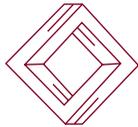


# CONTEMPORARY MATHEMATICS

476



**APORTACIONES MATEMÁTICAS**  
SOCIEDAD MATEMÁTICA MEXICANA

## Fourth Summer School in Analysis and Mathematical Physics

Topics in Spectral Theory  
and Quantum Mechanics

May 2005

Universidad Nacional Autónoma de México  
Cuernavaca, México

Carlos Villegas-Blas  
Editor



*This page intentionally left blank*

Fourth Summer School  
in Analysis and  
Mathematical Physics

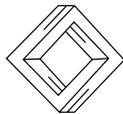
Topics in Spectral Theory  
and Quantum Mechanics

*This page intentionally left blank*

# CONTEMPORARY MATHEMATICS

---

476



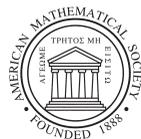
**APORTACIONES MATEMÁTICAS**  
SOCIEDAD MATEMÁTICA MEXICANA

## Fourth Summer School in Analysis and Mathematical Physics

Topics in Spectral Theory  
and Quantum Mechanics

May 2005  
Universidad Nacional Autónoma de México  
Cuernavaca, México

Carlos Villegas-Blas  
Editor



---

**American Mathematical Society**  
Providence, Rhode Island

## Editorial Board of Contemporary Mathematics

Dennis DeTurck, managing editor

George Andrews    Abel Klein    Martin J. Strauss

## Editorial Board of Aportaciones Matemáticas

Guillermo Pastor and Luz de Tesesa, Managing Editors

Marcelo Aguilar	Luis Hernández-Lamoneda	José Seade
José María González-Barrios	Max Neumann	Martha Takane
Luis Gorostiza	Sergio Rajsbaum	Jorge Velasco

2000 *Mathematics Subject Classification*. Primary 81–02, 82D30, 35P05, 81Q99.

---

### Library of Congress Cataloging-in-Publication Data

Summer School in analysis and Mathematical Physics (4th : 2005 : Cuernavaca, Mexico)  
Fourth Summer School in Analysis and Mathematical Physics : topics in spectral theory and quantum mechanics : Fourth Summer School in Analysis and Mathematical Physics, May 2005, Cuernavaca, Mexico / Carlos Villegas-Blas, editor.

p. cm. — (Contemporary mathematics, ISSN 0271-4132 ; 476) (Aportaciones matemáticas / Sociedad Matemática Mexicana)

Includes bibliographical references.

ISBN 978-0-8218-4064-1 (alk. paper)

1. Spectral theory (Mathematics)—Congresses. 2. Schrödinger operator—Congresses. 3. Quantum theory—Congresses. I. Villegas-Blas, Carlos, 1964– II. Sociedad Matemática Mexicana. III. Title. IV. Title: Topics in spectral theory and quantum mechanics.

QC20.7.S64S76 2005

515'.7222—dc22

2008028976

---

**Copying and reprinting.** Material in this book may be reproduced by any means for educational and scientific purposes without fee or permission with the exception of reproduction by services that collect fees for delivery of documents and provided that the customary acknowledgment of the source is given. This consent does not extend to other kinds of copying for general distribution, for advertising or promotional purposes, or for resale. Requests for permission for commercial use of material should be addressed to the Acquisitions Department, American Mathematical Society, 201 Charles Street, Providence, Rhode Island 02904-2294, USA. Requests can also be made by e-mail to [reprint-permission@ams.org](mailto:reprint-permission@ams.org).

Excluded from these provisions is material in articles for which the author holds copyright. In such cases, requests for permission to use or reprint should be addressed directly to the author(s). (Copyright ownership is indicated in the notice in the lower right-hand corner of the first page of each article.)

© 2008 by the American Mathematical Society. All rights reserved.

The American Mathematical Society retains all rights  
except those granted to the United States Government.

Copyright of individual articles may revert to the public domain 28 years  
after publication. Contact the AMS for copyright status of individual articles.

Printed in the United States of America.

∞ The paper used in this book is acid-free and falls within the guidelines  
established to ensure permanence and durability.

Visit the AMS home page at <http://www.ams.org/>

10 9 8 7 6 5 4 3 2 1    13 12 11 10 09 08

## Contents

Introduction	vii
Isoperimetric Inequalities for Eigenvalues of the Laplace Operator RAFAEL D. BENGURIA AND HELMUT LINDE	1
Lectures on Random Schrödinger Operators PETER D. HISLOP	41
Quantum Mechanics, The Stability of Matter and Quantum Electrodynamics ELLIOTT H. LIEB	131

*This page intentionally left blank*

## Introduction

Quantum mechanics started at the beginning of the last century as a revolutionary theory to explain certain experimental facts which were in contradiction with the classical physics theory. One of the very basic aspects of this new theory was to recognize the wave-like behavior of particles. As a consequence, the study of characteristic frequencies of a given system plays a key role in the theory. In mathematical terms, the spectral theory of Schrödinger operators turned out to be part of the base of Quantum Mechanics.

This volume is about three topics strongly related to the development of spectral theory. It consists of three expository articles written by outstanding researchers in mathematical physics and spectral theory: Rafael D. Benguria, Peter D. Hislop, Elliott H. Lieb and Helmut Linde. The articles are based on the lectures delivered by Benguria, Hislop and Lieb in the IV Summer School in analysis and mathematical physics that took place at the Institute of Mathematics, Universidad Nacional Autónoma de México, Cuernavaca, May 2005. The main goal of both the lectures and the articles is to link the basic knowledge of a graduate student in mathematics with three current research topics in mathematical physics: Isoperimetric inequalities for eigenvalues of the Laplace Operator, Random Schrödinger Operators and Stability of Matter. The above-mentioned authors have made decisive contributions to their corresponding research fields. Thus graduate students will find very well written articles guiding and introducing them into current research topics and specialist researchers will find information on recent progress in some areas of mathematical physics written by some of the people who have made it.

Finally let me mention that the article written by Elliott H. Lieb has already appeared in a previous publication: *The Stability of Matter and Quantum Electrodynamics*, the Jahresbericht of the German Math. Soc. JB **106**, 93-110 (2004) (Teubner). Professor Lieb based his lectures on this article and has been so kind to let us reproduce it in this volume in order to make it more complete as the proceedings of the above-mentioned summer school.

Carlos Villegas Blas, editor.  
Universidad Nacional Autónoma de México,  
Instituto de Matemáticas, Unidad Cuernavaca.  
email: villegas@matcuer.unam.mx

*This page intentionally left blank*

# Isoperimetric Inequalities for Eigenvalues of the Laplace Operator

Rafael D. Benguria and Helmut Linde

## CONTENTS

1. Introduction	2
2. Can one hear the shape of a drum?	3
3. Rearrangements	12
4. The Rayleigh–Faber–Krahn inequality	18
5. The Szegő–Weinberger inequality	24
6. The Payne–Pólya–Weinberger inequality	26
7. Appendix	36
References	38

---

2000 *Mathematics Subject Classification.* 35P15, 81Q10, 15A42.

*Key words and phrases.* Isoperimetric Inequalities, Laplace Operator.

The work of RB was partially supported by Fondecyt (CHILE) project 106–0651, and CONICYT/PBCT Proyecto Anillo de Investigación en Ciencia y Tecnología ACT30/2006.

The work of HL was partially supported by a doctoral fellowship from CONICYT (CHILE).

## 1. Introduction

The contents of this manuscript are based on a series of lectures that one of us (RB) gave in the *IV Escuela de Verano en Análisis y Física Matemática*. The Summer School took place at the Unidad Cuernavaca del Instituto de Matemáticas de la Universidad Nacional Autónoma de México. It is a pleasure to thank the organizers of the Summer School for their kind invitation and hospitality. Preliminary versions of these lectures were also given in the *Short Course in Isoperimetric Inequalities for Eigenvalues of the Laplacian*, given by one of us (RB) in February of 2004, as part of the *Thematic Program on Partial Differential Equations* held at the Fields Institute, in Toronto, and also as part of the course *Autovalores del Laplaciano y Geometría* given at the Department of Mathematics of the Universidad de Pernambuco, in Recife, Brazil, in August 2003.

Isoperimetric Inequalities have played an important role in mathematics since the times of the Ancient Greece. The first and best known isoperimetric inequality is the *classical isoperimetric inequality*

$$A \leq \frac{L^2}{4\pi},$$

relating the area  $A$  enclosed by a planar closed curve of perimeter  $L$  (i.e., *Queen Dido's problem* described in Virgilio's epic poem "The Aeneid"). After the introduction of Calculus in the XVII century, many new isoperimetric inequalities have been discovered in mathematics and physics (see, e.g., the review articles [B80, O80, P67, PSz51]). The eigenvalues of the Laplacian are "geometric objects" in the sense they do depend on the geometry of the underlying domain, and to some extent (see section 3) the knowledge of the domain characterizes the geometry of the domain. Therefore it is natural to pose the problem of finding isoperimetric inequalities for the eigenvalues of the Laplacian. The first one to consider this possibility was Lord Rayleigh in his monograph *The Theory of Sound* [R45]. In these lectures we will present some of the problems arising in the study of isoperimetric inequalities for the Laplacian, some of the tools needed in their proof and many bibliographic discussions about the subject. We start our review with the classical problem of Mark Kac, *Can one hear the shape of a drum*. In section three we review the definitions and basic facts about rearrangements of functions. Section 4 is devoted to the Rayleigh–Faber–Krahn inequality. In section 5 we review the Szegő–Weinberger inequality, which is an isoperimetric inequality for the first nontrivial Neumann eigenvalue of the Laplacian. In section 6 we review the Payne–Pólya–Weinberger isoperimetric inequality for the quotient of the first two Dirichlet eigenvalues of the Laplacian, as well as several recent extensions. There are many recent interesting isoperimetric results for the eigenvalues of the bi-harmonic operator, as well as many open problems in that area, which we have left out of this review.

We would like to thank the anonymous referee for many useful corrections and remarks.

## 2. Can one hear the shape of a drum?

...but it would baffle the most skillful  
 mathematician to solve the Inverse Problem,  
 and to find out the shape of a bell by means  
 of the sounds which is capable of sending out.  
 Sir Arthur Schuster (1882).

**2.1. Introduction.** In 1965, the Committee on Educational Media of the Mathematical Association of America produced a film on a mathematical lecture by Mark Kac (1914–1984) with the title: *Can one hear the shape of a drum?* One of the purposes of the film was to inspire undergraduates to follow a career in mathematics. An expanded version of that lecture was later published [K66]. Consider two different smooth, bounded domains, say  $\Omega_1$  and  $\Omega_2$  in the plane. Let  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  be the sequence of eigenvalues of the Laplacian on  $\Omega_1$ , with Dirichlet boundary conditions and, correspondingly,  $0 < \lambda'_1 < \lambda'_2 \leq \lambda'_3 \leq \dots$  be the sequence of Dirichlet eigenvalues for  $\Omega_2$ . Assume that for each  $n$ ,  $\lambda_n = \lambda'_n$  (i.e., both domains are *isospectral*). Then, Mark Kac posed the following question: *Are the domains  $\Omega_1$  and  $\Omega_2$  congruent in the sense of Euclidean geometry?*

In 1910, H. A. Lorentz, at the *Wolfskehl lecture* at the University of Göttingen, reported on his work with Jeans on the characteristic frequencies of the electromagnetic field inside a resonant cavity of volume  $\Omega$  in three dimensions. According to the work of Jeans and Lorentz, the number of eigenvalues of the electromagnetic cavity whose numerical values is below  $\lambda$  (this is a function usually denoted by  $N(\lambda)$ ) is given asymptotically by

$$(2.1) \quad N(\lambda) \approx \frac{|\Omega|}{6\pi^2} \lambda^{3/2},$$

for large values of  $\lambda$ , for many different cavities with simple geometry (e.g., cubes, spheres, cylinders, etc.) Thus, according to the calculations of Jeans and Lorentz, to leading order in  $\lambda$ , the *counting function*  $N(\lambda)$  seemed to depend only on the volume of the electromagnetic cavity  $|\Omega|$ . Apparently David Hilbert (1862–1943), who was attending the lecture, predicted that this conjecture of Lorentz would not be proved during his lifetime. This time, Hilbert was wrong, since his own student, Hermann Weyl (1885–1955) proved the conjecture less than two years after the Lorentz' lecture.

*Remark:* There is a nice account of the work of Hermann Weyl on the eigenvalues of a membrane in his 1948 *J. W. Gibbs Lecture* to the American Mathematical Society [We50].

In  $N$  dimensions, (2.1) reads,

$$(2.2) \quad N(\lambda) \approx \frac{|\Omega|}{(2\pi)^N} C_N \lambda^{N/2},$$

where  $C_N = \pi^{(N/2)} / \Gamma((N/2) + 1)$  denotes the volume of the unit ball in  $N$  dimensions.

Using Tauberian theorems, one can relate the behavior of the counting function  $N(\lambda)$  for large values of  $\lambda$  with the behavior of the function

$$(2.3) \quad Z_{\Omega}(t) \equiv \sum_{n=1}^{\infty} \exp\{-\lambda_n t\},$$

for small values of  $t$ . The function  $Z_{\Omega}(t)$  is the trace of the heat kernel for the domain  $\Omega$ , i.e.,  $Z_{\Omega}(t) = \text{tr exp}(\Delta t)$ . As we mention above,  $\lambda_n(\Omega)$  denotes the  $n$  Dirichlet eigenvalue of the domain  $\Omega$ .

### An example: the behavior of $Z_{\Omega}(t)$ for rectangles

With the help of the *Riemann Theta function*  $\Theta(x)$ , it is simple to compute the trace of the heat kernel when the domain is a rectangle of sides  $a$  and  $b$ , and therefore to obtain the leading asymptotic behavior for small values of  $t$ . The Riemann Theta function is defined by

$$(2.4) \quad \Theta(x) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 x},$$

for  $x > 0$ . The function  $\Theta(x)$  satisfies the following modular relation,

$$(2.5) \quad \Theta(x) = \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right).$$

*Remark:* This modular form for the Theta Function already appears in the classical paper of Riemann [Ri859] (manuscript where Riemann puts forward his famous *Riemann Hypothesis*). In that manuscript, the modular form is attributed to Jacobi.

The modular form (2.5) may be obtained from a very elegant application of Fourier Analysis (see, e.g., [CH53], pp. 75–76) which we reproduce here for completeness. Define

$$(2.6) \quad \varphi_x(y) = \sum_{n=-\infty}^{\infty} e^{-\pi(n+y)^2 x}.$$

Clearly,  $\Theta(x) = \varphi_x(0)$ . Moreover, the function  $\varphi_x(y)$  is periodic in  $y$  of period 1. Thus, we can express it as follows,

$$(2.7) \quad \varphi_x(y) = \sum_{k=-\infty}^{\infty} a_k e^{2\pi k i y},$$

where, the Fourier coefficients are

$$(2.8) \quad a_k = \int_0^1 \varphi_x(y) e^{-2\pi k i y} dy.$$

Replacing the expression (2.6) for  $\varphi_x(y)$  in (2.9), using the fact that  $e^{2\pi k i n} = 1$ , we can write,

$$(2.9) \quad a_k = \int_0^1 \sum_{n=-\infty}^{\infty} e^{-\pi(n+y)^2 x} e^{-2\pi i k(y+n)} dy.$$

Interchanging the order between the integral and the sum, we get,

$$(2.10) \quad a_k = \sum_{n=-\infty}^{\infty} \int_0^1 \left( e^{-\pi(n+y)^2 x} e^{-2\pi i k(y+n)} \right) dy.$$

In the  $n^{\text{th}}$  summand we make the change of variables  $y \rightarrow u = n + y$ . Clearly,  $u$  runs from  $n$  to  $n + 1$ , in the  $n^{\text{th}}$  summand. Thus, we get,

$$(2.11) \quad a_k = \int_{-\infty}^{\infty} e^{-\pi u^2 x} e^{-2\pi i k u} du.$$

i.e.,  $a_k$  is the Fourier transform of a Gaussian. Thus, we finally obtain,

$$(2.12) \quad a_k = \frac{1}{\sqrt{x}} e^{-\pi k^2/x}.$$

Since,  $\Theta(x) = \varphi_x(0)$ , from (2.7) and (2.12) we finally get,

$$(2.13) \quad \Theta(x) = \sum_{k=-\infty}^{\infty} a_k = \frac{1}{\sqrt{x}} \sum_{k=-\infty}^{\infty} e^{-\pi k^2/x} = \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right).$$

*Remarks:* i) The method exhibited above is a particular case of the *Poisson Summation Formula*. See [CH53], pp. 76–77; ii) It should be clear from (2.4) that  $\lim_{x \rightarrow \infty} \Theta(x) = 1$ . Hence, from the modular form for  $\Theta(x)$  we immediately see that

$$(2.14) \quad \lim_{x \rightarrow 0} \sqrt{x} \Theta(x) = 1.$$

Once we have the modular form for the Riemann Theta function, it is simple to get the leading asymptotic behavior of the trace of the heat kernel  $Z_{\Omega}(t)$ , for small values of  $t$ , when the domain  $\Omega$  is a rectangle. Take  $\Omega$  to be the rectangle of sides  $a$  and  $b$ . Its Dirichlet eigenvalues are given by

$$(2.15) \quad \lambda_{n,m} = \pi^2 \left[ \frac{n^2}{a^2} + \frac{m^2}{b^2} \right],$$

with  $n, m = 1, 2, \dots$ . In terms of the Dirichlet eigenvalues, the trace of the heat kernel,  $Z_{\Omega}(t)$  is given by

$$(2.16) \quad Z_{\Omega}(t) = \sum_{n,m=1}^{\infty} e^{-\lambda_{n,m} t}.$$

and using (2.15), and the definition of  $\Theta(x)$ , we get,

$$(2.17) \quad Z_{\Omega}(t) = \frac{1}{4} \left[ \theta\left(\frac{\pi t}{a^2}\right) - 1 \right] \left[ \theta\left(\frac{\pi t}{b^2}\right) - 1 \right].$$

Using the asymptotic behavior of the Theta function for small arguments, i.e., (2.14) above, we have

$$(2.18) \quad Z_{\Omega}(t) \approx \frac{1}{4} \left( \frac{a}{\sqrt{\pi t}} - 1 \right) \left( \frac{b}{\sqrt{\pi t}} - 1 \right) \approx \frac{1}{4\pi t} ab - \frac{1}{4\sqrt{\pi t}}(a+b) + \frac{1}{4} + O(t).$$

In terms of the area of the rectangle  $A = ab$  and its perimeter  $L = 2(a+b)$ , the expression  $Z_{\Omega}(t)$  for the rectangle may be written simply as,

$$(2.19) \quad Z_{\Omega}(t) \approx \frac{1}{4\pi t} A - \frac{1}{8\sqrt{\pi t}} L + \frac{1}{4} + O(t).$$

*Remark:* Using similar techniques, one can compute the small  $t$  behavior of  $Z_\Omega(t)$  for various simple regions of the plane (see, e.g., [McH94]).

The fact that the leading behavior of  $Z_\Omega(t)$  for  $t$  small, for any bounded, smooth domain  $\Omega$  in the plane is given by

$$(2.20) \quad Z_\Omega(t) \approx \frac{1}{4\pi t} A$$

was proven by Hermann Weyl [We11]. Here,  $A = |\Omega|$  denotes the area of  $\Omega$ . In fact, what Weyl proved in [We11] is the *Weyl Asymptotics* of the Dirichlet eigenvalues, i.e., for large  $n$ ,  $\lambda_n \approx (4\pi n)/A$ . Weyl's result (2.20) implies that *one can hear the area of the drum*.

In 1954, the Swedish mathematician, Åke Pleijel [Pj54] obtained the improved asymptotic formula,

$$Z(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}},$$

where  $L$  is the perimeter of  $\Omega$ . In other words, one *can hear* the area and the perimeter of  $\Omega$ . It follows from Pleijel's asymptotic result that if all the frequencies of a drum are equal to those of a circular drum then the drum must itself be circular. This follows from the classical isoperimetric inequality (i.e.,  $L^2 \geq 4\pi A$ , with equality if and only if  $\Omega$  is a circle). In other words, one *can hear* whether a drum is circular. It turns out that it is enough to hear the first two eigenfrequencies to determine whether the drum has the circular shape [AB91]

In 1966, Mark Kac obtained the next term in the asymptotic behavior of  $Z(t)$  [K66]. For a smooth, bounded, multiply connected domain on the plane (with  $r$  holes)

$$(2.21) \quad Z(t) \approx \frac{A}{4\pi t} - \frac{L}{8\sqrt{\pi t}} + \frac{1}{6}(1 - r).$$

Thus, one *can hear* the *connectivity* of a drum. The last term in the above asymptotic expansion changes for domains with corners (e.g., for a rectangular membrane, using the modular formula for the Theta Function, we obtained  $1/4$  instead of  $1/6$ ). Kac's formula (2.21) was rigorously justified by McKean and Singer [McKS67]. Moreover, for domains having corners they showed that each corner with interior angle  $\gamma$  makes an additional contribution to the constant term in (2.21) of  $(\pi^2 - \gamma^2)/(24\pi\gamma)$ .

An sketch of Kac's analysis for the first term of the asymptotic expansion is as follows (here we follow [K66, McH94]). If we imagine some substance concentrated at  $\vec{p} = (x_0, y_0)$  diffusing through the domain  $\Omega$  bounded by  $\partial\Omega$ , where the substance is absorbed at the boundary, then the concentration  $P_\Omega(\vec{p} \mid \vec{r}; t)$  of matter at  $\vec{r} = (x, y)$  at time  $t$  obeys the diffusion equation

$$\frac{\partial P_\Omega}{\partial t} = \Delta P_\Omega$$

with boundary condition  $P_\Omega(\vec{p} \mid \vec{r}; t) \rightarrow 0$  as  $\vec{r} \rightarrow \vec{a}$ ,  $\vec{a} \in \partial\Omega$ , and initial condition  $P_\Omega(\vec{p} \mid \vec{r}; t) \rightarrow \delta(\vec{r} - \vec{p})$  as  $t \rightarrow 0$ , where  $\delta(\vec{r} - \vec{p})$  is the Dirac delta function. The

concentration  $P_\Omega(\vec{p} \mid \vec{r}; t)$  may be expressed in terms of the Dirichlet eigenvalues of  $\Omega$ ,  $\lambda_n$  and the corresponding (normalized) eigenfunctions  $\phi_n$  as follows,

$$P_\Omega(\vec{p} \mid \vec{r}; t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(\vec{p}) \phi_n(\vec{r}).$$

For small  $t$ , the diffusion is slow, that is, it will not *feel* the influence of the boundary in such a short time. We may expect that

$$P_\Omega(\vec{p} \mid \vec{r}; t) \approx P_0(\vec{p} \mid \vec{r}; t),$$

as  $t \rightarrow 0$ , where  $\partial P_0 / \partial t = \Delta P_0$ , and  $P_0(\vec{p} \mid \vec{r}; t) \rightarrow \delta(\vec{r} - \vec{p})$  as  $t \rightarrow 0$ .  $P_0$  in fact represents the heat kernel for the whole  $\mathbb{R}$ , i.e., no boundaries present. This kernel is explicitly known. In fact,

$$P_0(\vec{p} \mid \vec{r}; t) = \frac{1}{4\pi t} \exp(-|\vec{r} - \vec{p}|^2 / 4t),$$

where  $|\vec{r} - \vec{p}|^2$  is just the Euclidean distance between  $\vec{p}$  and  $\vec{r}$ . Then, as  $t \rightarrow 0^+$ ,

$$P_\Omega(\vec{p} \mid \vec{r}; t) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(\vec{p}) \phi_n(\vec{r}) \approx \frac{1}{4\pi t} \exp(-|\vec{r} - \vec{p}|^2 / 4t).$$

Thus, when set  $\vec{p} = \vec{r}$  we get

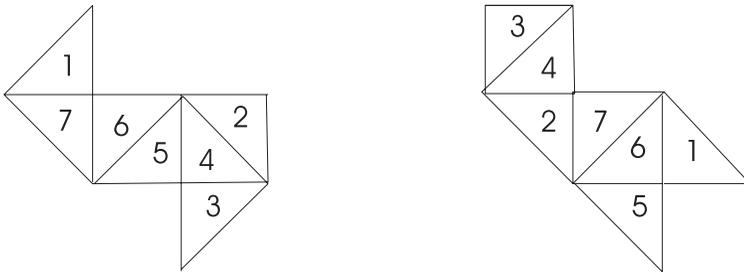
$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n^2(\vec{r}) \approx \frac{1}{4\pi t}.$$

Integrating both sides with respect to  $\vec{r}$ , using the fact that  $\phi_n$  is normalized, we finally get,

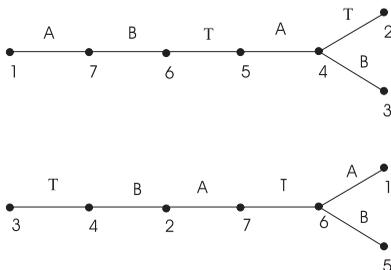
$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \approx \frac{|\Omega|}{4\pi t},$$

which is the first term in the expansion (2.21). Further analysis gives the remaining terms (see [K66]).

**2.2. One cannot hear the shape of a drum.** In the quoted paper of Mark Kac [K66] he says that he personally believed that one cannot hear the shape of a drum. A couple of years before Mark Kac's article, John Milnor [Mi64], had constructed two non-congruent sixteen dimensional tori whose Laplace–Beltrami operators have exactly the same eigenvalues. In 1985 Toshikazu Sunada [Su85], then at Nagoya University in Japan, developed an algebraic framework that provided a new, systematic approach of considering Mark Kac's question. Using Sunada's technique several mathematicians constructed isospectral manifolds (e.g., Gordon and Wilson; Brooks; Buser, etc.). See, e.g., the review article of Robert Brooks (1988) with the situation on isospectrality up to that date in [Br88]. Finally, in 1992, Carolyn Gordon, David Webb and Scott Wolpert [GWW92] gave the definite negative answer to Mark Kac's question and constructed two plane domains (henceforth called the GWW domains) with the same Dirichlet eigenvalues.

FIGURE 1. GWW Isospectral Domains  $D_1$  and  $D_2$ 

T

FIGURE 2. Sunada Graphs corresponding to Domains  $D_1$  and  $D_2$ 

### Proof of Isospectrality Using Transplantation:

The most elementary proof of isospectrality of the GWW domains is done using the method of *transplantation*. For the method of *transplantation* see, e.g., [Be92, Be93]. See also the expository article [Be93b] by the same author. The method also appears briefly described in the article of Sridhar and Kudrolli cited in the *Bibliographical Remarks, iii*) at the end of this section.

To conclude this section we will give the details of the proof of isospectrality of the GWW domains using transplantation. For that purpose label from 1 to 7 the congruent triangles that make the two GWW domains (see Figure 1). Each of this isosceles right triangles has two cathets, labeled  $A$  and  $B$  and the hypotenuse, labeled  $T$ . Each of the pieces (triangles) that make each one of the two domains is connected to one or more neighboring triangles through a side  $A$ , a side  $B$  or a side  $T$ . Each of the two isospectral domains has an associated graph, which are given in Figure 2.

These graphs have their origin in the algebraic formulation of Sunada [Su85]. The vertices in each graph are labeled according to the number that each of the pieces (triangles) has in each of the given domains. As for the edges joining two vertices in these graphs, they are labeled by either an  $A$ , a  $B$  or a  $T$  depending on the type of the common side of two neighboring triangles in Figure 1. In order to show that both domains are isospectral it is convenient to consider any function defined on each domain as consisting of seven parts, each part being the restriction of the original function to each one of the individual triangles that make the domain.

In this way, if  $\psi$  is a function defined on the domain 1, we will write as a vector with seven components, i.e.,  $\psi = [\psi_i]_{i=1}^7$ , where  $\psi_i$  is a scalar function whose support is triangle  $i$  on the domain 1. Similarly, a function  $\varphi$  defined over the domain 2 may be represented as a seven component vector  $\varphi = [\varphi_i]_{i=1}^7$ , with the equivalent meaning but referred to the second domain.

In order to show the isospectrality of the two domains we have to exhibit a mapping transforming the functions defined on the first domain into functions defined in the second domain. Given the decomposition we have made of the eigenfunctions as vectors of seven components, this transformation will be represented by a  $7 \times 7$  matrix. In order to show that the two domains have the same spectra we need this matrix to be orthogonal. This matrix is given explicitly by

$$(2.22) \quad T_D = \begin{pmatrix} -a & a & a & -a & b & -b & b \\ a & -b & -a & b & -a & a & -b \\ a & -a & -b & b & -b & a & -a \\ -a & b & b & -a & a & -b & a \\ -b & a & b & -a & a & -a & b \\ b & -a & -a & b & -a & b & -a \\ -b & b & a & -a & b & -a & a \end{pmatrix}.$$

For the matrix  $T_D$  to be orthogonal, we need that the parameters  $a$  and  $b$  satisfy the following relations:  $4a^2 + 3b^2 = 1$ ,  $2a^2 + 4ab + b^2 = 0$ , and  $4a + 3b = 1$ . Although we do not need the numerical values of  $a$  and  $b$  in the sequel, it is good to know that there is a solution to this system of equations, namely  $a = (1 - 3\sqrt{8}/4)/7$  and  $b = (1 + \sqrt{8})/7$ . The matrix  $T_D$  is orthogonal, i.e.,  $T_D T_D^t = 1$ . The label  $D$  used here refers to the fact that this matrix  $T_D$  is used to show isospectrality for the Dirichlet problem. A similar matrix can be constructed to show isospectrality for the Neumann problem. In order to show isospectrality it is not sufficient to show that the matrix  $T_D$  is orthogonal. It must fulfill two additional properties. On the one hand it should transform a function  $\psi$  that satisfies the Dirichlet conditions in the first domain in a function  $\varphi$  that satisfies Dirichlet boundary conditions on the second domain. Moreover, by elliptic regularity, since the functions  $\psi$  and its image  $\varphi$  satisfy the eigenvalue equation  $-\Delta u = \lambda u$ , they must be smooth (in fact they should be real analytic in the interior of the corresponding domains), and therefore they must be continuous at the adjacent edges connecting two neighboring triangles. Thus, while the function  $\psi$  is continuous when crossing the common edges of neighboring triangles in the domain 1, the function  $\varphi$  should be continuous when crossing the common edges of neighboring triangles in the domain 2. These two properties are responsible for the peculiar structure of  $a$ 's and  $b$ 's in the components of the matrix  $T_D$ .

To illustrate these facts, if the function  $\varphi$  is an eigenfunction of the Dirichlet problem for the domain 2, it must satisfy, among others, the following properties,

$$(2.23) \quad \varphi_2^A = \varphi_7^A,$$

and

$$(2.24) \quad \varphi_5^T = 0.$$

Here  $\varphi_2$  denotes the second component of  $\varphi$ , i.e., the restriction of the function  $\varphi$  to the second triangle in Domain 2 (see figure 2). On the other hand,  $\varphi_2^A$  denotes the restriction of  $\varphi_2$  on the edge  $A$  of triangle 2. Since on the domain 2, the triangles

2 and 7 are glued through a cathet of type  $A$ , (2.23) is precisely the condition that  $\varphi$  has to be smooth in the interior of 2. On the other hand,  $\varphi$  must be a solution of the Dirichlet problem for the domain 2 and as such it must satisfy zero boundary conditions. Since the hypotenuse  $T$  of triangle 5 is part of the boundary of the domain 2,  $\varphi$  must vanish there. This is precisely the condition (2.24). Let us check, as an exercise that if  $\psi$  is smooth and satisfies Dirichlet boundary conditions in the domain 1, its image  $\varphi = T_D\psi$  satisfies (2.24) over the domain 2. We let as an exercise to the reader to check (2.23), and all the other conditions on “smoothness” and boundary condition of  $\varphi$  (this is a long but straightforward task). From (2.22) we have that

$$(2.25) \quad \varphi_5^T = -b\psi_1^T + a\psi_2^T + b\psi_3^T - a\psi_4^T + a\psi_5^T - a\psi_6^T + b\psi_7^T.$$

Since all the sides of type  $T$  of the pieces 1, 3 and 7 in the domain 1 are part of the boundary of the domain (see figure 1),  $\psi_1^T = \psi_3^T = \psi_7^T = 0$ . On the other hand, since 2 and 7 are neighboring triangles in the domain 1, glued through a side of type  $T$ , we have  $\psi_2^T = \psi_4^T$ . By the same reasoning we have  $\psi_5^T = \psi_6^T$ . Using these three conditions on (2.25) we obtain (2.24). All the other conditions can be verified in a similar way. Collecting all these facts, we conclude with

**THEOREM 2.1** (P. Bérard). *The transformation  $T_D$  given by (2.22) is an isometry from  $L^2(D_1)$  into  $L^2(D_2)$  (here  $D_1$  and  $D_2$  are the two domains of Figures 1 and 2), which induces an isometry from  $H_0^1(D_1)$  into  $H_0^1(D_2)$ .*

and therefore we have

**THEOREM 2.2** (C. Gordon, D. Webb, S. Wolpert). *The domains  $D_1$  and  $D_2$  of figures 1 and 2 are isospectral.*

Although the proof by transplantation is straightforward to follow, it does not shed light on the rich geometric, analytic and algebraic structure of the problem initiated by Mark Kac. For the interested reader it is recommendable to read the papers of Sunada [Su85] and of Gordon, Webb and Wolpert [GWW92].

**2.3. Bibliographical Remarks.** i) The sentence of Arthur Schuster (1851–1934) quoted at the beginning of this section is cited in Reed and Simon’s book, volume IV [RSIV]. It is taken from the article A. Schuster, *The Genesis of Spectra*, in *Report of the fifty-second meeting of the British Association for the Advancement of Science* (held at Southampton in August 1882). Brit. Assoc. Rept., pp. 120–121, 1883. Arthur Schuster was a British physicist (he was a leader spectroscopist at the turn of the XIX century). It is interesting to point out that Arthur Schuster found the solution to the Lane–Emden equation with exponent 5, i.e., to the equation,

$$-\Delta u = u^5,$$

in  $\mathbb{R}^3$ , with  $u > 0$  going to zero at infinity. The solution is given by

$$u = \frac{3^{1/4}}{(1 + |x|^2)^{1/2}}.$$

(A. Schuster, *On the internal constitution of the Sun*, Brit. Assoc. Rept. pp. 427–429, 1883). Since the Lane–Emden equation for exponent 5 is the Euler–Lagrange equation for the minimizer of the Sobolev quotient, this is precisely the function that, modulo translations and dilations, gives the best Sobolev constant. For a nice autobiography of Arthur Schuster see A. Schuster, *Biographical fragments*, Mc Millan & Co., London, (1932).

ii) A very nice short biography of Marc Kac was written by H. P. McKean [*Mark Kac in Bibliographical Memoirs*, National Academy of Science, **59**, 214–235 (1990); available on the web (page by page) at <http://www.nap.edu/books/0309041988/html/214.html>]. The reader may want to read his own autobiography: Mark Kac, *Enigmas of Chance*, Harper and Row, NY, 1985 [reprinted in 1987 in paperback by The University of California Press]. For his article in the American Mathematical Monthly, *op. cit.*, Mark Kac obtained the 1968 Chauvenet Prize of the Mathematical Association of America.

iii) It is interesting to remark that the values of the first Dirichlet eigenvalues of the GWW domains were obtained experimentally by S. Sridhar and A. Kudrolli, *Experiments on Not “Hearing the Shape” of Drums*, Physical Review Letters, **72**, 2175–2178 (1994). In this article one can find the details of the physics experiments performed by these authors using very thin electromagnetic resonant cavities with the shape of the Gordon–Webb–Wolpert (GWW) domains. This is the first time that the approximate numerical values of the first 25 eigenvalues of the two GWW were obtained. The corresponding eigenfunctions are also displayed. A quick reference to the transplantation method of Pierre Berard is also given in this article, including the *transplantation matrix* connecting the two isospectral domains. The reader may want to check the web page of S. Sridhar’s Lab (<http://sagar.physics.neu.edu/>) for further experiments on resonating cavities, their eigenvalues and eigenfunctions, as well as on experiments on *quantum chaos*.

iv) The numerical computation of the eigenvalues and eigenfunctions of the pair of GWW isospectral domains was obtained by Tobin A. Driscoll, *Eigenmodes of isospectral domains*, SIAM Review **39**, 1–17 (1997).

v) In its simplified form, the Gordon–Webb–Wolpert domains (GWW domains) are made of seven congruent rectangle isosceles triangles. Certainly the GWW domains have the same area, perimeter and connectivity. The GWW domains are not convex. Hence, one may still ask the question whether one *can hear* the shape of a *convex drum*. There are examples of convex isospectral domains in higher dimension (see e.g. C. Gordon and D. Webb, *Isospectral convex domains in Euclidean Spaces*, Math. Res. Letts. **1**, 539–545 (1994), where they construct convex isospectral domains in  $\mathbb{R}^n$ ,  $n \geq 4$ ). *Remark*: For an update of the Sunada Method, and its applications see the article of Robert Brooks [*The Sunada Method*, in *Tel Aviv Topology Conference “Rothenberg Festschrift” 1998*, Contemporary Mathematics **231**, 25–35 (1999); electronically available at: <http://www.math.technion.ac.il/~rbrooks>]

### 3. Rearrangements

**3.1. Definition and basic properties.** For many problems of functional analysis it is useful to replace some function by an equimeasurable but more symmetric one. This method, which was first introduced by Hardy and Littlewood, is called *rearrangement* or *Schwarz symmetrization* [HLP64]. Among several other applications, it plays an important role in the proofs of isoperimetric inequalities like the Rayleigh–Faber–Krahn inequality or the Payne–Pólya–Weinberger inequality (see section 4 and section 6 below). In the following we present some basic definitions and theorems concerning spherically symmetric rearrangements.

We let  $\Omega$  be a measurable subset of  $\mathbb{R}^n$  and write  $|\Omega|$  for its Lebesgue measure, which may be finite or infinite. If it is finite we write  $\Omega^*$  for an open ball with the same measure as  $\Omega$ , otherwise we set  $\Omega^* = \mathbb{R}^n$ . We consider a measurable function  $u : \Omega \rightarrow \mathbb{R}$  and assume either that  $|\Omega|$  is finite or that  $u$  decays at infinity, i.e.,  $|\{x \in \Omega : |u(x)| > t\}|$  is finite for every  $t > 0$ .

DEFINITION 3.1. The function

$$\mu(t) = |\{x \in \Omega : |u(x)| > t\}|, \quad t \geq 0$$

is called *distribution function* of  $u$ .

From this definition it is straightforward to check that  $\mu(t)$  is a decreasing (non-increasing), right-continuous function on  $\mathbb{R}^+$  with  $\mu(0) = |\text{sprt } u|$  and  $\text{sprt } \mu = [0, \text{ess sup } |u|)$ .

DEFINITION 3.2.

- The *decreasing rearrangement*  $u^\sharp : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  of  $u$  is the distribution function of  $\mu$ .
- The *symmetric decreasing rearrangement*  $u^* : \Omega^* \rightarrow \mathbb{R}^+$  of  $u$  is defined by  $u^*(x) = u^\sharp(C_n |x|^n)$ , where  $C_n = \pi^{n/2} [\Gamma(n/2 + 1)]^{-1}$  is the measure of the  $n$ -dimensional unit ball.

Because  $\mu$  is a decreasing function, Definition 3.2 implies that  $u^\sharp$  is an essentially inverse function of  $\mu$ . The names for  $u^\sharp$  and  $u^*$  are justified by the following two lemmas:

LEMMA 3.3.

- (a) The function  $u^\sharp$  is decreasing,  $u^\sharp(0) = \text{ess sup } |u|$  and  $\text{sprt } u^\sharp = [0, |\text{sprt } u|)$
- (b)  $u^\sharp(s) = \min \{t \geq 0 : \mu(t) \leq s\}$
- (c)  $u^\sharp(s) = \int_0^\infty \chi_{[0, \mu(t))}(s) dt$
- (d)  $|\{s \geq 0 : u^\sharp(s) > t\}| = |\{x \in \Omega : |u(x)| > t\}|$  for all  $t \geq 0$ .
- (e)  $\{s \geq 0 : u^\sharp(s) > t\} = [0, \mu(t))$  for all  $t \geq 0$ .

PROOF. Part (a) is a direct consequence of the definition of  $u^\sharp$ , keeping in mind the general properties of distribution functions stated above. The representation formula in part (b) follows from

$$u^\sharp(s) = |\{w \geq 0 : \mu(w) > s\}| = \sup\{w \geq 0 : \mu(w) > s\} = \min\{w \geq 0 : \mu(w) \leq s\},$$

where we have used the definition of  $u^\sharp$  in the first step and then the monotonicity and right-continuity of  $\mu$ . Part (c) is a consequence of the ‘layer-cake formula’, see Theorem 7.1 in the appendix. To prove part (d) we need to show that

$$(3.1) \quad \{s \geq 0 : u^\sharp(s) > t\} = [0, \mu(t)).$$

Indeed, if  $s$  is an element of the left hand side of (3.1), then by Lemma 3.3, part (b), we have

$$\min\{w \geq 0 : \mu(w) \leq s\} > t.$$

But this means that  $\mu(t) > s$ , i.e.,  $s \in [0, \mu(t))$ . On the other hand, if  $s$  is an element of the right hand side of (3.1), then  $s < \mu(t)$  which implies again by part (b) that

$$u^\sharp(s) = \min\{w \geq 0 : \mu(w) \leq s\} \geq \min\{w \geq 0 : \mu(w) < \mu(t)\} > t,$$

i.e.,  $s$  is also an element of the left hand side. Finally, part (e) is a direct consequence from part (d).  $\square$

It is straightforward to transfer the statements of Lemma 3.3 to the symmetric decreasing rearrangement:

LEMMA 3.4.

- (a) *The function  $u^\star$  is spherically symmetric and radially decreasing.*
- (b) *The measure of the level set  $\{x \in \Omega^\star : u^\star(x) > t\}$  is the same as the measure of  $\{x \in \Omega : |u(x)| > t\}$  for any  $t \geq 0$ .*

From Lemma 3.3 (c) and Lemma 3.4 (b) we see that the three functions  $u$ ,  $u^\sharp$  and  $u^\star$  have the same distribution function and therefore they are said to be *equimeasurable*. Quite analogous to the decreasing rearrangements one can also define increasing ones:

DEFINITION 3.5.

- If the measure of  $\Omega$  is finite, we call  $u_\sharp(s) = u^\sharp(|\Omega| - s)$  the *increasing rearrangement* of  $u$ .
- The *symmetric increasing rearrangement*  $u_\star : \Omega^\star \rightarrow \mathbb{R}^+$  of  $u$  is defined by  $u_\star(x) = u_\sharp(C_n|x|^n)$

**3.2. Main theorems.** Rearrangements are a useful tool of functional analysis because they considerably simplify a function without changing certain properties or at least changing them in a controllable way. The simplest example is the fact that the integral of a function's absolute value is invariant under rearrangement. A bit more generally, we have:

THEOREM 3.6. *Let  $\Phi$  be a continuous increasing map from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  with  $\Phi(0) = 0$ . Then*

$$\int_{\Omega^\star} \Phi(u^\star(x)) \, dx = \int_{\Omega} \Phi(|u(x)|) \, dx = \int_{\Omega^\star} \Phi(u_\star(x)) \, dx.$$

PROOF. The theorem follows directly from Theorem 7.1 in the appendix: If we choose  $m(dx) = dx$ , the right hand side of (7.1) takes the same value for  $v = |u|$ ,  $v = u^\star$  and  $v = u_\star$ . The conditions on  $\Phi$  are necessary since  $\Phi(t) = \nu([0, t])$  must hold for some measure  $\nu$  on  $\mathbb{R}^+$ .  $\square$

For later reference we state a rather specialized theorem, which is an estimate on the rearrangement of a spherically symmetric function that is defined on an asymmetric domain:

THEOREM 3.7. *Assume that  $u_\Omega : \Omega \rightarrow \mathbb{R}^+$  is given by  $u_\Omega(x) = u(|x|)$ , where  $u : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is a non-negative decreasing (resp. increasing) function. Then  $u_\Omega^\star(x) \leq u(|x|)$  (resp.  $u_{\Omega^\star}(x) \geq u(|x|)$ ) for every  $x \in \Omega^\star$ .*

PROOF. Assume first that  $u$  is a decreasing function. The layer–cake representation for  $u_\Omega^*$  is

$$\begin{aligned} u_\Omega^*(x) &= u^\#(C_n|x|^n) = \int_0^\infty \chi_{[0,|\{x \in \Omega: u_\Omega(x) > t\}|)}(C_n|x|^n) dt \\ &\leq \int_0^\infty \chi_{[0,|\{x \in \mathbb{R}^n: u(|x|) > t\}|)}(C_n|x|^n) dt \\ &= \int_0^\infty \chi_{\{x \in \mathbb{R}^n: u(|x|) > t\}}(x) dt \\ &= u(|x|) \end{aligned}$$

□

The product of two functions changes in a controllable way under rearrangement:

THEOREM 3.8. *Suppose that  $u$  and  $v$  are measurable and non-negative functions defined on some  $\Omega \subset \mathbb{R}^n$  with finite measure. Then*

$$(3.2) \quad \int_{\mathbb{R}^+} u^\#(s) v^\#(s) ds \geq \int_\Omega u(x) v(x) dx \geq \int_{\mathbb{R}^+} u^\#(s) v_\#(s) ds$$

and

$$(3.3) \quad \int_{\Omega^*} u^*(x) v^*(x) dx \geq \int_\Omega u(x) v(x) dx \geq \int_{\Omega^*} u^*(x) v_*(x) dx.$$

PROOF. We first show that for every measurable  $\Omega' \subset \Omega$  and every measurable  $v: \Omega \rightarrow \mathbb{R}^+$  the relation

$$(3.4) \quad \int_0^{|\Omega'|} v^\#(s) ds \geq \int_{\Omega'} v dx \geq \int_0^{|\Omega'|} v_\#(s) ds$$

holds: We can assume without loss of generality that  $v$  is integrable. Then the layer–cake formula (see Theorem 7.1 in the appendix) gives

$$(3.5) \quad v = \int_0^\infty \chi_{\{x \in \Omega: v(x) > t\}} dt \quad \text{and} \quad v^\# = \int_0^\infty \chi_{[0, \mu(t))} dt.$$

Hence,

$$\begin{aligned} \int_{\Omega'} v dx &= \int_0^\infty |\Omega' \cap \{x \in \Omega : v(x) > t\}| dt, \\ \int_0^{|\Omega'|} v^\#(s) ds &= \int_0^\infty \min(|\Omega'|, |\{x \in \Omega : v(x) > t\}|) dt. \end{aligned}$$

The first inequality in (3.4) follows. The second inequality in (3.4) can be established with the help of the first:

$$\begin{aligned} \int_0^{|\Omega'|} v_\# ds &= \int_0^{|\Omega|} v_\# ds - \int_{|\Omega'|}^{|\Omega|} v_\# ds \\ &= \int_\Omega v dx - \int_0^{|\Omega| - |\Omega'|} v^\# ds \\ &\leq \int_\Omega v dx - \int_{\Omega \setminus \Omega'} v dx = \int_{\Omega'} v dx. \end{aligned}$$

Now assume that  $u$  and  $v$  are measurable, non-negative and - without losing generality - integrable. Since we can replace  $v$  by  $u$  in the equations (3.5), we have

$$\begin{aligned} \int_{\Omega} u(x)v(x) dx &= \int_0^{\infty} dt \int_{\{x \in \Omega: u(x) > t\}} v(x) dx, \\ \int_0^{\infty} u^{\sharp}(s)v^{\sharp}(s) ds &= \int_0^{\infty} dt \int_0^{\mu(t)} v^{\sharp}(s) ds, \end{aligned}$$

where  $\mu$  is the distribution function of  $u$ . On the other hand, the first inequality in (3.4) tells us that

$$\int_{\{x \in \Omega: u(x) > t\}} v(x) dx \leq \int_0^{\mu(t)} v^{\sharp}(s) ds$$

for every non-negative  $t$ , such that the first inequality in (3.2) follows. The second part of (3.2) can be proven completely analogously, and the inequalities (3.3) are a direct consequence of (3.2).  $\square$

**3.3. Gradient estimates.** The integral of a function's gradient over the boundary of a level set can be estimated in terms of the distribution function:

**THEOREM 3.9.** *Assume that  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz continuous and decays at infinity, i.e., the measure of  $\Omega_t := \{x \in \mathbb{R}^n : |u(x)| > t\}$  is finite for every positive  $t$ . If  $\mu$  is the distribution function of  $u$  then*

$$(3.6) \quad \int_{\partial\Omega_t} |\nabla u| H_{n-1}(dx) \geq -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)}.$$

*Remark:* Here  $H_n(A)$  denotes the  $n$ -dimensional Hausdorff measure of the set  $A$  (see, e.g., [Fe69]).

**PROOF.** On the one hand, by the classical isoperimetric inequality we have

$$(3.7) \quad \int_{\partial\Omega_t} H_{n-1}(dx) \geq n C_n^{1/n} |\Omega_t|^{1-1/n} = n C_n^{1/n} \mu(t)^{1-1/n}.$$

On the other hand, we can use the Cauchy-Schwarz inequality to get

$$\begin{aligned} \int_{\partial\Omega_t} H_{n-1}(dx) &= \int_{\partial\Omega_t} \frac{\sqrt{|\nabla u|}}{\sqrt{|\nabla u|}} H_{n-1}(dx) \\ &\leq \left( \int_{\partial\Omega_t} |\nabla u| H_{n-1}(dx) \right)^{1/2} \left( \int_{\partial\Omega_t} \frac{1}{|\nabla u|} H_{n-1}(dx) \right)^{1/2}. \end{aligned}$$

The last integral in the above formula can be replaced by  $-\mu'(t)$  according to Federer's coarea formula (see, [Fe69]). The result is

$$(3.8) \quad \int_{\partial\Omega_t} H_{n-1}(dx) \leq \left( \int_{\partial\Omega_t} |\nabla u| H_{n-1}(dx) \right)^{1/2} (-\mu'(t))^{1/2}.$$

Comparing the equations (3.7) and (3.8) yields Theorem 3.9.  $\square$

Integrals that involve the norm of the gradient can be estimated using the following important theorem:

**THEOREM 3.10.** *Let  $\Phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a Young function, i.e.,  $\Phi$  is increasing and convex with  $\Phi(0) = 0$ . Suppose that  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  is Lipschitz continuous and decays at infinity. Then*

$$\int_{\mathbb{R}^n} \Phi(|\nabla u^*(x)|) dx \leq \int_{\mathbb{R}^n} \Phi(|\nabla u(x)|) dx.$$

For the special case  $\Phi(t) = t^2$  Theorem 3.10 states that the ‘energy expectation value’ of a function decreases under symmetric rearrangement, a fact that is key to the proof of the Rayleigh–Faber–Krahn inequality (see section 4.1).

**PROOF.** Theorem 3.10 is a consequence of the following chain of (in)equalities, the second step of which follows from Lemma 3.11 below.

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(|\nabla u|) dx &= \int_0^\infty ds \frac{d}{ds} \int_{\{x \in \mathbb{R}^n : |u(x)| > u^*(s)\}} \Phi(|\nabla u|) dx \\ &\geq \int_0^\infty ds \Phi \left( -nC_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s) \right) \\ &= \int_{\mathbb{R}^n} \Phi(|\nabla u^*|) dx. \end{aligned}$$

□

**LEMMA 3.11.** *Let  $u$  and  $\Phi$  be as in Theorem 3.10. Then for almost every positive  $s$  holds*

$$(3.9) \quad \frac{d}{ds} \int_{\{x \in \mathbb{R}^n : |u(x)| > u^*(s)\}} \Phi(|\nabla u|) dx \geq \Phi \left( -nC_n^{1/n} s^{1-1/n} \frac{du^*}{ds}(s) \right).$$

**PROOF.** First we prove Lemma 3.11 for the special case of  $\Phi$  being the identity. If  $s > |\text{sprt } u|$  then (3.9) is clearly true since both sides vanish. Thus we can assume that  $0 < s < |\text{sprt } u|$ . For all  $0 \leq a < b < |\text{sprt } u|$  we show that

$$(3.10) \quad \int_{\{x \in \mathbb{R}^n : u^*(a) > |u(x)| > u^*(b)\}} |\nabla u(x)| dx \geq nC_n^{1/n} a^{1-1/n} (u^*(a) - u^*(b)).$$

The statement (3.10) is proven by the following chain of inequalities, in which we first use Federer’s coarea formula, then the classical isoperimetric inequality in  $\mathbb{R}^n$  and finally the monotonicity of the integrand:

$$\begin{aligned} \text{l.h.s. of (3.10)} &= \int_{u^*(b)}^{u^*(a)} H_{n-1} \{x \in \mathbb{R}^n : |u(x)| = t\} dt \\ &\geq \int_{u^*(b)}^{u^*(a)} nC_n^{1/n} |\{x \in \mathbb{R}^n : |u(x)| \geq t\}|^{1-1/n} dt \\ &\geq nC_n^{1/n} |\{x \in \mathbb{R}^n : |u(x)| \geq u^*(a)\}|^{1-1/n} \cdot (u^*(a) - u^*(b)) \\ &\geq \text{r.h.s. of (3.10)}. \end{aligned}$$

In the case of  $\Phi$  being the identity, Lemma 3.11 follows from (3.10): Replace  $b$  by  $a + \epsilon$  with some  $\epsilon > 0$ , multiply both sides by  $\epsilon^{-1}$  and then let  $\epsilon$  go to zero.

It remains to show that equation (3.9) holds for almost every  $s > 0$  if  $\Phi$  is not the identity but some general Young function. From the monotonicity of  $u^*$  follows that for almost every  $s > 0$  either  $\frac{du^*}{ds}$  is zero or there is a neighborhood of  $s$  where

$u^*$  is continuous and decreases strictly. In the first case there is nothing to prove, thus we can assume the second one. Then we have

$$(3.11) \quad |\{x \in \mathbb{R}^n : u^*(s) \geq |u(x)| > u^*(s + \epsilon)\}| = \epsilon$$

for small enough  $\epsilon > 0$ . Consequently, we can apply Jensen's inequality to get

$$\begin{aligned} \frac{1}{\epsilon} \int_{\{x \in \mathbb{R}^n : u^*(s) \geq |u(x)| > u^*(s+h)\}} \Phi(|\nabla u(x)|) dx \\ \geq \Phi \left( \frac{1}{\epsilon} \int_{\{x \in \mathbb{R}^n : u^*(s) \geq |u(x)| > u^*(s+h)\}} |\nabla u(x)| dx \right). \end{aligned}$$

Taking the limit  $\epsilon \downarrow 0$ , this yields

$$\frac{d}{ds} \int_{\{x \in \mathbb{R}^n : |u(x)| > u^*(s)\}} \Phi(|\nabla u(x)|) dx \geq \Phi \left( \frac{d}{ds} \int_{\{x \in \mathbb{R}^n : |u(x)| > u^*(s)\}} |\nabla u(x)| dx \right).$$

Since we have already proven Lemma 3.11 for the case of  $\Phi$  being the identity, we can apply it to the argument of  $\Phi$  on the right hand side of the above inequality. The statement of Lemma 3.11 for general  $\Phi$  follows.  $\square$

**3.4. Bibliographical Remarks.** i) Rearrangements of functions were introduced by G. Hardy and J. E. Littlewood. Their results are contained in the classical book, G.H. Hardy, J. E. Littlewood, J.E., and G. Pólya, *Inequalities*, 2d ed., Cambridge University Press, 1952. The fact that the  $L^2$  norm of the gradient of a function decreases under rearrangements was proven by Faber and Krahn (see references below). A more modern proof as well as many results on rearrangements and their applications to PDE's can be found in G. Talenti, *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3**, 697–718 (1976). (The reader may want to see also the article by E.H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. **57**, 93–105 (1976/77), for an alternative proof of the fact that the  $L^2$  norm of the gradient decreases under rearrangements using heat kernel techniques). An excellent expository review on rearrangements of functions (with a good bibliography) can be found in Talenti, G., *Inequalities in rearrangement invariant function spaces*, in *Nonlinear analysis, function spaces and applications, Vol. 5 (Prague, 1994)*, 177–230, Prometheus, Prague, 1994. (available at the website: <http://www.emis.de/proceedings/Praha94/>). The Riesz rearrangement inequality is the assertion that for nonnegative measurable functions  $f, g, h$  in  $\mathbb{R}^n$ , we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} f(y)g(x-y)h(x)dx dy \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} f^*(y)g^*(x-y)h^*(x)dx dy$$

. For  $n = 1$  the inequality is due to F. Riesz, *Sur une inégalité intégrale*, Journal of the London Mathematical Society **5**, 162–168 (1930). For general  $n$  is due to S.L. Sobolev, *On a theorem of functional analysis*, Mat. Sb. (NS) **4**, 471–497 (1938) [the English translation appears in AMS Translations (2) **34**, 39–68 (1963)]. The cases of equality in the Riesz inequality were studied by A. Burchard, *Cases of equality in the Riesz rearrangement inequality*, Annals of Mathematics **143** 499–627 (1996) (this paper also has an interesting history of the problem).

ii) Rearrangements of functions have been extensively used to prove symmetry properties of positive solutions of nonlinear PDE's. See, e.g., Kawohl, Bernhard, *Rearrangements and convexity of level sets in PDE*. Lecture Notes in Mathematics, 1150. Springer-Verlag, Berlin (1985), and references therein.

iii) There are different types of rearrangements of functions. For an interesting approach to rearrangements see, Brock, Friedemann and Solynin, Alexander Yu. *An approach to symmetrization via polarization*. Trans. Amer. Math. Soc. **352** 1759–1796 (2000). This approach goes back through Baernstein–Taylor (Duke Math. J. 1976), who cite Ahlfors (book on “Conformal invariants”, 1973), who in turn credits Hardy and Littlewood.

## 4. The Rayleigh–Faber–Krahn inequality

**4.1. The Euclidean case.** Many isoperimetric inequalities have been inspired by the question which geometrical layout of some physical system maximizes or minimizes a certain quantity. One may ask, for example, how matter of a given mass density must be distributed to minimize its gravitational energy, or which shape a conducting object must have to maximize its electrostatic capacity. The most famous question of this kind was put forward at the end of the 19<sup>th</sup> century by Lord Rayleigh in his work on the theory of sound [R45]: He conjectured that among all drums of the same area and the same tension the circular drum produces the lowest fundamental frequency. This statement was proven independently in the 1920s by Faber [F23] and Krahn [K25, K26].

To treat the problem mathematically, we consider an open bounded domain  $\Omega \subset \mathbb{R}^2$  which matches the shape of the drum. Then *the oscillation frequencies of the drum* are given by the eigenvalues of the Laplace operator  $-\Delta_D^\Omega$  on  $\Omega$  with Dirichlet boundary conditions, up to a constant that depends on the drum's tension and mass density. In the following we will allow the more general case  $\Omega \subset \mathbb{R}^n$  for  $n \geq 2$ , although the physical interpretation as a drum only makes sense if  $n = 2$ . We define the Laplacian  $-\Delta_D^\Omega$  via the quadratic-form approach, i.e., it is the unique self-adjoint operator in  $L^2(\Omega)$  which is associated with the closed quadratic form

$$h[\Psi] = \int_{\Omega} |\nabla \Psi|^2 dx, \quad \Psi \in H_0^1(\Omega).$$

Here  $H_0^1(\Omega)$ , which is a subset of the Sobolev space  $W^{1,2}(\Omega)$ , is the closure of  $C_0^\infty(\Omega)$  with respect to the form norm

$$(4.1) \quad |\cdot|_h^2 = h[\cdot] + \|\cdot\|_{L^2(\Omega)}^2.$$

For more details about the important question of how to define the Laplace operator on arbitrary domains and subject to different boundary conditions we refer the reader to [D96, BS87].

The spectrum of  $-\Delta_D^\Omega$  is purely discrete since  $H_0^1(\Omega)$  is, by Rellich's theorem, compactly imbedded in  $L^2(\Omega)$  (see, e.g., [BS87]). We write  $\lambda_1(\Omega)$  for the lowest eigenvalue of  $-\Delta_D^\Omega$ .

**THEOREM 4.1** (Rayleigh–Faber–Krahn inequality). *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with smooth boundary and  $\Omega^* \subset \mathbb{R}^n$  a ball with the same measure as  $\Omega$ . Then*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$$

*with equality if and only if  $\Omega$  itself is a ball.*

PROOF. With the powerful mathematical tool of rearrangements (see section 3) at hand, the proof of the Rayleigh–Faber–Krahn inequality is actually not difficult. Let  $\Psi$  be the positive normalized first eigenfunction of  $-\Delta_D^\Omega$ . Since the domain of a positive self-adjoint operator is a subset of its form domain, we have  $\Psi \in H_0^1(\Omega)$ . Then we have  $\Psi^* \in H_0^1(\Omega^*)$ . Thus we can apply first the min–max principle and then the Theorems 3.6 and 3.10 to obtain

$$\lambda_1(\Omega^*) \leq \frac{\int_{\Omega^*} |\nabla \Psi^*|^2 \, d^n x}{\int_{\Omega^*} |\Psi^*|^2 \, d^n x} \leq \frac{\int_{\Omega} |\nabla \Psi|^2 \, d^n x}{\int_{\Omega} \Psi^2 \, d^n x} = \lambda_1(\Omega).$$

□

The Rayleigh–Faber–Krahn inequality has been extended to a number of different settings, for example to Laplace operators on curved manifolds or with respect to different measures. In the following we shall give an overview of these generalizations.

**4.2. Schrödinger operators.** It is not difficult to extend the Rayleigh–Faber–Krahn inequality to Schrödinger operators, i.e., to operators of the form  $-\Delta + V(x)$ . Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain and  $V : \mathbb{R}^n \rightarrow \mathbb{R}^+$  a non-negative potential in  $L^1(\Omega)$ . Then the quadratic form

$$h_V[u] = \int_{\Omega} (|\nabla u|^2 + V(x)|u|^2) \, d^n x,$$

defined on

$$\text{Dom } h_V = H_0^1(\Omega) \cap \left\{ u \in L^2(\Omega) : \int_{\Omega} (1 + V(x))|u(x)|^2 \, d^n x < \infty \right\}$$

is closed (see, e.g., [D90, D96]). It is associated with the positive self-adjoint Schrödinger operator  $H_V = -\Delta + V(x)$ . The spectrum of  $H_V$  is purely discrete and we write  $\lambda_1(\Omega, V)$  for its lowest eigenvalue.

