq \sqrt{2}s;\ \phi(x_1) + \phi(x_2) \geq \sqrt{2}t\}\right] \\
&\leq (\mu_N \otimes \mu_N)\left[\{(x_1, x_2) \mid \min\{\phi(x_1); \phi(x_2)\} \geq (t-s)/\sqrt{2}\}\right] \\
&= \left[\mu_N\left(B^+_{(t-s)/\sqrt{2}}\right)\right]^2.
\end{aligned} \tag{1.3.80}$$

Here we have used the assumed properties of ϕ. Next we divide both sides of (1.3.80) by $[\mu_N(B_s^-)]^2$ and arrive at

$$\frac{\mu_N(B_t^+)}{\mu_N(B_s^-)} \leq \left[\frac{\mu_N\left(B^+_{(t-s)/\sqrt{2}}\right)}{\mu_N(B_s^-)}\right]^2. \tag{1.3.81}$$

By means of Chebyshev's inequality (1.3.29) we obtain from (1.3.76)

$$\mu_N(B_s^+) \leq b/s^2.$$

Thus, for

$$s \geq 2\sqrt{b}, \tag{1.3.82}$$

one gets $\mu_N(B_s^-) \geq 3/4$ and $\mu_N(B_s^+) \leq 1/4$ for all $N \in \mathbb{N}$. For this s, we define $t_n = \sqrt{2}t_{n-1} + s$, $n \in \mathbb{N}$, and $t_0 = s$. That is,

$$t_n = s(1 + \sqrt{2} + \cdots + 2^{n/2}), \quad n \in \mathbb{N}_0.$$

Thereby,

$$t_n < 2^{n/2}\gamma, \quad \text{where } \gamma = \frac{2\sqrt{2}}{\sqrt{2}-1} \cdot \sqrt{b}. \tag{1.3.83}$$

Iterating (1.3.81) we receive

$$\mu_N(B_{t_n}^+) \le \mu_N(B_s^-)\left[\frac{\mu_N(B_s^+)}{\mu_N(B_s^-)}\right]^{2^n} \le 3^{-2^n}.$$

Thus, in view of (1.3.83), it follows that

$$\mu_N\left(B_{2^{n/2}\gamma}^+\right) \le 3^{-2^n}. \tag{1.3.84}$$

Since

$$B_\gamma^+ = \bigcup_{n\in\mathbb{N}_0}\left(B_{2^{n/2}\gamma}^+ \setminus B_{2^{(n+1)/2}\gamma}^+\right),$$

we have for $\alpha < \varsigma/b$,

$$\begin{aligned}\sup_N\left\{\int_{B_\gamma^+}\exp\left[\alpha\phi^2(x)\right]\mu_N(\mathrm{d}x)\right\} &\le \sum_{n=0}^{\infty}\exp(2^{n+1}\alpha\gamma^2)\sup_N\left\{\mu_N\left(B_{2^{n/2}\gamma}^+\right)\right\}\\ &\le \sum_{n=0}^{\infty}3^{-2^n(1-\alpha b/\varsigma)},\end{aligned}$$

which yields (1.3.78). □

1.3.5 Probability Measures on Hilbert Spaces

Separable Hilbert spaces are Polish spaces and measures defined on such spaces possess the properties described in Subsection 1.3.4. At the same time, they have additional properties related to their specific structure, which are summarized in the current subsection. Here $\mathbb{H}$ stands for a real separable Hilbert space endowed with the scalar product $(\cdot,\cdot)_{\mathbb{H}}$, corresponding norm and with the standard Borel σ-algebra. Most of its properties coincide with the corresponding properties of complex Hilbert spaces described in Subsection 1.1.2. In this case, we shall refer to their description in this subsection.

Along with $\mathbb{H}$, we also consider a larger Hilbert space, $\mathbb{H}_-$. For such spaces, we suppose that the embedding operator $O: \mathbb{H} \to \mathbb{H}_-$ is of Hilbert–Schmidt type, see Definition 1.1.23, and that the image of $\mathbb{H}$ in $\mathbb{H}_-$ is dense in $\mathbb{H}_-$. The topology induced from $\mathbb{H}_-$ on $\mathbb{H}$ is then weaker than its own topology. We shall call $\mathbb{H}_-$ *a Hilbert–Schmidt extension* of the space $\mathbb{H}$. For a detailed description of such extensions, we refer to the book [62].

Given $\mu \in \mathcal{P}(\mathbb{H})$, we set

$$\varphi_\mu(z) = \int_{\mathbb{H}}\exp(\mathrm{i}(z,x)_{\mathbb{H}})\mu(\mathrm{d}x), \qquad z\in\mathbb{H}. \tag{1.3.85}$$

Since the function $\mathbb{H} \ni x \mapsto \exp(\mathrm{i}(z,x)_{\mathbb{H}}) \in \mathbb{C}$ is bounded and continuous for any $z \in \mathbb{H}$, the above integral is well defined. The function φ_μ is called *the characteristic* *function* (or *the Fourier transform*) of the measure μ. It determines the measure in the following sense.

Proposition 1.3.38. *Let $\mu, \mu' \in \mathcal{P}(\mathbb{H})$ be such that $\varphi_\mu = \varphi_{\mu'}$. Then $\mu = \mu'$.*

Proof. The family of functions $\{\exp(\mathrm{i}(z, \cdot)_{\mathbb{H}}) \mid z \in \mathbb{H}\}$ certainly satisfies the conditions of Theorem 1.3.26 and thereby is a measure-defining class. $\square$

Other properties of characteristic functions (1.3.85) are described by the renowned Minlos–Sazonov theorem (see [217] and e.g., page 15 of [283]), which is an infinite-dimensional version of the classical Bochner theorem. We recall that a function $\varphi\colon \mathbb{H} \to \mathbb{C}$ is called *positive definite* if for any $n \in \mathbb{N}$, arbitrary $x_1, \ldots, x_n \in \mathbb{H}$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{j,k=1}^{n} \varphi(x_j - x_k) c_j \bar{c}_k \geq 0.$$

Proposition 1.3.39 (Minlos–Sazonov theorem). *A function $\varphi\colon \mathbb{H} \to \mathbb{C}$ is the characteristic function of a measure $\mu \in \mathcal{P}(\mathbb{H})$ if and only if the following three conditions are satisfied:*

(a) $\varphi(0) = 1$;

(b) *φ is positive definite;*

(c) *there exists a Hilbert–Schmidt extension $\mathbb{H}_-$ of the space $\mathbb{H}$ such that the map $\mathbb{H}_- \supset \mathbb{H} \ni x \mapsto \varphi(x) \in \mathbb{C}$ is continuous with respect to the topology induced by $\mathbb{H}_-$ on $\mathbb{H}$.*

Since $\mathbb{H}$ is a metric space, the weak topology on the set $\mathcal{P}(\mathbb{H})$ is defined in the usual way (see Definition 1.3.29). Let $\mathcal{K} \subset \mathcal{P}(\mathbb{H})$ be such that for every $\mu \in \mathcal{K}$,

$$\int_{\mathbb{H}} \|x\|^2_{\mathbb{H}} \mu(\mathrm{d}x) < \infty. \tag{1.3.86}$$

Furthermore, let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{H}$ be an orthonormal basis of $\mathbb{H}$. The following statement, the proof of which can be found on page 154 of [239], gives a condition for $\mathcal{K}$ to be relatively compact in the weak topology.

Proposition 1.3.40. *If*

$$\lim_{N\to+\infty} \sup_{\mu\in\mathcal{K}} \int_{\mathbb{H}} \Big(\sum_{n=N+1}^{\infty} (x, x_n)^2_{\mathbb{H}} \Big) \mu(\mathrm{d}x) = 0, \tag{1.3.87}$$

then the set $\mathcal{K}$ is weakly relatively compact.

Another statement, also taken from [239], page 153, shows the role of the characteristic functions in establishing weak convergence. Here we employ positive bounded operators. In the case of real spaces, the corresponding property is defined as follows. A bounded linear operator $A\colon \mathbb{H} \to \mathbb{H}$ is called positive if it is: (a) self-adjoint, i.e., $(Ax, x)_{\mathbb{H}} = (x, Ax)_{\mathbb{H}}$ for all $x \in \mathbb{H}$; (b) $(x, Ax)_{\mathbb{H}} \geq 0$ for all $x \in \mathbb{H}$.

Proposition 1.3.41. *Let a sequence $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\mathbb{H})$ be weakly relatively compact and let the sequence of their characteristic functions $\{\varphi_{\mu_n}\}_{n\in\mathbb{N}}$ converge point-wise to a function $\varphi\colon \mathbb{H} \to \mathbb{C}$. Then this φ is the characteristic function of a measure $\mu \in \mathcal{P}(\mathbb{H})$ and the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ converges weakly to μ.*

A measure $\mu \in \mathcal{P}(\mathbb{H})$ is called Gaussian, if, cf. (1.3.11),

$$\varphi_\mu(x) = \exp\left[\mathrm{i}(a, x)_{\mathbb{H}} - \tfrac{1}{2}(x, Sx)_{\mathbb{H}}\right], \tag{1.3.88}$$

where $a \in \mathbb{H}$ and $S\colon \mathbb{H} \to \mathbb{H}$ is linear, positive, and trace-class, see Definitions 1.1.23 and 1.1.24. The vector a is called the mean value of μ, whereas S is its *covariance operator*.

Theorem 1.3.42. *Let a sequence of zero mean Gaussian measures $\{\gamma_m\}_{m\in\mathbb{N}}$ on a separable Hilbert space $\mathbb{H}$ be given. Let also each γ_m have covariance operator S_m. Suppose that the sequence $\{S_m\}_{m\in\mathbb{N}}$ converges in the trace norm to an operator $S\colon \mathbb{H} \to \mathbb{H}$, which by (1.3.88) defines a zero mean Gaussian measure γ. Then it follows that $\gamma_m \Rightarrow \gamma$.*

Proof. For a trace-class operator $A\colon \mathbb{H} \to \mathbb{H}$, its trace norm is

$$\|A\|_{\mathrm{trace}} = \mathrm{trace}|A| \stackrel{\mathrm{def}}{=} \sum_{n=1}^{\infty} (x_n, |A|x_n)_{\mathbb{H}}, \tag{1.3.89}$$

where $\{x_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathbb{H}$ and $|A| = \sqrt{A^*A}$, see (1.1.34). Note that the value of the sum in (1.3.89) is independent of the choice of the basis $\{x_n\}_{n\in\mathbb{N}}$. By the positivity of S_m, $\|S_m\|_{\mathrm{trace}} = \mathrm{trace}(S_m)$ for each $m \in \mathbb{N}$, see (1.1.58).

If $A^* = A$, then $A = A_+ - A_-$, where both $A_\pm$ are positive, see (1.1.31). Furthermore, $|A| = A_+ + A_-$, which immediately yields that

$$(x, Ax)_{\mathbb{H}} \le (x, |A|x)_{\mathbb{H}}.$$

Then for self-adjoint A and B, we have

$$(x, Ax)_{\mathbb{H}} = (x, Bx)_{\mathbb{H}} + (x, (A - B)x)_{\mathbb{H}} \le (x, Bx)_{\mathbb{H}} + (x, |A - B|x)_{\mathbb{H}}. \tag{1.3.90}$$

For a given $\varepsilon > 0$, let m_ε be such that for all $m > m_\varepsilon$,

$$\mathrm{trace}|S_m - S| < \varepsilon/2. \tag{1.3.91}$$

Then for any $N \in \mathbb{N}$, by (1.3.90) we have

$$\begin{aligned} \sum_{n=N+1}^{\infty} (x_n, S_m x_n)_{\mathbb{H}} &\le \sum_{n=N+1}^{\infty} (x_n, Sx_n)_{\mathbb{H}} + \sum_{n=N+1}^{\infty} (x_n, |S_m - S|x_n)_{\mathbb{H}} \\ &\le \sum_{n=N+1}^{\infty} (x_n, Sx_n)_{\mathbb{H}} + \mathrm{trace}|S_m - S|. \end{aligned}$$

For ε as in (1.3.91), one can pick N_ε such that for all $N > N_\varepsilon$,

$$\sum_{n=N+1}^{\infty} (x_n, Sx_n)_{\mathbb{H}} < \varepsilon/2,$$

which yields that for all $m > m_\varepsilon$ and $N > N_\varepsilon$,

$$\sum_{n=N+1}^{\infty} (x_n, S_m x_n)_{\mathbb{H}} < \varepsilon. \tag{1.3.92}$$

As every S_m is of trace-class, for each $m = 1, \dots, m_\varepsilon$, one can pick N_m such that (1.3.92) holds for each such m and $N > N_m$. Therefore, for

$$N > \max\{N_1, \dots, N_{m_\varepsilon}, N_\varepsilon\},$$

the inequality (1.3.92) holds for all $m \in \mathbb{N}$. Hence,

$$\lim_{N\to+\infty} \sup_{m\in\mathbb{N}} \sum_{n=N+1}^{\infty} (x_n, S_m x_n)_{\mathbb{H}} = \lim_{N\to+\infty} \sup_{m\in\mathbb{N}} \int_{\mathbb{H}} \Big(\sum_{n=N+1}^{\infty} (x_n, x)^2_{\mathbb{H}} \Big) \gamma_m(\mathrm{d}x) = 0.$$

This yields (1.3.87); thus, the sequence $\{\gamma_m\}_{m\in\mathbb{N}}$ is weakly relatively compact. Since each γ_m is Gaussian, its characteristic function is, cf. (1.3.88),

$$\varphi_{\gamma_m}(y) = \exp\Big(-\frac{1}{2}(y, S_m y)_{\mathbb{H}} \Big), \quad y \in \mathbb{H}.$$

As the trace norm convergence of $\{S_m\}_{m\in\mathbb{N}}$ implies its strong convergence, we have

$$\varphi_{\gamma_m}(y) \to \exp\Big(-\frac{1}{2}(y, S y)_{\mathbb{H}} \Big), \quad \text{for all } y \in \mathbb{H}.$$

Thereby, the assertion follows by Proposition 1.3.41. $\square$

Now let us consider measures on $\mathbb{H}$, which possess exponential moments.

Definition 1.3.43. By $\mathcal{M}(\mathbb{H})$ we denote the set of all probability measures on $\mathbb{H}$ for each of which, there exists $a > 0$, such that

$$\int_{\mathbb{H}} \exp(a\|x\|^2_{\mathbb{H}})\mu(\mathrm{d}x) < \infty. \tag{1.3.93}$$

Characteristic functions of such measures possess useful analytic properties. To describe them we should pass to complex variables by introducing the following extension of the space $\mathbb{H}$:

$$\mathbb{H}^c = \{z = x + \mathrm{i}y \mid x, y \in \mathbb{H}\}. \tag{1.3.94}$$

We endow this set with the natural linear operations over the field $\mathbb{C}$ and with the norm

$$\|z\|_{\mathbb{H}^c} = \sqrt{\|x\|^2_{\mathbb{H}} + \|y\|^2_{\mathbb{H}}},$$

which turns $\mathbb{H}^c$ into a complex Banach space. Furthermore, for a linear operator $T\colon \mathbb{H} \to \mathbb{H}$, (respectively, for $u \in \mathbb{H}$), and $z = x + \mathrm{i}y$, we set $Tz = Tx + \mathrm{i}Ty$ (respectively, $(u, z)_{\mathbb{H}} = (z, u)_{\mathbb{H}} = (u, x)_{\mathbb{H}} + \mathrm{i}(u, y)_{\mathbb{H}}$).

Now let us provide some facts from the infinite-dimensional holomorphy, taken mainly from the monographs [97], [221], [225].

Definition 1.3.44 ([97], pages 144, 152). Given a complex Banach space $\mathbb{E}$, let $U \subset \mathbb{E}$ be open. A function $\varphi\colon U \to \mathbb{C}$ is called G-holomorphic on U if for every $u \in U$ and $v \in \mathbb{E}$, the function of a single complex variable $\mathbb{C} \ni \zeta \mapsto \varphi(u + \zeta v) \in \mathbb{C}$ is holomorphic in some neighborhood of zero. A G-holomorphic function on U is called holomorphic on U if it is continuous. A function $f : \mathbb{E} \to \mathbb{C}$ is called holomorphic if it is holomorphic on $\mathbb{E}$.

By $\mathrm{Hol}(\mathbb{E})$ we denote the set of all holomorphic functions on $\mathbb{E}$. A function $f : \mathbb{E} \to \mathbb{C}$ is called *locally bounded* if it maps bounded subsets of $\mathbb{E}$ into bounded subsets of $\mathbb{C}$.

Proposition 1.3.45 ([97], page 153). *A G-holomorphic function on $\mathbb{E}$ is continuous and hence holomorphic if and only if it is locally bounded.*

Given $n \in \mathbb{N}$, let $\mathbb{E}^n$ stand for the n-th Cartesian power of $\mathbb{E}$. Let also $\Phi\colon \mathbb{E}^n \to \mathbb{C}$ be linear with respect to each of its arguments. Such functions are called *n-linear*. A function $\pi\colon \mathbb{E} \to \mathbb{C}$ is called an *n-monomial* if there exists an n-linear function Φ, such that $\pi(u) = \Phi(u, \ldots, u)$. By construction, an n-monomial is G-holomorphic on $\mathbb{E}$ and, by Proposition 1.3.45, it is continuous (and hence holomorphic) if and only if

$$\|\pi\| \stackrel{\text{def}}{=} \sup_{u \in B} |\pi(u)| < \infty, \tag{1.3.95}$$

where B is the unit ball in $\mathbb{E}$, i.e., $B = \{u \in \mathbb{E} |\ \|u\|_{\mathbb{E}} \leq 1\}$. The set of all continuous n-monomials $\Pi_n(\mathbb{E})$ endowed with the norm (1.3.95) becomes a Banach space (see page 22 of [97]). The set $\Pi_0(\mathbb{E})$, i.e., the set of all complex-valued constant functions on $\mathbb{E}$, will be identified with $\mathbb{C}$. A function $\varphi\colon \mathbb{E} \to \mathbb{C}$ is holomorphic if and only if for every $u \in \mathbb{E}$, there exists a sequence of monomials $\{\pi_n\}_{n \in \mathbb{N}_0}$, $\pi_n \in \Pi_n(\mathbb{E})$, dependent on u, such that

$$\varphi(u + v) = \sum_{n=0}^{\infty} \pi_n(v) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{n!} \varphi^{(n)}(u)(v), \quad \varphi^{(0)}(u) = \varphi(u), \tag{1.3.96}$$

and the series converges for all v belonging to some neighborhood of $0 \in \mathbb{E}$. It is the *Taylor expansion* of φ centered at u, whereas the monomials $\varphi^{(n)}(u)$ are called the *derivatives* of φ at u. Such monomials are characterized by the following statement – an infinite-dimensional analog of the Cauchy theorem.

Proposition 1.3.46 ([225], pages 21, 23). *Let $\varphi \in \mathrm{Hol}(\mathbb{E})$. Then for every $u, v \in \mathbb{E}$, $n \in \mathbb{N}$, and $r > 0$,*

$$\varphi^{(n)}(u)(v) = \frac{n!}{2\pi \mathrm{i}} \int_{|\zeta| = r} \frac{\varphi(u + \zeta v)}{\zeta^{n+1}} \mathrm{d}\zeta, \tag{1.3.97}$$

and hence,

$$\left|\varphi^{(n)}(u)(v)\right| \leq \frac{n!}{r^n} \sup_{\zeta\colon |\zeta|=r} |\varphi(u+\zeta v)|. \tag{1.3.98}$$

The latter estimate is called the *Cauchy inequality*.

Now let us turn to the analytic properties of characteristic functions of measures obeying (1.3.93).

Lemma 1.3.47. *For every $\mu \in \mathcal{M}(\mathbb{H})$, its characteristic function φ_μ can be extended to a function from* $\mathrm{Hol}(\mathbb{H}^c)$.

Proof. For $\mu \in \mathcal{M}(\mathbb{H})$ and $\zeta \in \mathbb{C}$, we set

$$A_\mu(\zeta) = \int_{\mathbb{H}} \exp(\zeta \|x\|^2_{\mathbb{H}})\mu(\mathrm{d}x). \tag{1.3.99}$$

By Definition 1.3.43 there exists $a \in (0, +\infty)$ such that $A_\mu(a) < \infty$. Let us show that A_μ is holomorphic in the disc $\{\zeta \in \mathbb{C} \mid |\zeta| < a\}$. To this end we fix $x \in \mathbb{H}$ and set $f(\zeta) = \exp(\zeta\|x\|^2_{\mathbb{H}})$. It is an entire function of a single complex variable and its derivatives at zero are $f^{(n)}(0) = \|x\|^{2n}_{\mathbb{H}}$, $n \in \mathbb{N}_0$. By the Cauchy inequality for f we have

$$\|x\|^{2n}_{\mathbb{H}} \leq \frac{n!}{a^n} \exp(a\|x\|^2_{\mathbb{H}}),$$

which immediately yields

$$A^{(n)}_\mu(0) = \int_{\mathbb{H}} \|x\|^{2n}_{\mathbb{H}} \mu(\mathrm{d}x) \leq \frac{n!}{a^n} \int_{\mathbb{H}} \exp(a\|x\|^2_{\mathbb{H}})\mu(\mathrm{d}x) = \frac{n!}{a^n} A_\mu(a).$$

Thus, A_μ is holomorphic in the disc $|\zeta| < a$. By the Cauchy–Schwarz inequality, see (1.1.15), it follows that

$$\int_{\mathbb{H}} \|x\|^n_{\mathbb{H}} \mu(\mathrm{d}x) \leq \left(\int_{\mathbb{H}} \|x\|^{2n}_{\mathbb{H}} \mu(\mathrm{d}x)\right)^{1/2} \leq \frac{1}{a^{n/2}} \sqrt{n! A_\mu(a)}. \tag{1.3.100}$$

Given $u, v \in \mathbb{H}^c$, we set $u_i = \Im(u)$ and

$$\begin{aligned} \varphi_\mu(u+\zeta v) &= \int_{\mathbb{H}} \exp(\mathrm{i}(u,x)_{\mathbb{H}} + \mathrm{i}\zeta(v,x)_{\mathbb{H}})\mu(\mathrm{d}x) \\ &= \sum_{n=0}^{\infty} \frac{(\mathrm{i}\zeta)^n}{n!} g_n(u,v), \end{aligned} \tag{1.3.101}$$

where

$$g_n(u,v) = \int_{\mathbb{H}} [\mathrm{i}(v,x)_{\mathbb{H}}]^n \exp(\mathrm{i}(u,x)_{\mathbb{H}})\mu(\mathrm{d}x). \tag{1.3.102}$$

Then by (1.3.100) and the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}
|g_n(u,v)| &\le \|v\|_{\mathbb{H}}^n \int_{\mathbb{H}} \|x\|_{\mathbb{H}}^n \exp(-(u_i,x)_{\mathbb{H}})\mu(\mathrm{d}x) \\
&\le \|v\|_{\mathbb{H}}^n \left(\int_{\mathbb{H}} \|x\|_{\mathbb{H}}^{2n}\mu(\mathrm{d}x)\right)^{1/2} \left(\int_{\mathbb{H}} \exp(-2(u_i,x)_{\mathbb{H}})\mu(\mathrm{d}x)\right)^{1/2} \\
&\le \|v\|_{\mathbb{H}}^n \left(\frac{n!}{a^n}A_\mu(a)\right)^{1/2} \left(\sum_{m=0}^{\infty} \frac{2^m}{m!}\|u\|_{\mathbb{H}}^m \int_{\mathbb{H}} \|x\|_{\mathbb{H}}^m \mu(\mathrm{d}x)\right)^{1/2} \\
&\le \|v\|_{\mathbb{H}}^n \left(\frac{n!}{a^n}\right)^{1/2} A_\mu(a) \left(\sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}}\left[\frac{2\|u\|_{\mathbb{H}}}{\sqrt{a}}\right]^m\right)^{1/2},
\end{aligned}$$

which means that the power series on the right-hand side of (1.3.101) converges uniformly on compact subsets of $\mathbb{C}$ to an entire function of ζ. Thus, the function φ_μ is G-holomorphic. Furthermore,

$$\begin{aligned}
|\varphi_\mu(z)| &\le \int_{\mathbb{H}} \exp(-(z_i,x)_{\mathbb{H}})\mu(\mathrm{d}x) \\
&= \exp\left(\frac{1}{4a}\|z_i\|_{\mathbb{H}}^2\right) \int_{\mathbb{H}} \exp\left(-\frac{1}{4a}\|z_i - 2ax\|_{\mathbb{H}}^2 + a\|x\|_{\mathbb{H}}^2\right)\mu(\mathrm{d}x) \quad (1.3.103) \\
&\le \exp\left(\frac{1}{4a}\|z_i\|_{\mathbb{H}}^2\right) A_\mu(a),
\end{aligned}$$

where $z_i = \Im(z)$. Hence, the function φ_μ is locally bounded, which in view of Proposition 1.3.45 completes the proof. □

By the statement just proven the function

$$f_\mu(z) = \int_{\mathbb{H}} \exp((z,x)_{\mathbb{H}})\mu(\mathrm{d}x), \quad z \in \mathbb{H}^c, \tag{1.3.104}$$

is also holomorphic whenever $\mu \in \mathcal{M}(\mathbb{H})$. In this case, the estimate (1.3.103) implies

$$\begin{aligned}
|\varphi_\mu(z)| &\le \exp\left(\frac{1}{4a}\|z\|_{\mathbb{H}}^2\right) A_\mu(a), \\
|f_\mu(z)| &\le \exp\left(\frac{1}{4a}\|z\|_{\mathbb{H}}^2\right) A_\mu(a),
\end{aligned} \tag{1.3.105}$$

and

$$|f_\mu(z)| \le \exp\left(\frac{1}{4a}\|z_r\|_{\mathbb{H}}^2\right) A_\mu(a), \quad z_r = \Re(z). \tag{1.3.106}$$

The function f_μ will be called the *Laplace transform* of the measure $\mu \in \mathcal{M}(\mathbb{H})$.

1.3.6 Properties of the Høegh-Krohn Process

The Høegh-Krohn process was constructed in Subsection 1.3.2; its canonical realization is the process on $(C_\beta, \mathcal{B}_{C_\beta}, \chi_\beta)$ described in Subsection 1.3.3. The measure χ_β is Gaussian and such that

$$\chi_\beta(C_\beta^\sigma) = 1, \quad \text{for any } \sigma \in (0, 1/2). \tag{1.3.107}$$

This measure is the Euclidean Gibbs measure for a single harmonic oscillator and thereby is the basic element of the construction of Euclidean Gibbs measure for the whole model. That is why we study this measure in detail.

The measure

The measure χ_β can also be considered as a measure on the Hilbert space L^2_β. We recall that the scalar product in L^2_β is

$$(x, y)_{L^2_\beta} = \sum_{j=1}^{\nu} \int_0^\beta x^{(j)}(\tau) y^{(j)}(\tau) \mathrm{d}\tau. \tag{1.3.108}$$

The measure χ_β is uniquely determined by its Fourier transform, cf. (1.3.88),

$$\int_{L^2_\beta} \exp(\mathrm{i}(x, y)_{L^2_\beta}) \chi_\beta(\mathrm{d}y) = \exp\left\{-\frac{1}{2}\left(x, S_\beta x\right)_{L^2_\beta}\right\}, \tag{1.3.109}$$

where $S_\beta : L^2_\beta \to L^2_\beta$ is a strictly positive trace-class operator. Its kernel is given in (1.3.18). If in this kernel one sets $\kappa = (am)^{1/4}$, $\delta = (a/m)^{1/2}$ (see (1.1.70), (1.1.72)), it will coincide with the matrix element (1.2.125) which by (1.2.123) and (1.2.124) determine the state ϱ_ℓ. The operator S_β can also be written in the form

$$S_\beta = \delta\kappa^{-2}\left(\mathbf{1} \otimes \left[-(\mathrm{d}/\mathrm{d}\tau)^2 + \delta^2\right]^{-1}\right), \tag{1.3.110}$$

where $\mathbf{1}$ is the identity operator in $\mathbb{R}^\nu$. There exists an orthonormal basis of L^2_β consisting of the eigenvectors of S_β. Let $\{\wp_j\}$, $j = 1, \ldots, \nu$, be the canonical basis of $\mathbb{R}^\nu$ (this means $\wp_{j'}^{(j)} = \delta_{jj'}$). For $k \in \mathcal{K}$ defined in (1.3.20), we set

$$e_k(\tau) = \begin{cases} (\sqrt{2/\beta}) \cos k\tau, & k > 0; \\ -(\sqrt{2/\beta}) \sin k\tau, & k < 0; \\ \sqrt{1/\beta}, & k = 0. \end{cases} \tag{1.3.111}$$

Then the basis of L^2_β mentioned above consists of $\epsilon_{j,k} = \wp_j \otimes e_k, k \in \mathcal{K},\ j = 1, \ldots \nu$, such that

$$S_\beta \epsilon_{j,k} = s_k(\kappa, \delta)\epsilon_{j,k}, \quad s_k(\kappa, \delta) = \frac{\delta\kappa^{-2}}{\delta^2 + k^2} = \frac{1}{mk^2 + a}. \tag{1.3.112}$$

By (1.3.21) the integral kernel $S_\beta^{jj'}(\tau,\tau')$ of the operator (1.3.110) is

$$\begin{aligned} S_\beta^{jj'}(\tau,\tau') &= \delta_{jj'} S_\beta(\tau,\tau'), \quad j,j'=1,\dots,\nu,\ \tau,\tau'\in[0,\beta], \\ S_\beta(\tau,\tau') &\stackrel{\text{def}}{=} \sum_{k\in\mathcal{K}} s_k(\kappa,\delta) e_k(\tau) e_k(\tau') = \Phi_\beta(|\tau-\tau'|_\beta), \end{aligned} \tag{1.3.113}$$

where for $\theta\in(0,\beta)$, we have set

$$\begin{aligned} \Phi_\beta(\theta) \stackrel{\text{def}}{=} & \frac{1}{2\sqrt{am}[1-\exp(-\beta\sqrt{a/m})]} \\ & \times \left\{ \exp[-\theta\sqrt{a/m}] + \exp[-(\beta-\theta)\sqrt{a/m}] \right\}. \end{aligned} \tag{1.3.114}$$

Clearly, $\Phi_\beta(\theta)\le\Phi_\beta(0)$; hence,

$$\begin{aligned} S_\beta(\tau,\tau') &\le S_\beta(\tau,\tau) \\ &= \frac{1+\exp(-\beta\sqrt{a/m})}{2\sqrt{am}[1-\exp(-\beta\sqrt{a/m})]} \\ &\stackrel{\text{def}}{=} \upsilon. \end{aligned} \tag{1.3.115}$$

Proposition 1.3.48. *The kernel* (1.3.113) *obeys the estimates*

$$\begin{aligned} S_\beta(0,0) - S_\beta(\tau,\tau') &\le \frac{1}{2m}\cdot|\tau-\tau'|_\beta, \\ \sup_{t\in[0,\beta]} \left| S_\beta(t,\tau) - S_\beta(t,\tau') \right| &\le \frac{1}{2m}\cdot|\tau-\tau'|_\beta, \end{aligned} \tag{1.3.116}$$

thus, for any $p\in\mathbb{N}$, *one has*

$$\left\langle \left| x(\tau) - x(\tau') \right|^{2p} \right\rangle_{\chi_\beta} \le \frac{\Gamma(\nu/2+p)}{\Gamma(\nu/2)} \left(\frac{2}{m}\right)^p \cdot |\tau-\tau'|_\beta^p. \tag{1.3.117}$$

Proof. For the function (1.3.114), one gets

$$\left|\Phi_\beta(\theta) - \Phi_\beta(\theta+\vartheta)\right| \le \sup_{t\in[0,\beta]} \left|\Phi_\beta'(t)\right| \cdot |\vartheta|_\beta = \frac{1}{2m}\cdot|\vartheta|_\beta, \tag{1.3.118}$$

which yields both estimates in (1.3.116). The estimate (1.3.117) is obtained from the identities

$$\left\langle \left[x^{(j)}(\tau) - x^{(j)}(\tau')\right]^{2p} \right\rangle_{\chi_\beta} = \frac{(2p)!}{p!} \cdot \left[S_\beta(0,0) - S_\beta(\tau,\tau')\right]^p$$

and

$$\left\langle \left\{ \sum_{j=1}^{\nu} \left[x^{(j)}(\tau) - x^{(j)}(\tau')\right]^2 \right\}^p \right\rangle_{\chi_\beta} = 2^{2p} \cdot \frac{\Gamma(\nu/2+p)}{\Gamma(\nu/2)} \cdot \left[S_\beta(0,0) - S_\beta(\tau,\tau')\right]^p,$$

where the first identity is a more detailed version of (1.3.55) for a one-dimensional Gaussian random variable, and the second identity follows from the first one after some calculations. □

Now we employ (1.3.117) to estimate $\langle \|x\|^2_{C^\sigma_\beta} \rangle_{\chi_\beta}$, which then can be used in estimating exponential moments of χ_β by means of Fernique's theorem. By (1.3.58) one has

$$\|x\|^2_{C^\sigma_\beta} \leq 2\big(|x(0)|^2 + [\beta^\sigma K_\sigma(x)]^2\big). \tag{1.3.119}$$

Furthermore, by (1.3.115),

$$\langle |x(0)|^2 \rangle_{\chi_\beta} = \nu S_\beta(0,0) = \nu\upsilon. \tag{1.3.120}$$

For every $p \in \mathbb{N}$,

$$\langle K^2_\sigma \rangle_{\chi_\beta} \leq \left[\langle K^{2p}_\sigma \rangle_{\chi_\beta}\right]^{1/p} = \left[\langle L^{2p}_{\sigma,\beta} \rangle_{\chi_\beta}\right]^{1/p}, \tag{1.3.121}$$

where the latter is defined by (1.3.49) and hence can be estimated in (1.3.50) with $q = p - 1$ and C taken from (1.3.117). Thereafter, one obtains

$$\langle \|x\|^2_{C^\sigma_\beta} \rangle_{\chi_\beta} \leq b_\sigma, \quad \sigma \in (0, 1/2), \tag{1.3.122}$$

where

$$\begin{aligned} b_\sigma &= 2\nu\upsilon + \frac{2^{8+3/p}(1+1/\sigma p)^2}{[(p-1-2\sigma p)(p-2\sigma p)]^{1/p}} \\ &\quad \times \left(\frac{\Gamma(\nu/2+p)}{\Gamma(\nu/2)}\right)^{1/p} \cdot \frac{\beta}{m}, \\ p &= [1/1-2\sigma] + 1. \end{aligned} \tag{1.3.123}$$

Here $[\cdot]$ denotes integral part. Now we apply Theorem 1.3.37 and obtain the following

Proposition 1.3.49. *For every $\sigma \in (0, 1/2)$ and $\lambda \in (0, \varsigma/b_\sigma)$, it follows that*

$$\int_{L^2_\beta} \exp\left(\lambda \|x\|^2_{C^\sigma_\beta}\right) \chi_\beta(\mathrm{d}x) \leq C_\sigma(\lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 3^{-2^n(1-\lambda b_\sigma/\varsigma)}. \tag{1.3.124}$$

Tightness

We fix the rigidity parameter $a > 0$ and consider the dependence of χ_β on the mass m, which we indicate by writing χ^m_β. Furthermore, by writing $\upsilon(m)$, $b_\sigma(m)$ we indicate the m-dependence of the parameters defined by (1.3.115) and (1.3.123), respectively, cf. Theorem 1.1.60.

Theorem 1.3.50. *For any $m_0 > 0$, the family $\{\chi^m_\beta\}_{m \in [m_0, +\infty)} \subset \mathcal{P}(C_\beta)$ is tight.*

Proof. By the Arzela–Ascoli theorem the balls

$$B_R^\sigma = \{x \in C_\beta \mid \|x\|_{C_\beta^\sigma} \le R\}, \quad \sigma > 0,\ R > 0, \tag{1.3.125}$$

are compact in C_β. For $m \ge m_0$, one readily gets from (1.3.115) that $\upsilon(m) \le \upsilon(m_0)$, and hence $b_\sigma(m) \le b_\sigma(m_0)$. Then

$$\langle \|x\|^2_{C_\beta^\sigma} \rangle_{\chi_\beta^m} \le b_\sigma(m_0),$$

which by Proposition 1.3.35 yields the proof. □

Finite-dimensional approximations

Here we fix also the mass m and construct approximations of the measure χ_β by measures concentrated on finite-dimensional subspaces of C_β, which possess certain useful properties. Then we show that the sequences of such approximating measures converge weakly, as measures on C_β, to χ_β. The construction is based on Proposition 1.3.33 and Theorem 1.3.42.

In view of (1.3.112), the covariance operator S_β defined by (1.3.110) can be written in the form

$$S_\beta = \sum_{j=1}^{\nu} \sum_{k \in \mathcal{K}} s_k(\kappa, \delta) P_{j,k}, \tag{1.3.126}$$

where $P_{j,k}$ is the orthogonal projection in L^2_β on the one-dimensional subspace spanned by the eigenvector $\epsilon_{j,k}$. Let $\{s^{(n)}\}_{n\in\mathbb{N}}$ be a sequence, each element of which $s^{(n)} = \{s_k^{(n)}\}_{k\in\mathcal{K}}$ is a summable sequence of positive numbers. For such $s^{(n)}$, we set

$$S_\beta^{(n)} = \sum_{j=1}^{\nu} \sum_{k \in \mathcal{K}} s_k^{(n)} P_{j,k}, \tag{1.3.127}$$

which is a positive trace-class operator $S_\beta^{(n)} : L^2_\beta \to L^2_\beta$. The sequence $\{S_\beta^{(n)}\}_{n\in\mathbb{N}}$ converges in the trace norm to the operator (1.3.126) if and only if the sequence $\{s^{(n)}\}_{n\in\mathbb{N}}$ converges in $l^1(\mathcal{K})$ to $\{s_k(\kappa,\delta)\}_{k\in\mathcal{K}}$. Each $S_\beta^{(n)}$ by (1.3.109) determines a zero mean Gaussian measure γ_n on L^2_β. Suppose that $\gamma_n(C_\beta) = 1$ for every $n \in \mathbb{N}$. Thus, each γ_n can be redefined as a Gaussian measure on the Banach space C_β. To construct the approximation in question we consider the following sequence of operators (1.3.127). Given $L \in \mathbb{N}$, we set $N = 2L$ and

$$\mathcal{K}_N = \{k = (2\pi/\beta)\varkappa \mid \varkappa = -(L-1), -(L-2), \ldots, 0, \ldots, L\}. \tag{1.3.128}$$

Thereafter, we set

$$\begin{aligned} s_k^{(N)} &= \frac{\delta\kappa^{-2}}{(2N/\beta)^2\,[\sin(\beta k/2N)]^2 + \delta^2}, && \text{for } k \in \mathcal{K}_N, \\ s_k^{(N)} &= 0, && \text{for } k \in \mathcal{K} \setminus \mathcal{K}_N. \end{aligned} \tag{1.3.129}$$

Let $S_\beta^{(N)}$ be defined by (1.3.127) with this $\{s_k^{(N)}\}_{k\in\mathcal{K}}$. Hence, each $S_\beta^{(N)}$ is a finite-rank operator, see Definition 1.1.6. Let also γ_N be the zero mean Gaussian measure on L^2_β for which $S_\beta^{(N)}$ is the covariance operator. Such a measure is concentrated on a finite-dimensional subspace of L^2_β consisting of trigonometric polynomials; thus, it can be redefined as a measure on the Banach space C_β.

Theorem 1.3.51. *For the sequence of measures introduced above, it follows that $\gamma_N \Rightarrow \chi_\beta$ in the sense of convergence of measures on the Banach space C_β.*

Remark 1.3.52. Since the topology of C_β is stronger than the topology induced on the set $C_\beta \subset L^2_\beta$ from the Hilbert space L^2_β, there exist fewer functions $f : C_\beta \to \mathbb{R}$, continuous in this induced topology than those continuous in the norm topology of the space C_β. Therefore, the weak topology in the sense of the Hilbert space L^2_β is weaker than the weak topology in the sense of the Banach space C_β. Hence, to prove Theorem 1.3.51 it is not enough to use Theorem 1.3.42 only.

The proof of Theorem 1.3.51 will be done in several steps. First, one observes that each $\alpha_k \stackrel{\text{def}}{=} (\beta k/2N)$, $k \in \mathcal{K}_N$, belongs to $(-\pi/2, \pi/2]$. Hence,

$$(\alpha_k \cos\alpha_k)^2 \le (\sin\alpha_k)^2 \le \alpha_k^2. \tag{1.3.130}$$

Therefore, for $k \in \mathcal{K}_N$,

$$s_k(\kappa,\delta) \le s_k^{(N)} \le c_k^{(N)} \stackrel{\text{def}}{=} \frac{\delta\kappa^{-2}}{k^2\,[\cos(\beta k/2N)]^2 + \delta^2}. \tag{1.3.131}$$

Lemma 1.3.53. *The sequence of the operators $\{S_\beta^{(N)}\}_{N\in 2\mathbb{N}}$ introduced in (1.3.127)–(1.3.129) converges in the trace norm to S_β.*

Proof. We have to show that

$$\lim_{N\to+\infty} \sum_{k\in\mathcal{K}} |s_k^{(N)} - s_k(\kappa,\delta)| = 0.$$

By (1.3.131) this is equivalent to the following, see (1.3.18),

$$\lim_{N\to+\infty} \sum_{k\in\mathcal{K}_N} s_k^{(N)} = \sum_{k\in\mathcal{K}} s_k(\kappa,\delta) = \sigma_{jj}(0,0) = \frac{1}{2\kappa^2}\cdot\frac{1+e^{-\beta\delta}}{1-e^{-\beta\delta}}. \tag{1.3.132}$$

Given $\varepsilon \in (0,1)$, we set

$$\begin{aligned} \mathcal{K}'_N &= \{k = (2\pi/\beta)\kappa \in \mathcal{K}_N \mid |\kappa| \le L(2/\pi)\arcsin\varepsilon\},\\ \mathcal{K}''_N &= \mathcal{K}_N \setminus \mathcal{K}'_N. \end{aligned} \tag{1.3.133}$$

Then

$$\begin{aligned} \forall k \in \mathcal{K}'_N: &\quad [\sin\alpha_k]^2 \le \varepsilon^2, \quad [\cos\alpha_k]^2 \ge 1-\varepsilon^2;\\ \forall k \in \mathcal{K}''_N: &\quad [\sin\alpha_k]^2 > \varepsilon^2, \quad [\cos\alpha_k]^2 < 1-\varepsilon^2. \end{aligned} \tag{1.3.134}$$

Taking this into account we get

$$\sum_{k\in\mathcal{K}''_N} s_k^{(N)} \le \sum_{k\in\mathcal{K}''_N} \frac{\delta\kappa^{-2}}{(2N/\beta)^2\,\varepsilon^2+\delta^2} < \frac{\delta\kappa^{-2}N}{(2N/\beta)^2\,\varepsilon^2+\delta^2} \to 0, \tag{1.3.135}$$

as $N \to +\infty$. On the other hand, by (1.3.131) and (1.3.134)

$$\begin{aligned}\sum_{k\in\mathcal{K}'_N} s_k(\kappa,\delta) &\le \sum_{k\in\mathcal{K}'_N} s_k^{(N)} \le \sum_{k\in\mathcal{K}'_N} c_k^{(N)}\\ &\le \sum_{k\in\mathcal{K}'_N} \frac{\delta\kappa^{-2}}{k^2(1-\varepsilon^2)+\delta^2}\\ &= \sum_{k\in\mathcal{K}'_N} s_k\big(\kappa[1-\varepsilon^2]^{1/4},\delta[1-\varepsilon^2]^{-1/2}\big).\end{aligned} \tag{1.3.136}$$

Since ε can be taken arbitrarily small, one gets from this estimate that

$$\lim_{N\to+\infty}\sum_{k\in\mathcal{K}_N} s_k^{(N)} = \lim_{N\to+\infty}\sum_{k\in\mathcal{K}'_N} s_k^{(N)} = \sum_{k\in\mathcal{K}} s_k(\kappa,\delta),$$

which completes the proof. □

Therefore, by Theorem 1.3.42 the sequence of measures $\{\gamma_N\}$ converges to the measure χ_β weakly in the Hilbert space L^2_β. Now we show that this sequence is tight in C_β which, by Proposition 1.3.33, will give us the convergence to be proven.

Lemma 1.3.54. *For every $\sigma \in (0,1/2)$, there exists $\tilde{b}_\sigma > 0$ such that for all $N \in \mathbb{N}$,*

$$\int_{L^2_\beta} \|x\|^2_{C^\sigma_\beta}\gamma_N(\mathrm{d}x) \le \tilde{b}_\sigma. \tag{1.3.137}$$

Thus, the sequence $\{\gamma_N\}_{N\in\mathbb{N}}$ is tight in the Banach space C_β.

Proof. By (1.3.127) for any γ_N and $j, j' = 1,\dots,\nu$, one obtains

$$\begin{aligned}\langle x^{(j)}(s)x^{(j')}(t)\rangle_{\gamma_N} &= \sum_{k\in\mathcal{K}}\Big(S_\beta^{(N)}\epsilon_{k,j}(s)\Big)\epsilon_{k,j'}(t)\\ &= \delta_{jj'}\sum_{k\in\mathcal{K}} s_k^{(N)} e_k(s)e_k(t).\end{aligned} \tag{1.3.138}$$

For a fixed $\varepsilon > 0$, by (1.3.131), (1.3.133), and (1.3.134) it follows that

$$\begin{aligned}
s_k^{(N)} &\le \frac{\delta\kappa^{-2}}{k^2(1-\varepsilon^2)+\delta^2}, && k\in\mathcal{K}'_N;\\
s_k^{(N)} &\le \frac{\delta\kappa^{-2}}{(2N/\beta)^2\varepsilon^2+\delta^2} < \frac{\delta\kappa^{-2}}{k^2(2\varepsilon/\pi)^2+\delta^2}, && k\in\mathcal{K}''_N;\\
s_k^{(N)} &= 0 \le \frac{\delta\kappa^{-2}c}{k^2+\delta^2 c}, && k\in\mathcal{K}\setminus\mathcal{K}_N,
\end{aligned}$$

where the latter estimate holds for any $c > 0$. Thereby, for all $k \in \mathcal{K}$,

$$s_k^{(N)} \leq \delta\kappa^{-2}c/(k^2 + \delta^2 c), \quad c = \max\{[1-\varepsilon^2]^{-1}, (\pi/2\varepsilon)^2\}. \tag{1.3.139}$$

The minimal value of c is achieved at $\varepsilon = \pi/\sqrt{\pi^2+4}$, in view of which we set

$$c = 1 + \pi^2/4. \tag{1.3.140}$$

Thereafter, by (1.3.138) and (1.3.112) we obtain

$$\begin{aligned} \langle |x(0)|^2 \rangle_{\gamma_N} &= \nu\beta^{-1} \sum_{k\in\mathcal{K}_N} s_k^{(N)} \leq \sum_{k\in\mathcal{K}} \frac{\nu\beta^{-1}\delta\kappa^{-2}c}{k^2+\delta^2 c} \\ &= \nu\beta^{-1} \sum_{k\in\mathcal{K}_N} \frac{1}{(m/c)k^2 + a}, \end{aligned} \tag{1.3.141}$$

where c is given in (1.3.140).

To estimate the integral of the second term in (1.3.119) we employ Proposition 1.3.12, as it was done in (1.3.121). Since the measures γ_N, $N \in \mathbb{N}$, are Gaussian, we can use the factorization (1.3.55) to obtain,

$$\langle |x(\tau) - x(\tau')|^{2p} \rangle_{\gamma_N} \leq 2^{2p} \frac{\Gamma(\nu/2+p)}{\Gamma(\nu/2)} A_N^p \cdot |\tau - \tau'|_\beta^p, \tag{1.3.142}$$

where the constant A_N has to be found from the $p = 1$ case. Applying (1.3.139) we get from (1.3.138) the estimate

$$\langle |x(\tau) - x(\tau')|^2 \rangle_{\gamma_N} \leq \nu\sqrt{c} \sum_{k\in\mathcal{K}} \frac{\tilde{\delta}\tilde{\kappa}^{-2}}{k^2+\tilde{\delta}^2} [e_k(\tau) - e_k(\tau')]^2,$$

with $\tilde{\kappa} = \kappa$ and $\tilde{\delta} = \delta\sqrt{c}$. Then by (1.3.117),

$$\begin{aligned} \langle |x(\tau) - x(\tau')|^2 \rangle_{\gamma_N} &\leq \nu\sqrt{c}\tilde{\delta}\tilde{\kappa}^{-2}|\tau - \tau'|_\beta \\ &= \frac{\nu c}{m} \cdot |\tau - \tau'|_\beta, \end{aligned} \tag{1.3.143}$$

which is independent of N. For a given $\sigma \in (0, 1/2)$, one sets $p = [1/1 - 2\sigma]$, cf. (1.3.123), and obtains, see (1.3.121),

$$\begin{aligned} \langle K_\sigma^2 \rangle_{\gamma_N} &\leq \left[\langle L_{\sigma,\beta/2}^{2p} \rangle_{\gamma_N} \right]^{1/p} \\ &\leq \frac{2^{7+3/p}(1+/\sigma p)^2}{[(p-1-2\sigma p)(p-2\sigma p)]^{1/p}} \cdot \left(\frac{\Gamma(\nu/2+p)}{\Gamma(\nu/2)} \right)^{1/p} \cdot \frac{c\beta^{1-2\sigma}}{m}. \end{aligned}$$

Thereafter, the estimate (1.3.137) follows from (1.3.141) and the latter estimate with $\tilde{b}$ as in (1.3.123) but with m replaced by m/c with c being as in (1.3.140). In particular, this means that υ in the first summand has to be replaced by, cf. (1.3.115),

$$\tilde{\upsilon} = \sqrt{\frac{c}{4am}} \cdot \frac{1 + \exp\left(-\beta\sqrt{ac/m}\right)}{1 - \exp\left(-\beta\sqrt{ac/m}\right)}.$$

Now the stated tightness follows from (1.3.137) by Proposition 1.3.35. $\square$

Proof of Theorem 1.3.51. As a tight sequence, $\{\gamma_N\}_{N\in\mathbb{N}}$ has accumulation points in $\mathcal{P}(C_\beta)$, each of which ought to be χ_β in view of Theorem 1.3.42 and Lemma 1.3.53. $\square$

In Chapter 2, we shall use the following extension of Theorem 1.3.51. Let $V : \mathbb{R} \to \mathbb{R}$ be continuous and such that $V(t) \geq b_V t^{2r} - c_V$, the constants b_V and c_V being the same as in (1.1.10). Consider

$$\begin{aligned}\sigma(\mathrm{d}x) &= \frac{1}{Z}\exp\left(-\int_0^\beta V\left(x(\tau)\right)\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}x),\\ Z &= \int_{C_\beta}\exp\left(-\int_0^\beta V\left(x(\tau)\right)\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}x),\end{aligned} \tag{1.3.144}$$

and

$$\begin{aligned}\sigma^N(\mathrm{d}x) &= \frac{1}{Z^N}\exp\left(-\frac{\beta}{N}\sum_{\lambda=0}^{N-1}V\left(x\left(\frac{\lambda}{N}\beta\right)\right)\right)\gamma_N(\mathrm{d}x),\\ Z^N &= \int_{C_\beta}\exp\left(-\frac{\beta}{N}\sum_{\lambda=0}^{N-1}V\left(x\left(\frac{\lambda}{N}\beta\right)\right)\right)\gamma_N(\mathrm{d}x).\end{aligned} \tag{1.3.145}$$

with $N = 2L$, $L \in \mathbb{N}$.

Theorem 1.3.55. *The sequence $\{\sigma^N\}$ of measures introduced in (1.3.145) converges weakly to the measure σ in the sense of convergence of measures on the Banach space C_β.*

Proof. For an even N, we set

$$F_N(x) = \exp\left(-\frac{\beta}{N}\sum_{\lambda=0}^{N-1}V\left(x\left(\frac{\lambda}{N}\beta\right)\right)\right), \quad x \in C_\beta.$$

All these functions F_N are bounded and continuous, and for all $x \in C_\beta$,

$$F_N(x) \to \exp\left(-\int_0^\beta V(x(\tau))\mathrm{d}\tau\right), \quad N \to +\infty.$$

For a bounded continuous function $G : C_\beta \to \mathbb{R}$ and even $N, M \in \mathbb{N}$, we set

$$a_{NM} = \int_{C_\beta} G(x)F_N(x)\gamma_M(\mathrm{d}x).$$

With the help of Theorem 1.3.51 and Lebesgue's dominated convergence theorem, see Proposition 1.3.1, one easily shows that

$$\begin{aligned}\lim_{N\to+\infty}\lim_{M\to+\infty} a_{NM} &= \lim_{M\to+\infty}\lim_{N\to+\infty} a_{NM}\\ &= \int_{C_\beta} G(x)\exp\left(-\int_0^\beta V\left(x(\tau)\right)\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}x).\end{aligned} \tag{1.3.146}$$

Therefrom, by a standard diagonal procedure one gets

$$a_{NN} \to \int_{C_\beta} G(x) \exp\left(- \int_0^\beta V(x(\tau))\mathrm{d}\tau \right) \chi_\beta(\mathrm{d}x), \quad N \to +\infty.$$

which yields the proof. □

As we shall see below, the property of χ_β described by Proposition 1.3.49 plays a key role in our theory. Along with this property we employ also a property of the measures γ_N, $N \in \mathbb{N}$, which can be thought of as a uniform integrability of $\exp\{\lambda \|x\|^2_{C^\sigma_\beta}\}$. We recall that the absolute constant ς was defined in (1.3.77).

Theorem 1.3.56. *For $\sigma \in (0, 1/2)$, let $\tilde{b}_\sigma$ be as in (1.3.137). Then for any $\lambda \in (0, \varsigma/\tilde{b}_\sigma)$ and $N \in \mathbb{N}$, the following holds:*

$$\int_{L^2_\beta} \exp\left(\lambda \|x\|^2_{C^\sigma_\beta}\right) \gamma_N(\mathrm{d}x) \le \widetilde{C}_\sigma(\lambda) \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} 3^{-2^n(1-\lambda\tilde{b}_\sigma/\varsigma)}. \tag{1.3.147}$$

Proof. The proof follows immediately from the estimate (1.3.137) by Theorem 1.3.37. □

Notably, in (1.3.147) we can integrate over the space C_β.

Corollary 1.3.57. *For $\sigma \in (0, 1/2)$, let $\tilde{b}_\sigma$ be as in (1.3.137). Let also υ be in $(0, \varsigma/\tilde{b}_\sigma)$ for some $\sigma \in (0, 1/2)$. Then the sequence of measures*

$$\begin{aligned} &\tilde{\gamma}_N(\mathrm{d}x) = \frac{1}{Z_N(\upsilon)} \exp(\upsilon \|x\|^2_{C_\beta}) \gamma_N(\mathrm{d}x), \\ &N = 2L, \;\; L \in \mathbb{N}, \;\; Z_N(\upsilon) = \int_{C_\beta} \exp(\upsilon \|x\|^2_{C_\beta}) \gamma_N(\mathrm{d}x), \end{aligned} \tag{1.3.148}$$

converges weakly, as measures on C_β, to the measure

$$\mu(\mathrm{d}x) = \exp(\upsilon \|x\|^2_{C_\beta}) \chi_\beta(\mathrm{d}x) / \int \exp(\upsilon \|x\|^2_{C_\beta}) \chi_\beta(\mathrm{d}x). \tag{1.3.149}$$

Proof. Fix any $\sigma \in (0, 1/2)$. By (1.3.59) and (1.3.147)

$$1 \le Z_N(\upsilon) \le \widetilde{C}_\sigma(\upsilon). \tag{1.3.150}$$

For every $\upsilon \in (0, \varsigma/\tilde{b}_\sigma)$, there exists $\epsilon > 0$ such that $\upsilon + \epsilon \in (0, \varsigma/\tilde{b}_\sigma)$. Then by (1.3.59) and (1.3.150) we get

$$\begin{aligned} &\int_{C_\beta} \exp(\epsilon \|x\|^2_{C^\sigma_\beta}) \tilde{\gamma}_N(\mathrm{d}x) \\ &\quad \le [Z_N(\upsilon)]^{-1} \int_{C_\beta} \exp[(\upsilon + \epsilon)\|x\|^2_{C^\sigma_\beta}] \gamma_N(\mathrm{d}x) \\ &\quad \le \widetilde{C}_\sigma(\upsilon + \epsilon). \end{aligned}$$

Thus, by Proposition 1.3.35 the sequence $\{\tilde{\gamma}_N\}_{N\in\mathbb{N}}$ is tight and hence relatively weakly compact. Let $\mu \in \mathcal{P}(C_\beta)$ be any of the accumulation points of this sequence and $\{N_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ be such that $\tilde{\gamma}_{N_n} \Rightarrow \mu$ as $n \to +\infty$. Set

$$f(x) = \exp(-\upsilon\|x\|^2_{C_\beta}),$$

$\upsilon > 0$ being the same as in (1.3.148). Then $f \in C_{\mathrm{b}}(C_\beta)$; hence,

$$\langle f\rangle_{\tilde{\gamma}_{N_n}} \to \langle f\rangle_\mu, \quad \text{as } n \to +\infty,$$

which yields

$$\lim_{n\to+\infty} 1/Z_{N_n}(\upsilon) = \int_{C_\beta} \exp(-\upsilon\|x\|^2_{C_\beta})\mu(\mathrm{d}x).$$

Now we take any $\phi \in C_{\mathrm{b}}(C_\beta)$ and set $g(x) = \phi(x)f(x)$. Then

$$\langle g\rangle_{\tilde{\gamma}_{N_n}} \to \langle g\rangle_\mu, \quad \text{as } n \to +\infty.$$

Thereby, taking into account Theorem 1.3.51 we obtain

$$\int_{C_\beta} \phi(x)\chi_\beta(\mathrm{d}x) = \int_{C_\beta} \phi(x)\exp(-\upsilon\|x\|^2_{C_\beta})\mu(\mathrm{d}x) \Big/ \int_{C_\beta} \exp(-\upsilon\|x\|^2_{C_\beta})\mu(\mathrm{d}x),$$

which holds for arbitrary $\phi \in C_{\mathrm{b}}(C_\beta)$. As the latter set is measure-defining, this yields that the measure μ is as in (1.3.149). □

Now let us return to the measures (1.3.144), (1.3.145). Similarly as above, we obtain the following corollary of Theorem 1.3.55.

Corollary 1.3.58. *Let the measures σ and σ^N be as in Theorem 1.3.55 and υ be as in Corollary 1.3.57. Then the for arbitrary $\varkappa > 0$, the sequence of measures*

$$\begin{aligned} \tilde{\sigma}^N(\mathrm{d}x) &= \frac{1}{Z_N(\upsilon,\varkappa)}\exp\left(\upsilon\|x\|^2_{C_\beta} + \varkappa\|x\|^2_{L^2_\beta}\right)\sigma^N(\mathrm{d}x), \\ Z_N(\upsilon,\varkappa) &= \int_{C_\beta}\exp\left(\upsilon\|x\|^2_{C_\beta} + \varkappa\|x\|^2_{L^2_\beta}\right)\sigma^N(\mathrm{d}x) \end{aligned} \tag{1.3.151}$$

converges weakly, as measures on C_β, to the measure

$$\begin{aligned} \mu(\mathrm{d}x) &= \frac{1}{Z(\upsilon,\varkappa)}\exp\left(\upsilon\|x\|^2_{C_\beta} + \varkappa\|x\|^2_{L^2_\beta}\right)\sigma(\mathrm{d}x), \\ Z(\upsilon,\varkappa) &= \int_{C_\beta}\exp\left(\upsilon\|x\|^2_{C_\beta} + \varkappa\|x\|^2_{L^2_\beta}\right)\sigma(\mathrm{d}x). \end{aligned} \tag{1.3.152}$$

Proof. By (1.3.145) and (1.3.151) it follows that

$$\tilde{\sigma}^N(\mathrm{d}x) = \frac{1}{\tilde{Z}_N(\upsilon,\varkappa)}\Psi_N(x)\tilde{\gamma}_N(\mathrm{d}x),$$

where $\tilde{\gamma}_N$ is the same as in (1.3.148), $\widetilde{Z}_N(\upsilon, \varkappa)$ is a normalizing factor, and

$$\Psi_N(x) = \exp\left\{\varkappa \|x\|^2_{L^2_\beta} - \frac{\beta}{N} \sum_{\lambda=0}^{N-1} V\left(x\left(\frac{\lambda}{N}\beta\right)\right)\right\}. \tag{1.3.153}$$

For every $x \in C_\beta$, we have

$$\Psi_N(x) \to \Psi(x) \stackrel{\text{def}}{=} \exp\left(\varkappa \|x\|^2_{L^2_\beta} - \int_0^\beta V(x(\tau))\mathrm{d}\tau\right), \quad N \to +\infty.$$

Since $V(t) \geq b_V |t|^{2r} - c_V$, by Jensen's inequality we obtain

$$\begin{aligned}\Psi(x) &\leq \exp\left\{\beta c_V + \varkappa\|x\|^2_{L^2_\beta} - b_V\left(\int_0^\beta |(x(\tau))|^2 \mathrm{d}\tau\right)^r\right\} \\ &\leq \exp\left\{\beta c_V + \frac{r-1}{r}\left(\frac{\varkappa^r}{b_V r}\right)^{1/(r-1)}\right\}.\end{aligned}$$

Thereafter, the proof follows by Corollary 1.3.57 with the help of the diagonal method used in the proof of Theorem 1.3.55. □

1.3.7 Harmonic Oscillators and the Høegh-Krohn Process

With the help of the decomposition (1.2.119) and the representation (1.3.109), the Gibbs state ϱ_ℓ of a ν-dimensional harmonic oscillator can be fully determined by the Høegh-Krohn process. This connection between the measure χ_β and the state ϱ_ℓ is described by the following theorem, cf. Corollary 1.2.39. We recall that the maximal commutative subalgebra $\mathfrak{M}_\Lambda$ of the algebra $\mathfrak{C}_\Lambda$ consists of multiplication operators by functions $F \in L^\infty(\mathbb{R}^{\nu|\Lambda|})$. By $\mathfrak{M}_\ell$ and $\mathfrak{C}_\ell$ we denote these algebras for $\Lambda = \{\ell\}$; $\Gamma^{\beta,\ell}_{A_1,\ldots,A_n}$ stands for the Matsubara function with $\Lambda = \{\ell\}$.

Theorem 1.3.59. *For any $F_1, \ldots F_n \in \mathfrak{M}_\ell$ and $\tau_1, \ldots \tau_n$, such that $0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \beta$, it follows that*

$$\begin{aligned}&\Gamma^{\beta,\ell}_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) \\ &\quad = \varrho_\ell\left(F_1 \exp(-(\tau_2 - \tau_1)H^{\text{har}}_\ell)F_2 \ldots F_n \exp(-(\tau_1 - \tau_n)H^{\text{har}}_\ell)\right) \\ &\quad = \int_{L^2_\beta} F_1(x(\tau_1)) \ldots F_n(x(\tau_n))\chi_\beta(\mathrm{d}x) \\ &\quad = \int_{C_\beta} F_1(x(\tau_1)) \ldots F_n(x(\tau_n))\chi_\beta(\mathrm{d}x).\end{aligned} \tag{1.3.154}$$

Proof. By Corollary 1.2.31 the family $\{Q_\ell(\lambda) \mid \lambda \in \mathbb{R}^\nu\}$ of multiplication operators defined in (1.2.121) is σ-weakly dense in $\mathfrak{M}_\ell$; hence, the theorem can be proven by showing that (1.3.154) holds for $F_i = Q_\ell(\lambda_i)$, $\lambda_i \in \mathbb{R}^\nu$, $i = 1, \ldots, n$. For such operators, by (1.3.11) and (1.3.12) one obtains

$$\begin{aligned}
&\int_{L^2_\beta} F_1(x(\tau_1)) \ldots F_n(x(\tau_n)) \chi_\beta(\mathrm{d}x) \\
&\quad = \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{\nu} \sigma_{kk}(\tau_i, \tau_j) \lambda_i^{(k)} \lambda_j^{(k)} \right\} \\
&\quad = \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{n} \sum_{k=1}^{\nu} \Gamma_2(\tau_i, \tau_j) \lambda_i^{(k)} \lambda_j^{(k)} \right\} \\
&\quad = \exp\left\{ -\frac{1}{2} \sum_{i,j=1}^{n} (\Theta(\tau_i, \tau_j)\lambda_i, \lambda_j) \right\} \\
&\quad = \Gamma^{\beta,\ell}_{Q(\lambda_1),\ldots,Q(\lambda_n)}(\tau_1, \ldots, \tau_n).
\end{aligned} \tag{1.3.155}$$

Here we have also used (1.2.105), (1.2.124), (1.2.125), and (1.3.18). □

Now let us extend the above theorem to the states of noninteracting harmonic oscillators in arbitrary $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$. Set

$$\begin{aligned}
L^2_{\beta,\Lambda} &= \prod_{\ell \in \Lambda} L^2_{\beta,\ell} = \{x_\Lambda = (x_\ell)_{\ell\in\Lambda} \mid x_\ell \in L^2_\beta\}, \\
C_{\beta,\Lambda} &= \prod_{\ell \in \Lambda} C_{\beta,\ell} = \{x_\Lambda = (x_\ell)_{\ell\in\Lambda} \mid x_\ell \in C_\beta\},
\end{aligned} \tag{1.3.156}$$

and

$$\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \bigotimes_{\ell\in\Lambda} \chi_\beta(\mathrm{d}x_\ell). \tag{1.3.157}$$

We equip $L^2_{\beta,\Lambda}$ with the scalar product, cf. (1.3.108),

$$(x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}} = \sum_{\ell\in\Lambda} (x_\ell, y_\ell)_{L^2_\beta} = \sum_{\ell\in\Lambda} \sum_{j=1}^{\nu} \int_0^\beta x_\ell^{(j)}(\tau) y_\ell^{(j)}(\tau) \mathrm{d}\tau, \tag{1.3.158}$$

and with the corresponding norm $\|\cdot\|_{L^2_{\beta,\Lambda}}$, which turns it into a real separable Hilbert space. In a similar way, we equip $C_{\beta,\Lambda}$ with the norm

$$\|x_\Lambda\|_{C_{\beta,\Lambda}} = \sup_{\ell\in\Lambda} \sup_{\tau\in[0,\beta]} |x_\ell(\tau)|, \tag{1.3.159}$$

where, as above, $|\cdot|$ stands for the Euclidean norm on $\mathbb{R}^\nu$.

Like χ_β, the measure $\chi_{\beta,\Lambda}$ can be considered as a probability measure on the Hilbert space $L^2_{\beta,\Lambda}$, concentrated on the set $C_{\beta,\Lambda}$, or as a probability measure on the Banach space $C_{\beta,\Lambda}$. The next statement is a straightforward corollary of Theorem 1.3.59.

Theorem 1.3.60. *For any $F_1, \ldots F_n \in \mathfrak{M}_\Lambda$ and $\tau_1, \ldots, \tau_n$, such that $0 \leq \tau_1 \leq \cdots \leq \tau_n \leq \beta$, it follows that*

$$\begin{aligned} &\varrho^{\mathrm{har}}_{\beta,\Lambda}\left\{F_1 \exp\left[-(\tau_2-\tau_1)H^{\mathrm{har}}_\Lambda\right] F_2 \ldots F_n \exp\left[-(\tau_1-\tau_n)H^{\mathrm{har}}_\Lambda\right]\right\} \\ &\qquad = \int_{L^2_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n))\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \\ &\qquad = \int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n))\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \end{aligned} \tag{1.3.160}$$

This theorem defines Euclidean Gibbs states for a model of noninteracting harmonic oscillators. Our next aim is to get them for the model (1.1.3), (1.2.5). This is the subject of the next section, where the states in question are constructed as perturbations of the states of noninteracting harmonic oscillators.

1.4 Local Gibbs States via Stochastic Analysis

The main aim of this section is to obtain the representation of local Matsubara functions corresponding to systems of interacting anharmonic oscillators, similar to (1.3.160). We begin by obtaining such a representation for the partition function $Z_{\beta,\Lambda} = \mathrm{trace}\,[\exp(-\beta H_\Lambda)]$. The main tool for this is a version of the Trotter–Kato product formula, adapted to the trace operation. As a consequence, we construct a probability measure on $C_{\beta,\Lambda}$, which plays the same role as the measure $\chi_{\beta,\Lambda}$ in the representation (1.3.160). Afterwards, we prove that the Matsubara functions constructed on multiplication operators by bounded Borel functions can be represented as integrals with respect to this measure. Such a representation is then naturally extended to all integrable functions, which opens the possibility to introduce Matsubara functions for unbounded multiplication operators, which will be used in the description of a number of physical properties of the model (1.1.3), (1.2.5). This is done in Subsection 1.4.1. For bounded operators, the properties of the corresponding Matsubara functions are described in Theorem 1.2.32. However, for unbounded operators, this theorem cannot be used directly and the only possible way is to employ the mentioned integral representation. This is realized in Subsection 1.4.2. In Subsection 1.4.3, we introduce and study the so-called periodic local Gibbs states, which describe translation-invariant versions of our model. Finally, Subsection 1.4.4 is dedicated to the study of some analytic properties of local Gibbs states based on the integral representation mentioned above. A typical result here is the statement that the partition function with $J_{\ell\ell'}$ replaced by $tJ_{\ell\ell'}$ can be extended to an entire function of $t \in \mathbb{C}$.

1.4.1 Local Euclidean Gibbs States

Given $\Lambda \in \mathfrak{L}_{\rm fin}$, we set

$$K_\Lambda = K_\Lambda(q_\Lambda) = -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda} V_\ell(q_\ell). \tag{1.4.1}$$

This operator acts in the Hilbert space $\mathcal{H}_\Lambda$, see (1.2.1) and (1.2.95), and is self-adjoint on $\mathcal{A}_\Lambda$, defined in (1.2.4). Then the Hamiltonian (1.2.5) can be written

$$H_\Lambda = H_\Lambda^{\rm har} + K_\Lambda. \tag{1.4.2}$$

Notice, that it is essentially self-adjoint on the set $C_0^\infty(\mathbb{R}^{\nu|\Lambda|})$ of all infinitely differentiable functions with compact support. We endow $\mathcal{H}_\Lambda$ with the following orthonormal basis. For $\mathsf{n} = (n_\ell^{(j)}) \in \mathbb{N}^{\nu|\Lambda|}$, we set

$$|\mathsf{n}| = \sum_{j=1}^{\nu}\sum_{\ell\in\Lambda} n_\ell^{(j)},$$

and

$$\Psi_{\mathsf{n}} = \bigotimes_{j=1}^{\nu}\bigotimes_{\ell\in\Lambda}\psi_{n_\ell^{(j)}}, \tag{1.4.3}$$

where the functions ψ_n, $n \in \mathbb{N}_0$ are the same as in (1.1.70). The set $\{\Psi_{\mathsf{n}}\}_{\mathsf{n}\in\mathbb{N}^{\nu|\Lambda|}}$ is the basis we aimed to get. The operator $H_\Lambda^{\rm har}$ is self-adjoint on the domain, cf. (1.1.87)

$$\mathrm{Dom}(H_\Lambda^{\rm har}) = \left\{\Psi \in \mathcal{H}_\Lambda \mid \textstyle\sum_{\mathsf{n}\in\mathbb{N}^{\nu|\Lambda|}} |\mathsf{n}|^2 \left|(\Psi, \Psi_{\mathsf{n}})_{\mathcal{H}_\Lambda}\right|^2 < \infty\right\}. \tag{1.4.4}$$

Clearly,

$$C_0^\infty(\mathbb{R}^{\nu|\Lambda|}) \subset \mathrm{Dom}(H_\Lambda^{\rm har}) \cap \mathcal{A}_\Lambda. \tag{1.4.5}$$

By definition, an essentially self-adjoint operator A on the Hilbert space $\mathcal{H}$ is lower bounded if there exists a real constant, say C, such that for all $\psi \in \mathrm{Dom}(A)$,

$$(\psi, A\psi)_{\mathcal{H}} \geq C(\psi, \psi)_{\mathcal{H}}. \tag{1.4.6}$$

As the anharmonic potentials V_ℓ, $\ell \in \Lambda$, obey Assumption 1.1.1, the above operator K_Λ, as well as the Hamiltonians $H_\Lambda^{\rm har}$, H_Λ, are lower bounded, cf. Theorem 1.2.1. To proceed further we need the Trotter–Kato product formula, but in a stronger version than the one given by Proposition 1.2.29. This version, presented below, was obtained in [150], Theorem 3.1 (see also [105], [158], [226], [227] for further developments).

Proposition 1.4.1. *Let A and B be lower bounded self-adjoint operators on a Hilbert space $\mathcal{H}$, such that the operator $A + B$ defined on* $\mathrm{Dom}(A) \cap \mathrm{Dom}(B)$ *is essentially self-adjoint. Given $t > 0$, let additionally the operator* $\exp(-tA)$ *be of trace class. Then* $\exp(-t(A + B))$ *is a trace-class operator and*

$$\lim_{n\to+\infty} \mathrm{trace}\left[\exp(-(t/n)A)\exp(-t/n)B)\right]^n = \mathrm{trace}\left[\exp(-t(A + B))\right].$$

Lemma 1.4.2. *For every $\beta > 0$ and $\Lambda \in \mathfrak{L}_{\text{fin}}$,*

$$\lim_{n\to+\infty} \operatorname{trace}\left[\exp(-(\beta/n)H_\Lambda^{\text{har}})\exp(-\beta/n)K_\Lambda)\right]^n = \operatorname{trace}\exp(-\beta H_\Lambda). \quad (1.4.7)$$

Proof. By (1.4.5) the operators H_Λ^{har}, K_Λ, and H_Λ obey the conditions of Proposition 1.4.1, by which (1.4.7) follows. □

For $x_\Lambda \in C_{\beta,\Lambda}$, we set

$$E_{\beta,\Lambda}(x_\Lambda) = -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}\int_0^\beta (x_\ell(\tau), x_{\ell'}(\tau))\mathrm{d}\tau + \sum_{\ell\in\Lambda}\int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau. \quad (1.4.8)$$

Proposition 1.4.3. *The function $E_{\beta,\Lambda} : C_{\beta,\Lambda} \to \mathbb{R}$ is continuous and lower bounded, that is, there exists $C_\Lambda \in \mathbb{R}$, such that*

$$\forall x_\Lambda \in C_{\beta,\Lambda}: \quad E_{\beta,\Lambda}(x_\Lambda) \geq C_\Lambda. \quad (1.4.9)$$

Proof. In view of (1.1.11), one obtains

$$\left|\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta}\right| \leq \hat{J}_0 \sum_{\ell\in\Lambda} \|x_\ell\|^2_{L^2_\beta}. \quad (1.4.10)$$

Therefore, the first summand in (1.4.8) is continuous as a map $L^2_{\beta,\Lambda} \to \mathbb{R}$ and hence continuous as a map $C_{\beta,\Lambda} \to \mathbb{R}$. The second summand is continuous in view of the continuity of the functions V_ℓ, see Assumption 1.1.1. By (1.1.10) and Jensen's inequality one gets

$$\int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau \geq -\beta c_V + \beta^{1-r} b_V \|x_\ell\|^{2r}_{L^2_\beta},$$

which along with (1.4.10) yields that the estimate (1.4.9) holds for

$$C_\Lambda = -\beta|\Lambda|\left[c_V + \frac{r-1}{b_V^{1/(r-1)}}\left(\frac{\hat{J}_0}{2r}\right)^{r/(r-1)}\right], \quad (1.4.11)$$

where b_V, c_V, and r are the same as in (1.1.10). □

Our next statement establishes an integral representation for the partition function corresponding to the Hamiltonian (1.2.5). By means of (1.1.73) and (1.2.93), (1.2.96) one obtains

$$\begin{aligned} Z^{\text{har}}_{\beta,\Lambda} &\stackrel{\text{def}}{=} \operatorname{trace}\exp(-\beta H_\Lambda^{\text{har}}) \\ &= \left[\operatorname{trace}\exp(-\beta H^{\text{har}})\right]^{\nu|\Lambda|} \\ &= \left[\frac{e^{-\beta\delta/2}}{1-e^{-\beta\delta}}\right]^{\nu|\Lambda|}. \end{aligned} \quad (1.4.12)$$

Proposition 1.4.4. *For any $\beta > 0$ and $\Lambda \in \mathfrak{L}_{\rm fin}$,*

$$\begin{aligned} Z_{\beta,\Lambda} &\stackrel{\rm def}{=} \mathrm{trace}\,[\exp(-\beta H_\Lambda)] \\ &= Z^{\rm har}_{\beta,\Lambda} \int_{C_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(x_\Lambda)\right) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \end{aligned} \tag{1.4.13}$$

Proof. By Proposition 1.4.1 and Lemma 1.4.2 one has

$$\begin{aligned} Z_{\beta,\Lambda} &= \lim_{n\to+\infty} \mathrm{trace}\left[\exp(-(\beta/n)K_\Lambda)\exp(-(\beta/n)H^{\rm har}_\Lambda)\right]^n \\ &= \lim_{n\to+\infty} \mathrm{trace}\left\{F\exp[-(\tau_2-\tau_1)H^{\rm har}_\Lambda]F\ldots F\exp[-(\tau_1-\tau_n+\beta)H^{\rm har}_\Lambda]\right\} \\ &= Z^{\rm har}_{\beta,\Lambda} \lim_{n\to+\infty} \varrho^{\rm har}_{\beta,\Lambda}\left\{F\exp[-(\tau_2-\tau_1)H^{\rm har}_\Lambda]F\ldots F\exp[-(\tau_1-\tau_n)H^{\rm har}_\Lambda]\right\}, \end{aligned} \tag{1.4.14}$$

where $\tau_k = \beta(k-1)/n$, $k = 1,\ldots,n$ and F is the multiplication operator by the function

$$\begin{aligned} &F(y) = \exp\left[-(\beta/n)K_\Lambda(y)\right], \quad y = (y_\ell)_{\ell\in\Lambda} \in \mathbb{R}^{\nu|\Lambda|}, \\ &K_\Lambda(y) \stackrel{\rm def}{=} -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(y_\ell,y_{\ell'}) + \sum_{\ell\in\Lambda} V_\ell(y_\ell). \end{aligned} \tag{1.4.15}$$

Now we apply Theorem 1.3.60 to the last line in (1.4.14) and obtain

$$Z_{\beta,\Lambda} = Z^{\rm har}_{\beta,\Lambda} \lim_{n\to+\infty} \Xi_n, \tag{1.4.16}$$

where

$$\Xi_n = \int_{C_{\beta,\Lambda}} \exp\left\{-\frac{\beta}{n}\sum_{k=0}^{n-1} K_\Lambda\left(x_\Lambda\left(\frac{k}{n}\beta\right)\right)\right\}\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{1.4.17}$$

Since the function $K_\Lambda(y)$ defined by (1.4.15) is lower bounded, the function under the integral is positive and bounded by a constant. For every $x_\Lambda \in C_{\beta,\Lambda}$, this function converges, as $n \to +\infty$ to the function

$$\exp\left(-E_{\beta,\Lambda}(x_\Lambda)\right);$$

hence, by Lebesgue's dominated convergence theorem, Proposition 1.3.1, one has

$$\lim_{n\to+\infty} \Xi_n = \int_{C_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(x_\Lambda)\right) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),$$

which completes the proof. □

Thereafter, for every $\Lambda \in \mathfrak{L}_{\rm fin}$, one can introduce the probability measure

$$\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \frac{1}{N_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(x_\Lambda)\right) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{1.4.18}$$

where

$$N_{\beta,\Lambda} = Z_{\beta,\Lambda}/Z^{\mathrm{har}}_{\beta,\Lambda} \tag{1.4.19}$$

is a normalization constant. Like the Gaussian measure $\chi_{\beta,\Lambda}$, this measure is defined on the Banach space $C_{\beta,\Lambda}$. At the same time, by Kuratowski's theorem it can be redefined as a measure on the Hilbert space $L^2_{\beta,\Lambda}$, concentrated on the subset consisting of periodic continuous functions.

The next statement is an analog of Theorem 1.3.59.

Theorem 1.4.5. *Let the probability measure $\nu_{\beta,\Lambda}$ be as in (1.4.18). Then for any $F_1,\dots,F_n \in \mathfrak{M}_\Lambda$, the Matsubara function constructed on these operators has the integral representation*

$$\Gamma^{\beta,\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1))\dots F_n(x_\Lambda(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{1.4.20}$$

Proof. The proof will be done by passing from taking trace to integration performed in the representation (1.2.92), exactly as it was done in the proof of Proposition 1.4.4. One observes that for a sequence of trace-class operators $\{T_n\}_{n\in\mathbb{N}}$, which converges in the trace norm to a certain T, and an arbitrary $A \in \mathfrak{C}_\Lambda$,

$$\lim_{n\to+\infty} \mathrm{trace}(AT_n) = \mathrm{trace}(AT). \tag{1.4.21}$$

In view of the periodicity property given by (1.2.90), in (1.4.20) we can set $\tau_1 = 0$. We also set

$$\theta_i = \tau_{i+1} - \tau_i, \quad i = 1,2,\dots,n-1, \qquad \theta_n = \beta - \tau_n. \tag{1.4.22}$$

By the representation (1.2.92),

$$\Gamma^{\beta,\Lambda}_{F_1,\dots,F_n}(0,\tau_2,\dots,\tau_n) = \frac{1}{Z_{\beta,\Lambda}}\mathrm{trace}(\Upsilon), \tag{1.4.23}$$

where

$$\Upsilon = F_1\exp\left(-\theta_1 H_\Lambda\right) F_2 \exp\left(-\theta_2 H_\Lambda\right)\dots F_n\exp\left(-\theta_n H_\Lambda\right). \tag{1.4.24}$$

For $m_1,\dots,m_n \in \mathbb{N}$, we define

$$L_i = \exp\left(-\frac{\theta_i}{m_i}K_\Lambda\right), \quad M_i = \exp\left(-\frac{\theta_i}{m_i}H^{\mathrm{har}}_\Lambda\right), \; i = 1,\dots,n, \tag{1.4.25}$$

and

$$\Upsilon_{m_1,\dots,m_n} = F_1\left(L_1M_1\right)^{m_1} F_2\left(L_2M_2\right)^{m_2}\dots F_n\left(L_nM_n\right)^{m_n}. \tag{1.4.26}$$

Then by (1.4.21) and Proposition 1.3.60 it follows that

$$\Upsilon = \lim_{m_1,\dots,m_n\to+\infty}\Upsilon_{m_1,\dots,m_n} \tag{1.4.27}$$

in the trace norm. Now let us rewrite (1.4.26) in the following way:

$$\Upsilon_{m_1,\dots,m_n} = \prod_{k=1}^{n} \Big[F_k \exp\big(-(t_1^{(k)} - t_{m_{k-1}+1}^{(k-1)}) H_\Lambda^{\mathrm{har}}\big) \times L_k \exp\big(-(t_2^{(k)} - t_1^{(k)}) H_\Lambda^{\mathrm{har}}\big) \dots L_k \exp\big(-(t_{m_k+1}^{(k)} - t_{m_k}^{(k)}) H_\Lambda^{\mathrm{har}}\big)\Big], \tag{1.4.28}$$

where we have set (cf. (1.4.25))

$$\begin{gathered} t_{m_0+1}^{(0)} = t_1^{(1)} = 0, \quad t_{m_{k-1}+1}^{(k-1)} = t_1^{(k)}, \\ t_s^{(k)} = \theta_1 + \dots + \theta_{k-1} + \left(\frac{s-1}{m_k}\right)\theta_k, \\ s = 1, 2, \dots, m_k + 1,\ k = 1, \dots, n,\ \theta_0 = 0. \end{gathered} \tag{1.4.29}$$

By (1.4.22) $\theta_1 + \dots + \theta_n = \beta$; hence, $t_{m_n+1}^{(n)} - t_{m_n}^{(n)} = t_1^{(1)} - t_{m_n}^{(n)} + \beta$ and (1.4.28) can be rewritten as

$$\begin{aligned} \Upsilon_{m_1,\dots,m_n} = \prod_{k=1}^{n-1} \Big[& F_k \exp\big(-(t_1^{(k)} - t_{m_{k-1}+1}^{(k-1)}) H_\Lambda^{\mathrm{har}}\big) \\ & \times L_k \exp\big(-(t_2^{(k)} - t_1^{(k)}) H_\Lambda^{\mathrm{har}}\big) \dots L_k \exp\big(-(t_{m_k+1}^{(k)} - t_{m_k}^{(k)}) H_\Lambda^{\mathrm{har}}\big)\Big] \\ & \times L_n \exp\big(-(t_2^{(n)} - t_1^{(n)}) H_\Lambda^{\mathrm{har}}\big) \dots L_n \exp\big(-(t_1^{(1)} - t_{m_n}^{(n)} + \beta) H_\Lambda^{\mathrm{har}}\big). \end{aligned}$$

Then by Theorem 1.3.60,

$$\begin{aligned} & \mathrm{trace}(\Upsilon_{m_1,\dots,m_n}) \\ & = Z_{\beta,\Lambda}^{\mathrm{har}} \int_{C_{\beta,\Lambda}} \Big[\prod_{k=1}^{n} F_k(x_\Lambda(\theta_1 + \dots + \theta_{k-1}))\Big] \\ & \quad \times \Big[\prod_{k=1}^{n}\prod_{s=1}^{m_k} L_k\Big(x_\Lambda\Big(\theta_1 + \dots + \theta_{k-1} + \frac{s-1}{m_k}\theta_k\Big)\Big)\Big]\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \\ & = Z_{\beta,\Lambda}^{\mathrm{har}} \int_{C_{\beta,\Lambda}} \Big[\prod_{k=1}^{n} F_k(x_\Lambda(\tau_k))\Big]\Psi_{m_1,\dots,m_n}(x_\Lambda)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \end{aligned} \tag{1.4.30}$$

where we have set

$$\begin{aligned} & \Psi_{m_1,\dots,m_n}(x_\Lambda) \\ & \quad = \exp\Big\{-\sum_{k=1}^{n}\sum_{s=1}^{m_k} \frac{\theta_k}{m_k} K_\Lambda\Big(x_\Lambda\Big(\theta_1 + \dots + \theta_{k-1} + \frac{s-1}{m_k}\theta_k\Big)\Big)\Big\}. \end{aligned} \tag{1.4.31}$$

One observes that the last line in (1.4.30) depends on $m_1, \dots, m_n$ only through the above $\Psi_{m_1,\dots,m_n}(x_\Lambda)$. At the same time, for any $x_\Lambda \in C_{\beta,\Lambda}$, the right-hand side of

(1.4.31) converges, as $\min\{m_1, \dots, m_n\} \to +\infty$, to the function

$$\exp\big(-E_{\beta,\Lambda}(x_\Lambda)\big).$$

Then by (1.4.30),

$$\mathrm{trace}(\Upsilon) = Z^{\mathrm{har}}_{\beta,\Lambda} \int_{C_{\beta,\Lambda}} \Big[\prod_{k=1}^{n} F_k(x_\Lambda(\tau_k))\Big] \exp\big(-E_{\beta,\Lambda}(x_\Lambda)\big)\, \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),$$

which by (1.4.18) and (1.4.23) gives (1.4.20). □

Since the measure (1.4.18) uniquely determines the local Gibbs states (1.2.12), we will call it a *local Euclidean Gibbs measure*. As linear functionals, such measures can also be referred to as *local Euclidean Gibbs states*. They are β-periodic and satisfy the Osterwalder–Schrader positivity (OS-positivity) condition (see [235], and also [137] and [176]). This means that for any $F_1, \dots F_n \in \mathfrak{M}_\Lambda$ and any $\tau_1, \dots \tau_n \in [0, \beta]$,

$$\begin{aligned} \int_{C_{\beta,\Lambda}} & \bar{F}_1(x_\Lambda(\beta - \tau_1)) \dots \bar{F}_n(x_\Lambda(\beta - \tau_n)) \\ & \times F_1(x_\Lambda(\tau_1)) \dots F_n(x_\Lambda(\tau_n)) \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \geq 0. \end{aligned} \tag{1.4.32}$$

The OS-positivity follows from the stochastic positivity of the local Gibbs states proven in Theorem 1.2.35 (see Theorem 6.1 in [176]).

1.4.2 Matsubara Functions for Unbounded Operators

We recall that the extension of the states (1.2.12) to unbounded operators was discussed in Subsection 1.1.2. There, by $\overline{\mathfrak{C}}_{\beta,\Lambda}$ we denoted the sets of operators to which the states (1.2.12) can be extended. Here we describe certain families of unbounded operators and the properties of the Matsubara functions constructed on such operators. Of course, we are going to include into these families the operators $q_\ell^{(j)}$, which play a special role in our theory (see Subsection 1.2.4).

Our first result in this direction is a generalization of Theorem 1.2.32.

Theorem 1.4.6. *Let $F_1, \dots, F_n : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$ be measurable functions, such that for every $\beta > 0$ and every $\tau \in [0, \beta]$, the functions $C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_i(x_\Lambda(\tau))$, $i = 1, \dots, n$, are $\nu_{\beta,\Lambda}$-integrable. Then the Green functions (1.2.50) constructed on the corresponding multiplication operators $F_1, \dots, F_n$ have the properties established by Theorem 1.2.32, except for claim* (b).

Proof. For every $\alpha > 0$, the functions $C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_i(x_\Lambda(0))$, $i = 1, \dots, n$, are $\nu_{\alpha,\Lambda}$-integrable. Employing this fact and (1.4.20) one can show that

$$\mathrm{trace}[F_i \exp(-\alpha H_\Lambda)] = Z_{\alpha,\Lambda} \int_{C_{\alpha,\Lambda}} F_i(x_\Lambda(0)) \nu_{\alpha,\Lambda}(\mathrm{d}x_\Lambda) < \infty. \tag{1.4.33}$$

Hence, the operators $\widehat{F}_i \stackrel{\text{def}}{=} F_i \exp(-\alpha H_\Lambda)$ are bounded. Let $\{\psi_s\}_{s\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}_\Lambda$, consisting of the eigenfunctions of H_Λ, the same as in (1.2.54). Then for every $\alpha > 0$, the matrix elements

$$F^{(i)}_{ss'} \stackrel{\text{def}}{=} (\psi_s, F_i\psi_{s'})_{\mathcal{H}_\Lambda} = \exp(\alpha E_{s'})(\psi_s, \widehat{F}_i\psi_{s'})_{\mathcal{H}_\Lambda} = \exp(\alpha E_{s'})\widehat{F}^{(i)}_{ss'}$$

exist for all $s, s' \in \mathbb{N}$. This yields that for the Green functions constructed on $F_1, \ldots, F_n$, one has the representation (1.2.55) which can also be rewritten in the form

$$\begin{aligned} G^{\beta,\Lambda}_{F_1,\ldots,F_n}&(t_1,\ldots,t_n) \\ &= \frac{1}{Z_{\beta,\Lambda}} \sum_{s_1,\ldots,s_n\in\mathbb{N}} \widehat{F}^{(1)}_{s_1s_2}\exp[\mathrm{i}(t_2-t_1-\mathrm{i}\alpha_1)E_{s_2}] \\ &\qquad \times\cdots\times \widehat{F}^{(n-1)}_{s_{n-1}s_n}\exp[\mathrm{i}(t_n-t_{n-1}-\mathrm{i}\alpha_{n-1})E_{s_n}] \\ &\qquad \times \widehat{F}^{(n)}_{s_ns_1}\exp[\mathrm{i}(t_1-t_n-\mathrm{i}\alpha_n+\mathrm{i}\beta)E_{s_1}], \end{aligned} \tag{1.4.34}$$

where the positive numbers $\alpha_1, \ldots, \alpha_n$ may be arbitrarily small. Since the operators $\widehat{F}_1, \ldots, \widehat{F}_n$ are bounded, the right-hand side of (1.4.34) is analytic in the domain

$$\begin{aligned} \widehat{D}^{\beta}_{\alpha_1,\ldots,\alpha_n} = \{(t_1,\ldots,t_n)\in\mathbb{C}^n \mid 0 < \Im(t_1) < \Im(t_2)-\alpha_1 < \Im(t_3)-\alpha_2 \\ < \cdots < \Im(t_n)-\alpha_{n-1} < \beta-\alpha_n\}, \end{aligned}$$

which is nonempty for sufficiently small $\alpha_1, \ldots, \alpha_n$. Moreover, the Dirichlet series in (1.4.34) converges uniformly on the closure of each such a $\widehat{D}^{\beta}_{\alpha_1,\ldots,\alpha_n}$. Since for every compact subset of $\mathcal{D}^{\beta}_n$, one finds positive $\alpha_1, \ldots, \alpha_n$, such that this subset is contained in the closure of the corresponding $\widehat{D}^{\beta}_{\alpha_1,\ldots,\alpha_n}$, the series (1.4.34) converges uniformly on compact subsets of $\mathcal{D}^n_\beta$. □

Definition 1.4.7. The family $\mathfrak{P}^{(\nu)}_\Lambda$ consists of continuous functions $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, such that for all $\alpha > 0$, the functions

$$\mathbb{R}^{\nu|\Lambda|} \ni u_\Lambda \mapsto |F(u_\Lambda)|\exp\Big(-\alpha\sum_{\ell\in\Lambda}|u_\ell|^2\Big) \tag{1.4.35}$$

are bounded.

In the case of one-element subsets $\Lambda = \{\ell\}$, we write simply $\mathfrak{P}^{(\nu)}$. It is worth noting that $\mathfrak{P}^{(\nu)}_\Lambda$ is a $*$-algebra.

Corollary 1.4.8. *For any $F_1, \ldots, F_n \in \mathfrak{P}^{(\nu)}_\Lambda$, the Green functions constructed on the corresponding multiplication operators have the properties described by Theorem 1.4.6.*

Proof. Given $F_i \in \mathfrak{P}_\Lambda^{(\nu)}$ and $\alpha_i > 0$, $\tau_i \in [0, \beta]$, $i = 1, \ldots, n$, let the function $G_i : C_{\beta,\Lambda} \to \mathbb{C}$ be defined as

$$G_i(x_\Lambda) = F_i(x_\Lambda(\tau_i)) \exp\Big(- \alpha_i \sum_{\ell \in \Lambda} |x_\ell(\tau_i)|^2 \Big).$$

Then the function

$$G_i(x_\Lambda) \exp\Big(\alpha_i \sum_{\ell \in \Lambda} |x_\ell(\tau_i)|^2 - E_{\beta,\Lambda}(x_\Lambda) \Big)$$

is bounded on $C_{\beta,\Lambda}$ and measurable, thus, it is $\chi_{\beta,\Lambda}$-integrable. This means that the function $C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_i(x_\Lambda(\tau_i))$ satisfies the conditions of Theorem 1.4.6. □

We recall that the Hilbert space $L^2_{\beta,\Lambda}$ and the corresponding scalar product $(\cdot,\cdot)_{L^2_{\beta,\Lambda}}$ were defined in (1.3.156) and (1.3.158). For reasons which will become clear in the next section, we modify the measures (1.4.18) as follows:

$$\begin{aligned} \nu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda) &= \frac{1}{N_{\beta,\Lambda}(y_\Lambda)} \exp\Big(-E_{\beta,\Lambda}(x_\Lambda) + (x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}} \Big) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \\ N_{\beta,\Lambda}(y_\Lambda) &= \int_{C_{\beta,\Lambda}} \exp\Big(-E_{\beta,\Lambda}(x_\Lambda) + (x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}} \Big) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \end{aligned} \tag{1.4.36}$$

Here $y_\Lambda \in C_{\beta,\Lambda}$ and hence the function $(\cdot, y_\Lambda)_{L^2_{\beta,\Lambda}}$ is measurable on $C_{\beta,\Lambda}$. Clearly, the integral in (1.4.36) exists; thus, $\nu_{\beta,\Lambda}^{y_\Lambda}$ is a probability measure on $C_{\beta,\Lambda}$. We stress that it is a local Euclidean Gibbs measure only if the components of $y_\Lambda = (y_\ell)_{\ell \in \Lambda}$ are constant functions of $\tau \in [0, \beta]$. For $(\tau_1, \ldots, \tau_n) \in [0, \beta]^n$, we set

$$\Gamma_{F_1,\ldots,F_n}^{\beta,\Lambda}(\tau_1, \ldots, \tau_n; y_\Lambda) = \int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)) \nu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda). \tag{1.4.37}$$

Theorem 1.4.9. *For any $y_\Lambda \in C_{\beta,\Lambda}$ and arbitrary $F_1, \ldots, F_n \in \mathfrak{P}_\Lambda^{(\nu)}$, the functions* (1.4.37) *are continuous in $(\tau_1, \ldots, \tau_n)$.*

Proof. We rewrite (1.4.37) in the form

$$\begin{aligned} \Gamma_{F_1,\ldots,F_n}^{\beta,\Lambda}(\tau_1, \ldots, \tau_n; y_\Lambda) = \int_{C_{\beta,\Lambda}} & F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)) \\ & \times \Psi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \end{aligned} \tag{1.4.38}$$

where

$$\Psi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) = \frac{1}{N_{\beta,\Lambda}(y_\Lambda)} \exp\Big(-E_{\beta,\Lambda}(x_\Lambda) + (x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}} \Big). \tag{1.4.39}$$

As all F_i's are continuous, all the functions $C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_i(x_\Lambda(\tau_i))$, as well as the functions $[0, \beta]^n \ni (\tau_1, \ldots, \tau_n) \mapsto F_i(x_\Lambda(\tau_i))$ are continuous. Set

$$R(x_\Lambda) = \max_{i=1,\ldots,n} \sup_{\tau_i \in [0,\beta]} |F_i(x_\Lambda(\tau_i))|. \tag{1.4.40}$$

Since all F_i's belong to $\mathfrak{P}_\Lambda^{(\nu)}$, the function

$$[R(x_\Lambda)]^n \, \Psi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) \geq 0$$

is $\chi_{\beta,\Lambda}$-integrable; hence,

$$\mu(\mathrm{d}x_\Lambda) = [R(x_\Lambda)]^n \, \Psi_{\beta,\Lambda}(x_\Lambda; y_\Lambda)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \tag{1.4.41}$$

is a finite Borel measure on $C_{\beta,\Lambda}$. It is tight since $C_{\beta,\Lambda}$ is a Polish space. Therefore, for any $\varepsilon > 0$, there exists a compact subset $Y^\varepsilon \subset C_{\beta,\Lambda}$, such that

$$\mu\left(C_{\beta,\Lambda} \setminus Y^\varepsilon\right) < \varepsilon/4. \tag{1.4.42}$$

Given $\delta > 0$, we set

$$\Xi_\delta = \sup\left|\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n; y_\Lambda) - \Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau'_1,\ldots,\tau'_n; y_\Lambda)\right|, \tag{1.4.43}$$

where the supremum is taken over the subsets of $[0,\beta]^n$ defined by the condition, cf. (1.3.23),

$$\max_{i=1,\ldots,n} |\tau_i - \tau'_i| < \delta.$$

Then for such δ and a fixed $x_\Lambda \in C_{\beta,\Lambda}$, we set

$$W_\delta(x_\Lambda) = \max_{i=1,\ldots,n} \sup_{|\tau_i-\tau'_i|<\delta} \left|F_i(x_\Lambda(\tau_i)) - F_i(x_\Lambda(\tau'_i))\right|. \tag{1.4.44}$$

Since all $F_i : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$ are continuous, in order that Y^ε be compact it is necessary and sufficient that the following conditions be satisfied simultaneously (see page 213 of [239]):

$$\begin{aligned} &\lim_{\delta\downarrow 0} \sup_{x_\Lambda\in Y^\varepsilon} W_\delta(x_\Lambda) = 0, \\ &\sup_{x_\Lambda\in Y^\varepsilon} R(x_\Lambda) < \infty, \end{aligned} \tag{1.4.45}$$

where R was defined in (1.4.40). Now let us estimate Ξ_δ. By (1.4.40), (1.4.41), (1.4.43), and (1.4.44) one obtains

$$\Xi_\delta \leq n\int_{Y^\varepsilon} W_\delta(x_\Lambda)[R(x_\Lambda)]^{n-1}\Psi_{\beta,\Lambda}(x_\Lambda; y_\Lambda)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) + 2\mu(C_{\beta,\Lambda} \setminus Y^\varepsilon).$$

In view of (1.4.45), one can choose δ so small, that the first summand on the right-hand side is less than $\varepsilon/2$. The second one is also less than $\varepsilon/2$ by (1.4.42), which completes the proof. □

Unlike in Theorem 1.4.6, here we have no information about the analytic properties of the functions (1.4.38) if the components of y_Λ are not constant.

1.4.3 Periodic Local Gibbs States

If $\mathbb{L}$ is a crystal lattice, the corresponding Gibbs states possess additional properties, which can be used in the analysis of such states. Recall that in this case, we suppose that $\mathbb{L}$ is the lattice $\mathbb{Z}^d$ endowed with the metric (1.1.1).

Definition 1.4.10. The model (1.1.3), (1.1.8) is said to be translation-invariant if the anharmonic potentials are the same at each site, i.e., $V_\ell = V$, and the dynamical matrix $(J_{\ell\ell'})$ is invariant with respect to the translations of the lattice. The latter means that for every $\ell_0 \in \mathbb{Z}^d$, one has $J_{(\ell+\ell_0)(\ell'+\ell_0)} = J_{\ell\ell'}$ for all ℓ, ℓ'.

For $\ell^s = (\ell_1^s, \ldots, \ell_d^s) \in \mathbb{Z}^d$, $s = 0, 1$, such that $\ell_j^0 < \ell_j^1$ for all $j = 1, \ldots, d$, we set

$$\Lambda = \{\ell = (\ell_1, \ldots, \ell_d) \mid \ell_j^0 \leq \ell_j \leq \ell_j^1, \ \ j = 1, \ldots, d\}. \tag{1.4.46}$$

Such sets will be called *boxes*. As usual, we define

$$\Lambda + \ell = \{\ell' + \ell \mid \ell' \in \Lambda\}.$$

For a box Λ, by $\mathbb{P}(\Lambda)$ we denote the family of boxes which has the following properties:

$$\begin{aligned}
&\text{(a)} \quad \mathbb{P}(\Lambda) = \{\Lambda' \mid \exists \ell \in \mathbb{Z}^d : \Lambda' = \Lambda + \ell\}; \\
&\text{(b)} \quad \mathbb{Z}^d = \bigcup_{\Lambda' \in \mathbb{P}(\Lambda)} \Lambda'; \\
&\text{(c)} \quad \forall\, \Lambda', \Lambda'' \in \mathbb{P}(\Lambda): \quad \Lambda' \neq \Lambda'' \ \Rightarrow \ \Lambda' \cap \Lambda'' = \emptyset.
\end{aligned} \tag{1.4.47}$$

Therefore, $\mathbb{P}(\Lambda)$ is the partition of the lattice by the translates of Λ. Clearly the lattice $\mathbb{Z}^d$ is an additive group and those ℓ which appear in the definition of $\mathbb{P}(\Lambda)$ constitute its subgroup, which we denote by $\mathbb{Z}^d_\Lambda$. In what follows, $\mathbb{P}(\Lambda) = \{\Lambda + \ell \mid \ell \in \mathbb{Z}^d_\Lambda\}$. The factor-group $\mathbb{Z}^d/\mathbb{Z}^d_\Lambda$ may be interpreted as the group of all translations of the torus which one obtains by identifying the opposite walls of the box Λ. The distance between the points of this torus is set to be

$$|\ell - \ell'|_\Lambda = \min_{\tilde{\ell} \in \mathbb{Z}^d_\Lambda} |\ell - (\ell' + \tilde{\ell})|. \tag{1.4.48}$$

Given $\ell, \ell' \in \Lambda$ and $j = 1, \ldots, d$, we write $|\ell_j - \ell'_j|_\Lambda$ meaning the above distance between ℓ and $(\ell_1, \ldots, \ell_{j-1}, \ell'_j, \ell_{j+1}, \ldots, \ell_d) \in \Lambda$.

If the dynamical matrix $(J_{\ell\ell'})$ is translation-invariant, its elements depend on the differences $|\ell_j - \ell'_j|$, $j = 1, \ldots, d$, only, i.e., there exists a function, $f : \mathbb{Z}^d \to \mathbb{R}$, such that

$$J_{\ell\ell'} = f(|\ell_1 - \ell'_1|, \ldots, |\ell_d - \ell'_d|). \tag{1.4.49}$$

For this function and a box Λ, we set

$$J^\Lambda_{\ell\ell'} = f(|\ell_1 - \ell'_1|_\Lambda, \ldots, |\ell_d - \ell'_d|_\Lambda), \quad \ell, \ell' \in \Lambda. \tag{1.4.50}$$

Then the matrix $(J^{\Lambda}_{\ell\ell'})_{\ell,\ell'\in\Lambda}$ is invariant under the action of the factor-group. By means of it we introduce the *periodic Hamiltonian*

$$H^{\rm per}_{\Lambda} = -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}(q_\ell, q_{\ell'}) + \sum_{\ell\in\Lambda}\left[H^{\rm har}_{\ell} + V(q_\ell)\right], \tag{1.4.51}$$

where $H^{\rm har}_{\ell}$ is the same as in (1.1.3). By means of the local Hamiltonian we introduce the *periodic local Gibbs state* $\varrho^{\rm per}_{\beta,\Lambda}$, defined by the same formula (1.2.12). Note that such states can be defined for boxes only. Clearly, they possess all the properties established for the states $\varrho_{\beta,\Lambda}$. In particular, we can represent them by means of periodic local Euclidean Gibbs measures

$$\begin{aligned}
\nu^{\rm per}_{\beta,\Lambda}(\mathrm{d}x) &= \frac{1}{N^{\rm per}_{\beta,\Lambda}}\exp[-E^{\rm per}_{\beta,\Lambda}(x_\Lambda)]\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),\\
E^{\rm per}_{\beta,\Lambda}(x_\Lambda) &= -\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_{\beta,\Lambda}} + \sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau,\\
N^{\rm per}_{\beta,\Lambda} &= \int_{C_{\beta,\Lambda}} \exp[-E^{\rm per}_{\beta,\Lambda}(x_\Lambda)]\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda).
\end{aligned} \tag{1.4.52}$$

The periodic Matsubara functions constructed according to (1.4.20) with the measure $\nu^{\rm per}_{\beta,\Lambda}$ have the same relationship with the Green functions corresponding to the state $\varrho^{\rm per}_{\beta,\Lambda}$. In addition, the moments

$$\langle x^{(j_1)}_{\ell_1}(\tau_1)\dots x^{(j_n)}_{\ell_n}(\tau_n)\rangle_{\nu^{\rm per}_{\beta,\Lambda}}, \quad \ell_1,\dots,\ell_n\in\Lambda,\ j_1,\dots,j_n = 1,\dots,\nu, \tag{1.4.53}$$

are invariant with respect to the action of $\mathbb{Z}^d/\mathbb{Z}^d_\Lambda$.

1.4.4 Analytic Properties of Local Gibbs States

Given $y_\Lambda\in L^2_{\beta,\Lambda}$, let us consider, cf. (1.4.36),

$$\begin{aligned}
\Psi_{\beta,\Lambda}(t,\theta) \stackrel{\rm def}{=} \int_{C_{\beta,\Lambda}} &\exp\left(\frac{t}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} + \theta(x_\Lambda, y_\Lambda)_{L^2_\beta}\right)\\
&\times\exp\left(-\sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\right)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),
\end{aligned} \tag{1.4.54}$$

where t and θ are complex variables. Then $\Psi_{\beta,\Lambda}(1,1) = N_{\beta,\Lambda}(y_\Lambda)$. Similarly, for a box Λ, one defines $\Psi^{\rm per}_{\beta,\Lambda}(t,\theta)$. By letting t vary in the interval $[0,1]$ we obtain an interpolation between the zero and the full interaction. Likewise, one can switch on and off the linear term. Our aim is to show that the dependence of $\Psi_{\beta,\Lambda}$ on t and θ is analytic. To this end we use the Vitali theorem which we present here in the version suitable for our purpose. More details on this theorem can be found in [273].

Proposition 1.4.11. *Given a domain $\mathcal{O} \subset \mathbb{C}$, let a sequence of functions $f_n : \mathcal{O} \to \mathbb{C}$, $n \in \mathbb{N}$, holomorphic in $\mathcal{O}$, have the following properties:* (a) *for every compact subset $K \subset \mathcal{O}$, there exists $C_K > 0$, such that $\sup_{z\in K} |f_n(z)| \leq C_K$ for all $n \in \mathbb{N}$;* (b) *there exists a function $f : \mathcal{O} \to \mathbb{C}$ and a subset $S \subset \mathcal{O}$, which has an accumulation point $z_0 \in \mathbb{C}$, such that $f_n(z) \to f(z)$, as $n \to +\infty$, for every $z \in S$. Then this sequence $\{f_n\}_{n\in\mathbb{N}}$ converges to f uniformly on compact subsets of $\mathcal{O}$ and hence f is also holomorphic in $\mathcal{O}$.*

Now we present our result.

Proposition 1.4.12. *The above introduced $\Psi_{\beta,\Lambda}$ and $\Psi^{\rm per}_{\beta,\Lambda}$ are entire functions of both variables.*

Proof. We prove that $\Psi_{\beta,\Lambda}$ is an entire function of one variable, say t, at a fixed value of the other one. Then the claimed property of $\Psi_{\beta,\Lambda}$ will follow by Hartogs' theorem, see e.g., [269]. The case of $\Psi^{\rm per}_{\beta,\Lambda}$ is completely analogous.

Thus, we fix $\theta \in \mathbb{C}$ and consider

$$\begin{aligned}\Psi^{(n)}_{\beta,\Lambda}(t,\theta) &\stackrel{\rm def}{=} \int_{C_{\beta,\Lambda}} \Big(1 + \frac{t}{2n} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_{\beta,\Lambda}}\Big)^n \\ &\qquad \times \exp\Big(\theta(x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}} - \sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Big)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda).\end{aligned} \tag{1.4.55}$$

For each $n \in \mathbb{N}$, $\Psi^{(n)}_{\beta,\Lambda}$ is a polynomial in t. Clearly, for big enough n, one has

$$\begin{aligned}&\left|\Big(1 + \frac{t}{2n} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta}\Big)^n \exp\Big(\theta(x_\Lambda, y_\Lambda)_{L^2_{\beta,\Lambda}}\Big)\right| \\ &\quad \leq \exp\Big(\frac{|t|}{2}\sum_{\ell,\ell'\in\Lambda} |J_{\ell\ell'}|(\|x_\ell\|^2_{L^2_\beta} + \|x_{\ell'}\|^2_{L^2_\beta}) + |\theta|\cdot\|x_\Lambda\|_{L^2_{\beta,\Lambda}}\cdot\|y_\Lambda\|_{L^2_{\beta,\Lambda}}\Big),\end{aligned} \tag{1.4.56}$$

which holds for all $t \in \mathbb{C}$. By Assumption 1.1.1 the function on the right-hand side of (1.4.56) is integrable with respect to the measure

$$\exp\Big(-\sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Big)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda).$$

Thus, by Lebesgue's dominated convergence theorem, see Proposition 1.3.1, at every fixed $t \in \mathbb{R}$ one has

$$\Psi^{(n)}_{\beta,\Lambda}(t,\theta) \to \Psi_{\beta,\Lambda}(t,\theta). \tag{1.4.57}$$

On the other hand, the estimate (1.4.56) yields for $\{\Psi^{(n)}_{\beta,\Lambda}\}$ the bounds necessary for the Vitali theorem to be applied. Then the proof follows. The claimed property with respect to the variable θ is obtained in the same way. $\square$

Definition 1.4.13. Given $\Lambda \in \mathfrak{L}_{\rm fin}$, the family $\mathfrak{E}_\Lambda$ consists of continuous functions $f : C_{\beta,\Lambda} \to \mathbb{C}$, such that for all $\upsilon > 0$ and $x_\Lambda \in C_{\beta,\Lambda}$,

$$|f(x_\Lambda)| \leq D_f \exp\Big(\upsilon \sum_{\ell\in\Lambda} \|x_\ell\|^2_{C_\beta}\Big), \tag{1.4.58}$$

with a certain positive D_f, which may depend on υ. For a one-point $\Lambda = \{\ell\}$, we write $\mathfrak{E}_{\{\ell\}} = \mathfrak{E}_\ell$.

Clearly, for any $F_1, \ldots, F_n \in \mathfrak{P}^\nu_\Lambda$, see Definition 1.4.7, and arbitrary $\tau_1, \ldots, \tau_n \in [0, \beta]$, the function

$$C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n))$$

belongs to $\mathfrak{E}_\Lambda$. By Proposition 1.3.49, all elements of the latter family are integrable with respect to the measures (1.4.18) and (1.4.52). For $t, \theta \in \mathbb{C}$ and $f \in \mathfrak{E}_\Lambda$, we set

$$\begin{aligned} \Psi_{\beta,\Lambda}(t,\theta|f) = \big[\Psi_{\beta,\Lambda}(t,\theta)\big]^{-1} & \int_{C_{\beta,\Lambda}} f(x_\Lambda) \\ & \times \exp\Big(\frac{t}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} + \theta(x_\Lambda, y_\Lambda)_{L^2_\beta}\Big) \\ & \times \exp\Big(-\sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Big)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \end{aligned} \tag{1.4.59}$$

where $\Psi_{\beta,\Lambda}(t,\theta)$ is the same as in (1.4.54).

Theorem 1.4.14. *For any $f \in \mathfrak{E}_\Lambda$, the function (1.4.59) is meromorphic in a domain, which contains the set $\{(t,\theta) \in \mathbb{C}^2 \mid t, \theta \in \mathbb{R}\}$. If Λ is a box, the same property is exhibited by the function which one obtains by replacing $J_{\ell\ell'}$ with $J^\Lambda_{\ell\ell'}$ defined by* (1.4.50).

Proof. As in the proof of Proposition 1.4.12, it is enough to show that (1.4.59) is a meromorphic function of t at any fixed $\theta \in \mathbb{R}$. For real t and θ, the function (1.4.54) is positive. For $\theta \in \mathbb{R}$, by Proposition 1.4.12, $\big[\Psi_{\beta,\Lambda}(t,\theta)\big]^{-1}$ is meromorphic, as a function of t, in a domain, which contains $\mathbb{R}$. Furthermore, the integral in (1.4.59) is an entire function of (t,θ), which can be proven similarly as Proposition 1.4.12. Hence, as a product of an entire and a meromorphic functions, the function (1.4.59) is also meromorphic. The case of the periodic measures (1.4.52) is analogous. □

1.5 Comments and Bibliographic Notes

Section 1.1: The model (1.1.3), (1.1.8) with $\mathbb{L}$ being a crystal lattice and with finite range interaction is widely used in theoretical and mathematical physics. Its derivation from more 'physical' models can be found in [86] and [284]. The metric $|\ell - \ell'|$, $\ell, \ell' \in \mathbb{L}$, see (1.1.1), which we use in the condition (1.1.2) can easily be replaced by another metric. Further discussions of the connection of the model (1.1.3), (1.1.8) with real physical objects will be given in Chapter 4.1.

There exist at least two reasons for studying systems of interacting oscillators – both classical and quantum. First, they are used in solid state physics as models of crystalline substances, see [86], [179], [284], [309]. Second, they are employed as lattice approximations in Euclidean quantum field theory, see [135], [273]. The case of harmonic oscillators, which is much simpler, was studied in detail in [60], [149], [310], [316], [311], [312]. Here we mention also the recent research in [119], where a system of quantum harmonic oscillators with randomly distributed rigidities and with anharmonic interaction terms was studied.

The material presented in Subsection 1.1.2 is quite standard, with the only exception that we payed a little bit more attention to the notion of a state and purity of vector states, see Theorem 1.1.15. This is important for understanding the nature of a Gibbs state, which is a mixture of vector states. For more details on the theory of linear operators in Hilbert spaces we refer to such well-known books as [172], [209], [214], [255], [256]. See also B. Simon's survey [278], where the latest results in the theory of Schrödinger operators are discussed.

Subsection 1.1.3 contains an updated theory of a single quantum oscillator, based among others on quite recent results of [48] (see also [270]). Due to these results, in Theorem 1.1.47 we have established the class of anharmonic potentials for which we know exactly the domain of self-adjointness of the Hamiltonian (1.1.109). Among others benefits, by Theorem 1.1.47 we obtain that the eigenfunctions of the operator (1.1.109) are strict (classical) solutions of the Schrödinger equation, see Theorem 1.1.51. With the help of Theorem 1.1.47 we also control the domains of analytic families of operators employed in the proof of Theorem 1.1.60. More on Sturm's theorem and its applications can be found in B. Simon's article [279]. A detailed study of the eigenvalues of the Hamiltonian (1.1.123) is presented in the paper [49].

Due to quantum mechanical tunneling, quantum particles move between the regions separated by potential barriers, which is classically forbidden. It is an important example, which demonstrates how unusual the behavior of quantum particles can be from the point of view of classical mechanics. Quite often, see [88], [148], [166], [275], [276], a mathematical theory of this phenomenon is concentrated on semiclassical expansions for the gap parameter $E_1 - E_0$, which in our case would be in negative powers of m, see (1.1.7), and hence reasonable for big values of m. Our analysis given in Theorem 1.1.60 is essentially different. First, we prove that the gap Δ_m (which needs not be $E_1 - E_0$, see (1.1.162)) is a continuous function of m. Then we obtain the small m asymptotic formula (1.1.163), which will be used in Part II in the description of quantum effects in our model, where, in Theorem 7.1.1 we obtain also the bound $\Delta_m \leq C m^{-1}$, valid for some types of the anharmonic potentials V. The tunneling effects are well-known in modern physics and widely used in technology. We refer to the recent physical monographs [42], [299], where detailed explanations and a wide variety of applications of tunneling effects are presented.

Section 1.2: Fundamentals of the theory of local Gibbs states of quantum systems can be found in [262], [263], see also [77] for a more recent setting. In [76], [77], as well as in [114], [145], [244], [265], one can find material related to Subsection 1.2.2. The Høegh-Krohn theorem, in a weaker version, was first proven in [156]; its present form has been taken from [195], see also [196] for a more general setting. It is worth noting that without such results there is no information about the σ-weak closure of the algebra $\mathfrak{A}(\mathfrak{M})$, generated by the operators evolving from the algebra of all multiplication operators $\mathfrak{M}$. A typical condition imposed on this closure, see e.g., page 332 in [175], is that it is 'large enough' to be a C^*-algebra of observables representing the underlying physical system. Theorem 1.2.32 establishes a relationship between local states (1.2.12) and their realizations as local Euclidean Gibbs measures. The main property described by this theorem is known as *multiple-time analyticity*. A similar statement appeared already in [43], see also [44], [45]. For the model considered here, statements (a) and (b) of Theorem 1.2.32 were formulated in [5]. At the same time, the uniqueness related to statement (c) has never been discussed before explicitly[8]. In Theorem 3.2 of [176], it was proven that every KMS state possesses a kind of multiple analyticity. Here we stress that all the three properties stated in Theorem 1.2.32 are crucial. Due to these properties and to the density established in Theorem 1.2.24, we have obtained a complete characterization of the local state $\varrho_{\beta,\Lambda}$ by the set of Matsubara functions constructed on multiplication operators constituting a complete family.

The approach to describing Gibbs states of quantum systems in terms of functional integrals can be traced back to works by K. Symanzik [296], [297], [298] and J. Ginibre [134]. E. Nelson in [229], [230] proposed to use the Markov property of the Euclidean field for continuing it back to the Minkowski world. More about the beginnings of the Euclidean strategy in quantum field theory can be found in F. Guerra's article [141]. A more abstract and systematic approach to functional integrals is presented in [96], see also [5], [82], [83].

In K. Symanzik's works [296], [297], [298], the representation of particle correlation functions in terms of functional integrals arises in connection with the (heuristic) construction of Euclidean quantum fields, whereas J. Ginibre in [134] uses such representations directly in the statistical mechanics of quantum particles. Later on, the development of this approach was continued in [156], [5]. The key concept of the latter works was based on the fact that the Hamiltonian of the free particle $H^{\text{free}} = p^2/2m$ generates a Brownian motion, which means that this process is defined by the Markov semi-group $\exp(-tH^{\text{free}})$, $t \geq 0$. Adding a potential energy, as in the Hamiltonian (1.1.3), can be handled by means of the celebrated Feynman–Kac formula, see also [274]. An inconvenience of this approach is that the trace of $\exp(-tH^{\text{free}})$ is infinite for any $t > 0$. In our approach, we start with the Hamiltonian H^{har}, which generates the process introduced and studied by R. Høegh-Krohn in [156]. From the very beginning this allows us to establish the right probabilistic framework for the construction of our main objects – the Euclidean Gibbs measures of the model. In Subsection 1.2.5, we de-

[8] In fact, claim (c) provides a key argument for proving Proposition 3.3 in [176].

scribe the Gibbs states of a single scalar harmonic oscillator in detail. Complementary material can be found in [255] and in [77].

Section 1.3: As the Høegh-Krohn process, which is one of the main elements of our theory, is a periodic Markov Gaussian process, we begin Subsection 1.3.2 by introducing the general notion of a Gaussian process. Here we also present standard examples, including Høegh-Krohn's process itself. Our approach is quite standard; more details on this subject can be found in [94], [163], [274]. Periodic Gaussian Markov processes were studied in [177]. The construction performed in Subsection 1.3.3 is also standard. Since we crucially use continuous and Hölder continuous realizations of the Høegh-Krohn process and those constructed from them, we describe this property in detail. This construction employs Kolmogorov's extension theorem and the Garsia–Rodemich–Rumsey (GRR) lemma. In view of their importance and aiming to make the book self-contained, we give detailed proofs of these statements. The proof of the first one is quite standard, see e.g., [47], [274]. The proof of the GRR lemma is based on the original source [127] and on the version given in [292]. We remark that the GRR lemma was employed in [3] to establish the Hölder continuity of paths in models of Euclidean quantum field theory.

The stochastic processes we use in our theory can be realized as processes in the Hilbert spaces L^2 or in the spaces of continuous or Hölder-continuous periodic paths, which is a very useful fact for this theory. Therefore, we study canonical realizations of stochastic processes in detail employing the Kuratowski theorem [204] in the version given in [239]. In Subsection 1.3.4, we present relevant facts from the theory of measures on complete separable metric spaces, based on the books [64], [65], [222], [239], [308]. General aspects of the theory of measures on topological spaces are taken from [47], [57], [75], [107], [147], [180]. Special attention is given to the weak topology on spaces of probability measures on Polish spaces, based on [64], [75], [65], [239]. We also present here Fernique's theorem and prove its generalization for sequences of measures. Here we use the original source [118] and the proof of this theorem given in [95]. Fernique's theorem is crucially used in proving the convergence of finite-dimensional approximations of the local Gibbs measures, as well as in establishing support properties of the global Euclidean Gibbs measures. As was mentioned above, we employ the realizations of our main stochastic processes in the Hilbert space L^2. Thus, in Subsection 1.3.5 we present a number of facts from the theory of measures on real Hilbert spaces, taken mainly from the books [62] and [283]. Since the measures we use are sub-Gaussian, their Fourier transforms have remarkable analytic properties, which we describe employing a number of facts on infinite-dimensional holomorphy, taken mostly from the monographs [97], [221], [225]. In Subsection 1.3.6, the properties of the Høegh-Krohn process are studied in detail. Special attention is given to the dependence of this process on the particle mass, which is used in the sequel to describe the classical limit $m \to +\infty$ and quantum effects. Notably, the use of Feynman functional integrals for obtaining the classical limit $\hbar \to 0$ can be traced back to the very beginning of the theory of such methods, see [96] and the corresponding references therein.

In the subsequent parts of the present book, one of the main tools in studying the Euclidean Gibbs measures is their approximation by Gibbs measures of certain classical systems of 'unbounded spins'. This approximation, performed in Chapter 2, is based on the corresponding approximation of the Høegh-Krohn measure, which is constructed in Subsection 1.3.6 with the help of the GRR lemma (Theorem 1.3.11 and Proposition 1.3.12) and of the generalized Fernique theorem mentioned above. This approximation can be considered as an approximation of the Høegh-Krohn process by Markov chains. In the context of the Euclidean quantum field theory, such a technique is called *lattice approximation,* see [142] and Chapter IX of [273]. A similar technique is also known for the Wiener processes on manifolds [40].

In Subsection 1.3.7, we perform the main construction of Section 1.3 which yields the realization of the Gibbs states of a single harmonic oscillator in terms of the Høegh-Krohn measure. It is the starting point in the realization of the local Gibbs states of interacting anharmonic oscillators in terms of path measures.

Section 1.4: In this section, we construct the mentioned realization of the local Gibbs states of the model (1.1.3), (1.1.8) in terms of measures on path spaces, which we call Euclidean local Gibbs measures. Here we use the analogous construction elaborated above for noninteracting harmonic oscillators and a version of the Feynman–Kac formula. With the help of stronger versions of the Trotter–Kato product formula, obtained recently in [105], [158], [226], [227], we prove the main result of this section – Theorem 1.4.5, which gives a representation of the Matsubara functions corresponding to bounded multiplication operators in the form of moments of the local Euclidean Gibbs measures. In Subsection 1.4.2, we extend the above result to unbounded operators, which will allow us to include such operators into the theory. In Subsection 1.4.3, we introduce a special type of local Gibbs states, which can be constructed if $\mathbb{L}$ is a crystal lattice. These are periodic states, which one obtains by imposing periodic conditions on the boundaries of the subsets where such measures are constructed. In the sequel, periodic local Gibbs states play an important role, e.g., in establishing the existence of phase transitions. Many of the technical results which we use throughout the book are based on the properties of the local Euclidean Gibbs measures connected with their dependence on the interaction intensities $J_{\ell\ell'}$, on the anharmonic potentials V_ℓ, as well as on 'external fields', which may depend on ℓ and τ. The study of these properties begins in Subsection 1.4.4. Its main result, Theorem 1.4.14, is then extensively employed for deriving differential equations describing the corresponding properties of the moments of such measures.

Chapter 2
Lattice Approximation and Applications

A characteristic feature of the local Euclidean Gibbs measures studied in this book is that the corresponding 'one-particle' space C_β is infinite-dimensional. In classical statistical mechanics, Gibbs measures of lattice models have finite-dimensional one-particle spaces. A typical example here is the model with unbounded spins where such a space is $\mathbb{R}^\nu$, see [59], [206]. Our aim in this chapter is to create a framework, in which local Euclidean Gibbs measures are approximated by local Gibbs measures of classical lattice models. This will allow us to apply here a number of methods elaborated for the latter models. Among them there are those based on correlation inequalities and analytic properties of the Laplace transforms of local Gibbs measures. In the mentioned approximation, the 'imaginary time' variable $\tau \in [0, \beta]$ is discretized and turned into an extra dimension of the lattice. The corresponding approach is then called the *lattice approximation of Euclidean Gibbs measures.* In Section 2.1, we construct such approximating measures and prove that they converge weakly to the corresponding Euclidean Gibbs measures. Thereby, we rederive basic correlation inequalities, known for the corresponding classical Gibbs measures (Section 2.2), as well as the logarithmic Sobolev inequality (Section 2.3). In Section 2.4, we analyze how the Lee–Yang property can be established for Euclidean Gibbs measures. In Section 2.5, we derive a number of correlation inequalities, specific to the path measures we consider.

2.1 Lattice Approximation

Since we are going to use the lattice approximations in various situations, we construct them for formally new local Euclidean measures, which, however, have the form of (1.4.36). For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $y_\Lambda \in C_{\beta,\Lambda}$, we consider the following probability measure,

$$\begin{aligned}
\mu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda) &= \Phi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) \bigotimes_{\ell\in\Lambda} \sigma_\ell(\mathrm{d}x_\ell), \\
\Phi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) &= \frac{1}{Y_{\beta,\Lambda}(y_\Lambda)} \exp\left\{\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} I_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}(x_\ell, y_\ell)_{L^2_\beta}\right\}, \qquad (2.1.1) \\
\sigma_\ell(\mathrm{d}x_\ell) &= \frac{1}{Z_\ell} \exp\left(-\int_0^\beta W_\ell(x_\ell(\tau))\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}x_\ell).
\end{aligned}$$

Here Z_ℓ and $Y_{\beta,\Lambda}(y_\Lambda)$ are normalization factors, χ_β is the Høegh-Krohn measure, the functions $W_\ell \colon \mathbb{R}^\nu \to \mathbb{R}$ and the interaction intensities $I_{\ell\ell'}$ are supposed to obey

Assumption 1.1.1 (see p. 17). The properties of the measure χ_β were described in Subsection 1.3.6. In particular, according to Theorem 1.3.51 it can be approximated by the finite-dimensional Gaussian measures γ_N, $N = 2L$, $L \in \mathbb{N}$, see (1.3.129). Correspondingly, the measure σ_ℓ can be approximated by

$$\sigma_\ell^N(\mathrm{d}x_\ell) = \frac{1}{Z_\ell^N}\exp\left\{-\frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_\ell\left(x_\ell\left(\frac{\lambda}{N}\beta\right)\right)\right\}\gamma_N(\mathrm{d}x_\ell), \tag{2.1.2}$$

see (1.3.144) and (1.3.145). Thus, for even $N \in \mathbb{N}$, we set

$$\sigma_\Lambda^N(\mathrm{d}x_\Lambda) = \bigotimes_{\ell\in\Lambda}\sigma_\ell^N(\mathrm{d}x_\ell) \quad \text{and} \quad \sigma_\Lambda(\mathrm{d}x_\Lambda) = \bigotimes_{\ell\in\Lambda}\sigma_\ell(\mathrm{d}x_\ell). \tag{2.1.3}$$

For $f \in \mathfrak{C}_\Lambda$, see Definition 1.4.13, the integrals

$$\langle f\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} = \int_{C_{\beta,\Lambda}} f(x_\Lambda)\mu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda) \tag{2.1.4}$$

will be approximated by

$$\begin{aligned}&\int_{C_{\beta,\Lambda}} f(x_\Lambda)\Phi_{\beta,\Lambda}(x_\Lambda;y_\Lambda)\sigma_\Lambda^N(\mathrm{d}x_\Lambda)\\ &\qquad = \int_{C_{\beta,\Lambda}} f^{(N)}(x_\Lambda)\Phi_{\beta,\Lambda}^{(N)}(x_\Lambda;y_\Lambda)\sigma_\Lambda^N(\mathrm{d}x_\Lambda).\end{aligned} \tag{2.1.5}$$

Here

$$\begin{aligned} f^{(N)}(x_\Lambda) &\stackrel{\mathrm{def}}{=} f(x_\Lambda^{(N)}),\\ \Phi_{\beta,\Lambda}^{(N)}(x_\Lambda;y_\Lambda) &\stackrel{\mathrm{def}}{=} \Phi_{\beta,\Lambda}(x_\Lambda^{(N)};y_\Lambda),\end{aligned} \tag{2.1.6}$$

and

$$\begin{aligned} x_\Lambda^{(N)} &= \left(x_\ell^{(N)}\right)_{\ell\in\Lambda}, \qquad x_\ell^{(N)} = (x_\ell^{(1,N)},\dots,x_\ell^{(\nu,N)}),\\ x_\ell^{(j,N)} &\stackrel{\mathrm{def}}{=} \sum_{k\in\mathcal{K}_N} P_{j,k}x_\ell, \quad j=1,\dots,\nu.\end{aligned} \tag{2.1.7}$$

Here $P_{j,k}$, $j = 1,\dots,\nu$, $k \in \mathcal{K}$, are the orthogonal projections onto the eigenvectors $\epsilon_{j,k}$, see (1.3.126), (1.3.127), as well as (1.3.20), (1.3.111).

We begin by proving the first of the statements which constitute the base of the lattice approximation of the measure (2.1.1).

Theorem 2.1.1. *For every $f \in \mathfrak{C}_\Lambda$ and all $y_\Lambda \in C_{\beta,\Lambda}$, the following convergence holds:*

$$\begin{aligned}&\int_{C_{\beta,\Lambda}} f^{(N)}(x_\Lambda)\Phi^{(N)}(x_\Lambda;y_\Lambda)\sigma_\Lambda^N(\mathrm{d}x_\Lambda)\\ &\qquad\longrightarrow \int_{C_{\beta,\Lambda}} f(x_\Lambda)\Phi(x_\Lambda;y_\Lambda)\sigma_\Lambda(\mathrm{d}x_\Lambda), \quad N\to+\infty.\end{aligned} \tag{2.1.8}$$

Proof. Let $\upsilon > 0$ be as in Corollary 1.3.58. Then the function

$$C_{\beta,\Lambda} \ni x_\Lambda \mapsto f(x_\Lambda) \exp\Big(-\upsilon \sum_{\ell\in\Lambda} \|x_\ell\|^2_{C_\beta} \Big)$$

is bounded, see Definition 1.4.13. Furthermore, for

$$\varkappa > \frac{1}{2}\Big(1 + \sup_{\ell\in\Lambda} \sum_{\ell'\in\Lambda} |I_{\ell\ell'}| \Big),$$

the function

$$C_{\beta,\Lambda} \ni x_\Lambda \mapsto \Phi(x_\Lambda; y_\Lambda) \exp\Big(-\varkappa \sum_{\ell\in\Lambda} \|x_\ell\|^2_{L^2_\beta} \Big)$$

is also bounded, cf. (1.4.10). Then the proof follows by Corollary 1.3.58 and (2.1.5). $\square$

In order to exploit the result just proven we have to rewrite the right-hand side of (2.1.5) as an integral over a finite-dimensional space with respect to the Gibbs measure of a classical lattice model. An important class of functions for which such lattice approximations of the corresponding integrals will be constructed constitute the functions

$$C_{\beta,\Lambda} \ni x_\Lambda \mapsto F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)), \quad \tau_1, \ldots, \tau_n \in [0, \beta], \tag{2.1.9}$$

with $F_1, \ldots, F_n \in \mathfrak{P}^{(\nu)}_\Lambda$, see Definition 1.4.7. They belong to $\mathfrak{C}_\Lambda$ and are used in

$$\begin{aligned}
&\Gamma_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) \\
&\quad = \int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)) \mu^{y_\Lambda}_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \\
&\quad = \langle F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)) \rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}}.
\end{aligned} \tag{2.1.10}$$

By Theorem 1.4.9 the functions $\Gamma_{F_1,\ldots,F_n}$ are continuous on $[0, \beta]^n$. Since the above introduced $x^{(N)}_\ell$, $\ell \in \Lambda$, belong to finite-dimensional subspaces of L^2_β, they can be written as linear combinations of $x_\ell(\lambda\beta/N)$, $\lambda = 0, 1, \ldots, N-1$, cf. (1.3.145), which can be chosen as variables for the mentioned finite-dimensional integrals. For f given by (2.1.9), it is convenient to construct the lattice approximations if the arguments $\tau_1, \ldots, \tau_n$ belong to the set $\mathcal{Q}_\beta \subset [0, \beta]$ consisting of the values of τ for which τ/β is rational. Then, for given $\tau_1, \ldots, \tau_n \in \mathcal{Q}_\beta$, one finds $\lambda_1, \ldots, \lambda_n \in \{0, \ldots, N-1\}$, such that $\tau_s = (\lambda_s/N)\beta$, $s = 1, \ldots, n$. In this case, we obtain the lattice approximations of the functions (2.1.10) only for the arguments belonging to $\mathcal{Q}_\beta$. But in view of their continuity this will be enough for recovering the corresponding properties for all values of their arguments.

In what follows, we choose $n \in \mathbb{N}$, $\tau_1, \ldots, \tau_n \in \mathcal{Q}_\beta$, $y_\Lambda = (y_\ell)_{\ell\in\Lambda} \in C_{\beta,\Lambda}$, and keep them fixed. Then we pick the sequences of integers $\{N^{(k)}\}_{k\in\mathbb{N}}$, $N^{(k)}$ being even,

$\{\lambda_s^{(k)}\}_{k\in\mathbb{N}}$, $s = 1, \ldots, n$, such that $N^{(k)} \to +\infty$, $\lambda_s^{(k)} \to +\infty$ as $k \to +\infty$, and for all $k \in \mathbb{N}$, the following holds:

$$\tau_s = \left(\frac{\lambda_s^{(k)}}{N^{(k)}}\right)\beta, \qquad s = 1, \ldots, n. \tag{2.1.11}$$

In the sequel, we drop the superscript (k) assuming that N and λ_s, $s = 1, \ldots, n$, tend to infinity in such a way that (2.1.11) holds. The set of values of such N is denoted by $\mathcal{N}(\tau_1, \ldots, \tau_n)$. Then by writing $N \to +\infty$ we assume that N tends to infinity in this set. Now we are in a position to represent the right-hand side of (2.1.5) as a finite-dimensional integral with respect to a Gibbs measure of a classical system of unbounded spins $S_\ell(\lambda) \in \mathbb{R}^\nu$, $\ell \in \Lambda$, $\lambda = 0, \ldots, N-1$. For such ℓ and λ and y_ℓ as in (2.1.1), we set

$$h_\ell^{(j)}(\lambda) = \sqrt{\frac{\beta}{N}}\, y_\ell^{(j)}\left(\frac{\lambda}{N}\beta\right), \qquad j = 1, \ldots, \nu, \tag{2.1.12}$$

and introduce

$$\begin{aligned} P_{\beta,\Lambda}^{h_\Lambda}(\mathrm{d}S_\Lambda) = \frac{1}{D_{\beta,\Lambda}(h_\Lambda)} \exp\Bigg\{ &\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} I_{\ell\ell'} \sum_{\lambda=0}^{N-1} (S_\ell(\lambda), S_{\ell'}(\lambda)) \\ &- \frac{mN^2}{2\beta^2} \sum_{\ell\in\Lambda}\sum_{\lambda=0}^{N-1} |S_\ell(\lambda+1) - S_\ell(\lambda)|^2| \\ &+ \sum_{\ell\in\Lambda}\sum_{\lambda=0}^{N-1} (S_\ell(\lambda), h_\ell(\lambda))\Bigg\} \bigotimes_{\ell\in\Lambda}\bigotimes_{\lambda=0}^{N-1} Q_\ell^N(\mathrm{d}S_\ell(\lambda)), \end{aligned} \tag{2.1.13}$$

where $S_\Lambda = (S_\ell)_{\ell\in\Lambda}$, $S_\ell = (S_\ell(\lambda))_{\lambda=0,\ldots,N-1}$, addition $\lambda + 1$ is modulo N, m is as in (1.1.3), the interaction intensities $I_{\ell\ell'}$ are the same as in (2.1.1), $D_{\beta,\Lambda}(h_\Lambda)$ is a normalization factor, and, cf. (2.1.2),

$$Q_\ell^N(\mathrm{d}S_\ell(\lambda)) = \exp\left\{-\frac{\beta}{N} W_\ell\left(\sqrt{\frac{N}{\beta}}\, S_\ell(\lambda)\right) - \frac{a}{2}|S_\ell(\lambda)|^2\right\} \mathrm{d}S_\ell(\lambda), \tag{2.1.14}$$

$a > 0$ being the same as in (1.1.3).

Theorem 2.1.2. *For every $F_1, \ldots, F_n \in \mathfrak{P}_\Lambda^{(\nu)}$, $\tau_1, \ldots, \tau_n \in \mathcal{Q}_\beta$, $y_\Lambda \in C_{\beta,\Lambda}$, and $N \in \mathcal{N}(\tau_1, \ldots, \tau_n)$, the following representation holds:*

$$\begin{aligned} &\int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1)) \ldots F_n(x_\Lambda(\tau_n)) \Phi(x_\Lambda; y_\Lambda) \sigma_\Lambda^N(\mathrm{d}x_\Lambda) \\ &\quad = K_{\beta,\Lambda}^{(N)} \int_{\mathbb{R}^{\nu N|\Lambda|}} F_1(S_\Lambda(\lambda_1)) \ldots F_n(S_\Lambda(\lambda_n))\, P_{\beta,\Lambda}^{h_\Lambda}(\mathrm{d}S_\Lambda) \\ &\quad = K_{\beta,\Lambda}^{(N)} \langle F_1(S_\Lambda(\lambda_1)) \ldots F_n(S_\Lambda(\lambda_n))\rangle_{P_{\beta,\Lambda}^{h_\Lambda}}. \end{aligned} \tag{2.1.15}$$

Here $K^{(N)}_{\beta,\Lambda}$ is a positive constant, $\lambda_s = (\tau_s/\beta)N$, $s = 1,\ldots,n$, and the probability measure $P^{h_\Lambda}_{\beta,\Lambda}$ is given by (2.1.13).

Proof. We will deal with the following types of functions $C_{\beta,\Lambda} \to \mathbb{R}$:

$$\begin{aligned} &\text{(i)} \quad x_\Lambda \mapsto x^{(j)}_\ell(\tau), \quad \ell \in \Lambda,\ j = 1,\ldots,\nu,\ \tau \in \mathcal{Q}_\beta, \\ &\text{(ii)} \quad x_\Lambda \mapsto (x_\ell, x_{\ell'})_{L^2_\beta}, \quad x_\Lambda \mapsto (x_\ell, y_\ell)_{L^2_\beta}, \quad \ell,\ell' \in \Lambda. \end{aligned} \tag{2.1.16}$$

To change the variables in the integral on the right-hand side of (2.1.5) we use the Fourier transform

$$\begin{aligned} x^{(j)}_\ell(\tau) &= \sum_{k\in\mathcal{K}} \hat{x}^{(j)}_\ell(k) e_k(\tau), \quad j = 1,\ldots,\nu, \\ \hat{x}^{(j)}_\ell(k) &= (\epsilon_{j,k}, x_\ell)_{L^2_\beta} = \int_0^\beta x^{(j)}_\ell(\tau) e_k(\tau)\mathrm{d}\tau, \end{aligned} \tag{2.1.17}$$

where the functions e_k, $k \in \mathcal{K}$, are defined in (1.3.111). Then for $\tau \in \mathcal{Q}_\beta$, one finds $\lambda \in \{0,\ldots,N-1\}$ such that $\tau = (\lambda/N)\beta$. Recall that $N = 2L$, $L \in \mathbb{N}$. Thus, the function of type (i) taken at $x^{(N)}_\Lambda = (x^{(1,N)}_\Lambda,\ldots,x^{(\nu,N)}_\Lambda)$ can be written

$$\begin{aligned} x^{(j,N)}_\ell(\tau) &= \sum_{k\in\mathcal{K}_N} \hat{x}^{(j)}_\ell(k) e_k\left((\lambda/N)\beta\right) \\ &= \sqrt{\frac{N}{\beta}} \sum_{p\in\mathcal{P}_N} \hat{x}^{(j)}_\ell\left((N/\beta)p\right) \varepsilon_p(\lambda), \end{aligned} \tag{2.1.18}$$

where

$$\mathcal{P}_N = \{p = (2\pi/N)\kappa \mid \kappa = -(L-1),\ldots,-1,0,1,\ldots,L\}, \tag{2.1.19}$$

and, cf. (1.3.111),

$$\varepsilon_p(\lambda) = \begin{cases} \sqrt{2/N}\cdot\cos(\lambda p), & \text{if } p > 0; \\ -\sqrt{2/N}\cdot\sin(\lambda p), & \text{if } p < 0; \\ 1/\sqrt{N}, & \text{if } p = 0. \end{cases} \tag{2.1.20}$$

For the functions of type (ii) taken at $x^{(N)}_\Lambda$, we have

$$\begin{aligned} (x^{(N)}_\ell, x^{(N)}_{\ell'})_{L^2_\beta} &= \sum_{k\in\mathcal{K}_N} \sum_{j=1}^{\nu} \hat{x}^{(j)}_\ell(k)\hat{x}^{(j)}_{\ell'}(k) \\ &= \sum_{p\in\mathcal{P}_N} \sum_{j=1}^{\nu} \hat{x}^{(j)}_\ell((N/\beta)p)\hat{x}^{(j)}_{\ell'}((N/\beta)p), \end{aligned} \tag{2.1.21}$$

and

$$
\begin{aligned}
(x_\ell^{(N)}, y_\ell)_{L_\beta^2} &= \sum_{k\in\mathcal{K}_N}\sum_{j=1}^{\nu} \hat{x}_\ell^{(j)}(k)\hat{y}_\ell^{(j)}(k),\\
\hat{y}_\ell^{(j)}(k) &= \int_0^\beta y_\ell^{(j)}(\tau)e_k(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{2.1.22}
$$

Now we pass to the spin variables $\widehat{S}_\ell^{(j)}(p)$, $j = 1,\dots,\nu$, $\lambda = 0,1,\dots,N-1$, $p\in\mathcal{P}_N$, by setting

$$
\widehat{S}_\ell^{(j)}(p) = \hat{x}_\ell^{(j)}((N/\beta)p),
\tag{2.1.23}
$$

for which we introduce the Fourier transform, see (2.1.19) and (2.1.20),

$$
\begin{aligned}
S_\ell^{(j)}(\lambda) &= \sum_{p\in\mathcal{P}_N}\widehat{S}_\ell^{(j)}(p)\varepsilon_p(\lambda),\\
\widehat{S}_\ell^{(j)}(p) &= \sum_{\lambda=0}^{N-1} S_\ell^{(j)}(\lambda)\varepsilon_p(\lambda).
\end{aligned}
\tag{2.1.24}
$$

Then by (2.1.18) we have

$$
x_\ell^{(j,N)}((\lambda/N)\beta) = \sqrt{\frac{N}{\beta}}S_\ell^{(j)}(\lambda),
\tag{2.1.25}
$$

and hence

$$
\frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_\ell\left(x_\ell^{(N)}(\tau)\right)\mathrm{d}\tau = \frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_\ell\left(\sqrt{\frac{N}{\beta}}S_\ell(\lambda)\right).
\tag{2.1.26}
$$

By means of (2.1.23) and (2.1.24) we obtain

$$
\left(x_\ell^{(N)}, x_{\ell'}^{(N)}\right)_{L_\beta^2} = \sum_{p\in\mathcal{P}_N}\left(\widehat{S}_\ell(p), \widehat{S}_{\ell'}(p)\right) = \sum_{\lambda=0}^{N-1}\left(S_l(\lambda), S_{l'}(\lambda)\right),
\tag{2.1.27}
$$

and, cf. (2.1.12),

$$
\begin{aligned}
\left(x_\ell^{(N)}, y_\ell\right)_{L_\beta^2} &= \sum_{k\in\mathcal{K}_N}\left(\hat{x}_\ell(k), \hat{y}_\ell(k)\right) = \sum_{\lambda=0}^{N-1}\left(S_\ell(\lambda), h_\ell(\lambda)\right),\\
h_\ell^{(j)}(\lambda) &\stackrel{\mathrm{def}}{=} \sum_{p\in\mathcal{P}_N}\hat{y}_\ell^{(j)}\left((N/\beta)p\right)\varepsilon_p(\lambda),\\
\hat{y}_\ell^{(j)}(k) &\stackrel{\mathrm{def}}{=} (\epsilon_{j,k}, y_\ell)_{L_\beta^2}, \quad j = 1,\dots,\nu.
\end{aligned}
\tag{2.1.28}
$$

The next step is to construct the measure (2.1.13). We begin by introducing a finite-dimensional analog of the measure γ_N. This is the following Gaussian measure on $\mathbb{R}^{\nu N}$:

$$\pi^{(N)}(\mathrm{d}\widehat{S}_\ell) = \bigotimes_{p\in\mathcal{P}_N} \pi_p^{(N)}(\mathrm{d}\widehat{S}_\ell(p)), \tag{2.1.29}$$

where $\pi_p^{(N)}$ is the isotropic Gaussian measure on $\mathbb{R}^\nu$, such that

$$\begin{aligned} &\int_{\mathbb{R}^\nu} \exp\left(\mathrm{i}(u,v)\right)\pi_p^{(N)}(\mathrm{d}u) = \exp\left(-\frac{1}{2}\theta_p^{(N)}(v,v)\right), \\ &\theta_p^{(N)} = \frac{1}{m\,(2N/\beta)^2\,[\sin(p/2)]^2 + a}, \end{aligned} \tag{2.1.30}$$

i.e., $\theta_p^{(N)} = s_{(N/\beta)p}^{(N)}$, see (1.3.129). In view of (2.1.24), the measure $\pi^{(N)}$ can also be written in the coordinates $S_\ell(\lambda)$, $\lambda \in \{0, 1, \ldots, N-1\}$. Here we remark that the map $\widehat{S}_\ell \mapsto S_\ell$ defined by (2.1.24) is an orthogonal transformation in the Euclidean space $\mathbb{R}^{\nu N}$. In what follows, we have

$$\begin{aligned} \pi^{(N)}(\mathrm{d}S_\ell) = \frac{1}{C_N}\exp\Bigg\{ &-\frac{mN^2}{2\beta^2}\sum_{\lambda=0}^{N-1} |S_\ell(\lambda+1) - S_\ell(\lambda)|^2 \\ &-\frac{a}{2}\sum_{\lambda=0}^{N-1}|S_\ell(\lambda)|^2\Bigg\} \bigotimes_{\lambda=0}^{N-1}\mathrm{d}S_\ell(\lambda), \end{aligned} \tag{2.1.31}$$

with the convention that $S_\ell(N) = S_\ell(0)$. For every bounded continuous function $f : C_\beta \to \mathbb{R}$ and a given $N = 2L$, one finds a bounded continuous function $\phi : \mathbb{R}^{\nu N} \to \mathbb{R}$, such that $f(x_\ell^{(N)}) = \phi(S_\ell)$, where $x_\ell^{(N)}$ and $S_\ell = (S_\ell(\lambda))_{\lambda\in\{0,\ldots,N-1\}}$, are related to each other by (2.1.19), (2.1.23), and (2.1.24). Then for such functions, we have

$$\int_{C_\beta} f(x_\ell)\gamma_N(\mathrm{d}x_\ell) = \int_{C_\beta} f(x_\ell^{(N)})\gamma_N(\mathrm{d}x_\ell) = \int_{\mathbb{R}^{\nu N}} \phi(S_\ell)\pi^{(N)}(\mathrm{d}S_\ell).$$

Furthermore, for such f and ϕ, by (2.1.2) and (2.1.26) it follows that

$$\begin{aligned} \int_{C_\beta} f(x_\ell)\sigma_\ell^N(\mathrm{d}x_\ell) &= \int_{C_\beta} f(x_\ell^{(N)})\sigma_\ell^N(\mathrm{d}x_\ell) \\ &= \frac{1}{Z_\ell^N}\int_{C_\beta} f(x_\ell^{(N)})\exp\left\{-\frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_\ell\left(x_\ell\left(\frac{\lambda}{N}\beta\right)\right)\right\}\gamma_N(\mathrm{d}x_\ell) \\ &= \frac{1}{Z_\ell^N}\int_{\mathbb{R}^{\nu N}} \phi(S_\ell)\exp\left\{-\frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_\ell\left(\sqrt{\frac{N}{\beta}}S_\ell(\lambda)\right)\right\}\pi^{(N)}(\mathrm{d}S_\ell). \end{aligned} \tag{2.1.32}$$

Taking into account (2.1.2), (2.1.3), (2.1.5)–(2.1.7) and (2.1.31), (2.1.32), we conclude that

$$\int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1))\ldots F_n(x_\Lambda(\tau_n))\Phi(x_\Lambda; y_\Lambda)\sigma_\Lambda^N(\mathrm{d}x_\Lambda)$$
$$= \frac{D_{\beta,\Lambda}(h)}{Y_{\beta,\Lambda}(y_\Lambda)}\int_{\mathbb{R}^{\nu N|\Lambda|}} F_1\left(S_\Lambda(\lambda_1)\right)\ldots F_n\left(S_\Lambda(\lambda_n)\right) P_{\beta,\Lambda}^{h_\Lambda}(\mathrm{d}S_\Lambda),$$

where

$$P_{\beta,\Lambda}^{h_\Lambda}(\mathrm{d}S_\Lambda) = \frac{1}{D_{\beta,\Lambda}(h)}\exp\left\{\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} I_{\ell\ell'}\sum_{\lambda=0}^{N-1}(S_\ell(\lambda, S_{\ell'}(\lambda))\right.$$
$$+\sum_{\ell,\ell'\in\Lambda}\sum_{\lambda=0}^{N-1}(S_\ell(\lambda, h_\ell(\lambda))$$
$$\left.-\frac{\beta}{N}\sum_{\ell\in\Lambda}\sum_{\lambda=0}^{N-1} W_\ell\left(\sqrt{\frac{N}{\beta}}S_\ell(\lambda)\right)\right\}\bigotimes_{\ell\in\Lambda}\pi^{(N)}(\mathrm{d}S_\ell),$$

i.e., it is as in (2.1.13). □

2.2 Basic Correlation Inequalities

Correlation inequalities proved to be useful in studying classical lattice systems. Typically, they involve moments of Gibbs measures. The approximation established by Theorems 2.1.1 and 2.1.2 can be employed to derive such inequalities also for the moments of the measure (2.1.1). This will be done in the current section.

The inequalities for classical lattice systems which we are going to use describe the scalar ferromagnetic case. Thus, in this section we assume that, cf. Definition 1.1.2,

$$\nu = 1 \quad \text{and} \quad J_{\ell\ell'} \geq 0 \quad \text{for all } \ell, \ell'.$$

In the statements below, we derive the basic correlation inequalities for a number of particular versions of the measure (2.1.1). The scheme of the proof of each of these statements is the same. One approximates this version by the corresponding classical Gibbs measure for which the inequality in question holds. Then by Theorems 2.1.1 and 2.1.2 this inequality is obtained for the considered version of the measure (2.1.1). In view of this, below we just mention the source statements concerning the classical measure. Each time the functions W_ℓ obey Assumption 1.1.1 and the particular cases of the measure (2.1.1) are specified by further restrictions imposed on these functions.

Before stating our first result we introduce some new notions. By writing $x_\Lambda \leq x'_\Lambda$ we mean that $x_\ell(\tau) \leq x'_\ell(\tau)$ for all $\ell \in \Lambda$ and $\tau \in [0, \beta]$ (we recall that all x_ℓ are continuous functions from $[0, \beta]$ to $\mathbb{R}$). A function $f : C_{\beta,\Lambda} \to \mathbb{R}$ is called *increasing* if $x_\Lambda \leq x'_\Lambda$ implies $f(x_\Lambda) \leq f(x'_\Lambda)$.

Our first statement follows from Theorem VIII.16, page 280 in [273]. It holds since the approximating measure (2.1.13) is the Gibbs measure of a general ferromagnet. We recall that the family $\mathfrak{C}_\Lambda$ was introduced in Definition 1.4.13.

Theorem 2.2.1 (FKG Inequality). *If the functions $f, g \in \mathfrak{C}_\Lambda$ are increasing, then the inequality*

$$\langle fg\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \geq \langle f\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \langle g\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \tag{2.2.1}$$

holds for all $y_\Lambda \in C_{\beta,\Lambda}$.

One observes that in Theorem 2.2.1 we do not suppose any special properties of the functions W_ℓ in addition to Assumption 1.1.1. However, in the next statement we require something more. Clearly, every $W_\ell \colon \mathbb{R} \to \mathbb{R}$ can be decomposed into its even and odd parts, that is,

$$\begin{aligned} W_\ell(u) &= W_\ell^{\mathrm{o}}(u) + W_\ell^{\mathrm{e}}(u), \\ W_\ell^{\mathrm{e}}(u) &= \frac{1}{2}(W_\ell(u) + W_\ell(-u)), \\ W_\ell^{\mathrm{o}}(u) &= \frac{1}{2}(W_\ell(u) - W_\ell(-u)). \end{aligned} \tag{2.2.2}$$

Our next statement follows from Theorem VIII.14 of [273] (see also pages 120–122 of [274]). We recall that the family of functions $\mathfrak{P}_\Lambda^{(\nu)}$ was introduced in Definition 1.4.7.

Theorem 2.2.2 (GKS Inequalities). *Given $\ell_1, \ldots, \ell_n, \ldots, \ell_{n+m} \in \Lambda$, $n, m \in \mathbb{N}$, let the functions $F_i \in \mathfrak{P}^{(1)}_{\{\ell_i\}}$, $i = 1, \ldots, n, \ldots, n+m$ be either even or odd and positive on $\mathbb{R}_+$. Suppose also that all W_ℓ^{o}'s, are negative on $\mathbb{R}_+$ and $y_\Lambda \geq 0$. Then*

$$\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n))\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \geq 0. \tag{2.2.3}$$

If in addition, one assumes that all F_i's are increasing and all W_ℓ^{o}'s are decreasing, then

$$\begin{aligned} &\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n)) \\ &\qquad \times F_{n+1}(x_{\ell_{n+1}}(\tau_{n+1})) \ldots F_{n+m}(x_{\ell_{n+m}}(\tau_{n+m}))\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \\ &\quad \geq \langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n))\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}} \\ &\qquad \times \langle F_{n+1}(x_{\ell_{n+1}}(\tau_{n+1})) \ldots F_{n+m}(x_{\ell_{n+m}}(\tau_{n+m}))\rangle_{\mu_{\beta,\Lambda}^{y_\Lambda}}. \end{aligned} \tag{2.2.4}$$

Remark 2.2.3. If all W_ℓ's are even, the above inequalities hold also for $y_\Lambda \leq 0$.

The next statements require more specific properties of the functions W_ℓ.

Definition 2.2.4. The anharmonic potential W_ℓ is said to be of Brydges–Fröhlich–Spencer (BFS) type if it can be written as $W_\ell(u) = w_\ell(u^2)$, where the function w_ℓ is convex on $\mathbb{R}_+$. The measure (2.1.1) is of Brydges–Fröhlich–Spencer (BFS) type if all W_ℓ, $\ell \in \mathbb{L}$, are of BFS type.

The anharmonic potential W_ℓ is said to be of Ellis–Monroe (EM) type if $W_\ell(u) = w_\ell(u^2)$ with

$$\begin{aligned} &w_\ell(u) = b_\ell^{(1)} t + \cdots + b_\ell^{(r)} t^r, \quad r \in \mathbb{N}, \\ &b_\ell^{(1)} \in \mathbb{R}, \quad b_\ell^{(s)} \geq 0, \ s = 2, \ldots, r-1, \quad b_\ell^{(r)} > 0, \ r \geq 2. \end{aligned} \tag{2.2.5}$$

The anharmonic potential W_ℓ is said to be of Ellis–Monroe–Newman (EMN) type if W_ℓ is an even continuously differentiable function, such that W'_ℓ is convex on $[0, +\infty)$. The measure (2.1.1) is said to be of Ellis–Monroe (EM) type (respectively, of Ellis–Monroe–Newman (EMN) type) if all W_ℓ, $\ell \in \mathbb{L}$, are of EM type (respectively, of EMN type)

Clearly, every measure of EM type is also of EMN and BFS types. The next result follows from the inequality (12.192), page 254 in [117]. In Theorem 2.2.5 and in the sequel, $\mu^0_{\beta,\Lambda}$ is the measure (2.1.1) with $y_\Lambda = 0$.

Theorem 2.2.5. *Let the measure $\mu^{y_\Lambda}_{\beta,\Lambda}$ be of BFS type and $y_\Lambda \geq 0$. Then for all $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$,*

$$\langle x_\ell(\tau) x_{\ell'}(\tau')\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} - \langle x_\ell(\tau)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell'}(\tau')\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \leq \langle x_\ell(\tau) x_{\ell'}(\tau')\rangle_{\mu^0_{\beta,\Lambda}}. \tag{2.2.6}$$

For zero-mean Gaussian random variables $X_1, \ldots, X_{2n}$, $n \in \mathbb{N}$, the following is known:

$$\langle X_1 \ldots X_{2n}\rangle = \sum_\sigma \prod_{k=1}^n \langle X_{\sigma(2k-1)} X_{\sigma(2k)}\rangle, \tag{2.2.7}$$

cf. (1.3.55). Here the sum is taken over all partitions of the set $\{1, \ldots, 2n\}$ onto unordered pairs of its elements. If $X_1, \ldots, X_{2n}$ are such that the inequality obtained from the latter expression by replacing "$=$" with "$\leq$", these variables are said to obey the *Gaussian upper bound principle*. The result below follows from the fact that this principle holds for the measures (2.1.13), see Section 12.1, page 230 in [117].

Theorem 2.2.6 (Gaussian Upper Bound). *Let the measure (2.1.1) be of BFS type. Then the inequality*

$$\langle x_{\ell_1}(\tau_1) \ldots x_{\ell_{2n}}(\tau_{2n})\rangle_{\mu^0_{\beta,\Lambda}} \leq \sum_\sigma \prod_{k=1}^n \langle x_{\ell_{\sigma(2k-1)}} x_{\ell_{\sigma(2k)}}\rangle_{\mu^0_{\beta,\Lambda}} \tag{2.2.8}$$

holds for all $\ell_1, \ldots, \ell_{2n} \in \Lambda$ and $\tau_1, \ldots, \tau_{2n} \in [0, \beta]$.

The next result can be obtained by means of the corresponding statement proven in [140] (see also [111], [112], [113], [295], and page 260 in [117]). It is known as the Griffiths–Hurst–Sherman (GHS) inequality.

Theorem 2.2.7 (GHS Inequality). *Let the measure* (2.1.1) *be of EMN type and* $y_\Lambda \geq 0$. *Then the inequality*

$$\begin{aligned}
&\langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\
&\quad \leq \langle x_{\ell_1}(\tau_1)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\
&\qquad + \langle x_{\ell_2}(\tau_2)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_1}(\tau_1)x_{\ell_3}(\tau_3)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\
&\qquad + \langle x_{\ell_3}(\tau_3)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\
&\qquad - 2\langle x_{\ell_1}(\tau_1)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_2}(\tau_2)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_3}(\tau_3)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}}
\end{aligned} \tag{2.2.9}$$

holds for all $\ell_1, \ell_2, \ell_3 \in \Lambda$ *and* $\tau_1, \tau_2, \tau_3 \in [0, \beta]$.

Now we set

$$f_{\beta,\Lambda}(x_\Lambda; y_\Lambda) = \int_{C_{\beta,\Lambda}} \exp\left((x_\Lambda, \xi_\Lambda)_{L^2_{\beta,\Lambda}}\right) \mu^{y_\Lambda}_{\beta,\Lambda}(\mathrm{d}\xi_\Lambda), \tag{2.2.10}$$

where $\mu^{y_\Lambda}_{\beta,\Lambda}$ is the measure (2.1.1), and the real Hilbert space $L^2_{\beta,\Lambda}$ and its scalar product are as defined in (1.3.156) and (1.3.158), respectively. By the Kuratowski theorem (Proposition 1.3.18) the measure $\mu^{y_\Lambda}_{\beta,\Lambda}$ can be redefined on $L^2_{\beta,\Lambda}$ with the property $\mu^{y_\Lambda}_{\beta,\Lambda}(C_{\beta,\Lambda}) = 1$. By Assumption 1.1.1,

$$\int_{\Omega_{\beta,\Lambda}} \exp\left(a(x_\Lambda, x_\Lambda)_{L^2_{\beta,\Lambda}}\right) \mu^{y_\Lambda}_{\beta,\Lambda}(\mathrm{d}x_\Lambda) < \infty,$$

for all $a > 0$, which means that $\mu^{y_\Lambda}_{\beta,\Lambda} \in \mathcal{M}(L^2_{\beta,\Lambda})$, see Definition 1.3.43. Then by Lemma 1.3.47 the function $f_{\beta,\Lambda}(\cdot; y_\Lambda)$ can be extended to a function holomorphic on the complexification of $L^2_{\beta,\Lambda}$. Therefore, it can be expanded into a convergent series as follows, see (1.3.96):

$$f_{\beta,\Lambda}(x_\Lambda; y_\Lambda) = \sum_{n=0}^{+\infty} \frac{1}{n!} f^{(n)}_{\beta,\Lambda}(0; y_\Lambda)(x_\Lambda) \tag{2.2.11}$$

where

$$\begin{aligned}
f^{(n)}_{\beta,\Lambda}(0; y_\Lambda)(x_\Lambda) = \sum_{\ell_1,\ldots,\ell_n \in \Lambda} \int_{[0,\beta]^n} &\langle \xi_{\ell_1}(\tau_1)\ldots\xi_{\ell_n}(\tau_n)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\
&\times x_{\ell_1}(\tau_1)\ldots x_{\ell_n}(\tau_n)\mathrm{d}\tau_1, \ldots \mathrm{d}\tau_n.
\end{aligned} \tag{2.2.12}$$

The coefficients in the latter representation are the Matsubara functions (2.1.10) with $F_j(\xi_\Lambda) = \xi_{\ell_j}$, $j = 1, \ldots, n$. They are continuous with respect to $\tau_1, \ldots, \tau_n \in [0, \beta]$. By (2.1.1) and (2.2.10) for any $y'_\Lambda \in C_{\beta,\Lambda}$,

$$f_{\beta,\Lambda}(x_\Lambda + y'_\Lambda; y_\Lambda - y'_\Lambda) = [Y_{\beta,\Lambda}(y_\Lambda)/Y_{\beta,\Lambda}(y_\Lambda - y'_\Lambda)] f_{\beta,\Lambda}(x_\Lambda; y_\Lambda). \tag{2.2.13}$$

Since $f_{\beta,\Lambda}(0; y_\Lambda) = 1$, the function

$$\begin{aligned}\varphi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) &= \log f_{\beta,\Lambda}(x_\Lambda; y_\Lambda),\\ &= -\log[Y_{\beta,\Lambda}(y_\Lambda)/Y_{\beta,\Lambda}(\xi-\eta)] + \varphi_{\beta,\Lambda}(y_\Lambda+\eta|\xi-\eta)\end{aligned} \tag{2.2.14}$$

is holomorphic in x_Λ in some neighborhood of the origin of the complexification of $L^2_{\beta,\Lambda}$ and hence in some domain which contains $L^2_{\beta,\Lambda}$. Thus, one can expand $\varphi_{\beta,\Lambda}(\cdot; y_\Lambda)$ into the convergent Taylor series, similarly to (2.2.11), that is,

$$\varphi_{\beta,\Lambda}(x_\Lambda; y_\Lambda) = \sum_{n=1}^{+\infty} \frac{1}{n!}\varphi^{(n)}_{\beta,\Lambda}(0; y_\Lambda)(x_\Lambda), \tag{2.2.15}$$

with

$$\begin{aligned}\varphi^{(n)}_{\beta,\Lambda}(0; y_\Lambda)(x_\Lambda) = \sum_{\ell_1,\dots,\ell_n\in\Lambda} \int_{[0,\beta]^n} & U_{\ell_1,\dots,\ell_n}(\tau_1,\dots,\tau_n; y_\Lambda)\\ &\times x_{\ell_1}(\tau_1)\dots x_{\ell_n}(\tau_n)\mathrm{d}\tau_1,\dots\mathrm{d}\tau_{l_n},\end{aligned} \tag{2.2.16}$$

where *the Ursell functions* $U_{\ell_1,\dots,\ell_n}(\tau_1,\dots,\tau_n; y_\Lambda)$ can be expressed in terms of the above Matsubara functions, e.g.,

$$\begin{aligned}U_{\ell_1,\ell_2}(\tau_1,\tau_2; y_\Lambda) = {}& \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}}\\ & - \langle x_{\ell_1}(\tau_1)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell_2}(\tau_2)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}}.\end{aligned} \tag{2.2.17}$$

Set

$$e_\Lambda = (e_\ell)_{\ell\in\Lambda} \in \Omega_{\beta,\Lambda}, \quad e_\ell(\tau) = 1, \quad \text{for all } \ell\in\Lambda,\ \tau\in[0,\beta], \tag{2.2.18}$$

and

$$\phi_{\beta,\Lambda}(z) = \varphi_{\beta,\Lambda}(ze_\Lambda; y_\Lambda), \quad z\in\mathbb{C}. \tag{2.2.19}$$

Corollary 2.2.8. *For every $y_\Lambda \in C_{\beta,\Lambda}$, $\phi_{\beta,\Lambda}$ is a holomorphic function on some domain of $\mathbb{C}$, which contains the real line $\mathbb{R}$. Its restriction to $\mathbb{R}$ is convex.*

Proof. The holomorphy of $\phi_{\beta,\Lambda}$ can be derived from the corresponding property of $\varphi_{\beta,\Lambda}(\cdot; y_\Lambda)$. By (2.2.13), (2.2.17), and (2.2.18) it follows that

$$\phi''_{\beta,\Lambda}(z) = \sum_{\ell_1,\ell_2\in\Lambda} \int_{[0,\beta]^2} U_{\ell_1\ell_2}(\tau_1,\tau_2; y_\Lambda + ze_\Lambda)\mathrm{d}\tau_1\mathrm{d}\tau_2 \ge 0, \tag{2.2.20}$$

where we have used the FKG inequality (2.2.1), which holds for all $z\in\mathbb{R}$. Therefrom, the stated convexity follows. □

For $y_\Lambda = 0$, we have

$$\begin{aligned} U_{\ell_1,\ldots,\ell_4}(\tau_1,\ldots,\tau_4;0) = {} & \langle x_{\ell_1}(\tau_1)\ldots x_{\ell_4}(\tau_4)\rangle_{\mu^0_{\beta,\Lambda}} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)\rangle_{\mu^0_{\beta,\Lambda}} \cdot \langle x_{\ell_3}(\tau_3)x_{\ell_4}(\tau_4)\rangle_{\mu^0_{\beta,\Lambda}} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_3}(\tau_3)\rangle_{\mu^0_{\beta,\Lambda}} \cdot \langle x_{\ell_2}(\tau_2)x_{\ell_4}(\tau_4)\rangle_{\mu^0_{\beta,\Lambda}} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_4}(\tau_4)\rangle_{\mu^0_{\beta,\Lambda}} \cdot \langle x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\mu^0_{\beta,\Lambda}}. \end{aligned} \tag{2.2.21}$$

The following inequality is a particular case of the Gaussian upper bound (2.2.8). It can also be proven by means of Theorem 2.4 and Corollary 2.5 of [295] (see also page 231 of [117]).

Theorem 2.2.9 (Lebowitz Inequality). *If $\mu_{\beta,\Lambda}$ is of BFS type, the inequality*

$$U_{\ell_1,\ldots,\ell_4}(\tau_1,\ldots,\tau_4;0) \leq 0 \tag{2.2.22}$$

holds for all $\ell_1,\ldots,\ell_4 \in \Lambda$ and $\tau_1,\ldots,\tau_4 \in [0,\beta]$.

The next statement gives a lower bound for the above Ursell function. It holds for a more special choice of the functions W_ℓ and can be deduced from the inequality obtained in [81] (see equation (3.15) in that paper).

Theorem 2.2.10. *Let all W_ℓ in (2.1.1) be of the form*

$$W_\ell(u) = W(u) = b_1u^2 + b_2u^4, \quad b_1 \in \mathbb{R},\ b_2 > 0. \tag{2.2.23}$$

Then the estimate

$$\begin{aligned} U_{\ell_1,\ldots,\ell_4}(\tau_1,\ldots,\tau_4;0) \geq -4!b_2 \sum_{\ell\in\Lambda} \int_0^\beta & U_{\ell_1\ell}(\tau_1,\tau;0)U_{\ell_2\ell}(\tau_2,\tau;0) \\ & \times U_{\ell_3\ell}(\tau_3,\tau;0)U_{\ell_4\ell}(\tau_4,\tau;0)\mathrm{d}\tau \end{aligned} \tag{2.2.24}$$

holds for all $\ell_1,\ldots,\ell_4 \in \Lambda$ and $\tau_1,\ldots,\tau_4 \in [0,\beta]$.

The next statement gives a sign rule for all Ursell functions. It also holds for W_ℓ being of the form of (2.2.23) and can be deduced from Shlosman's result [271], valid for the Ising model by means of the classical Ising approximation (for more details see Chapter IX in [273]).

Theorem 2.2.11 (Shlosman Sign Rule). *Let the functions W_ℓ in (2.1.1) be as in (2.2.23). Then the sign rule*

$$(-1)^{n-1}U_{\ell_1,\ldots,\ell_{2n}}(\tau_1,\ldots,\tau_{2n};0) \geq 0 \tag{2.2.25}$$

hold for all $n \in \mathbb{N}$, $\ell_1,\ldots,\ell_{2n} \in \Lambda$, and $\tau_1,\ldots,\tau_{2n} \in [0,\beta]$.

2.3 The Logarithmic Sobolev Inequality

The logarithmic Sobolev inequality, as well as the Poincaré inequality which is close to it, have various applications in the theory of lattice systems. Given $n \in \mathbb{N}$, let μ be a probability measure on the Euclidean space $\mathbb{R}^n$. For this μ and an appropriate function $f : \mathbb{R}^n \to \mathbb{R}_+$, we set

$$\mathrm{Var}_\mu(f) = \langle f^2 \rangle_\mu - \langle f \rangle_\mu^2, \tag{2.3.1}$$

and

$$\mathrm{Ent}_\mu(f) = \langle f \log f \rangle_\mu - \langle f \rangle_\mu \cdot \log \langle f \rangle_\mu. \tag{2.3.2}$$

Suppose now that the function f has *weak gradient* in $L^2(\mathbb{R}^n, \mu)$, cf. (1.1.93), and let $|\nabla f|$ be its Euclidean norm. Then the Poincaré inequality is

$$\mathrm{Var}_\mu(f) \le \varkappa \cdot \langle |\nabla f|^2 \rangle_\mu. \tag{2.3.3}$$

The least constant $\varkappa$ such that (2.3.3) holds for all functions $f : \mathbb{R}^n \to \mathbb{R}_+$ with the properties mentioned above is called the spectral gap constant $C_{\mathrm{SG}}(\mu)$. The logarithmic Sobolev inequality is

$$\mathrm{Ent}_\mu(f^2) \le 2\varkappa \cdot \langle |\nabla f|^2 \rangle_\mu. \tag{2.3.4}$$

The least constant $\varkappa$ such that (2.3.4) holds for all functions $f : \mathbb{R}^n \to \mathbb{R}$ having gradient is called the log-Sobolev constant $C_{\mathrm{LS}}(\mu)$. The normalization in (2.3.4) is chosen such that the classical inequality

$$C_{\mathrm{SG}}(\mu) \le C_{\mathrm{LS}}(\mu) \tag{2.3.5}$$

holds.

Our aim in this section is to prove that the measure (2.1.1) obeys similar inequalities and to estimate the constant (2.3.5) corresponding to this measure in the simplest case of a one-point set $\Lambda = \{\ell\}$. That is, we consider

$$\begin{aligned} \mu^{y_\ell}(\mathrm{d}x_\ell) &= \frac{1}{Y(y_\ell)} \exp\left((x_\ell, y_\ell)_{L^2_\beta} - \int_0^\beta W_\ell(x_\ell(\tau))\mathrm{d}\tau \right) \chi_\beta(\mathrm{d}x_\ell) \\ &= \Phi_{\beta,\ell}(x_\ell; y_\ell)\sigma_\ell(\mathrm{d}x_\ell), \end{aligned} \tag{2.3.6}$$

which is a probability measure on the Hilbert space $L^2_\beta = L^2([0,\beta] \to \mathbb{R}^\nu)$, possessing the property $\mu^{y_\ell}(C_\beta) = 1$. Note that here, unlike the preceding section, we consider the case of general $\nu \in \mathbb{N}$. By definition, a function $f : L^2_\beta \to \mathbb{R}$ has gradient $f'(x) \in L^2_\beta$, if for every x_ℓ and ξ_ℓ in L^2_β,

$$\left(\frac{\partial}{\partial t} f(x_\ell + t\xi_\ell) \right)_{t=0} = (f'(x_\ell), \xi_\ell)_{L^2_\beta}, \quad t \in \mathbb{R}, \tag{2.3.7}$$

cf. (1.1.93). Recall that the family $\mathfrak{C}_\Lambda$ was introduced in Definition 1.4.13.

Definition 2.3.1. The measure μ^{y_ℓ} obeys the Poincaré inequality (respectively, the logarithmic Sobolev inequality) if there exists positive $\varkappa$ (respectively, $\varkappa'$) such that for all $f \in \mathfrak{E}_\ell$ having gradient, the estimate (i) below (respectively, (ii) below) holds:

$$\begin{aligned} &\text{(i)} \quad \mathrm{Var}_{\mu^{y_\ell}}(f^2) \le \varkappa \langle \| f' \|^2_{L^2_\beta} \rangle_{\mu^{y_\ell}}, \\ &\text{(ii)} \quad \mathrm{Ent}_{\mu^{y_\ell}}(f^2) \le 2\varkappa' \langle \| f' \|^2_{L^2_\beta} \rangle_{\mu^{y_\ell}}. \end{aligned} \tag{2.3.8}$$

The least values of $\varkappa$ and $\varkappa'$ will be called the spectral gap $C_{\mathrm{SG}}(\mu^{y_\ell})$ and the logarithmic Sobolev $C_{\mathrm{LS}}(\mu^{y_\ell})$ constants, respectively.

Let us decompose W_ℓ in (2.3.6) as

$$W_\ell = W_{1,\ell} + W_{2,\ell}, \tag{2.3.9}$$

where $W_{1,\ell} \in C^2(\mathbb{R}^\nu)$ is such that

$$-a < b \stackrel{\text{def}}{=} \inf_{u,v \in \mathbb{R}^\nu,\ v \neq 0} \Big(W''_{1,\ell}(u)v, v \Big) / |v|^2 < +\infty, \tag{2.3.10}$$

a being the same as in (1.1.3). As for the second term, we set

$$0 \le \delta(W_{2,\ell}) = \mathrm{Osc}(W_{2,\ell}) \stackrel{\text{def}}{=} \sup_{u \in \mathbb{R}^\nu} W_{2,\ell} - \inf_{u \in \mathbb{R}^\nu} W_{2,\ell} \le +\infty. \tag{2.3.11}$$

Theorem 2.3.2. *For any $y_\ell \in C_\beta$, the measure* (2.3.6) *obeys the logarithmic Sobolev inequality with the constant*

$$C_{\mathrm{LS}}(\mu^{y_\ell}) \le \frac{\exp[\beta\delta(W_{2,\ell})]}{(a+b)}. \tag{2.3.12}$$

Proof. By (2.1.13) and (2.1.14) the lattice approximation of (2.3.6) is the following measure:

$$\begin{aligned} P_N^{h_\ell}(\mathrm{d}S_\ell) = \frac{1}{D(h_\ell)} \exp\Bigg\{ &-\frac{mN^2}{2\beta^2} \sum_{\lambda=0}^{N-1} |S_\ell(\lambda+1) - S_\ell(\lambda)|^2 \\ &+ \sum_{\lambda=0}^{N-1} (S_\ell(\lambda), h_\ell(\lambda)) - \frac{\beta}{N} \sum_{\lambda=0}^{N-1} W_{2,\ell}\left(\sqrt{\frac{N}{\beta}} S_\ell(\lambda) \right) \Bigg\} \\ &\times \exp\left\{ -\frac{\beta}{N} \sum_{\lambda=0}^{N-1} W_{1,\ell}\left(\sqrt{\frac{N}{\beta}} S_\ell(\lambda) \right) - \frac{a}{2} \sum_{\lambda=0}^{N-1} |S_\ell(\lambda)|^2 \right\} \bigotimes_{\lambda=1}^{N-1} \mathrm{d}S_\ell(\lambda), \end{aligned} \tag{2.3.13}$$

where $N = 2L$, $L \in \mathbb{N}$, and $h_\ell(\lambda)$ is the same as in (2.1.28). In the convex case where $W_{2,\ell} \equiv 0$, the logarithmic Sobolev constant of $P_N^{h_\ell}$ does not exceed $1/($Hessian of the potential energy), that is $1/(a+b)$. The non-convex part contributes in the form of the following factor, see equation (2.9), page 202 in [291]:

$$\exp\left[\operatorname{Osc}\left(\frac{\beta}{N}\sum_{\lambda=0}^{N-1} W_{2,\ell}\left(\sqrt{\frac{N}{\beta}}S_\ell\right)\right)\right] \le \exp[\beta\delta(W_{2,\ell})].$$

Note that the latter estimate is uniform in N. Therefore,

$$C_{\mathrm{LS}}(P_N^{h_\ell}) \le \exp[\beta\delta(W_{2,\ell})]/(a+b). \tag{2.3.14}$$

By (2.3.7)

$$\phi(x_\ell) \stackrel{\mathrm{def}}{=} \|f'\|^2_{L^2_\beta} = \sum_{j=1}^{\nu}\sum_{k\in\mathcal{K}}\left[(f'(x_\ell),\epsilon_{j,k})_{L^2_\beta}\right]^2. \tag{2.3.15}$$

Given $N=2L$, $L\in\mathbb{N}$, let S_ℓ and $x_\ell^{(N)}$ be connected with each other by (2.1.18), (2.1.23), and 2.1.24). Let also $g\colon\mathbb{R}^{\nu N}\to\mathbb{R}$ be such that $f(x_\ell^{(N)})=g(S_\ell)$ for all such $x_\ell^{(N)}$ and S_ℓ. As in [291], page 202, we use the inequality

$$\mathrm{Ent}_{\mu^{y_\ell}}(f^2) \le \langle[f^2\log f^2 - f^2\log t - f^2 + t]\rangle_{\mu^{y_\ell}},$$

which holds for all $t>0$. Then, for $f\in\mathfrak{C}_\ell$ by Theorem 2.1.1 and (2.3.14) we have

$$\begin{aligned}
\mathrm{Ent}_{\mu^{y_\Lambda}}(f^2) &\le \lim_{N\to+\infty}\int_{C_\beta}\Big[f^2(x_\ell)\log f^2(x_\ell) - f^2(x_\ell)\log\langle g^2\rangle_{P_N^{h_\ell}} \\
&\qquad\qquad - f^2(x_\ell) + \langle g^2\rangle_{P_N^{h_\ell}}\Big]\Phi(x_\ell,y_\ell)\chi_\beta^{(N)}(\mathrm{d}x_\ell) \\
&= \lim_{N\to+\infty}\int_{\mathbb{R}^{\nu N}}\Big[g^2(S_\ell)\log g^2(S_\ell) - g^2(S_\ell)\log\langle g^2\rangle_{P_N^{h_\ell}} \\
&\qquad\qquad - g^2(S_\ell) + \langle g^2\rangle_{P_N^{h_\ell}}\Big]P_N^{h_\ell}(\mathrm{d}S_\ell) \\
&\le \big\{\exp[\beta\delta(W_{2,\ell})]/2(a+b)\big\}\cdot\lim_{N\to+\infty}\langle|\nabla g|^2\rangle_{P_N^{h_\ell}}.
\end{aligned} \tag{2.3.16}$$

Similarly as in (2.3.15)

$$\begin{aligned}
|\nabla g|^2 &= \sum_{j=1}^{\nu}\sum_{p\in\mathcal{P}_N}\left[\varepsilon_p(\lambda)(\partial g/\partial S_\ell^j(\lambda))\right]^2 \\
&= \sum_{j=1}^{\nu}\sum_{k\in\mathcal{K}_N}\left[(f'(x_\ell^{(N)}),\epsilon_{j,k})_{L^2_\beta}\right]^2 \le \sum_{j=1}^{\nu}\sum_{k\in\mathcal{K}}\left[(f'(x_\ell^{(N)}),\epsilon_{j,k})_{L^2_\beta}\right]^2.
\end{aligned}$$

Employing this estimate in (2.3.16) we arrive at

$$\mathrm{Ent}_{\mu^{y_\Lambda}}(f^2) \le \frac{\exp[\beta\delta(W_{2,\ell})]}{2(a+b)}\cdot\lim_{N\to+\infty}\int_{C_\beta}\phi(x_\ell)\Phi_\ell^{(N)}(x_\ell;y_\ell)\sigma_\ell^N(\mathrm{d}x_\ell).$$

If f is such that $\phi\in\mathfrak{C}_\ell$, by Theorem 2.1.1 the latter limit exists and is equal to the right-hand side of (2.3.8). □

2.4 The Lee–Yang Property

In this section, we prove that the measure (2.1.1) has a useful property if the functions W_ℓ belong to a certain class. To introduce it we need the Laguerre entire functions, see [159] and [192], [201].

Definition 2.4.1. The Laguerre entire functions (of first kind) are either polynomials of a single complex variable with real non-positive zeros only, or the uniform limits of sequences of such polynomials taken on compact subsets of $\mathbb{C}$.

By $\mathcal{F}^{\mathrm{Laguerre}}$ we denote the set of all such functions. It is known that $\varphi \in \mathcal{F}^{\mathrm{Laguerre}}$ if and only if it possesses the representation

$$\begin{aligned} &\varphi(z) = Ce^{\kappa_0 z} z^m \prod_{j=1}^{+\infty}(1+\kappa_j z), \quad C \in \mathbb{C}, \\ &\sum_{j=1}^{+\infty} \kappa_j < \infty, \qquad \kappa_j \geq 0, \ j \in \mathbb{N}_0. \end{aligned} \tag{2.4.1}$$

If $\varphi(0) \neq 0$, then $m = 0$; if $\kappa_0 = 0$, the function φ is of order at most 1 with minimal type.

Definition 2.4.2. A rotation-invariant probability measure μ on $\mathbb{R}^\nu$ possesses the Lee–Yang property if there exists $\varphi_\mu \in \mathcal{F}^{\mathrm{Laguerre}}$ such that

$$\varphi_\mu(|\hat{h}|^2) = \int_{\mathbb{R}^\nu} \exp\left[(\hat{h}, u)\right]\mu(\mathrm{d}u), \quad \hat{h} \in \mathbb{R}^\nu. \tag{2.4.2}$$

Such a measure is called sub-Gaussian if the parameter κ_0 in the representation (2.4.1) of φ_μ is equal to zero.

As the measure μ is rotation-invariant, one can choose the direction of $\hat{h}$ arbitrarily, e.g., $\hat{h} = (h, 0, \ldots, 0)$, $h \in \mathbb{R}$. Then the function $f_\mu(h) \stackrel{\mathrm{def}}{=} \varphi_\mu(h^2)$ can be written

$$f_\mu(h) = \exp(\kappa_0 h^2) \prod_{j=1}^{+\infty}(1 + \kappa_j h^2) \tag{2.4.3}$$

and hence extended to an even entire function of $h \in \mathbb{C}$. This function is ridge, which means that

$$|f_\mu(x + \mathrm{i}y)| \leq f_\mu(x), \quad \text{for all } x, y \in \mathbb{R}, \tag{2.4.4}$$

see [139] for the concept and properties on ridge functions. The latter fact follows directly from the definition (2.4.2). Also from (2.4.2) one easily derives the following property.

Proposition 2.4.3. *The function* $f_\mu(h) = \varphi_\mu(h^2)$ *is such that the function*

$$p_\mu(h) \stackrel{\mathrm{def}}{=} \log f_\mu(h),$$

is convex as it has the property

$$p''_\mu(h) \geq 0, \quad \textit{for all } h \in \mathbb{R}. \tag{2.4.5}$$

So far, no necessary conditions for measures to possess the Lee–Yang property are known. The most general sufficient condition for measures having the form $C \exp[-V(u)]\,\mathrm{d}u$ to possess the Lee–Yang property which is known so far, reads as follows, see [185] and [192].

Proposition 2.4.4. *Suppose that*

$$\mu(\mathrm{d}u) = C \exp[-V(u)]\,\mathrm{d}u, \qquad V(u) = v(|u|^2), \quad u \in \mathbb{R}^\nu, \tag{2.4.6}$$

where v *is such that, for a certain* $b \geq 0$*, the function* $b + v'$ *belongs to* $\mathcal{F}^{\mathrm{Laguerre}}$. *Then* μ *possesses the Lee–Yang property. The parameter* κ_0 *in the representation* (2.4.1) *of* φ_μ *equals zero if* v' *is an increasing function.*

If v is a polynomial of degree 1, the measure (2.4.6) is Gaussian and its φ_μ is represented according to (2.4.1) with $\kappa_j = 0$, $j \in \mathbb{N}$. If v is a polynomial of higher degree, the order of growth of φ_μ is $\deg v/(2\deg v - 1)$, which implies that $\kappa_0 = 0$. It is easy to see that the polynomial $v(t) = b_1 t + b_2 t^2$ (cf. (2.2.23)) obeys the above condition for any $b_1 \in \mathbb{R}$. If v is a transcendental entire function, the corresponding φ_μ is of order $1/2$ and maximal type.

Given $\nu \in \mathbb{N}$, if a measure μ on $\mathbb{R}^\nu$ has the Lee–Yang property, then for all $b \geq 0$, the measure

$$\mu_b(\mathrm{d}u) = \exp(b|u|^2)\mu(\mathrm{d}u) \Big/ \int \exp(b|u|^2)\mu(\mathrm{d}u) \tag{2.4.7}$$

also has this property, see Theorem 3.2 in [192]. For $\nu = 1$, the class of measures μ such that the corresponding measures μ_b possess the Lee–Yang property for all $b \in \mathbb{R}$ was described in Theorem 2 of the paper [231].

Proposition 2.4.5 (Newman). *Given probability measure* μ *on* $\mathbb{R}$*, let the measure* μ_b *possess the Lee–Yang property for all real* b*. Then either* $\mu(\mathrm{d}u) = [\delta_{-a}(\mathrm{d}u) + \delta_a(\mathrm{d}u)]/2$*,* δ_a *being the Dirac measure centered at* $a > 0$*, or else* $\mu(\mathrm{d}u) = \phi_\mu(u)\mathrm{d}u$ *with density*

$$\begin{aligned} \phi_\mu(u) &= K u^{2m} \exp\left(-\alpha u^4 - \beta u^2\right) \varPhi(u^2), \\ \varPhi(\theta) &= \prod_{j\in J} \left[\left(1 + \gamma_j \theta\right) \exp\left(-\theta\gamma_j\right)\right], \end{aligned} \tag{2.4.8}$$

where $K > 0$ *is a normalization constant;* $m \in \mathbb{N}_0$*; the set* J *may be void, finite, or countable; the numbers* $\gamma_j > 0$ *obey the condition* $\sum_{j\in J} \gamma_j^2 < +\infty$. *Finally,* $\alpha \geq 0$*; if* $\alpha > 0$*, then* β *may be arbitrary real. For* $\alpha = 0$*, one demands* $\beta + \sum_{j\in J} \gamma_j > 0$.

Under the conditions imposed on the numbers γ_j, $j \in J$, Φ_μ can be extended to an entire function. Therefore, if μ, being of the form (2.4.6), has the property described by Proposition 2.4.5, then the function V has to be

$$V(u) = v(u^2) = \alpha u^4 + \beta u^2 - \log \Phi(u), \tag{2.4.9}$$

with α, β, and Φ as in (2.4.8). Note that in this case, if μ is not Gaussian,

$$v''(t) = 2\alpha + \sum_{j \in J} \left(\frac{\gamma_j}{1 + \gamma_j t} \right)^2 > 0. \tag{2.4.10}$$

Turning back to the measure (2.1.1) we conclude that for $\nu = 1$, if all W_ℓ have the form (2.4.9), the corresponding measure is of EM- and hence of BFS type.

The main result of this section is contained in the following statement.

Theorem 2.4.6. *Suppose that $\nu = 1, 2$ and* (a) *for $\nu = 1$, each W_ℓ has the form* (2.4.9)*;* (b) *for $\nu = 2$, $W_\ell(u) = \alpha_\ell |u|^4 + \beta_\ell |u|^2$, with $\alpha_\ell > 0$ and $\beta_\ell \in \mathbb{R}$ for all ℓ. Then the Laplace transform $f_{\beta,\Lambda}$* (2.2.10) *of the measure* (2.1.1)*, which by Lemma* 1.3.47 *can be extended to a holomorphic function on the complexification of $L^2_{\beta,\Lambda}$, has the following property – the function of $z \in \mathbb{C}$ defined by*

$$g_{\beta,\Lambda}(z) = f_{\beta,\Lambda}(z^2 e_\Lambda; 0) \tag{2.4.11}$$

can be written in the form (2.4.1)*. Here $e_\Lambda = (e_\ell)_{\ell \in \Lambda} \in C_{\beta,\Lambda}$ is such that, for all $\ell \in \Lambda$ and $\tau \in [0, \beta]$, $e_\ell(\tau) = 1$.*

The proof of this theorem will be done in several steps. First we prove an auxiliary statement.

Lemma 2.4.7. *Let the measure* (2.1.1) *be as in Theorem* 2.4.6*. Then the Laplace transform*

$$F^{(N)}_{\beta,\Lambda}(h_\Lambda) = \int_{\mathbb{R}^{N|\Lambda|}} \exp\Big(\sum_{\ell \in \Lambda} \sum_{\lambda=0}^{N-1} h_\ell(\lambda) S^{(1)}_\ell(\lambda) \Big) P^0_{\beta,\Lambda}(\mathrm{d}S_\Lambda) \tag{2.4.12}$$

of the approximating measure (2.1.13) *can be extended to an entire function of $h_\Lambda = (h_l(\lambda)) \in \mathbb{C}^{N|\Lambda|}$ such that*

$$F^{(N)}_{\beta,\Lambda}(h_\Lambda) \neq 0, \quad \textit{if } \Re[h_\ell(\lambda)] > 0, \textit{ for all } \lambda = 0, \ldots, N-1;\ \ell \in \Lambda. \tag{2.4.13}$$

Proof. The measure (2.1.13) can be rewritten in the form

$$P^0_{\beta,\Lambda}(\mathrm{d}S_\Lambda) = \frac{1}{D_{\beta,\Lambda}(0)} \exp\Big\{ \frac{1}{2} \sum_{\ell,\ell' \in \Lambda} \sum_{\lambda,\lambda'=0}^{N-1} K^N_{\ell\ell'}(\lambda,\lambda')(S_\ell(\lambda), S_{\ell'}(\lambda')) \Big\} \bigotimes_{\ell \in \Lambda} \bigotimes_{\lambda=0}^{N-1} \tilde{Q}^{(N)}(\mathrm{d}S_\ell(\lambda)), \tag{2.4.14}$$

with

$$K^N_{\ell\ell'}(\lambda,\lambda') = I_{\ell\ell'}\delta_{\lambda\lambda'} + \frac{mN^2}{\beta^2}\delta_{\ell\ell'}\delta_{\lambda\lambda'-1} \geq 0 \tag{2.4.15}$$

and

$$\widetilde{Q}^{(N)}(\mathrm{d}S_\ell(\lambda)) = \exp\Big(-\frac{mN^2}{\beta^2}|S_\ell(\lambda)|^2\Big)Q^{(N)}(\mathrm{d}S_\ell(\lambda)), \tag{2.4.16}$$

where $Q^{(N)}$ is as in (2.1.14). For $\nu = 1$, the measure $\widetilde{Q}^{(N)}$ has the Lee–Yang property for all N by Proposition 2.4.5. For $\nu = 2$, the same holds by Proposition 2.4.4. Thereby, in view of the positivity (2.4.15) the proof of the lemma follows by Corollaries 3.3 and 4.4 in [208]. □

Corollary 2.4.8. *Let the measure* (2.1.1) *be as in Theorem* 2.4.6*. Then the function*

$$G^{(N)}_{\beta,\Lambda}(z) = F^{(N)}_{\beta,\Lambda}(z^2 I_\Lambda), \quad z \in \mathbb{C}, \tag{2.4.17}$$

can be represented in the form of (2.4.1)*, in which* $\kappa_0 = 0$ *if the entire functions* w_ℓ *grow at infinity faster than a polynomial of degree* 1*. Here* $I_\Lambda = (I_\ell(\lambda))_{\ell\in\Lambda}$ *with* $I_\ell(\lambda) = 1$ *for all* $\lambda = 0, \ldots, N-1$ *and* $\ell \in \Lambda$*.*

From the latter statement, the proof of Theorem 2.4.6 will be obtained by passing to the limit $N \to +\infty$. Here we employ the Vitali theorem, see Proposition 1.4.11.

Proof of Theorem 2.4.6*.* By (2.4.12), (2.4.17) and Corollary 2.4.8, for all $x, y \in \mathbb{R}$, one has

$$\begin{aligned} \big|F^{(N)}_{\beta,\Lambda}((x+\mathrm{i}y)I_\Lambda)\big| &\leq F^{(N)}_{\beta,\Lambda}(xI_\Lambda), \\ F^{(N)}_{\beta,\Lambda}(xI_\Lambda) &\leq F^{(N)}_{\beta,\Lambda}(x'I_\Lambda) \quad \text{if } 0 < x < x'. \end{aligned} \tag{2.4.18}$$

By (2.4.17), (2.4.11), and Theorems 2.1.1, 2.1.2 for any $x > 0$, the sequence of positive numbers $\{F^{(N)}_{\beta,\Lambda}(xI_\Lambda)\}_{N\in\mathbb{N}}$ converges to $f_{\beta,\Lambda}(xe_\Lambda; 0)$. Thus, by (2.4.18) the sequence $\{F^{(N)}_{\beta,\Lambda}(zI_\Lambda)\}_{N\in\mathbb{N}}$ is uniformly bounded in the strip $\{z \in \mathbb{C} \mid |\Re(z)| \leq x\}$; hence, it is uniformly bounded on every compact $K \subset \mathbb{C}$ by

$$\sup_{N\in\mathbb{N}} F^{(N)}_{\beta,\Lambda}(I_\Lambda \sup_{z\in K}|z|),$$

which by the Vitali theorem gives the convergence $F^{(N)}_{\beta,\Lambda}(zI_\Lambda) \to f_{\beta,\Lambda}(ze_\Lambda; 0)$, uniform on compact subsets of $\mathbb{C}$. Thereby, $F^{(N)}_{\beta,\Lambda}(z^2 I_\Lambda) \to f_{\beta,\Lambda}(z^2 e_\Lambda; 0)$ in the same sense. This yields the result to be proven as the family $\mathcal{F}^{\text{Laguerre}}$ is closed with respect to this uniform convergence (see e.g., [201]). □

2.5 More Inequalities

In this section, we derive a number of more specific inequalities, which we are going to use in the study of our Euclidean Gibbs measures.

2.5.1 The Vector Case: Scalar Domination

Here ν in (2.1.1) will be an arbitrary positive integer number. Let $\Gamma_{F_1,\ldots,F_n}$ be the function (2.1.10). The measure (2.1.1) is supposed to be invariant with respect to the group of all orthogonal transformations $O(\nu)$, which holds if and only if each W_ℓ can be written in the form

$$W_\ell(u) = w_\ell(|u|^2), \quad u \in \mathbb{R}^\nu, \tag{2.5.1}$$

where $w_\ell : [0, +\infty) \to \mathbb{R}$ is continuous. Our first result is a version of the GKS inequality (2.2.3). Below and in the sequel, by writing $y_\Lambda \geq 0$ we mean $y_\ell^{(j)}(\tau) \geq 0$ for all ℓ, τ and $j = 1, \ldots, \nu$. We recall that the family $\mathfrak{P}_\Lambda^{(\nu)}$ of functions $F : \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, which appear in the Matsubara functions (2.1.10), was introduced in Definition 1.4.7.

Theorem 2.5.1. *Let all W_ℓ's in (2.1.1) be rotation-invariant. Suppose also that the functions $F_i \in \mathfrak{P}^{(\nu)}_{\{\ell_i\}}$, $\ell_i \in \Lambda$, $i = 1, \ldots, n$, depend on $x_{\ell_i}^{(j)}$ only, with one and the same $j \in \{1, \ldots, \nu\}$, and, as functions of such $x_{\ell_i}^{(j)}$, they obey the conditions of Theorem 2.2.2. Then for all $\tau_1, \ldots, \tau_n \in [0, \beta]$ and $y_\Lambda \geq 0$, it follows that*

$$\Gamma_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) \geq 0. \tag{2.5.2}$$

The proof of this and the next theorem will be given below. The following result is based on the comparison of the Matsubara functions from the previous theorem with the corresponding functions defined by the measure (2.1.1) with $\nu = 1$. This will give us a *scalar domination estimate.* For this, however, we have to impose additional conditions on the functions W_ℓ. To distinguish between the scalar and vector cases, in case $\nu = 1$, all quantities will be supplied by the tilde. For example, the measure (2.1.1) and the corresponding Matsubara functions are denoted by $\tilde{\mu}_{\beta,\Lambda}$ and $\widetilde{\Gamma}_{F_1,\ldots,F_n}$, respectively. Those with arbitrary $\nu \in \mathbb{N}$ will be written as before, i.e., without the tilde.

Theorem 2.5.2. *Suppose that the functions w_ℓ in (2.5.1) are convex. Suppose also that the functions $F_i \in \mathfrak{P}^{(\nu)}_{\{\ell_i\}}$, $\ell_i \in \Lambda$, $i = 1, \ldots, n$, are as in Theorem 2.5.1, and for each F_i, there exists $\widetilde{F}_i \in \mathfrak{P}^{(1)}_{\{\ell_i\}}$, which obeys the conditions of Theorem 2.2.2, such that $F_i(x_{\ell_i}) = \widetilde{F}_i(x_{\ell_i}^{(j)})$. Finally, let $y_\Lambda \geq 0$ and the measure $\tilde{\mu}_{\beta,\Lambda}^{y_\Lambda^{(j)}}$ be defined by (2.1.1) with $\nu = 1$ and $\tilde{y}_\Lambda = y_\Lambda^{(j)}$, j being the same as above. Then for arbitrary $\tau_1, \ldots, \tau_n \in [0, \beta]$, the functions (2.1.10) corresponding to these measures obey the estimate*

$$\Gamma_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) \leq \widetilde{\Gamma}_{\widetilde{F}_1,\ldots,\widetilde{F}_n}(\tau_1, \ldots, \tau_n). \tag{2.5.3}$$

Remark 2.5.3. It is important that in the above theorems all F_i's depend on their $x_{\ell_i}^{(j)}$ with one and the same j. Obviously, (2.5.3) is the scalar domination estimate.

To prove (2.5.3) we use the convexity of the w_ℓ's. For a function $\varphi : \mathbb{R} \to \mathbb{R}$, we denote its corresponding one-sided derivatives at a given $t \in \mathbb{R}$ by $D_\pm \varphi(t)$. It is known, see e.g., [277], pages 34–37 and Lemma 1 in [250], that convex functions have the following properties.

Proposition 2.5.4. *For a convex function* $\varphi\colon \mathbb{R} \to \mathbb{R}$, *it follows that:*

(a) *the one-sided derivatives* $D_\pm\varphi(t)$ *exist for every* $t \in \mathbb{R}$; *the set* $\{t \in \mathbb{R} \mid D_+\varphi(t) \neq D_-\varphi(t)\}$ *is at most countable;*

(b) *for every* $t \in \mathbb{R}$ *and* $\theta > 0$,

$$D_+\varphi(t) \leq D_-\varphi(t+\theta) \leq D_+\varphi(t+\theta); \tag{2.5.4}$$

(c) *the point-wise limit* φ *of a sequence of convex functions* $\{\varphi_n\}_{n\in\mathbb{N}}$ *is a convex function; if* φ *and all* φ_n*'s are differentiable at a given* t, $\varphi_n'(t) \to \varphi'(t)$ *as* $n \to +\infty$;

(d) *if a sequence of convex functions* $\{f_n\}$ *is such that* $0 \leq f_n(x) \leq C$ *for some* $C > 0$ *all* $n \in \mathbb{N}$ *and* $x \in (a,b)$, $a < b$, *then it contains a subsequence* $\{f_{n_k}\}$, *which converges point-wise on* (a,b).

Proof of Theorems 2.5.1 *and* 2.5.2. We recall that the space $C_{\beta,\Lambda}$ was defined in (3.1.5). For j being as in the hypothesis of the theorems, we decompose $x_\Lambda = (\bar{x}_\Lambda, x_\Lambda^{(j)})$, where

$$\bar{x}_\ell \in C([0,\beta] \to \mathbb{R}^{\nu-1}), \quad x_\ell^{(j)} \in C([0,\beta] \to \mathbb{R}), \quad \ell \in \Lambda.$$

Thereby,

$$C_{\beta,\Lambda} = \bar{C}_{\beta,\Lambda} \times \widetilde{C}_{\beta,\Lambda}.$$

Likewise, $y_\Lambda = (\bar{y}_\Lambda, y_\Lambda^{(j)})$ and $u = (\bar{u}, u^{(j)})$ for $u \in \mathbb{R}^\nu$. Then the Høegh-Krohn measure χ_β (which is a Gaussian measure) can also be decomposed according to

$$\chi_\beta(\mathrm{d}x_\ell) = \left(\bar{\chi}_\beta \otimes \tilde{\chi}\right)(\mathrm{d}\bar{x}_\ell, \mathrm{d}x_\ell^{(j)}), \tag{2.5.5}$$

where the Gaussian measures $\bar{\chi}_\beta$ and $\tilde{\chi}$ are defined in (1.3.109) as measures on the spaces $L^2([0,\beta] \to \mathbb{R}^{\nu-1})$ and $L^2([0,\beta] \to \mathbb{R})$, respectively (see Subsection 1.3.6). Thereby, the functions W_ℓ can be written, see (2.5.1),

$$W_\ell(u) = w_\ell\big(|\bar{u}|^2 + (u^{(j)})^2\big). \tag{2.5.6}$$

For $\theta, t \in \mathbb{R}_+$, we set

$$\begin{aligned} B_\ell(\theta,t) &= w_\ell(\theta+t) - w_\ell(\theta) - w_\ell(t), \\ Q_\ell(\bar{x}_\ell, x_\ell^{(j)}) &= \int_0^\beta B_\ell\big(|\bar{x}_\ell(\tau)|^2, (x_\ell^{(j)}(\tau))^2\big)\mathrm{d}\tau. \end{aligned} \tag{2.5.7}$$

Then by means of the decomposition (2.5.5) the measure (2.1.1) can be written

$$\begin{aligned} \mu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda) = D_{\beta,\Lambda} \exp\Big\{ - \sum_{\ell\in\Lambda} Q_\ell(\bar{x}_\ell, x_\ell^{(j)})\Big\} \\ \times \Big(\bar{\mu}_{\beta,\Lambda}^{\bar{y}_\Lambda} \otimes \tilde{\mu}_{\beta,\Lambda}^{y_\Lambda^{(j)}}\Big)(\mathrm{d}\bar{x}_\Lambda, \mathrm{d}x_\Lambda^{(j)}), \end{aligned} \tag{2.5.8}$$

where $D_{\beta,\Lambda}$ is a normalization constant and the Gibbs measures $\bar{\mu}_{\beta,\Lambda}^{\bar{y}_\Lambda}$ and $\tilde{\mu}_{\beta,\Lambda}^{\tilde{y}_\Lambda}$ describe the systems of $\nu-1$- and one-dimensional interacting anharmonic oscillators, respectively. This allows for rewriting the Matsubara function in the following way

$$\begin{aligned}\Gamma_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = D_{\beta,\Lambda}\int_{\bar{C}_{\beta,\Lambda}} &\Xi(1|\bar{x}_\Lambda;\tau_1,\ldots,\tau_n)\\ &\times\Theta(1|\bar{x}_\Lambda)\bar{\mu}_{\beta,\Lambda}^{\bar{y}_\Lambda}(\mathrm{d}\bar{x}_\Lambda),\end{aligned} \tag{2.5.9}$$

where for $\vartheta\in[0,1]$, we have set

$$\begin{aligned}\Xi(\vartheta|\bar{x}_\Lambda;\tau_1,\ldots,\tau_n) = \frac{1}{\Theta(\vartheta|\bar{x}_\Lambda)}\int_{\tilde{C}_{\beta,\Lambda}} &F_1(x_{\ell_1}(\tau_1))\ldots F_n(x_{\ell_n}(\tau_n))\\ &\times\exp\Big\{-\vartheta\sum_{\ell\in\Lambda}Q_\ell(\bar{x}_\ell,x_\ell^{(j)})\Big\}\tilde{\mu}_{\beta,\Lambda}^{y_\Lambda^{(j)}}(\mathrm{d}x_\Lambda^{(j)}),\end{aligned} \tag{2.5.10}$$

and

$$\Theta(\vartheta|\bar{x}_\Lambda) = \int_{\tilde{C}_{\beta,\Lambda}}\exp\Big\{-\vartheta\sum_{\ell\in\Lambda}Q_\ell(\bar{x}_\ell,x_\ell^{(j)})\Big\}\tilde{\mu}_{\beta,\Lambda}^{y_\Lambda^{(j)}}(\mathrm{d}x_\Lambda^{(j)}). \tag{2.5.11}$$

For every fixed $\bar{x}_\ell$, the function $x_\ell^{(j)}\mapsto Q_\ell(\bar{x}_\ell,x_\ell^{(j)})$, defined by (2.5.7), is even. Thus, for every $\vartheta\in[0,1]$ and $\bar{x}_\ell\in\bar{C}_{\beta,\Lambda}$, the integral in (2.5.10) is taken with respect to a probability measure, which satisfies the conditions of Theorem 2.2.2. Then (2.5.2) follows from (2.2.3).