THEOREM 4.2. *Under the assumptions stated above,*

$$\lambda_1(\Omega^*, V_*) \leq \lambda_1(\Omega, V).$$

PROOF. Let  $u_1 \in \text{Dom } h_V$  be the positive normalized first eigenfunction of  $H_V$ . Then we have  $u_1^* \in H_0^1(\Omega^*)$  and by Theorem 3.8

$$\int_{\Omega^*} (1 + V_*)u_1^{*2} \, d^n x \leq \int_{\Omega} (1 + V)u_1^2 \, d^n x < \infty.$$

Thus  $u_1^* \in \text{Dom } h_{V_*}$  and we can apply first the min–max principle and then Theorems 3.6, 3.8 and 3.10 to obtain

$$\begin{aligned} \lambda_1(\Omega^*, V_*) &\leq \frac{\int_{\Omega^*} (|\nabla u_1^*|^2 + V_* u_1^{*2}) \, d^n x}{\int_{\Omega^*} |u_1^*|^2 \, d^n x} \\ &\leq \frac{\int_{\Omega} (|\nabla u_1|^2 + V u_1^2) \, d^n x}{\int_{\Omega} u_1^2 \, d^n x} = \lambda_1(\Omega, V). \end{aligned}$$

□

**4.3. Gaussian Space.** Consider the space  $\mathbb{R}^n$  ( $n \geq 2$ ) endowed with the measure  $d\mu = \gamma(x) d^n x$ , where

$$(4.2) \quad \gamma(x) = (2\pi)^{-n/2} e^{-\frac{|x|^2}{2}},$$

is the standard Gaussian density. Since  $\gamma(x)$  is a Gauss function we will call  $(\mathbb{R}^n, d\mu)$  the *Gaussian space*. For any Lebesgue-measurable  $\Omega \subset \mathbb{R}^n$  we define the Gaussian perimeter of  $\Omega$  by

$$P_\mu(\Omega) = \sup \left\{ \int_\Omega ((\nabla - x) \cdot v(x)) \gamma(x) dx : v \in C_0^1(\Omega, \mathbb{R}^n), \|v\|_\infty \leq 1 \right\}.$$

If  $\partial\Omega$  is sufficiently well-behaved then

$$P_\mu(\Omega) = \int_{\partial\Omega} \gamma(x) dH^{n-1},$$

where  $H^{n-1}$  is the  $(n-1)$ -dimensional Hausdorff measure [Fe69]. It has been shown by Borell that in Gaussian space there is an analog to the classical isoperimetric inequality. Yet the sets that minimize the surface (i.e., the Gaussian perimeter) for a given volume (i.e., Gaussian measure) are not balls, as in Euclidean space, but half-spaces [B75]. More precisely:

**THEOREM 4.3.** *Let  $\Omega \subset \mathbb{R}^n$  be open and measurable. Let further  $\Omega^\sharp$  be the half-space  $\{\vec{x} \in \mathbb{R}^n : x_1 > a\}$ , where  $a \in \mathbb{R}$  is chosen such that  $\mu(\Omega) = \mu(\Omega^\sharp)$ . Then*

$$P_\mu(\Omega) \geq P_\mu(\Omega^\sharp)$$

*with equality only if  $\Omega = \Omega^\sharp$  up to a rotation.*

Next we define the Laplace operator for domains in Gaussian space. We choose an open domain  $\Omega \subset \mathbb{R}^n$  with  $\mu(\Omega) < \mu(\mathbb{R}^n) = 1$  and consider the function space

$$H^1(\Omega, d\mu) = \left\{ u \in W_{\text{loc}}^{1,1}(\Omega) \text{ such that } (u, |\nabla u|) \in L^2(\Omega, d\mu) \times L^2(\Omega, d\mu) \right\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega, d\mu)} = \|u\|_{L^2(\Omega, d\mu)} + \|\nabla u\|_{L^2(\Omega, d\mu)}.$$

We define the quadratic form

$$h[u] = \int_\Omega |\nabla u|^2 d\mu$$

on the closure of  $C_0^\infty(\Omega)$  in  $H^1(\Omega, d\mu)$ . Since  $H^1$  is complete,  $\text{Dom } h$  is also complete under its form norm, which is equal to  $\|\cdot\|_{H^1(\Omega, d\mu)}$ . The quadratic form  $h$  is therefore closed and associated with a unique positive self-adjoint operator  $-\Delta_G$ .  $\text{Dom } h$  is embedded compactly in  $L^2(\Omega, d\mu)$  and therefore the spectrum of  $-\Delta_G$  is discrete. Its eigenfunctions and eigenvalues are solutions of the boundary value problem

$$(4.3) \quad \begin{aligned} -\sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \gamma(x) \frac{\partial u}{\partial x_j} \right) &= \lambda \gamma(x) u && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

The analog of the Rayleigh–Faber–Krahn inequality for *Gaussian Spaces* is the following theorem.

**THEOREM 4.4.** *Let  $\lambda_1(\Omega)$  be the lowest eigenvalue of  $-\Delta_G$  on  $\Omega$  and let  $\Omega'$  be a half-space of the same Gaussian measure as  $\Omega$ . Then*

$$\lambda_1(\Omega') \leq \lambda_1(\Omega).$$

*Equality holds if and only if  $\Omega$  itself is a half-space.*

**4.4. Spaces of constant curvature.** Differential operators can not only be defined for functions in Euclidean space, but also for the more general case of functions on Riemannian manifolds. It is therefore natural to ask whether the isoperimetric inequalities for the eigenvalues of the Laplacian can be generalized to such settings as well. In this section we will state Rayleigh–Faber–Krahn type theorems for the spaces of constant non-zero curvature, i.e., for the sphere and the hyperbolic space. Isoperimetric inequalities for the second Laplace eigenvalue in these curved spaces will be discussed in section 6.7.

To start with, we define the Laplacian in hyperbolic space as a self-adjoint operator by means of the quadratic form approach. We realize  $\mathbb{H}^n$  as the open unit ball  $B = \{(x_1, \dots, x_n) : \sum_{j=1}^n x_j^2 < 1\}$  endowed with the metric

$$(4.4) \quad ds^2 = \frac{4|dx|^2}{(1-|x|^2)^2}$$

and the volume element

$$(4.5) \quad dV = \frac{2^n dx}{(1-|x|^2)^n},$$

where  $|\cdot|$  denotes the Euclidean norm. Let  $\Omega \subset \mathbb{H}^n$  be an open domain and assume that it is bounded in the sense that  $\Omega$  does not touch the boundary of  $B$ . The quadratic form of the Laplace operator in hyperbolic space is the closure of

$$(4.6) \quad h[u] = \int_{\Omega} g^{ij}(\partial_i u)(\partial_j u) dV, \quad u \in C_0^\infty(\Omega).$$

It is easy to see that the form (4.6) is indeed closeable: Since  $\Omega$  does not touch the boundary of  $B$ , the metric coefficients  $g^{ij}$  are bounded from above on  $\Omega$ . They are also bounded from below by  $g^{ij} \geq 4$ . Consequently, the form norms of  $h$  and its Euclidean counterpart, which is the right hand side of (4.6) with  $g^{ij}$  replaced by  $\delta^{ij}$ , are equivalent. Since the ‘Euclidean’ form is well known to be closeable,  $h$  must also be closeable.

By standard spectral theory, the closure of  $h$  induces a unique positive self-adjoint operator  $-\Delta_{\mathbb{H}}$  which we call the Laplace operator in hyperbolic space. Equivalence between corresponding norms in Euclidean and hyperbolic space implies that the imbedding  $\text{Dom } h \rightarrow L^2(\Omega, dV)$  is compact and thus the spectrum of  $-\Delta_{\mathbb{H}}$  is discrete. For its lowest eigenvalue the following Rayleigh–Faber–Krahn inequality holds.

**THEOREM 4.5.** *Let  $\Omega \subset \mathbb{H}^n$  be an open bounded domain with smooth boundary and  $\Omega^* \subset \mathbb{H}^n$  an open geodesic ball of the same measure. Denote by  $\lambda_1(\Omega)$  and  $\lambda_1(\Omega^*)$  the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$$

*with equality only if  $\Omega$  itself is a geodesic ball.*

The Laplace operator  $-\Delta_{\mathbb{S}}$  on a domain which is contained in the unit sphere  $\mathbb{S}^n$  can be defined in a completely analogous fashion to  $-\Delta_{\mathbb{H}}$  by just replacing the metric  $g^{ij}$  in (4.6) by the metric of  $\mathbb{S}^n$ .

**THEOREM 4.6.** *Let  $\Omega \subset \mathbb{S}^n$  be an open bounded domain with smooth boundary and  $\Omega^* \subset \mathbb{S}^n$  an open geodesic ball of the same measure. Denote by  $\lambda_1(\Omega)$  and  $\lambda_1(\Omega^*)$  the lowest eigenvalue of the Dirichlet-Laplace operator on the respective domain. Then*

$$\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$$

with equality only if  $\Omega$  itself is a geodesic ball.

The proofs of the above theorems are similar to the proof for the Euclidean case and will be omitted here. A more general Rayleigh–Faber–Krahn theorem for the Laplace operator on Riemannian manifolds and its proof can be found in the book of Chavel [C84].

**4.5. Robin Boundary Conditions.** Another generalization of the Rayleigh–Faber–Krahn inequality holds for the boundary value problem

$$(4.7) \quad \begin{aligned} -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} u &= \lambda u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \beta u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

on a bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$  with the outer unit normal  $\nu$  and some constant  $\beta > 0$ . This so-called Robin boundary value problem can be interpreted as a mathematical model for a vibrating membrane whose edge is coupled elastically to some fixed frame. The parameter  $\beta$  indicates how tight this binding is and the eigenvalues of (4.7) correspond to the resonant vibration frequencies of the membrane. They form a sequence  $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$  (see, e.g., [M85]).

The Robin problem (4.7) is more complicated than the corresponding Dirichlet problem for several reasons. For example, the very useful property of domain monotonicity does not hold for the eigenvalues of the Robin–Laplacian. That is, if one enlarges the domain  $\Omega$  in a certain way, the eigenvalues may go up. It is known though, that a very weak form of domain monotonicity holds, namely that  $\lambda_1(B) \leq \lambda_1(\Omega)$  if  $B$  is ball that contains  $\Omega$ . Another difficulty of the Robin problem, compared to the Dirichlet case, is that the level sets of the eigenfunctions may touch the boundary. This makes it impossible, for example, to generalize the proof of the Rayleigh–Faber–Krahn inequality in a straightforward way. Nevertheless, such an isoperimetric inequality holds, as proven by Daners:

**THEOREM 4.7.** *Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a bounded Lipschitz domain,  $\beta > 0$  a constant and  $\lambda_1(\Omega)$  the lowest eigenvalue of (4.7). Then  $\lambda_1(\Omega^*) \leq \lambda_1(\Omega)$ .*

For the proof of Theorem 4.7, which is not short, we refer the reader to [D06].

**4.6. Bibliographical Remarks.** i) The Rayleigh–Faber–Krahn inequality is an isoperimetric inequality concerning the lowest eigenvalue of the Laplacian, with Dirichlet boundary condition, on a bounded domain in  $\mathbb{R}^n$  ( $n \geq 2$ ). Let  $0 < \lambda_1(\Omega) < \lambda_2(\Omega) \leq \lambda_3(\Omega) \leq \dots$  be the Dirichlet eigenvalues of the Laplacian in  $\Omega \subset \mathbb{R}^n$ , i.e.,

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on the boundary of } \Omega. \end{aligned}$$

If  $n = 2$ , the Dirichlet eigenvalues are proportional to the square of the eigenfrequencies of an elastic, homogeneous, vibrating membrane with fixed boundary. The Rayleigh–Faber–Krahn inequality for the membrane (i.e.,  $n = 2$ ) states that

$$\lambda_1 \geq \frac{\pi j_{0,1}^2}{A},$$

where  $j_{0,1} = 2.4048\dots$  is the first zero of the Bessel function of order zero, and  $A$  is the area of the membrane. Equality is obtained if and only if the membrane is circular. In other words, *among all membranes of given area, the circle has the lowest fundamental frequency*. This inequality was conjectured by Lord Rayleigh (see, Rayleigh, J.W.S., **The Theory of Sound**, second edition, London, 1894/1896, pp. 339–340). In 1918, Courant (see R. Courant, *Math. Z.* **1**, 321–328 (1918)) proved the weaker result that among all membranes of the same perimeter  $L$  the circular one yields the least lowest eigenvalue, i.e.,

$$\lambda_1 \geq \frac{4\pi^2 j_{0,1}^2}{L^2},$$

with equality if and only if the membrane is circular. Rayleigh’s conjecture was proven independently by Faber [F23] and Krahn [K25]. The corresponding isoperimetric inequality in dimension  $n$ ,

$$\lambda_1(\Omega) \geq \left(\frac{1}{|\Omega|}\right)^{2/n} C_n^{2/n} j_{n/2-1,1},$$

was proven by Krahn [K26]. Here  $j_{m,1}$  is the first positive zero of the Bessel function  $J_m$ ,  $|\Omega|$  is the volume of the domain, and  $C_n = \pi^{n/2}/\Gamma(n/2 + 1)$  is the volume of the  $n$ -dimensional unit ball. Equality is attained if and only if  $\Omega$  is a ball. For more details see, R.D. Benguria, *Rayleigh–Faber–Krahn Inequality*, in *Encyclopaedia of Mathematics*, Supplement III, Managing Editor: M. Hazewinkel, Kluwer Academic Publishers, pp. 325–327, (2001).

ii) A natural question to ask concerning the Rayleigh–Faber–Krahn inequality is the question of stability. If the lowest eigenvalue of a domain  $\Omega$  is within  $\epsilon$  (positive and sufficiently small) of the isoperimetric value  $\lambda_1(\Omega^*)$ , how close is the domain  $\Omega$  to being a ball? The problem of stability for (convex domains) concerning the Rayleigh–Faber–Krahn inequality was solved by Antonios Melas (Melas, A.D., *The stability of some eigenvalue estimates*, *J. Differential Geom.* **36**, 19–33 (1992)). In the same reference, Melas also solved the analogous stability problem for convex domains with respect to the PPW inequality (see section 6 below). The work of Melas has been extended to the case of the Szegő–Weinberger inequality (for the first nontrivial Neumann eigenvalue) by Xu, Youyu, *The first nonzero eigenvalue of Neumann problem on Riemannian manifolds*, *J. Geom. Anal.* **5** 151–165 (1995), and to the case of the PPW inequality on spaces of constant curvature by Andrés Avila, *Stability results for the first eigenvalue of the Laplacian on domains in space forms*, *J. Math. Anal. Appl.* **267**, 760–774 (2002). In this connection it is worth mentioning related results on the isoperimetric inequality of R. Hall, *A quantitative isoperimetric inequality in  $n$ -dimensional space*, *J. Reine Angew. Math.* **428** (1992), 161–176, as well as recent results of Maggi, Pratelli and Fusco (recently reviewed by F. Maggi in *Bull. Amer. Math. Soc.* **45** (2008), 367–408).

iii) The analog of the Faber–Krahn inequality for domains in the sphere  $S^n$  was proven by Sperner, Emanuel, Jr. *Zur Symmetrisierung von Funktionen auf Sphären*, *Math. Z.* **134** (1973), 317–327

iv) For isoperimetric inequalities for the lowest eigenvalue of the Laplace–Beltrami operator on manifolds, see, e.g., the book by Chavel, Isaac, **Eigenvalues in Riemannian**

**geometry.** Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984, (in particular sections IV and V), and also the articles, Chavel, I. and Feldman, E. A. *Isoperimetric inequalities on curved surfaces*. Adv. in Math. **37**, 83–98 (1980), and Bandle, Catherine, *Konstruktion isoperimetrischer Ungleichungen der mathematischen Physik aus solchen der Geometrie*, Comment. Math. Helv. 46, 182–213 (1971).

## 5. The Szegő–Weinberger inequality

In analogy to the Rayleigh–Faber–Krahn inequality for the Dirichlet–Laplacian one may ask which shape of a domain maximizes certain eigenvalues of the Laplace operator with Neumann boundary conditions. Of course, this question is trivial for the lowest Neumann eigenvalue, which is always zero. In 1952 Kornhauser and Stakgold [KS52] conjectured that the ball maximizes the first non-zero Neumann eigenvalue among all domains of the same volume. This was first proven in 1954 by Szegő [S54] for two-dimensional simply connected domains, using conformal mappings. Two years later his result was generalized general domains in any dimension by Weinberger [W56], who came up with a new strategy for the proof.

Although the Szegő–Weinberger inequality appears to be the analog for Neumann eigenvalues of the Rayleigh–Faber–Krahn inequality, its proof is completely different. The reason is that the first non-trivial Neumann eigenfunction must be orthogonal to the constant function, and thus it must have a change of sign. The simple symmetrization procedure that is used to establish the Rayleigh–Faber–Krahn inequality can therefore not work.

In general, when dealing with Neumann problems, one has to take into account that the spectrum of the respective Laplace operator on a bounded domain is very unstable under perturbations. One can change the spectrum arbitrarily much by only a slight modification of the domain, and if the boundary is not smooth enough, the Laplacian may even have essential spectrum. A sufficient condition for the spectrum of  $-\Delta_N^\Omega$  to be purely discrete is that  $\Omega$  is bounded and has a Lipschitz boundary [D96]. We write  $0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \dots$  for the sequence of Neumann eigenvalues on such a domain  $\Omega$ .

**THEOREM 5.1** (Szegő–Weinberger inequality). *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain with smooth boundary such that the Laplace operator on  $\Omega$  with Neumann boundary conditions has purely discrete spectrum. Then*

$$(5.1) \quad \mu_1(\Omega) \leq \mu_1(\Omega^*),$$

where  $\Omega^* \subset \mathbb{R}^n$  is a ball with the same  $n$ -volume as  $\Omega$ . Equality holds if and only if  $\Omega$  itself is a ball.

**PROOF.** By a standard separation of variables one shows that  $\mu_1(\Omega^*)$  is  $n$ -fold degenerate and that a basis of the corresponding eigenspace can be written in the form  $\{g(r)r_j r^{n-1}\}_{j=1, \dots, n}$ . The function  $g$  can be chosen to be positive and satisfies the differential equation

$$(5.2) \quad g'' + \frac{n-1}{r}g' + \left( \mu_1(\Omega^*) - \frac{n-1}{r^2} \right) g = 0, \quad 0 < r < r_1,$$

where  $r_1$  is the radius of  $\Omega^*$ . Further,  $g(r)$  vanishes at  $r = 0$  and its derivative has its first zero at  $r = r_1$ . We extend  $g$  by defining  $g(r) = \lim_{r' \uparrow r_1} g(r')$  for  $r \geq r_1$ . Then  $g$  is differentiable on  $\mathbb{R}$  and if we set  $f_j(\vec{r}) := g(r)r_j r^{n-1}$  then  $f_j \in W^{1,2}(\Omega)$  for  $j = 1 \dots, n$ . To apply the min-max principle with  $f_j$  as a test function for  $\mu_1(\Omega)$

we have to make sure that  $f_j$  is orthogonal to the first (trivial) eigenfunction, i.e., that

$$(5.3) \quad \int_{\Omega} f_j \, d^n r = 0, \quad j = 1, \dots, n.$$

We argue that this can be achieved by some shift of the domain  $\Omega$ : Since  $\Omega$  is bounded we can find a ball  $B$  that contains  $\Omega$ . Now define the vector field  $\vec{b} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  by its components

$$b_j(\vec{v}) = \int_{\Omega + \vec{v}} f_j(\vec{r}) \, d^n r, \quad \vec{v} \in \mathbb{R}^n.$$

For  $\vec{v} \in \partial B$  we have

$$\begin{aligned} \vec{v} \cdot \vec{b}(\vec{v}) &= \int_{\Omega + \vec{v}} \frac{\vec{v} \cdot \vec{r}}{r} g(r) \, d^n r \\ &= \int_{\Omega} \frac{\vec{v} \cdot (\vec{r} + \vec{v})}{|\vec{r} + \vec{v}|} g(|\vec{r} + \vec{v}|) \, d^n r \\ &\geq \int_{\Omega} \frac{|\vec{v}|^2 - |\vec{v}| \cdot |r|}{|\vec{r} + \vec{v}|} g(|\vec{r} + \vec{v}|) \, d^n r > 0. \end{aligned}$$

Thus  $\vec{b}$  is a vector field that points outwards on every point of  $\partial B$ . By an application of the Brouwer's fixed-point theorem (see Theorem 7.3 in the Appendix) this means that  $\vec{b}(\vec{v}_0) = 0$  for some  $\vec{v}_0 \in B$ . Thus, if we shift  $\Omega$  by this vector, condition (5.3) is satisfied and we can apply the min-max principle with the  $f_j$  as test functions for the first non-zero eigenvalue:

$$\begin{aligned} \mu_1(\Omega) &\leq \frac{\int_{\Omega} |\nabla f_j| \, d^n r}{\int_{\Omega} f_j^2 \, d^n r} \\ &= \frac{\int_{\Omega} \left( g'^2(r) r_j^2 r^{-2} + g^2(r) (1 - r_j^2 r^{-2}) r^{-2} \right) \, d^n r}{\int_{\Omega} g^2 r_j^2 r^{-2} \, d^n r}. \end{aligned}$$

We multiply each of these inequalities by the denominator and sum up over  $j$  to obtain

$$(5.4) \quad \mu_1(\Omega) \leq \frac{\int_{\Omega} B(r) \, d^n r}{\int_{\Omega} g^2(r) \, d^n r}$$

with  $B(r) = g'^2(r) + (n-1)g^2(r)r^{-2}$ . Since  $r_1$  is the first zero of  $g'$ , the function  $g$  is non-decreasing. The derivative of  $B$  is

$$B' = 2g'g'' + 2(n-1)(rgg' - g^2)r^{-3}.$$

For  $r \geq r_1$  this is clearly negative since  $g$  is constant there. For  $r < r_1$  we can use equation (5.2) to show that

$$B' = -2\mu_1(\Omega^*)gg' - (n-1)(rg' - g)^2r^{-3} < 0.$$

If the following we will use the method of rearrangements, which was described in section 3. To avoid confusions, we use a more precise notation at this point: We introduce  $B_{\Omega} : \Omega \rightarrow \mathbb{R}$ ,  $B_{\Omega}(\vec{r}) = B(r)$  and analogously  $g_{\Omega} : \Omega \rightarrow \mathbb{R}$ ,  $g_{\Omega}(\vec{r}) = g(r)$ . Then equation (5.4) yields, using Theorem 3.7 in the third step:

$$(5.5) \quad \mu_1(\Omega) \leq \frac{\int_{\Omega} B_{\Omega}(\vec{r}) \, d^n r}{\int_{\Omega} g_{\Omega}^2(\vec{r}) \, d^n r} = \frac{\int_{\Omega^*} B_{\Omega}^*(\vec{r}) \, d^n r}{\int_{\Omega^*} g_{\Omega}^2(\vec{r}) \, d^n r} \leq \frac{\int_{\Omega^*} B(r) \, d^n r}{\int_{\Omega^*} g^2(r) \, d^n r} = \mu_1(\Omega^*)$$

Equality holds obviously if  $\Omega$  is a ball. In any other case the third step in (5.5) is a strict inequality.  $\square$

It is rather straightforward to generalize the Szegő–Weinberger inequality to domains in hyperbolic space. For domains on spheres, on the other hand, the corresponding inequality has not been established yet in full generality. At present, the most general result is due to Ashbaugh and Benguria: In [AB95] they show that an analog of the Szegő–Weinberger inequality holds for domains that are contained in a hemisphere.

**5.1. Bibliographical Remarks.** i) In 1952, Kornhauser and Stakgold [Journal of Mathematics and Physics **31**, 45–54 (1952)] conjectured that the lowest nontrivial Neumann eigenvalue for a smooth bounded domain  $\Omega$  in  $\mathbb{R}^2$  satisfies the isoperimetric inequality

$$\mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{\pi p^2}{A},$$

where  $\Omega^*$  is a disk with the same area as  $\Omega$ , and  $p = 1.8412\dots$  is the first positive zero of the derivative of the Bessel function  $J_1$ . This conjecture was proven by G. Szegő in 1954, using conformal maps [see, G. Szegő, *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3**, 343–356 (1954)]. The extension to  $n$  dimensions was proven by H. Weinberger [H. F. Weinberger, J. Rational Mech. Anal. **5**, 633–636 (1956)].

ii) For the case of mixed boundary conditions, Marie-Helene Bossel [ see *Membranes élastiquement liées inhomogènes ou sur une surface: une nouvelle extension du théoreme isopérimétrique de Rayleigh–Faber–Krahn*, Z. Angew. Math. Phys. **39**, 733–742 (1988)] proved the analog of the Rayleigh–Faber–Krahn inequality.

iii) Very recently, A. Girouard, N. Nadirashvili and I. Polterovich proved that the second positive eigenvalue of a bounded simply connected planar domain of a given area does not exceed the first positive Neumann eigenvalue on a disk of a twice smaller area (see, *Maximization of the second positive Neumann eigenvalue for planar domains*, preprint (2008)). For a review of optimization of eigenvalues with respect to the geometry of the domain, see the recent monograph of A. Henrot [H06].

## 6. The Payne–Pólya–Weinberger inequality

**6.1. Introduction.** A further isoperimetric inequality is concerned with the second eigenvalue of the Dirichlet–Laplacian on bounded domains. In 1955 Payne, Pólya and Weinberger (PPW) showed that for any open bounded domain  $\Omega \subset \mathbb{R}^2$  the bound  $\lambda_2(\Omega)/\lambda_1(\Omega) \leq 3$  holds [PPW55, PPW56]. Based on exact calculations for simple domains they also conjectured that the ratio  $\lambda_2(\Omega)/\lambda_1(\Omega)$  is maximized when  $\Omega$  is a circular disk, i.e., that

$$(6.1) \quad \frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)} = \frac{j_{1,1}^2}{j_{0,1}^2} \approx 2.539 \quad \text{for } \Omega \subset \mathbb{R}^2.$$

Here,  $j_{n,m}$  denotes the  $m^{\text{th}}$  positive zero of the Bessel function  $J_n(x)$ . This conjecture and the corresponding inequalities in  $n$  dimensions were proven in 1991 by

Ashbaugh and Benguria [AB91, AB92a, AB92b]. Since the Dirichlet eigenvalues on a ball are inversely proportional to the square of the ball's radius, the ratio  $\lambda_2(\Omega^*)/\lambda_1(\Omega^*)$  does not depend on the size of  $\Omega^*$ . Thus we can state the PPW inequality in the following form:

**THEOREM 6.1** (Payne–Pólya–Weinberger inequality). *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain and  $S_1 \subset \mathbb{R}^n$  a ball such that  $\lambda_1(\Omega) = \lambda_1(S_1)$ . Then*

$$(6.2) \quad \lambda_2(\Omega) \leq \lambda_2(S_1)$$

*with equality if and only if  $\Omega$  is a ball.*

Here the subscript 1 on  $S_1$  reflects the fact that the ball  $S_1$  has the same first Dirichlet eigenvalue as the original domain  $\Omega$ . The inequalities (6.1) and (6.2) are equivalent in Euclidean space in view of the mentioned scaling properties of the eigenvalues. Yet when one considers possible extensions of the PPW inequality to other settings, where  $\lambda_2/\lambda_1$  varies with the radius of the ball, it turns out that an estimate in the form of Theorem 6.1 is the more natural result. In the case of a domain on a hemisphere, for example,  $\lambda_2/\lambda_1$  on balls is an increasing function of the radius. But by the Rayleigh–Faber–Krahn inequality for spheres the radius of  $S_1$  is smaller than the one of the spherical rearrangement  $\Omega^*$ . This means that an estimate in the form of Theorem 6.1, interpreted as

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \leq \frac{\lambda_2(S_1)}{\lambda_1(S_1)}, \quad \Omega, S_1 \subset \mathbb{S}^n,$$

is stronger than an inequality of the type (6.1).

On the other hand, we will see that in the hyperbolic space  $\lambda_2/\lambda_1$  on balls is a strictly decreasing function of the radius. In this case we can apply the following argument to see that an estimate of the type (6.1) cannot possibly hold true: Consider a domain  $\Omega$  that is constructed by attaching very long and thin tentacles to the ball  $B$ . Then the first and second eigenvalues of the Laplacian on  $\Omega$  are arbitrarily close to the ones on  $B$ . The spherical rearrangement of  $\Omega$  though can be considerably larger than  $B$ . This means that

$$\frac{\lambda_2(\Omega)}{\lambda_1(\Omega)} \approx \frac{\lambda_2(B)}{\lambda_1(B)} > \frac{\lambda_2(\Omega^*)}{\lambda_1(\Omega^*)}, \quad B, \Omega \subset \mathbb{H}^n,$$

clearly ruling out any inequality in the form of (6.1).

The proof of the PPW inequality (6.2) is somewhat similar to that of the Szegő–Weinberger inequality (see section 5), but considerably more difficult. The additional complications mainly stem from the fact that in the Dirichlet case the first eigenfunction of the Laplacian is not known explicitly, while in the Neumann case it is just constant. We will give the full proof of the PPW inequality in the following three sections. Since it is quite long, a brief outline is in order:

The proof is organized in six steps. In the first one we use the min–max principle to derive an estimate for the eigenvalue gap  $\lambda_2(\Omega) - \lambda_1(\Omega)$ , depending on a test function for the second eigenvalue. In the second step we define such a function and then show in the third step that it actually satisfies all requirements to be used in the gap formula. In the fourth step we put the test function into the gap inequality and then estimate the result with the help of rearrangement techniques. These depend on the monotonicity properties of two functions  $g$  and  $B$ , which are to be defined in the proof, and on a Chiti comparison argument. The later is a special comparison result which establishes a crossing property between the symmetric

decreasing rearrangement of the first eigenfunction on  $\Omega$  and the first eigenfunction on  $S_1$ . We end up with the inequality  $\lambda_2(\Omega) - \lambda_1(\Omega) \leq \lambda_2(S_1) - \lambda_1(S_1)$ , which yields (6.2). In the remaining two steps we prove the mentioned monotonicity properties and the Chiti comparison result. We remark that from the Rayleigh–Faber–Krahn inequality follows  $S_1 \subset \Omega^*$ , a fact that is used in the proof of the Chiti comparison result. Although it enters in a rather subtle manner, the Rayleigh–Faber–Krahn inequality is an important ingredient of the proof of the PPW inequality.

**6.2. Proof of the Payne–Pólya–Weinberger inequality.** *First step:* We derive the ‘gap formula’ for the first two eigenvalues of the Dirichlet–Laplacian on  $\Omega$ . We call  $u_1 : \Omega \rightarrow \mathbb{R}^+$  the positive normalized first eigenfunction of  $-\Delta_\Omega^D$ . To estimate the second eigenvalue we will use the test function  $Pu_1$ , where  $P : \Omega \rightarrow \mathbb{R}$  is chosen such that  $Pu_1$  is in the form domain of  $-\Delta_\Omega^D$  and

$$(6.3) \quad \int_{\Omega} Pu_1^2 \, dr^n = 0.$$

Then we conclude from the min–max principle that

$$(6.4) \quad \begin{aligned} \lambda_2(\Omega) - \lambda_1(\Omega) &\leq \frac{\int_{\Omega} (|\nabla(Pu_1)|^2 - \lambda_1 P^2 u_1^2) \, dr^n}{\int_{\Omega} P^2 u_1^2 \, dr^n} \\ &= \frac{\int_{\Omega} (|\nabla P|^2 u_1^2 + (\nabla P^2) u_1 \nabla u_1 + P^2 |\nabla u_1|^2 - \lambda_1 P^2 u_1^2) \, dr^n}{\int_{\Omega} P^2 u_1^2 \, dr^n} \end{aligned}$$

If we perform an integration by parts on the second summand in the numerator of (6.4), we see that all summands except the first cancel. We obtain the gap inequality

$$(6.5) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega} |\nabla P|^2 u_1^2 \, dr^n}{\int_{\Omega} P^2 u_1^2 \, dr^n}.$$

*Second step:* We need to fix the test function  $P$ . Our choice will be dictated by the requirement that equality should hold in (6.5) if  $\Omega$  is a ball, i.e., if  $\Omega = S_1$  up to translations. We assume that  $S_1$  is centered at the origin of our coordinate system and call  $R_1$  its radius. We write  $z_1(r)$  for the first eigenfunction of the Dirichlet Laplacian on  $S_1$ . This function is spherically symmetric with respect to the origin and we can take it to be positive and normalized in  $L^2(S_1)$ . The second eigenvalue of  $-\Delta_{S_1}^D$  in  $n$  dimensions is  $n$ -fold degenerate and a basis of the corresponding eigenspace can be written in the form  $z_2(r)r_j r^{-1}$  with  $z_2 \geq 0$  and  $j = 1, \dots, n$ . This is the motivation to choose not only one test function  $P$ , but rather  $n$  functions  $P_j$  with  $j = 1, \dots, n$ . We set

$$P_j = r_j r^{-1} g(r)$$

with

$$g(r) = \begin{cases} \frac{z_2(r)}{z_1(r)} & \text{for } r < R_1, \\ \lim_{r' \uparrow R_1} \frac{z_2(r')}{z_1(r')} & \text{for } r \geq R_1. \end{cases}$$

We note that  $P_j u_1$  is a second eigenfunction of  $-\Delta_\Omega^D$  if  $\Omega$  is a ball which is centered at the origin.

*Third step:* It is necessary to verify that the  $P_j u_1$  are admissible test functions. First, we have to make sure that condition (6.3) is satisfied. We note that  $P_j$  changes when  $\Omega$  (and  $u_1$  with it) is shifted in  $\mathbb{R}^n$ . Since these shifts do not change

$\lambda_1(\Omega)$  and  $\lambda_2(\Omega)$ , it is sufficient to show that  $\Omega$  can be moved in  $\mathbb{R}^n$  such that (6.3) is satisfied for all  $j \in \{1, \dots, n\}$ . To this end we define the function

$$\vec{b}(\vec{v}) = \int_{\Omega + \vec{v}} u_1^2(|\vec{r} - \vec{v}|) \frac{\vec{r}}{r} g(r) \, dr^n \quad \text{for } \vec{v} \in \mathbb{R}^n.$$

Since  $\Omega$  is a bounded domain, we can choose some closed ball  $D$ , centered at the origin, such that  $\Omega \subset D$ . Then for every  $\vec{v} \in \partial D$  we have

$$\begin{aligned} \vec{v} \cdot \vec{b}(\vec{v}) &= \int_{\Omega} \vec{v} \cdot u_1^2(r) \frac{\vec{r} + \vec{v}}{|\vec{r} + \vec{v}|} g(|\vec{r} + \vec{v}|) \, dr^n \\ &> \int_{\Omega} u_1^2(r) \frac{|\vec{v}|^2 - |\vec{v}| \cdot |\vec{r}|}{|\vec{r} + \vec{v}|} g(|\vec{r} + \vec{v}|) \, dr^n > 0 \end{aligned}$$

Thus the continuous vector-valued function  $\vec{b}(\vec{v})$  points strictly outwards everywhere on  $\partial D$ . By Theorem 7.3, which is a consequence of the Brouwer fixed-point theorem, there is some  $\vec{v}_0 \in D$  such that  $\vec{b}(\vec{v}_0) = 0$ . Now we shift  $\Omega$  by this vector, i.e., we replace  $\Omega$  by  $\Omega - \vec{v}_0$  and  $u_1$  by the first eigenfunction of the shifted domain. Then the test functions  $P_j u_1$  satisfy the condition (6.3).

The second requirement on  $P_j u_1$  is that it must be in the form domain of  $-\Delta_{\Omega}^D$ , i.e., in  $H_0^1(\Omega)$ : Since  $u_1 \in H_0^1(\Omega)$  there is a sequence  $\{v_n \in C^1(\Omega)\}_{n \in \mathbb{N}}$  of functions with compact support such that  $|\cdot|_h - \lim_{n \rightarrow \infty} v_n = u_1$ , using the definition (4.1) of  $|\cdot|_h$ . The functions  $P_j v_n$  also have compact support and one can check that  $P_j v_n \in C^1(\Omega)$  ( $P_j$  is continuously differentiable since  $g'(R_1) = 0$ ). We have  $|\cdot|_h - \lim_{n \rightarrow \infty} P_j v_n = P_j u_1$  and thus  $P_j u_1 \in H_0^1(\Omega)$ .

*Fourth step:* We multiply the gap inequality (6.5) by  $\int P^2 u_1^2 \, dx$  and put in our special choice of  $P_j$  to obtain

$$\begin{aligned} (\lambda_2 - \lambda_1) \int_{\Omega} \frac{r_j^2}{r^2} g^2(r) u_1^2(r) \, dr^n &\leq \int_{\Omega} \left| \nabla \left( \frac{r_j}{r} g(r) \right) \right|^2 u_1^2(r) \, dr^n \\ &= \int_{\Omega} \left( \left| \nabla \frac{r_j}{r} \right|^2 g^2(r) + \frac{r_j^2}{r^2} g'(r)^2 \right) u_1^2(r) \, dr^n. \end{aligned}$$

Now we sum these inequalities up over  $j = 1, \dots, n$  and then divide again by the integral on the left hand side to get

$$(6.6) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega} B(r) u_1^2(r) \, dr^n}{\int_{\Omega} g^2(r) u_1^2(r) \, dr^n}$$

with

$$(6.7) \quad B(r) = g'(r)^2 + (n-1)r^{-2}g(r)^2.$$

If the following we will use the method of rearrangements, which was described in section 3. To avoid confusions, we use a more precise notation at this point: We introduce  $B_{\Omega} : \Omega \rightarrow \mathbb{R}$ ,  $B_{\Omega}(\vec{r}) = B(r)$  and analogously  $g_{\Omega} : \Omega \rightarrow \mathbb{R}$ ,  $g_{\Omega}(\vec{r}) = g(r)$ . Then equation (6.6) can be written as

$$(6.8) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega} B_{\Omega}(\vec{r}) u_1^2(\vec{r}) \, dr^n}{\int_{\Omega} g_{\Omega}^2(\vec{r}) u_1^2(\vec{r}) \, dr^n}.$$

Then by Theorem 3.8 the following inequality is also true:

$$(6.9) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega^*} B_{\Omega^*}^*(\vec{r}) u_1^*(\vec{r})^2 \, dr^n}{\int_{\Omega^*} g_{\Omega^*}^2(\vec{r}) u_1^*(\vec{r})^2 \, dr^n}.$$

Next we use the very important fact that  $g(r)$  is an increasing function and  $B(r)$  is a decreasing function, which we will prove in step five below. These monotonicity properties imply by Theorem 3.7 that  $B_{\Omega^*}^{\star}(\bar{r}) \leq B(r)$  and  $g_{\Omega^*}(\bar{r}) \geq g(r)$ . Therefore

$$(6.10) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega^*} B(r) u_1^*(r)^2 dr^n}{\int_{\Omega^*} g^2(r) u_1^*(r)^2 dr^n}.$$

Finally we use the following version of Chiti's comparison theorem to estimate the right hand side of (6.10):

LEMMA 6.2 (Chiti comparison result). *There is some  $r_0 \in (0, R_1)$  such that*

$$\begin{aligned} z_1(r) &\geq u_1^*(r) && \text{for } r \in (0, r_0) \text{ and} \\ z_1(r) &\leq u_1^*(r) && \text{for } r \in (r_0, R_1). \end{aligned}$$

We remind the reader that the function  $z_1$  denotes the first Dirichlet eigenfunction for the Laplacian defined on  $S_1$ . Applying Lemma 6.2, which will be proven below in step six, to (6.10) yields

$$(6.11) \quad \lambda_2(\Omega) - \lambda_1(\Omega) \leq \frac{\int_{\Omega^*} B(r) z_1(r)^2 dr^n}{\int_{\Omega^*} g^2(r) z_1(r)^2 dr^n} = \lambda_2(S_1) - \lambda_1(S_1).$$

Since  $S_1$  was chosen such that  $\lambda_1(\Omega) = \lambda_1(S_1)$  the above relation proves that  $\lambda_2(\Omega) \leq \lambda_2(S_1)$ . It remains the question: When does equality hold in (6.2)? It is obvious that equality does hold if  $\Omega$  is a ball, since then  $\Omega = S_1$  up to translations. On the other hand, if  $\Omega$  is not a ball, then (for example) the step from (6.10) to (6.11) is not sharp. Thus (6.2) is a strict inequality if  $\Omega$  is not a ball.

**6.3. Monotonicity of  $B$  and  $g$ .** *Fifth step:* We prove that  $g(r)$  is an increasing function and  $B(r)$  is a decreasing function. In this step we abbreviate  $\lambda_i = \lambda_i(S_1)$ . The functions  $z_1$  and  $z_2$  are solutions of the differential equations

$$(6.12) \quad \begin{aligned} -z_1'' - \frac{n-1}{r} z_1' - \lambda_1 z_1 &= 0, \\ -z_2'' - \frac{n-1}{r} z_2' + \left( \frac{n-1}{r^2} - \lambda_2 \right) z_2 &= 0 \end{aligned}$$

with the boundary conditions

$$(6.13) \quad z_1'(0) = 0, \quad z_1(R_1) = 0, \quad z_2(0) = 0, \quad z_2(R_1) = 0.$$

We define the function

$$(6.14) \quad q(r) := \begin{cases} \frac{rg'(r)}{g(r)} & \text{for } r \in (0, R_1), \\ \lim_{r' \downarrow 0} q(r') & \text{for } r = 0, \\ \lim_{r' \uparrow R_1} q(r') & \text{for } r = R_1. \end{cases}$$

Proving the monotonicity of  $B$  and  $g$  is thus reduced to showing that  $0 \leq q(r) \leq 1$  and  $q'(r) \leq 0$  for  $r \in [0, R_1]$ . Using the definition of  $g$  and the equations (6.12), one can show that  $q(r)$  is a solution of the Riccati differential equation

$$(6.15) \quad q' = (\lambda_1 - \lambda_2)r + \frac{(1-q)(q+n-1)}{r} - 2q \frac{z_1'}{z_1}.$$

It is straightforward to establish the boundary behavior

$$q(0) = 1, \quad q'(0) = 0, \quad q''(0) = \frac{2}{n} \left( \left( 1 + \frac{2}{n} \right) \lambda_1 - \lambda_2 \right)$$

and

$$q(R_1) = 0.$$

LEMMA 6.3. *For  $0 \leq r \leq R_1$  we have  $q(r) \geq 0$ .*

PROOF. Assume the contrary. Then there exist two points  $0 < s_1 < s_2 \leq R_1$  such that  $q(s_1) = q(s_2) = 0$  but  $q'(s_1) \leq 0$  and  $q'(s_2) \geq 0$ . If  $s_2 < R_1$  then the Riccati equation (6.15) yields

$$0 \geq q'(s_1) = (\lambda_1 - \lambda_2)s_1 + \frac{n-1}{s_1} > (\lambda_1 - \lambda_2)s_2 + \frac{n-1}{s_2} = q'(s_2) \geq 0,$$

which is a contradiction. If  $s_2 = R_1$  then we get a contradiction in a similar way by

$$0 \geq q'(s_1) = (\lambda_1 - \lambda_2)s_1 + \frac{n-1}{s_1} > (\lambda_1 - \lambda_2)R_1 + \frac{n-1}{R_1} = 3q'(R_1) \geq 0.$$

□

In the following we will analyze the behavior of  $q'$  according to (6.15), considering  $r$  and  $q$  as two independent variables. For the sake of a compact notation we will make use of the following abbreviations:

$$\begin{aligned} p(r) &= z_1'(r)/z_1(r) \\ N_y &= y^2 - n + 1 \\ Q_y &= 2y\lambda_1 + (\lambda_2 - \lambda_1)N_y y^{-1} - 2(\lambda_2 - \lambda_1) \\ M_y &= N_y^2/(2y) - (n-2)^2 y/2 \end{aligned}$$

We further define the function

$$(6.16) \quad T(r, y) := -2p(r)y - \frac{(n-2)y + N_y}{r} - (\lambda_2 - \lambda_1)r.$$

Then we can write (6.15) as

$$q'(r) = T(r, q(r)).$$

The definition of  $T(r, y)$  allows us to analyze the Riccati equation for  $q'$  considering  $r$  and  $q(r)$  as independent variables. For  $r$  going to zero,  $p$  is  $\mathcal{O}(r)$  and thus

$$T(r, y) = \frac{1}{r}((n-1+y)(1-y)) + \mathcal{O}(r) \quad \text{for } y \text{ fixed.}$$

Consequently,

$$\begin{aligned} \lim_{r \rightarrow 0} T(r, y) &= +\infty && \text{for } 0 \leq y < 1 \text{ fixed,} \\ \lim_{r \rightarrow 0} T(r, y) &= 0 && \text{for } y = 1 \text{ and} \\ \lim_{r \rightarrow 0} T(r, y) &= -\infty && \text{for } y > 1 \text{ fixed.} \end{aligned}$$

The partial derivative of  $T(r, y)$  with respect to  $r$  is given by

$$(6.17) \quad T' = \frac{\partial}{\partial r} T(r, y) = -2yp' + \frac{(n-2)y}{r^2} + \frac{N_y}{r^2} - (\lambda_2 - \lambda_1).$$

In the points  $(r, y)$  where  $T(r, y) = 0$  we have, by (6.16),

$$(6.18) \quad p|_{T=0} = -\frac{n-2}{2r} - \frac{N_y}{2yr} - \frac{(\lambda_2 - \lambda_1)r}{2y}.$$

From (6.12) we get the Riccati equation

$$(6.19) \quad p' + p^2 + \frac{n-1}{r}p + \lambda_1 = 0.$$

Putting (6.18) into (6.19) and the result into (6.17) yields

$$(6.20) \quad T'|_{T=0} = \frac{M_y}{r^2} + \frac{(\lambda_2 - \lambda_1)^2}{2y} r^2 + Q_y.$$

LEMMA 6.4. *There is some  $r_0 > 0$  such that  $q(r) \leq 1$  for all  $r \in (0, r_0)$  and  $q(r_0) < 1$ .*

PROOF. Suppose the contrary, i.e.,  $q(r)$  first increases away from  $r = 0$ . Then, because  $q(0) = 1$  and  $q(R_1) = 0$  and because  $q$  is continuous and differentiable, we can find two points  $s_1 < s_2$  such that  $\hat{q} := q(s_1) = q(s_2) > 1$  and  $q'(s_1) > 0 > q'(s_2)$ . Even more, we can choose  $s_1$  and  $s_2$  such that  $\hat{q}$  is arbitrarily close to one. Writing  $\hat{q} = 1 + \epsilon$  with  $\epsilon > 0$ , we can calculate from the definition of  $Q_y$  that

$$Q_{1+\epsilon} = Q_1 + \epsilon n (\lambda_2 - (1 - 2/n) \lambda_1) + \mathcal{O}(\epsilon^2).$$

The term in brackets can be estimated by

$$\lambda_2 - (1 - 2/n)\lambda_1 > \lambda_2 - \lambda_1 > 0.$$

We can also assume that  $Q_1 \geq 0$ , because otherwise  $q''(0) = \frac{2}{n^2} Q_1 < 0$  and Lemma 6.4 is immediately true. Thus, choosing  $R_1$  and  $r_2$  such that  $\epsilon$  is sufficiently small, we can make sure that  $Q_{\hat{q}} > 0$ .

Now consider  $T(r, \hat{q})$  as a function of  $r$  for our fixed  $\hat{q}$ . We have  $T(s_1, \hat{q}) > 0 > T(s_2, \hat{q})$  and the boundary behavior  $T(0, \hat{q}) = -\infty$ . Consequently,  $T(r, \hat{q})$  changes its sign at least twice on  $[0, R_1]$  and thus we can find two zeros  $0 < \hat{s}_1 < \hat{s}_2 < R_1$  of  $T(r, \hat{q})$  such that

$$(6.21) \quad T'(\hat{s}_1, \hat{q}) \geq 0 \quad \text{and} \quad T'(\hat{s}_2, \hat{q}) \leq 0.$$

But from (6.20), together with  $Q_{\hat{q}} > 0$ , one can see easily that this is impossible, because the right hand side of (6.20) is either positive or increasing (depending on  $M_{\hat{q}}$ ). This is a contradiction to our assumption that  $q$  first increases away from  $r = 0$ , proving Lemma 6.4.  $\square$

LEMMA 6.5. *For all  $0 \leq r \leq R_1$  the inequality  $q'(r) \leq 0$  holds.*

PROOF. Assume the contrary. Then, because of  $q(0) = 1$  and  $q(R_1) = 0$ , there are three points  $s_1 < s_2 < s_3$  in  $(0, R_1)$  with  $0 < \hat{q} := q(s_1) = q(s_2) = q(s_3) < 1$  and  $q'(s_1) < 0$ ,  $q'(s_2) > 0$ ,  $q'(s_3) < 0$ . Consider the function  $T(r, \hat{q})$ , which coincides with  $q'(r)$  at  $s_1, s_2, s_3$ . Taking into account its boundary behavior at  $r = 0$ , it is clear that  $T(r, \hat{q})$  must have at least the sign changes positive-negative-positive-negative. Thus  $T(r, \hat{q})$  has at least three zeros  $\hat{s}_1 < \hat{s}_2 < \hat{s}_3$  with the properties

$$T'(\hat{s}_1, \hat{q}) \leq 0, \quad T'(\hat{s}_2, \hat{q}) \geq 0, \quad T'(\hat{s}_3, \hat{q}) \leq 0.$$

Again one can see from (6.20) that this is impossible, because the term on the right hand side is either a strictly convex or a strictly increasing function of  $r$ . We conclude that Lemma 6.5 is true.  $\square$

Altogether we have shown that  $0 \leq q(r) \leq 1$  and  $q'(r) \leq 0$  for all  $r \in (0, R_1)$ , which proves that  $g$  is increasing and  $B$  is decreasing.

**6.4. The Chiti comparison result.** *Sixth step:* We prove Lemma 6.2: Here and in the sequel we write short-hand  $\lambda_1 = \lambda_1(\Omega) = \lambda_1(S_1)$ . We introduce a change of variables via  $s = C_n r^n$ , where  $C_n$  is the volume of the  $n$ -dimensional unit ball. Then by Definition 3.2 we have  $u_1^\sharp(s) = u_1^*(r)$  and  $z_1^\sharp(s) = z_1(r)$ .

LEMMA 6.6. *For the functions  $u_1^\sharp(s)$  and  $z_1^\sharp(s)$  we have*

$$(6.22) \quad -\frac{du_1^\sharp}{ds} \leq \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s u_1^\sharp(w) dw,$$

$$(6.23) \quad -\frac{dz_1^\sharp}{ds} = \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s z_1^\sharp(w) dw.$$

PROOF. We integrate both sides of  $-\Delta u_1 = \lambda_1 u_1$  over the level set  $\Omega_t := \{\vec{r} \in \Omega : u_1(\vec{r}) > t\}$  and use Gauss' Divergence Theorem to obtain

$$(6.24) \quad \int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(dr) = \int_{\Omega_t} \lambda_1 u_1(\vec{r}) d^n r,$$

where  $\partial\Omega_t = \{\vec{r} \in \Omega : u_1(\vec{r}) = t\}$ . Now we define the distribution function  $\mu(t) = |\Omega_t|$ . Then by Theorem 3.9 we have

$$(6.25) \quad \int_{\partial\Omega_t} |\nabla u_1| H_{n-1}(dr) \geq -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)}.$$

The left sides of (6.24) and (6.25) are the same, thus

$$\begin{aligned} -n^2 C_n^{2/n} \frac{\mu(t)^{2-2/n}}{\mu'(t)} &\leq \int_{\Omega_t} \lambda_1 u_1(\vec{r}) d^n r \\ &= \int_0^{(\mu(t)/C_n)^{1/n}} n C_n r^{n-1} \lambda_1 u_1^*(r) dr. \end{aligned}$$

Now we perform the change of variables  $r \rightarrow s$  on the right hand side of the above chain of inequalities. We also chose  $t$  to be  $u_1^\sharp(s)$ . Using the fact that  $u_1^\sharp$  and  $\mu$  are essentially inverse functions to one another, this means that  $\mu(t) = s$  and  $\mu'(t)^{-1} = (u_1^\sharp)'(s)$ . The result is (6.22). Equation (6.23) is proven analogously, with equality in each step.  $\square$

Lemma 6.6 enables us to prove Lemma 6.2. The function  $z_1^\sharp$  is continuous on  $(0, |S_1|)$  and  $u_1^\sharp$  is continuous on  $(0, |\Omega^*|)$ . By the normalization of  $u_1^\sharp$  and  $z_1^\sharp$  and because  $S_1 \subset \Omega^*$  it is clear that either  $z_1^\sharp \geq u_1^\sharp$  on  $(0, |S_1|)$  or  $u_1^\sharp$  and  $z_1^\sharp$  have at least one intersection on this interval. In the first case there is nothing to prove, simply setting  $r_0 = R_1$  in Lemma 6.2. In the second case we have to show that there is no intersection of  $u_1^\sharp$  and  $z_1^\sharp$  such that  $u_1^\sharp$  is greater than  $z_1^\sharp$  on the left and smaller on the right. So we assume the contrary, i.e., that there are two points  $0 \leq s_1 < s_2 < |S_1|$  such that  $u_1^\sharp(s) > z_1^\sharp(s)$  for  $s \in (s_1, s_2)$ ,  $u_1^\sharp(s_2) = z_1^\sharp(s_2)$  and either  $u_1^\sharp(s_1) = z_1^\sharp(s_1)$  or  $s_1 = 0$ . We set

$$(6.26) \quad v^\sharp(s) = \begin{cases} u_1^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_1^\sharp(s) ds > \int_0^{s_1} z_1^\sharp(s) ds, \\ z_1^\sharp(s) & \text{on } [0, s_1] \text{ if } \int_0^{s_1} u_1^\sharp(s) ds \leq \int_0^{s_1} z_1^\sharp(s) ds, \\ u_1^\sharp(s) & \text{on } [s_1, s_2], \\ z_1^\sharp(s) & \text{on } [s_2, |S_1|]. \end{cases}$$

Then one can convince oneself that because of (6.22) and (6.23)

$$(6.27) \quad -\frac{dv^\sharp}{ds} \leq \lambda_1 n^{-2} C_n^{-2/n} s^{n/2-2} \int_0^s v^\sharp(s') ds'$$

for all  $s \in [0, |S_1|]$ . Now define the test function  $v(r) = v^\sharp(C_n r^n)$ . Using the Rayleigh-Ritz characterization of  $\lambda_1$ , then (6.27) and finally an integration by parts, we get (if  $z_1$  and  $u_1$  are not identical)

$$\begin{aligned} \lambda_1 \int_{S_1} v^2(r) dx &< \int_{S_1} |\nabla v|^2 dx = \int_0^{|S_1|} (nC_n r^{n-1} v^\sharp(s))^2 ds \\ &\leq - \int_0^{|S_1|} v^\sharp(s) \lambda_1 \int_0^s v^\sharp(s') ds' ds \\ &= \lambda_1 \int_0^{|S_1|} v^\sharp(s)^2 ds - \lambda_1 \left[ v^\sharp(s) \int_0^s v^\sharp(s') ds' \right]_0^{|S_1|} \\ &\leq \lambda_1 \int_{S_1} v^2(r) dx \end{aligned}$$

Comparing the first and the last term in the above chain of (in)equalities reveals a contradiction to our assumption that the intersection point  $s_2$  exists, thus proving Lemma 6.2.

**6.5. Schrödinger operators.** Theorem 6.1 can be extended in several directions. One generalization, which has been considered by Benguria and Linde in [BL06], is to replace the Laplace operator on the domain  $\Omega \subset \mathbb{R}^n$  by a Schrödinger operator  $H = -\Delta + V$ . In this case the question arises which is the most suitable comparison operator for  $H$ . In analogy to the PPW inequality for the Laplacian, it seems natural to compare the eigenvalues of  $H$  to those of another Schrödinger operator  $\tilde{H} = -\Delta + \tilde{V}$ , which is defined on a ball and has the same lowest eigenvalue as  $H$ . The potential  $\tilde{V}$  should be spherically symmetric and it should reflect some properties of  $V$ , but it will also have to satisfy certain requirements in order for the PPW type estimate to hold. The precise result is stated in Theorem 6.7 below, which can be considered as a natural generalization of Theorem 6.1 to Schrödinger operators.

We assume that  $\Omega$  is open and bounded and that  $V : \Omega \rightarrow \mathbb{R}^+$  is a non-negative potential from  $L^1(\Omega)$ . Then we can define the Schrödinger operator  $H_V = -\Delta + V$  on  $\Omega$  in the same way as we did in section 4.2, i.e.,  $H_V$  is positive and self-adjoint in  $L^2(\Omega)$  and has purely discrete spectrum. We call  $\lambda_i(\Omega, V)$  its  $i$ -th eigenvalue and, as usual, we write  $V_\star$  for the symmetric increasing rearrangement of  $V$ .

**THEOREM 6.7.** *Let  $S_1 \subset \mathbb{R}^n$  be a ball centered at the origin and of radius  $R_1$  and let  $\tilde{V} : S_1 \rightarrow \mathbb{R}^+$  be a radially symmetric non-negative potential such that  $\tilde{V}(r) \leq V_\star(r)$  for all  $0 \leq r \leq R_1$  and  $\lambda_1(\Omega, V) = \lambda_1(S_1, \tilde{V})$ . If  $\tilde{V}(r)$  satisfies the conditions*

- a)  $\tilde{V}(0) = \tilde{V}'(0) = 0$  and
- b)  $\tilde{V}'(r)$  exists and is increasing and convex,

then

$$(6.28) \quad \lambda_2(\Omega, V) \leq \lambda_2(S_1, \tilde{V}).$$

If  $V$  is such that  $V_*$  itself satisfies the conditions a) and b) of the theorem, the best bound is obtained by choosing  $\tilde{V} = V_*$  and then adjusting the size of  $S_1$  such that  $\lambda_1(\Omega, V) = \lambda_1(S_1, V_*)$  holds. (Note that  $S_1 \subset \Omega^*$  by Theorem 4.2). In this case Theorem 6.7 is a typical PPW result and optimal in the sense that equality holds in (6.28) if  $\Omega$  is a ball and  $V = V_*$ . For a general potential  $V$  we still get a non-trivial bound on  $\lambda_2(\Omega, V)$  though it is not sharp anymore.

For further reference we state the following theorem, which is a direct consequence of Theorem 6.7 and Theorem 3.7:

**THEOREM 6.8.** *Let  $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}^+$  be a radially symmetric positive potential that satisfies the conditions a) and b) of Theorem 6.1. Further, assume that  $\Omega \subset \mathbb{R}^n$  is an open bounded domain and that  $S_1 \subset \mathbb{R}^n$  be the open ball (centered at the origin) such that  $\lambda_1(\Omega, \tilde{V}) = \lambda_1(S_1, \tilde{V})$ . Then*

$$\lambda_2(\Omega, \tilde{V}) \leq \lambda_2(S_1, \tilde{V}).$$

The proof of Theorem 6.7 is similar to the one of Theorem 6.1 and can be found in [BL06]. One of the main differences occurs in step five (see section 6.3), since the potential  $\tilde{V}(r)$  now appears in the Riccati equation for  $p$ . It turns out that the conditions a) and b) in Theorem 6.7 are required to establish the monotonicity properties of  $q$ . A second important difference is that a second eigenfunction of a Schrödinger operator with a spherically symmetric potential can not necessarily be written in the form  $u_2(r)r_j r^{-1}$ . It has been shown by Ashbaugh and Benguria [AB88] that it can be written in this form if  $rV(r)$  is convex. On the other hand, the second eigenfunction is radially symmetric (with a spherical nodal surface) if  $rV(r)$  is concave. This fact, which is also known as the Baumgartner–Grosse–Martin Inequality [BGM84], is another reason why the conditions a) and b) of Theorem 6.7 are needed.

**6.6. Gaussian space.** Theorem 6.8 has direct consequences for the eigenvalues of the Laplace operator  $-\Delta_G$  in Gaussian space, which had been defined in section 4.3. In this section we write  $\lambda_i^-(\Omega)$  for the  $i$ -th eigenvalue of  $-\Delta_G$  on some domain  $\Omega$ .

**THEOREM 6.9.** *Let  $\Omega \subset \mathbb{R}^n$  be an open bounded domain and assume that  $S_1 \subset \mathbb{R}^n$  is a ball, centered at the origin, such that  $\lambda_1^-(\Omega) = \lambda_1^-(S_1)$ . Then*

$$\lambda_2^-(\Omega) \leq \lambda_2^-(S_1).$$

**PROOF.** If  $\Psi$  is an eigenfunction of  $-\Delta_G$  on  $\Omega$  then  $\Psi e^{-r^2/2}$  is an eigenfunction of the Dirichlet-Schrödinger operator  $-\Delta + r^2$  on  $\Omega$ , and vice versa. The eigenvalues are related by

$$\lambda_i^-(\Omega) = \lambda(\Omega, r^2 - n),$$

where we keep using the notation from section 6.5. Theorem 6.9 now follows directly from Theorem 6.8, setting  $\tilde{V}(r) = r^2$ .  $\square$

**6.7. Spaces of constant curvature.** There are generalizations of the Payne–Pólya–Weinberger inequality to spaces of constant curvature. Ashbaugh and Benguria showed in [AB01] that Theorem 6.1 remains valid if one replaces the Euclidean space  $\mathbb{R}^n$  by a hemisphere of  $S^n$  and ‘ball’ by ‘geodesic ball’. Similar to the Szegő–Weinberger inequality, it is still an open problem to prove a Payne–Pólya–Weinberger result for the whole sphere. Although there seem to be no counterexamples known that rule out such a generalization, the original scheme of proving

the PPW inequality is not likely to work. One reason is that numerical studies show the function  $g$  to be not monotone on the whole sphere.

For the hyperbolic space, on the other hand, things are settled. Following the general lines of the original proof, Benguria and Linde established in [BL07] a PPW type inequality that holds in any space of constant negative curvature.