For the one-sided derivative of the function (2.5.7) with respect to t, by (2.5.4) we have

$$D_+B_\ell(\theta,t) = D_+w_\ell(t+\theta) - D_+w_\ell(t)\geq 0,\quad \text{for } \theta\geq 0; \tag{2.5.12}$$

hence, it is increasing in t for any fixed $\theta\geq 0$. Let the functions $\tilde{F}_1,\ldots,\tilde{F}_n$ be as supposed in Theorem 2.5.2. Then

$$\begin{aligned}\Xi(0|\bar{x}_\Lambda;\tau_1,\ldots,\tau_n) &= \int_{\Omega_{\beta,\Lambda}}\tilde{F}_1(x_{\ell_1}^{(j)}(\tau_1))\ldots\tilde{F}_n(x_{\ell_n}^{(j)}(\tau_n))\tilde{\mu}_{\beta,\Lambda}^{\tilde{y}_\Lambda}(\mathrm{d}x_\Lambda^{(j)})\\ &= \tilde{\Gamma}_{\tilde{F}_1,\ldots,\tilde{F}_n}^{\beta,\Lambda}(\tau_1,\ldots,\tau_n).\end{aligned} \tag{2.5.13}$$

As in Theorem 1.4.14, one can prove that for every fixed $x_\Lambda\in C_{\beta,\Lambda}$, the expression (2.5.11) defines an entire function of ϑ. Then, since the function of $x_\Lambda^{(j)}$ under the integral in (2.5.10) belongs to $\mathfrak{E}_\Lambda$, the right-hand side of (2.5.10) is a holomorphic function of ϑ in a domain which contains the real line. Thus, we can compute the derivative

$$\frac{\partial}{\partial \vartheta} \Xi(\vartheta|\bar{x}_\Lambda; \tau_1, \ldots, \tau_n)$$
$$= -\sum_{\ell\in\Lambda} \int_0^\beta \Big\{ \Big\langle \widetilde{F}_1(x_{\ell_1}^{(j)}(\tau_1)) \ldots \widetilde{F}_n(x_{\ell_n}^{(j)}(\tau_n)) B_\ell\Big(|\bar{x}_\ell(\tau)|^2, \big(x_\ell^{(j)}(\tau)\big)^2\Big)\Big\rangle_\phi$$
$$- \Big\langle \widetilde{F}_1(x_{\ell_1}^{(j)}(\tau_1)) \ldots \widetilde{F}_n(x_{\ell_n}^{(j)}(\tau_n))\Big\rangle_\phi \cdot \Big\langle B_\ell\Big(|\bar{x}_\ell(\tau)|^2, \big(x_\ell^{(j)}(\tau)\big)^2\Big)\Big\rangle_\phi \Big\} \mathrm{d}\tau. \tag{2.5.14}$$

Here for a fixed $\bar{x}_\Lambda \in \bar{C}_{\beta,\Lambda}$, the measure ϕ is defined on $\widetilde{C}_{\beta,\Lambda}$ as

$$\phi(\mathrm{d}x_\Lambda^{(j)}) = \frac{1}{\Theta(\vartheta|\bar{x}_\Lambda)} \exp\Big\{ -\vartheta \sum_{\ell\in\Lambda} Q_\ell(\bar{x}_\ell, x_\ell^{(j)})\Big\} \tilde{\mu}_{\beta,\Lambda}^{\tilde{y}_\Lambda}(\mathrm{d}x_\Lambda^{(j)}).$$

Since the measure $\tilde{\mu}_{\beta,\Lambda}^{\tilde{y}_\Lambda}$ and the functions

$$\widetilde{F}_i \quad \text{and} \quad x_{\ell_i}^{(j)} \mapsto B_{\ell_i}\Big(\big|\bar{x}_{\ell_i}(\tau)\big|^2, \big(x_{\ell_i}^{(j)}(\tau)\big)^2\Big), \quad i = 1, \ldots, n,$$

satisfy the conditions of Theorem 2.2.2, the estimates (2.2.4) and (2.5.12) yield in (2.5.14)

$$\frac{\partial}{\partial \vartheta} \Xi(\vartheta|\bar{x}_\Lambda; \tau_1, \ldots, \tau_n) \le 0,$$

for all $\vartheta \in \mathbb{R}$, $\bar{x}_\Lambda \in \bar{C}_{\beta,\Lambda}$, and $\tau_1, \ldots, \tau_n \in [0, \beta]$. The latter fact and the estimate (2.2.3) yield in turn

$$\begin{aligned} \Xi(1|\bar{x}_\Lambda; \tau_1, \ldots, \tau_n) &\le \Xi(0|\bar{x}_\Lambda; \tau_1, \ldots, \tau_n) \\ &= \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{F}_1,\ldots,\widetilde{F}_n}(\tau_1, \ldots, \tau_n). \end{aligned} \tag{2.5.15}$$

Using this inequality in (2.5.9) we get

$$\begin{aligned} \Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) \le \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{F}_1,\ldots,\widetilde{F}_n}(\tau_1, \ldots, \tau_n) & \\ \times D_{\beta,\Lambda} \int_{\bar{C}_{\beta,\Lambda}} \int_{\widetilde{C}_{\beta,\Lambda}} & \exp\Big\{ -\sum_{\ell\in\Lambda} Q_\ell(\bar{x}_\ell, x_\ell^{(j)})\Big\} \\ \times & \Big(\bar{\mu}_{\beta,\Lambda}^{\bar{y}_\Lambda} \otimes \tilde{\mu}_{\beta,\Lambda}^{\tilde{y}_\Lambda}\Big)(\mathrm{d}\bar{x}_\Lambda, \mathrm{d}x_\Lambda^{(j)}) \\ = \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{F}_1,\ldots,\widetilde{F}_n}(\tau_1, \ldots, \tau_n) & \int_{C_{\beta,\Lambda}} \mu_{\beta,\Lambda}^{y_\Lambda}(\mathrm{d}x_\Lambda) \\ = \widetilde{\Gamma}^{\beta,\Lambda}_{\widetilde{F}_1,\ldots,\widetilde{F}_n}(\tau_1, \ldots, \tau_n), & \end{aligned}$$

which completes the proof. □

The above theorem has the following

Corollary 2.5.5. *Let the conditions of Theorem 2.5.2 be satisfied. Then for every $\mu^{y_\Lambda}_{\beta,\Lambda}$-integrable function $G\colon C_{\beta,\Lambda} \to \mathbb{R}_+$, which does not depend on $x^{(j)}_\Lambda$, the inequalities*

$$\begin{aligned} 0 &\leq \langle F_1(x_{\ell_1}(\tau_1)) \dots F_n(x_{\ell_n}(\tau_n)) G(x_\Lambda)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\ &\leq \langle \widetilde{F}_1(x_{\ell_1}(\tau_1)) \dots \widetilde{F}_n(x_{\ell_n}(\tau_n))\rangle_{\tilde{\mu}^{\tilde{y}_\Lambda}_{\beta,\Lambda}} \cdot \langle G(x_\Lambda)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \end{aligned} \tag{2.5.16}$$

hold for all $\tau_1, \dots, \tau_n \in [0, \beta]$.

To prove this result one writes, cf. (2.5.9),

$$\begin{aligned} &\langle F_1(x_\Lambda(\tau_1)) \dots F_n(x_\Lambda(\tau_n)) G(x_\Lambda)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \\ &\quad = D_{\beta,\Lambda} \int_{\bar{C}_{\beta,\Lambda}} \Xi(1|\bar{x}_\Lambda; \tau_1, \dots, \tau_n) G(x_\Lambda) \Theta(1|\bar{x}_\Lambda) \bar{\mu}^{\bar{y}_\Lambda}_{\beta,\Lambda}(\mathrm{d}\bar{x}_\Lambda), \end{aligned}$$

which by (2.5.15) yields (2.5.16).

2.5.2 Zero Field Domination

Here we set again $\nu = 1$. In this case, Theorem 2.2.5 states that the zero external field correlation function dominates the corresponding correlation function with $y_\Lambda \geq 0$ if the measure (2.1.1) is of BFS type. In this subsection, we are going to get a similar result, but for arbitrary y_Λ. For this, however, we have to impose further restrictions on the functions W_ℓ. Namely, we suppose them to be even, see (2.5.1), and such that the functions w_ℓ are the polynomials (2.2.5), i.e., the measure (2.1.1) has to be of EM type. The correlation functions of this measure will be compared with the ones corresponding to the measure $\hat{\mu}^0_{\beta,\Lambda}$, given by (2.1.1), with the functions W_ℓ replaced by

$$\widehat{W}_\ell(u) \stackrel{\mathrm{def}}{=} 2W_\ell\big(u/\sqrt{2}\big) = b^{(1)}_\ell u^2 + \sum_{s=2}^{r} 2^{1-s} b^{(s)}_\ell u^{2s}, \quad u \in \mathbb{R}. \tag{2.5.17}$$

Such functions obey the relation

$$W_\ell\left(\frac{u+v}{\sqrt{2}}\right) + W_\ell\left(\frac{u-v}{\sqrt{2}}\right) = \widehat{W}_\ell(u) + \widehat{W}_\ell(v) + \sum_{s=1}^{r-1} b^{(s)}_\ell(u) v^{2s}, \tag{2.5.18}$$

where

$$b^{(s)}_\ell(u) = \sum_{p=s+1}^{r} \binom{2p}{2s} 2^{1-p} b^{(p)}_\ell u^{2(p-s)}. \tag{2.5.19}$$

To simplify notation we write $\hat{\mu}_{\beta,\Lambda}$ for $\hat{\mu}^{y_\Lambda}_{\beta,\Lambda}$ with $y_\Lambda = 0$. Consider

$$K^{y_\Lambda}_{\ell\ell'}(\tau,\tau') = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} - \langle x_\ell(\tau)\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}} \cdot \langle x_{\ell'}(\tau')\rangle_{\mu^{y_\Lambda}_{\beta,\Lambda}}. \tag{2.5.20}$$

For the same $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0,\beta]$, we set

$$\widehat{K}_{\ell\ell'}(\tau,\tau') = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\hat{\mu}_{\beta,\Lambda}}. \tag{2.5.21}$$

Theorem 2.5.6. *For arbitrary $y_\Lambda \in C_{\beta,\Lambda}$ and all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0,\beta]$,*

$$0 \leq K^{y_\Lambda}_{\ell\ell'}(\tau,\tau') \leq \widehat{K}_{\ell\ell'}(\tau,\tau'). \tag{2.5.22}$$

Proof. We rewrite (2.5.20) in the form

$$\begin{aligned}
K^{y_\Lambda}_{\ell\ell'}(\tau,\tau') = \frac{1}{\left[Y_{\beta,\Lambda}(y_\Lambda)\right]^2} \iint_{C_{\beta,\Lambda}\times C_{\beta,\Lambda}} & \frac{x_\ell(\tau)-\tilde{x}_\ell(\tau)}{\sqrt{2}} \cdot \frac{x_{\ell'}(\tau')-\tilde{x}_{\ell'}(\tau')}{\sqrt{2}} \\
& \times \exp\Big\{ \sum_{\ell_1\in\Lambda} (x_{\ell_1}+\tilde{x}_{\ell_1}, y_{\ell_1})_{L^2_\beta} \\
& + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda} I_{\ell_1\ell_2}[(x_{\ell_1},x_{\ell_2})_{L^2_\beta} + (\tilde{x}_{\ell_1},\tilde{x}_{\ell_2})_{L^2_\beta}] \\
& - \sum_{\ell_1\in\Lambda}\int_0^\beta [W_{\ell_1}(x_{\ell_1}(t)) + W_{\ell_1}(\tilde{x}_{\ell_1}(t))]\mathrm{d}t\Big\} \\
& \bigotimes_{\ell_1\in\Lambda} \left(\chi_\beta\otimes\chi_\beta\right)(\mathrm{d}x_{\ell_1},\mathrm{d}\tilde{x}_{\ell_1}).
\end{aligned}$$

Then we apply the orthogonal transformation

$$z_{\ell_1}(t) = [x_{\ell_1}(t) - \tilde{x}_{\ell_1}(t)]/\sqrt{2}, \quad \tilde{z}_{\ell_1}(t) = [x_{\ell_1}(t) + \tilde{x}_{\ell_1}(t)]/\sqrt{2}, \tag{2.5.23}$$

$\ell_1 \in \Lambda$, $t \in [0,\beta]$, which yields

$$\begin{aligned}
K^{y_\Lambda}_{\ell\ell'}(\tau,\tau') = \frac{1}{\left[Y_{\beta,\Lambda}(y_\Lambda)\right]^2} \iint_{C_{\beta,\Lambda}\times C_{\beta,\Lambda}} & z_\ell(\tau)z_{\ell'}(\tau') \\
& \times \exp\Big\{ \sum_{\ell_1\in\Lambda} \sqrt{2}(\tilde{z}_{\ell_1}, y_{\ell_1})_{L^2_\beta} \\
& + \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda} I_{\ell\ell'}[(z_{\ell_1},z_{\ell_2})_\beta + (\tilde{z}_{\ell_1},\tilde{z}_{\ell_2})_{L^2_\beta}] - \sum_{\ell_1\in\Lambda} Q_{\ell_1}(z_{\ell_1},\tilde{z}_{\ell_1}) \\
& - \sum_{\ell_1\in\Lambda}\int_0^\beta [\widehat{W}_{\ell_1}(z_{\ell_1}(t)) + \widehat{W}_{\ell_1}(\tilde{z}_{\ell_1}(t))]\mathrm{d}t\Big\} \\
& \times \bigotimes_{\ell_1\in\Lambda} \left(\chi_\beta\otimes\chi_\beta\right)(\mathrm{d}z_{\ell_1},\mathrm{d}\tilde{z}_{\ell_1}),
\end{aligned} \tag{2.5.24}$$

where we have set, see (2.5.18) and (2.5.19),

$$Q_{\ell_1}(z_{\ell_1}, \tilde{z}_{\ell_1}) = \sum_{s=1}^{r-1} \int_0^\beta b_{\ell_1}^{(s)}(\tilde{z}_{\ell_1}(t))[z_{\ell_1}(t)]^{2s} \mathrm{d}t. \tag{2.5.25}$$

Since all $b_\ell^{(s)}$ in (2.5.19) are nonnegative, all the coefficients in (2.5.25) are nonnegative as well, which holds for all $\tilde{z}_{\ell_1}(t)$. For $\vartheta \in [0, 1]$, we set

$$\Xi_{\ell\ell'}(\vartheta|\tilde{z}_\Lambda, \tau, \tau') = \langle z_\ell(\tau) z_{\ell'}(\tau')\rangle_{\phi_\vartheta(\cdot|\tilde{z}_\Lambda)}, \tag{2.5.26}$$

where the expectation is taken with respect to the measure

$$\begin{aligned}\phi_\vartheta(\mathrm{d}z_\Lambda|\tilde{z}_\Lambda) = \frac{1}{\Theta(\vartheta|\tilde{z}_\Lambda)} \exp\Big\{ &- \vartheta \sum_{\ell_1\in\Lambda} Q_{\ell_1}(z_{\ell_1}, \tilde{z}_{\ell_1}) \\ &+ \frac{1}{2} \sum_{\ell_1,\ell_2\in\Lambda} I_{\ell_1\ell_2}(z_{\ell_1}, z_{\ell_2})_{L^2_\beta} \\ &- \sum_{\ell_1\in\Lambda} \int_0^\beta \widehat{W}_{\ell_1}(z_{\ell_1}(\tau))\mathrm{d}\tau \Big\} \bigotimes_{\ell_1\in\Lambda} \chi_\beta(\mathrm{d}z_{\ell_1}),\end{aligned} \tag{2.5.27}$$

with

$$\begin{aligned}\Theta(\vartheta|\tilde{z}_\Lambda) = \int_{C_{\beta,\Lambda}} \exp\Big\{ &- \vartheta \sum_{\ell_1\in\Lambda} Q_{\ell_1}(z_{\ell_1}, \tilde{z}_{\ell_1}) \\ &+ \frac{1}{2} \sum_{\ell_1,\ell_2\in\Lambda} I_{\ell_1\ell_2}(z_{\ell_1}, z_{\ell_2})_{L^2_\beta} \\ &- \sum_{\ell_1\in\Lambda} \int_0^\beta \widehat{W}_{\ell_1}(z_{\ell_1}(\tau))\mathrm{d}\tau \Big\} \bigotimes_{\ell_1\in\Lambda} \chi_\beta(\mathrm{d}z_{\ell_1}).\end{aligned} \tag{2.5.28}$$

As in Subsection 2.5.1, both Ξ and Θ as functions of ϑ are differentiable on the interval $(0, 1)$ and continuous at its endpoints. Thus, one can calculate the derivative

$$\begin{aligned}\frac{\partial}{\partial\vartheta} \Xi_{\ell\ell'}(\vartheta|\tilde{z}_\Lambda, \tau, \tau') = -\frac{1}{\Theta(\vartheta|\tilde{z}_\Lambda)} &\sum_{s=1}^{r-1} \sum_{\ell_1\in\Lambda} \int_0^\beta b_{\ell_1}^{(s)}(\tilde{z}_{\ell_1}(t)) \\ &\times \Big\{\Big\langle \big[z_{\ell_1}(t)\big]^{2s} z_\ell(\tau) z_{\ell'}(\tau')\Big\rangle_{\phi_\vartheta(\cdot|\tilde{z}_\Lambda)} \\ &- \Big\langle \big[z_{\ell_1}(t)\big]^{2s} \Big\rangle_{\phi_\vartheta(\cdot|\tilde{z}_\Lambda)} \Big\langle z_\ell(\tau) z_{\ell'}(\tau')\Big\rangle_{\phi_\vartheta(\cdot|\tilde{z}_\Lambda)} \Big\} \mathrm{d}t.\end{aligned}$$

For every $\tilde{z}_\Lambda \in C_{\beta,\Lambda}$, the measure (2.5.27) has the form (2.1.1) with $y_\Lambda = 0$, thus the GKS inequalities (2.2.3), (2.2.4) hold for its moments. Then by (2.2.4), it follows that

$$\frac{\partial}{\partial\vartheta} \Xi_{\ell\ell'}(\vartheta|\tilde{z}_\Lambda, \tau, \tau') \le 0,$$

hence

$$0 \leq \Xi_{\ell\ell'}(1|\tilde{z}_\Lambda, \tau, \tau') \leq \Xi_{\ell\ell'}(0|\tilde{z}_\Lambda, \tau, \tau')\widehat{K}_{\ell\ell'}(\tau, \tau'), \tag{2.5.29}$$

which holds for all $\tilde{z}_\Lambda$, $\ell, \ell' \in \Lambda$, and $\tau, \tau' \in [0, \beta]$. On the other hand, the representation (2.5.24) may be rewritten as

$$\begin{aligned} K_{\ell\ell'}^{y_\Lambda}(\tau, \tau') = {} & \frac{1}{\left[Y_{\beta,\Lambda}(y_\Lambda)\right]^2} \int_{C_{\beta,\Lambda}} \Xi(1|\tilde{z}_\Lambda, \tau, \tau')\Theta(1|\tilde{z}_\Lambda) \\ & \times \exp\Big\{ \sum_{\ell_1 \in \Lambda} \sqrt{2}(\tilde{z}_{\ell_1}, y_{\ell_1})_{L^2_\beta} + \sum_{\ell_1, \ell_2 \in \Lambda} I_{\ell_1\ell_2}(\tilde{z}_{\ell_1}, \tilde{z}_{\ell_2})_{L^2_\beta} \\ & - \sum_{\ell_1 \in \Lambda} \int_0^\beta \widehat{W}_{\ell_1}(\tilde{z}_{\ell_1}(\tau))\mathrm{d}\tau \Big\} \bigotimes_{\ell_1 \in \Lambda} \chi_\beta(\mathrm{d}\tilde{z}_{\ell_1}). \end{aligned}$$

Applying here (2.5.29) and taking into account (2.5.28) one arrives at (2.5.22). □

2.5.3 Estimates of Moments and Correlation Functions

In this subsection, we derive some estimates for the moments and correlation functions of the Euclidean local Gibbs measures (1.4.18), (1.4.52) with $\nu = 1$. As we shall see in Part II, correlation functions like (2.5.20) play a significant role in the physical applications. First, employing the correlation inequalities obtained in the previous section, we prove two general statements. Here we study the dependence of the moments and correlation functions of the measures (2.1.1) on the interaction intensities $I = (I_{\ell,\ell'})_{\ell,\ell' \in \Lambda}$. Until the end of this subsection, in order to indicate this dependence we shall write $\mu_{\beta,\Lambda}^{I,y_\Lambda}$ and $\mu_{\beta,\Lambda}^{I}$ if $y_\Lambda = 0$. Correspondingly, we set

$$K_{\ell\ell'}^{y_\Lambda}(I|\tau, \tau') = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\mu_{\beta,\Lambda}^{I,y_\Lambda}} - \langle x_\ell(\tau)\rangle_{\mu_{\beta,\Lambda}^{I,y_\Lambda}} \cdot \langle x_{\ell'}(\tau')\rangle_{\mu_{\beta,\Lambda}^{I,y_\Lambda}}, \tag{2.5.30}$$

with $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$. For two matrices $I = (I_{\ell\ell'})_{\ell,\ell' \in \Lambda}$ and $\Upsilon = (\Upsilon_{\ell\ell'})_{\ell,\ell' \in \Lambda}$, we set $\Upsilon \geq I$ if for all $\ell, \ell' \in \Lambda$, one has $\Upsilon_{\ell\ell'} \geq I_{\ell\ell'}$.

Theorem 2.5.7. *Let the measures $\mu_{\beta,\Lambda}^{I,y_\Lambda}$ and $\mu_{\beta,\Lambda}^{\Upsilon,y_\Lambda}$ be defined by (2.1.1) with $\Upsilon \geq I \geq 0$ and with the same $y_\Lambda \in C_{\beta,\Lambda}$, $y_\Lambda \geq 0$. Suppose also that all W_ℓ's are of EMN type. Then for all $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$,*

$$0 \leq K_{\ell\ell'}^{y_\Lambda}(I|\tau, \tau') \leq K_{\ell\ell'}^{y_\Lambda}(\Upsilon|\tau, \tau'). \tag{2.5.31}$$

Proof. For $\vartheta \in [0, 1]$, we set

$$\Theta(\vartheta) = \int_{C_{\beta,\Lambda}} \exp\Big\{\frac{\vartheta}{2} \sum_{\ell,\ell' \in \Lambda} [\Upsilon_{\ell\ell'} - I_{\ell\ell'}](x_\ell, x_{\ell'})_{L^2_\beta}\Big\} \mu_{\beta,\Lambda}^{I,y_\Lambda}(\mathrm{d}x_\Lambda), \tag{2.5.32}$$

$$\phi_\vartheta(\mathrm{d}x_\Lambda) = \frac{1}{\Theta(\vartheta)} \exp\Big\{\frac{\vartheta}{2} \sum_{\ell,\ell' \in \Lambda} [\Upsilon_{\ell\ell'} - I_{\ell\ell'}](x_\ell, x_{\ell'})_{L^2_\beta}\Big\} \mu_{\beta,\Lambda}^{I,y_\Lambda}(\mathrm{d}x_\Lambda), \tag{2.5.33}$$

and

$$\Xi_{\ell\ell'}(\vartheta|\tau,\tau') = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\phi_\vartheta} - \langle x_\ell(\tau)\rangle_{\phi_\vartheta}\cdot\langle x_{\ell'}(\tau')\rangle_{\phi_\vartheta}. \tag{2.5.34}$$

One observes that the measure (2.5.33) is of the same type as the one in (2.1.1); hence, the inequalities established in Section 2.2 hold for its moments. Clearly,

$$\Xi_{\ell\ell'}(0|\tau,\tau') = K^{y_\Lambda}_{\ell\ell'}(I|\tau,\tau'), \quad \Xi_{\ell\ell'}(1|\tau,\tau') = K^{y_\Lambda}_{\ell\ell'}(\Upsilon|\tau,\tau'). \tag{2.5.35}$$

As above, to prove the theorem we are going to exploit the interpolation (2.5.35). Since $y_\Lambda \geq 0$, by (2.2.3) we get

$$\langle x_\ell(\tau)\rangle_{\phi_\vartheta} \geq 0, \quad \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\phi_\vartheta} \geq 0, \tag{2.5.36}$$

which holds for all $\vartheta \in [0,1]$. We use this fact and the GHS inequality (2.2.8) to show that

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\langle x_\ell(\tau)\rangle_{\phi_\vartheta} = \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda}[\Upsilon_{\ell_1\ell_2} - I_{\ell_1\ell_2}]\int_0^\beta \big\{&\langle x_\ell(\tau)x_{\ell_1}(t)x_{\ell_2}(t)\rangle_{\phi_\vartheta} \\ &- \langle x_\ell(\tau)\rangle_{\phi_\vartheta}\cdot\langle x_{\ell_1}(t)x_{\ell_2}(t)\rangle_{\phi_\vartheta}\big\}\,\mathrm{d}t \leq 0.\end{aligned}$$

Then by (2.5.36) and the latter estimate,

$$\begin{aligned}\frac{\partial}{\partial\vartheta}\Xi_{\ell\ell'}(\vartheta|\tau,\tau') = \frac{1}{2}\sum_{\ell_1,\ell_2\in\Lambda}[\Upsilon_{\ell_1\ell_2} - I_{\ell_1\ell_2}]\int_0^\beta &\big\{\langle x_\ell(\tau)x_{\ell'}(\tau')x_{\ell_1}(t)x_{\ell_2}(t)\rangle_{\phi_\vartheta} \\ &- \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\phi_\vartheta}\cdot\langle x_{\ell_1}(t)x_{\ell_2}(t)\rangle_{\phi_\vartheta}\big\}\,\mathrm{d}t \\ &- \langle x_\ell(\tau)\rangle_{\phi_\vartheta}\cdot\frac{\partial}{\partial\vartheta}\langle x_{\ell'}(\tau')\rangle_{\phi_\vartheta} \\ &- \langle x_{\ell'}(\tau')\rangle_{\phi_\vartheta}\cdot\frac{\partial}{\partial\vartheta}\langle x_\ell(\tau)\rangle_{\phi_\vartheta} \geq 0.\end{aligned} \tag{2.5.37}$$

Here we have taken into account that $\Upsilon \geq I$ and that the term in $\{\ldots\}$ is nonnegative, which follows from the GKS inequality (2.2.4). The lower bound in (2.5.31) follows from the GKS inequality (2.2.3). □

Remark 2.5.8. Since all W_ℓ's are supposed to be even, the estimate (2.5.31) holds also if $y_\Lambda \leq 0$.

Our second statement describes the moments of the measure $\mu^{I,y_\Lambda}_{\beta,\Lambda}$.

Theorem 2.5.9. *Let y_Λ and the functions W_ℓ, $F_i \in \mathfrak{P}^{(1)}_{\{\ell_i\}}$, $i = 1,\ldots,n$, be as in Theorem 2.2.2. Then for all $\tau_1,\ldots,\tau_n \in [0,\beta]$,*

$$\big\langle F_1(x_{\ell_1}(\tau_1))\ldots F_n(x_{\ell_n}(\tau_n))\big\rangle_{\mu^{I,y_\Lambda}_{\beta,\Lambda}} \leq \big\langle F_1(x_{\ell_1}(\tau_1))\ldots F_n(x_{\ell_n}(\tau_n))\big\rangle_{\mu^{\Upsilon,y_\Lambda}_{\beta,\Lambda}}, \tag{2.5.38}$$

whenever $I \leq \Upsilon$.

Proof. To prove this statement one considers

$$\left\langle F_1(x_{\ell_1}(\tau_1))\dots F_n(x_{\ell_n}(\tau_n))\right\rangle_{\phi_\vartheta}$$

and shows by Theorem 2.2.2, as it was done directly above, that this moment is an increasing function of ϑ. $\square$

A particular case of this statement may be related to Theorem 2.5.7.

Corollary 2.5.10. *If all W_ℓ's are even and $y_\Lambda = 0$, then* (2.5.31) *holds whenever $I \le \Upsilon$.*

Note that here we do not suppose W_ℓ's to be of EM type. Similarly, one can compare the moments of the measures (2.1.1) with different anharmonic potentials W_ℓ. Suppose one has two such potentials, say W_ℓ and $\widetilde{W}_\ell$, $\ell \in \Lambda$, both obeying Assumption 1.1.1. The measure (2.1.1) with the latter potential is denoted by $\tilde{\mu}^{y_\Lambda}_{\beta,\Lambda}$.

Theorem 2.5.11. *Let both W_ℓ and $\widetilde{W}_\ell$, $\ell \in \Lambda$, be even and such that for every $\ell \in \Lambda$,*

$$W_\ell(u_\ell) - \widetilde{W}_\ell(u_\ell) \le W_\ell(\tilde{u}_\ell) - \widetilde{W}_\ell(\tilde{u}_\ell), \quad \text{whenever } u_\ell^2 \le \tilde{u}_\ell^2. \tag{2.5.39}$$

Then for any $y_\Lambda \ge 0$, $\ell_1, \dots, \ell_n \in \Lambda$, and $\tau_1, \dots, \tau_n \in [0, \beta]$,

$$\langle x_{\ell_1}(\tau_1)\dots x_{\ell_n}(\tau_n)\rangle_{\mu^{y_\ell}_{\beta,\Lambda}} \le \langle x_{\ell_1}(\tau_1)\dots x_{\ell_n}(\tau_n)\rangle_{\tilde{\mu}^{y_\ell}_{\beta,\Lambda}}. \tag{2.5.40}$$

Proof. The functions $F_\ell(u_\ell) = W_\ell(u_\ell) - \widetilde{W}_\ell(u_\ell)$ obey the conditions of Theorem 2.2.2. Then for $\vartheta \in [0,1]$ we introduce

$$\Theta(\vartheta) = \int_{C_{\beta,\Lambda}} \exp\left\{\vartheta \sum_{\ell\in\Lambda}\int_0^\beta \left[W_\ell(x_\ell(\tau)) - \widetilde{W}_\ell(x_\ell(\tau))\mathrm{d}\tau\right]\right\}\mu^{y_\Lambda}_{\beta,\Lambda}(\mathrm{d}x_\Lambda),$$

$$\phi_\vartheta(\mathrm{d}x_\Lambda) = \frac{1}{\Theta(\vartheta)}\exp\left\{\vartheta \sum_{\ell\in\Lambda}\int_0^\beta \left[W_\ell(x_\ell(\tau)) - \widetilde{W}_\ell(x_\ell(\tau))\mathrm{d}\tau\right]\right\}\mu^{y_\Lambda}_{\beta,\Lambda}(\mathrm{d}x_\Lambda),$$

and employ the interpolation based on the inequalities (2.2.4), as it was in the proof of Theorem 2.5.7. $\square$

Now let us employ the statements proven above to study the moments and correlation functions of the local Euclidean Gibbs measures for $\nu = 1$. Recall that the measure $\nu_{\beta,\Lambda}$ corresponding to the zero boundary conditions was defined by (1.4.18), whereas the measure $\nu^{\mathrm{per}}_{\beta,\Lambda}$ corresponding to the periodic conditions on the boundaries of the box Λ was defined by (1.4.52). We set

$$K^\Lambda_{\ell\ell'}(\tau,\tau'|p) = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} - \langle x_\ell(\tau)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} \cdot \langle x_{\ell'}(\tau')\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}. \tag{2.5.41}$$

Let also $K^\Lambda_{\ell\ell'}(\tau,\tau')$ stand for the correlation function (2.5.20) corresponding to the measure $\nu_{\beta,\Lambda}$.

Theorem 2.5.12. *Suppose that for all ℓ, $V_\ell(u) = -h_\ell u + v_\ell(u^2)$, where $h_\ell \geq 0$ and the functions v_ℓ have the form (2.2.5); or, alternatively, for all ℓ, V_ℓ's are even functions satisfying Assumption 1.1.1. Suppose also that $J_{\ell\ell'} \geq 0$ for all ℓ, ℓ'. Then:*

(a) *for any box Λ,*

$$0 \leq K^{\Lambda}_{\ell\ell'}(\tau, \tau') \leq K^{\Lambda}_{\ell\ell'}(\tau, \tau'|p); \tag{2.5.42}$$

(b) *for any $\Lambda, \Lambda' \in \mathfrak{L}_{\text{fin}}$, such that $\Lambda \subset \Lambda'$,*

$$0 \leq K^{\Lambda}_{\ell\ell'}(\tau, \tau') \leq K^{\Lambda'}_{\ell\ell'}(\tau, \tau'). \tag{2.5.43}$$

Both estimates hold for all ℓ, ℓ' and $\tau, \tau' \in [0, \beta]$.

Proof. By the definition of the periodic potential $J^{\Lambda}_{\ell\ell'}$, see (1.4.50), it follows that $J^{\Lambda} \geq J$; hence, (2.5.42) follows from Theorem 2.5.7 and Corollary 2.5.10. To prove claim (b) we set, for $\ell, \ell' \in \Lambda'$,

$$I_{\ell\ell'} = \begin{cases} J_{\ell\ell'} & \text{if } \ell, \ell' \in \Lambda; \\ 0 & \text{otherwise.} \end{cases} \tag{2.5.44}$$

Then $J = (J_{\ell\ell'})_{\ell,\ell' \in \Lambda'} \geq I = (I_{\ell\ell'})_{\ell,\ell' \in \Lambda'}$ and (2.5.43) also follows from Theorem 2.5.7 and Corollary 2.5.10. $\square$

Finally, from Theorem 2.5.9 we have

Theorem 2.5.13. *Let the functions V_ℓ, $F_i \in \mathfrak{P}^{(1)}_{\{\ell_i\}}$, $i = 1, \ldots, n$, be as in Theorem 2.2.2. Then for all $\tau_1, \ldots, \tau_n \in [0, \beta]$ and all boxes Λ, it follows that*

$$\left\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n)) \right\rangle_{\nu_{\beta,\Lambda}} \leq \left\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n)) \right\rangle_{\nu^{\text{per}}_{\beta,\Lambda}}. \tag{2.5.45}$$

Furthermore, for any $\Lambda, \Lambda' \in \mathfrak{L}_{\text{fin}}$, such that $\Lambda \subset \Lambda'$,

$$\left\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n)) \right\rangle_{\nu_{\beta,\Lambda}} \leq \left\langle F_1(x_{\ell_1}(\tau_1)) \ldots F_n(x_{\ell_n}(\tau_n)) \right\rangle_{\nu_{\beta,\Lambda'}}. \tag{2.5.46}$$

2.5.4 Estimates for Ursell Functions

Recall that the Ursell function $U_{\ell_1,\ldots,\ell_4}(\tau_1, \ldots, \tau_4; 0)$ is defined in (2.2.21).

Theorem 2.5.14. *Let the measure (2.1.1) be ferromagnetic and of EM type. Then the inequality*

$$\begin{aligned} &\int_0^\beta \int_0^\beta U_{\ell_1,\ldots,\ell_4}(\tau, \tau, \tau_1, \tau_2; 0) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ &\qquad \leq \int_0^\beta \int_0^\beta U_{\ell_1,\ldots,\ell_4}(\tau, \tau', \tau_1, \tau_2; 0) \mathrm{d}\tau_1 \mathrm{d}\tau_2 \end{aligned} \tag{2.5.47}$$

holds for all $\ell_1, \ldots, \ell_4 \in \Lambda$ and $\tau, \tau' \in [0, \beta]$.

The proof of this theorem is based on an inequality for classical spin systems, which may have interesting applications in its own right. Thus, we derive this inequality now. Let a finite set Ξ and a matrix $M_{\kappa\kappa'} = M_{\kappa'\kappa} \geq 0$, $\kappa, \kappa' \in \Xi$, have the following properties. There exists a bijection $\rho\colon \Xi \to \Xi$, $\rho \circ \rho = \mathrm{id}$, such that Ξ admits a disjoint decomposition $\Xi_+ \cup \Xi_-$, for which $\rho\colon \Xi_+ \to \Xi_-$. Furthermore, we suppose that:

$$\begin{array}{llll} \text{(i)} & M_{\kappa\kappa'} = M_{\rho(\kappa)\rho(\kappa')}, & \text{for all } \kappa, \kappa' \in \Xi, & \text{(invariance);} \\ \text{(ii)} & M_{\kappa\kappa'} \geq M_{\kappa\rho(\kappa')}, & \text{for all } \kappa, \kappa' \in \Xi_+, & \text{(decay).} \end{array}$$

Now for $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $\kappa \in \Xi$, let

$$\nu(\mathrm{d}u_\Lambda^{(\kappa)}) = \exp\left\{ \frac{1}{2} \sum_{\ell,\ell' \in \Lambda} I_{\ell\ell'} u_\ell^{(\kappa)} u_{\ell'}^{(\kappa)} - \sum_{\ell \in \Lambda} w_\ell([u_\ell^{(\kappa)}]^2) \right\} \mathrm{d}u_\Lambda^{(\kappa)}, \tag{2.5.48}$$

where $I_{\ell\ell'} \geq 0$ and w_ℓ are as in (2.2.5). Thus, ν is a finite measure on $\mathbb{R}^{|\Lambda|}$. Set

$$\mu(\mathrm{d}u) = \frac{1}{Z} \exp\left\{ \frac{1}{2} \sum_{\ell \in \Lambda} \sum_{\kappa,\kappa' \in \Xi} M_{\kappa\kappa'} u_\ell^{(\kappa)} u_\ell^{(\kappa')} \right\} \bigotimes_{\kappa \in \Xi} \nu(\mathrm{d}u_\Lambda^{(\kappa)}), \tag{2.5.49}$$

where $u = (u_\ell^{(\kappa)})_{\ell \in \Lambda,\, \kappa \in \Xi}$ and $1/Z_\Delta$ is a normalization constant. Let $\langle \cdot \rangle_\mu$ be the expectation with respect to this measure and

$$\begin{aligned} U_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2,\kappa_3,\kappa_4) = {} & \langle u_{\ell_1}^{(\kappa_1)} u_{\ell_2}^{(\kappa_2)} u_{\ell_3}^{(\kappa_3)} u_{\ell_4}^{(\kappa_4)} \rangle_\mu \\ & - \langle u_{\ell_1}^{(\kappa_1)} u_{\ell_2}^{(\kappa_2)} \rangle_\mu \cdot \langle u_{\ell_3}^{(\kappa_3)} u_{\ell_4}^{(\kappa_4)} \rangle_\mu \\ & - \langle u_{\ell_1}^{(\kappa_1)} u_{\ell_3}^{(\kappa_3)} \rangle_\mu \cdot \langle u_{\ell_2}^{(\kappa_2)} u_{\ell_4}^{(\kappa_4)} \rangle_\mu \\ & - \langle u_{\ell_1}^{(\kappa_1)} u_{\ell_4}^{(\kappa_4)} \rangle_\nu \cdot \langle u_{\ell_2}^{(\kappa_2)} u_{\ell_3}^{(\kappa_3)} \rangle_\mu. \end{aligned} \tag{2.5.50}$$

Thereby, we set

$$D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) = \sum_{\kappa_3,\kappa_4 \in \Xi} U_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2,\kappa_3,\kappa_4). \tag{2.5.51}$$

Obviously,

$$D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) = D_{\ell_1\ell_2\ell_3\ell_4}(\rho(\kappa_1),\rho(\kappa_2)) \tag{2.5.52}$$

for all $\ell_1, \ldots, \ell_4 \in \Lambda$ and $\kappa_1, \kappa_2 \in \Xi$.

Proposition 2.5.15. *For all $\ell_1, \ldots, \ell_4 \in \Lambda$ and $\kappa_1, \kappa_2 \in \Xi_+$, it follows that*

$$D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) \leq D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\rho(\kappa_2)). \tag{2.5.53}$$

Proof. Note that $D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) \leq 0$, which follows by the Lebowitz inequality (2.2.22). Taking into account (2.5.52) one concludes that (2.5.53) is equivalent to

$$\widehat{D}_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) \leq 0, \tag{2.5.54}$$

where

$$\widehat{D}_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) \stackrel{\text{def}}{=} D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) + D_{\ell_1\ell_2\ell_3\ell_4}(\rho(\kappa_1),\rho(\kappa_2)) - D_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\rho(\kappa_2)) - D_{\ell_1\ell_2\ell_3\ell_4}(\rho(\kappa_1),\kappa_2).$$

For $\ell \in \Lambda$ and $\kappa \in \Xi_+$, we introduce

$$\theta_\ell^{(\kappa)} = \frac{1}{\sqrt{2}}\left(u_\ell^{(\kappa)} + u_\ell^{(\rho(\kappa))}\right), \quad t_\ell^{(\kappa)} = \frac{1}{\sqrt{2}}\left(u_\ell^{(\kappa)} - u_\ell^{(\rho(\kappa))}\right). \tag{2.5.55}$$

Then

$$\widehat{D}_{\ell_1\ell_2\ell_3\ell_4}(\kappa_1,\kappa_2) = 2\Big\{\langle t_{\ell_1}^{(\kappa_1)} t_{\ell_2}^{(\kappa_2)}\Theta_{\ell_3}\Theta_{\ell_4}\rangle_\mu - \langle t_{\ell_1}^{(\kappa_1)} t_{\ell_2}^{(\kappa_2)}\rangle_\mu \cdot \langle \Theta_{\ell_3}\Theta_{\ell_4}\rangle_\mu - \langle t_{\ell_1}^{(\kappa_1)}\Theta_{\ell_3}\rangle_\mu \cdot \langle t_{\ell_2}^{(\kappa_2)}\Theta_{\ell_4}\rangle_\mu - \langle t_{\ell_1}^{(\kappa_1)}\Theta_{\ell_4}\rangle_\mu \cdot \langle t_{\ell_2}^{(\kappa_2)}\Theta_{\ell_3}\rangle_\mu\Big\}, \tag{2.5.56}$$

where

$$\Theta_\ell = \sum_{\kappa\in\Xi} u_\ell^{(\kappa)} = \sqrt{2}\sum_{\kappa\in\Xi_+} \theta_\ell^{(\kappa)}.$$

In the new variables $\theta = (\theta_\ell^{(\kappa)})_{\ell\in\Lambda,\,\kappa\in\Xi_+}$, $t = (t_\ell^{(\kappa)})_{\ell\in\Lambda,\,\kappa\in\Xi_+}$, the measure (2.5.49) takes the form, cf. (2.5.24),

$$\begin{aligned}\mu(\mathrm{d}\theta,\mathrm{d}t) = \frac{1}{Z}\exp\Bigg\{&\frac{1}{2}\sum_{\ell\in\Lambda}\sum_{\kappa,\kappa'\in\Xi_+}\Big[M^+_{\kappa\kappa'}\theta_\ell^{(\kappa)}\theta_\ell^{(\kappa')} + M^-_{\kappa\kappa'}t_\ell^{(\kappa)}t_\ell^{(\kappa')}\Big] \\ &+ \frac{1}{2}\sum_{\kappa\in\Xi_+}\sum_{\ell,\ell'\in\Lambda} I_{\ell\ell'}\Big[\theta_\ell^{(\kappa)}\theta_{\ell'}^{(\kappa)} + t_\ell^{(\kappa)}t_{\ell'}^{(\kappa)}\Big] - \sum_{\kappa\in\Xi_+}\sum_{\ell\in\Lambda} Q_\ell(\theta_\ell^{(\kappa)}, t_\ell^{(\kappa)}) \\ &- \sum_{\kappa\in\Xi_+}\sum_{\ell\in\Lambda}\Big[\hat{w}_\ell\Big(\Big[\theta_\ell^{(\kappa)}\Big]^2\Big) + \hat{w}_\ell\Big(\Big[t_\ell^{(\kappa)}\Big]^2\Big)\Big]\Bigg\}\mathrm{d}\theta\mathrm{d}t,\end{aligned} \tag{2.5.57}$$

where

$$M^\pm_{\kappa\kappa'} = M_{\kappa\kappa'} \pm M_{\kappa\rho(\kappa')} \geq 0$$

and

$$\hat{w}_\ell(\vartheta) = b_\ell^{(1)}\vartheta + \sum_{s=2}^{r} 2^{1-s} b_\ell^{(s)}\vartheta^s, \tag{2.5.58}$$

cf. (2.5.17), (2.5.18), and (2.5.19),

$$Q_\ell(\gamma,\delta) = \sum_{s=1}^{r-1}\left(\sum_{p=s+1}^{r}\binom{2p}{2s}2^{1-p}b_\ell^{(p)}\gamma^{2(p-s)}\right)\delta^{2s}, \quad \gamma,\delta\in\mathbb{R}. \tag{2.5.59}$$

As the off-diagonal coefficients $M^\pm_{\kappa\kappa'}$ and $I_{\ell\ell'}$ in $\exp\{\cdot\}$ in (2.5.57) are non-negative, we have that, for all $\ell,\ell'\in\Lambda$ and $\kappa\in\Xi_+$,

$$\langle t_\ell^{(\kappa)}\Theta_{\ell'}\rangle_\mu \geq 0, \tag{2.5.60}$$

which is a kind of Ginibre inequality, see Theorem 2.2 in [295]. Furthermore, in view of the mentioned property of the non-diagonal coefficients and of the form of the functions w_ℓ, we have

$$\langle t_{\ell_1}^{(\kappa_1)} t_{\ell_2}^{(\kappa_2)} \Theta_{\ell_3} \Theta_{\ell_4} \rangle_\mu \leq \langle t_{\ell_1}^{(\kappa_1)} t_{\ell_2}^{(\kappa_2)} \rangle_\mu \cdot \langle \Theta_{\ell_3} \Theta_{\ell_4} \rangle_\mu, \tag{2.5.61}$$

which is a version of the inequality (16c) in [295]. Now we use (2.5.60) and (2.5.61) in (2.5.56) and get (2.5.54). □

Proof of Theorem 2.5.14. Obviously, the measure (2.1.13) may be written in the form of (2.5.49). Therefore, the inequality (2.5.47) is obtained from (2.5.53) with $\kappa_2 = \kappa_1$ in the limit $N \to +\infty$. □

2.6 Comments and Bibliographic Notes

Section 2.1: In the integration theory on infinite-dimensional spaces, finite-dimensional approximations are quite natural. In Euclidean quantum field theory, they are known under the name *lattice approximations*, by means of which a number of techniques of classical statistical mechanics were adapted and employed there, see [143], Chapter 9 in [135], and Chapter VIII in [273]. For Wiener integrals on Riemannian manifolds, a similar approach was developed in [40]. A general scheme of simplicial approximations of random fields on Riemannian manifolds was elaborated in [39].

In our context, main elements of the lattice approximation scheme were elaborated in [15]. The most important fact here is that the approximating measure (2.1.13) is ferromagnetic. It allowed for transferring the main correlation inequalities known for classical ferromagnetic lattice models. Note that the version presented here is not unique – one can develop another one based on the integral kernels of the operator $\exp(-\tau H_{\ell,j}^{\mathrm{har}})$, see (1.2.93).

Section 2.2: Here we present only the inequalities which are used in the subsequent parts of the book. In principle, all correlation inequalities which hold for measures like (2.1.13) can be derived. As a standard source for such inequalities we recommend the books [117], [274].

Section 2.3: There exists a very extensive literature on the logarithmic Sobolev inequality and its applications in the theory of Gibbs measures of classical lattice models, see [207], [291], [293], [294], [319], [320]. By the lattice approximation developed in this chapter, the results obtained by N. Yoshida in the latter two papers can be proven also for the corresponding local Euclidean Gibbs measures.

Section 2.4: The Lee–Yang property proved to be useful in statistical mechanics, in the theory of phase transitions in particular, see e.g., [157], [262] and Chapter 4 in [135]. This property has also been used in Euclidean quantum field theory, see Chapter IX in [273]. In the papers [185], [186], [198], the Laguerre entire functions, see [159], [192], [201], [202], were employed to describe this property. In the case $\nu = 1$, the measures possessing the Lee–Yang property are exactly those whose characteristic functions can be extended to even entire functions non-vanishing outside the real axis. A complete description of such measures is an old problem of complex analysis, which is still unsolved. As a particular item, it includes the problem of the zeros of the Riemann ζ-function. For a detailed discussion of this problem, see [159], [304]. A partial result in this domain was obtained by G. Pólya in [247], where he studied the zeros of Riemannian entire functions. He showed that the characteristic function of the measure (2.4.6) with $V(u) = a \cosh u, a > 0$ is an entire function possessing real zeros only. Here we note that Proposition 2.4.4 includes this Pólya's result. Characteristic functions are ridge in the sense that $|\phi(x + \mathrm{i}y)| \leq \phi(\mathrm{i}y)$ as the ridge for such functions is the imaginary axis, whereas for the functions (2.4.3), (2.4.4), it is the real axis. In their brilliant paper, A. A. Goldberg and I. V. Osrtovskii [139] proved that a ridge entire function ϕ, which is of finite order, is non-vanishing outside the real axis if and only if it has the following infinite-product representation, cf. (2.4.3),

$$\phi(z) = Ce^{-\kappa_0 z^2} \prod_{j=1}^{\infty} (1 - \kappa_j z^2), \quad \kappa_j \geq 0, \ j \in \mathbb{N}_0, \tag{2.6.1}$$

which means, in particular, that its order is at most 2. This establishes the form of the functions f_μ for the measures possessing the Lee–Yang property. Putting all these facts together one concludes that the Laguerre entire functions constitute a proper setting for developing the notion of the Lee–Yang property. More on this subject can be found in [192].

It turns out that an even entire function, which is ridge and non-vanishing, has the property described by Proposition 2.4.3. It follows from the fact that the numbers κ_j, $j \in \mathbb{N}_0$ have the property

$$\sum_{j=1}^{+\infty} \frac{\kappa_j(1 - \kappa_j t)}{(1 + \kappa_j t)^2} \geq 0, \quad \text{for all } t \geq 0, \tag{2.6.2}$$

which can be proven as follows. Since ϕ is ridge, for any $x, y \in \mathbb{R}$,

$$\prod_{j=1}^{+\infty} \left[(1 + \kappa_j(x^2 - y^2))^2 + 4\kappa_j x^2 y^2\right] \leq \prod_{j=1}^{+\infty} \left[1 + \kappa_j x^2\right]^2 .$$

Therefrom, we get

$$\sum_{j=1}^{+\infty} \log\left[1 + 2\kappa_j t(1 - \vartheta) + \kappa_j^2 t^2 (1 + \vartheta)^2\right] \leq 2 \sum_{j=1}^{+\infty} \log(1 + \kappa_j t),$$

which holds for all $t, \vartheta \geq 0$. Then (2.6.2) is obtained as the condition for the derivative of the left-hand side of the latter inequality at $\vartheta = 0$ to be non-positive. The convexity of the function $\log \phi(\mathrm{i}h)$ can be checked by computing its second derivative and taking into account (2.6.2).

The main result of this section, Theorem 2.4.6, will be used in Chapter 6 where we study phase transitions in the model (1.1.3), (1.1.8). For the $(\phi^4)_2$ model of Euclidean quantum field theory, a similar statement was proven by B. Simon and R. B. Griffiths, see Theorem 6 in [280]. As will be shown in Section 4.2, the $m \to +\infty$ limit of measures like (2.1.1) are the local Gibbs measures of a classical anharmonic crystal. In this limit, Theorem 2.4.6 can be formulated in a stronger version, see Theorem 4.2.3.

Section 2.5: Along with the standard inequalities proven in Section 2.2, in the study of the Euclidean Gibbs measures we use a number of new inequalities, which are proven in this section. The scalar domination in the form of Theorem 2.5.2 gives us the only tool to describe e.g., quantum effects in the vector case. A weaker version of this theorem, valid for EM potentials, was proven in [188], [189]. The zero field domination (2.5.22) was proven and used in [14], see also [15], [16]. Estimates proven in Subsection 2.5.3 play an important role in the construction of the Euclidean Gibbs states in the next chapter. The estimate (2.5.47) along with Theorem 2.4.6 will be useful in the description of the critical point of a version of the model (1.1.3), (1.1.8), which we perform in Chapter 6.

Chapter 3

Euclidean Gibbs Measures of Quantum Crystals

In Chapter 1, the local properties of the model (1.1.8) were described by means of the local Hamiltonians (1.2.5) and (1.4.51), which define the local Gibbs states $\varrho_{\beta,\Lambda}$ and $\varrho^{\text{per}}_{\beta,\Lambda}$ The main result in this direction is the representation of the local states by means of probability measures – local Euclidean Gibbs measures, see (1.4.18), (1.4.15). The key elements of this construction are the Høegh-Krohn and the multiple-time analyticity theorems which establish a one-to-one correspondence between the local states and the corresponding Euclidean Gibbs measures. In the current chapter, we employ this correspondence to construct global Gibbs measures, which describe thermodynamic properties of the whole infinite model. This construction is carried out in the framework of the DLR approach – a standard tool of classical statistical mechanics, see [129], [249], adapted here to the case of infinite-dimensional spins and infinite-range interactions.

We start by discussing the thermodynamic limit and limiting Gibbs states (Subsections 3.1.1 and 3.1.2). Then, in Subsection 3.1.3, we introduce the spaces of all configurations Ω_β, and tempered configurations Ω^{t}_β. The space Ω_β is constructed from the spaces of local configurations in a natural way. We equip Ω_β with the product topology that turns it into a Polish space. This fact is essential in view of the DLR techniques which we are going to use. The introduction of tempered configurations is connected with the necessity to control the spatial growth of $\|x_\ell\|_{L^2_\beta}$-norms in order to exclude situations where infinite forces act on a given oscillator. Such situations can occur in view of the infinite range of interactions. By definition, tempered Euclidean Gibbs measures are to be supported by Ω^{t}_β. The use of tempered configurations is a standard procedure in the DLR theory of Gibbs measures of lattice models with 'unbounded spins'. The mentioned growth control is carried out by means of a family of weights. This family $\{w_\alpha\}_{\alpha\in\mathcal{I}}$ consists of functions $w_\alpha : \mathbb{L}\times\mathbb{L}\to(0,1]$, which among other features have the property that every $-\log w_\alpha$, $\alpha\in\mathcal{I}$, is a metric on $\mathbb{L}$. We equip Ω^{t}_β with the projective limit topology, determined by the weights, so that it becomes a Polish space as well. Thereafter, in Subsection 3.1.4 we construct the local Gibbs specification corresponding to our model. The key element of this construction is the local energy functional $E_{\beta,\Lambda}(x|\xi)$, which describes also the influence of the configuration outside Λ on the configuration in Λ. By means of the local Gibbs specification we define the Euclidean Gibbs measures of our model. In Section 3.2, we describe a number of properties of the local Gibbs specification, which play a key role in constructing and studying tempered Euclidean Gibbs measures. In Section 3.3, we prove that the set of such measures $\mathcal{G}^{\text{t}}_\beta$ is non-void and weakly compact. We also prove a number of statements characterizing $\mathcal{G}^{\text{t}}_\beta$. Next, in Section 3.4, we develop an alternative approach to the construction of Euclidean Gibbs measures, based on the Radon–Nikodym characterization. In this approach, $\mathscr{G}^{\mathrm{t}}_\beta$ is defined as the set of measures obeying an integration-by-parts formula. Subsequently, we study in detail the case of local interactions (Section 3.5), where the intensities $J_{\ell\ell'}$ have finite range, and the translation-invariant case (Section 3.6), where $\mathbb{L} = \mathbb{Z}^d$, $V_\ell = V$, and $J_{\ell\ell'}$ are invariant with respect to the translations of $\mathbb{L}$. In the latter case, the set $\mathscr{G}^{\mathrm{t}}_\beta$ among others contains the so-called periodic elements. Finally, in Section 3.7, for $J_{\ell\ell'} \geq 0$ and $\nu = 1$, we introduce a stochastic order on $\mathscr{G}^{\mathrm{t}}_\beta$, with respect to which $\mathscr{G}^{\mathrm{t}}_\beta$ has a minimal element, μ_-, and a maximal element, μ_+. By means of these elements we derive a uniqueness criterion for $\mathscr{G}^{\mathrm{t}}_\beta$.

3.1 Gibbs States and Euclidean Gibbs Measures

We begin this section by introducing the framework for the description of bulk properties of our model and by analyzing the possibilities to obtain the limiting Gibbs states and to relate them to the Euclidean Gibbs measures. Since from now on we deal with the whole 'lattice' $\mathbb{L}$, it is convenient to adopt the following simplifications. If we say that something holds for all ℓ, we mean it holds for all $\ell \in \mathbb{L}$; expressions like $\sum_\ell$ mean $\sum_{\ell\in\mathbb{L}}$. We recall that $\mathfrak{L}_{\mathrm{fin}}$ (respectively, $\mathfrak{L}$) stands for the family of all non-void finite (respectively, all) subsets of $\mathbb{L}$.

3.1.1 Thermodynamic Limit

Given $\Lambda, \Lambda' \in \mathfrak{L}_{\mathrm{fin}}$, such that $\Lambda \subset \Lambda'$, by definition the map $\varkappa\colon \mathfrak{C}_\Lambda \to \mathfrak{C}_{\Lambda'}$ acts as follows:

$$\mathfrak{C}_\Lambda \ni A \mapsto \varkappa(A) = A \otimes I'' \in \mathfrak{C}_{\Lambda'},$$

where I'' is the unit element of the algebra $\mathfrak{C}_{\Lambda''}$, $\Lambda'' = \Lambda' \setminus \Lambda$. Then $\varkappa(\mathfrak{C}_\Lambda)$ is a subalgebra of $\mathfrak{C}_{\Lambda'}$, it is isomorphic to $\mathfrak{C}_\Lambda$. Usually, the distinction between $\varkappa(\mathfrak{C}_\Lambda)$ and $\mathfrak{C}_\Lambda$ is ignored and the latter algebra is considered as a subalgebra of $\mathfrak{C}_{\Lambda'}$. Thus, one can speak about the family $\{\mathfrak{C}_\Lambda \mid \Lambda \in \mathfrak{L}_{\mathrm{fin}}\}$ which by constructions has the following properties:

(a) if $\Lambda \subset \Lambda'$, then $\mathfrak{C}_\Lambda \subset \mathfrak{C}_{\Lambda'}$;

(b) if $\Lambda \cap \Lambda' = \emptyset$, then the algebras $\mathfrak{C}_\Lambda, \mathfrak{C}_{\Lambda'}$ commute.

Then the union

$$\mathfrak{C}^{\mathrm{loc}} = \bigcup_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}} \mathfrak{C}_\Lambda \tag{3.1.1}$$

is a $*$-algebra, which consists of local observables. It can be normalized by letting $\|A\|$ be the norm of A in the corresponding $\mathfrak{C}_\Lambda$. The norm completion of this algebra of local observables is a C^*-algebra, called the *quasi-local algebra* and denoted by $\mathfrak{C}$, see

Section 2.6 in [76] for more details on this topic. Clearly, to construct this algebra one can use families of local algebras, smaller than $\{\mathfrak{C}_\Lambda\}_{\Lambda \in \mathfrak{L}_{\text{fin}}}$. Let $\mathcal{L}$ be an increasing sequence of finite subsets $\Lambda \subset \mathbb{L}$, which exhausts $\mathbb{L}$. This means that $\mathcal{L}$ is ordered by inclusion and that any finite subset of $\mathbb{L}$ is contained in some element of $\mathcal{L}$. Such sequences will be called *cofinal*. Then the algebra $\mathfrak{C}$ can be obtained as the norm completion of the union, as in (3.1.1), taken over any cofinal sequence $\mathcal{L}$.

For a cofinal sequence $\mathcal{L}$ and an appropriate sequence $\{A_\Lambda\}_{\Lambda \in \mathcal{L}}$, by $\lim_{\mathcal{L}} A_\Lambda$ we mean the limit of the net $\{A_\Lambda\}_{\Lambda \in \mathcal{L}}$. By $\lim_{\Lambda \nearrow \mathbb{L}} A_\Lambda$ we mean the limit of the net $\{A_\Lambda\}_{\Lambda \in \mathfrak{L}_{\text{fin}}}$. Such limits, if they exist, are called *thermodynamic limits*.

If $\mathbb{L} = \mathbb{Z}^d$, among all cofinal sequences one can distinguish the sequence of boxes

$$\{\Lambda_L\}_{L \in \mathbb{N}}, \quad \Lambda_L = (-L, L]^d \cap \mathbb{Z}^d, \quad L \in \mathbb{N}. \tag{3.1.2}$$

Sometimes, a more precise control of the way in which the elements of a cofinal sequence $\mathcal{L}$ grow is needed. Let $\mathbb{P}(\Lambda_L)$ be the partition of the lattice $\mathbb{Z}^d$ defined by (1.4.47). Given $\Lambda \in \mathfrak{L}_{\text{fin}}$, let $N_L^+(\Lambda)$ (respectively, $N_L^-(\Lambda)$) be the number of elements of $\mathbb{P}(\Lambda_L)$ which have nonempty intersection with Λ (respectively, which are contained in Λ).

Definition 3.1.1. The sequence $\mathcal{L}$ is a van Hove sequence (i.e., it tends to $\mathbb{Z}^d$ in the sense of van Hove) if

$$\lim_{\mathcal{L}} N_L^-(\Lambda) = +\infty, \quad \lim_{\mathcal{L}} \left(N_L^-(\Lambda)/N_L^+(\Lambda)\right) = 1, \tag{3.1.3}$$

for every $L \in \mathbb{N}$.

Given $\ell \in \mathbb{Z}^d$, its *nearest neighbors* are those $\ell' \in \mathbb{Z}^d$, for which $|\ell - \ell'| = 1$. For $a \in \mathbb{R}^d$ and $\Delta \subset \mathbb{R}^d$, by the distance between a and Δ we mean

$$\text{dist}(a, \Delta) = \inf_{b \in \Delta} |a - b|.$$

Definition 3.1.2. Given $\Lambda \in \mathfrak{L}_{\text{fin}}$, the boundary $\partial\Lambda$ consists of the elements of Λ, each of which has a nearest neighbor outside Λ.

One can easily prove the following

Proposition 3.1.3. *Given a cofinal sequence $\mathcal{L}$, the following statements are equivalent:*

(i) *$\mathcal{L}$ is a van Hove sequence;*

(ii) $\inf_{\mathcal{L}} |\partial\Lambda|/|\Lambda| = 0$*;*

(iii) *there exist sequences $\{L_n^\pm\}_{n \in \mathbb{N}} \subset \mathbb{N}$, tending to infinity with $L_n^- < L_n^+$ for all $n \in \mathbb{N}$, such that, for any $n \in \mathbb{N}$, one finds $\Lambda \in \mathcal{L}$, such that the following holds:*

$$\Lambda_{L_n^-} \subset \Lambda \subset \Lambda_{L_n^+}, \quad \lim_{n \to +\infty} \left(L_n^+/L_n^-\right) = 1. \tag{3.1.4}$$

3.1.2 Gibbs States

By construction, every state on the quasi-local algebra $\mathfrak{C}$ can be restricted to any local algebra $\mathfrak{C}_\Lambda$. Since states on a C^*-algebra are norm-continuous, a state on the algebra $\mathfrak{C}$ is completely determined by its restrictions to the local algebras $\mathfrak{C}_\Lambda$. We recall that a state on $\mathfrak{C}_\Lambda$ is normal if it is defined by means of the trace operation (1.2.29) (see Definition 1.2.12 and Proposition 1.2.15); thus, the local Gibbs states (1.2.12), (1.4.52) are normal. The way of constructing Gibbs states as thermodynamic limits of the states (1.2.12) and (1.4.52) can be based on the following arguments. Since such a state is determined by its values on local algebras, we may fix some Λ_0 and consider a cofinal sequence $\mathcal{L}$, each element of which contains Λ_0 as a subset. Suppose that for every $A \in \mathfrak{C}_{\Lambda_0}$, the sequence $\{\varrho_{\beta,\Lambda}(A)\}_{\mathcal{L}}$ converges. Then the limits of all such sequences define a linear functional $\phi_{\Lambda_0} : \mathfrak{C}_{\Lambda_0} \to \mathbb{C}$, which is a *weak** limit of the sequence of states $\{\varrho_{\beta,\Lambda}\}_{\mathcal{L}}$. We recall that a net of states $\{\omega_\alpha\}$ on a C^*-algebra $\mathfrak{A}$ converges uniformly to a state ω if

$$\sup_{A\in\mathfrak{A}:\ \|A\|\leq 1} |\omega(A) - \omega_\alpha(A)| \to 0.$$

Since in general a weak* limit need not be continuous, we cannot claim that ϕ_{Λ_0} is a state. In this situation we can use the following facts, see Theorem 2.6.16, page 133 in [76],

Proposition 3.1.4. *The uniform topology and the weak* topology restricted to the set of all normal states on any $\mathfrak{C}_\Lambda$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ coincide. The set of all normal states is closed in both of these topologies.*

Thus, the functional ϕ_{Λ_0} is a normal state. By construction the system of all such states is consistent, which means that if $\Lambda \subset \Lambda'$, then the restriction of $\phi_{\Lambda'}$ to $\mathfrak{C}_\Lambda$ coincides with ϕ_Λ. Therefore, the family $\{\phi_\Lambda\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$ defines a state on the quasi-local algebra $\mathfrak{C}$, the restrictions of which to any $\mathfrak{C}_\Lambda$ coincides with the corresponding ϕ_Λ. This state is *locally normal*. It can be called *limiting Gibbs state* of our model. In principle, this state may depend on the sequence $\mathcal{L}$ along which the limit was taken. In what follows, the whole variety of the limiting Gibbs states can be obtained by taking limits along various cofinal sequences of the local states (1.2.12) and (1.4.52). If all local states are rotation-invariant, the limiting Gibbs states should also be rotation-invariant. Thus, we have no possibility to describe phase transitions by making use of such limiting Gibbs states only.

As was mentioned above, the second way of constructing Gibbs state relies on the KMS conditions. To formulate them one needs the group of time automorphisms which determines the dynamics of the whole infinite system. These automorphisms should be obtained as limits of local time automorphisms (1.2.13), (1.2.14), that is, as

$$\mathfrak{a}^t(A) = \lim_{\mathcal{L}} \mathfrak{a}^t_{\Lambda'}(A),$$

which should be obtained for all $t \in \mathbb{R}$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $A \in \mathfrak{C}_\Lambda$. If these limits exist, one can try to construct from them a strongly continuous group defined on the algebra of quasi-local observables $\mathfrak{C}$. However, for our model there is no way to get such limits. In this case, it is quite natural to use the representation of the local Gibbs states by means of the Euclidean Gibbs measures elaborated in Section 1.4. In this approach, equilibrium states are constructed as probability measures with the help of equilibrium conditions, analogous to the KMS ones, which are known as the Dobrushin–Lanford–Ruelle conditions. Realization of this idea will be presented in the subsequent sections of this chapter. Additional comments on the problem of constructing Gibbs states for our model are given in Section 3.8.

3.1.3 Configuration Spaces

We recall that $\mathfrak{L}$ (respectively, $\mathfrak{L}_{\mathrm{fin}}$) stands for the family of all subsets (respectively, of all finite subsets) of $\mathbb{L}$ and C_β is the Banach space of continuous periodic functions from $[0,\beta]$ to $\mathbb{R}^\nu$. For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, in (1.3.156) we introduced the Banach space $C_{\beta,\Lambda}$ as the space of 'vectors' $x_\Lambda = (x_\ell)_{\ell\in\Lambda}$. Now we need such 'vectors' with infinite Λ. Thus, we set

$$\Omega_{\beta,\Lambda} = \{x_\Lambda = (x_\ell)_{\ell\in\Lambda} \mid x_\ell \in C_\beta\}, \quad \Omega_\beta = \{x = (x_\ell)_{\ell\in\mathbb{L}} \mid x_\ell \in C_\beta\}. \tag{3.1.5}$$

In order to relate the spaces $\Omega_{\beta,\Lambda}$ with different Λ to each other we introduce projections. For $\Lambda \subset \Lambda'$, we set $\pi_{\Lambda,\Lambda'}(x_{\Lambda'}) = x_\Lambda$, where the components of both x_Λ, $x_{\Lambda'}$ indexed by the same $\ell \in \Lambda$ coincide. We equip each $\Omega_{\beta,\Lambda'}$ with the component-wise real linear operations and with the product topology, i.e., the weakest topology in which all the projections

$$\Omega_{\beta,\Lambda'} \ni x_{\Lambda'} \mapsto \pi_{\Lambda,\Lambda'}(x) = x_\Lambda \in C_{\beta,\Lambda}, \quad \Lambda \in \mathfrak{L}_{\mathrm{fin}}, \tag{3.1.6}$$

are continuous. Thereafter, we can also define Borel σ-algebras $\mathcal{B}_{\beta,\Lambda}$ and $\mathcal{B}_\beta = \mathcal{B}_{\beta,\mathbb{L}}$. The elements of Ω_β are called *configurations*. In the case of $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, the space $\Omega_{\beta,\Lambda}$ is nothing but the Banach spaces $C_{\beta,\Lambda}$, defined in (1.3.156). For such spaces, we use both notations; however, for infinite Λ, we use the notation $\Omega_{\beta,\Lambda}$ only.

As above, for $\Lambda = \Lambda' \cup \Lambda''$, such that $\Lambda' \cap \Lambda'' = \emptyset$, we write $x_\Lambda = x_{\Lambda'} \times x_{\Lambda''}$, meaning that $x_{\Lambda'} = \pi_{\Lambda',\Lambda}(x_\Lambda)$ and $x_{\Lambda''} = \pi_{\Lambda'',\Lambda}(x_\Lambda)$. Given $\Lambda' \subset \Lambda \in \mathfrak{L}$, we define the map

$$\Omega_{\beta,\Lambda'} \ni x_{\Lambda'} \mapsto x_{\Lambda'} \times 0_{\Lambda''} \in \Omega_{\beta,\Lambda}, \tag{3.1.7}$$

where $\Lambda'' = \Lambda \setminus \Lambda'$ and $0_{\Lambda''}$ is the zero element of $\Omega_{\beta,\Lambda''}$. It is a linear continuous embedding. As usual, we identify $\Omega_{\beta,\Lambda'}$ with its image and consider it as a subspace of the space $\Omega_{\beta,\Lambda}$. Correspondingly, $\mathcal{B}_{\beta,\Lambda'}$ is considered as a subalgebra of $\mathcal{B}_{\beta,\Lambda}$.

The product topology on Ω_β can also be defined in the following way. For $x \in \Omega_\beta$ and $\ell \in \mathbb{L}$, we set

$$p_\ell(x) = \sup_{\tau\in[0,\beta]} |x_\ell(\tau)|. \tag{3.1.8}$$

It is a semi-norm on Ω_β and the system $\{p_\ell\}_{\ell\in\mathbb{L}}$ separates the points of Ω_β. Thus, this system $\{p_\ell\}_{\ell\in\mathbb{L}}$ defines a locally convex topology on Ω_β, which is equivalent to the product topology introduced above. Since $\{p_\ell\}_{\ell\in\mathbb{L}}$ is countable, the topological space Ω_β is metrizable, and the corresponding metric can be

$$d(x,x') = \sum_\ell 2^{-|\ell|} \cdot \frac{p_\ell(x-x')}{1+p_\ell(x-x')}. \tag{3.1.9}$$

The configuration space Ω_β equipped with this metric is complete and separable; hence, the set of probability measures $\mathcal{P}(\Omega_\beta)$ possesses all the properties described in Subsection 1.3.4. Correspondingly, the weak topology on $\mathcal{P}(\Omega_\beta)$ is the topology defined by the neighborhoods (1.3.65) with $f \in C_{\mathrm{b}}(\Omega_\beta)$.

As was mentioned above, for $\Lambda' \subset \Lambda$ we consider the Borel σ-algebra $\mathcal{B}_{\beta,\Lambda'}$ as a subalgebra of $\mathcal{B}_{\beta,\Lambda}$. Thus, we set

$$\mathcal{B}_\beta^{\mathrm{loc}} = \bigcup_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}} \mathcal{B}_{\beta,\Lambda}. \tag{3.1.10}$$

Obviously, $\mathcal{B}_\beta^{\mathrm{loc}}$ can be obtained by taking the union over any cofinal sequence $\mathcal{L}$.

Definition 3.1.5. A function $f : \Omega_\beta \to \mathbb{R}$ is called local if it is measurable with respect to $\mathcal{B}_\beta^{\mathrm{loc}}$. A bounded function $f : \Omega_\beta \to \mathbb{R}$ is called quasi-local if it is a limit, uniform on Ω_β, of a sequence of bounded local functions. By $C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta)$ (respectively, $C_{\mathrm{b}}^{\mathrm{qloc}}(\Omega_\beta)$) we denote the set of all local (respectively, all quasi-local) continuous bounded functions $f : \Omega_\beta \to \mathbb{R}$.

Proposition 3.1.6. *The σ-algebra $\mathcal{B}_\beta^{\mathrm{loc}}$, as well as the set of all local continuous bounded functions $C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta)$, are measure determining. That is, if for any $\mu,\nu \in \mathcal{P}(\Omega_\beta)$, one has*

$$\forall B \in \mathcal{B}_\beta^{\mathrm{loc}}: \quad \mu(B) = \nu(B), \quad \textit{or} \quad \forall f \in C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta): \quad \int_{\Omega_\beta} f\,\mathrm{d}\mu = \int_{\Omega_\beta} f\,\mathrm{d}\nu,$$

then $\mu = \nu$.

Proof. We recall that the indicator function $\mathbb{I}_B$ was defined by (1.1.39). Set

$$\mathcal{I}^{\mathrm{loc}} = \{\mathbb{I}_B \mid B \in \mathcal{B}_\beta^{\mathrm{loc}}\}. \tag{3.1.11}$$

Clearly, for any $\mu \in \mathcal{P}(\Omega_\beta)$,

$$\mu(B) = \int_{\Omega_\beta} \mathbb{I}_B \mathrm{d}\mu.$$

Then the proof follows by the fact that both sets $\mathcal{I}^{\mathrm{loc}}$ and $C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta)$ satisfy the conditions of Theorem 1.3.26. □

As we show in the next statement, the set $C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta)$ has one more useful property.

Proposition 3.1.7. *A sequence* $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega_\beta)$ *weakly converges to a measure* $\mu \in \mathcal{P}(\Omega_\beta)$ *if and only if*

$$\forall f \in C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta): \quad \lim_{n\to+\infty} \int_{\Omega_\beta} f \mathrm{d}\mu_n = \int_{\Omega_\beta} f \mathrm{d}\mu. \tag{3.1.12}$$

Proof. By (1.3.66) the above holds if $\mu_n \Rightarrow \mu$. Let us show that any $f \in C_{\mathrm{b}}(\Omega_\beta)$, which is uniformly continuous with respect to the metric (3.1.9), can uniformly be approximated by functions from $C_{\mathrm{b}}^{\mathrm{loc}}(\Omega_\beta)$. For such a function f and a finite subset Λ, we set $f_\Lambda(x) = f(x_\Lambda \times 0_{\Lambda^c})$. Since f is uniformly continuous, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall x, x' \in \Omega_\beta: \quad d(x, x') < \delta \text{ implies } |f(x) - f(x')| < \varepsilon. \tag{3.1.13}$$

Then for any $\varepsilon > 0$, one finds $\Lambda_\varepsilon \in \mathcal{D}$, such that

$$\sum_{\ell\in\Lambda_\varepsilon^c} 2^{-|\ell|} < \delta;$$

hence, by (3.1.9) for all $\Lambda \supset \Lambda_\varepsilon$,

$$\|f - f_\Lambda\|_{C_{\mathrm{b}}(\Omega_\beta)} \stackrel{\mathrm{def}}{=} \sup_{x\in\Omega_\beta} |f(x) - f_\Lambda(x)| < \varepsilon.$$

Since the measures μ_n, as linear functionals on $C_{\mathrm{b}}(\Omega_\beta)$, are continuous in the above norm, the weak convergence of the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ follows from (3.1.12) by Proposition 1.3.32. □

In the construction of Euclidean Gibbs measures, we will deal with functions of the following kind, cf. (1.4.8),

$$(x_\Lambda, \xi) \mapsto \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(x_\ell, \xi_{\ell'})_{L^2_\beta}, \quad \Lambda \in \mathfrak{L}_{\mathrm{fin}}. \tag{3.1.14}$$

Clearly, such a function can be defined for all $\xi \in \Omega_\beta$ only if the interaction has finite range. Otherwise, one should restrict ξ to a subset of Ω_β, naturally defined by the condition

$$\forall \ell \in \mathbb{L}: \quad \sum_{\ell'} |J_{\ell\ell'}| \cdot |(x_\ell, \xi_{\ell'})_{L^2_\beta}| < \infty, \tag{3.1.15}$$

which can be rewritten in terms of growth restrictions imposed on $\{|\xi_\ell|_{L^2_\beta}\}_{\ell\in\mathbb{L}}$. Of course, these restrictions should be defined by the decay of $J_{\ell\ell'}$. Configurations obeying such restrictions are called tempered. To introduce them we use weights.

Definition 3.1.8. Weights are the maps $w_\alpha : \mathbb{L} \times \mathbb{L} \to (0, +\infty)$, indexed by

$$\alpha \in \mathcal{I} = (\underline{\alpha}, \overline{\alpha}), \quad 0 \le \underline{\alpha} < \overline{\alpha} \le +\infty, \tag{3.1.16}$$

which satisfy the following conditions:

(a) for any $\alpha \in \mathcal{I}$ and ℓ, $w_\alpha(\ell, \ell) = 1$;

(b) for any $\alpha \in \mathcal{I}$ and ℓ_1, ℓ_2, ℓ_3,

$$w_\alpha(\ell_1, \ell_2) \cdot w_\alpha(\ell_2, \ell_3) \leq w_\alpha(\ell_1, \ell_3); \tag{3.1.17}$$

(c) for any $\alpha, \alpha' \in \mathcal{I}$, such that $\alpha < \alpha'$, and arbitrary ℓ, ℓ',

$$w_{\alpha'}(\ell, \ell') \leq w_\alpha(\ell, \ell'), \qquad \lim_{|\ell - \ell'| \to +\infty} w_{\alpha'}(\ell, \ell')/w_\alpha(\ell, \ell') = 0. \tag{3.1.18}$$

The concrete choice of w_α depends on the decay of $J_{\ell\ell'}$, which will be subject to the following

Assumption 3.1.9. For all $\alpha \in \mathcal{I}$,

$$\sup_\ell \sum_{\ell'} \log(1 + |\ell - \ell'|) \cdot w_\alpha(\ell, \ell') < \infty, \tag{3.1.19}$$

$$\hat{J}_\alpha \stackrel{\text{def}}{=} \sup_\ell \sum_{\ell'} |J_{\ell\ell'}| \cdot \left[w_\alpha(\ell, \ell')\right]^{-1} < \infty. \tag{3.1.20}$$

Given $\delta > 0$, which is a parameter of the theory, there exists $\alpha \in \mathcal{I}$, such that

$$\hat{J}_\alpha - \hat{J}_0 < \delta. \tag{3.1.21}$$

One observes that the conditions (3.1.19) and (3.1.20) are competitive. Let us present now the basic examples which will be used in the sequel.

Suppose that

$$\sup_\ell \sum_{\ell'} |J_{\ell\ell'}| \cdot \exp\left(\alpha|\ell - \ell'|\right) < \infty, \qquad \text{for a certain } \alpha > 0. \tag{3.1.22}$$

Then by $\bar{\alpha}$ we denote the supremum of α obeying (3.1.22) and set

$$w_\alpha(\ell, \ell') = \exp\left(-\alpha|\ell - \ell'|\right), \quad \alpha \in \mathcal{I} = (0, \bar{\alpha}). \tag{3.1.23}$$

In this case, $\lim_{\alpha \to 0} \hat{J}_\alpha = \hat{J}_0$; hence, (3.1.21) is satisfied for any $\delta > 0$.

In the second case, instead of (3.1.22) we suppose that

$$\sup_\ell \sum_{\ell'} |J_{\ell\ell'}| \cdot \left(1 + |\ell - \ell'|\right)^{\alpha d} < \infty, \tag{3.1.24}$$

for a certain $\alpha > 1$. Then $\bar{\alpha}$ is set to be the supremum of α obeying (3.1.24) and

$$w_\alpha(\ell, \ell') = \left(1 + \varepsilon|\ell - \ell'|\right)^{-\alpha d}, \quad \mathcal{I} = (1, \bar{\alpha}), \tag{3.1.25}$$

where the parameter $\varepsilon > 0$ is chosen for (3.1.21) to be satisfied. If

$$|J_{\ell\ell'}| \leq J(1 + |\ell - \ell'|)^{-d-\gamma}, \quad \gamma > 0,$$

then $\bar{\alpha} = \gamma/d$, which implies $\gamma > d$.

Given $\ell_0 \in \mathbb{L}$, $s = (s_\ell)_{\ell\in\mathbb{L}} \in \mathbb{R}^{\mathbb{L}}$, and $\alpha \in \mathcal{I}$, we set

$$|s|_{l^1(\ell_0,\alpha)} = \sum_{\ell} |s_\ell| w_\alpha(\ell_0,\ell), \quad |s|_{l^\infty(\ell_0,\alpha)} = \sup_{\ell} \{|s_\ell| w_\alpha(\ell_0,\ell)\},$$

and introduce the Banach spaces

$$l^p(\ell_0,\alpha) = \left\{ s \in \mathbb{R}^{\mathbb{L}} \ \big| \ |s|_{l^p(\ell_0,\alpha)} < \infty \right\}, \quad p = 1, +\infty. \tag{3.1.26}$$

By means of (3.1.17) one can prove that the topologies defined on $l^p(\ell_0,\alpha)$, $p = 1, +\infty$, by the corresponding norms with different ℓ_0 are equivalent.

Remark 3.1.10. By (3.1.18) for $\alpha < \alpha'$, the embedding $l^1(\ell_0,\alpha) \hookrightarrow l^1(\ell_0,\alpha')$ is compact. By (3.1.20) for every $\alpha \in \mathcal{I}$, the operator $s \mapsto Js$, where

$$(Js)_\ell = \sum_{\ell'} J_{\ell\ell'} s_{\ell'},$$

is bounded in both $l^p(\ell_0,\alpha)$, $p = 1, +\infty$. Its norm does not exceed $\hat{J}_\alpha$.

We recall that L^2_β is the real Hilbert space $L^2\left([0,\beta] \to \mathbb{R}^\nu\right)$. For $\alpha \in \mathcal{I}$ and $\ell_0 \in \mathbb{L}$, we define

$$\|x\|_{\ell_0,\alpha} = \left[\sum_{\ell} \|x_\ell\|^2_{L^2_\beta} w_\alpha(\ell_0,\ell) \right]^{1/2}, \tag{3.1.27}$$

and

$$\Omega_\beta^{\ell_0,\alpha} = \left\{ x \in \Omega_\beta \ \big| \|x\|_{\ell_0,\alpha} < \infty \right\}. \tag{3.1.28}$$

Then we endow $\Omega_\beta^{\ell_0,\alpha}$ with the metric

$$\rho_{\ell_0,\alpha}(x,y) = \|x-y\|_{\ell_0,\alpha} + \sum_{\ell} 2^{-|\ell_0-\ell|} \cdot \frac{\|x_\ell - y_\ell\|_{C_\beta}}{1 + \|x_\ell - y_\ell\|_{C_\beta}}, \tag{3.1.29}$$

which turns it into a Polish space. The set of tempered configurations is defined to be

$$\Omega_\beta^{\mathrm{t}} = \bigcap_{\alpha\in\mathcal{I}} \Omega_\beta^{\ell_0,\alpha}. \tag{3.1.30}$$

For any $\ell_0, \ell_1 \in \mathbb{L}$ and $\alpha \in \mathcal{I}$, by the triangle inequality (3.1.17) one obtains

$$[w_\alpha(\ell_0,\ell_1)]^{1/2} \|x\|_{\ell_1,\alpha} \le \|x\|_{\ell_0,\alpha} \le [w_\alpha(\ell_0,\ell_1)]^{-1/2} \|x\|_{\ell_1,\alpha}; \tag{3.1.31}$$

hence, all $\Omega_\beta^{\ell,\alpha}$, $\ell \in \mathbb{L}$, are homeomorphic to each other. Therefore, the definition (3.1.30) is independent of the particular choice of ℓ_0, but it certainly depends on the choice of the weights. Equipped with the projective limit topology, $\Omega_\beta^{\mathrm{t}}$ becomes a Polish space as well. For any $\alpha \in \mathcal{I}$ and ℓ_0, we have continuous dense embeddings $\Omega_\beta^{\mathrm{t}} \hookrightarrow \Omega_\beta^{\ell_0,\alpha} \hookrightarrow \Omega_\beta$. Then by the Kuratowski theorem (Proposition 1.3.18), it follows that $\Omega_\beta^{\ell_0,\alpha}, \Omega_\beta^{\mathrm{t}} \in \mathcal{B}_\beta$ and the Borel σ-algebras of all these Polish spaces coincide with the ones induced on them by $\mathcal{B}_\beta$. Thus, we set

$$\mathcal{B}(\Omega_\beta^{\mathrm{t}}) = \{B \cap \Omega_\beta^{\mathrm{t}} \mid B \in \mathcal{B}_\beta\}. \tag{3.1.32}$$

Definition 3.1.11. A measure $\mu \in \mathscr{P}(\Omega_\beta)$ is called tempered, if $\mu(\Omega^{\mathrm{t}}_\beta) = 1$.

Now let us turn to the model we consider.

3.1.4 Tempered Euclidean Gibbs Measures

Given $\xi \in \Omega^{\mathrm{t}}_\beta$ and $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, we set, cf. (3.1.14) and (3.1.15),

$$E_{\beta,\Lambda}(x|\xi) = E_{\beta,\Lambda}(x_\Lambda) - \sum_{\ell\in\Lambda,\ell'\in\Lambda^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}, \tag{3.1.33}$$

where $E_{\beta,\Lambda}(x_\Lambda)$ was defined in (1.4.8). Note that $x = x_\Lambda \times x_{\Lambda^c}$; hence,

$$E_{\beta,\Lambda}(x|\xi) = E_{\beta,\Lambda}(x_\Lambda \times 0_{\Lambda^c}|0_\Lambda \times \xi_{\Lambda^c}). \tag{3.1.34}$$

We call $E_{\beta,\Lambda}(x|\xi)$ the *energy functional*, corresponding to the boundary condition ξ. Among other properties, it has the one of describing the connection of the configuration in Λ, i.e., x_Λ, with the one outside Λ, i.e., ξ_{Λ^c}.

Lemma 3.1.12. *For every $\ell_0 \in \mathbb{L}$, $\alpha \in \mathcal{I}$, and $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, the map*

$$\Omega^{\ell_0,\alpha}_\beta \times \Omega^{\ell_0,\alpha}_\beta \ni (x,\xi) \mapsto E_{\beta,\Lambda}(x|\xi),$$

is continuous. Furthermore, for every ball

$$B_{\ell_0,\alpha}(R) = \{x \in \Omega^{\ell_0,\alpha}_\beta \mid \rho_{\ell_0,\alpha}(0,x) < R\}, \quad R > 0,$$

it follows that

$$\begin{aligned} \inf_{x\in\Omega_\beta,\ \xi\in B_{\ell_0,\alpha}(R)} E_{\beta,\Lambda}(x|\xi) &> -\infty, \\ \sup_{x,\xi\in B_{\ell_0,\alpha}(R)} \left|E_{\beta,\Lambda}(x|\xi)\right| &< +\infty. \end{aligned} \tag{3.1.35}$$

Proof. As the functions $V_\ell\colon \mathbb{R}^\nu \to \mathbb{R}$ are continuous, the map $(x,\xi) \mapsto E_{\beta,\Lambda}(x_\Lambda)$ is continuous and locally bounded. Furthermore,

$$\begin{aligned} \left|\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\right| &\le \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} |J_{\ell\ell'}|\cdot\|x_\ell\|_{L^2_\beta}\cdot\|\xi_{\ell'}\|_{L^2_\beta} \\ &= \sum_{\ell\in\Lambda} \|x_\ell\|_{L^2_\beta}[w_\alpha(\ell_0,\ell)]^{-1/2} \\ &\qquad \times \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}|\left[w_\alpha(\ell_0,\ell)/w_\alpha(0,\ell')\right]^{1/2} \\ &\qquad \cdot \|\xi_{\ell'}\|_{L^2_\beta}[w_\alpha(\ell_0,\ell')]^{1/2} \end{aligned} \tag{3.1.36}$$

$$\leq \sum_{\ell\in\Lambda} \|x_\ell\|_{L^2_\beta}[w_\alpha(\ell_0,\ell)]^{-1/2}$$
$$\times \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}|\cdot[w_\alpha(\ell,\ell')]^{-1/2}$$
$$\cdot \|\xi_{\ell'}\|_{L^2_\beta}[w_\alpha(\ell_0,\ell')]^{1/2}$$
$$\leq \hat{J}_\alpha\|x\|_{\ell_0,\alpha}\|\xi\|_{\ell_0,\alpha}\sum_{\ell\in\Lambda}[w_\alpha(\ell_0,\ell)]^{-1},$$

where we used the triangle inequality (3.1.17). Similarly, by means of the Minkowski inequality and (3.1.17) we get

$$\begin{aligned}\left|\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\right| &\leq \frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}|J_{\ell\ell'}|\cdot\|x_\ell\|^2_{L^2_\beta} \\ &\quad + \frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}|J_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta} \\ &\leq \frac{1}{2}\hat{J}_0\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta} + \frac{1}{2}\hat{J}_\alpha\|\xi\|^2_{\ell_0,\alpha}\sum_{\ell\in\Lambda}\frac{1}{w_\alpha(\ell_0,\ell)}.\end{aligned} \tag{3.1.37}$$

The estimate (3.1.36) and Proposition 1.4.3 yield the stated continuity and the upper bound in (3.1.35). To prove the lower bound we employ Jensen's inequality and the super-quadratic growth of V_ℓ assumed in (1.1.10). This yields

$$\begin{aligned}E_{\beta,\Lambda}(x|\xi) &\geq -c_V\beta|\Lambda| + b_V\beta^{1-r}\sum_{\ell\in\Lambda}\|x_\ell\|^{2r}_{L^2_\beta} \\ &\quad - \frac{1}{2}\sum_{\ell,\ell'\in\Lambda}J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} - \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}.\end{aligned}$$

Therefrom, for a fixed $\alpha\in\mathcal{I}$, by (3.1.37) for arbitrary $x\in\Omega_\beta$ and $\xi\in B_{\ell_0,\alpha}(R)$, we get

$$\begin{aligned}E_{\beta,\Lambda}(x|\xi) &\geq -c_V\beta|\Lambda| + b_V\beta^{1-r}\sum_{\ell\in\Lambda}\|x_\ell\|^{2r}_{L^2_\beta} \\ &\quad - \frac{1}{2}\hat{J}_\alpha\|\xi\|^2_{\ell_0,\alpha}\sum_{\ell\in\Lambda}[w_\alpha(\ell_0,\ell)]^{-1} - \hat{J}_0\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta} \\ &\geq -\beta|\Lambda|\left[c_V + \frac{r-1}{b_V^{1/(r-1)}}\left(\frac{\hat{J}_0}{r}\right)^{r/(r-1)}\right] \\ &\quad - \frac{1}{2}\hat{J}_\alpha R^2\sum_{\ell\in\Lambda}[w_\alpha(\ell_0,\ell)]^{-1},\end{aligned} \tag{3.1.38}$$

cf. (1.4.11). □

Now for $\Lambda \in \mathfrak{L}_{\rm fin}$ and $\xi \in \Omega^{\rm t}_\beta$, we introduce

$$N_{\beta,\Lambda}(\xi) = \int_{C_{\beta,\Lambda}} \exp\left[-E_{\beta,\Lambda}(x_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{3.1.39}$$

which is the relative partition function corresponding to the state in Λ, cf. (1.4.13). An immediate corollary of the estimates (1.3.124) and (3.1.38) is the following

Proposition 3.1.13. *For every $\Lambda \in \mathfrak{L}_{\rm fin}$, the function $\Omega^{\rm t}_\beta \ni \xi \mapsto N_{\beta,\Lambda}(\xi) \in (0, +\infty)$ is continuous. Moreover, for any $R > 0$,*

$$\inf_{\xi \in B_{\ell_0,\alpha}(R)} N_{\beta,\Lambda}(\xi) > 0, \qquad \sup_{\xi \in B_{\ell_0,\alpha}(R)} N_{\beta,\Lambda}(\xi) < \infty. \tag{3.1.40}$$

Now by means of the energy functional $E_{\beta,\Lambda}(x|\xi)$ we introduce the local Euclidean Gibbs measure which corresponds to the boundary condition defined by the configuration ξ. For the sake of brevity, we will call it the measure with the boundary condition ξ. Thus, for $\xi \in \Omega^{\rm t}_\beta$, we set, see (3.1.34),

$$\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi) = \frac{1}{N_{\beta,\Lambda}(\xi)} \exp\left[-E_{\beta,\Lambda}(x_\Lambda \times 0_{\Lambda^c}|\xi)\right] \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{3.1.41}$$

and $\nu_{\beta,\Lambda}(\cdot|\xi) = 0$ if $\xi \in \Omega_\beta \setminus \Omega^{\rm t}_\beta$. By Lemma 3.1.12 we get the following property of this measure.

Remark 3.1.14. For every fixed $\xi \in \Omega^{\rm t}_\beta$, the Radon–Nikodym derivative of $\nu_{\beta,\Lambda}(\cdot|\xi)$ with respect to the measure $\chi_{\beta,\Lambda}$ is a bounded continuous function on $C_{\beta,\Lambda}$.

For the measure (3.1.41), we define the Matsubara functions, cf. (1.4.20),

$$\Gamma^{\beta,\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n|\xi) = \int_{C_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1))\dots F_n(x_\Lambda(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi), \tag{3.1.42}$$

where $\tau_1,\dots,\tau_n \in [0,\beta]$ and $F_1,\dots,F_n \in \mathfrak{M}_\Lambda$. Recall that the family of functions $\mathfrak{P}^{(\nu)}_\Lambda$ was introduced in Definition 1.4.7. As a corollary of Theorem 1.4.9, we have the following property of the functions (3.1.42).

Proposition 3.1.15. *For any $\xi \in \Omega^{\rm t}_\beta$, the Matsubara functions (3.1.42) are continuous in $(\tau_1,\dots,\tau_n) \in [0,\beta]^n$. They can also be defined for multiplication operators by functions from the family $\mathfrak{P}^{(\nu)}_\Lambda$. These extensions are continuous in $(\tau_1,\dots,\tau_n) \in [0,\beta]^n$ as well.*

Proof. For $\xi \in \Omega^{\rm t}_\beta$, we set

$$y_\ell = \sum_{\ell' \in \Lambda^c} J_{\ell\ell'}\xi_{\ell'}. \tag{3.1.43}$$

As in (3.1.36), one proves that for every ℓ_0 and $\alpha \in \mathcal{I}$, the map $x \mapsto y_\ell$ defined by (3.1.43) continuously maps $\Omega_{\ell_0,\alpha}$ into L^2_β. Hence, $y_\Lambda = (y_\ell)_{\ell\in\Lambda}$ is an element of $L^2_{\beta,\Lambda}$ and the stated continuity follows by Theorem 1.4.9. □

Let $(\Omega, \mathcal{F})$ be a measurable space. A map $\pi : \mathcal{F} \times \Omega \to \mathbb{R}$ is called *a measure kernel* (respectively, *a probability kernel*) if: (a) for any $\omega \in \Omega$, the map $\pi(\cdot|\omega)$ is a measure (respectively, a probability measure); (b) for any $B \in \mathcal{F}$, the map $\pi(B|\cdot)$ is measurable. A typical example of a probability kernel is the indicator function (1.1.39). On the measure space $(\Omega_\beta, \mathcal{B}_\beta)$ we define the measure kernels

$$\pi_{\beta,\Lambda}(B|\xi) = \int_{C_{\beta,\Lambda}} \mathbb{I}_B(x_\Lambda \times \xi_{\Lambda^c}) \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi), \tag{3.1.44}$$

where $\xi \in \Omega_\beta$ and $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$. By the definition of the measure $\nu_{\beta,\Lambda}(\cdot|\xi)$ for all $B \in \mathcal{B}_\beta$, it follows that

$$\pi_{\beta,\Lambda}(B|\xi) = 0 \quad \text{if } \xi \in \Omega_\beta \setminus \Omega^{\mathrm{t}}_\beta; \tag{3.1.45}$$

hence, $\pi_{\beta,\Lambda}$ is a measure kernel on $(\Omega_\beta, \mathcal{B}_\beta)$. By construction, the kernels $\pi_{\beta,\Lambda}$ with different $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ satisfy the following consistency condition. For any $\Lambda' \subset \Lambda \in \mathfrak{L}_{\mathrm{fin}}$,

$$\int_{\Omega_\beta} \pi_{\beta,\Lambda'}(B|x) \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = \pi_{\beta,\Lambda}(B|\xi), \tag{3.1.46}$$

which holds for every $B \in \mathcal{B}_\beta$ and $\xi \in \Omega^{\mathrm{t}}_\beta$. Furthermore, by (3.1.38) it follows that for any $\xi \in \Omega_\beta$, $\sigma \in (0, 1/2)$, $\ell_0 \in \mathbb{L}$, $\varkappa > 0$, and $\lambda < \varsigma/b_\sigma$,

$$\int_{\Omega_\beta} \exp\left\{ \sum_{\ell \in \Lambda} \left(\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta} \right) \right\} \pi_\Lambda(\mathrm{d}x|\xi) < \infty, \tag{3.1.47}$$

where ς and b_σ are the same as in (1.3.124).

Definition 3.1.16. The set $\Pi_\beta = \{\pi_{\beta,\Lambda}\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ of the kernels (3.1.44) is called the local Gibbs specification for the model (1.1.8).

By $C_{\mathrm{b}}(\Omega^{\ell_0,\alpha}_\beta)$ (respectively, by $C_{\mathrm{b}}(\Omega^{\mathrm{t}}_\beta)$) we denote the Banach space of all bounded continuous functions $f : \Omega^{\ell_0,\alpha}_\beta \to \mathbb{R}$ (respectively, $f : \Omega^{\mathrm{t}}_\beta \to \mathbb{R}$), equipped with the supremum norm. In view of (3.1.31), the sets $C_{\mathrm{b}}(\Omega^{\ell_0,\alpha}_\beta)$ are the same for all $\ell_0 \in \mathbb{L}$. For every $\alpha \in \mathcal{I}$, one has a natural continuous embedding $C_{\mathrm{b}}(\Omega^{\ell_0,\alpha}_\beta) \hookrightarrow C_{\mathrm{b}}(\Omega^{\mathrm{t}}_\beta)$.

Lemma 3.1.17 (Feller Property). *The specification Π_β is such that for every $\alpha \in \mathcal{I}$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, and for any $f \in C_{\mathrm{b}}(\Omega_{\ell_0,\alpha})$, the function*

$$\Omega^{\ell_0,\alpha}_\beta \ni \xi \mapsto \pi_{\beta,\Lambda}(f|\xi) \stackrel{\mathrm{def}}{=} \int_{\Omega_{\beta,\Lambda}} f(x_\Lambda \times \xi_{\Lambda^c}) \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi) \tag{3.1.48}$$

belongs to $C_{\mathrm{b}}(\Omega^{\ell_0,\alpha}_\beta)$. The linear operator $f \mapsto \pi_{\beta,\Lambda}(f|\cdot)$ is a contraction on $C_{\mathrm{b}}(\Omega^{\ell_0,\alpha}_\beta)$.

Proof. By Lemma 3.1.12 and Proposition 3.1.13 the integrand

$$G^f_\Lambda(x_\Lambda|\xi) \stackrel{\text{def}}{=} f(x_\Lambda \times \xi_{\Lambda^c}) \exp\left[-E_{\beta,\Lambda}(x_\Lambda \times 0_{\Lambda^c}|\xi)\right]/N_{\beta,\Lambda}(\xi)$$

is continuous in both variables. Moreover, by (3.1.35) and (3.1.40) the map

$$\Omega_\beta^{\ell_0,\alpha} \ni \xi \mapsto \sup_{x_\Lambda \in \Omega_{\beta,\Lambda}} |G^f_\Lambda(x_\Lambda|\xi)|$$

is bounded on every ball $B_{\ell_0,\alpha}(R)$. This allows us to apply Lebesgue's dominated convergence theorem, which yields the stated continuity. Obviously,

$$\sup_{\xi\in\Omega_\beta^{\ell_0,\alpha}} \left|\pi_{\beta,\Lambda}(f|\xi)\right| \leq \sup_{\xi\in\Omega_\beta^{\ell_0,\alpha}} |f(\xi)|, \tag{3.1.49}$$

which completes the proof. □

Note that for $\xi \in \Omega^{\mathrm{t}}_\beta$, any $\alpha \in \mathcal{I}$, and $f \in C_{\mathrm{b}}(\Omega_\beta^{\ell_0,\alpha})$, by (3.1.44) one has

$$\pi_{\beta,\Lambda}(f|\xi) = \int_{\Omega_\beta} f(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi). \tag{3.1.50}$$

For $\mu \in \mathcal{P}(\Omega_\beta)$, the map $\mu \mapsto \mu\pi_{\beta,\Lambda}$, where

$$\left(\mu\pi_{\beta,\Lambda}\right)(B) = \int_{\Omega_\beta} \pi_{\beta,\Lambda}(B|x)\mu(\mathrm{d}x), \quad B \in \mathcal{B}_\beta, \tag{3.1.51}$$

defines a new probability measure on Ω_β. If $\mu\pi_{\beta,\Lambda} = \mu$ for any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, then one can say that this μ is consistent with the set Π_β.

Definition 3.1.18. A measure $\mu \in \mathcal{P}(\Omega_\beta)$ is said to be a tempered Euclidean Gibbs measure of the model (1.1.8) if for any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, it satisfies the condition

$$\mu\pi_{\beta,\Lambda} = \mu. \tag{3.1.52}$$

This condition considered as an equation, which defines tempered Euclidean Gibbs measures, is called the Dobrushin–Lanford–Ruelle (DLR) equation.

The set of all measures which solve the equation (3.1.52), i.e., the set of all tempered Euclidean Gibbs measures existing at a given β, is denoted by $\mathcal{G}^{\mathrm{t}}_\beta$. *A priori*, one does not know whether this set is non-void. In Section 3.3, we prove that it is indeed non-void. So far we can prove the following assertion.