## 7. Appendix

### 7.1. The layer-cake formula.

**THEOREM 7.1.** *Let  $\nu$  be a measure on the Borel sets of  $\mathbb{R}^+$  such that  $\Phi(t) := \nu([0, t])$  is finite for every  $t > 0$ . Let further  $(\Omega, \Sigma, m)$  be a measure space and  $v$  a non-negative measurable function on  $\Omega$ . Then*

$$(7.1) \quad \int_{\Omega} \Phi(v(x))m(dx) = \int_0^{\infty} m(\{x \in \Omega : v(x) > t\})\nu(dt).$$

*In particular, if  $m$  is the Dirac measure at some point  $x \in \mathbb{R}^n$  and  $\nu(dt) = dt$  then (7.1) takes the form*

$$(7.2) \quad v(x) = \int_0^{\infty} \chi_{\{y \in \Omega : v(y) > t\}}(x) dt.$$

**PROOF.** Since  $m(\{x \in \Omega : v(x) > t\}) = \int_{\Omega} \chi_{\{v > t\}}(x)m(dx)$  we have, using Fubini's theorem,

$$\int_0^{\infty} m(\{x \in \Omega : v(x) > t\})\nu(dt) = \int_{\Omega} \left( \int_0^{\infty} \chi_{\{v > t\}}(x)\nu(dt) \right) m(dx).$$

Theorem 7.1 follows from observing that

$$\int_0^{\infty} \chi_{\{v > t\}}(x)\nu(dt) = \int_0^{v(x)} \nu(dt) = \Phi(v(x)).$$

□

### 7.2. A consequence of the Brouwer fixed-point theorem.

**THEOREM 7.2** (Brouwer's fixed-point theorem). *Let  $B \subset \mathbb{R}^n$  be the unit ball for  $n \geq 0$ . If  $f : B \rightarrow B$  is continuous then  $f$  has a fixed point, i.e., there is some  $x \in B$  such that  $f(x) = x$ .*

The proof appears in many books on topology, e.g., in [M75]. Brouwer's theorem can be applied to establish the following result:

**THEOREM 7.3.** *Let  $B \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a closed ball and  $\vec{b}(\vec{r})$  a continuous map from  $B$  to  $\mathbb{R}^n$ . If  $\vec{b}$  points strictly outwards at every point of  $\partial B$ , i.e., if  $\vec{b}(\vec{r}) \cdot \vec{r} > 0$  for every  $\vec{r} \in \partial B$ , then  $\vec{b}$  has a zero in  $B$ .*

**PROOF.** Without losing generality we can assume that  $B$  is the unit ball centered at the origin. Since  $\vec{b}$  is continuous and  $\vec{b}(\vec{r}) \cdot \vec{r} > 0$  on  $\partial B$ , there are two constants  $0 < r_0 < 1$  and  $p > 0$  such that  $\vec{b}(\vec{r}) \cdot \vec{r} > p$  for every  $\vec{r}$  with  $r_0 < |\vec{r}| \leq 1$ . We show that there is a constant  $c > 0$  such that

$$| -c\vec{b}(\vec{r}) + \vec{r} | < 1$$

for all  $\vec{r} \in B$ : In fact, for all  $\vec{r}$  with  $|\vec{r}| \leq r_0$  the constant  $c$  can be any positive number below  $(\sup_{\vec{r} \in B} |\vec{b}(\vec{r})|)^{-1}(1 - r_0)$ . The supremum exists because  $|\vec{b}|$  is continuously defined on a compact set and therefore bounded. On the other hand, for all  $\vec{r} \in B$  with  $|\vec{r}| > r_0$  we have

$$\begin{aligned} | -c\vec{b}(\vec{r}) + \vec{r} |^2 &= c^2 |\vec{b}(\vec{r})|^2 - 2c\vec{b}(\vec{r}) \cdot \vec{r} + |\vec{r}|^2 \\ &\leq c^2 \sup_{\vec{r} \in B} |\vec{b}|^2 - 2cp + 1, \end{aligned}$$

which is also smaller than one if one chooses  $c > 0$  sufficiently small. Now set

$$\vec{g}(\vec{r}) = -c\vec{b}(\vec{r}) + \vec{r} \quad \text{for } \vec{r} \in B.$$

Then  $\vec{g}$  is a continuous mapping from  $B$  to  $B$  and by Theorem 7.2 it has some fixed point  $\vec{r}_1 \in B$ , i.e.,  $\vec{g}(\vec{r}_1) = \vec{r}_1$  and  $\vec{b}(\vec{r}_1) = 0$ .  $\square$

## References

- [AB88] M.S. Ashbaugh, R.D. Benguria: *Log-concavity of the ground state of Schrödinger operators: A new proof of the Baumgartner-Grosse-Martin inequality*, Physical Letters A **131**, 273–276 (1988).
- [AB91] M. S. Ashbaugh and R. D. Benguria, *Proof of the Payne–Pólya–Weinberger conjecture*, Bull. Amer. Math. Soc. **25**, 19–29 (1991).
- [AB92a] M. S. Ashbaugh and R. D. Benguria, *A sharp bound for the ratio of the first two eigenvalues of Dirichlet Laplacians and extensions*, Annals of Math. **135**, 601–628 (1992).
- [AB92b] M. S. Ashbaugh and R. D. Benguria, *A second proof of the Payne–Pólya–Weinberger conjecture*, Commun. Math. Phys. **147**, 181–190 (1992).
- [AB95] M. S. Ashbaugh and R. D. Benguria, *Sharp Upper Bound to the First Nonzero Neumann Eigenvalue for Bounded Domains in Spaces of Constant Curvature*, Journal of the London Mathematical Society (2) **52**, 402–416 (1995).
- [AB01] M. S. Ashbaugh and R. D. Benguria, *A Sharp Bound for the Ratio of the First Two Dirichlet Eigenvalues of a Domain in a Hemisphere of  $\mathbb{S}^n$* , Transactions of the American Mathematical Society **353**, 1055–1087 (2001).
- [B80] C. Bandle: *Isoperimetric Inequalities and Applications*, Pitman Monographs and Studies in Mathematics, vol. **7**, Pitman, Boston (1980).
- [BGM84] B. Baumgartner, H. Grosse, A. Martin: *The Laplacian of the potential and the order of energy levels*, Physics Letters **146B**, 363–366 (1984).
- [BL06] R.D. Benguria, H. Linde: *A second eigenvalue bound for the Dirichlet Schrödinger operator*, Commun. Math. Phys. **267**, 741–755 (2006).
- [BL07] R.D. Benguria, H. Linde: *A second eigenvalue bound for the Dirichlet Laplacian in hyperbolic space*, Duke Mathematical Journal **140**, 245–279 (2007).
- [Be92] Pierre Bérard, *Transplantation et isospectralité I*, Math. Ann. **292**, 547–559 (1992).
- [Be93] Pierre Bérard, *Transplantation et isospectralité II*, J. London Math. Soc. **48**, 565–576 (1993).
- [Be93b] Pierre Bérard, *Domaines plans isospectraux a la Gordon–Web–Wolpert: une preuve élémentaire*, Afrika Math. **1**, 135–146 (1993).
- [BS87] M. S. Birman, M. Z. Solomjak: *Spectral theory of self-adjoint operators in Hilbert Space*, D. Reidel Publishing Company, Dordrecht (1987).
- [B75] C. Borell: *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. **30**, 207–211 (1975).
- [Br88] Robert Brooks, *Constructing Isospectral Manifolds*, Amer. Math. Monthly **95**, 823–839 (1988).
- [B92] P. Buser: *Geometry and spectra of compact Riemannian surfaces*, Progress in Mathematics **106**, Birkhäuser, Boston (1992).
- [C84] I. Chavel: *Eigenvalues in Riemannian geometry*, Academic Press, Inc., NY (1984).
- [CH53] R. Courant and D. Hilbert, *Methods of Mathematical Physics, vol. 1*, Interscience Publishers, New York (1953).
- [D06] D. Daners: *A Faber-Krahn inequality for Robin problems in any space dimension*, Mathematische Annalen **335**, 767–785 (2006).
- [D90] E.B. Davies: *Heat kernels and spectral theory*, paperback edition, Cambridge University Press, Cambridge, UK (1990).
- [D96] E.B. Davies: *Spectral theory and differential operators*, Cambridge University Press, Cambridge, UK (1996).
- [F23] G. Faber: *Beweis, dass unter allen homogenen Membranen von gleicher Fläche und gleicher Spannung die kreisförmige den tiefsten Grundton gibt*, Sitzungberichte der mathematisch-physikalischen Klasse der Bayerischen Akademie der Wissenschaften zu München Jahrgang, pp. 169–172 (1923).
- [Fe69] H. Federer, *Geometric Measure Theory*, Springer Verlag, New York (1969).
- [GWW92] C. Gordon, D. Webb, and S. Wolpert, *Isospectral plane domains and surfaces via Riemannian orbifolds*, Invent. Math. **110**, 1–22 (1992).
- [H06] A. Henrot, *Extremum Problems for Eigenvalues of Elliptic Operators*, Collection Frontiers in Mathematics, Birkhauser, Basel (2006).
- [HLP64] G.H. Hardy, J.E. Littlewood and G. Pólya: *Inequalities*, Cambridge Univ. Press, Cambridge, UK (1964).

- [K66] Mark Kac, *Can one hear the shape of a drum?*, American Mathematical Monthly **73**, 1–23 (1966).
- [KS52] E. T. Kornhauser, I. Stakgold: *A variational theorem for  $\nabla^2 u + \lambda u = 0$  and its applications*, J. Math. and Physics **31**, 45–54 (1952).
- [K25] E. Krahn: *Über eine von Rayleigh formulierte Minimaleigenschaft des Kreises*, Math. Ann. **94**, 97–100 (1925).
- [K26] E. Krahn: *Über Minimaleigenschaften der Kugel in drei und mehr Dimensionen*, Acta Comm. Univ. Tartu (Dorpat) **A9**, 1–44 (1926). [English translation: *Minimal properties of the sphere in three and more dimensions*, **Edgar Krahn 1894–1961: A Centenary Volume**, Ü. Lumiste and J. Peetre, editors, IOS Press, Amsterdam, The Netherlands, pp. 139–174 (1994).]
- [LL97] E. H. Lieb, M. Loss: *Analysis*, Graduate Studies in Mathematics Vol. **14**, American Mathematical Society, Providence, RI (1997).
- [McH94] K. P. McHale, *Eigenvalues of the Laplacian*, “Can you Hear the Shape of a Drum?”, Master’s Project, Mathematics Department, University of Missouri, Columbia, MO (1994).
- [McKS67] H.P. McKean and I.M. Singer, *Curvature and the eigenvalues of the Laplacian*, Journal of Differential Geometry **1**, 662–670 (1967).
- [M85] V. G. Maz’ja: *Sobolev spaces*, Springer Series in Soviet Mathematics. Springer-Verlag, Berlin (1985). Translated from the Russian by T.O. Shaposhnikova.
- [Mi64] John Milnor, *Eigenvalues of the Laplace operator on certain manifolds*, Proc. Nat. Acad. Sc. **51**, 542 (1964).
- [M75] J.R. Munkres: *Topology, A first course*, Prentice-Hall, Englewood Cliffs (1975).
- [O80] R. Osserman, *Isoperimetric inequalities and eigenvalues of the Laplacian*. in **Proceedings of the International Congress of Mathematicians (Helsinki, 1978)**, pp. 435–442, Acad. Sci. Fennica, Helsinki, (1980).
- [P67] L.E. Payne, *Isoperimetric inequalities and their applications*, SIAM Review **9**, 453–488 (1967).
- [PPW55] L. E. Payne, G. Pólya, and H. F. Weinberger, *Sur le quotient de deux fréquences propres consécutives*, Comptes Rendus Acad. Sci. Paris **241**, 917–919 (1955).
- [PPW56] L. E. Payne, G. Pólya, and H. F. Weinberger, *On the ratio of consecutive eigenvalues*, J. Math. and Phys. **35**, 289–298 (1956).
- [Pj54] Å. Pleijel, *A study of certain Green’s functions with applications in the theory of vibrating membranes*, Arkiv för Matematik, **2**, 553–569 (1954).
- [PSz51] G. Pólya and G. Szegő, *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, NJ (1951).
- [R45] J. W. S. Rayleigh: *The Theory of Sound*, 2nd. ed. revised and enlarged (in 2 vols.), Dover Publications, New York, (1945) (republication of the 1894/1896 edition).
- [RSI] M. Reed, B. Simon: *Mathematical Physics*, Vol. I, Academic Press Inc., NY (1980).
- [RSIII] M. Reed, B. Simon, **Methods of Modern Mathematical Physics, Vol. 3**, Academic Press, NY (1979).
- [RSIV] M. Reed and B. Simon, **Methods of Modern Mathematical Physics, Vol. 4: Analysis of Operators**, Academic Press, NY (1978).
- [Ri859] B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, pp. 671–680 (1859).
- [SrKu94] S. Sridhar and A. Kudrolli, *Experiments on Not “Hearing the Shape” of Drums*, Physical Review Letters **72**, 2175–2178 (1994).
- [Su85] T. Sunada, *Riemannian Coverings and Isospectral Manifolds*, Annals of Math. **121**, 169–186 (1985).
- [S54] G. Szegő: *Inequalities for certain eigenvalues of a membrane of given area*, J. Rational Mech. Anal. **3**, 343–356 (1954).
- [T76] G. Talenti: *Elliptic equations and rearrangements*, Ann. Scuola Norm. Sup. Pisa (4) **3**, 697–718 (1976).
- [W56] H. F. Weinberger: *An isoperimetric inequality for the  $n$ -dimensional free membrane problem*, J. Rational Mech. Anal. **5**, 633–636 (1956).
- [We11] H. Weyl, *Über die asymptotische Verteilung der Eigenwerte*, Nachr. Akad. Wiss. Göttingen Math.-Phys., Kl. II 110–117 (1911).
- [We50] H. Weyl, *Ramifications, old and new, of the eigenvalue problem* Bull. Am. Math. Soc. **56**, 115–139 (1950).

DEPARTAMENTO DE FÍSICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CASILLA 306, SANTIAGO 22, CHILE

*E-mail address:* `rbenguri@fis.puc.cl`

DEPARTAMENTO DE FÍSICA, PONTIFICIA UNIVERSIDAD CATÓLICA DE CHILE, CASILLA 306, SANTIAGO 22, CHILE

*Current address:* Global Management Office, SAP SI AG, Albert Einstein Allee 3, D-64625 Bensheim, Germany

*E-mail address:* `Helmut.Linde@sap.com`

## Lectures on Random Schrödinger Operators

Peter D. Hislop

*This paper is dedicated to Jean-Michel Combes on the occasion of his sixty-fifth birthday.*

ABSTRACT. These notes are based on lectures given in the IV Escuela de Verano en Análisis y Física Matemática at the Instituto de Matemáticas, Cuernavaca, Mexico, in May 2005. It is a complete description of Anderson localization for random Schrödinger operators on  $L^2(\mathbb{R}^d)$  and basic properties of the integrated density of states for these operators. A discussion of localization for randomly perturbed Landau Hamiltonians and their relevance to the integer quantum Hall effect completes the lectures.

### CONTENTS

1. Models of Random Media	42
2. Basic Results in the Theory of Random Operators	55
3. One-Parameter Families of Operators: Spectral Averaging and Perturbation of Singular Spectra	64
4. The Wegner Estimate and the Integrated Density of States	77
5. Resolvent Estimates and Multiscale Analysis	89
6. Localization for Families of Random Schrödinger Operators	98
7. Random Magnetic Schrödinger Operators and the Integer Quantum Hall Effect	103
8. Fixed-Energy Multiscale Analysis	119
References	126

---

April 18, 2007.

2000 *Mathematics Subject Classification*. Primary 81Q10, 35P05; Secondary 47A55.

*Key words and phrases*. Random operators, Localization, Wegner estimate, Density of states.

The author was supported in part by NSF Grant #0503784.

## 1. Models of Random Media

**1.1. Introduction.** In this chapter, we present an overview of the spectral theory of random media discussed in these notes. We are primarily concerned with the phenomena of localization for electrons and for classical waves propagating in randomly perturbed media. By *localization*, we mean that the permissible states of the system, with energies in a specified range, do not propagate in time and are localized in space. These states correspond to the existence of point spectrum for the corresponding self-adjoint operators describing the system. The fundamental idea is that the random perturbations cause states, which would otherwise be extended throughout all space, to effectively localize in a finite region due to multiple backscattering off of the random potential. Spectrally, these models are interesting because the spectrum in an interval does not consist of discrete eigenvalues but rather the eigenvalues are dense in the spectrum in the interval with probability one. Dynamically, states with energies localized in such intervals are characterized by the fact that the corresponding moments of the time-dependent position operator are bounded for all time. This effect is referred to as *dynamical localization*.

**1.2. Physical Considerations from Condensed Matter Physics.** The one-electron model of condensed matter physics is a simplification which has had some remarkable success in describing certain properties of crystals. The basic notions of insulating, conducting, and semiconducting materials can be explained by the energy spectrum of a single electron moving under the influence of a periodic array of atoms, and the Pauli exclusion principle. The latter determines the occupation of states by noninteracting electrons. In the one-electron model, the interaction between the electrons is neglected and the spectral and transport properties of the material is described by a one-particle Schrödinger operator  $H_0$ . We emphasize that another approximation implicit in the one-electron model is that the media is infinitely-extended in space. Under this assumption, we can neglect any boundary effects and use techniques common in statistical mechanics. In particular, we will often approximate the infinitely-extended system by a finitely-extended one. We will study the dependence of relevant properties of the system on the volume of the region, and then take the volume to infinity to recover the properties of the extended system. We refer to this as the *thermodynamic limit*.

The actual one-electron model is determined by the choice of the potential  $V_0$ , and the presence of external fields. For example, a perfect crystal is described by a periodic potential function  $V_0$ . The one-electron model is given by the Schrödinger operator  $H_0 \equiv K + V_0$ , where  $K$  describes the kinetic energy of the particle. For example, the choice  $K \equiv -\Delta$  describes the kinetic energy of a free particle. We can make other choices for the operator  $K$ . A particle moving in the presence of an external electric field is described by  $K \equiv -\Delta + F \cdot x$ , where  $F$  is a constant vector representing the electric field. A particle in a magnetic field is described by the Schrödinger operator  $K \equiv (-i\nabla - A)^2$ , for a vector potential  $A$ , so that the magnetic field is  $B = dA$  (that is,  $B = \nabla \times A$  in two- and three-dimensions).

We will always assume that the spectral properties of the background Schrödinger operator  $H_0 = K + V_0$ , for operators  $K$  given above, are, in some sense, completely known. We are interested in studying perturbations of  $H_0$ . As we consider systems that are infinitely-extended in space, we will be primarily concerned with perturbations which do not decay at infinity. Such perturbations have the possibility of

radically changing the spectrum of  $H_0$ , as we shall see. There are other, interesting phenomena associated with compactly-supported perturbations of  $H_0$ , such as the occurrence of impurity levels in band gaps (see, for example [37, 49]). Our most important example is the description of impurities in an infinitely-extended crystal. The impurities are described by a perturbation  $V_1$ . Thus, one is lead to the study of Schrödinger operators of the form  $H = K + V_0 + V_1 = H_0 + V_1$ .

Let us consider the simple case  $H_0 = -\Delta + V_0$  further. It is well-known [113] that in the absence of any perturbation, a Schrödinger operator  $H_0$ , with  $K = -\Delta$  and a continuous, periodic potential  $V_0$ , has purely absolutely continuous spectrum. As a set, the spectrum of  $H_0$ , which we denote by  $\sigma(H_0)$ , is a union of closed intervals  $B_n$ , called bands. An energy  $E \in B_n$  represents an energy at which an electron can propagate through the crystal. If two bands are separated, the intervening open interval is called a spectral gap. The occurrence of a gap in the spectrum, which represents forbidden energies at which an electron cannot propagate, provides one of the bases of our understanding of the electronic properties of crystals.

We point out that our terminology of energy bands and gaps includes the following cases. Many Schrödinger operators of the form  $H_0 = -\Delta + V_0$  have spectrum filling the half-line  $[M_0, \infty)$ , for some constant  $M_0 \in \mathbb{R}$ . This is the case, for example, if  $V_0$  is positive, bounded, and has compact support. In this case, we say that  $H_0$  has one band  $[M_0, \infty)$  and one gap  $G \equiv (-\infty, M_0)$ . The free Stark Hamiltonian  $H_0 = -\Delta + F \cdot x$  has spectrum equal to  $\mathbb{R}$ . This operator has one band  $B_1 = \mathbb{R}$  and no gap. As another extreme case, the Landau Hamiltonian  $H_A = (-i\nabla - A)^2$ , in two-dimensions, with vector potential  $A(x_1, x_2) = (B/2)(-x_2, x_1)$ , has only pure point spectrum with infinitely-degenerate eigenvalues given by  $E_n(B) = (2n+1)B$ . We call the infinitely-degenerate energy levels  $E_n(B)$  the bands and the open intervals  $G_0 \equiv (-\infty, E_1(B))$ , and  $G_n \equiv (E_n(B), E_{n+1}(B))$ , for  $n \geq 1$ , the gaps.

Returning to our description of the one-electron model for a perfect crystal, let us consider why we are interested in perturbations of  $H_0$  that do not decay at infinity. The goal in the study of electrons in crystals is a description of finite conductivity. Classically, conductivity  $\sigma$  is the constant of proportionality between an applied electric field  $E$  and the induced current  $J$  so that we have Ohm's law  $J = \sigma E$ . The current  $J$  corresponding to an electron state  $\psi$  in a one-electron model is the electric charge  $e$  times the matrix element of the velocity operator  $v = \frac{dX(t)}{dt}$ , so we have  $J = e\langle\psi, v\psi\rangle$ . The operator  $X(t)$  is the Heisenberg position operator (see (1.2) below). In a perfect crystal, the static conductivity is defined to be  $\sigma \equiv \lim_{|E| \rightarrow 0} |J(E)|/|E|$ . It is known that the static conductivity for a perfect crystal is infinite [74]. Anderson [6] proposed that impurities and defects might explain the observed finiteness of electric conductivity of crystals at zero temperatures (at finite temperatures electron-electron and electron-phonon collisions become important mechanisms in resistivity). In the one-electron picture, impurities are described by a nonperiodic perturbation  $V_1$  of the periodic atomic potential. Thus, one is lead to the study of Schrödinger operators of the form  $H = -\Delta + V_0 + V_1$ , where  $V_0$  is periodic and  $V_1$  represents the impurities.

How shall we choose the perturbation  $V_1$ ? It follows from the basic stability of the absolutely continuous spectrum [75] that the impurity potential  $V_1$  must be extended in space and nondecaying if it is to change the qualitative nature of the absolutely continuous spectrum or the spectral type. If the impurity potential  $V_1$  has compact support or is rapidly decaying as  $|x| \rightarrow \infty$ , then the absolutely

continuous spectrum of  $H_0$  is unchanged [112], and at most some eigenvalues are added in the spectral gaps (as mentioned above, this is an interesting problem in itself). For example, if the potential  $V_1$  is such that the pair of Hamiltonians  $(H_0, H)$  has a conjugate operator  $A$  in the sense of Mourre theory, then  $H$  is also *ballistic* [24] meaning that  $X^2(t) \sim t^2$ , and the electron motion resembles free motion. Clearly, such motion corresponds to infinite conductivity which is what we are trying to eliminate through the perturbation  $V_1$ . It follows that our impurity potential  $V_1$  cannot be concentrated in space or decaying at infinity, but must be present throughout the infinite crystal. The investigation of such perturbations is difficult and requires new tools.

In these notes, we will concentrate on the study of the spectral properties of Schrödinger operators  $H_{V_1} = H_0 + V_1$ , where  $H_0 = K + V_0$  is assumed to have known spectral properties, and the perturbation  $V_1$  belongs to a class of infinitely-extended perturbations. We will consider the following questions.

### 1. Spectral Properties of $H_{V_1}$ .

- (1) What is the spectrum of  $H_{V_1}$  as a set?
- (2) What is the spectral type of  $H_{V_1}$ ?
- (3) What is the nature of the eigenfunctions, if there are any?

The study of transport properties of the one-electron model involves the determination of the long-time asymptotic behavior of the matrix elements of the moments of the position operator  $X(t)$ . Basic questions which are of interest here include

### 2. Transport Properties of $H_{V_1}$ .

- (1) Is the motion generated by  $H_{V_1}$  ballistic, diffusive, or localized?
- (2) What is the conductivity of the material characterized by  $H_{V_1}$ ?
- (3) How stable are the transport exponents under perturbations?

As can be imagined, we need to restrict the class of allowable perturbations in order to address these questions. It turns out that if  $V_1$  belongs to a family of perturbations which form a stochastic process  $V_\omega$ , and we consider the *family* of operators  $H_\omega = H_0 + V_\omega$ , then we can provide detailed answers to the above questions. The introduction of a family of perturbations, rather than just one perturbation, allows us to make a general spectral analysis which holds for *almost every* perturbation in the family. The ability to average over potentials in the family provides us with a powerful tool for spectral analysis. The general framework for our investigations, then, will be a family of self-adjoint operators  $H_\omega$ , where  $\omega \in \Omega$  describes a possible realization of the random media. These realizations occur with a probability given by a measure  $\mathbb{P}$  on the space of all possible configurations  $\Omega$ . We will make this picture more precise in chapter 2.

**1.3. Geometric Considerations from Deterministic Models.** As an introduction to the manner in which an infinitely-extended perturbation (that is, a potential  $V$  which does not vanish at infinity) influences the spectral properties of the Laplacian, we review some results of a geometric character. Many spectral and transport properties of a Schrödinger operator  $H_V = -\Delta + V$  depend on geometric characteristics of the potential  $V$ . There now exist 1) geometric conditions on  $V$  which imply the absence of absolutely continuous spectrum, 2) geometric conditions on  $V$  which imply the absence of diffusion, and 3) lower bounds on the moments of the position operator in terms of the fractal dimension of the spectrum for a general self-adjoint operator  $A$ .

The absence of absolutely continuous spectrum and the absence of diffusion require geometric conditions on the potential  $V$  which insure that the particle does not freely propagate to infinity. In particular, its motion does not resemble that of a free particle. Such potentials are characterized by regions where they are large in the sense of the Agmon metric for a given interval of energy. The general results indicate that these effective barrier regions can be sparsely distributed provided they dominate the potential in the complementary region. The conditions on the potential depend on the dimension. Examples of such geometric conditions can be found in the papers [103, 124].

This type of result on the absence of absolutely continuous spectra, and the absence of diffusion discussed next, require careful estimates on the boundary-value of the resolvent  $R(E + i\epsilon)$  as  $\epsilon \rightarrow 0$ . Absence of absolutely continuous spectrum follows from the estimate

$$(1.1) \quad \sup_{\epsilon \rightarrow 0} \|R(E + i\epsilon)\phi\| < C_0,$$

for a finite constant  $C_0$  and any function of compact support  $\phi$ . The main techniques used to establish this type of estimate include the geometric resolvent formula, the spectral stability results of Briet, Combes, and Duclos [18], Agmon exponential decay results [1], and deBrange's theorem on the boundary-values of operator-valued Herglotz functions.

There are far fewer results on sufficient conditions for the absence of diffusion for Schrödinger operators. Montcho [104] has discovered particularly nice, dimension-independent conditions on the potential  $V$  which guarantee the absence of diffusion. For a Schrödinger operator  $H$ , the dynamics is generated by the unitary group  $U(t) = e^{-iHt}$ . For a state  $\psi$ , the mean-square displacement is defined by

$$(1.2) \quad \langle \psi, U(t)^* X^2 U(t) \psi \rangle \equiv \langle X^2(t) \rangle_\psi.$$

Roughly speaking, the dynamics is called *diffusive* if there exists a nonzero constant  $D$  so that

$$(1.3) \quad \langle X^2(t) \rangle_\psi \sim Dt.$$

The absence of diffusion means that  $D = 0$ , that is, the left side of (3) grows more slowly than  $t$ . Let the time-averaged mean-square displacement be denoted by

$$(1.4) \quad \langle \langle X^2 \rangle \rangle_T \equiv T^{-1} \int_0^T \langle X^2(t) \rangle_\psi dt.$$

There are sufficient geometric conditions on the potential  $V$  so that

$$(1.5) \quad \limsup_{T \rightarrow \infty} T^{-1} \langle \langle X^2 \rangle \rangle_T = 0$$

which implies the absence of diffusion.

Another topic concerns the relation between the fractal properties of the spectrum of a self-adjoint operator  $A$  and the long-time asymptotic behavior of the time evolution  $U(t) = e^{-itA}$  generated by  $A$ . Given any state  $\psi$ , a spectral measure  $\mu_{A,\psi}$  for  $A$  and  $\psi$  is defined by the spectral theory through the representation

$$(1.6) \quad \langle \psi, U(t)\psi \rangle = \int_{\mathbb{R}} e^{-it\lambda} d\mu_{A,\psi}(\lambda),$$

Barbaroux, Combes, and Montcho [10], and Last [98] proved that the Hausdorff dimension of the measure  $\mu_{A,\psi}$  determines a lower bound on the  $\alpha$  moment of the

position operator  $X$  defined as in (4) with  $\alpha$  replacing 2. They prove that for sufficiently regular states  $\psi$  and for any  $\epsilon > 0$ , there is a constant  $C_{\psi, \epsilon}$ , so that

$$(1.7) \quad \langle \langle X^\alpha \rangle \rangle_T \geq C_{\psi, \epsilon} T^{\frac{\alpha}{d}(A_\psi - \epsilon)},$$

where  $A_\psi$  is the Hausdorff dimension of the spectral measure  $\mu_{A, \psi}$ . This work requires a firm understanding of spectral theory and the classical analysis of measures. Many recent works are devoted to estimating the diffusion exponent from below, and we mention [23, 12].

**1.4. Anderson Tight-Binding Model.** The simplest model that describes the propagation of a single electron in a randomly perturbed crystal is called the Anderson tight-binding model. Atoms are considered to act on the electron at a single site only and short distances are eliminated by restricting the electron motion to the points of the lattice  $\mathbb{Z}^d$ . This eliminates short distances and provides a high-energy cut-off for the free kinetic energy.

We consider an operator of the form

$$(1.8) \quad H = \Delta + \lambda V_\omega^A, \text{ on } l^2(\mathbb{Z}^d) \text{ for } d \geq 1.$$

The finite-difference Laplacian  $\Delta$  is defined by

$$(1.9) \quad (\Delta f)(n) = \sum_{|n-m|=1} f(m).$$

This operator describes the kinetic energy of an electron moving on the lattice. By the Fourier transform, one finds that the spectrum is purely absolutely continuous and equal to

$$(1.10) \quad \sigma(\Delta) = [-2d, 2d].$$

The Anderson-type potential  $V_\omega^A$  is constructed as follows. Let the family

$$(1.11) \quad \{\omega_i\}, \quad i \in \mathbb{Z}^d,$$

be a stochastic process indexed by  $\mathbb{Z}^d$ . Each random variable is distributed identically with a distribution function

$$(1.12) \quad g(\omega) \geq 0,$$

which we assume has bounded support. We consider the random variables to be independent. For simplicity, we take

$$(1.13) \quad \text{supp } g = [-1, 1],$$

The potential  $V_\omega^A$  is defined by

$$(1.14) \quad (V_\omega^A f)(m) = \omega_m f(m), \text{ for } m \in \mathbb{Z}^d.$$

It is often convenient to add a coupling constant  $\lambda$  in front of the potential which measures the strength of the interaction and as such is a measure of the *disorder* of the system. More generally, the *disorder* of a random system is usually taken to be  $\|g\|^{-1}$ , where  $g$  is the normalized density. The range of the scaled potential  $\lambda V_\omega^A$  is  $[-\lambda, \lambda]$ . The totality of the independent random variables allows us to construct a probability space  $[-1, 1]^{\mathbb{Z}^d}$ , with the measure  $P$  induced by the product measure.

The Anderson tight-binding model is

$$(1.15) \quad H_\omega(\lambda) = \Delta + \lambda V_\omega^A.$$

This Hamiltonian  $\mathbb{Z}^d$ -ergodic with respect to the translation group on the lattice. By Pastur's theorem, the family of operators has a deterministic spectrum  $\Sigma$ . This means that there exists a fixed subset  $\Sigma \subset \mathbb{R}$  such that

$$(1.16) \quad \mathbb{P} - \text{a.s. } \sigma(H_\omega) = \Sigma.$$

**THEOREM 1.1.** *For  $\mathbb{P}$ -almost every  $\omega \in \Omega$ , the spectrum  $\sigma(H_\omega) = \Sigma$ , and  $\Sigma = [-2d, 2d] + \lambda \text{ supp } g = [-2d - \lambda, 2d + \lambda]$ .*

Our object of study is the *spectral type* of the random family  $H_\omega$  for almost every  $\omega \in \Omega$ . We can already make some preliminary observations with respect to the spectral type.

(i) The spectrum  $\sigma(\Delta) = [-2d, 2d]$  is absolutely continuous since the operator  $\Delta$  is unitarily equivalent to multiplication by  $2 \sum_{j=1}^d \cos(2\pi k_j)$ ,  $k_j \in [0, 1]$ , by the Fourier transform.

(ii)  $\sigma(V_\omega)$  is pure point. The vector  $\delta_n(m) \equiv \begin{cases} 1 & n = m, \\ 0 & \text{other,} \end{cases}$  satisfies

$$(V(\delta_n))(k) = \begin{cases} \omega_n & n = k, \\ 0 & \text{other,} \end{cases}$$

that is

$$V_\omega \delta_n = \omega_n \delta_n,$$

so each vector  $\delta_n$  is an eigenfunction for  $V_\omega$ . Hence, the almost sure spectrum of  $V_\omega$  is  $[-1, 1]$  and dense pure point.

Although this model seems extremely simple, there are some major outstanding open questions.

**Conjectures on  $\Sigma_\lambda \equiv \sigma(H_\omega(\lambda))$  almost surely.**

- (1) Fix  $\lambda > 0$ .  $\exists E_0 > 2d$  such that  $\Sigma_\lambda \cap \{(-\infty, -E_0] \cup [E_0, \infty)\}$  is dense pure point with  $2d < E_0 < 2d + \lambda$ .
- (2)  $\exists \lambda_0$  such that  $\lambda > \lambda_0$ ,  $\Sigma$  is dense pure point.
- (3) i. dimension  $d = 1$ ,  $\Sigma_\lambda$  is always dense pure point.  
 ii. dimension  $d = 2$ ,  $\Sigma_\lambda$  is always dense pure point.  
 iii. dimension  $d \geq 3$ , for small  $\lambda \geq 0$ ,  $\exists E_m = E_m(\lambda) < E_0$  of (1) such that  $\Sigma_\lambda \cap [-E_m, E_m]$  is absolutely continuous.

We refer to Case 1 as spectral localization near the band edges, Case 2 as localization at large disorder. These are expected to hold independently of the dimension. Case 3 summarizes the dimension-dependent conjectures.

Statements (1), (2) and (3i) are known. This work on the lattice Anderson model is presented in the books of Carmona and Lacroix [21], and Pastur and Figotin [107]. In the multidimensional lattice case, the key original articles are those of Fröhlich and Spencer [56], Martinelli and Scoppola [102], Simon and Wolff [119], Kotani and Simon [94], Delyon, Soulliard, and [38], von Dreifus and Klein [132], and Aizenman and Molchanov [4].

Problems (3ii)-(3iii) remain open. Klein [88] proved the existence of extended states (a band of absolutely continuous spectrum near zero energy) for the Anderson model on the Bethe lattice. Another proof of this result was recently given by Froese, Hasler, and Spitzer [57], and a related stability result for the absolutely continuous spectrum was given by Aizenman, Sims, and Warzel [5]. The existence of extended states and a mobility edge  $E_m$  for small  $\lambda \geq 0$  for the lattice Anderson

model, with decaying randomness, was also recently established by Kirsch, Krishna, and Obermeit [84]

**1.5. Single Electron Models.** There are several models of physical relevance which describe the behavior of a single electron in randomly perturbed media. We have the choice of the background operator  $H_0$  and of the random potential  $V_\omega$ . The most well known situation is when the background operator  $H_0 = -\Delta + V_{per}$ , where  $V_{per}$  is a periodic function on  $R^d$ . This unperturbed operator describes a single electron moving in an infinite, periodic lattice. The basic random perturbation is the Anderson-type potential. It is described by a family of functions  $u_i$ , with  $i \in \mathbb{Z}^d$ , called the *single-site potentials*, and coupling constants  $\{\omega_i \mid i \in \mathbb{Z}^d\}$ . The potential is defined by

$$(1.17) \quad V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} \omega_i u_i(x).$$

The family of coupling constants  $\{\omega_i \mid i \in \mathbb{Z}^d\}$  forms a stochastic process on  $\mathbb{Z}^d$ . In the simplest model, the coupling constants are a collection of independent, identically distributed (*iid*) random variables. In the *iid* case, the random operators are ergodic (see chapter 2) if  $u_i(x) = u(x - i)$ , for all  $i \in \mathbb{Z}^d$ . More complicated models treat the case of correlations between the random variables (see, for example [31, 132]). We can also introduce another family of vector-valued random variables  $\{\xi_i \mid i \in \mathbb{Z}^d\}$  with  $\xi_i \in B_R(0)$ ,  $0 < R < \frac{1}{2}$ . We assume that the random variable  $\xi_i$  has an absolutely continuous distribution, for example, a uniform distribution. These random variables will model thermal fluctuations of the scatterers with random strengths about the lattice points  $\mathbb{Z}^d$ . The random potential has the form

$$(1.18) \quad V_{\omega, \omega'}(x) \equiv \sum_{i \in \mathbb{Z}^d} \omega_i u_i(x - \xi_i).$$

The Anderson model on the lattice has been extensively studied. We refer to the books by Carmona and Lacroix [21] and by Pastur and Figotin [107] for the references. The basic papers, of relevance to the approaches described here for the continuous cases, include [56, 76, 119, 120, 131]. There are now many results on localization at the bottom of the spectrum and at the band edges for the continuous Anderson model. These include [8, 25, 90, 83, 86, 87]. These papers use the method of multiscale analysis explained in these lectures. Recently, the method of fractional moments was extended to continuum models and we refer the reader to [3, 20]. The case of correlated random variables for lattice models is described in [132] and for the continuous models in [31].

In general, the background operator for electrons may have the form

$$(1.19) \quad H_0 = (p - A_0)^2 + V_0,$$

where  $p \equiv -i\nabla$ , the vector-valued function  $A_0$  is a reasonable vector potential and  $V_0$  is a background potential, cf. [8]. The Landau Hamiltonian  $H_A$  in two dimensions is a special case with a constant magnetic field  $B$  generated by the vector potential  $A = (B/2)(-x_2, x_1)$ , and  $V_0 = 0$ . Localization and the integrated density of states for this model have been extensively studied, see [9, 26, 40, 41, 63, 133, 134], because of the role it plays in the theory of the integer quantum Hall effect (cf. [95]).

There are other models of randomness of physical interest. We give a brief description of them here. In sections 3–5, we will concentrate on results for the Anderson-type potential, although they also hold for some of the other models.

1.5.1. *Breather Model.* Let  $u > 0$  be a single-site potential of compact support. We require a repulsive condition on  $u$ :  $-x \cdot \nabla u(x) \geq 0$ , and a relative boundedness condition on the Hessian of  $u$ . We assume that the random variables  $\omega_i$  are *iid* with a common density supported some positive interval. The breather potential has the form

$$(1.20) \quad V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} u(\omega_i(x - i)).$$

One can show that the random family  $H_\omega = -\Delta + V_\omega$  has an interval of localized states  $[0, E_1]$ , almost surely, for some  $E_1 > 0$  [30].

1.5.2. *Random Displacement Model.* The random potential for this model has the form

$$(1.21) \quad V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} u(x - i - \xi_i),$$

where the vector-valued random variables  $\{\xi_i(\omega)\}$  are distributed in a ball of radius  $R < 1/2$ , as above. This model was studied in the case of  $u \leq 0$  so that the single-site potential represents a potential well. Klopp [91] showed that in this case the model exhibits localization at negative energies provided the semiclassical parameter  $h$ , appearing in the Hamiltonian as  $H(h) = -h^2\Delta + V_\omega$ , is sufficiently small. Quantum tunneling plays a major role in the localization of states for this model. Localization in the general case is an open problem.

1.5.3. *Poisson Model.* One of the most realistic models of impurities randomly distributed in a perfect crystal is given by a Poisson potential. Let  $X_i(\omega)$ ,  $i \in \mathbb{Z}$ , represent the points of a Poisson process in  $\mathbb{R}^d$ . Suppose that  $u \geq 0$  is a single-site potential of compact support. The Poisson potential is given by

$$(1.22) \quad V_\omega(x) \equiv \sum_{i \in \mathbb{Z}} u(x - X_i(\omega)).$$

Surprisingly, until recently, very little was known about these potentials except in one dimension. Stolz [125] recently proved that all states are localized for the one-dimensional model. In arbitrary dimensions, Tip [127] has proved that for a class of repulsive potentials  $u$ , the integrated density of states is absolutely continuous at high energies. Most recently, Germinet, Klein and the author [59, 60, 61] proved Anderson and dynamical localization for this model. The work is related to recent work of Bourgain and Kenig [19] on the Anderson model with Bernoulli distributed random variables.

1.5.4. *Gaussian Models.* This random family of Schrödinger operators has the form (1.1), with  $H_0 = -\Delta$  and the random potential  $V_\omega(x)$  a Gaussian process indexed by  $\mathbb{R}^d$ . This model has been recently studied [52, 53, 54]. These authors prove a Wegner estimate, show the absence of absolutely continuous spectrum, and prove localization at negative energies.

**1.6. Classical Wave Propagation.** The localization of acoustic waves and of light is of theoretical and practical importance. Because of the absence of the electron-electron interaction, which might tend to obscure localization effects for electrons, it may be easier to detect the localization of light experimentally. For a

review of these questions, we refer to [13, 80]. The techniques used to prove band-edge localization for electrons can also be used for various models of classical waves propagating in inhomogeneous media. Some of the early work on localization for lattice models describing waves was done by Faris [43, 44] and Figotin and Klein [46]. We describe here the models studied in [33].

(1) **Acoustic waves.** The wave equation for acoustic waves propagating in a medium with sound speed  $C$  and density  $\rho$  is

$$(1.23) \quad \partial_t^2 \psi + \hat{H} \psi = 0,$$

where the propagation operator  $\hat{H}$  is given by

$$(1.24) \quad \hat{H} \equiv -C^2 \rho \nabla \rho^{-1} \nabla.$$

By a standard unitary transformation, it suffices to consider the operator  $H$ , unitarily equivalent to  $\hat{H}$ , given by

$$(1.25) \quad \begin{aligned} H &\equiv -C \Delta C - \frac{1}{2} \left\{ \frac{C^2 \Delta \rho}{\rho} - \frac{3}{2} \frac{C^2 |\nabla \rho|^2}{\rho^2} \right\} \\ &= -(C^2 \rho)^{\frac{1}{2}} \nabla \cdot \rho^{-1} \nabla (C^2 \rho)^{\frac{1}{2}}, \end{aligned}$$

acting on the Hilbert space  $\mathcal{H} = L^2(\mathbf{R}^d)$ ,  $d \geq 1$ . We consider perturbed sound speeds of the form

$$(1.26) \quad C_\omega(g) \equiv (1 + g\tilde{C}_\omega)^{-1/2} C_0,$$

for  $g \geq 0$ . To relate this to (1.2), we factor out the unperturbed sound speed  $C_0$  and define the unperturbed acoustic wave propagation operator  $H_0$  by

$$(1.27) \quad H_0 \equiv -C_0 \rho^{\frac{1}{2}} \nabla \cdot \rho^{-1} \nabla \rho^{\frac{1}{2}} C_0.$$

The coefficient  $A_\omega$  appearing in (1.2) is given by

$$(1.28) \quad A_\omega \equiv (1 + g\tilde{C}_\omega).$$

(2) **Electromagnetic waves.** The wave equation for electromagnetic waves can be written in the form of equation (1.2) for vector-valued functions  $\psi$ . In this case, the operator  $H$  describing the propagation of electromagnetic waves in a medium characterized by a dielectric function  $\epsilon$  and a magnetic permeability  $\mu = 1$  is given by

$$(1.29) \quad H \equiv -\epsilon^{-1/2} \Delta \Pi \epsilon^{-1/2},$$

acting on the Hilbert space  $\mathcal{H} = L^2(\mathbf{R}^3, \mathcal{C}^3)$ . The matrix-valued operator  $\Pi$  is the orthogonal projection onto the subspace of transverse modes. We consider random perturbations of a background medium described by a dielectric function  $\epsilon_0$  and given by

$$(1.30) \quad \epsilon_\omega(g) \equiv 1 + \epsilon_0 + g\tilde{\epsilon}_\omega,$$

where  $\tilde{\epsilon}_\omega$  is a stochastic process. The unperturbed operator describing the background medium is defined by

$$(1.31) \quad H_0 \equiv -(1 + \epsilon_0)^{-1/2} \Delta \Pi (1 + \epsilon_0)^{-1/2},$$

and the coefficient  $A_\omega$  in (1.2) is given by by a dielectric function  $\epsilon_0$  and given by

$$(1.32) \quad \epsilon_\omega(g) \equiv 1 + \epsilon_0 + g\tilde{\epsilon}_\omega,$$

where  $\tilde{\epsilon}_\omega$  is a stochastic process. The unperturbed operator describing the background medium is defined by

$$(1.33) \quad H_0 \equiv -(1 + \epsilon_0)^{-1/2} \Delta \Pi (1 + \epsilon_0)^{-1/2},$$

and the coefficient  $A_\omega$  in (1.2) is given by

$$(1.34) \quad A_\omega \equiv (1 + g(1 + \epsilon_0)^{-1} \tilde{\epsilon}_\omega).$$

We note that  $A_\omega$  is the velocity of light for the realization  $\omega$ .

There have been several recent papers proving band-edge localization for these and related models. Figotin and Klein [47, 48] have results on band edge localization quite similar to ours. Stollmann [121] recently studied random perturbations of metrics. The random processes  $\tilde{C}_\omega$  and  $\tilde{\epsilon}_\omega$  can be of any of the type described in section 2.

**1.7. The Main Results on Band-Edge Localization.** We now list our hypotheses necessary to prove localization at energies near the unperturbed band-edges  $B_\pm$ . We will begin with conditions of the deterministic operator  $H_0$ . this operator, which describes the unperturbed media, is given in (1.19) in the Schrödinger case, (1.27) in the acoustic case, and in (1.31) in the Maxwell case. When it is necessary to distinguish these three cases, we will write  $H_0^X$ , with  $X = S, A$ , or  $M$  for the Schrödinger acoustic, or Maxwell case, respectively.

- (H1) The self-adjoint operator  $H_0^X$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ , for  $X = S$  and  $X = A$ , and on  $C_0^\infty(\mathbb{R}^3, \mathcal{C}^3)$  for  $X = M$ . The operator  $H_0^X$  is semi-bounded and has an open spectral gap. That is, there exist constants  $-\infty < -C_0 \leq B_- < B_+ \leq \infty$  so that

$$\sigma(H_0) \subset (-C_0, B_-] \cup [B_+, \infty).$$

- (H2) The operator  $H_0^X$  is strongly locally compact in the sense that for any  $f \in L^\infty(\mathbb{R}^d)$ , for  $X = S$  and  $A$ , or for any  $f \in L^\infty(\mathbb{R}^3, \mathcal{C}^3)$  for  $X = M$ , with compact support, the operator  $f(H_0)(H_0 + C_0 + 1)^{-1} \in \mathcal{I}_q$ , for some even integer  $q$ , with  $1 \leq q < \infty$ . Here,  $\mathcal{I}_q$  denotes the  $q^{\text{th}}$ -Schatten class, cf. [114].

- (H3) Let  $\rho(x) \equiv (1 + \|x\|^2)^{1/2}$ . The operator

$$H_0(\alpha) \equiv e^{i\alpha\rho} H_0 e^{-i\alpha\rho},$$

defined for  $\alpha \in \mathbb{R}$ , admits an analytic continuation as a type-A analytic family to a strip

$$S(\alpha_0) \equiv \{x + iy \in \mathcal{C} \mid |y| < \alpha_0\},$$

for some  $\alpha_0 > 0$ .

Hypothesis (H1) is a condition on the unperturbed potential  $V_0$  and the vector potential  $A$  in the Schrödinger case, and on the unperturbed medium, as described by  $C_0$  and  $\epsilon_0$ , for the classical wave case. Schrödinger operators with periodic potentials provide examples of operators  $H_0$  with open spectral gaps, cf. [113]. As for classical waves, certain models of photonic crystals are known to have open gaps, cf. [50, 51, 128]. Hypothesis (H3) is satisfied for a large class of operators  $H_0$ , see [8, 33].

We now give hypotheses on the random potential given in (1.17)–(1.18) and in (1.28) and (1.34). The random potentials of Anderson-type are specified by giving conditions on the single-site potentials  $u_j$ , the coupling constants  $\omega_j$ , and

the random variables  $\xi_j$ . We will use the following notation. We denote by  $B_r(x)$  the ball of radius  $r$  centered at  $x \in \mathbb{R}^d$ . A cube of side length  $\ell$  centered at  $x \in \mathbb{R}^d$  is denoted by  $\Lambda_\ell(x)$ . The characteristic function of a subset  $A \subset \mathbb{R}^d$  is denoted by  $\chi_A$ .

(H4) The single-site potentials  $u_i \geq C_i \chi_{B_{r_i}(0)}$  for some constant  $C_i > 0$  and some radius  $r_i > 0$ . Furthermore, we assume that

$$(1.35) \quad \sum_{j \in \mathbb{Z}^d} \left\{ \int_{\Lambda_1(0)} |u_j(x)|^p \right\}^{1/p} < \infty,$$

for  $p \geq d$  when  $d \geq 2$  and  $p = 2$  when  $d = 1$ .

We will assume that the random variables  $\omega_j$ , appearing in the Anderson-type potential (1.17), form a stationary stochastic process indexed by  $\mathbb{Z}^d$ . The probability space for this process is  $\Omega = [-m, M]^{\mathbb{Z}^d}$ , for some constants  $m \neq M$  and  $0 \leq m, M \leq \infty$ . In the case that the range of the random variables is unbounded, we will need to control some of the moments of  $\omega_j$ .

(H5) The random variables  $\omega_j$  have  $p$  finite moments:

$$(1.36) \quad \mathbb{E}\{\omega_j^k\} < \infty, \quad k = 1, 2, \dots, p,$$

where  $p$  is the dimension-dependent constant given in (H4).

We refer to the review article of Kirsch [82] for a proof of the fact that hypotheses (H4)–(H5) imply the essential self-adjointness of  $H_\omega$  on  $C_0^\infty(\mathbb{R}^d)$ .

(H6) The conditional probability distribution of  $\omega_0$ , conditioned on  $\omega_0^\perp \equiv \{\omega_i \mid i \neq 0\}$ , is absolutely continuous with respect to Lebesgue measure. It has a density  $h_0$  satisfying  $\|h_0\|_\infty < \infty$ .

Hypothesis (H5) implies that the correlation function  $C(i, j) \equiv \mathbb{E}\{\omega_i \omega_j\} - \mathbb{E}\{\omega_i\} \mathbb{E}\{\omega_j\}$  exists and is finite. An example of a process satisfying (H5) is a Gaussian process on  $\mathbb{Z}^d$  with each local covariance function  $C(i, j)$ ,  $i, j \in \Lambda$  being a bounded, invertible matrix. In the case that the random coupling constants are *iid* random variables, hypothesis (H6) reduces to the usual assumption that a density exists as a bounded, compactly-supported function. **For simplicity, in the remainder of these notes, we will usually assume that  $h_0$  of (H6) is compactly-supported.** This automatically implies that (H5) is satisfied.

(H7) The density  $h_0$  decays sufficiently rapidly near  $-m$  and near  $M$  in the following sense:

$$0 < \mathbb{P}\{|\omega + m| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

$$0 < \mathbb{P}\{|\omega - M| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

for some  $\beta > 0$ .

Recent work of Klopp [92] on the existence of internal Lifshitz tails may allow us to remove hypothesis (H7).

We need to assume the existence of deterministic spectrum  $\Sigma$  for families of randomly perturbed operators as in (1.1)–(1.2) with  $H_0$  satisfying (H1)–(H7). If  $H_0$  is periodic with respect to the translation group  $\mathbb{Z}^d$ , and for Anderson-type perturbations described above, the random families of operators  $H_\omega$  are measurable,

self-adjoint, and  $\mathbb{Z}^d$ -ergodic. In this case, it is known (cf. [21, 107]) that the spectrum of the family is deterministic.

(H8) There exists constants  $B'_\pm$  satisfying  $B_- < B'_- < B'_+ < B_+$  such that

$$\Sigma \cap \{(B_-, B'_-) \cup (B'_+, B_+)\} \neq \emptyset, \text{ and } \Sigma \cap (B'_-, B'_+) = \emptyset.$$

In light of hypothesis (H8), we define the band edges of the almost sure spectrum  $\Sigma$  near the gap  $G$ , as follows:

$$\tilde{B}_- \equiv \sup\{E \leq B'_- \mid E \in \Sigma\},$$

and

$$\tilde{B}_+ \equiv \inf\{E \geq B'_+ \mid E \in \Sigma\},$$

Examples of operators satisfying these conditions may be found in [8, 33, 47, 48]. The main results are the following two theorems. We refer to the first theorem as *Band-Gap Localization*.

**THEOREM 1.2.** *Assume (H1) - (H8). There exist constants  $E_\pm$  satisfying  $B_- \leq E_- < \tilde{B}_-$  and  $\tilde{B}_+ < E_+ \leq B_+$  such that  $\Sigma \cap (E_-, E_+)$  is pure point with exponentially decaying eigenfunctions. The eigenvalues have finite multiplicity almost surely.*

For lattice models, the eigenvalues in the localization regime are known to be simple (multiplicity one), cf. [89, 116]. The finite multiplicity of the eigenvalues for the continuum models is an open question, although the finiteness of the multiplicity is proved in [25], as discussed in section 3.4, and another proof is given in [65]. One can also establish dynamical localization for the regions of  $\Sigma$  where Theorem 1.2 holds. We refer the reader to [58, 62].

We are able to prove a strong result on the IDS using hypotheses that are weaker than (H1)–(H8). In chapter 4, we list a weaker set of hypotheses (A1)–(A4) that are necessary to prove the following theorem. The full strength of the result on the IDS is given in [28].

**THEOREM 1.3.** *Assume (A1) - (A4) of chapter 4. The integrated density of states is locally uniformly Lipschitz continuous on  $\mathbb{R}$ .*

### 1.8. Description of the Contents: A Road Map to Localization.

In order to give the reader a global picture of the theory of localization for continuous, random systems, we give an outline of the proof of localization for additively and for multiplicatively perturbed operators. This will serve as a guide to the contents of the chapters of these notes.

We use the fixed-energy approach (see Figotin and Klein [45], von Dreifus and Klein [131], and Germinet and Klein [62], for the energy-interval approach) which uses spectral averaging and Kotani's trick. The arguments apply to both additive and multiplicative perturbations.

- a. The first goal is to prove almost sure exponential decay of the resolvent of the Hamiltonian  $H_\omega$  for Lebesgue almost energies in an interval  $I_\delta \equiv [\tilde{B}_- - \delta/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta/2]$ , for a fixed  $\delta > 0$ , near the edges of the spectral gap of  $\Sigma$ . This estimate has the form

$$(1.37) \quad \mathbb{P}\{\sup_{\epsilon > 0} \|\chi_x R_\omega(E + i\epsilon)\chi_y\| \leq e^{-m_0|x-y|}\} = 1,$$

for a constant  $m_0 > 0$ , depending on the interval  $I_\delta$ , and for all  $x, y \in \mathbb{R}^d$  such that  $\|x - y\|$  is sufficiently large. The functions  $\chi_x$  and  $\chi_y$  are compactly supported in a small neighborhood of  $x$  and  $y$ , respectively.

To prove this estimate, we need to establish two estimates on finite-volume Hamiltonians. These Hamiltonians,  $H_\Lambda$ , for bounded regular regions  $\Lambda \subset \mathbb{R}^d$ , are local perturbations of the background operator  $H_0$ . As such, they have only discrete spectra in the unperturbed spectral gap  $G$ . We must control, with a good probability depending on  $|\Lambda|$ , the location of these eigenvalues. This is done in chapters 4 and 5. First, in chapter 4, we prove a Wegner estimate which states that for any energy  $E \in G$ , and for any  $\eta > 0$  such that  $[E - \eta, E + \eta] \subset G$ ,

$$(1.38) \quad \mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E) \leq \eta\} \leq C_W \eta |\Lambda|.$$

Second, we show in chapter 5 that there exists  $\delta_\Lambda > 0$  such that

$$(1.39) \quad \sigma(H_\Lambda) \cap [\tilde{B}_- - \delta_\Lambda, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta_\Lambda] = \emptyset,$$

with a probability which tends to 1 as  $|\Lambda| \rightarrow \infty$ . Using a refined Combes-Thomas estimate, which is proven in [8], we prove that this result implies that the resolvent of the local Hamiltonian  $H_\Lambda$  decays exponentially at all energies in the interval  $I_\delta \equiv [\tilde{B}_- - \delta_\Lambda/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta_\Lambda/2]$ , with the same probability, provided  $|\Lambda|$  is large enough.

These two estimates for the finite-volume Hamiltonians  $H_\Lambda$  are the starting point of a multiscale analysis as developed in [25]. We present this analysis in chapters 5 and 9, with the modifications of this analysis necessary for multiplicatively-perturbed operators given in chapter 8. This leads directly to the fixed-energy exponential decay result stated in (1.19) for Lebesgue almost-every energy in  $I_\delta = [\tilde{B}_- - \delta/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta/2]$ , for an appropriate  $\delta = \delta_{\Lambda_0}$ , where  $\Lambda_0$  is sufficiently large.

- b. The second main step is to extend this fixed-energy result to prove localization in the interval  $I_\delta$ . To do this, we need a result which is referred to as *spectral averaging* (this result is also used in the proof of the Wegner estimate presented in chapter 4). The main consequence of this technical result states, roughly speaking, that the expectation of a spectral measure associated with the random family  $H_\omega$  is absolutely continuous with respect to Lebesgue measure. This is often referred to as Kotani's trick in the literature. This trick allows us to transfer information about the exponential decay of the resolvent at Lebesgue almost every energy in  $I_\delta$  to almost every energy with respect to the spectral measure for  $H_\omega$  for almost every  $\omega$ . We prove these results for additive perturbations in chapter 3 and present the modifications necessary for multiplicative perturbations. This result, together with the fixed energy resolvent estimates discussed above, are then combined with some probabilistic arguments of [25] to prove that  $\Sigma \cap \{[\tilde{B}_- - \delta/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta/2]\}$  is pure point. This established localization. A separate argument as in [25] proves that the eigenfunctions decay exponentially.
- c. One can also prove the absence of diffusion, in the sense that

$$(1.40) \quad \limsup_{t \rightarrow \infty} t^{-1} \langle x^2(t) \rangle_\psi = 0,$$

using an argument of Barbaroux [11]. In fact, recent work of DeBièvre and Germinet [58] (see also [62]) establish dynamical localization for the models discussed in these notes.

**Acknowledgments.** I thank Carlos Villegas-Blas for the invitation to lecture at the IV Escuela de Verano en Análisis y Física Matemática at the Instituto de Matemáticas, Cuernavaca, Mexico, in May 2005, and for all his support. The basic results presented in these notes were obtained in collaboration with Jean-Michel Combes, whom I warmly thank for enjoyable collaborations. I also thank the many colleagues who contributed to my understanding of the topics discussed in these notes, including Michael Aizenman, Jean-Marie Barbaroux, Jean Bellissard, Stephan DeBièvre, Francois Germinet, Werner Kirsch, Abel Klein, Frédéric Klopp, Hajo Leschke, Eric Mourre, Peter Müller, Shu Nakamura, Peter Stollmann, Gunter Stolz, and Adriaan Tip.

## 2. Basic Results in the Theory of Random Operators

**2.1. Introduction.** We give a review of certain aspects of the theory of random operators that will be used in this book. Extensive discussions of these topics can be found in the books of Carmona and Lacroix [21], Pastur and Figotin [107], Stollmann [123], and the review article of Kirsch [82]. In the first part, we review the questions of self-adjointness and the deterministic spectrum. We are mostly interested in techniques of computing the deterministic spectrum. The second part of this chapter is devoted to the integrated density of states (IDS). We introduce the notion of *finite-volume operators* that will figure significantly in the other chapters.

**2.2. A Summary of Basic Results on Self-Adjoint Operators.** The topics of self-adjointness, deterministic spectrum, and computation of the deterministic spectrum are discussed in this section.

The random families of Schrödinger operators  $\{H_\omega \mid \omega \in \Omega\}$  that we deal with are all symmetric operators on the dense domain  $C_0^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ , for almost every realization  $\omega \in \Omega$ . We will not treat the most general situations of self-adjointness for  $H_\omega$ , but will concentrate on certain models. An extensive discussion of self-adjointness for Schrödinger operators can be found in [111, 113].

We are interested in the stability of self-adjointness under perturbations. The unperturbed operator for Schrödinger operators is the Laplacian  $-\Delta$  on  $L^2(\mathbb{R}^d)$ , which is self-adjoint on the domain  $H^2(\mathbb{R}^d)$ . Let us recall four basic topics concerning self-adjointness for linear operators, and its stability under perturbations: 1) relative-boundedness results, and 2) results for positive potentials, 3) results applicable to Schrödinger operators with magnetic and electric fields, and 4) stability of self-adjointness under multiplicative perturbations. This latter part is applicable to the study of classical waves.

### Relatively-Bounded Perturbations

We will first review relatively-bounded operators. Let  $A$  be a closed operator on a Hilbert space  $\mathcal{H}$ . We consider perturbations of  $A$  by a class of linear operators  $B$  on the Hilbert space  $\mathcal{H}$ .

**DEFINITION 2.1.** An operator  $B$  is called  $A$ -bounded if  $D(B) \supset D(A)$ .

It is obvious, but important to note, that any bounded operator  $B \in \mathcal{L}(\mathcal{H})$  is  $A$ -bounded for any closed linear operator  $A$ . Relative-boundedness implies an important boundedness relation between the operators.

PROPOSITION 2.1. If  $\rho(A) \neq \phi$  and  $B$  is  $A$ -bounded, then there exist nonnegative constants  $a$  and  $b$  such that

$$(2.1) \quad \|Bu\| \leq a \|Au\| + b \|u\|,$$

for all  $u \in D(A)$ .

Let us recall that the smallest nonnegative constant  $a$  for which (2.1) holds for all  $u \in D(A)$  is called the *relative  $A$ -bound of  $B$* . The fundamental result on the stability of self-adjointness under relatively-bounded perturbations is the Kato-Rellich Theorem.

THEOREM 2.2. (*Kato-Rellich Theorem*). *Let  $A$  be self-adjoint and let  $B$  be a symmetric and  $A$ -bounded operator with relative  $A$ -bound less than one. Then  $A + B$  is self-adjoint on  $D(A)$ .*

The basic application of this result to Schrödinger operators is the following theorem.

THEOREM 2.3. *Let the real-valued function  $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ . Then the operator  $H_V \equiv -\Delta + V$ , defined on  $D(\Delta) = H^2(\mathbb{R}^d)$ , is self-adjoint if  $p \geq d/2$  for  $d \geq 5$ , if  $p > 2$ , for  $d = 4$ , and if  $p \geq 2$  for  $d \leq 3$ . Furthermore,  $H_V$  is essentially self-adjoint on  $C_0(\mathbb{R}^d)$  under these conditions.*

Note that the conditions on  $V$  in Theorem 2.3 allow two types of behavior at infinity. If  $V$  decays at infinity like  $\|x\|^\alpha$ , for  $\alpha > d/p$ , then  $H_V$  is self-adjoint on the domain  $H^2(\mathbb{R}^d)$ . At the other extreme, if  $V$  is everywhere bounded, then  $H_V$  is self-adjoint on the domain  $H^2(\mathbb{R}^d)$ . This will be the case for most random potentials of Anderson-type.

### Positive Potentials

When the potential  $V$  is nonnegative and not-necessarily bounded, self-adjointness can be established using the Kato inequality. We will first state this inequality for the Laplacian on  $\mathbb{R}^d$ .

THEOREM 2.4. *Let  $u \in L^1_{loc}(\mathbb{R}^n)$  and suppose that the distributional Laplacian  $\Delta u \in L^1_{loc}(\mathbb{R}^n)$ . Then*

$$(2.2) \quad \Delta|u| \geq \operatorname{Re} [(sgn u)\Delta u],$$

*in the distributional sense.*

We now consider a real potential  $V \in L^2_{loc}(\mathbb{R}^d)$ . We define  $H_V = -\Delta + V$  on  $D(H_V) \equiv D(\Delta) \cap D(V)$ , where  $D(\Delta) = H^2(\mathbb{R}^n)$  and

$$D(V) = \{f \in L^2(\mathbb{R}^n) \mid \int |Vf|^2 < \infty\}.$$

Note that  $C^\infty_0(\mathbb{R}^n) \subset D(H_V)$ , so  $H_V$  is densely defined.

THEOREM 2.5. *Let  $V \in L^2_{loc}(\mathbb{R}^d)$  and  $V \geq 0$ . Then, the Schrödinger operator  $H_V = -\Delta + V$  is essentially self-adjoint on  $C^\infty_0(\mathbb{R}^d)$ .*

### Electric and Magnetic Fields

We will consider as our background operators  $H_0$  Schrödinger operators with electric and magnetic potentials. Let  $A$  be a  $C^1$ -vector potential on  $\mathbb{R}^d$ . The pure magnetic Schrödinger operator is defined by  $H_A \equiv (-i\nabla - A)^2$ . We will treat the special two-dimensional case of a constant magnetic field in detail in chapter 7. A

vector potential in two dimensions  $(x_1, x_2)$  which describes a constant magnetic field of strength  $B > 0$  in the perpendicular  $x_3$ -direction is given by

$$(2.3) \quad A = \frac{B}{2}(x_2, -x_1).$$

The choice of a vector potential  $A$  for a given magnetic field is not unique since the transformation  $A \rightarrow A + \nabla\phi$  results in the same magnetic field. A consequence of this is the gauge invariance of the magnetic Schrödinger operator  $H_A$ . For a real-valued differentiable function  $\phi$ , we have the identity:

$$(2.4) \quad e^{-i\phi} H_{A+\nabla\phi} e^{i\phi} = H_A.$$

In the presence of an external potential  $V$ , the Hamiltonian  $H_{V,A}$  has the form

$$(2.5) \quad H_{V,A} = (p - A)^2 + V,$$

where  $p \equiv -i\nabla$ . The basic self-adjointness theorem is due to Leinfelder and Simader [99]. In order to state this theorem, we need to define the Kato class of potentials  $K_d(\mathbb{R}^d)$ .