Proposition 3.1.19. *For every $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ and $\sigma \in (0, 1/2)$, it follows that*

$$\mu(\Omega^{\mathrm{t}}_\beta) = 1, \quad \mu(\{x \in \Omega^{\mathrm{t}}_\beta \mid \forall \ell\colon\ x_\ell \in C^\sigma_\beta\}) = 1. \tag{3.1.53}$$

Proof. By (3.1.44) and (3.1.45),

$$\pi_{\beta,\Lambda}(\Omega_\beta \setminus \Omega^{\mathrm{t}}_\beta|\xi) = 0, \quad \text{for every } \Lambda \in \mathfrak{L}_{\mathrm{fin}} \text{ and } \xi \in \Omega_\beta.$$

Then by (3.1.52),

$$\mu(\Omega_\beta \setminus \Omega^{\mathrm{t}}_\beta) = 0 \implies \mu(\Omega^{\mathrm{t}}_\beta) = 1.$$

The second equality in (3.1.53) follows from (3.1.47). □

Given $\alpha \in \mathcal{I}$, by $\mathcal{W}_\alpha$ we denote the usual weak topology on the set of all probability measures $\mathcal{P}(\Omega_\beta^{\ell_0,\alpha})$, defined by means of $C_{\mathrm{b}}(\Omega_\beta^{\ell_0,\alpha})$. In view of (3.1.31), it is independent of the choice of ℓ_0. By $\mathcal{W}^{\mathrm{t}}$ we denote the weak topology on $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$. With these topologies the sets $\mathcal{P}(\Omega_\beta^{\ell_0,\alpha})$ and $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ become Polish spaces (Theorem 6.5, page 46 of [239]). In general, the convergence of $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, does not yet imply its $\mathcal{W}^{\mathrm{t}}$-convergence. However, as we show in Lemma 3.2.6 and Corollary 3.3.3 below, the topologies induced on $\mathcal{G}^{\mathrm{t}}_\beta$ by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ coincide.

Lemma 3.1.20. *For each $\alpha \in \mathcal{I}$, every $\mathcal{W}_\alpha$-accumulation point $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ of the family $\{\pi_{\beta,\Lambda}(\cdot|\xi) \mid \Lambda \in \mathfrak{L}_{\mathrm{fin}},\ \xi \in \Omega^{\mathrm{t}}\}$ is a tempered Euclidean Gibbs measure.*

Proof. For each $\alpha \in \mathcal{I}$, $C_{\mathrm{b}}(\Omega_\beta^{\ell_0,\alpha})$ is a measure-defining class for $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$. Then a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ solves (3.1.52) if and only if for any $f \in C_{\mathrm{b}}(\Omega_\beta^{\ell_0,\alpha})$ and all $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$,

$$\int_{\Omega^{\mathrm{t}}_\beta} f(x)\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_\beta} \pi_{\beta,\Lambda}(f|x)\mu(\mathrm{d}x). \tag{3.1.54}$$

Let $\{\pi_{\beta,\Lambda_k}(\cdot|\xi_k)\}_{k\in\mathbb{N}}$ converge in $\mathcal{W}_\alpha$ to some $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$. For every $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, one finds $k_\Lambda \in \mathbb{N}$ such that $\Lambda \subset \Lambda_k$ for all $k > k_\Lambda$. Then by (3.1.46) one has

$$\int_{\Omega^{\mathrm{t}}_\beta} f(x)\pi_{\beta,\Lambda_k}(\mathrm{d}x|\xi_k) = \int_{\Omega^{\mathrm{t}}_\beta} \pi_{\beta,\Lambda}(f|x)\pi_{\beta,\Lambda_k}(\mathrm{d}x|\xi_k).$$

Now by Lemma 3.1.17, one can pass to the limit $k \to +\infty$ and get (3.1.54). □

In the remaining part of this section we present a number of facts about the DLR equation and its solutions. One may observe that for any $\mu, \nu \in \mathcal{G}^{\mathrm{t}}_\beta$ and $\theta \in (0, 1)$, the measure $\theta\mu + (1-\theta)\nu$ belongs to $\mathcal{G}^{\mathrm{t}}_\beta$. This means that the latter set is convex. We recall that a state is extreme (pure) if it cannot be written as a nontrivial combination of other states, see Definition 1.1.13.

Definition 3.1.21. An element of $\mathcal{G}^{\mathrm{t}}_\beta$ is called extreme if it cannot be written as a nontrivial combination of other elements of $\mathcal{G}^{\mathrm{t}}_\beta$. The set of all such elements will be denoted by $\mathrm{ex}(\mathcal{G}^{\mathrm{t}}_\beta)$.

Along with the family of local events (3.1.10) we introduce the family

$$\mathcal{T}_\beta = \bigcap_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}} \mathcal{B}_{\beta,\Lambda^c}, \tag{3.1.55}$$

called the *tail σ-algebra*. The following fact is known, see Theorem 7.7 on page 118 in [129].

Proposition 3.1.22. *A measure, $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, is extreme in $\mathcal{G}^{\mathrm{t}}_\beta$ if and only if it is trivial on $\mathcal{T}_\beta$, i.e., for any $A \in \mathcal{T}_\beta$, either $\mu(A) = 1$ or $\mu(A) = 0$.*

Some of Gibbs measures (but not all of them) can serve as mathematical models of equilibrium states of physical systems. According to the basic principles of statistical mechanics, in equilibrium states macroscopic quantities are non-random. Thereby, a Gibbs measure μ which corresponds to such a state, should have the property that tail-measurable functions are constant μ-almost surely. Hence, Gibbs measures describing equilibrium of the underlying physical system should be trivial on $\mathcal{T}_\beta$. A more detailed discussion of the connection between tail triviality and equilibrium states can be found in [129], see Comment 7.8 on page 119 therein.

Now let us turn to the case where the model (1.1.3), (1.1.8) is translation-invariant, see Definition 1.4.10. Here the lattice $\mathbb{L} = \mathbb{Z}^d$ is considered as an additive group. For $\ell_0 \in \mathbb{L}$, $\Lambda \in \mathfrak{L}$, and $x \in \Omega_\beta$, we set

$$\Lambda + \ell_0 = \{\ell + \ell_0 \mid \ell \in \Lambda\}, \qquad t_{\ell_0}(x) = (y^{\ell_0}_\ell)_{\ell \in \mathbb{L}}, \qquad y^{\ell_0}_\ell = x_{\ell - \ell_0}. \tag{3.1.56}$$

Furthermore, for $B \in \mathcal{B}(\Omega_\beta)$, we also set

$$t_\ell(B) = \{t_\ell(x) \mid x \in B\}. \tag{3.1.57}$$

Clearly, $t_\ell(B) \in \mathcal{B}(\Omega_\beta)$ and $t_\ell(\Omega^{\mathrm{t}}_\beta) = \Omega^{\mathrm{t}}_\beta$ for all ℓ.

Definition 3.1.23. A probability measure $\mu \in \mathcal{P}(\Omega_\beta)$ is said to be translation-invariant if $\mu(t_\ell(B)) = \mu(B)$ for every ℓ and $B \in \mathcal{B}_\beta$.

Proposition 3.1.24. *The Gibbs specification $\{\pi_{\beta,\Lambda}\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ of the translation-invariant model (1.1.3), (1.1.8) is translation-invariant, which means that*

$$\pi_{\beta,\Lambda}(t_\ell(B)|\xi) = \pi_{\beta,\Lambda+\ell}(B|t_\ell(\xi)), \quad B \in \mathcal{B}(\Omega_\beta), \ \xi \in \Omega^{\mathrm{t}}_\beta. \tag{3.1.58}$$

The proof follows directly from the translation invariance of the Hamiltonians (1.1.3), (1.2.5).

Remark 3.1.25. The translation invariance of the local Gibbs specification $\Pi_\beta = \{\pi_{\beta,\Lambda}\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ does not mean that each probability kernel $\pi_{\beta,\Lambda}$ is translation-invariant as a measure. Moreover, it does not mean that all Euclidean Gibbs measures defined by this specification are translation-invariant. One can only claim that if the set $\mathcal{G}^{\mathrm{t}}_\beta$ consists of one element only, this element is translation-invariant.

Set

$$\mathcal{B}_\beta^{\mathrm{inv}} = \{B \in \mathcal{B}_\beta \mid \forall \ell : \ t_\ell(B) = B\}, \tag{3.1.59}$$

i.e., $\mathcal{B}_\beta^{\mathrm{inv}}$ is the set of all translation-invariant events. By construction, $\Omega^{\mathrm{t}}_\beta$ belongs to $\mathcal{B}_\beta^{\mathrm{inv}}$. We say that a measure $\mu \in \mathcal{P}(\Omega)$ is trivial on $\mathcal{B}_\beta^{\mathrm{inv}}$ if for every $B \in \mathcal{B}_\beta^{\mathrm{inv}}$, one has $\mu(B) = 0$ or $\mu(B) = 1$. By $\mathcal{P}^{\mathrm{inv}}(\Omega_\beta)$ we denote the set of all translation-invariant probability measures on $(\Omega_\beta, \mathcal{B}_\beta)$.

Definition 3.1.26. A probability measure $\mu \in \mathcal{P}^{\mathrm{inv}}(\Omega_\beta)$ is said to be ergodic (with respect to the group $\mathbb{Z}^d$) if it is trivial on $\mathcal{B}_\beta^{\mathrm{inv}}$.

Ergodic measures are characterized by a mixing property, which we formulate here as in the book [277], see Theorem III.1.8 on page 244 therein. Let Λ_L, $L \in \mathbb{N}$, be the box (3.1.2).

Proposition 3.1.27 (Von Neumann Ergodic Theorem). *Given $\mu \in \mathcal{P}^{\mathrm{inv}}(\Omega_\beta)$, the following statements are equivalent:*

(i) *μ is ergodic;*

(ii) *for all $f, g \in L^2(\Omega_\beta, \mu)$,*

$$\lim_{L \to +\infty} \frac{1}{|\Lambda_L|} \left\{ \sum_{\ell \in \Lambda_L} \left(\int_{\Omega_\beta} f(x) g(t_\ell(x)) \mu(\mathrm{d}x) - \langle f \rangle_\mu \cdot \langle g \rangle_\mu \right) \right\} = 0. \tag{3.1.60}$$

There exists a connection between ergodicity and triviality on the tail algebra (3.1.55), see e.g., Proposition 14.9 in [129], page 293.

Proposition 3.1.28. *Let $\mu \in \mathcal{P}^{\mathrm{inv}}(\Omega_\beta)$. Then for each $A \in \mathcal{B}_\beta^{\mathrm{inv}}$, there exists $B \in \mathcal{T}_\beta$, such that $\mu[(A \setminus B) \cup (B \setminus A)] = 0$. In particular, μ is ergodic if it is trivial on $\mathcal{T}_\beta$.*

Corollary 3.1.29. *For a translation-invariant model* (1.1.3), (1.1.8), *if the set $\mathcal{G}^{\mathrm{t}}_\beta$ is a singleton, its unique element is ergodic.*

Important information about the measures $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ is contained in their Matsubara functions. Let $F_1, \ldots, F_n$ be local bounded multiplication operators. This means that there exists $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, such that $F_1, \ldots, F_n \in \mathfrak{M}_\Lambda$. For these F_j's and a $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$, we set

$$\Gamma^\mu_{F_1,\ldots,F_n}(\tau_1, \ldots, \tau_n) = \int_{\Omega^{\mathrm{t}}_\beta} F_1(x(\tau_1)) \ldots F_n(x(\tau_n)) \mu(\mathrm{d}x), \tag{3.1.61}$$

where $\tau_1, \ldots, \tau_n \in [0, \beta]$. As the set of functions F's corresponding to all local multiplication operators is a measure-defining class, the Matsubara functions constructed for all such operators uniquely determine the measure μ. For local $F_1, \ldots, F_n$, one can construct the local Matsubara functions $\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n|\xi)$ defined by (3.1.42). Then by (3.1.52),

$$\begin{aligned}\Gamma^{\mu}_{F_1,\ldots,F_n}&(\tau_1,\ldots,\tau_n)\\ &= \int_{\Omega^{\mathrm{t}}_\beta}\int_{\Omega^{\mathrm{t}}_\beta} F_1(x(\tau_1))\ldots F_n(x(\tau_n))\pi_{\beta,\Lambda}(\mathrm{d}x|\xi)\mu(\mathrm{d}\xi)\\ &= \int_{\Omega^{\mathrm{t}}_\beta}\left\{\int_{\Omega_{\beta,\Lambda}} F_1((x_\Lambda\times 0_{\Lambda^c})(\tau_1))\ldots F_n((x_\Lambda\times 0_{\Lambda^c})(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi)\right\}\mu(\mathrm{d}\xi)\\ &= \int_{\Omega^{\mathrm{t}}_\beta}\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n|\xi)\mu(\mathrm{d}\xi).\end{aligned} \tag{3.1.62}$$

As we know, the local Matsubara functions (3.1.42) corresponding to $\xi = 0$ coincide with the functions (1.4.20) and hence with (1.2.84). The latter functions have the property (1.2.90) connected with the KMS property of the local Gibbs states. For nonzero ξ, one cannot expect the invariance (1.2.90) for the functions (3.1.42). Thus, the same is true also for (3.1.61).

Definition 3.1.30. A measure $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ is said to be τ-shift-invariant if its Matsubara functions (3.1.61) possess the property that for all $\tau \in [0,\beta]$,

$$\Gamma^{\mu}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n) = \Gamma^{\mu}_{F_1,\ldots,F_n}(\tau_1+\tau,\ldots,\tau_n+\tau), \tag{3.1.63}$$

where addition is modulo β.

3.2 Properties of Gibbs Specifications

As was already noted, so far we have no knowledge about the existence of tempered Euclidean Gibbs measures corresponding to the model (1.1.3), (1.1.8). In this section, we prove statements which later will allow us to prove that these measures do exist, as well as to establish a number of properties of their set $\mathcal{G}^{\mathrm{t}}_\beta$. These statements establish exponential moment estimates for the probability kernels $\pi_{\beta,\Lambda}$ defined by (3.1.44), (3.1.45). Note that here we do not suppose any additional properties of the model, i.e., we consider the model (1.1.3), (1.1.8) satisfying Assumption 1.1.1 only.

3.2.1 Exponential Moment Estimates

To simplify notation, in this section we write π_ℓ meaning the kernel $\pi_{\beta,\Lambda}$ with $\Lambda = \{\ell\}$.

Lemma 3.2.1. *For any $\varkappa$, $\vartheta > 0$, and $\sigma \in (0, 1/2)$, there exists $C_{3.2.1} > 0$ such that for all $\ell \in \mathbb{L}$ and $\xi \in \Omega^{\mathrm{t}}_\beta$,*

$$\int_{\Omega_\beta}\exp\left\{\lambda\|x_\ell\|^2_{C^\sigma_\beta}+\varkappa\|x_\ell\|^2_{L^2_\beta}\right\}\pi_\ell(\mathrm{d}x|\xi)\le\exp\left\{C_{3.2.1}+\vartheta\sum_{\ell'}|J_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}\right\}. \tag{3.2.1}$$

Here $\lambda > 0$ is the same as in (1.3.124) and (3.1.47).

Proof. Note that by (3.1.47) the left-hand side is finite and the second term in $\exp\{\cdot\}$ on the right-hand side is also finite since $\xi \in \Omega^{\mathrm{t}}_\beta$. For any $\vartheta > 0$, by the Minkowski inequality one has, see (1.1.11),

$$\Big| \sum_{\ell'} J_{\ell\ell'} (x_\ell, \xi_{\ell'})_{L^2_\beta} \Big| \leq \frac{\hat{J}_0}{2\vartheta} \|x_\ell\|^2_{L^2_\beta} + \frac{\vartheta}{2} \sum_{\ell'} |J_{\ell\ell'}| \cdot \|\xi_{\ell'}\|^2_{L^2_\beta}, \tag{3.2.2}$$

which holds for all $x, \xi \in \Omega^{\mathrm{t}}_\beta$. By this estimate and (1.4.8), (3.1.33), (3.1.39), (3.1.44) we get

$$\begin{aligned} \mathrm{LHS}(3.2.1) &\leq [1/Y_\ell(\vartheta)] \cdot \exp\Big\{ \vartheta \sum_{\ell'} |J_{\ell\ell'}| \cdot \|\xi_{\ell'}\|^2_{L^2_\beta} \Big\} \\ &\times \int_{C_\beta} \exp\Big\{ \lambda \|x_\ell\|^2_{C^\sigma_\beta} + (\varkappa + \hat{J}_0/2\vartheta)\|x_\ell\|^2_{L^2_\beta} - \int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau \Big\} \chi_\beta(\mathrm{d}x_\ell), \end{aligned} \tag{3.2.3}$$

where

$$Y_\ell(\vartheta) = \int_{C_\beta} \exp\Big\{ -\frac{\hat{J}_0}{2\vartheta} \cdot \|x_\ell\|^2_{L^2_\beta} - \int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau \Big\} \chi_\beta(\mathrm{d}x_\ell).$$

Thereafter, (3.2.1) follows from the estimate (3.2.3) with

$$C_{3.2.1} = -\log Y(\vartheta) + \beta c_V + \log C_\sigma(\lambda) + \frac{\beta(r-1)}{b_V^{1/(r-1)}} \Big(\frac{\varkappa}{r} + \frac{\hat{J}_0}{2\vartheta r} \Big)^{r/r-1} \tag{3.2.4}$$

and

$$Y(\vartheta) = \int_{C_\beta} \exp\Big\{ -\frac{\hat{J}_0}{2\vartheta} \cdot \|x_\ell\|^2_{L^2_\beta} - \int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau \Big\} \chi_\beta(\mathrm{d}x_\ell), \tag{3.2.5}$$

where b_V, c_V, and V are the same as in (1.1.10), and $C_\sigma(\lambda)$ is defined in (1.3.124). □

Remark 3.2.2. As the right-hand side of (3.2.4) is an increasing function of $\hat{J}_0$, the constant $C_{3.2.1}$ is uniform for all interaction potentials $\tilde{J}_{\ell\ell'}$, for which $\sup_\ell \sum_{\ell'} |\tilde{J}_{\ell\ell'}|$ does not exceed the parameter $\hat{J}_0$ appearing in (3.2.4), (3.2.5).

By Jensen's inequality we readily get from (3.2.1) the following Dobrushin-type bound.

Corollary 3.2.3. *For all ℓ and $\xi \in \Omega^{\mathrm{t}}$, the kernels $\pi_\ell(\cdot|\xi)$, obey the estimate*

$$\int_{\Omega_\beta} h(x_\ell)\pi_\ell(\mathrm{d}x|\xi) \leq C_{3.2.1} + (\vartheta/\varkappa) \sum_{\ell'} |J_{\ell\ell'}| \cdot h(\xi_{\ell'}), \tag{3.2.6}$$

with

$$h(x_\ell) = \lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta}. \tag{3.2.7}$$

For translation-invariant lattice systems with the single-spin space $\mathbb{R}$ and ferromagnetic pair interactions, integrability estimates like

$$\log \left\{ \int_{\mathbb{R}^{\mathbb{L}}} \exp(\lambda |x_\ell|) \pi_\ell(\mathrm{d}x|y) \right\} < A + \sum_{\ell'} I_{\ell\ell'} |y_{\ell'}|$$

were first obtained by J. Bellissard and R. Høegh-Krohn, see Proposition III.1 and Theorem III.2 in [59]. Dobrushin's type estimates like (3.2.6) were also obtained and used in [85], [282]. The methods employed there were essentially based on the properties of the model and hence cannot be of use in our situation. Our approach is much simpler; at the same time, it is applicable in both cases – classical and quantum. Its peculiarities are: (a) first we prove the exponential integrability (3.2.1) and then derive the Dobrushin bound (3.2.6) rather than prove it directly; (b) the function (3.2.7) consists of two additive terms, the first of which guarantees the compactness while the second one controls the interaction.

Now by means of (3.2.1) we obtain moment estimates for the kernels $\pi_{\beta,\Lambda}$ with arbitrary $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$. Let the parameters σ, $\varkappa$, and λ be as in (3.2.1). For $\ell \in \Lambda \in \mathfrak{L}_{\mathrm{fin}}$, we define

$$n_\ell(\Lambda|\xi) = \log \left\{ \int_{\Omega_\beta} \exp\left(\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta} \right) \pi_{\beta,\Lambda}(\mathrm{d}\omega|\xi) \right\}, \tag{3.2.8}$$

which is finite in view of (3.1.47).

Lemma 3.2.4. *For every $\alpha \in \mathcal{I}$, there exists $C_{3.2.9}(\alpha) > 0$, which obviously depends also on σ, λ, and $\varkappa$, such that for all ℓ_0 and $\xi \in \Omega^{\mathrm{t}}_\beta$,*

$$\limsup_{\Lambda \nearrow \mathbb{L}} \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0, \ell) \leq C_{3.2.9}(\alpha). \tag{3.2.9}$$

Hence,

$$\limsup_{\Lambda \nearrow \mathbb{L}} n_{\ell_0}(\Lambda|\xi) \leq C_{3.2.9}(\alpha), \quad \textit{for any } \alpha \in \mathcal{I}. \tag{3.2.10}$$

Thereby, for all $\xi \in \Omega^{\mathrm{t}}_\beta$, there exists $C_{3.2.11}(\ell, \xi) > 0$ such that for all $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ containing ℓ,

$$n_\ell(\Lambda|\xi) \leq C_{3.2.11}(\ell, \xi). \tag{3.2.11}$$

Proof. Given $\varkappa > 0$ and $\alpha \in \mathcal{I}$, we fix $\vartheta > 0$, such that

$$\vartheta \sum_{\ell'} |J_{\ell\ell'}| \leq \vartheta \hat{J}_0 \leq \vartheta \hat{J}_\alpha < \varkappa. \tag{3.2.12}$$

Then integrating both sides of (3.2.1) with respect to the measure $\pi_{\beta,\Lambda}(\cdot|\xi)$ we get

$$\begin{aligned} n_\ell(\Lambda|\xi) &\leq C_{3.2.1} + \vartheta \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}| \cdot \|\xi_{\ell'}\|^2_{L^2_\beta} \\ &\quad + \log\left\{ \int_{\Omega_\beta} \exp\left(\vartheta \sum_{\ell'\in\Lambda} |J_{\ell\ell'}| \cdot \|x_{\ell'}\|^2_{L^2_\beta}\right) \pi_{\beta,\Lambda}(\mathrm{d}x|\xi)\right\} \\ &\leq C_{3.2.1} + \vartheta \sum_{\ell'\in\Lambda^c} |J_{\ell\ell'}| \cdot \|\xi_{\ell'}\|^2_{L^2_\beta} + \vartheta/\varkappa \sum_{\ell'\in\Lambda} |J_{\ell\ell'}| \cdot n_{\ell'}(\Lambda|\xi). \end{aligned} \tag{3.2.13}$$

Here we have used (3.2.12) and the multiple Hölder inequality

$$\int \left(\prod_{i=1}^n \varphi_i^{\kappa_i}\right) \mathrm{d}\mu \leq \prod_{i=1}^n \left(\int \varphi_i \,\mathrm{d}\mu\right)^{\kappa_i}, \tag{3.2.14}$$

in which μ is a probability measure, $\varphi_i \geq 0$ (respectively, $\kappa_i \geq 0$), $i = 1, \ldots, n$, are functions (respectively, numbers such that $\sum_{i=1}^n \kappa_i \leq 1$). Then (3.2.13) yields

$$\begin{aligned} n_{\ell_0}(\Lambda|\xi) &\leq \sum_{\ell\in\Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) \\ &\leq \frac{1}{1-\vartheta \hat{J}_\alpha/\varkappa}\left[C_{3.2.1} \sum_{\ell'\in\Lambda} w_\alpha(\ell_0,\ell') + \vartheta \hat{J}_\alpha \sum_{\ell'\in\Lambda^c} \|\xi_{\ell'}\|^2_{L^2_\beta} w_\alpha(\ell_0,\ell')\right]. \end{aligned} \tag{3.2.15}$$

Therefrom, for all $\xi \in \Omega^{\mathrm{t}}_\beta$, we get

$$\begin{aligned} \limsup_{\Lambda\nearrow\mathbb{L}} n_{\ell_0}(\Lambda|\xi) &\leq \limsup_{\Lambda\nearrow\mathbb{L}} \sum_{\ell\in\Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) \\ &\leq \frac{C_{3.2.1}}{1-\vartheta \hat{J}_\alpha/\varkappa} \sum_\ell w_\alpha(\ell_0,\ell) \stackrel{\mathrm{def}}{=} C_{3.2.9}(\alpha), \end{aligned} \tag{3.2.16}$$

which gives (3.2.9) and (3.2.10). The proof of (3.2.11) is straightforward. □

We recall that the norm $\|\cdot\|_{\ell_0,\alpha}$ was defined in (3.1.27). Given $\alpha \in \mathcal{I}$, $\ell_0 \in \mathbb{L}$, and $\sigma \in (0, 1/2)$, we set

$$\|\xi\|_{\ell_0,\alpha,\sigma} = \left[\sum_\ell \|\xi_\ell\|^2_{C^\sigma_\beta} w_\alpha(\ell_0,\ell)\right]^{1/2}. \tag{3.2.17}$$

Lemma 3.2.5. *For every $\alpha \in \mathcal{I}$, $\ell_0 \in \mathbb{L}$, and $\xi \in \Omega^{\mathrm{t}}_\beta$, one finds a positive $C_{3.2.18}(\xi)$, such that for all $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$,*

$$\int_{\Omega_\beta} \|x\|^2_{\ell_0,\alpha} \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \leq C_{3.2.18}(\xi). \tag{3.2.18}$$

Furthermore, for every $\alpha \in \mathcal{I}$, $\ell_0 \in \mathbb{L}$, $\sigma \in (0, 1/2)$, and $\xi \in \Omega^{\mathrm{t}}_\beta$ for which the norm (3.2.17) is finite, one finds a $C_{3.2.19}(\xi) > 0$, such that for all $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$,

$$\int_{\Omega_\beta} \|x\|^2_{\ell_0,\alpha,\sigma} \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \leq C_{3.2.19}(\xi). \tag{3.2.19}$$

The right-hand sides of both estimates, (3.2.18) *and* (3.2.19)*, are the same for all interaction potentials* $\tilde{J}_{\ell\ell'}$ *for which the parameters* (3.1.20) (*respectively,* $\sup_\ell \sum_{\ell'} |\tilde{J}_{\ell\ell'}|$) *do not exceed* $\hat{J}_\alpha$ (*respectively,* $\hat{J}_0$)*, appearing in* (3.2.16) (*respectively, in* (3.2.4)*,* (3.2.5)).

Proof. For any fixed $\xi \in \Omega_\beta^{\mathrm{t}}$, by the Jensen inequality and (3.2.15) one has

$$\begin{aligned}
&\limsup_{\Lambda \nearrow \mathbb{L}} \int_{\Omega_\beta} \|x\|^2_{\ell_0,\alpha} \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \\
&\quad \leq \limsup_{\Lambda \nearrow \mathbb{L}} \left[\frac{1}{\varkappa} \sum_{\ell \in \Lambda} n_\ell(\Lambda|\xi) w_\alpha(\ell_0,\ell) + \sum_{\ell \in \Lambda^c} \|\xi_\ell\|^2_{L^2_\beta} w_\alpha(\ell_0,\ell) \right] \\
&\quad \leq C_{3.2.9}(\alpha)/\varkappa.
\end{aligned} \tag{3.2.20}$$

Hence, the set consisting of the left-hand sides of (3.2.18) indexed by $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ is bounded. The proof of (3.2.19) is analogous. The uniformity stated in the concluding part of the lemma follows by Remark 3.2.2 and by the fact that the constant $C_{3.2.9}$ is an increasing function of $\hat{J}_\alpha$. $\square$

3.2.2 Weak Convergence of Tempered Measures

Recall that $f : \Omega_\beta \to \mathbb{R}$ is called a local function if it is $\mathcal{B}_{\beta,\Lambda}/\mathcal{B}(\mathbb{R})$ measurable for a certain $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$.

Lemma 3.2.6. *Let* $\{\mu_n\}_{n\in\mathbb{N}} \subset \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ *have the following properties:* (a) *for every* $\alpha \in \mathcal{I}$ *and* $\ell_0 \in \mathbb{L}$*, each of its elements obeys the estimate*

$$\int_{\Omega^{\mathrm{t}}_\beta} \|x\|^2_{\ell_0,\alpha} \mu_n(\mathrm{d}x) \leq C_{3.2.21}(\ell_0,\alpha), \tag{3.2.21}$$

with one and the same constant $C_{3.2.21}(\ell_0,\alpha)$*;* (b) *for every local* $f \in C_{\mathrm{b}}(\Omega^{\mathrm{t}}_\beta)$*,* $\{\langle f\rangle_{\mu_n}\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *is a Cauchy sequence. Then* $\{\mu_n\}_{n\in\mathbb{N}}$ *converges in* $\mathcal{W}^{\mathrm{t}}$ *to a certain* $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$.

Proof. The topology of the space $\Omega^{\mathrm{t}}_\beta$ is consistent with the following metric, cf. (3.1.29),

$$\rho(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{\|x-y\|_{\ell_0,\alpha_k}}{1+\|x-y\|_{\ell_0,\alpha_k}} + \sum_{\ell} 2^{-|\ell_0-\ell|} \frac{\|x_\ell - y_\ell\|_{C_\beta}}{1+\|x_\ell-y_\ell\|_{C_\beta}}, \tag{3.2.22}$$

where $\{\alpha_k\}_{k\in\mathbb{N}} \subset \mathcal{I} = (\underline{\alpha}, \overline{\alpha})$ is a monotone strictly decreasing sequence convergent to $\underline{\alpha}$. In fact, the metric (3.2.22) may depend on the choice of this sequence, as well as on the choice of ℓ_0, but all such metrics define the same topology of the space $\Omega^{\mathrm{t}}_\beta$. Let us denote by $C^{\mathrm{u}}_{\mathrm{b}}(\Omega^{\mathrm{t}}_\beta;\rho)$ the set of all bounded functions $f : \Omega^{\mathrm{t}}_\beta \to \mathbb{R}$, which are uniformly continuous with respect to the metric (3.2.22). Thus, to prove the lemma it suffices to show that, under its conditions, $\{\langle f\rangle_{\mu_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence for every $f \in C_{\mathrm{b}}^{\mathrm{u}}(\Omega_\beta^{\mathrm{t}};\rho)$, see Proposition 1.3.32. Given $\delta > 0$, we choose $\Lambda_\delta \in \mathfrak{L}_{\mathrm{fin}}$ and $k_\delta \in \mathbb{N}$, such that

$$\sum_{\ell\in\Lambda_\delta^c} 2^{-|\ell_0-\ell|} < \delta/3, \qquad \sum_{k=k_\delta}^{\infty} 2^{-k} = 2^{-k_\delta+1} < \delta/3. \tag{3.2.23}$$

For this δ and a certain $R > 0$, we pick $\Lambda_\delta(R) \in \mathfrak{L}_{\mathrm{fin}}$, such that

$$\sup_{\ell\in\mathbb{L}\setminus\Lambda_\delta(R)} \left\{ w_{\alpha_{k_\delta-1}}(\ell_0,\ell)/w_{\alpha_{k_\delta}}(\ell_0,\ell) \right\} < \frac{\delta}{3R^2}, \tag{3.2.24}$$

which is possible in view of (3.1.18). Finally, for $R > 0$, we set

$$B_R = \{x \in \Omega_\beta^{\mathrm{t}} \mid \|x\|_{\ell_0,\alpha_{k_\delta}} \le R\}. \tag{3.2.25}$$

By (3.2.21) and the Chebyshev inequality one has that for all $n \in \mathbb{N}$,

$$\mu_n\left(\Omega_\beta^{\mathrm{t}} \setminus B_R\right) \le C_{3.2.21}(\ell_0,\alpha_{k_\delta})/R^2. \tag{3.2.26}$$

Now for $f \in C_{\mathrm{b}}^{\mathrm{u}}(\Omega_\beta^{\mathrm{t}};\rho)$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, and $n, m \in \mathbb{N}$, we have

$$\begin{aligned}\left|\langle f\rangle_{\mu_n} - \langle f\rangle_{\mu_m}\right| \le & \left|\langle f_\Lambda\rangle_{\mu_n} - \langle f_\Lambda\rangle_{\mu_m}\right| \\ & + 2\max\{\langle|f - f_\Lambda|\rangle_{\mu_n}; \langle|f - f_\Lambda|\rangle_{\mu_m}\},\end{aligned} \tag{3.2.27}$$

where we have set $f_\Lambda(x) = f(x_\Lambda \times 0_{\Lambda^c})$. By (3.2.26)

$$\begin{aligned}\langle|f - f_\Lambda|\rangle_{\mu_n} \le & \, 2C_{3.2.21}(\ell_0,\alpha_{k_\delta})\|f\|_\infty/R^2 \\ & + \int_{B_R} |f(x) - f(x_\Lambda \times 0_{\Lambda^c})|\,\mu_n(\mathrm{d}x).\end{aligned} \tag{3.2.28}$$

For chosen $f \in C_{\mathrm{b}}^{\mathrm{u}}(\Omega^{\mathrm{t}};\rho)$ and $\varepsilon > 0$, one finds $\delta > 0$, such that for all $x, y \in \Omega_\beta^{\mathrm{t}}$,

$$|f(x) - f(y)| < \varepsilon/6, \quad \text{ whenever } \rho(x,y) < \delta.$$

For these f, ε, and δ, one picks $R(\varepsilon,\delta) > 0$, such that

$$C_{3.2.21}(\ell_0,\alpha_{k_\delta})\|f\|_\infty / \left[R(\varepsilon,\delta)\right]^2 < \varepsilon/12. \tag{3.2.29}$$

Now one takes $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, which contains both Λ_δ and $\Lambda_\delta[R(\varepsilon,\delta)]$ defined by (3.2.23), (3.2.24). For this Λ, $x \in B_{R(\varepsilon,\delta)}$, and $k = 1, 2, \ldots, k_\delta - 1$,

$$\begin{aligned}&\|x - x_\Lambda \times 0_{\Lambda^c}\|^2_{\ell_0,\alpha_k} \\ &\quad = \sum_{\ell\in\Lambda^c} \|x_\ell\|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(\ell_0,\ell)\left[w_{\alpha_k}(\ell_0,\ell)/w_{\alpha_{k_\delta}}(\ell_0,\ell)\right] \\ &\quad \le \frac{\delta}{3\left[R(\varepsilon,\delta)\right]^2} \sum_{\ell\in\Lambda^c} \|x_\ell\|^2_{L^2_\beta} w_{\alpha_{k_\delta}}(\ell_0,\ell) < \frac{\delta}{3},\end{aligned} \tag{3.2.30}$$

see (3.2.24) and (3.2.25). Then by (3.2.22), (3.2.23), it follows that

$$\forall x \in B_{R(\varepsilon,\delta)}: \quad \rho(x, x_\Lambda \times 0_{\Lambda^c}) < \delta, \tag{3.2.31}$$

which together with (3.2.29) yields in (3.2.28),

$$\langle |f - f_\Lambda| \rangle_{\mu_n} < \frac{\varepsilon}{6} + \frac{\varepsilon}{6}\mu_n\left(B_{R(\varepsilon,\delta)}\right) \leq \frac{\varepsilon}{3}.$$

By assumption (b) of the lemma, one finds N_ε such that for all $n, m > N_\varepsilon$,

$$\left|\langle f_\Lambda \rangle_{\mu_n} - \langle f_\Lambda \rangle_{\mu_n}\right| < \frac{\varepsilon}{3}.$$

Applying the latter two estimates in (3.2.27) we get that $\{\mu_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the topology $\mathcal{W}^{\mathrm{t}}$ in which the space $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ is complete. □

Remark 3.2.7. The weak convergence of the projections of μ_n onto $C_{\beta,\Lambda}$ can be established with the help of the Lipschitz functions $f \in BL(C_{\beta,\Lambda}, \|\cdot\|_{C_{\beta,\Lambda}})$, see (1.3.69) and (1.3.159).

3.3 Properties of Tempered Euclidean Gibbs Measures

In this section, we describe general properties of the set $\mathcal{G}^{\mathrm{t}}_\beta$ corresponding to the model (1.1.3), (1.1.8) which satisfies the conditions of Assumption 1.1.1 only. More specific cases where the model has some extra properties will be described in the subsequent sections.

The exponential moment estimates for the kernels $\pi_{\beta,\Lambda}$ proven above give us a tool for establishing the existence of the tempered Euclidean Gibbs measures, as well as a number of their properties. We begin by deriving an a priori integrability estimate, similar to (3.2.1), (1.3.124). Recall that the Hölder norm $\|\cdot\|_{C^\sigma_\beta}$ was defined in (1.3.58).

Theorem 3.3.1. *For every $\sigma \in (0, 1/2)$, $\lambda \in (0, \varsigma/b_\sigma)$, and $\varkappa > 0$, there exists a positive constant, $C_{3.3.1}(\sigma, \lambda, \varkappa)$, such that for any ℓ and for all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$,*

$$\int_{\Omega_\beta} \exp\left(\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta}\right)\mu(\mathrm{d}x) \leq C_{3.3.1}(\sigma, \lambda, \varkappa), \tag{3.3.1}$$

where ς and b_σ are the same as in (1.3.124).

Remark 3.3.2. According to (3.3.1), the one-site projections of each $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ are sub-Gaussian. The bound $C_{3.3.1}(\sigma, \lambda, \varkappa)$ does not depend on ℓ and is the same for all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$. For $\lambda = 0$, this bound is independent of σ. The estimate (3.3.1) plays a crucial role in the study of the set $\mathcal{G}^{\mathrm{t}}_\beta$.

Proof. Let us show that every $\mu \in \mathcal{P}(\Omega)$ which solves the DLR equation (3.1.52) ought to obey the estimate (3.3.1) with one and the same $C_{3.3.1}$. To this end we apply the bounds for the kernels $\pi_{\beta,\Lambda}(\cdot|\xi)$ obtained above. Consider the functions

$$G_N(x_\ell) \stackrel{\text{def}}{=} \exp\left(\min\left\{\lambda\|x_\ell\|^2_{C^\sigma_\beta} + \varkappa\|x_\ell\|^2_{L^2_\beta}; N\right\}\right), \quad N \in \mathbb{N}.$$

By (3.1.52), Fatou's lemma, and the estimate (3.2.10) with an arbitrarily chosen $\alpha \in \mathcal{I}$ we get

$$\begin{aligned}\int_{\Omega_\beta} G_N(x_\ell)\mu(\mathrm{d}x) &= \limsup_{\Lambda\nearrow\mathbb{L}} \int_{\Omega_\beta}\left[\int_{\Omega_\beta} G_N(x_\ell)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi)\right]\mu(\mathrm{d}\xi)\\ &\leq \int_{\Omega_\beta}\left[\limsup_{\Lambda\nearrow\mathbb{L}}\int_{\Omega_\beta}\exp\left(\lambda\|x_\ell\|^2_{C^\sigma_\beta} + \varkappa\|x_\ell\|^2_{L^2_\beta}\right)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi)\right]\mu(\mathrm{d}\xi)\\ &\leq \exp C_{3.2.9}(\alpha) \stackrel{\text{def}}{=} C_{3.3.1}(\sigma,\lambda,\varkappa).\end{aligned}$$

In view of the second support property in (3.1.53) of any measure solving the equation (3.1.52) we can pass here to the limit $N \to +\infty$ and obtain (3.3.1). □

Corollary 3.3.3. *For every $\alpha \in \mathcal{I}$, the topologies induced on $\mathcal{G}^{\mathrm{t}}_\beta$ by $\mathcal{W}_\alpha$ and $\mathcal{W}^{\mathrm{t}}$ coincide.*

Proof. The assertions in this corollary follow immediately by Lemma 3.2.6 and the estimate (3.3.1). □

In the sequel, we will use yet another moment estimate for the tempered Euclidean Gibbs measures. It is a direct corollary of (3.3.1).

Corollary 3.3.4. *For any $\lambda > 0$, $\alpha \in \mathcal{I}$, and ℓ_0, there exists a positive constant $C_{3.3.2}(\lambda)$ such that, for every $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$,*

$$\int_{\Omega_\beta} \exp\left(\lambda\|x\|^2_{\ell_0,\alpha}\right)\mu(\mathrm{d}x) \leq C_{3.3.2}(\lambda). \tag{3.3.2}$$

Proof. For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, we set

$$I_\Lambda = \int_{\Omega_\beta} \exp\left(\lambda\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta} w_\alpha(\ell_0,\ell)\right)\mu(\mathrm{d}x), \tag{3.3.3}$$

and

$$w_\alpha = \sum_\ell w_\alpha(\ell_0,\ell), \tag{3.3.4}$$

which is finite by (3.1.20). Since each $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, if it exists, should be a probability measure, the bound $C_{3.3.1}$ in (3.3.1) is greater than 1 for any $\varkappa > 0$. Taking this into account we apply in (3.3.3) the Hölder inequality (3.2.14) with $\kappa_\ell = w_\alpha(\ell_0, \ell)/w_\alpha$ and the estimate (3.3.1) with $\varkappa = \lambda w_\alpha$. In this way, we obtain

$$I_\Lambda \leq C_{3.3.1}(\sigma, 0, \lambda w_\alpha).$$

Obviously, $I_\Lambda \leq I_{\Lambda'}$ if $\Lambda \subset \Lambda'$; hence, the sequence $\{I_\Lambda\}_{\Lambda \in \mathfrak{L}}$ converges to the left-hand side of (3.3.2), which ought to be bounded by the same $C_{3.3.1}(\sigma, 0, \lambda w_\alpha)$. Thus, the estimate (3.3.2) holds true with $C_{3.3.2}(\lambda) = C_{3.3.1}(\sigma, 0, \lambda w_\alpha)$. Note that such $C_{3.3.2}(\lambda)$ is independent of σ, see Remark 3.3.2. $\square$

By repetition of the above argument with replacing the bound (3.3.1) by (3.2.11) we prove the following

Corollary 3.3.5. *For any $\lambda > 0$, $\alpha \in \mathcal{I}$, ℓ_0, and $\xi \in \Omega_\beta$, there exists a positive $C_{3.3.5}(\lambda, \xi)$, such that for every $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$,*

$$\int_{\Omega_\beta} \exp\left(\lambda \|x\|^2_{\ell_0,\alpha}\right) \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \leq C_{3.3.5}(\lambda, \xi). \tag{3.3.5}$$

Now we are in a position to prove the existence of tempered Euclidean Gibbs measures and compactness of their set $\mathcal{G}^{\mathrm{t}}_\beta$. For models with non-compact spins, here they are even infinite-dimensional, such a property is far from being evident.

Theorem 3.3.6. *For every $\beta > 0$, the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}_\beta$ is non-void and $\mathcal{W}^{\mathrm{t}}$- compact.*

Proof. Consider the following scale of Banach spaces (cf. (3.1.28)):

$$\Omega_\beta^{\ell_0,\alpha,\sigma} = \left\{x \in \Omega_\beta \;\middle|\; \|x\|_{\ell_0,\alpha,\sigma} < \infty\right\}, \quad \ell_0 \in \mathbb{L},\ \sigma \in (0, 1/2),\ \alpha \in \mathcal{I}, \tag{3.3.6}$$

where the norm $\|\cdot\|_{\ell_0,\alpha,\sigma}$ was defined in (3.2.17). For any pair $\alpha, \alpha' \in \mathcal{I}$, such that $\alpha < \alpha'$, the embedding $\Omega_\beta^{\ell_0,\alpha,\sigma} \hookrightarrow \Omega_\beta^{\ell_0,\alpha'}$ is compact, see Remark 3.1.10. This fact and the estimate (3.2.19), which holds for any $\xi \in \Omega_\beta^{\ell_0,\alpha,\sigma}$, imply by Prokhorov's criterion the relative compactness of the set $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ in $\mathcal{W}_{\alpha'}$. Therefore, the set $\{\pi_{\beta,\Lambda}(\cdot|0)\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ is relatively compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$. Then Lemma 3.1.20 yields that $\mathcal{G}^{\mathrm{t}}_\beta \neq \emptyset$. By the same Prokhorov criterion and the estimate (3.3.1), we get the $\mathcal{W}_\alpha$-relative compactness of $\mathcal{G}^{\mathrm{t}}_\beta$. Then in view of the Feller property (Lemma 3.1.17), the set $\mathcal{G}^{\mathrm{t}}_\beta$ is closed and hence compact in every $\mathcal{W}_\alpha$, $\alpha \in \mathcal{I}$, which by Corollary 3.3.3 yields the result to be proven. $\square$

For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, we have introduced the families of functions $\mathfrak{C}_\Lambda$, see Definition 1.4.13. For a pair $\Lambda \subset \Lambda'$, one can define a natural embedding $\mathfrak{C}_\Lambda \hookrightarrow \mathfrak{C}_{\Lambda'}$. Then we set

$$\mathfrak{C} = \bigcup_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}} \mathfrak{C}_\Lambda. \tag{3.3.7}$$

It is clear that for every $f \in \mathfrak{E}$, there exist $\Lambda_f \in \mathfrak{L}_{\text{fin}}$ and $D_f > 0$, such that

$$|f(x)| \leq D_f \sum_{\ell \in \Lambda_f} \exp\left(\lambda \|x_\ell\|^2_{C_\beta}\right), \quad \text{for all } x \in \Omega^{\text{t}}_\beta, \tag{3.3.8}$$

which holds with any $\lambda > 0$. We recall that $\text{ex}(\mathcal{G}^{\text{t}}_\beta)$ stands for the set of all extreme elements of $\mathcal{G}^{\text{t}}_\beta$.

Lemma 3.3.7. *For every $\mu \in \text{ex}(\mathcal{G}^{\text{t}}_\beta)$ and any cofinal sequence $\mathcal{L}$, the following holds:*
(a) *the sequence $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda\in\mathcal{L}}$ converges in $\mathcal{W}^{\text{t}}$ to this μ for μ-almost all $\xi \in \Omega^{\text{t}}_\beta$;*
(b) *for every $f \in \mathfrak{E}$ and μ-almost all $\xi \in \Omega^{\text{t}}_\beta$,*

$$\lim_{\mathcal{L}} \int_{\Omega_\beta} f(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = \langle f\rangle_\mu.$$

Proof. Claim (c) of Theorem 7.12, page 122 in [129], implies that for any local $f \in C_{\text{b}}(\Omega^{\text{t}}_\beta)$,

$$\lim_{\mathcal{L}} \int_{\Omega_\beta} f(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = \langle f\rangle_\mu, \quad \text{for } \mu\text{-almost all } \xi \in \Omega^{\text{t}}_\beta. \tag{3.3.9}$$

Then the convergence stated in claim (a) follows from Lemmas 3.2.5 and 3.2.6. Given $f \in \mathfrak{E}$ and $N \in \mathbb{N}$, we set $\Omega^{(N)}_\beta = \{x \in \Omega_\beta \mid |f(x)| > N\}$ and

$$f_N(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq N; \\ Nf(x)/|f(x)| & \text{otherwise.} \end{cases}$$

Each f_N belongs to $C_{\text{b}}(\Omega^{\text{t}}_\beta)$ and $f_N \to f$ point-wise as $N \to +\infty$. Then by (3.3.9) there exists a Borel set $\Xi_\mu \subset \Omega^{\text{t}}_\beta$, such that $\mu(\Xi_\mu) = 1$ and for every $N \in \mathbb{N}$,

$$\lim_{\mathcal{L}} \pi_{\beta,\Lambda}(f_N|\xi) = \mu(f_N), \quad \text{for all } \xi \in \Xi_\mu. \tag{3.3.10}$$

Note that by (3.2.8), (3.2.11), (1.3.59), and (3.3.8) for any $\xi \in \Xi_\mu$ one finds a positive $C_{3.3.11}(f,\xi)$ such that for all $\Lambda \in \mathfrak{L}_{\text{fin}}$, which contain Λ_f, it follows that

$$\int_{\Omega_\beta} |f(x)|^2 \pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \leq C_{3.3.11}(f,\xi). \tag{3.3.11}$$

Hence

$$\begin{aligned} |\pi_{\beta,\Lambda}(f|\xi) - \pi_{\beta,\Lambda}(f_N|\xi)| &\leq 2\int_{\Omega^{(N)}_\beta} |f(x)|\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \\ &\leq \frac{2}{N}\cdot\int_{\Omega_\beta} |f(x)|^2\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) \leq \frac{2}{N}\cdot C_{3.3.11}(f,\xi). \end{aligned}$$

Similarly, by means of (1.3.59), (3.3.8), and Theorem 3.3.1, one gets

$$\langle |f - f_N| \rangle_\mu \leq \frac{2}{N} \cdot D_f C_{3.3.1}.$$

The latter two inequalities and (3.3.10) allow for estimating $|\pi_{\beta,\Lambda}(f|\xi) - \langle f \rangle_\mu|$, which completes the proof. □

Given $\ell_1, \ldots, \ell_n$ and $\tau_1, \ldots, \tau_n$, not necessarily distinct, we consider the monomial

$$f(x) = x_{\ell_1}(\tau_1) \ldots x_{\ell_n}(\tau_n), \tag{3.3.12}$$

which obviously belongs to the family $\mathfrak{C}$ defined by (1.4.58) and (3.3.7). Let $\mathcal{F}$ be the family of all such monomials. It is closed with respect to multiplication and separates the points of $\Omega^{\mathrm{t}}_\beta$. Then, if for $\mu_1, \mu_2 \in \mathcal{G}^{\mathrm{t}}_\beta$, one has $\langle f \rangle_{\mu_1} = \langle f \rangle_{\mu_2}$ for all $f \in \mathcal{F}$, then $\mu_1 = \mu_2$, see Theorem 1.3.26. On the other hand, for such monomials we have the GKS inequalities (2.2.3), (2.2.4). Combining these two facts we prove the following

Theorem 3.3.8. *If the model* (1.1.3), (1.1.8) *is scalar and ferromagnetic, the family* $\{\pi_{\beta,\Lambda}(\cdot|0)\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ *has only one* $\mathcal{W}^{\mathrm{t}}$*-accumulation point.*

Proof. As was established in the proof of Theorem 3.3.6, the family $\{\pi_{\beta,\Lambda}(\cdot|0)\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ is $\mathcal{W}^{\mathrm{t}}$-relatively compact; hence, it possesses accumulation points, which are elements of $\mathcal{G}^{\mathrm{t}}_\beta$. Let $\mathcal{L}$ be any cofinal sequence and f be a monomial (3.3.12). Then there exists $\Lambda_0 \in \mathcal{L}$, such that for all $\Lambda \supset \Lambda_0$,

$$\langle f \rangle_{\pi_{\beta,\Lambda}(\cdot|0)} = \langle f \rangle_{\nu_{\beta,\Lambda}},$$

see (3.1.44). Then by Theorem 2.5.13, the sequence $\{\langle f \rangle_{\pi_{\beta,\Lambda}(\cdot|0)}\}_{\Lambda \in \mathcal{L}}$ is monotone increasing. On the other hand, by means of (3.2.19) we easily see that this sequence is bounded and hence convergent. Let its limit be $c(f, \mathcal{L})$. Now we take two cofinal sequences $\mathcal{L}$ and $\mathcal{L}'$. For any $\Lambda \in \mathcal{L}$, one finds $\Lambda' \in \mathcal{L}'$ such that $\Lambda \subset \Lambda'$. Then by (2.5.46),

$$\langle f \rangle_{\pi_{\beta,\Lambda}(\cdot|0)} \leq \langle f \rangle_{\pi_{\beta,\Lambda'}(\cdot|0)},$$

which yields $c(f, \mathcal{L}) \leq c(f, \mathcal{L}')$. By interchanging the sequences $\mathcal{L}$ and $\mathcal{L}'$ we see that the opposite inequality also holds, which completes the proof. □

Next we describe some regularity properties of the elements of $\mathcal{G}^{\mathrm{t}}_\beta$. For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, let μ_Λ be the projection of μ onto $C_{\beta,\Lambda}$. By this we mean the measure on $C_{\beta,\Lambda}$, such that for every Borel subset $B \subset C_{\beta,\Lambda}$,

$$\mu_\Lambda(B) = \mu(B \times \Omega_{\beta,\Lambda^c}). \tag{3.3.13}$$

Theorem 3.3.9. *For every* $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ *and any* $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$*, the projection* μ_Λ *is absolutely continuous with respect to the measure* $\chi_{\beta,\Lambda}$*. The corresponding Radon–Nikodym* *derivative* $\rho_\Lambda : C_{\beta,\Lambda} \to \mathbb{R}$ *is a continuous function, which obeys the estimates*

$$\begin{aligned}\exp\left(-\log N^{+}_{\beta,\Lambda} - E_{\beta,\Lambda}(x_\Lambda)\right) &\leq \rho_\Lambda(x_\Lambda) \leq C_{3.3.2}(\lambda) \\ &\times \exp\left(-\log N^{-}_{\beta,\Lambda} - E_{\beta,\Lambda}(x_\Lambda) + \frac{\hat{J}_0}{2}\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta}\right),\end{aligned} \tag{3.3.14}$$

where α *is an element of* $\mathcal{I}$,

$$\lambda = \hat{J}_\alpha \sum_{\ell\in\Lambda} 1/w_\alpha(\ell_0,\ell), \tag{3.3.15}$$

and

$$N^{\pm}_{\beta,\Lambda} = \int_{C_{\beta,\Lambda}} \exp\left(-E_{\beta,\Lambda}(x_\Lambda) \pm \frac{\hat{J}_0}{2}\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta}\right)\chi_{\beta,\Lambda}(dx_\Lambda). \tag{3.3.16}$$

Proof. One observes that both $N^{\pm}_{\beta,\Lambda}$ are finite in view of the super-quadratic growth of V_ℓ, see (1.1.10). For any bounded continuous function $f : C_{\beta,\Lambda} \to \mathbb{R}$, we have, by (3.1.54),

$$\int_{\Omega_\beta} f(x_\Lambda)\mu(dx) = \int_{\Omega_\beta} f(x_\Lambda)\mu_\Lambda(dx_\Lambda) = \int_{C_{\beta,\Lambda}}\int_{\Omega_\beta} f(x_\Lambda)\nu_{\beta,\Lambda}(x_\Lambda|\xi)\mu(d\xi).$$

Thus,

$$\rho_\Lambda(x_\Lambda) = \exp\left(-E_{\beta,\Lambda}(x_\Lambda)\right)\psi_\Lambda(x_\Lambda), \tag{3.3.17}$$

where, see (3.1.39) and (3.1.41),

$$\psi_\Lambda(x_\Lambda) = \int_{\Omega_\beta} \exp\left(-\log N_{\beta,\Lambda}(\xi) + \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\right)\mu(d\xi). \tag{3.3.18}$$

Then we use in (3.1.39) the estimate (3.2.2) with $\vartheta = 1$ and obtain

$$\begin{aligned}\exp\left(-\frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}|J_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}\right)N^{-}_{\beta,\Lambda} &\leq N_{\beta,\Lambda}(\xi) \\ &\leq \exp\left(\frac{1}{2}\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}|J_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}\right)N^{+}_{\beta,\Lambda}.\end{aligned} \tag{3.3.19}$$

Now we combine in (3.3.18) the estimate (3.2.2) with the latter double inequality and arrive at

$$\begin{aligned}\left(N^{+}_{\beta,\Lambda}\right)^{-1} \leq \psi_\Lambda(x_\Lambda) &\leq \left(N^{-}_{\beta,\Lambda}\right)^{-1}\exp\left(\frac{\hat{J}_0}{2}\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta}\right) \\ &\times \int_{\Omega_\beta}\exp\left(\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c}|J_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}\right)\mu(d\xi).\end{aligned} \tag{3.3.20}$$

Employing the triangle inequality (3.1.17), as it was done in (3.1.36), we get

$$\sum_{\ell \in \Lambda,\ \ell' \in \Lambda^c} |J_{\ell\ell'}| \cdot \|\xi_{\ell'}\|^2_{L^2_\beta} \leq \sum_{\ell \in \Lambda} \frac{1}{w_\alpha(\ell_0, \ell)} \sum_{\ell'} \frac{|J_{\ell\ell'}|}{w_\alpha(\ell, \ell')} \cdot \|\xi_{\ell'}\|^2_{L^2_\beta} w_\alpha(\ell_0, \ell')$$
$$\leq \hat{J}_\alpha \cdot \|\xi\|^2_{\ell_0,\alpha} \sum_{\ell \in \Lambda} \frac{1}{w_\alpha(\ell_0, \ell)} = \lambda \|\xi\|^2_{\ell_0,\alpha},$$

see (3.3.15). Applying this estimate in (3.3.20) together with the integrability estimate (3.3.2) we finally arrive at (3.3.14). The continuity of ψ_Λ follows by the Lebesgue dominated convergence theorem and the estimate (3.2.2). The continuity of $E_{\beta,\Lambda}$ has been discussed in the proof of Lemma 3.1.12. □

Employing the estimate (3.3.5) in place of (3.3.2) one can prove the following

Corollary 3.3.10. *For every $\Lambda \subset \Delta \in \mathfrak{L}_{\mathrm{fin}}$ and any $\xi \in \Omega_\beta$, the projection of the measure $\pi_{\beta,\Delta}(\cdot|\xi)$ onto $C_{\beta,\Lambda}$ is absolutely continuous with respect to $\chi_{\beta,\Lambda}$. The corresponding Radon–Nikodym derivative $\rho^\xi_{\Lambda,\Delta} : C_{\beta,\Lambda} \to \mathbb{R}$ is a continuous function, which obeys*

$$\begin{aligned} &\exp\left(-\log N^+_{\beta,\Lambda} - E_{\beta,\Lambda}(x_\Lambda)\right) \leq \rho^\xi_{\Lambda,\Delta}(x_\Lambda) \\ &\quad \leq C_{3.3.5}(\lambda, \xi) \times \exp\Big(-\log N^+_{\beta,\Lambda} - E_{\beta,\Lambda}(x_\Lambda) + \frac{\hat{J}}{2} \sum_{\ell \in \Lambda} \|x_\ell\|^2_{L^2_\beta}\Big), \end{aligned} \tag{3.3.21}$$

where λ and $N^\pm_{\beta,\Lambda}$ are the same as in (3.3.14).

The set of tempered configurations $\Omega^{\mathrm{t}}_\beta$ was introduced in (3.1.28), (3.1.30) by means of rather slack conditions, imposed on the L^2_β-norms of x_ℓ. The elements of $\mathcal{G}^{\mathrm{t}}_\beta$ are supported by $\Omega^{\mathrm{t}}_\beta$, see (3.1.53). Let us show now that these elements have a much smaller support, similar to the one occurring in the J. L. Lebowitz and E. Presutti paper [206]. Recall that the 'lattice' $\mathbb{L}$ is endowed with the metric (1.1.1) which has the property (1.1.2). Given $b > 0$ and $\sigma \in (0, 1/2)$, we define

$$\begin{aligned} \Xi(b, \sigma) = \{\xi \in \Omega_\beta \mid (\forall \ell_0 \in \mathbb{L})\ (\exists \Lambda_{\xi,\ell_0} \in \mathfrak{L}_{\mathrm{fin}})\ (\forall \ell \in \Lambda^c_{\xi,\ell_0}): \\ \|\xi_\ell\|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|)\}, \end{aligned} \tag{3.3.22}$$

which in view of (3.1.19) is a Borel subset of $\Omega^{\mathrm{t}}_\beta$.

Theorem 3.3.11. *For every $\sigma \in (0, 1/2)$, there exists $b > 0$, which depends on σ and on the parameters of the model only, such that for all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$,*

$$\mu(\Xi(b, \sigma)) = 1. \tag{3.3.23}$$

Proof. To some extent we shall follow the line of arguments used in the proof of Lemma 3.1 in [206]. Given $\ell, \ell_0, b > 0$, $\sigma \in (0, 1/2)$, and $\Lambda \subset \mathbb{L}$, we introduce

$$\begin{aligned} \Xi_\ell(\ell_0, b, \sigma) &= \{\xi \in \Omega_\beta \mid \|\xi_\ell\|^2_{C^\sigma_\beta} \leq b \log(1 + |\ell - \ell_0|)\}, \\ \Xi_\Lambda(\ell_0, b, \sigma) &= \bigcap_{\ell \in \Lambda} \Xi_\ell(\ell_0, b, \sigma). \end{aligned} \tag{3.3.24}$$

For a cofinal sequence $\mathcal{L}$, we set

$$\Xi(\ell_0, b, \sigma) = \bigcup_{\Lambda \in \mathcal{L}} \Xi_{\Lambda^c}(\ell_0, b, \sigma), \quad \Xi(b, \sigma) = \bigcap_{\ell_0 \in \mathbb{L}} \Xi(\ell_0, b, \sigma). \tag{3.3.25}$$

The latter $\Xi(b, \sigma)$ is a subset of $\Omega^{\mathrm{t}}_\beta$ and is the same as the one given by (3.3.22). To prove the theorem let us show that for any $\sigma \in (0, 1/2)$, there exists $b > 0$ such that for all ℓ_0 and $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$,

$$\mu\left(\Omega_\beta \setminus \Xi(\ell_0, b, \sigma)\right) = 0. \tag{3.3.26}$$

By (3.3.24) we have

$$\begin{aligned} &\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma) \\ &\quad = \{\xi \in \Omega_\beta \mid (\exists \ell \in \Lambda^c): \ \|\xi_\ell\|^2_{C^\sigma_\beta} > b \log(1 + |\ell - \ell_0|)\} \\ &\quad \subset \{\xi \in \Omega_\beta \mid (\exists \ell \in \Delta^c): \ \|\xi_\ell\|^2_{C^\sigma_\beta} > b \log(1 + |\ell - \ell_0|)\}, \end{aligned} \tag{3.3.27}$$

for any $\Lambda \subset \Delta$. Therefore,

$$\mu\Big(\bigcap_{\Lambda \in \mathcal{L}} \left[\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right]\Big) = \lim_{\mathcal{L}} \mu\left(\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right), \tag{3.3.28}$$

which holds for any cofinal sequence $\mathcal{L}$. By (3.3.27),

$$\begin{aligned} \mu\left(\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right) &= \mu\Big(\bigcup_{\ell \in \Lambda^c} \left[\Omega_\beta \setminus \Xi_\ell(\ell_0, b, \sigma)\right]\Big) \\ &\leq \sum_{\ell \in \Lambda^c} \mu\Big(\{\xi \mid \|\xi_\ell\|^2_{C^\sigma_\beta} > b \log(1 + |\ell - \ell_0|)\}\Big) \\ &= \sum_{\ell \in \Lambda^c} \mu\Big(\{\xi \mid \exp\left(\lambda \|\xi_\ell\|^2_{C^\sigma_\beta}\right) > (1 + |\ell - \ell_0|)^{b\lambda_\sigma}\}\Big). \end{aligned}$$

Applying here the Chebyshev inequality and the estimate (3.3.1) we get

$$\mu\left(\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right) \leq C_{3.3.1} \sum_{\ell \in \Lambda^c} (1 + |\ell - \ell_0|)^{-b\lambda}.$$

Then, by (1.1.2) and (3.3.28),

$$\mu\left(\Omega_\beta \setminus \Xi(\ell_0, b, \sigma)\right) = \lim_{\mathcal{L}} \mu\left(\left[\Omega_\beta \setminus \Xi_{\Lambda^c}(\ell_0, b, \sigma)\right]\right) = 0,$$

whenever $b > d/\lambda$. Thus, (3.3.26) holds. $\square$

3.4 Analytic Approach to Euclidean Gibbs Measures

In this section, we develop yet another approach to the description of the Euclidean Gibbs measures of the model (1.1.3), (1.1.8). It is based on the flow characterization of the Euclidean Gibbs measures in terms of their Radon–Nikodym derivatives with respect to the shift transformations of the configuration space Ω_β. Then we characterize such measures in terms of their logarithmic derivatives, which allow us to derive integration by parts formulas. Obviously, here we have to assume, in addition to Assumption 1.1.1, that the anharmonic potentials V_ℓ are differentiable.

3.4.1 Flow Description of Euclidean Gibbs Measures

To harmonize notation we introduce the following compound index $\mathrm{i} = (j, k, \ell)$, where $j = 1, \dots, \nu$, $k \in \mathcal{K}$ (defined by (1.3.20)) and $\ell \in \mathbb{L}$. For such i, we set

$$\phi_{\mathrm{i}} = \wp_j \otimes e_k \otimes \delta_\ell = \epsilon_{j,k} \otimes \delta_\ell, \tag{3.4.1}$$

where $\{\wp_j\}_{j=1,\dots\nu}$ is the canonical basis of $\mathbb{R}^\nu$, e_k were defined in (1.3.111), and $\delta_\ell \colon \mathbb{L} \to \{0, 1\}$ is $\delta_\ell(\ell') = \delta_{\ell\ell'}$. The set of all such i will be denoted by $\mathfrak{I}$. Then $\{\phi_{\mathrm{i}}\}_{\mathrm{i}\in\mathfrak{I}}$ is a subset of Ω_β. For $t \in \mathbb{R}$, $\mathrm{i} \in \mathfrak{I}$, and $B \in \mathcal{B}_\beta$, we set

$$B + t\phi_{\mathrm{i}} = \{x + t\phi_{\mathrm{i}} \mid x \in B\}. \tag{3.4.2}$$

Then for a given $\mu \in \mathcal{P}(\Omega_\beta)$, we define

$$\mu^{t\phi_{\mathrm{i}}}(B) = \mu(B + t\phi_{\mathrm{i}}). \tag{3.4.3}$$

If this new measure is absolutely continuous with respect to μ, its Radon–Nikodym derivative can give us important information about μ itself. As an example, let us consider the Gaussian measure $\chi = \chi_{\beta,\mathbb{L}}$ defined by (1.3.157), which exists as a product measure. By standard methods one calculates

$$\begin{aligned}
\frac{\mathrm{d}\chi^{t\phi_{\mathrm{i}}}}{\mathrm{d}\chi}(x) &= \frac{\mathrm{d}\chi_\beta^{t\epsilon_{j,k}}}{\mathrm{d}\chi_\beta}(x_\ell) \\
&= \exp\left(-\frac{t^2}{2}(S_\beta^{-1}\epsilon_{j,k}, \epsilon_{j,k})_{L^2_\beta} - t(S_\beta^{-1}\epsilon_{j,k}, x_\ell)_{L^2_\beta} \right) \\
&= \exp\left(-\frac{t^2}{2} \cdot \frac{1}{mk^2 + a} - \frac{t}{mk^2 + a}(\epsilon_{j,k}, x_\ell)_{L^2_\beta} \right) \stackrel{\mathrm{def}}{=} \mathbf{a}^{(0)}_{t\phi_{\mathrm{i}}}(x).
\end{aligned} \tag{3.4.4}$$

These Radon–Nikodym derivatives uniquely determine the measures $\chi_{\beta,\ell}$ and hence χ by the properties of Gaussian measures.

Proposition 3.4.1. *Let a probability measure μ on the real Hilbert space L^2_β be such that for any $t \in \mathbb{R}$ and $j = 1, \dots, \nu$, $k \in \mathcal{K}$, the shifted measure $\mu^{t\epsilon_{j,k}}$ is absolutely* *continuous with respect to μ with the Radon–Nikodym derivative*

$$\frac{\mathrm{d}\mu^{t\epsilon_{j,k}}}{\mathrm{d}\mu}(\xi) = \exp\left(- \frac{t^2}{2}(S_\beta^{-1}\epsilon_{j,k}, \epsilon_{j,k})_{L^2_\beta} - t(S_\beta^{-1}\epsilon_{j,k}, \xi)_{L^2_\beta}\right), \quad \xi \in L^2_\beta. \tag{3.4.5}$$

Then $\mu = \chi_\beta$, where the latter is the Høegh-Krohn measure described in Subsection 1.3.6; in particular, its Fourier transform has the form (1.3.109).

Proof. First of all, one observes that the vectors $\{\epsilon_{j,k}\}_{j=1,\ldots,\nu;\ k\in\mathcal{K}}$ constitute a basis of the space L^2_β. Let $L^2_{\beta,\mathrm{fin}}$ be the set of all finite linear combinations of these vectors. It is a dense linear subset of L^2_β. For any $\eta \in L^2_{\beta,\mathrm{fin}}$, one finds $n \in \mathbb{N}$, $t_1, \ldots, t_n \in \mathbb{R}$, $j_1, \ldots, j_n = 1, \ldots, \nu$ and $k_1, \ldots, k_n \in \mathcal{K}$ such that

$$\eta = t_1\epsilon_{j_1,k_1} + \cdots + t_n\epsilon_{j_n,k_n}.$$

Then, see (1.3.112),

$$S_\beta^{-1}\eta = (t_1/s_{k_1})\epsilon_{j_1,k_1} + \cdots + (t_n/s_{k_n})\epsilon_{j_n,k_n},$$

which means $L^2_{\beta,\mathrm{fin}} \subset \mathrm{Dom}(S_\beta^{-1})$ and $S_\beta^{-1} : L^2_{\beta,\mathrm{fin}} \to L^2_{\beta,\mathrm{fin}}$. Applying (3.4.5) n times one gets

$$\frac{\mathrm{d}\mu^{\eta}}{\mathrm{d}\mu}(\xi) = \exp\left(- \frac{1}{2}(S_\beta^{-1}\eta, \eta)_{L^2_\beta} - (S_\beta^{-1}\eta, \xi)_{L^2_\beta}\right). \tag{3.4.6}$$

This immediately yields that the Fourier transform of the measure μ, i.e., the function

$$\varphi_\mu(\zeta) = \int_{L^2_\beta} \exp\left(\mathrm{i}(\zeta, \xi)_{L^2_\beta}\right) \mu(\mathrm{d}\xi), \quad \zeta \in L^2_\beta, \tag{3.4.7}$$

can be extended to $L^2_\beta \oplus \mathrm{i}L^2_{\beta,\mathrm{fin}}$. Indeed, for $\zeta \in L^2_\beta$ and $\zeta' \in L^2_{\beta,\mathrm{fin}}$, we set $\eta = S_\beta\zeta'$ and obtain by (3.4.7) and (3.4.6) that

$$\begin{aligned}\varphi_\mu(\zeta + \mathrm{i}\zeta') &= \int_{L^2_\beta} \exp\left(\mathrm{i}(\zeta, \xi)_{L^2_\beta} - (\zeta', \xi)_{L^2_\beta}\right) \mu(\mathrm{d}\xi) \\ &= \exp\left(\frac{1}{2}(S_\beta^{-1}\eta, \eta)_{L^2_\beta}\right) \int_{L^2_\beta} \exp\left(\mathrm{i}(\zeta, \xi)_{L^2_\beta}\right)\mu^{\eta}(\mathrm{d}\xi) \\ &= \exp\left(\frac{1}{2}(S_\beta\zeta', \zeta')_{L^2_\beta}\right) \varphi_{\mu^\eta}(\zeta).\end{aligned} \tag{3.4.8}$$

In particular,

$$\varphi_\mu(\mathrm{i}\zeta) = \exp\left(\frac{1}{2}(S_\beta\zeta, \zeta)_{L^2_\beta}\right), \quad \zeta \in L^2_{\beta,\mathrm{fin}},$$

which yields

$$\varphi_\mu(\zeta) = \exp\left(-\frac{1}{2}(S_\beta\zeta, \zeta)_{L^2_\beta}\right), \quad \zeta \in L^2_{\beta,\mathrm{fin}}. \tag{3.4.9}$$

By the Minlos–Sazonov theorem, the characteristic function φ_μ is continuous on L^2_β (it is continuous even in a weaker topology, see Proposition 1.3.39). Then the representation (3.4.9) can be extended to the whole space L^2_β. Comparing (3.4.9) and (1.3.109), by Proposition 1.3.38 one gets $\mu = \chi_\beta$. □

Given $x \in \Omega_\beta$, $\mathrm{i} = (j, k, \ell)$, $j = 1, \ldots, \nu$, $k \in \mathcal{K}$, $\ell \in \mathbb{L}$ and $t \in \mathbb{R}$, we set

$$W_{\mathrm{i}}(x_\ell, t) = \int_0^\beta \left[V_\ell(x_\ell(\tau) + t\epsilon_{j,k}(\tau)) - V_\ell(x_\ell)\right] \mathrm{d}\tau, \tag{3.4.10}$$

and

$$\mathbf{a}^V_{t\phi_{\mathrm{i}}}(x) = \exp\left(t \sum_{\ell'} J_{\ell\ell'}(\epsilon_{j,k}, x_{\ell'})_{L^2_\beta} - W_{\mathrm{i}}(x_\ell, t)\right). \tag{3.4.11}$$

As in the proof of Proposition 3.1.15, one shows that the map

$$\Omega^{\mathrm{t}}_\beta \ni x \mapsto \sum_{\ell'} J_{\ell\ell'}(\epsilon_{j,k}, x_{\ell'})_{L^2_\beta} \in \mathbb{R}$$

is continuous. We recall that all V_ℓ's are continuous; hence, the above $\mathbf{a}^V_{t\phi_{\mathrm{i}}}$ is a continuous function on $\Omega^{\mathrm{t}}_\beta$. Now, having in mind (3.4.4), we set

$$\mathbf{a}_{t\phi_{\mathrm{i}}}(x) = \mathbf{a}^{(0)}_{t\phi_{\mathrm{i}}}(x) \cdot \mathbf{a}^V_{t\phi_{\mathrm{i}}}(x). \tag{3.4.12}$$

Definition 3.4.2. Given $\mathrm{i} \in \mathfrak{I}$ and $t \in \mathbb{R}$, by $\mathcal{M}_{\mathbf{a}_{t\phi_{\mathrm{i}}}}$ we denote the set of measures $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ which are quasi-invariant with respect to the shifts (3.4.3) and are such that

$$\frac{\mathrm{d}\mu^{t\phi_{\mathrm{i}}}}{\mathrm{d}\mu}(x) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x), \quad x \in \Omega^{\mathrm{t}}_\beta. \tag{3.4.13}$$

Furthermore, we set

$$\mathcal{M}_{\mathbf{a}} \stackrel{\mathrm{def}}{=} \bigcap_{t\in\mathbb{R},\ \mathrm{i}\in\mathfrak{I}} \mathcal{M}_{\mathbf{a}_{t\phi_{\mathrm{i}}}}. \tag{3.4.14}$$

The main result of this subsection is the following theorem which gives a flow characterization of the set of tempered Euclidean Gibbs measures.

Theorem 3.4.3. *For the model considered, it follows that*

$$\mathcal{G}^{\mathrm{t}}_\beta = \mathcal{M}_{\mathbf{a}}. \tag{3.4.15}$$

Proof. We recall that for a given $\ell \in \mathbb{L}$, we have set $\Lambda_\ell = \mathbb{L} \setminus \{\ell\}$. Every $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ can be disintegrated with respect to its projection μ_{Λ_ℓ} onto $\Omega_{\beta,\Lambda_\ell}$, that is,

$$\mu(\mathrm{d}x_\ell, \mathrm{d}x_{\Lambda_\ell}) = \sigma_\mu(\mathrm{d}x_\ell|x_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}x_{\Lambda_\ell}), \tag{3.4.16}$$

where $\sigma_\mu(\mathrm{d}x_\ell|x_{\Lambda_\ell})$ is a probability measure on the Banach space $C_{\beta,\{\ell\}} = C_\beta$. It is not unique, which means that the measure μ can be decomposed like (3.4.16) with the same projection measure μ_{Λ_ℓ} and with some other measure $\tilde{\sigma}_\mu$, which, however, should satisfy the condition

$$\forall B \in \mathcal{B}_{\beta,\{\ell\}}: \quad \sigma_\mu(B|x_{\Lambda_\ell}) = \tilde{\sigma}_\mu(B|x_{\Lambda_\ell}), \quad \mu_{\Lambda_\ell}\text{-almost surely.} \tag{3.4.17}$$

Let ℓ in (3.4.16) be the same as in $\mathrm{i} = (j,k,\ell)$. Then, obviously,

$$\mu^{t\phi_{\mathrm{i}}}(\mathrm{d}x_\ell, \mathrm{d}x_{\Lambda_\ell}) = \sigma_\mu^{t\epsilon_{j,k}}(\mathrm{d}x_\ell|x_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}x_{\Lambda_\ell}). \tag{3.4.18}$$

If $\mu \in \mathcal{G}_\beta^{\mathrm{t}}$, then one can take the measure (3.1.41) with $\Lambda = \{\ell\}$ as σ_μ. Indeed, by (3.1.44), for any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, one has

$$\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi) \bigotimes_{\ell' \in \Lambda^c} \delta_{\xi_{\ell'}}(\mathrm{d}x_{\ell'}), \tag{3.4.19}$$

where $\delta_{\xi_{\ell'}}(\mathrm{d}x_{\ell'})$ is the Dirac measure concentrated at $x_{\ell'} = \xi_{\ell'}$. Now we insert the measure μ, as given by (3.4.16), and the kernel $\pi_{\beta,\Lambda}$, given by (3.4.19) with $\Lambda = \{\ell\}$, into the DLR equation (3.1.52). This yields

$$\begin{aligned}
\sigma_\mu(\mathrm{d}x_\ell|x_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}x_{\Lambda_\ell}) &= \int_{\Omega_\beta} \nu_{\beta,\{\ell\}}(\mathrm{d}x_\ell|\xi_{\Lambda_\ell}) \bigotimes_{\ell' \in \Lambda^c} \delta_{\xi_{\ell'}}(\mathrm{d}x_{\ell'}) \\
&\qquad \times \sigma_\mu(\mathrm{d}\xi_\ell|\xi_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}\xi_{\Lambda_\ell}) \\
&= \nu_{\beta,\{\ell\}}(\mathrm{d}x_\ell|x_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}x_{\Lambda_\ell}) \\
&\qquad \times \int_{C_\beta} \sigma_\mu(\mathrm{d}\xi_\ell|x_\Lambda) \\
&= \nu_{\beta,\{\ell\}}(\mathrm{d}x_\ell|x_{\Lambda_\ell})\mu_{\Lambda_\ell}(\mathrm{d}x_{\Lambda_\ell}),
\end{aligned}$$

where we have taken into account that the conditional measure (3.1.41) depends only on the configuration ξ_{Λ^c} (see (3.1.33)) and that the conditional measure σ_μ is μ_{Λ_ℓ}-almost surely a probability measure. Thus, for μ_{Λ_ℓ}-almost all $x_{\Lambda_\ell} \in \Omega_{\beta,\Lambda_\ell}$ both conditional measures coincide.

By (3.1.33), (3.1.41),

$$\frac{\mathrm{d}(\nu_{\beta,\{\ell\}})^{t\epsilon_{j,k}}(\cdot|x_{\Lambda_\ell})}{\mathrm{d}\nu_{\beta,\{\ell\}}(\cdot|x_{\Lambda_\ell})}(x_\ell) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x), \tag{3.4.20}$$

which, by (3.4.18), yields $\mathcal{G}_\beta^{\mathrm{t}} \subset \mathcal{M}_{\mathbf{a}}$. In order to prove the inverse inclusion we take an arbitrary $\mu \in \mathcal{M}_{\mathbf{a}}$ and represent it in the form of (3.4.16). Then, by (3.4.18)

$$\frac{\mathrm{d}\sigma_\mu^{t\epsilon_{j,k}}(\cdot|x_{\Lambda_\ell})}{\mathrm{d}\sigma_\mu(\cdot|x_{\Lambda_\ell})}(x_\ell) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x). \tag{3.4.21}$$

Now let us introduce the following conditional measure on C_β:

$$\gamma(\mathrm{d}x_\ell|x_{\Lambda_\ell}) = \exp\left(-\sum_{\ell' \in \Lambda_\ell} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} + \int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau \right) \sigma_\mu(\mathrm{d}x_\ell|x_{\Lambda_\ell}). \tag{3.4.22}$$

For μ_{Λ_ℓ}-almost all x_{Λ_ℓ} it is a Borel measure, though, it may not be finite. By (3.4.21),

$$\frac{\mathrm{d}\gamma^{t\epsilon_{j,k}}(\cdot|x_{\Lambda_\ell})}{\mathrm{d}\gamma(\cdot|x_{\Lambda_\ell})}(x_\ell) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x)\cdot\left[\mathbf{a}^V_{t\phi_{\mathrm{i}}}(x)\right]^{-1} = \mathbf{a}^{(0)}_{t\phi_{\mathrm{i}}}(x). \tag{3.4.23}$$

Then by Proposition 4 of [260], $\gamma(C_\beta|x_{\Lambda_\ell}) < \infty$ for μ_{Λ_ℓ}-almost all x_{Λ_ℓ}. Thus, up to a normalization constant, γ coincides with the measure χ_β, which follows from Proposition 3.4.1. Then, by (3.4.22), one immediately gets $\sigma_\mu(\cdot|x_{\Lambda_\ell}) = \nu_{\beta,\{\ell\}}(\cdot|x_{\Lambda_\ell})$ for μ_{Λ_ℓ}-almost all x_{Λ_l}, which completes the proof. □

3.4.2 Integration by Parts

As we shall show in this subsection, there exists one more property of Gibbs measures, which is equivalent to the quasi-invariance described above and which can be used to characterize the set $\mathscr{G}^{\mathrm{t}}_\beta$. This property is described in terms of the logarithmic derivatives. Clearly, here we should assume, in addition to Assumption 1.1.1, that the anharmonic potentials V_ℓ are of class $C^1(\mathbb{R}^\nu)$.

We start by making precise the notion of the derivative of a function $f : \Omega_\beta \to \mathbb{R}$ at a given $x \in \Omega_\beta$ in the direction $\phi \in \Omega_\beta$. Namely, we say that such a function is differentiable in the above sense if the function $\mathbb{R} \ni t \mapsto f(x+t\phi) \in \mathbb{R}$ is differentiable at $t = 0$. In this case we write

$$\partial_\phi f(x) = \left(\frac{\partial}{\partial t} f(x+t\phi)\right)_{t=0} = \lim_{t\to 0}\left(f(x+t\phi) - f(x)\right)/t. \tag{3.4.24}$$

We shall also use one-sided derivatives

$$\partial^\pm_\phi f(x) = \lim_{t\to\pm 0}\left(f(x+t\phi) - f(x)\right)/t. \tag{3.4.25}$$

Let ϕ_{i}, $\mathrm{i} \in \mathfrak{I}$ be as in (3.4.1) and

$$\Phi_\ell(x) \stackrel{\mathrm{def}}{=} \int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau, \tag{3.4.26}$$

where ℓ is the same as in $\mathrm{i} = (j,k,\ell)$. Then

$$\partial_{\phi_{\mathrm{i}}}\Phi_\ell(x) \stackrel{\mathrm{def}}{=} (F_\ell(x), \epsilon_{j,k})_{L^2_\beta}, \tag{3.4.27}$$

where $F_\ell \colon \Omega^{\mathrm{t}}_\beta \to C_\beta$ is defined by its values

$$\left(F_\ell(x)\right)(\tau) = \left(\nabla V_\ell\right)(x_\ell(\tau)). \tag{3.4.28}$$

Given ϕ_{i}, $\mathrm{i} \in \mathfrak{I}$, we set

$$\mathbf{b}_{\mathrm{i}}(x) = \left(\frac{\partial}{\partial t}\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\right)_{t=0}. \tag{3.4.29}$$

Then by (3.4.4), (3.4.10) and (3.4.26)-(3.4.28),

$$\mathbf{b}_{\mathrm{i}}(x) = -(S_\beta^{-1}\epsilon_{j,k}, x_\ell)_{L^2_\beta} + \sum_{\ell'} J_{\ell\ell'}(\epsilon_{j,k}, x_{\ell'})_{L^2_\beta} - (F_\ell(x), \epsilon_{j,k})_{L^2_\beta}. \tag{3.4.30}$$

On the other hand, by the chain rule

$$\begin{aligned}\mathbf{a}_{(t+\theta)\phi_{\mathrm{i}}}(x) &= \frac{\mathrm{d}\mu^{\theta\phi_{\mathrm{i}}}}{\mathrm{d}\mu}(x + t\phi_{\mathrm{i}}) \cdot \frac{\mathrm{d}\mu^{t\phi_{\mathrm{i}}}}{\mathrm{d}\mu}(x)\\ &= \mathbf{a}_{\theta\phi_{\mathrm{i}}}(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mathbf{a}_{t\phi_{\mathrm{i}}}(x + \vartheta\phi_{\mathrm{i}})\mathbf{a}_{\vartheta\phi_{\mathrm{i}}}(x),\end{aligned}$$

we get

$$\frac{\partial}{\partial t}\mathbf{a}_{t\phi_{\mathrm{i}}}(x) = \left(\frac{\partial}{\partial\theta}\mathbf{a}_{(t+\theta)\phi_{\mathrm{i}}}(x)\right)_{\theta=0} = \mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mathbf{b}_{\mathrm{i}}(x + t\phi_{\mathrm{i}}), \tag{3.4.31}$$

and

$$\begin{aligned}\partial_{\phi_{\mathrm{i}}}\mathbf{a}_{t\phi_{\mathrm{i}}}(x) &= \left(\frac{\partial}{\partial\vartheta}\mathbf{a}_{t\phi_{\mathrm{i}}}(x + \vartheta\phi_{\mathrm{i}})\right)_{\vartheta=0} = \left(\frac{\partial}{\partial\vartheta}\left(\frac{\mathbf{a}_{(t+\vartheta)\phi_{\mathrm{i}}}(x)}{\mathbf{a}_{\vartheta\phi_{\mathrm{i}}}(x)}\right)\right)_{\vartheta=0}\\ &= \frac{\partial}{\partial t}\mathbf{a}_{t\phi_{\mathrm{i}}}(x) - \mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mathbf{b}_{\mathrm{i}}(x) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x)\left[\mathbf{b}_{\mathrm{i}}(x + t\phi_{\mathrm{i}}) - \mathbf{b}_{\mathrm{i}}(x)\right].\end{aligned} \tag{3.4.32}$$

One can consider (3.4.31) as a differential equation subject to the initial condition $\mathbf{a}_0(x) = 1$, the solution of which is

$$\mathbf{a}_{t\phi_{\mathrm{i}}}(x) = \exp\left(\int_0^t \mathbf{b}_{\mathrm{i}}(x + \vartheta\phi_{\mathrm{i}})\mathrm{d}\vartheta\right). \tag{3.4.33}$$

Definition 3.4.4. Given $\mathrm{i} = (j, k, l) \in \mathfrak{I}$, the family $\mathcal{F}_{\mathrm{i}}$ consists of the functions $f : \Omega^{\mathrm{t}}_\beta \to \mathbb{R}$ possessing the following properties: (a) at every $x \in \Omega^{\mathrm{t}}_\beta$, there exists the derivative $\partial_{\phi_{\mathrm{i}}} f(x)$; (b) both f and $\partial_{\phi_{\mathrm{i}}} f$ are continuous functions on $\Omega^{\mathrm{t}}_\beta$; (c) there exist $\lambda, \varkappa > 0$, $n \in \mathbb{N}$, $n > r$, where r is the same as in (1.1.10), and a bounded continuous local function $g : \Omega_\beta \to \mathbb{R}$, such that

$$f(x) = g(x)\exp\left(-\lambda\int_0^\beta |x_\ell(\tau)|^{2n}\mathrm{d}\tau - \varkappa\sum_{\ell'} |J_{\ell\ell'}| \cdot \|x_{\ell'}\|^2_{L^2_\beta}\right). \tag{3.4.34}$$

Obviously, for every $t \in \mathbb{R}$ and $f \in \mathcal{F}_{\mathrm{i}}$, the functions

$$\partial_{\phi_{\mathrm{i}}} f(x), \quad f(x)\mathbf{b}_{\mathrm{i}}(x + t\phi_{\mathrm{i}}), \quad f(x)\mathbf{a}_{t\phi_{\mathrm{i}}}(x), \quad f(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x) \tag{3.4.35}$$

belong to $\mathcal{F}_{\mathrm{i}}$ as well. Moreover, the family $\mathcal{F}_{\mathrm{i}}$ is closed under multiplication and separates points of $\Omega^{\mathrm{t}}_\beta$. Then, since $\Omega^{\mathrm{t}}_\beta$ is a Polish space, this family is a measure-defining set in the sense of Theorem 1.3.26.

Let $\mathcal{M}_{\mathbf{b}_{\mathrm{i}}}$ stand for the class of measures $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$ which satisfy the integration-by-parts formula

$$\int_{\Omega^{\mathrm{t}}_{\beta}} \partial_{\phi_{\mathrm{i}}} f(x)\mu(\mathrm{d}x) = -\int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{b}_{\mathrm{i}}(x)\mu(\mathrm{d}x), \tag{3.4.36}$$

for all $f \in \mathcal{F}_{\mathrm{i}}$. Recall that the set $\mathcal{M}_{t\mathbf{a}_{\phi_{\mathrm{i}}}}$ was introduced in Definition 3.4.2.

Theorem 3.4.5. *For every* $\mathrm{i} \in \mathfrak{I}$,

$$\mathcal{M}_{\mathbf{b}_{\mathrm{i}}} = \bigcap_{t\in\mathbb{R}} \mathcal{M}_{t\mathbf{a}_{\phi_{\mathrm{i}}}}.$$

Therefore,

$$\mathcal{G}^{\mathrm{t}}_{\beta} = \mathcal{M}_{\mathbf{a}} = \mathcal{M}_{\mathbf{b}} \stackrel{\mathrm{def}}{=} \bigcap_{\mathrm{i}\in\mathfrak{I}} \mathcal{M}_{\mathbf{b}_{\mathrm{i}}}.$$

Proof. First we show that every μ satisfying (3.4.13) with a certain $t \in \mathbb{R}$ obeys also (3.4.36). For any $f \in \mathcal{F}_{\mathrm{i}}$,

$$\int_{\Omega^{\mathrm{t}}_{\beta}} f(x - t\phi_{\mathrm{i}})\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mu^{t\phi_{\mathrm{i}}}(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mu(\mathrm{d}x).$$

Thus, for $t \neq 0$, we have

$$\int_{\Omega^{\mathrm{t}}_{\beta}} [(f(x - t\phi_{\mathrm{i}}) - f(x))/t]\,\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\left[(\mathbf{a}_{t\phi_{\mathrm{i}}}(x) - 1)/t\right]\mu(\mathrm{d}x). \tag{3.4.37}$$

In view of (3.4.24) and (3.4.31), the equality (3.4.37) may be rewritten in the form

$$-\int_{\Omega^{\mathrm{t}}_{\beta}} \partial_{\phi_{\mathrm{i}}} f(x - \vartheta\phi_{\mathrm{i}})\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{a}_{\theta\phi_{\mathrm{i}}}(x)\mathbf{b}_{\mathrm{i}}(x + \theta\phi_{\mathrm{i}})\mu(\mathrm{d}x), \tag{3.4.38}$$

with certain $\theta, \vartheta \in \mathbb{R}$ such that $0 \leq |\theta| < |t|$, $0 \leq |\vartheta| < |t|$. Both functions under the integrals in (3.4.38) are bounded and continuous on Ω_{β}. Moreover, for any $x \in \Omega^{\mathrm{t}}_{\beta}$,

$$\partial_{\phi_{\mathrm{i}}} f(x - \vartheta\phi_{\mathrm{i}}) \to \partial_{\phi_{\mathrm{i}}} f(x), \quad \mathbf{a}_{\theta\phi_{\mathrm{i}}}(x)\mathbf{b}_{\mathrm{i}}(x + \theta\phi_{\mathrm{i}}) \to \mathbf{b}_{\mathrm{i}}(x),$$

as $t \to 0$. Thus, by Lebesgue's dominated convergence theorem, the limit of (3.4.38) is (3.4.36).

Now let us show that a measure μ, which satisfies (3.4.36) with every $f \in \mathcal{F}_{\mathrm{i}}$, is quasi-invariant with respect to the shifts (3.4.2) with any $t \in \mathbb{R}$ and the corresponding Radon–Nikodym derivatives are given by (3.4.13). In view of (3.4.35), the function $\tilde{f}_t(x) \stackrel{\mathrm{def}}{=} f(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x)$ belongs to $\mathcal{F}_{\mathrm{i}}$ for any $f \in \mathcal{F}_{\mathrm{i}}$ and $t \in \mathbb{R}$; hence, the integration-by-parts formula (3.4.36) may be applied to this function. This yields

$$\int_{\Omega^{\mathrm{t}}_{\beta}} \left[\left(\partial_{\phi_{\mathrm{i}}} f\right)(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x) + f(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mathbf{b}_{\mathrm{i}}(x + t\phi_{\mathrm{i}})\right]\mu(\mathrm{d}x) = 0, \tag{3.4.39}$$

where we have also used (3.4.32). In the same way as we have proven that (3.4.37) tends to (3.4.36) as $t \to 0$, we can show that

$$\frac{\partial}{\partial t}\int_{\Omega^{\mathrm{t}}_{\beta}} \tilde{f}_t(x)\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} \frac{\partial \tilde{f}_t(x)}{\partial t}\mu(\mathrm{d}x),$$

which, by (3.4.39), yields

$$\frac{\partial}{\partial t}\int_{\Omega^{\mathrm{t}}_{\beta}} \tilde{f}_t(x)\mu(\mathrm{d}x) = 0.$$

Therefore, for all $t \in \mathbb{R}$,

$$\int_{\Omega^{\mathrm{t}}_{\beta}} f(x + t\phi_{\mathrm{i}})\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x + t\phi_{\mathrm{i}})\mu^{t\phi_{\mathrm{i}}}(\mathrm{d}x), \tag{3.4.40}$$

which holds for every $f \in \mathcal{F}_{\mathrm{i}}$. Since $\mathcal{F}_{\mathrm{i}}$ is a measure-defining set, the integrals in (3.4.40) uniquely determine the measure (see Theorem 1.3.26). This yields $\mu^{t\phi_{\mathrm{i}}}(\mathrm{d}x) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x)\mu(\mathrm{d}x)$. □

In what follows, Theorem 3.4.5 gives one more characterization of Euclidean Gibbs measures. Namely, a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$ is a Euclidean Gibbs measure of the model described by the heuristic Hamiltonian (1.1.8) if and only if it satisfies the integration-by-parts formulas (3.4.36), (3.4.41) with the prescribed logarithmic derivatives $\mathbf{b}_{\mathrm{i}}$, for all $\mathrm{i} \in \mathfrak{I}$. These logarithmic derivatives (3.4.30) are expressed through the potentials $J_{\ell\ell'}$ and V_ℓ which determine the model (1.1.8); hence, they are the same for all elements of $\mathcal{G}^{\mathrm{t}}_{\beta}$.

Remark 3.4.6. Let $f : \Omega^{\mathrm{t}}_{\beta} \to \mathbb{R}$ be as in Definition 3.4.4 but possessing only bounded continuous one-sided derivatives $\partial^{\pm}_{\phi_{\mathrm{i}}} f$. Then for any $\mu \in \mathcal{M}_{\mathbf{b}_{\mathrm{i}}}$ and any such f, the integration-by-parts formula (3.4.36) holds in the form

$$\int_{\Omega^{\mathrm{t}}_{\beta}} \partial^{\pm}_{\phi_{\mathrm{i}}} f(x)\mu(\mathrm{d}x) = -\int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{b}_{\mathrm{i}}(x)\mu(\mathrm{d}x). \tag{3.4.41}$$

The proof of (3.4.41) is exactly the same as for Theorem 3.4.5. Here one should take into account that the function $\mathbf{a}_{t\phi_{\mathrm{i}}}$ is differentiable, hence, $\partial^{+}_{\phi_{\mathrm{i}}}\mathbf{a}_{t\phi_{\mathrm{i}}} = \partial^{-}_{\phi_{\mathrm{i}}}\mathbf{a}_{t\phi_{\mathrm{i}}}$.

We conclude this subsection by establishing an analog of the formulas (3.4.36), (3.4.41) for the kernels (3.1.44), which, for any fixed $\xi \in \Omega^{\mathrm{t}}_{\beta}$, are elements of $\mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$. Given $\mathrm{i} = (j, k, \ell)$, let $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ contain ℓ. Then by (3.1.41), (3.1.44),

$$\frac{\mathrm{d}(\pi_{\beta,\Lambda})^{t\phi_{\mathrm{i}}}}{\mathrm{d}\pi_{\beta,\Lambda}}(x|\xi) = \mathbf{a}_{t\phi_{\mathrm{i}}}(x), \tag{3.4.42}$$

which holds for all $\xi \in \Omega^{\mathrm{t}}_{\beta}$, $t \in \mathbb{R}$ and for $\pi_{\beta,\Lambda}(\cdot|\xi)$-almost all $x \in \Omega^{\mathrm{t}}_{\beta}$. As above, this immediately yields that for all $f \in L^1(\Omega_\beta, \pi_{\beta,\Lambda}(\cdot|\xi))$,

$$\int_{\Omega^{\mathrm{t}}_{\beta}} f(x - t\phi_{\mathrm{i}})\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = \int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{a}_{t\phi_{\mathrm{i}}}(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi).$$

Proceeding as in the proof of Theorem 3.4.5, from this identity one can show that the kernel $\pi_{\beta,\Lambda}(\cdot|\xi)$ satisfies integration-by-parts formulas like (3.4.36), (3.4.41), e.g.,

$$\int_{\Omega^{\mathrm{t}}_{\beta}} \partial^{\pm}_{\phi_{\mathrm{i}}} f(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) = -\int_{\Omega^{\mathrm{t}}_{\beta}} f(x)\mathbf{b}_{\mathrm{i}}(x)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi), \tag{3.4.43}$$

with the same functions $f \in \mathcal{F}_{\mathrm{i}}$ and $\mathbf{b}_{\mathrm{i}}$ and with $\mathrm{i} = (j,k,\ell)$ such that $\ell \in \Lambda$. Therefore, $\pi_{\beta,\Lambda}(\cdot|\xi) \in \mathcal{M}_{\mathbf{b}_{\mathrm{i}}}$ with such i. Here one has to remark that these facts do not yield that $\pi_{\beta,\Lambda}$ belongs to the classes $\mathcal{M}_{\mathbf{a}} = \mathcal{M}_{\mathbf{b}}$, because (3.4.42) and the corresponding integration-by-parts formulas hold *not for all* $\mathrm{i} \in \mathfrak{I}$.

Now let $\mathcal{L}$ be a tending to infinity sequence of finite subsets $\Lambda \subset \mathbb{L}$. Given $\xi \in \Omega^{\mathrm{t}}_{\beta}$, suppose that the sequence of probability measures $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\mathcal{L}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$. Since for $f \in \mathcal{F}_{\mathrm{i}}$, the functions under the integrals in (3.4.43) are bounded, both sides of this equation converge, as $\Lambda \to \mathbb{L}$, to the integration-by-parts formula for the measure μ. This formula holds *for all* $\mathrm{i} \in \mathfrak{I}$, because the sequence $\mathcal{L}$ exhausts the lattice; hence, $\mu \in \mathcal{M}_{\mathbf{b}} = \mathcal{G}^{\mathrm{t}}_{\beta}$. Therefore, if for a given $\xi \in \Omega^{\mathrm{t}}_{\beta}$ and a cofinal sequence $\mathcal{L}$, the sequence of probability kernels $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\mathcal{L}}$ converges in $\mathcal{W}^{\mathrm{t}}$ to a measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$, this μ should be a tempered Euclidean Gibbs measure.

3.5 The Case of Local Interactions

Here we consider the model (1.1.8) with interaction intensities which have finite range, see Definition 1.1.2. In this case, for a given oscillator, the set of oscillators interacting directly with this oscillator is finite, which agrees with the notion of locality used in this book. Then the second term in the energy functional (3.1.33) makes sense for all $\xi \in \Omega_{\beta}$. Therefore, one can define *a priori* a wider class of probability measures satisfying the DLR equation (3.1.52).

For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, we introduce the set

$$\partial^{+}\Lambda = \{\ell' \in \Lambda^{c} \mid \exists \ell \in \Lambda : \ J_{\ell\ell'} \neq 0\}, \tag{3.5.1}$$

which also belongs to $\mathfrak{L}_{\mathrm{fin}}$. Thus,

$$\Big| \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^{c}} J_{\ell\ell'}(x_{\ell},\xi_{\ell'})_{L^2_{\beta}} \Big| \leq |\Lambda|\hat{J}_0 \sup_{\ell\in\Lambda}\left(\|x_{\ell}\|_{L^2_{\beta}}\right)\cdot \sup_{\ell'\in\partial^{+}\Lambda}\left(\|\xi_{\ell'}\|_{L^2_{\beta}}\right). \tag{3.5.2}$$

This proves the following statement, cf. Lemma 3.1.12.

Lemma 3.5.1. *If the interaction has finite range, the map*

$$\Omega_{\beta}\times\Omega_{\beta} \ni (x,\xi) \mapsto E_{\beta,\Lambda}(x|\xi), \tag{3.5.3}$$

is continuous for any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$. *Furthermore, for every* $R > 0$,

$$\begin{aligned} \inf_{x\in\Omega_{\beta},\ \xi\in B_{\partial^{+}\Lambda}(R)} E_{\beta,\Lambda}(x|\xi) &> -\infty, \\ \sup_{x\in B_{\Lambda}(R),\ \xi\in B_{\partial^{+}\Lambda}(R)} E_{\beta,\Lambda}(x|\xi) &< +\infty, \end{aligned} \tag{3.5.4}$$

where

$$B_\Lambda(R) = \{x \in \Omega_\beta \mid \sup_{\ell \in \Lambda} \|x_\ell\|_{L^2_\beta} \leq R\}. \tag{3.5.5}$$

Then the partition function (3.1.39) can be defined for all $\xi \in \Omega_\beta$. It is then continuous and obeys the estimates (3.1.40) with $B_{\ell_0,\alpha}$ replaced by B given by (3.5.5). Correspondingly, the local Gibbs measure (3.1.41) can also be defined for all $\xi \in \Omega_\beta$, which allows us to introduce, cf. (3.1.44),

$$\widetilde{\pi}_{\beta,\Lambda}(B|\xi) = \int_{C_{\beta,\Lambda}} \mathbb{I}_B(x_\Lambda \times \xi_{\Lambda^c}) \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda|\xi), \quad \xi \in \Omega_\beta, \ \Lambda \in \mathfrak{L}_{\mathrm{fin}}, \tag{3.5.6}$$

which is a probability kernel. The set $\widetilde{\Pi}_\beta = \{\widetilde{\pi}_{\beta,\Lambda}\}_{\Lambda \in \mathfrak{L}_{\mathrm{fin}}}$ is called the local Gibbs specification for the model (1.1.8). Exactly as in Lemma 3.1.17 one proves that the map $f \mapsto \widetilde{\pi}_{\beta,\Lambda}(f|\cdot)$ defined by (3.1.48) is a contraction on $C_{\mathrm{b}}(\Omega_\beta)$.

Definition 3.5.2. A measure $\mu \in \mathcal{P}(\Omega_\beta)$ is called a Euclidean Gibbs measure for the model (1.1.8), if for any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, it solves the DLR equation, see (3.1.51),

$$\mu \widetilde{\pi}_{\beta,\Lambda} = \mu. \tag{3.5.7}$$

The set of all such measures will be denoted by $\widetilde{\mathcal{G}}_\beta$.

Unlike the tempered Euclidean Gibbs measures introduced in Definition 3.1.18, the elements of $\widetilde{\mathcal{G}}_\beta$ have no prescribed supporting properties. As a result, the latter set may contain measures which have no physical relevance and hence should be excluded from the consideration. We perform this by imposing restrictions on the support of 'proper' elements of $\widetilde{\mathcal{G}}_\beta$. To this end we use weights and tempered configurations. Let $\{w_\alpha\}_{\alpha \in \mathcal{I}}$ be any system of weights possessing the properties established by Definition 3.1.8. Since the interaction $J_{\ell\ell'}$ has finite range, the condition (3.1.20) is automatically satisfied; hence, the only condition left is (3.1.19). Its meaning is that the configurations $x(x_\ell)_{\ell \in \mathbb{L}}$ with logarithmic growth of $\|x_\ell\|^2_{L^2_\beta}$, see (3.1.27), should be among the tempered configurations. On the other hand, the bigger set of tempered configurations we would like to have, the stronger decay of the weights $w_\alpha(\ell, \ell')$, as $|\ell - \ell'| \to +\infty$, we should admit.