DEFINITION 2.6. A real-valued potential  $V \in K_d(\mathbb{R}^d)$ , for  $d \geq 3$  if

$$(2.6) \quad \lim_{\alpha \rightarrow 0} \sup_x \int_{\|x-y\| \leq \alpha} \|x-y\|^{2-d} |V(y)| d^d y = 0,$$

and for  $d = 2$  if the same identity holds with the kernel  $\|x-y\|^{2-d}$  replaced by  $\log \|x-y\|$ .

THEOREM 2.7. Let  $V \in L^2_{loc}(\mathbb{R}^d)$ ,  $V_- \in K_d(\mathbb{R}^d)$ , and the vector potential  $A \in L^4_{loc}(\mathbb{R}^d)^d$ . The Hamiltonian  $H_{V,A} = (p - A)^2 + V$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ .

The Schrödinger operator describing a particle in a constant external electric field  $E$  and potential  $V$  is given by

$$(2.7) \quad H_{V,E} \equiv -\Delta + x \cdot E + V.$$

Because the potential  $x \cdot E$  is unbounded, self-adjointness of the Hamiltonian  $H_{0,E}$  requires a separate treatment. The fundamental theorem is due to Faris and Lavine. Rather than cite the theorem in complete generality (cf. [111]), we give a version which covers (2.7).

THEOREM 2.8. Let  $V$  be a real-valued potential in  $L^p(\mathbb{R}^d)$ , where  $p \geq 2$  if  $d \leq 3$ ,  $p > 2$  if  $d = 4$ , or  $p \geq d/2$  if  $d \geq 5$ . Then the Stark Hamiltonian  $H_{V,E} \equiv -\Delta + x \cdot E + V$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ .

**2.3. Basic Results on Families of Random Operators.** The basic techniques for proving essential self-adjointness of a random family on  $C_0^\infty(\mathbb{R}^d)$ , for almost every  $\omega$ , is to prove that the random potential  $V_\omega$  satisfies the conditions of one of the above theorems with probability one. The Borel-Cantelli Lemmas are quite useful for establishing this.

Since a random family  $\{H_\omega \mid \omega \in \Omega\}$  of self-adjoint operators consists, in general, of uncountably-many self-adjoint operators, the study of the spectrum of any individual operator  $H_\omega$  may not shed light on the properties of the family. An illuminating example is given in Chapter 1, section 1.4, for which the Hamiltonian is a pure random potential  $H_\omega = V_\omega$  on  $\ell(\mathbb{Z}^d)$ . Let us suppose that the *iid* (independent, identically distributed) random variables take values in  $[0, 1]$  with a

uniform distribution. For any fixed value of  $\omega$ , the operator  $V_\omega$  has pure point spectrum consisting of the closure of the set  $\{\omega_n \mid n \in \mathbb{Z}^d\}$ . Note that this set changes enormously as  $\omega$  varies. For example, if  $\omega_n = \lambda \in [0, 1]$ , for every  $n \in \mathbb{Z}^d$ , then the operator has a single, infinitely degenerate eigenvalue  $\lambda$  with each localized delta function  $\delta_{n,m}$ , for  $n$  fixed, an  $\ell^2$ -eigenvector. On the other hand, suppose that  $\phi : \mathbb{Z}^d \rightarrow [0, 1] \cap Q$  is a bijection. If we take a representation  $H_\omega$  for which  $\omega_n = \phi(n)$ , then the spectrum of  $H_\omega$  is the closed interval  $[0, 1]$ , the range of the potential  $V_\omega$ . In this case, the spectrum consists of only eigenvalues, and is an example of *dense pure point spectrum*. Clearly, the spectrum of an individual  $H_\omega$  varies greatly as the realization  $\omega$  changes. We would like to find a set, which we call the *deterministic spectrum* which is the spectrum of the operators  $H_\omega$ , for almost every  $\omega$ .

Pastur [106] seems to have been the first to realize that the family  $\{H_\omega\}$  has a notion of spectrum associated with it provided it is *ergodic* in a certain sense. We will not go into the general theory here. The probability spaces  $\Omega$  of interest to us are product spaces of the form  $\prod_{n \in \mathbb{Z}^d} J$ , where  $J \subset \mathbb{R}$  is the support of the probability distribution.

DEFINITION 2.9. Let  $(\Omega, \mathbb{P})$  be a complete probability space. We suppose that the family of maps  $\{\alpha_x : \Omega \rightarrow \Omega \mid x \in I\}$  is a subgroup of the automorphism of  $(\Omega, \mathbb{P})$ . We say that it acts ergodically on  $(\Omega, \mathbb{P})$  if 1) it is measure-preserving, that is,  $\mathbb{P}(\alpha_x^{-1}(A)) = \mathbb{P}(A)$ , for all measurable subsets  $A \subset \Omega$  and  $x \in I$ , and, 2) if for any measurable subset  $A \subset \Omega$ , the condition  $\alpha_x(A) = A$ , for all  $x \in I$ , implies either  $\mathbb{P}(A) = 1$  or  $\mathbb{P}(A) = 0$ .

In all cases of interest, the index set  $I$  will be the additive group  $\mathbb{Z}^d$  or  $\mathbb{R}^d$ . We note that the family  $\{\alpha_x : \Omega \rightarrow \Omega \mid x \in I\}$  need only form a semigroup. Given that our measure spaces are products, the maps  $\alpha_x$  are translations. In fact, by using the canonical representation of a stochastic process, we can always represent these maps  $\alpha_x$  by translations. These translations induce a natural action on  $L^2(\mathbb{R}^d)$  or  $\ell^2(\mathbb{Z}^d)$  by unitary operators,

$$(2.8) \quad U_x f(y) = f(y - x).$$

We are interested in the quantity  $U_x H_\omega U_x^{-1}$ . Before we give the main definition, we clarify the notion of measurability for self-adjoint operators.

DEFINITION 2.10. A family of self-adjoint operators  $\{H_\omega \mid \omega \in \Omega\}$  is called measurable if the functions  $\langle \phi, (H_\omega - z)^{-1} \psi \rangle$ , for any  $\phi, \psi \in \mathcal{H}$ , are measurable for some  $z$ , with  $\text{Im}(z) \neq 0$ .

DEFINITION 2.11. A measurable family of self-adjoint operators  $\{H_\omega\}$  is ergodic with respect to an ergodic action of the translations  $\{\alpha_x : \Omega \rightarrow \Omega \mid x \in I\}$  if it satisfies the covariance identity

$$(2.9) \quad H_{\alpha_x^{-1}(\omega)} = U_x H_\omega U_x^{-1}$$

for all  $x \in I$ .

To simplify terminology, we will call a measurable, ergodic family of self-adjoint operators simply an ergodic family of self-adjoint operators. The fundamental result is the following theorem.

**THEOREM 2.12.** (*Pastur's Theorem [106].*) *Let  $\{H_\omega \mid \omega \in \Omega\}$  be an ergodic family of self-adjoint operators. Then, there exists a set  $\Sigma \subset \mathbb{R}$  such that  $\Sigma = \sigma(H_\omega)$   $\mathbb{P}$ -almost surely.*

Another way to phrase this result is the following. There exists a set  $\Omega_1 \subset \Omega$ , with  $\mathbb{P}(\Omega_1) = 1$ , so that  $\sigma(H_\omega) = \Sigma$ , for all  $\omega \in \Omega_1$ . We will call the set  $\Sigma$  the *deterministic spectrum* of the family of self-adjoint operators.

The spectrum of any self-adjoint operator has a natural decomposition into three components consisting of the pure point, absolutely continuous, and singular continuous spectra. It is natural to ask if the deterministic spectrum has a similar, deterministic decomposition. This result was proved by Kunz and Soulliard [97], and by Kirsch and Martinelli [85].

**THEOREM 2.13.** *Let  $\{H_\omega \mid \omega \in \Omega\}$  be an ergodic family of self-adjoint operators. Then, there exist sets  $\Sigma_{pp}$ ,  $\Sigma_{ac}$ , and  $\Sigma_{sc}$ , and a set  $\Omega_1 \subset \Omega$ , with  $\mathbb{P}(\Omega_1) = 1$ , such that*

$$\begin{aligned}\sigma_{pp}(H_\omega) &= \Sigma_{pp} \quad \forall \omega \in \Omega_1, \\ \sigma_{ac}(H_\omega) &= \Sigma_{ac} \quad \forall \omega \in \Omega_1, \\ \sigma_{sc}(H_\omega) &= \Sigma_{sc} \quad \forall \omega \in \Omega_1.\end{aligned}$$

**2.3.1. Techniques for Computing the Deterministic Spectrum.** We mention two methods commonly used for computing the deterministic spectrum of ergodic families of self-adjoint operators. We will use these in the later chapters.

**Sets of Positive Probability Measure.** The most immediate consequence of Theorem 2.13 is the following approach to computing the deterministic spectrum  $\Sigma$  for an ergodic family of self-adjoint operators  $\{H_\omega \mid \omega \in \Omega\}$ . Suppose that a set  $I \subset \sigma(H_\omega)$ , for  $\omega \in \Omega_0 \subset \Omega$ , and that  $\mathbb{P}(\Omega_0) > 0$ . Then, it follows that  $I \subset \Sigma$ . It is often easy to construct such a set of configurations  $\Omega_0$ .

**Determining Configurations and Periodic Approximations.** A second approach to the computation of  $\Sigma$  consists of approximating the random potential  $V_\omega$  by periodic operators. This result follows from a more general one on the deterministic spectrum. A *base* for the probability space  $(\Omega, \mathbb{P})$  is a measurable subset  $\Omega_0 \subset \Omega$  such that 1)  $\mathbb{P}(\Omega_0) = 1$ , 2) the operator  $H_\omega$  is self-adjoint for any  $\omega \in \Omega_0$ , and 3) the map  $\omega \in \Omega_0 \rightarrow H_\omega$  is strong resolvent continuous. This means that for any sequence  $\omega_n \rightarrow \omega$  in  $\Omega_0$ , we have  $\|\{(H_{\omega_n} - z)^{-1} - (H_\omega - z)^{-1}\}\phi\| = 0$ , for any  $\phi \in \mathcal{H}$ .

**THEOREM 2.14.** *Let  $\{H_\omega \mid \omega \in \Omega\}$  be an ergodic family of self-adjoint operators with base  $\Omega_0$ . If  $\Omega_1 \subset \Omega_0$  is dense in the support of the probability measure  $\mathbb{P}$ , then*

$$(2.10) \quad \Sigma = \overline{\bigcup_{\omega \in \Omega_1} \sigma(H_\omega)}.$$

The interesting aspect of this theorem is that it might very well be that  $\mathbb{P}(\Omega_1) = 0$ . For the Anderson model with a continuous density  $g$ , it can be shown that  $\Omega_1 \equiv \{\omega \mid V_\omega \text{ is periodic}\}$  is dense in the support of the probability measure. Hence, the deterministic spectrum can be computed from the closure of the union of the spectrum of all periodic realizations of the potential.

**2.4. The Integrated Density of States.** For a quantum mechanical system described by a Hamiltonian  $H$ , the *density of states* (DOS) is a measure of the average number of states of the system per unit volume. It is defined through a

limiting process as the volume of the system becomes infinite. This limiting process is called the *thermodynamic limit*. To describe it, we denote by  $\Lambda_\ell(x)$  the cube of side length  $\ell$  centered at the point  $x \in \mathbb{R}^d$ ,

$$(2.11) \quad \Lambda_\ell(x) \equiv \{y \in \mathbb{R}^d \mid |x_i - y_i| < \ell/2, i = 1, \dots, d\}.$$

We will denote an arbitrary, bounded, open, simply connected subset of  $\mathbb{R}^d$  by  $\Lambda$ .

The DOS is well-defined for a system in a finite volume region  $\Lambda$ . Let  $H_\Lambda^X$  be defined as the self-adjoint extension of  $H \mid C_0^\infty(\Lambda)$ , with boundary conditions  $X$  imposed on  $\partial\Lambda$ , the boundary of  $\Lambda$ . We will work with Dirichlet boundary conditions ( $X = D$ ), and Neumann boundary conditions ( $X = N$ ). We assume that the spectrum is semi-bounded below, although more general situations can be treated. For regular regions  $\Lambda$ , the operators  $H_\Lambda^X$  have only discrete spectrum, which we denote by  $\{E_j(\Lambda)\}$ . The set of eigenvalues is ordered so that  $-\infty < E_1(\Lambda) < E_2(\Lambda) \leq E_3(\Lambda), \dots$ , where we repeat an eigenvalue according to its multiplicity. We define the *finite-volume integrated density of states* for the system in the region  $\Lambda$  by

$$(2.12) \quad N_\Lambda^X(E) \equiv \frac{1}{|\Lambda|} \#\{j \mid E_j(\Lambda) \leq E, \text{ which includes multiplicities}\}.$$

It is useful, for later purposes, to write these expressions using the spectral projectors of  $H_\Lambda^X$ . For any  $I \subset \mathbb{R}$ , let  $E_\Lambda^X(I)$  denote the spectral projector for  $H_\Lambda^X$  and the interval  $I$ . We denote by  $\text{Tr}_\Lambda$  the trace on the Hilbert space  $L^2(\Lambda)$ . Then, we have

$$(2.13) \quad N_\Lambda^X(I) = |\Lambda|^{-1} \text{Tr}_\Lambda(E_\Lambda^X(I)).$$

Because the spectrum of  $H_\Lambda^X$  is discrete, the corresponding density of states defines a pure point measure, called the *finite-volume density of states measure*. It is given by

$$(2.14) \quad dN_\Lambda^X(E) \equiv \frac{1}{|\Lambda|} \sum_j \delta(E - E_j(\Lambda)) dE.$$

The finite-volume integrated density of states  $N_\Lambda^X(E)$  is the distribution function corresponding to this measure,

$$(2.15) \quad N_\Lambda^X(E) = \int_{-\infty}^E dN_\Lambda^X(\lambda).$$

We now pass to the thermodynamic limit of  $N_\Lambda^X(E)$ . Formally, the integrated density of states (IDS)  $N^X(E)$ , is defined by

$$(2.16) \quad N^X(E) = \lim_{\Lambda \rightarrow \mathbb{R}^d} N_\Lambda^X(E),$$

whenever the limit exists. If the IDS exists, as defined in (2.16), then the DOS measure is defined in the usual way starting with the measure of an interval  $I = [I_-, I_+]$ , for which  $N^X(I) \equiv N^X(I_+) - N^X(I_-)$ , provided these are points of continuity for  $N^X(E)$ . Alternatively, the DOS measure is the vague limit of the finite-volume DOS measures defined in (2.14). That is, for any  $f \in C_0^\infty(\mathbb{R}^d)$ , we have,

$$(2.17) \quad \int f(E) dN^X(E) = \lim_{\Lambda \rightarrow \mathbb{R}^d} \int f(E) dN_\Lambda^X(E).$$

It is easy to compute the high energy asymptotics of IDS for the Laplacian on  $\mathbb{R}^d$ . For either boundary condition, one finds that

$$(2.18) \quad N_{\Lambda}^X(E) \sim E^{d/2}.$$

For an elliptic operator of second order, like a Schrödinger operator with a locally bounded potential, the Weyl estimate of the asymptotic distribution of eigenvalues guarantees that

$$(2.19) \quad N_{\Lambda}^X(E) \leq \frac{\omega_d E^{d/2}}{(2\pi)^{d/2}},$$

where  $\omega_d$  is the volume of the unit ball in  $d$ -dimensions so that

$$(2.20) \quad \limsup_{\Lambda \rightarrow \mathbb{R}^d} N_{\Lambda}^X(E)$$

always exists.

2.4.1. *Sketch of the Existence of the IDS for Ergodic Operators.* Let us now consider an ergodic family of self-adjoint operators  $\{H_{\omega} \mid \omega \in \Omega\}$ . With each operator we associate a finite-volume IDS  $N_{\Lambda, \omega}^X(E)$ . This is a random variable for each fixed  $E$  and volume  $\Lambda$ . The ergodicity is essential as it introduces self-averaging which implies the existence of a *deterministic* IDS for the family. We will show that the IDS exists using the superadditive ergodic theorem.

Let us consider the Dirichlet case  $X = D$ . First, we note that for fixed  $E$ , the family  $N_{\Lambda, \omega}^D(E)$  is ergodic in the sense that

$$(2.21) \quad N_{\Lambda, T_x \omega}^D(E) = N_{T_x \Lambda, \omega}^D(E),$$

where  $T_x$  is the translation by  $x \in \mathbb{Z}^d$  or  $x \in \mathbb{R}^d$ . Second, suppose  $\Lambda_1$  and  $\Lambda_2$  are two regions with nonintersecting interiors and let  $\Lambda \equiv \Lambda_1 \cup \Lambda_2$ . It follows from the Dirichlet boundary conditions that

$$(2.22) \quad N_{\Lambda, \omega}^D(E) \geq N_{\Lambda_1, \omega}^D(E) + N_{\Lambda_2, \omega}^D(E).$$

This shows that the family of random variables  $N_{\Lambda, \omega}^D(E)$ , indexed by  $\Lambda$ , is a *superadditive ergodic process*. Let  $\Lambda_n$  denote the cubes  $\Lambda_n(0)$ . The Superadditive Ergodic Theorem states that if

$$(2.23) \quad \sup_n \mathbb{E}(N_{\Lambda_n, \omega}^D(E)) < \infty,$$

then

$$(2.24) \quad \lim_{n \rightarrow \infty} N_{\Lambda_n, \omega}^D(E) = \lim_{n \rightarrow \infty} \mathbb{E}(N_{\Lambda_n, \omega}^D(E)) \equiv N^D(E)$$

exist  $\mathbb{P}$ -almost surely. For locally bounded potentials, the hypothesis (2.23) is easily verified. For more general potentials, the verification is given, for example, in [82]. As a consequence, the limit  $N^D(E)$  is independent of the configuration.

For each energy  $E$ , there is a set  $\Omega_E$  of  $\mathbb{P}$ -measure one for which the limit in (2.24) exists. The function  $N^D(E)$  is a monotone increasing function. Let  $S \subset \mathbb{R}$  be a countable set of points of continuity of  $N^D(E)$ . The set  $\Omega_0 \equiv \cap_{E \in S} \Omega_E$  is a set of  $\mathbb{P}$ -measure one. The monotonicity of  $N^D(E)$  implies that for any point of continuity of  $N^D(E)$ , the limit on the left side of (2.24) exists for  $\omega \in \Omega_0$ . Hence, with probability one, the limit of the left side of (2.24) exists except for at most a countable number of points of discontinuity.

Since the limit function  $N^D(E)$  is monotone increasing and continuous from the left, it is the distribution function of a measure  $dN^D$  which is the DOS measure.

We are often interested in proving more about the smoothness of the IDS which implies the absolute continuity of the measure  $dN^D$  with respect to Lebesgue measure. For example, if  $N^D(E)$  is locally Lipschitz continuous, i. e. if the function  $N^D(E)$  satisfies a condition such as

$$(2.25) \quad |N^D(E) - N^D(E')| \leq C_E |E - E'|,$$

for some constant  $C_E > 0$ , then the measure  $dN^D$  is absolutely continuous with respect to Lebesgue measure. The Radon-Nikodym derivative of this measure is the actual density of states.

*2.4.2. Pastur-Shubin Formula for the IDS.* There is another expression for the DOS measure which is often convenient for calculations. Let  $\chi_\Lambda$  be the characteristic function on the region  $\Lambda$ . We consider the mapping

$$(2.26) \quad f \in C_0(\mathbb{R}) \rightarrow \text{Tr}(\chi_\Lambda f(H)\chi_\Lambda),$$

where  $C_0(\mathbb{R})$  is the set of continuous functions of compact support on  $\mathbb{R}$ , and  $\text{Tr}$  denotes the trace on  $L^2(\mathbb{R}^d)$ . For a bounded region  $\Lambda$ , the operator  $\chi_\Lambda f(H)\chi_\Lambda$  is trace class provided the potential  $V$  is reasonable. To see this, let  $g \in C_0^\infty(\mathbb{R})$  be another function such that  $g \cdot f = f$ . We can then write

$$(2.27) \quad \begin{aligned} \chi_\Lambda f(H)\chi_\Lambda &= \chi_\Lambda g(H)f(H)\chi_\Lambda \\ &= [\chi_\Lambda(1+H)^{-k}][(1+H)^k g(H)f(H)\chi_\Lambda], \end{aligned}$$

for some  $k > 0$ . Standard estimates show that for  $k > d$ , the operator  $\chi_\Lambda(1+H)^{-k}$  is in the trace class. Consequently, we have the bound,

$$(2.28) \quad |\text{Tr}(\chi_\Lambda f(H)\chi_\Lambda)| \leq \|\chi_\Lambda(1+H)^{-k}\|_1 \|(1+H)^k g(H)\| \|f\|_\infty.$$

It follows from (2.28) that the mapping defined in (2.26) is continuous. The Riesz-Markov Theorem [110] states that there exists a positive Baire measure  $\mu_\Lambda$  such that

$$(2.29) \quad \text{Tr}(\chi_\Lambda f(H)\chi_\Lambda) = \int f(E) d\mu_\Lambda(E).$$

This measure is the analog of the finite-volume density of states measure (2.14). However, it is defined with respect to the Hamiltonian  $H$ , rather than the finite-volume operator  $H_\Lambda$ . We can show, however, that the difference, being localized near the boundary of  $\Lambda$  vanishes in the infinite-volume limit.

LEMMA 2.15. *Suppose that for  $f \in C_0^\infty(\mathbb{R})$ , we have*

$$(2.30) \quad \lim_{\Lambda \rightarrow \mathbb{R}^d} |\Lambda|^{-1} \text{Tr}(\chi_\Lambda f(H)\chi_\Lambda) \equiv N(f)$$

*exists. Then, we have*

$$(2.31) \quad N(f) = \lim_{\Lambda \rightarrow \mathbb{R}^d} |\Lambda|^{-1} \text{Tr}(f(H_\Lambda)).$$

**Proof.** It suffices to show that

$$(2.32) \quad \lim_{\Lambda \rightarrow \mathbb{R}^d} |\Lambda|^{-1} |\text{Tr}\{\chi_\Lambda(f(H) - f(H_\Lambda))\chi_\Lambda\}| = 0.$$

We use the Helffer-Sjöstrand formula (cf. [36]) to express  $f(H)$  as an integral over the complex plane. Let  $\tilde{f}$  be an almost analytic extension of  $f$  vanishing to order  $N$

as  $y \rightarrow 0$ . An almost analytic extension of  $f$ , that vanishing to order  $N$  as  $y \rightarrow 0$ , can be constructed from the Taylor coefficients of  $f$  by

$$(2.33) \quad \tilde{f}(x + iy) = \sum_{k=0}^N (iy)^k \chi(y) \frac{f^{(k)}(x)}{k!},$$

where  $\chi$  is a smooth, nonnegative function of compact support equal to one in a neighborhood of zero. The formula for  $f(H)$  is

$$(2.34) \quad f(H) = \frac{1}{i\pi} \int_{\mathcal{G}} dx dy \partial_{\bar{z}} \tilde{f}(z) (H - z)^{-1}.$$

Let us assume that  $\chi_{\Lambda}(H+1)^{-k}$  is in the trace class for  $k$  suitably large. Replacing  $f$  by  $g(z) = (z+1)^k f(z)$ , we can write

$$\chi_{\Lambda}(f(H) - f(H_{\Lambda}))\chi_{\Lambda} =$$

$$(2.35) \quad \frac{1}{i\pi} \int_{\mathcal{G}} dx dy \partial_{\bar{z}} \tilde{g}(z) \{ \chi_{\Lambda} [(H+1)^{-k} (H-z)^{-1} - (H_{\Lambda}+1)^{-k} (H_{\Lambda}-z)^{-1}] \chi_{\Lambda} \}.$$

We write the operator appearing in this expression as

$$(2.36) \quad \begin{aligned} & \chi_{\Lambda} (H+1)^{-k} (H-z)^{-1} \{ \chi_{\Lambda} (H_{\Lambda}+1)^k (H_{\Lambda}-z) \\ & - (H+1)^k (H-z)^{-1} \chi_{\Lambda} \} (H_{\Lambda}+1)^{-k} (H_{\Lambda}-z)^{-1} \chi_{\Lambda}. \end{aligned}$$

Since  $H_{\Lambda}\chi_{\Lambda} = H\chi_{\Lambda}$ , we see that the quantity in the parentheses involves commutators of  $H$  with  $\chi_{\Lambda}$ . Let us now choose  $\chi_{\Lambda}$  to be of the form  $\chi(x/R)$  where  $\chi$  is smooth, has support in the ball of radius 2, and is equal to one on the unit ball. The commutator of  $\chi_R$  with  $H$  is  $O(R^{-1})$ . Hence, since the trace norms are  $O(R^d)$ , the extra decay from the commutator proves that the limit vanishes as  $R \rightarrow \infty$ .  $\square$

**2.4.3. Finite-Volume Operators.** In the discussion of the DOS above, the finite-volume operators  $H_{\Lambda}^X$  are defined with boundary conditions  $X$  on  $\partial\Lambda$ . Finite-volume operators also play an important role in the multiscale analysis discussed in chapters 5 and 9. If we are concerned only with energies near the bottom of the the deterministic spectrum  $\Sigma$ , then the Dirichlet operators  $H_{\Lambda}^X$  are suitable for the DOS and the multi-scale analysis. However, for energies near the band-edges of the deterministic spectrum  $\Sigma$ , another choice of the finite-volume operators is more appropriate. For example, we will study Schrödinger operators of the form  $H_{\omega} = H_0 + V_{\omega}$ , where  $H_0$  is a periodic Schrödinger operator with band spectrum. Under certain conditions, the deterministic spectrum of the random family will also have open spectral gaps near the gaps  $G$  in the spectrum of  $H_0$ . In this case, we want to exploit the fact that the energies we are interested in lie in the unperturbed gap  $G$  in the spectrum of  $H_0$ .

Let  $V_{\Lambda} \equiv V_{\omega} |_{\Lambda}$  be the local potential associated with  $\Lambda$  and denote by  $H_{\Lambda}$  the local Hamiltonian  $H_0 + V_{\Lambda}$ . Note that since the local potential  $V_{\Lambda}$  is a relatively-compact perturbation of  $H_0$ , the essential spectrum is unchanged. The effect of the local potential is to create at most finitely-many eigenvalues in the unperturbed gap  $G$ . We will also make use of the exponential decay of the resolvent of  $H_0$  at energies in  $G$ . The same type of argument as in the proof of Lemma 2.15 shows that the DOS measure, and therefore the IDS, calculated from the local Hamiltonians  $H_{\Lambda} = H_0 + V_{\Lambda}$  agrees with the usual definition.

### 3. One-Parameter Families of Operators: Spectral Averaging and Perturbation of Singular Spectra

**3.1. Introduction.** We begin with a discussion of a part of the theory which is of interest in its own right: one-parameter families of perturbations of a self-adjoint operator. Let  $A$  be a self-adjoint operator and consider a self-adjoint perturbation  $V$  that is relatively  $A$ -bounded with relative bound less than one. We will study one-parameter family of perturbations of  $A$  by  $V$  of the form  $A(\lambda) = A + \lambda V$ . The operators  $A(\lambda)$  are self-adjoint on the domain  $D(A)$  provided  $|\lambda| < 1$ . As we will see, even the case when  $V$  is a rank-one operator is of interest. The introduction of the parameter  $\lambda$  allows us to work with averaged quantities. For example, if  $\mu_\lambda$  is the spectral measure for  $A(\lambda)$ , we can study the average spectral measure  $\int \mu_\lambda d\lambda$ . Notice that this is a form of an expectation if we consider  $\lambda$  as a random variable with a uniform, constant density. In certain cases, we will derive properties of the averaged measure. This will allow us to conclude that  $\mu_\lambda$  has similar properties for almost every  $\lambda$ .

The random families of operators that we have considered are multiparameter perturbations of the background operator  $H_0$ . We reduce the problem to a one-parameter family in the following manner using the fact that the process is translation invariant. Let us consider a random family of Schrödinger operators  $H_\omega = H_0 + V_\omega$ , where  $V_\omega$  is an Anderson-type potential. We consider the set of configurations  $\Omega_0$  for which all the coupling constants  $\omega_i$ ,  $i \neq 0$ , are kept constant, but the coupling constant  $\omega_0$  varies within the support of the density  $g$ . We can write the Hamiltonian for these configurations  $\omega$  as

$$(3.1) \quad H_\omega = H_0 + V_{\omega^\perp} + \omega_0 u,$$

where  $V_{\omega^\perp}(x) \equiv \sum_{i \neq 0} \omega_i u(x - i)$  denotes the potential with the term at the zero site omitted. We now vary  $\omega \rightarrow \tilde{\omega}$  within the class of configurations  $\Omega_0$  for which  $\omega_i = \tilde{\omega}_i$ , for  $i \neq 0$ . We then have

$$(3.2) \quad \begin{aligned} H_{\tilde{\omega}} &= H_0 + V_\omega + (\tilde{\omega}_0 - \omega_0)u \\ &= H_\omega + \lambda u, \end{aligned}$$

where we defined a new parameter

$$(3.3) \quad \lambda \equiv \tilde{\omega}_0 - \omega_0.$$

Note that as operators, the Hamiltonian  $H_{\tilde{\omega}}$  and  $H_\omega$  have exactly the same form and differ only in the coupling constant for the zero site. Hence, we can write equation (3.2) as

$$(3.4) \quad H_\lambda = H + \lambda u,$$

and study the effect of varying only one coupling constant. Of course, in the ergodic case, the choice of one site is equivalent to any other, so we can choose to study the effects of the variation of the coupling constant at the zero site without any loss of generality.

It was Simon and Wolff [119] who realized the importance of considering the variation of the coupling constant at one site in the proof of localization for random families of multidimensional Schrödinger operators on the lattice  $\mathbb{Z}^d$ ,  $d > 1$ . This allowed them to prove localization given the exponential decay estimates of Fröhlich and Spencer [56] on the resolvent. For the one dimensional case, Kotani [93] had also used the variation of a single boundary condition to obtain information about

the spectral type of the deterministic spectrum. The classic studies of the effects on the spectrum of one-parameter families of (rank one) perturbations are due to Aronszajn [7] and Donoghue [39]. A nice review of the rank one theory and its application to random Schrödinger operators on the lattice is given in Simon's Vancouver lectures [114, 115].

In this chapter, we discuss extension of the results necessary for the proof of localization for multidimensional random operators on  $\mathbb{R}^d$  for which the rank one theory is insufficient. We begin with spectral averaging results. Spectral averaging techniques play an important role in controlling the singular continuous spectrum of families of self-adjoint operators. Such methods have been used in the theory of random Schrödinger operators (cf. [21]), and in some approaches to quantum stability of time-dependent models [77]. We first present the simple spectral averaging method of Kotani and Simon [94], that they applied to the IDS, and used by Combes and the author [25] in the proof of localization. We will use this technique when we discuss the Wegner estimate and the Lipschitz continuity of the IDS in section 4. The method of Kotani and Simon [94] requires some analyticity of the potential in the random variables. We next present a technique applicable to more general families of potentials depending only differentiably on some parameter. The main tool is the method of differential inequalities. Finally, we present a refined spectral averaging result of [28] that is used to prove a Wegner estimate for general probability measures.

We then present a summary of the classical Aronszajn-Donoghue theory on rank-one perturbations. This is a beautiful theory and provides motivation for the second main result of this chapter on the perturbation of the singular spectrum. Our result presents an extension of this rank-one theory to a family of relatively compact perturbations arising in the theory. This allows us to eliminate the singular continuous spectrum for families of random operators.

A nice presentation of the classical Aronszajn-Donoghue theory and its applications to localization for random Schrödinger operators on the lattice is given by Simon in his Vancouver lectures, recently reprinted at the end of the reprinting of his trace ideals text [114].

**3.2. Spectral Averaging.** The basic idea behind spectral averaging is the following. Let  $\mu_\lambda$ ,  $\lambda \in \Gamma$ , be a one-parameter family of real-valued measures on  $\mathbb{R}$  with a common  $\sigma$ -algebra of measurable sets  $\mathcal{B}$ . Suppose that  $g$  is a bounded, real-valued function on  $\Gamma$ . We can construct an averaged measure from the family in the following manner. For any subset  $A \in \mathcal{B}$ , define

$$(3.5) \quad \nu(A) \equiv \int_{\Gamma} \mu_\lambda(A) g(\lambda) d\lambda.$$

The measure  $\nu$  is obtained by averaging the measures  $\mu_\lambda$  with respect to the weight function  $g$ . The main result of this section is that if the measures  $\mu_\lambda$  are spectral measures of certain families of one-parameter self-adjoint operators, then the averaged measure  $\nu$  is absolutely continuous with respect to Lebesgue measure. This result is often referred to as *Kotani's trick* in the literature about random Schrödinger operators.

**3.2.1. The Kotani-Simon Spectral Averaging Method.** The simplest spectral averaging result in the context of Schrödinger operators goes back to Kotani [93] who studied one-dimensional models. The result we present here appears in [94] and

[25]. We consider the one-parameter family of operators  $H_\lambda = H_0 + \lambda V$ , on a Hilbert space  $\mathcal{H}$ , where  $H_0$  is self-adjoint and  $V$  is a bounded operator satisfying

$$(3.6) \quad 0 < c_0 B^2 \leq V,$$

for some bounded self-adjoint operator  $B$  and a finite, positive constant  $c_0 > 0$ .

**THEOREM 3.1.** *Let  $H_\lambda$  be as above and suppose that  $V$  satisfies (3.6). For  $\Im z < 0$  and  $\Im \lambda \geq 0$ , we define the operator*

$$(3.7) \quad K(\lambda, z) \equiv B(H_\lambda - z)^{-1}B,$$

and for  $t > 0$  and  $\epsilon \geq 0$ , we define

$$(3.8) \quad F_t(\epsilon, z) \equiv \int_{\mathbb{R}} \frac{1}{1 + t\lambda^2} K(\lambda + i\epsilon, z) d\lambda.$$

Then, for  $c_0 > 0$  as in (3.6), we have

$$(3.9) \quad \sup_{\epsilon > 0} \|F_t(\epsilon, z)\| \leq \pi c_0^{-1}.$$

**Proof.** Note that for  $\Im z < 0$  and  $\Im \lambda = 0$ , the operator  $K(\lambda, z)$  is bounded. We now establish an *a priori* bound on  $K(\lambda, z)$ . For  $\Im z < 0$  and  $\Im \lambda \geq 0$ , we easily verify that

$$(3.10) \quad -\Im K(\lambda, z) \geq c_0(\Im \lambda)K(\lambda, z)^*K(\lambda, z),$$

that implies

$$(3.11) \quad \|K(\lambda, z)\| \leq \min((c_0 \Im \lambda)^{-1}, |\Im z|^{-1} \|B\|^2).$$

We next note that the operator  $K(\lambda, z)$  is analytic in  $\lambda$  for  $\Im \lambda > 0$ , and for fixed  $z$ , with  $\Im z < 0$ . Furthermore, this operator is bounded in the upper-half complex  $\lambda$ -plane as in (3.11). Consequently, we can apply the Residue Theorem to evaluate the integral in the definition (3.8) and obtain

$$(3.12) \quad F_t(\epsilon, z) = \pi t^{-1/2} K(it^{-1/2} + \epsilon, z).$$

The result (3.9) now follows from (3.12) and (3.11).  $\square$

An immediate consequence of the bound (3.9) is the following estimate that we will use in chapter 4. Suppose that  $g \in L_0^\infty(\mathbb{R})$  is a nonnegative function. Then for any  $\phi \in \mathcal{H}$ , we have

$$(3.13) \quad \left| \int_{\mathbb{R}} g(\lambda) \langle \phi, (H_\lambda - z)^{-1} \phi \rangle \right| \leq \pi c_0^{-1} \|g\|_\infty \|\phi\|^2.$$

We obtain this from (3.9) by replacing the  $g$  by  $(1 + t\lambda^2)^{-1}$  on the left in (3.13) at the cost of a factor  $\sup_\lambda (g(\lambda)(1 + t\lambda^2))$ , for any  $t \geq 0$ . Since the bound is independent of  $t$ , we can take  $t = 0$ .

**3.2.2. A Differential Inequality and Spectral Averaging.** We now present a more local version of spectral averaging that does not require that the operator  $H_\lambda$  depend linearly on  $\lambda$  (or, more generally, analytically on  $\lambda$ ). This is the case for the breather model (1.5.1). As in the proof in section 3.2.1, the proof of the spectral averaging theorem requires an *a priori* bound on the resolvent of the operators  $H_\lambda$ . We obtain this using a differential inequality method reminiscent of the method of Mourre [105] used to prove the limiting absorption principle for Schrödinger operators.

**THEOREM 3.2.** *Let  $H_\lambda$ ,  $\lambda \in \Gamma \equiv (\lambda_0, \lambda_1)$ , be a  $C^2$ -family of self-adjoint operators on a separable Hilbert space  $\mathcal{H}$  such that  $D(H_\lambda) = D_0 \subset \mathcal{H}$ ,  $\forall \lambda \in \Gamma$ , and such that  $R_\lambda(z) \equiv (H_\lambda - z)^{-1}$  is twice strongly differentiable in  $\lambda$  for all  $z$ ,  $\text{Im } z \neq 0$ . Assume that there exist finite positive constants  $C_j$ ,  $j = 0, 1$ , and bounded operator  $B$ , such that on  $D_0$ ,*

$$(D1) \quad \dot{H}_\lambda \equiv \frac{dH_\lambda}{d\lambda} \geq C_0 B B^* > 0; \quad (D2) \quad |\ddot{H}_\lambda| \equiv \left| \frac{d^2 H_\lambda}{d\lambda^2} \right| \leq C_1 \dot{H}_\lambda.$$

*Then  $\forall E \in \mathbb{R}$  and  $\forall$  real, positive  $g \in C_0^2(\Gamma)$ , there exists a finite positive constant  $C$ , depending only on  $\|g^{(j)}\|_1$ ,  $j = 0, 1, 2$ , and the constants  $C_j$ ,  $j = 0, 1$  of (D1) and (D2), such that  $\forall \phi \in \mathcal{H}$ ,*

$$(3.14) \quad \sup_{\delta > 0} \left| \int_\Gamma g(\lambda) \langle \phi, B(H_\lambda - E - i\delta)^{-1} B\phi \rangle \right| \leq C \|\phi\|^2.$$

**Remarks:**

- (1) Although the proof of this abstract theorem requires some derivatives on the density  $g$ , there is a version in [25] that requires no derivatives on  $g$ .
- (2) We will prove this theorem only for the linear case when  $H_\lambda = H_0 + \lambda u$ . For this case,  $\dot{H}_\lambda = u \geq 0$ , and we can take  $C_0 = 1$  and  $B = u^{1/2}$  in (D1). Note that  $\ddot{H}_\lambda = 0$  so (D2) is automatically satisfied in this case. The proof is simpler if this holds because there is no error term to estimate.

**Proof of Theorem 3.2 in the Linear Case.**

The basic idea of the proof is to use differential inequality techniques (cf. [35, 105]).

1. *a priori estimates.* For  $\epsilon > 0$  and  $0 < \delta < 1$ , we define a modified resolvent by

$$(3.15) \quad R(\lambda, \epsilon, \delta) \equiv (H_\lambda - E + i\delta + i\epsilon \dot{H}_\lambda)^{-1}.$$

Since the operator  $\dot{H}_\lambda$  is positive in the sense of (D1), we will use this to control the  $\delta \rightarrow 0$  limit. We study the limiting behavior of the  $B$ -weighted resolvent

$$(3.16) \quad K(\lambda, \epsilon, \delta) = B^* R(\lambda, \epsilon, \delta) B.$$

To see that the operator  $K$  is well-defined, we compute a bound depending on  $\epsilon$  only. For  $\phi \in \mathcal{H}$ ,  $\|\phi\| = 1$ , the Cauchy-Schwartz inequality gives

$$(3.17) \quad \begin{aligned} \|K(\lambda, \epsilon, \delta)\phi\| &\geq -\text{Im} \langle \phi, K(\lambda, \epsilon, \delta)\phi \rangle \\ &\geq \langle \phi, B^* R^*(\lambda, \epsilon, \delta) (\delta + \epsilon \dot{H}_\lambda) R(\lambda, \epsilon, \delta) B\phi \rangle \\ &\geq C_0 \epsilon \|K(\lambda, \epsilon, \delta)\phi\|^2, \end{aligned}$$

where we used (D1) to replace  $\dot{H}_\lambda$  by its lower bound  $C_0 B B^*$ . Hence, we obtain

$$(3.18) \quad \|K(\lambda, \epsilon, \delta)\| \leq (C_0 \epsilon)^{-1}.$$

We also need a related result for  $K(\lambda, \epsilon, \delta)^*$ . Note that

$$K(\lambda, \epsilon, \delta)^* = K(\lambda, -\epsilon, -\delta),$$

so that in place of (3.12), we have

$$(3.19) \quad \begin{aligned} \|K(\lambda, \epsilon, \delta)^*\phi\| &\geq \text{Im} \langle \phi, K(\lambda, -\epsilon, -\delta)\phi \rangle \\ &\geq \epsilon C_0 \|K(\lambda, \epsilon, \delta)^*\phi\|^2. \end{aligned}$$

2. *First differential inequality.* Let  $g \in C_0^2(\Gamma)$ , and define

$$(3.20) \quad F(\epsilon, \delta) \equiv \int_{\Gamma} g(\lambda) \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle d\lambda.$$

By (3.13), we obtain the bound

$$(3.21) \quad |F(\epsilon, \delta)| \leq (C\epsilon)^{-1} \|g\|_1.$$

We first compute the  $\epsilon$ -derivative of  $F$ ,

$$(3.22) \quad i \frac{dF}{d\epsilon}(\epsilon, \delta) = \int_{\Gamma} g(\lambda) \langle \phi, BR(\lambda, \epsilon, \delta) \dot{H}_{\lambda} R(\lambda, \epsilon, \delta) B \phi \rangle d\lambda.$$

On the other hand, we note that

$$(3.23) \quad \frac{d}{d\lambda} \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle = -\langle B \phi, R \dot{H}_{\lambda} R B \phi \rangle,$$

using the fact that  $\ddot{H}_{\lambda} = 0$  in this case. We substitute this into the right side of (3.17) to obtain

$$(3.24) \quad i \frac{dF}{d\epsilon}(\epsilon, \delta) = - \int_{\Gamma} g(\lambda) \frac{d}{d\lambda} \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle.$$

We integrate by parts with respect to  $\lambda$  in the first term on the right to obtain

$$(3.25) \quad i \frac{dF}{d\epsilon}(\epsilon, \delta) = \int_{\Gamma} g'(\lambda) \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle.$$

Using the *a priori* estimates, we obtain

$$(3.26) \quad \left| \frac{dF}{d\epsilon}(\epsilon, \delta) \right| \leq \int_{\Gamma} g'(\lambda) \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle | \leq (\epsilon C_0)^{-1} \|g'\|_1.$$

Integrating this differential inequality yields an improved estimate for  $F$ ,

$$(3.27) \quad |F(\epsilon, \delta)| \leq C_3 |\log \epsilon| + |F(1, \delta)|,$$

where  $C_3$  is independent of  $\delta$  and  $|F(1, \delta)|$  is uniformly bounded as  $\delta \downarrow 0$ .

3. *Iteration.* With the new bound (3.27) for  $F(\epsilon, \delta)$ , we repeat the above procedure in order to obtain an improved upper bound for  $F(\epsilon, \delta)$  which remains finite as  $\epsilon \rightarrow 0$ . To do this, we need a better estimate on

$$(3.28) \quad \int_{\Gamma} g'(\lambda) \langle \phi, K \phi \rangle d\lambda.$$

We define another function  $\tilde{F}(\epsilon, \delta)$  by

$$(3.29) \quad \tilde{F}(\epsilon, \delta) = \int_{\Gamma} g'(\lambda) \langle \phi, K(\lambda, \epsilon, \delta) \phi \rangle d\lambda.$$

As in (3.16),  $\tilde{F}$  satisfies

$$(3.30) \quad |\tilde{F}(\epsilon, \delta)| \leq (C_0 \epsilon)^{-1} \|g'\|_1.$$

We now repeat the arguments of part 2 of the proof for this function  $\tilde{F}$ . We obtain an estimate similar to (3.26),

$$(3.31) \quad \left| \frac{d\tilde{F}}{d\epsilon}(\epsilon, \delta) \right| \leq (\epsilon C_0)^{-1} C_1 \|g'\|_1.$$

Integration of this inequality leads to,

$$(3.32) \quad |\tilde{F}(\epsilon, \delta)| \leq C_4 |\log \epsilon| + |\tilde{F}(1, \delta)|.$$

4. *Conclusion.* With this new estimate (3.32), we return to (3.27) and obtain

$$(3.33) \quad \left| \frac{dF}{d\epsilon}(\epsilon, \delta) \right| \leq C_5 |\log \epsilon| + C_6,$$

where  $C_5$  is independent of  $\delta$  and depends on  $\|g^{(p)}\|_1$ ,  $p = 0, 1$  and  $C_6$  depends on  $|F(1, \delta)|$  and  $|\tilde{F}(1, \delta)|$ , which are bounded, independent of  $\delta$ . We now integrate the inequality (3.33) and obtain,

$$(3.34) \quad |F(\epsilon, \delta)| \leq C,$$

where  $C$  depends on  $\|g^{(p)}\|_1$ ,  $p = 0, 1$  and is independent of  $\delta$  and uniform in  $\epsilon$ ,  $0 < \epsilon < 1$ . The proof of the theorem now follows from the fact that  $R(\lambda, \epsilon, \delta)$  converges weakly to  $R(\lambda, \delta)$ , as  $\epsilon \rightarrow 0$ , provided  $\delta > 0$ , and the dominated convergence theorem since

$$\left| \int_{\Gamma} g(\lambda) \langle \phi, BR(\lambda, \epsilon, \delta)B\phi \rangle d\lambda \right| \leq C,$$

by (3.34).  $\square$

3.2.3. *A General Spectral Averaging Method.* If we want to treat more general probability measures, like Hölder continuous measures, we need a more refined and local version of spectral averaging in order to prove a Wegner estimate, as discussed in chapter 4. Let us consider a probability measure  $\mu$  and define the Levy concentration of the measure, for any  $\epsilon > 0$ , by

$$(3.35) \quad s(\epsilon) \equiv \sup_{E \in \mathbb{R}} \mu([E, E + \epsilon]).$$

If  $\mu$  is Hölder continuous with exponent  $0 < \alpha \leq 1$ , then  $s(\epsilon) \sim \epsilon^\alpha$ . Spectral averaging with respect to such general probability measures relies on the following basic result of [28] that is a discrete version of spectral averaging.

PROPOSITION 3.1. Let  $A$  and  $B$  be two self-adjoint operators on a separable Hilbert space  $\mathcal{H}$ , and suppose that  $B \geq 0$  is bounded. Then, for any  $\phi \in \mathcal{H}$ , we have the bound

$$(3.36) \quad \sum_{n \in \mathbb{Z}} \sup_{y \in [0, 1]} \langle B\phi, \frac{1}{(A + (n + y)B)^2 + 1} B\phi \rangle \leq \pi(\|B\| + \|B\|^2) \|\phi\|^2.$$

We do not prove this abstract result here but refer to [28], mentioning that the proof uses results from the theory of maximally dissipative operators. The application of this result to random Schrödinger operators is given in the following proposition.

THEOREM 3.3. Let  $H_\lambda = H_0 + \lambda V$  be self-adjoint on a separable Hilbert space  $\mathcal{H}$  with  $V \geq 0$  bounded and suppose that  $\lambda$  is distributed according to a probability measure  $\mu$  with Levy concentration  $s(\epsilon)$ , defined in (3.35). For any  $\epsilon > 0$ , let  $\Delta_\epsilon \subset \mathbb{R}$  be an interval with  $|\Delta_\epsilon| = \epsilon$ . We have the following bound on the average of the matrix element of the imaginary part of the resolvent:

$$(3.37) \quad \int_{\Delta_\epsilon} dE \int_{\mathbb{R}} d\mu(\lambda) \Im \langle \phi, V \left( \frac{1}{H_\lambda - E - i\epsilon} \right) V \phi \rangle \leq \pi \|V\| (1 + \|V\|) s(\epsilon) \|\phi\|^2.$$

**Proof:** The imaginary part of the matrix element in (3.37) is

$$(3.38) \quad \langle V\phi, \frac{\epsilon}{(H_0 - E + \lambda V)^2 + \epsilon^2} V\phi \rangle = \frac{1}{\epsilon} \langle V\phi, \frac{1}{\epsilon^{-2}(H_0 - E + \lambda V)^2 + 1} V\phi \rangle.$$

To apply Proposition 3.1, we choose  $B = V$  and define a self-adjoint operator  $A \equiv \epsilon^{-1}(H_0 - E)$  so the matrix element in (3.38) may be written as

$$(3.39) \quad \langle V\phi, \frac{1}{(A + \epsilon^{-1}\lambda V)^2 + 1} V\phi \rangle.$$

We divide the integration over  $\lambda$  into a sum over intervals  $[n\epsilon, (n+1)\epsilon]$ , and change variables letting  $\lambda/\epsilon = n + y$ , so that  $y \in [0, 1]$ . We then obtain

$$(3.40) \quad \begin{aligned} & \int_{\mathbb{R}} d\mu(\lambda) \langle V\phi, \frac{1}{(A + \epsilon^{-1}\lambda V)^2 + 1} V\phi \rangle \\ &= \sum_n \int_{n\epsilon}^{(n+1)\epsilon} d\mu(\lambda) \langle V\phi, \frac{1}{(A + (n+y)V)^2 + 1} V\phi \rangle \\ &\leq \left( \sup_{m \in \mathbb{Z}} \mu([m\epsilon, (m+1)\epsilon]) \right) \left\{ \sum_n \sup_{y \in [0,1]} \langle V\phi, \frac{1}{(A + (n+y)V)^2 + 1} V\phi \rangle \right\}. \end{aligned}$$

We apply Proposition 3.1 to the last line in (3.40) and obtain

$$(3.41) \quad \int_{\mathbb{R}} d\mu(\lambda) \langle V\phi, \frac{1}{(A + \epsilon^{-1}\lambda V)^2 + 1} V\phi \rangle \leq \pi \|V\| (1 + \|V\|) s(\epsilon) \|\phi\|^2$$

This provides a bound for the average over  $\lambda$  of (3.38). Integrating in energy over  $\Delta_\epsilon$ , and recalling the factor of  $\epsilon^{-1}$  in (3.38), we obtain the estimate (3.37).  $\square$

We note that when  $\mu$  has a density  $g \in L^\infty(\mathbb{R})$ , we recover the estimate in Theorem 3.1.

**3.2.4. A Result on Averaged Spectral Projections.** An immediate application of all three spectral averaging theorems, Theorem 3.1, Theorem 3.2, or Theorem 3.3, is the following result on the averaged spectral projectors of the Hamiltonian  $H_\lambda$ . Let  $E_\lambda(A)$  be the spectral projection for the self-adjoint operator  $H_\lambda$  and the subset  $A \subset \mathbb{R}$ .

**COROLLARY 3.1.** Let  $H_\lambda$  be a one-parameter family of self-adjoint operators as in Theorem 3.1, Theorem 3.2, or Theorem 3.3. Let  $\lambda$  be distributed with a density  $0 \leq g \in L^\infty(\mathbb{R})$ , or  $g \in C_0^2(\Gamma)$ , or a general probability measure  $\mu$ , respectively. Then, for the cases of Theorems 3.1 and 3.2, we have that for any Borel set  $J \subset \mathbb{R}$ , there exists a finite positive constant  $C_g > 0$ , depending only on  $\|g\|_\infty$  for Theorem 3.1, or on  $\|g^{(j)}\|_1$ ,  $j = 0, 1, 2$  for Theorem 3.2, such that

$$(3.42) \quad \left\| \int_{\Gamma} g(\lambda) B^* E_\lambda(J) B \right\| \leq C_g |J|.$$

As a consequence, for any  $\phi \in \mathcal{H}$ , with  $\|\phi\| = 1$ , we have

$$(3.43) \quad \int_{\Gamma} g(\lambda) \langle B\phi, E_\lambda(J) B\phi \rangle \leq C_g |J|.$$

For the general case of Theorem 3.3, the analogues of (3.42) and (3.43) hold with  $|J|$  replaced by  $s(|J|)$ , for  $|J|$  sufficiently small.

**Proof.** One uses Stone's formula for spectral projections to write

$$(3.44) \quad \langle \phi, BE_\lambda(J)B\phi \rangle \leq \frac{1}{\pi} \lim_{\delta \downarrow 0} \operatorname{Im} \int_J dE \langle \phi, B^*(H_\lambda - E - i\delta)^{-1} B\phi \rangle.$$

It is easy to show that

$$(3.45) \quad B^*(H_\lambda - E - i\delta)^{-1} B = n \lim_{\epsilon \downarrow 0} K(\lambda, \epsilon, \delta).$$

Substituting  $K$  into the right side of (3.2.4), the result follows from any of the spectral averaging Theorems 3.1, 3.2, or 3.3.  $\square$

3.2.5. *Kotani's Trick.* This technique is based on the theorem that the averaged spectral measure is absolutely continuous with respect to Lebesgue measure. Let  $E_\lambda(\cdot)$  be the spectral family for  $H_\lambda$ .

**COROLLARY 3.2.** In addition to the hypotheses of Theorem 3.2, assume that  $\operatorname{Ran} B$  is cyclic  $\forall H_\lambda$ ,  $\lambda \in \Gamma$ , in the sense that  $\{f(H_\lambda)B\phi \mid f \in L^\infty(\mathbb{R}), \phi \in \mathcal{H}\}$  is dense in  $\mathcal{H}$ . Then for any Borel set  $J \subset \mathbb{R}$  with  $|J| = 0$ , one has  $E_\lambda(J) = 0$  almost every  $\lambda \in \Gamma$ .

**Proof.** Corollary 3.1 shows that

$$(3.46) \quad \int_\Gamma h(\lambda) \langle \phi, B^* E_\lambda(J) B\phi \rangle d\lambda = 0,$$

implies that

$$(3.47) \quad \langle \phi, B^* E_\lambda(J) B\phi \rangle = 0,$$

for almost every  $\lambda \in \Gamma$ . This set of  $\lambda$ , however, depends on  $\phi$ , and we denote it by  $\Gamma_\phi$ . The  $H$ -cyclicity assumption on  $B$  implies that for each  $\psi$  in a dense set there exists a set  $\Gamma_\psi$  such that  $\langle \psi, E_\lambda(J)\psi \rangle = 0$ , for all  $\lambda \in \Gamma_\psi$ , and  $|\Gamma_\psi| = |\Gamma|$ . Now let  $\{\psi_n\}$  be a complete orthonormal basis for  $\mathcal{H}$  and set  $\Gamma_\infty \equiv \bigcap_n \Gamma_{\psi_n}$ . Then  $|\Gamma_\infty| = |\Gamma|$  and  $\langle \psi, E_\lambda(J)\psi \rangle = 0$ ,  $\forall \lambda \in \Gamma_\infty$  and  $\forall \psi$  in a dense set in  $\mathcal{H}$ . By standard arguments, this can be extended to all  $\psi \in \mathcal{H}$ . Since  $E_\lambda$  is a projection, this shows  $E_\lambda(J) = 0$  a.e.  $\lambda \in \Gamma$ .  $\square$

**3.3. Review of Aronszajn-Donoghue Theory.** The theory of one-parameter families presented section 3.2 is inspired by the theory of rank one perturbations developed by Aronszajn [7] and Donoghue [39]. We review this theory here because of its interest and transparency. A comprehensive discussion of recent developments in the theory of rank-one perturbations is given by Simon in [115]. Let  $\mu$  be a real, nonnegative measure on  $\mathbb{R}$  satisfying

$$(3.48) \quad \int \frac{d\mu(\lambda)}{1 + |\lambda|}.$$

**DEFINITION 3.4.** The Borel transform  $F_\mu(z)$  of a real, nonnegative measure on  $\mathbb{R}$ , satisfying (3.48), for  $z = E + i\epsilon \in \mathcal{C}$  with  $\operatorname{Im} z = \epsilon \neq 0$ , is defined by

$$(3.49) \quad F_\mu(z) \equiv \int \frac{d\mu(\lambda)}{\lambda - z}.$$

It is useful to record here the real and imaginary parts of the Borel transform, both of which exist provided  $\epsilon \neq 0$ :

$$(3.50) \quad \operatorname{Im} F_\mu(E + i\epsilon) = \epsilon \int \frac{d\mu(\lambda)}{(\lambda - E)^2 + \epsilon^2},$$

and

$$(3.51) \quad \operatorname{Re} F_\mu(E + i\epsilon) = \int \frac{(\lambda - E)\mu(\lambda)}{(\lambda - E)^2 + \epsilon^2}.$$

In preparation for the Aronszajn-Donoghue Theorem, our first goal is to relate the Lebesgue decomposition of a measure  $\mu$  into its singular  $\mu_s$  and absolutely continuous  $\mu_{ac}$  parts to the behavior of the boundary-value of the Borel transform of the measure  $F_\mu$ . In the applications of interest to us, the measure  $\mu$  will be a spectral measure of a self-adjoint operator. We will see that the Borel transform is related to matrix elements of the resolvent of that self-adjoint operator.

We recall that a measure  $\nu$  is *supported* on a subset  $S \subset \mathbb{R}$  if  $\nu(\mathbb{R} \setminus S) = 0$ . We will also need the function  $B_\mu(E + i\epsilon)$  defined by

$$(3.52) \quad B(E + i\epsilon) \equiv \left[ \int \frac{d\mu(\lambda)}{(E - \lambda)^2 + \epsilon^2} \right]^{-1}.$$

PROPOSITION 3.2. Let  $\mu$  be a real, nonnegative measure on  $\mathbb{R}$  satisfying (3.48). We define the following subsets of  $\mathbb{R}$ :

$$(3.53) \quad S_\mu \equiv \{E \mid \lim_{\epsilon \rightarrow 0} \operatorname{Im} F_\mu(E + i\epsilon) = \infty\};$$

$$(3.54) \quad C_\mu \equiv \{E \mid \lim_{\epsilon \rightarrow 0} \operatorname{Im} F_\mu(E + i\epsilon) > 0 \text{ and finite}\};$$

$$(3.55) \quad P_\mu \equiv \{E \mid E \in S_\mu, B_\mu(E) = 0, \text{ and } \lim_{\epsilon \rightarrow 0} \epsilon \operatorname{Im} F_\mu(E + i\epsilon) \neq 0\}.$$

Then,  $\mu$  is supported on  $S_\mu \cup C_\mu$ , the measure  $\mu|_{S_\mu}$  is singular with respect to Lebesgue measure, the measure  $\mu|_{C_\mu}$  is absolutely continuous with respect to Lebesgue measure, and  $\mu|_{P_\mu}$  is a pure point measure. Furthermore, the density of  $\mu|_{C_\mu}$  is given by

$$(3.56) \quad \frac{d\mu_{ac}}{dE}(E) = \lim_{\epsilon \rightarrow 0} \frac{1}{\pi} \operatorname{Im} F_\mu(E + i\epsilon),$$

and for any  $E_0$ ,

$$(3.57) \quad \mu(\{E_0\}) = \lim_{\epsilon \rightarrow 0} \epsilon \operatorname{Im} F_\mu(E + i\epsilon).$$

### Sketch of the Proof.

1. For any continuous function  $f$  of compact support, it is easy to show that

$$(3.58) \quad \lim_{\epsilon \rightarrow 0} \int f(E) \frac{1}{\pi} \operatorname{Im} F_\mu(E + i\epsilon) dE = \int f(E) d\mu(E).$$

Consequently, the weak limit as  $\epsilon \rightarrow 0$  of  $\operatorname{Im} F_\mu(E + i\epsilon)$  exists for Lebesgue almost every  $E \in \mathbb{R}$ . Furthermore, it is a classical result, called Fatou's Lemma, that the pointwise  $\lim_{\epsilon \rightarrow 0} \operatorname{Im} F_\mu(E + i\epsilon)$  exists almost everywhere. The limit is exactly the Radon-Nikodym derivative  $\frac{d\mu_{ac}}{dE}(E)$  of  $\mu$  with respect to Lebesgue measure. The theorem of de la Vallée Poussin states that the singular part of the measure is supported on the set  $S_\mu$ .

2. An application of the Lebesgue Dominated Convergence Theorem shows that

$$(3.59) \quad \mu(\{E_0\}) = \lim_{\epsilon \rightarrow 0} \epsilon \operatorname{Im} F_\mu(E_0 + i\epsilon).$$

If this is nonzero, it follows that  $E_0 \in S_\mu$ . Now suppose the measure  $\mu$  has an atom at  $E_0$  with weight  $C(E_0)$ . Then the function  $B_\mu(E)$  satisfies

$$(3.60) \quad \lim_{\epsilon \rightarrow 0} \epsilon B_\mu(E_0 + i\epsilon)^{-1} = C(E_0).$$

Hence, the set  $P_\mu$  supports the pure point part of the measure  $\mu$ .  $\square$

We mention a related criterion for absolute continuity of a measure which was recently proved and used by Klein [88] in the study of extended states for the Anderson model on the Bethe lattice.

**PROPOSITION 3.3.** Let  $\mu$  be a real, nonnegative, measure  $\mu$  on  $\mathbb{R}$  satisfying (3.48) with Borel transform  $F_\mu(z)$ . If

$$(3.61) \quad \lim_{\epsilon \rightarrow 0^+} \int_a^b |F_\mu(E + i\epsilon)|^2 dE < \infty,$$

then  $\mu|(a, b)$  is absolutely continuous with respect to Lebesgue measure.

Another proof of this result, and related results concerning the relation between properties of the measure and the boundary behavior of the  $L^p$ -norm of the Borel transform, are given in Simon [117].

The subject of the Aronszajn-Donoghue theory is the instability of the singular spectrum under rank-one perturbations. It is well-known (cf. [81]) that the absolutely continuous spectrum is stable under a trace class perturbation. Let  $H_0$  be a self-adjoint operator on a separable Hilbert space  $\mathcal{H}$ . We assume that  $H_0$  has a cyclic vector  $\phi \in \mathcal{H}$  and form the rank-one, orthogonal projector  $P \equiv |\phi\rangle\langle\phi|$ . We consider the one-parameter family of operators

$$(3.62) \quad H_\lambda \equiv H_0 + \lambda P.$$

The resolvent of this family is denoted  $R_\lambda(z) = (H_\lambda - z)^{-1}$ . Since  $\phi$  is cyclic for  $H_0$ , the spectral measure  $\mu_{0,\phi}$ , defined by

$$(3.63) \quad \langle\phi, R_0(z)\phi\rangle = \int \frac{d\mu_{0,\phi}(\eta)}{\eta - z},$$

determines the operator  $H_0$ . Similarly, the measure  $\mu_{\lambda,\phi}$  determines  $H_\lambda$  since  $\phi$  is also cyclic for each  $H_\lambda$ .

We have seen that information about a measure can be recovered from the boundary-value of its Borel transform. We want to obtain information about the perturbed measure  $\mu_{\lambda,\phi}$ , and hence about the perturbed operator  $H_\lambda$ , from knowledge of the boundary behavior of the Borel transform of the measure  $\mu_{0,\phi}$ , corresponding to  $H_0$ . In the rank-one perturbation case, the Borel transform of the spectral measure is just the matrix element of the resolvent. Hence, we define

$$(3.64) \quad F_\lambda(z) \equiv \langle\phi, R_\lambda(z)\phi\rangle.$$

A key role is played by the equation relating  $F_\lambda(z)$  to  $F_0(z)$ . To derive this equation, we begin with the second resolvent formula,

$$(3.65) \quad R_\lambda(z) = R_0(z) - \lambda R_\lambda(z) P R_0(z).$$

Upon taking the matrix element of this equation in the state  $\phi$ , we find

$$(3.66) \quad F_\lambda(z)(1 + \lambda F_0(z)) = F_0(z).$$

When the factor  $(1 + \lambda F_0(z))$  is invertible, we can deduce the behavior of  $F_\lambda(E + i\epsilon)$ , as  $\epsilon \rightarrow 0$ , from the behavior of  $F_0(E + i\epsilon)$  in the same limit. Note that the beauty of a rank one perturbation is that this equation (3.22) is simply a relation between functions which are analytic in the upper half-plane.

The first main result of the Aronszajn-Donoghue analysis is the following theorem.

THEOREM 3.5. *Relative to the function  $F_0(E + i\epsilon)$  defined above, we define the following sets for  $\lambda \neq 0$ ,*

$$(3.67) \quad C \equiv \{E \mid \lim_{\epsilon \rightarrow 0} \text{Im } F_0(E + i\epsilon) = \Phi(E) > 0 \text{ and finite } \};$$

$$(3.68) \quad S_\lambda \equiv \{E \mid \lim_{\epsilon \rightarrow 0} F_0(E + i\epsilon) = -1/\lambda\};$$

*Then, the set  $C$  is the support of  $\mu_{\lambda,ac}$ , for all  $\lambda$ , and the set  $S_\lambda$  is the support of  $\mu_{\lambda,s}$ , for  $\lambda \neq 0$ .*

Of course, the absolutely continuous spectrum is invariant under finite-rank perturbations, and this fact is reflected by the first part of the theorem. The singular spectrum, however, is very unstable with respect to the perturbation. One of the most interesting consequences of Theorem 3.3 concerns the singular spectrum of  $H_\lambda$ .

COROLLARY 3.3. Under the conditions of Theorem 3.5,

- (1) The absolutely continuous parts of  $H_\lambda$  and  $H_\beta$  are unitarily equivalent;
- (2) The singular components of the spectral measures of  $H_\lambda$  and  $H_\beta$ , for  $\lambda \neq \beta$ , are mutually singular.

We recall that the second part of the corollary means that the if  $\mu_{\lambda_i,s}$  are the singular parts of the spectral measures for  $H_{\lambda_1}$  and  $H_{\lambda_2}$ , for  $0 \neq \lambda_1 \neq \lambda_2 \neq 0$ , then they are mutually singular if  $\mu_{\lambda_i,s}(S_{\lambda_j}) = 0$ , for  $i \neq j$ , where  $S_{\lambda_j}$  is the set defined in (3.68).

With regard to the eigenvalues of  $H_\lambda$ , for  $\lambda \neq 0$ , we have the following finer decomposition of the singular set  $S_\lambda$  in (3.68).

COROLLARY 3.4. As in framework of Theorem 3.5, we define the function  $B_0(E)$  by

$$(3.69) \quad B_0(E) \equiv \left[ \int_I R \frac{d\mu_0(s)}{(s-E)^2} \right]^{-1}.$$

Suppose that  $E$  is not an eigenvalue of  $H_0$ . Then, the energy  $E$  is an eigenvalue of  $H_\lambda$  if and only if  $E \in S_\lambda$ , where  $S_\lambda$  is defined in (3.68), and  $B_0(E) > 0$ . Consequently,  $E \in \sigma_{sc}(H_\lambda)$  if and only if  $E \in S_\lambda$  and  $B_0(E) = 0$ .

**3.4. Perturbation of Singular Spectra.** Our goal is to extend the ideas of the Aronszajn-Donoghue theory to relatively compact perturbations. This is essential for the our treatment of multidimensional, continuous random operators. A similar analysis for Schrödinger operators was performed by Howland [76].

Analogous to (3.2)–(3.4), we consider one parameter families of the form

$$(3.70) \quad H_\lambda = H_0 + \lambda V, \quad \lambda \in \mathbb{R}.$$

We require that  $V$  is a nonzero, bounded, self-adjoint operator and that it admits a factorization of the form  $V = CD^*$ . In the applications,  $V = u$ , the single-site potential, so we can take  $C = D = u^{1/2}$ . As in section 3.2, we set  $R_\lambda(z) = (H_\lambda - z)^{-1}$ , for  $\lambda \in \mathbb{R}$ . We consider a fixed energy interval  $I \subset \mathbb{R}$ . We need the

following assumptions

$$(A1) \quad \Gamma(z) \equiv D^*R_0(z)C \text{ is compact } \forall z, \text{ Im } z \neq 0;$$

$$(A2) \quad \text{there exists a subset } I_0 \subset I, |I_0| = |I|, \text{ and constants } C(E) < \infty, \text{ such that}$$

$$\sup_{\epsilon \neq 0} \|R_0(E + i\epsilon)X\| \leq C(E) < \infty$$

$$\forall E \in I_0 \text{ and } X = C \text{ and } D.$$

We show that (A2) is a sufficient condition for the absence of singular continuous spectrum of  $H_\lambda$  in  $I$  for almost-every  $\lambda$  in the case that  $C = \pm D$ . This is a variation of the instability of singular spectrum result presented in Theorem 3.5. As above, let  $\tilde{\mathcal{H}}$  be the cyclic subspace of  $\mathcal{H}$  generated by  $C$  and  $H_\lambda$ :

$$\tilde{\mathcal{H}} = [f(H_\lambda)C\phi, f \in L^\infty(\mathbb{R}), \phi \in \mathcal{H}]^{\text{closure}}.$$

We write  $\tilde{H}_\lambda$  for the restriction of  $H_\lambda$  to the invariant subspace  $\tilde{\mathcal{H}}$ , which is independent of  $\lambda$ .

**THEOREM 3.6.** *Let us assume that the operator  $H_0$  and the perturbation  $V = CD^*$  satisfy (A1) and (A2). We then have*

$$(1) \quad \sigma_{ac}(\tilde{H}_\lambda) \cap I = \emptyset, \forall \lambda \in \mathbb{R};$$

$$(2) \quad \sigma(\tilde{H}_\lambda) \cap I_0 \text{ is pure point, } \forall \lambda \in \mathbb{R} \setminus \{0\}.$$

If, in addition,  $C = \pm D$ , then we have

$$(3) \quad \sigma_{sc}(\tilde{H}_\lambda) \cap I = \emptyset \text{ for Lebesgue almost every } \lambda \in \mathbb{R}.$$

Consequently, the spectrum  $\tilde{H}_\lambda$ , for Lebesgue almost-every  $\lambda \in \mathbb{R}$ , is pure point in  $I$  with finitely-degenerate eigenvalues.

**Proof.** 1. We omit the tildes for simplicity of notation. From the second resolvent equation, we can express  $R_\lambda(z)$  in terms of  $R_0(z)$  and  $\Gamma(z) \equiv D^*R_0(z)C$ . We obtain

$$(3.71) \quad D^*R_\lambda(z)C(1 + \lambda\Gamma(z)) = \Gamma(z),$$

which is the analogue of the Aronszajn-Donoghue formula (3.66) in our case. In accordance with the Aronszajn-Donoghue theory, we want to study the boundary values of  $R_\lambda(E + i\epsilon)$ , as  $\epsilon \rightarrow 0^+$ , given the information about the boundary values of  $R_0(E + i\epsilon)$  given in (A2). We first note that for all  $\lambda \in \mathbb{R}$ , the operator  $(1 + \lambda\Gamma(z))$  is invertible for  $z \in \mathcal{C}$  with  $\text{Im } z \neq 0$ . If it were not invertible for some  $\lambda$  and  $z \in \mathcal{C}$ , with  $\text{Im } z \neq 0$ , then condition (A1) and the Fredholm alternative for compact operators implies that  $\exists \psi \in \mathcal{H}$  such that

$$(3.72) \quad (1 + \lambda\Gamma(z))\psi = 0.$$

This equation is equivalent to

$$\lambda D^* \frac{1}{H_0 - z} C \psi = -\psi.$$

Multiplying both sides by  $C$  and defining  $\xi = (H_0 - z)^{-1}C\psi \in \mathcal{H}$ , we get

$$-\lambda V \xi = (H_0 - z)\xi,$$

or,

$$(H_\lambda - z)\xi = 0,$$

which shows  $H_\lambda$  has a complex eigenvalue, which is impossible since  $H_\lambda$  is self-adjoint.

2. We next study the behavior of the boundary-values of  $\Gamma(E + i\epsilon)$ . For  $E \in I_0$ , condition (A2) implies that  $n - \lim_{\epsilon \rightarrow 0} \Gamma(E + i\epsilon) \equiv \Gamma(E + i0)$  exists. This operator is also compact by (A1). It now follows by the Bounded Inverse Theorem and the Fredholm alternative that for  $E \in I_0$ ,

$$(3.73) \quad n - \lim_{\epsilon \rightarrow 0^+} (1 + \lambda\Gamma(E + i\epsilon))^{-1} = (1 + \lambda\Gamma(E + i0))^{-1}$$

exists if and only if there is no  $\eta \in \mathcal{H}$  such that

$$(3.74) \quad (1 + \lambda\Gamma(E + i0^+))\eta = 0.$$

We will show that this equation holds only at energies  $E \in I_0$  that are eigenvalues of  $H_\lambda$ . The idea is as in (1) but now one has to control the limit of  $\eta_\epsilon \equiv R_0(E + i\epsilon)C\eta$ . Since condition (A2) implies that  $\|\eta_\epsilon\| \leq C(E)$ , we have that  $\eta_0 \equiv w - \lim_{\epsilon \rightarrow 0} \eta_\epsilon$  exists. We now need to show that  $\eta_0 \neq 0$ . For this, we have for any  $\phi \in D(H_0)$ ,

$$(3.75) \quad \langle (H_0 - E)\phi, \eta_0 \rangle = \lim_{\epsilon \rightarrow 0} \langle (H_0 - E)\phi, \eta_\epsilon \rangle = \langle \phi, C\eta \rangle.$$

Since  $D(H_0)$  is dense and  $C\eta \neq 0$ , this shows that  $\eta_0 \neq 0$ . It follows from (3.74) that

$$(3.76) \quad \begin{aligned} \langle C^*\phi, \eta \rangle &= -\lambda \langle C^*\phi, \Gamma(E + i0)\eta \rangle \\ &= -\lim_{\epsilon \rightarrow 0} \lambda \langle C^*\phi, D^*\eta_\epsilon \rangle \\ &= -\lambda \langle \phi, V\eta_0 \rangle. \end{aligned}$$

Inserting this equation into (3.75), we obtain

$$(3.77) \quad \langle (H_\lambda - E)\phi, \eta_0 \rangle = 0,$$

which holds for all  $\phi \in D(H_0) = D(H_\lambda)$ . It follows that  $(H_\lambda - E)\eta_0 = 0$ , so that the energy  $E$  is an eigenvalue of  $H_\lambda$ . Let us recall that the number of eigenvalues of  $H_\lambda$  is countable. Hence, for each  $\lambda \in \mathbb{R}$ , there exists a subset  $I_\lambda \subset I_0$ , with  $|I_\lambda| = |I_0|$  so that  $n - \lim_{\epsilon \rightarrow 0} (1 + \lambda\Gamma(E + i\epsilon))^{-1}$  exists for all  $E \in I_\lambda$ .

3. Using the second resolvent equation as in (3.71), we have for all  $E \in I_\lambda$ ,

$$(3.78) \quad R_\lambda(E + i\epsilon)C = R_0(E + i\epsilon)C(1 + \lambda\Gamma(E + i\epsilon))^{-1}.$$

By the result of part (2) and (A2), which controls the boundary value of  $R_0(E + i\epsilon)C$ , we obtain

$$(3.79) \quad \sup_{\epsilon > 0} \|R_\lambda(E + i\epsilon)C\| < C(E) < \infty, \quad \forall E \in I_\lambda.$$

Hence, we have obtained control on the boundary values of the resolvent of the perturbed operator given control on the boundary values of the unperturbed operator as in (A2).

4. We now exploit this control in order to make conclusions about the spectrum of  $H_\lambda$  in  $I$ . First, with regards to the absolutely continuous spectrum, the existence of the bound (3.79) implies that for all  $\phi \in \mathcal{H}$ , and for all  $E \in I_\lambda$ ,

$$(3.80) \quad \lim_{\epsilon \rightarrow 0} \operatorname{Im} \langle \phi, C^*R_\lambda(E + i\epsilon)C\phi \rangle = 0.$$

This proves  $\sigma_{ac}(\tilde{H}_\lambda) \cap I = \emptyset$ . Concerning the point spectrum in  $I_0$ , we know that the countably-many points  $E \in I_0$  where the limit in (3.73) does not exist are eigenvalues of  $\tilde{H}_\lambda$ . At the other points in  $I_0$ , the bound (3.79) holds and, as in Theorem 3.3, this set cannot support the singular continuous component of the spectral measure of  $\tilde{H}_\lambda$ . This proves the first part of the theorem.

5. As for singular continuous spectrum, it can only be supported in  $I \setminus I_0$ , which has  $|I \setminus I_0| = 0$ . We now use Corollary 3.2 taking  $h(\lambda) = (1 + \lambda^2)^{-1}$  (one can check that this type of function can be used in the proof). Since we have  $C = \pm D$ , we set  $B = C$  in Corollary 3.2 and obtain,

$$(3.81) \quad \left\| \int \frac{d\lambda}{1 + \lambda^2} C E_\lambda(I \setminus I_0) C \right\| \leq C |I \setminus I_0| = 0.$$

Consequently, for Lebesgue almost-every  $\lambda \in \mathbb{R}$ , we have that  $\|C E_\lambda(I \setminus I_0) C\| = 0$ . Due to the generating nature of the operator  $C$ , this proves that the singular component of the spectral measure of  $\tilde{H}_\lambda$  is not supported on  $I \setminus I_0$ . Accordingly, the singular spectrum of  $\tilde{H}_\lambda$  in  $I$  consists only of eigenvalues in  $I_0$ , for almost-every  $\lambda \in \mathbb{R}$ . The finite degeneracy of the eigenvalues follows from compactness of  $\Gamma(E + i0)$ .  $\square$

## 4. The Wegner Estimate and the Integrated Density of States

**4.1. Overview.** The term *Wegner estimate* refers to an upper bound on the probability that a given energy  $E$  is separated from the spectrum of a local Hamiltonian  $H_\Lambda$ . Equivalently, this is the probability that the local Hamiltonian  $H_\Lambda$  has an eigenvalue in a given interval. A *good* Wegner estimate is one for which the upper bound depends linearly on the volume  $|\Lambda|$  and on the distance from the energy  $E$  to the spectrum of  $H_\Lambda$ . In this chapter, we will present two proofs of Wegner's estimate in order to illustrate the ideas and difficulties involved. Wegner's original proof [136] introduced the clever device of interchanging differentiation with respect to energy with differentiation with respect to the random variables on which the random potential depends. Since an expectation is taken in the course of the proof, these derivatives can be removed by an integration-by-parts with respect to the random variables. In some way, all subsequent proofs of the Wegner estimate depend on this trick.

We begin in section 1 by sketching a proof of Wegner's estimate for energies in a spectral gap of  $H_0$  which follows the original argument. The proof is transparent but unfortunately leads to a  $|\Lambda|^2$ -volume dependence. This is sufficient for the proof of Anderson localization but provides no information in the IDS. We then give a simple version of the proof of a good Wegner's estimate from [28] that works for a wide variety of models at all energies. We will show how this implies the Lipschitz continuity of the integrated density of states (IDS). We remark that one can use the spectral shift function (SSF) (cf. [27, 78]) in the proof of Wegner's estimate, but the result is not as strong.

We mention that the Wegner estimate discussed here depends crucially on the fact that the single-site potential  $u$  is sign-definite. There have been some extensions to the nonsign-definite case, see [71, 129], and the review [130] for related results. The nonsign-definite case is important for applications to the proof of Anderson localization for Schrödinger operators with random vector potentials [68, 71, 72].

**4.2. Wegner's Argument: A First Proof of Wegner's Estimate.** Here we give a proof of Wegner's estimate for energies in the gap of the unperturbed operator  $H_0$ . The proof of the Wegner estimate begins by replacing the probability with an expectation which, in general, is easier to compute. Let us assume that  $H_0$  has a spectral gap  $G$  and that  $E \in G$ . Since the local potential  $V_\Lambda$  is a relatively compact perturbation of  $H_0$ , its effect is to introduce at most finitely-many eigenvalues  $E_j(\Lambda)$  into the gap  $G$ . We label these eigenvalues in increasing order, including multiplicity. Let  $\eta > 0$  be chosen so that  $I_\eta \equiv [E - \eta/2, E + \eta/2] \subset G$ . We want to estimate

$$(4.1) \quad \mathbb{P}\{\text{dist}(E, \sigma(H_\Lambda)) < \eta\}.$$

This probability is expressible in term of the finite-rank spectral projector for the interval  $I_\eta$  and  $H_\Lambda$ , which we write as  $E_\Lambda(I_\eta)$ . This projection is a random variable, depending on the process restricted to the region  $\Lambda$  in the *iid* case, but we suppress this in the notation. We have

$$(4.2) \quad \mathbb{P}_\Lambda\{\text{dist}(E, \sigma(H_\Lambda)) < \eta\} = \mathbb{P}_\Lambda\{\text{Tr}(E_\Lambda(I_\eta)) \geq 1\}.$$

We recall that Chebyshev's inequality for a random variable  $X$  is

$$(4.3) \quad \mathbb{P}\{X \geq \eta\} \leq \frac{1}{\eta} \mathbb{E}\{X\}.$$

We apply this to the random variable  $\text{Tr}(E_\Lambda(I_\eta))$  and obtain

$$(4.4) \quad \mathbb{P}_\Lambda\{\text{dist}(E, \sigma(H_\Lambda)) < \eta\} \leq \mathbb{E}_\Lambda\{\text{Tr}(E_\Lambda(I_\eta))\}.$$

We now proceed to estimate the expectation of the trace on the right in (4.4). We follow the original argument of Wegner [136] as modified by Kirsch [83] and using some results of [86]. Let  $\rho$  be a smooth, monotone increasing function such that  $\rho(x) = 1$ , for  $x > \eta/2$ , and  $\rho(x) = 0$ , for  $x < -\eta/2$ . We choose  $E \in G$  as above and  $\eta$  such that  $[E - 3\eta/2, E + 3\eta/2] \subset G$ . By the functional calculus, we have

$$(4.5) \quad \rho(H_\Lambda - E + \eta) - \rho(H_\Lambda - E - \eta) = \sum_j \{\rho(E_j(\Lambda) - E + \eta) - \rho(E_j(\Lambda) - E - \eta)\} P_j(\Lambda),$$

where the sum is over eigenvalues  $E_j(\Lambda)$  of  $H_\Lambda$  in  $[E - 3\eta/2, E + 3\eta/2]$ . The operators  $P_j(\Lambda)$  are the projectors onto the corresponding eigenspaces. The difference of the two operators is therefore trace class. Note that this difference  $\rho(H_\Lambda - E + \eta) - \rho(H_\Lambda - E - \eta)$  is, roughly, the number of eigenvalues of  $H_\Lambda$  less than  $E + 3\eta/2$  minus the number of eigenvalues of  $H_\Lambda$  less than  $E - 3\eta/2$ . However, the operator  $H_\Lambda$  may have continuous spectrum to the left of  $E - 3\eta/2$  so the operator  $\rho(H_\Lambda - E - \eta)$  by itself is not trace class.

The coefficient in (4.5) is always nonnegative and precisely equal to one for  $E_j(\Lambda) \in I_\eta$ , so we have

$$(4.6) \quad \text{Tr}(E_\Lambda(I_\eta)) \leq \text{Tr}(\rho(H_\Lambda - E + \eta) - \rho(H_\Lambda - E - \eta)).$$

Thus, the right side will give us an upper bound for the right side of (4.4). There is one other advantage to the introduction of the smooth function  $\rho$ . The counting function for eigenvalues is not differentiable since it is a step function. We will use the differentiability of  $\rho$  below.

Returning to the expectation of the trace in (4.4), we can now bound it above by

$$(4.7) \quad \mathbb{E}_\Lambda \{ \text{Tr} [\rho(H_\Lambda - E + \eta) - \rho(H_\Lambda - E - \eta)] \} \leq \mathbb{E}_\Lambda \left\{ \text{Tr} \left[ \int_{-\eta}^{\eta} \frac{d}{dt} \rho(H_\Lambda - E - t) \right] \right\}.$$

Since the support of  $\rho'(s)$  lies in the interval  $[-\eta/2, \eta/2]$ , we write

$$(4.8) \quad \text{Tr} \{ \rho'(H_\Lambda - E - t) \} = \sum_j \rho'(E_j(\Lambda) - E - t),$$

where, as above, the sum is over the eigenvalues of  $H_\Lambda$  in the interval  $[E - 3\eta/2, E + 3\eta/2]$  including multiplicity. The first part of Wegner's trick is to consider the eigenvalues as functions of the random coupling constants and rewrite the differentiation with respect to the energy  $t$  in terms of differentiation with respect to the coupling constants  $\lambda_j \in \tilde{\Lambda}$ . By the chain rule for differentiation, we have

$$(4.9) \quad \sum_{k \in \tilde{\Lambda}} \frac{\partial}{\partial \lambda_k} \rho(E_j(\Lambda) - E - t) = \rho'(E_j(\Lambda) - E - t) \sum_{k \in \tilde{\Lambda}} \frac{\partial}{\partial \lambda_k} E_j(\Lambda).$$

In the next step, we need to estimate the sum  $\sum_{k \in \tilde{\Lambda}} \frac{\partial}{\partial \lambda_k} E_j(\Lambda)$  from below so we can solve (4.9) for an upper bound on  $\rho'(E_j(\Lambda) - E - t)$ . We use the Feynman-Hellman formula for the variation of an eigenvalue with respect to a parameter.

**THEOREM 4.1.** (*Feynman-Hellman Theorem.*) *Let  $H(s)$  be a one-parameter family of self-adjoint operators for  $s \in I$ , a neighborhood of zero. Suppose that  $H(s)$  has a simple eigenvalue  $E(s) \in C^1(I)$  with eigenfunction  $\phi(s) \in C^1(I)$ . We then have*

$$(4.10) \quad \frac{d}{ds} E(s) = \langle \phi(s), \left( \frac{d}{ds} H(s) \right) \phi(s) \rangle.$$

**Proof:** We begin with the simple identity, valid for  $s \in I$ ,

$$(4.11) \quad 0 = \langle \phi(s), (H(s) - E(s))\phi(s) \rangle,$$

and differentiate each side. Note that the eigenvalue equation implies that

$$(4.12) \quad 0 = \left\langle \frac{d}{ds} \phi(s), (H(s) - E(s))\phi(s) \right\rangle,$$

and similarly for the conjugate term. Since  $\|\phi(s)\| = 1$ , we obtain the result from the term involving  $\frac{d}{ds}(H(s) - E(s))$ .  $\square$

We will use this identity again in chapter 5. The theorem is particularly useful in cases of analytic perturbation theory for which the hypotheses of the theorem are satisfied.

Returning to our discussion of the Wegner estimate, we apply the Feynman-Hellman Theorem to the eigenvalues  $E_j(\Lambda)$ , which depend on the coupling constants  $\lambda_k, k \in \tilde{\Lambda}$ , and obtain

$$(4.13) \quad \sum_{k \in \tilde{\Lambda}} \frac{\partial}{\partial \lambda_k} E_j(\Lambda) = \sum_{k \in \tilde{\Lambda}} \langle \phi_j, \frac{\partial}{\partial \lambda_k} H_\Lambda \phi_j \rangle.$$

The Anderson-type potential depends linearly on the coupling constants so that

$$(4.14) \quad \frac{\partial H_\Lambda}{\partial \lambda_k} = u_k.$$

It follows from this and (4.12) that we need to estimate from below the sum

$$(4.15) \quad \sum_{k \in \bar{\Lambda}} \langle \phi_j, u_k \phi_j \rangle,$$

for an eigenfunction  $\phi_j$  of  $H_\Lambda$ .

Let us now suppose for simplicity that the single-site potential  $u$  is bounded and has support in  $B_{R_0}(0) \subset \Lambda_1(0)$ . We assume that  $u \geq 0$  and satisfies  $u \geq \epsilon_1$  on  $B_{R_1}(0) \subset \Lambda_1(0)$  for some  $\epsilon_1 > 0$  and some  $R_1 > 0$ . Let us define a subset of  $\Lambda$  by  $\Lambda' \equiv \cup_{k \in \bar{\Lambda}} B_{R_1}(k)$ . We then have

$$(4.16) \quad \sum_{k \in \bar{\Lambda}} u_k \geq \epsilon_1 \chi_{\Lambda'}.$$

We estimate from below the quantity  $\langle \phi_j, \chi_{\Lambda'} \phi_j \rangle$  using the Comparison Theorem of Kirsch, Stollmann, and Stolz (KSS) [86]. Let  $H_0$  be a background Hamiltonian with an open spectral gap  $G$ , which is perturbed by a localized potential  $V_\Lambda$ . For any region  $\mathcal{O} \subset \mathbb{R}^d$ , let  $\chi_{\mathcal{O}}$  be the characteristic function of the region.

**THEOREM 4.2.** *Let  $H_0$  and  $V_\Lambda$  be as above and  $H_\Lambda \phi = E \phi$  with  $E \in G$  and  $\phi \in L^2(\mathbb{R}^d)$ . Suppose that the following two conditions are satisfied:*

- (1) *There exists a potential  $V_0$  such that, with  $H_\Lambda^0 \equiv H_0 + V_0$ , we have  $E \in \rho(H_\Lambda^0)$ ;*
- (2) *There exists a subset  $F \subset \Lambda$  and a constant  $\theta > 0$  so that  $\text{dist}(F \cup \Lambda^c, \{x \mid V_\Lambda(x) \neq V_0(x)\}) > \theta > 0$ .*

*Then, there exists a constant  $C_0$ , depending on  $\text{dist}(E, \sigma(H_0))^{-1}$ , such that*

$$(4.17) \quad \|\phi\| \leq C_0(1 + \|(H_\Lambda^0 - E)^{-1} W_1\|) \|\chi_F \phi\|,$$

*where  $W_1 \equiv [H_0, \chi_1]$ . Here,  $\chi_1$  is a smoothed characteristic function of compact support with  $\chi_1 = 1$  on  $\text{supp}(1 - \chi_F)$ , and  $\chi_1 = 0$  on  $\mathcal{D} \equiv \{x \mid V_\Lambda(x) \neq V_0(x)\}$ .*

Referring to this theorem, we must construct a comparison potential  $V_0$ . We do this as follows. We choose a radius  $0 < R_2 < R_1$  and an  $\epsilon_2 \geq \epsilon_1$  so that  $u \mid B_{R_2}(0)^c \leq \epsilon_2$ . We define a subset of  $\Lambda$  by  $\mathcal{D} \equiv \cup_{k \in \bar{\Lambda}} B_{R_2}(k)$ . We define the comparison potential  $V_0$  as follows

$$(4.18) \quad \begin{aligned} V_0(x) &= V_\Lambda(x) \text{ for } x \in \mathcal{D}^c \cap \Lambda \\ &= 0 \text{ for } x \in \mathcal{D} \end{aligned}$$

With this definition, we can take the set  $F$  to be  $F \equiv (\Lambda')^c \cap \Lambda$ . Note that the distance from the set  $\mathcal{D}$  on which the potentials are not equal to  $F$  is  $(R_1 - R_2) > 0$ . Furthermore, we have  $|V_0(x)| \leq \epsilon_2$ . Consequently, the spectrum of  $H_\Lambda^0$  is contained in the complement of the gap  $\tilde{G} \equiv (B_- + \epsilon_2, B_+ - \epsilon_2)$ . Given an energy  $E \in G$  and an  $\eta$  as above, we choose  $R_2$  and  $R_1$  sufficiently close to  $R_0$  so that  $\epsilon_2$  is sufficiently small in order to guarantee that the  $3\eta/2$ -interval around  $E$  does not intersect the gap  $\tilde{G}$ . This insures that any eigenvalue of  $H_\Lambda$  in the interval is not in the spectrum of the comparison operator  $H_\Lambda^0$ .

Having verified all the hypotheses of the KSS Comparison Theorem, we obtain the estimate

$$(4.19) \quad \langle \phi_j, \chi_{\bar{\Lambda}} \phi_j \rangle \geq C_0 \|\phi_j\|^2,$$

for a constant depending on  $\text{dist}(E, \sigma(H_0))$ . Returning to (4.13), we have the lower bound,

$$(4.20) \quad \sum_{k \in \bar{\Lambda}} \frac{\partial E_j(\Lambda)}{\partial \lambda_k} \geq C_0,$$

so we obtain

$$(4.21) \quad \rho'(E_j(\Lambda) - E - t) \leq \frac{1}{C_0} \sum_{k \in \bar{\Lambda}} \frac{\partial \rho}{\partial \lambda_k} (E_j(\Lambda) - E - t).$$

With this estimate, the right side of (4.7) can be bounded above by

$$(4.22) \quad \frac{1}{C_0} \sum_j \sum_{k \in \bar{\Lambda}} \int_{-\eta}^{\eta} \mathbb{E}_{\Lambda} \left\{ \frac{\partial \rho}{\partial \lambda_k} (E_j(\Lambda) - E - t) \right\} dt.$$

The expectation is the integral over the random variables  $\lambda_l$  with the product measure. Let us integrate over one of these variables, say the  $k^{\text{th}}$  one. Because of the positivity of  $\rho'$  and of the density  $g$ , we obtain

$$(4.23) \quad \int d\lambda_k g(\lambda_k) \frac{\partial \rho}{\partial \lambda_k} (E_j(\Lambda) - E - t) \leq \|g\|_{\infty} \{ \rho(E_j^{M,k}(\Lambda) - E - t) - \rho(E_j^{m,k}(\Lambda) - E - t) \},$$

where  $E_j^{M,k}$  is the  $j^{\text{th}}$ -eigenvalue of the local Hamiltonian  $H_{\Lambda}$  with the coupling constant at the  $k^{\text{th}}$ -site fixed at its maximum value. Similarly, the small  $m$  denotes the minimum value. Consequently, we are left with the task of estimating

$$(4.24) \quad \frac{\|g\|_{\infty}}{C_0} \sum_{k \in \bar{\Lambda}} \int_{-\eta}^{\eta} dt \int \prod_{l \neq k} g(\lambda_l) d\lambda_l \text{Tr} \{ \rho(H_{\Lambda}^{M,k} - E - t) - \rho(H_{\Lambda}^{m,k} - E - t) \}.$$

The expression involving the trace is basically the number of eigenvalues created in the interval by increasing the  $k^{\text{th}}$ -coupling constant from the minimum to the maximum value. In fact, the trace can be rewritten in terms of the *spectral shift function* (SSF). Estimates on the SSF are key to a good Wegner estimate and we will discuss this in the next section. Let us note here that Weyl's upper bound for the eigenvalue counting function gives a crude estimate in terms of the volume:

$$(4.25) \quad |\text{Tr} \{ \rho(H_{\Lambda}^{M,k} - E - t) - \rho(H_{\Lambda}^{m,k} - E - t) \}| \leq C_1 |\Lambda|.$$

Finally, this estimate and equations (4.4) and (4.24), lead us to the result

$$(4.26) \quad \mathbb{P} \{ \text{dist}(E, \sigma(H_{\Lambda})) < \eta \} \leq 2\eta \frac{\|g\|_{\infty} C_1}{C_0} |\Lambda|^2.$$

This estimate is sufficient for the proof of localization for the models we will discuss.

There are two defects in the simple proof sketched above. First, the restriction to the spectral gap of  $H_0$  was necessary for the crucial lower bound estimate (4.19) following from Theorem 4.2. This lower bound is a form of the unique continuation theorem for solutions to elliptic equations. Roughly, if the left side of (4.19) is zero, this means that the eigenfunction  $\phi_j$  vanishes on an open set. However, if  $\phi_j$  were analytic, this would imply that  $\phi_j = 0$  identically. A refined and quantitative version of this type of result will allow us to get a good lower bound without using Theorem 4.2. Secondly, the crude estimate on (4.25) is too big. The upper bound should depend on the size of the support of the perturbation  $u$ , not the total volume  $|\Lambda|$ .

**4.3. The Wegner Estimate: Preliminaries.** The optimal form of the Wegner estimate has an upper bound that depends on the first power of the volume and the first power of the length of the interval. This first power of the volume will insure that the thermodynamic limit exists, whereas the first power of the interval size will prove the Lipschitz continuity of the IDS via the bound, for  $E \leq E'$ ,

$$(4.27) \quad 0 \leq N(E') - N(E) \leq \liminf_{|\Lambda \rightarrow \infty} \mathbb{E}\{Tr(E_\Lambda([E, E']))\}.$$

We will prove, then, for any small energy interval,

$$(4.28) \quad \mathbb{E}\{Tr E_\Lambda([E, E'])\} \leq C_I |E - E'| |\Lambda|,$$

for some locally uniform constant  $C_I$ . In order to prove the Lipschitz continuity, we will assume that the single-site probability distribution has a bounded density with compact support. Much more general situations are discussed in [28].

In order to control the background operator, we make the assumption that  $H_0 = (-i\nabla - A_0)^2 + V_0$  is a periodic Schrödinger operator with a real-valued, periodic, potential  $V_0$ , and a periodic vector potential  $A_0$ . We assume that  $V_0$  and  $A_0$  are sufficiently regular so that  $H_0$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ . We assume that both  $V_0$  and  $A_0$  are periodic with respect to the group  $\Gamma = \mathbb{Z}^d$  because of the form of the Anderson-type potential. We note that we could work with a nondegenerate lattice  $\Gamma$ , by defining a corresponding Anderson-type potential, but we will explicitly treat the case  $\Gamma = \mathbb{Z}^d$ .

Concerning the Anderson-type random potential, We will always make the following four assumptions:

- (A1): The background operator  $H_0 = (-i\nabla - A_0)^2 + V_0$  is a lower semi-bounded,  $\mathbb{Z}^d$ -periodic Schrödinger operator with a real-valued,  $\mathbb{Z}^d$ -periodic, potential  $V_0$ , and a  $\mathbb{Z}^d$ -periodic vector potential  $A_0$ . We assume that  $V_0$  and  $A_0$  are sufficiently regular so that  $H_0$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ .
- (A2): The periodic operator  $H_0$  has the unique continuation property (UCP), that is, for any  $E \in \mathbb{R}$  and for any function  $\phi \in H_{loc}^2(\mathbb{R}^d)$ , if  $(H_0 - E)\phi = 0$ , and if  $\phi$  vanishes on an open set, then  $\phi \equiv 0$ .
- (A3): The nonzero, nonnegative, compactly-supported, single-site potential  $u \in L_0^\infty(\mathbb{R}^d)$ , and it is strictly positive on a nonempty open set.
- (A4): The random coupling constants  $\{\lambda_j(\omega) \mid j \in \mathbb{Z}^d\}$ , are independent and identically distributed. The distribution has a density  $h_0 \in L^\infty(\mathbb{R})$  with  $\text{supp } h_0 \subset [0, 1]$ .

We make three important comments on these hypotheses:

- (1) For a given density  $h_0$  of compact support  $[m, M_1]$ , we can always add the periodic potential  $m \sum_j u_j$ , with  $u_j(x) = u(x - j)$ , to the background potential  $V_0$ , so that the random coupling constants take their value in an interval  $[0, M]$ , with  $M = M_1 - m$ .
- (2) Because of the explicit disorder parameter  $\lambda > 0$ , we can rescale the coupling constants so that, without loss of generality, the support of  $h_0$  is included in the interval  $[0, 1]$ .
- (3) Hypotheses (A1) and (A2) imply the following. There exists a finite constant  $C_1(u, M, d) > 0$ , depending only on the single-site potential  $u$ , and

the dimension  $d \geq 1$ , so that for all  $\Lambda \subset \mathbb{R}^d$ ,

$$(4.29) \quad 0 \leq V_\Lambda^2 \leq C_1(u, d)\tilde{V}_\Lambda,$$

where  $V_\Lambda$  and  $\tilde{V}_\Lambda$  are defined in (1.3) and (1.4), respectively. This simple inequality is used in the proof of Theorem 4.3.

These hypotheses overlap with the hypotheses of chapter 1. As we have discussed, the Wegner estimate and the continuity of the IDS are very closely related. We first give the result on the Wegner estimate.

**THEOREM 4.3.** *Assume hypotheses (A1)–(A4). Let  $\Delta \subset \mathbb{R}$  be a bounded, closed interval. Then, there exists a finite constant  $C_W > 0$ , locally uniform in energy, and depending on  $d$ ,  $u$ , and  $\lambda > 0$ , so that*

$$(4.30) \quad \mathbb{E}\{\text{Tr}E_\Lambda(\Delta)\} \leq C_W|\Lambda| |\Delta|.$$

We note that one can show that  $C_W \sim 1/\lambda$ . It is possible to obtain a weaker Wegner estimate for which the constant  $C_W$  is independent of  $\lambda$  (cf. [27] and an application to continuity of the IDS with respect to the disorder in [73]). This has consequences for the continuity, with respect to the disorder  $\lambda$  of the IDS [73]. Under the same conditions, we obtain the following Lipschitz continuity result on the IDS.

**THEOREM 4.4.** *Assume hypotheses (A1)–(A4). Then, the IDS  $N(E)$  for the random family  $H_\omega(\lambda) = H_0 + \lambda V_\omega$ , for  $\lambda \neq 0$ , is locally Lipschitz continuous on  $\mathbb{R}$ .*

**4.3.1. Quantitative Unique Continuation Principle.** Before we sketch the proof of Theorem 4.3, we must discuss an important consequence of hypothesis (A2) on the background operator  $H_0$  that we refer to as a *quantitative UCP*. We first recall the Floquet decomposition of a  $\Gamma$ -periodic operator. Let  $\Gamma^*$  denote the dual lattice, that is,  $\Gamma^* = \{\gamma' \mid \gamma \cdot \gamma' \in 2\pi\mathbb{Z}, \text{ for all } \gamma \in \Gamma\}$ . We let  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$  be the torus, and  $(\mathbb{T}^d)^* = (\mathbb{R}^d)^*/(\mathbb{Z}^d)^*$  be the dual torus. We denote by  $C_0$  the unit cell for  $\Gamma = \mathbb{Z}^d$ , and by  $C_0^*$  the unit cell for  $\Gamma^* = (\mathbb{Z}^d)^*$ . The Floquet decomposition of  $H_0$  yields a family of operators  $H_0(\theta)$ , for  $\theta \in (\mathbb{R}^d)^*$ . Each operator  $H_0(\theta)$  is self-adjoint on  $L^2(\mathbb{T}^d)$ , and has a compact resolvent. We denote the eigenvalues of  $H_0(\theta)$  by  $E_n(\theta)$ . The spectrum of  $H_0$  is given by

$$(4.31) \quad \sigma(H_0) = \bigcup_{\theta \in (\mathbb{T}^d)^*} \sigma(H_0(\theta)) = \bigcup_{n \in \mathbb{N}} \bigcup_{\theta \in (\mathbb{T}^d)^*} E_n(\theta).$$

For an open, relatively compact interval  $\Delta \subset \mathbb{R}$ , we let  $E_0(\Delta, \theta)$  denote the spectral projector of  $H_0(\theta)$  onto the eigenspace of  $H_0(\theta)$  spanned by its eigenfunctions with eigenvalues  $E_n(\theta) \in \Delta$ . Because of the discreteness of the spectrum of  $H_0(\theta)$ , the dimension of  $\text{Ran}E_0(\Delta, \theta)$  is finite and locally constant.

We also need to consider  $H_0$  as an  $n\Gamma$ -periodic operator for any  $n \in \mathbb{N}$ . Let  $H_0^n(\theta)$  be the operator  $H_0$  restricted to the torus  $\mathbb{R}^d/(n\Gamma)$ , with  $\theta$ -quasi-periodic boundary conditions. For  $I \subset \mathbb{R}$ , an interval, let  $\mathbf{E}_0^n(I, \theta)$  denote the spectral projection onto the interval  $I$  for  $H_0^n(\theta)$ . We remark that in section 2 we took  $I = \tilde{\Delta}$ . Finally, for any  $\Gamma$ -periodic function  $g$ , we write  $g^{(n)}$  for the same function understood as an  $n\Gamma$ -periodic function.

**THEOREM 4.5.** *Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded, real-valued,  $\Gamma$ -periodic function. Consider a bounded interval  $I \subset \mathbb{R}$ . Then, if there exists a finite constant  $C(I, V) >$*

0 such that, for all  $\theta \in \mathbb{R}^d$ , one has

$$(4.32) \quad \mathbf{E}_0(I, \theta) V \mathbf{E}_0(I, \theta) \geq C(I, V) \mathbf{E}_0(I, \theta),$$

then, for all  $n \geq 1$  and all  $\theta \in \mathbb{R}^d$ , one has, with the same constant  $C(I, V)$ ,

$$(4.33) \quad \mathbf{E}_0^n(I, \theta) V^{(n)} \mathbf{E}_0^n(I, \theta) \geq C(I, V) \mathbf{E}_0^n(I, \theta),$$

where  $V$  is considered as a  $\Gamma$ -periodic function in (4.1), and  $V^{(n)}$  represents the corresponding  $n\Gamma$ -periodic function in (4.2).

We next show that, in fact, condition (4.1) holds for a wide family of periodic potentials  $V$ . We remind the reader that in the applications, we will take the potential  $V$  appearing in (4.1) to be the single-site potential  $u$ , restricted to the unit cell  $\Lambda_1(0)$ , viewed as a  $\Gamma$ -periodic function. The  $n\Gamma$ -periodic function appearing in (4.2) is  $\tilde{V}(x) = \sum_j u(x - j)$ , restricted to  $\Lambda^{(n)}$ , where  $\Lambda^{(n)}$  is the basic  $n\Gamma$ -periodic cell.

**THEOREM 4.6.** *Let  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  be a bounded,  $\Gamma$ -periodic, nonnegative function. Suppose that  $V > 0$  on some open set and  $H_0$  has the unique continuation property. Then, condition (4.1) holds for any compact interval  $I \subset \mathbb{R}$  with a finite constant  $C(I, V) > 0$ .*

As a summary of these results, we have the following. Let  $\Lambda \subset \mathbb{R}^d$  be a  $\mathbb{Z}^d$ -periodic cell. There exists a finite constant  $C_0 > 0$ , depending only on  $u$  and the dimension  $d$ , so that for any closed, bounded interval  $I \subset \mathbb{R}$ , we have

$$(4.34) \quad E_0^\Lambda(I) \tilde{V}_\Lambda E_0^\Lambda(I) \geq C_0 E_0^\Lambda(I).$$

This is the quantitative UCP. It plays a key role in the proof of Theorem 4.3.

**4.3.2. Spectral Averaging.** The second ingredient is the spectral averaging result, Theorem 3.1 of chapter 3, and its Corollary 3.1. We use this estimate (3.43) as follows. Let  $E_0 \in \mathbb{R}$  be fixed and arbitrary. We consider an interval  $\Delta_\epsilon = [E_0, E_0 + \epsilon]$ , for some fixed  $0 < \epsilon < \infty$ . A simple use of the spectral theorem for a self-adjoint operator  $H$  with spectral family  $E_H(\cdot)$  shows that

$$(4.35) \quad \begin{aligned} & \int_{\Delta_\epsilon} dE \langle \phi, \Im(H - E - i\epsilon)^{-1} \phi \rangle \\ &= \langle \phi, \left[ \tan^{-1} \left( \frac{E_0 + \epsilon - H}{\epsilon} \right) - \tan^{-1} \left( \frac{E_0 - H}{\epsilon} \right) \right] \phi \rangle \\ &\geq (\tan^{-1} 1) \langle \phi, E_H(\Delta_\epsilon) \phi \rangle = (\pi/4) \langle \phi, E_H(\Delta_\epsilon) \phi \rangle, \end{aligned}$$

so that

$$(4.36) \quad \langle \phi, E_H(\Delta_\epsilon) \phi \rangle \leq \frac{4}{\pi} \int_{\Delta_\epsilon} dE \langle \phi, \Im(H - E - i\epsilon)^{-1} \phi \rangle.$$

We combine (4.36) with Theorem 3.1 to obtain

$$(4.37) \quad \mathbb{E} \{ \langle \phi, u_j E_\Lambda(\Delta) u_j \phi \rangle \} \leq 8|\epsilon| \|\phi\|^2.$$

We remark that Theorem 3.3 leads to a similar upper bound for more general probability distributions with the factor  $\epsilon$  replaced by the Levy concentration  $s(\epsilon)$  of the probability measure as described in (3.35) and Corollary (3.1), see also [28].

**4.4. Proof of Theorem 4.3.** 1. As in section 2, we need to estimate

$$(4.38) \quad \mathbb{E}\{Tr E_\Lambda(\Delta)\}.$$

We begin with a decomposition relative to the spectral projectors  $E_0^\Lambda(\cdot)$  for the operator  $H_0^\Lambda$ . We write

$$(4.39) \quad Tr E_\Lambda(\Delta) = Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) + Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c).$$

We require  $\Delta \subset \tilde{\Delta}$  satisfy  $|\Delta| < 1$  and that the distance from the interval  $\Delta$  to the unbounded set  $\tilde{\Delta}^c$  be strictly positive, that is,  $d_\Delta > 0$ . If  $\tilde{\Delta}$ , and consequently  $\Delta$ , lies in a spectral gap of  $H_0$ , then only the second term on the right in (4.39) contributes and the result follows from (4.49). Hence, we only need to consider the case when  $\Delta$  does not lie in a spectral gap of  $H_0$ .

2. The term involving  $\tilde{\Delta}^c$  is estimated as follows. Since  $E_\Lambda(\Delta)$  is trace class, let  $\{\phi_m^\Lambda\}$  be the set of normalized eigenfunctions in its range. We expand the trace in these eigenfunctions and obtain

$$(4.40) \quad Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) = \sum_m \langle \phi_m^\Lambda, E_0^\Lambda(\tilde{\Delta}^c) \phi_m^\Lambda \rangle.$$

From the eigenfunction equation  $(H_\omega^\Lambda - E_m) \phi_m^\Lambda = 0$ , we easily obtain

$$-(H_0^\Lambda - E_m)^{-1} E_0^\Lambda(\tilde{\Delta}^c) V_\Lambda \phi_m^\Lambda = E_0^\Lambda(\tilde{\Delta}^c) \phi_m^\Lambda.$$

Substituting this into the right side of (4.40), and resumming to obtain a trace, we find

$$(4.41) \quad Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) = \sum_m \langle \phi_m^\Lambda, \left( V_\Lambda \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda - E_m)^2} V_\Lambda \right) \phi_m^\Lambda \rangle.$$

We next want to replace the energy  $E_m \in \Delta$  in the resolvent in (4.41) by a fixed number, say  $-M$ , assuming  $H_0^\Lambda > -M > -\infty$ . To do this, we define an operator  $K$  by

$$(4.42) \quad K \equiv \left( \frac{H_0^\Lambda + M}{H_0^\Lambda - E_m} \right)^2 E_0^\Lambda(\tilde{\Delta}^c),$$

and note that  $K$  is bounded, independent of  $m$ , by

$$\|K\| \leq K_0 \equiv \left[ 1 + \frac{2(M + \Delta_+)}{d_\Delta} + \frac{(M + \Delta_+)^2}{d_\Delta^2} \right],$$

where  $\Delta = [\Delta_-, \Delta_+]$ . Now, for any  $\psi \in L^2(\mathbb{R}^d)$ ,

$$(4.43) \quad \begin{aligned} \left\langle \psi, \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda - E_m)^2} \psi \right\rangle &\leq \left\langle \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)} \psi, K \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)} \psi \right\rangle \\ &\leq K_0 \left\langle \psi, \frac{E_0^\Lambda(\tilde{\Delta}^c)}{(H_0^\Lambda + M)^2} \psi \right\rangle \\ &\leq K_0 \left\langle \psi, \frac{1}{(H_0^\Lambda + M)^2} \psi \right\rangle, \end{aligned}$$

since  $E_0^\Lambda(\tilde{\Delta}^c) \leq 1$ . We use the bound (4.43) on the right in (4.41) and expand the potential. To facilitate this, let  $\chi \geq 0$  be a function of compact support slightly

larger than the support of  $u$ , and so that  $\chi u = u$ . We set  $\chi_j(x) = \chi(x - j)$ , for  $j \in \mathbb{Z}^d$ . Returning to (4.41), we obtain the bound

$$\begin{aligned}
Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) &\leq K_0 Tr E_\Lambda(\Delta) \left( V_\Lambda \frac{1}{(H_0^\Lambda + M)^2} V_\Lambda \right) \\
&\leq K_0 \sum_{i,j \in \tilde{\Lambda}} |\omega_i \omega_j| \left| Tr \left[ u_j E_\Lambda(\Delta) u_i \cdot \left( \chi_i \frac{1}{(H_0^\Lambda + M)^2} \chi_j \right) \right] \right| \\
&\leq K_0 \sum_{i,j \in \tilde{\Lambda}} \left| Tr \left[ u_j E_\Lambda(\Delta) u_i \cdot \left( \chi_i \frac{1}{(H_0^\Lambda + M)^2} \chi_j \right) \right] \right|.
\end{aligned}
\tag{4.44}$$

3. Let us make the simplifying assumption that  $u_i u_j = 0$ , if  $i \neq j$ . This is the case when  $u$  has support inside a unit cube. It is also the most difficult case. We refer to [28] for the general case. We note that the operator  $K_{ij} \equiv \chi_i (H_0^\Lambda + M)^{-2} \chi_j$  in (4.41) is trace class for  $d = 1, 2, 3$ . It is proved in [28] that the operator  $K_{ij}$  is trace class in all dimensions when  $\chi_i \chi_j = 0$ , and the trace norm  $\|K_{ij}\|_1$  decays exponentially in  $\|i - j\|$  as

$$(4.45) \quad \|K_{ij}\|_1 = \|\chi_i (H_0^\Lambda + M)^{-2} \chi_j\|_1 \leq C_0 e^{-c_0 \|i - j\|}, \quad i \neq j,$$

for positive constants  $C_0, c_0 > 0$  depending on  $M$ . This exponential decay is crucial for controlling the double sum in (4.44) in order to obtain just one power of the volume. Omitting some technical details that arise in dimensions  $d > 3$ , we write the trace on the last line of (4.44) as

$$(4.46) \quad \sum_{i,j \in \tilde{\Lambda}} Tr [u_j E_\Lambda(\Delta) u_i \cdot K_{ij}],$$

with  $K_{ij}$  trace class and satisfying the decay estimate (4.45). The canonical representation of  $K_{ij}$  (where we write  $j$  for  $j_n$ ) is

$$\tilde{K}(n)_{ij} = \sum_l \mu_l^{(ij)} |\phi_l^{(ij)}\rangle \langle \psi_l^{(ij)}|$$

where  $(\phi_l^{(ij)})_l, (\psi_l^{(ij)})_l$  are orthonormal families and  $\sum_l |\mu_l^{(ij)}| < +\infty$ . Inserting this into (4.46), we obtain

$$\begin{aligned}
\sum_{i,j \in \tilde{\Lambda}} Tr [u_j E_\Lambda(\Delta) u_i \cdot K_{ij}] &\leq \sum_{i,j \in \tilde{\Lambda}} \sum_l \mu_l^{(ij)} \langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_i \phi_l^{(ij)} \rangle \\
&\leq \sum_{i,j \in \tilde{\Lambda}} \sum_l \mu_l^{(ij)} \left\{ \langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(ij)} \rangle + \right. \\
&\quad \left. \langle \phi_l^{(ij)}, u_i E_\Lambda(\Delta) u_i \phi_l^{(ij)} \rangle \right\}.
\end{aligned}
\tag{4.47}$$

It follows from (4.37) that the expectation of the matrix elements in (4.47) satisfy the following bound

$$(4.48) \quad \mathbb{E} \left\{ \langle \psi_l^{(ij)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(ij)} \rangle \right\} \leq 8|\Delta|.$$

Returning to (4.44), we obtain

$$(4.49) \quad Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \leq K_0 C(u, m) |\Delta| |\Lambda|.$$

3. As for the first term on the right in (4.39), we use the fundamental assumption (4.34). As in [25], we will use the spectral projector  $E_0(\tilde{\Delta})$  of  $H_0^\Lambda$  in order to control the trace. We have

$$\begin{aligned}
 \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) &\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \right\} \\
 &\leq \frac{1}{C(\tilde{\Delta}, u)} \left\{ \text{Tr} E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \right. \\
 (4.50) \qquad \qquad \qquad &\left. - \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \right\}.
 \end{aligned}$$

We estimate the second term on the right in (4.50). Using the Hölder inequality for trace norms, we have, for any  $\kappa_0 > 0$ ,

$$\begin{aligned}
 &|\text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \\
 &\leq \|E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c)\|_2 \|\tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_2 \\
 &\leq \frac{1}{2\kappa_0} \text{Tr} E_0^\Lambda(\tilde{\Delta}^c) E_\Lambda(\Delta) + \frac{\kappa_0}{2} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta). \\
 (4.51)
 \end{aligned}$$

We next estimate the second term on the right in (4.51). Let  $D_0$  be a finite constant so that  $\tilde{V}_\Lambda^2 \leq D_0 \tilde{V}_\Lambda$ . Using this, we find that for any  $\kappa_1 > 0$ ,

$$\begin{aligned}
 &\text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) \\
 &\leq D_0 \|E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda\|_2 \|E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_2 \\
 &\leq \frac{D_0 \kappa_1}{2} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) + \frac{D_0}{2\kappa_1} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}).
 \end{aligned}$$

We choose  $\kappa_1 = 1/D_0 > 0$  so that  $(1 - D_0 \kappa_1/2) = 1/2$ . Consequently, we obtain

$$(4.52) \quad \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda^2 E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) \leq D_0^2 \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}).$$

Inserting this into (4.51), we find

$$(4.53) \quad |\text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \leq \frac{1}{2\kappa_0} \text{Tr} E_0^\Lambda(\tilde{\Delta}^c) E_\Lambda(\Delta) + \frac{\kappa_0 D_0^2}{2} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}).$$

As a consequence of (4.53), we obtain for the first term on the right in (4.39),

$$\begin{aligned}
 &\left(1 - \frac{\kappa_0 D_0^2}{2C(\tilde{\Delta}, u)}\right) \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \\
 &\leq \frac{1}{C(\tilde{\Delta}, u)} |\text{Tr} E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| + \frac{1}{2\kappa_0 C(\tilde{\Delta}, u)} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c).
 \end{aligned}$$

We choose  $\kappa_0 = C(\tilde{\Delta}, u)/D_0^2$  so that we have

$$(4.54) \quad \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) \leq \frac{2}{C(\tilde{\Delta}, u)} |\text{Tr} E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| + \frac{D_0^2}{C(\tilde{\Delta}, u)^2} \text{Tr} E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c).$$

As for the first term on the right in (4.54), we use Hölder's inequality and write

$$\begin{aligned}
& |Tr E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})| \\
& \leq \|E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_2 \|E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\|_2 \\
& \leq \frac{1}{2\sigma} \|E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta)\|_2^2 + \frac{\sigma}{2} \|E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta})\|_2^2 \\
(4.55) \quad & \leq \frac{1}{2\sigma} Tr E_0^\Lambda(\tilde{\Delta}) E_\Lambda(\Delta) + \frac{\sigma}{2} Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}),
\end{aligned}$$

for any constant  $\sigma > 0$ . In light of the coefficient in (4.54), we choose  $\sigma = 2/C(\tilde{\Delta}, u)$  and obtain from (4.54) and (4.55),

$$\begin{aligned}
Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}) & \leq \frac{4}{C(\tilde{\Delta}, u)^2} Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\
(4.56) \quad & + \frac{2D_0^2}{C(\tilde{\Delta}, u)^2} Tr E_\Lambda(\Delta) E_0^\Lambda(\tilde{\Delta}^c).
\end{aligned}$$

The second term on the right in (4.56) is bounded above as in (4.47) and (4.49).

4. We estimate the first term on the right in the last line of (4.56). Let  $f_\Delta \in C_0^\infty(\mathbb{R})$  be a smooth, compactly-supported, nonnegative function  $0 \leq f \leq 1$ , with  $f_\Delta \chi_\Delta = \chi_\Delta$ , where  $\chi_\Delta$  is the characteristic function on  $\Delta$ . Note that we can take  $|\text{supp } f| \sim 1$  so that the derivatives of  $f$  are order one. By positivity, we have the bound

$$\begin{aligned}
& Tr E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \\
& = Tr E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \\
(4.57) \quad & \leq Tr E_\Lambda(\Delta) \tilde{V}_\Lambda f_\Delta(H_0^\Lambda) \tilde{V}_\Lambda E_\Lambda(\Delta).
\end{aligned}$$

Recall that  $\chi_j$  is a compactly-supported function so that  $u_j \chi_j = u_j$ . Upon expanding the potential  $\tilde{V}_\Lambda$ , the term on the right in (4.57) is

$$(4.58) \quad \sum_{j,k \in \tilde{\Lambda}} Tr u_k E_\Lambda(\Delta) u_j \cdot \chi_j f_\Delta(H_0^\Lambda) \chi_k.$$

The operator  $\chi_j f_\Delta(H_0^\Lambda) \chi_k$  is a nonrandom, trace class operator. As with the operator  $K_{ij}$  in (4.44), it admits a canonical representation

$$(4.59) \quad \chi_j f_\Delta(H_0^\Lambda) \chi_k = \sum_l \lambda_l^{(jk)} |\phi_l^{(jk)}\rangle \langle \psi_l^{(jk)}|,$$

for orthonormal functions  $\phi_l^{(jk)}$  and  $\psi_l^{(jk)}$ . This operator also satisfies a decay estimate of the type

$$(4.60) \quad \|\chi_j f_\Delta(H_0^\Lambda) \chi_k\|_1 \leq C_N(f) (1 + \|k - j\|^2)^{-N},$$

for any  $N \in \mathbb{N}$  and a finite positive constant depending on  $\|f^{(j)}\|$  independent of  $|\Delta|$ . This can be proved using the Helffer-Sjöstrand formula, see, for example, [64].

Expanding the trace in (4.58) as in (4.47), we can bound (4.58) from above by

$$\begin{aligned}
 \text{Tr} E_0^\Lambda(\tilde{\Delta}) \tilde{V}_\Lambda E_\Lambda(\Delta) \tilde{V}_\Lambda E_0^\Lambda(\tilde{\Delta}) &= \sum_l \sum_{j,k \in \bar{\Lambda}} \lambda_l^{(jk)} \langle \psi_l^{(jk)}, u_j E_\Lambda(\Delta) u_k \phi_l^{(jk)} \rangle \\
 &\leq \sum_l \sum_{j,k \in \bar{\Lambda}} \lambda_l^{(jk)} \{ \langle \psi_l^{(jk)}, u_j E_\Lambda(\Delta) u_j \psi_l^{(jk)} \rangle \\
 (4.61) \quad &+ \langle \phi_l^{(jk)}, u_k E_\Lambda(\Delta) u_k \phi_l^{(jk)} \rangle \}.
 \end{aligned}$$

It is now clear that each matrix element can be estimated as in part 3 applying (4.37). This completes the proof.  $\square$

## 5. Resolvent Estimates and Multiscale Analysis

**5.1. Introduction.** In chapter 3 we derived a necessary condition for the absence of singular and absolutely continuous spectrum for a one-parameter perturbation of an operator. We showed that the models discussed in chapter 1 fit into this framework if we consider the variation of one-coupling constant while holding the others fixed. In order to apply this theory, we must show that our family of Hamiltonians satisfy the condition (A2) of section 3.4 for Lebesgue almost-every energy in an interval near the band edges. Since the perturbed and unperturbed Hamiltonians have structurally the same form (they differ only in the realization of the perturbation), we must verify that

$$(5.1) \quad \sup_{\epsilon \neq 0} \|R(E + i\epsilon)u\| \leq C(E),$$

for some finite constant and for almost-every energy in an interval near  $\tilde{B}_-$  and  $\tilde{B}_+$ .

This is an estimate on the infinite-volume Hamiltonian which we expect to have dense pure point spectrum in the energy interval we are considering. Thus, we cannot expect such an estimate to hold at every energy. What saves the situation is the fact that we are dealing with a random family of operators. Since the probability that any fixed energy  $E$  is an eigenvalue of  $H_\omega$  is zero, we can expect that a bound like (5.1) might hold with probability one.

How can we expect to prove such an infinite-volume estimate? Following the notion of thermodynamic limit in statistical mechanics, Fröhlich and Spencer [56] proved an almost-sure fixed energy resolvent bound for the multidimensional lattice Anderson model through an iterative process starting with estimates on finite-volume Hamiltonians. This technique is called *multiscale analysis* (MSA). The work of Fröhlich and Spencer opened the way to proving localization for multidimensional models. The technique was refined in the work of von Dreifus and Klein [131], Spencer [120], and extended to continuous models in Holden and Martinelli [101], Combes and Hislop [25], Barbaroux, Combes, and Hislop [8], and Kirsch, Stollmann, and Stolz [86].

In this chapter, we provide a general framework for the fixed energy multiscale analysis. We show that infinite-volume estimates such as (5.1) can be derived from two basic estimates on the finite-volume Hamiltonians. These two estimates are the Wegner estimate, proved in chapter 4, and the initial length scale estimate, referred to as  $[H1](\ell_0, \gamma_0)$ . The estimate  $[H1](\ell_0, \gamma_0)$  on the finite-volume Hamiltonians is proved for additive perturbations in chapter 6.

We also show how to improve these estimates in order to obtain exponential localization. The details of the inductive step in the multiscale analysis are presented in chapter 6. Finally, we must show how to turn these fixed-energy estimates for the resolvent with respect to one coupling coefficient into an almost sure estimate with respect to the realizations of the potential. We present this probabilistic argument in the last section of this chapter.

**5.2. Some Probability Theory: The Borel-Cantelli Lemmas.** We will use repeatedly the two Borel-Cantelli Lemmas that we recall here. Let  $\{A_n\}$  be a family of subsets of a given set  $X$ . We define two subsets of  $X$ ,  $\limsup A_n$  and the  $\liminf A_n$ , by

$$(5.2) \quad \liminf_n A_n \equiv \bigcup_{k=1}^{\infty} \left( \bigcap_{j \geq k} A_j \right),$$

and

$$(5.3) \quad \limsup_n A_n \equiv \bigcap_{k=1}^{\infty} \left( \bigcup_{j \geq k} A_j \right).$$

Let us recall that  $x \in \liminf_n A_n$  if and only if  $x$  belongs to all but finitely many of the sets  $A_n$ . Similarly,  $x \in \limsup_n A_n$  if and only if  $x$  belongs infinitely many of the sets  $A_n$ . As subsets of  $X$ , it is clear that  $\liminf_n A_n \subset \limsup_n A_n$ .