In what follows, for a given system of weights $\{w_\alpha\}_{\alpha \in \mathcal{I}}$, which obeys the conditions of Definition 3.1.8 and (3.1.19), let the set of tempered configurations be defined by (3.1.30). Let also $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ be as in Definition 3.1.11. Then we set

$$\widetilde{\mathcal{G}}^{\mathrm{t}}_\beta = \widetilde{\mathcal{G}}_\beta \cap \mathcal{P}(\Omega^{\mathrm{t}}_\beta), \tag{3.5.8}$$

which need not coincide with the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}_\beta$ introduced in Definition 3.1.18. However, simple arguments based on the DLR equation lead to the conclusion that

$$\widetilde{\mathcal{G}}^{\mathrm{t}}_\beta = \mathcal{G}^{\mathrm{t}}_\beta. \tag{3.5.9}$$

But, for the elements of $\mathcal{G}^{\rm t}_\beta$, we have proven Theorem 3.3.11, i.e., these elements have much smaller support than $\Omega^{\rm t}_\beta$. Therefore, any restriction of the support of the elements of $\widetilde{\mathcal{G}}_\beta$, i.e., imposed by any system of weights obeying (3.1.16)–(3.1.18), (3.1.19), leads to (3.3.23). On the other hand, it is easy to show that, for any countable set of configurations $\mathcal{X} \subset \Omega_\beta$, there exists a system of weights with the properties just mentioned, such that the set $\Omega^{\rm t}_\beta$ defined by means of these weights contains $\mathcal{X}$.

Let us summarize the arguments presented above. For models with non-local interactions, the restrictions on the supporting properties of the Gibbs measures should be imposed *ab initio*, since this is the only way to define the second term in (3.1.33) and hence the kernels (3.1.44). These restrictions are imposed by means of weights, obeying the conditions (3.1.16)–(3.1.18), (3.1.19), (3.1.20). The latter two conditions are competitive, and, in principle, there exist models for which they contradict each other. If there exists a system of weights obeying all these conditions, then the set $\mathcal{G}^{\rm t}_\beta$ is non-empty, Theorem 3.3.6. *A posteriori* its elements have much smaller support than $\Omega^{\rm t}_\beta$, see (3.3.23). Significantly, this support does not depend on the particular choice of the weights. The same scheme of constructing Gibbs measures by imposing a priori growth restrictions with the help of weights can also be applied to models with local interactions. At the same time, for such models one can define Gibbs measures without any restriction on the support of its elements. In this case, the set of Gibbs measures $\widetilde{\mathcal{G}}_\beta$ might contain 'physically irrelevant' elements, which have to be excluded from the theory. This is done by imposing support conditions like (3.5.8), which, however, can be chosen in such a way that any given countable set of configurations belongs to $\Omega^{\rm t}_\beta$. Once the support restrictions are imposed, the set of Gibbs measures obeying these conditions is exactly the same as the one constructed according to the scheme which yields $\mathcal{G}^{\rm t}_\beta$.

3.6 Periodic Euclidean Gibbs Measures

In this section, we set $\mathbb{L} = \mathbb{Z}^d$ and consider a translation-invariant version of the model (1.1.8), see Definition 1.4.10, with $J_{\ell\ell'}$ being of general type. By means of the measures (1.4.52) we define, cf. (3.1.44),

$$\pi^{\rm per}_{\beta,\Lambda}(B) = \frac{1}{N^{\rm per}_{\beta,\Lambda}} \int_{\Omega_{\beta,\Lambda}} \exp\big(-E^{\rm per}_{\beta,\Lambda}(x_\Lambda)\big)\mathbb{I}_B(x_\Lambda \times 0_{\Lambda^c})\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \qquad (3.6.1)$$

where Λ is a box, see (1.4.46). The set of all such boxes will be denoted by $\mathfrak{L}_{\rm box}$. Comparing (3.6.1) with (1.4.52) one gets, cf. (3.4.19),

$$\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}(x_\Lambda \times x_{\Lambda^c})) = \nu^{\rm per}_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \bigotimes_{\ell'\in\Lambda^c} \delta_{0_{\ell'}}(\mathrm{d}x_{\ell'}), \qquad (3.6.2)$$

where $0_{\ell'}$ is the zero vector in the Banach space C_β. We note that $\pi^{\rm per}_{\beta,\Lambda}$ is a measure on the space $\Omega^{\rm t}_\beta$, whereas $\nu^{\rm per}_{\beta,\Lambda}$ is a measure on $C_{\beta,\Lambda}$. At the same time, the projection of $\pi^{\rm per}_{\beta,\Lambda}$ onto $C_{\beta,\Lambda}$ is exactly $\nu^{\rm per}_{\beta,\Lambda}$.

For a certain $\ell \in \Lambda$, we set $\Delta_\ell = \mathbb{L} \setminus \{\ell\}$. Let also $\rho^\ell_{\beta,\Lambda}$ be the projection of $\pi^{\rm per}_{\beta,\Lambda}$ onto $\mathcal{B}_{\beta,\Delta_\ell}$. On the single-spin space C_β we define the measure

$$\sigma^\ell_{\beta,\Lambda}(\mathrm{d}x_\ell|\xi) = \frac{1}{N_\ell(\xi)} \exp\left\{\frac{1}{2}\sum_{\ell'\in\Lambda} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} - \int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\right\}\chi_\beta(\mathrm{d}x_\ell), \tag{3.6.3}$$

where $\xi \in \Omega^{\rm t}_\beta$ and

$$N_\ell(\xi) = \int_{C_\beta} \exp\left\{\frac{1}{2}\sum_{\ell'\in\Lambda} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} - \int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\right\}\chi_\beta(\mathrm{d}x_\ell). \tag{3.6.4}$$

Like in (3.4.16) we disintegrate

$$\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}x) = \sigma^\ell_{\beta,\Lambda}(\mathrm{d}x_\ell|x_{\Delta_\ell})\rho^\ell_{\beta,\Lambda}(\mathrm{d}x_{\Delta_\ell}). \tag{3.6.5}$$

Let us prove that the measure (3.6.3) satisfies the exponential moment estimate (3.2.1) with the same constant $C_{3.2.1}$ and parameters $\varkappa$ and ϑ. As the interaction potential $J^\Lambda_{\ell\ell'}$, given by (1.4.50), is invariant under the action of the group $\mathbb{Z}^d/\mathbb{Z}^d_\Lambda$, the sum $\sum_{\ell'\in\Lambda}|J^\Lambda_{\ell\ell'}|$ is independent of ℓ. Furthermore, there exists $\ell_0 \in \Lambda$ such that, for any $\ell \in \Lambda$, $|\ell-\ell_0|_\Lambda = |\ell-\ell_0|$, see (1.4.48). Thus,

$$\sum_{\ell'\in\Lambda}|J^\Lambda_{\ell\ell'}| = \sum_{\ell'\in\Lambda}|J_{\ell_0\ell'}| \le \hat{J}_0. \tag{3.6.6}$$

Then, as in (3.2.2), we have

$$\left|\sum_{\ell'\in\Lambda} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\right| \le \frac{\hat{J}_0}{2\vartheta}\|x_\ell\|^2_{L^2_\beta} + \frac{\vartheta}{2}\sum_{\ell'\in\Lambda}|J^\Lambda_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}.$$

Applying this estimate in (3.6.3) and (3.6.4) we obtain the following analog of (3.2.1)

$$\begin{aligned}\int_{C_\beta} & \exp\left\{\lambda\|x_\ell\|^2_{C^\sigma_\beta} + \varkappa\|x_\ell\|^2_{L^2_\beta}\right\}\sigma^\ell_{\beta,\Lambda}(\mathrm{d}x_\ell|\xi) \\ & \le \exp\left\{C_{3.2.1} + \vartheta\sum_{\ell'\in\Lambda}|J^\Lambda_{\ell\ell'}|\cdot\|\xi_{\ell'}\|^2_{L^2_\beta}\right\}.\end{aligned} \tag{3.6.7}$$

We recall that the norm $\|x\|_{\ell_0,\alpha,\sigma}$ was defined in (3.2.17).

Lemma 3.6.1. *For every $\alpha \in \mathcal{I}$, $\sigma \in (0,1/2)$, and ℓ_0, there exists a constant $C_{3.6.8} > 0$, such that for all boxes Λ,*

$$\int_{\Omega^{\rm t}_\beta}\|x\|^2_{\ell_0,\alpha,\sigma}\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}x) \le C_{3.6.8}. \tag{3.6.8}$$

Thereby, the family of all measures (3.6.1) is $\mathcal{W}^{\rm t}$-relatively compact.

Proof. For $\ell \in \Lambda$, we set, cf. (3.2.8),

$$n_\ell(\Lambda) = \log\left\{ \int_{\Omega^{\rm t}_\beta} \exp\left[\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta}\right] \pi_{\beta,\Lambda}(\mathrm{d}x)\right\}. \tag{3.6.9}$$

As the measure $\pi_{\beta,\Lambda}(\mathrm{d}x)$ is translation-invariant, the above quantity is independent of ℓ. Then we take $\varkappa$ and ϑ such that $\vartheta \hat{J}_0 < \varkappa$, cf. (3.2.12), and perform the calculations which led to the estimate (3.2.13). Thereafter, we get

$$\begin{aligned} n_\ell(\Lambda) &\leq C_{3.2.1} + (\vartheta/\varkappa) \sum_{\ell' \in \Lambda} |J^\Lambda_{\ell\ell'}| \cdot n_{\ell'}(\Lambda) \\ &= C_{3.2.1} + (\vartheta \hat{J}_0/\varkappa) \cdot n_\ell(\Lambda), \end{aligned}$$

and hence

$$n_\ell(\Lambda) \leq \frac{C_{3.2.1}}{1 - \vartheta \hat{J}_0/\varkappa}. \tag{3.6.10}$$

By the Jensen inequality, we have

$$\int_{\Omega^{\rm t}_\beta} \|x_\ell\|^2_{C^\sigma_\beta} \pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}x) \leq n_\ell(\Lambda)/\varkappa,$$

which yields (3.6.8) with

$$C_{3.6.8} = \mathrm{RHS}(3.6.10) \cdot \sum_\ell w_\alpha(\ell_0, \ell).$$

The $\mathcal{W}^{\rm t}$-relative compactness of the family of all measures $\pi^{\rm per}_{\beta,\Lambda}$ follows from the estimate (3.6.8) by the arguments used in the proof of Theorem 3.3.6. $\square$

Let $\mathcal{M}^{\rm per}_\beta \subset \mathcal{P}(\Omega^{\rm t}_\beta)$ be the set of all accumulation points of the family $\{\pi^{\rm per}_{\beta,\Lambda}\}_{\Lambda \in \mathfrak{L}_{\rm box}}$. Our next goal is to prove that $\mathcal{M}^{\rm per}_\beta \subset \mathcal{G}^{\rm t}_\beta$. Note that each $\mu \in \mathcal{M}^{\rm per}_\beta$ can be obtained as a $\mathcal{W}^{\rm t}$-limit of a sequence $\{\pi^{\rm per}_{\beta,\Lambda}\}_{\Lambda \in \mathcal{L}_{\rm box}}$ taken along a cofinal sequence of boxes $\mathcal{L}_{\rm box}$.

Lemma 3.6.2. *Given $\sigma \in (0, 1/2)$, let λ and $\varkappa$ be as in (3.3.1). Then there exists $C_{3.6.11} > 0$, such that for every ℓ and $\mu \in \mathcal{M}^{\rm per}_\beta$,*

$$\int_{\Omega^{\rm t}_\beta} \exp\left(\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta}\right) \mu(\mathrm{d}x) \leq C_{3.6.11}. \tag{3.6.11}$$

Proof. By (3.6.9), (3.6.10), for every $\Lambda \in \mathfrak{L}_{\rm box}$ and $\ell \in \Lambda$,

$$\int_{\Omega^{\rm t}_\beta} \exp\left(\lambda \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa \|x_\ell\|^2_{L^2_\beta}\right) \pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}x) \leq \exp\left(\frac{C_{3.2.1}}{1 - \vartheta \hat{J}_0/\varkappa}\right).$$

Passing here to the limit along the sequence $\mathcal{L}_{\rm box}$ for which $\lim_{\mathcal{L}_{\rm box}} \pi^{\rm per}_{\beta,\Lambda} = \mu$, we arrive at (3.6.11). $\square$

As in the proof of Theorem 3.3.11, by means of the latter statement we get the following

Corollary 3.6.3. *Let $\sigma \in (0, 1/2)$, $b > 0$, and $\Xi(b, \sigma)$ be as in (3.3.23). Then for every $\pi^{\rm per}_{\beta,\Lambda}$, $\Lambda \in \mathfrak{L}_{\rm box}$, as well as for every $\mu \in \mathcal{M}^{\rm per}_\beta$, it follows that*

$$\pi^{\rm per}_{\beta,\Lambda}(\Xi(b,\sigma)) = \mu(\Xi(b,\sigma)) = 1. \tag{3.6.12}$$

Theorem 3.6.4. *Every accumulation point of the family $\{\pi^{\rm per}_{\beta,\Lambda}\}_{\Lambda\in\mathfrak{L}_{\rm box}}$ is a tempered Euclidean Gibbs measure of the model (1.1.3), (1.1.8), i.e., $\mathcal{M}^{\rm per}_\beta \subset \mathcal{G}^{\rm t}_\beta$. The elements of $\mathcal{M}^{\rm per}_\beta$, called periodic Euclidean Gibbs measures, are translation-invariant.*

Proof. For given $\Lambda \in \mathfrak{L}_{\rm box}$ and $\Delta \subset \Lambda$, we disintegrate $\pi_{\beta,\Lambda}$, cf. (3.6.5), according to

$$\pi_{\beta,\Lambda}(\mathrm{d}x) = \sigma^{\Delta}_{\beta,\Lambda}(\mathrm{d}x_\Delta|x_{\Delta^c})\rho^{\Delta}_{\beta,\Lambda}(\mathrm{d}x_{\Delta^c}). \tag{3.6.13}$$

Exactly as in the proof of Lemma 3.1.17, one shows that for any ℓ_0, $\alpha \in \mathcal{I}$, and $f \in C_{\rm b}(\Omega^{\ell_0,\alpha}_\beta)$, the function

$$\Omega^{\ell_0,\alpha}_\beta \ni \xi \mapsto \langle f\rangle_{\sigma^{\Delta}_{\beta,\Lambda}(\cdot|\xi_{\Delta^c})} = \int_{\Omega^{\rm t}_\beta} f(x_\Delta \times \xi_{\Lambda\setminus\Delta} \times 0_{\Lambda^c})\sigma^{\Delta}_{\beta,\Lambda}(\mathrm{d}x_\Delta|\xi_{\Delta^c}) \tag{3.6.14}$$

belongs to $f \in C_{\rm b}(\Omega^{\ell_0,\alpha}_\beta)$ and, cf. (3.1.49),

$$|\langle f\rangle_{\sigma^{\Delta}_{\beta,\Lambda}(\cdot|\xi_{\Delta^c})}| \le |f(\xi)|. \tag{3.6.15}$$

Given $\mu \in \mathcal{M}^{\rm per}_\beta$, let $\mathcal{L}_{\rm box}$ be the sequence such that $\lim_{\mathcal{L}_{\rm box}} \pi^{\rm per}_{\beta,\Lambda} = \mu$. To prove the theorem we have to show that for any $\Delta \in \mathfrak{L}_{\rm fin}$, $\alpha \in \mathcal{I}$, and $f \in C_{\rm b}(\Omega^{\ell_0,\alpha}_\beta)$, one has, cf. (3.1.54),

$$\begin{aligned}
&\int_{\Omega^{\rm t}_\beta} f(x)\pi_{\beta,\Delta}(\mathrm{d}x|\xi)\mu(\mathrm{d}\xi) - \int_{\Omega^{\rm t}_\beta} f(x)\mu(\mathrm{d}x)\\
&\quad = \lim_{\mathcal{L}_{\rm box}}\left\{\int_{\Omega^{\rm t}_\beta} f(x)\pi_{\beta,\Delta}(\mathrm{d}x|\xi)\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}\xi) - \int_{\Omega^{\rm t}_\beta} f(x)\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}x)\right\}\\
&\quad = \lim_{\mathcal{L}_{\rm box}}\left\{\int_{\Omega^{\rm t}_\beta} f(x_\Delta\times\xi_{\Delta^c})\big[\pi_{\beta,\Delta}(\mathrm{d}x|\xi) - \sigma^{\Delta}_{\beta,\Lambda}(\mathrm{d}x_\Delta|\xi_{\Delta^c})\big]\pi^{\rm per}_{\beta,\Lambda}(\mathrm{d}\xi)\right\}\\
&\quad = 0.
\end{aligned} \tag{3.6.16}$$

In view of the estimate (3.6.8) and Lemma 3.2.6, the function f in (3.6.16) can be taken from $C_{\rm b}(\Omega_{\beta,\Delta})$. As the sequence $\{\pi^{\rm per}_{\beta,\Lambda}\}_{\mathcal{L}_{\rm box}}$ is $\mathcal{W}^{\rm t}$-convergent, it is tight, see Proposition 1.3.33. Thus, for any $\varepsilon > 0$, one finds a compact subset $K_\varepsilon \subset \Omega^{\rm t}_\beta$, such that for all $\pi^{\rm per}_{\beta,\Lambda}$, $\Lambda \in \mathcal{L}_{\rm box}$,

$$\pi^{\rm per}_{\beta,\Lambda}(K^c_\varepsilon) < \varepsilon. \tag{3.6.17}$$

In view of (3.6.12), one can take $K_\varepsilon \subset \Xi(b,\sigma)$; hence, there exist $\alpha \in \mathcal{I}$, $\sigma \in (0, 1/2)$, ℓ_0, and $R_\varepsilon > 0$ and such that

$$K_\varepsilon \subset B_\varepsilon \stackrel{\mathrm{def}}{=} \{x \in \Omega^{\mathrm{t}}_\beta \mid \|x\|_{\ell_0,\alpha,\sigma} \le R_\varepsilon\},$$

see the proof of Theorem 3.3.6. Then we have

$$\begin{aligned}
&\int_{\Omega^{\mathrm{t}}_\beta} f(x_\Delta)\left[\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) - \sigma^\Delta_{\beta,\Lambda}(\mathrm{d}x_\Delta|\xi_{\Delta^c})\right]\pi^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}\xi)\\
&\quad = \int_{K_\varepsilon} f(x_\Delta)\left[\pi_{\beta,\Lambda}(\mathrm{d}x|\xi) - \sigma^\Delta_{\beta,\Lambda}(\mathrm{d}x_\Delta|\xi_{\Delta^c})\right]\pi^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}\xi) \qquad (3.6.18)\\
&\qquad + \int_{K^c_\varepsilon} f(x_\Delta)\pi_{\beta,\Lambda}(\mathrm{d}x|\xi)\pi^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}\xi) - \int_{K^c_\varepsilon} f(x_\Delta)\pi^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}x).
\end{aligned}$$

The second summand in the latter expression can be estimated by means of (3.1.49) and (3.6.17). Similarly, one estimates the third summand. Let us estimate the first one. By (1.4.48) and (1.4.50), it follows that for a given Δ, one finds $\Lambda \in \mathcal{L}_{\mathrm{box}}$, big enough, such that $J^\Lambda_{\ell\ell'} = J_{\ell\ell'}$ for all $\ell, \ell' \in \Delta$, which holds also for all Λ', such that $\Lambda \subset \Lambda'$. Then for such big Λ, by (3.6.13) we have, cf. (3.6.3),

$$\begin{aligned}
\sigma^\Delta_{\beta,\Lambda}(\mathrm{d}x_\Delta|\xi_{\Delta^c}) &= \frac{1}{\Sigma^\Delta_{\beta,\Lambda}(\xi_{\Delta^c})}\exp\Bigg\{\frac{1}{2}\sum_{\ell,\ell'\in\Delta} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} \qquad (3.6.19)\\
&\quad + \sum_{\ell\in\Lambda}\sum_{\ell'\in\Lambda\setminus\Delta} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} - \sum_{\ell\in\Delta}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_{\beta,\Delta}(\mathrm{d}x_\Delta),
\end{aligned}$$

where $\Sigma^\Delta_{\beta,\Lambda}(\xi_{\Delta^c})$ is a normalization constant. For $\vartheta \in [0,1]$, we set

$$\begin{aligned}
\phi^{(\vartheta)}(\mathrm{d}x_\Delta) &= \frac{1}{\Theta(\vartheta)}\exp\Bigg\{\vartheta\Bigg[\sum_{\ell\in\Delta,\ \ell'\in\Delta^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} - \sum_{\ell\in\Lambda}\sum_{\ell'\in\Lambda\setminus\Delta} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\Bigg]\\
&\quad + \frac{1}{2}\sum_{\ell,\ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}\sum_{\ell'\in\Lambda\setminus\Delta} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} \qquad (3.6.20)\\
&\quad - \sum_{\ell\in\Delta}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_{\beta,\Delta}(\mathrm{d}x_\Delta),
\end{aligned}$$

where

$$\begin{aligned}
\Theta(\vartheta) &= \int_{\Omega_{\beta,\Delta}} \exp\Bigg\{\vartheta\Bigg[\sum_{\ell\in\Delta,\ \ell'\in\Delta^c} J_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} - \sum_{\ell\in\Lambda}\sum_{\ell'\in\Lambda\setminus\Delta} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta}\Bigg]\\
&\quad + \frac{1}{2}\sum_{\ell,\ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda}\sum_{\ell'\in\Lambda\setminus\Delta} J^\Lambda_{\ell\ell'}(x_\ell,\xi_{\ell'})_{L^2_\beta} \qquad (3.6.21)\\
&\quad - \sum_{\ell\in\Delta}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_{\beta,\Delta}(\mathrm{d}x_\Delta).
\end{aligned}$$

Thereafter, we set

$$\varphi(\vartheta;\xi) = \langle f \rangle_{\phi^{(\vartheta)}}, \tag{3.6.22}$$

where f is the same as in (3.6.18). Then

$$\left| \langle f \rangle_{\pi_{\beta,\Delta}(\cdot|\xi)} - \langle f \rangle_{\sigma^{\Delta}_{\beta,\Lambda}(\cdot|\xi_{\Delta^c})} \right| \leq \sup_{\vartheta \in [0,1]} |\varphi'(\vartheta;\xi)|. \tag{3.6.23}$$

By (3.6.20) and (3.6.21), the derivative can be calculated:

$$\varphi'(\vartheta;\xi) = \sum_{\ell' \in \Lambda \setminus \Delta} (\xi_{\ell'}, \eta^{\Lambda}_{\ell'})_{L^2_\beta} + \sum_{\ell' \in \Lambda^c} (\xi_{\ell'}, \zeta_{\ell'})_{L^2_\beta}, \tag{3.6.24}$$

where

$$\begin{aligned} \eta^{\Lambda}_{\ell'}(\tau) &= \sum_{\ell \in \Delta} \left[J_{\ell\ell'} - J^{\Lambda}_{\ell\ell'} \right] \Upsilon_\vartheta(\tau; f), \\ \zeta_{\ell'}(\tau) &= \sum_{\ell \in \Delta} J_{\ell\ell'} \Upsilon_\vartheta(\tau; f), \end{aligned} \tag{3.6.25}$$

and

$$\Upsilon_\vartheta(\tau; f) = \langle f(x_\Delta) x_\ell(\tau) \rangle_{\phi^{(\vartheta)}} - \langle f(x_\Delta) \rangle_{\phi^{(\vartheta)}} \cdot \langle x_\ell(\tau) \rangle_{\phi^{(\vartheta)}}. \tag{3.6.26}$$

Since f is bounded and ξ is supposed to belong to K_ε, we can estimate

$$|\Upsilon_\vartheta(\tau; f)| \leq \Upsilon_{\max} \tag{3.6.27}$$

uniformly with respect to ξ, τ, and Λ.

Given $\ell \in \Delta$, we set $\Lambda_\ell = \{\ell' \in \Lambda \mid |\ell - \ell'|_\Lambda < |\ell - \ell'|\}$. As Λ is big enough, $\Lambda_\ell \subset \Lambda \setminus \Delta$ for all $\ell \in \Delta$. Let $\Lambda_\Delta \in \mathcal{L}_{\text{box}}$ be the biggest element of $\mathcal{L}_{\text{box}}$ such that

$$\bigcup_{\ell \in \Delta} \Lambda_\ell \subset \Lambda^c_\Delta. \tag{3.6.28}$$

It is clear that Λ_Δ is bigger for bigger Λ and $\Lambda_\Delta \to \mathbb{L}$ as $\Lambda \to \mathbb{L}$. Keeping these facts in mind we estimate the derivative (3.6.24) for $\xi \in B_\varepsilon$

$$|\varphi'(\vartheta;\xi)| \leq 3\beta R_\varepsilon \Upsilon_{\max} \sum_{\ell \in \Delta} \sum_{\ell' \in \Lambda^c_\Delta} |J_{\ell\ell'}|, \tag{3.6.29}$$

where the right-hand side can be made arbitrarily small by choosing big enough Λ. Hence, for any $\varepsilon > 0$, the first summand in (3.6.18) is small for big Λ, which completes the proof. $\square$

3.7 FKG order

In this section, we set $\nu = 1$ and $J_{\ell\ell'} \geq 0$. The anharmonic potentials V_ℓ are supposed to obey Assumption 1.1.1 only. That is, they need not be even and may be site-dependent. In this case, the moments of the conditional Euclidean Gibbs measures (3.1.41), and hence of the kernels (3.1.44), obey the FKG inequality, see Theorem 2.2.1. The aim of this section is to employ this fact for the study of the corresponding Euclidean Gibbs measures.

For $x, \tilde{x} \in \Omega_\beta$, we write $x \leq \tilde{x}$ if $x_\ell(\tau) \leq \tilde{x}_\ell(\tau)$ for all ℓ and $\tau \in [0, \beta]$. This defines an order on the set of configurations Ω_β. By means of this order we are going to order $\mathcal{G}^{\mathrm{t}}_\beta$. To this end we introduce the following family of increasing functions, see (3.3.7) and (3.3.8),

$$\mathcal{K}_+(\Omega^{\mathrm{t}}_\beta) = \{f \in \mathfrak{C} \mid f(x) \leq f(\tilde{x}), \text{ if } x \leq \tilde{x}\}. \tag{3.7.1}$$

We recall that measure-defining families of bounded continuous functions are described in Theorem 1.3.26. The family (3.7.1) does not meet the conditions of this theorem. However, as we are going to show now, this family is measure-defining as well.

A continuous function $f : \Omega^{\mathrm{t}}_\beta \to \mathbb{R}$ is said to be a cylinder function if it can be written

$$f(\omega) = \phi(x_{\ell_1}(\tau_1), \ldots, x_{\ell_n}(\tau_n)), \tag{3.7.2}$$

with certain $n \in \mathbb{N}$, $\ell_1, \ldots, \ell_n$, $\tau_1, \ldots, \tau_n$, and a continuous $\phi : \mathbb{R}^n \to \mathbb{R}$.

Lemma 3.7.1. *Given $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}_\beta$, suppose that*

$$\langle f \rangle_\mu = \langle f \rangle_{\tilde{\mu}}, \quad \textit{for all } f \in \mathcal{K}_+(\Omega^{\mathrm{t}}_\beta). \tag{3.7.3}$$

Then $\mu = \tilde{\mu}$.

Proof. By $\mathcal{K}^{\mathrm{cyl}}_+(\Omega^{\mathrm{t}}_\beta)$ we denote the subset of $\mathcal{K}_+(\Omega^{\mathrm{t}}_\beta)$ consisting of cylinder functions. Suppose that the equality (3.7.3) holds for all $f \in \mathcal{K}^{\mathrm{cyl}}_+(\Omega^{\mathrm{t}}_\beta)$. Then

$$\int_{\Omega^{\mathrm{t}}_\beta} x_\ell(\tau)\mu(\mathrm{d}x) = \int_{\Omega^{\mathrm{t}}_\beta} x_\ell(\tau)\tilde{\mu}(\mathrm{d}x), \quad \text{for all } \ell, \tau. \tag{3.7.4}$$

For fixed $\ell_1, \ldots, \ell_n$, $\tau_1, \ldots, \tau_n$, let P and $\widetilde{P}$ be the projections of the measures μ and $\tilde{\mu}$ on $\mathbb{R}^n$, respectively. That is, each of P and $\widetilde{P}$ obeys

$$\int_{\Omega^{\mathrm{t}}_\beta} f(x)\mu(\mathrm{d}x) = \int_{\mathbb{R}^n} \phi(u_1, \ldots, u_n) P(\mathrm{d}u),$$

for f and ϕ as in (3.7.2). Then by (3.7.3), it follows that

$$\int_{\mathbb{R}^n} \phi(u_1, \ldots, u_n) P(\mathrm{d}u) \leq \int_{\mathbb{R}^n} \phi(u_1, \ldots, u_n) \widetilde{P}(\mathrm{d}u), \tag{3.7.5}$$

for all increasing ϕ. Let $\widehat{P}$ be the probability measure on $\mathbb{R}^{2n}$ such that

$$P(\mathrm{d}u) = \int_{\mathbb{R}^n} \widehat{P}(\mathrm{d}u, \mathrm{d}v), \quad \widetilde{P}(\mathrm{d}v) = \int_{\mathbb{R}^n} \widehat{P}(\mathrm{d}u, \mathrm{d}v).$$

That is, $\widehat{P}$ is a *coupling* of P and $\widetilde{P}$. Of course, the above equalities do not determine $\widehat{P}$ uniquely. The Wasserstein distance between P and $\widetilde{P}$ can be defined as follows, see e.g., [107],

$$D(P, \widetilde{P}) = \inf \int_{\mathbb{R}^{2n}} |u - v| \widehat{P}(\mathrm{d}u, \mathrm{d}v), \tag{3.7.6}$$

where the infimum is taken over all couplings of P and $\widetilde{P}$. Consider

$$M = \{(u, v) \in \mathbb{R}^{2n} \mid u_i \leq v_i, \text{ for all } i = 1, \ldots, n\}.$$

This set is closed in $\mathbb{R}^{2n}$. Then by Strassen's theorem, see page 129 of [210], from (3.7.5) it follows that there exists a coupling $\widehat{P}_*$, such that

$$\widehat{P}_*(M) = 1. \tag{3.7.7}$$

Thereby,

$$\begin{aligned} D(P, \widetilde{P}) &\leq \int_M |u - v| \widehat{P}_*(\mathrm{d}u, \mathrm{d}v) \\ &\leq \sum_{i=1}^{n} \int_{\mathbb{R}^{2n}} (v_i - u_i) \widehat{P}_*(\mathrm{d}u, \mathrm{d}v) \\ &= \sum_{i=1}^{n} \int_{\mathbb{R}^n} u_i \left[\widetilde{P}(\mathrm{d}u) - P(\mathrm{d}u) \right] = 0. \end{aligned}$$

The latter equality follows from (3.7.4). Since the subset of $\mathfrak{C}$ consisting of all cylinder functions (3.7.2) is a defining family for $\mathcal{G}^{\mathrm{t}}_\beta$, see Theorem 1.3.26, the equality of all the projections of μ and $\tilde{\mu}$ yields $\mu = \tilde{\mu}$. □

Thereby, for $\mu, \tilde{\mu} \in \mathcal{G}^{\mathrm{t}}_\beta$, we set $\mu \leq \tilde{\mu}$ if

$$\langle f \rangle_\mu \leq \langle f \rangle_{\tilde{\mu}}, \quad \text{for all } f \in \mathcal{K}_+(\Omega^{\mathrm{t}}_\beta). \tag{3.7.8}$$

In view of Lemma 3.7.1, this defines an order on $\mathcal{G}^{\mathrm{t}}_\beta$. Obviously, the corresponding order can be introduced on the subset of $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$ consisting of measures, for which the first moments exist, cf. (3.7.4).

Now we study the properties of $\mathcal{G}^{\mathrm{t}}_\beta$ related to the order just introduced. First we prove that the measures (3.1.41) possess a property, arising from Theorem 2.2.1, which obviously holds for such measures.

Proposition 3.7.2. *In the scalar ferromagnetic case $\nu = 1$ and $J_{\ell\ell'} \geq 0$, the measures $\nu_{\beta,\Lambda}(\cdot|\xi)$ have the property:*

$$\nu_{\beta,\Lambda}(\cdot|\xi) \leq \nu_{\beta,\Lambda}(\cdot|\tilde{\xi}), \quad \textit{whenever } \xi \leq \tilde{\xi}.$$

Proof. For an increasing function $f \in \mathfrak{C}_\Lambda$ and $\vartheta \in [0,1]$, we set $\zeta_\vartheta = \xi + \vartheta(\tilde{\xi} - \xi)$ and

$$\varphi(\vartheta) = \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\zeta_\vartheta)}. \tag{3.7.9}$$

This function in differentiable in $\vartheta \in (0,1)$ and continuous at the endpoints of $[0,1]$, see Theorem 1.4.14. Furthermore,

$$\langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\tilde{\xi})} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\xi)} = \varphi'(\theta), \tag{3.7.10}$$

for a certain $\theta \in (0,1)$. The derivative φ' can be calculated explicitly:

$$\varphi'(\vartheta) = \sum_{\ell \in \Lambda} \int_0^\beta \left\{ \langle f(x_\Lambda) x_\ell(\tau) \rangle_{\nu_{\beta,\Lambda}(\cdot|\zeta_\vartheta)} - \langle f \rangle_{\nu_{\beta,\Lambda}(\cdot|\zeta_\vartheta)} \cdot \langle x_\ell(\tau) \rangle_{\nu_{\beta,\Lambda}(\cdot|\zeta_\vartheta)} \right\} d\tau.$$

By (2.2.1), the expression in $\{\dots\}$ is non-negative; hence, $\varphi'(\vartheta) \geq 0$ for all $\vartheta \in (0,1)$. $\square$

We recall that the kernels (3.1.44) are probability measures on $\Omega^{\mathrm{t}}_\beta$ for every $\xi \in \Omega^{\mathrm{t}}_\beta$. Then we have

Corollary 3.7.3. *For any $\xi, \tilde{\xi} \in \Omega^{\mathrm{t}}_\beta$, the kernels* (3.1.44) *of the scalar ferroelectric model* (1.1.3), (1.1.8) *have the property*

$$\pi_{\beta,\Lambda}(\cdot|\xi) \leq \pi_{\beta,\Lambda}(\cdot|\tilde{\xi}), \quad \textit{whenever } \xi \leq \tilde{\xi}.$$

Now let us turn to the study of the set $\mathcal{G}^{\mathrm{t}}_\beta$ related to the order (3.7.8). We recall that the invariance with respect to translations and τ-shifts were defined in Definitions 3.1.23 and 3.1.30, respectively, see also Definitions 1.4.10 and 3.1.21.

Theorem 3.7.4. *Let the model* (1.1.3), (1.1.8) *be scalar and ferromagnetic. Then for every tempered Euclidean Gibbs measures $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ and any $f, g \in \mathcal{K}_+(\Omega^{\mathrm{t}}_\beta)$, it follows that*

$$\langle f \cdot g \rangle_\nu \geq \langle f \rangle_\mu \langle g \rangle_\mu. \tag{3.7.11}$$

Furthermore, the set $\mathcal{G}^{\mathrm{t}}_\beta$ has a maximal element, μ_+, and a minimal element, μ_-, in the sense of the order (3.7.8). *These elements are extreme and τ-shift invariant. They are translation-invariant if the model is translation-invariant; if all V_ℓ's are even, then $\mu_+(B) = \mu_-(-B)$ for all $B \in \mathcal{B}(\Omega^{\mathrm{t}}_\beta)$.*

Proof. The FKG inequality (3.7.11) follows from Theorem 2.2.1 and Lemma 3.3.7 since the FKG inequality (2.2.1) holds for the measures $\pi_{\beta,\Lambda}(\cdot|\xi)$ for *all* $\xi \in \Omega^{\mathrm{t}}_\beta$.

Let us show that for every $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, there exists $\tilde{\mu} \in \mathcal{G}^{\mathrm{t}}_\beta$, such that $\mu \leq \tilde{\mu}$. Clearly, it is enough to prove this property for extreme elements of $\mathcal{G}^{\mathrm{t}}_\beta$ only. Given $\mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}}_\beta)$, let $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda \in \mathfrak{L}}$ converge to this μ, see Lemma 3.3.7. For this ξ, one takes $\tilde{\xi} \geq \xi$. Then the sequence $\{\pi_{\beta,\Lambda}(\cdot|\tilde{\xi})\}_{\Lambda \in \tilde{\mathfrak{L}}}$, $\tilde{\mathfrak{L}} \subset \mathfrak{L}$, converges to a certain $\tilde{\mu} \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}}_\beta)$, which obviously dominates μ.

Now let $\mathcal{M} \subset \mathcal{G}^{\mathrm{t}}_\beta$ be a totally ordered set, which can be considered as a net $\{\mu_\alpha\}_{\alpha\in\mathcal{I}}$ with the totally ordered index set $\mathcal{I}$ (as $\mathcal{I}$ one can take $\mathcal{M}$ itself). As a subset of the set $\mathcal{G}^{\mathrm{t}}_\beta$, which is compact in the topology $\mathcal{W}^{\mathrm{t}}$ (see Theorem 3.3.6), this net is relatively compact; hence, it contains a subnet $\{\mu_{\alpha(\gamma)}\}_{\gamma\in\mathcal{J}}$ converging in $\mathcal{W}^{\mathrm{t}}$ to a certain $\mu_* \in \mathcal{G}^{\mathrm{t}}_\beta$. Since the subnet is also totally ordered, it follows that

$$\mu_* = \sup_{\gamma\in\mathcal{J}} \mu_{\alpha(\gamma)},$$

i.e., μ_* is an upper bound of $\{\mu_{\alpha(\gamma)}\}_{\gamma\in\mathcal{J}}$. Then, if $\mathcal{M}$ were unbounded, it would contain a subset, $\mathcal{M}'$, each subnet of which does not converge, which is impossible in view of the compactness of $\mathcal{G}^{\mathrm{t}}_\beta$. Thereby, every totally ordered subset of $\mathcal{G}^{\mathrm{t}}_\beta$ is bounded; hence, by the Kuratowski–Zorn lemma, $\mathcal{G}^{\mathrm{t}}_\beta$ has maximal elements. Let μ_1 and μ_2 be two such elements. For any $f \in \mathcal{K}_+(\Omega^{\mathrm{t}}_\beta)$, both $\langle f\rangle_{\mu_1} \leq \langle f\rangle_{\mu_2}$ and $\langle f\rangle_{\mu_2} \leq \langle f\rangle_{\mu_1}$ should hold. This yields

$$\langle f\rangle_{\mu_1} = \langle f\rangle_{\mu_2},$$

which by Lemma 3.7.1 yields $\mu_1 = \mu_2$, that proves uniqueness. Thereby, we denote the unique maximal element of $\mathcal{G}^{\mathrm{t}}_\beta$ by μ_+. The proof of the existence and uniqueness of the minimal element μ_- is similar.

Now let us prove that both these elements are extreme. To this end we use claim (a) of Lemma 3.3.7. For some $\ell_0 \in \mathbb{L}$, we set

$$\xi^{\ell_0}_\ell(\tau) = [b\log(1+|\ell-\ell_0|)]^{1/2}. \tag{3.7.12}$$

The set $\{\pi_{\beta,\Lambda}(\cdot|\xi^{\ell_0})\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$ is $\mathcal{W}^{\mathrm{t}}$-relatively compact, see Lemma 3.2.6 and the proof of Theorem 3.3.6. Let $\mu^{\ell_0} \in \mathcal{G}^{\mathrm{t}}_\beta$ be its accumulation point. The above ξ^{ℓ_0} belongs to $\Xi(\ell_0, b, \sigma)$ for any $\sigma \in (0, 1/2)$, see (3.3.25). Thus, for any $\xi \in \Xi(\ell_0, b, \sigma)$, one finds $\Lambda_\xi \in \mathfrak{L}_{\mathrm{fin}}$ such that $\xi_\ell(\tau) \leq \xi^{\ell_0}_\ell(\tau)$ for all $\tau \in [0,\beta]$ and $\ell \in \Lambda^c_\xi$. Therefore, for any cofinal sequence $\mathcal{L}$, one finds $\Lambda \in \mathcal{L}$, which contains this Λ_ξ, and hence $\pi_{\beta,\Lambda'}(\cdot|\xi) \leq \pi_{\beta,\Lambda'}(\cdot|\xi^{\ell_0})$ for all $\Lambda' \supset \Lambda$. Now we fix $\sigma \in (0, 1/2)$ and choose b in (3.7.12) such that (3.3.23) holds. Then passing to the infinite volume limit along $\mathcal{L}$ we get

$$\forall \mu \in \mathrm{ex}(\mathcal{G}^{\mathrm{t}}_\beta): \quad \mu \leq \mu^{\ell_0}, \tag{3.7.13}$$

which yields $\mu^{\ell_0} = \mu_+$, since the maximal element is unique. Similarly, as μ_- one may take an accumulation point of the set $\{\pi_{\beta,\Lambda}(\cdot| -\xi^{\ell_0})\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$. Since the configuration (3.7.12) is independent of τ, Matsubara functions $\Gamma^{\beta,\Lambda}_{F_1,\ldots,F_n}(\tau_1,\ldots,\tau_n|\pm\xi^{\ell_0})$, defined by (3.1.42), have the property (1.2.90). Then these functions corresponding to the accumulation points $\mu^{\pm\xi_{\ell_0}}$ also have this property, which means that the latter measures are τ-shift invariant.

Given $\ell_0 \in \mathbb{L}$, we set

$$t_{\ell_0}(\omega) = (\omega_{\ell-\ell_0})_{\ell\in\mathbb{L}}, \quad t_{\ell_0}(B) = \{t_{\ell_0}(\omega) \mid \omega \in B\}.$$

Then if the model is translation-invariant, one has

$$\pi_{\beta,\Lambda+\ell}(t_\ell(B)|t_\ell(\xi)) = \pi_{\beta,\Lambda}(B|\xi), \tag{3.7.14}$$

which ought to hold for all ℓ, B, and ξ. For $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, we define

$$t^*_{\ell_0}(\mu)(B) = \mu(t_{\ell_0}(B)), \quad B \in \mathcal{B}(\Omega_\beta).$$

This $t^*_{\ell_0}(\mu)$ is an element of $\mathcal{G}^{\mathrm{t}}_\beta$. So, we have a map which preserves the latter set. It also preserves the order (3.7.8), which can easily been proven by its definition. Then, as μ_+ is maximal, $t^*_{\ell_0}(\mu_+) \leq \mu_+$ and $(t^*_{\ell_0})^{-1}(\mu_+) \leq \mu_+$. Applying to the latter relation $t^*_{\ell_0}$, we get $\mu_+ \leq t^*_{\ell_0}(\mu_+)$, which means $t^*_{\ell_0}(\mu_+) = \mu_+$. Similarly, $t^*_{\ell_0}(\mu_-) = \mu_-$. The proof of the property connected with the parity of V_ℓ's may be done in the same way. □

For the potentials V_ℓ being of the EMN type, see Definition 2.2.4, another property connected with the FKG order can be proven.

Proposition 3.7.5. *Let all V_ℓ be of EMN type. Then for every $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, $\ell \in \Lambda$, $\tau \in [0,\beta]$, and any $\theta \in [0,1]$, it follows that*

$$\begin{aligned} &\theta \cdot \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)} + (1-\theta)\cdot\langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|0)} \\ &\qquad \leq \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \leq \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)}, \end{aligned} \tag{3.7.15}$$

which holds for all $\xi \geq 0$ and $h \geq 0$.

Proof. The upper bound for $\langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|t\xi)}$ follows directly from Proposition 3.7.2. To prove the lower bound we use the GHS inequality, i.e., Theorem 2.2.7. One observes that $\langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}$ is a two times differentiable function of $\theta \in \mathbb{R}$ and

$$\begin{aligned} &\frac{\partial^2}{\partial\theta^2}\langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \\ &\quad = \sum_{\ell_1\in\Lambda,\ \ell_2\in\Lambda^c} J_{\ell_1\ell_2} \sum_{\ell_1'\in\Lambda,\ \ell_2'\in\Lambda^c} J_{\ell_1'\ell_2'} \\ &\qquad \times \int_0^\beta\int_0^\beta \xi_{\ell_2}(\tau_1)\xi_{\ell_2'}(\tau_1')\Big\{\langle x_\ell(\tau)x_{\ell_1}(\tau_1)x_{\ell_1'}(\tau_1')\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \\ &\qquad - \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\cdot\langle x_{\ell_1}(\tau_1)x_{\ell_1'}(\tau_1')\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \\ &\qquad - \langle x_{\ell_1}(\tau_1)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\cdot\langle x_\ell(\tau)x_{\ell_1'}(\tau_1')\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \\ &\qquad - \langle x_{\ell_1'}(\tau_1')\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\cdot\langle x_{\ell_1}(\tau_1)x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)} \\ &\qquad + 2\langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\cdot\langle x_{\ell_1}(\tau_1)\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\cdot\langle x_{\ell_1'}(\tau_1')\rangle_{\pi_{\beta,\Lambda}(\cdot|\theta\xi)}\Big\}\mathrm{d}\tau_1\mathrm{d}\tau_1' \leq 0, \end{aligned}$$

where we used (2.2.9). Therefrom, the lower bound in (3.7.15) follows by the concavity of the function $t \mapsto \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|t\xi)}$ just shown. □

3.8 Comments and Bibliographic Notes

Section 3.1: For quantum systems described by bounded local Hamiltonians, the Gibbs states are constructed as KMS states on algebras of quasi-local observables, see Section 6.2 in [77]. The time automorphisms employed to define the corresponding KMS condition, cf. Definition 1.2.6, are obtained as the limits $\Lambda \nearrow \mathbb{L}$ of the local automorphisms (1.2.13). However, for quantum anharmonic crystals, such limits do not exist and hence there is no way to define the Gibbs states of the whole model as KMS states. If one needs to consider also unbounded operators, see [248], [267], as it is the case here, the situation becomes much more complicated and the construction of Gibbs states even for simple models turns into a very hard task; for more details see the discussion in [160], pages 169, 170, and in [161]. Thus, the construction of Gibbs states of a quantum crystal by means of the Euclidean Gibbs measures performed in Chapter 3 seems to be the only possible way to describe equilibrium thermodynamic properties of such models. This construction, initiated by the papers [5], [156] was then continued in [136], [137], [50], [51], [52], [53], [54]. Our presentation is mainly based on the results of [11], [12], [13], [14], [15], [18], [19], [20], [22], [23], [28], [29], [30], [32], [191], [199], [200]. Certain ideas on how to construct Gibbs measures of systems of unbounded spins, developed in the articles [59], [85], [206], [264], have also been taken into account. Examples based on the DLR approach in Euclidean quantum field theory can be found in [7], [130].

For every Euclidean Gibbs measure μ, one can define corresponding Matsubara functions (3.1.61). If the measure is shift-invariant, these functions possess the property (3.1.63). There exist methods, see [66], [131], [132], [133], which allow one to construct a von Neumann algebra of observables and a KMS state from a ‘complete’ family of Matsubara functions. In our case, as such a family one can take the set of all Matsubara functions

$$\Gamma^{\mu}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \int_{\Omega^{\mathrm{t}}_{\beta}} F_1(x_{\Lambda_1}(\tau_1))\dots F_n(x_{\Lambda_n}(\tau_n))\mu(\mathrm{d}x) \tag{3.8.1}$$

corresponding to a given $\mu \in \mathscr{G}^{\mathrm{t}}_{\beta}$. Here *all* means the functions (3.8.1) with all $n \in \mathbb{N}$, all choices of subsets $\Lambda_i \in \mathfrak{L}_{\mathrm{fin}}$ and of bounded continuous functions $F_i : \mathbb{R}^{\nu|\Lambda_i|} \to \mathbb{C}$. The realization of this construction for the model studied in this book is left for the future.

Section 3.2: The main property of local Gibbs specifications established here is the integrability estimate (3.2.1), by which we have proven a Dobrushin-like estimate (3.2.6). This integrability estimate, together with the fact, established by Lemma 3.2.6, that sequences of measures obeying such estimates converge weakly if they converge weakly locally, allow one to prove the existence of tempered Euclidean Gibbs measures. The estimates (3.2.1), (3.2.6) were deduced from the corresponding results of [199], [200]. This way seems to be completely new as it has never been used in similar constructions before.

Section 3.3: Theorem 3.3.1 gives an a priori uniform integrability estimate for tempered Euclidean Gibbs measures in terms of model parameters. For classical systems of unbounded spins, the problem of deriving such estimates was first posed in the paper [59], see the discussion following Corollary 3.2.3. For quantum anharmonic crystals, similar estimates were obtained in the analytic approach, described in Section 3.4, see [21], [22], [24], [31]. The method we used to prove Theorem 3.3.1 is much more elementary than the one based on the analytic approach. At the same time, this theorem gives an improvement of the corresponding results of [21], [22] because: (a) the estimate (3.3.1) gives a much sharper bound; (b) we do not suppose that the potentials V_ℓ are differentiable (a key element of the analytic approach). Theorem 3.3.6 establishes the existence of tempered Euclidean Gibbs measures. A standard tool for proving such results for classical models is the celebrated Dobrushin criterion, see Theorem 1 in [101]. To apply it in our case one has to find a compact[1] function h on the single-spin space C_β such that for all ℓ and $\xi \in \Omega^{\mathrm{t}}_\beta$, the following estimate holds, cf. (3.2.6),

$$\int_{\Omega_\beta} h(x_\ell)\pi_\ell(\mathrm{d}x|\xi) \leq A + \sum_{\ell'} I_{\ell\ell'} h(\xi_{\ell'}), \tag{3.8.2}$$

where

$$A > 0, \quad I_{\ell\ell'} \geq 0 \quad \text{for all } \ell, \ell', \text{ and } \sup_\ell \sum_{\ell'} I_{\ell\ell'} < 1.$$

Then the estimate (3.8.2) would yield that for any $\xi \in \Omega^{\mathrm{t}}_\beta$, such that

$$\sup_\ell h(\xi_\ell) < \infty,$$

the family $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$ is relatively compact in the weak topology on $\mathcal{P}(\Omega_\beta)$, but not yet in $\mathcal{W}_\alpha$, $\mathcal{W}^{\mathrm{t}}$. As a next step one has to show that any accumulation point of $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$ is a Gibbs measure, which is much stronger than the fact established by our Lemma 3.1.20. Such a scheme was used in [59], [85], [282] where the existence of Gibbs measures for lattice systems with the single-spin space $\mathbb{R}$ was proven. The methods used there heavily employed specific properties of the models, like attractiveness of the interaction and translation invariance. A direct extension of that scheme to quantum models seems to be impossible. The scheme we used to prove Theorem 3.3.6 is based on the compactness of the family $\{\pi_{\beta,\Lambda}(\cdot|\xi)\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$ in the topologies $\mathcal{W}_\alpha$, $\mathcal{W}^{\mathrm{t}}$. After obvious modifications it can be applied to similar models with more general interaction potentials. Theorem 3.3.9 establishes a regularity of the elements of $\mathcal{G}^{\mathrm{t}}_\beta$. A weaker property of this kind, which is exhibited by tempered Gibbs measures of classical systems of unbounded spins, was proven in Theorem 4.1 in the paper [206]. The support property (3.3.23) is a stronger version of the property established in [59], [206]. Thus, one can call $\Xi(b,\sigma)$ a Lebowitz–Presutti type support. In [206], the support property was proven by means of Ruelle's superstability estimates [264], applicable to translation-invariant models only. The extension of the Lebowitz–Presutti

[1] See Definition 1.3.34

result to translation-invariant quantum models was done in the papers [237], [238], where superstable Gibbs measures were specified by the support property

$$\sup_{N\in\mathbb{N}} \left\{(1+2N)^{-d} \sum_{\ell:\,|\ell|\le N} \|x_\ell\|^2_{L^2_\beta}\right\} \le C(x), \quad \mu\text{-a.s.}$$

Regarding this property we remark that by the Birkhoff–Khinchine ergodic theorem, for any translation-invariant measure $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_\beta)$, which obeys the estimate (3.3.1), it follows that for every $\sigma \in (0, 1/2)$, $\varkappa > 0$, and μ-almost all x,

$$\sup_{N\in\mathbb{N}} \left\{(1+2N)^{-d} \sum_{\ell:\,|\ell|\le N} \exp\left(\lambda_\sigma \|x_\ell\|^2_{C^\sigma_\beta} + \varkappa\|x_\ell\|^2_{L^2_\beta}\right)\right\} \le C(\sigma, \varkappa, x). \quad (3.8.3)$$

In particular, every periodic Euclidean Gibbs measure constructed in Section 3.6 has this property.

Section 3.4: As was mentioned above, the analytic approach in the version presented here was elaborated in [21], [22], [31]. In a more general context, its basic elements were elaborated in [71], [72], [73]. In comparison to the one based on the DLR equation, the analytic approach gives an alternative definition of the tempered Euclidean Gibbs measures. It is based on the assumption of differentiability of the anharmonic potentials. Along with deriving integrability estimates like (3.3.1), this approach can be used to obtain correlation inequalities, which is very important if the model is of a more general type, where the ferromagnetic inequalities do not hold.

Section 3.5: Here we only remark that the set $\widetilde{\mathcal{G}}_\beta$ is too big. As was shown in [60], for a one-dimensional harmonic system such a set contains infinitely many elements. At the same time, one can easily show that in this case there exists only one tempered Gibbs measure.

Section 3.6: In the second part of this book, periodic Euclidean Gibbs measures will be used for proving the existence of phase transitions. By construction, any of these measures is translation-invariant. If there exists a non-ergodic periodic Gibbs measure, the set $\mathcal{G}^{\mathrm{t}}_\beta$ contains more than one element – a fact which corresponds to the presence of a phase transition.

Section 3.7: The FKG order, which we introduce on the set $\mathcal{G}^{\mathrm{t}}_\beta$ for scalar ferromagnetic quantum crystals, proved to be a useful tool in the study of this set. The results obtained in the second part of the book with the use of the FKG order are much stronger. In the theory of Gibbs random fields, this notion seems to have first been introduced in [249], see also [251]. Connections between the FKG inequality and the properties of a random field were studied in [232], [233]. A wider look at the notion of stochastic order is presented in the book [223].

Part II

Physical Applications

Chapter 4
Quantum Anharmonic Crystal as a Physical Model

In Part I, we have developed a version of the mathematical theory of equilibrium thermodynamic properties of a quantum anharmonic crystal. Its cardinal element is the use of Euclidean Gibbs measures. In this theory, the quantum anharmonic crystal is described as a rather nonphysical system of interacting 'classical' spins, which are unbounded and infinite-dimensional. A natural question arising afterwards might be to which extent this theory is adequate and hence useful in studying real physical systems. In view of this, the main aim of the second part of this book is to apply our theory to the description of equilibrium thermodynamic properties of the physical substances which can be modeled with the use of quantum anharmonic crystals. For obvious reasons, such a description can only be qualitative; hence, we shall mostly concentrate on fundamental properties, such as phase transitions, caused by the interaction between the particles, and quantum effects, competing with phase transitions. These spectacular phenomena are well studied in both experimental and theoretical physics. However, their rigorous mathematical description is still a challenging task.

The current chapter has a preparatory character. We begin it by discussing which physical substances can be modeled by the Hamiltonians (1.1.3), (1.1.8). Then we analyze the classical limit of the model, which should be in agreement with basic physical concepts. Finally, we prove that at high temperatures and/or weak interactions the set of all tempered Euclidean Gibbs measures is a singleton, which corresponds to the absence of phase transitions. In the subsequent chapters, we study phase transitions and the role of quantum effects in detail.

4.1 Modeling Physical Substances

In theoretical physics, the quantum anharmonic oscillator is a model of a quantum particle moving in a potential field with possibly multiple minima, which has a sufficient growth at infinity and hence localizes the particle. Most of the models of interacting quantum oscillators are related with solids such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, or quantum crystals consisting entirely of such particles. For instance, a potential field with multiple minima is seen by a helium atom located at the center of the crystal cell in bcc helium, see page 11 in [179]. The same situation exists in other quantum crystals, He, H_2 and to some extent Ne, where even zero point oscillations create large anharmonic effects. An example of the ionic crystal with localized quantum particles moving in a double-well potential field is given by a KDP-type ferroelectric with hydrogen bounds in which such particles are protons or deuterons performing one-dimensional oscillations along the bounds, see [70], [284], [306], [309]. It is believed that structural phase transitions in such ferroelectrics are triggered by the ordering of protons. Another relevant physical object of this kind is a system of apex oxygen ions in YBaCuO-type high-temperature superconductors, see [123], [224], [288], [289], [290]. Quantum anharmonic oscillators are also used in models describing interaction of vibrating quantum particles with a radiation (photon) field, see [146], [151], [234], or strong electron-electron correlations caused by the interaction of electrons with vibrating ions, see [120], [121], responsible for such phenomena as superconductivity, and charge density waves. Finally, we mention systems of light atoms, like Li, inculcated into ionic crystals, like KCl. The quantum particles in this system are not necessarily regularly distributed. For more information on this subject, we refer to the survey [153].

In the aforementioned systems, collective phenomena, like phase transitions, are triggered by the corresponding effects in the subsystem of light quantum particles. Often such particles carry electric charges – a proton (a deuteron) in a KDP-type ferroelectric may serve as an example. In view of this fact, the displacement of the particle from its equilibrium point produces a dipole moment; hence, the main contribution to the two-particle interaction is proportional to the product of the displacements of the particles and is of long range. These arguments have been taken into account in the choice of the interaction term in the Hamiltonian (1.1.8). However, a slight modification of our technique could allow for extending most of our results (at least those regarding the existence of Euclidean Gibbs measures) to more general types of interaction. For a detailed explanation of how to derive a model like (1.1.3), (1.1.8) from physical models of concrete substances, we refer the reader to the survey [284].

In classical systems undergoing phase transitions, ordering caused by the interaction between the particles is achieved in competition with thermal fluctuations. In quantum systems, ordering competes also with quantum effects, which can be strong, especially at low temperatures. This possibility was first discussed in [266]. Later on a number of publications dedicated to the study of quantum effects in such systems had appeared, see e.g., [218], [315] and the references therein. For better understanding, illuminating exactly solvable models of systems of interacting quantum anharmonic oscillators were introduced and studied, see [246], [285], [313], [314]. In these works, the quantity $\hbar^2/m_{\rm ph}$, see (1.1.7), was used as a parameter describing the rate of quantum effects. Such effects became strong in the small mass limit, which was in agreement with the experimental observations, e.g., on the isotopic effect in the ferroelectrics with hydrogen bounds, see [70], [309], see also [224] for the data on the isotopic effect in the YBaCuO-type high-temperature superconductors. However, in those works no other mechanisms of the appearance of quantum effects were discussed. At the same time, experimental data, see e.g., the table on page 11 in the monograph [70] or the article [303], show that high hydrostatic pressure applied to KDP-type ferroelectrics prevents them from ordering by strengthening quantum effects. It is believed that the pressure shortens the hydrogen bounds and makes the tunneling motion of the quantum particles more intensive. This can be taken into account in the model (1.1.3), (1.1.8) by a corresponding modification of the anharmonic potentials V_ℓ. One of our main tasks in Part II is to give a complete qualitative theory of this phenomenon.

There exists one more possibility to relate the theory developed in Part I with fundamental properties of the corresponding physical substances. According to the Bohr–Heisenberg correspondence principle, in the classical limit $\hbar \to 0$ a quantum model should turn into its classical analog. This means that the limiting model should behave exactly as the corresponding classical model, which in our case is the model of interacting classical anharmonic oscillators (classical anharmonic crystal). However, in concrete situations it is quite hard to implement the mentioned principle as the classical and quantum theories have essentially different mathematical structures. The fact that in the Euclidean approach quantum anharmonic oscillators are described as 'classical' spins opens the possibility to get a deeper insight into this problem. In view of (1.1.7), the parameter responsible for the classical limit is the reduced particle mass $m = m_{\rm ph}/\hbar^2$. One observes that the limit $m \to +\infty$ excludes the kinetic energy term in (1.1.3), which is in agreement with the fact that the kinetic energy of a classical model plays no role in the description of its equilibrium thermodynamic properties. In Theorem 4.2.1 below, we show that the local Euclidean Gibbs measures converge, as $m \to +\infty$, to the corresponding local Gibbs measures of the classical anharmonic crystal.

In Theorem 3.3.6, we have proven the existence of tempered Euclidean Gibbs measures for our model. Then the next natural question would be how many of such measures may exist at certain fixed values of β and the model parameters. Besides its purely mathematical significance, the answer to this question would give important information about physical properties of the model. Namely, the multiplicity (respectively, uniqueness) of the elements of $\mathcal{G}^{\rm t}_\beta$ corresponds to the multiplicity (respectively, uniqueness) of equilibrium phases of the underlying physical system. Therefore, a physical notion of a phase transition can be associated with a property of the set of tempered Euclidean Gibbs states, which is a mathematical object. Then the study of this property can clarify the connections between the theory of the model (1.1.3), (1.1.8) developed here and the properties of the corresponding physical systems. In this chapter, we prove that for high temperatures (i.e., for small β), there exists exactly one tempered Euclidean Gibbs measure. This high-temperature uniqueness holds also for small values of the interaction parameter $\hat{J}_0$, defined in (1.1.11). For $\hat{J}_0 = 0$, the set $\mathcal{G}^{\rm t}_\beta$ is a singleton, which can readily be proven with the help of Kolmogorov's extension theorem, Proposition 1.3.3. Thus, the high-temperature uniqueness may be viewed as the stability of the latter property under small perturbations[1] of $\hat{J}_0$. It holds for all values of the mass m; hence, also in the classical limit $m \to +\infty$. In the subsequent chapters, we clarify the role of the mass m in the appearance of phase transitions, that is, in the appearance of non-uniqueness for the Euclidean Gibbs states.

[1] In [174], there is given an example of an 'unstable' spin model, for which arbitrary perturbations of the Gibbs state with zero interaction destroys uniqueness.

4.2 The Classical Limit

The classical analog of the model (1.1.3), (1.1.8) is the classical anharmonic crystal. Its basic element is the classical anharmonic oscillator with the same potential energy as in (1.1.3), (1.1.8). The natural framework for the construction of Gibbs states of a classical anharmonic crystal is the DLR approach. Let us briefly summarize its main steps. For $J_{\ell\ell'}$, V_ℓ as in (1.1.3), (1.1.8), and given $\beta > 0$, $\Lambda \in \mathfrak{L}_{\rm fin}$, we set, cf. (1.4.8) and (3.1.33),

$$E^{\rm cl}_{\beta,\Lambda}(u_\Lambda) = -\frac{\beta}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(u_\ell, u_{\ell'}) + \beta\sum_{\ell\in\Lambda} V_\ell(u_\ell), \quad u_\Lambda \in \mathbb{R}^{\nu|\Lambda|}, \tag{4.2.1}$$

and

$$\begin{aligned} \gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda) &= \bigotimes_{\ell\in\Lambda}\gamma_\beta(\mathrm{d}u_\ell), \\ \gamma_\beta(\mathrm{d}u_\ell) &= (a\beta/2\pi)^{\nu/2}\exp\left[-a\beta|u_\ell|^2/2\right]\mathrm{d}u_\ell, \end{aligned} \tag{4.2.2}$$

where a is the same as in (1.1.3). By definition, the local Gibbs measure of our classical anharmonic crystal is

$$\begin{aligned} \nu^{\rm cl}_{\beta,\Lambda}(\mathrm{d}u_\Lambda) &= \frac{1}{N^{\rm cl}_{\beta,\Lambda}}\exp\left[-E^{\rm cl}_{\beta,\Lambda}(u_\Lambda)\right]\gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda), \\ N^{\rm cl}_{\beta,\Lambda} &= \int_{\mathbb{R}^{\nu|\Lambda|}}\exp\left[-E^{\rm cl}_{\beta,\Lambda}(u_\Lambda)\right]\gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda). \end{aligned} \tag{4.2.3}$$

Let the weights w_α, $\alpha \in \mathcal{I}$, be as in (3.1.27). Then, for $u = (u_\ell)_{\ell\in\mathbb{L}} \in (\mathbb{R}^\nu)^{\mathbb{L}}$, $\alpha \in \mathcal{I}$, and $\ell_0 \in \mathbb{L}$, we set, cf. (3.1.27),

$$\|u\|^2_{\alpha,\ell_0} = \sum_\ell |u_\ell|^2 w_\alpha(\ell_0,\ell), \tag{4.2.4}$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^\nu$. Thereby, we introduce

$$\Omega^{\ell_0,\alpha}_{\rm cl} \stackrel{\rm def}{=} \{u \in (\mathbb{R}^\nu)^{\mathbb{L}} \mid \|u\|_{\alpha,\ell_0} < \infty\}, \quad \Omega^{\rm t}_{\rm cl} \stackrel{\rm def}{=} \bigcap_{\alpha\in\mathcal{I}}\Omega^{\ell_0,\alpha}_{\rm cl}. \tag{4.2.5}$$

Like the set of tempered temperature loops (3.1.30), the set of 'classical' tempered configurations $\Omega^{\rm t}_{\rm cl}$ is independent of ℓ_0. We equip $\Omega^{\ell_0,\alpha}_{\rm cl}$ with the metric $\rho^{\rm cl}_{\ell_0,\alpha}(u,v) = \|u - v\|_{\ell_0,\alpha}$, which turns it into a Polish space. Then the set $\Omega^{\rm t}_{\rm cl}$ is equipped with the projective limit topology, in which it is a Polish space as well. For $v \in \Omega^{\rm t}_{\rm cl}$, we set

$$E^{\rm cl}_{\beta,\Lambda}(u|v) = E^{\rm cl}_{\beta,\Lambda}(u) - \beta\sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} J_{\ell\ell'}(u_\ell, v_{\ell'}). \tag{4.2.6}$$

As in Lemma 3.1.12, one proves that the map

$$\Omega^{\ell_0,\alpha}_{\rm cl} \times \Omega^{\ell_0,\alpha}_{\rm cl} \ni (u,v) \mapsto E^{\rm cl}_{\beta,\Lambda}(u|v)$$

is continuous and the estimates

$$\inf_{u\in\Omega^{\mathrm{t}}_{\mathrm{cl}},\ v\in B_{\ell_0,\alpha}} E^{\mathrm{cl}}_{\beta,\Lambda}(u|v) > -\infty, \qquad \sup_{u,v\in B_{\ell_0,\alpha}} E^{\mathrm{cl}}_{\beta,\Lambda}(u|v) < +\infty$$

hold for any bounded $B_{\ell_0,\alpha} \subset \Omega^{\ell_0,\alpha}_{\mathrm{cl}}$. Then one defines, cf. (3.1.39),

$$N^{\mathrm{cl}}_{\beta,\Lambda}(v) = \int_{\mathbb{R}^{\nu|\Lambda|}} \exp\big[-E^{\mathrm{cl}}_{\beta,\Lambda}(u_\Lambda \times 0_{\Lambda^c}|v)\big]\gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda), \tag{4.2.7}$$

where the Gaussian measure $\gamma_{\beta,\Lambda}$ is as in (4.2.2). Next we define the stochastic kernels, cf. (3.1.44),

$$\begin{aligned} \pi^{\mathrm{cl}}_{\beta,\Lambda}(B|v) &= \int_{(\mathbb{R}^\nu)^\Lambda} \mathbb{I}_B(u_\Lambda \times v_{\Lambda^c})\nu^{\mathrm{cl}}_{\beta,\Lambda}(\mathrm{d}u_\Lambda|v), \quad B \in \mathcal{B}(\Omega^{\mathrm{t}}_{\mathrm{cl}}), \\ \nu^{\mathrm{cl}}_{\beta,\Lambda}(\mathrm{d}u_\Lambda|v) &= \frac{1}{N^{\mathrm{cl}}_{\beta,\Lambda}(v)} \exp\big[-E^{\mathrm{cl}}_{\beta,\Lambda}(u_\Lambda \times 0^c_\Lambda|v)\big]\gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda). \end{aligned} \tag{4.2.8}$$

Here $\mathcal{B}(\Omega^{\mathrm{t}}_{\mathrm{cl}})$ is the corresponding Borel σ-algebra and $\nu^{\mathrm{cl}}_{\beta,\Lambda}(\cdot|v)$ is a probability measure on the Euclidean space $\mathbb{R}^{\nu|\Lambda|}$. It is the conditional local Gibbs measure of the classical analog of the model (1.1.3), (1.1.8). By means of the kernels (4.2.8) we define the set of tempered Gibbs measures $\mathcal{G}^{\mathrm{cl}}_\beta$, exactly as it was done in Definition 3.1.18.

Let us now return to the quantum case. To describe the limit $m \to +\infty$ we introduce the following sets of configurations

$$\Omega^{\mathrm{const}}_{\beta,\Lambda} = \{x_\Lambda \in \Omega_{\beta,\Lambda} \mid \exists u_\Lambda \in \mathbb{R}^{\nu|\Lambda|}\ \ \forall \ell \in \Lambda\ \forall \tau \in [0,\beta]:\ x_\ell(\tau) = u_\ell\}, \tag{4.2.9}$$

where $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$. Clearly, $\Omega^{\mathrm{const}}_{\beta,\Lambda}$ is isomorphic to $\mathbb{R}^{\nu|\Lambda|}$. For $C \subset \Omega^{\mathrm{const}}_{\beta,\Lambda}$ and $A \in \mathcal{B}(\mathbb{R}^{\nu|\Lambda|})$, we write $C \simeq A$ if, for every $x_\Lambda \in C$, there exists $u_\Lambda \in A$ such that $x_\Lambda(\tau) = u_\Lambda$ for all τ; and for every $u_\Lambda \in A$, one finds $x_\Lambda \in C$ with the same property. For $f \in C_{\mathrm{b}}(\Omega_{\beta,\Lambda})$, let $\phi_f \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$ be such that for any $x_\Lambda \in \Omega^{\mathrm{const}}_{\beta,\Lambda}$, such that $x_\Lambda(\tau) = u_\Lambda$ for all $\tau \in [0,\beta]$, one has

$$f(x_\Lambda) = \phi_f(u_\Lambda). \tag{4.2.10}$$

The main result of this section is given by the following statement, in which we write χ^m_β, $\chi^m_{\beta,\Lambda}$, $\nu^m_{\beta,\Lambda}$ to indicate the dependence on the mass parameter (1.1.7).

Theorem 4.2.1. *For every $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $f \in C_{\mathrm{b}}(\Omega_{\beta,\Lambda})$, it follows that*

$$\langle f \rangle_{\nu^m_{\beta,\Lambda}} \to \langle \phi_f \rangle_{\nu^{\mathrm{cl}}_{\beta,\Lambda}}, \quad \textit{as } m \to +\infty, \tag{4.2.11}$$

where ϕ_f is as in (4.2.10).

Proof. By (1.3.112) the covariance operator S^m_β of the measure χ^m_β converges, in the trace norm in L^2_β, to the operator S^∞_β with the same eigenvectors and with the eigenvalues $s^\infty_k = a^{-1}\delta_{k,0}$, $k \in \mathcal{K}$. This yields that the limiting measure χ^∞_β is concentrated on constant paths $x(\tau) = \hat{x}_0/\sqrt{\beta}$, with $\langle \hat{x}_0^{(j)} \hat{x}_0^{(j')} \rangle_{\chi_\beta^\infty} = \delta_{jj'}/a$. Therefore, it is connected with the measure γ_β defined by (4.2.2) as follows. For $B \in \mathcal{B}_{\beta,\{\ell\}}$, let $A \in \mathcal{B}(\mathbb{R}^\nu)$ be such that $A \simeq B \cap \Omega^{\mathrm{const}}_{\beta,\{\ell\}}$. Then

$$\chi_\beta^\infty(B) = \chi_\beta^\infty\left(B \cap \Omega^{\mathrm{const}}_{\beta,\{\ell\}}\right) = \gamma_\beta(A);$$

hence,

$$\chi^\infty_{\beta,\Lambda}(B) = \gamma_{\beta,\Lambda}(A), \tag{4.2.12}$$

where

$$\chi^\infty_{\beta,\Lambda} \stackrel{\mathrm{def}}{=} \bigotimes_{\ell\in\Lambda} \chi_\beta^\infty$$

and $B \in \mathcal{B}_{\beta,\Lambda}$ and $A \in \mathcal{B}(\mathbb{R}^{\nu|\Lambda|})$ are such that $A \simeq B \cap \Omega^{\mathrm{const}}_{\beta,\Lambda}$. By Theorem 1.3.50 for any $m_0 > 0$, the net $\{\chi^m_{\beta,\Lambda}\}_{m\geq m_0} \subset \mathcal{P}(\Omega_{\beta,\Lambda})$ is tight, which implies $\chi^m_{\beta,\Lambda} \Rightarrow \chi^\infty_{\beta,\Lambda}$ as $m \to +\infty$. Since the function $\exp(-E_{\beta,\Lambda})$ belongs to $C_{\mathrm{b}}(\Omega_{\beta,\Lambda})$, the aforementioned weak convergence of $\chi^m_{\beta,\Lambda}$ yields

$$\int_{\Omega_{\beta,\Lambda}} f(x_\Lambda) \exp\left[\frac{1}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'} \int_0^\beta (x_\ell(\tau), x_{\ell'}(\tau))\mathrm{d}\tau - \sum_{\ell\in\Lambda}\int_0^\beta V_\ell(x_\ell(\tau))\mathrm{d}\tau\right]\chi^m_{\beta,\Lambda}(\mathrm{d}x_\Lambda)$$

$$\xrightarrow{m+\infty} \int_{\mathbb{R}^{\nu|\Lambda|}} \phi_f(u_\Lambda)\exp\left[\frac{\beta}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(u_\ell, u_{\ell'}) - \beta\sum_{\ell\in\Lambda} V_\ell(u_\ell)\right]\gamma_{\beta,\Lambda}(\mathrm{d}u_\Lambda).$$

The convergence $N^m_{\beta,\Lambda} \to N^{\mathrm{cl}}_{\beta,\Lambda}$ follows from the above one by setting $f = 1$. This proves (4.2.11). □

In the statement below we write $\varrho^m_{\beta,\Lambda}$ to indicate the m-dependence of the state (1.2.12). Its proof readily follows from the Euclidean representation (1.4.20) and the convergence established in Theorem 4.2.1.

Corollary 4.2.2. *For every $\beta > 0$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, and any $F \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$, it follows that*

$$\lim_{m\to+\infty} \varrho^m_{\beta,\Lambda}(F) = \int_{\mathbb{R}^{\nu|\Lambda|}} F(u_\Lambda)\nu^{\mathrm{cl}}_{\beta,\Lambda}(\mathrm{d}u_\Lambda). \tag{4.2.13}$$

The result just proven allows us to strengthen Theorem 2.4.6.

Theorem 4.2.3. *Let each $V_\ell(u) = v_\ell(|u|^2)$ in (4.2.1) be such that for a certain $b_\ell \geq 0$, $b_\ell + a/2 + v'_\ell(\zeta) \in \mathcal{F}^{\mathrm{Laguerre}}$. Then, for $\nu = 1, 2$ and any $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, the function*

$$f_{\beta,\Lambda}(h_\Lambda) = \int_{\mathbb{R}^{\nu|\Lambda|}} \exp\left(\sum_{\ell\in\Lambda} h_\ell u_\ell^{(1)}\right)\nu^{\mathrm{cl}}_{\beta,\Lambda}(\mathrm{d}u_\Lambda) \tag{4.2.14}$$

can be extended to an entire function of $h_\Lambda = (h_\ell)_{\ell\in\Lambda} \in \mathbb{C}^{|\Lambda|}$ such that

$$f_{\beta,\Lambda}(h_\Lambda) \neq 0, \quad \textit{if } \Re(h_\ell) \geq 0, \textit{ for all } \ell \in \Lambda.$$

Proof. To prove this theorem one just repeats the arguments the proof of Lemma 2.4.7 was based on. Namely, by Proposition 2.4.4, the single-site measure

$$\nu_\ell^{\mathrm{cl}}(\mathrm{d}u_\ell) \stackrel{\mathrm{def}}{=} \exp\left(-\frac{\beta a}{2}|u_\ell|^2 - \beta v_\ell(|u_\ell|^2)\right)\mathrm{d}u_\ell$$

possesses the Lee–Yang property. Then the measure (4.2.3) takes the form

$$\nu_{\beta,\Lambda}^{\mathrm{cl}}(\mathrm{d}u_\Lambda) = \frac{1}{Z_{\beta,\Lambda}^{\mathrm{cl}}} \exp\left(\frac{\beta}{2}\sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(u_\ell, u_{\ell'})\right)\bigotimes_{\ell\in\Lambda}\nu_\ell^{\mathrm{cl}}(\mathrm{d}u_\ell).$$

As $J_{\ell\ell'} \geq 0$ for all $\ell, \ell' \in \Lambda$, the property stated follows by Corollaries 3.3 and 4.4 of [208]. □

Now let us make some summarizing comments. In general, one cannot expect that the Euclidean Gibbs measures of the initial model of quantum anharmonic oscillators will converge to a given classical Gibbs measures, as $m \to +\infty$. All the more, as we shall see in Chapters 6 and 7, for certain values of the model parameters and β, the set $\mathcal{G}_\beta^{\mathrm{t}}$ is a singleton for small values of m, and is not a singleton for large m. Therefore, it would be more correct to speak about the dependence on m of the whole set $\mathcal{G}_\beta^{\mathrm{t}}$ rather than of its single elements. Additional information on this item can be found in Section 4.4 below. On the level of local states, the $m \to +\infty$ limit is described by Theorem 4.2.1. Its corollary describes the classical limits of the states (1.2.12) restricted to the commutative algebra of multiplication operators by bounded continuous functions. It is worthwhile to remark here that one can hardly imagine how to get the $m \to +\infty$ limits of such states directly from their definition (1.2.12).

4.3 The High-Temperature Uniqueness

In the theory of Gibbs states it is very important to establish under which conditions the set $\mathcal{G}_\beta^{\mathrm{t}}$ consists of exactly one element. If $|\mathcal{G}_\beta^{\mathrm{t}}| > 1$, the model has a phase transition, studied in detail in Chapter 6. If the oscillators do not interact, $\mathcal{G}_\beta^{\mathrm{t}}$ is a singleton. Indeed, in this case the measures (3.1.41), (3.1.44) are merely the products of the one-point measures describing individual anharmonic oscillators. As a result, the condition (3.1.46) turns into the consistency condition the Kolmogorov theorem is based on, see Proposition 1.3.3. Therefore, it would be reasonable to expect the uniqueness of the Euclidean Gibbs measures for small values of the interaction parameter $\hat{J}_0$, see (1.1.11). As we shall see below, the same effect can be obtained by choosing small values of the inverse temperature β – that is why the uniqueness of this kind is called high-temperature uniqueness. As was mentioned above, we are going to obtain a condition for such a uniqueness, which is independent of the mass m and hence would be applicable also in the classical limit. To this end we employ the logarithmic Sobolev inequality proven in Section 2.3. We recall that the anharmonic potentials

V_ℓ obey Assumption 1.1.1. Thus, as in (2.3.9) we decompose each V_ℓ into the sum $V_{\ell,1} + V_{\ell,2}$, where $V_{\ell,1} \in C^2(\mathbb{R}^\nu)$ is such that

$$-a \le b \stackrel{\mathrm{def}}{=} \inf_{\ell} \inf_{u,v\in\mathbb{R}^\nu,\ v\neq 0} \big(V''_{1,\ell}(u)v, v\big)/|v|^2 < +\infty, \tag{4.3.1}$$

where a is the same as in (1.1.3). For the second summand, we set, cf. (2.3.11),

$$0 \le \omega \stackrel{\mathrm{def}}{=} \sup_{\ell} \mathrm{Osc}(V_{2,\ell}). \tag{4.3.2}$$

We recall that the interaction is said to have finite range if $J_{\ell\ell'} = 0$ whenever $|\ell - \ell'| > R$ for some $R > 0$, see Definition 1.1.2. The main result of this section is the following

Theorem 4.3.1. *Let the interaction $J_{\ell\ell'}$ have finite range. Then the set $\mathcal{G}^{\mathrm{t}}_\beta$ is a singleton whenever the following condition is satisfied:*

$$\frac{e^{\beta\omega}}{a+b} < \frac{1}{\hat{J}_0}. \tag{4.3.3}$$

Before proving this statement let us make some comments. The uniqueness condition (4.3.3) certainly holds in the limit $\hat{J}_0 \to 0$. If

$$\omega = 0 \quad \text{and} \quad \hat{J}_0 < a + b, \tag{4.3.4}$$

the condition (4.3.3) holds for all β. In this case the potential energy of every local Hamiltonian (1.2.5) is a convex function of $u_\Lambda \in \mathbb{R}^{\nu|\Lambda|}$. The second inequality in (4.3.4) is a stability condition. In the anharmonic case, the model is stable since the anharmonic potential is super-quadratic. Hence, the inequality (4.3.4) is just a sufficient condition. In the harmonic case, it turns into a necessary condition. The discussion of the stability problems will be continued in Chapter 7.

If the inequality (4.3.4) holds and $\omega > 0$, the condition (4.3.3) is satisfied in the high-temperature limit $\beta \to 0$. Note that (4.3.3) does not contain the particle mass m; hence, the property stated holds also in the classical limit.

The proof of Theorem 4.3.1 will be done by means of the Dobrushin uniqueness criterion, which can be applied to a model with finite range interactions only. This criterion is based on estimating the *Dobrushin matrix*. This matrix is expressed by means of the Wasserstein distance between the kernels $\pi_{\beta,\Lambda}(\cdot|\xi)$ with different ξ.

The Wasserstein distance between measures on Euclidean spaces was introduced in Section 3.7, see (3.7.6). Here we introduce it in a more general context. Let $\mathbb{X}$ be a Polish space with metric d. Recall that by $\mathcal{P}(\mathbb{X})$ we denote the set of all probability measures on $\mathbb{X}$. Consider

$$\mathcal{P}_1 \stackrel{\mathrm{def}}{=} \left\{\mu \in \mathcal{P}(\mathbb{X}) \mid \textstyle\int d(x, x_0)\mu(\mathrm{d}x) < \infty\right\} \tag{4.3.5}$$

for some and hence for all $x_0 \in \mathbb{X}$. By definition, for $\mu, \nu \in \mathcal{P}_1$, the Wasserstein distance is

$$D(\mu,\nu) \stackrel{\mathrm{def}}{=} \inf \int_{\mathbb{X}^2} d(x,x')P(\mathrm{d}x, \mathrm{d}x'), \tag{4.3.6}$$

where the infimum is taken over the set of all probability measures on $\mathbb{X} \times \mathbb{X}$, which marginal distributions are μ and ν, respectively. One can show that D is a metric on $\mathcal{P}_1$. By the Kantorovich–Rubinstein duality the Wasserstein distance has yet another form

$$D(\mu, \nu) = \sup_{f \in \mathrm{Lip}_1(\mathbb{X})} \left| \int f \mathrm{d}\mu - \int f \mathrm{d}\nu \right|, \tag{4.3.7}$$

where

$$\mathrm{Lip}_1(\mathbb{X}) = \{ f : \mathbb{X} \to \mathbb{R} \mid \|f\|_L \leq 1 \},$$

see (1.3.67). In view of Proposition 1.3.31, the metric topology defined on $\mathcal{P}_1$ by the Wasserstein distance is equivalent to the weak topology induced from $\mathcal{P}(\mathbb{X})$.

Now let us turn to our problem. As was mentioned above, the Dobrushin criterion employs the Dobrushin matrix. In our context, it is

$$C_{\ell\ell'}^{\mathrm{Dob}} = \sup \left\{ \frac{D[\pi_\ell(\cdot|\xi), \pi_\ell(\cdot|\eta)]}{\|\xi_{\ell'} - \eta_{\ell'}\|_{L^2_\beta}} \right\}, \tag{4.3.8}$$

where the supremum is taken over all $\xi, \eta \in \Omega^{\mathrm{t}}_\beta$ which differ only at ℓ'. Here by π_ℓ we denoted the kernel (3.1.44) corresponding to $\Lambda = \{\ell\}$. According to the Dobrushin criterion, the uniqueness stated will follow if

$$\sup_\ell \sum_{\ell' :\, \ell' \neq \ell} C_{\ell\ell'}^{\mathrm{Dob}} < 1. \tag{4.3.9}$$

In view of (3.1.47), the map

$$L^2_\beta \ni \xi_{\ell'} \mapsto \phi_f(\xi_{\ell'}) \stackrel{\mathrm{def}}{=} \int_{\Omega_\beta} f(x_\ell) \pi_\ell(\mathrm{d}x|\xi), \quad f \in \mathrm{Lip}_1(L^2_\beta) \tag{4.3.10}$$

has the following derivative in direction $\zeta \in L^2_\beta$:

$$\begin{aligned} (\nabla\phi_f(\xi_{\ell'}), \zeta)_{L^2_\beta} = J_{\ell\ell'} \bigg\{ &\int_{\Omega_\beta} f(x_\ell)(x_\ell, \zeta)_{L^2_\beta} \pi_\ell(\mathrm{d}x|\xi) \\ &- \int_{\Omega_\beta} f(x_\ell)\pi_\ell(\mathrm{d}x|\xi) \cdot \int_{\Omega_\beta} (x_\ell, \zeta)_{L^2_\beta} \pi_\ell(\mathrm{d}x|\xi) \bigg\}. \end{aligned} \tag{4.3.11}$$

The expression in $\{\ldots\}$ is the covariance of the functions f and $\varphi_\zeta(x_\ell) = (x_\ell, \zeta)_{L^2}$. Thus, applying in (4.3.11) the Cauchy–Schwarz inequality we get

$$\left| (\nabla\phi_f(\xi_{\ell'}), \zeta)_{L^2_\beta} \right| \leq |J_{\ell\ell'}| \cdot [\mathrm{Var}(f)]^{1/2} \cdot \left[\mathrm{Var}(\varphi_\zeta)\right]^{1/2}, \tag{4.3.12}$$

where the variances, see (2.3.1), are taken with respect to the measure $\pi_\ell(\cdot|\xi)$. The function φ_ζ has the derivative (2.3.7)

$$\varphi'_\zeta(x_\ell) = \zeta;$$

hence, by (2.3.8) and (2.3.5),

$$\mathrm{Var}(\varphi_\zeta) \leq C_{\mathrm{LS}}[\pi_\ell(\cdot|\xi)] \cdot \|\zeta\|^2_{L^2_\beta}, \tag{4.3.13}$$

where C_{LS} is the logarithmic Sobolev constant of the corresponding measure. For every $\xi \in \Omega^{\mathrm{t}}_\beta$, the measure $\pi_\ell(\cdot|\xi)$ has the form (2.3.6) with $V_\ell = W_\ell$; hence, its constant C_{LS} obeys the bound (2.3.12), i.e.,

$$C_{\mathrm{LS}}[\pi_\ell(\cdot|\xi)] \leq e^{\beta\omega}/(a+b), \tag{4.3.14}$$

where a, b, and ω are the same as in (4.3.3). Now let us estimate the variance of f. Here we cannot apply (2.3.8) directly as the function f may not be differentiable. Nevertheless, after some additional efforts we will be able to do this.

Lemma 4.3.2. *For every* $f \in \mathrm{Lip}_1(L^2_\beta)$, *the variance in* (4.3.12) *has the bound*

$$\mathrm{Var}(f) \leq e^{\beta\omega}/(a+b). \tag{4.3.15}$$

Proof. Let $\mathfrak{C}$ be the family $\mathfrak{C}_\Lambda$ with $\Lambda = \{\ell\}$, see Definition 1.4.13. Obviously, for each $f \in \mathrm{Lip}_1(L^2_\beta)$, its restriction to C_β belongs to $\mathfrak{C}$. Then by Theorem 2.1.1,

$$\mathrm{Var}(f) = \lim_{N\to+\infty} \mathrm{Var}_{P_N^{h_\ell}}(g), \tag{4.3.16}$$

where, as in (2.3.16), the function $g\colon \mathbb{R}^{\nu N} \to \mathbb{R}$ is such that $f(x_\ell^{(N)}) = g(S_\ell)$ for $x_\ell^{(N)}$ and S_ℓ related to each other by (2.1.18), (2.1.23), (2.1.24). The measure $P_N^{h_\ell}$ is given by (2.3.13) with h_ℓ being as in (2.1.28), in which $y_\ell = \sum_{\ell'} J_{\ell\ell'}\xi_{\ell'}$. By (2.1.18), (2.1.23), (2.1.24), it follows that

$$\|x_\ell^{(N)}\|^2_{L^2_\beta} = \sum_{k\in\mathcal{K}_N} |\tilde{x}_\ell(k)|^2 = \sum_{p\in\mathcal{P}_N} |\tilde{x}_\ell((N/\beta)p)|^2 = \sum_{p\in\mathcal{P}_N} \left|\tilde{S}_\ell(p)\right|^2 = |S_\ell|^2 .$$

For distinct x_ℓ, y_ℓ, and for the corresponding S_ℓ^x, S_ℓ^y, one has

$$\frac{|g(S_\ell^x) - g(S_\ell^y)|}{|S_\ell^x - S^y{}_\ell|} = \frac{|f(x_\ell^{(N)}) - f(y_\ell^{(N)})|}{\|x_\ell^{(N)} - y_\ell^{(N)}\|_{L^2_\beta}};$$

hence, $g \in \mathrm{Lip}_1(\mathbb{R}^{\nu N})$. If for every S_ℓ, such a function has a gradient, one finds a unit vector $e \in \mathbb{R}^{\nu N}$ such that

$$|(\nabla g)(S_\ell)| = ((\nabla g)(S_\ell), e) = \lim_{t\to 0} \frac{1}{t}\left[g(S_\ell + te) - g(S_\ell)\right] \leq 1. \tag{4.3.17}$$

Then for this function, by (2.3.14), (2.3.5), and (2.3.3) we have

$$\mathrm{Var}_{P_N^{h_\ell}}(g) \leq e^{\beta\omega}/(a+b). \tag{4.3.18}$$

In the general case, g can be approximated by its *regularizations*

$$(\varrho_n * g)(S_\ell) = \int_{\mathbb{R}^{\nu N}} g(S_\ell - s)\varrho_n(s)\mathrm{d}s, \quad n \in \mathbb{N},$$

where

$$\varrho_n(s) = \frac{n^{\nu N}\varrho(ns)}{\int_{\mathbb{R}^{\nu N}} \varrho(s)\mathrm{d}s}$$

is a *mollifier*. The function $\varrho \geq 0$ is infinitely differentiable with compact support. Then the function $(\varrho_n * g)$ has gradient and belongs to $\mathrm{Lip}_1(\mathbb{R}^{\nu N})$, see Lemma 1.8, page 21 in [307], where one can also find that the sequence $\{(\varrho_n * g)\}_{n\in\mathbb{N}}$ converges to g uniformly on $\mathbb{R}^{\nu N}$. Putting all these facts together we conclude that the variance of g obeys (4.3.18) also in this case. Thus, by (4.3.16) we get the bound which was to be proven. □

Proof of Theorem 4.3.1. By (4.3.7), (4.3.10), (4.3.12), and the mean value theorem we have

$$\begin{aligned} D[\pi_\ell(\cdot|\xi), \pi_\ell(\cdot|\eta)] &= \sup_{f\in \mathrm{Lip}_1(L^2_\beta)} \left| \left(\nabla\phi_f(\vartheta\xi_{\ell'} + (1-\vartheta)\eta_{\ell'}), \xi_{\ell'} - \eta_{\ell'}\right)_{L^2_\beta} \right| \\ &\leq |J_{\ell\ell'}| \cdot \|\xi_{\ell'} - \eta_{\ell'}\|_{L^2_\beta} \sqrt{C_{\mathrm{LS}}[\pi_\ell(\cdot|\xi)]} \times \sup_{f\in \mathrm{Lip}_1(L^2_\beta)} [\mathrm{Var}(f)]^{1/2}. \end{aligned}$$

Taking here into account (4.3.14), (4.3.15), and (4.3.8) we get

$$C^{\mathrm{Dob}}_{\ell\ell'} \leq |J_{\ell\ell'}| \cdot e^{\beta\omega}/(a+b). \tag{4.3.19}$$

Thus, the uniqueness condition (4.3.9) is satisfied if (4.3.3) holds. □

4.4 Comments and Bibliographic Notes

Section 4.1: As was mentioned at the beginning of this section, the model (1.1.3), (1.1.8) is widely used in the description of physical systems, in the thermodynamic behavior of which localized light particles play an important role. Usually, the anharmonic potential is taken as a polynomial (1.1.12), the simplest form of which is the so-called ϕ^4 polynomial with $r = 2$. A 'derivation' of this model from a more realistic physical model is given in the review paper [284]. Various physical substances modeled in this way are described in the monographs [70], [80], [153], [179], [309]. Here we mention also the monograph [212] and the paper [149] where a system of harmonic oscillators was used to describe the dynamics of crystals. Systems of oscillators – both classical and quantum – were studied in [86] and in a series of papers [310], [311], [312], [316]. Systems of interacting anharmonic oscillators were used to model the interaction of electrons with vibrating ions in solids, see [120], [121], [286], [287].

Finally, we mention the papers [146], [151], [197] where quantum oscillators were used to describe the interaction of the photon field with solids.

Gibbs measures in general, and those we study in this book in particular, can be heuristically seen as equilibrium measures of certain systems of stochastic differential equations (SDE's). This is the interpretation initiated by G. Parisi and Y. S. Wu in [236] and called 'stochastic quantization'. The evolution time entering the SDE is often called 'computer time', inasmuch as the SDE can be used to simulate 'computer time' phenomena described by the equilibrium measure. The rigorous study of stochastic quantization was done in connection with one- and two-dimensional Euclidean quantum fields, see e.g., [164], [90], [261], [173] for the one-dimensional case, and [167], [35], [4], [92], [91], [216] for the two-dimensional case. For some extensions to the three-dimensional case, see [33]. Stochastic quantization was also studied in connection with polymer measures [36]. For lattice and continuous models with unbounded spins, stochastic quantization (also called Glauber dynamics) was studied in [2], [26], [27], [28], [32], [93].

Section 4.2: Theorem 4.2.1 establishes the classical limit of the local Euclidean Gibbs measures, which is then used to obtain this limit for the corresponding local Gibbs states, restricted to the algebra of multiplication operators by bounded continuous functions. As was noted in the concluding part of this section, on the global level it would be more correct to speak about classical limits of the sets $\mathcal{G}^{\rm t}_\beta$ rather than their individual members. In this case one could use the fact that each $\mathcal{G}^{\rm t}_\beta$ is a compact subset of the Polish space $\mathcal{P}(\Omega^{\rm t}_\beta)$, see Theorem 3.3.6. Let $\mathfrak{K}$ be the family of all compact subsets of $\mathcal{P}(\Omega^{\rm t}_\beta)$. One equips it with the Hausdorff metric h defined as follows. For $A, B \in \mathfrak{K}$,

$$h(A, B) = \inf\{\delta > 0 \mid A \subset O_\delta(B) \ \text{ and } \ B \subset O_\delta(A)\},$$

where $O_\delta(A)$ is the union of the balls of radii δ, centered at the elements of A. Recall that $\mathcal{P}(\Omega^{\rm t}_\beta)$ is a metric space. Then the convergence $\mathcal{G}^{\rm t}_\beta \to \mathcal{G}^{\rm cl}_\beta$, as $m \to +\infty$, can be studied in this metric. Furthermore, it is clear that two convex sets $A, B \in \mathfrak{K}$ coincide if and only if their extreme boundaries coincide. Then, in view of Lemma 3.3.7, to study the classical limit of the sets $\mathcal{P}(\Omega^{\rm t}_\beta)$ one has to evaluate the Wasserstein distance between two $\pi_{\beta,\Lambda}(\cdot|\xi)$ with different values of m and with the same Λ and ξ. We leave this problem for future investigations.

Path integrals provide an ideal tool for describing the relationship between classical and quantum mechanics. In fact, asymptotic developments in powers of relevant parameters measuring the 'quantumness' of the system can be achieved by infinite-dimensional versions of the stationary phase method, respectively, Laplace or saddle point method. This leads in particular to detailed studies of the approach to classical mechanics from quantum mechanics formulated in terms of rigorous Feynman path integrals, see for example, [5]. For related works with respect to the Wiener measure see e.g., [34], [37], [38] and for applications in statistical mechanics see [110]. The classical limit of a simpler version of the model (1.1.3), (1.1.8) was first described in [13].

Section 4.3: The proof of Theorem 4.3.1 was done by means of an extension of the method originally used in [28], [29], [30], see also [23] for further generalizations. All these results were obtained by means of a combination of the logarithmic Sobolev inequality and a contraction condition. The latter is widely known as the Dobrushin uniqueness criterion, see Theorem 4 in [101] as well as [103] for more sophisticated versions. This criterion was derived for Gibbs states of classical lattice models with finite-range interactions and bounded spins. Due to the latter fact, one can estimate the Wasserstein distance $D[\pi_\ell(\cdot|\xi), \pi_\ell(\cdot|\eta)]$ uniformly with respect to the configurations ξ and η. Later on, a similar method was elaborated for Gibbs states of systems of interacting finite-state spins on graphs with locally bounded degree, see [56] and [174] for the recent development. For unbounded spins, proving uniqueness, especially by means of cluster expansions, see [218], encounters essential problems. There exists a modification of the original Dobrushin criterion, in which one needs to estimate the coefficients of the Dobrushin matrix only for ξ and η belonging to some compact sets. It was proposed by R. A. Dobrushin and E. A. Pechersky in [102] and later applied in [58], [243] also to Gibbs states of systems of classical particles in continuum. A detailed analysis of all these methods, as well as their extensions to super-quadratic interactions and to interactions with infinite range, can be found in Sections 2.3 and 4.4 of [242], see also [241].

Chapter 5
Thermodynamic Pressure

In this chapter, we set $\mathbb{L} = \mathbb{Z}^d$ and study thermodynamic pressure. As we shall see below, pressure is very useful for the study of the set $\mathcal{G}^{\mathrm{t}}_\beta$. It is defined for local states and then obtained in the limit $\Lambda \nearrow \mathbb{L}$. The limiting pressure is a thermodynamic function; up to a factor it coincides with *the free energy density* and contains information about the macroscopic properties of the model.

5.1 The Existence of Thermodynamic Pressure

In this section, we again consider the translation-invariant version of the model (1.1.3), (1.1.8), see Definition 1.4.10. Besides this condition, the interaction potential $J_{\ell\ell'}$ and the function V are supposed to obey Assumption 1.1.1 only.