**THEOREM 5.1.** (*The First Borel-Cantelli Lemma*) Let  $\{A_n \in \mathcal{F}\}$  be a countable family of measurable subsets of  $\Omega$  such that  $\sum_n \mathbb{P}(A_n) < \infty$ . Then, the measure of the set of points that lie in infinitely-many of the sets  $A_n$  (that is, which occur infinitely-often) is zero. That is,  $\mathbb{P}\{\limsup_n A_n\} = 0$ .

**THEOREM 5.2.** (*The Second Borel-Cantelli Lemma*) Let  $\{A_n \in \mathcal{F}\}$  be a countable family of measurable subsets of  $\Omega$  which are independent and for which  $\sum_n \mathbb{P}(A_n) = \infty$ . Then, the measure of the points that lie in infinitely-many of the sets  $A_n$  (that is, which occur infinitely-often) is one. That is,  $\mathbb{P}\{\limsup_n A_n\} = 1$ .

**5.3. The Geometric Resolvent Equation.** The basic idea which makes the multiscale analysis practical for continuous models (see [101] for a presentation using boundary conditions) is to use geometric methods for analysis of resolvents which have been used extensively in semiclassical analysis (see, for example, [75]). We review the notation that we have used in previous chapters. Let  $\Lambda_l$  be a cube of side  $l$  centered at the origin

$$(5.4) \quad \Lambda_l = \{x \in \mathbb{R}^d \mid |x_i| < l/2, \ i = 1, \dots, d\}.$$

We fix some  $\delta > 0$ . In practice, this  $\delta$  is determined by  $\text{diam}(\text{supp } u)$ . Let  $\Lambda_{\ell, \delta} \subset \Lambda_l$  be the subcube defined by

$$(5.5) \quad \Lambda_{\ell, \delta} \equiv \{x \in \Lambda_l \mid \text{dist}(x, \partial\Lambda_l) > \delta\}.$$

Note that  $\delta$  is independent of  $l$ . Let  $\Lambda_l(x)$  be a cube centered at  $x \in \mathbb{R}^d$ . We need smoothed characteristic functions  $\chi_l$  such that  $\chi_l \geq 0$ ,  $\chi_l|_{\Lambda_{\ell, \delta}} = 1$ , and  $\text{supp}(\nabla\chi_l) \subset \Lambda_l \setminus \Lambda_{\ell, \delta}$ . When working with length scales  $l_k$ , we write  $\chi_k$  and  $\Lambda_k(x)$ , for notational convenience. We also define  $\Lambda_\delta$  to be the subset of  $\Lambda$  defined as in (5.5).

We treat the additive case in this chapter. We consider a random family  $H_\omega = H_0 + V_\omega$  of self-adjoint operators with a common dense domain  $D_0 \subset L^2(\mathbb{R}^d)$ . For any regular region  $\Lambda \subset \mathbb{R}^d$ , we define  $H_\Lambda$  to be the local Hamiltonian associated with  $\Lambda$  by either

$$(5.6) \quad H_\Lambda = H_0 + V_\Lambda, \text{ on } L^2(\mathbb{R}^d),$$

where

$$(5.7) \quad V_\Lambda \equiv V|_\Lambda,$$

or by

$$(5.8) \quad H_\Lambda = (H_0 + V_\omega)|_\Lambda, \text{ on } L^2(\Lambda),$$

with self-adjoint boundary conditions on  $\partial\Lambda$ . It is often convenient to use one form or the other of the local Hamiltonian. The MSA is the same regardless of the form used except that the equations hold on  $L^2(\mathbb{R}^d)$  in the first case, or  $L^2(\Lambda)$  in the second case. We will write  $\mathcal{H}$  for either space.

Let  $R_\Lambda(z) = (H_\Lambda - z)^{-1}$  be the resolvent of  $H_\Lambda$  on  $\mathcal{H}$ . For two regions  $\Lambda \subset \Lambda' \subset \mathbb{R}^d$ , we can compare the resolvents  $R_\Lambda(z)$  and  $R_{\Lambda'}(z)$  associated with the local Hamiltonians as follows. Let  $W_\Lambda$  be the first-order differential operator defined by

$$(5.9) \quad W_\Lambda = [H_0, \chi_\Lambda].$$

We recall that  $\text{supp}(\nabla\chi_\Lambda) \subset \Lambda \setminus \Lambda_\delta$  so that  $W_\Lambda$  is localized near  $\partial\Lambda$ . To compare the two resolvents,  $R_\Lambda(z)$  and  $R_{\Lambda'}(z)$ , we write

$$(5.10) \quad \chi_\Lambda(H_{\Lambda'} - z) = (H_\Lambda - z)\chi_\Lambda - W_\Lambda,$$

from which it follows that

$$(5.11) \quad \chi_\Lambda R_{\Lambda'}(z) = R_\Lambda(z)\chi_\Lambda + R_\Lambda(z)W(\chi_\Lambda)R_{\Lambda'}(z),$$

acting as an operator on  $L^2(\mathbb{R}^d)$  or on  $L^2(\Lambda')$ , where  $\chi_\Lambda$  serves as the natural injection. By taking the adjoint of (5.11) and replacing  $\bar{z}$  by  $z$ , we can also write this equation as

$$(5.12) \quad R_{\Lambda'}(z)\chi_\Lambda = \chi_\Lambda R_\Lambda(z) - R_{\Lambda'}(z)W(\chi_\Lambda)R_\Lambda(z).$$

We used the fact that  $V_{\Lambda'}|_\Lambda = V_\Lambda$ . This last relation  $V_{\Lambda'}|_\Lambda = V_\Lambda$  does not always hold, especially if the support of the single-site potential extends beyond the unit cube  $\Lambda_1(0)$ . The difference, however, can be easily incorporated into the MSA. We refer to either of the geometric resolvent equations, (5.11) or (5.12), as the GRE.

**5.4. The ELF Theorem.** The name of this theorem was inspired by the French oil company ELF, since it is what makes the MSA work. We also name the ingredients  $\epsilon$ , length-scale  $\ell$ , and  $f$ . This is the basic theorem that allows us to control the iteration of resolvent estimates for finite-volume Hamiltonians over several increasing length scales. We need the following:

- (i)  $\{\epsilon_n\}$ , a monotone decreasing sequence of positive numbers with  $\lim_{n \rightarrow \infty} \epsilon_n = 0$ ;
- (ii)  $\{\ell_k\}$ , a monotone increasing length scale:  $\ell_k \uparrow \infty$ ;
- (iii)  $f > 0$ , is a non-decreasing function with  $(1/f) \in L^2_{\text{loc}}(\mathbb{R})$ .

For example, we can take  $f(s) = \langle s \rangle^\alpha$ , for any  $\alpha < 1/2$ .

We need two basic hypotheses on the local Hamiltonians  $H_k \equiv H_{\Lambda_{\ell_k}}$ , where  $\ell_k$  is a length scale as in (ii) above, and their resolvents  $R_k(z) \equiv (H_k - z)^{-1}$ :

**[H1].** For each  $k \in \mathbb{N}$ , there exists a probability  $\tilde{P}_k$ , with  $(1 - \tilde{P}_k) \in \ell^1(\mathbb{N})$ , such that

$$(5.13) \quad \mathbb{P}\{\sup_{\epsilon > 0} \|W(\chi_k)R_k(\epsilon)\chi_{k-1}\| \leq \epsilon_{k+1}^2\} \geq \tilde{P}_k.$$

Note that this requires estimates over multiple length scales, and is not practical from a computational point of view. We will improve this below. The second hypothesis is a Wegner estimate on each length scale:

**[H2].** For some sequences  $\{l_k\}$  and  $\{\epsilon_k\}$ , as in [H1], we have  $\forall k \in \mathbb{N}$ ,

$$(5.14) \quad P_k \equiv \mathbb{P}\{\text{dist}(\sigma(H_k), E) < f(\epsilon_k)\} \in \ell^1(\mathbb{N}).$$

Given these two hypotheses, the basic result is

**THEOREM 5.3.** Let  $\{l_k\}$ ,  $\{\epsilon_k\}$ , and  $f$  be as above and suppose  $c$  is a function of compact support. If [H1] and [H2] hold, then for  $\mathbb{P}$ -almost every  $\omega$  there exists a constant  $0 < d_\omega < \infty$  such that

$$(5.15) \quad \sup_{\epsilon > 0} \|(H - E - i\epsilon)^{-1}c\| \leq d_\omega \|c\|_\infty.$$

**Proof.** Since  $c$  has compact support, there is a  $k_0$  so that for all  $k > k_0$ , we have  $\chi_k c = c$ . We choose  $k$  large enough so  $\chi_{k-1} c = c$ . Let  $R(\epsilon_k) \equiv (H - E - i\epsilon_k)^{-1}$ . Applying the geometric resolvent equation twice, we obtain,

$$(5.16) \quad \begin{aligned} R(\epsilon_{k+1})R(\epsilon_k)c &= R(\epsilon_{k+1})\{\chi_k R_k(\epsilon_k) + R(\epsilon_k)W_k R_k(\epsilon_k)\}c \\ &= \chi_k R_k(\epsilon_{k+1})R_k(\epsilon_k)c + R(\epsilon_{k+1})W_k R_k(\epsilon_{k+1})R_k(\epsilon_k)c \\ &\quad + R(\epsilon_{k+1})R(\epsilon_k)W_k R_k(\epsilon_k)c \\ &= I + II + III. \end{aligned}$$

By hypothesis [H1],

$$(5.17) \quad \|III\| \leq \|c\|_\infty \epsilon_{k+1}^2 \cdot \left(\frac{1}{\epsilon_{k+1}}\right) \left(\frac{1}{\epsilon_k}\right) \leq \|c\|_\infty,$$

with probability  $\geq \tilde{P}_k$ . As for  $II$ , we use the first resolvent formula for  $R_k$  and obtain,

$$(5.18) \quad \begin{aligned} &\|R(\epsilon_{k+1})W_k[R_k(\epsilon_{k+1}) - R_k(\epsilon_k)](\epsilon_{k+1} - \epsilon_k)^{-1}c\| \\ &\leq \frac{1}{\epsilon_{k+1}} \cdot \frac{1}{(\epsilon_k - \epsilon_{k+1})} \cdot (\|W_k R_k(\epsilon_{k+1})c\| + \|W_k R_k(\epsilon_k)c\|). \end{aligned}$$

By choosing a subsequence, if necessary, we assume that  $\epsilon_k > 2\epsilon_{k+1}$ . Then, the bound in (5.18) is less than

$$(5.19) \quad 2 \cdot \frac{1}{\epsilon_{k+1}^2} \cdot \epsilon_{k+1}^2 \cdot \|c\|_\infty \leq 2\|c\|_\infty,$$

with a probability  $\geq \tilde{P}_k$ . As for  $I$ , [H2] implies that

$$(5.20) \quad \|I\| \leq \frac{1}{f(\epsilon_{k+1})} \frac{1}{f(\epsilon_k)} \|c\|_\infty \leq \frac{1}{f(\epsilon_k)^2} \|c\|_\infty,$$

with a probability  $\geq 1 - P_k$ . Hence, the event

$$(5.21) \quad B_k \equiv \left\{ \omega \in \Omega \mid \|R(\epsilon_{k+1})R(\epsilon_k)c\| \leq \frac{2\|c\|_\infty}{f(\epsilon_k)^2} \right\},$$

occurs with probability

$$(5.22) \quad \mathbb{P}(B_k) \geq \tilde{P}_k - P_k,$$

or

$$(5.23) \quad \mathbb{P}(B_k^c) \leq 1 - \tilde{P}_k + P_k.$$

But, by our assumptions,

$$(5.24) \quad \sum_k \mathbb{P}(B_k^c) \leq \sum_k (1 - \tilde{P}_k) + \sum_k P_k < \infty.$$

By the first Borel Cantelli lemma 5.1, it follows that  $\mathbb{P}\{\limsup_k B_k^c\} = 0$ . As  $(\limsup_k B_k^c)^c = \liminf_k B_k$ , we have that  $\mathbb{P}(\liminf_k B_k) = 1$ . So for any  $\omega \in \liminf_k B_k$ , there exists a  $k_0(\omega)$ , so that for all  $k > k_0(\omega)$ , the event on the right in (5.21) occurs. Consequently, for  $k > k_0(\omega)$ , we write

$$(5.25) \quad \begin{aligned} R(\epsilon_k) &= R(\epsilon_{k_0}) + \sum_{l=k_0}^{k-1} (R(\epsilon_{l+1}) - R(\epsilon_l)) \\ &= R(\epsilon_{k_0}) + \sum_{l=k_0}^{k-1} (\epsilon_{l+1} - \epsilon_l) R(\epsilon_{l+1}) R(\epsilon_l). \end{aligned}$$

Since  $\omega \in B_l$ , we use the estimate on the right in (5.21), and obtain

$$(5.26) \quad \|R(\epsilon_k)c\| \leq \|R(\epsilon_{k_0})c\| + \sum_{l=k_0}^{k-1} \left( 2\|c\|_\infty \frac{(\epsilon_l - \epsilon_{l+1})}{f(\epsilon_l)^2} \right).$$

The sum is finite and bounded by a constant due to the fact that  $(1/f)^2 \in L^1(\mathbb{R})$ . As a consequence, for each  $\omega \in \liminf_k B_k$ , a set of full measure, there is a finite constant  $C(c, \omega) > 0$  so that

$$(5.27) \quad \limsup_{k \rightarrow \infty} \|R(\epsilon_k)c\| \leq C(c, \omega) < \infty,$$

with probability one.  $\square$

**5.5. Reduction to an Initial Length Scale Estimate.** We next want to show how to simplify [H1] by reducing it to an initial estimate at one length scale. We have two new hypotheses modifying [H1]-[H2].

[H1]( $\gamma_0, l_0$ ). *For some  $\gamma_0 > 0$  and length  $l_0 \gg 1$ , such that  $\gamma_0 l_0 \gg 1$ , there exists an exponent  $\xi > 2d$  such that*

$$(5.28) \quad \mathbb{P} \left\{ \sup_{\epsilon > 0} \|W(\chi_{l_0})R_{\Lambda_{l_0}}(E + i\epsilon)\chi_{l_0/3}\| \leq e^{-\gamma_0 \epsilon l_0} \right\} \geq 1 - l_0^{-\xi}.$$

The next version of [H2] is satisfied by the models for which a Wegner's estimate can be proved.

[W] *There exists constants  $0 < C_W < \infty$ , exponent  $1 \leq \tau < \text{infy}$ , and  $0 < q \leq 1$ , such that for all sufficiently large  $\Lambda$ , and all  $\eta > 0$ ,*

$$(5.29) \quad \mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E) < \eta\} \leq C_W |\Lambda|^\tau \eta^q.$$

The main technique result, which requires the multiscale analysis, consists in proving

$$[\text{H1}](\gamma_0, l_0) + [\text{W}] \Rightarrow [\text{H1}].$$

This is given in the following lemma, whose proof we give in chapter 8.

LEMMA 5.4. *Let  $\gamma_0 > 0$  and assume [W]. There exists a minimum length scale  $l^* = l^*(\gamma_0, C_W, \tau, q, d)$ , such that if  $[\text{H1}](\gamma_0, l_0)$  holds for  $l_0 > l^*$ , then for length scales  $l_k \equiv l_0^{(3/2)^k}$ ,  $\xi > 2d$ ,  $\exists 0 < \kappa < \infty$  such that for each  $k$*

$$P\{\|W_{l_k} R_{l_k}(E + i\epsilon)\chi_{l_k/3}\| \leq e^{-\kappa\gamma_0 l_k}\} \geq 1 - l_k^{-\xi}.$$

A consequence of this lemma is an improvement in the hypotheses of Theorem 5.3.

THEOREM 5.5. *Let  $\gamma_0 > 0$  and assume [W]. There exists a minimum length scale  $l^* = l^*(\gamma_0, C_W, \tau, q, d)$ , such that if  $[\text{H1}](\gamma_0, l_0)$  holds for  $l_0 > l^*$ , and if  $c$  is any function of compact support, then for almost every  $\omega$ , there exists a finite constant  $d_\omega(c) < \infty$  such that*

$$(5.30) \quad \sup_{\epsilon > 0} \|(H - E - i\epsilon)^{-1}c\| \leq d_\omega \delta(c).$$

**Proof.** From Lemma 5.4, we have  $[\text{H1}]$  in a strong form with:

$$\epsilon_{k+1} = (l_k)^{-(3+\delta_1)\tau d}, \quad 0 < \delta_1 \ll 1;$$

$$l_k = l_0^{(3/2)^k};$$

$$\tilde{P}_k = 1 - l_k^{-\xi}, \quad \xi > 2d.$$

We apply Theorem 5.1 with  $f(\epsilon) = \epsilon^{1/2-\delta}$ ,  $0 < \delta \ll \delta_1$ . Then, we see that  $1 - \tilde{P}_k \in l^1$ ,  $f^{-1} \in L_{\text{loc}}^2$  and

$$|\Lambda_k|^\tau f(\epsilon_k) = l_k^{-\sigma}$$

for some  $\sigma > 0$ . This verifies the assumptions of the theorem.  $\square$

**5.6. Exponential Decay Estimates.** We now obtain exponential decay estimates on the infinite-volume resolvents with good probability. We observe that Theorem 5.5 suffices to prove localization by condition (A2), Theorem 3.4, and the results of the next section. If we want exponential decay of the eigenfunctions, we need stronger estimates on the Green's function. In the case that  $\text{supp } u$  is not compact, we can only get polynomial decay estimates.

**THEOREM 5.6.** *Let  $\chi_x$  be the characteristic function of a unit cube centered at  $x$ . Then, under the assumptions of Theorem 5.5, we have*

(1) *If  $c$  has compact support, then with probability one there exists a constant  $0 < d_\omega < \infty$  such that for all large  $\|x\|$ ,*

$$(5.31) \quad \sup_{\epsilon > 0} \|\chi_x R(E + i\epsilon)c\| \leq d_\omega \|c\|_\infty e^{-\frac{1}{6\sqrt{d}}\gamma_0\|x\|}.$$

(2) *If  $c$  satisfies  $|c(x)| < \delta(c)(1 + \|x\|)^{-\beta}$ , with  $\beta > 9d\tau/2$ , then with probability one there exists a constant  $0 < d_\omega < \infty$  such that for  $\|x\|$  large,*

$$(5.32) \quad \sup_{\epsilon > 0} \|\chi_x R(E + i\epsilon)c\| \leq d_\omega \delta(c) \|x\|^{-\beta/2}$$

**Proof.** We take  $\|x\|$  large so  $3l_{k-1} < |x| < 3l_k$  for some  $k$ . Applying the GRE (5.11) to  $R$  and  $R_{8l_k}$ , we obtain

$$(5.33) \quad \chi_x Rc = \chi_x R_{8l_k} \chi_{8l_k} c + \chi_x R_{8l_k} W_{8l_k} Rc.$$

Let  $c_k \equiv c|_{\Lambda_{l_{k-1}}}$ . We estimate (5.33) by

$$(5.34) \quad \|\chi_x Rc\| \leq \|\chi_x R_{8l_k} c_k\| + \|\chi_x R_{8l_k} \chi_{8l_k} (c - c_k)\| + c_0 \|\chi_x R_{8l_k} W_{8l_k}\|,$$

where we used Theorem 5.3 to bound  $\|Rc\|$  with probability one. We now bound the first and third terms using a variation of Lemma 5.4. When  $\text{supp } c$  is non-compact one must use the decay of  $c$  to estimate the second term.  $\square$

Theorem 5.6 reduces the proof of exponential localization to the verification of [H1]( $\gamma_0, l_0$ ) and [W]. The proof of [W] is given in chapter 4. For the additive models of chapter 1, the proof of [H1]( $\gamma_0, l_0$ ) is presented in chapter 6.

**5.7. Probabilistic Estimates for Localization.** Let us fix a closed energy interval  $I$  for which we can verify conditions [H1]( $\gamma_0, \ell_0$ ) and [W] for the finite-volume Hamiltonians. According to Theorem 5.3, we can then verify the key hypothesis of (A2) on the perturbation of one-parameter families at each energy  $E \in I$ . Recall that we consider fixing all but one random variable, say  $\lambda_0$ , and study the spectrum of the resulting one-parameter family of operators. The main result of Theorem 3.4 states that under the condition (5.28) on the unperturbed Hamiltonian, the spectrum of the one-parameter family in the interval  $I$  is dense pure point.

What we have proved is the following: for  $\mathcal{P}$ -almost every configuration  $\omega$ , there is a set  $J_\omega$  of values of the random variable  $\lambda_0$  for which the resulting one-parameter family of operators  $H_{\omega_0, \omega^\perp}$  has pure point spectrum in the interval  $I$ . The problem is that the set of  $\lambda_0$  values  $J_\omega$  depends on the initial configuration  $\omega$  chosen. Another way to think of this is as follows. Consider the configurations  $\hat{\omega}$  having the property that  $\lambda_0(\hat{\omega}) = 0$ . There is a set of values  $\lambda_0$  for which the configurations  $\omega = (\hat{\omega}, \lambda_0)$  for which (5.28) holds at almost-every energy in  $I$ . We will show that configurations of this form have full measure. This will imply the result. Hence, we need to turn this result into a result for almost-every configuration  $\omega$ .

In order to do this, we introduce a device of extending the probability space  $(\Omega, \mathcal{P})$  in the following manner. Let us define subsets  $\Omega_0 \subset \Omega$  and  $\Omega_1 \subset \Omega$  by

$$(5.35) \quad \Omega_0 \equiv \{\omega \mid \sup_{\epsilon > 0} \|(H - E - i\epsilon)^{-1}c\| \leq \delta(c)C_{\omega, E} < \infty \forall E \in I\},$$

and for any function  $c$  as in Theorem 5.6, and

$$(5.36) \quad \Omega_1 \equiv \{\omega \mid H_\omega \text{ is pure point in } I \text{ with finitely degenerate eigenvalues}\}.$$

The weak measurability of the spectral projectors show that the set  $\Omega_1$  is measurable. Our goal is to show that  $\mathbb{P}(\Omega_1) = 1$ . Let  $\omega \in \Omega_0$  and consider the variation of the random variable  $\lambda_0$ . As in chapter 3, we write

$$(5.37) \quad H_{\lambda,\omega} = H_\omega + \lambda u.$$

By Theorem 3.2, there exists a set  $S_\omega \subset \text{supp } g$ , such that  $\int_{S_\omega} g(\lambda) d\lambda = 1$ , for which the following holds:

$$(5.38) \quad \lambda \in S_\omega \rightarrow \{\tilde{H}_{\lambda,\omega} \text{ has pure point spectrum in } I \\ \text{with finite multiplicities}\}.$$

Here we write  $\tilde{H}_{\lambda,\omega}$  for the restriction of the operator in (5.37) to the cyclic subspace generated by the action of  $f(H)$  on the set  $\{u^{1/2}\phi \mid \phi \in L^2(\mathbb{R}^d)\}$ . In our models, this cyclic subspace is the entire Hilbert spaces, so we will drop the tilde from the notation. We define a third subset of  $\Omega$  by

$$(5.39) \quad \Omega_2 \equiv \{(\omega + \lambda\delta_{0i}) \mid \omega \in \Omega_0 \text{ and } \lambda \in S_\omega\}.$$

The notation  $\omega + \lambda\delta_{0i}$  means  $\lambda_j(\omega + \lambda\delta_{0i}) = \lambda_j(\omega)$ , for  $j \neq 0$ , and  $\lambda_0(\omega + \lambda\delta_{0i}) = \lambda_0(\omega) + \lambda$ . Note that  $\Omega_2 \subset \Omega_1$ .

We now consider the quotient of each of the subspaces  $\Omega_i$  by  $\mathbb{R}$  along the zero site. We denote by  $\hat{\omega}$  those configurations in  $\Omega$  for which  $\lambda_0(\hat{\omega}) = 0$ . We define the sets

$$(5.40) \quad \tilde{\Omega}_i \equiv \{\hat{\omega} \mid \exists \mu \in \mathbb{R} \text{ so that } \hat{\omega} + \mu\delta_{0i} \in \Omega_i\}.$$

We note an important identity:  $\tilde{\Omega}_0 = \Omega_2/\mathbb{R} \subset \tilde{\Omega}_1$ . Given an element  $\tilde{\omega} \in \tilde{\Omega}_i$ , we define a section  $S(\tilde{\omega} \mid \Omega_i)$  by

$$(5.41) \quad S(\tilde{\omega} \mid \Omega_i) \equiv \{\lambda \in \mathbb{R} \mid (\hat{\omega}, \lambda) \in \Omega_i\}.$$

These subsets of  $\mathbb{R}$  are measurable with respect to the measure  $g(\lambda)d\lambda$  and have full measure.

We now compute the probability measure of the set  $\Omega_1$  using Fubini's Theorem. For this, we note that  $\Omega_1 = \tilde{\Omega}_1 \times S(\tilde{\omega} \mid \Omega_1)$ , that  $\tilde{\Omega}_0 \subset \tilde{\Omega}_1$ , with  $\mathbb{P}(\tilde{\Omega}_0) = 1$ , and that  $S(\tilde{\omega} \mid \Omega_1) \subset S(\tilde{\omega} \mid \Omega_2)$ . Using these relations, we can write

$$(5.42) \quad \begin{aligned} \int_{\Omega_1} d\mathbb{P} &= \int_{\tilde{\Omega}_1} d\tilde{\mathbb{P}} \int_{S(\tilde{\omega} \mid \Omega_1)} g(\lambda) d\lambda \\ &\geq \int_{\tilde{\Omega}_0} d\tilde{\mathbb{P}} \int_{S(\tilde{\omega} \mid \Omega_2)} g(\lambda) d\lambda \\ &= 1. \end{aligned}$$

Given the result on the  $H$ -cyclicity of the set  $\{u^{1/2}\phi \mid \phi \in L^2(\mathbb{R}^d)\}$ , we have proven that the set  $\Omega_1$  has full probability measure.

**5.8. Probabilistic Estimates for Exponential Decay.** We show how to use the refined decay estimates of Theorem 5.6 to prove exponential decay of the eigenfunctions for those  $\omega$  exhibiting localization. The argument is similar to the one presented in subsection 5.7. For the simplicity of presentation, we consider only the case when the single-site potential  $u$  has compact support.

Let  $\chi_n$  be the characteristic function for the unit cube centered at  $n \in \mathbb{Z}^d$ . We define a function  $f_E(n)$  by

$$(5.43) \quad f_E(n) = e^{-\gamma_E \|n\|}.$$

We define another set of configurations  $\Omega_0 \subset \Omega$  as follows.

$$(5.44) \quad \begin{aligned} \Omega_0 &\equiv \{ \omega \mid \sup_{\epsilon > 0} \|\chi_n(H - E - i\epsilon)^{-1}c\| \leq \delta(c)C_{\omega,E}f_E(n) \\ &\text{for a. e. } E \in I(\delta_0) \text{ and for all } n \text{ large } \}, \end{aligned}$$

and for any function  $c$  as in Theorem 5.6. It follows from Theorem 5.6 that  $\mathbb{P}(\Omega_0) = 1$ .

We now use the one-parameter theory to show that the eigenfunctions of  $H_\omega$  decay exponentially. We use the notation  $H_{\lambda,\omega}$  as in subsection 5.6. For any  $\omega \in \Omega_0$ , Theorem 3.4 implies that there exists a subset  $S(\omega) \subset \mathbb{R}$ , of full measure, such that  $\lambda \in S(\omega)$  implies that  $H_{\lambda,\omega}$  has pure point spectrum in  $I(\delta_0)$ . For any  $E$  which is an eigenvalue of  $H_{\lambda,\omega}$  with eigenfunction  $\psi_E$ , it follows from the analysis in chapter 3 that

$$(5.45) \quad -s - \lim_{\epsilon \rightarrow 0} \lambda(H_\omega - E - i\epsilon)^{-1}V_\omega\psi_E = \psi_E.$$

Since  $\omega \in \Omega_0$ , we obtain for any  $n \in \mathbb{Z}^d$ ,

$$(5.46) \quad \|\chi_n\psi_E\| \leq C_\omega(E)f_E(n),$$

where we used the normalization  $\|\psi_E\| = 1$ .

Let  $P_\omega^E$  be the spectral projector for  $H_\omega$  onto the eigenspace for the eigenvalue  $E$ . We now define another subset  $\Omega_1 \subset \Omega_0$  by

$$(5.47) \quad \begin{aligned} \Omega_1 &\equiv \{ \omega \in \Omega \mid \sigma(H_\omega) \cap I(\delta_0) \text{ is pure point and for any } E \in \sigma(H_\omega) \cap I(\delta_0), \\ &\text{and for any } \delta > 0, \sum_n \|n\|^{-1-\delta} f_E(n)^{-1} \|\chi_n P_\omega^E\| < \infty \}. \end{aligned}$$

Since the projector is weakly measurable and bounded, it follows that  $\|\chi_n P_\omega^E\|$  is measurable, which implies that the set  $\Omega_1$  is measurable.

To prove that  $\mathbb{P}(\Omega_1) = 1$ , we proceed as above. We define another set  $\Omega_2 \subset \Omega$  by

$$(5.48) \quad \Omega_2 \equiv \{ \omega + \lambda\delta_{0_i} \mid \omega \in \Omega_0, \lambda \in S(\omega) \},$$

so that  $\Omega_2 \subset \Omega_1$ . We now consider the quotient of each of the subspaces  $\Omega_i$  by  $\mathbb{R}$  along the zero site. We denote by  $\hat{\omega}$  those configurations in  $\Omega$  for which  $\lambda_0(\hat{\omega}) = 0$ . We define the sets

$$(5.49) \quad \tilde{\Omega}_i \equiv \{ \hat{\omega} \mid \exists \mu \in \mathbb{R} \text{ so that } \hat{\omega} + \mu\delta_{0_i} \in \Omega_i \}.$$

By construction, we have  $\tilde{\Omega}_0 = \tilde{\Omega}_2 \subset \tilde{\Omega}_1$ . Given an element  $\tilde{\omega} \in \tilde{\Omega}_i$ , we define a section  $S(\tilde{\omega} \mid \Omega_i)$  by

$$(5.50) \quad S(\tilde{\omega} \mid \Omega_i) \equiv \{ \lambda \in \mathbb{R} \mid (\hat{\omega}, \lambda) \in \Omega_i \}.$$

These subsets of  $\mathbb{R}$  are measurable with respect to the measure  $g(\lambda)d\lambda$  and have full measure.

We can now compute the probability measure of the set  $\Omega_1$  using Fubini's Theorem, as above,

$$\begin{aligned} \int_{\Omega_1} d\mathbb{P} &= \int_{\tilde{\Omega}_1} d\tilde{\mathbb{P}} \int_{S(\tilde{\omega} | \Omega_1)} g(\lambda)d\lambda \\ &\geq \int_{\tilde{\Omega}_2} d\tilde{\mathbb{P}} \int_{S(\tilde{\omega} | \Omega_2)} g(\lambda)d\lambda \\ (5.51) \qquad &= 1. \end{aligned}$$

Hence, for any  $\omega \in \Omega_2$ , and for any  $E \in \sigma(H_\omega) \cap I(\delta_0)$  with eigenfunction  $\psi_E$ , there exists a constant  $C_\omega(E)$  so that for any  $\delta > 0$ ,

$$(5.52) \qquad \|\chi_n \psi_E\| \leq C_\omega(E) f(n) \|n\|^{\delta/2}.$$

Now let  $J_k$  be the characteristic function on the set  $\mathcal{A}_k \equiv \{x \in \mathbb{R}^d \mid k-1 < \|x\| < k, k \in \mathbb{N}\}$ . We define a function  $F$  by

$$(5.53) \qquad F(x) \equiv e^{(\gamma-\epsilon)\|x\|}.$$

We then have

$$\begin{aligned} \int |F(x)\psi_E(x)|^2 &\leq \sum_k \int F(x)^2 J_k(x) |\psi_E(x)|^2 \\ (5.54) \qquad &\leq \sum_k C_\omega(E) F(k+1)^2 k^{d+\delta} f(k)^2, \end{aligned}$$

which is finite. We complete the proof of exponential decay by using subsolution estimates, as in [35], to pass from  $L^2$ -bounds to pointwise bounds.

## 6. Localization for Families of Random Schrödinger Operators

**6.1. Introduction.** We are now in position to prove localization at band-edge energies for random families of Schrödinger operators. In chapter 5, we established two conditions on the local Hamiltonians which will guarantee localization. First, we must establish an initial length-scale estimate for the local Hamiltonians at the energies of interest:

**[H1]**  $(\gamma_0, \ell_0)$ . *For some  $\gamma_0 > 0$ , and for some length scale  $\ell_0 > 0$ , so that  $\gamma_0 \ell_0 \gg 1$ , there exists an exponent  $\xi > 2d$  such that*

$$(6.1) \qquad \mathbb{P}\{\sup_{\epsilon > 0} \|W(\chi_{\ell_0})R_{\Lambda_{\ell_0}}(E+i\epsilon)\chi_{\ell_0/3}\| \leq e^{-\gamma_0 \ell_0}\} \geq 1 - \ell_0^{-\xi}.$$

Second, we must prove a Wegner estimate at those energies:

**[W].** *There exist constants  $0 < C_W < \infty$ ,  $\tau > 1$ , and  $\sigma > 0$ , so that for all  $|\Lambda|$  sufficiently large and for all  $\eta > 0$ ,*

$$(6.2) \qquad \mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E) < \eta\} \leq C_W(|\Lambda|^\tau \eta + e^{-|\Lambda|^\sigma}).$$

We note that for the Anderson and breather models, the probability estimate in the Wegner estimate is much simpler: one can replace the right side of (6.2) by  $C_W \eta |\Lambda|$ .

We already proved [W] in chapter 4. We prove condition [H1] $(\gamma_0, \ell_0)$  in this chapter for additive perturbations. We refer the reader to the literature (for example, [33]) for the case of multiplicative perturbations. Our general approach to

proving this estimate is to study the dependence of the eigenvalues of the local Hamiltonian  $H_\Lambda$  on the random variables in  $\Lambda$ . We will prove that with a good, nonzero probability, the eigenvalues will stay away from a small interval of energies near the band edges  $\tilde{B}_-$  and  $\tilde{B}_+$ . We can then apply the refined version of the Combes-Thomas exponential decay estimate given in [8] in order to establish the exponential decay of the localized resolvent at these energies. We mention that whereas the Wegner estimate holds at all energies in the unperturbed spectral gap  $G = [B_-, B_+]$ , it is the estimate  $[H1](\gamma_0, \ell_0)$  which determines the interval of energies for which we have localization.

**6.2. Eigenvalues of the Local Hamiltonians.** The goal of this section is to prove the hypothesis  $[H1](\gamma_0, \ell_0)$  for finite-volume Hamiltonians corresponding to the additively perturbed models introduced in chapter 1. Let us recall the construction of the finite-volume Hamiltonians. As in previous chapters, we let  $\Lambda \subset \mathbb{R}^d$  denote a bounded open region, and  $\Lambda_\ell(x_0) \equiv \{x \in \mathbb{R}^d \mid |x_i - x_{0,i}| < \ell/2, i = 1, \dots, d\}$ . When  $x_0 = 0$ , we will write  $\Lambda_\ell$  for simplicity. The potential depending only on the  $\omega_i$  in a region  $\Lambda$  is denoted  $V_\Lambda = (V_\omega|_\Lambda)$ . The finite-volume Hamiltonians  $H_{\Lambda, \omega}$  are defined as  $H_{\Lambda, \omega} \equiv H_0 + V_\Lambda$ . For notational simplicity, we will omit the  $\omega$  when writing  $H_\Lambda$ , when no particular configuration is considered. Since  $V_\Lambda$  has compact support, it is a relatively compact perturbation of  $H_0$  and hence  $\sigma_{ess}(H_0) = \sigma_{ess}(H_\Lambda)$ . One of our first tasks is to locate precisely the eigenvalues of  $H_\Lambda$  in the unperturbed spectral gap  $G = (B_-, B_+)$  with good probability.

The condition  $[H1](\gamma_0, \ell_0)$  in (6.1) is an estimate on the decay of the on the resolvent of  $H_\Lambda$ , which we write as  $R_\Lambda(z) = (H_\Lambda - z)^{-1}$ , when it exists. We recall that the most general form of  $H_0$  is  $H_0 = (p - A)^2 + V_0$ , where  $p \equiv -i\nabla$ , and  $A$  and  $V_0$  are periodic vector-valued and scalar functions, respectively. For any  $\chi \in C^2$ , define the first order differential operator  $W(\chi)$  by

$$(6.3) \quad W(\chi) \equiv [H_0, \chi] = -i(p - A) \cdot \nabla \chi - i\nabla \chi \cdot (p - A).$$

This operator is localized on the support of  $\nabla \chi$ . We choose any  $\delta > 0$  small, and let  $\Lambda_{\ell, \delta} \equiv \{x \in \Lambda_\ell \mid \text{dist}(\partial \Lambda_\ell, \chi) > \delta\}$ . We will use  $\chi_\ell$  to denote a function satisfying  $\chi_\ell|_{\Lambda_{\ell, \delta}} = 1$ ,  $\text{supp } \chi_\ell \subset \Lambda_\ell$ , and  $0 \leq \chi_\ell \leq 1$ . It follows that  $\text{supp } \nabla \chi_\ell \subset \Lambda_\ell \setminus \Lambda_{\ell, \delta}$ , and  $W(\chi_\ell)$  is also localized in this region. With these definitions, the condition we must verify is

**[H1]** $(\gamma_0, \ell_0)$ . *There exist  $\gamma_0 > 0$  and  $\ell_0 \gg 1$ , such that  $\gamma_0 \ell_0 \gg 1$ , and*

$$(6.4) \quad \mathbb{P}\{ \|W(\chi_\ell)R_{\Lambda_\ell}(E + i\varepsilon)\chi_{\ell/3}\| \leq e^{-\gamma_0 \ell_0} \| \} \geq 1 - \ell_0^{-\xi},$$

for  $E$  near the band edges  $\tilde{B}_\pm$  and for some  $\xi > 2d$ .

We do this in two steps. We first prove that for  $\delta_\Lambda > 0$  small,  $\text{dist}(\sigma(H_\Lambda), \tilde{B}_\pm) > \delta_\Lambda$ , with good probability. We can then apply the Combes-Thomas result of Appendix 2 to conclude exponential decay at energies  $E \in (\tilde{B}_- - \delta_\Lambda/2, \tilde{B}_-) \cup (\tilde{B}_+, \tilde{B}_+ + \delta_\Lambda/2)$  with a good probability. We then verify  $[H1](\gamma_0, \ell_0)$  for an appropriate choice of  $\gamma_0$  and  $\ell_0$ .

We now discuss the location of the spectrum of the finite-volume Hamiltonians  $H_\Lambda$  in the unperturbed spectral gap. Recall that by (H8) the family  $\{H_\omega\}$  has an almost sure spectrum  $\Sigma$  with an open spectral gap  $(\tilde{B}_-, \tilde{B}_+)$ . The probability space for the models is  $\Omega = (\text{supp } g)^{\mathbb{Z}^d}$ .

LEMMA 6.1. *Suppose  $\mu \equiv \mu_{\Lambda, \omega_0} \in \sigma_d(H_{\Lambda, \omega_0}) \cap (B_-, B_+)$ , for some  $\omega_0 \in \Omega$ , then  $\mu \in \Sigma$ .*

**Proof:** Let  $\psi_{\omega_0}$  be an eigenfunction of  $H_{\Lambda, \omega_0}$  with eigenvalue  $\mu_{\Lambda, \omega_0} \equiv \mu : H_{\Lambda, \omega_0} \psi_{\omega_0} = \mu \psi_{\omega_0}$ ,  $\|\psi_{\omega_0}\| = 1$ . For any  $R$  such that  $\Lambda \subset \subset \Lambda_R$ , and for any  $\nu > 0$ , we define the following events:

$$I_{R, \nu} \equiv \left\{ \omega \in \Omega \mid |\omega_0 - \omega_i| \leq \nu(6|\Lambda| \|u\|_\infty)^{-1}, \forall i \in \tilde{\Lambda} \right\},$$

and

$$E_{R, \nu} \equiv \left\{ \omega \in \Omega \mid |\omega_i| < \nu(6|\tilde{\Lambda}_R \setminus \tilde{\Lambda}| \|u\|_\infty)^{-1}, \forall i \in \tilde{\Lambda}_R \setminus \tilde{\Lambda} \right\}.$$

Set  $B_{R, \nu} \equiv I_{R, \nu} \cap E_{R, \nu}$ . Let  $\chi \in C^2$  be a smoothed characteristic function with  $\text{supp } \chi \subset \Lambda_2$ ,  $\chi \leq 1$ , and  $\chi|_{\Lambda_1} = 1$ . For  $R > 1$ , set  $\chi_R(x) \equiv \chi(R^{-1}x)$  so that  $\|\partial^\alpha \chi_R\| = \mathcal{O}(R^{-|\alpha|})$ , for  $|\alpha| = 0, 1, 2$ . Choose  $R_1$  sufficiently large so  $\|\chi_{R_1} \psi_{\omega_0}\| > \frac{1}{2}$ , and for  $R > R_1$  define  $\psi_R \equiv \|\chi_R \psi_{\omega_0}\|^{-1} \chi_R \psi_{\omega_0}$  so  $\|\psi_R\| = 1$ . Then, by the definition of  $\psi_R$  and the local Hamiltonians,

$$(H_\omega - \mu)\psi_R = (H_{\Lambda, \omega_0} - \mu)\psi_R + \sum_{i \in \tilde{\Lambda}} (\omega_i - \omega_0) u_i \psi_R + \sum_{i \in \tilde{\Lambda}_R \setminus \Lambda} \omega_i u_i \psi_R,$$

and it follows that for all  $\omega \in B_{R, \nu}$ ,

$$(6.5) \quad \|(H_\omega - \mu)\psi_R\| \leq 2\| [H_0, \chi_R] \psi_{\omega_0} \| + \frac{1}{3}\nu$$

The commutator is estimated as follows. As  $H_0 = (p - A)^2 + V_0$ , we have

$$(6.6) \quad [(p - A)^2, \chi_R] \psi_{\omega_0} = -2i\Delta \chi_R (p - A) \psi_{\omega_0} - (\Delta \chi_R) \psi_{\omega_0}.$$

Now  $\psi_{\omega_0}$  is an eigenfunction of  $H_{\Lambda, \omega_0}$  and, in particular,  $\psi_{\omega_0} \in D(H_0)$ , so

$$(p - A)_j \psi_{\omega_0} = (p - A)_j (H_0 - z)^{-1} (\mu - z - V_{\Lambda, \omega_0}) \psi_{\omega_0}.$$

Setting  $z = i\delta$ ,  $\delta > 0$ , we obtain

$$(6.7) \quad \|(p - A)_j \psi_{\omega_0}\| \leq \delta^{-1} \|\mu - i\delta - V_{\Lambda, \omega_0}\|_\infty.$$

which is independent of  $R$ . Hence, by taking  $R$  sufficiently large, it follows from (6.5) that

$$\|(H_\omega - \mu)\psi_R\| \leq \frac{2}{3}\nu.$$

This shows that for any  $\nu > 0$ ,  $\sigma(H_\omega) \cap [\mu - \nu, \mu + \nu] \neq \emptyset$  with probability  $\mathbb{P}(B_{R, \nu}) = \mathbb{P}(E_{R, \nu})\mathbb{P}(I_{R, \nu}) > 0$ . Since the spectrum of  $\{H_\omega\}$  is deterministic, this implies  $\mu \in \Sigma$ .  $\square$

LEMMA 6.2. *Let  $\mu_{\Lambda, \omega} \equiv \mu \in \sigma_d(H_{\Lambda, \omega}) \cap (B_-, B_+)$ , with eigenfunction  $\phi_\omega$ ,  $\|\phi_\omega\| = 1$ . Assume that  $V_{\Lambda, \omega} \geq 0$ . Then we have*

$$\langle \phi_\omega, V_{\Lambda, \omega} \phi_\omega \rangle \geq [\text{dist}(\mu, \sigma(H_0))]^2 M_\infty^{-1}.$$

**Proof:** Since  $M_\infty V_{\Lambda, \omega} \geq (V_{\Lambda, \omega})^2$  under the hypothesis that  $V_{\Lambda, \omega} \geq 0$ , we have

$$\begin{aligned} \langle \phi_\omega, V_{\Lambda, \omega} \phi_\omega \rangle &= M_\infty^{-1} \langle \phi_\omega, M_\infty V_{\Lambda, \omega} \phi_\omega \rangle \\ &\geq M_\infty^{-1} \|V_{\Lambda, \omega} \phi_\omega\|^2. \end{aligned}$$

The eigenvalue equation gives  $V_{\Lambda, \omega} \phi_\omega = -(H_0 - \mu)\phi_\omega$ , so that

$$\begin{aligned} \langle \phi_\omega, V_{\Lambda, \omega} \phi_\omega \rangle &\geq M_\infty^{-1} \|(H_0 - \mu)\phi_\omega\|^2 \\ &\geq M_\infty^{-1} [\text{dist}(\sigma(H_0), \mu)]^2, \end{aligned}$$

which proves the estimate.  $\square$

**PROPOSITION 6.1.** Let  $\delta_{\pm} \equiv \frac{1}{2}|\tilde{B}_{\pm} - B_{\pm}|$ , and for any  $0 < \delta < \frac{1}{2}M_{\infty}^{-1} \min(\delta_{+}, \delta_{-})$ , assume that  $\omega_i < (1 - \delta M_{\infty}[\min(\delta_{+}, \delta_{-})]^{-2})M, \forall i \in \tilde{\Lambda}$ . Then we have

$$\sup \left\{ \sigma(H_{\Lambda, \omega}) \cap (-\infty, \tilde{B}_{-}) \right\} < \tilde{B}_{-} - \delta$$

and

$$\inf \left\{ \sigma(H_{\Lambda, \omega}) \cap (\tilde{B}_{+}, \infty) \right\} > \tilde{B}_{+} + \delta .$$

**Proof:** Without loss of generality, we assume  $H_{\Lambda, \omega}$  has an eigenvalue  $\mu_{\Lambda, \omega} \equiv \mu \in [\tilde{B}_{-} - \delta, \tilde{B}_{-}]$ . Furthermore, we can assume that  $V_{\Lambda, \omega} \geq 0$ , since by Lemma 6.1, we always have  $\mu \leq \tilde{B}_{-}$  and the eigenvalues of  $H_{\Lambda, \omega}$  are increasing functions of the coupling constants  $\{\omega_i | i \in \tilde{\Lambda}\}$ . This fact follows, for example, from the Feynman-Hellman formula, Theorem 4.1, and the positivity of  $u$ . Indeed, if  $\phi_{\omega}$  is an eigenfunction of  $H_{\Lambda, \omega}$ , so that  $H_{\Lambda, \omega}\phi_{\omega} = \mu\phi_{\omega}$ , then

$$\begin{aligned} \frac{\partial \mu_{\Lambda, \omega}}{\partial \omega_i} &= \left\langle \phi_{\omega}, \frac{\partial H_{\Lambda, \omega}}{\partial \omega_i} \phi_{\omega} \right\rangle \\ &= \left\langle \phi_{\omega}, u_i \phi_{\omega} \right\rangle > 0 . \end{aligned}$$

The family  $T(\theta) \equiv H_0 + \theta V_{\Lambda, \omega}$ , for  $\theta$  in a small neighborhood of  $\theta_0 = 1$ , is an analytic type  $A$  family which is self-adjoint for  $\theta$  real. If  $\mu$  has multiplicity  $m$ , there are at most  $m$  functions  $\mu^{(k)}(\theta)$ , analytic in  $\theta$  for  $\theta$  near  $\theta_0 = 1$ , and which satisfy  $\lim_{\theta \rightarrow \theta_0=1} \mu^{(k)}(\theta) = \mu$ . Let  $\phi^{(k)}(\theta)$  be an eigenfunction for  $\mu^{(k)}(\theta)$ , with  $\|\phi^{(k)}(\theta)\| = 1$  for  $\theta$  real and  $|\theta - 1|$  small. Applying the Feynman-Hellman formula again, we find

$$(6.8) \quad \begin{aligned} \frac{d\mu^{(k)}(\theta)}{d\theta} &= \left\langle \phi(\theta), V_{\Lambda, \omega} \phi(\theta) \right\rangle \\ &= \theta^{-1} \left\langle \phi(\theta), (\theta V_{\Lambda, \omega}) \phi(\theta) \right\rangle . \end{aligned}$$

We now assume  $\omega_i < (1 - \delta M_{\infty}[\min(\delta_{+}, \delta_{-})]^{-2})M, \forall i \in \tilde{\Lambda}$ , and fix

$$\begin{aligned} \theta_1 = \min_{i \in \tilde{\Lambda}} \left( \frac{M}{\omega_i} \right) &\geq \left( 1 - \delta M_{\infty} [\min(\delta_{+}, \delta_{-})]^{-2} \right)^{-1} \\ &> 1 . \end{aligned}$$

Applying Lemma 6.2 to  $V_{\Lambda, \omega}$  under these conditions yields

$$\frac{d\mu^{(k)}(\theta)}{d\theta} \geq \theta^{-1} M_{\infty}^{-1} \left[ \text{dist}(\mu^{(k)}(\theta), \sigma(H_0)) \right]^2 ,$$

Upon integrating over  $[1, \theta_1]$ , we get, by monotonicity of  $\mu^{(k)}(\theta)$ :

$$\begin{aligned} \mu^{(k)}(\theta_1) &\geq \mu + (\log \theta_1) M_{\infty}^{-1} \min \left\{ \left[ \text{dist}(\mu^{(k)}(\theta_1), \sigma(H_0)) \right]^2, \left[ \text{dist}(\mu, \sigma(H_0)) \right]^2 \right\} \\ &\geq \mu + \delta > \tilde{B}_{-} . \end{aligned}$$

This shows that  $(H_0 + \sum_{i \in \tilde{\Lambda}} M u_i)$  has an eigenvalue outside of  $\Sigma$  which contradicts Lemma 6.1.  $\square$

This proposition is the main technical result. We can now easily compute the probability that  $\text{dist}(\sigma(H_{\Lambda, \omega}), \tilde{B}_{\pm}) > \delta_{\Lambda}$ .

COROLLARY 6.1. For  $0 < \delta < \frac{1}{2}M_\infty^{-1}\min(\delta_+, \delta_-)$ , we have

$$\sup \left\{ \sigma(H_{\Lambda, \omega}) \cap (-\infty, \tilde{B}_-) \right\} < \tilde{B}_- - \delta,$$

and

$$\inf \left\{ \sigma(H_{\Lambda, \omega}) \cap (\tilde{B}_+, \infty) \right\} > \tilde{B}_+ + \delta,$$

with a probability larger than

$$1 - |\Lambda| \max_{X=m, M} \left| \int_{1-\delta M_\infty [\min(\delta_+, \delta_-)]^{-2} X}^X g(s) ds \right|$$

**Proof:** The probability that  $\omega_i < (1 - \delta M_\infty [\min(\delta_+, \delta_-)]^{-2})M$ ,  $\forall i \in \tilde{\Lambda}$ , is given by  $\left[ 1 - \int_{(1-\delta M_\infty \Delta^{-2})M}^M h(s) ds \right]^{|\Lambda|}$ . The corollary now follows by expanding this probability and from Proposition 6.1.

**6.3. Verification of [H1]( $\gamma_0, \ell_0$ ).** We verify [H1]( $\gamma_0, \ell_0$ ) by combining Corollary 6.1 on the location of the spectrum of  $H_{\Lambda_\ell, \omega}$  and the exponential decay estimate of [8]. We note that hypothesis (H6) on the decay of the tail of the density  $h$  near the endpoints of its support  $m$  and  $M$  is essential in order to control the probability in Corollary 6.1. We first give the decay estimate for the localized resolvent and then comment on the gradient term.

PROPOSITION 6.2. Let  $\chi_i, i = 1, 2$ , be two functions with  $\|\chi_i\|_\infty \leq 1$ ,  $\text{supp } \chi_1 \subset \Lambda_{\ell_0/3}$  and  $\text{supp } \chi_2$  localized near  $\partial\Lambda_\ell$  and  $\delta_\pm \equiv \frac{1}{2}|\tilde{B}_+ - \tilde{B}_-|$ . For  $\beta > 0$  as in (H6), consider any  $\nu > 0$  such that  $0 < \nu < 4\beta(2\beta + 3d)^{-1}$ . Then  $\exists \ell_0^* \equiv \ell_0^*(M_\infty, \delta_+, \delta_-, M)$  such that  $\forall \ell_0 > \ell_0^*$  and  $\forall E \in (\tilde{B}_- - \ell_0^{\nu-2}, \tilde{B}_-) \cup [\tilde{B}_+, \tilde{B}_+ + \ell_0^{\nu-2})$ ,

$$\sup_{\varepsilon > 0} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| \leq e^{-\ell_0^{\nu/3}},$$

with probability  $\geq 1 - \ell_0^{-\xi}$ , for some  $\xi > 2d$ .

**Proof:** From Corollary 6.1 and (H7), we compute the probability that  $\sigma(H_{\Lambda_{\ell_0}, \omega})$  is at a distance  $\delta = 2\ell_0^{\nu-2}$  from  $\tilde{B}_\pm$ ,

$$IP \left\{ \text{dist} \left( \sigma(H_{\Lambda_{\ell_0}, \omega}), \tilde{B}_\pm \right) > 2\delta \right\} \geq 1 - \ell_0^d \left( 2\ell_0^{\nu-2} M_\infty [\min(\delta_+, \delta_-)]^{-2} X \right)^{3d/2+\beta},$$

where  $X = m$  for  $\tilde{B}_-$  and  $X = M$  for  $\tilde{B}_+$ . A simple computation shows that the right side of (6.9) is bounded below by  $1 - \ell_0^{-\xi}$  for some  $\xi > 2d$  provided  $\nu$  satisfies  $0 < \nu < 4\beta(2\beta + 3d)^{-1}$ . We now apply the Combes-Thomas exponential decay estimate [8]  $H_{\Lambda_{\ell_0}, \omega}$ . Since  $\text{dist}(\text{supp } \chi_2, \text{supp } \chi_1) \geq \ell_0/3$  (in dimension  $d > 9$ , this is no longer true; one has to replace  $\ell_0/3$  by  $\ell_0/(3\sqrt{d})$ , for the diameter of the inner cube), we obtain

$$\begin{aligned} \|\chi_2 R_{\Lambda_{\ell_0}}(E + i\varepsilon)\chi_1\| &\leq C_2 \sup \left( |\tilde{B}_+ - \tilde{B}_-|^{-1}, \ell_0^{2-\nu} \right) \\ &\times e^{-\inf(\alpha_0, C_1 \ell_0^{\nu/2-1} |\tilde{B}_+ - \tilde{B}_-|^{1/2}) \ell_0/6} \end{aligned}$$

The result follows by taking  $\ell_0$  large.  $\square$

COROLLARY 6.2. There exists a length scale  $\ell_0^* > 0$  such that  $\forall \ell_0 > \ell_0^*$ , hypothesis [H1] ( $\gamma_0, \ell_0$ ) holds  $\forall E \in (\tilde{B}_- - \ell_0^{\nu-2}, \tilde{B}_-) \cup [\tilde{B}_+, \tilde{B}_+ + \ell_0^{\nu-2})$  and any  $\nu$  satisfying  $0 < \nu < 4\beta(2\beta + 3d)^{-1}$ ,  $\beta$  as in (H6).

**Proof:** As in (8.23) of chapter 8, we write

$$(6.10) \quad \|W(\chi_{\ell,v})R_{\Lambda_\ell}\chi_{\ell/3}\| \leq \|(\Delta\chi_{\ell,v})R_{\Lambda_\ell}\chi_{\ell/3}\| + 2\sum_{j=1}^d \|(\partial_j\chi_{\ell,v})(p-A)_jR_{\Lambda_\ell}\chi_{\ell/3}\|$$

for a function  $\chi_{\ell,v}$  localized within distance  $v$  of  $\partial\Lambda_\ell$ . Let  $\chi_i$ ,  $i = 1, 2$ , be smooth functions such that  $\chi_i\chi_{\ell,v} = \chi_{\ell,v}$ ,  $\chi_2\chi_1 = \chi_1$ , and  $\text{supp } \chi_i$  is localized within a distance  $2v$  for  $i = 1$  and  $3v$  for  $i = 2$ , of  $\partial\Lambda_\ell$ . Then, we write for each  $j$  and any  $u \in L^2(\mathbb{R}^d)$ ,

$$\begin{aligned} \|(\partial_j\chi_{\ell,v})(p-A)_jR_{\Lambda_\ell}u\|^2 &\leq C_0\langle(p-A)_jR_{\Lambda_\ell}u, \chi_1(p-A)_j\chi_2R_{\Lambda_\ell}u\rangle \\ &\leq C_0\|\chi_2R_{\Lambda_\ell}u\| \|(p-A)_j\chi_1(p-A)_jR_{\Lambda_\ell}u\| \end{aligned}$$

Taking  $u = \chi_{\ell/3}f$ , we see that (6.10) is bounded above as in Proposition 6.2 (taking  $\ell_0^*$  larger) provided we have  $\|(p-A)^2R_{\Lambda_{\ell_0}}u\|$  bounded. This follows with a probability  $\geq 1 - \ell_0^{-\xi}$ , since  $V_0$  is relatively bounded and  $V_\omega^{\Lambda_{\ell_0}}$  is bounded.  $\square$

## 7. Random Magnetic Schrödinger Operators and the Integer Quantum Hall Effect

**7.1. Overview: Integer Quantum Hall Effect.** This chapter is devoted to the study of the family of random Schrödinger operators describing an electron moving in two-dimensions and subject to a random potential and a constant, transverse magnetic field, and the application of this model to the integral quantum Hall Effect (IQHE). The classic Hall effect for a thin conductor of width  $\delta$  and infinite spatial extent in the  $(x_1, x_2)$ -directions, may be described as follows (we follow [15]). The conductor is viewed as a collection of fixed, positively charged centers and a sea of negatively charged, noninteracting, electrons with charge  $q$  and density  $n$ . The sample is subject to a transverse, constant magnetic field  $B = (0, 0, B)$ . In this ideal experiment, suppose that a constant electric field  $E = (E, 0, 0)$  is applied in the  $x_1$ -direction. This creates a current  $J$  of electrons moving in the  $x_1$ -direction. The Lorentz force, proportional to  $J \times B$ , acts on the electrons in the  $x_2$ -direction. As a result, a potential difference develops in the  $x_2$ -direction with a corresponding electric field, the Hall field  $E_H$ , in the  $x_2$ -direction. If the system is in equilibrium, the force on the electrons from  $E_H$  and the Lorentz force must balance, giving rise to the equation

$$(7.1) \quad nqE_H + J \times B = 0.$$

We can solve this equation by taking the curl with the magnetic field  $B$  and obtain,

$$(7.2) \quad J = \sigma E_H,$$

Restricting to the  $(x_1, x_2)$ -plane, and changing to surface density  $\delta n$ , the  $2 \times 2$  conductivity tensor  $\sigma$  has zero diagonal elements and nonvanishing off-diagonal elements given by

$$(7.3) \quad \sigma_{21} = \frac{nq\delta}{B} = -\sigma_{12},$$

We call  $\sigma_{12}$  the *Hall conductance*  $\sigma_H$ . It is proportional to the electron density in the plane  $n\delta$ . This is the classic Hall effect and has been observed since the end of

the nineteenth century. For comparison with the IQHE, we define the filling factor by

$$(7.4) \quad \nu \equiv \frac{n\delta h}{Bq},$$

where  $h$  is Planck's constant. In terms of this, the Hall conductance (7.3) is

$$(7.5) \quad \sigma_H = \nu \frac{q^2}{h},$$

which is proportional to the filling factor.

In the 1980's, experimental techniques had advanced to the point where the Hall effect experiments could be performed at very low temperatures. As is now widely known, the law in (7.3) is not observed, but rather von Klitzing and co-workers observed a quantization of the Hall conductivity. In particular, they found that

$$(7.6) \quad \sigma_H \in \frac{q^2}{h} \mathbb{N},$$

That is, the Hall conductance is quantized in integer multiples of  $q^2/h$ . As the filling factor increases, the Hall conductance remains constant and then increases by integer amounts. The experiments are extremely accurate. This behavior of the Hall conductance can now be explained in terms of the Kubo formula for the conductance and properties of one-particle random Schrödinger operators, see, for example [15].

**7.2. Landau Hamiltonians with Random Potentials.** We consider a one-particle Hamiltonian which describes an electron in two-dimensions  $(x_1, x_2)$  subject to a constant magnetic field of strength  $B > 0$  in the perpendicular  $x_3$ -direction, and a random potential  $V_\omega$ . The Hamiltonian  $H_\omega$  has the form

$$(7.7) \quad H_\omega = (p - A)^2 + V_\omega = H_A + V_\omega,$$

on the Hilbert space  $L^2(\mathbb{R}^2)$ , where  $p \equiv -i\nabla$ , and the vector potential  $A$  is

$$(7.8) \quad A = \frac{B}{2}(x_2, -x_1),$$

so the magnetic field  $B = \nabla \times A$  is in the  $x_3$ -direction. The random potential  $V_\omega$  is Anderson-like having the form

$$(7.9) \quad V_\omega(x) = \sum_{i \in \mathbb{Z}^2} \omega_i u(x - i).$$

We denote by  $H_A \equiv (p - A)^2$ , the Landau Hamiltonian. As is well-known, the spectrum of  $H_A$  consists of an increasing sequence  $\{E_n(B)\}$  of eigenvalues, each of infinite multiplicity, given by

$$(7.10) \quad E_n(B) = (2n + 1)B, \quad n = 0, 1, 2, \dots$$

We will call  $E_n(B)$  the  $n^{\text{th}}$  Landau level and denote by  $P_n$  the projection onto the corresponding subspace. The orthogonal projection is denoted by  $Q_n \equiv 1 - P_n$ .

This family of random Schrödinger operators plays a key role in the theory of the *integer quantum Hall effect* (IQHE) described in section 7.1. Several recent discussions of the integer quantum Hall effect (IQHE) [14, 15, 70, 96, 108, 109, 126] require certain spectral properties of this family of random, one-particle Schrödinger operators. As proved in [15], the one-particle model of a free electron moving in two

dimensions in the presence of a constant, transverse magnetic field will not exhibit the IQHE. Rather, the conductance follows the classical law (7.3). At very low temperatures, when dissipative effects are small, it is commonly believed that random impurities in the sample provide a necessary mechanism for the quantization of the Hall conductance and the appearance of the plateaus. The key hypothesis on the spectral properties of one-particle Hamiltonian is the existence of regions of localized states between the Landau levels.

In this chapter, we discuss the existence of localized states for Landau Hamiltonians with random perturbations. We provide the necessary arguments for the Wegner estimate and the initial length scale estimate that are necessary for the proof of the existence of localized states at energies between the Landau levels  $E_n(B)$ , and to within  $\mathcal{O}(\log B B^{-1})$  of the Landau energies, for fixed disorder and large magnetic field depending on the Landau level  $n$ . The actual proof that these estimates suffice to prove Anderson localization requires a more refined MSA than that presented in chapter 8 here and we refer the reader to the original references [62, 63].