Given $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, the pressure in Λ corresponding to the boundary condition $\xi \in \Omega^{\mathrm{t}}_\beta$ is

$$p_\Lambda(\xi) = \frac{1}{|\Lambda|} \log N_{\beta,\Lambda}(\xi), \tag{5.1.1}$$

where $N_{\beta,\Lambda}(\xi)$ is given in (3.1.39). If Λ is a box, one can define the pressure corresponding to the periodic boundary conditions, see (1.4.52),

$$p^{\mathrm{per}}_\Lambda = \frac{1}{|\Lambda|} \log N^{\mathrm{per}}_{\beta,\Lambda}. \tag{5.1.2}$$

For $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, we set

$$p^\mu_\Lambda = \int_{\Omega_\beta} p_\Lambda(\xi)\mu(\mathrm{d}\xi). \tag{5.1.3}$$

If for a cofinal sequence $\mathcal{L}$, the limit

$$p^\mu \stackrel{\mathrm{def}}{=} \lim_{\mathcal{L}} p^\mu_\Lambda \tag{5.1.4}$$

exists, we shall call it the pressure in the state μ. Note that it may depend on $\mathcal{L}$. Finally, we introduce

$$p = \lim_{\mathcal{L}} p_\Lambda(0), \qquad p^{\mathrm{per}} = \lim_{\mathcal{L}_{\mathrm{box}}} p^{\mathrm{per}}_\Lambda, \tag{5.1.5}$$

if these limits exist for some cofinal sequences $\mathcal{L} \subset \mathfrak{L}_{\mathrm{fin}}$ and $\mathcal{L}_{\mathrm{box}} \subset \mathfrak{L}_{\mathrm{box}}$. Good candidates to study the convergence in (5.1.4), (5.1.5) are van Hove sequences, see Definition 3.1.1. To see this, for $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ we set

$$\hat{J}(\Lambda) = \frac{1}{|\Lambda|} \sum_{\ell\in\Lambda,\ \ell'\in\Lambda^c} |J_{\ell\ell'}|. \tag{5.1.6}$$

Proposition 5.1.1. *If $\mathcal{L}$ is a van Hove sequence, then*

$$\lim_{\mathcal{L}} \hat{J}(\Lambda) = 0. \tag{5.1.7}$$

Proof. Given a van Hove sequence $\mathcal{L}$, let $\{\Lambda_{L_n^{\pm}}\}_{n\in\mathbb{N}}$ be the sequences of boxes (3.1.2) with the properties described by claim (ii) of Proposition 3.1.3. For a given $\Lambda \in \mathcal{L}$, we pick $n \in \mathbb{N}$ such that $\Lambda_{L_n^-} \subset \Lambda \subset \Lambda_{L_n^+}$, see (3.1.4). Then

$$\begin{aligned}
\hat{J}(\Lambda) &\leq \frac{1}{|\Lambda_{L_n^-}|} \sum_{\ell\in\Lambda_{L_n^+}} \sum_{\ell'\in\Lambda^c_{L_n^-}} |J_{\ell\ell'}| \\
&\leq \frac{1}{|\Lambda_{L_n^-}|} \sum_{\ell\in\Lambda_{L_n^-}} \sum_{\ell'\in\Lambda^c_{L_n^-}} |J_{\ell\ell'}| + \frac{1}{|\Lambda_{L_n^-}|} \sum_{\ell\in\Lambda_{L_n^+}\setminus\Lambda_{L_n^-}} \sum_{\ell'} |J_{\ell\ell'}| \\
&= \frac{1}{|\Lambda_{L_n^-}|} \sum_{\ell\in\Lambda_{L_n^-}} \sum_{\ell'\in\Lambda^c_{L_n^-}} |J_{\ell\ell'}| + \hat{J}_0 \cdot \left[\left(\frac{L_n^+}{L_n^-} \right)^d - 1 \right].
\end{aligned}$$

In view of (3.1.4), to prove the statement we have to show that the first summand in the latter expression tends to zero as $n \to +\infty$, i.e., to prove (5.1.7) for sequences of boxes (3.1.2) only. By (1.1.11) for given $\ell \in \mathbb{L}$ and $\varepsilon > 0$, one finds $L^\varepsilon \in \mathbb{N}$, such that

$$\sum_{\ell'\in K^c_{\varepsilon,\ell}} |J_{\ell\ell'}| < \varepsilon/2, \tag{5.1.8}$$

where $K_{\varepsilon,\ell} = \{\ell' \mid |\ell - \ell'| \leq L^\varepsilon\}$. Note that $N_\varepsilon = |K_{\varepsilon,\ell}|$ is independent of ℓ. Now let us take a sequence $\{\Lambda_{L_n}\}_{n\in\mathbb{N}}$ of boxes (3.1.2) and pick $n \in \mathbb{N}$ such that $L_n > L^\varepsilon$. If $\ell' \in \Lambda^c_{L_n}$ and $\ell \in \Lambda_{L_n - L^\varepsilon}$, then $|\ell - \ell'| > L^\varepsilon$. Thus, only for the points $\ell \in \Lambda_{L_n} \setminus \Lambda_{L_n - L^\varepsilon}$, the intersections of the corresponding $K_{\varepsilon,\ell}$ with $\Lambda^c_{L_n}$ are non-empty. The number of pairs of such points does not exceed the number of the points in $\Lambda_{L_n} \setminus \Lambda_{L_n - L^\varepsilon}$ times the cardinality of $K_{\varepsilon,\ell}$, that is, it does not exceed $2dL^\varepsilon(2L_n)^{d-1} \cdot N_\varepsilon$. Thus, we have

$$\begin{aligned}
\hat{J}(\Lambda_{L_n}) &= \frac{1}{|\Lambda_{L_n}|} \sum_{\ell\in\Lambda_{L_n}\setminus\Lambda_{L_n-L^\varepsilon}} \sum_{\ell'\in\Lambda^c_{L_n}\cap K_{\varepsilon,\ell}} |J_{\ell\ell'}| \\
&\quad + \frac{1}{|\Lambda_{L_n}|} \sum_{\ell\in\Lambda_{L_n}} \sum_{\ell'\in\Lambda^c_{L_n}\cap K^c_{\varepsilon,\ell}} |J_{\ell\ell'}| \\
&\leq 2dL^\varepsilon(2L_n)^{d-1} \cdot N_\varepsilon \cdot \hat{J}_0/(2L_n)^d + \varepsilon/2,
\end{aligned}$$

where we have taken into account (5.1.8). Now we make the first summand in the latter expression less than $\varepsilon/2$ by taking big enough n, which completes the proof as $\varepsilon > 0$ is arbitrarily small. □

Now we are in a position to prove the following

Theorem 5.1.2. *For every van Hove sequence $\mathfrak{L}$, the limiting pressure p defined in* (5.1.5) *exists and is independent of the particular choice of $\mathfrak{L}$.*

Proof. Let us recall that we are considering the translation-invariant case, where all V_ℓ coincide. For $\vartheta \in [0,1]$, $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$, and $\Delta \subseteq \Lambda$, we set, cf. (2.5.32), (2.5.33),

$$\begin{aligned}\Theta_{\Delta,\Lambda}(\vartheta) = \int_{\Omega_{\beta,\Lambda}} \exp\Bigg\{\frac{1}{2}\sum_{\ell,\ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} \\ + \vartheta\Bigg(\frac{1}{2}\sum_{\ell,\ell'\in\Lambda\setminus\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda\setminus\Delta,\ \ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta}\Bigg) \\ - \sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),\end{aligned} \tag{5.1.9}$$

and

$$\theta_{\Delta,\Lambda}(\vartheta) = \log\Theta_{\Delta,\Lambda}(\vartheta)/|\Lambda|. \tag{5.1.10}$$

For $\Delta = \emptyset$,

$$\theta_{\emptyset,\Lambda}(0) = \log\left\{\int_{C_\beta}\exp\left(-\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\right)\chi_\beta(\mathrm{d}x_\ell)\right\}, \tag{5.1.11}$$

which is independent of Λ. For $\Delta \neq \emptyset$, we have

$$\begin{aligned}\theta_{\Delta,\Lambda}(0) &= \frac{|\Delta|}{|\Lambda|}\cdot p_\Delta(0) + \frac{|\Lambda|-|\Delta|}{|\Lambda|}\cdot\theta_{\emptyset,\Lambda}(0), \\ \theta_{\Delta,\Lambda}(1) &= p_\Lambda(0),\end{aligned} \tag{5.1.12}$$

which extends also to the case $\Delta = \emptyset$. Thereafter, we set

$$\begin{aligned}\nu^{(\vartheta)}_{\Delta,\Lambda}(\mathrm{d}x_\Lambda) = \frac{1}{\Theta_{\Delta,\Lambda}(\vartheta)}\exp\Bigg\{\frac{1}{2}\sum_{\ell,\ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} \\ + \vartheta\Bigg(\frac{1}{2}\sum_{\ell,\ell'\in\Lambda\setminus\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} + \sum_{\ell\in\Lambda\setminus\Delta,\ \ell'\in\Delta} J_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta}\Bigg) \\ - \sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Bigg\}\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda).\end{aligned} \tag{5.1.13}$$

Obviously, $\nu^{(1)}_{\Delta,\Lambda}$ is independent of Δ and coincides with the measure (1.4.18). For the same reason as in Subsection 2.5.3, $\theta_{\Delta,\Lambda}(\vartheta)$ is differentiable and

$$\begin{aligned}\theta'_{\Delta,\Lambda}(\vartheta) &= \frac{1}{2|\Lambda|}\sum_{\ell,\ell'\in\Lambda\setminus\Delta} J_{\ell\ell'}\langle (x_\ell, x_{\ell'})_{L^2_\beta}\rangle_{\nu^{(\vartheta)}_{\Delta,\Lambda}} \\ &\quad + \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda\setminus\Delta,\ \ell'\in\Delta} J_{\ell\ell'}\langle (x_\ell, x_{\ell'})_{L^2_\beta}\rangle_{\nu^{(\vartheta)}_{\Delta,\Lambda}} \\ &\stackrel{\mathrm{def}}{=} \Pi^{(1)}_{\Delta,\Lambda}(\vartheta) + \Pi^{(2)}_{\Delta,\Lambda}(\vartheta).\end{aligned} \tag{5.1.14}$$

For $\Delta = \emptyset$, we set $\Pi^{(2)}_{\emptyset,\Lambda}(\vartheta) \equiv 0$. Then, by (5.1.11),

$$|p_\Lambda(0) - \theta_{\emptyset,\Lambda}(0)| \le \sup_{\vartheta\in[0,1]} |\Pi^{(1)}_{\emptyset,\Lambda}(\vartheta)|. \tag{5.1.15}$$

For $\Delta \neq \emptyset$, by (5.1.12),

$$\begin{aligned}\frac{|\Delta|}{|\Lambda|}\cdot|p_\Delta(0) - p_\Lambda(0)| \le \sup_{\vartheta\in[0,1]} &\left\{|\Pi^{(1)}_{\Delta,\Lambda}(\vartheta)| + |\Pi^{(2)}_{\Delta,\Lambda}(\vartheta)|\right\} \\ &+ \frac{|\Lambda| - |\Delta|}{|\Lambda|}\cdot|p_\Lambda(0) - \theta_{\emptyset,\Lambda}(0)|\,.\end{aligned} \tag{5.1.16}$$

By simple calculations we get

$$\begin{aligned}|\Pi^{(1)}_{\emptyset,\Lambda}(\vartheta)| &\le \frac{1}{4|\Lambda|}\sum_{\ell,\ell'\in\Lambda} |J_{\ell\ell'}|\cdot\langle \|x_\ell\|^2_{L^2_\beta} + \|x_{\ell'}\|^2_{L^2_\beta}\rangle_{\nu^{(\vartheta)}_{\beta,\Lambda}} \\ &\le \hat{J}_0 C_{3.2.18}(0)/2,\end{aligned} \tag{5.1.17}$$

where we have taken into account also the fact that the estimate (3.2.18) is uniform in $\vartheta \in [0, 1]$. Employing the latter estimate in (5.1.15) we get

$$\theta_{\emptyset,\Lambda}(0) - \hat{J}_0 C_{3.2.18}(0)/2 \le p_\Lambda(0) \le \theta_{\emptyset,\Lambda}(0) + \hat{J}_0 C_{3.2.18}(0)/2, \tag{5.1.18}$$

where both bounds are independent of Λ, see (5.1.11). This means that for any cofinal sequence $\mathcal{L}$, the sequence $\{p_\Lambda(0)\}_{\Lambda\in\mathcal{L}}$ contains a convergent subsequence. For the sequence of boxes (3.1.2), let $\{L_n\}_{n\in\mathbb{N}}$ be such that there exists

$$p \stackrel{\mathrm{def}}{=} \lim_{n\to+\infty} p_{\Lambda_{L_n}}(0). \tag{5.1.19}$$

Now we show that this p is the limit (5.1.5) for any van Hove sequence $\mathcal{L}$. Given such a sequence $\mathcal{L}$, for an element $\Lambda \in \mathcal{L}$, and a fixed $n \in \mathbb{N}$, let $\mathfrak{L}^-_n(\Lambda) \subset \mathbb{P}(\Lambda_{L_n})$ (respectively, $\mathfrak{L}^+_n(\Lambda) \subset \mathbb{P}(\Lambda_{L_n})$) be the set of the translates of Λ_{L_n} which are contained in Λ (respectively, which have nonempty intersections with Λ). Here $\mathbb{P}(\Lambda_{L_n})$ is the same as in (1.4.47). We also set

$$\Lambda^\pm_n = \bigcup_{\Delta\in\mathfrak{L}^\pm_n} \Delta. \tag{5.1.20}$$

For $\Lambda \neq \Lambda_n^-$, let us estimate the difference between $p_\Lambda(0)$ and $p_{\Lambda_n^-}(0)$. To this end we use (5.1.16) with $\Delta = \Lambda_n^-$; hence, we have to estimate $\Pi^{(i)}_{\Lambda_n^-,\Lambda}$, $i = 1, 2$. As in (5.1.17), we get

$$\begin{aligned}
|\Pi^{(1)}_{\Lambda_n^-,\Lambda}(\vartheta)| &\leq [C_{3.2.18}(0)/2] \cdot \frac{|\Lambda| - |\Lambda_n^-|}{|\Lambda|} \cdot \frac{1}{|\Lambda| - |\Lambda_n^-|} \sum_{\ell,\ell' \in \Lambda \setminus \Lambda_n^-} |J_{\ell\ell'}| \\
&\leq [C_{3.2.18}(0)/2] \cdot \frac{|\Lambda_n^+| - |\Lambda_n^-|}{|\Lambda_n^-|} \cdot \frac{1}{|\Lambda| - |\Lambda_n^-|} \sum_{\ell \in \Lambda \setminus \Lambda_n^-,\ \ell' \in \mathbb{L}} |J_{\ell\ell'}| \\
&= \left[\hat{J}_0 C_{3.2.18}(0)/2\right] \cdot \left(\frac{N^+_{L_n}(\Lambda)}{N^-_{L_n}(\Lambda)} - 1 \right).
\end{aligned}$$

Similarly,

$$\begin{aligned}
|\Pi^{(2)}_{\Lambda_n^-,\Lambda}(\vartheta)| &\leq \frac{C_{3.2.18}(0)|\Lambda_n^-|}{|\Lambda_n^+|} \cdot \frac{1}{|\Lambda_n^-|} \sum_{\ell \in \Lambda_n^-,\ \ell' \in \mathbb{L} \setminus \Lambda_n^-} |J_{\ell\ell'}| \\
&\leq C_{3.2.18}(0) \hat{J}(\Lambda_n^-).
\end{aligned}$$

In what follows, taking into account (5.1.16), (5.1.15), (5.1.7), and the latter two estimates we arrive at

$$\begin{aligned}
|p_\Lambda(0) - p_{\Lambda_n^-}(0)| &\leq \frac{|\Lambda|}{|\Lambda_{L_n}^-|} \sup_{\vartheta \in [0,1]} \left\{ \left| \Pi^{(1)}_{\Lambda_{L_n}^-,\Lambda}(\vartheta) \right| + \left| \Pi^{(2)}_{\Lambda_{L_n}^-,\Lambda}(\vartheta) \right| \right\} \\
&\quad + \frac{|\Lambda| - |\Lambda_{L_n}^-|}{|\Lambda_{L_n}^-|} \sup_{\vartheta \in [0,1]} \left| \Pi^{(1)}_{\Lambda_{L_n}^-,\Lambda}(\vartheta) \right| \\
&\leq \left(\frac{N^+_{L_n}(\Lambda)}{N^-_{L_n}(\Lambda)} - 1 \right) \cdot \frac{N^+_{L_n}(\Lambda)}{N^-_{L_n}(\Lambda)} \cdot \hat{J}_0 C_{3.2.18}(0) \\
&\quad + \frac{N^+_{L_n}(\Lambda)}{N^-_{L_n}(\Lambda)} \cdot \hat{J}(\Lambda_n^-) C_{3.2.18}(0).
\end{aligned} \tag{5.1.21}$$

Now let us estimate the difference between $p_{\Lambda_n^-}(0)$ and $p_{\Lambda_{L_n}}(0)$. By the translation invariance of the model, the latter quantity coincides with the pressure at every translate of the box Λ_{L_n}. To get the estimate in question we introduce

$$\begin{aligned}
\Theta_n(\vartheta) = \int_{\Omega_{\beta,\Lambda}} \exp \Bigg\{ &\frac{1}{2} \sum_{\Delta \in \mathfrak{L}_n^-(\Lambda)} \sum_{\ell,\ell' \in \Delta} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} \\
&+ \vartheta \sum_{\Delta,\Delta' \in \mathfrak{L}_n^-(\Lambda),\ \Delta \neq \Delta'} \sum_{\ell \in \Delta} \sum_{\ell' \in \Delta'} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} \\
&- \sum_{\ell \in \Lambda_n^-} \int_0^\beta V(x_\ell(\tau)) \mathrm{d}\tau \Bigg\} \chi_{\beta,\Lambda_n^-}(\mathrm{d}x_{\Lambda_n^-}),
\end{aligned} \tag{5.1.22}$$

and

$$\theta_n(\vartheta) = \log \Theta_n(\vartheta)/|\Lambda_n^-|, \quad \vartheta \in [0,1]. \tag{5.1.23}$$

Then

$$\theta_n(0) = \frac{|\Lambda_{L_n}|}{|\Lambda_n^-|} \sum_{\Delta \in \mathfrak{L}_n^-(\Lambda)} p_\Delta(0) = p_{\Lambda_{L_n}}(0), \quad \theta_n(1) = p_{\Lambda_n^-}(0), \tag{5.1.24}$$

and

$$|p_{\Lambda_{L_n}}(0) - p_{\Lambda_n^-}(0)| \le \sup_{\vartheta \in [0,1]} |\theta_n'(\vartheta)|. \tag{5.1.25}$$

The latter derivative can be calculated from (5.1.22), (5.1.23),

$$\theta_n'(\vartheta) = \frac{1}{|\Lambda_n^-|} \sum_{\Delta,\Delta' \in \mathfrak{L}_n^-(\Lambda),\ \Delta \neq \Delta'} \sum_{\ell \in \Delta} \sum_{\ell' \in \Delta'} J_{\ell\ell'} \cdot \langle (x_\ell, x_{\ell'})_{L^2_\beta} \rangle_{\nu_n^{(\vartheta)}}, \tag{5.1.26}$$

where the expectation is taken with respect to the probability measure

$$\begin{aligned} \nu_n^{(\vartheta)}(\mathrm{d}x_{\Lambda_n^-}) = \frac{1}{\Theta_n(\vartheta)} \exp\Bigg\{ & \frac{1}{2} \sum_{\Delta \in \mathfrak{L}_n^-(\Lambda)} \sum_{\ell,\ell' \in \Delta} J_{\ell\ell'} (x_\ell, x_{\ell'})_{L^2_\beta} \\ & + \vartheta \sum_{\Delta,\Delta' \in \mathfrak{L}_n^-(\Lambda),\ \Delta \neq \Delta'} \sum_{\ell \in \Delta} \sum_{\ell' \in \Delta'} J_{\ell\ell'} (x_\ell, x_{\ell'})_{L^2_\beta} \\ & - \sum_{\ell \in \Lambda_n^-} \int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau \Bigg\} \chi_{\beta,\Lambda_n^-}(\mathrm{d}x_{\Lambda_n^-}). \end{aligned}$$

Proceeding as above, e.g., in obtaining (5.1.17), we get

$$\begin{aligned} |p_{\Lambda_{L_n}}(0) - p_{\Lambda_n^-}(0)| &\le \frac{C_{3.2.18}(0)}{|\Lambda_n^-|} \sum_{\Delta,\Delta' \in \mathfrak{L}_n^-(\Lambda),\ \Delta \neq \Delta'} \sum_{\ell \in \Delta} \sum_{\ell' \in \Delta'} |J_{\ell\ell'}| \\ &\le \frac{C_{3.2.18}(0)}{|\Lambda_n^-|} \sum_{\Delta \in \mathfrak{L}_n^-(\Lambda)} \sum_{\ell \in \Delta} \sum_{\ell' \in \Delta^c} |J_{\ell\ell'}| \\ &= C_{3.2.18}(0) \hat{J}(\Lambda_{L_n}). \end{aligned} \tag{5.1.27}$$

Since the sequence of boxes $\{\Lambda_{L_n}\}_{n\in\mathbb{N}}$ itself is a van Hove sequence, for a given $\varepsilon > 0$, we can choose n such that $C_{3.2.18}(0)\hat{J}(\Lambda_{L_n}) < \varepsilon/2$, see (5.1.7). As the sequence $\mathcal{L}$ under consideration is also a van Hove sequence, we can choose Λ big enough and make the right-hand side of (5.1.21) also less than $\varepsilon/2$. Thus, for these n and Λ,

$$|p_\Lambda(0) - p_{\Lambda_{L_n}}(0)| < \varepsilon,$$

which in view of (5.1.19) completes the proof. $\square$

Our next aim is to show that all p^{μ}'s defined by (5.1.4), as well as $p^{\rm per}$ given by (5.1.5), exist and coincide with p. Thereby, the pressure p given by (5.1.5) will become a universal characteristic of the set $\mathcal{G}^{\rm t}_{\beta}$.

Theorem 5.1.3. *For every $\mu \in \mathcal{G}^{\rm t}_{\beta}$ and any van Hove sequence $\mathcal{L}$, the limiting pressure in the state μ exists and coincides with the pressure p given by (5.1.5). For any van Hove sequence $\mathcal{L}_{\rm box}$, the limiting pressure $p^{\rm per}$ exists and also coincides with p.*

Proof. Let us prove the first part of the theorem. By the Jensen inequality we obtain from (3.1.39) that for $t_1, t_2 \in \mathbb{R}$ and $\xi \in \Omega^{\rm t}_{\beta}$,

$$N_{\beta,\Lambda}((t_1 + t_2)\xi) \geq N_{\beta,\Lambda}(t_1\xi) \exp\Big\{t_2 \sum_{\ell\in\Lambda} \sum_{\ell'\in\Lambda^c} J_{\ell\ell'} \langle (x_\ell, \xi_{\ell'})_{L^2_\beta} \rangle_{\pi_{\beta,\Lambda}(\cdot|t_1\xi)}\Big\}.$$

We set here first $t_1 = 0$, $t_2 = 1$, then $t_1 = -t_2 = 1$, and obtain

$$\begin{aligned} p_\Lambda(0) &+ \frac{1}{|\Lambda|} \sum_{\ell\in\Lambda} \sum_{\ell'\in\Lambda^c} J_{\ell\ell'} \langle (x_\ell, \xi_{\ell'})_{L^2_\beta} \rangle_{\pi_{\beta,\Lambda}(\cdot|0)} \\ &\leq p_\Lambda(\xi) \leq p_\Lambda(0) + \frac{1}{|\Lambda|} \sum_{\ell\in\Lambda} \sum_{\ell'\in\Lambda^c} J_{\ell\ell'} \langle (x_\ell, x_{\ell'})_{L^2_\beta} \rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)}, \end{aligned} \tag{5.1.28}$$

where we used the fact that

$$\langle (x_\ell, \xi_{\ell'})_{L^2_\beta} \rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)} = \langle (x_\ell, x_{\ell'})_{L^2_\beta} \rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)},$$

see (3.1.46). Thereby, we integrate (5.1.28) with respect to the considered measure $\mu \in \mathcal{G}^{\rm t}_{\beta}$, take into account (3.1.51), and after some calculations arrive at

$$|p^{\mu}_{\Lambda} - p_\Lambda(0)| \leq C \hat{J}(\Lambda), \tag{5.1.29}$$

where

$$C = \max\{\sqrt{C_{3.2.18}\varkappa^{-1} \log C_{3.3.1}};\ \varkappa^{-1} \log C_{3.3.1}\},$$

where $\varkappa$ is the same as in (3.3.1). Then by (5.1.7) and Theorem 5.1.2 we get the proof of the first part.

In the case of periodic boundary conditions, we will proceed in the spirit of the proof of Theorem 5.1.2. Thus, we introduce the function

$$\begin{aligned} \theta_\Lambda(\vartheta) = \frac{1}{|\Lambda|} \log \Big\{ \int_{\Omega_{\beta,\Lambda}} \exp\Big(&\frac{\vartheta}{2} \sum_{\ell,\ell'\in\Lambda} [J^{\Lambda}_{\ell\ell'} - J_{\ell\ell'}](x_\ell, x_{\ell'})_{L^2_\beta} \\ &+ \frac{1}{2} \sum_{\ell,\ell'\in\Lambda} J_{\ell\ell'}(x_\ell, x_{\ell'})_{L^2_\beta} - \sum_{\ell\in\Lambda} \int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau \Big) \chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda) \Big\}, \end{aligned} \tag{5.1.30}$$

where Λ is a box and $\vartheta \in [0, 1]$. It is differentiable on $\vartheta \in (0, 1)$ and obeys the boundary conditions

$$\theta_\Lambda(1) = p^{\rm per}_{\Lambda}, \quad \theta_\Lambda(0) = p_\Lambda(0). \tag{5.1.31}$$

The derivative is

$$\theta'_\Lambda(\vartheta) = \sum_{\ell\in\Lambda}\sum_{\ell'\in\Delta'_\ell(\Lambda)} [J^\Lambda_{\ell\ell'} - J_{\ell\ell'}]\cdot\langle (x_\ell, x_{\ell'})_{L^2_\beta}\rangle_{\nu_\vartheta}, \tag{5.1.32}$$

where, cf. (1.4.48), (1.4.50),

$$\Delta'_\ell(\Lambda) = \{\ell' \in \Lambda \mid |\ell - \ell'|_\Lambda < |\ell - \ell'|\},$$

and ν_ϑ is the probability measure whose density with respect to $\chi_{\beta,\Lambda}$ is the function under the integral in (5.1.30), properly normalized. Then, similarly as above, one estimates

$$|\langle (x_\ell, x_{\ell'})_{L^2_\beta}\rangle_{\nu_\vartheta}| \le C,$$

where the constant is independent of Λ and ϑ. For a given $\varepsilon > 0$ and any $\ell \in \Lambda$, $\ell' \in \Delta'_\ell(\Lambda)$, both $|\ell - \ell'|$ and $|\ell - \ell'|_\Lambda$ exceed L^ε if the box Λ is big enough. Then we obtain from (5.1.32) the estimate

$$\begin{aligned}|\theta'_\Lambda(\vartheta)| &\le \frac{C}{|\Lambda|}\sum_{\ell\in\Lambda}\sum_{\ell'\in\Delta'_\ell(\Lambda)} |J^\Lambda_{\ell\ell'} - J_{\ell\ell'}| \\ &\le \frac{2C}{|\Lambda|}\sum_{\ell\in\Lambda}\sum_{\ell'\in K_{\varepsilon,\ell}} |J_{\ell\ell'}| \le \varepsilon C,\end{aligned}$$

where we have taken into account (5.1.8). Thereby, the proof of the second part of the theorem follows by (5.1.31) and Theorem 5.1.2. □

5.2 Dependence on the External Field

In this section, we assume that the anharmonic potential contains an external field term, i.e., it is $V(u) - (\hat{h}, u)$, $\hat{h} \in \mathbb{R}^\nu$. Without loss of generality, one may set $\hat{h} = (h, \dots, 0)$, $h \in \mathbb{R}$. Thus, we study the properties of the pressure as a function of h. To indicate the h-dependence we write $p_\Lambda(h, \xi)$, $p_\Lambda(h) \stackrel{\text{def}}{=} p_\Lambda(h, 0)$, $p^{\text{per}}_\Lambda(h)$, $p^\mu_\Lambda(h)$, $p(h)$. For $\Lambda \in \mathfrak{L}_{\text{fin}}$, the local Euclidean Gibbs measure (1.4.18) can be written in the form

$$\nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \left(\frac{N_{\beta,\Lambda}(0)}{N_{\beta,\Lambda}(h)}\right)\nu^{(0)}_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{5.2.1}$$

where the latter measure corresponds to $h = 0$. Then

$$N_{\beta,\Lambda}(h) = N_{\beta,\Lambda}(0)\int_{\Omega_{\beta,\Lambda}} \exp\left(h\sum_{\ell\in\Lambda}\int_0^\beta x^{(1)}_\ell(\tau)\mathrm{d}\tau\right)\nu^{(0)}_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{5.2.2}$$

The same representation can also be written for the periodic measure (1.4.52). By (5.1.1) and Proposition 1.4.12, $p_\Lambda(h)$ and $p^{\text{per}}_\Lambda(h)$, as functions of h, are analytic in a subset of $\mathbb{C}$, which contains the real line. Thus, one can compute the derivatives and obtain by (1.2.12), (1.4.20) and (1.4.52) the following expressions:

$$\frac{\partial}{\partial h} p_\Lambda(h) = \beta M_\Lambda(h), \quad \frac{\partial}{\partial h} p_\Lambda^{\rm per}(h) = \beta M_\Lambda^{\rm per}(h), \tag{5.2.3}$$

where

$$M_\Lambda(h) \stackrel{\rm def}{=} \frac{1}{|\Lambda|} \sum_{\ell \in \Lambda} \varrho_{\beta,\Lambda}\big[q_\ell^{(1)}\big], \quad M_\Lambda^{\rm per}(h) \stackrel{\rm def}{=} \varrho_{\beta,\Lambda}^{\rm per}\big[q_\ell^{(1)}\big] \tag{5.2.4}$$

are local *magnetizations*, corresponding to the zero and periodic boundary conditions, respectively. Furthermore, the second derivatives can be written in the form

$$\begin{aligned}
&\frac{\partial^2}{\partial h^2} p_\Lambda(h) \\
&\quad = \frac{1}{|\Lambda|} \int_{\Omega_{\beta,\Lambda}} \int_{\Omega_{\beta,\Lambda}} \Big[\sum_{\ell \in \Lambda} \int_0^\beta (x_\ell(\tau) - \tilde{x}_\ell(\tau))\, \mathrm{d}\tau\Big]^2 \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda)\nu_{\beta,\Lambda}(\mathrm{d}\tilde{x}_\Lambda), \\
&\frac{\partial^2}{\partial h^2} p_\Lambda^{\rm per}(h) \\
&\quad = \frac{1}{|\Lambda|} \int_{\Omega_{\beta,\Lambda}} \int_{\Omega_{\beta,\Lambda}} \Big[\sum_{\ell \in \Lambda} \int_0^\beta (x_\ell(\tau) - \tilde{x}_\ell(\tau))\, \mathrm{d}\tau\Big]^2 \nu_{\beta,\Lambda}^{\rm per}(\mathrm{d}x_\Lambda)\nu_{\beta,\Lambda}^{\rm per}(\mathrm{d}\tilde{x}_\Lambda).
\end{aligned} \tag{5.2.5}$$

Therefore, both $p_\Lambda(h)$ and $p_\Lambda^{\rm per}(h)$ are convex functions of h. Now let $p(h)$ be the pressure, the existence of which was established in Theorem 5.1.3. Then by Proposition 2.5.4 it has the following properties.

Theorem 5.2.1. *The pressure $p(h)$ is a convex function of $h \in \mathbb{R}$. Therefore, the set*

$$\mathcal{R} \stackrel{\rm def}{=} \{h \in \mathbb{R} \mid D_- p(h) < D_+ p(h)\} \tag{5.2.6}$$

is at most countable. For any $h \in \mathcal{R}^c$ and any van Hove sequence $\mathcal{L}$ (hence, for any sequence of boxes $\mathcal{L}_{\rm box}$), it follows that

$$\lim_{\mathcal{L}} M_\Lambda(h) = \lim_{\mathcal{L}_{\rm box}} M_\Lambda^{\rm per}(h) = \beta^{-1} p'(h) \stackrel{\rm def}{=} M(h). \tag{5.2.7}$$

By this theorem, the global magnetization $M(h)$ is a nondecreasing function of $h \in \mathcal{R}^c$; it is continuous on each open connected component of this set. That is, $M(h)$ is continuous on the intervals $(a_-, a_+) \subset \mathcal{R}^c$, where $a_\pm$ are two consecutive elements of $\mathcal{R}$. At each such $a_\pm$, the global magnetization is discontinuous. One observes, however, that the set $\mathcal{R}^c$ may have empty interior; hence, $M(h)$ may be nowhere continuous.

Theorem 5.2.2. *Let the model be ferromagnetic and translation-invariant. Then there exists $m_* > 0$, which may depend on β but is independent of the interaction intensities, with the following property. For every $m > m_*$, there exist $h_\pm$, $h_- < h_+$, which may also depend on the model parameters and β, such that*

$$M(h) < 0, \quad \text{for } h \in \mathcal{R}^c \cap (-\infty, h_-); \quad M(h) > 0, \quad \text{for } h \in \mathcal{R}^c \cap (h_+ + \infty).$$

Proof. Let us consider the following measure on C_β:

$$\begin{aligned} \sigma_\beta(\mathrm{d}x) &= \frac{1}{N_\beta} \exp\left(- \int_0^\beta V(x(\tau))\mathrm{d}\tau \right) \chi_\beta(\mathrm{d}x), \\ N_\beta &= \int_{C_\beta} \exp\left(- \int_0^\beta V(x(\tau))\mathrm{d}\tau \right) \chi_\beta(\mathrm{d}x), \end{aligned} \tag{5.2.8}$$

where χ_β is Høegh-Krohn's measure. By (1.3.117) it follows that

$$\left\langle \left| x(\tau) - x(\tau') \right|^{2p} \right\rangle_{\sigma_\beta} \leq Q_V(\beta, \nu, p) m^{-p} |\tau - \tau'|^p, \tag{5.2.9}$$

where

$$Q_V(\beta, \nu, p) = \frac{2^p \exp\left(-\beta c_V\right) \Gamma(\nu/2 + p)}{N_\beta \Gamma(\nu/2)},$$

c_V being the same as in (1.1.10). Let us fix some $p \in \mathbb{N} \setminus \{1\}$ and $\alpha \in (0, 1/2 - 1/2p)$. Thereby, for $\vartheta \in (0, \beta)$, we set, cf. (1.3.49),

$$\lambda_\vartheta(x) = \sup\left\{ \frac{|x(\tau) - x(\tau')|^{2p}}{|\tau - \tau'|^{2\alpha p}} \;\Big|\; |\tau - \tau'| \leq \vartheta \right\}. \tag{5.2.10}$$

Then by Proposition 1.3.12 it follows from (5.2.9) that

$$\langle \lambda_\vartheta \rangle_{\sigma_\beta} \leq D_V(\alpha, \nu, p) m^{-p} \vartheta^{p(1-2\alpha)}, \tag{5.2.11}$$

where, see (1.3.53),

$$D_V(\alpha, \nu, p) \stackrel{\text{def}}{=} \frac{2^{3(2p+1)}(1 + 1/\alpha p)^{2p}}{(p - 1 - 2p\alpha)(p - 2p\alpha)} \cdot Q_V(\beta, \nu, p).$$

For $c > 0$ and $n \in \mathbb{N}$, we set

$$C_\beta^\pm(n; c) = \{x \in C_\beta \mid \pm x^{(j)}(k\beta/n) \geq c, \; j = 1, \ldots, \nu; \; k = 0, 1, \ldots, n\}. \tag{5.2.12}$$

Thereby, for $\varepsilon \in (0, c)$ and $n \geq 2$, we also set

$$\begin{aligned} A(c; \varepsilon) &= \{x \in C_\beta \mid \lambda_{1/n}(x) \leq (c - \varepsilon)^{2p} n^{2\alpha p}\}, \\ B^\pm(\varepsilon, c) &= A(c; \varepsilon) \cap C_\beta^\pm(n; c). \end{aligned} \tag{5.2.13}$$

Then for any $\tau \in [0, \beta]$, one finds $k \in \mathbb{N}$ such that $|\tau - k\beta/n| \le 1/n$, and hence for any $j = 1, \ldots, \nu$,

$$|x^{(j)}(\tau) - x^{(j)}(k\beta/n)| \le \left[\lambda_{1/n}(x)\right]^{1/2p} n^{-\alpha},$$

which yields $\pm x^{(j)}(\tau) \ge \varepsilon$ if $x \in B^{\pm}(\varepsilon, c)$. Let us estimate $\sigma_\beta[B^{\pm}(\varepsilon, c)]$. By (5.2.11) and Chebyshev's inequality (1.3.29) one gets

$$\begin{aligned}\sigma_\beta\left(C_\beta \setminus A(c;\varepsilon)\right) &\le \frac{n^{-2\alpha p}}{(c-\varepsilon)^{2p}} \langle \lambda_{1/n}\rangle_{\sigma_\beta} \\ &\le \frac{D_V(\alpha, \nu, p)}{[mn(c-\varepsilon)^2]^p}.\end{aligned}$$

Set

$$\Sigma(n;c) = \min\left\{\sigma_\beta\left(C_\beta^+(n;c)\right); \sigma_\beta\left(C_\beta^-(n;c)\right)\right\}.$$

Thereby,

$$\begin{aligned}\sigma_\beta\left[B^{\pm}(\varepsilon, c)\right] &= \sigma_\beta\left[C^{\pm}(n;c) \setminus \left(C_\beta \setminus A(c;\varepsilon)\right)\right] \\ &\ge \Sigma(n;c) - \sigma_\beta\left(C_\beta \setminus A(c;\varepsilon)\right) \\ &\ge \Sigma(n;c) - \frac{D_V(\alpha, \nu, p)}{[mn(c-\varepsilon)^2]^p} \\ &\stackrel{\text{def}}{=} \gamma(m),\end{aligned} \tag{5.2.14}$$

which is positive for all

$$m \ge m_* \stackrel{\text{def}}{=} \frac{1}{n(c-\varepsilon)^2} \cdot \left(\frac{D_V(\alpha, \nu, p)}{\Sigma(n;c)}\right)^{1/p}. \tag{5.2.15}$$

Now for a box Λ, we introduce the following functions on $\Omega_{\beta,\Lambda}$:

$$\begin{aligned}Y_\Lambda(x_\Lambda) &= \frac{1}{2}\sum_{\ell,\ell' \in \Lambda} J^\Lambda_{\ell\ell'} \sum_{j=1}^{\nu} \int_0^\beta x_\ell^{(j)}(\tau) x_{\ell'}^{(j)}(\tau) \mathrm{d}\tau, \\ X_\Lambda(x_\Lambda) &= \sum_{\ell \in \Lambda} \int_0^\beta x_\ell^{(1)}(\tau) \mathrm{d}\tau.\end{aligned} \tag{5.2.16}$$

Thereby, from (1.4.52) one gets

$$\begin{aligned}p_\Lambda^{\text{per}}(h) &= \log N_\beta \\ &+ \frac{1}{|\Lambda|} \log\left\{\int_{\Omega_{\beta,\Lambda}} \exp\left[Y_\Lambda(x_\Lambda) + hX_\Lambda(x_\Lambda)\right] \prod_{\ell \in \Lambda} \sigma_\beta(\mathrm{d}x_\ell)\right\}.\end{aligned} \tag{5.2.17}$$

Suppose now that $h > 0$. Then restricting the integration in (5.2.17) to $[B^+(\varepsilon, c)]^\Lambda$, we get

$$p_\Lambda^{\rm per}(h) \geq h\beta\varepsilon + \log N_\beta + \frac{1}{2}\beta\nu\varepsilon^2 \sum_{\ell'\in\Lambda} J_{\ell\ell'}^\Lambda + \log \sigma_\beta[B^+(\varepsilon, c)]$$
$$\geq h\beta\varepsilon + \log N_\beta + \log\gamma(m). \qquad (5.2.18)$$

As the right-hand side of the latter estimate is independent of Λ, the inequality holds also for the limiting pressure p. For any positive $h \in \mathcal{R}^c$, by the convexity of p one has

$$M(h) \geq \beta^{-1}\,[p(h) - p(0)]\,/\,h$$
$$\geq \varepsilon + \frac{1}{\beta h}\left\{-p(0) + \log N_\beta + \log\gamma(m)\right\}.$$

Picking h big enough we get the positivity stated. The negativity can be proven by similar arguments. □

In the above theorem, the anharmonic potential was of general type. However, for a certain specific kind of V, one can get much more precise information about the set $\mathcal{R}$.

Theorem 5.2.3. *Let $\nu = 1, 2$ and assume that the model is ferromagnetic, translation-invariant, and such that the anharmonic potential V obeys the conditions of Theorem 2.4.6. Then for all $\beta > 0$, the pressure $p(h)$ is infinitely differentiable at each $h \neq 0$. That is, the set (5.2.6) is either $\mathcal{R} = \emptyset$ or $\mathcal{R} = \{0\}$.*

Proof. The proof will be done by showing that $p(h)$ can be extended to a holomorphic function in a domain, which contains both positive and negative half-lines. To this end we use Theorem 2.4.6 and the convergence established in the previous section.

Up to a multiplicative constant, $N_{\beta,\Lambda}(h)$, given by (5.2.2), is the Laplace transform (2.2.10) of the measure $\nu_{\beta,\Lambda}^{(0)}(\mathrm{d}x_\Lambda)$, which obviously is sub-Gaussian. Thereby, according to Theorem 2.4.6, $N_{\beta,\Lambda}(h)$ can be extended to an entire function of $h \in \mathbb{C}$ possessing the representation

$$N_{\beta,\Lambda}(h) = N_{\beta,\Lambda}(0)\prod_{j=1}^{\infty}(1 + \kappa_j h^2)$$

with positive κ_j's obeying the summability condition (2.4.1). This yields

$$\frac{p'_\Lambda(h)}{h} = \sum_{j=1}^{\infty}\frac{2\kappa_j}{1 + \kappa_j h^2}. \qquad (5.2.19)$$

Therefore, all the functions $p_\Lambda(h)$ are holomorphic in the domain $\mathbb{C}\setminus A$, where

$$A = A_+ \cup A_-, \quad A_\pm = \{z = \pm it \mid t \in [(\kappa_1)^{-1/2}, +\infty)\},$$

which includes the whole real line $\mathbb{R}$. Then, for $h \in \mathbb{C} \setminus A$, $h = x + \mathrm{i}y$, one has

$$\left| \frac{2\kappa_j}{1+\kappa_j h^2} \right|^2 = \frac{4\kappa_j^2}{[1+\kappa_j(x^2-y^2)]^2 + 4\kappa_j^2 x^2 y^2} \\ \leq \frac{4\kappa_j^2}{[1+\kappa_j(x^2-y^2)]^2}.$$

For $\theta > 0$, we set

$$B_\theta = \{z = x + iy \in \mathbb{C} \mid x \geq 0,\ x^2 - y^2 \geq \theta^2\}.$$

Applying the above estimate in (5.2.19) we get for $h \in B_\theta$,

$$\left| \frac{p'_\Lambda(h)}{h} \right| \leq \sum_{j=1}^{\infty} \frac{2\kappa_j}{1+\kappa_j\theta^2} \frac{p'_\Lambda(\theta)}{\theta}. \tag{5.2.20}$$

By Corollary 2.4.3 all $p_\Lambda(h)$'s are convex; hence, by Proposition 2.5.4 and Theorem 5.1.2, the limiting pressure $p(h)$ is convex on $\mathbb{R}$. Thus, it is not differentiable on a subset $E \subset \mathbb{R}$, which is at most countable. This means that for any $\varepsilon > 0$, the interval $(0, \varepsilon) \subset \mathbb{R}$ contains points at which $p'(h)$ exists. Moreover, by the same statement at each such h, $p'_\Lambda(h) \to p'(h)$, as $\Lambda \to \mathbb{Z}^d$ along a van Hove sequence $\mathcal{L}$. Thus, we take an arbitrary ε and pick $\theta \in (0, \varepsilon)$ such that $p'(\theta)$ exists. As the sequence $\{p'_\Lambda(\theta)\}$ converges to $p'(\theta)$, it is bounded. Now we take $t > \theta$ and set

$$B_{\theta,t} = \{z = x + \mathrm{i}y \in \mathbb{C} \mid x^2 - y^2 \geq \theta^2,\ x \in [0, t]\}. \tag{5.2.21}$$

This set contains $[\theta, t] \subset \mathbb{R}$. Then, for $h \in B_{\theta,t}$, one has

$$|h| = \sqrt{x^2 + y^2} \leq \sqrt{2x^2 - \theta^2} \leq \sqrt{2t^2 - \theta^2},$$

and, by the estimate (5.2.20),

$$|p'_\Lambda(h)| \leq \left(\sqrt{2(t/\theta)^2 - 1} \right) p'_\Lambda(\theta).$$

Since the sequence $\{p'_\Lambda(\theta)\}_{\Lambda \in \mathcal{L}}$ is bounded, the sequence of holomorphic in $B_{\theta,t}$ functions $\{p'_\Lambda\}_{\Lambda \in \mathcal{L}}$ is uniformly bounded on $B_{\theta,t}$. Moreover, one has $p'_\Lambda(z) \to p'(z)$ for all $z \in [\theta, t]$ except possibly for a countable subset of this interval. Thus, the subset of $[\theta, t]$ on which $p'_\Lambda(z) \to p'(z)$ has an accumulation point, which yields by Proposition 1.4.11 that p' is holomorphic on $B_{\theta,t}$; hence, p is infinitely differentiable on (θ, t). Since this is true for any $t > \theta$ and θ may be chosen arbitrarily close to zero (recall that $\theta \in (0, \varepsilon)$ with any $\varepsilon > 0$), this is true for all $h \in (0, +\infty)$. As all the functions p_Λ and p are even, the same is true also for $h \in (-\infty, 0)$. Thus, the only point where the stated differentiability of p may fail to hold is $h = 0$. □

For $\nu = 1$, the class of anharmonic potentials for which the pressure is differentiable at each $h \neq 0$, can be essentially extended. We recall that the EMN class of anharmonic potentials was introduced in Definition 2.2.4.

Theorem 5.2.4. *Let $\nu = 1$ and assume that the model is ferromagnetic, translation-invariant, and with the anharmonic potential of the EMN type. Then for all $\beta > 0$, the pressure $p(h)$ is differentiable at each $h \neq 0$.*

Proof. Here we employ the GHS inequality, see Theorem 2.2.7. By (5.2.3) and (5.2.4) we get

$$\frac{\partial^2}{\partial h^2}\beta M_\Lambda^{\mathrm{per}}(h) = \frac{1}{|\Lambda|}\sum_{\ell_1,\ell_2,\ell_3\in\Lambda}\int_0^\beta\int_0^\beta\int_0^\beta U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1,\tau_2,\tau_3;h)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3, \quad (5.2.22)$$

where Λ is a box and

$$\begin{aligned} U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1,\tau_2,\tau_3;h) = \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} & \\ - \langle x_{\ell_1}(\tau_1)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}\cdot\langle x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} & \\ - \langle x_{\ell_2}(\tau_2)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}\cdot\langle x_{\ell_1}(\tau_1)x_{\ell_3}(\tau_3)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} & \\ - \langle x_{\ell_3}(\tau_3)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}\cdot\langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} & \\ + 2\langle x_{\ell_1}(\tau_1)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}\cdot\langle x_{\ell_2}(\tau_2)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}\cdot\langle x_{\ell_3}(\tau_3)\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} & \end{aligned}$$

is an Ursell function, cf. (2.2.16). If the anharmonic potential in $\nu^{\mathrm{per}}_{\beta,\Lambda}$ contains non-negative h, by Theorem 2.2.7 we have

$$U^\Lambda_{\ell_1\ell_2\ell_3}(\tau_1,\tau_2,\tau_3;h) \leq 0, \quad h \geq 0,$$

which by (5.2.22) yields that $-M_\Lambda^{\mathrm{per}}(h)$ is convex on $(0,+\infty)$. By (3.6.8) one obtains that for every fixed h, $M_\Lambda^{\mathrm{per}}(h) \leq C(h)$, where the constant is independent of Λ. By (5.2.5) $M_\Lambda^{\mathrm{per}}(h)$ is increasing; hence, $M_\Lambda^{\mathrm{per}}(h) \leq C(h_*)$ for all $h \in (0,h_*)$. Therefore, by claims (c) and (d) of Proposition 2.5.4 the sequence $\{M_\Lambda^{\mathrm{per}}(h)\}_{\mathcal{L}_{\mathrm{box}}}$ contains a subsequence, such that $M_{\Lambda_n}^{\mathrm{per}}(h) \to M(h)$ for all $h \in (0,h_*)$. Then, by claim (c) of Proposition 2.5.4, $p(h)$ is differentiable at each $h \in (0,h_*)$ and hence at each $h > 0$ since h_* is arbitrary. The case of negative h is handled in the same way. $\square$

5.3 Uniqueness Criteria

In Section 4.3, we have proven that the set $\mathcal{G}^{\mathrm{t}}_\beta$ is a singleton for high temperatures and/or weak interactions. Here we obtain a condition for the uniqueness connected with the FKG order and its modification based on the use of the pressure. Then we apply them to our model assuming that the anharmonic potential V is as in Proposition 2.4.4.

In the statement below, we suppose that the model (1.1.3), (1.1.8) is ferromagnetic and scalar, i.e., $\nu = 1$, but not necessarily translation-invariant. For such models, we know from Theorem 3.7.4 that the set $\mathcal{G}^{\mathrm{t}}_\beta$ has maximal and minimal elements, μ_+ and μ_-, respectively.

Theorem 5.3.1. *The set $\mathscr{G}^{\mathrm{t}}_\beta$ is a singleton if and only if, for all ℓ,*

$$\int_{\Omega_\beta} x_\ell(0)\mu_+(\mathrm{d}x) = \int_{\Omega_\beta} x_\ell(0)\mu_-(\mathrm{d}x). \tag{5.3.1}$$

Proof. Certainly, (5.3.1) holds if $|\mathscr{G}^{\mathrm{t}}_\beta| = 1$. Let us prove the converse. Since the measures $\mu_\pm$ are shift-invariant, (5.3.1) yields

$$\forall \tau \in [0,\beta]: \qquad \int_{\Omega_\beta} x_\ell(\tau)\mu_\pm(\mathrm{d}x) = \int_{\Omega_\beta} x_\ell(0)\mu_\pm(\mathrm{d}x),$$

and thereby (3.7.3). Hence, by Lemma 3.7.1, $\mu_+ = \mu_-$, which gives the uniqueness in question. Note that, if the model is translation-invariant, (5.3.1) holding at a some ℓ already yields the uniqueness stated. $\square$

Theorems 3.7.4 and 5.3.1 have the following

Corollary 5.3.2. *If $V_\ell(x) = V_\ell(-x)$ for all ℓ, the set $\mathscr{G}^{\mathrm{t}}_\beta$ is a singleton if and only if $\langle x_\ell(0)\rangle_{\mu_+} = 0$ for all ℓ.*

Now we derive a uniqueness condition based on Theorem 5.3.1 and the use of the pressure. Thus, we have to assume again the translation invariance of the model. By (3.1.39), Proposition 1.4.12, and (5.1.1), for every $\xi \in \Omega^{\mathrm{t}}_\beta$, the pressure $p_\Lambda(h,\xi)$ is an infinitely differentiable function of h and, cf. (5.2.3),

$$\frac{\partial}{\partial h} p_\Lambda(h,\xi) = \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)}\mathrm{d}\tau, \tag{5.3.2}$$

$$\frac{\partial^2}{\partial h^2} p_\Lambda(h,\xi) = \frac{1}{|\Lambda|}\sum_{\ell,\ell'\in\Lambda}\int_0^\beta\int_0^\beta K^\Lambda_{\ell\ell'}(\tau,\tau'|\xi)\mathrm{d}\tau\mathrm{d}\tau', \tag{5.3.3}$$

where, cf. (2.5.41),

$$\begin{aligned} K^\Lambda_{\ell\ell'}(\tau,\tau'|\xi) \stackrel{\mathrm{def}}{=} &\langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)} \\ &- \langle x_\ell(\tau)\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)}\cdot\langle x_{\ell'}(\tau')\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)} \end{aligned} \tag{5.3.4}$$

is a correlation function. By (2.2.1), for all $\xi \in \Omega^{\mathrm{t}}_\beta$,

$$K^\Lambda_{\ell\ell'}(\tau,\tau'|\xi) \geq 0, \tag{5.3.5}$$

for all ℓ,ℓ' and τ,τ'. Therefore, the pressure $p_\Lambda(h,\xi)$ is a convex function of h, cf. Theorem 5.2.1.

Theorem 5.3.3. *If the limiting pressure of the scalar ferromagnetic model is differentiable at a given $h \in \mathbb{R}$, then the set of all tempered Euclidean Gibbs states $\mathscr{G}^{\mathrm{t}}_\beta$ of this model is a singleton.*

Proof. By Theorem 5.1.3, $p = p^{\mu_+} = p^{\mu_-}$. By claim (a) of Proposition 2.5.4, p is convex. For every $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $\mu \in \mathcal{G}^{\mathrm{t}}_{\beta}$,

$$\begin{aligned}\frac{\partial}{\partial h} p_{\Lambda}^{\mu_\pm}(h) &= \int_{\Omega_\beta} \frac{\partial}{\partial h} p_{\Lambda}(h,\xi)\mu_\pm(\mathrm{d}\xi) \\ &= \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \left(\int_{\Omega_\beta} \langle x_{\ell'}(\tau')\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi)}\mathrm{d}\tau\right)\mu_\pm(\mathrm{d}\xi) \\ &= \frac{1}{|\Lambda|}\sum_{\ell\in\Lambda}\int_0^\beta \langle x_{\ell'}(\tau)\rangle_{\mu_\pm}\mathrm{d}\tau = \beta\langle x_{\ell'}(0)\rangle_{\mu_\pm},\end{aligned} \tag{5.3.6}$$

where we have taken into account that both extreme measures $\mu_\pm$ are translation and shift invariant, see Theorem 3.7.4. Then passing in (5.3.6) to the limit along a van Hove sequence we get (5.3.1) and hence the uniqueness stated. □

Remark 5.3.4. The fact that the pressure is the same in all states, crucial for the proof of the theorem above, follows from the existence of van Hove sequences. The latter property is related to the amenability of the graph $(\mathbb{Z}^d, E)$, where E is the set of edges connecting nearest neighbors. For nonamenable graphs, e.g., for Cayley trees, phase transitions at nonzero h are possible, see [168], unlike here, see Theorem 5.3.5 below which we prove based on Theorem 5.3.3.

The Lee–Yang property of the local Euclidean Gibbs measures described by Theorem 2.4.6 can be used to establish the corresponding properties of the limiting pressure.

Theorem 5.3.5. *Let the model be scalar, ferromagnetic, translation-invariant, and with the anharmonic potential V obeying the conditions of Theorem 2.4.6 or Theorem 5.2.4. Then $\mathcal{G}^{\mathrm{t}}_{\beta}$ is a singleton for all $h \neq 0$.*

Proof. Follows immediately from Theorems 5.2.3, 5.2.4, and 5.3.3. □

5.4 Comments and Bibliographic Notes

Section 5.1: The role of the pressure in the theory of Gibbs states of classical models is studied in great detail in [162], [258], [277]. The proof of the main statement of this section – Theorem 5.1.2 – is crucially based on the moment estimates obtained in Sections 3.2 and 3.3. Here we have taken into account the arguments of J. L. Lebowitz and E. Presutti employed in [206] for the same purposes. Instead of the moment estimates, the authors of [206] used a method based on the superstability arguments developed by D. Ruelle in [264]. In our case, such arguments were incorporated in the method by which the moment estimates were obtained. This made the proof of Theorem 5.1.2 more automatic. We note that a similar result was proven in [59]. Theorems 5.1.2 and 5.1.3 establish the existence of the pressure as a thermodynamic function describing the whole set $\mathcal{G}^{\mathrm{t}}_{\beta}$. Here we mention the paper [41] where for an isotropic ν-vector version of our model, a $1/\nu$ power expansion of the pressure was constructed. This fact that the limiting pressure is independent of the sequence of subsets $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ along which it is obtained is closely related to the properties of the lattice $\mathbb{L}$ which secure the existence of van Hove sequences and hence the validity of Proposition 5.1.1. For the Ising model on a nonamenable graph where van Hove sequences do not exist, it is possible to have a nonuniqueness of Gibbs states for a nonzero external field h, see [168], [211]. On the lattice $\mathbb{L} = \mathbb{Z}^d$, this would be impossible by Theorem 5.3.5, which is applicable to the classical ϕ^4 and hence to the Ising model, see Theorem 4.2.3.

Section 5.2: The existence of the pressure as a thermodynamic function allows one to introduce one more thermodynamic function – the global magnetization. It is defined at each value of the external field for which the pressure is differentiable. For Gibbs random fields with bounded interaction, there exists a very powerful technique of studying these fields by means of the pressure, see [162], [277]. A particular result of this theory states that each nondifferentiability of the pressure corresponds to the nonuniqueness of the Gibbs states. Unfortunately, the case considered here is much more robust, so we failed to get a similar result for the set $\mathcal{G}^{\mathrm{t}}_{\beta}$. For classical lattice systems with unbounded spins, a statement like our Theorem 5.2.2 is obtained almost for granted, cf. the very end of the proof of Theorem 3.5 in [124]. In our case where the 'spins' are paths, the only way to fix their signs is to apply the GRR lemma, Theorem 1.3.11, and thereby the GRR estimate (1.3.50). In Section 6.3, with the help of Theorem 5.2.2 we prove the existence of phase transitions for asymmetric anharmonic potentials. For $h \in \mathcal{R}^c$, the pressure is differentiable and hence the global magnetization exists. In general, the only information about this set is that its complement $\mathcal{R}$ is at most countable, which does not exclude the possibility for $\mathcal{R}^c$ to have empty interior and hence for the magnetization $M(h)$ to be nowhere continuous. Thus, it would be much desirable to know more on this subject. It turns out, that the only possibility available so far is to show that $\mathcal{R}$ can contain at most zero, which occurs only for the EMN potentials, see Theorem 5.2.4, or in the case where the Lee–Yang theory can be applied. The latter case was described in Theorem 5.2.3. Its proof is performed by means of an appropriate analytic function technique since the pointwise convergence of the sequences $\{p_\Lambda(h)\}_{\Lambda \in \mathfrak{L}}$ for $h \in \mathbb{R}$ is not enough to secure the analyticity desirable. The technique is based on the Vitali theorem and on the boundedness of the sequences $\{p_\Lambda(h)\}_{\Lambda \in \mathfrak{L}}$ on compact subsets of $\mathbb{C}$ obtained from the Lee–Yang property of the model. The scheme used here is similar to that used in [193].

Section 5.3: The use of the pressure combined with the FKG order introduced in Section 3.7 allowed for obtaining a very strong uniqueness condition for Euclidean Gibbs measures – for each pair $\mu_1, \mu_2 \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$, such that $\mu_1 \leq \mu_2$, the coincidence of the first moments only already implies the coincidence of the measures, see Lemma 3.7.1. The proof of this property is based on the Strassen theorem, see Theorem 2.4 in [210]. A consequence of Lemma 3.7.1 is Theorem 5.3.3 which gives the uniqueness criterion based on the differentiability of the pressure. As was mentioned above, we failed to prove the converse, i.e., that the nondifferentiability of p implies the nonuniqueness of tempered Euclidean Gibbs measures.

Chapter 6
Phase Transitions

The study of phase transitions is among the main problems of equilibrium statistical mechanics, both classical and quantum. There exist several approaches to describe these phenomena. Their common point is that the macroscopic equilibrium properties of a statistical mechanical model can be different at the same values of the model parameters. In the language adopted in this book, the model has a phase transition if the set of its tempered Euclidean Gibbs states contains more than one element. That is, one speaks about the possibility for the multiple states to occur rather than the transition between these states or between their uniqueness and multiplicity. The description of the dynamics of phase transitions is beyond the scope of equilibrium statistical mechanics.

If the considered model is translation-invariant, the multiplicity of its Euclidean Gibbs states is equivalent to the existence of non-ergodic elements of $\mathscr{G}^{\mathrm{t}}_{\beta}$, see Corollary 3.1.29. Thus, to prove that this model has a phase transition it is enough to show that there exists an element of $\mathscr{G}^{\mathrm{t}}_{\beta}$, which fails to obey the condition (3.1.60). We shall use this fact in the translation-invariant case. For models which are not translation-invariant, we shall use a comparison method, based on correlation inequalities. Its main idea is that the model has a phase transition if the translation-invariant model with which we compare it has a phase transition. Of course, all the models we are discussing here are particular realizations of the general model considered in the book.

One of the most effective tools for establishing phase transitions in a translation-invariant lattice model is the method based on exploiting the *reflection positivity* of the model. This property of the model (1.1.3), (1.1.8) is connected with a special choice of the interaction intensities $J_{\ell\ell'}$. In particular, the model with nearest neighbor interactions (i.e., with $J_{\ell\ell'} = J$, for $|\ell - \ell'| = 1$, and $J_{\ell\ell'} = 0$ otherwise) has this property. By means of the reflection positivity one derives an estimate, called *infrared bound*, according to which for large $J > 0$, the decay of correlations in a periodic Euclidean Gibbs state (studied in Section 3.6) is so slow that the condition (3.1.60) does not hold.

The present chapter is organized as follows. In Section 6.1, we give a precise definition of the notion and study its connection with more 'physical' definitions of phase transitions. We also show how to prove the non-ergodicity of the periodic Euclidean Gibbs states. Then in Section 6.2, we give a detailed presentation of the reflection positivity method in the context of the Euclidean approach. In Section 6.3, the above methods are applied to the proof of phase transitions in various ferromagnetic versions of our model. Section 6.4 is dedicated to the description of the critical point of a hierarchical version of the model, in which the interaction has a special metric property.

6.1 Phase Transitions and Order Parameters

Let us begin by introducing the main notion of this chapter.

Definition 6.1.1. The model described by the Hamiltonians (1.1.3), (1.1.8) has a phase transition if $|\mathcal{G}^{\mathrm{t}}_\beta| > 1$ at certain values of β and the model parameters.

We note that here we demand the existence of multiple *tempered* Euclidean Gibbs measures. For models with finite range interactions, there may exist Euclidean Gibbs measures, which are not tempered, see Section 3.5. Such measures should not be taken into account. Another observation is that in Definition 6.1.1 we do not assume any symmetry of the model, the translation invariance including. If the model is rotation-invariant (symmetric for $\nu = 1$, see Definition 1.1.2), the unique element of $\mathcal{G}^{\mathrm{t}}_\beta$ should have the same symmetry. If $|\mathcal{G}^{\mathrm{t}}_\beta| > 1$, the symmetry can be 'distributed' among the elements of $\mathcal{G}^{\mathrm{t}}_\beta$. In this case, the phase transition is connected with *symmetry breaking*. In the sequel, we consider mostly phase transitions of this type. However, in Subsection 6.3.3 we study phase transitions without symmetry breaking.

As was mentioned above, for translation-invariant models one can establish multiplicity of the elements of $\mathcal{G}^{\mathrm{t}}_\beta$ by showing the existence of a translation-invariant state $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, which fails to have the mixing property (3.1.60). To present an idea of how to do this, let us consider the simplest case where the interaction is of nearest neighbor type with intensity $J > 0$ and the model is rotation-invariant. Then the measures (3.6.2), and hence the accumulation points of the sequence $\{\pi^{\mathrm{per}}_{\beta,\Lambda}\}_{\Lambda\in\mathcal{L}_{\mathrm{box}}}$, are rotation-invariant as well. Given ℓ and $j = 1,\dots,\nu$, we set

$$f^{(j)}_\ell(x) = \int_0^\beta x^{(j)}_\ell(\tau)\mathrm{d}\tau. \tag{6.1.1}$$

By Theorem 3.3.1 and Lemma 3.2.5, this function is square integrable with respect to all $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ and all local Gibbs measures. Thus, for a periodic Gibbs measure μ, which by construction is translation and rotation-invariant, we set

$$D^\mu_{\ell\ell'} = \sum_{j=1}^{\nu} \langle f^{(j)}_\ell f^{(j)}_{\ell'}\rangle_\mu. \tag{6.1.2}$$

By (2.5.2) and (3.3.1),

$$0 \le D^\mu_{\ell\ell'} \le C, \tag{6.1.3}$$

where C is the same for all such measures. Furthermore, by the rotational symmetry we have $\langle f^{(j)}_\ell\rangle_\mu = 0$; thus, the phase transition occurs if

$$\lim_{L\to+\infty}\frac{1}{|\Lambda_L|}\sum_{\ell'\in\Lambda_L} D^\mu_{\ell\ell'} = \lim_{L\to+\infty}\frac{1}{|\Lambda_L|^2}\sum_{\ell,\ell'\in\Lambda_L} D^\mu_{\ell\ell'} > 0. \tag{6.1.4}$$

In view of the translation invariance, $D^{\mu}_{\ell\ell'}$ is a function of the distance (1.1.1) and hence can be written

$$D^{\mu}_{\ell\ell'} = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \widehat{D}^{\mu}_{p} e^{\mathrm{i}(p,\ell-\ell')} \mathrm{d}p, \quad \mathrm{i} = \sqrt{-1}, \tag{6.1.5}$$

where $\widehat{D}^{\mu}$ is defined by the Fourier series

$$\widehat{D}^{\mu}_{p} = \sum_{\ell'} D^{\mu}_{\ell\ell'} e^{-\mathrm{i}(p,\ell-\ell')}, \quad p \in (-\pi,\pi]^d. \tag{6.1.6}$$

In general, $\widehat{D}^{\mu}$ is a distribution. By the lower bound in (6.1.3), the possible singularity of $\widehat{D}^{\mu}$ occurs at $p = 0$ if $D^{\mu}_{\ell\ell'}$ is not summable. On the other hand, the upper bound in (6.1.3) allows one to anticipate the singularity type. Suppose one has a pair, $D_{\ell\ell'}$ and $\widehat{D}_p$, connected by (6.1.5), (6.1.6). If $D_{\ell\ell'}$ is constant, i.e., $D_{\ell\ell'} = \varkappa > 0$, then $\widehat{D}_p = (2\pi)^d \varkappa\delta(p)$, δ being the Dirac delta-function. Therefore, it would be reasonable to set

$$\widehat{D}^{\mu}_{p} = (2\pi)^d \varkappa\delta(p) + g(p), \tag{6.1.7}$$

where g is measurable and finite almost everywhere on $(-\pi,\pi]^d$. Its role is to produce the eventual spatial decay of $D^{\mu}_{\ell\ell'}$. Then the phase transition occurs if $\varkappa > 0$, see (6.1.4).

Suppose that the following two estimates hold. The first one is the infrared bound

$$g(p) \leq \gamma/J|p|^2, \quad \text{for almost all } p \in (-\pi,\pi]^d, \tag{6.1.8}$$

where $\gamma > 0$ is a constant. Its name is connected with the 'infrared' singularity of the right-hand side. The $J \to +\infty$ asymptotics corresponds to a 'totally correlated' case of $D^{\mu}_{\ell\ell'} = \varkappa$. The second mentioned estimate is

$$D^{\mu}_{\ell\ell} \geq \vartheta > 0, \tag{6.1.9}$$

where ϑ should be independent of J. By these two estimates and (6.1.5), (6.1.6) we get

$$\varkappa \geq \vartheta - \frac{\gamma}{(2\pi)^d J} \int_{(-\pi,\pi]^d} \mathrm{d}p/|p|^2. \tag{6.1.10}$$

For $d \geq 3$, the latter integral exists. Therefore, $\varkappa > 0$ and hence the state μ is non-ergodic for large enough J.

The main difficulty with deriving estimates like (6.1.8), (6.1.9) is that they correspond to limiting objects, whereas the techniques available are applicable rather to local Euclidean Gibbs states. Therefore, for practical use we develop a 'local' version of the above arguments. For Λ as in (3.1.2), we set

$$D^{\Lambda}_{\ell\ell'} = \sum_{j=1}^{\nu} \langle f^{(j)}_{\ell} f^{(j)}_{\ell'} \rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}, \tag{6.1.11}$$

which can be rewritten

$$D^{\Lambda}_{\ell\ell'} = \beta \int_0^{\beta} K^{\Lambda}_{\ell\ell'}(\tau, \tau'|p)\mathrm{d}\tau', \tag{6.1.12}$$

where the correlation function is, cf. (2.5.41),

$$K^{\Lambda}_{\ell\ell'}(\tau, \tau'|p) = \big\langle \big(x_\ell(\tau), x_{\ell'}(\tau')\big) \big\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}. \tag{6.1.13}$$

The right-hand side in (6.1.12) is independent of τ due to the periodicity (1.2.90). To introduce the Fourier transform in the box Λ we employ the conjugate set Λ_* (Brillouin zone), consisting of the vectors $p = (p_1, \ldots, p_d)$, such that

$$p_j = -\pi + \frac{\pi}{L}s_j, \quad s_j = 1, \ldots, 2L, \; j = 1, \ldots, d. \tag{6.1.14}$$

Then the Fourier transform is

$$\begin{aligned} x^{(j)}_\ell(\tau) &= \frac{1}{|\Lambda|^{1/2}} \sum_{p \in \Lambda_*} \hat{x}^{(j)}_p(\tau) e^{\mathrm{i}(p,\ell)}, \\ \hat{x}^{(j)}_p(\tau) &= \frac{1}{|\Lambda|^{1/2}} \sum_{\ell \in \Lambda} x^{(j)}_\ell(\tau) e^{-\mathrm{i}(p,\ell)}, \quad j = 1, \ldots, \nu. \end{aligned} \tag{6.1.15}$$

In order that $x^{(j)}_\ell(\tau)$ be real, the Fourier coefficients should satisfy

$$\overline{\hat{x}^{(j)}_p(\tau)} = \hat{x}^{(j)}_{-p}(\tau).$$

By the rotation invariance of the state $\langle \cdot \rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}}$, as well as by its invariance with respect to the translations of the torus Λ, it follows that

$$\langle \hat{x}^{(j)}_p(\tau) \hat{x}^{(j')}_{p'}(\tau') \rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} = \delta_{jj'} \delta(p + p') \sum_{\ell' \in \Lambda} \langle x^{(j)}_\ell(\tau) x^{(j)}_{\ell'}(\tau') \rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} e^{\mathrm{i}(p,\ell'-\ell)}.$$

In view of this property, we set

$$\begin{aligned} \widehat{K}^{\Lambda}_p(\tau, \tau'|p) &= \sum_{\ell' \in \Lambda} K^{\Lambda}_{\ell\ell'}(\tau, \tau'|p) e^{\mathrm{i}(p,\ell'-\ell)}, \\ K^{\Lambda}_{\ell\ell'}(\tau, \tau'|p) &= \frac{1}{|\Lambda|} \sum_{p \in \Lambda_*} \widehat{K}^{\Lambda}_p(\tau, \tau'|p) e^{\mathrm{i}(p,\ell-\ell')}, \end{aligned} \tag{6.1.16}$$

and

$$\widehat{D}^{\Lambda}_p = \beta \int_0^{\beta} \widehat{K}^{\Lambda}_p(\tau, \tau'|p)\mathrm{d}\tau' = \sum_{\ell' \in \Lambda} D^{\Lambda}_{\ell\ell'} e^{\mathrm{i}(p,\ell'-\ell)}. \tag{6.1.17}$$

One observes that the latter representation of $\widehat{D}^{\Lambda}$ can be extended to all $p \in (-\pi, \pi]^d$. From (6.1.16) and (6.1.17) one gets

$$\widehat{D}^{\Lambda}_p = \widehat{D}^{\Lambda}_{-p} = \sum_{\ell' \in \Lambda} D^{\Lambda}_{\ell\ell'} \cos(p, \ell' - \ell), \tag{6.1.18}$$

and

$$D^{\Lambda}_{\ell\ell'} = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*}\widehat{D}^{\Lambda}_p e^{\mathrm{i}(p,\ell-\ell')} = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*}\widehat{D}^{\Lambda}_p \cos(p,\ell-\ell'). \tag{6.1.19}$$

Furthermore, for $u_\Lambda = (u_\ell)_{\ell\in\Lambda}$, $u_\ell \in \mathbb{R}$, we have

$$\begin{aligned}\left(u_\Lambda, D^{\Lambda}u_\Lambda\right)_{l^2(\Lambda)} &\stackrel{\text{def}}{=} \sum_{\ell,\ell'\in\Lambda} D^{\Lambda}_{\ell\ell'}u_\ell u_{\ell'} \\ &= \sum_{j=1}^{\nu}\left\langle\left[\sum_{\ell\in\Lambda}u_\ell\int_0^\beta x^{(j)}_\ell(\tau)\mathrm{d}\tau\right]^2\right\rangle_{\nu^{\mathrm{per}}_{\beta,\Lambda}} \geq 0.\end{aligned} \tag{6.1.20}$$

Therefore, the operator $D^{\Lambda}: l^2(\Lambda) \to l^2(\Lambda)$ is strictly positive; hence, all its eigenvalues (6.1.17), (6.1.18) are also strictly positive.

Suppose now that we are given a continuous function $\widehat{B}: (-\pi,\pi]^d \to (0,+\infty]$ with the following properties:

$$\begin{aligned}&\text{(i)} \quad \int_{(-\pi,\pi]^d}\widehat{B}(p)\mathrm{d}p < \infty, \\ &\text{(ii)} \quad \widehat{D}^{\Lambda}_p \leq \widehat{B}(p), \quad \text{for all } p\in\Lambda_*\setminus\{0\},\end{aligned} \tag{6.1.21}$$

holding for all boxes Λ. One observes that the latter estimate is a local version of the infrared bound (6.1.8). Then we set

$$B_{\ell\ell'} = \frac{1}{(2\pi)^d}\int_{(-\pi,\pi]^d}\widehat{B}(p)\cos(p,\ell-\ell')\mathrm{d}p, \quad \ell,\ell'\in\mathbb{L}, \tag{6.1.22}$$

and, for a box Λ,

$$B^{\Lambda}_{\ell\ell'} = \frac{1}{|\Lambda|}\sum_{p\in\Lambda_*\setminus\{0\}}\widehat{B}(p)\cos(p,\ell-\ell'), \quad \ell,\ell'\in\Lambda. \tag{6.1.23}$$

We also set $B^{\Lambda}_{\ell\ell'} = 0$ if either of ℓ,ℓ' belongs to Λ^c.

Proposition 6.1.2. *For every ℓ,ℓ', it follows that $B^{\Lambda}_{\ell\ell'} \to B_{\ell\ell'}$ as $L\to+\infty$.*

Proof. By (6.1.21), $\widehat{B}(p)\cos(p,\ell-\ell')$ is an absolutely integrable function in the sense of the improper Riemann integral. The right-hand side of (6.1.23) is its integral sum; thereby, the convergence stated follows in a standard way. □

Now we get a local version of (6.1.10).

Lemma 6.1.3. *For every box Λ and any $\ell,\ell'\in\Lambda$, one has*

$$D^{\Lambda}_{\ell\ell'} \geq \left(D^{\Lambda}_{\ell\ell} - B^{\Lambda}_{\ell\ell}\right) + B^{\Lambda}_{\ell\ell'}. \tag{6.1.24}$$

Proof. By (6.1.19), (6.1.23), and claim (ii) of (6.1.21), one has

$$\begin{aligned} D^\Lambda_{\ell\ell} - D^\Lambda_{\ell\ell'} &= \frac{2}{|\Lambda|} \sum_{p\in\Lambda_*\setminus\{0\}} \widehat{D}^\Lambda_p \sin^2(p,\ell-\ell') \\ &\le \frac{2}{|\Lambda|} \sum_{p\in\Lambda_*\setminus\{0\}} \widehat{B}(p) \sin^2(p,\ell-\ell') \\ &= B^\Lambda_{\ell\ell} - B^\Lambda_{\ell\ell'}, \end{aligned}$$

which yields (6.1.24). □

Corollary 6.1.4. *For every $\mu \in \mathcal{M}^{\mathrm{per}}_\beta$, it follows that*

$$D^\mu_{\ell\ell'} \ge \left(D^\mu_{\ell\ell} - B_{\ell\ell}\right) + B_{\ell\ell'}, \tag{6.1.25}$$

holding for any ℓ, ℓ'.

Proof. Given $\mu \in \mathcal{M}^{\mathrm{per}}_\beta$, one finds the sequence $\{L_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$, such that $D^{\Lambda_{L_n}}_{\ell\ell'} \to D^\mu_{\ell\ell'}$, see Theorem 3.6.4 and Lemma 3.6.1. Then (6.1.25) follows from (6.1.24) and Proposition 6.1.2. □

One observes that the first summand in (6.1.25) is independent of ℓ, whereas the second one is a priori neither positive nor summable. Suppose now that there exists a positive ϑ, such that for any box Λ,

$$D^\Lambda_{\ell\ell} \ge \vartheta, \tag{6.1.26}$$

which clearly would yield (6.1.9). Then the phase transition occurs if

$$\begin{aligned} &\text{(i)} \quad \vartheta > B_{\ell\ell}; \\ &\text{(ii)} \quad \lim_{|\ell-\ell'|\to+\infty} B_{\ell\ell'} = 0. \end{aligned} \tag{6.1.27}$$

We return to these conditions later after we find the function $\widehat{B}$ obeying (6.1.21).

Now let us consider other possibilities to describe phase transitions in translation- and rotation-invariant versions of our model. Given a box Λ, we introduce

$$\begin{aligned} P_\Lambda &= \frac{1}{(\beta|\Lambda|)^2} \sum_{\ell,\ell'\in\Lambda} D^\Lambda_{\ell\ell'} \\ &= \int_{\Omega_{\beta,\Lambda}} \left| \frac{1}{\beta|\Lambda|} \sum_{\ell\in\Lambda} \int_0^\beta x_\ell(\tau)\mathrm{d}\tau \right|^2 \nu^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \end{aligned} \tag{6.1.28}$$

Definition 6.1.5. The order parameter of the model is set to be

$$P \stackrel{\mathrm{def}}{=} \limsup_{\Lambda\nearrow\mathbb{Z}^d} P_\Lambda. \tag{6.1.29}$$

If $P > 0$ for given values of β and the model parameters, then there exists a long-range order.

One observes that P is positive if both conditions (6.1.27) are satisfied. The appearance of the long-range order, which in a more 'physical' context is identified with phase transition, does not imply the phase transition in the sense of Definition 6.1.1. At the same time, the latter definition describes models without translation invariance. On the other hand, Definition 6.1.5 is based on the local states only and hence can be formulated without employing Euclidean Gibbs states. Yet another 'physical' approach to phase transitions in translation-invariant models like (1.1.3), (1.1.8) is based on the properties of the pressure p, which by Theorems 5.1.2 and 5.1.3 exists and is the same in every state. It does not employ the set $\mathcal{G}_\beta^{\mathrm{t}}$ and is based on the continuity of the global magnetization (5.2.7), that is, on the differentiability of p as a function of the external field h.

Definition 6.1.6 (Landau Classification). The model has a first-order phase transition if p' is discontinuous at a certain h_*. The model has a second-order phase transition if there exists $h_* \in \mathbb{R}$ such that p' is continuous but p'' is discontinuous at $h = h_*$.

As in Definition 6.1.1, here we do not assume any symmetry of the model. The relationship between the first-order phase transition and the appearance of the long-range order can be established with the help of the following result, the proof of which will be done by a slight modification of the arguments used in [109], see Theorem 1.1 and its corollaries. Let $\{\mu_n\}_{N\in\mathbb{N}}$ (respectively, $\{M_n\}_{n\in\mathbb{N}}$) be a sequence of probability measures on $\mathbb{R}$ (respectively, positive real numbers, $\lim M_n = +\infty$) such that, for every $y \in \mathbb{R}$,

$$f(y) = \lim_{n\to+\infty} \frac{1}{M_n} \log \int e^{yu} \mu_n(\mathrm{d}u) \tag{6.1.30}$$

exists and is finite. By the Hölder inequality (3.2.14), for every $\theta \in [0, 1]$, $x, y \in \mathbb{R}$, and $n \in \mathbb{N}$, it follows that

$$\begin{aligned} f_n(\theta x + (1-\theta)y) &\stackrel{\text{def}}{=} \frac{1}{M_n} \log \int \exp\left[\theta x u + (1-\theta) y u\right] \mu_n(\mathrm{d}u) \\ &\leq \theta f_n(x) + (1-\theta) f_n(y). \end{aligned}$$

Thus, each f_n and hence f are convex. This yields that f has one-sided derivatives $f'_\pm(0)$, see Proposition 2.5.4.

Theorem 6.1.7 (Griffiths). *Let the sequence of measures $\{\mu_n\}_{N\in\mathbb{N}}$ be as above. If $f'_+(0) = f'_-(0) = \phi$ (i.e., f is differentiable at $y = 0$), then*

$$\lim_{n\to+\infty} \int g(u/M_n)\mu_n(\mathrm{d}u) = g(\phi), \tag{6.1.31}$$

for any continuous $g\colon \mathbb{R} \to \mathbb{R}$, such that $|g(u)| \leq \lambda e^{\varkappa |u|}$ with certain $\lambda, \varkappa > 0$. Furthermore, for each such a function g,

$$\limsup_{n\to+\infty} \int g(u/M_n)\mu_n(\mathrm{d}u) \leq \max_{z\in[f'_-(0), f'_+(0)]} g(z). \tag{6.1.32}$$

In particular, if $f'_-(0) = -f'_+(0)$, then for any $k \in \mathbb{N}$,

$$f'_+(0) \geq \limsup_{n \to +\infty} \left(\int (u/M_n)^{2k} \mu_n(\mathrm{d}u) \right)^{1/2k}. \tag{6.1.33}$$

The proof of this theorem will be based on the following lemma. For a fixed $\delta > 0$, we set

$$a_\pm = f'_\pm(0) \pm \delta, \quad b_n^+ = \int_{a_+ M_n}^{+\infty} \mu_n(\mathrm{d}u), \quad b_n^- = \int_{-\infty}^{a_- M_n} \mu_n(\mathrm{d}u), \tag{6.1.34}$$

where f, M_n, and μ_n are as in (6.1.30).

Lemma 6.1.8. *For the above sequences, it follows that*

$$\limsup_{n \to +\infty} \left(b_n^\pm\right)^{1/M_n} < 1. \tag{6.1.35}$$

Proof. For any $n \in \mathbb{N}$ and $y \in \mathbb{R}$,

$$\begin{aligned} \frac{1}{M_n} \log \int e^{yu} \mu_n(\mathrm{d}u) &\geq \frac{1}{M_n} \log \int_{a_+ M_n}^{+\infty} e^{yu} \mu_n(\mathrm{d}u) \\ &\geq \frac{1}{M_n} \log e^{a_+ M_n y} \int_{a_+ M_n}^{+\infty} \mu_n(\mathrm{d}u) \\ &= a_+ y + \log \left(b_n^+\right)^{1/M_n}. \end{aligned}$$

Then if (6.1.35) fails to hold, we have $f(y) \geq a_+ y$. Since $f(0) = 0$, the latter contradicts the definition of a_+. The second part of the proof can be done in the same way. $\square$

By (6.1.35), it follows that there exists $b \in (0, 1)$, such that for all $n \in \mathbb{N}$,

$$b_n^\pm \leq b^{M_n}. \tag{6.1.36}$$

Proof of Theorem 6.1.7. For any $n \in \mathbb{N}$, one has

$$\int_{a_- M_n}^{a_+ M_n} g(u/M_n) \mu_n(\mathrm{d}u) \leq \max_{z \in [a_-, a_+]} g(z). \tag{6.1.37}$$

Then

$$\int g(u/M_n) \mu_n(\mathrm{d}u) \leq B_n^+(g) + B_n^-(g) + \max_{z \in [a_-, a_+]} g(z), \tag{6.1.38}$$

where

$$B_n^+(g) = \int_{a_+ M_n}^{+\infty} g(u/M_n) \mu_n(\mathrm{d}u), \quad B_n^-(g) = \int_{-\infty}^{a_- M_n} g(u/M_n) \mu_n(\mathrm{d}u).$$

We recall that $|g(u)| \leq \lambda \exp(\varkappa |u|)$ and f is given by (6.1.30). For $\varepsilon > 0$, we pick $\gamma > a_+$ such that

$$\lambda \exp\left[-\gamma\varkappa + f(2\varkappa)\right] < \varepsilon/2. \tag{6.1.39}$$

For this γ, we set

$$\Upsilon(g, \gamma) = \max_{z \in [a_+, \gamma]} |g(z)|. \tag{6.1.40}$$

Then

$$\begin{aligned} |B_n^+(g)| &\leq \int_{a_+ M_n}^{\gamma M_n} |g(u/M_n)| \mu_n(\mathrm{d}u) + \int_{\gamma M_n}^{+\infty} |g(u/M_n)| \mu_n(\mathrm{d}u) \\ &\leq \Upsilon(g, \gamma) b_n^+ + \lambda e^{-\gamma\varkappa} \int_{\gamma M_n}^{+\infty} \exp\left(2\varkappa u/M_n\right) \mu_n(\mathrm{d}u) \\ &\leq \Upsilon(g, \gamma) b^{M_n} + \lambda e^{-\gamma\varkappa} \left(\int \exp\left(2\varkappa u\right) \mu_n(\mathrm{d}u) \right)^{1/M_n}. \end{aligned}$$

Here we used (6.1.36) and the Hölder inequality (3.2.14). Now we take into account (6.1.30) and (6.1.39), as well as the fact that $b < 1$, and choose n big enough to make each summand in the latter expression less than $\varepsilon/2$, which finally yields $|B_n^+(g)| < \varepsilon$ for such n. Similarly, one estimates $|B_n^-(g)|$. Applying this in (6.1.38) we arrive at (6.1.32) since $\delta > 0$ is arbitrary. If $f'_+(0) = f'_-(0) = \phi$, then by (6.1.37),

$$\limsup_{n \to +\infty} \int g(u/M_n) \mu_n(\mathrm{d}u) \leq g(\phi). \tag{6.1.41}$$

On the other hand,

$$\begin{aligned} \int g(u/M_n) \mu_n(\mathrm{d}u) &\geq \int_{(\phi-\delta)M_n}^{(\phi+\delta)M_n} g(u/M_n) \mu_n(\mathrm{d}u) \\ &= g(\phi + t\delta) \int_{(\phi-\delta)M_n}^{(\phi+\delta)M_n} \mu_n(\mathrm{d}u), \end{aligned}$$

for some $t \in [-1, 1]$. As δ is arbitrary, this yields

$$\liminf_{n \to +\infty} \int g(u/M_n) \mu_n(\mathrm{d}u) \geq g(\phi),$$

which along with (6.1.41) leads to (6.1.31). The estimate (6.1.33) readily follows from (6.1.32). □

Let the anharmonic potential be of the form $V(u) = v(|u|^2) - hu^{(1)}$. Then we can write, cf. (5.2.2),

$$N_{\beta,\Lambda}^{\mathrm{per}}(h) = N_{\beta,\Lambda}^{\mathrm{per}}(0) \int_{\Omega_{\beta,\Lambda}} \exp\left(h \sum_{\ell \in \Lambda} \int_0^\beta x_\ell^{(1)}(\tau) \mathrm{d}\tau \right) \nu_{\beta,\Lambda}^{0,\mathrm{per}}(\mathrm{d}x_\Lambda), \tag{6.1.42}$$

where $\nu^{0,\mathrm{per}}_{\beta,\Lambda}$ is the local periodic Euclidean Gibbs measure with $h = 0$. Now let $\{L_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ be such that the sequences of local measures $\{\nu^{0,\mathrm{per}}_{\beta,\Lambda_{L_n}}\}$ and $\{\nu^{\mathrm{per}}_{\beta,\Lambda_{L_n}}\}$ converge to the corresponding periodic Euclidean Gibbs measures μ^0 and μ, respectively. Set

$$\mathcal{X}_n = \left\{ x_{\Lambda_{L_n}} \in \Omega_{\beta,\Lambda_{L_n}} \,\middle|\, \exists u \in \mathbb{R}\colon\ \textstyle\sum_{\ell\in\Lambda_{L_n}} \int_0^\beta x^{(1)}_\ell(\tau)\mathrm{d}\tau = u \right\}. \tag{6.1.43}$$

Clearly, each such $\mathcal{X}_n$ is measurable and isomorphic to $\mathbb{R}$. Let μ_n, $n \in \mathbb{N}$, be the projection of $\{\nu^{0,\mathrm{per}}_{\beta,\Lambda_{L_n}}\}$ onto this $\mathcal{X}_n$. Then

$$p(h) = p(0) + f(h), \tag{6.1.44}$$

where f is given by (6.1.30) with such μ_n and $M_n = |\Lambda_{L_n}| = (2L_n)^d$. Then by means of Theorem 6.1.7 we have that in the general case where the model (1.1.3), (1.1.8) is just rotation and translation-invariant, the existence of the long-range order implies first-order phase transition. For certain cases, we have more specific information about the relationships between all types of phase transitions introduced above. We still suppose that the model is rotation and translation-invariant.

Theorem 6.1.9. *If $\nu = 1$, differentiability of the pressure at $h = 0$ (i.e., the lack of first-order phase transitions) yields $|\mathcal{G}^{\mathrm{t}}_\beta| = 1$ and the lack of the long-range order. If $d \geq 3$ and the interaction is ferromagnetic and of nearest neighbor type, and if the estimates (6.1.21), (6.1.26) hold, then for $J > J_*$, the long-range order exists and both phase transitions – of first order and in the sense of Definition 6.1.1 – take place.*

Proof. The first part of this theorem merely repeats Theorem 5.3.3. For $d \geq 3$, $B_{\ell\ell}$ is finite, see (6.1.21), (6.1.22). Therefore, by (6.1.26) and (6.1.25), there exists $\mu \in \mathcal{M}^{\mathrm{per}}_\beta$, for which $D^\mu_{\ell\ell'}$ does not tend to zero as $|\ell - \ell'| \to +\infty$, which implies the existence of a non-ergodic $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$ and hence the fact $|\mathcal{G}^{\mathrm{t}}_\beta| > 1$. □

In the subsequent sections, we prove that the estimates (6.1.21), (6.1.26) do hold if the model parameters satisfy certain conditions and study also the cases of nonlocal interactions and non translation-invariant models.

Now let us turn to phase transitions of second order in rotation-invariant models. For $\alpha \in (0, 1]$, we set, cf. (6.1.28),

$$P^{(\alpha)}_\Lambda = \frac{\beta^{-2}}{|\Lambda|^{1+\alpha}} \int_{\Omega_{\beta,\Lambda}} \left| \sum_{\ell\in\Lambda} \int_0^\beta x_\ell(\tau)\mathrm{d}\tau \right|^2 \nu^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{6.1.45}$$

where Λ is a box. Then $P^{(1)}_\Lambda = P_\Lambda$ and the existence of a positive limit (6.1.29) yields a first-order phase transition in the general case where the model is just rotation and translation-invariant. Also in this case, we have the following

Proposition 6.1.10. *If there exists $\alpha \in (0,1)$, such that for a sequence $\{L_n\}$, there exists the finite limit*

$$\lim_{n\to+\infty} P^{(\alpha)}_{\Lambda_n} \stackrel{\text{def}}{=} P^{(\alpha)} > 0, \tag{6.1.46}$$

then the model has at $h = 0$ a second-order phase transition.

Proof. One observes that

$$P^{(\alpha)}_{\Lambda} = p''_{\Lambda}(0)/\beta^2|\Lambda|^{\alpha}. \tag{6.1.47}$$

As the first derivative $p'(0)$ exists, we get

$$p''(0) = \lim_{\delta\to 0+} \left[p'(\delta) - p'(-\delta)\right]/2\delta = \lim_{\delta\to 0+} \lim_{L\to+\infty} p''_{\Lambda_L}(h),$$

for a certain $h \in (-\delta, \delta)$. By Proposition 1.4.12, $p''_{\Lambda}(h)$ is continuous in h for every fixed Λ; hence, by (6.1.46), (6.1.47), it follows that $p''_{\Lambda_L}(h)$ can be made arbitrarily big by choosing a sufficiently big Λ_L and a sufficiently small δ. □

It is easy to understand that Proposition 6.1.10 remains true if one replaces in (6.1.45) the periodic local measure $\nu^{\text{per}}_{\beta,\Lambda}$ by the one corresponding to the zero boundary condition, i.e., by $\nu_{\beta,\Lambda}$. Then the limit in (6.1.46) may be taken along any van Hove sequence $\mathcal{L}$. We recall that Proposition 6.1.10 describes the rotation-invariant case. The existence of a positive $P^{(\alpha)}$ with $\alpha > 0$ can be interpreted as follows. According to the central limit theorem for independent identically distributed random variables, for our model with $J_{\ell\ell'} = 0$ and $V_\ell = V$, the only possibility to have a finite positive limit in (6.1.46) is to set $\alpha = 0$. If $P^{(0)} > 0$ for nonzero interaction, one can say that the dependence between the temperature loops is weak; it holds for small $\hat{J}_0$. Of course, in this case $P^{(\alpha)} = 0$ for any $\alpha > 0$. If $P^{(\alpha)}$ gets positive for a certain $\alpha \in (0,1)$, one can say that strong dependence between the loops appears. In this case, the central limit theorem holds with an abnormal normalization. However, this dependence is not so strong as to make p' discontinuous, which occurs at $\alpha = 1$, where a new law of large numbers comes to power. In statistical physics, the point at which $P^{(\alpha)} > 0$ for $\alpha \in (0,1)$ is called a *critical point*. The quantity $P^{(0)}$ is called the *susceptibility*, it gets discontinuous at the critical point. Its singularity at this point is connected with the value of α for which $P^{(\alpha)} > 0$. The above analysis allows one to extend the notion of the critical point to models that are not translation-invariant.

Definition 6.1.11. The rotation-invariant model has a critical point if there exist a van Hove sequence $\mathcal{L}$ and $\alpha \in (0,1)$ such that

$$\lim_{\mathcal{L}} \frac{1}{|\Lambda|^{1+\alpha}} \int_{\Omega_{\beta,\Lambda}} \left| \sum_{\ell\in\Lambda} \int_0^\beta x_\ell(\tau)\mathrm{d}\tau \right|^2 \nu_{\beta,\Lambda}(\mathrm{d}x_\Lambda) > 0 \tag{6.1.48}$$

for certain values of the model parameters, including h, and β.

In the translation-invariant case, a critical point and a second-order phase transition mean the same – a fact that follows from Proposition 6.1.10. Below we study the critical point in models of the type of (1.1.3), (1.1.8), in which the set $\mathbb{L}$ is equipped with a special (hierarchical) metric instead of the Euclidean metric (1.1.1). Therefore, instead of the translation invariance, such models have a special symmetry determined by this metric.

6.2 Reflection Positivity and Infrared Bound

As was mentioned above, the method which we are going to use for obtaining the infrared estimates (6.1.8), (6.1.21) is based on a special property of the local Euclidean Gibbs states of the version of our model for which $\mathbb{L} = \mathbb{Z}^d$, the interaction is of nearest neighbor type, and $V_\ell = V$. From the mathematical point of view the infrared estimate can be characterized as a certain sophisticated version of the Hölder inequality (3.2.14). The proof of phase transitions in more general versions of our model (1.1.3), (1.1.8) will be made by comparing these models with reference models, for which the phase transition is established by means of the infrared estimates. The comparison is based on correlation inequalities.

6.2.1 Reflection positive functionals

Let $\Lambda \in \mathfrak{L}_{\rm fin}$, consisting of an even number of points, be given. It may be the box (3.1.2), which contains $|\Lambda| = (2L)^d$ points. Suppose we are given a bijection $\rho\colon \Lambda \to \Lambda$, $\rho \circ \rho = \mathrm{id}$, such that Λ falls into two disjoint parts $\Lambda_\pm$ with the property that $\rho\colon \Lambda_+ \to \Lambda_-$. We shall call ρ *reflection*. For $x_\Lambda \in \Omega_{\beta,\Lambda}$, we set $\varrho(x_\Lambda) = (x_{\rho(\ell)})_{\ell\in\Lambda}$. We recall that

$$\Omega_{\beta,\Lambda} = \{x_\Lambda = (x_\ell)_{\ell\in\Lambda} \mid x_\ell\colon [0,\beta] \to \mathbb{R}^\nu,\ x_\ell(0) = x_\ell(\beta)\}, \quad \nu \in \mathbb{N},$$

is the Banach space of continuous temperature loops, and that $\mathfrak{C}_\Lambda$ is the set of continuous functions $f : \Omega_{\beta,\Lambda} \to \mathbb{R}$ obeying the estimate (1.4.58). $\mathfrak{C}_\Lambda$ is closed with respect to the multiplication and linear operations over $\mathbb{R}$, i.e., it is a real algebra. On this algebra we define the map $\theta_\rho\colon \mathfrak{C}_\Lambda \to \mathfrak{C}_\Lambda$, also called *reflection*, by setting

$$\theta_\rho(f)(x_\Lambda) = f[\varrho(x_\Lambda)]. \tag{6.2.1}$$

Clearly, for any $f, g \in \mathfrak{C}_\Lambda$ and $\kappa, \lambda \in \mathbb{R}$,

$$\theta_\rho(\kappa f + \lambda g) = \kappa\theta_\rho(f) + \lambda\theta_\rho(g), \qquad \theta_\rho(f \cdot g) = \theta_\rho(f)\cdot\theta_\rho(g). \tag{6.2.2}$$

Along with the functions $f \in \mathfrak{C}_\Lambda$, we consider the affine functions

$$\Omega_{\beta,\Lambda_+} \ni x_{\Lambda_+} \mapsto \xi_{\Lambda_+} = (\xi_\ell)_{\ell\in\Lambda_+}, \quad \xi_\ell = \alpha_\ell x_\ell + a_\ell,\ \alpha_\ell \in \mathbb{R},\ a_\ell \in L^2_\beta, \tag{6.2.3}$$

and set

$$\theta_\rho(\xi_\ell) = \alpha_\ell x_{\rho(\ell)} + a_\ell, \quad \ell \in \Lambda_+. \tag{6.2.4}$$

By $\mathfrak{C}_\Lambda^+$ (respectively, $\mathfrak{C}_\Lambda^-$) we denote the subalgebra of $\mathfrak{C}_\Lambda$ consisting of functions dependent on x_{Λ_+} (respectively, x_{Λ_-}) only. Then $\theta_\rho(\mathfrak{C}_\Lambda^+) = \mathfrak{C}_\Lambda^-$ and $\theta_\rho \circ \theta_\rho = \mathrm{id}$.

Definition 6.2.1. A linear functional $\omega\colon \mathfrak{C}_\Lambda \to \mathbb{R}$ is said to be reflection positive (RP) with respect to θ_ρ, if

$$\forall f \in \mathfrak{C}_\Lambda^+: \quad \omega[f \cdot \theta_\rho(f)] \geq 0, \tag{6.2.5}$$

and for any affine function (6.2.3),

$$\forall\, \ell \in \Lambda_+: \quad \omega\Big[(\xi_\ell, \theta_\rho(\xi_\ell))_{L^2_\beta}\Big] \geq 0. \tag{6.2.6}$$

Let χ be a finite measure on C_β. For $\Delta \subseteq \Lambda$, we set

$$\chi_\Delta(\mathrm{d}x_\Lambda) = \bigotimes_{\ell\in\Delta} \chi(\mathrm{d}x_\ell), \tag{6.2.7}$$

and suppose that each $f \in \mathfrak{C}_\Lambda$ is χ_Λ-integrable. Finally, let ρ and θ_ρ be any of the maps possessing the properties described above.

Proposition 6.2.2. *The functional*

$$\mathfrak{C}_\Lambda \ni f \mapsto \omega_\chi(f) = \int_{\Omega_{\beta,\Lambda}} f(x_\Lambda)\chi_\Lambda(\mathrm{d}x_\Lambda) \tag{6.2.8}$$

is RP with respect to this θ_ρ.

Proof. Given $f \in \mathfrak{C}_\Lambda^+$, let $\varphi\colon \Omega_{\beta,\Lambda_+} \to \mathbb{R}$ be such that $f(x_\Lambda) = \varphi(x_{\Lambda_+})$. Then

$$\begin{aligned}
\omega_\chi[f \cdot \theta_\rho(f)] &= \int_{\Omega_{\beta,\Lambda_+}} \varphi(x_{\Lambda_+})\chi_{\Lambda_+}(\mathrm{d}x_{\Lambda_+}) \\
&\qquad \times \int_{\Omega_{\beta,\Lambda_-}} \varphi(x_{\Lambda_-})\chi_{\Lambda_-}(\mathrm{d}x_{\Lambda_-}) \\
&= \left[\int_{\Omega_{\beta,\Lambda_+}} \varphi(x_{\Lambda_+})\chi_{\Lambda_+}(\mathrm{d}x_{\Lambda_+})\right]^2 \geq 0.
\end{aligned}$$

Likewise,

$$\omega_\chi\Big[(\xi_\ell, \theta_\rho(\xi_\ell))_{L^2_\beta}\Big] = \|\langle \xi_\ell\rangle_\chi\|^2_{L^2_\beta} \geq 0.$$ □

In the above example, the multiplicative structure of the measure (6.2.7) is crucial. It results in the positivity of ω_χ with respect to all reflections $\rho\colon \Lambda \to \Lambda$. One observes that the functional (6.2.8) is symmetric, i.e.,

$$\omega_\chi[f \cdot \theta_\rho(g)] = \omega_\chi[g \cdot \theta_\rho(f)], \quad \text{for all } f, g \in \mathfrak{C}_\Lambda^+, \tag{6.2.9}$$

and similarly with respect to its values on the functions (6.2.3). In the sequel, we shall suppose that all RP functionals are symmetric. Therefore, all such functionals obey the following Schwarz-type inequality.

Lemma 6.2.3. *If a state ω on $\mathfrak{C}_\Lambda$ is symmetric and RP with respect to a given θ_ρ, then for any $f, g \in \mathfrak{C}_\Lambda^+$,*

$$\left\{\omega[f \cdot \theta_\rho(g)]\right\}^2 \leq \omega[f \cdot \theta_\rho(f)] \cdot \omega[g \cdot \theta_\rho(g)], \tag{6.2.10}$$

and for any two affine maps (6.2.3),

$$\left\{\omega\Big[(\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big]\right\}^2 \leq \omega\Big[(\xi_\ell, \theta_\rho(\xi_\ell))_{L^2_\beta}\Big] \cdot \omega\Big[(\eta_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big]. \tag{6.2.11}$$

Proof. For $\kappa \in \mathbb{R}$, by (6.2.2) we have

$$\omega[(f + \kappa g) \cdot \theta_\rho(f + \kappa g)] = \omega[(f + \kappa g) \cdot (\theta_\rho(f) + \kappa \theta_\rho(g))] \geq 0.$$

As ω is a linear functional, the latter can be written as a 3-nomial, whose positivity for all $\kappa \in \mathbb{R}$ is equivalent to (6.2.10). The proof of (6.2.11) is analogous. □

Our next step is to obtain a generalization of the inequality (6.2.11), which we then use to obtain the Gaussian domination estimate and thereby the infrared bound. Let us consider a doubled system of temperature loop. This means that for every $\ell \in \Lambda$, we have a pair $z_\ell = (x_\ell, y_\ell) \in C_\beta \times C_\beta$. One can consider this pair as one vector $z_\ell \colon C[0, \beta] \to \mathbb{R}^{2\nu}$. By $\widetilde{\mathfrak{C}}_\Lambda$ we denote the algebra of functions $f(z_\ell)$ obeying the estimate (1.4.58) with respect to this doubled vector. As above, we have a reflection $\rho \colon \Lambda \to \Lambda$, which defines also the reflection θ_ρ acting according to (6.2.1) and (6.2.4) on both components of z_ℓ's. The functions $f \in \widetilde{\mathfrak{C}}_\Lambda$ which depend on z_{Λ_+} only constitute the subalgebra $\widetilde{\mathfrak{C}}_\Lambda^+$. For $\lambda_\ell \in \mathbb{R}$ and $a_\ell \in L^2_\beta$, $\ell \in \Lambda$, we set

$$\xi_\ell(z_\ell) = \lambda_\ell x_\ell, \quad \eta_\ell(z_\ell) = \lambda_\ell(y_\ell + a_\ell), \quad \ell \in \Lambda_+. \tag{6.2.12}$$

Let the functions $f, g, \in \widetilde{\mathfrak{C}}_\Lambda^+$ be such that $\exp[f + \theta_\rho(g)] \in \widetilde{\mathfrak{C}}_\Lambda$. Then for the affine functions (6.2.12), it follows that

$$\exp\Big[f + \theta_\rho(g) + \sum_{\ell \in \Lambda_+} (\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big] \in \widetilde{\mathfrak{C}}_\Lambda. \tag{6.2.13}$$

Suppose that a given linear RP functional $\tilde{\omega} \colon \widetilde{\mathfrak{C}}_\Lambda \to \mathbb{R}$ is such that the series

$$\begin{aligned} \sum_{k_1,\ldots,k_n=0}^{+\infty} \frac{1}{k_1! \ldots k_n!} \cdot \tilde{\omega}\Big\{ (F \cdot \theta_\rho(G)) \cdot \Big[(\xi_{\ell_1}, \theta_\rho(\xi_{\ell_1}))_{L^2_\beta}\Big]^{k_1} \\ \times \cdots \times \Big[(\xi_{\ell_n}, \theta_\rho(\xi_{\ell_n}))_{L^2_\beta}\Big]^{k_n}\Big\}, \quad n = |\Lambda_+|, \ F = e^f, \ G = e^g, \end{aligned} \tag{6.2.14}$$

as well as the one with ξ_ℓ replaced by η_ℓ, are absolutely convergent.

Lemma 6.2.4. *Let the functional $\tilde{\omega}$ and the functions $f, g \in \widetilde{\mathfrak{E}}^+_\Lambda$, ξ_ℓ, η_ℓ be as above. Then*

$$\begin{aligned}
&\Big\{\tilde{\omega}\Big[\exp\Big(f + \theta_\rho(g) + \sum_{\ell\in\Lambda_+}(\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big)\Big]\Big\}^2 \\
&\quad\le \tilde{\omega}\Big[\exp\Big(f + \theta_\rho(f) + \sum_{\ell\in\Lambda_+}(\xi_\ell, \theta_\rho(\xi_\ell))_{L^2_\beta}\Big)\Big] \\
&\qquad\times \tilde{\omega}\Big[\exp\Big(g + \theta_\rho(g) + \sum_{\ell\in\Lambda_+}(\eta_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big)\Big].
\end{aligned} \tag{6.2.15}$$

Proof. By the above assumptions,

$$\begin{aligned}
&\tilde{\omega}\Big[\exp\Big(f + \theta_\rho(g) + \sum_{\ell\in\Lambda_+}(\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big)\Big] \\
&\quad = \omega\Big[F\cdot\theta_\rho(G)\prod_{\ell\in\Lambda_+}\exp\Big((\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big)\Big] \\
&\quad = \sum_{k_1,\ldots,k_n=0}^{+\infty}\frac{1}{k_1!\ldots k_n!}\cdot\tilde{\omega}\Big\{\Big(F\cdot\theta_\rho(G)\Big)\cdot\Big[\Big(\xi_{\ell_1}, \theta_\rho(\eta_{\ell_1})\Big)_{L^2_\beta}\Big]^{k_1} \\
&\qquad\qquad \times\cdots\times\Big[\Big(\xi_{\ell_n}, \theta_\rho(\eta_{\ell_n})\Big)_{L^2_\beta}\Big]^{k_n}\Big\},
\end{aligned} \tag{6.2.16}$$

and the latter series is absolutely convergent. Then by (6.2.10), (6.2.11) and the Schwarz inequality for sums we get

$$\begin{aligned}
&\mathrm{LHS}(6.2.16) = |\mathrm{LHS}(6.2.16)| \\
&\quad\le \sum_{k_1,\ldots,k_n=0}^{+\infty}\frac{1}{k_1!\ldots k_n!}\cdot\Big|\tilde{\omega}\Big\{(F\cdot\theta_\rho(G))\cdot\Big[(\xi_{\ell_1}, \theta_\rho(\eta_{\ell_1}))_{L^2_\beta}\Big]^{k_1} \\
&\qquad\qquad \times\cdots\times\Big[(\xi_{\ell_n}, \theta_\rho(\eta_{\ell_n}))_{L^2_\beta}\Big]^{k_n}\Big\}\Big| \\
&\quad\le \sum_{k_1,\ldots,k_n=0}^{+\infty}\Big\{\frac{1}{k_1!\ldots k_n!}\cdot\tilde{\omega}\Big\{(F\cdot\theta_\rho(F))\cdot\Big[(\xi_{\ell_1}, \theta_\rho(\xi_{\ell_1}))_{L^2_\beta}\Big]^{k_1} \\
&\qquad\qquad \times\cdots\times\Big[(\xi_{\ell_n}, \theta_\rho(\xi_{\ell_n}))_{L^2_\beta}\Big]^{k_n}\Big\}\Big\}^{1/2} \\
&\qquad \times\Big\{\frac{1}{k_1!\ldots k_n!}\tilde{\omega}\Big\{(G\cdot\theta_\rho(G))\cdot\Big[(\eta_{\ell_1}, \theta_\rho(\eta_{\ell_1}))_{L^2_\beta}\Big]^{k_1} \\
&\qquad\qquad \times\cdots\times\Big[(\eta_{\ell_n}, \theta_\rho(\eta_{\ell_n}))_{L^2_\beta}\Big]^{k_n}\Big\}\Big\}^{1/2} \\
&\quad\stackrel{\mathrm{def}}{=} \Omega(f, g).
\end{aligned}$$

Thereby,

$$\begin{aligned}
\Omega(f,g) \leq &\left\{ \sum_{k_1,\ldots,k_n=0}^{+\infty} \frac{1}{k_1!\ldots k_n!} \cdot \tilde{\omega}\left\{ \left(F \cdot \theta_\rho(F)\right) \cdot \left[\left(\xi_{\ell_1}, \theta_\rho(\xi_{\ell_1})\right)_{L^2_\beta}\right]^{k_1} \right.\right. \\
&\left.\left. \times \cdots \times \left[\left(\xi_{\ell_n}, \theta_\rho(\xi_{\ell_n})\right)_{L^2_\beta}\right]^{k_n} \right\}\right\}^{1/2} \\
&\times \left\{ \sum_{k_1,\ldots,k_n=0}^{+\infty} \frac{1}{k_1!\ldots k_n!} \cdot \tilde{\omega}\left\{ \left(G \cdot \theta_\rho(G)\right) \cdot \left[\left(\eta_{\ell_1}, \theta_\rho(\eta_{\ell_1})\right)_{L^2_\beta}\right]^{k_1} \right.\right. \\
&\left.\left. \times \cdots \times \left[\left(\eta_{\ell_n}, \theta_\rho(\eta_{\ell_n})\right)_{L^2_\beta}\right]^{k_n} \right\}\right\}^{1/2} \\
= &\left\{\tilde{\omega}\left[\exp\left(f + \theta_\rho(f) + \sum_{\ell\in\Lambda_+} (\xi_\ell, \theta_\rho(\xi_\ell))_{L^2_\beta}\right)\right]\right\}^{1/2} \\
&\times \left\{\tilde{\omega}\left[\exp\left(g + \theta_\rho(g) + \sum_{\ell\in\Lambda_+} (\eta_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\right)\right]\right\}^{1/2},
\end{aligned}$$

which yields (6.2.15). □

6.2.2 Gaussian Domination

First we prove an auxiliary result. Given $\Delta \in \mathfrak{L}_{\mathrm{fin}}$, let Δ' be its nonempty subset and μ, ν be finite Borel measures on $\Omega_{\beta,\Delta}$, such that every $f \in \mathfrak{E}_\Delta$ is integrable with respect to both ones.