There are two cases to consider. In the first case, the random potential is bounded. For bounded perturbations, a large magnetic field means that there are spectral gaps between the Landau bands of length  $\mathcal{O}(B)$  since the Landau levels broaden no more than  $\|V_\omega\|_\infty$ , which is bounded independent of  $B$ . The corresponding eigenfunctions for energies in the localized regime decay exponentially with respect to distance and the magnetic field strength  $B$ . Kunz's proof [96] of the IQHE requires the existence of spectral gaps and localized states near the band-edges with finite localization length. A consequence of his analysis is a proof that the localization length diverges at some energy in each band. It is believed that for neutral samples, the localization length should diverge precisely at the Landau energy.

In the second case, the random potential is unbounded. Typically, the spectral gaps no longer exist, having been completely filled-in by spectrum. The importance of this case is discussed in a review article by Bellissard, van Elst, and Schulz-Baldes [15]. These authors point out that in experiments, the disorder is strong enough to fill the gap between the Landau levels. In their proof of the IQHE, they require regions of localized states between the Landau energies in order that the quantum Hall conductivity exhibits a plateau region. They do not require a spectral gap. They prove that if the quantum Hall conductivity jumps by an integer (as a function of the filling factor), then the localization length must diverge at some energy between the localized state regions.

The second result presented in this chapter concerns the integrated density of states. Using a proof simpler than the one presented in chapter 4, we prove that, in general, the integrated density of states (IDS) is Lipschitz continuous at all energies but the Landau energies. If, in addition, the support of the single site potential  $u$  includes the unit cell, then the IDS is Lipschitz continuous at all energies. It is now known that the IDS for the random Landau Hamiltonian is Lipschitz continuous at all energies, [28, 29, 66], but the proof requires a different version of the quantitative unique continuation principle. Much less is known about the regularity of the IDS. W. M. Wang [134, 135] proved the smoothness of the IDS at energies away from the Landau energies provided the single-site probability density is smooth.

From the point of view of models, the randomly perturbed Landau Hamiltonian is interesting because the proof of localization requires the use of nontrivial results from the theory of bond percolation. In particular, the critical probability for bond percolation, the probability above which an infinite cluster exists almost surely, enters in a key way.

**7.3. A Description of the Model and the Main Results.** We make the following assumptions on the single-site potential  $u$  and the coupling constants  $\{\omega_i\}$  appearing in the Anderson-type potential (7.7).

- (V1)  $u \geq 0$ ,  $u \in C^2$ ,  $\text{supp } u \subset B(0, 1/\sqrt{2})$ , and  $\exists C_0 > 0$  and  $r_0 > 0$  s.t.  $u|_{B(0, r_0)} > C_0$ .
- (V2)  $\{\omega_i\}$  is an independent, identically distributed family of random variables with common distribution  $g \in L^\infty([-M, M])$ , for some  $0 < M \leq \infty$ , s.t.  $g(-\omega) = g(\omega)$ ,  $g(\omega) > 0$  Lebesgue a.e.  $\omega \in [-M, M]$ , and for some  $\epsilon > 0$ ,  $\sup_{\omega \in \mathbb{R}} \{\omega^{3+\epsilon} g(\omega)\} < \infty$ .

The condition on the decay of  $g$  implies that the first two moments are finite, i.e.  $\int |\omega|^k g(\omega) d\omega < \infty$ , for  $k = 0, 1, 2$ . The above condition does not require any differentiability of the density  $g$ . If we require  $g$  to be twice differentiable, we can replace (V2) by another condition.

- (V2)'  $\{\omega_i\}$  is an independent, identically distributed family of random variables with common distribution  $g \in C^2([-M, M])$ , for some  $0 < M \leq \infty$ , s.t.  $g(-\omega) = g(\omega)$ ,  $g(\omega) > 0$  Lebesgue a.e.  $\omega \in [-M, M]$ ,  $\|g^{(p)}\|_1 < \infty$ , for  $p = 0, 1, 2$ , and  $\int_{-M}^M \omega^2 g(\omega) < \infty$ .

We remark for future use that either condition (V2) or (V2)' implies that

$$(7.11) \quad \mathbb{P}\{\omega \geq \xi > 0\} = \mathcal{O}(\xi^{-N}), \text{ for all } \xi > 0 \text{ large and some } N \geq 2.$$

An important example is the case of Gaussian distributed coupling-constants for which  $g(\omega) = (\alpha\pi)^{-1/2} e^{-\alpha\omega^2}$ , for some  $\alpha > 0$ .

Our main theorem concerning localization for random Landau Hamiltonians is the following [26, 63, 133].

**THEOREM 7.1.** *Let  $H_\omega$  be the family given in (7.7) with vector potential  $A$  as in (7.8) with  $B > 0$ , and the random potential  $V_\omega$  as in (7.9), and satisfying (V1) and either (V2) or (V2)' Let  $I_n(B)$  denote the unbounded set of energies*

$$I_n(B) \equiv (-\infty, B - \mathcal{O}(\log B \cdot B^{-1})] \cup \bigcup_{j=0}^n [E_j(B) + \mathcal{O}(\log B \cdot B^{-1}), E_{j+1}(B) - \mathcal{O}(\log B \cdot B^{-1})],$$

where the term  $\mathcal{O}(\log B \cdot B^{-1})$  depends on  $n$ . For each integer  $n > 0$ , there exists  $B_n \gg 0$  such that for  $B > B_n$ ,

$$\Sigma \cap I_n(B)$$

is pure point and the corresponding eigenfunctions decay exponentially.

We remark that we can also prove localization at energies between the Landau levels for fixed, nonzero  $B > 0$  for the family  $H_\omega(g) \equiv H_A + gV_\omega$ , with a coupling constant  $g > 0$ , provided that we work in the small coupling regime. This result does not require percolation theory, but we cannot control the spectrum to within  $\mathcal{O}(B^{-1})$  of the Landau levels. We refer the reader to section 6 of [8] where results of this type are proved in general situations. As with previous work, the Wegner estimate for the finite-area Hamiltonians allows us to control the integrated density

of states. However, we no longer need to take  $B$  large as in [26]. The proof of the following theorem is given in [28] based on results in [29, 66].

**THEOREM 7.2.** *Let  $H_\omega$  be the family given in (7.7) with vector potential  $A$  as in (7.8) with  $B > 0$ , and the random potential  $V_\omega$  as in (7.9) and satisfying (V1) and either (V2) or (V2)'. Then, the integrated density of states is locally uniformly Lipschitz continuous on  $\mathbb{R}$ .*

**7.4. The Wegner Estimate.** The Wegner estimate for this model follows the same lines of the proof of the Wegner estimate given in chapter 4. There are some simplifications which we will mention due to the explicit structure of the unperturbed Landau Hamiltonian  $H_A$ . As these are of interest in themselves, we give a proof of the Wegner that works for all energies away from the Landau levels. The proof of the Lipschitz continuity of the IDS at all energies is given in [28]. We will use the spectral averaging result of chapter 3. The local Hamiltonians  $H_\Lambda$  are defined as in chapter 4. For a bounded region  $\Lambda \subset \mathbb{R}^2$ , we define the local potential  $V_\Lambda$  by  $V_\omega|_\Lambda$ . The local Hamiltonian is  $H_\Lambda \equiv H_A + V_\Lambda$ . As discussed in chapter 4, the essential spectrum of  $H_A$  is invariant under this perturbation. The Wegner estimate for these local Hamiltonians takes the following form.

**THEOREM 7.3.** *There exists a constant  $B_0 > 0$  and a constant  $C_W > 0$  such that for all  $B > B_0$  and for any  $E \notin \sigma(H_A)$  and  $\eta > 0$  so that  $[E - \eta, E + \eta] \cap \sigma(H_A) = \emptyset$ , we have,*

$$IP_\Lambda \{ \text{dist}(\sigma(H_\Lambda), E) < \eta \} \leq C_W [\text{dist}(\sigma(H_A), E) - \delta]^{-2} \|g\|_\infty \eta B |\Lambda|.$$

In the proof of this theorem, the exponential decay of the resolvent of the unperturbed Hamiltonian is replaced by the explicit exponential decay of the projectors onto the the Landau levels when localized between disjoint sites. Let  $P_n$  be the orthogonal projection on the  $n^{\text{th}}$  Landau level of  $H_A$ . The projector  $P_n$  has an integral kernel given by

$$(7.12) \quad P_n(x, y) = B e^{-i\frac{B}{2}x \wedge y} p_n \left( B^{1/2}(x - y) \right),$$

where  $p_n(x)$  is of the form

$$(7.13) \quad p_n(x) = \left\{ n^{\text{th}} \text{ degree polynomial in } x \right\} e^{-|x|^2/2},$$

and independent of  $B$ . We define the projector  $Q_n$  as  $Q_n \equiv 1 - P_n$ . We will make repeated use of the following two elementary lemmas. The proof of Lemma 7.4 follows by direct calculation using the kernel (7.12)–(7.13).

**LEMMA 7.4.** *Let  $\chi_1, \chi_2$  be functions of disjoint, not necessarily compact, support with  $|\chi_i| \leq 1$ , and let  $\delta \equiv \text{dist}(\text{supp } \chi_1, \text{supp } \chi_2) > 0$ . Then,*

- (1)  $\|\chi_1 P_n \chi_1\|_1 \leq C_n B |\text{supp } \chi_1|$ ;
- (2)  $\|\chi_1 P_n \chi_2\|_{HS} \leq C_n B^{1/2} e^{-B\delta^2/8} \inf \{ |\text{supp } \chi_1|, |\text{supp } \chi_2| \}^{1/2}$ ,

where  $C_n$  varies from line to line and depends only on  $n$ , and  $HS$  denotes the Hilbert Schmidt norm.

The next lemma allows us to estimate the size of the overlap between the perturbed band given by the range of  $E_\Delta$  and the orthogonal complement of the first unperturbed Landau level.

LEMMA 7.5. *The restriction of the projection  $Q_0$  to the first band satisfies*

$$(7.14) \quad \|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-2} (1 - (2d_\Delta)^{-1}|\Delta|)^{-2} M_0^2,$$

where  $d_\Delta \equiv \text{dist}(\sigma(H_A) \setminus \{B\}, \Delta) = \mathcal{O}(B)$ , and  $M_0 \equiv \|V_\omega\|_\infty \leq M\|u\|_\infty$ .

**Proof.** Let  $E_m \in \Delta$  be the center of the interval. We then can write

$$\begin{aligned} E_\Delta Q_0 E_\Delta &\leq [\text{dist}(\sigma(H_A) \setminus \{B\}, \Delta)]^{-1} (E_\Delta (H_A - E_m) Q_0 E_\Delta) \\ &\leq d_\Delta^{-1} \{E_\Delta (H_A - E_m) Q_0 E_\Delta + E_\Delta V_\Lambda Q_0 E_\Delta\}. \end{aligned}$$

This implies that

$$\|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-1} \{|\Delta|/2 \|E_\Delta Q_0 E_\Delta\| + M_0 \|Q_0 E_\Delta\|\}.$$

Since  $d_\Delta = \mathcal{O}(B)$ , it is clear that for all  $B$  sufficiently large  $(2d_\Delta)^{-1}|\Delta| \ll 1$ , so

$$\|E_\Delta Q_0 E_\Delta\| \leq d_\Delta^{-1} (1 - (2d_\Delta)^{-1}|\Delta|)^{-1} M_0 \|E_\Delta Q_0 E_\Delta\|^{1/2},$$

and the result follows.  $\square$

We will compute the Wegner estimate under the assumption that the closest point in  $\sigma(H_A)$  to the energy  $E$  is the first Landau level  $E_0(B)$ . The arguments apply to any Landau level but the constants depend on the index  $n$ .

**Proof of Theorem 7.3.**

1. As in the proof of Theorem 4.3, we must estimate  $\mathbb{I}E_\Delta(Tr E_\Delta)$ . Working with the first Landau level for simplicity, we separate the trace into a piece depending upon  $P_0$ , and one depending on  $Q_0$ ,

$$(7.15) \quad Tr E_\Delta = Tr E_\Delta P_0 E_\Delta + Tr E_\Delta Q_0 E_\Delta.$$

We make the simple estimate

$$(7.16) \quad Tr E_\Delta Q_0 E_\Delta \leq \|E_\Delta Q_0 E_\Delta\| (Tr E_\Delta),$$

where we used the fact that  $E_\Delta Q_0 E_\Delta \geq 0$ . Now by Lemma 7.3,  $\|E_\Delta Q_0 E_\Delta\| = \mathcal{O}(B^{-2})$ , so it follows that for all  $B$  sufficiently large,

$$(7.17) \quad Tr E_\Delta \leq 2Tr(P_0 E_\Delta P_0).$$

2. Let us now suppose  $\inf \Delta > B$  for definiteness. From (7.7), and positivity we obtain

$$(7.18) \quad \begin{aligned} Tr E_\Delta P_0 E_\Delta &\leq Tr(E_\Delta (H_\Lambda - B) P_0 (H_\Lambda - B) E_\Delta) \cdot [\text{dist}(\Delta, B)]^{-2} \\ &\leq Tr(P_0 V_\Lambda E_\Delta V_\Lambda P_0) \cdot [\text{dist}(\Delta, B)]^{-2}. \end{aligned}$$

We now expand the potential  $V_\Lambda$ , writing  $V_\Lambda = \sum_i \omega_i u_i$  for short. The trace in

(7.18) is

$$(7.19) \quad \sum_{i,j} \omega_i \omega_j Tr(P_0 u_i E_\Delta u_j P_0),$$

where  $i, j \in \Lambda \cap \mathbb{Z}^2$ . Defining  $A^{ij} \equiv u_i^{1/2} A u_j^{1/2}$ , for any  $A \in B(\mathcal{H})$ , we have from (7.7.19),

$$(7.20) \quad \sum_{i,j} \omega_i \omega_j Tr\left(P_0^{ji} E_\Delta^{ij}\right).$$

We must estimate

$$(7.21) \quad \mathbb{E}_\Lambda \left( \left| \sum_{i,j} \omega_i \omega_j \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \right) \leq \sum_{i,j} \mathbb{E}_\Lambda \left( |\omega_i \omega_j| \left| \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \right) \\ \leq M^2 \sum_{i,j} \mathbb{E}_\Lambda \left( \left| \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \right).$$

3. To estimate  $\mathbb{E}_\Lambda \left( \left| \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \right)$ , we write the spectral decomposition of the trace-class operator  $P_0^{ji}$  as

$$P_0^{ji} = \sum_{n=1}^{\infty} \mu_n \langle \psi_n, \cdot \rangle \phi_n,$$

where  $\{\mu_n\}$  are the deterministic, non-negative eigenvalues of  $|P_0^{ji}|$ , so that  $\sum_n \mu_n = \|P_0^{ji}\|_1$ , and the set  $\{\psi_n\}$  (respectively,  $\{\phi_n\}$ ) is the orthonormal basis of eigenvectors for  $|P_0^{ji}|$  (respectively,  $|(P_0^{ji})^*| = |P_0^{ij}|$ ). Expanding the trace in (7.21) in this basis, we obtain,

$$(7.22) \quad \left| \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \leq \sum_n \mu_n |\langle \psi_n, E_\Delta^{ij} \phi_n \rangle| \\ \leq \frac{1}{2} \sum_n \mu_n \left( \langle \phi_n, E_\Delta^{jj} \phi_n \rangle + \langle \psi_n, E_\Delta^{ii} \psi_n \rangle \right).$$

The expectation on the right side of (7.22) can be bounded above by

$$(7.23) \quad \mathbb{E}_\Lambda \left( \left| \text{Tr} \left( P_0^{ji} E_\Delta^{ij} \right) \right| \right) \leq \frac{1}{2} \|P_0^{ji}\|_1 \sup_n \mathbb{E}_\Lambda \left( \langle \phi_n, E_\Delta^{jj} \phi_n \rangle + \langle \psi_n, E_\Delta^{ii} \psi_n \rangle \right).$$

We estimate the two inner products on the right side of (7.23) using the spectral averaging result Corollary 3.1, with  $B$  equal to  $u_j^{1/2}$  and  $u_i^{1/2}$ , respectively. Consequently, (7.20) is bounded above by

$$(7.24) \quad M^2 \|g\|_\infty |\Delta| \sum_{i,j} \|P_0^{ji}\|_1,$$

since  $C_0 = 1$ .

4. To evaluate the sum, we first consider those indices  $i$  and  $j$  for which  $|i - j| < 2$ . Let  $\chi_{ij}$  be the characteristic function for  $\text{supp} (u_i + u_j)$ . Then the contribution from these indices to the sum in (7.24) is

$$(7.25) \quad \sum_{|i-j|<2} \|P_0^{ji}\|_1 \leq \|u\|_\infty^2 \sum_{|i-j|<2} \|\chi_{ij} P_0 \chi_{ij}\|_1 \\ \leq C_1 B |\Lambda| |\text{supp } u|,$$

by Lemma 7.4, part (1). Next, in order to estimate the sum over the complimentary set of indices, we define the function  $\chi_{ij}^+$  to be the characteristic function for the set  $\{x \in \mathbb{R}^2 \mid |x - i| < |x - j|\}$ , and write  $\chi_{ij}^- \equiv 1 - \chi_{ij}^+$ . Using the inequality

$$(7.26) \quad \|AB\|_1 \leq \|A\|_{HS} \|B\|_{HS},$$

we obtain

$$(7.27) \quad \|P_0^{ji}\|_1 \leq \|u_j^{1/2} P_0 \chi_{ij}^+\|_{HS} \|\chi_{ij}^+ P_0 u_i^{1/2}\|_{HS} + \|u_j^{1/2} P_0 \chi_{ij}^-\|_{HS} \|\chi_{ij}^- P_0 u_i^{1/2}\|_{HS}.$$

If  $|i - j| \geq 2$ , condition (V1) on the support of  $u_j$  implies that

$$(7.28) \quad \begin{aligned} \text{dist}(\text{supp } \chi_{ij}^+, \text{supp } u_j) &\geq \frac{|i - j|}{2} - \frac{1}{\sqrt{2}} \\ &\geq a|i - j|, \end{aligned}$$

for some strictly positive constant  $a$ . A similar inequality holds for  $\text{dist}(\text{supp } \chi_{ij}^-, \text{supp } u_i)$ . By part (2) of Lemma 7.4, one obtains from (7.27) and (7.28),

$$(7.29) \quad \|P_0^{ji}\|_1 \leq 2C_0^2 B |\text{supp } u| e^{-aB|i-j|^2/8}.$$

from which it follows that

$$(7.30) \quad \sum_{|i-j| \geq 2} \|P_0^{ji}\|_1 \leq C_2 |\text{supp } u| |\Lambda|.$$

Combining (7.25) and (7.30) in (7.24), we obtain an upper bound for all  $B$  large enough,

$$(7.31) \quad \mathbb{E}_\Lambda(\text{Tr} E_\Delta) \leq C_W [\text{dist}(B, E) - \delta]^{-2} B \|g\|_\infty |\Delta| |\Lambda|,$$

where  $C_W$  depends on  $M$ ,  $\|u\|_\infty$ , and  $\text{supp } u$ . This proves the theorem.  $\square$

**7.5. Percolation Theory.** The most interesting aspect of the proof of localization for the randomly perturbed Landau Hamiltonian is the use of classical bond percolation theory. This is used in order to prove the initial length-scale exponential decay estimate [H1]( $\gamma_0, \ell_0$ ). We begin by giving a brief overview of percolation theory in two dimensions. These are standard results on percolation theory which can be found in [69] and [22].

Let  $\mathbb{Z}^2$  be the square lattice (the length of the side plays no role in the calculations). A *bond* (edge) of  $\mathbb{Z}^2$  is said to be *occupied* with probability  $p$ ,  $0 \leq p \leq 1$ , and *empty* with probability  $1 - p$ . We are interested in the case when the bonds are independent. This is called Bernoulli bond percolation. The critical percolation probability  $p_c$  is defined as follows. Let  $P_\infty(p)$  be the probability that the origin belongs to an infinite, connected, cluster of occupied bonds. Then, we define

$$p_c \equiv \inf\{p | P_\infty(p) > 0\}.$$

For 2-dimensional Bernoulli bond percolation, it is a major result that  $p_c = 1/2$ . Hence if  $p > p_c$ , occupied bonds percolate to infinity. That is, we can find a connected cluster of occupied bonds running off to infinity with non-zero probability.

Of importance for us are the results concerning the existence of closed circuits of occupied bonds. Let  $r_{n,\ell}$  be a rectangle in  $\mathbb{Z}^2$  of width  $\ell$  and length  $n\ell$ . Let  $R_{n,\ell}$  be the probability that there is a crossing of  $r_{n,\ell}$ , the long way, by a connected path of occupied bonds. This probability is controlled by an exponential factor  $m(p)$ , which is strictly positive for  $p < p_c$  and  $m(p) \searrow 0$  as  $p \nearrow p_c$ . This factor measures the probability that the origin  $0 \in \mathbb{Z}^2$  is connected to  $x \in \mathbb{Z}^2$  by a path of occupied bonds

$$(7.32) \quad P_{0x}(p) \leq e^{-m(p)|x|}.$$

The basic result on the probability that a rectangle  $r_{n,\ell}$  is crossed, the long way, by a cluster of occupied bonds is the following theorem.

**THEOREM 7.6.** *For  $p > p_c$ ,  $R_{n,\ell} \geq 1 - C_0 n \ell e^{-m(1-p)\ell}$ , for some constant  $C_0$ .*

We will relate the existence of an effective potential barrier, which prevents an electron from percolating to infinity, to the existence of annular regions, about any point in space, which contain a closed cluster of occupied sites. From Theorem 7.6, we can derive the probability that such closed clusters exist in annular regions.

Let us write  $r_\ell$  for  $r_{1,\ell}$ , the square box of side length  $\ell$ . An annular region of width  $\ell$ , between two concentric boxes, is denoted by  $a_\ell \equiv r_{3\ell} \setminus r_\ell$ . A closed circuit of occupied bonds in  $a_\ell$  is a connected path of occupied bonds lying entirely within  $a_\ell$ . Using Theorem 7.6 and the *FGK* inequality, one can compute the probability  $A_\ell$  of a closed circuit of occupied bonds in  $a_\ell$  for  $p > p_c$ .

**THEOREM 7.7.** *For any  $p \in [0, 1]$ ,  $A_\ell \geq [R_{3,\ell}(p)]^4$ . In particular, if  $p > p_c$ ,  $\exists 0 < C_0 < \infty$  as in Theorem 7.6, such that*

$$(7.33) \quad A_\ell \geq 1 - 12C_0\ell e^{-m(1-p)\ell}.$$

**7.6. Exponential Decay Estimates from Percolation Theory.** Some of the earlier work on randomly perturbed Landau Hamiltonians concentrated on the *one-band approximation*. Instead of studying the full Hamiltonian  $H_\omega$  directly, one considers the restricted Hamiltonian  $P_0 H_\omega P_0$  on the subspace  $P_0 L^2(\mathbb{R}^2)$  (cf. [40, 41]). In the situation when the random potential is bounded, this approximation is reasonable for large magnetic fields since the separation of the Landau bands is  $\mathcal{O}(B)$ . Hence, one expects the interband interaction to be small. We will prove that this is true. In this way, the one-band approximation will give us some insight into the nature of localization.

Formally, if one neglects the band interaction, the effective Hamiltonian for an electron at energy  $E$ , near the first Landau level  $E_0(B) = B$ , is  $E \sim P_0 H_\omega P_0 = B + P_0 V_\omega P_0 \sim B + V_\omega(x)$ . We assume that the interaction terms,  $P_0 V_\omega Q_0$ , and its adjoint, and the term coming from the other bands,  $Q_0 V_\omega Q_0$ , are small. Consequently, in this approximation, the electron motion is along equipotential lines  $V(x) + B - E = 0$ . Since the potential  $V$  is random, it is natural to estimate the probability that these equipotential lines percolate through a given box. If there is no percolation, the (classical) electron will remain confined to bounded regions. One can expect that the interband interaction will not change this picture. We will first show how to reformulate our problem as a problem in bond percolation. We will then show that the Green's function decays exponentially in  $x$  and  $B$  through regions where  $|V(x) + B - E| > a > 0$ .

Recall that  $V_\omega(x) = \sum_{i \in \mathbb{Z}^2} \omega_i u(x - i)$ , where the single-site potential  $u \geq 0$  and

has support inside a ball of radius  $r_u < 1/\sqrt{2}$ . We define  $r_u$  to be the smallest radius such that  $\text{supp } u \subset B(0, r_u)$ . Consider a new square lattice  $\Gamma \equiv e^{i\pi/4} \sqrt{2} \mathbb{Z}^2$ . The midpoint of each bond of  $\Gamma$  is a site of  $\mathbb{Z}^2$ . We will denote by  $b_j$  the bond of  $\Gamma$  having  $j \in \mathbb{Z}^2$  as its midpoint. For definiteness, we assume  $E \in (B, B + M_0)$ . The other energy interval  $(B - M_0, B)$  can be treated similarly.

**DEFINITION 7.8.** The bond  $b_j$  of  $\Gamma$  is **occupied** if  $\omega_j < (E - B)/2$ . The probability  $\mathbb{P}\{\omega_j < E - B/2\} \equiv p$  is the probability that  $b_j$  is occupied ( $p$  is independent of  $j$  by the *iid* assumption).

Let us assume that the bond  $b_j$  is occupied and consider,

$$(7.34) \quad \mathcal{R}_j \equiv \left\{ x \mid \text{dist}(x, b_j) < 1/\sqrt{2} - r_u \equiv r_1 \right\}.$$

Obviously,  $\mathcal{R}_j$  does not intersect the support of the other single-site potentials centered on  $\mathbb{Z}^2 \setminus \{j\}$  so that  $V(x) = \omega_j u(x-j) \forall x \in \mathcal{R}_j$ . Then, if  $b_j$  is occupied, one has  $V(x) < (E-B)/2 \forall x \in \mathcal{R}_j$  (recall that  $(E-B)/2 > 0$ ). We now assume that there is a closed circuit of occupied bonds  $\mathcal{C} \equiv \bigcup_{j \in \gamma} b_j$ ,  $\gamma \subset \mathbb{Z}^2$  (i.e. a connected union of occupied bonds). We call  $\mathcal{R} \equiv \bigcup_{j \in \gamma} \mathcal{R}_j$  the closed *ribbon* associated with  $\mathcal{C}$ . For all  $x \in \mathcal{R}$ , we have  $V(x) < (E-B)/2$ . If we take  $a \equiv (E-B)/2$ , then

$$(7.35) \quad V(x) + B - E < -a \forall x \in \mathcal{R}.$$

The existence of a closed ribbon  $\mathcal{R}$  so that  $V$  satisfies condition (7.35) is a consequence of the existence of a closed circuit  $\mathcal{C}$  in  $\Gamma$  of occupied bonds. We saw in the previous section how to estimate the probability that  $\mathcal{C}$  exists. We now apply the result of Theorem 7.7 to our reformulation of the equipotential line problem. First, we need a definition. For a subset  $\mathcal{O} \subset \mathbb{R}^2$ , the *in-radius* of  $\mathcal{O}$  is defined to be  $\sup\{R > 0 \mid B_R \subset \mathcal{O}\}$ , where  $B_R$  denotes a ball of radius  $R$ . We will write  $\text{Inrad } \mathcal{O}$  for the in-radius of  $\mathcal{O}$ .

On the lattice  $\Gamma$ , the probability that any bond is occupied is given by

$$p = \int_{-M}^a g(\omega) d\omega,$$

so, under our assumptions on the density  $g$ , if  $a > 0$  then  $p > p_c = 1/2$ , and we are above the critical percolation threshold  $p_c = 1/2$ . Note that when  $E = B$ ,  $a = 0$  so  $p = 1/2 = p_c$ , the critical probability. It follows from Theorem 7.7 that any annular region  $a_\ell \equiv r_{3\ell} \setminus r_\ell$  in  $\Gamma$  of in-radius  $\sqrt{2}\ell \equiv 1/2(3\sqrt{2}\ell - \sqrt{2}\ell)$  and sides parallel to the bonds of  $\Gamma$  contains a closed circuit of occupied bonds with probability given by (7.33). By the argument above, there is a ribbon  $\mathcal{R}$  associated with  $\mathcal{C}$  in  $a_\ell$  whose properties we summarize in the next proposition.

**PROPOSITION 7.1.** Assume (V1) and (V2) and suppose that  $\text{supp } u \subset B_{r_u}(0)$ . Let  $\ell > \sqrt{2}$ ,  $E \in \sigma_0 \setminus \{B\}$ , and  $a > 0$ . Then for  $m(1-p)$  and  $C_0$  as in Theorem 7.8, there exists a ribbon  $\mathcal{R}$  satisfying

$$(7.36) \quad \text{inrad } \mathcal{R} \geq 2 \left( 1/\sqrt{2} - r_u \right);$$

$$(7.37) \quad \begin{aligned} \text{dist}(\mathcal{R}, \partial r_{3\ell}), \text{dist}(\mathcal{R}, \partial r_\ell) &\geq 1/\sqrt{2} + r_u; \\ \mathcal{R} &\subset a_\ell, \end{aligned}$$

and such that

$$(7.38) \quad V(x) + B - E < -a, \forall x \in \mathcal{R},$$

with a probability larger than

$$(7.39) \quad 1 - 12C_0\ell e^{-m(1-p)\ell},$$

where

$$(7.40) \quad p \equiv \int_{-M}^a g(\omega) d\omega.$$

It is important to note that the probability  $p$  defined in (7.40) is a function of  $a$ . In the applications, the constant  $0 < a = (E-B)/2$ , for  $E > B$ . As  $E$  approaches the Landau level  $B$ ,  $a \rightarrow 0$  and  $p \rightarrow p_c$ , so it is important to have a lower bound

on the effective mass  $m(p(a))$ , defined in (7.32), in order to obtain both the region of localization to within  $O(\log B/B)$  of the Landau levels, and good exponential bounds on the eigenfunctions. We refer to [64] for a detailed discussion of this question.

The ribbon  $\mathcal{R}$  i described in Proposition 7.1 provides an effective barrier for the equipotential lines along which a classical electron at energy  $E$  would move. The addition of quantum effects results in exponential decay of the resolvent of the Hamiltonian at energy  $E$  through the ribbon region. We now show how to prove such an estimate. This will lead to our verification of [H1]( $\gamma_0, \ell_0$ ). By the geometric resolvent equation, we will show that it suffices to consider the following ideal situation, where for some  $a > 0$ ,

$$(7.41) \quad V(x) + B - E < -a, \quad \forall x \in \mathbb{R}^2,$$

or, alternately,

$$(7.42) \quad V(x) + B - E > a, \quad \forall x \in \mathbb{R}^2.$$

A condition such as (7.41) with  $E > B$  is satisfied, with a probability given in Proposition 7.1, by a smoothing of the potential  $V_{\mathcal{R}}$  defined as

$$(7.43) \quad V_{\mathcal{R}}(x) = \begin{cases} V(x) & x \in \mathcal{R} \\ 0 & x \in \mathbb{R}^2 \setminus \mathcal{R}. \end{cases}$$

Here we obtain decay estimates on the model Hamiltonian

$$(7.44) \quad H_{\mathcal{R}} \equiv H_A + V_{\mathcal{R}},$$

with  $V_{\mathcal{R}}$  having compact support with nonempty interior and satisfying (7.41) or (7.42).

As is often the case, we will derive exponential decay estimates through the introduction of analytic families of operators associated with  $H_{\omega}$ , and the projectors  $P_0$  and  $Q_0$ . We will obtain *a priori* bounds on these operators using condition (7.41) or (7.42). The type of argument used in the proof of the next lemma is similar to that used in the proof of the Combes-Thomas estimate given in [8], so we will just sketch it.

Let  $\mathcal{O}$  be an open, bounded, connected set in  $\mathbb{R}^2$  with smooth boundary and define  $\rho(x) = \text{dist}(x, \mathcal{O})$ . Let  $\eta \in C_0^\infty(\mathbb{R}^2)$  with  $\eta > 0$  and  $\text{supp } \eta \subset B_1(0)$ . For any  $\epsilon > 0$ , define  $\eta_\epsilon(x) = \eta(x/\epsilon)$ . We consider the smoothed distance function  $\rho_\epsilon(x) \equiv (\eta_\epsilon \star \rho)(x)$ ;  $\text{supp } \rho_\epsilon \subset \mathbb{R}^2 \setminus \{x \mid \text{dist}(x, \mathcal{O}^c) < \epsilon\}$ . We fix  $\epsilon > 0$  small and write  $\rho$  for  $\rho_\epsilon$  below for simplicity. We have  $\|\nabla \rho\|_\infty < C_0/\epsilon$  and  $\|\Delta \rho\|_\infty < C_1/\epsilon^2$ , for constants  $C_0, C_1 > 0$  depending only on  $\eta$  and  $\mathcal{O}$ . This  $\epsilon$  will play no role in the analysis below and, consequently, we absorb it into the constants  $C_0$  and  $C_1$ . We consider one-parameter families of operators defined for  $\alpha \in \mathbb{R}$  as

$$(7.45) \quad H_A(\alpha) \equiv e^{i\alpha\rho} H_A e^{-i\alpha\rho};$$

$$(7.46) \quad H(\alpha) \equiv H_A(\alpha) + V;$$

$$(7.47) \quad P(\alpha) \equiv e^{i\alpha\rho} P e^{-i\alpha\rho}, \text{ etc..}$$

Here, we write  $P$  for the projector  $P_0$  and  $Q \equiv 1 - P$ . For  $\alpha \in \mathbb{R}$ , these families are unitarily equivalent with the  $\alpha = 0$  operators.

LEMMA 7.9. *The family  $H(\alpha)$ ,  $\alpha \in \mathbb{R}$ , has an analytic continuation into the strip*

$$(7.48) \quad S \equiv \{\alpha \in \mathcal{C} \mid |\text{Im } \alpha| < \eta_\rho B^{1/2}, \}$$

as a type  $A$  analytic family with domain  $D(H)$ . The positive constant  $\eta_\rho$  depends only on the distance function  $\rho$ . Furthermore, in this strip  $S$ , one has  $P(\alpha)^2 = P(\alpha)$ , and for some constant  $C_1$  independent of  $\alpha$ ,

$$(7.49) \quad \|P(\alpha)\| < C_1$$

and

$$(7.50) \quad \|Q(\alpha)(H_A(\alpha) - z)^{-1}\| < C_1 B^{-1}, \text{ if } \text{dist}(z, B) \leq B.$$

**Sketch of the Proof.** For  $\alpha \in \mathbb{R}$ , one has

$$(7.51) \quad \begin{aligned} H_A(\alpha) &= (-i\nabla - \alpha\nabla\rho - A)^2 \\ &= H_A - \alpha[\nabla\rho \cdot (p - A) + (p - A) \cdot \nabla\rho] + \alpha^2|\nabla\rho|^2 \\ &= H_A + \alpha^2|\nabla\rho|^2 + i\alpha\Delta\rho - 2\alpha\nabla\rho \cdot (p - A). \end{aligned}$$

We first show that  $H_A(\alpha)$  is a type  $A$  analytic family on the strip  $S$ . For this, it suffices to show that

$$\{\alpha^2|\nabla\rho|^2 + i\alpha\Delta\rho - 2\alpha\nabla\rho \cdot (p - A)\}(H_A - z)^{-1},$$

has norm less than 1 for some  $z \notin \sigma(H_A)$  and  $|\text{Im}\alpha| < \eta_\rho B^{1/2}$  (cf [81], in particular, Theorem IV.1.1 and Chapter VII.2). We now proceed to bound each term for  $z \in C(B) \equiv \{z \mid |z - B| = B\}$ , circle of radius  $B$  centered at  $B$ . Since type  $A$  analyticity is stable under bounded perturbations, it follows that  $H(\alpha)$  is a type  $A$  analytic family of operators on  $S$ . For the projectors, we use the integral representation

$$(7.52) \quad P(\alpha) = \frac{-1}{2\pi i} \int_{C(B)} (H_A(\alpha) - z)^{-1} dz.$$

□

**THEOREM 7.10.** *Assume that  $(V, E, B)$  satisfy (7.41) or (7.42) for some  $a > 0$  and  $E \in \sigma_0 \setminus \{B\}$ . Furthermore, assume that  $\text{supp } V$  is compact with non-empty interior. There exists constants  $C_2 \leq \eta_\rho$ ,  $C_3$ , and  $B_1$ , depending only on  $M_0 \equiv \|V\|_\infty$ ,  $\|\nabla\rho\|_\infty$ , and  $\|\nabla V\|_\infty$ , such that if we define  $\gamma \equiv C_2 \min\{B^{1/2}, aB\}$ , and  $u$  is a solution of*

$$(7.53) \quad (H_A + V - z)u = v, \quad z \equiv E + i\epsilon, \quad \epsilon > 0, \quad E > 0,$$

for some  $v \in D(e^{\gamma\rho})$ , then for  $B > B_1$ ,  $\forall \alpha \in \mathcal{C}$ ,  $|\text{Im}\alpha| < \gamma$ , we have

$$(7.54) \quad u \in D(e^{i\rho\alpha}),$$

$$(7.55) \quad \|e^{i\alpha\rho}Pu\| \leq C_3 a^{-1} \|e^{i\alpha\rho}v\|,$$

and

$$(7.56) \quad \|e^{i\alpha\rho}Qu\| \leq C_3 B^{-1} \|e^{i\alpha\rho}v\|.$$

**Proof.** Let  $v(\alpha) = e^{i\alpha\rho}v$ , so that  $v(\alpha)$  is analytic in the strip  $|\text{Im}(\alpha)| < \gamma$ . Let  $u(\alpha) = e^{i\alpha\rho}u$ ,  $\alpha \in \mathbb{R}$ , i.e.,

$$(7.57) \quad u(\alpha) = (H(\alpha) - z)^{-1}v(\alpha) \equiv ((H - z)^{-1}v)(\alpha),$$

with  $H \equiv H_A + V$ , as above. Since  $V$  is  $H_A$ -compact by assumption,  $H$  has point spectrum, and according to Lemma 7.9 and standard arguments,  $H(\alpha)$  has real spectrum independent of  $\alpha$  in the strip  $S$  defined in (7.48). Then,  $u(\alpha)$  has an analytic continuation in  $|\text{Im}(\alpha)| < \gamma$ , which proves (7.54). Furthermore, this

continuation satisfies (7.53) in the whole strip  $S$ , i.e.  $(H(\alpha) - z)u(\alpha) = v(\alpha)$ , for all  $\alpha \in S$ . Projecting this equation along  $P(\alpha)$  gives

$$(7.58) \quad (B + V - z)(Pu)(\alpha) = (Pv)(\alpha) + ([QVP - PVQ]u)(\alpha),$$

where  $(Pu)(\alpha) \equiv P(\alpha)u(\alpha)$ , etc. Taking the scalar product of (7.58) with  $(Pu)(\alpha)$  results in the inequality,

$$(7.59) \quad \begin{aligned} a\|(Pu)(\alpha)\|^2 &\leq \|(Pu)(\alpha)\| \|(Pv)(\alpha)\| \\ &+ \{ \|(P^*Q)(\alpha)\| \|(QVP)(\alpha)\| \} \|(Pu)(\alpha)\|^2 \\ &+ \|(PVQ)(\alpha)\| \|(Pu)(\alpha)\| \|(Qu)(\alpha)\|. \end{aligned}$$

One can prove that for  $B$  large enough,

$$(7.60) \quad \|(QVP)(\alpha)\| \leq C_4 B^{-1/2},$$

and

$$(7.61) \quad \|(P^*Q)(\alpha)\| \leq C_5 |\operatorname{Im}\alpha| B^{-1/2}.$$

With these estimates, we obtain from (7.59),

$$(7.62) \quad \begin{aligned} (a - C_6\gamma B^{-1})\|(Pu)(\alpha)\|^2 &\leq \|(Pu)(\alpha)\| \|(Pv)(\alpha)\| \\ &+ C_7 B^{-1/2} \|(Pu)(\alpha)\| \|(Qu)(\alpha)\|, \end{aligned}$$

where the constants  $C_6$  and  $C_7$  depend only on  $\|V\|_\infty$ ,  $\|\nabla V\|_\infty$ , and  $\|\nabla\rho\|_\infty$ . To estimate  $\|(Qu)(\alpha)\|$ , it follows from the resolvent equation and (7.57) that

$$(7.63) \quad \begin{aligned} \|(Qu)(\alpha)\| &\leq \|(Q(H_A - z)^{-1}v)(\alpha)\| + \|\{Q(H_A - z)^{-1}QV(Q + P)u\}(\alpha)\| \\ &\leq C_1 B^{-1} \|v(\alpha)\| + C_1 B^{-1} M_0 \|(Qu)(\alpha)\| \\ &+ C_1 B^{-1} M_0 \|(QVPu)(\alpha)\|, \end{aligned}$$

with  $M_0 \equiv \|V\|_\infty < \infty$ . Using an estimate on  $QVP$  similar to (7.60), and taking  $B > 2M_0C_1$ , we obtain,

$$(7.64) \quad \|(Qu)(\alpha)\| \leq 2C_1 B^{-1} \|v(\alpha)\| + C_8 B^{-3/2} \|(Pu)(\alpha)\|,$$

where  $C_8 \equiv 2M_0C_1C_2$ . Substituting (7.64) into (7.62), we obtain

$$(7.65) \quad (a - C_6\gamma B^{-1} - C_7C_8B^{-2})\|(Pu)(\alpha)\| \leq (C_1 + 2C_1C_7B^{-3/2})\|v(\alpha)\|.$$

This proves (7.55) for  $B$  large enough. Inserting (7.55) into (7.64) yields (7.56).  $\square$

**COROLLARY 7.1.** Let  $\mathcal{O}$  be an open, connected, bounded subset of  $\mathbb{R}^2$  with smooth boundary, and suppose  $\mathcal{E} \subset \mathbb{R}^2 \setminus \mathcal{O}$ . Let  $E \in \sigma_0 \setminus \{B\}$  and assume that  $(B, E, V)$  satisfy (7.41) or (7.42) for some  $a > 0$ . Let  $\chi_X$ ,  $X = \mathcal{O}$  and  $\mathcal{E}$ , be bounded functions with support in  $X$  and s.t.  $\|\chi_X\|_\infty \leq 1$ . Then,

$$(7.66) \quad \sup_{\epsilon \neq 0} \|\chi_{\mathcal{E}}(H_A + V - E - i\epsilon)^{-1}\chi_{\mathcal{O}}\| \leq C \max\{a^{-1}, B^{-1}\}e^{-\gamma d},$$

where  $C$  and  $\gamma$  are as in Theorem 7.10 and  $d \equiv \operatorname{dist}(\mathcal{O}, \mathcal{E})$ .

**Proof.** This is an immediate consequence of Theorem 7.10. We set  $\rho(x) \equiv \text{dist}(x, \mathcal{O})$  and choose  $v \equiv \chi_{\mathcal{O}}v$ . Then,  $e^{i\alpha\rho}v = v$ ,  $\forall \alpha \in \mathcal{C}$ . For  $u$  a solution of  $(H_A + V - E - i\epsilon)u = \chi_{\mathcal{O}}v$ , one has  $\forall \alpha \in \mathcal{C}$ ,  $|\text{Im } \alpha| < \gamma$ ,

$$\begin{aligned} & \| \chi_{\mathcal{E}}(H_A + V - E - i\epsilon)^{-1} \chi_{\mathcal{O}}v \| = \| \chi_{\mathcal{E}}(P + Q)u \| \\ & \leq e^{-d(\text{Im } \alpha)} \{ \| e^{-i\alpha\rho}Pu \| + \| e^{-i\alpha\rho}Qu \| \} \\ & \leq e^{-d(\text{Im } \alpha)} C \max \{ a^{-1}, B^{-1} \} \| v \|, \end{aligned}$$

by Theorem 7.17. Taking  $\text{Im } \alpha \rightarrow \gamma$ , we obtain (7.66).  $\square$

**7.7. Verification of Hypothesis [H1]( $\gamma_0, \ell_0$ ).** It remains to verify the initial length-scale hypothesis. The proof of localization then follows the lines as for Schrödinger operators as given in chapter 6, but requires the bootstrap MSA of Germinet and Klein [62, 63]. In particular, we see from (7.67) that the exponential decay of the localized resolvent at initial length scale  $\ell_0$  is approximately  $\sim C_1 \min(aB, B^{1/2})/\ell_0$ . From this we see that  $\ell_0$  cannot be too big relative to  $B$  and  $a$ . On the other hand, the MSA requires the initial length scale to be large enough. These two opposing demands require a more refined multiscale analysis than the one presented in chapter 8 and we refer to [62, 63]. The underlying ideas, however, are similar.

**PROPOSITION 7.2.** Let  $\chi_2$  be any function,  $\|\chi_2\|_{\infty} \leq 1$ , supported on  $\Lambda_{\ell} \cap \text{Ext}\mathcal{R}$ , where  $\text{Ext}\mathcal{R} \equiv \{x \in \mathbb{R}^2 \mid \lambda x \notin \mathcal{R} \forall \lambda \geq 1\}$ , so that, in particular,  $\text{supp } \chi_2 \cap \mathcal{R} = \emptyset$ . For any  $E \in \sigma_0 \setminus \{B\}$ ,  $\delta > 0$ ,  $\epsilon > 0$ , and  $a > 0$ , we have

$$(7.67) \quad \sup_{\epsilon \neq 0} \| \chi_2 R_{\Lambda_{\ell}}(E + i\epsilon) \chi_{\ell/3} \| \leq C e^{-\gamma d} \max \{ a^{-1}, B^{-1} \} \cdot \max \{ \delta^{-1}, (2M_0 + |E|)\delta^{-2} \},$$

where  $C$  depends on  $C_3$  of Theorem 7.10, the constants  $\chi_2$  and  $\gamma$  are defined in Theorem 7.10, and  $d \equiv (r_1 - 3\epsilon)/2$  ( $r_1 \equiv \text{inrad}\mathcal{R}$ ), with a probability larger than

$$(7.68) \quad 1 - \{ C \ell e^{-m\ell} + C_W [\text{dist}(E, B) - \delta]^{-2} \|g\|_{\infty} \delta B \ell^2 \}.$$

In particular, for  $\chi_{\ell, \delta}$  defined above and  $E \in \sigma_0$  with  $a = (E - B)/2 = \mathcal{O}(B^{-1+\sigma})$ , any  $\sigma > 0$ , we have that for any  $\ell_0 > \sqrt{2}$  and large enough, and any  $\xi > 4$ ,  $\exists B(\ell_0) > 0$  such that  $\forall B > B(\ell_0)$ , [H1] ( $\gamma_0, \ell_0$ ) holds for some  $\gamma_0 > \gamma d/4\ell_0 > 0$ , so that  $\gamma_0 = \mathcal{O}\{\min(B^{1/2}, B^{\sigma})\}$ .

**Proof.**

1. By Proposition 7.1, there exists a constant  $B_0$  such that  $B > B_0$  implies there exists a ribbon  $\mathcal{R} \subset \Lambda_{\ell} \setminus \Lambda_{\ell/3}$  (with a probability given by (7.39) satisfying

$$(7.69) \quad \text{dist}(\mathcal{R}, \partial\Lambda_{\ell}), \text{ and } \text{dist}(\mathcal{R}, \partial\Lambda_{\ell/3}) > 1/\sqrt{2} + r_u > 0,$$

and

$$(7.70) \quad r_1 \equiv \text{inrad}\mathcal{R} > 2 \left( 1/\sqrt{2} - r_u \right),$$

and such that

$$(7.71) \quad V(x) + B - E > -a \forall x \in \mathcal{R}, \quad a = E - B/2.$$

We assume  $E > B$ ; similar arguments hold for  $E < B$ . For any  $\epsilon > 0$ ,  $3\epsilon \ll r_1$ , define the border of  $\mathcal{R}$  by

$$\mathcal{R}_\epsilon \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \partial\mathcal{R}) < \epsilon\}.$$

Then  $\mathcal{R}_\epsilon \equiv \mathcal{R}_\epsilon^+ \cup \mathcal{R}_\epsilon^-$ , where  $\mathcal{R}_\epsilon^\pm$  are two disjoint, connected subsets of  $\mathcal{R}$ . Let  $\mathcal{C} \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \mathcal{R}_\epsilon^+) = \text{dist}(x, \mathcal{R}_\epsilon^-)\}$ ;  $\mathcal{C}$  is a closed, connected path in  $\mathcal{R}$ . Let  $\mathcal{C}_\epsilon \equiv \{x \in \mathcal{R} \mid \text{dist}(x, \mathcal{C}) < \epsilon/2\} \subset \mathcal{R}$ , so that  $\text{inrad}\mathcal{C}_\epsilon = \epsilon$  and

$$(7.72) \quad \text{dist}(\mathcal{C}_\epsilon, \mathcal{R}_\epsilon^\pm) \geq (r_1 - 3\epsilon)/2.$$

This is strictly positive. Because of this, we can adjust  $\mathcal{C}_\epsilon$  so that  $\partial\mathcal{C}_\epsilon$  is smooth. We need two,  $C^2$ , positive cut-off functions. Let  $\chi_{\mathcal{R}} > 0$  satisfy  $\chi_{\mathcal{R}}|_{\mathcal{C}_\epsilon} = 1$  and  $\text{supp}|\nabla\chi_{\mathcal{R}}| \subset \mathcal{R}_\epsilon$ . Let  $\chi_1$  satisfy  $\chi_1|_{\Lambda_{\ell/3}} = 1$  and  $\text{supp}|\nabla\chi_1| \subset \mathcal{C}_\epsilon$ . By simple commutation, we have (with  $\chi_2$  as in the proposition),

$$(7.73) \quad \begin{aligned} \chi_2 R_{\Lambda_\ell}(E + i\epsilon)\chi_{\ell/3} &= \chi_2 R_{\Lambda_\ell} \chi_1 \chi_{\ell/3} \\ &= \chi_2 R_{\Lambda_\ell} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3} \\ &= \chi_2 R_{\Lambda_\ell} \chi_{\mathcal{R}} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3}. \end{aligned}$$

Next, denote by  $R_{\mathcal{R}}$  the resolvent of  $H_{\mathcal{R}}$  defined in (7.44). The *GRE* relating  $R_{\Lambda_\ell}$  and  $R_{\mathcal{R}}$  is

$$(7.74) \quad R_{\Lambda_\ell} \chi_{\mathcal{R}} = \chi_{\mathcal{R}} R_{\mathcal{R}} + R_{\Lambda_\ell} W(\chi_{\mathcal{R}}) R_{\mathcal{R}}.$$

Substituting (7.74) into (7.73) and noting that  $\chi_2 \chi_{\mathcal{R}} = 0$ , we obtain

$$(7.75) \quad \chi_2 R_{\Lambda_\ell} \chi_{\ell/3} = \chi_2 R_{\Lambda_\ell} W(\chi_{\mathcal{R}}) R_{\mathcal{R}} W(\chi_1) R_{\Lambda_\ell} \chi_{\ell/3}.$$

Note that from (7.72) and the choice of  $\chi_{\mathcal{R}}$  and  $\chi_1$ , we obtain that

$$(7.76) \quad \text{dist}(\text{supp}W(\chi_{\mathcal{R}}), \text{supp}W(\chi_1)) \geq (r_1 - 3\epsilon)/2.$$

We apply Wegner's estimate, Theorem 7.4 to control the two  $R_{\Lambda_\ell}$  factors in (7.75), and the decay estimate, Corollary 7.1, to control the factor  $R_{\mathcal{R}}$ , which is possible due to the localization of  $W(\chi_{\mathcal{R}})$  and  $W(\chi_1)$  and (7.76).

2. To estimate the  $R_{\mathcal{R}}(E + i\epsilon)$  contribution, we use Corollary 7.1 with  $\mathcal{O} \equiv \mathcal{C}_\epsilon$  and  $\mathcal{E} = \mathcal{R}_\epsilon$ . Let  $\chi_X$ ,  $X = \mathcal{O}$  and  $\mathcal{E}$ , be a characteristic function on these sets. Then  $W(\chi_{\mathcal{R}})\chi_{\mathcal{E}} = W(\chi_{\mathcal{R}})$  and  $\chi_{\mathcal{O}}W(\chi_1) = W(\chi_1)$ . Inserting these localization functions into (7.75), we obtain from Corollary 7.1,

$$(7.77) \quad \|\chi_{\mathcal{E}} R_{\mathcal{R}}(E + i\epsilon)\chi_{\mathcal{O}}\| \leq C \max\{a^{-1}, B^{-1}\} e^{-\gamma d},$$

with probability larger than

$$(7.78) \quad 1 - C\ell e^{-m\ell},$$

for some  $m = m(1-p) > 0$  (see (7.39) and  $0 < C < \infty$ ). The factor  $d$  satisfies

$$(7.79) \quad d \geq (r_1 - 3\epsilon)/2,$$

where  $r_1 \equiv \text{inrad}\mathcal{R}$  as in (7.36).

3. Next, we turn to

$$(7.80) \quad W(\chi_1) R_{\ell}(E + i\epsilon)\chi_{\ell/3},$$

and

$$(7.81) \quad \chi_2 R_{\ell}(E + i\epsilon)W(\chi_{\mathcal{R}}),$$

where we write  $R_\ell$  for  $R_{\Lambda_\ell}$  for short. We will bound  $R_\ell$  by Wegner's estimate and the terms (7.80)–(7.81) via Proposition 7.2. From Theorem 7.4, we have for any  $\delta > 0$ ,

$$(7.82) \quad \|R_\ell(E + i\epsilon)\| < \delta^{-1},$$

with probability larger than

$$(7.83) \quad 1 - C_W[\text{dist}(E, B) - \delta]^{-2} \|g\|_\infty \delta B \ell^2.$$

From Proposition 7.2, both (7.80) and (7.81) are bounded above by

$$(7.84) \quad C_4 \max \left\{ \delta^{-1/2}, (2M + |E|)^{1/2} \delta^{-1} \right\},$$

with probability at least (7.39). The constant  $C_4$  depends on  $\partial^\alpha \chi_1$ , for  $|\alpha| = 0, 1, 2$ . 4. Using the estimate  $P(A \cap B) \geq P(A) + P(B) - 1$ , and (7.75)–(7.77), (7.77), and (7.82)–(7.83), we find

$$(7.85) \quad \begin{aligned} \|\chi_2 R_\ell(E + i\epsilon) \chi_{\ell/3}\| &\leq 2C \max \{a^{-1}, B^{-1}\} \cdot \max \{\delta^{-1}, \\ &(2M_0 + |E|) \delta^{-2}\} \cdot e^{-\gamma d}, \end{aligned}$$

with probability at least

$$(7.86) \quad 1 - \{C \ell e^{-m\ell} + C_W[\text{dist}(E, B) - \delta]^{-2} \|g\|_\infty \delta B \ell^2\}.$$

This proves the first part of the proposition.

5. To estimate  $W(\chi_{\ell, \delta}) R_\ell \chi_{\ell/3}$ , we again use Proposition 7.2 which gives

$$(7.87) \quad \begin{aligned} \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\|^2 &\leq \|\chi_2 R_\ell \chi_{\ell/3}\| + (2M + |E|) \|\chi_2 R_\ell \chi_{\ell/3}\|^2 \\ &+ 2 \max_{i=1,2} \|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\| \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\|. \end{aligned}$$

Since  $\partial_i \chi_2$  satisfies the same condition as  $\chi_2$ , the factor  $\|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\|$  in (7.87) satisfies the estimate (7.85) with possibly a different constant. Solving the quadratic inequality (7.87), we obtain

$$(7.88) \quad \begin{aligned} \|\chi_2(p - A)_i R_\ell \chi_{\ell/3}\| &\leq \max_{i=1,2} \left\{ \|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\| \right. \\ &+ \left[ \|(\partial_i \chi_2) R_\ell \chi_{\ell/3}\|^2 + (\|\chi_2 R_\ell \chi_{\ell/3}\| \right. \\ &\left. \left. + (2M_0 + |E|) \|\chi_2 R_\ell \chi_{\ell/3}\|^2) \right]^{1/2} \right\}, \end{aligned}$$

which can be estimated as in (7.85). Finally, we write

$$(7.89) \quad \|W(\chi_{\ell, \delta}) R_\ell \chi_{\ell/3}\| \leq \|(\Delta \chi_{\ell, \delta}) R_\ell \chi_{\ell/3}\| + 2 \sum_{j=1}^2 \|(\partial_j \chi_{\ell, \delta})(p - A)_j R_\ell \chi_{\ell/3}\|,$$

which can be estimated from (7.77) with  $\chi_2 \equiv \Delta \chi_{\ell, \delta}$  and (7.88) with  $\chi_2 \equiv (\partial_j \chi_{\ell, \delta})$ .

6. We now show that for any  $\ell_0$  large enough,  $\exists B_0 \equiv B_0(\ell_0)$  such that for all  $B > B_0$ , condition [H1]( $\ell_0, \gamma_0$ ) is satisfied with  $\gamma_0 = \mathcal{O}\{\min(B^{1/2}, B^\sigma)\}$ . We take  $E \in [B - M_0, B - \mathcal{O}(B^{-1})] \cup I_0(B) \cap \sigma_0$  and  $a = (E - B)/2 = \mathcal{O}(B^{-1+\sigma})$ , for any  $\sigma > 0$ . First, we require that (7.66) be bounded above by  $e^{-\gamma d/2}$ . This leads to the condition

$$(7.90) \quad C \delta^{-2} B^{2-\sigma} e^{-\gamma d} \leq e^{-\gamma d/2},$$

where  $\gamma = C_2 \min\{B^{1/2}, B^\sigma\}$ . This condition implies that we must choose  $\delta$  in (7.90) to satisfy

$$(7.91) \quad \delta > B^{1-(\sigma/2)} e^{-\gamma d/4}.$$

If we now define  $\gamma_0 \equiv \gamma d/4\ell_0$ , we find that

$$(7.92) \quad \|W(\chi_{\ell,\delta}) R_\ell \chi_{\ell/3}\| \leq e^{-\gamma_0 \ell_0}.$$

Next, in order to obtain the probability estimate (5.28), it follows from (7.67) that we must require that

$$(7.93) \quad C\ell e^{-m\ell_0} + C_2 B^{3-2\sigma} \delta \ell_0^2 \leq \ell_0^{-\xi},$$

or, for all  $\ell_0$  large,

$$(7.94) \quad C_3 B^{3-2\sigma} \delta \ell_0^2 \leq \ell_0^{-\xi},$$

for some  $\xi > 4$ . We can choose  $\delta$  so that both conditions (7.90) and (7.94) are satisfied provided the condition

$$(7.95) \quad \ell_0^{\xi+2} < B^{3/2-(5/2)\sigma} e^{\gamma d/4},$$

is satisfied for some  $\xi > 4$ . It is clear from the definition of  $\gamma$ , that for any  $\ell_0$ , there exists a  $B_0 \equiv B_0(\ell_0)$  such that condition (5.28) is satisfied for all  $B > B_0$ . We remind the reader again that  $m$  is a function of  $a = (E - B)/2$ , so this must be kept in mind while making these estimates. This completes the proof of the theorem.  $\square$

## 8. Fixed-Energy Multiscale Analysis

**8.1. Introduction.** In this chapter, we complete the discussion of chapter 5 on the multiscale analysis (MSA) by giving the proof of Lemma 5.4. We introduced two hypotheses on the finite-volume Hamiltonians, called [H1]( $\gamma_0, \ell_0$ ) and the Wegner estimate [W], in chapter 5. We show in this chapter that these two hypotheses, [H1]( $\gamma_0, \ell_0$ ) and [W], allow us to prove the hypotheses [H1] and [H2] of chapter 5. Hypotheses [H1] and [H2] are required for Theorems 5.3 and 5.6. As a result of the proofs in this chapter, we will have shown that these two hypotheses [H1]( $\gamma_0, \ell_0$ ) and [W] are the starting points for deriving the almost sure exponential decay of the infinite-volume Green's function. We saw in chapter 3 that this estimate on the resolvent is essential for eliminating the continuous singular spectrum almost surely. The proof given here is a simplified and modified version of the multiscale analysis for lattice models developed in the work of Fröhlich and Spencer [56], Spencer [120], and von Dreifus and Klein [132]. A summary of this method for lattice models is given in the book of Carmona and Lacroix [21]. The proof for random operators on  $\mathbb{R}^d$  requires the use of geometric methods common in this book. The MSA used here is a *fixed-energy* MSA. There are other methods, pioneered by von Dreifus and Klein [132] in which all energies in a fixed interval are treated simultaneously. In this method, the techniques of chapter 3 are replaced by an analysis of the generalized eigenfunctions. This is discussed in the next section. The energy-interval method for Schrödinger operators on  $L^2(\mathbb{R}^d)$  is presented in [62]. Let us first recall Lemma 5.4.

LEMMA 8.1. *Let  $\gamma_0 > 0$  and assume [W]. There exists a minimum length scale  $l^* = l^*(\gamma_0, q, d, C_W)$  such that if [H1]( $\gamma_0, l_0$ ) holds for  $l_0 > l^*$ , then for  $l_k \equiv l_0^{(3/2)^k}$ , and for some  $\xi > 2d$ , there exists a finite constant  $\kappa > 0$  such that for each  $k$*

$$(8.1) \quad \mathbb{P}\{\sup_{\epsilon > 0} \|W_{l_k} R_{l_k}(E + i\epsilon)\chi_{l_k/3}\| \leq e^{-\kappa\gamma_0 l_k}\} \geq 1 - l_k^{-\xi}.$$

This lemma shows that the two assumptions [H1]( $\gamma_0, l_0$ ) and [W] are sufficient to prove the hypotheses of Theorem 5.3. Theorem 5.3 is the almost sure boundedness of the boundary-value of  $\|(H_\omega - E - i0^+)^{-1}c\|$ , for suitable functions of compact support  $c$ .

**8.2. Fixed Energy vs. Energy Interval Hypotheses.** As has been discussed in previous chapters, we take the fixed energy approach to proving localization. Another approach, introduced in Fröhlich, Spencer, Martinelli, and Scoppola [55], and simplified in the work of von Dreifus and Klein [132], considers working in a fixed-energy interval. The technical aspects of these two approaches differ in two respects. First, the energy interval approach does not use the results on the perturbation of singular spectra presented in chapter 3. Instead, the approach uses the result that the spectrum of a Hamiltonian is characterized by the existence of polynomially-bounded solutions to the eigenvalue equation  $H_\omega\psi_\omega = E\psi_\omega$ , for spectrally almost all energies and fixed  $\omega$ . Secondly, the probabilistic estimates required for the inductive step, presented in section 8.3, are different. We present these to contrast and compare them with our hypotheses [H1]( $\gamma_0, l_0$ ) and [W].

The basic finite-volume Hamiltonian assumption used by von Dreifus and Klein [132] is the following.

[vDK1]. *Let  $I \in \mathbb{R}$  be an energy interval, and let  $p > d$ ,  $L_0 > 0$ , and take  $1 < \alpha < 2p/d$ . We set  $L_{k+1} \equiv L_k^\alpha$ . Then, for any  $k = 0, 1, \dots$ , we have*

$$(8.2) \quad \mathbb{P}\{\text{for any } E \in I, \text{ either } \Lambda_k(x) \text{ or } \Lambda_k(y) \text{ is } \gamma\text{-good at energy } E\} \geq 1 - L_k^{-p},$$

for any  $x, y \in \Gamma_k$  with  $\|x - y\| > L_k$ .

Note that unlike our hypotheses, control over the resolvent is required at all energies  $E \in I$ . Given hypothesis [vDK1], the main step replacing the Theorem 5.3, and the perturbation of singular spectra analysis of chapter 3, is the use of the generalized eigenfunctions associated with  $H_\omega$ . A *generalized eigenfunction* at energy  $E$  is a nonzero, polynomially bounded solution of the equation  $H\psi = E\psi$ . An energy  $E$  for which there exists a generalized eigenfunction is called a *generalized eigenvalue*. The main result about these generalized eigenfunctions is the following theorem. A proof of this theorem can be found, for example, in [35].