Lemma 6.2.5. *Let the sets Δ, Δ' and the measures μ, ν be as above. Then for every $\ell \in \Delta'$ and $a_\ell \in L^2_\beta$, it follows that*

$$\begin{aligned}
&\left\{\int_{\Omega_{\beta,\Delta}}\int_{\Omega_{\beta,\Delta}} \exp\left(-\frac{1}{2}\sum_{\ell\in\Delta'} \|x_\ell - y_\ell - a_\ell\|^2_{L^2_\beta}\right)\mu(\mathrm{d}x_\Delta)\nu(\mathrm{d}y_\Delta)\right\}^2 \\
&\quad\leq \int_{\Omega_{\beta,\Delta}}\int_{\Omega_{\beta,\Delta}} \exp\left(-\frac{1}{2}\sum_{\ell\in\Delta'} \|x_\ell - y_\ell\|^2_{L^2_\beta}\right)\mu(\mathrm{d}x_\Delta)\mu(\mathrm{d}y_\Delta) \qquad (6.2.17)\\
&\qquad\times \int_{\Omega_{\beta,\Delta}}\int_{\Omega_{\beta,\Delta}} \exp\left(-\frac{1}{2}\sum_{\ell\in\Delta'} \|x_\ell - y_\ell\|^2_{L^2_\beta}\right)\nu(\mathrm{d}x_\Delta)\nu(\mathrm{d}y_\Delta).
\end{aligned}$$

Proof. We take two copies of Δ and denote them by Λ_+ and Λ_-, respectively. Then by $\Lambda'_\pm \subset \Lambda_\pm$ we denote the subsets consisting of the elements of Δ'. For an $\ell \in \Lambda_+$, by $\rho(\ell)$ we denote its counterpart in Λ_-. Then ρ is a reflection and $\rho(\Lambda'_+) = \rho(\Lambda'_-)$. Finally, we set $\Lambda = \Lambda_+ \cup \Lambda_-$ and $\Lambda' = \Lambda'_+ \cup \Lambda'_-$. Now let $\widetilde{\mathfrak{E}}_{\Lambda'}$ be the algebra of functions $f(z_{\Lambda'})$, $z_{\Lambda'} = (x_{\Lambda'}, y_{\Lambda'}) \in \Omega_{\beta,\Lambda'} \times \Omega_{\beta,\Lambda'}$, the same as in

(6.2.13) and Lemma 6.2.4. Correspondingly, $\widetilde{\mathfrak{C}}^{+}_{\Lambda'}$ (respectively, $\widetilde{\mathfrak{C}}^{-}_{\Lambda'}$) will stand for the subalgebras consisting of functions dependent on $z_{\Lambda'_+}$ (respectively, $z_{\Lambda'_+}$) only. Thereby, we introduce the measures

$$\begin{aligned}\hat{\mu}(\mathrm{d}x_\Delta) &= \exp\Big(-\frac{1}{2}\sum_{\ell\in\Delta'}\|x_\ell\|^2_{L^2_\beta}\Big)\mu(\mathrm{d}x_\Delta),\\ \hat{\nu}(\mathrm{d}x_\Delta) &= \exp\Big(-\frac{1}{2}\sum_{\ell\in\Delta,}\|x_\ell\|^2_{L^2_\beta}\Big)\nu(\mathrm{d}x_\Delta),\end{aligned} \tag{6.2.18}$$

and define the functional

$$\begin{aligned}&\widetilde{\mathfrak{C}}_{\Lambda'} \ni f \mapsto \tilde{\omega}(f)\\ &\quad = \int_{\Omega_{\beta,\Lambda}}\int_{\Omega_{\beta,\Lambda}} f(x_{\Lambda'}, y_{\Lambda'})\hat{\mu}(\mathrm{d}x_{\Lambda_+})\hat{\nu}(\mathrm{d}x_{\Lambda_+})\hat{\mu}(\mathrm{d}x_{\Lambda_-})\hat{\nu}(\mathrm{d}x_{\Lambda_-}).\end{aligned} \tag{6.2.19}$$

This functional has the same structure as the one described by Lemma 6.2.4; hence, it is positive with respect to the reflection θ_ρ defined by (6.2.1), (6.2.4). Set

$$\Upsilon_\mu = \int_{\Omega_{\beta,\Delta}}\hat{\mu}(\mathrm{d}x_\Delta), \quad \Upsilon_\nu = \int_{\Omega_{\beta,\Delta}}\hat{\nu}(\mathrm{d}x_\Delta), \tag{6.2.20}$$

and

$$\begin{aligned}&f(x_{\Lambda'}, y_{\Lambda'}) \equiv 0, \quad g(x_{\Lambda'}, y_{\Lambda'}) = -\sum_{\ell\in\Lambda'_+}\Big[(y_\ell, a_\ell)_{L^2_\beta} + \frac{1}{2}\|a_\ell\|^2_{L^2_\beta}\Big],\\ &\xi_\ell(x_\ell, y_\ell) = x_\ell, \quad \eta_\ell(x_\ell, y_\ell) = y_\ell + a_\ell, \quad \text{for } \ell\in\Lambda'_+,\\ &\xi_\ell(x_\ell, y_\ell) = \eta_\ell(x_\ell, y_\ell) = 0, \quad \text{for } \ell\in\Lambda_+\setminus\Lambda'_+.\end{aligned} \tag{6.2.21}$$

The latter affine functions are of the type of (6.2.12). Thereby,

$$\mathrm{LHS}(6.2.17) = \frac{1}{(\Upsilon_\mu\Upsilon_\nu)^2}\Big\{\tilde{\omega}\Big[\exp\Big(f + \theta_\rho(g) + \sum_{\ell\in\Lambda_+}(\xi_\ell, \theta_\rho(\eta_\ell))_{L^2_\beta}\Big)\Big]\Big\}^2.$$

Applying here the estimate (6.2.15) and taking into account (6.2.20) we arrive at

$$\begin{aligned}&\mathrm{LHS}(6.2.17)\\ &\quad\le \frac{1}{(\Upsilon_\mu\Upsilon_\nu)^2}\int_{\Omega_{\beta,\Lambda}}\int_{\Omega_{\beta,\Lambda}}\exp\Big(\sum_{\ell\in\Lambda'_+}(x_\ell, x_{\rho(\ell)})_{L^2_\beta}\Big)\\ &\qquad\qquad\times\hat{\mu}(\mathrm{d}x_{\Lambda_+})\hat{\nu}(\mathrm{d}x_{\Lambda_+})\hat{\mu}(\mathrm{d}x_{\Lambda_-})\hat{\nu}(\mathrm{d}x_{\Lambda_-})\\ &\qquad\times\int_{\Omega_{\beta,\Lambda}}\int_{\Omega_{\beta,\Lambda}}\exp\Big(\sum_{\ell\in\Lambda'_+}(y_\ell, y_{\rho(\ell)})_{L^2_\beta}\Big)\\ &\qquad\qquad\times\hat{\mu}(\mathrm{d}x_{\Lambda_+})\hat{\nu}(\mathrm{d}x_{\Lambda_+})\hat{\mu}(\mathrm{d}x_{\Lambda_-})\hat{\nu}(\mathrm{d}x_{\Lambda_-}),\end{aligned}$$

which is exactly the right-hand side of (6.2.17). $\square$

Now we return to the model (1.1.3), (1.1.8), which we suppose here to be translation-invariant, ferromagnetic, and of nearest neighbor type. The latter means that $J_{\ell\ell'} = J > 0$ if the distance (1.1.1) equals one, and $J_{\ell\ell'} = 0$ otherwise, cf. Definition 1.1.2. As we are going to show, the model with such interaction intensities is refection positive. This is the simplest case of $J_{\ell\ell'}$ with such a property. Possible generalizations are discussed in Section 6.5 below.

Let Λ be the box (3.1.2). We impose periodic conditions on its boundaries; hence, 'the periodic interaction' (1.4.50) in this case is $J^{\Lambda}_{\ell\ell'} = J$ if the distance (1.4.48) equals 1, and $J^{\Lambda}_{\ell\ell'} = 0$ otherwise. The periodic local Euclidean Gibbs measure of this model is, cf. (1.4.52),

$$\nu^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \frac{1}{Z_\Lambda(0)} \exp\Big(-\frac{J}{2}\sum_{\langle\ell,\ell'\rangle\in E} \|x_\ell - x_{\ell'}\|^2_{L^2_\beta}\Big)\sigma_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{6.2.22}$$

where E is the set of unordered pairs $\langle\ell,\ell'\rangle$, $\ell,\ell'\in\Lambda$, such that $|\ell-\ell'|_\Lambda = 1$. One can consider E as the set of edges of the graph (Λ, E) in which the vertices are elements of Λ. Furthermore,

$$\sigma_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \exp\Big(Jd\sum_{\ell\in\Lambda}\|x_\ell\|^2_{L^2_\beta} - \sum_{\ell\in\Lambda}\int_0^\beta V(x_\ell(\tau))\mathrm{d}\tau\Big)\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda), \tag{6.2.23}$$

and

$$Z_\Lambda(0) = \int_{\Omega_{\beta,\Lambda}} \exp\Big(-\frac{J}{2}\sum_{\langle\ell,\ell'\rangle\in E}\|x_\ell - x_{\ell'}\|^2_{L^2_\beta}\Big)\sigma_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{6.2.24}$$

Note that we do not suppose here $V(u)$ to be rotation-invariant. With every edge $\langle\ell,\ell'\rangle\in E$ we associate $b_{\ell\ell'}\in L^2_\beta$ and consider

$$Z_\Lambda(b) = \int_{\Omega_{\beta,\Lambda}} \exp\Big(-\frac{J}{2}\sum_{\langle\ell,\ell'\rangle\in E}\|x_\ell - x_{\ell'} - b_{\ell\ell'}\|^2_{L^2_\beta}\Big)\sigma_{\beta,\Lambda}(\mathrm{d}x_\Lambda). \tag{6.2.25}$$

Definition 6.2.6. The model considered here admits Gaussian domination if

$$Z_\Lambda(b) \le Z_\Lambda(0), \tag{6.2.26}$$

for any $b = (b_{\ell\ell'})_{\langle\ell,\ell'\rangle\in E}$, $b_{\ell\ell'}\in L^2_\beta$.

The set of edges E has the following structure. For $j\in\{1,\dots,d\}$ and $l\in\{1,\dots,L\}$, let us consider the hyperplanes

$$\begin{aligned}\Pi^{(j)}_{l,>} &= \{\ell\in\Lambda \mid \ell = (\ell_1,\dots,\ell_{j-1}, l, \ell_{j+1},\dots,\ell_d)\},\\ \Pi^{(j)}_{l,<} &= \{\ell\in\Lambda \mid \ell = (\ell_1,\dots,\ell_{j-1}, l-L, \ell_{j+1},\dots,\ell_d)\}.\end{aligned} \tag{6.2.27}$$

We also set

$$\Pi^{(j)}_{L+1,\gtrless} = \Pi^{(j)}_{1,\lessgtr}, \tag{6.2.28}$$

and introduce

$$E^{(j)}_{l,\gtrless} = \big\{\langle \ell, \ell'\rangle \mid \ell \in \Pi^{(j)}_{l,\gtrless} \text{ and } \ell' \in \Pi^{(j)}_{l+1,\gtrless}\big\}, \\ E^{(j)}_{l} = E^{(j)}_{l,>} \cup E^{(j)}_{l,<}. \tag{6.2.29}$$

Then, for every fixed j, we have L different families $E^{(j)}_1, \dots, E^{(j)}_L$, each consisting of $2 \cdot (2L)^{d-1}$ edges, and such that

$$E = \bigcup_{l=1,\dots,L;\ j=1,\dots,d} E^{(j)}_l. \tag{6.2.30}$$

Furthermore, for every such $E^{(j)}_l$, the graph $(\Lambda, E \setminus E^{(j)}_l)$ is disconnected and falls into two connected components, $(\Lambda_+(j,l), E_+(j,l))$ and $(\Lambda_-(j,l), E_-(j,l))$, where

$$\Lambda_+(j,l) = \Big(\bigcup_{l'=1}^{l} \Pi^{(j)}_{l',<}\Big) \cup \Big(\bigcup_{l'=l+1}^{L} \Pi^{(j)}_{l',>}\Big), \\ \Lambda_-(j,l) = \Big(\bigcup_{l'=l+1}^{L} \Pi^{(j)}_{l',<}\Big) \cup \Big(\bigcup_{l'=1}^{l} \Pi^{(j)}_{l',>}\Big), \tag{6.2.31}$$

and

$$E_\pm(j,l) = \{\langle \ell, \ell'\rangle \in E \mid \ell, \ell' \in \Lambda_\pm(j,l)\}. \tag{6.2.32}$$

By (6.2.31),

$$\Lambda_+(j,l) = \big\{(\ell_1, \dots, \ell_d) \in \Lambda \mid \ \ell_{j'} \in \{-L+1, \dots, L\}, \text{ for } j' \neq j; \\ \ell_j \in \{-L+1, \dots, -L+l\} \cup \{l+1, \dots, L\}\big\}. \tag{6.2.33}$$

For $l \in \{1, \dots, L\}$ and $k \in \{-L+1, \dots, -L+l\} \cup \{l+1, \dots, L\}$, we set

$$\rho_l(k) = 2l + 1 - k \mod 2L. \tag{6.2.34}$$

Then we define the map $\rho_{jl} : \Lambda_+(j,l) \to \Lambda_-(j,l)$ by setting

$$\rho_{j,l} : (\ell_1, \dots, \ell_{j-1}, \ell_j, \ell_{j+1}, \dots, \ell_d) \\ \to (\ell_1, \dots, \ell_{j-1}, \rho_l(\ell_j), \ell_{j+1}, \dots, \ell_d). \tag{6.2.35}$$

For $\ell \in \Lambda_-(j,l)$, we set $\rho_{j,l}(\ell) = \rho^{-1}_{j,l}(\ell)$; hence, such a $\rho_{j,l}$ is a reflection. If $\langle \ell, \ell'\rangle \in E^{(j)}_l$ and $\ell \in \Lambda_+(j,l)$, then $\rho_{j,l}(\ell) = \ell'$. By $\theta_{j,l}$ we denote $\theta_{\rho_{j,l}}$, see (6.2.1) and (6.2.4).

Theorem 6.2.7. *Let the model (1.1.3), (1.1.8) be translation-invariant, but not necessarily rotation-invariant, with ferromagnetic interaction of nearest neighbor type. Then it admits Gaussian domination.*

Proof. Let j, l be as above and $o = \pm 1$. Then for $b = (b_{\ell\ell'})_{\langle \ell,\ell'\rangle \in E}$, we set

$$\left(T^o_{j,l} b\right)_{\ell\ell'} = \begin{cases} b_{\ell\ell'} & \text{if } \langle \ell, \ell'\rangle \in E_o(j,l); \\ b_{\rho_{j,l}(\ell)\rho_{j,l}(\ell')} & \text{if } \langle \ell, \ell'\rangle \in E_{-o}(j,l); \\ 0 & \text{if } \langle \ell, \ell'\rangle \in E^{(j)}_l. \end{cases} \tag{6.2.36}$$

With the help of Lemma 6.2.5 we prove (6.2.26) by comparing $Z_\Lambda(b)$ with the one with fewer nonzero $b_{\ell\ell'}$. Let us do this for $b_{\ell\ell'}$ with $\langle \ell, \ell'\rangle$ belonging to $E^{(j)}_l$ with $j = 1$ and $l = 1$. To this end we rewrite (6.2.25) as follows:

$$\begin{aligned} Z_\Lambda(b) = \int_{\Omega_{\beta,\Lambda_+(1,1)}} \int_{\Omega_{\beta,\Lambda_-(1,1)}} & \exp\left(-\frac{J}{2} \sum_{\langle \ell,\ell'\rangle \in E^{(1)}_1} \|x_\ell - x_{\ell'} - b_{\ell\ell'}\|^2_{L^2_\beta} \right) \\ & \times \mu^+_{\Lambda_+(1,1)}(\mathrm{d}x_{\Lambda_+(1,1)}) \mu^-_{\Lambda_-(1,1)}(\mathrm{d}x_{\Lambda_-(1,1)}), \end{aligned} \tag{6.2.37}$$

where

$$\begin{aligned} & \mu^o_{\Lambda_o(1,1)}(\mathrm{d}x_{\Lambda_o(1,1)}) \\ & \quad = \exp\left(-\frac{J}{2} \sum_{\langle \ell,\ell'\rangle \in E_o(1,1)} \|x_\ell - x_{\ell'} - b_{\ell\ell'}\|^2_{L^2_\beta} \right) \sigma_{\beta,\Lambda_o(1,1)}(\mathrm{d}x_{\Lambda_o(1,1)}). \end{aligned} \tag{6.2.38}$$

Furthermore, we set

$$\begin{aligned} & \mu^-_{\Lambda_+(1,1)}(\mathrm{d}x_{\Lambda_+(1,1)}) \\ & \quad = \exp\left(-\frac{J}{2} \sum_{\langle \ell,\ell'\rangle \in E_+(1,1)} \|x_\ell - x_{\ell'} - b_{\rho_{1,1}(\ell)\rho_{1,1}(\ell')}\|^2_{L^2_\beta} \right) \\ & \qquad \times \sigma_{\beta,\Lambda_+(1,1)}(\mathrm{d}x_{\Lambda_+(1,1)}), \end{aligned} \tag{6.2.39}$$

and

$$\begin{aligned} & \mu^+_{\Lambda_-(1,1)}(\mathrm{d}x_{\Lambda_-(1,1)}) \\ & \quad = \exp\left(-\frac{J}{2} \sum_{\langle \ell,\ell'\rangle \in E_+(1,1)} \|x_{\rho_{1,1}(\ell)} - x_{\rho_{1,1}(\ell')} - b_{\ell\ell'}\|^2_{L^2_\beta} \right) \\ & \qquad \times \sigma_{\beta,\Lambda_-(1,1)}(\mathrm{d}x_{\Lambda_-(1,1)}). \end{aligned} \tag{6.2.40}$$

Now we apply Lemma 6.2.5 with

$$\Delta' = \Lambda'_+ = \Pi^{(1)}_{2,>} = \{\ell \in \Lambda_+(1,0) \mid \langle \ell, \ell'\rangle \in E^{(1)}_1\},$$

to (6.2.37) and obtain

$$\begin{aligned}
[Z_\Lambda(b)]^2 \leq & \int_{\Omega_{\beta,\Lambda_+(1,1)}} \int_{\Omega_{\beta,\Lambda_-(1,1)}} \exp\Big(-\frac{J}{2} \sum_{\langle \ell,\ell' \rangle \in E_0^{(1)}} \|x_\ell - x_{\ell'}\|^2_{L^2_\beta} \Big) \\
& \times \mu^+_{\Lambda_+(1,1)}(\mathrm{d}x_{\Lambda_+(1,1)}) \mu^+_{\Lambda_-(1,1)}(\mathrm{d}x_{\Lambda_-(1,1)}) \\
& \times \int_{\Omega_{\beta,\Lambda_+(1,1)}} \int_{\Omega_{\beta,\Lambda_-(1,1)}} \exp\Big(-\frac{J}{2} \sum_{\langle \ell,\ell' \rangle \in E_1^{(1)}} \|x_\ell - x_{\ell'}\|^2_{L^2_\beta} \Big) \\
& \times \mu^-_{\Lambda_+(1,1)}(\mathrm{d}x_{\Lambda_+(1,1)}) \mu^-_{\Lambda_-(1,1)}(\mathrm{d}x_{\Lambda_-(1,1)}) \\
= & \, Z_\Lambda(T^+_{1,1} b) Z_\Lambda(T^-_{1,1} b).
\end{aligned} \tag{6.2.41}$$

In the same way we estimate both $Z_\Lambda(T^\pm_{1,1} b)$, employing this time $E_2^{(1)}$ and $T^o_{1,2}$. Repeating this procedure an appropriate number of times we finally arrive at

$$[Z_\Lambda(b)]^{2^{dL}} \leq \prod_{o_{1,1},\ldots,o_{d,L}=\pm 1} Z_\Lambda\big(T^{o_{d,L}}_{d,L} \ldots T^{o_{1,1}}_{1,1} b\big) = [Z_\Lambda(0)]^{2^{dL}}. \tag{6.2.42}$$

Here we have taken into account that $T^{o_{d,L}}_{d,L} \ldots T^{o_{1,1}}_{1,1} b = 0$ for any b and any sequence $o_{1,1}, \ldots, o_{d,L} = \pm 1$, which readily follows from (6.2.30) and (6.2.36). □

6.2.3 Infrared Bound

Now we apply Theorem 6.2.7 to obtain the function $\widehat{B}$ which gives the bound (6.1.21). Here we again assume that the model is rotation-invariant.

Let (Λ, E) be the graph introduced in the previous subsection. Let also $\mathcal{X}_E$ be the real Hilbert space

$$\mathcal{X}_E = \{ b = (b_{\ell\ell'})_{\langle \ell,\ell' \rangle \in E} \mid b_{\ell\ell'} \in L^2_\beta \}, \tag{6.2.43}$$

equipped with the scalar product

$$(b, c)_{\mathcal{X}_E} = \sum_{\langle \ell,\ell' \rangle \in E} (b_{\ell\ell'}, c_{\ell\ell'})_{L^2_\beta}. \tag{6.2.44}$$

To simplify notation we shall denote elements of E by e, i.e., $e = \langle \ell, \ell' \rangle$ is the edge which connects the vertices ℓ and ℓ'.

A bounded linear operator $Q : \mathcal{X}_E \to \mathcal{X}_E$ can be defined by means of its kernel $Q^{jj'}_{ee'}(\tau, \tau')$, $j, j' = 1, \ldots, \nu$, $e, e' \in E$, and $\tau, \tau' \in [0, \beta]$. That is,

$$(Qb)^{(j)}_e(\tau) = \sum_{j'=1}^{d} \sum_{e' \in E} \int_0^\beta Q^{jj'}_{ee'}(\tau, \tau') b^{(j')}_{e'}(\tau') \mathrm{d}\tau'. \tag{6.2.45}$$

The object of our study in this subsection is the operator defined by the kernel

$$Q^{jj'}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(\tau,\tau') = \left\langle \left[x^{(j)}_{\ell_1}(\tau) - x^{(j)}_{\ell_1'}(\tau)\right]\cdot\left[x^{(j')}_{\ell_2}(\tau') - x^{(j')}_{\ell_2'}(\tau')\right]\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}}, \quad (6.2.46)$$

where the expectation is taken with respect to the measure (6.2.22), which we suppose here to be rotation-invariant as well. This operator is positive. Indeed,

$$(b, Qb)_{\mathcal{X}_E} = \left\langle \Big[\sum_{\langle \ell,\ell'\rangle\in E} (x_\ell - x_{\ell'}, b_{\ell\ell'})_{L^2_\beta}\Big]^2\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}} \geq 0.$$

The kernel (6.2.46) can be expressed in terms of the periodic Matsubara functions; thus, as a function of τ,τ', it has the periodicity property (1.2.90) and hence can be written in the form (1.3.113), (1.3.115). Let us obtain such a representation. By the periodicity (1.2.90) and rotation invariance,

$$\langle \hat{x}^{(j)}_\ell(k)\hat{x}^{(j')}_{\ell'}(k')\rangle_{\nu^{\rm per}_{\beta,\Lambda}} = 0 \quad \text{if } k \neq k', \text{ and } j \neq j',$$

where $\hat{x}^{(j)}_\ell(k)$ is the same as in (2.1.17). Taking this into account we employ in (6.2.46) the Fourier transformation (2.1.17) and obtain

$$Q^{jj'}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(\tau,\tau') = \delta_{jj'}\sum_{k\in\mathcal{K}} \widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k)e_k(\tau)e_k(\tau'), \quad (6.2.47)$$

with

$$\widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k) = \left\langle \left[\hat{x}^{(j)}_{\ell_1}(k) - \hat{x}^{(j)}_{\ell_1'}(k)\right]\cdot\left[\hat{x}^{(j)}_{\ell_2}(k) - \hat{x}^{(j)}_{\ell_2'}(k)\right]\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}}. \quad (6.2.48)$$

In view of the periodic conditions imposed on the boundaries of the box Λ, the latter kernel, as well as the one given by (6.2.46), are invariant with respect to the translations of the corresponding torus, see Subsection 1.4.3 for more details. This allows us to 'diagonalize' the kernel (6.2.48) by means of a spatial Fourier transformation (6.1.14), (6.1.15). Then the spatial periodicity of the state $\langle\cdot\rangle_{\nu^{\rm per}_{\beta,\Lambda}}$ yields

$$\langle \hat{x}^{(j)}(p,k)\hat{x}^{(j)}(p',k)\rangle_{\nu^{\rm per}_{\beta,\Lambda}} = 0 \quad \text{if } p + p' \neq 0. \quad (6.2.49)$$

Taking this into account we obtain

$$\begin{aligned}\widehat{Q}_{\langle \ell_1,\ell_1'\rangle\langle \ell_2,\ell_2'\rangle}(k) = \sum_{p\in\Lambda_*} &\langle \hat{x}^{(j)}(p,k)\hat{x}^{(j)}(-p,k)\rangle_{\nu^{\rm per}_{\beta,\Lambda}} \\ &\times \left(e^{{\rm i}(p,\ell_1)} - e^{{\rm i}(p,\ell_1')}\right)/|\Lambda|^{1/2} \\ &\times \left(e^{-{\rm i}(p,\ell_2)} - e^{{\rm i}(-p,\ell_2')}\right)/|\Lambda|^{1/2}.\end{aligned} \quad (6.2.50)$$

Since the summand corresponding to $p = 0$ equals zero, the sum can be restricted to $\Lambda_* \setminus \{0\}$. This representation, however, cannot serve as a spectral decomposition, similar to (6.2.47), because the eigenfunctions here are not normalized. Indeed,

$$\sum_{\langle \ell,\ell' \rangle \in E} \left(e^{\mathrm{i}(p,\ell)} - e^{\mathrm{i}(p,\ell')}\right)/|\Lambda|^{1/2} \times \left(e^{-\mathrm{i}(p,\ell)} - e^{-\mathrm{i}(p,\ell')}\right)/|\Lambda|^{1/2} = 2\mathcal{E}(p),$$

where

$$\mathcal{E}(p) \stackrel{\mathrm{def}}{=} \sum_{j=1}^{d} [1 - \cos p_j]. \tag{6.2.51}$$

Then we set

$$\sigma_{\ell\ell'}(p) = \left(e^{\mathrm{i}(p,\ell)} - e^{\mathrm{i}(p,\ell')}\right)/\sqrt{2|\Lambda|\mathcal{E}(p)}, \quad p \in \Lambda_* \setminus \{0\}, \tag{6.2.52}$$

and

$$\widehat{Q}(p,k) = 2\mathcal{E}(p)\langle \hat{x}^{(j)}(p,k)\hat{x}^{(j)}(-p,k)\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}}, \quad p \in \Lambda_* \setminus \{0\}. \tag{6.2.53}$$

Thereby,

$$\begin{aligned} &Q_{\langle \ell_1,\ell_1' \rangle \langle \ell_2,\ell_2' \rangle}(\tau,\tau') \\ &\quad = \sum_{p \in \Lambda_* \setminus \{0\}} \sum_{k \in \mathcal{K}} \widehat{Q}(p,k)\sigma_{\ell_1\ell_1'}(p)\sigma_{\ell_2\ell_2'}(-p)e_k(\tau)e_k(\tau'), \end{aligned} \tag{6.2.54}$$

which is the spectral decomposition of the operator (6.2.46). Now we show that the eigenvalues (6.2.53) have a specific upper bound[1].

Theorem 6.2.8. *For every $p \in \Lambda_* \setminus \{0\}$ and $k \in \mathcal{K}$, the eigenvalues* (6.2.53) *obey the estimate*

$$\widehat{Q}(p,k) \leq 1/J, \tag{6.2.55}$$

where J is the same as in (6.2.22). *From this estimate one gets*

$$\langle \hat{x}^{(j)}(p,k)\hat{x}^{(j)}(-p,k)\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}} \leq \frac{1}{2J\mathcal{E}(p)}, \quad p \in \Lambda_* \setminus \{0\}. \tag{6.2.56}$$

Proof. The estimate in question will be obtained from the Gaussian domination (6.2.26). For $t \in \mathbb{R}$ and a given $b \in \mathcal{X}_E$, we consider the function $\phi(t) = Z_\Lambda(tb)$. By Theorem 6.2.7 it has a maximum at $t = 0$; hence, $\phi''(0) \leq 0$. Computing the derivative from (6.2.25) we get

$$\phi''(0) = J(b, Qb)_{\mathcal{X}_E} - \|b\|^2_{\mathcal{X}_E},$$

where the operator Q is defined by its kernel (6.2.46). Then the estimate (6.2.55) is immediate. □

[1]Their natural lower bound is zero as the operator (6.2.46) is positive.

By (1.3.111), (6.2.47), and (6.2.53), we readily obtain

$$\langle(\hat{x}_p(\tau), \hat{x}_{-p}(\tau'))\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}} = \frac{\nu}{2\beta\mathcal{E}(p)} \sum_{k\in\mathcal{K}} \widehat{Q}(p,k)\cos[k(\tau-\tau')], \quad p \neq 0,$$

which yields, see (6.1.16), (6.1.17), and (6.2.55),

$$\widehat{D}_p^{\Lambda} = \frac{\beta\nu}{2\mathcal{E}(p)}\widehat{Q}(p,0) \leq \frac{\beta\nu}{2J\mathcal{E}(p)}, \quad p \neq 0. \tag{6.2.57}$$

Comparing this estimate with (6.1.21) we have the following

Corollary 6.2.9. *If the model (1.1.3), (1.1.8) is translation and rotation-invariant with ferromagnetic interaction of nearest neighbor type, then the infrared estimate (6.1.21) for this model holds with*

$$\widehat{B}(p) = \frac{\beta\nu}{2J\mathcal{E}(p)}, \quad p \in (-\pi,\pi]^d \setminus \{0\}. \tag{6.2.58}$$

Proposition 6.2.10. *For $d \geq 3$, $B_{\ell\ell'}$ defined by (6.1.22) with $\widehat{B}(p)$ given by (6.2.58) has the property: $B_{\ell\ell'} \to 0$ as $|\ell-\ell'| \to +\infty$.*

Proof. Though the validity of this statement follows from a standard fact (the Riemann–Lebesgue lemma, see page 116 in [209]), we give its direct proof here as the same method will also be used below. For $p = (p_1, \bar{p})$, $\bar{p} = (p_2, \ldots, p_d) \in (-\pi,\pi]^{d-1}$, we set

$$a(\bar{p}) = 1 + \sum_{j=2}^{d}(1-\cos p_j). \tag{6.2.59}$$

For $s \in \mathbb{N}$, we also set

$$\begin{aligned} \phi(s,\bar{p}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos p_1 s}{a(\bar{p}) - \cos p_1} \mathrm{d}p_1, \\ b(s) &= \int_{(-\pi,\pi]^{d-1}} \phi(s,\bar{p})\mathrm{d}\bar{p}. \end{aligned} \tag{6.2.60}$$

Thereby, the property in question will be proven if we show that the sequence $\{s^{d-2}b(s)\}_{s\in\mathbb{N}}$ is bounded. To get an explicit form for $\phi(s,\bar{p})$ we use the following known formula

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\lambda l}\mathrm{d}\lambda}{a-\cos\lambda} &= \frac{1}{\sqrt{a^2-1}}\exp(-\gamma|l|), \\ \gamma &= \log(a+\sqrt{a^2-1}), \end{aligned} \tag{6.2.61}$$

which holds for all $a > 1$ and $l \in \mathbb{Z}$. Then

$$\phi(s,\bar{p}) = \frac{1}{\sqrt{[a(\bar{p})]^2-1}}\exp(-\alpha(\bar{p})s), \tag{6.2.62}$$

where

$$\alpha(\bar{p}) = \log\left[a(\bar{p}) + \sqrt{[a(\bar{p})]^2 - 1}\right].$$

By (6.2.59) for $\bar{p} \in (-\pi, \pi]^{d-1}$, one can show that

$$a \cdot |\bar{p}| \leq \sqrt{[a(\bar{p})]^2 - 1}, \qquad \alpha \cdot |\bar{p}| \leq \alpha(\bar{p})$$

with certain positive a and α. Here $|\bar{p}|$ is the Euclidean norm. Then by (6.2.62),

$$\begin{aligned} s^{d-2} b(s) &\leq s^{d-2} \int_{(-\pi,\pi]^{d-1}} \frac{1}{a|\bar{p}|} \exp\left(-\alpha|\bar{p}|s\right) \mathrm{d}\bar{p} \\ &\leq s^{d-2} \int_{\mathbb{R}^{d-1}} \frac{1}{a|\bar{p}|} \exp\left(-\alpha|\bar{p}|s\right) \mathrm{d}\bar{p} \\ &\leq c_d \alpha^{2-d} a^{-1} \int_0^{+\infty} t^{d-3} e^{-t} \mathrm{d}t = c_d \alpha^{2-d} a^{-1} \Gamma(d-2), \end{aligned}$$

where c_d is the Lebesgue measure of the unit sphere in $\mathbb{R}^{d-1}$. □

6.3 Examples of Phase Transitions

In this section, we apply the technique developed above to prove phase transitions in concrete versions of the model (1.1.3), (1.1.8). The version obeying the conditions of Corollary 6.2.9 has phase transition if the corresponding $D^{\Lambda}_{\ell\ell}$ obeys (6.1.26) with

$$\vartheta > \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\beta\nu}{2J\mathcal{E}(p)} \mathrm{d}p. \tag{6.3.1}$$

Extensions of this result to the ferromagnetic models which are not translation-invariant and/or have interactions of general type will be made by means of correlation inequalities, proven in Chapter 2. However, as most of them are valid only for $\nu = 1$ and for special types of the anharmonic potentials, our results in this direction will be restricted to the scalar models with symmetric V_ℓ. In Subsection 6.3.3, we prove that the scalar models with asymmetric anharmonic potentials have first-order phase transitions.

6.3.1 Phase Transitions in Rotation Invariant Models

In this subsection, we prove the existence of phase transitions in the model described by Corollary 6.2.9. For ϕ^4-type anharmonic potentials,

$$V(u) = -b|u|^2 + b_2|u|^4, \quad b > a/2, \; b_2 > 0, \tag{6.3.2}$$

where a is as in (1.1.3), the bound (6.1.26) will be found explicitly. We begin by considering this special case.

Lemma 6.3.1. *Let V be as in (6.3.2). Then for every $\Lambda \in \mathfrak{L}_{\rm box}$,*

$$K^{\Lambda}_{\ell\ell}(\tau,\tau|p) \geq \frac{(2b-a)\nu}{4b_2(\nu+2)} \stackrel{\rm def}{=} \vartheta_*. \tag{6.3.3}$$

Proof. Let A be an operator, affiliated with the algebra $\mathfrak{C}_\Lambda$, see Definition 1.2.13, and let A^* be its adjoint. Then, cf. (1.2.55), we have

$$\begin{aligned}
&\varrho^{\rm per}_{\beta,\Lambda}\left([A^*,[H^{\rm per}_\Lambda, A]]\right)\\
&\quad = \varrho^{\rm per}_{\beta,\Lambda}\left(A^* H^{\rm per}_\Lambda A + A H^{\rm per}_\Lambda A^* - A^* A H^{\rm per}_\Lambda - H^{\rm per}_\Lambda A A^*\right)\\
&\quad = \frac{1}{Z^{\rm per}_{\beta,\Lambda}} \sum_{s,s'\in\mathbb{N}} \left|A^*_{ss'}\right|^2 \left(E^{\rm per}_{s'} - E^{\rm per}_{s}\right)\left\{\exp\left[-\beta E^{\rm per}_{s}\right] - \exp\left[-\beta E^{\rm per}_{s'}\right]\right\}\\
&\quad \geq 0.
\end{aligned} \tag{6.3.4}$$

Here $E^{\rm per}_s$, $s\in\mathbb{N}$, are the eigenvalues of the periodic Hamiltonian (1.4.51), $A_{ss'}$ are the corresponding matrix elements, $\varrho^{\rm per}_{\beta,\Lambda}$ is the periodic local Gibbs state which was used in Subsection 1.4.3 to construct the measure $\nu^{\rm per}_{\beta,\Lambda}$. By the Euclidean representation (1.4.20)

$$K^{\Lambda}_{\ell\ell}(\tau,\tau|p) = \sum_{j=1}^{\nu}\left\langle\left(x^{(j)}_\ell(0)\right)^2\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}} = \sum_{j=1}^{\nu}\varrho^{\rm per}_{\beta,\Lambda}\left[\left(q^{(j)}_\ell\right)^2\right].$$

Now we set in (6.3.4) $A = A^* = p^{(j)}_\ell$, $j = 1,\ldots,\nu$, make use of the commutation relation (1.1.6), and take into account the rotation invariance. Following this way we arrive at

$$\begin{aligned}
\varrho^{\rm per}_{\beta,\Lambda}\left([A^*,[H^{\rm per}_\Lambda, A]]\right) &= \varrho^{\rm per}_{\beta,\Lambda}\left(-2b + a + 2b_2|q_\ell|^2 + 4b_2(q^{(j)}_\ell)^2\right)\\
&= -2b + a + 4b_2(\nu+2)\left\langle\left[x^{(j)}_\ell(0)\right]^2\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}}\\
&\geq 0,
\end{aligned} \tag{6.3.5}$$

which yields (6.3.3). ☐

Now let us turn to the general case where V is just rotation-invariant, which in particular means that the external field h equals zero. To indicate the dependence of the pressure on the interaction intensity we write $p^{\rm per}_\Lambda(J)$. Take any $\varepsilon > 0$. Then the pressure has the representation (5.2.17) (where one sets $h = 0$), by which we get, cf. (5.2.18),

$$p^{\rm per}_\Lambda(J) - p^{\rm per}_\Lambda(0) \geq \beta\nu J d\varepsilon^2 + \log\gamma(m). \tag{6.3.6}$$

Clearly, $p^{\rm per}_\Lambda(J)$ is convex and differentiable; its derivative can be computed from (5.2.17). Then

$$\begin{aligned}
\frac{J}{|\Lambda|}\sum_{\langle\ell,\ell'\rangle\in E}\left\langle(x_\ell,x_{\ell'})_{L^2_\beta}\right\rangle_{\nu^{\rm per}_{\beta,\Lambda}} &= J\frac{\partial}{\partial J}p^{\rm per}_\Lambda(J)\\
&\geq p^{\rm per}_\Lambda(J) - p^{\rm per}_\Lambda(0),
\end{aligned} \tag{6.3.7}$$

where E is the same as in (6.2.43). By the first GKS inequality, the left-hand side is positive. Then the Cauchy–Schwarz inequality and the translation invariance of $\nu_{\beta,\Lambda}^{\mathrm{per}}$ yield

$$\left\langle (x_\ell, x_{\ell'})_{L^2_\beta} \right\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}} \leq \left\langle (x_\ell, x_\ell)_{L^2_\beta} \right\rangle_{\nu_{\beta,\Lambda}^{\mathrm{per}}} = \beta K_{\ell\ell}^{\Lambda}(\tau, \tau|p).$$

Combining the latter estimate with (6.3.6), (6.3.7), we get the following

Lemma 6.3.2. *For every $\theta > 0$, there exist positive m_* and J_*, which may depend on β, θ, and on the potential V, such that for $m > m_*$ and $J > J_*$,*

$$K_{\ell\ell}^{\Lambda}(\tau, \tau|p) \geq \theta. \tag{6.3.8}$$

Proof. As was just shown,

$$K_{\ell\ell}^{\Lambda}(\tau, \tau|p) \geq \nu\varepsilon^2 + \log\gamma(m)/\beta J d,$$

where $\gamma(m)$ may depend on ε, see (5.2.14). Given $\theta > 0$, one picks $\varepsilon > \sqrt{\theta/\nu}$, which yields m_*, see (5.2.15). For $m > m_*$, $\gamma(m) > 0$; hence, one can find J_* such that the right-hand side of the latter estimate equals θ for $J = J_*$. □

To proceed further we need the function $f : [0, +\infty) \to [0, 1)$ defined implicitly by

$$f(u \tanh u) = u^{-1} \tanh u, \quad \text{for } u > 0 \quad \text{and} \quad f(0) = 1. \tag{6.3.9}$$

It is differentiable, convex, and monotone decreasing on $(0, +\infty)$, such that $tf(t) \to 1$. For $t \geq 6$, $f(t) \approx 1/t$ to five-place accuracy, see Theorem A.2 in [109]. By direct calculation,

$$\frac{f'(u\tau)}{f(u\tau)} = -\frac{1}{u\tau} \cdot \frac{\tau - u(1-\tau^2)}{\tau + u(1-\tau^2)}, \qquad \tau = \tanh u. \tag{6.3.10}$$

Proposition 6.3.3. *For every fixed $\alpha > 0$, the function*

$$\phi(t) = t\alpha f(t/\alpha), \quad t > 0, \tag{6.3.11}$$

is differentiable and monotone increasing to α^2 as $t \to +\infty$.

Proof. By (6.3.10),

$$\phi'(t) = \frac{1}{u\tau} \cdot \frac{2u(1-\tau^2)}{\tau + u(1-\tau^2)} > 0, \qquad u\tau = u \tanh u = t/\alpha.$$

The limit α^2 is obtained from the corresponding asymptotic property of f. □

Next, we need the following fact, known as Inequality of Bruch and Falk, see e.g., Theorem IV.7.5 in [277]. In view of its importance, we present a complete proof here.

Proposition 6.3.4. *Let A and A^* be as in (6.3.4). Let also*

$$b(A) = \beta^{-1} \int_0^\beta \varrho^{\mathrm{per}}_{\beta,\Lambda} \left(A^* \exp[-\tau H^{\mathrm{per}}_\Lambda] A \exp[\tau H^{\mathrm{per}}_\Lambda] \right) \mathrm{d}\tau,$$
$$g(A) = \varrho^{\mathrm{per}}_{\beta,\Lambda}(A^*A + AA^*)/2, \quad c(A) = \varrho^{\mathrm{per}}_{\beta,\Lambda}\left([A^*, [\beta H^{\mathrm{per}}_\Lambda, A]]\right).$$

Then

$$b(A) \geq g(A) f\left(\frac{c(A)}{4g(A)}\right), \tag{6.3.12}$$

where f is given by (6.3.9).

Proof. Set

$$\varphi(\tau) = \varrho^{\mathrm{per}}_{\beta,\Lambda} \left(A^* \exp[-\tau H^{\mathrm{per}}_\Lambda] A \exp[\tau H^{\mathrm{per}}_\Lambda] \right), \quad \tau \in [0, \beta]. \tag{6.3.13}$$

Clearly,

$$g(A) = [\varphi(0) + \varphi(\beta)]/2,$$

and, see (6.3.4),

$$\begin{aligned} \varphi(\tau) &= \frac{1}{Z^{\mathrm{per}}_{\beta,\Lambda}} \sum_{s,s' \in \mathbb{N}} \left|A^*_{ss'}\right|^2 \exp\left[-\tau\left(E^{\mathrm{per}}_{s'} - E^{\mathrm{per}}_s\right)\right] \exp\left(-\beta E^{\mathrm{per}}_s\right) \\ &\stackrel{\mathrm{def}}{=} \sum_{n=1}^\infty a_n e^{-\lambda_n \tau}, \quad \lambda_{n+1} > \lambda_n > \cdots > \lambda_1 \geq 0, \;\; a_n > 0. \end{aligned} \tag{6.3.14}$$

Thereby,

$$b(A) = \beta^{-1} \int_0^\beta \varphi(\tau)\mathrm{d}\tau, \quad c(A) = \beta\left[\varphi'(\beta) - \varphi'(0)\right], \tag{6.3.15}$$

and

$$g(A) = [\varphi(0) + \varphi(\beta)]/2 = \sum_{n=1}^\infty a_n(1 + e^{-\lambda_n \beta})/2 \stackrel{\mathrm{def}}{=} \sum_{n=1}^\infty b_n. \tag{6.3.16}$$

Taking into account the latter representation we rewrite (6.3.15) as

$$\begin{aligned} b(A) &= \sum_{n=1}^\infty b_n \cdot \left(\frac{2}{\beta\lambda_n} \tanh \frac{\beta\lambda_n}{2}\right), \\ c(A) &= 4\sum_{n=1}^\infty b_n \cdot \left(\frac{\beta\lambda_n}{2} \tanh \frac{\beta\lambda_n}{2}\right). \end{aligned} \tag{6.3.17}$$

By (6.3.16),

$$\sum_{n=1}^\infty \theta_n = 1, \quad \text{where } \theta_n \stackrel{\mathrm{def}}{=} b_n/g(A).$$

Therefrom,

$$\begin{aligned}
\mathrm{RHS}(6.3.12) &= g(A) f\left(\sum_{n=1}^{\infty} \theta_n \cdot \left(\frac{\beta\lambda_n}{2} \tanh \frac{\beta\lambda_n}{2}\right)\right) \\
&\leq g(A) \sum_{n=1}^{\infty} \theta_n \cdot f\left(\frac{\beta\lambda_n}{2} \tanh \frac{\beta\lambda_n}{2}\right) \\
&= \sum_{n=1}^{\infty} b_n \cdot \left(\frac{2}{\beta\lambda_n} \tanh \frac{\beta\lambda_n}{2}\right) \\
&= b(A).
\end{aligned}$$

Here we have used the convexity of f and the definition (6.3.9). □

Set

$$\mathcal{J}(d) = \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\mathrm{d}p}{\mathcal{E}(p)}, \tag{6.3.18}$$

where $\mathcal{E}(p)$ is given by (6.2.51). The exact value of $\mathcal{J}(3)$ can be expressed in terms of complete elliptic integrals, see [318] and also [169] for more recent developments. For $d \geq 4$, the following property was proven in [106] as Theorem 5.1.

Proposition 6.3.5. *For $d \geq 4$, one has*

$$\frac{1}{d - 1/2} < \mathcal{J}(d) < \frac{1}{d - \alpha(d)} < \frac{1}{d-1}, \tag{6.3.19}$$

where $\alpha(d) \to 1/2$ as $d \to +\infty$.

Recall that m is the reduced particle mass (1.1.7).

Theorem 6.3.6. *Let $d \geq 3$, the interaction be of nearest neighbor type, and the anharmonic potential be of the form (6.3.2), which determines the parameter ϑ_*. Let also the following condition be satisfied:*

$$8m\vartheta_*^2 J > \mathcal{J}(d). \tag{6.3.20}$$

Then for every $\beta > \beta_*$, where β_* is the unique solution of the equation

$$2\beta J \vartheta_* f(\beta/4m\vartheta_*) = \mathcal{J}(d), \tag{6.3.21}$$

the model has a phase transition in the sense of Definition 6.1.1.

Proof. One observes that

$$[q_\ell^{(j)}, [H_\Lambda^{\mathrm{per}}, q_\ell^{(j)}]] = 1/m, \quad \ell \in \Lambda. \tag{6.3.22}$$

Now we employ (6.3.12) with $A = q_\ell^{(j)}$ and obtain

$$b(A) = \beta^{-2}\Big\langle \Big(f_\ell^{(j)} \Big)^2 \Big\rangle_{\nu_{\beta,\Lambda}^{\rm per}} \geq \Big\langle \Big(x_\ell^{(j)}(0) \Big)^2 \Big\rangle_{\nu_{\beta,\Lambda}^{\rm per}} f\left(\frac{\beta}{4m\big\langle (x_\ell^{(j)}(0))^2 \big\rangle_{\nu_{\beta,\Lambda}^{\rm per}}} \right),$$

where $f_\ell^{(j)}$ is as in (6.1.1) and (6.1.11). By (6.3.9), $\vartheta f(\beta/4m\vartheta)$ is an increasing function of ϑ. Thus, by (6.3.3) and (6.1.11),

$$D_{\ell\ell}^{\Lambda} \geq \beta^2 \nu \vartheta_* f(\beta/4m\vartheta_*), \tag{6.3.23}$$

which yields the bound (6.1.26). Thereby, the condition (6.3.1) takes the form

$$\vartheta_* f\ (\beta/4m\vartheta_*) > \mathcal{J}(d)/2\beta J. \tag{6.3.24}$$

By Proposition 6.3.3, the function

$$\phi(\beta) = 2\beta J \vartheta_* f(\beta/4m\vartheta_*)$$

is monotone increasing and hits the level $\mathcal{J}(d)$ at certain β_*. For $\beta > \beta_*$, the estimate (6.3.24) holds, which yields $|\mathcal{G}_\beta^{\rm t}| > 1$. □

We remark that $f(\beta/4m\vartheta_*) \to 1$ as $m \to +\infty$. In this limit, the condition (6.3.20) turns into the corresponding condition for a classical model of ϕ^4 anharmonic oscillators, which is in agreement with the conclusions of Chapter 4. Now let us turn to a more general case.

Theorem 6.3.7. *Let $d \geq 3$, the interaction be of nearest neighbor type, and the anharmonic potential be rotation-invariant. Then for every $\beta > 0$, there exist positive m_* and J_*, which may depend on β and on the anharmonic potential, such that $|\mathcal{G}_\beta^{\rm t}| > 1$ for $m > m_*$ and $J > J_*$.*

Proof. Given positive β and θ, one has the bound (6.3.8) for big enough m and J. Then one applies Proposition 6.3.4, which yields that (6.3.1) is satisfied if

$$\theta f(\beta/4m\theta) > \mathcal{J}(d)/2\beta J.$$

Then one sets m_* to be as in (5.2.15) and J_* to be the smallest value of J for which both (6.3.8) and the latter inequality hold. □

6.3.2 Phase Transitions in Symmetric Scalar Models

For $\nu = 1$, we can extend the above results to much more general $J_{\ell\ell'}$ and V_ℓ; however, certain assumptions beyond Assumption 1.1.1 should be made.

As the basic element of our technique is the reflection positivity method, we still suppose that $\mathbb{L} = \mathbb{Z}^d$. At the same time, the model need not be translation-invariant.

Suppose also that the interaction between the nearest neighbors is uniformly nonzero, that is,

$$\inf_{|\ell-\ell'|=1} J_{\ell\ell'} \stackrel{\mathrm{def}}{=} J > 0. \tag{6.3.25}$$

Next we suppose that all V_ℓ's are even continuous functions and the upper bound in (1.1.10) can be chosen to obey the following conditions:

(a) for every ℓ,

$$V(u_\ell) - V_\ell(u_\ell) \le V(\tilde{u}_\ell) - V_\ell(\tilde{u}_\ell), \quad \text{whenever } u_\ell^2 \le \tilde{u}_\ell^2; \tag{6.3.26}$$

(b) the function V has the form

$$V(u_\ell) = \sum_{s=1}^{r} b^{(s)} u_\ell^{2s}, \quad 2b^{(1)} < -a;\ b^{(s)} \ge 0,\ s \ge 2, \tag{6.3.27}$$

where a is as in (1.1.3) and $r \ge 2$ is either positive integer or infinite;

(c) if $r = +\infty$, the series

$$\Phi(\vartheta) = \sum_{s=2}^{+\infty} \frac{(2s)!}{2^{s-1}(s-1)!} b^{(s)} \vartheta^{s-1}, \tag{6.3.28}$$

converges at some $\vartheta > 0$.

Since $2b^{(1)} + a < 0$, the equation

$$a + 2b^{(1)} + \Phi(\vartheta) = 0, \tag{6.3.29}$$

has a unique solution $\vartheta_* > 0$. By the above assumptions all V_ℓ are 'uniformly double-welled'. If $V_\ell(u_\ell) = v_\ell(u_\ell^2)$ and v_ℓ are differentiable, the condition (6.3.26) can be formulated as an upper bound for v'_ℓ. Note that the pressure as a unified characteristic of all Euclidean Gibbs states makes sense for translation-invariant models only. Thus, Definition 6.1.6 is not applicable to the versions of models that do not have this property.

The main result of this subsection is given by the following statement.

Theorem 6.3.8. *Let the model be as just described. Let also the condition* (6.3.20) *with ϑ_* defined by the equation* (6.3.29) *and J defined by* (6.3.25) *be satisfied. Then for every $\beta > \beta_*$, where β_* is defined by the equation* (6.3.21)*, the model has a phase transition in the sense of Definition* 6.1.1*. If the model is translation-invariant, the long-range order and the first-order phase transition take place at such β.*

Proof. The proof will be done by comparing the model considered with the reference model which is the scalar model with the nearest neighbor interaction of intensity (6.3.25) and with the anharmonic potential (6.3.27). In view of (6.3.26), the reference model is more stable; hence, the phase transition in this model implies the same for the model considered, which will be shown by means of the correlation inequalities. The reference model is defined by its local periodic Hamiltonians

$$H_\Lambda^{\rm low} = \sum_{\ell\in\Lambda}\left[H_\ell^{\rm har} + V(q_\ell)\right] - J\sum_{\langle\ell,\ell'\rangle\in E} q_\ell q_{\ell'}, \tag{6.3.30}$$

where Λ is a box, for which E is as in (6.2.43), (6.2.44), and $H_\ell^{\rm har}$ is the same as in (1.1.3). The local Hamiltonians corresponding to arbitrary $\Lambda \in \mathfrak{L}_{\rm fin}$ are defined in the usual way. For the reference model, we have the infrared estimate (6.2.57) with $\nu = 1$. Let us obtain the lower bound, see (6.3.3). To this end we use the inequalities (6.3.4) and (6.3.5), which yields

$$\begin{aligned} 0 &\leq a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)b^{(s)}\left\langle [x_\ell(0)]^{2(s-1)}\right\rangle_{\nu_{\beta,\Lambda}^{\rm low}} \\ &\leq a + 2b^{(1)} + \sum_{s=2}^{r} 2s(2s-1)\frac{(2s-2)!}{2^{s-1}(s-1)!}\cdot b^{(s)}\left[\left\langle (x_\ell(0))^2\right\rangle_{\nu_{\beta,\Lambda}^{\rm low}}\right]^{s-1}. \end{aligned} \tag{6.3.31}$$

Here $\nu_{\beta,\Lambda}^{\rm low}$ is the periodic Gibbs measure for the model (6.3.30). To get the second line we used the Gaussian upper bound inequality (2.2.8), which is possible since all $b^{(s)}$, $s \geq 2$ are nonnegative. The solution of the latter inequality is

$$\left\langle (x_\ell(0))^2\right\rangle_{\nu_{\beta,\Lambda}^{\rm low}} \geq \vartheta_*.$$

Then the proof of the phase transitions in the model (6.3.30) goes along the line of arguments used in Theorem 6.3.6. Thus, for $\beta > \beta_*$, $\langle x_\ell(0)\rangle_{\mu_+^{\rm low}} > 0$, see Corollary 5.3.2. Let $\pi_{\beta,\Lambda}^{\rm low}$, $\pi_{\beta,\Lambda}$, $\Lambda \in \mathfrak{L}_{\rm fin}$, be the kernels (3.1.44) for the model (6.3.30) and the model in question, respectively. Then by Theorem 2.5.11 for every ℓ, ℓ_0,

$$\left\langle x_\ell(0)\right\rangle_{\pi_{\beta,\Lambda}^{\rm low}(\cdot|\xi^{\ell_0})} \leq \left\langle x_\ell(0)\right\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi^{\ell_0})},$$

where ξ^{ℓ_0} is given by (3.7.12). We pass here to the limit $\Lambda \nearrow \mathbb{L}$ and obtain that for any ℓ,

$$\left\langle x_\ell(0)\right\rangle_{\mu_+} \geq \left\langle x_\ell(0)\right\rangle_{\mu_+^{\rm low}} > 0,$$

which yields that $|\mathcal{G}_\beta^{\rm t}| > 1$, see Corollary 5.3.2. If the model is translation-invariant, by (2.5.40) we get

$$P_\Lambda \geq P_\Lambda^{\rm low}.$$

Then the long-range order and the first-order phase transition follow. □

6.3.3 Phase Transition in Asymmetric Scalar Models

The phase transitions proven so far have a common feature – the spontaneous symmetry breaking. This means that the symmetry, e.g., rotation invariance, proper to the model and hence to the measure $\mu \in \mathcal{G}^{\rm t}_\beta$ if $|\mathcal{G}^{\rm t}_\beta| = 1$, is no longer possessed by the multiple Gibbs measures appearing as the result of the phase transition. In this subsection, we show that the translation-invariant scalar ferromagnetic version of the model (1.1.3), (1.1.8) has a phase transition without symmetry breaking. However, we restrict ourselves to the case of first-order phase transitions, see Definition 6.1.6. The reason for this restriction can be explained as follows. The fact that $D^\mu_{\ell\ell'}$, given by (6.1.2), does not decay to zero as $|\ell - \ell'| \to +\infty$, see (6.1.4), implies that μ is non-ergodic only if μ is symmetric. Otherwise, to show that μ is non-ergodic one should prove that the difference $D^\mu_{\ell\ell'} - \langle f_\ell\rangle_\mu \cdot \langle f_{\ell'}\rangle_\mu$ does not decay to zero, which cannot be done by means of our methods based on the infrared estimate.

In what follows, the only conditions imposed on the anharmonic potential are those from Assumption 1.1.1. Obviously, we have to include the external field, that is, the anharmonic potential is now $V(u)-hu$. Since we are not going to impose any conditions on the odd part of V, here we cannot apply the GKS inequalities (2.2.3), (2.2.4), the comparison methods are based on. In view of this fact, we suppose that the interaction is of nearest neighbor type. Thus, for a box Λ, the periodic local Hamiltonian of the model has the form (6.3.30).

In accordance with Definition 6.1.6, our goal is to show that the model parameters (except for h) and the inverse temperature β can be chosen in such a way that the set $\mathcal{R}$, defined by (5.2.6), is non-void. The main idea on how to do this can be explained as follows. First we find a condition, independent of h, under which $D^\mu_{\ell\ell'}$ does not decay to zero for a certain periodic μ. By Theorem 5.2.2, for fixed values of the parameters mentioned above, there exist $h_\pm$, $h_- < h_+$, such that the magnetization (5.2.7) has the property: $M(h) > 0$ for $h > h_+$ and $M(h) < 0$ for $h < h_-$. Then, if $\mathcal{R}$ were void, one would find $h_* \in (h_-, h_+)$ such that $M(h_*) = 0$. At such h_*, the aforementioned property of D^μ would yield the non-ergodicity of μ and hence the first-order phase transition, see Theorem 6.3.8. The realization of this scheme is based on the following

Lemma 6.3.9. *For every $\beta > 0$ and ϑ_*, there exist positive m_* and J_*, which may depend on $\beta > 0$ and ϑ_* but are independent of h, such that, for any box Λ and any $h \in \mathbb{R}$,*

$$\left\langle [x_\ell(0)]^2 \right\rangle_{\nu^{\rm per}_{\beta,\Lambda}} \geq \vartheta_*, \quad \textit{if } J > J_* \textit{ and } m > m_*. \tag{6.3.32}$$

Proof. For a fixed h, we set

$$\begin{aligned} \sigma^h_\beta(\mathrm{d}x) &= \frac{1}{N^h_\beta} \exp\left(h\int_0^\beta x(\tau)\mathrm{d}\tau\right)\sigma_\beta(\mathrm{d}x), \\ N^h_\beta &= \int_{C_\beta} \exp\left(h\int_0^\beta x(\tau)\mathrm{d}\tau\right)\sigma_\beta(\mathrm{d}x), \end{aligned} \tag{6.3.33}$$

where σ_β is as in (5.2.8). We also set

$$\sigma^h_{\beta,\Lambda}(\mathrm{d}x_\Lambda) = \bigotimes_{\ell\in\Lambda} \sigma^h_\beta(\mathrm{d}x_\ell), \quad \Lambda \in \mathfrak{L}_{\rm fin}.$$

Then the local pressure corresponding to the Hamiltonian (6.3.30) by (5.2.17) can be written

$$p_\Lambda^{\rm per}(J,h) = p_\Lambda^{\rm per}(0,h) + \frac{1}{|\Lambda|}\log\big\langle \exp(Y_\Lambda)\big\rangle_{\sigma^h_{\beta,\Lambda}}. \tag{6.3.34}$$

It is clearly a convex function of J for every fixed h. Thus, by (5.2.16), cf. (6.3.7),

$$\begin{aligned} J\frac{\partial}{\partial J} p_\Lambda^{\rm per}(J,h) &= \frac{J}{|\Lambda|}\sum_{\langle \ell,\ell'\rangle\in E}\int_0^\beta \big\langle x_\ell(\tau)x_{\ell'}(\tau)\big\rangle_{\nu^{\rm per}_{\beta,\Lambda}}\mathrm{d}\tau \\ &\geq p_\Lambda^{\rm per}(J,h) - p_\Lambda^{\rm per}(0,h). \end{aligned} \tag{6.3.35}$$

As in (6.1.20), one can show that $\widehat{K}^\Lambda_p(\tau,\tau|p)\geq 0$; hence, cf. (6.1.19),

$$\big\langle x_\ell(\tau)x_{\ell'}(\tau)\big\rangle_{\nu^{\rm per}_{\beta,\Lambda}} = K^\Lambda_{\ell\ell'}(\tau,\tau|p) \leq K^\Lambda_{\ell\ell}(\tau,\tau|p) = \big\langle [x_\ell(0)]^2\big\rangle_{\nu^{\rm per}_{\beta,\Lambda}}.$$

Thereby, (6.3.35) yields

$$\big\langle [x_\ell(0)]^2\big\rangle_{\nu^{\rm per}_{\beta,\Lambda}} \geq \big[p_\Lambda^{\rm per}(J,h) - p_\Lambda^{\rm per}(0,h)\big]/\beta J d. \tag{6.3.36}$$

Let n, α, c, ε, and p be as in (5.2.9)–(5.2.14) with $\nu = 1$. Then, for $\pm h\geq 0$, from (6.3.34) we get, cf. (5.2.18),

$$p_\Lambda^{\rm per}(J,h) - p_\Lambda^{\rm per}(0,h) \geq \beta J d\varepsilon^2 + \log\sigma^h_\beta\big[B^\pm(\varepsilon,c)\big]. \tag{6.3.37}$$

Let us show now that for $\pm h\geq 0$,

$$\sigma^h_\beta\big[B^\pm(\varepsilon,c)\big] \geq \sigma_\beta\big[B^\pm(\varepsilon,c)\big]. \tag{6.3.38}$$

For $h\geq 0$, let $I(x)$ be the indicator function of the set $C^+_\beta(n;c)$, see (5.2.12). For $\delta>0$ and $t\in\mathbb{R}$, we set

$$\iota_\delta(t) = \begin{cases} 0 & t\leq c, \\ (t-c)/\delta & t\in(c,c+\delta], \\ 1 & c\geq c+\delta, \end{cases}$$

and

$$I_\delta(x) \stackrel{\rm def}{=} \prod_{k=0}^{n}\iota_\delta[x(k\beta/n)].$$

By Lebesgue's dominated convergence theorem,

$$\begin{aligned} N^h_\beta\sigma^h_\beta[C^+_\beta(n;c)] &= \int_{C_\beta} I(x)\exp\left(h\int_0^\beta x(\tau)\mathrm{d}\tau\right)\sigma_\beta(\mathrm{d}x) \\ &= \lim_{\delta\downarrow 0}\int_{C_\beta} I_\delta(x)\exp\left(h\int_0^\beta x(\tau)\mathrm{d}\tau\right)\sigma_\beta(\mathrm{d}x). \end{aligned} \tag{6.3.39}$$

As the function ι_δ is continuous and increasing, by the FKG inequality, see Theorem 2.2.1, it follows that

$$\int_{C_\beta} I_\delta(x) \exp\left(h \int_0^\beta x(\tau)\mathrm{d}\tau\right)\sigma_\beta(\mathrm{d}x) \geq N_\beta^h \int_{C_\beta} I_\delta(x)\sigma_\beta(\mathrm{d}x).$$

Passing here to the limit we obtain from (6.3.39)

$$\sigma_\beta^h\Big[C_\beta^+(n;c)\Big] \geq \sigma_\beta\Big[C_\beta^+(n;c)\Big],$$

which obviously yields (6.3.38). For $h \leq 0$, one just changes the signs of h and x. Now let m_* be as in (5.2.15). If we fix β and the parameters that appear in (5.2.14), then for $m > m_*$ given by (5.2.15), the right-hand side of (5.2.14) gets positive. Therefore, we pick any positive $\vartheta_* < \varepsilon^2$ and obtain (6.3.32) from (6.3.36)–(6.3.38) with, cf. (5.2.14),

$$J_* = (\varepsilon^2 - \vartheta_*)\left[\beta d \log\left(\Sigma(n;c) - \frac{D_V(\alpha, 1, p)}{[mn(c-\varepsilon)^2]}\right)\right]^{-1}. \tag{6.3.40}$$

□

Now we can prove the main statement of this subsection.

Theorem 6.3.10. *Let the model be scalar, translation-invariant, and with the nearest-neighbor ferromagnetic interaction. Let also $d \geq 3$. Then for every β, there exist $m_* > 0$ and $J_* > 0$ such that, for all $m > m_*$ and $J > J_*$, there exists $h_* \in \mathbb{R}$, possibly dependent on m, β, and J, such that $p'(h)$ gets discontinuous at h_*, i.e., the model has a first-order phase transition.*

Proof. Let m_* be as in (5.2.15) and J_*, ϑ_* be as in Lemma 6.3.9. Fix any $\beta > 0$ and $m > m_*$. Then, for $J > J_*$, the estimate (6.3.32) holds, which yields the validity of (6.3.23) for all boxes Λ with such β, m, and $\nu = 1$. Thereby, we increase J, if necessary, up to the value at which (6.3.24) holds. Afterwards, all the parameters, except for h, are set fixed. In this case, there exists a periodic state $\mu \in \mathcal{G}_\beta^{\mathrm{t}}$ such that the first summand in (6.1.25) is positive; hence, $D_{\ell\ell'}^\mu$ does not decay to zero as $|\ell - \ell'| \to +\infty$, see Proposition 6.2.10. If $p(h)$ is everywhere differentiable, i.e., if $\mathcal{R} = \emptyset$, then by Theorem 5.2.2 there should exist h_* such that $M(h_*) = 0$; hence, the state μ with such h_* is non-ergodic, which yields $|\mathcal{G}_\beta^{\mathrm{t}}| > 1$ and hence a first-order phase transition. Otherwise, $\mathcal{R} \neq \emptyset$. □

6.4 Critical Point of a Hierarchical Model

According to Definition 6.1.11, the critical point of the model corresponds to an abnormal normalization in (6.1.45), i.e., the one with $\alpha > 0$. In general, to find such an α and to prove (6.1.47) is a very hard problem, even for classical models. However, there exists a class of models, for which such an α is given from the very beginning. These are the so-called *hierarchical models*, where the underlying set $\mathbb{L}$ is equipped with a special metric. The first hierarchical model was introduced by F. J. Dyson in [108].

6.4.1 The Model and the Main Result

The model we study in this section is a version of the general model (1.1.8) with $\nu = 1$ and $\mathbb{L} = \mathbb{N}_0$. The interaction intensities $J_{\ell\ell'}$ are defined by means of a certain metric on $\mathbb{L}$, due to which they possess a specific symmetry, reflected in the name of the model.

Given $\varkappa \in \mathbb{N} \setminus \{1\}$, we set

$$\Lambda_{n,s} = \{\ell \in \mathbb{N}_0 \mid \varkappa^n s \leq \ell \leq \varkappa^n (s+1) - 1\}, \quad s, n \in \mathbb{N}_0. \tag{6.4.1}$$

Then, for $n \in \mathbb{N}$, one has

$$\Lambda_{n,s} = \bigcup_{\ell \in \Lambda_{k,s}} \Lambda_{n-k,\ell}, \quad k = 1, 2, \ldots, n. \tag{6.4.2}$$

The collection of families $\{\Lambda_{n,s}\}_{s \in \mathbb{N}_0}$, $n \in \mathbb{N}_0$, is called a *hierarchical structure.* Given $\ell, \ell' \in \mathbb{L}$, we set

$$n(\ell, \ell') = \min\{n \mid \exists \Lambda_{n,s} \colon \ \ell, \ell' \in \Lambda_{n,s}\}, \quad d(\ell, \ell') = \varkappa^{n(\ell,\ell')} - 1. \tag{6.4.3}$$

Then any triple $\{\ell_1, \ell_2, \ell_3\} \subset \mathbb{L}$ contains two elements, say ℓ_1, ℓ_2, such that $d(\ell_1, \ell_3) = d(\ell_2, \ell_3)$. Thus, $d(\ell, \ell')$ is a metric on $\mathbb{L}$. In our model, the interaction potential is

$$J_{\ell\ell'} = J \left[d(\ell, \ell') + 1 \right]^{-1-\delta}, \quad J, \delta > 0. \tag{6.4.4}$$

In order to employ the symmetry imposed by the hierarchical structure, as finite subsets of $\mathbb{L}$ we shall use the sets (6.4.2) only. A standard choice is the sequence of $\Lambda_{n,0} \stackrel{\text{def}}{=} \Lambda_n$, $n \in \mathbb{N}_0$. The formal Hamiltonian (1.1.8) of the model with the interaction potential (6.4.4) can be written also as

$$H = -\frac{\theta}{2} \sum_{n=0}^{\infty} \varkappa^{-n(1+\delta)} \sum_{\ell} \Big(\sum_{\ell' \in \Lambda_{n,\ell}} q_{\ell'} \Big)^2 + \sum_{\ell} H_\ell, \tag{6.4.5}$$

where $\theta = J(1 - \varkappa^{-(1+\delta)}) > 0$ and H_ℓ is as in (1.1.3). The anharmonic potential will be taken in the form of (6.3.2), (2.2.23). Due to this choice, we shall be able to use inequalities like (2.2.25). One more simplification can be made if we set

$$\theta = \varkappa^\delta - 1, \tag{6.4.6}$$

which affects the scale of β only and hence can be done without any loss of generality.

The main peculiarity of the model (6.4.5) is that it is invariant with respect to the transformations of $\mathbb{L}$ which preserve the hierarchical structure. Therefore, we define the local Hamiltonians for sets Λ having the form (6.4.2) only by setting

$$\begin{aligned} H_{n,\ell} &= -\frac{\theta}{2} \sum_{k=0}^{n} \varkappa^{-n(1+\delta)} \Big(\sum_{\ell' \in \Lambda_{n,\ell}} q_{\ell'} \Big)^2 + \sum_{\ell} H_\ell \\ &= -\frac{1}{2} \theta \varkappa^{-n(1+\delta)} \Big(\sum_{\ell' \in \Lambda_{n,\ell}} q_{\ell'} \Big)^2 + \sum_{\ell' \in \Lambda_{1,\ell}} H_{n-1,\ell'}, \end{aligned} \tag{6.4.7}$$

with the initial element $H_{0,\ell} = H_\ell$. For these Hamiltonians, we introduce the local Euclidean Gibbs measures $\nu_{n,\ell}$, see (1.4.8), (1.4.18), employing the following energy functions:

$$\begin{aligned} E_{n,\ell}(x_{\Lambda_{n,\ell}}) = &-\frac{1}{2}\theta\varkappa^{-n(1+\delta)}\int_0^\beta \Big(\sum_{\ell'\in\Lambda_{n,\ell}} x_{\ell'}(\tau)\Big)^2 \mathrm{d}\tau \\ &+ \sum_{\ell'\in\Lambda_{1,\ell}} E_{n-1,\ell'}(x_{\Lambda_{n-1,\ell'}}), \end{aligned} \tag{6.4.8}$$

with

$$E_{0,\ell} = \int_0^\beta \big(-b[x_\ell(\tau)]^2 + b_2[x_\ell(\tau)]^4\big)\,\mathrm{d}\tau. \tag{6.4.9}$$

According to Definition 6.1.11, we consider

$$P_{n,\ell}^{(\alpha)} = \frac{\beta^{-2}}{|\Lambda_{n,\ell}|^{1+\alpha}}\int_{\Omega_{\beta,\Lambda}}\Big[\sum_{\ell'\in\Lambda_{n,\ell}}\int_0^\beta x_{\ell'}(\tau)\mathrm{d}\tau\Big]^2 \nu_{n,\ell}(\mathrm{d}\tau). \tag{6.4.10}$$

We also set $P_n^{(\alpha)} = P_{n,0}^{(\alpha)}$. Then the model has a critical point if there exists $\alpha \in (0,1)$ such that

$$\lim_{n\to+\infty} P_n^{(\alpha)} \stackrel{\mathrm{def}}{=} P^{(\alpha)} > 0. \tag{6.4.11}$$

Thus, the main object of our study will be the *fluctuation operator*

$$Q_{n,\ell}^{(\alpha)} = \frac{1}{|\Lambda_{n,\ell}|^{(1+\alpha)/2}}\sum_{\ell'\in\Lambda_{n,\ell}} q_{\ell'} = \frac{1}{\varkappa^{n(1+\alpha)/2}}\sum_{\ell'\in\Lambda_{n,\ell}} q_{\ell'}, \quad \alpha\in(0,1), \tag{6.4.12}$$

which is characterized by an abnormal normalization. An evident candidate for the index α is the parameter δ which describes the decay of the interaction (6.4.4). To simplify notation we write

$$Q_{n,\ell} \stackrel{\mathrm{def}}{=} Q_{n,\ell}^{(\delta)}, \tag{6.4.13}$$

and the corresponding Matsubara functions (1.2.84), (1.4.20),

$$\begin{aligned} &\Gamma^{\beta,\Lambda_{n,\ell}}_{Q^{(\alpha)}_{n,\ell},\dots,Q^{(\alpha)}_{n,\ell}}(\tau_1,\dots,\tau_{2k}) = \Gamma^{(\alpha,n)}_{2k}(\tau_1,\dots,\tau_{2k}), \quad k\in\mathbb{N}, \\ &\Gamma^{\beta,\Lambda_{n,\ell}}_{Q_{n,\ell},\dots,Q_{n,\ell}}(\tau_1,\dots,\tau_{2k}) = \Gamma^{(n)}_{2k}(\tau_1,\dots,\tau_{2k}), \end{aligned} \tag{6.4.14}$$

where we have taken into account that all such functions with the same n but different ℓ coincide by the hierarchical symmetry. Thereby,

$$P_n \stackrel{\mathrm{def}}{=} P_n^{(\delta)} = \beta^{-1}\int_0^\beta \Gamma_2^{(n)}(\tau,\tau')\mathrm{d}\tau', \tag{6.4.15}$$

which is independent of τ by (1.2.90). Now we can formulate the main result of this section.

Theorem 6.4.1. *For the model (6.4.5) one can choose the parameters m, $b_1 \stackrel{\text{def}}{=} b - a/2$, and b_2 in such a way that there will exist $\beta_* > 0$, dependent on b_1, b_2, m, with the following properties:* (a) *if $\beta = \beta_*$, then for all $k \in \mathbb{N}$, the convergence*

$$\Gamma_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}) \to \Gamma_{2k}^{\infty}(\tau_1, \ldots, \tau_{2k}) \stackrel{\text{def}}{=} \frac{(2k)!}{k!2^k \beta_*^k} \tag{6.4.16}$$

holds uniformly on $(\tau_1, \ldots, \tau_{2k}) \in [0, \beta]^{2k}$; (b) *if $\beta < \beta_*$, for all $\alpha > 0$ the functions $\Gamma_{2k}^{(\alpha,n)}$, $k \in \mathbb{N}$, converge to zero in the same sense.*

Remark 6.4.2. In fact, for $\beta < \beta_*$ one can prove that the sequences of Matsubara functions (6.4.14) corresponding to the operators (6.4.12) with $\alpha = 0$ (i.e., to the normal fluctuations) are bounded. As stated above, for $\beta = \beta_*$, the limiting Matsubara functions are Gaussian, cf. (2.2.7), and independent of τ_j, which means that the critical fluctuations are Gaussian and non-quantum, cf. Theorem 4.2.1, like in quantum hierarchical spin models [219].

Remark 6.4.3. The construction of tempered Euclidean Gibbs measures for the model (6.4.5) is beyond the technical abilities of the theory developed above, see the discussion between the definition of the weights (3.1.25) and Assumption 3.1.9 (see p. 198). Therefore, the only aspect of the theory of this model related to phase transitions we are able to establish is the one provided by Theorem 6.4.1. Note that the definition of the critical point, as well as the definition of the first- and second-order phase transitions, do not involve the set $\mathcal{G}_\beta^{\mathrm{t}}$.

The proof of Theorem 6.4.1 will be performed in several steps. First we transform the problem into the problem of controlling two particular sequences of numbers. Then we make a preliminary investigation of these sequences. Afterwards, we formulate a number of lemmas, by means of which we then prove the theorem. Later we prove the lemmas.

6.4.2 Preliminary Statements

To prove our theorem we only need the Matsubara functions corresponding to the operators $Q_{n,\ell}$. Thus, we shall study the measures describing distributions of $Q_{n,\ell}$ given by (6.4.12), (6.4.13). By (6.4.8), such measures $\mu_{n,\ell} = \mu_n$ are defined by the recursion relation[2]

$$\mu_n(\mathrm{d}x) = \frac{1}{Z_n} \exp\left(\frac{\theta}{2}\|x\|_{L_\beta^2}^2\right) \mu_{n-1}^{\star\varkappa}(\varkappa^{(1+\delta)/2}\mathrm{d}x), \tag{6.4.17}$$

$$\mu_0(\mathrm{d}x) = \frac{1}{Z_0} \exp\left(-E_{0,s}(x)\right) \chi_\beta(\mathrm{d}x), \tag{6.4.18}$$

[2] Transformations of measures of this kind were studied in [184].

where the function $E_{0,s}$ is given by (6.4.9), Z_n, $n \in \mathbb{N}$, are normalizing constants, and $\star$ stands for convolution. In order to simplify notation, we drop the labels ℓ and s. Next we set, cf. (2.2.10),

$$\begin{aligned} f_n(x) &= \int_{L^2_\beta} \exp\left((x,\xi)_{L^2_\beta}\right)\mu_n(\mathrm{d}\xi) \\ &= \int_{C_\beta} \exp\left((x,\xi)_{L^2_\beta}\right)\mu_n(\mathrm{d}\xi), \quad x \in C_\beta. \end{aligned} \tag{6.4.19}$$

Expanding its logarithm into the series (2.2.15) we obtain the Ursell functions, cf. (2.2.16),

$$U_{2k}^{(n)}(\tau_1,\ldots,\tau_{2k}) \stackrel{\text{def}}{=} U_{2k}^{\delta,\beta,\Lambda_{n,s}}(\tau_1,\ldots,\tau_{2k}), \quad k \in \mathbb{N}. \tag{6.4.20}$$

Each function $U_{2k}^{(n)}$ can be written as a polynomial of the Matsubara functions $\Gamma_{2s}^{(n)}$, $s = 1,2,\ldots,k$ and vice versa. In particular,

$$\begin{aligned} U_2^{(n)}(\tau_1,\tau_2) &= \Gamma_2^{(n)}(\tau_1,\tau_2), \\ U_4^{(n)}(\tau_1,\ldots,\tau_4) &= \Gamma_4^{(n)}(\tau_1,\ldots,\tau_4) - \Gamma_2^{(n)}(\tau_1,\tau_2)\Gamma_2^{(n)}(\tau_3,\tau_4) \\ &\quad - \Gamma_2^{(n)}(\tau_1,\tau_3)\Gamma_2^{(n)}(\tau_2,\tau_4) - \Gamma_2^{(n)}(\tau_1,\tau_4)\Gamma_2^{(n)}(\tau_2,\tau_3). \end{aligned} \tag{6.4.21}$$

Obviously, for every $\alpha > 0$ and $k \in \mathbb{N}$,

$$\begin{aligned} U_{2k}^{\alpha,\beta,\Lambda_{n,s}}(\tau_1,\ldots,\tau_{2k}) &= |\Lambda_{n,s}|^{\delta-\alpha} U_{2k}^{(n)}(\tau_1,\ldots,\tau_{2k}) \\ &= \varkappa^{n(\delta-\alpha)} U_{2k}^{(n)}(\tau_1,\ldots,\tau_{2k}). \end{aligned} \tag{6.4.22}$$

Note that the Ursell functions just introduced obey the sign rule (2.2.25).

In view of (1.2.90), the Matsubara and Ursell functions depend only on the periodic distances $|\tau_i - \tau_j|_\beta = \min\{|\tau_i - \tau_j|, \beta - |\tau_i - \tau_j|\}$. By Theorems 2.5.14 and 2.2.11, we have the following property.

Lemma 6.4.4. *For all $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, the following estimates hold for all values of $\tau, \tau', \tau_1, \ldots, \tau_{2k} \in [0,\beta]$:*

$$\int_{\mathcal{I}^2_\beta} U_4^{(n)}(\tau,\tau,\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2 \leq \int_{\mathcal{I}^2_\beta} U_4^{(n)}(\tau,\tau',\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2, \tag{6.4.23}$$

$$(-1)^{k-1} U_{2l}^{(n)}(\tau_1,\tau_2,\ldots,\tau_{2k}) \geq 0. \tag{6.4.24}$$

Set

$$\begin{aligned} \hat{u}_n(k) &= \int_0^\beta U_2^{(n)}(\tau',\tau)\cos(k\tau)\mathrm{d}\tau \\ &= \int_0^\beta U_2^{(n)}(0,\tau)\cos(k\tau)\mathrm{d}\tau, \quad k \in \mathcal{K},\ n \in \mathbb{N}_0, \end{aligned} \tag{6.4.25}$$

where $\mathcal{K}$ is the same as in (1.3.20). Then

$$U_2^{(n)}(\tau_1, \tau_2) = \frac{1}{\beta} \sum_{k \in \mathcal{K}} \hat{u}_n(k) \cos[k(\tau_1 - \tau_2)]. \tag{6.4.26}$$

Furthermore, we set

$$\mathrm{u}_{2k}^{(n)} = \int_0^\beta \ldots \int_0^\beta U_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{2k}. \tag{6.4.27}$$

Then

$$\hat{u}_n \stackrel{\text{def}}{=} \hat{u}_n(0) = \beta^{-1} \mathrm{u}_2^{(n)}. \tag{6.4.28}$$

Lemma 6.4.5. *For every $n \in \mathbb{N}_0$ and $k \in \mathcal{K}$, $\hat{u}_n(k)$ is a continuous function of β, obeying the estimates*

$$0 \le \hat{u}_n(k) \le \hat{u}_n; \tag{6.4.29}$$

$$\hat{u}_n(k) \le \varkappa^{-n\delta}/mk^2, \quad k \neq 0. \tag{6.4.30}$$

Proof. By (6.4.25), (6.4.21), (6.4.14), (1.2.84), and (1.2.50) one obtains

$$U_2^{(n)}(0, \tau) = \frac{1}{Z_{n,\ell}} \mathrm{trace} \left\{ Q_{n,\ell} \exp\left[-\tau H_{n,\ell}\right] Q_{n,\ell} \exp\left[-(\beta - \tau) H_{n,\ell}\right] \right\}.$$

Let $E_s^{(n)}$, $s \in \mathbb{N}$, be the eigenvalues of $H_{n,\ell}$, see (1.2.54). Let also $\psi_s^{(n)}$ be the corresponding eigenfunctions and set

$$Q_{ss'}^{(n)} = (\psi_s^{(n)}, Q_{n,\ell} \psi_{s'}^{(n)})_{\mathcal{H}_{\Lambda_{n,\ell}}}.$$

Then the above representation of $U_2^{(n)}$ can be rewritten

$$U_2^{(n)}(0, \tau) = \frac{1}{Z_{n,\ell}} \sum_{s,s' \in \mathbb{N}} \left| Q_{ss'}^{(n)} \right|^2 \exp\left[-\beta E_s^{(n)} + \tau(E_s^{(n)} - E_{s'}^{(n)})\right],$$

which by (6.4.25) yields

$$\begin{aligned} \hat{u}_n(k) &= \frac{1}{Z_{n,\ell}} \sum_{s,s' \in \mathbb{N}} \left| Q_{ss'}^{(n)} \right|^2 \frac{E_s^{(n)} - E_{s'}^{(n)}}{k^2 + (E_s^{(n)} - E_{s'}^{(n)})^2} \\ &\quad \times \left(\exp[-\beta E_{s'}^{(n)}] - \exp[-\beta E_s^{(n)}] \right), \\ Z_{n,\ell} &= \sum_{s \in \mathbb{N}} \exp[-\beta E_s^{(n)}]. \end{aligned} \tag{6.4.31}$$

Both above series converge uniformly, as functions of β, on compact subsets of $(0, +\infty)$, which yields continuity and positivity. The upper bound (6.4.29) follows from (6.4.31) or from (6.4.25). To prove (6.4.30) we estimate the denominator in (6.4.31) by $k^2 \neq 0$ and obtain

$$\begin{aligned} \hat{u}_n(k) \leq & \frac{1}{k^2} \frac{1}{Z_{n,\ell}} \sum_{s,s'} \left| Q_{ss'}^{(n)} \right|^2 (E_s^{(n)} - E_{s'}^{(n)}) \\ & \times \left(\exp[-\beta E_{s'}^{(n)}] - \exp[-\beta E_s^{(n)}] \right) \\ = & \frac{1}{k^2} \frac{1}{Z_{n,\ell}} \mathrm{trace} \left\{ \left[Q_{n,\ell}, \left[H_{n,\ell}, Q_{n,\ell} \right] \right] \exp\left(-\beta H_{n,\ell} \right) \right\}. \end{aligned} \tag{6.4.32}$$

The double commutator is equal to $|\Lambda_{n,\ell}|^{-\delta}/m$, which yields (6.4.30). □

The main idea of our method is that the convergence stated in the theorem will be obtained by controlling the sequence $\{\hat{u}_n\}_{n \in \mathbb{N}_0}$ and 'something else'. It turns out that this 'something else' is just one more sequence of numbers. This significant feature of our model comes from the Lee–Yang property of the measures $\nu_{n,\ell}$ and hence of μ_n, $n \in \mathbb{N}_0$, which they possess since the anharmonic potential (6.3.2) with any b and $b_2 > 0$ meets the conditions of Proposition 2.4.4. Therefore, the functions

$$g_n(\zeta) = f_n(\zeta x), \quad \zeta \in \mathbb{C},\ n \in \mathbb{N}_0,$$

where f_n is given by (6.4.19), can be represented in the following way, see Theorem 2.4.6 and (2.4.1):

$$g_n(\zeta) = \prod_{j=1}^{\infty} (1 + c_j^{(n)} \zeta^2), \quad c_1^{(n)} \geq c_2^{(n)} \geq \cdots > 0, \quad \sum_{j=1}^{\infty} c_j^{(n)} < \infty. \tag{6.4.33}$$

Therefrom, the parameters (6.4.27) are

$$\mathrm{u}_{2s}^{(n)} = 2(2s-1)!(-1)^{s-1} \sum_{j=1}^{\infty} \left[c_j^{(n)} \right]^s; \tag{6.4.34}$$

hence,

$$\begin{aligned} |\mathrm{u}_{2s}^{(n)}| & \leq 2(2s-1)! \left[c_1^{(n)} \right]^{s-2} \sum_{j=1}^{\infty} \left[c_j^{(n)} \right]^2, \quad s \geq 2, \\ |\mathrm{u}_{2s}^{(n)}| & \leq (2s-1)! \left[c_1^{(n)} \right]^{s-1} \mathrm{u}_2^{(n)}, \quad s \in \mathbb{N}, \end{aligned} \tag{6.4.35}$$

and

$$\begin{aligned} |\mathrm{u}_{2s}^{(n)}| & \leq (2s-1)!(2^{1-s}/3) \left[\mathrm{u}_2^{(n)} \right]^{s-2} |\mathrm{u}_4^{(n)}|, \quad s \geq 2, \\ |\mathrm{u}_{2s}^{(n)}| & \leq (2s-1)! 2^{1-s} \left[\mathrm{u}_2^{(n)} \right]^s, \quad s \in \mathbb{N}. \end{aligned} \tag{6.4.36}$$

We have thus proven the following statement.

Lemma 6.4.6. *The parameters* (6.4.27) *obey the estimates, see* (6.4.28)*,*

$$|\mathrm{u}_{2s}^{(n)}| \leq (2s-1)!2^{1-s}\,[\beta\hat{u}_n]^s\,, \quad s \in \mathbb{N}; \tag{6.4.37}$$

$$|\mathrm{u}_{2s}^{(n)}| \leq \frac{(2s-1)!}{3\cdot 2^{s-1}}\,[\beta\hat{u}_n]^{s-2}\cdot|\mathrm{u}_4^{(n)}|, \quad s \geq 2. \tag{6.4.38}$$

From this we see that all the sequences $\{\mathrm{u}_{2s}^{(n)}\}_{n\in\mathbb{N}_0}$, $s \in \mathbb{N}$, are controlled by just two of them: $\{\mathrm{u}_2^{(n)}\}$ and $\{\mathrm{u}_4^{(n)}\}$.

6.4.3 Proof of Theorem 6.4.1

For $n \in \mathbb{N}_0$, we set

$$\lambda_n = -\int_0^\beta\int_0^\beta U_4^{(n)}(\tau,\tau,\tau_1,\tau_2)\mathrm{d}\tau_1\mathrm{d}\tau_2. \tag{6.4.39}$$

Then by Theorem 2.5.14 and the sign rule (2.2.25), one has

$$0 < \beta^{-2}|\mathrm{u}_4^{(n)}| \leq \lambda_n, \quad \text{for all } n \in \mathbb{N}_0, \tag{6.4.40}$$

thus, we can control the sequence $\{\mathrm{u}_4^{(n)}\}$ by controlling $\{\lambda_n\}$. We can now state the main lemma for the proof of Theorem 6.4.1.

Lemma 6.4.7. *For the model* (6.4.5) *with* $\delta \in (0, 1/2)$*, one can choose the parameters* b_1*,* b_2*, and* m *in such a way that there will exist* $\beta_* > 0$*, dependent on* b_1*,* b_2*, and* m *only, with the following properties:* (a) *for* $\beta \leq \beta_*$, $\lim_{n\to+\infty}\lambda_n = 0$*;* (b) *for* $\beta = \beta_*$*,* $\lim_{n\to+\infty}\hat{u}_n = 1$*;* (c) *for* $\beta < \beta_*$*, there exists* $K(\beta) > 0$ *such that, for all* $n \in \mathbb{N}_0$*,*

$$\hat{u}_n \leq K(\beta)\varkappa^{-n\delta}. \tag{6.4.41}$$

The proof of this lemma will be given in Subsection 6.4.4 below. Lemmas 6.4.5 and 6.4.7 have two important corollaries.

Corollary 6.4.8. *For every* $\beta \leq \beta_*$ *and* $s \in \mathbb{N}$*, the sequences* $\{\Gamma_{2s}^{(n)}\}_{n\in\mathbb{N}_0}$*,* $\{U_{2s}^{(n)}\}_{n\in\mathbb{N}_0}$ *are relatively compact in the topology of uniform convergence on* $[0,\beta]^{2s}$.

Proof. Since the Ursell function $U_{2s}^{(n)}$ can be expressed as a polynomial of $\Gamma_{2k}^{(n)}$ with $k = 1,\ldots,s$ and vice versa, it is enough to prove this statement for the Matsubara functions only. By the Arzela–Ascoli theorem (see e.g., [221], p. 72) we have to show that the sequence $\{\Gamma_{2s}^{(n)}\}_{n\in\mathbb{N}_0}$ is point-wise bounded and equicontinuous. By (6.4.30) and (6.4.26),

$$\Gamma_2^{(n)}(\tau,\tau') \leq \Gamma_2^{(n)}(0,0) \leq \frac{1}{\beta}\hat{u}_n(0) + \frac{\varkappa^{-n\delta}}{\beta m}\sum_{k\in\mathcal{K}\setminus\{0\}}\frac{1}{k^2}. \tag{6.4.42}$$

For $\beta \le \beta_*$, the sequence $\{\hat{u}_n(0)\}_{n\in\mathbb{N}_0}$ is bounded by Lemma 6.4.7. Together with the Gaussian upper bound (2.2.8), this yields the uniform boundedness of $\Gamma_{2s}^{(n)}$ on $[0,\beta]^{2s}$. Furthermore,

$$\begin{aligned}
&\Gamma_{2s}^{(n)}(\tau_1,\ldots,\tau_{2s}) - \Gamma_{2s}^{(n)}(\vartheta_1,\ldots,\vartheta_{2s}) \\
&\quad = \int_{C_\beta}\sum_{k=1}^{2s} x(\tau_1)\ldots x(\tau_{k-1})\,[x(\tau_k)-x(\vartheta_k)]\,x(\vartheta_{k+1})\ldots x(\vartheta_{2s})\mu_n(\mathrm{d}x).
\end{aligned} \tag{6.4.43}$$

Applying here the Cauchy–Schwarz inequality (as to the scalar product in $L^2(C_\beta,\mu_n)$ of $[x(\tau_k)-x(\vartheta_k)]$ and the rest of x's), the Gaussian upper bound (2.2.8) and the left-hand inequality in (6.4.42), one gets

$$\begin{aligned}
&|\Gamma_{2s}^{(n)}(\tau_1,\ldots,\tau_{2s}) - \Gamma_{2s}^{(n)}(\vartheta_1,\ldots,\vartheta_{2s})|^2 \\
&\quad \le \left(\Gamma_2^{(n)}(0,0)-\Gamma_2^{(n)}(\tau,\vartheta)\right)\cdot\frac{8s^2(4s-2)!}{(2s-1)!2^{2s-1}}\left(\Gamma_2^{(n)}(0,0)\right)^{2s-1},
\end{aligned} \tag{6.4.44}$$

where (τ,ϑ) is chosen amongst the pairs (τ_k,ϑ_k), $k=1,\ldots,2s$, such that $|\tau-\vartheta|_\beta = \max_k|\tau_k-\vartheta_k|_\beta$. But by (6.4.26) and (6.4.30),

$$\begin{aligned}
\Gamma_2^{(n)}(0,0)-\Gamma_2^{(n)}(\tau,\vartheta) &= \frac{2}{\beta}\sum_{k\in\mathcal{K}}\hat{u}_n(k)\,\{\sin[(k/2)\,(\tau-\vartheta)]\}^2 \\
&\le \frac{2\varkappa^{-n\delta}}{\beta m}\sum_{k\in\mathcal{K}\setminus\{0\}}\frac{1}{k^2}\{\sin[(k/2)\,(\tau-\vartheta)]\}^2 \\
&\le C\varkappa^{-n\delta}|\tau-\vartheta|_\beta,
\end{aligned}$$

with an appropriate $C>0$. □

The next fact follows directly from (6.4.30) and (1.3.20).

Corollary 6.4.9. *For every* β,

$$\sum_{k\in\mathcal{K}\setminus\{0\}}\hat{u}_n(k)\to 0,\quad n\to+\infty.$$

Proof of Theorem 6.4.1. By Lemma 6.4.7, (6.4.38) and (6.4.39), (6.4.40), one obtains that for all $s\ge 2$ and $\beta\le\beta_*$, $\lim_{n\to+\infty}\mathrm{u}_{2s}^{(n)}=0$. Then by the sign rule (2.2.25), for all $s\ge 2$, the sequences $\{U_{2s}^{(n)}\}_{n\in\mathbb{N}_0}$ converge to zero for almost all $(\tau_1,\ldots,\tau_{2s})\in[0,\beta]^{2s}$, which, by Corollary 6.4.8, yields their uniform convergence to zero. By (6.4.26)–(6.4.30), Corollary 6.4.9, and Lemma 6.4.7, one has for $\beta=\beta_*$,

$$U_2^{(n)}(\tau_1,\tau_2)=\frac{1}{\beta}\hat{u}_n(0)+\frac{1}{\beta}\sum_{k\in\mathcal{K}\setminus\{0\}}\hat{u}_n(k)\cos[k(\tau_1-\tau_2)]\to 1/\beta, \tag{6.4.45}$$

uniformly with respect to τ_1, τ_2. Now one can express each $\Gamma_{2s}^{(n)}$ polynomially by $U_{2k}^{(n)}$ with $k = 1, \ldots, s$ and obtain the convergence (6.4.16) for $\beta = \beta_*$. For $\beta < \beta_*$, we have the estimate (6.4.41), which yields, cf. (6.4.42) and (6.4.22),

$$\begin{aligned} \Gamma_2^{\alpha,\beta,\Lambda_{n,s}}(\tau_1, \tau_2) &\leq \Gamma_2^{\alpha,\beta,\Lambda_{n,s}}(0,0) \\ &\leq \frac{\varkappa^{-n\alpha}}{\beta}\Big[K(\beta) + \frac{1}{m}\sum_{k\in\mathcal{K}\setminus\{0\}}\frac{1}{k^2}\Big]; \end{aligned} \tag{6.4.46}$$

hence, $\Gamma_2^{\alpha,\beta,\Lambda_{n,s}}(\tau_1, \tau_2) \to 0$ as $n \to +\infty$, uniformly on $[0,\beta]^2$. The convergence of the Matsubara functions $\Gamma_{2k}^{\alpha,\beta,\Lambda_{n,s}}$ with $k \geq 2$ follows from the Gaussian upper bound (2.2.8). $\square$

6.4.4 Proof of Lemma 6.4.7

In the proof of Lemma 6.4.7, an important role is played by a statement about differential equations, which will serve us also in the subsequent section. We derive it here based on another fact, known in the theory of ordinary differential equations.

For a domain $\mathcal{D} \subset \mathbb{R}^2$, let $f : \mathcal{D} \to \mathbb{R}$ be a continuous function. Suppose $(0, u_0) \in \mathcal{D}$. Then one can consider the initial value problem

$$\dot{u} = f(t,u), \quad u(0) = u_0, \quad \dot{u} \stackrel{\text{def}}{=} \mathrm{d}u/\mathrm{d}t. \tag{6.4.47}$$

For an appropriate function v, its defect D_v with respect to the equation (6.4.47) is defined to be

$$D_v(t) = \dot{v}(t) - f(t, v(t)).$$

Given functions v, w, suppose that the following holds: (a) $w(0) < v(0)$; (b) $D_w(t) < D_v(t)$ for all t belonging to a certain $\mathcal{I} \subset \mathbb{R}$, which contains the initial point $t = 0$. Then, according to Theorem V, page 65 of the book [317], it follows that

$$w(t) < v(t), \quad \text{for all } t \in \mathcal{I}. \tag{6.4.48}$$

Proposition 6.4.10. *Let $\mathcal{I} = [0,1]$ and f be continuous on $[0,1] \times \mathbb{R}$. Let also $r\colon [0,1] \to (-\infty, 0]$ be continuous and $r(t) < 0$ for $t > 0$. Suppose we are given functions $u, v\colon [0,1] \to \mathbb{R}$, which solve the following initial value problems:*

$$\begin{aligned} \dot{u} &= f(t,u) + r(t), & u(0) &= a, \\ \dot{v} &= f(t,v), & v(0) &= a. \end{aligned} \tag{6.4.49}$$

Then $u(t) < v(t)$ for all $t > 0$.

Proof. Given $\epsilon > 0$, we set $w(t) = u(t) - \epsilon$. Let D_u be the defect with respect to the second equation in (6.4.49). Then $D_v = 0$ and

$$D_w(t) = f(t, u(t)) + r(t) - f(t, u(t) - \epsilon).$$

As $r(t) < 0$ and both $f(t, u(t))$, $f(t, u(t) - \epsilon)$ are continuous on $[0, 1]$, one can choose ϵ so small that $D_w(t) < 0$ for all $t \in [0, 1]$. This yields (6.4.48) and hence $u(t) \leq v(t)$. Now it remains to exclude the case $u(t_*) = v(t_*)$ for some $t_* > 0$. Consider the function $y(t) = v(t) - u(t)$, for which one has $y(t) \geq 0$. If $y(t_*) = 0$, then

$$\dot{y}(t_*) = -r(t_*) > 0,$$

hence, $y(t_* - \delta) < 0$ for small enough $\delta > 0$, which is impossible. □

Set

$$\sigma(\upsilon) = \frac{\varkappa^{-\delta}}{1 - (1 - \varkappa^{-\delta})\upsilon}, \quad \upsilon \in \big(0, 1/(1 - \varkappa^{-\delta})\big), \tag{6.4.50}$$

and

$$\phi(\upsilon) = \varkappa^{2\delta - 1}[\sigma(\upsilon)]^4, \quad \psi(\upsilon) = \frac{1}{2}\varkappa^{2\delta - 1}(1 - \varkappa^{-\delta})[\sigma(\upsilon)]^3. \tag{6.4.51}$$

Lemma 6.4.11. *Given $n \in \mathbb{N}$, let the condition*

$$(1 - \varkappa^{-\delta})\hat{u}_{n-1} < 1 \tag{6.4.52}$$

be satisfied. Then

$$\hat{u}_n < \sigma(\hat{u}_{n-1})\hat{u}_{n-1}, \tag{6.4.53}$$

$$\hat{u}_n \geq \sigma(\hat{u}_{n-1})\hat{u}_{n-1} - \lambda_{n-1}\psi(\hat{u}_{n-1}), \tag{6.4.54}$$

$$0 < \lambda_n \leq \phi(\hat{u}_{n-1})\lambda_{n-1}, \tag{6.4.55}$$

where λ_n is defined by (6.4.39).

Proof. For $t \in [0, \theta]$, $\theta = \varkappa^\delta - 1$, $x \in L^2_\beta$, and $n \in \mathbb{N}$, we set (cf. (6.4.19))

$$f_n(x|t) = \frac{1}{Z_n}\int_{L^2_\beta} \exp\Big((x, \xi)_{L^2_\beta} + \frac{t}{2}\|\xi\|^2_{L^2_\beta}\Big)\mu^{\star\varkappa}_{n-1}\big(\varkappa^{(1+\delta)/2}\mathrm{d}\xi\big), \tag{6.4.56}$$

where Z_n is as in (6.4.17). Then

$$f_n(x|\theta) = f_n(x), \quad f_n(x|0) = Z_n^{-1}\left[f_{n-1}\big(\varkappa^{-(1+\delta)/2}x\big)\right]^\varkappa. \tag{6.4.57}$$

For every $t \in [0, \theta]$, the function (6.4.56) can be expanded in the series, cf. (2.2.11), (2.2.12),

$$f_n(x|t) = 1 + \sum_{k=1}^{+\infty}\frac{1}{(2k)!}\int_0^\beta \dots \int_0^\beta f^{(n)}_{2k}(\tau_1, \dots, \tau_{2k}|t)x(\tau_1)\dots x(\tau_{2k})\mathrm{d}\tau_1 \dots \mathrm{d}\tau_{2k}, \tag{6.4.58}$$

with the coefficients

$$f_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}|t) = \frac{1}{Z_n} \int_{L_\beta^2} \xi(\tau_1) \ldots \xi(\tau_{2k}) \exp\left(\frac{t}{2}\|\xi\|^2_{L^2_\beta}\right) \times \mu_{n-1}^{\star\varkappa}\left(\varkappa^{(1+\delta)/2} \mathrm{d}\xi\right), \tag{6.4.59}$$

which for $t = \theta$ coincide with the corresponding Matsubara functions $\Gamma_{2k}^{(n)}$. By Theorem 1.4.14 for every fixed $(\tau_1, \ldots, \tau_{2k}) \in [0, \beta]^{2k}$, as functions of t, the coefficients (6.4.59) are differentiable and continuous on $[0, \theta]$. The corresponding derivatives are

$$\begin{aligned} \frac{\partial}{\partial t} f_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}|t) &\stackrel{\text{def}}{=} \dot{f}_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}|t) \\ &= \frac{1}{2} \int_0^\beta f_{2k+2}^{(n)}(\tau_1, \ldots, \tau_{2k}, \tau, \tau|t) \mathrm{d}\tau. \end{aligned} \tag{6.4.60}$$

Now we expand $\log f_n(x|t)$, cf. (2.2.15), (2.2.16),

$$\log f_n(x|t) = \sum_{k=1}^{+\infty} \frac{1}{(2k)!} \int_0^\beta \ldots \int_0^\beta u_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}|t) x(\tau_1) \ldots x(\tau_{2k}) \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_{2k}. \tag{6.4.61}$$

The derivatives of the Ursell function $u_{2k}^{(n)}(\tau_1, \ldots, \tau_{2k}|t)$ with respect to t can be calculated from (6.4.60). In particular,

$$\dot{u}_2^{(n)}(\tau_1, \tau_2|t) = \frac{1}{2} \int_0^\beta u_4^{(n)}(\tau_1, \tau_2, \tau, \tau|t) \mathrm{d}\tau + \int_0^\beta u_2^{(n)}(\tau_1, \tau|t) u_2^{(n)}(\tau_2, \tau|t) \mathrm{d}\tau; \tag{6.4.62}$$

$$\begin{aligned} \dot{u}_4^{(n)}(\tau_1, \tau_2, \tau_3, \tau_4|t) = {} & \frac{1}{2} \int_0^\beta u_6^{(n)}(\tau_1, \tau_2, \tau_3, \tau_4, \tau, \tau|t) \mathrm{d}\tau \\ & + \int_0^\beta u_4^{(n)}(\tau_1, \tau_2, \tau_3, \tau|t) u_2^{(n)}(\tau_4, \tau|t) \mathrm{d}\tau \\ & + \int_0^\beta u_4^{(n)}(\tau_1, \tau_2, \tau_4, \tau|t) u_2^{(n)}(\tau_3, \tau|t) \mathrm{d}\tau \\ & + \int_0^\beta u_4^{(n)}(\tau_1, \tau_3, \tau_4, \tau|t) u_2^{(n)}(\tau_2, \tau|t) \mathrm{d}\tau \\ & + \int_0^\beta u_4^{(n)}(\tau_2, \tau_3, \tau_4, \tau|t) u_2^{(n)}(\tau_1, \tau|t) \mathrm{d}\tau. \end{aligned} \tag{6.4.63}$$

Then for

$$v_n(t) \stackrel{\text{def}}{=} \int_0^\beta u_2^{(n)}(\tau_1, \tau_2|t) \mathrm{d}\tau_2 = \int_0^\beta u_2^{(n)}(0, \tau|t) \mathrm{d}\tau \tag{6.4.64}$$

we obtain the following system of equations:

$$\dot{\upsilon}_n(t) = \frac{1}{2}U(t) + [\upsilon_n(t)]^2\,, \tag{6.4.65}$$

$$\begin{aligned}\dot{U}(t) &= \frac{1}{2}V(t) + 2\upsilon_n(t)U(t)\\ &\quad + 2\int_0^\beta\int_0^\beta\int_0^\beta u_2^{(n)}(\tau_2,\tau_3|t)u_4^{(n)}(0,\tau_1,\tau_2,\tau_3|t)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3,\end{aligned} \tag{6.4.66}$$

subject to the initial conditions (see (6.4.57))

$$\begin{aligned}\upsilon_n(0) &= \varkappa^{-\delta}\hat{u}_{n-1}(0),\\ U(0) &= \varkappa^{-2\delta-1}\int_0^\beta\int_0^\beta u_4^{(n)}(0,\tau_1,\tau_2,\tau_2|t)\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &= -\varkappa^{-2\delta-1}\lambda_{n-1}.\end{aligned} \tag{6.4.67}$$

Here

$$\begin{aligned}U(t) &\stackrel{\text{def}}{=} \int_0^\beta\int_0^\beta u_4^{(n)}(0,\tau_1,\tau_2,\tau_2|t)\mathrm{d}\tau_1\mathrm{d}\tau_2\\ &= \quad \int_0^\beta\int_0^\beta u_4^{(n)}(\tau,\tau,\tau_1,\tau_2|t)\mathrm{d}\tau_1\mathrm{d}\tau_2,\\ V(t) &\stackrel{\text{def}}{=} \int_0^\beta\int_0^\beta\int_0^\beta u_6^{(n)}(0,\tau_1,\tau_2,\tau_2,\tau_3,\tau_3|t)\mathrm{d}\tau_1\mathrm{d}\tau_2\mathrm{d}\tau_3.\end{aligned} \tag{6.4.68}$$

Along with the problem (6.4.65)–(6.4.67) we consider the problem

$$\dot{y}(t) = [y(t)]^2, \quad y(0) = \upsilon_n(0) = \varkappa^{-\delta}\hat{u}_{n-1}(0). \tag{6.4.69}$$

Under the condition (6.4.52) its solution is

$$y(t) = \frac{\varkappa^{-\delta}\hat{u}_{n-1}(0)}{1 - t\varkappa^{-\delta}\hat{u}_{n-1}(0)} = \sigma\left[(t/\theta)\hat{u}_{n-1}(0)\right]\cdot\hat{u}_{n-1}(0), \quad t\in[0,\theta]. \tag{6.4.70}$$

The sign rule (2.2.25) is valid for the above $u_{2k}^{(n)}$ for all $t \in [0,\theta]$, which yields $U(t) < 0$, $V(t) > 0$. Then by Proposition 6.4.10, the solution of (6.4.65) will be dominated by (6.4.70), i.e.,

$$\hat{u}_n(0) = \upsilon_n(\theta) < y(\theta) = \sigma(\hat{u}_{n-1}(0))\hat{u}_{n-1}(0),$$

which gives (6.4.53).

With the help of (6.4.23), (6.4.24) the third term on the right-hand side of (6.4.66) can be estimated as

$$
\begin{aligned}
2\int_0^\beta \int_0^\beta & u_2^{(n)}(\tau_2,\tau_3|t)\left(\beta^{-1}\int_0^\beta \int_0^\beta u_4^{(n)}(\tau,\tau_1,\tau_2,\tau_3|t)\mathrm{d}\tau\mathrm{d}\tau_1\right)\mathrm{d}\tau_2\mathrm{d}\tau_3 \\
&\geq 2\left(\beta^{-1}\int_0^\beta \int_0^\beta u_4^{(n)}(\tau,\tau_1,\tau_2,\tau_2|t)\mathrm{d}\tau\mathrm{d}\tau_1\right) \\
&\quad \times \int_0^\beta \int_0^\beta u_2^{(n)}(\tau_2,\tau_3|t)\mathrm{d}\tau_2\mathrm{d}\tau_3 = 2\upsilon_n(t)U(t).
\end{aligned}
$$

Applying this in (6.4.66) we arrive at (recall that $U(t) < 0$ and $V(t) > 0$)

$$
\frac{\dot{U}(t)}{U(t)} \leq 4y(t) = \frac{4\varkappa^{-\delta}\hat{u}_{n-1}}{1-t\varkappa^{-\delta}\hat{u}_{n-1}}, \quad \forall t \in [0,\theta]. \tag{6.4.71}
$$

Integrating this one gets

$$
U(t) \geq \frac{U(0)}{[1-t\varkappa^{-\delta}\hat{u}_{n-1}]^4}, \quad \forall t \in [0,\theta], \tag{6.4.72}
$$

which yields in turn

$$
U(\theta) = -\lambda_n \geq -\varkappa^{2\delta-1}\left[\sigma(\hat{u}_{n-1})\right]^4 \lambda_{n-1} = -\phi(\hat{u}_{n-1})\lambda_{n-1},
$$

and thereby (6.4.55). Now we set

$$
h(t) = \frac{1}{[1+t\varkappa^{-\delta}\hat{u}_{n-1}]^2}\cdot \upsilon_n\left(\frac{t}{1+t\varkappa^{-\delta}\hat{u}_{n-1}}\right) - \frac{\varkappa^{-\delta}\hat{u}_{n-1}}{1+t\varkappa^{-\delta}\hat{u}_{n-1}},
$$

where $t \in [0, t_{\max}]$, $t_{\max} = \theta\varkappa^{\delta}\sigma(\hat{u}_{n-1})$. For this function we obtain from (6.4.65) the equation

$$
\dot{h}(t) = \frac{1}{2[1+t\varkappa^{-\delta}\hat{u}_{n-1}]^4}\cdot U\left(\frac{t}{1+t\varkappa^{-\delta}\hat{u}_{n-1}}\right) + [h(t)]^2, \tag{6.4.73}
$$

subject to the boundary conditions

$$
h(0) = 0, \quad h(t_{\max}) = [1-\theta\varkappa^{-\delta}\hat{u}_{n-1}]^2\cdot\left[\upsilon_n(\theta) - \sigma(\hat{u}_{n-1})\hat{u}_{n-1}\right]. \tag{6.4.74}
$$

By means of (6.4.71), one can show that the first term on the right-hand side of (6.4.73) is a monotone increasing function of $t \in [0, t_{\max}]$, which yields

$$
h(t_{\max}) - h(0) \geq t_{\max}U(0)/2.
$$

Taking into account (6.4.74) and (6.4.67) one obtains from the latter

$$
\begin{aligned}
\upsilon_n(\theta) - \sigma(\hat{u}_{n-1})\hat{u}_{n-1} &= \hat{u}_n - \sigma(\hat{u}_{n-1})\hat{u}_{n-1} \\
&\geq -\frac{1}{2}(1-\varkappa^{-\delta})[\sigma(\hat{u}_{n-1})]^3\varkappa^{2\delta-1}\lambda_{n-1},
\end{aligned}
$$

which gives (6.4.54). □

Now we prove a statement that will allow us to control the initial elements of the sequences $\{\hat{u}_n\}$ and $\{\lambda_n\}$. With its help we shall choose the parameters b_1, b_2, and m in such a way that $\hat{u}_0$ and λ_0 possess the properties securing the corresponding convergence of these sequences. Set

$$\varsigma = \varsigma(\beta, b_1, b_2, m) = \varrho_{\beta,\Lambda_{0,\ell}}(q_\ell^2) = \langle x_\ell^2 \rangle_{\mu_0}. \tag{6.4.75}$$

From now on we suppose that $b_1 = b - a/2 < 0$, cf. (6.3.2). Recall that the function f was introduced in (6.3.9).

Lemma 6.4.12. *The initial elements of the sequences $\{\hat{u}_n\}$, $\{\lambda_n\}$ obey the following estimates:*

$$\left(\frac{\beta b_1}{6 b_2}\right) f\left(\frac{3\beta b_2}{2 m b_1}\right) \leq \hat{u}_0 \leq \min\left\{\beta\varsigma;\ \frac{\beta b_1}{4 b_2}\left[1 + \sqrt{1 + (8 b_2/\beta b_1)}\,\right]\right\}, \tag{6.4.76}$$

$$\lambda_0 \leq 4! b_2 [\hat{u}_0]^4 \left[f\left(\frac{3\beta b_2}{2 m b_1}\right)\right]^{-1}. \tag{6.4.77}$$

Proof. One observes, see (6.4.25), that $\hat{u}_0$ is exactly $\beta^{-1} D^{\Lambda}_{\ell\ell}$ for $\Lambda = \{\ell\}$; hence, the lower bound in (6.4.76) is nothing else but the estimate (6.3.23) with ϑ_* being as in (6.3.3) and $b_1 = 2b - a$. The bound $\hat{u}_0 \leq \beta\varsigma$ follows from the estimate (6.4.29) (positivity) and the definition (6.4.26). To obtain the other upper bound we use the integration-by-parts formula (3.4.36), in which we take i to be just $k \in \mathcal{K}$ and ϕ_{i} to be e_k, see (3.4.1). The logarithmic derivative of the measure μ_0 in this direction is, cf. (3.4.30),

$$\mathbf{b}_k(x) = -(mk^2 - 2b_1) \int_0^\beta e_k(\tau) x(\tau) \mathrm{d}\tau - 4 b_2 \int_0^\beta e_k(\tau) [x(\tau)]^3 \mathrm{d}\tau.$$

Then we apply (3.4.36) to the function

$$g(x) = \int_0^\beta e_k(\tau) x(\tau) \mathrm{d}\tau, \tag{6.4.78}$$

and obtain for $k = 0$ the equation

$$1 = -2 b_1 \hat{u}_0 + \frac{4 b_2}{\beta} \int_0^\beta \int_0^\beta \Gamma_4^{(0)}(\tau, \tau, \tau, \tau') \mathrm{d}\tau \mathrm{d}\tau'. \tag{6.4.79}$$

By the GKS inequality (2.2.4), it follows that

$$\Gamma_4^{(0)}(\tau,\tau,\tau,\tau') \geq \Gamma_2^{(0)}(\tau,\tau) \cdot \Gamma_2^{(0)}(\tau,\tau').$$

From this by the estimate $\hat{u}_0 \leq \beta\varsigma$ we have, in (6.4.79),

$$1 \geq -2 b_1 \hat{u}_0 + 4 b_2 \varsigma \hat{u}_0 \geq -2 b_1 \hat{u}_0 + 4 b_2 \beta^{-1} [\hat{u}_0]^2,$$

which yields the second upper bound in (6.4.76).

By (2.2.10) with $\Lambda = \{\ell\}$, which obviously is applicable in the case considered, we get

$$-U_4^{(0)}(\tau_1, \tau_2, \tau_3, \tau_4) \le 4! b_2 \int_0^\beta U_2^{(0)}(\tau_1, \tau) U_2^{(0)}(\tau_2, \tau) U_2^{(0)}(\tau_3, \tau) U_2^{(0)}(\tau_1, \tau) \mathrm{d}\tau,$$

which yields

$$\begin{aligned} \lambda_0 &\le 4! b_2 [\hat{u}_0]^2 \int_0^\beta \left[U_2^{(0)}(\tau, \tau')\right]^2 \mathrm{d}\tau \le 4! b_2 [\hat{u}_0]^3 \beta\varsigma \\ &\le 4! b_2 [\hat{u}_0]^4 \left[f\left(\frac{\beta}{4m\varsigma}\right)\right]^{-1}, \end{aligned} \tag{6.4.80}$$

where we have used the upper bound for $\beta\varsigma$ obtained from

$$\hat{u}_0 \ge \beta\varsigma f\left(\frac{\beta}{4m\varsigma}\right),$$

which is the estimate (6.3.12) written in the present context. As was mentioned above, the function f is monotone decreasing; hence, if we replace ς by its lower bound $b_1/6b_2$, see (6.3.3), we obtain from (6.4.80) exactly the bound to be proven. □

Now let us return to the functions (6.4.50) and (6.4.51). Recall that $\delta \in (0, 1/2)$. Given $\epsilon \in (0, (1-2\delta)/4)$, we define $\upsilon(\epsilon)$ by the condition

$$\sigma[\upsilon(\epsilon)] = \varkappa^{\epsilon}, \tag{6.4.81}$$

which yields

$$\upsilon(\epsilon) = \frac{\varkappa^\delta - \varkappa^{-\epsilon}}{\varkappa^\delta - 1} = 1 + \frac{1 - \varkappa^{-\epsilon}}{\varkappa^\delta - 1}. \tag{6.4.82}$$

Therefrom,

$$\phi(\upsilon) \le \varkappa^{2\delta + 4\epsilon - 1} < 1, \quad \text{for } \upsilon \in [1, \upsilon(\epsilon)]. \tag{6.4.83}$$

Furthermore, we set

$$\omega(\epsilon) = 2\varkappa^{1-\delta-2\epsilon} \cdot \frac{(\varkappa^\delta - \varkappa^{-\epsilon})(1 - \varkappa^{-\epsilon})}{(\varkappa^\delta - 1)^2}, \tag{6.4.84}$$

$$\omega_{\max} = \sup_{\epsilon \in (0, (1-2\delta)/4)} \omega(\epsilon). \tag{6.4.85}$$

The function $\epsilon \mapsto \omega(\epsilon)$ is continuous, and hence, for every $\omega < \omega_{\max}$, one finds $\varepsilon \in (0, (1-2\delta)/4)$ such that $\omega < \omega(\varepsilon)$. Thus, we set $\bar{\upsilon} = \upsilon(\varepsilon)$ and $\bar{\omega} = \omega(\varepsilon)$. Therefore, for $\omega < \bar{\omega}$, one has

$$-\psi(\upsilon)\omega + \upsilon\sigma(\upsilon) > \upsilon, \quad \text{for } \upsilon \in [1, \bar{\upsilon}]. \tag{6.4.86}$$

Now by means of Lemma 6.4.12 we find the interval where the 'critical value' β_* is located.

Lemma 6.4.13. *The parameters b_1, b_2, and m can be chosen in such a way that there will exist $\varepsilon \in (0, (1-2\delta)/4)$ and the numbers $\beta_0^{\pm}$, $0 < \beta_0^- < \beta_0^+$ with the following properties:* (a) $\hat{u}_0 = 1$ *for* $\beta = \beta_0^-$ *and* $\hat{u}_0 < 1$ *for* $\beta < \beta_0^-$; (b) $\hat{u}_0 = \bar{\upsilon} = \upsilon(\varepsilon)$ *for* $\beta = \beta_0^+$ *and* $\hat{u}_0 < \bar{\upsilon}$ *for* $\beta < \beta_0^+$; (c) $\lambda_0 < \bar{\omega} = \omega(\varepsilon)$ *for all* $\beta \in [\beta_0^-, \beta_0^+]$.

Proof. The function f defined in (6.3.9) has the following bound, see Theorem A.4 in [109],

$$f(u) \geq u^{-1}(1 - e^{-u}). \tag{6.4.87}$$

Then by (6.4.76), one has

$$4m\vartheta_*^2 \left[1 - \exp\left(-\frac{\beta}{4m\vartheta_*}\right)\right] \leq \hat{u}_0 \leq \frac{3}{2}\beta\vartheta_* \left[1 + \sqrt{1 + \frac{4}{3\beta\vartheta_*}}\right], \tag{6.4.88}$$

where $\vartheta_* = b_1/6b_2$, cf. (6.3.3). This immediately yields that $\hat{u}_0 \to 0$ as $\beta \to 0$. On the other hand, by taking $4m\vartheta_*^2 > \bar{\upsilon}$ (big mass and/or deep wells) one gets $\hat{u}_0 > \bar{\upsilon}$ for sufficiently large β. By Lemma 6.4.5, $\hat{u}_0$ depends on β continuously; hence, $\beta_0^{\pm}$ with the properties stated do exist. By Proposition 6.3.3, the factor $[f(\beta/4m\vartheta_*)]^{-1}$ on the right-hand side of (6.4.77) is bounded. For $\beta \in (0, \beta_0^+]$, $\hat{u}_0 \leq \bar{\upsilon}$. Thus, one can pick b_2 such that the right-hand side of (6.4.77) is less than $\bar{\omega}$. □

Next we prove the following inductive statement.

Lemma 6.4.14. *Let $\mathfrak{S}_n = (\mathfrak{s}_n^1, \mathfrak{s}_n^2, \mathfrak{s}_n^3)$, $n \in \mathbb{N}_0$, be the triple of the following statements:*

$$\begin{aligned}
\mathfrak{s}_n^1 &= \{\exists \beta_n^+ \in [\beta_0^-, \beta_0^+] : \ \hat{u}_n = \bar{\upsilon}, \ \textit{for } \beta = \beta_n^+; \ \hat{u}_n < \bar{\upsilon}, \ \textit{for } \beta < \beta_n^+\},\\
\mathfrak{s}_n^2 &= \{\exists \beta_n^- \in [\beta_0^-, \beta_0^+] : \ \hat{u}_n = 1, \ \textit{for } \beta = \beta_n^-; \ \hat{u}_n < 1, \ \textit{for } \beta < \beta_n^-\},\\
\mathfrak{s}_n^3 &= \{\forall \beta \in (0, \beta_n^+) : \ \lambda_n < \bar{\omega}\}.
\end{aligned}$$

Then (i) $\mathfrak{S}_0$ *is true;* (ii) $\mathfrak{S}_{n-1}$ *implies* $\mathfrak{S}_n$.

Proof. $\mathfrak{S}_0$ is true by Lemma 6.4.13. For $\beta = \beta_n^+$, by (6.4.50) $\sigma(\hat{u}_n) = \varkappa^{\varepsilon}$ and $\sigma(\hat{u}_n) < \varkappa^{\varepsilon}$ for $\beta < \beta_n^+$. If $\beta = \beta_{n-1}^+$, by (6.4.86), (6.4.84), (6.4.54), and $\mathfrak{s}_{n-1}^3$ it follows that

$$\begin{aligned}
\hat{u}_n &\geq \varkappa^{\varepsilon}\bar{\upsilon} - \frac{1}{2}(1 - \varkappa^{-\delta})\varkappa^{3\varepsilon}\varkappa^{2\delta-1}\lambda_{n-1}\\
&> \varkappa^{\varepsilon}\bar{\upsilon}\left[1 - \varkappa^{2(\varepsilon-1)+\delta}(\varkappa^{\delta} - 1)\frac{\bar{\omega}}{\bar{\upsilon}}\right] = \bar{\upsilon}.
\end{aligned} \tag{6.4.89}$$

For $\beta = \beta_{n-1}^-$, the estimates (6.4.53) yield

$$\hat{u}_n < 1. \tag{6.4.90}$$

Taking into account Lemma 6.4.5 (continuity) and the estimates (6.4.89), (6.4.90), one concludes that there exists at least one value $\tilde{\beta}_n^+ \in (\beta_{n-1}^-, \beta_{n-1}^+)$ such that $\hat{u}_n = \bar{\upsilon}$.

Then we set $\beta_n^+ = \min \tilde{\beta}_n^+$. The mentioned continuity of $\hat{u}_n$ yields also $\hat{u}_n < \bar{\upsilon}$ for $\beta < \beta_n^+$. Thus, $\mathfrak{s}_n^1$ is true. The existence of $\beta_n^- \in [\beta_{n-1}^-, \beta_{n-1}^+)$ can be proven in the same way. For $\beta < \beta_n^+ < \beta_{n-1}^+$, we have $\sigma(\hat{u}_{n-1}) < \varkappa^{\varepsilon}$, which yields

$$\lambda_n < \varkappa^{2\delta-1}\varkappa^{4\varepsilon}\lambda_{n-1} \leq \lambda_{n-1} < \bar{\omega};$$

hence, $\mathfrak{s}_n^3$ is true as well. The proof is concluded by remarking that

$$[\beta_n^-, \beta_n^+] \subset [\beta_{n-1}^-, \beta_{n-1}^+] \subset [\beta_0^-, \beta_0^+]. \qquad \square$$

Lemma 6.4.15. *There exists $\beta_* \in [\beta_0^-, \beta_0^+]$ such that, for $\beta = \beta_*$, the following estimates hold for all $n \in \mathbb{N}_0$:*

$$1 \leq \hat{u}_n < \bar{\upsilon}. \tag{6.4.91}$$

For $\beta < \beta_*$, the above upper estimate, as well as the estimate (6.4.41), hold.

Proof. Consider the set $\Delta_n \stackrel{\text{def}}{=} \{\beta \in (0, \beta_n^+) \mid 1 \leq \hat{u}_n < \bar{\upsilon}\}$. Just above we have shown that it is nonempty and $\Delta_n \subseteq [\beta_n^-, \beta_n^+)$. Let us prove that $\Delta_n \subseteq \Delta_{n-1}$. Suppose that there exists some $\beta \in \Delta_n$, which does not belong to Δ_{n-1}. For this β, either $\hat{u}_{n-1} < 1$ or $\hat{u}_{n-1} \geq \bar{\upsilon}$. Hence, either $\hat{u}_n < 1$ or $\hat{u}_n > \bar{\upsilon}$ (which can be proven as above), which is in conflict with the assumption $\beta \in \Delta_n$. Now let D_n be the closure of Δ_n. Then

$$D_n = \{\beta \in [\beta_n^-, \beta_n^+] \mid 1 \leq \hat{u}_n \leq \upsilon(\delta)\}, \tag{6.4.92}$$

which is a nonempty closed set. Furthermore, $D_n \subseteq D_{n-1} \subseteq \cdots \subset [\beta_0^-, \beta_0^+]$. Set

$$D_* = \bigcap_n D_n,$$

then $D_* \subset [\beta_0^-, \beta_0^+]$ is also nonempty and closed. Now let us show that, for every $\beta \in D_*$, the sharp upper bound in (6.4.91) holds for all $n \in \mathbb{N}$. Suppose $\hat{u}_n = \bar{\upsilon}$ for some $n \in \mathbb{N}$. Then (6.4.89) yields $\hat{u}_m > \hat{\upsilon}$ for all $m > n$, which means that this β does not belong to all D_m, and hence to D_*. Set $\beta_* = \min D_*$. Then both inequalities (6.4.91) hold for $\beta = \beta_*$. Let us prove (6.4.41). Take $\beta < \beta_*$. If $\hat{u}_n > 1$ for all $n \in \mathbb{N}$, then either (6.4.91) holds or there exists such an n_0 that $\hat{u}_{n_0} \geq \bar{\upsilon}$. Therefore, either $\beta \in D_*$ or $\beta > \inf \beta_n^+$. Both these cases contradict the assumption $\beta < \beta_*$. Thus, there exists $n_0 \in \mathbb{N}$ such that $\hat{u}_{n_0-1} \leq 1$ and hence $\hat{u}_n < 1$ for all $n \geq n_0$. In what follows, the definition (6.4.50) and the estimate (6.4.53) imply that the sequences $\{\hat{u}_n\}_{n \geq n_0}$ and $\{\sigma(\hat{u}_n)\}_{n \geq n_0}$ are strictly decreasing. Then for all $n > n_0$, one has (see (6.4.53))

$$\hat{u}_n < \sigma(\hat{u}_{n-1})\hat{u}_{n-1} < \cdots < \sigma(\hat{u}_{n-1})\sigma(\hat{u}_{n-2})\ldots\sigma(\hat{u}_{n_0})\hat{u}_{n_0} < \left[\sigma(\hat{u}_{n_0})\right]^{n-n_0}.$$

Since $\sigma(\hat{u}_{n_0}) < 1$, one gets

$$\sum_{n=0}^{\infty} \hat{u}_n < \infty.$$

Thus,

$$\prod_{n=1}^{\infty} \left[1-(1-\varkappa^{-\delta})\hat{u}_{n-1}\right]^{-1} \stackrel{\text{def}}{=} K_0 < \infty.$$

Finally, we apply (6.4.53) once again and obtain

$$\begin{aligned} \hat{u}_n &< \varkappa^{-n\delta}\left[1-(1-\varkappa^{-\delta})\hat{u}_{n-1}\right]^{-1} \ldots \left[1-(1-\varkappa^{-\delta})\hat{u}_0\right]^{-1}\hat{u}_0 \\ &< \varkappa^{-n\delta} K_0 \bar{v} \stackrel{\text{def}}{=} K(\beta)\varkappa^{-n\delta}. \end{aligned}$$ □

Proof of Lemma 6.4.7. The existence of β_* has been proven in Lemma 6.4.15. Consider the case $\beta = \beta_*$ where the estimates (6.4.91) hold. First we show that $\lambda_n \to 0$. Making use of (6.4.55) we obtain

$$0 < \lambda_n \leq \varkappa^{2\delta-1} \left[\sigma(\hat{u}_{n-1})\right]^4 \lambda_{n-1} < \lambda_{n-1} < \lambda_{n-2} < \cdots < \bar{\omega}.$$

Therefore, the sequence $\{\lambda_n\}$ is strictly decreasing and bounded; hence, it converges and its limit, say λ_*, obeys the condition $\lambda_* < \lambda_0 < \bar{\omega}$. Assume that $\lambda_* > 0$. Then (6.4.55) yields $\sigma(\hat{u}_n) \to \varkappa^{\varepsilon}$; hence, $\hat{u}_n \to \hat{u}_\infty \geq \bar{v}$. Passing to the limit $n \to \infty$ in (6.4.54) one obtains $\lambda_* \geq \bar{\omega}$ which contradicts the above condition. Thus $\lambda_* = 0$. To prove that $\hat{u}_n \to 1$ we set

$$\eta_n = -\frac{1}{2}(1-\varkappa^{-\delta})\left[\sigma(\hat{u}_{n-1})\right]^3 \varkappa^{2\delta-1}\lambda_{n-1}. \tag{6.4.93}$$

Combining (6.4.53) and (6.4.54) we obtain

$$0 \geq \hat{u}_n - \sigma(\hat{u}_{n-1})\hat{u}_{n-1} \geq \eta_n \to 0. \tag{6.4.94}$$

For $\beta = \beta_*$, one has by Lemma 6.4.15 that $\{\hat{u}_n\} \subset [1, \bar{v})$. By (6.4.94), all the accumulation points of $\{\hat{u}_n\}$ in $[1, \bar{v}]$ ought to solve the equation

$$u - \sigma(u)u = 0.$$

There is only one such point: $u_* = 1$, which hence is the limit of $\{\hat{u}_n\}$. For $\beta < \beta_*$, the estimate (6.4.41) yields $\hat{u}_n \to 0$, then $\sigma(\hat{u}_n(0)) \to \varkappa^{-\delta}$. The latter and (6.4.55) imply $\lambda_n \to 0$. □

6.5 Comments and Bibliographic Notes

Section 6.1: According to the main definition of phase transition, this phenomenon occurs if $|\mathcal{G}^{\mathrm{t}}_{\beta}| > 1$. In the translation-invariant case, the main way of proving the existence of phase transitions is to show that the set $\mathcal{G}^{\mathrm{t}}_{\beta}$ contains a non-ergodic element. Usually, it is a periodic Euclidean Gibbs measure. In the general case, the model under consideration is compared with a translation-invariant model, which is more stable with respect to phase transitions. The comparison is performed by means of the correlation inequalities valid for the model under consideration. This fact imposes certain restrictions on the choice of the latter. For classical lattice models, the method of proving phase transitions by showing the non-ergodicity of periodic Gibbs states was developed in [124]. At more or less the same time, an alternative rigorous theory of phase transitions in classical lattice spin models was proposed. This is the Pirogov–Sinai theory elaborated in [245], see also [282], based on contour estimates for the Ising model originated by Peierls, see e.g., [282]. Later, the Pirogov–Sinai theory was essentially extended and generalized into an abstract sophisticated method, applicable also to classical lattice models with unbounded spins, see [321] and the references therein. Finally, we mention that other techniques for the study of phase transitions, based on statistical arguments, like the large deviation principle, entropy methods, etc., see [89], [129] and the references therein.

For quantum lattice models, the theory of phase transitions has essential peculiarities, which distinguish it from the corresponding theory of classical systems. In this context, it suffices to mention that the existence of phase transitions in the three-dimensional isotropic quantum Heisenberg model has not yet been proven. Most of the results in this domain were obtained by means of quantum versions of the method of infrared bounds developed in [124]. The first publication in which the infrared estimates were applied to quantum spin models seems to be the article [109]. After certain modifications this method was applied to a number of models with unbounded Hamiltonians [52], [53], [106], [181], [240], the main characteristic feature of which was the Z_2-symmetry broken by the phase transition. In [106], [240], by means of the infrared estimates the appearance of the long-range order was proven for ϕ^4 models, similar to the version of our model with V as in (6.3.2). In this case, inequalities like (6.3.4) allow one to get the estimate (6.1.26) and thereby to prove that the correlation function does not decay to zero if J is big enough. By (6.1.28) this yields $P > 0$. In [52], [53], [181], the estimate analogous to (6.1.26) was obtained for general symmetric V by means of the representation of the correlation function as a path integral. More or less the same approach was used in [15]. Here we also mention a version of contour techniques, which allows for establishing the long-range order in quantum crystals, see [25]. However, in all these papers, phase transitions in the sense of Definition 6.1.1 could not be even discussed. The first papers where such definition was used are [199], [200].

As was already mentioned, in our approach the system of quantum anharmonic oscillators is described as a system of 'classical' infinite-dimensional spins. This allows us to apply here the original version of the method of infrared estimates elaborated in [124] adapted to the infinite-dimensional case. Thus, in deriving the estimate (6.1.10) we mostly follow the line of arguments taken from the latter paper. The 'local' version of such arguments leading to the conditions (6.1.26), (6.1.27) was developed in [171], where most of the material of this chapter concerning phase transitions can be found.

Our definition of phase transition employs the set $\mathcal{G}^{\mathrm{t}}_{\beta}$. In the physical literature, there exist alternative definitions, based directly on the thermodynamic properties of the system. These are the definition employing the differentiability of the pressure

(Definition 6.1.6, which is applicable to translation-invariant models only), and the definition based on the long-range order. The relationship between the latter two notions is established by means of the Griffiths theorem, Theorem 6.1.7, the proof of which mostly repeats the arguments used for this purpose in [109]. As was mentioned above, for translation-invariant models with bounded interaction, non-differentiability of the pressure corresponds to the non-uniqueness of the Gibbs states, see [162], [277]. For our model, the problem of establishing this relation remains open.

Section 6.2: There exist several descriptions of the reflection positivity method. First of all, one has to mention the original paper [124] as well as further developments in [125], [126], [272]. Then we mention also the monographs [129], [253], [254], [277], [282]. The version presented in subsections 6.2.1 and 6.2.2 is close to the text of [183], where the related bibliography can also be found. With regard to the infrared bound obtained in Subsection 6.2.3 we note that, in accordance with the conditions (6.1.21), (6.1.26), and (6.1.27), it was obtained for the Duhamel function, see (6.2.57), rather than for

$$\widehat{K}_p^{\Lambda}(\tau, \tau) = \sum_{\ell' \in \Lambda} K_{\ell\ell'}^{\Lambda}(\tau, \tau | p) \cos(p, \ell - \ell'),$$

which was used in the papers [15], [52], [53], [181].

Section 6.3: The simplest case of the ϕ^4 model with the nearest neighbor interaction is described by Theorem 6.3.6. The sufficient condition for the phase transition to occur which we obtained in the form of (6.3.20) contains three parameters: the particle mass m; the interaction intensity J; the anharmonicity parameter ϑ_*. The 'softer' is the localizing field, the bigger is ϑ_*. For a more general symmetric scalar version of the model, this description with obvious modifications remains valid, which we prove by comparing the model considered with its translation-invariant version. These two cases are studied in subsections 6.3.1 and 6.3.2, based on the corresponding materials of the papers [199], [200].

In symmetric, respectively, rotation-invariant, models, the phase transitions are accompanied by the symmetry breaking. Due to this symmetry, the condition (6.3.20), as well as the corresponding conditions in a more general symmetric case, are quite simple. Furthermore, the conditions (6.1.21), (6.1.26), and (6.1.27) correspond to the models symmetric with respect to the transformation $q_\ell \to -q_\ell$, for all ℓ. In the case of asymmetric V_ℓ, so far there have no results been obtained about phase transitions in quantum models. In classical models with unbounded spins, for proving phase transitions by means of the infrared estimates, symmetry was not especially important, see Theorem 3.5 in [124] and the discussion preceding this theorem. There might be two explanations of such a discrepancy: (a) the symmetry was the key element but only of the methods employed therein, and, as in the classical case, its lack does not imply the lack of phase transitions; (b) the symmetry is crucial in view of e.g., quantum effects, which stabilize the system, see Chapter 7. In Subsection 6.3.3, we prove that the first-order phase transition without symmetry breaking occurs in our model under the condition that the oscillator mass is sufficiently big and hence quantum effects are not so strong. As a consequence, the dilemma mentioned above has been solved in favor of explanation (a).

Section 6.4: In the language of limit theorems of probability theory, the long-range order, see Definition 6.1.5, corresponds to the fact that a new law of large numbers comes to power, see Theorem 6.1.7 and the discussion preceding Definition 6.1.11. The critical point of the model corresponds to the case where the law of large numbers still holds in its classical form (in the translation-invariant case this means absence of the first-order phase transitions), but the central limit theorem holds true with an abnormal normalization. Thus, the critical point can be detected by means of the fluctuation operators corresponding to this normalization, which for the model considered in this section is given by (6.4.12). In a general context, algebras of abnormal fluctuation operators were studied in [79]. In application to quantum crystals, the corresponding study was performed in [313], [314], where the reader can find a more detailed discussion of this subject as well as the corresponding bibliography.

In classical statistical mechanics, hierarchical models have been known since 1969, when F. J. Dyson introduced in [108] his model and used it as a tool for the study of the one-dimensional Ising model with long-range interactions. A little bit later it was understood that this model is interesting in its own right as due to the hierarchical symmetry this model is self-similar. The latter property is related to the presence of critical points, see [281]. As a consequence, a complete theory of critical points of classical hierarchical models was developed. We refer the reader to the original works by P. Bleher and Ya. Sinai [67], [69], to the survey article [68] and the monograph [282], as well as to the publications [87] and [137], [185], [186], [187].

The reason why we consider the hierarchical version rather than e.g., the translation-invariant one is the same as in the case of classical hierarchical models. Due to the self-similarity of the model, the recursion relations between the Matsubara functions (6.4.14) with consecutive n are relatively simple. Our choice of the anharmonic potential (of ϕ^4 type) was due to the following reasons: (a) Shlosman's sign rule (2.2.25) allowed us to study the sequences of Ursell functions (6.4.20) by studying the corresponding sequences of numbers (6.4.27); (b) by means of Theorem 2.4.6 we obtained the estimates (6.4.37), (6.4.38), which reduced the study of an infinite number of the latter sequences to the study of merely two sequences, see Lemma 6.4.7. As a consequence, we managed to prove the main statement – Theorem 6.4.1. A preliminary study of the model considered in this section was performed in [9], [10], [11]. The approach presented in this section mostly repeats the one given in [8]. Here we point out that this statement is the only one describing the critical point convergence in a quantum model known in the literature. In the paper [219], the authors showed that the critical point of a hierarchical quantum spin model is the same as that of its classical counterpart.

Chapter 7
Quantum Effects

7.1 Quantum Stabilization

A spontaneous change of the structure of a crystalline substance is called a *structural phase transition*. It occurs when the existing structure of the crystal becomes unstable due to the interaction between the structural elements. In a number of physical systems, ordering of light particles moving in multi-welled potential fields can trigger structural phase transitions. This has been confirmed by the fact that the phase transitions were suppressed by application of strong hydrostatic pressure, which makes the wells closer to each other and increases tunneling of the light particles between the wells. The same effect can be observed when the particles are replaced by similar particles but with smaller mass, see [70], [303], [309] for the corresponding data on KDP-type ferroelectrics and [224] for the data on YBaCuO-type high-temperature superconductors. Since the mechanism of this effect is purely quantum, it can be called *quantum stabilization*. The aim of this section is to describe such effects in the model studied in this book in the framework of the theory developed in Part I.

7.1.1 The Stability of Quantum Crystals

To understand what makes a crystal stable or unstable let us first consider the scalar harmonic version of the model (1.1.8) with finite-range interactions – a quantum harmonic crystal. In this case, the one-particle Hamiltonian is given by (1.1.68). Its spectrum consists of the eigenvalues $E_n^{\rm har} = (n + 1/2)\delta$, $n \in \mathbb{N}_0$, $\delta = \sqrt{a/m}$, where a is the oscillator rigidity, see (1.1.73) and (1.1.72). Then the gap parameter (1.1.162), i.e.,

$$\varDelta^{\rm har} = \min_{n\in\mathbb{N}}(E_n^{\rm har} - E_{n-1}^{\rm har})$$

in this case is $\varDelta^{\rm har} = \delta$. Thereby, cf. (1.1.165),

$$a = m\varDelta_{\rm har}^2. \tag{7.1.1}$$

The set of tempered Euclidean Gibbs measures of the harmonic crystal can be constructed similarly as it was done in Chapter 3, but with one exception. Such measures exist only if the stability condition

$$\sup_{\ell}\sum_{\ell'}|J_{\ell\ell'}| = \hat{J}_0 < a = m\varDelta_{\rm har}^2 \tag{7.1.2}$$

is satisfied. In this case, $\mathscr{G}_\beta^{\rm t}$ is a singleton at all β, which readily follows from Theorem 4.3.1. At $\hat{J}_0 = a$, the harmonic crystal becomes unstable with respect to spatial translations; for $d \geq 3$, the set $\mathcal{G}^{\mathrm{t}}_{\beta}$ is still a singleton. For $d = 1, 2$, $\mathcal{G}^{\mathrm{t}}_{\beta}$ is empty. As the right-hand side of (7.1.2) is independent of m, the same stability condition is applicable to the classical harmonic crystal which is obtained in the limit $m \to +\infty$, see Chapter 4. According to Assumption 1.1.1, the anharmonic potentials V_ℓ have a super-quadratic growth, due to which the tempered Euclidean Gibbs measures of anharmonic crystals exist for all $\hat{J}_0$. In this case, the instability of the crystal is connected with the change of equilibrium positions and hence with a structural phase transition. A sufficient condition for the models described in Section 6.3 to have a phase transition can be derived from the equation (6.3.24). It is

$$2\beta J \vartheta_* f(\beta/4m\vartheta_*) > \mathcal{J}(d), \tag{7.1.3}$$

which in the classical limit $m \to +\infty$ takes the form

$$2\beta J \vartheta_* > \mathcal{J}(d).$$

For $d \geq 3$ and hence finite $\mathcal{J}(d)$, the latter condition can be satisfied by picking a big enough β. Therefore, the classical anharmonic crystals always have phase transitions – no matter how small is the interaction parameter J. For finite m, the left-hand side of (7.1.3) is bounded by $8m\vartheta_*^2 J$, and the bound is achieved in the limit $\beta \to +\infty$. If for given values of the interaction parameter J, the mass m, and the parameter ϑ_* which characterizes the anharmonic potential, this bound does not exceed $\mathcal{J}(d)$, the condition (7.1.3) will not be satisfied for any β. Although this condition is only sufficient, one might expect that the phase transition cannot occur at all β if the compound parameter $8m\vartheta_*^2 J$ is small enough. This effect, if it really exists, could be called *quantum stabilization*, since it is impossible in the classical analog of the model.

According to the arguments presented above, one might expect that the stability condition for quantum anharmonic crystals, at least for their ferromagnetic scalar versions with the anharmonic potentials independent of ℓ, has the form of (7.1.2), in which the rigidity parameter a is replaced by its anharmonic analog (1.1.165). That is,

$$\hat{J}_0 < \mathfrak{r}_m. \tag{7.1.4}$$

In this section, we show that the model with the one-particle Hamiltonian (1.1.145) obeying the condition (7.1.4) never has phase transitions. This means that the corresponding set of tempered Euclidean Gibbs measures $\mathcal{G}^{\mathrm{t}}_{\beta}$ is a singleton at all β. The proof is based on the methods developed mostly in Chapters 5 and 6. Then by means of the correlation inequalities proven in Chapter 2 we show that a similar stability condition yields the same uniqueness result for quite general ferromagnetic scalar anharmonic crystals, not necessarily translation-invariant. For ferromagnetic translation-invariant quantum anharmonic crystals with arbitrary $\nu \in \mathbb{N}$, under a condition similar to (7.1.4) we prove that the pressure is two times differentiable with respect to the external field $h \in \mathbb{R}$. This means that the phase transitions in the sense of Definition 6.1.6, as well as the long-range order, are suppressed at all β and h. All these results follow from the fact that under the condition (7.1.4), at all $\beta > 0$ the pair correlation function $K^{\Lambda}_{\ell\ell'}(\tau, \tau'|0)$, see (5.3.4), of the corresponding model decreases properly as $|\ell - \ell'| \to \infty$.

According to Theorem 1.1.60, the quantum rigidity $\mathfrak{r}_m$ of the oscillator with the Hamiltonian (1.1.145) is a continuous function of m, which tends to $+\infty$ as $m \to 0$. Therefore, the condition (7.1.4) certainly holds in the small-mass limit. Its large-mass behavior is described by the following statement, where we suppose that the Hamiltonian has the form (1.1.145), (1.1.146) with $b_s \geq 0$ for all $s = 2, \ldots, r-1$, i.e., its anharmonic potential is exactly as in (6.3.27) with finite $r \in \mathbb{N}$. We recall that for this potential, the parameter $\vartheta_* > 0$ is the unique solution of the equation (6.3.29).

Theorem 7.1.1. *Let $\nu = 1$ and V be as in (6.3.27). Then the gap parameter Δ_m and the quantum rigidity $\mathfrak{r}_m$ of the Hamiltonian with such V obey the estimates*

$$\Delta_m \leq \frac{1}{2m\vartheta_*}, \quad \mathfrak{r}_m \leq \frac{1}{4m\vartheta_*^2}. \tag{7.1.5}$$

Proof. Let ϱ_m be the local Gibbs state (1.2.12) corresponding to the one-particle Hamiltonian with the assumed anharmonic potential. Then by means of the inequality (6.3.4) and the Gaussian upper bound (2.2.8) we get, see (6.3.31),

$$a + 2b^{(1)} + \Phi\left(\varrho_m(q^2)\right) \geq 0,$$

by which

$$\varrho_m(q^2) \geq \vartheta_*. \tag{7.1.6}$$

Let ψ_n, $n \in \mathbb{N}_0$ be the eigenfunctions of the Hamiltonian H_m corresponding to the eigenvalues E_n. By Theorem 1.1.50, to each E_n there corresponds exactly one such ψ_n. Set

$$Q_{nn'} = (\psi_n, q\psi_{n'})_{L^2(\mathbb{R})}, \quad n, n' \in \mathbb{N}_0.$$

As ψ_n are either even or odd, we have $Q_{nn} = 0$ for any $n \in \mathbb{N}_0$. Let us consider

$$\Gamma(\tau, \tau') = \varrho_m\left[q \exp\left(-(\tau' - \tau)H_m\right) q \exp\left(-(\tau - \tau')H_m\right)\right], \quad \tau, \tau' \in [0, \beta],$$

which is the Matsubara function (1.2.92) corresponding to the state ϱ_m and the operators $F_1 = F_2 = q$. Set

$$\hat{u}(k) = \int_0^\beta \Gamma(0, \tau) \cos k\tau \mathrm{d}\tau, \quad k \in \mathcal{K} = \{(2\pi/\beta)\kappa \mid \kappa \in \mathbb{Z}\}. \tag{7.1.7}$$

Then, cf. (6.4.31),

$$\hat{u}(k) = \frac{1}{Z_m} \sum_{n,n'=0}^{+\infty} |Q_{nn'}|^2 \frac{E_n - E_{n'}}{k^2 + (E_n - E_{n'})^2} \times \left\{\exp(-\beta E_{n'}) - \exp(-\beta E_n)\right\},$$

where $Z_m = \operatorname{trace}\exp(-\beta H_m)$. The term $(E_n - E_{n'})^2$ in the denominator may be estimated from below by (1.1.162), which yields, cf. (6.4.32),

$$\begin{aligned}
\hat{u}(k) &\leq \frac{1}{k^2 + \Delta_m^2} \cdot \frac{1}{Z_m} \sum_{n,n'=0}^{+\infty} |Q_{nn'}|^2 (E_n - E_{n'}) \\
&\quad \times \{\exp(-\beta E_n) - \exp(-\beta E_{n'})\} \\
&\leq \frac{1}{k^2 + \Delta_m^2} \cdot \varrho_m([q, [H_m, q]]) \\
&= \frac{1}{m(k^2 + \Delta_m^2)}.
\end{aligned} \tag{7.1.8}$$

By this estimate we get

$$\begin{aligned}
\varrho_m(q^2) &= \Gamma(0,0) = \frac{1}{\beta} \sum_{k \in \mathcal{K}} \hat{u}(k) \\
&\leq \frac{1}{\beta} \sum_{k \in \mathcal{K}} \frac{1}{m(k^2 + \Delta_m^2)} = \frac{1}{2m\Delta_m} \coth(\beta\Delta_m/2).
\end{aligned} \tag{7.1.9}$$

Combining the latter inequality with (7.1.6) we obtain

$$\Delta_m \tanh(\beta\Delta_m/2) < 1/(2m\vartheta_*),$$

which yields (7.1.5) in the limit $\beta \to +\infty$. □

Now let us analyze the quantum stability condition (7.1.4) in the light of the latter results. The first conclusion is that unlike the case of harmonic oscillators, this condition can be satisfied for all $\hat{J}_0$ by letting the mass be small enough. For the nearest-neighbor interaction, one has $\hat{J}_0 = 2dJ$; hence, if (7.1.4) holds, then

$$8dm\vartheta_*^2 J < 1. \tag{7.1.10}$$

This can be compared with the estimate

$$8dm\vartheta_*^2 J > d\mathcal{J}(d),$$

guaranteeing a phase transition, which one derives from (7.1.3). For finite d, $d\mathcal{J}(d) > 1$, see Proposition 6.3.5; hence, there is a gap between the latter estimate and (7.1.10), which, however, diminishes as $d \to +\infty$, since

$$\lim_{d \to +\infty} d\mathcal{J}(d) = 1.$$

7.2 Decay of Correlations

In this section, we show that for ferromagnetic quantum crystals, both scalar and vector, a stability condition like (7.1.4) yields a sufficient decay of the pair correlation function. In the scalar case, this decay guarantees the uniqueness of tempered Euclidean Gibbs measures. However, in the vector case it yields a weaker result – suppression of the long-range order and of the phase transitions of any order in the sense of Definition 6.1.6. The discrepancy arises from the fact that criteria like Theorem 5.3.1, based on the FKG inequalities, are applicable to scalar models only.

7.2.1 The Main Statement

In this subsection, we consider the model (1.1.3), (1.1.8) which is: (a) translation-invariant; (b) scalar; (c) the anharmonic potential is $V(q) = v(q^2)$ with v being convex on $\mathbb{R}_+$, i.e., of BFS type, see Definition 2.2.4.

Let Λ be a box (3.1.2), Λ_* be its conjugate (6.1.14), and $K^{\Lambda}_{\ell\ell'}(\tau,\tau'|p)$ be the periodic correlation function (2.5.41). Recall that the periodic interaction potential $J^{\Lambda}_{\ell\ell'}$ was defined in (1.4.50). For the one-particle Hamiltonian (1.1.3), let $\hat{u}(k)$ be as in (7.1.7).

Theorem 7.2.1. *Let the model be as described above. If*

$$\hat{u}(0)\hat{J}_0 < 1, \tag{7.2.1}$$

then

$$K^{\Lambda}_{\ell\ell'}(\tau,\tau'|p) \leq \frac{1}{\beta|\Lambda|}\sum_{p\in\Lambda_*}\sum_{k\in\mathcal{K}} \frac{\exp\left[\mathrm{i}(p,\ell-\ell') + \mathrm{i}k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1} - \hat{J}^{\Lambda}_0 + \Upsilon^{\Lambda}(p)}, \tag{7.2.2}$$

where

$$\hat{J}^{\Lambda}_0 = \sum_{\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}, \tag{7.2.3}$$

and

$$\begin{aligned}\Upsilon^{\Lambda}(p) &= \hat{J}^{\Lambda}_0 - \sum_{\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}\exp[\mathrm{i}(p,\ell-\ell')] \\ &= 2\sum_{\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}\sin^2(p,\ell-\ell').\end{aligned} \tag{7.2.4}$$

Proof. The main tool for the proof of the estimate (7.2.2) will be a multidimensional version of Proposition 6.4.10. Along with the periodic local Gibbs measure (1.4.52) we introduce

$$\begin{aligned}&\nu^{\mathrm{per}}_{\beta,\Lambda}(\mathrm{d}x_\Lambda|t) \\ &\quad = \frac{1}{N^{\mathrm{per}}_{\beta,\Lambda}(t)}\exp\left\{\frac{t}{2}\sum_{\ell,\ell'\in\Lambda} J^{\Lambda}_{\ell\ell'}(x_\ell,x_{\ell'})_{L^2_\beta} - \int_0^\beta \sum_{\ell\in\Lambda} V(x_\ell(\tau))\mathrm{d}\tau\right\}\chi_{\beta,\Lambda}(\mathrm{d}x_\Lambda),\end{aligned} \tag{7.2.5}$$

where $t \in [0, 1]$ and $N^{\rm per}_{\beta,\Lambda}(t)$ is the corresponding normalization factor, which is just $\Psi_{\beta,\Lambda}(t, 0)$, see (1.4.54) and Proposition 1.4.12 for the properties of $\Psi_{\beta,\Lambda}$. Thereby, we set

$$X_{\ell\ell'}(\tau, \tau'|t) = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)}, \quad \ell, \ell' \in \Lambda. \tag{7.2.6}$$

By Theorem 1.4.14, it is a meromorphic function of t. Then we get, cf. (6.4.62),

$$\begin{aligned}\frac{\partial}{\partial t} X_{\ell\ell'}(\tau, \tau'|t) = \frac{1}{2} \sum_{\ell_1,\ell_2\in\Lambda} J^\Lambda_{\ell_1\ell_2} \int_0^\beta U_{\ell\ell'\ell_1\ell_2}(\tau, \tau', \tau'', \tau''|t)\mathrm{d}t \\ + \sum_{\ell_1,\ell_2\in\Lambda} J^\Lambda_{\ell_1\ell_2} \int_0^\beta X_{\ell\ell_1}(\tau, \tau''|t)X_{\ell_2\ell'}(\tau', \tau''|t)\mathrm{d}t,\end{aligned} \tag{7.2.7}$$

where

$$\begin{aligned}U_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4|t) = {}& \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)x_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_2}(\tau_2)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \cdot \langle x_{\ell_3}(\tau_3)x_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_3}(\tau_3)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \cdot \langle x_{\ell_2}(\tau_2)x_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \\ & - \langle x_{\ell_1}(\tau_1)x_{\ell_4}(\tau_4)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)} \cdot \langle x_{\ell_2}(\tau_2)x_{\ell_3}(\tau_3)\rangle_{\nu^{\rm per}_{\beta,\Lambda}(\cdot|t)}.\end{aligned}$$

As the anharmonic potential in (7.2.5) is of BFS type, by the Lebowitz inequality we have

$$U_{\ell_1\ell_2\ell_3\ell_4}(\tau_1, \tau_2, \tau_3, \tau_4|t) \le 0, \tag{7.2.8}$$

holding for all values of its arguments, see Theorem 2.2.9. As in the proof of Lemma 6.4.11, we consider (7.2.7) as an integro-differential equation subject to the initial condition

$$X_{\ell\ell'}(\tau, \tau'|0) = \delta_{\ell\ell'}\Gamma(\tau, \tau') = (\delta_{\ell\ell'}/\beta) \sum_{k\in\mathcal{K}} \hat{u}(k) \cos k(\tau - \tau'). \tag{7.2.9}$$

Besides, we also have

$$X_{\ell\ell'}(\tau, \tau'|1) = K^\Lambda_{\ell\ell'}(\tau, \tau'|p). \tag{7.2.10}$$

Along with the Cauchy problem (7.2.7), (7.2.9) we consider the following equation:

$$\frac{\partial}{\partial t} Y_{\ell\ell'}(\tau, \tau'|t) = \sum_{\ell_1,\ell_2\in\Lambda} \left[J^\Lambda_{\ell_1\ell_2} + \frac{\varepsilon}{|\Lambda|}\right] \int_0^\beta Y_{\ell\ell_1}(\tau, \tau''|t)Y_{\ell_2\ell'}(\tau'', \tau'|t)\mathrm{d}\tau'', \tag{7.2.11}$$

where $\varepsilon > 0$ is a parameter, subject to the initial condition

$$Y_{\ell\ell'}(\tau, \tau'|0) = X_{\ell\ell'}(\tau, \tau'|0) = (\delta_{\ell\ell'}/\beta) \sum_{k\in\mathcal{K}} \hat{u}(k) \cos k(\tau - \tau'). \tag{7.2.12}$$

Let us show that under the condition (7.2.1) there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the problem (7.2.11), (7.2.12), $t \in [0, 1]$, has the unique solution

$$Y_{\ell\ell'}(\tau, \tau'|t) = \frac{1}{\beta|\Lambda|} \sum_{p\in\Lambda_*} \sum_{k\in\mathcal{K}} \frac{\exp\left[\mathrm{i}(p, \ell - \ell') + \mathrm{i}k(\tau - \tau')\right]}{[\hat{u}(k)]^{-1} - t[\hat{J}_0^\Lambda + \varepsilon\delta_{p,0}] + t\Upsilon^\Lambda(p)}, \tag{7.2.13}$$

where $\hat{J}_0$, $\Upsilon^\Lambda(p)$ are the same as in (7.2.3) and $\delta_{p,0}$ is the Kronecker symbol with respect to each component of p. By means of the Fourier transformation

$$\begin{aligned} Y_{\ell\ell'}(\tau, \tau'|t) &= \frac{1}{\beta|\Lambda|} \sum_{p\in\Lambda_*} \sum_{k\in\mathcal{K}} \hat{Y}(p, k|t) \exp\left[\mathrm{i}(p, \ell - \ell') + \mathrm{i}k(\tau - \tau')\right], \\ \hat{Y}(p, k|t) &= \sum_{\ell'\in\Lambda} \int_0^\beta Y_{\ell\ell'}(\tau, \tau'|t) \exp\left[-\mathrm{i}(p, \ell - \ell') - \mathrm{i}k(\tau - \tau')\right] \mathrm{d}\tau', \end{aligned} \tag{7.2.14}$$

we bring (7.2.11), (7.2.12) into the form

$$\frac{\partial}{\partial t}\hat{Y}(p, k|t) = \left[\hat{J}^\Lambda(p) + \varepsilon\delta_{p,0}\right] \cdot \left[\hat{Y}(p, k|t)\right]^2, \quad \hat{Y}(p, k|0) = \hat{u}(k), \tag{7.2.15}$$

where, see (7.2.3),

$$\hat{J}^\Lambda(p) = \sum_{\ell'\in\Lambda} J^\Lambda_{\ell\ell'} \exp\left[\mathrm{i}(p, \ell - \ell')\right] = \hat{J}_0^\Lambda - \Upsilon^\Lambda(p). \tag{7.2.16}$$

By (3.6.6) one has $\hat{J}_0^\Lambda \leq \hat{J}_0$. Clearly, $|\hat{J}^\Lambda(p)| \leq \hat{J}_0^\Lambda$ and $\hat{u}(k) \leq \hat{u}(0)$. Then in view of (7.2.1), one finds $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$,

$$\left[\hat{J}^\Lambda(p) + \varepsilon\delta_{p,0}\right]\hat{u}(k) < 1$$

holds for all $p \in \Lambda_*$ and $k \in \mathcal{K}$. Thus, the problem (7.2.15) can be solved explicitly, which via the transformation (7.2.14) yields (7.2.13).

Given $\theta \in (0, 1)$, we set

$$Y^{(\theta)}_{\ell\ell'}(\tau, \tau'|t) = Y_{\ell\ell'}(\tau, \tau'|t + \theta), \quad t \in [0, 1 - \theta]. \tag{7.2.17}$$

Obviously, the latter function obeys the equation (7.2.11) for $t \in [0, 1 - \theta]$ with the initial condition

$$Y^{(\theta)}_{\ell\ell'}(\tau, \tau'|0) = Y_{\ell\ell'}(\tau, \tau'|\theta) > Y_{\ell\ell'}(\tau, \tau'|0) = X_{\ell\ell'}(\tau, \tau'|0). \tag{7.2.18}$$

The latter inequality is due to the positivity of the right-hand side of (7.2.11). Therefore,

$$Y^{(\theta)}_{\ell\ell'}(\tau, \tau'|t) > 0, \tag{7.2.19}$$

for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0, \beta]$, and $t \in [0, 1 - \theta]$.

Let us show that under the condition (7.2.1), for all $\theta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_0)$,

$$X_{\ell\ell'}(\tau, \tau'|t) < Y^{(\theta)}_{\ell\ell'}(\tau, \tau'|t), \tag{7.2.20}$$

also for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0, \beta]$, and $t \in [0, 1-\theta]$. To this end we introduce

$$Z^{\pm}_{\ell\ell'}(\tau, \tau'|t) \stackrel{\text{def}}{=} Y^{(\theta)}_{\ell\ell'}(\tau, \tau'|t) \pm X_{\ell\ell'}(\tau, \tau'|t), \quad t \in [0, 1-\theta]. \tag{7.2.21}$$

Then one has from (7.2.7), (7.2.11) that

$$\begin{aligned} \frac{\partial}{\partial t} Z^{-}_{\ell\ell'}(\tau, \tau'|t) = \frac{1}{2} \sum_{\ell_1, \ell_2 \in \Lambda} J^{\Lambda}_{\ell_1\ell_2} \int_0^{\beta} \Big\{ & Z^{+}_{\ell\ell_1}(\tau, \tau''|t) Z^{-}_{\ell'\ell_2}(\tau', \tau''|t) \\ & + Z^{-}_{\ell\ell_1}(\tau, \tau''|t) Z^{+}_{\ell'\ell_2}(\tau', \tau''|t) \Big\} \mathrm{d}\tau'' \\ & + \frac{\varepsilon}{|\Lambda|} \sum_{\ell_1, \ell_2 \in \Lambda} \int_0^{\beta} Y^{(\theta)}_{\ell\ell_1}(\tau, \tau''|t) Y^{(\theta)}_{\ell'\ell_2}(\tau', \tau''|t) \mathrm{d}\tau'' - S_{\ell\ell'}(\tau, \tau'|t), \end{aligned} \tag{7.2.22}$$

where $S_{\ell\ell'}(\tau, \tau'|t)$ stands for the first term on the right-hand side of (7.2.7). By (7.2.21) and (7.2.18),

$$Z^{-}_{\ell\ell'}(\tau, \tau'|0) = Y_{\ell\ell'}(\tau, \tau'|\theta) - X_{\ell\ell'}(\tau, \tau'|0) > 0, \tag{7.2.23}$$

which holds for all $\ell, \ell' \in \Lambda$, $\tau, \tau' \in [0, \beta]$. For every $\ell, \ell' \in \Lambda$, both $Y_{\ell\ell'}(\tau, \tau'|t)$, $X_{\ell\ell'}(\tau, \tau'|t)$, and hence $Z^{\pm}_{\ell\ell'}(\tau, \tau'|t)$, are continuous functions of their arguments. For $Y_{\ell\ell'}$, this follows from (7.2.14). For $X_{\ell\ell'}$, this follows from its analyticity as a function of t proven in Theorem 1.4.14 and from the fact that it is a Matsubara function of τ, τ', see Theorem 1.2.32. Set

$$\zeta(t) = \inf \left\{ Z^{-}_{\ell\ell'}(\tau, \tau'|t) \mid \ell, \ell' \in \Lambda, \ \ \tau, \tau' \in [0, \beta] \right\}. \tag{7.2.24}$$

By (7.2.23), it follows that $\zeta(0) > 0$. Suppose now that $\zeta(t_0) = 0$ at some $t_0 \in [0, 1-\theta]$ and $\zeta(t) > 0$ for all $t \in [0, t_0)$. Then by continuity of $Z^{-}_{\ell\ell'}$, there exist $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$ such that

$$Z^{-}_{\ell\ell'}(\tau, \tau'|t_0) = 0 \quad \text{and} \quad Z^{-}_{\ell\ell'}(\tau, \tau'|t) > 0 \quad \text{for all } t < t_0.$$

For these $\ell, \ell' \in \Lambda$ and $\tau, \tau' \in [0, \beta]$, the derivative $(\partial/\partial t) Z^{-}_{\ell\ell'}(\tau, \tau'|t)$ at $t = t_0$ is positive since on the right-hand side of (7.2.22) the third term is positive and the remaining terms are non-negative. But a differentiable function which is positive at $t \in [0, t_0)$ and zero at $t = t_0$ cannot increase at $t = t_0$. Thus, $\zeta(t) > 0$ for all $t \in [0, 1-\theta]$, which yields (7.2.20). By the latter estimate, we have

$$\begin{aligned} & X_{\ell\ell'}(\tau, \tau'|1-\theta) < Y_{\ell\ell'}(\tau, \tau'|1) \\ & \quad = \frac{1}{\beta|\Lambda|} \sum_{p \in \Lambda_*} \sum_{k \in \mathcal{K}} \frac{\exp\left[\mathrm{i}(p, \ell - \ell') + \mathrm{i}k(\tau - \tau')\right]}{[\hat{u}(k)]^{-1} - t[\hat{J}^{\Lambda}_0 + \varepsilon\delta_{p,0}] + t\Upsilon^{\Lambda}(p)}. \end{aligned}$$

All the functions above depend on θ and ε continuously. Hence, passing to the limit $\theta = \varepsilon \to 0$ and taking into account (7.2.10) we obtain (7.2.2). □

7.2.2 Decay of Correlations in Gibbs States

By Lemma 3.6.1 and Theorem 3.6.4 the set of $\mathcal{W}^{\mathrm{t}}$-accumulation points of the family $\{\pi^{\mathrm{per}}_{\beta,\Lambda}\}_{\Lambda\in\mathfrak{L}_{\mathrm{box}}}$ is a non-void subset of $\mathcal{G}^{\mathrm{t}}_{\beta}$. Such accumulation points are called periodic Gibbs states. In the case considered here, they are symmetric, i.e., invariant under the transformation $\Omega^{\mathrm{t}}_{\beta} \ni x \mapsto -x$. Let $\mathcal{L}_{\mathrm{box}}$ be the sequence of boxes (3.1.2) such that $\{\pi^{\mathrm{per}}_{\beta,\Lambda}\}_{\Lambda\in\mathcal{L}_{\mathrm{box}}}$ converges to a given periodic state μ^{per}. Then the sequence of corresponding correlation functions (2.5.41) converges, at least point-wise, to the correlation function in this state

$$K_{\ell\ell'}(\tau,\tau'|\mu^{\mathrm{per}}) = \langle x_\ell(\tau)x_{\ell'}(\tau')\rangle_{\mu^{\mathrm{per}}}. \tag{7.2.25}$$

Recall that according to Theorem 3.3.8, there exists only one $\mu_0 \in \mathcal{G}^{\mathrm{t}}_{\beta}$ which is the limit of $\{\pi_{\beta,\Lambda}(\cdot|0)\}_{\Lambda\in\mathcal{L}}$, for any cofinal sequence $\mathcal{L}$. In the case considered here, μ_0 is also symmetric. By $K_{\ell\ell'}(\tau,\tau'|\mu_0)$ we denote its correlation function. For $p \in (-\pi,\pi]^d$, we also set

$$\Upsilon(p) = \hat{J}_0 - \sum_{\ell'} J_{\ell\ell'}\exp[\mathrm{i}(p,\ell-\ell')] = 2\sum_{\ell'} J_{\ell\ell'}\sin^2(p,\ell-\ell'). \tag{7.2.26}$$

The next corollary of Theorem 7.2.1 gives an upper bound for correlation functions mentioned above.

Theorem 7.2.2. *Let the stability condition (7.1.4) be satisfied. Then for every periodic tempered Euclidean Gibbs state, the correlation function (7.2.25) has the bound*

$$\begin{aligned} K_{\ell\ell'}(\tau,\tau'|\mu^{\mathrm{per}}) &\leq Y_{\ell\ell'}(\tau,\tau') \\ &\stackrel{\mathrm{def}}{=} \frac{1}{\beta(2\pi)^d}\sum_{k\in\mathcal{K}}\int_{(-\pi,\pi]^d}\frac{\exp\left[\mathrm{i}(p,\ell-\ell')+\mathrm{i}k(\tau-\tau')\right]}{[\hat{u}(k)]^{-1}-\hat{J}_0+\Upsilon(p)}\mathrm{d}p \\ &\leq \frac{1}{\beta(2\pi)^d}\sum_{k\in\mathcal{K}}\int_{(-\pi,\pi]^d}\frac{\exp\left[\mathrm{i}(p,\ell-\ell')+\mathrm{i}k(\tau-\tau')\right]}{m(\Delta_m^2+k^2)-\hat{J}_0+\Upsilon(p)}\mathrm{d}p. \end{aligned} \tag{7.2.27}$$

The same estimate is also satisfied for the correlation function $K_{\ell\ell'}(\tau,\tau'|\mu_0)$.

Proof. By (7.1.8), the estimate (7.2.1) holds if (7.1.4) is satisfied. Then the first estimate in (7.2.27) is obtained by passing in (7.2.2) to the limit $\Lambda \nearrow \mathbb{L}$ along the corresponding sequence $\mathcal{L}_{\mathrm{box}}$. The second one follows by (7.1.8), according to which $[\hat{u}(k)]^{-1} \geq m(\Delta_m^2+k^2)$. The second part of the theorem follows from the estimate (2.5.45). □

Remark 7.2.3. The last line in (7.2.27) is the infinite volume correlation function for the quantum harmonic crystal discussed at the beginning of Subsection 7.1.1. Thus, under the condition (7.2.1) the decay of the correlation functions in the periodic states and in μ_0 is at least as strong as it is in the stable quantum harmonic crystal.

Theorem 7.2.4. *If the function (7.2.26) is such that $\Upsilon(p) \sim \Upsilon_0|p|^2$, $\Upsilon_0 > 0$, as $p \to 0$, the upper bound in (7.2.27) has an exponential spatial decay.*

Proof. For $\Upsilon(p) \sim \Upsilon_0|p|^2$, $\Upsilon_0 > 0$, as $p \to 0$, the asymptotics of the bound in (7.2.27) as $\sqrt{|\ell - \ell'|^2 + |\tau - \tau'|^2}$ will be the same as for the $d+1$-dimensional free field, which is well known, see claim (c) of Proposition 7.2.1, page 162 of [135]. Nevertheless, let us show that for $J_{\ell\ell'} = J\delta_{|\ell-\ell'|,1}$, we have

$$Y_{\ell\ell'}(\tau,\tau') \leq C_\beta \exp(-\gamma_0|\ell - \ell'|), \tag{7.2.28}$$

with

$$\begin{aligned} \gamma_0 &= d^{-1/2} \log\left(1 + \frac{m\Delta_m^2 - 2dJ}{J}\right), \\ C_\beta &= \frac{1}{\beta}\sum_{k\in\mathcal{K}} \frac{1}{m\Delta_m^2 - 2dJ + mk^2}. \end{aligned} \tag{7.2.29}$$

Note that in this case $\Upsilon(p) = 2J\mathcal{E}(p)$, see (6.2.51). This is the same decay (with $\Delta_m\delta$, see (7.1.1)) as for the correlation function of the stable harmonic crystal.

To prove (7.2.28) we use (6.2.61). Suppose that $|\ell_1 - \ell_1'| \geq |\ell_j - \ell_j'|$ for all $j = 2, \ldots, d$. Set $\hat{\ell} = (\ell_2, \ldots, \ell_d)$ and $\hat{p} = (p_2, \ldots, p_d)$. Then by (6.2.61),

$$\begin{aligned} Y_{\ell\ell'}(\tau,\tau') = &\frac{1}{2\beta J(2\pi)^{d-1}} \sum_{k\in\mathcal{K}} \int_{(-\pi,\pi]^{d-1}} \exp\left\{\mathrm{i}k(\tau-\tau') + \mathrm{i}(\hat{p}, \hat{\ell} - \hat{\ell}')\right\} \\ &\times \frac{1}{\sqrt{a^2-1}} \exp\left(-\gamma|\ell_1 - \ell_1'|\right) \mathrm{d}\hat{p}, \end{aligned}$$

where γ is as in (6.2.61) with

$$a = \frac{1}{2J\hat{u}(k)} - \sum_{j=2}^{d} \cos p_j > 1, \tag{7.2.30}$$

which holds by (7.2.1). By means of the estimate (7.1.8) and the following evident inequalities

$$|\ell_1 - \ell_1'| \geq d^{-1/2}|\ell - \ell'|, \quad \sqrt{a^2 - 1} > a - 1, \text{ for } a > 1,$$

we finally get (7.2.28), (7.2.29). □

Let us now turn to the vector case. We suppose that the model is ferromagnetic and translation-invariant, but $\nu \in \mathbb{N}$ is arbitrary. The anharmonic potential is supposed to have the form $V(u) = v(|u|^2)$ with v being convex. By this assumption the model is rotation-invariant. In the vector case, the eigenvalues of the one-point Hamiltonian are no longer simple; hence, one cannot introduce a gap parameter like Δ_m, which would play a similar role in the quantum stabilization of the model. Thus, the only possibility we have here is to compare the vector model with its scalar version, that is, with the scalar model with the anharmonic potential $v(u^2)$, $u \in \mathbb{R}$. To this end we will use Theorem 2.5.2.

Theorem 7.2.5. *Let the vector model (1.1.3), (1.1.8) be as just described; let also Δ_m be the gap parameter of the scalar one-particle Hamiltonian with the anharmonic potential $v(u^2)$. Then, if the stability condition (7.1.4) is satisfied (in which $\hat{J}_0$ is the interaction parameter of the model under consideration), the correlation function*

$$K^{jj}_{\ell\ell'}(\tau,\tau'|\mu) = \langle x^{(j)}_\ell(\tau) x^{(j)}_{\ell'}(\tau')\rangle_\mu, \quad j \in \{1,\ldots,\nu\}, \tag{7.2.31}$$

corresponding to any periodic Gibbs state $\mu \in \mathcal{G}^{\mathrm{t}}_\beta$, as well as to any Gibbs state being the accumulation point of the family $\{\pi_{\beta,\Lambda}(\cdot|0)\}_{\Lambda\in\mathfrak{L}_{\mathrm{fin}}}$, obeys the estimate (7.2.27) with $\hat{u}(k)$ calculated according to (7.1.7) for the scalar one-particle Hamiltonian with the anharmonic potential $v(u^2)$.

Proof. The proof follows immediately from the estimates (2.5.3) and (7.2.27). □

7.3 Suppression of Phase Transitions

From the 'physical' point of view, the decay of correlations (7.2.28) already shows the lack of any phase transition. However, in the mathematical theory, one should show this by means of the definition of a phase transition. The most general one is Definition 6.1.1 according to which the lack of phase transitions corresponds to the uniqueness of tempered Euclidean Gibbs states. Properties like differentiability of the pressure, cf. Definition 6.1.6, or the lack of the order parameter, see Definition 6.1.5, may also indicate the suppression of phase transitions, but in a weaker sense (if they do not imply the uniqueness, cf. Theorem 5.3.3). The aim of this section is to demonstrate that the decay of correlations of ferromagnetic versions of the model (1.1.3), (1.1.8) caused by the quantum stabilization yields the two-times differentiability of the pressure, which in the scalar case yields the uniqueness. This latter result is then extended to ferromagnetic models, which are not necessarily translation-invariant.

7.3.1 The Uniqueness for Scalar Ferromagnets

The most general result for the scalar ferromagnets is the following

Theorem 7.3.1. *Let the model (1.1.3), (1.1.8) be scalar and ferromagnetic, and $\mathbb{L} = \mathbb{Z}^d$. Let also the anharmonic potentials V_ℓ be even and such that there exists a convex function $v\colon \mathbb{R}_+ \to \mathbb{R}$, for which, and for any V_ℓ,*

$$V_\ell(u_\ell) - v(u_\ell^2) \le V_\ell(\tilde{u}_\ell) - v(\tilde{u}_\ell^2) \quad \textit{whenever } u_\ell^2 < \tilde{u}_\ell^2. \tag{7.3.1}$$

For such v, let Δ_m be the gap parameter of the one-particle Hamiltonian (1.1.3) with the anharmonic potential $v(q^2)$. Then the set of tempered Euclidean Gibbs measures of this model is a singleton if the stability condition (7.1.4) involving Δ_m and the interaction parameter $\hat{J}_0$ of the considered model is satisfied.

The proof of this theorem will be done in several steps. As in the proof of Theorem 6.3.8, we compare the considered model, which we call *basic*, with the translation-invariant model (*reference model*) for which the anharmonic potential is $v(q^2)$, and the other elements remain the same. The reference model is also called an up-model; it is less stable than the basic model, cf. (6.3.26). All the elements of the theory corresponding to the up-model will have the superscript up. The main idea of the proof is that the quantum stabilization of the up-model will imply the same effect for the basic model. Recall that in the scalar case, we have introduced the FKG order on the set of tempered Euclidean Gibbs measures $\mathcal{G}^{\rm t}_\beta$, with respect to which it has maximal and minimal elements $\mu_\pm$, see Theorem 3.7.4.

Lemma 7.3.2. *For every ℓ, it follows that*

$$\langle x_\ell(0)\rangle_{\mu_+} \leq \langle x_\ell(0)\rangle_{\mu_+^{\rm up}}. \tag{7.3.2}$$

Proof. Take $\sigma \in (0, 1/2)$ and $b > 0$ such that $\Xi(b, \sigma)$, see (3.3.22), is a full measure set for all tempered Euclidean Gibbs measures of both reference and basic models. Thereby, we take ℓ_0 and the configuration ξ^{ℓ_0} defined by (3.7.12). Then for any Λ, such that $\ell \in \Lambda$, by (2.5.40) it follows that

$$\langle x_\ell(0)\rangle_{\pi_{\beta,\Lambda}(\cdot|\xi^{\ell_0})} \leq \langle x_\ell(0)\rangle_{\pi^{\rm up}_{\beta,\Lambda}(\cdot|\xi^{\ell_0})}, \tag{7.3.3}$$

since for all ℓ' and τ, we have $\xi^{\ell_0}_{\ell'}(\tau) > 0$. As was established in the proof of Theorem 3.7.4, for any cofinal sequence $\mathcal{L}$, the $\mathcal{W}^{\rm t}$-limits of $\{\pi_{\beta,\Lambda}(\cdot|\xi^{\ell_0})\}_{\Lambda\in\mathcal{L}}$ and $\{\pi^{\rm up}_{\beta,\Lambda}(\cdot|\xi^{\ell_0})\}_{\Lambda\in\mathcal{L}}$ are the corresponding maximal elements μ_+ and $\mu_+^{\rm up}$, respectively. Then passing to this limit in the latter inequality we get (7.3.2). $\square$

One observes that (7.3.2) justifies the name given to the up-model. In view of this estimate and Corollary 5.3.2, for proving Theorem 7.3.1 it is enough to show that $\langle x_\ell(0)\rangle_{\mu_+^{\rm up}} = 0$ for a certain, and hence for all, ℓ as the measure $\mu_+^{\rm up}$ is translation-invariant, see Theorem 3.7.4.

We recall that for given $\tilde{\ell}$ and $\alpha \in \mathcal{I}$, the Banach space $l^\infty(\tilde{\ell}, \alpha)$ was defined in (3.1.26). Now for $\Lambda \in \mathfrak{L}_{\rm fin}$, we introduce the matrix $(M^\Lambda_{\ell\ell'})_{\ell\ell'\in\mathbb{L}}$ as follows. For $\ell, \ell' \in \mathbb{L}$, we set

$$M^\Lambda_{\ell\ell'} = \sum_{\ell_1\in\Lambda} J_{\ell\ell_1} \int_0^\beta \langle x_{\ell_1}(\tau_1) x_{\ell'}(\tau')\rangle_{\pi^{\rm up}_{\beta,\Lambda}(\cdot|0)} d\tau'. \tag{7.3.4}$$

Since the measure $\pi^{\rm up}_{\beta,\Lambda}(\cdot|0)$ is shift-invariant, the above integral is independent of τ.

Lemma 7.3.3. *Let the stability condition (7.1.4) for the* up*-model be satisfied. Then there exist $\alpha \in \mathcal{I}$ and $C_\alpha > 0$ such that for any $\Lambda \in \mathfrak{L}_{\rm fin}$ and $\tilde{\ell} \in \mathbb{L}$, the norm of the operator $M^\Lambda\colon l^\infty(\tilde{\ell}, \alpha) \to l^\infty(\tilde{\ell}, \alpha)$, defined by the matrix $(M^\Lambda_{\ell\ell'})_{\ell\ell'\in\mathbb{L}}$, obeys the estimate*

$$\|M^\Lambda\|_\alpha \stackrel{\rm def}{=} \sup_\ell \sum_{\ell'} |M^\Lambda_{\ell\ell'}| \cdot [w_\alpha(\ell,\ell')]^{-1} \leq C_\alpha. \tag{7.3.5}$$

Proof. First, let us show that there exists $\alpha \in \mathcal{I}$ such that

$$\hat{J}_0 < \hat{J}_\alpha < m\varDelta_m. \tag{7.3.6}$$

We recall that $\hat{J}_\alpha$ was defined in (3.1.20). Here the cases (3.1.22) and (3.1.24) should be considered separately. In the first case, we have $\mathcal{I} = (0, \bar{\alpha})$ and the α in question exists since

$$\lim_{\alpha \to 0+} \hat{J}_\alpha = \hat{J}_0, \tag{7.3.7}$$

which readily follows from (3.1.22) and (3.1.23). In the second case, the weights are given by (3.1.25) with a positive ε, which we are going to pick now to secure (7.3.6). To indicate the dependence of $\hat{J}_\alpha$ on this ε we write $\hat{J}_\alpha^{(\varepsilon)}$. Simple calculations yield

$$0 < \hat{J}_\alpha^{(\varepsilon)} - \hat{J}_0 < \varepsilon \alpha d \hat{J}_\alpha^{(1)}.$$

Thereby, we fix α and take ε obeying $\varepsilon < m\varDelta_m^2 / \alpha d \hat{J}_\alpha^{(1)}$, which yields (7.3.6).

Let us now estimate the norm of $\|M^\Lambda\|_\alpha$, defined in (7.3.5). By (3.1.26), (3.1.20), and (3.1.17) one has

$$\|M^\Lambda\|_\alpha \leq \hat{J}_\alpha \cdot \sup_\ell \sum_{\ell'} L_{\ell\ell'}^\Lambda \cdot [w_\alpha(\ell, \ell')]^{-1}, \tag{7.3.8}$$

where

$$L_{\ell\ell'}^\Lambda \stackrel{\text{def}}{=} \int_0^\beta \langle x_\ell(\tau) x_{\ell_1}(\tau') \rangle_{\pi_{\beta,\Lambda}^{\text{up}}(\cdot|0)} \mathrm{d}\tau'. \tag{7.3.9}$$

For any cofinal sequence $\mathcal{L}$, the sequence $\{L_{\ell\ell'}^\Lambda\}_{\Lambda \in \mathcal{L}}$ is monotone increasing, see (2.5.46), and its limit obeys the estimate (7.2.27), see the second part of Theorem 7.2.2. Therefrom we obtain

$$\begin{aligned} L_{\ell\ell'}^\Lambda &\leq \int_0^\beta Y_{\ell\ell'}(\tau, \tau') \mathrm{d}\tau' \\ &= \frac{1}{(2\pi)^d} \int_{(-\pi,\pi]^d} \frac{\exp[\mathrm{i}(p, \ell - \ell')]}{[\hat{u}(0)]^{-1} - \hat{J}_0 + \varUpsilon(p)} \mathrm{d}p. \end{aligned} \tag{7.3.10}$$

The matrix on the right-hand side of (7.3.10) determines an operator, which can be written by means of the von Neumann series

$$\text{RHS}(7.3.10) = \hat{u}(0) \sum_{n=0}^\infty I^n, \quad I_{\ell\ell'} = \hat{u}(0) J_{\ell\ell'}.$$

As $\hat{u}(0) < m\varDelta_m^2$, see (7.1.8), by (7.3.6) one has

$$\|I\|_\alpha \leq m\varDelta_m^2 \hat{J}_\alpha < 1.$$

Employing these estimates in (7.3.8) we get

$$\|M^\Lambda\|_\alpha \leq \frac{m\varDelta_m^2 \hat{J}_\alpha}{1 - m\varDelta_m^2 \hat{J}_\alpha},$$

which yields the bound (7.3.5). □

Proof of Theorem 7.3.1. Let Λ, ℓ, and ℓ_0 be as in (7.3.3). Set

$$\psi_\Lambda(t) = \int_{\Omega_\beta} x_\ell(0)\pi^{\mathrm{up}}_{\beta,\Lambda}(\mathrm{d}x|t\xi^{\ell_0}), \quad t \in \mathbb{R}. \tag{7.3.11}$$

This function is obviously differentiable at any $t \in \mathbb{R}$, and $\psi_\Lambda(0) = 0$. Then

$$0 \le \langle x_\ell(0)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|\xi^{\ell_0})} = \psi_\Lambda(1) \le \sup_{t\in[0,1]} \psi'(t). \tag{7.3.12}$$

The derivative can be calculated explicitly:

$$\psi'_\Lambda(t) = \sum_{\ell_1\in\Lambda,\ \ell_2\in\Lambda^c} J_{\ell_1\ell_2} \cdot \int_0^\beta K^\Lambda_{\ell\ell_1}(0,\tau|t\xi^{\ell_0})\xi^{\ell_0}_{\ell_2}(\tau)\mathrm{d}\tau, \tag{7.3.13}$$

$$\begin{aligned} K^\Lambda_{\ell\ell_1}(0,\tau|t\xi^{\ell_0}) &= \langle x_\ell(0)x_{\ell_1}(\tau)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|t\xi^{\ell_0})} \\ &\quad - \langle x_\ell(0)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|t\xi^{\ell_0})} \cdot \langle x_{\ell_1}(\tau)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|t\xi^{\ell_0})}, \end{aligned}$$

where $\xi^{\ell_0}_{\ell_2}(\tau) = \sqrt{b\log(1+|\ell_2-\ell_0|)}$, which is positive and independent of τ, cf. (3.7.12). The correlation function in (7.3.13) can be considered as the local correlation function (2.5.30) in the external field

$$y_\ell = \sum_{\ell'\in\Lambda^c} J_{\ell\ell'}\xi^{\ell_0}_{\ell'}(\tau) > 0.$$

Then by (2.2.6), it follows that for all $t \in [0,1]$,

$$K^\Lambda_{\ell\ell_1}(0,\tau|t\xi^{\ell_0}) \le K^\Lambda_{\ell\ell_1}(0,\tau|0) = \langle x_\ell(0)x_{\ell_1}(\tau)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|0)}. \tag{7.3.14}$$

Applying this estimate in (7.3.13) and taking into account (7.3.4) we arrive at

$$\psi'_\Lambda(t) \le \sum_{\ell'\in\Lambda^c} M^\Lambda_{\ell\ell'}\xi^{\ell_0}_{\ell'}, \tag{7.3.15}$$

where by the assumption (3.1.19), the configuration ξ^{ℓ_0} belongs to $l^\infty(\ell_0,\alpha)$ for any $\alpha \in \mathcal{I}$. Then by (7.3.5) the right-hand side of (7.3.15) can be made arbitrarily small by taking Λ to be big enough, which by (7.3.12) yields

$$\langle x_\ell(0)\rangle_{\pi^{\mathrm{up}}_{\beta,\Lambda}(\cdot|\xi^{\ell_0})} \to 0, \quad \text{as } \Lambda \nearrow \mathbb{L}.$$

Hence, by (7.3.2) and Corollary 5.3.2 the uniqueness to be proven follows. □

7.3.2 Normality of Fluctuations in Vector Ferromagnets

In this subsection, we consider the model (1.1.3), (1.1.8) which is translation-invariant and ferromagnetic, but with the dimension of the displacements ν can be arbitrary.

Furthermore, we suppose that the anharmonic potential is $V(q) = v(|q|^2)$ with v being convex on $\mathbb{R}_+$. For this model, we show that quantum stabilization prevents it from having phase transitions of the first and second order, which corresponds to a certain *normality* of fluctuations. We recall that the mentioned phase transitions were introduced in Definition 6.1.6.

To describe fluctuations we introduce the *fluctuation operators*, cf. (6.4.12),

$$Q_\Lambda^{(j)} = \frac{1}{\sqrt{|\Lambda|}} \sum_{\ell \in \Lambda} q_\ell^{(j)}, \quad \Lambda \in \mathfrak{L}_{\mathrm{fin}}, \; j = 1, \dots, \nu, \tag{7.3.16}$$

which correspond to *normal fluctuations*.

Definition 7.3.4. The fluctuations of the displacements of oscillators are called normal if the Matsubara functions (1.4.37) constructed on the operators $F_1 = Q^{(j_1)}, \dots, F_n = Q^{(j_n)}$, remain bounded as $\Lambda \nearrow \mathbb{L}$.

If Λ is a box, the parameter (6.1.45) can be written

$$P_\Lambda^{(\alpha)} = \frac{1}{\beta^2 |\Lambda|^\alpha} \sum_{j=1}^{\nu} \int_0^\beta \int_0^\beta \Gamma^{\beta,\Lambda}_{Q_\Lambda^{(j)}, Q_\Lambda^{(j)}}(\tau, \tau') \mathrm{d}\tau \mathrm{d}\tau'. \tag{7.3.17}$$

Thus, if the fluctuations are normal, phase transitions of the second order (and all the more of the first order) are suppressed.

As in the proof of Theorem 7.2.5, the model considered (basic model) will be compared with the scalar ferromagnetic (reference) model with the same particle mass and the anharmonic potential $v(q^2)$. Then the gap parameter Δ_m which we use in the sequel is the one calculated for the reference model. Our main result is presented in the following

Theorem 7.3.5. *Let the model be the same as in Theorem 7.2.5 and let the stability condition involving the interaction parameter $\hat{J}_0$ of the model and the gap parameter Δ_m corresponding to its scalar analog be satisfied. Then the fluctuations of the displacements of oscillators remain normal at all temperatures.*

Proof. For $\Lambda \in \mathfrak{L}_{\mathrm{fin}}$ and $n \in \mathbb{N}$, let us consider the Matsubara functions constructed on the operators $F_1 = Q^{(j_1)}, \dots, F_n = Q^{(j_n)}$. By the rotational invariance, the functions with odd n disappear. For given $F_1 = Q^{(j_1)}, \dots, F_{2n} = Q^{(j_{2n})}$, we split the set

$$\{1, 2, \dots, 2n\} = \bigcup_{\kappa=1}^{\sigma} \Sigma_\kappa, \quad \sigma \leq \nu,$$

into the groups

$$\Sigma_\kappa = \{l \in \{1, 2, \dots, 2n\} \mid j_l = i_\kappa\}, \quad i_\kappa \in \{1, 2, \dots, \nu\}.$$

That is, each Σ_κ contains as many elements as the number of times the given j_κ appears in the collection $\{j_1, j_2, \dots, j_{2n}\}$. The groups Σ_κ are numbered in such a way that

$$|\Sigma_\kappa| \stackrel{\mathrm{def}}{=} s_\kappa \geq s_{\kappa+1}, \quad \kappa = 1, 2, \dots, \sigma.$$

Clearly, $s_1+\cdots+s_\sigma = 2n$. Then the corresponding Matsubara function can be written as

$$\Gamma^{\beta,\Lambda}_{Q^{(j_1)}_\Lambda,\ldots,Q^{(j_{2n})}_\Lambda}(\tau_1,\ldots,\tau_{2n}) = \langle X_1 \ldots X_k\rangle_{\pi_{\beta,\Lambda}(\cdot|0)}, \tag{7.3.18}$$

where we have gathered into each X_κ all the $x^{(j)}$'s with the same $j = j_\kappa$, that is,

$$X_\kappa = \prod_{i\in\Sigma_\kappa}\left(\frac{1}{|\Lambda|}\sum_{\ell\in\Lambda} x_\ell^{(j_\kappa)}(\tau_i)\right). \tag{7.3.19}$$

Then we iterate in (7.3.18) the Cauchy–Schwarz inequality

$$\left|\langle XY\rangle_\mu\right| \le \left[\langle X^2\rangle_\mu\right]^{1/2}\cdot\left[\langle Y^2\rangle_\mu\right]^{1/2}$$

a sufficient number of times and obtain

$$\begin{aligned}&\left|\Gamma^{\beta,\Lambda}_{Q^{(j_1)}_\Lambda,\ldots,Q^{(j_{2n})}_\Lambda}(\tau_1,\ldots,\tau_{2n})\right| \\ &\quad\le \left[\langle X_\sigma^{2^{\sigma-1}}\rangle_{\pi_{\beta,\Lambda}(\cdot|0)}\right]^{2^{-(\sigma-1)}}\cdot\prod_{\kappa=1}^{\sigma-1}\left[\langle X_\kappa^{2^\kappa}\rangle_{\pi_{\beta,\Lambda}(\cdot|0)}\right]^{2^{-\kappa}}.\end{aligned} \tag{7.3.20}$$

Each Matsubara function $\langle X_\kappa^{2^\kappa}\rangle_{\pi_{\beta,\Lambda}(\cdot|0)}$ contains $Q^{(j)}_\Lambda$ with one and the same j; hence, it can be estimated according to (2.5.3) by the corresponding Matsubara function in the scalar model. The latter in turn can be estimated according to (2.2.7), which finally yields

$$\langle X_\kappa^{2^\kappa}\rangle_{\pi_{\beta,\Lambda}(\cdot|0)} \le (2^\kappa s_\kappa - 1)!!\left[\tilde{\Gamma}^{\beta,\tilde{\Lambda}}(0,0)\right]^{2^{\kappa-1}s_\kappa}, \tag{7.3.21}$$

where $(2N-1)!! = 1\cdot 3\cdot 5\ldots(2N-1)$, $\tilde{\Lambda}$ is a box (3.1.2), containing Λ, and

$$\tilde{\Gamma}^{\beta,\tilde{\Lambda}}(0,0) = \sum_{\ell\in\tilde{\Lambda}}\langle x_0(0)x_\ell(0)\rangle_{\tilde{\pi}^{\mathrm{per}}_{\beta,\tilde{\Lambda}}} \tag{7.3.22}$$

is a Matsubara function in the scalar model, which can be estimated by (7.2.2), cf. (7.2.29),

$$\tilde{\Gamma}^{\beta,\tilde{\Lambda}}(0,0) \le \tilde{C}_\beta \stackrel{\mathrm{def}}{=} \frac{1}{\beta}\sum_{k\in\mathcal{K}}\frac{1}{m\Delta_m^2 - \hat{J}_0 + mk^2}. \tag{7.3.23}$$

On the other hand, for all $n,\nu\in\mathbb{N}$ and all combinations of $j_1, j_2, \ldots, j_{2n}$, the following holds:

$$\frac{1}{(2n)}!\left[(2^\sigma s_\sigma - 1)!!\right]^{2^{-(\sigma-1)}}\cdot\prod_{\kappa=1}^{\sigma-1}\left[(2^\kappa s_\kappa - 1)!!\right]^{2^{-\kappa}} \le \left\{2\left[(2^{\nu-2})!\right]^{2^{2-\nu}}\right\}^n \stackrel{\mathrm{def}}{=} c_\nu^n,$$

by which we get in (7.3.20)

$$\left|\Gamma^{\beta,\Lambda}_{Q^{(j_1)}_\Lambda,\ldots,Q^{(j_{2n})}_\Lambda}(\tau_1,\ldots,\tau_{2n})\right| \le (2n)!\left[c_\nu\tilde{C}_\beta\right]^n.$$

This completes the proof. □

7.4 Comments and Bibliographic Notes

Section 7.1: In an ionic crystal, the ions usually form massive complexes the dynamics of which determine the physical properties of the crystal, including its instability with respect to structural phase transitions. Such massive complexes can be considered as classical particles; hence, the phase transitions are described in the framework of classical statistical mechanics. At the same time, in a number of ionic crystals containing localized light ions, see Section 4.1, certain aspects of the phase transitions are apparently unusual from the point of view of classical physics. Their presence can only be explained in a quantum-mechanical context, which points out the essential role of the light ions in such phase transitions. This influence of the quantum effects on phase transition was detected experimentally already in the early 1970s. Here we just mention the data presented in [70], [303] on the KDP-type ferroelectrics and in [224] on the YBaCuO-type superconductors. These data were then used for justifying the corresponding theoretical models. On the theoretical level, the influence of quantum effects on the structural phase transitions in ionic crystals was first discussed in the paper [266], where the particle mass was chosen as the only parameter responsible for these effects. The conclusion, obtained there by means of rather heuristic arguments, was that the long-range order, see Definition 6.1.5, becomes impossible at all temperatures if the mass is sufficiently small. Later a number of rigorous studies of quantum effects inspired by this result as well as by the corresponding experimental data have appeared, see [218], [315] and the references therein. For better understanding the mechanism which produces these effects, a number of illuminating exactly soluble models were introduced and studied, see [246], [285], [313], [314]. As in [266], in these works the reduced mass (1.1.7) was considered as the only parameter responsible for the effects. The result obtained was that the long-range order is suppressed at all temperatures in the light mass limit $m \to 0$. Based on the study of the quantum crystals performed in [11], [12], [14], [16], [18], in the paper [17] a mechanism for the appearance of quantum effects was proposed. Its key parameter is $m\Delta_m^2$, where Δ_m is the least difference between the eigenvalues of the single-particle Hamiltonian H_ℓ (which surely depends on m). This difference is well defined in the translation-invariant case and is surely positive if $\nu = 1$. In the harmonic case, $m\Delta_m^2$ is merely the oscillator rigidity and the stability of the crystal corresponds to large values of this quantity. That is why the parameter $m\Delta_m^2$ was called quantum rigidity and the effect was called quantum stabilization. If the tunneling between the wells gets more intensive (closer minima), or if the mass diminishes, $m\Delta_m^2$ gets bigger and the particle 'forgets' about the details of the potential energy in the vicinity of the origin (including instability) and oscillates as if its equilibrium at zero were stable, as in the harmonic case. In Subsection 7.1.1, we justify this approach and study the properties of the rigidity $m\Delta_m^2$. A preliminary investigation of these properties was done in [12], [194].

Section 7.2: As we show in this section, the quantum stabilization results in a sufficiently fast decay of correlations in the translation-invariant scalar case. The proof of Theorem 7.2.1, which is the main statement of the section, is based on a method from the theory of differential equations and inequalities. Its idea can be explained as follows. The studied local correlation function is obtained as the solution of the integro-differential equation (7.2.7). Along with this equation we construct another, (7.2.11), which: (a) can be solved explicitly; (b) the solution dominates that of (7.2.7). The point is that the equation (7.2.11) corresponds to a harmonic crystal, whose rigidity is exactly the quantum rigidity of the initial model. This gives the decay of correlations in the local periodic state, which afterwards is extended to describe the decay of correlations in the tempered Euclidean Gibbs states. The corresponding results were mostly obtained in [11], [12], [14], [16], [18] as well as in more recent publications [182], [194]. Here we also mention the articles [205], [252] where the decay of correlations in related classical systems was studied.

Section 7.3: Here we prove that the decay of correlations which occurs under the quantum stability condition (7.1.4) yields the uniqueness of tempered Euclidean Gibbs measures for the scalar ferromagnetic version of our model, which need not be translation-invariant. The only condition is (7.3.1) which allows us to compare the model with a translation-invariant reference model (*up*-model), which is less stable with respect to the phase transition, see Lemma 7.3.2. Then the stabilization of this reference model implies the same property for the model considered. In the vector case where the FKG cannot be introduced, the comparison of the model considered with its scalar version yields only the absence of any abnormal fluctuations at any temperature. The corresponding results were published in [188], [189], [190].

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List of Symbols

$\mathfrak{C}(\mathcal{H})$ algebra of bounded linear operators on a Hilbert space $\mathcal{H}$, page 20

$\mathfrak{C}^{+}(\mathcal{H})$ set of all positive elements of $\mathfrak{C}(\mathcal{H})$, page 23

$\mathfrak{C}(\mathcal{H}, \mathbb{C})$ space of bounded linear functionals, page 22

$\mathfrak{C}^{*}(\mathcal{H})$ set of all norm-continuous linear functionals $\theta\colon \mathfrak{C}(\mathcal{H}) \to \mathbb{C}$, page 24

$\mathfrak{C}_{\Lambda}$ algebra of observables, consists of all bounded $A\colon \mathcal{H}_{\Lambda} \to \mathcal{H}_{\Lambda}$, page 65

$\mathfrak{C}_{U}(\mathcal{H})$ group of unitary operators, page 22

$D(P, \widetilde{P})$ Wasserstein distance between measures P and $\widetilde{P}$, defined by (3.7.6), see also (4.3.7), page 239

Δ_{m} gap parameter, see (1.1.162), page 58

E_{s} eigenvalues of H_{Λ}, page 63

$E_{\beta,\Lambda}$ energy functional, see (3.1.33), page 200

$\mathcal{E}(p)$ function used in infrared estimates, see (6.2.51), page 302

$\mathfrak{E}$ family of local functions obeying (3.3.8), page 216

$\mathfrak{E}_{\Lambda}$ family of functions obeying (1.4.58), page 150

$\mathcal{F}^{\mathrm{Laguerre}}$ set of Laguerre entire functions, page 171

$G^{\beta,\Lambda}_{A_{1},\ldots,A_{n}}$ Green function, page 78

$\mathcal{G}^{\mathrm{t}}_{\beta}$ set of all tempered Euclidean Gibbs measures, page 204

$\Gamma^{\beta,\Lambda}_{A_{1},\ldots,A_{n}}$ Matsubara function, page 84

Γ^{μ} Matsubara function for $\mu \in \mathcal{P}(\Omega^{\mathrm{t}}_{\beta})$, see (3.1.61), page 207

H model Hamiltonian, page 17

H_{ℓ} Hamiltonian of a single anharmonic oscillator, page 16

H^{har}_{ℓ} Hamiltonian of a single harmonic oscillator, page 16

H_{Λ} local Hamiltonian, page 63

$H^{\mathrm{per}}_{\Lambda}$ periodic Hamiltonian, page 148

$H_{n,\ell}$ hierarchical local Hamiltonian, see (6.4.7), page 315

$\mathcal{H}_{\ell}$ one-site physical Hilbert space ($= L^{2}(\mathbb{R}^{\nu})$), page 16

$\mathcal{H}_{\Lambda}$ local physical Hilbert space, page 63

$\pi^{\mathrm{per}}_{\beta,\Lambda}$ periodic local Euclidean Gibbs measure, see (3.6.1), page 232

$Q^{(\alpha)}_{n,\ell}$ fluctuation operator, see (6.4.12), page 316

$\mathfrak{r}_m$ quantum rigidity, see (1.1.165), page 59

$\varrho_{\beta,\Lambda}$ local Gibbs state, page 65

S_β covariance operator for χ_β with $\nu = 1$, page 125

$\mathcal{S}(\mathbb{R}^\nu)$ Schwartz space, page 30

$\mathcal{T}_\beta$ tail σ-algebra, see (3.1.55), page 206

$\|T\|$ operator norm, page 20

V_ℓ anharmonic potential, page 16

$\mathcal{W}^{2,2}(\mathbb{R}^\nu)$ Sobolev space, page 42

$\mathcal{W}^{\mathrm{t}}$ weak topology on $\mathcal{P}(\Omega^{\mathrm{t}}_\beta)$, page 205

w_α, $\alpha \in \mathcal{I}$ system of weights, see Definition 3.1.8, page 197

$\Xi(b,\sigma)$ set of tempered configurations defined by (3.3.22), see also (3.3.23), page 220

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