THEOREM 8.2. *Let  $H$  be a self-adjoint operator on a separable Hilbert space. With respect to the spectral measure of  $H$ , almost every energy  $E \in \mathbb{R}$  is a generalized eigenvalue.*

One defines a set  $\Omega_0$  of configurations with  $\mathbb{P}(\Omega_0) = 1$ , and for which not too many singular sites occur for the corresponding local Hamiltonians. One then proves that for  $\omega \in \Omega_0$ , the generalized eigenfunctions of  $H_\omega$  decay, in fact, exponentially fast. Hence, they are eigenfunctions.

Condition (8.2) is not easy to prove directly. As in chapter 5, it is reduced to a consequence of two fixed length-scale hypotheses. The inductive part of the multiscale analysis then involves showing that these two fixed length-scale hypotheses

imply condition (8.2). The fixed length-scale hypotheses of von Dreifus and Klein [132] are the following.

[P1]. *There exists an  $L_0 > 0$ , so that for some  $E_0 \in \mathbb{R}$ , we have*

$$(8.3) \quad \mathbb{P}\{\Lambda_{L_0}(0) \text{ is } \gamma\text{-good at energy } E_0\} \geq 1 - L_0^{-p},$$

for some  $p > d$ .

[P2]. *For some  $\eta > 0$ , and for all  $E \in (E_0 - \eta, E_0 + \eta)$ , and for all  $L > L_0$ , there is a  $\beta$  satisfying  $0 < \beta < 1$ , and a  $q > 4p + 6d$ , such that*

$$(8.4) \quad \mathbb{P}\{\text{dist}(E, \sigma(H_{\Lambda_L}(0))) < e^{L^\beta}\} \leq L^{-q}.$$

These two hypotheses correspond to our hypotheses [H1]( $\gamma_0, \ell_0$ ) and [W], respectively. It is easy to show by using the first resolvent equation that condition [P1] can be extended to a small interval of energy about  $E_0$ . Condition [P2] is similar to the Wegner estimate [W]. A Wegner estimate at any fixed energy can always be extended to an interval about that energy.

**8.3. The Inductive Step.** The key to the proof of Lemma 5.4 is an inductive step. The proof of this step goes back to the fundamental work of Fröhlich and Spencer [56], and was successively refined by Spencer [120] and von Dreifus and Klein [132]. Like most of the proofs in the field of random operators, it has two components. First, one makes an assumption about the scarcity of regions which will produce eigenvalues for the local Hamiltonians close to the energy of interest. These regions are called resonant regions. Their presence causes small denominators in the iterated geometric resolvent equation. Second, one computes the probability that the bad, resonant regions occur.

For a given  $l$ , we will consider a covering of  $\mathbb{R}^d$  by boxes  $\Lambda_{l/3}(x)$ , centered at points  $x$  on a scaled lattice  $\Gamma_\ell \equiv \frac{l}{3}\ell\mathbb{Z}^d$ . In particular, the box  $\Lambda_l(0)$  is covered by  $3^d$  such boxes of side length  $\ell/3$ . We define a *family of  $n$  chained  $l$  boxes* to be a sequence of boxes  $\Lambda_l(x_k), k = 1, \dots, n$ , such that  $x_k \in \Gamma_l \cap \Lambda_l(x_{k-1})$  and  $x_k \neq x_{k-1}$ . For later purposes, we note that  $d(x_k, \partial\Lambda_l(x_{k-1})) = \frac{1}{6}l$ , and that  $|x_n - x_1| \leq \frac{\sqrt{2}}{3}(n-1)l$ . Whenever convenient, we write  $\chi_l, R_l$ , etc. for  $\chi_{\Lambda_l}, R_{\Lambda_l}$ , etc. Let us recall that for a fixed  $\delta > 0$ , we define a subbox  $\tilde{\Lambda}_l$  of  $\Lambda_l$  by  $\tilde{\Lambda}_l \equiv \{x \in \Lambda_l \mid \text{dist}(x, \partial\Lambda_l) > \delta\}$ .

We single out a family of boxes for which the resolvent of the local Hamiltonian has exponential decay across the box. This condition is roughly equivalent to saying that the eigenvalues of  $H_{\Lambda_l}$  are far separated from the energy  $E$  of interest.

DEFINITION 8.3. A box  $\Lambda_l$  is called a  $\gamma$ -good  $l$ -box at energy  $E$ , for some  $\gamma > 0$ , if

$$(8.5) \quad \sup_{\epsilon > 0} \|W(\chi_l)R_{\Lambda_l}(E + i\epsilon)\chi_{\Lambda_{l/3}}\| < e^{-\gamma l}.$$

We show that if there are not too many disjoint  $\gamma$ -bad  $l$ -boxes  $\Lambda_l$  in a bigger box  $\Lambda_{l'}$ , then the assumption that  $\Lambda_l$  is  $\gamma$ -good implies that  $\Lambda_{l'}$  is  $\gamma'$ -good without losing too much in the exponent  $\gamma$ .

LEMMA 8.4. *Let  $l' \gg 4l$  be chosen large enough and assume [W]. If  $\gamma \gg d/l$  and*

$$(8.6) \quad \mathbb{P}\{\Lambda_{l'} \text{ is a } \gamma\text{-good box}\} \geq 1 - \eta,$$

then we have

$$(8.7) \quad \mathbb{P}\{\Lambda_{l'} \text{ is a } \gamma'\text{-good box}\} \geq 1 - \eta',$$

where, for a constant  $c_d$  depending only on  $d$  and any  $\xi > 2d$ ,

$$(8.8) \quad \eta' = c_d \left(\frac{l'}{l}\right)^{2d} \eta^2 + \frac{1}{2}(l')^{-\xi},$$

and

$$(8.9) \quad \gamma' = \left(\gamma - \frac{c_1}{l}\right) \left(1 - \frac{4l}{l'}\right) - c_2 \frac{\log l'}{l'},$$

for some constants  $c_1, c_2 > 0$  depending on  $d, \xi$ , and  $C_W$ .

**Proof.** Without restricting generality we consider the cube  $\Lambda_{l'}$  centered at the origin.

1. We first assume that  $\Lambda_{q l'}$ , for  $q \geq 2$ , does not contain two non-overlapping  $\gamma$ -bad boxes  $\Lambda_l$  centered on the scaled lattice  $\Gamma_l$ . This means that there may be only one cluster  $\Lambda_b$  of overlapping  $\gamma$ -bad  $l$ -cubes in  $\Lambda_{q l'}$  centered on  $\Gamma_l$ . The diameter of this bad region  $\Lambda_b$  is at most  $\frac{5\sqrt{2}}{3}l$ . We now consider a covering of  $\Lambda_{l'/3}(0)$  by boxes  $\Lambda_{l/3}$  with centers in  $\Gamma_l \cap \Lambda_{l'/3}(0)$ . There are  $(l'/l)^d$  such centers and we call  $x$  one of them. We also consider a covering of  $\Lambda_{l'}(0) \setminus \tilde{\Lambda}_{l'}(0)$  by such cubes  $\Lambda_{l/3}$  with centers in  $\Lambda_{q l'}(0) \cap \Gamma_l$ . There are  $(d\delta/l')(3l'/l)^d$  such centers. Let  $y$  be one of them so that  $\|x - y\| \geq (l'/3 - \delta/2)$ . We want to estimate

$$(8.10) \quad \|\chi_{l/3}(x)R_{q l'}(E + i\epsilon)\chi_{l/3}(y)\|,$$

for the multiplier  $q \geq 2$ . We need the estimate for  $q \geq 2$  in order to compensate for the boundary, as we will explain below.

2. We estimate this norm by constructing suitable families of chained  $l$ -boxes, which are  $\gamma$ -good  $l$ -boxes with a certain probability, starting at  $x_1 \equiv x$  or  $y_1 \equiv y$ . By iterating the geometric resolvent equation (GRE) presented in section 5.1 (5.11) along such a chain, we obtain an estimate for the decay of the operator in (8.10) due to (8.1) and an a priori estimate derived from the Wegner estimate. In order to begin this process, we write the GRE as

$$(8.11) \quad \chi_{l/3}(x)R_{q l'} - R_l \chi_{l/3}(x) = R_l W_l(x_1)R_{q l'}.$$

Substituting this into (8.10), and using the fact that  $\chi_{l/3}(x)\chi_{l/3}(y) = 0$ , we obtain

$$(8.12) \quad \chi_{l/3}(x_1)R_{q l'}\chi_{l/3}(y) = \chi_{l/3}(x_1)R_l W_l(x_1)R_{q l'}\chi_{l/3}(y).$$

3. Let  $\chi_l^{(m)}$  be a covering of  $\partial\Lambda_l(x_1) \equiv \text{supp } W_l(x_1)$  by cubes  $\Lambda_{l/3}$  centered on  $\Gamma_l \cap \Lambda_l(x_1)$ . One needs  $b_d \leq (d\delta/l)3^d < 3^d$  such cubes. We decompose the perturbation  $W_\ell(x_1)$  appearing in (8.12) as

$$(8.13) \quad W_l(x_1) = \sum_{m=1}^{b_d} W_l(x_1)\chi_{l/3}^{(m)}.$$

Substituting this into (8.12), one obtains from (8.10)

$$(8.14) \quad \begin{aligned} & \|\chi_{l/3}(x_1)R_{q l'}\chi_{l/3}(y_1)\| \\ & \leq b_d \|\chi_{l/3}(x_1)R_l W_l(x_1)\| \left\{ \max_m \|\chi_{l/3}^{(m)} R_{q l'}\chi_{l/3}(y_1)\| \right\}. \end{aligned}$$

If  $x_1 \notin \Lambda_b$  and if  $x_2$  denotes the center in  $\Gamma_l$  for which the maximum on the right hand side of (8.14) is reached, one obtains

$$(8.15) \quad \|\chi_{l/3}(x_1)R_{q^{l'}}\chi_{l/3}(y_1)\| \leq b_d e^{-\gamma l} \|\chi_{l/3}(x_2)R_{q^{l'}}\chi_{l/3}(y_1)\|.$$

4. If  $x_2 \notin \Lambda_b$  and  $\Lambda_l(x_2) \cap \Lambda_{l/3}(y_1) = \emptyset$ , one can continue this procedure and construct in this way a family of  $n$  chained  $\gamma$ -good boxes  $\Lambda_l(x_k)$ ,  $k = 1, \dots, n$ . The chain is stopped when  $d(x_n, \Lambda_b) = l/6$  or  $\|x_n - y\| = l/3$ . In the case that  $\|x_n - y\| = l/3$ , the number of elements in the chain satisfies  $n \geq 3 \|x - y\|/l$  so that, assuming  $\gamma > \log b_d/l$ :

$$(8.16) \quad \|\chi_{l/3}(x_1)R_{q^{l'}}\chi_{l/3}(y)\| \leq e^{-3(\gamma - (\log b_d)/l)\|x-y\|} \|\chi_{l/3}(x_n)R_{q^{l'}}\chi_{l/3}(y)\|.$$

In the first case when  $d(x_n, \Lambda_b) = l/6$ , one starts a new chain  $(y_1, \dots, y_m)$  from  $y_1 = y$ . Once again, the chain is stopped when  $d(y_m, \Lambda_b) = l/6$  or  $|x_n - y_m| = l/3$ . In this last case, the number of sites in the chain starting at  $y_1$  satisfies  $m \geq 3 |x_n - y_1|/l$ , so that  $m + n \geq 3\|x - y\|/l$ , and an inequality like (8.16), with  $y_m$  instead of  $y$ , still holds. In the first case for which  $d(y_m, \Lambda_b) = l/6$ , we find that

$$(8.17) \quad (m + n - 2)\frac{l}{3} + 2l \geq \|x - y\|,$$

and one obtains

$$(8.18) \quad \|\chi_{l/3}(x)R_{q^{l'}}\chi_{l/3}(y)\| \leq e^{-(\gamma - (\log b_d/l)[3\|x-y\| - 4l])} \|R_{q^{l'}}\|.$$

Note that  $\|x - y\| \geq l'/3$ . We now turn this estimate into an estimate for

$$(8.19) \quad \|\chi_{l'/3}R_{q^{l'}}(E + i\epsilon)\tilde{\chi}_{l'}\|.$$

We use our covering of  $\Lambda_{l'/3}$  and of the boundary region  $\Lambda_{l'} \setminus \tilde{\Lambda}_{l'}$ . Since each term satisfies an identical estimate, we obtain

$$(8.20) \quad \|\chi_{l'/3}R_{q^{l'}}(E + i\epsilon)\tilde{\chi}_{l'}\| \leq \frac{\delta d}{l'} 3^d \left(\frac{l'}{l}\right)^{2d} e^{-(\gamma - (\log b_d)/l)(l' - 4l)} \|R_{q^{l'}}(E + i\epsilon)\|,$$

where  $\tilde{\chi}_{l'}$  is a characteristic function of the boundary region  $\Lambda_{l'} \setminus \tilde{\Lambda}_{l'}$ .

5. The main reason for introducing  $q \geq 2$  is twofold. First, we must make sure we can cover all of the boundary region  $\Lambda_{l'} \setminus \tilde{\Lambda}_{l'}$  with cubes  $\Lambda_{l/3}$  centered on the lattice  $\Gamma_l$ . Secondly, in the case when the chain starting at  $x_1$  stops because it meets the bad regions  $\Lambda_b$ , we must be sure that we can iterate the chain starting at  $y_1$  sufficiently far enough so that the combination  $(n + m)$  satisfies an estimate like (8.17). For example, it might occur that a chain starting at  $y_1$  in the boundary region has to be stopped because it hits the boundary of  $\Lambda_{l'}$  before  $m$  is large enough. To avoid this, we continue to iterate into the expanded region  $\Lambda_{q^{l'}}$ , if necessary. The price that we pay for this is not serious: we obtain a factor  $q^d$  in the coefficient of the resolvent estimate and in the probability estimate. Since these constants are independent of the length scale, they do not affect the results.

6. From (8.20), one obtains a similar bound for  $\|\chi_{l'/3}R_{q^{l'}}(E + i\epsilon)W_{l'}\|$ , with an extra multiplicative constant which only depends on  $\|\nabla\chi_{l'}\|_\infty$  and  $\|\Delta\chi_{l'}\|_\infty$ . This

is easily seen by writing  $W_{l'} = -2\nabla \cdot \nabla \chi_{l'} - \Delta \chi_{l'}$ , and using the following inequality:

$$\begin{aligned} \|\nabla(\nabla \chi_{l'} u)\|^2 &= 2\operatorname{Re} \langle -\Delta u, |\nabla \chi_{l'}|^2 u \rangle + 2\|(\Delta \chi_{l'})u\|^2 \\ &\leq 2|\operatorname{Re} \langle (H_{\Lambda_{q_{l'}}} - E - i\epsilon)u, |\nabla \chi_{l'}|^2 u \rangle| + 2\sum_{i=1}^d \|\nabla \partial_i \chi_{l'} u\|^2 \\ (8.21) \quad &+ 2|\operatorname{Re} \langle (E - V_0 - V_{\Lambda_{q_{l'}}})u, |\nabla \chi_{l'}|^2 u \rangle|. \end{aligned}$$

This inequality can be proved using the equality

$$(8.22) \quad \|\nabla(\psi u)\|^2 = \operatorname{Re} \langle \nabla u, \nabla(\psi^2 u) \rangle + \|\nabla \psi u\|^2,$$

which can be found in the book by Agmon [1] on page 20, equation (1.16). Taking  $u = R_{q_{l'}}(E+i\epsilon)\chi_{l'/3}\psi$ , for any  $\psi \in L^2(\mathbb{R}^d)$ , and using the fact that  $|\nabla \chi_{l'}| \chi_{l'/3} = 0$ , we finally obtain

$$\begin{aligned} \|W_{l'} R_{q_{l'}}(E+i\epsilon)\chi_{l'/3}\| &\leq \left( 2\sum_{i,j=1}^d \|\partial_i \partial_j \chi_{l'}\|_\infty + 2\|E - V_{\Lambda_{l'}} - V_0\|_\infty^{1/2} \|\nabla \chi_{l'}\|_\infty^2 \right. \\ &\quad \left. + 2\|\nabla \chi_{l'}\|_\infty \|\Delta \chi_{l'}\|_\infty \right) \\ (8.23) \quad &\times \|\tilde{\chi}_{l'} R_{q_{l'}}(E+i\epsilon)\chi_{l'/3}\|. \end{aligned}$$

Here, we take a function  $\tilde{\chi}_{l'}$  with a slightly larger support so that  $\tilde{\chi}_{l'} \chi_{l'} = \chi_{l'}$ .

7. Next, we use the GRE to pass from the result for  $R_{q_{l'}}$  to one for  $R_{l'}$ . As we have

$$(8.24) \quad \chi_{l'} R_{q_{l'}} - R_{l'} \chi_{l'} = R_{l'} W_{l'} R_{q_{l'}},$$

we obtain

$$(8.25) \quad W_{l'} R_{q_{l'}} \chi_{l'/3} = W_{l'} R_{l'} \chi_{l'/3} + W_{l'} R_{l'} W_{l'} R_{q_{l'}} \chi_{l'/3}.$$

We rewrite this equation to obtain a bound involving some constant  $C$ , depending only on  $\|\nabla \chi_{l'}\|_\infty$ ,  $\|\Delta \chi_{l'}\|_\infty$ , and  $E$ , similar to the coefficient on the right side of (8.23),

$$(8.26) \quad \|W_{l'} R_{l'}(E+i\epsilon)\chi_{l'/3}\| \leq C(1+\|R_l(E+i\epsilon)\chi_{l'/3}\|)\|\tilde{\chi}_{l'} R_{q_{l'}} \chi_{l'/3}\| + \|W_{l'} R_{q_{l'}}(E+i\epsilon)\chi_{l'/3}\|.$$

8. Finally, we use the Wegner estimate [W] to bound the norm of the resolvent on the right side of (8.23) and of (8.26) by

$$(8.27) \quad \|R_{p_{l'}}(E+i\epsilon)\| \leq 2C_W p^d (l')^{\xi+d}, \text{ for } p = 1 \text{ or } q.$$

This estimate holds with probability at least  $1 - \frac{1}{2}(l')^{-\xi}$ , provided  $l$ , and hence  $l'$ , is large enough. We have obtained the following estimate on the resolvent of the local Hamiltonian  $H_{\Lambda_{l'}}$  from (8.26)–(8.27),

$$(8.28) \quad \|W_{l'} R_{l'}(E+i\epsilon)\chi_{l'/3}\| \leq e^{-\gamma' l'},$$

with

$$(8.29) \quad \gamma' = \left( \gamma - \frac{\log b_d}{l} \right) \left( 1 - \frac{4l}{l'} \right) - c \frac{\log l'}{l'},$$

with a constant  $c$  depending only on  $d, C_W$ , and  $\xi$ . Recall that  $\xi > 2d$ , so  $\xi$  can be taken to be a simply a multiple of  $d$ . This holds provided the resolvent estimate (8.27) is satisfied, and there are not two disjoint or more  $\gamma$ -bad  $\Lambda_l$  boxes centered in  $\Gamma_l \cap \Lambda_{q_{l'}}$ . The probability of this last event is larger than  $1 - c_d (l'/l)^{2d} \eta^2$ , where  $c_d$  is a combinatorial factor depending only on  $d$  and  $q$ . Here, we use the independence of events associated with disjoint regions. Consequently, our localized resolvent

estimate (8.28) holds with probability  $\geq 1 - c_d \eta^2 (l'/l)^{2d} - \frac{1}{2} (l')^{-\xi}$ . This concludes the proof of Lemma 8.4.  $\square$

**8.4. Consequences of the Inductive Step.** We prove Lemma 5.4 from the inductive step described in Lemma 8.4.

there

**Proof of Lemma 5.4.** Assuming  $l_0$  is large enough, say  $l_0 \geq l_1^*$ , one can define  $\gamma_k$  inductively according to (8.9) in Lemma 8.4,

$$(8.30) \quad \gamma_{k+1} = \gamma_k \left( 1 - \frac{4l_k}{l_{k+1}} \right) - c_k,$$

where

$$c_k = \frac{c_1}{l_k} \left( 1 - \frac{4l_k}{l_{k+1}} \right) + c_2 \frac{\log l_{k+1}}{l_{k+1}}.$$

We find

$$(8.31) \quad \gamma_{k+1} = \gamma_0 \prod_{j=0}^k (1 - N_j) - \sum_{l=0}^k \prod_{j=0}^{(l-1)} (1 - N_j) c_{k-l},$$

with  $N_j = 4l_j/l_{j+1}$ . It is easy to check that  $\prod_{j=0}^k (1 - N_j)$  is uniformly bounded below

by a strictly positive number  $\kappa$  uniformly in  $l_0$  if  $l_0 \geq 2$ . The subtracted term in (8.31) is certainly  $\leq \kappa \gamma_0/2$  if  $l_0$  is large enough, say  $l_0 \geq l_2^*$ , so that  $\gamma_k \geq \kappa \gamma_0/2$  for all  $k$ . According to Lemma 8.4, suppose  $\Lambda_k$  is  $\gamma_k$ -good with probability  $\geq 1 - \eta_k$ , where  $\eta_k = l_k^{-\xi}$ . We compute an upper bound on  $\eta_{k+1}$  using the definition given in (8.8),

$$(8.32) \quad \begin{aligned} \eta_{k+1} &= c_d l_k^{(d-2\xi)} + \frac{1}{2} l_{k+1}^{-\xi} = c_d (l_{k+1})^{2(d-2\xi)/3} + \frac{1}{2} l_{k+1}^{-\xi} \\ &\leq \left( \frac{1}{2} + c_d l_{k+1}^{(2d-\xi/3)} \right) l_{k+1}^{-\xi}. \end{aligned}$$

This is less than  $l_{k+1}^{-\xi}$  since  $\xi > 2d$  for  $l_0 > l_3^* = (2c_d)^{(\xi-2d)/3}$ . In these last estimates, we have used the identity  $l_{k+1} = l_k^{3/2}$ . We obtain the result with  $l^* = \max \{l_i^*, i = 1, 2, 3\}$ .  $\square$

We now give a refinement of these lemmas needed in the proof of Theorem 5.4.

**LEMMA 8.5.** *Let  $\gamma_0 > 0$  and assume [W]. Let  $l^*$  be the minimum length scale of Lemma 5.4 and suppose [H1]( $\gamma_0, l_0$ ) holds for  $l_0 > l^*$ . Then, there exists a constant  $C_p = C_p(\gamma_0, C_W, d)$ , such that for any cube  $\Lambda \subset \mathbb{R}^d$  sufficiently large so that  $\Lambda_{ql_k} \subset \Lambda$ , for some large  $k$  and  $q \geq 2$ , and  $x, y \in \Lambda$  such that*

$$(8.33) \quad \text{dist}(x \text{ or } y, \partial\Lambda) > \|x - y\| > ql_k,$$

and

$$(8.34) \quad ql_{k+1} > \|x - y\| > ql_k,$$

one has

$$(8.35) \quad \mathbb{P} \left\{ \sup_{\epsilon > 0} \|\chi_x R_\Lambda(E + i\epsilon) \chi_y\| \leq e^{-(1/2\sqrt{d})\gamma_0 \|x-y\|} \right\} \geq 1 - C_p |\Lambda| q^{2d} \|x - y\|^{-\xi},$$

where  $\chi_x, \chi_y$  are characteristic functions of unit cubes centered at  $x$  or  $y$  and we write  $\gamma_0$  for  $\kappa\gamma_0/2$  of Lemma 8.4 for simplicity.

**Proof.** Let us suppose  $\text{dist}(x, \partial\Lambda) \geq \text{dist}(y, \partial\Lambda)$ ; in the other case we begin the construction at  $y$ . We assume without restricting generality that  $x, y \in \Gamma_{l_k}$ . As in Lemma 8.4 we can find a least  $n$  chained boxes of side  $l_k$  centered on points in  $\Gamma_{l_k}$  starting at  $x$  before reaching  $\partial\Lambda$  or  $y$  where  $n \geq 3\|x-y\|/l_k$ . By Lemma 8.4, the probability that each box  $\Lambda_{l_k}$  is  $\gamma_0$ -good is  $\geq 1 - l_k^{-\xi}$ . As in the proof of Lemma 8.4,

$$(8.36) \quad \sup_{\epsilon > 0} \|\chi_{l_k/3}(x)R_\Lambda(E + i\epsilon)\chi_{l_k/3}(y)\| \leq e^{-(3/4\sqrt{d})\gamma_0\|x-y\|} \|R_\Lambda(E + i\epsilon)\|,$$

with probability  $\geq 1 - \eta l_k^{-\xi} \geq 1 - l_k^{-\xi} |\Lambda| c_d$ , where  $c_d$  depends only on  $d$ . Here we have assumed that  $k$  is large enough so that

$$\gamma_0 - \frac{\log b_d}{l_k} > \frac{3\gamma_0}{4}.$$

We use Wegner's lemma [W] to estimate  $\|R_\Lambda\|$  and obtain

$$(8.37) \quad \mathbb{P}\{\|R_\Lambda(E + i\epsilon)\| \leq e^{(1/4\sqrt{d})\gamma_0 l_k}\} \geq 1 - C_W |\Lambda| e^{-(1/4\sqrt{d})\gamma_0 l_k}.$$

Estimates (8.36)–(8.37) yield

$$\sup_{\epsilon > 0} \|\chi_{l_k/3}(x)R(\Lambda)(E + i\epsilon)\chi_{l_k/3}(y)\| \leq e^{-(1/2\sqrt{d})\gamma_0\|x-y\|},$$

with probability  $\geq 1 - |\Lambda|\{C_W e^{-(1/4\sqrt{d})\gamma_0 l_k} + l_k^{-\xi}\} \geq 1 - c_p |\Lambda| q^{2d} \|x-y\|^{-2d}$ , where we have used the identity  $l_k = (l_{k+1})^{2/3}$ .  $\square$

## References

- [1] S. Agmon, *Lectures on exponential decay of solutions of second-order elliptic equations*, Mathematical Notes 29, Princeton University Press, Princeton, NJ, 1982.
- [2] Aizenman:1994 M. Aizenman: Localization at weak disorder: some elementary bounds, *Rev. Math. Phys.* **6**, No. 5a, 1163–1182 (1994).
- [3] M. Aizenman, A. Elgart, S. Naboko, J. H. Schenker, G. Stolz: Moment analysis for localization in random Schrödinger operators, *Invent. Math.* **163**, 343–413 (2006).
- [4] M. Aizenman, S. Molchanov: Localization at large disorder and at extreme energies: an elementary derivation, *Commun. Math. Phys.* **157**, 245–278 (1993).
- [5] M. Aizenman, R. Sims, S. Warzel: Stability of the absolutely continuous spectrum of random Schrödinger operators on tree graphs, *Probab. Theory Related Fields* **136**, 363–394 (2006).
- [6] P. W. Anderson, Absence of diffusion in certain random lattices, *Physical Review* **109**, 1492–1505 (1958).
- [7] N. Aronszajn: On a problem of Weyl, *Amer. J. Math.* **79**, 596–620 (1957).
- [8] J.-M. Barbaroux, J. M. Combes, and P. D. Hislop: Localization near band edges for random Schrödinger operators, *Helv. Phys. Acta* **70**, 16–43 (1997).
- [9] J.-M. Barbaroux, J. M. Combes, and P. D. Hislop: Landau Hamiltonians with unbounded random potentials, *Letts. Math. Phys.* **40**, 355–369 (1997).
- [10] J.-M. Barbaroux, J. M. Combes, and R. Montcho, Remarks on the relation between quantum dynamics and fractal spectra, *J. Math. Anal. Appl.* **213**, 698–722 (1997).
- [11] J.-M. Barbaroux, thèse, Université de Toulon et du Var 1997.
- [12] J.-M. Barbaroux, F. Germinet, S. Tcheremchantsev: Fractal dimensions and the phenomenon of intermittency in quantum dynamics, *Duke Math. J.* **110**, 161–193 (2001).
- [13] D. Belitz and T. R. Kirkpatrick: The Anderson-Mott Transition, *Rev. of Modern Phys.* **66**, 261 (1994).
- [14] J. Bellissard: Ordinary quantum Hall effect and non-commutative cohomology, in *Localization in Disordered Systems*, W. Weller and P. Zieche, eds., Leipzig: Teubner 1988.

- [15] J. Bellissard, A. van Elst, and H. Schulz-Baldes, *The noncommutative geometry of the quantum Hall effect*, *J. Math. Phys.* **35**, 5373–5451 (1994).
- [16] M. S. Birman, M. Z. Solomjak: *Spectral Theory of Self-Adjoint Operators in Hilbert Space*, Dordrecht: D. Reidel Publishing Co. 1987.
- [17] M. S. Birman, D. R. Yafaev: The spectral shift function: The work of M. G. Krein and its further development, *St. Petersburg Mathematics Journal* **4**, 833–870 (1992).
- [18] Ph. Briet, J. M. Combes, P. Duclos: Spectral stability under tunneling, *Commun. Math. Phys.*, **126**, 133–186 (1989).
- [19] J. Bourgain, C. Kenig: On localization in the continuous Anderson-Bernoulli model in higher dimension, *Invent. Math.* **161**, 389–426 (2005).
- [20] A. Boutet de Monvel, S. Naboko, P. Stollmann, G. Stolz: Localization near fluctuation boundaries via fractional moments and applications, *J. d'Anal.* (2007).
- [21] R. Carmona, J. Lacroix: *Spectral theory of random Schrödinger operators*, Birkhäuser, Boston, 1990.
- [22] J. L. Chayes and L. Chayes, *Percolation and Random Media*, in Critical Phenomena, Random Systems, and Gauge Theories, Proc. les Houches Summer School, Session XLIII, 1000 - 1142 (1984).
- [23] J. M. Combes: Connections between quantum dynamics and spectral properties of time-evolution operators, in *Differential equations with applications to mathematical physics*, 59–68, *Math. Sci. Engrg.*, **192**, Academic Press, Boston, MA, 1993
- [24] J. M. Combes, P. D. Hislop: Some Transport and Spectral Properties of Disordered Media, in *Schrödinger Operators: The Quantum Many-body Problem*, E. Balslev (ed.), 16-47, Berlin: Springer Verlag (1992).
- [25] J. M. Combes, P. D. Hislop: Localization for some continuous random Hamiltonians in  $d$ -dimensions, *J. Funct. Anal.* , **124**, 149-180 (1994).
- [26] J. M. Combes, P. D. Hislop: Landau Hamiltonians with random potentials: Localization and density of states, *Commun. Math. Phys.*, **177**, 603–629 (1996).
- [27] J. M. Combes, P. D. Hislop, F. Klopp: Hölder continuity of the integrated density of states for some random operators at all energies, *Int. Math. Res. Not.* **2003**, no. 4, 179–209.
- [28] J. M. Combes, P. D. Hislop, F. Klopp: An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators, to appear in the *Duke Mathematical Journal*, 2007.
- [29] J. M. Combes, P. D. Hislop, F. Klopp, G. Raikov: Global continuity of the integrated density of states for random Landau Hamiltonians, *Comm. Partial Differential Equations* **29**, 1187–1213 (2004).
- [30] J. M. Combes, P. D. Hislop, E. Mourre: Spectral averaging, perturbation of singular spectra, and localization, *Trans. Am. Math. Soc.*, **348**, 4883–4894 (1996).
- [31] J. M. Combes, P. D. Hislop, E. Mourre: Correlated Wegner inequalities for random Schrödinger operators, in *Advances in Differential Equations and Mathematical Physics*, Proceedings Int. Conf. Diff. Eqns. and Math. Phys., 1997, E. Carlen, E. M. Harrell, and M. Loss, eds., Contemporary Mathematics Vol. 217, American Mathematical Society, Providence, RI, 1998.
- [32] J. M. Combes, P. D. Hislop, S. Nakamura: The  $L^p$ -theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random operators, *Comm. Math. Phys.* **218**, 113–130 (2001).
- [33] J. M. Combes, P. D. Hislop, and A. Tip: Band edge localization and the density of states for electromagnetic and acoustic waves in random media, *Ann. Inst. H. Poincaré* **70**, 381–428 (1999).
- [34] J. M. Combes, L. Thomas: Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators, *Commun. Math. Phys.* **34**, 251-270 (1973).
- [35] H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon: *Schrödinger operators*. Berlin : Springer-Verlag, 1987.
- [36] E. B. Davies: *Spectral theory and differential operators*. Cambridge: Cambridge University Press, 1995.
- [37] P.A. Deift, R. Hempel: On the existence of eigenvalues of the Schrödinger operator  $H + \lambda W$  in a gap of  $\sigma(H)$ , *Commun. Math. Phys.* **103**, 461-490 (1986).
- [38] F. Delyon, Y. Levy, B. Souillard: Anderson localization for multidimensional systems at large disorder or large energy, *Comm. Math. Phys.* **100**, 463–470 (1985).

- [39] W. Donoghue: On the perturbation of spectra, *Comm. Pure Appl. Math.* **18**, 559–579 (1965).
- [40] T. C. Dorlas, N. Macris, J. V. Pulé: Localization in a Single-Band Approximation to Random Schrödinger Operators with Magnetic Field, *Helv. Phys. Acta.* **68**, 329–364 (1995).
- [41] T. C. Dorlas, N. Macris, J. V. Pulé: Localization in single Landau Bands, *J. Math. Phys.* **37**, 1574–1595 (1996).
- [42] H. J. S. Dorren and A. Tip, *Maxwell's equations for non-smooth media; fractal-shaped and pointlike objects*, *J. Math. Phys.* **32**, 3060 (1991).
- [43] W. Faris: Localization for a random discrete wave equation, in *Random Media*, IMA Volume 7, G. Papanicolaou, ed., Springer-Verlag, New York, 1987.
- [44] W. Faris: A localization principle for multiplicative perturbations, *J. Funct. Anal.* **67** (1986), 105–114.
- [45] A. Figotin, A. Klein: Localization phenomenon in gaps of the spectrum of random lattice operators, *J. Stat. Phys.* **75**, 997–1021 (1994).
- [46] A. Figotin and A. Klein: Localization of Electromagnetic and Acoustic Waves in Random Media: Lattice Models, *J. Stat. Phys.* **76**, 985–1003 (1994).
- [47] A. Figotin and A. Klein: Localization of Classical Waves I: Acoustic Waves, *Commun. Math. Phys.* **180**, 439–482 (1996).
- [48] A. Figotin, and A. Klein: Localization of Classical Waves II: Electromagnetic Waves, *Commun. Math. Phys.* **184**, 411–441 (1997).
- [49] A. Figotin, and A. Klein: Midgap defect modes in dielectric and acoustic media, *SIAM J. Appl. Math.* **58**, 1748–1773 (1998).
- [50] A. Figotin and P. Kuchment: Band-Gap structure of spectra of periodic dielectric and acoustic media. I. scalar Model, *SIAM J. Appl. Math.* **56**, No. 1, 68–88 (1996).
- [51] A. Figotin and P. Kuchment: Band-Gap structure of spectra of periodic dielectric and acoustic media. II. 2D Photonic Crystals, *SIAM J. Appl. Math.* **56**, No. 6, 1561–1620 (1996).
- [52] W. Fischer, H. Leschke, P. Müller: Towards localization by Gaussian random potentials in multidimensional continuous space, *Letts. Math. Phys.* **38**, 343–348 (1996).
- [53] W. Fischer, T. Hupfer, H. Leschke, P. Müller: Existence of the density of states for multidimensional continuum Schrödinger operators with Gaussian random potentials, *Commun. Math. Phys.* **190**, 133–141 (1997).
- [54] W. Fischer, H. Leschke, P. Müller: Spectral localization by Gaussian random potentials in multi-dimensional continuous space, *J. Statist. Phys.* **101**, 935–985 (2000).
- [55] J. Fröhlich, F. Martinelli, E. Scoppola, T. Spencer: A constructive proof of localization in Anderson tight binding model, *Commun. Math. Phys.* **101**, 21–46 (1985).
- [56] J. Fröhlich and T. Spencer: Absence of diffusion in the Anderson tight binding model for large disorder or low energy, *Commun. Math. Phys.* **88**, 151–184 (1983).
- [57] R. Froese, D. Hasler, W. Spitzer: Absolutely continuous spectrum for the Anderson model on a tree: a geometric proof of Klein's theorem, *Commun. Math. Phys.* **269**, 239–257 (2007).
- [58] F. Germinet, S. DeBièvre: Dynamical localization for discrete and continuous random Schrödinger operators, *Commun. Math. Phys.* **194**, 323–341 (1998).
- [59] F. Germinet, P. D. Hislop, A. Klein: On localization for the Schrödinger operator with a Poisson random potential, *C. R. Math. Acad. Sci. Paris* **341**, 525–528 (2005).
- [60] F. Germinet, P. D. Hislop, A. Klein: Localization for Schrödinger operators with Poisson random potential, to appear in *J. European Math. Soc.* (2007).
- [61] F. Germinet, P. D. Hislop, A. Klein: Localization at low energies for attractive Poisson random Schrödinger operators, to appear in *CRM Lect. Proc.* (2007).
- [62] F. Germinet, A. Klein: Bootstrap multiscale analysis and localization in random media, *Commun. Math. Phys.* **222**, 415–448 (2001).
- [63] F. Germinet, A. Klein: Explicit finite volume criteria for localization in continuous random media and applications, *Geom. Funct. Anal.* **13**, 1201–1238 (2003).
- [64] F. Germinet, A. Klein: Operator kernel estimates for functions of generalized Schrödinger operators, *Proc. Amer. Math. Soc.* **131**, 911–920 (2003).
- [65] F. Germinet, A. Klein: New characterizations of the region of complete localization for random Schrödinger operators, *J. Stat. Phys.* **122**, 73–94 (2006).

- [66] F. Germinet, A. Klein, J. Schenker: Dynamical delocalization in random Landau Hamiltonians, to appear in *Ann. Math.*.
- [67] F. Gesztesy: *Some applications of commutation methods*, in Schrödinger operators: Proc. of the Nordic Summer School in Mathematics held at Sandbjerg Slot, Sønderborg, Denmark, 1988, Lecture Notes in Physics No. **345**, H. Holden and A. Jensen, eds., Berlin: Springer-Verlag 1989.
- [68] F. Ghribi, P. D. Hislop, F. Klopp: Localization for Schrödinger operators with random vector potentials, preprint.
- [69] G. Grimmett: *Percolation*. Springer-Verlag, Berlin, 1980.
- [70] B. I. Halperin, *Quantized Hall conductance, current-carrying edge states, and the existence of extended states in a two-dimensional disordered potential*, *Phys. Rev.* **B 25**, 2185 (1982).
- [71] P. D. Hislop, F. Klopp: The integrated density of states for some random operators with nonsign definite potentials, *J. Funct. Anal.* **195**, 12–47 (2002).
- [72] F. Klopp, S. Nakamura, F. Nakano, Y. Nomura: Anderson localization for 2D discrete Schrödinger operators with random magnetic fields, *Ann. Henri Poincaré* **4**, 795–811 (2003).
- [73] P. D. Hislop, F. Klopp, J. Schenker: Continuity with respect to disorder of the integrated density of states, *Illinois J. Math.* **49**, 893–904 (2005).
- [74] P. D. Hislop, and S. Nakamura: Stark Hamiltonians with Unbounded Random Potentials, *Reviews in Math. Phys.* **2**, 479–494 (1991).
- [75] P. D. Hislop, I. M. Sigal, *Introduction to Spectral Theory, with Applications to Schrödinger Operators*, Applied Mathematical Sciences series volume 113, Springer, New York, 1996.
- [76] J. Howland: Perturbation theory of dense point spectra, *J. Funct. Anal.* **74**, 52–80 (1987).
- [77] J. Howland: Quantum stability, in Schrödinger Operators, E. Balslev, ed., Lecture Notes in Physics No. **403**, Springer-Verlag, 1992, pp. 100–122.
- [78] D. Hundertmark, R. Killip, S. Nakamura, P. Stollmann, I. Veselić: Bounds on the spectral shift function and the density of states, *Comm. Math. Phys.* **262**, 489–503 (2006).
- [79] D. Hundertmark, B. Simon: An optimal  $L^p$ -bound on the Krein spectral shift function, *J. Anal. Math.* **87**, 199–208 (2002).
- [80] S. John: The localization of light and other classical waves in disordered media, *Comments Cond. Mat. Phys.* **14** (1988), 193–230.
- [81] T. Kato: *Perturbation Theory for Linear Operators*, Second Edition, Grundlehren der mathematischen Wissenschaften 132, Berlin: Springer-Verlag, 1980.
- [82] W. Kirsch: Random Schrödinger Operators: a course, in Schrödinger Operators, Sønderborg DK 1988, ed. H. Holden and A. Jensen. Lecture Notes in Physics Vol. **345**, Berlin: Springer 1989.
- [83] W. Kirsch: Wegner estimates and localization for alloy-type potentials, *Math. Zeit.* **221**, 507–512 (1996).
- [84] W. Kirsch, Krishna M. J. Obermeit: Anderson model with decaying randomness: mobility edge, *Math. Z.* **235**, 421–433 (2000).
- [85] W. Kirsch and F. Martinelli, *On the spectrum of Schrödinger operators with a random potential*, *Commun. Math. Phys.* **85**, 329–350 (1982).
- [86] W. Kirsch, P. Stollmann, and G. Stolz: Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equations* **6**, 241–268 (1998).
- [87] W. Kirsch, P. Stollmann, and G. Stolz: Anderson localization for random Schrödinger operators with long range interactions, *Comm. Math. Phys.* **195**, 495–507 (1998).
- [88] A. Klein: Extended states for the Anderson model on the Bethe lattice, *Adv. Math.* **133**, 163–184 (1998).
- [89] A. Klein, S. Molchanov: Simplicity of eigenvalues in the Anderson model, *J. Stat. Phys.* **122**, 95–99 (2006).
- [90] F. Klopp: Localization for some continuous random Schrödinger operators, *Commun. Math. Phys.* **167**, 553–569 (1995).
- [91] F. Klopp: Localization for semi-classical continuous random Schrödinger operators II: the random displacement model, *Helv. Phys. Act.* **66**, 810–841 (1993).
- [92] F. Klopp: Internal Lifshits tails for random perturbations of periodic Schrödinger operators, *Duke Math. J.* **98**, 335–396 (1999).

- [93] S. Kotani: Lyapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrödinger operators, *Stochastic Analysis*, K. Ito, ed., North Holland, Amsterdam, 1984, 225–248.
- [94] S. Kotani, B. Simon: Localization in general one dimensional systems. II, *Commun. Math. Phys.* **112**, 103–120 (1987).
- [95] P. Kuchment, *Floquet Theory for Partial Differential Equations*, Basel: Birkhäuser Verlag 1993.
- [96] H. Kunz: The quantum Hall effect for electrons in a random potential, *Commun. Math. Phys.* **112**, 121 - 145 (1987).
- [97] H. Kunz, B. Souillard, *Sur le spectre des opérateurs aux différences finis aléatoires*, *Commun. math. Phys.* **78**, 201–246 (1980).
- [98] Y. Last: Quantum dynamics and decompositions of singular continuous spectra, *J. Funct. Anal.* **142**, 406–445 (1996).
- [99] H. Leinfelder, C. G. Simader: Schrödinger operators with singular magnetic vector potentials *Math. Z.* **176**, 1–19 (1981).
- [100] F. H. Lin: A uniqueness equation for parabolic equations, *Comm. Pure Appl. Math.* **43**, 127–136 (1990).
- [101] F. Martinelli, H. Holden: On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator on  $L^2(R^{\nu})$ , *Commun. Math. Phys.* **93**, 197–217 (1984).
- [102] F. Martinelli, E. Scoppola: Remark on the absence of absolutely continuous spectrum for  $d$ -dimensional Schrödinger operators with random potential for large disorder or low energy, *Commun. Math. Phys.* **97**, 465–471 (1985).
- [103] I. McGillivray, P. Stollmann, G. Stolz: Absence of absolutely continuous spectra for multidimensional Schrödinger operators with high barriers, *Bull. London Math. Soc.* **27**, 162–168 (1995).
- [104] R. Montcho, thèse, Université de Toulon et du Var, 1997.
- [105] E. Mourre: Absence of singular continuous spectrum for certain self-adjoint operators, *Commun. Math. Phys.* **78**, 391–408 (1983).
- [106] L. Pastur: *Spectra of random self-adjoint operators*, *Usp. Math. Nauk.* **28**, 3 (1973).
- [107] L. Pastur, A. Figotin: *Spectra of random and almost-periodic operators*. Berlin: Springer-Verlag, 1992.
- [108] R. E. Prange and S. M. Girvin (eds.): *The Quantum Hall Effect*, Graduate Texts in Contemporary Physics, New York: Springer-Verlag (1987).
- [109] R. Joynt and R. E. Prange: *Conditions for the quantum Hall effect*, *Phys. Rev.* **B 29**, 3303 - 3317 (1984).
- [110] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, I: Functional Analysis, revised and enlarged edition*. New York: Academic Press, 1980.
- [111] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, II: Fourier Analysis, Self-Adjointness*. New York: Academic Press, 1975.
- [112] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, III: Scattering Theory*. New York: Academic Press, 1979.
- [113] M. Reed, B. Simon: *Methods of Modern Mathematical Physics, IV: Analysis of Operators*. New York: Academic Press 1981.
- [114] B. Simon: *Trace ideals and their applications, second edition*, *Mathematical Surveys and Monographs* **120**, American Mathematical Society, 2005.
- [115] B. Simon: Spectral Analysis of Rank One Perturbations and Applications, Vancouver lectures, 1993, in *Mathematical Quantum Theory, II. Schrödinger operators*, CRM Proc. Lecture Notes **8**, 109–149. Providence: AMS, 1995; reprinted in the second edition of *Trace Ideals*.
- [116] B. Simon: Cyclic vectors in the Anderson model. Special issue dedicated to Elliott H. Lieb, *Rev. Math. Phys.* **6**, 1183–1185 (1994).
- [117] B. Simon:  $L^p$ -norms of the Borel transform and the decomposition of measures, *Proc. Amer. Math. Soc.* **123**, 3749–3755 (1995).
- [118] B. Simon: Spectral averaging and the Krein spectral shift, *Proc. Amer. Math. Soc.* **126**, 1409–1413 (1998).
- [119] B. Simon and T. Wolff: Singular continuous spectrum under rank one perturbation and localization for random Hamiltonians, *Comm. Pure Appl. Math.* **39**, 75–90 (1986).

- [120] T. Spencer: Localization for random and quasi-periodic potentials: *J. Stat. Phys.* **51**, 1009-1019 (1988).
- [121] P. Stollmann: Localization for random perturbations of anisotropic periodic media, *Israel J. Math.* **107**, 125–139 (1997).
- [122] P. Stollmann: Wegner estimates and localization for continuum Anderson models with some singular distributions, *Arch. Math. (Basel)* **75** 307–311 (2000).
- [123] P. Stollmann: *Caught by disorder. Bound states in random media*, Progress in Mathematical Physics, **20**, Birkhäuser Boston, Inc., Boston, MA, 2001.
- [124] G. Stolz: Localization for Schrödinger operators with effective barriers, *J. Funct. Anal.* **146**, 416–429 (1997).
- [125] G. Stolz: Localization for random Schrödinger operators with Poisson potential, *Ann. Inst. Henri Poincaré* **63**, 297–314 (1997).
- [126] D. J. Thouless, *Localization and the two-dimensional Hall effect*, *J. Phys. C* **14** , 3475 - 3480 (1981).
- [127] A. Tip, Absolute continuity of the integrated density of states of the quantum Lorentz gas for a class of repulsive potentials, *J. Phys. A: Math. Gen.* **27** (1994) 1057–1069.
- [128] D. V. van Coevorden, R. Sprik, A. Tip, and A. Lagendijk: Photonic Bandstructure of Atomic Lattices, *Phys. Rev. Letters* **77**, 2412 (1996).
- [129] I. Veselić: Existence of the density of states for some alloy type models with single site potentials that change sign, in *Applied mathematics and scientific computing* (Dubrovnik, 2001), 301–311, Kluwer/Plenum, New York, 2003.
- [130] I. Veselić: Integrated density of states and Wegner estimates for random Schrödinger operators, in *Spectral theory of Schrödinger operators*, Contemp. Math. **340**, 97–183, Amer. Math. Soc., Providence, RI, 2004.
- [131] H. von Dreifus and A. Klein: A new proof of localization in the Anderson tight binding model, *Commun. Math. Phys.* **124**, (1989), 285-299.
- [132] H. von Dreifus and A. Klein: Localization for random Schrödinger operators with correlated potentials, *Commun. Math. Phys.* **140**, 133–147 (1991).
- [133] W. M. Wang: Microlocalization, Percolation, and Anderson localization for the Magnetic Schrödinger Operator with a Random Potential, *J. Funct. Anal.* **146**, 1–26 (1997).
- [134] W. M. Wang: Asymptotic expansion for the density of states of the magnetic Schrödinger operator with a random potential, *Comm. Math. Phys.* **172**, 401–425 (1995).
- [135] W. M. Wang: Supersymmetry and density of states of the magnetic Schrödinger operator with a random potential revisited, *Comm. Partial Differential Equations* **25**, 601–679 (2000).
- [136] F. Wegner: Bounds on the density of states in disordered systems, *Z. Phys.* **B22**, (1981), 9–15.
- [137] D. R. Yafaev: *Mathematical Scattering Theory: General Theory*, Translations of Mathematical Monographs **105**, Providence, RI: American Mathematical Society, 1992.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KY 40506–0027  
USA

*E-mail address:* hislop@ms.uky.edu

*This page intentionally left blank*

# Quantum Mechanics, The Stability of Matter and Quantum Electrodynamics

Elliott H. Lieb

ABSTRACT. Much progress has been made in the last few decades in developing the necessary mathematics for understanding the full implications of the quantum-mechanical many-body problem and why the material world appears to be as stable as it is despite the serious  $-1/|x|$  singularity of the Coulomb potential that attracts negative electrons to positive atomic nuclei. Many problems remain, however, especially the understanding of the interaction of matter and the quantized radiation field discovered by Planck in 1900. A short review of some of the main topics is given.

## 1. Introduction

This paper is a brief survey of the quantum-mechanical many-body problem, especially the question of the interaction of matter with radiation. The quantum-mechanical revolution of the 1920's brought with it many successes, but also a few problems that have yet to be resolved. The realization that there were a few problems with the simple textbook theory surfaced three or four decades ago. Since then some of the mathematical questions have been answered, but some big ones remain. This brief overview might, it is hoped, encourage some mathematicians to look into this fascinating topic.

We begin with a sketch of the topics that will concern us here.

**1.1. Triumph of Quantum Mechanics.** One of the basic problems of classical physics (after the discovery of the point electron by Thomson and of the (essentially) point nucleus by Rutherford) was the stability of atoms. Why do the electrons in an atom not fall into the nucleus? Quantum mechanics explained this fact. It starts with the classical Hamiltonian of the system (nonrelativistic kinetic energy for the electrons plus Coulomb's law of electrostatic energy among

---

This paper is an extended version of a talk at the DMV Jahrestagung in Rostock, 17 November, 2003 [1].

Reproduction of this article, in its entirety, by any means is permitted for non-commercial purposes.

2000 *Mathematics Subject Classification.* Primary 81Q99; Secondary 82D99.

*Key words and phrases.* Quantum mechanics, Stability of Matter, Quantum Electrodynamics.

Work partially supported by U.S. National Science Foundation grant PHY 0139984-A01.

©2008 Elliott H. Lieb. This work may be reproduced in its entirety for non-commercial purposes.

the charged particles). By virtue of the non-commutativity of the kinetic and potential energies in quantum mechanics the stability of an atom – in the sense of a finite lower bound to the energy – was a consequence of the fact that any attempt to make the electrostatic energy very negative would require the localization of an electron close to the nucleus and this, in turn, would result in an even greater, positive, kinetic energy.

Thus, the basic stability problem for an atom was solved by an inequality that says that  $\langle 1/|x| \rangle$ , the expected value of  $1/|x|$ , can be made large only at the expense of making the kinetic energy, which is proportional to  $\langle p^2 \rangle$ , even larger. A fundamental hypothesis of quantum mechanics is that  $p$  is represented by the differential operator  $-i\hbar\nabla$  with  $\hbar = h/2\pi$  and  $h = \text{Planck's constant}$ . In elementary presentations of the subject it is often said that the mathematical inequality that ensures this fact is the famous uncertainty principle of Heisenberg (proved by Weyl), which states that  $\langle p^2 \rangle \langle x^2 \rangle \geq (9/8)\hbar^2$ .

While this principle is mathematically rigorous it is actually insufficient for the purpose, as explained, e.g., in [19, 21], and thus gives only a heuristic explanation of the power of quantum mechanics to prevent collapse. A more powerful inequality, such as Sobolev's inequality (2.6), is needed (see, e.g., [23]). The utility of the latter is made possible by Schrödinger's representation of quantum mechanics (which earlier was a somewhat abstract theory of operators on a Hilbert space) as a theory of differential operators on the space of square integrable functions on  $\mathbb{R}^3$ . The importance of Schrödinger's representation is sometimes underestimated by formalists, but it is of crucial importance because it permits the use of functional analytic methods, especially inequalities such as Sobolev's, which are not easily visible on the Hilbert space level. These methods are essential for the developments reported here.

To summarize, the understanding of the stability of atoms and ordinary matter requires a formulation of quantum mechanics with two ingredients:

- A Hamiltonian formulation in order to have a clear notion of a lowest possible (ground state) energy. Lagrangian formulations, while popular, do not always lend themselves to the identification of that quintessential quantum mechanical notion of a ground state energy. In quantum mechanics a Hamiltonian is not a function on phase space but rather a (pseudo-) differential operator.
- A formulation in terms of concrete function spaces instead of abstract Hilbert spaces so that the power of mathematical analysis can be fully exploited.

**1.2. Some Basic Definitions.** As usual, we shall denote the lowest energy (eigenvalue) of a quantum mechanical system by  $E_0$ . (More generally,  $E_0$  denotes the infimum of the spectrum of the Hamiltonian  $H$  in case this infimum is not an eigenvalue of  $H$  or is  $-\infty$ .) Our intention is to investigate arbitrarily large systems, not just atoms. In general we suppose that the system is composed of  $N$  electrons and  $K$  nuclei of various kinds. Of course we could include other kinds of particles but  $N$  and  $K$  will suffice here.  $N = 1$  for a hydrogen atom and  $N = 10^{23}$  for a mole of hydrogen. We shall use the following terminology for two notions of stability:

- |       |                  |                              |
|-------|------------------|------------------------------|
| (1.1) | $E_0 > -\infty$  | Stability of the first kind, |
| (1.2) | $E_0 > C(N + K)$ | Stability of the second kind |

for some constant  $C \leq 0$  that is independent of  $N$  and  $K$ , but which may depend on the physical parameters of the system (such as the electron charge and mass). Usually,  $C < 0$ , which means that there is a positive binding energy per particle.

Stability of the second kind is absolutely essential if quantum mechanics is going to reproduce some of the basic features of the ordinary material world: The energy of ordinary matter is extensive (i.e., it is proportional to the number of particles), the thermodynamic limit exists (i.e., the  $N \rightarrow \infty$  limit exists) and the laws of thermodynamics hold. Bringing two stones together might produce a spark, but not an explosion with a release of energy comparable to the energy in each stone. Stability of the second kind does not guarantee the existence of the thermodynamic limit for the free energy, but it is an essential ingredient [22] [19, Sect. V].

It turns out that stability of the second kind cannot be taken for granted, as Dyson discovered [9]. If Coulomb forces are involved, then *the Pauli exclusion principle is essential*. (This means that the  $L^2$  functions of  $N$  variables,  $\Psi(x_1, x_2, \dots, x_N)$ ,  $x_i \in \mathbb{R}^3$ , is antisymmetric under all transpositions  $x_i \leftrightarrow x_j$ . Particles, like electrons, whose functions  $\Psi$  obey this principle are called *fermions*. Particles whose  $\Psi$  functions are symmetric under permutations are called *bosons*.)

Charged bosons are *not stable* because for them  $E_0 \sim -N^{7/5}$  (nonrelativistically) and  $E_0 = -\infty$  for large, but finite  $N$  (relativistically, see Sect. 3.2). While positively charged bosons exist in the form of atomic nuclei, negatively charged, long-lived bosons do not exist in nature. This is a good thing in view of the instability just mentioned.

**1.3. The Electromagnetic Field.** A second big problem handed down from classical physics was the ‘electromagnetic mass’ of the electron. This poor creature has to drag around an infinite amount of electromagnetic energy that Maxwell burdened it with. Moreover, the electromagnetic field itself is quantized – indeed, that fact alone started the whole revolution [34].

While quantum mechanics accounted for stability with Coulomb forces and Schrödinger led us to think seriously about the ‘wave function of the universe’, physicists shied away from talking about the wave function of the particles in the universe *and* the electromagnetic field in the universe. It is noteworthy that physicists are happy to discuss the quantum mechanical many-body problem with external electromagnetic fields non-perturbatively, but this is rarely done with the quantized field. The quantized field cannot be avoided because it is needed for a correct description of atomic radiation, the laser, etc. However, the interaction of matter with the quantized field is almost always treated perturbatively or else in the context of highly simplified models (e.g., with two-level atoms for lasers).

The quantized electromagnetic field greatly complicates the stability of matter question. It requires, ultimately, mass and charge renormalizations. At present such a complete theory does not exist, but a theory *must* exist because matter exists and because we have strong experimental evidence about the manner in which the electromagnetic field interacts with matter, i.e., we know the essential features of a Hamiltonian that adequately accounts for the low energy processes that exist in every day life. In short, nature tells us that it must be possible to formulate a self-consistent quantum electrodynamics (QED) *non-perturbatively*, (perhaps with an ultraviolet, or high frequency, cutoff of the field at a few MeV). It should not be necessary to have recourse to quantum chromodynamics (QCD) or some other high energy theory to explain ordinary matter.

Physics and other natural sciences are successful because physical phenomena associated with each range of energy and other parameters are explainable to a good, if not perfect, accuracy by an appropriate self-consistent theory. This is true whether it be hydrodynamics, celestial dynamics, statistical mechanics, etc. If low energy physics (atomic and condensed matter physics) is not explainable by a self-consistent, non-perturbative theory on its own level one can speak of an epistemological crisis.

Some readers might say that QED is in good shape. After all, it accurately predicts the outcome of some very high precision experiments (Lamb shift,  $g$ -factor of the electron). But the theory does not really work well when faced with the problem, which is explored here, of understanding the many-body ( $N \approx 10^{23}$ ) problem and the stable low energy world in which we spend our everyday lives.

**1.4. Relativistic Mechanics.** When the classical kinetic energy of a particle,  $p^2/2m$ , is replaced by its relativistic version  $\sqrt{p^2c^2 + m^2c^4}$  the stability question becomes much more complicated, as will be seen later. It turns out that even stability of the first kind is not easy to obtain and it depends on the values of the physical constants, notably the fine structure constant

$$(1.3) \quad \alpha = e^2/\hbar c = 1/137.04 ,$$

where  $-e$  is the electric charge of the electron.

For ordinary matter relativistic effects are not dominant but they are noticeable. In large atoms these effects severely change the innermost electrons and this has a noticeable effect on the overall electron density profile. Therefore, some version of relativistic mechanics is needed, which means, presumably, that we must know how to replace  $p^2/2m$  by the Dirac operator (see (5.1)).

The combination of relativistic mechanics plus the electromagnetic field (in addition to the Coulomb interaction) makes the stability problem difficult and uncertain. Major aspects of this problem have been worked out in the last few years (about 35) and that is the subject of this paper.

## 2. Nonrelativistic Matter without the Magnetic Field

Maxwell's equations define the electric and magnetic fields in terms of potentials. While the equations determine the fields, the potentials are not determined uniquely; the choice of potentials is called the choice of gauge. We work in the 'Coulomb' gauge for the electromagnetic field. Despite the assertion that quantum mechanics and quantum field theory are gauge invariant, it seems to be essential to use this gauge, even though its relativistic covariance is not as transparent as that of the Lorentz gauge. The reason is the following.

The Coulomb gauge is the gauge in which electrostatic part of the interaction of matter with the electromagnetic field is just the conventional Coulomb "action at a distance" potential  $V_c$  given by (2.1) below (in energy units  $mc^2$  and length units the Compton wavelength  $\hbar/mc$ ). This part of the interaction depends only on the coordinates of the particles and not on their velocities. The dependence of the interaction on velocities, or currents, comes about through the magnetic part of the interaction. Despite appearances, this picture is fully Lorentz invariant (even

if it is not gauge invariant).

$$(2.1)V_c = - \sum_{i=1}^N \sum_{k=1}^K \frac{Z_k}{|\mathbf{x}_i - \mathbf{R}_k|} + \sum_{1 \leq i < j \leq N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|} + \sum_{1 \leq k < l \leq K} \frac{Z_k Z_l}{|\mathbf{R}_k - \mathbf{R}_l|} .$$

The first sum is the interaction of the electrons (with dynamical coordinates  $\mathbf{x}_i$ ) and fixed nuclei located at  $\mathbf{R}_k$  of positive charge  $Z_k$  times the (negative) electron charge  $e$ . The second is the electron-electron repulsion and the third is the nucleus-nucleus repulsion. The nuclei are fixed because they are so massive relative to the electron that their motion is irrelevant. It could be included, however, but it would change nothing essential. Likewise, there is no nuclear structure factor because if it were essential for stability then the size of atoms would be the size of nuclei, about  $10^{-13}$  cm, instead of about  $10^{-8}$  cm, contrary to what is observed.

Although the nuclei are fixed points the constant  $C$  in the stability of matter (1.2) is required to be independent of the  $\mathbf{R}_k$ 's. Likewise (1.1) requires that  $E_0$  have a finite lower bound that is independent of the  $\mathbf{R}_k$ 's.

For simplicity of exposition we shall assume here that all the  $Z_k$  are identical, i.e.,  $Z_k = Z$ .

The magnetic field, which will be introduced later, is described by a vector potential  $\mathbf{A}(x)$  which is a dynamical variable in the Coulomb gauge. The magnetic field is  $\mathbf{B} = \text{curl} \mathbf{A}$ .

There is a basic physical distinction between electric and magnetic forces which does not seem to be well known, but which motivates this choice of gauge. In electrostatics “like charges repel” while in magnetostatics “like currents attract”. A consequence of these facts is that the correct magnetostatic interaction energy can be obtained by minimizing the energy functional  $\frac{1}{2} \int B^2 - \int \mathbf{j} \cdot \mathbf{A}$  with respect to the vector field  $\mathbf{A}$ , where  $\mathbf{j}$  is the electric current density. The positive electrostatic energy, on the other hand, *cannot* be obtained by a minimization principle with respect to the field (e.g., minimizing  $\frac{1}{2} \int |\nabla \phi|^2 - \int \phi \rho$  with respect to  $\phi$ ).

The Coulomb gauge, which puts in the electrostatics correctly, by hand, so to speak, and allows us to minimize the total energy with respect to the  $\mathbf{A}$  field, is the gauge that gives us the correct physics and is consistent with the “quintessential quantum mechanical notion of a ground state energy” mentioned in Sect. 1.1. In any other gauge one would have to look for a critical point of a Hamiltonian rather than a true global minimum.

The type of Hamiltonian that we wish to consider in this section is

$$(2.2) \quad H_N = T_N + \alpha V_c .$$

Here,  $T_N$  is the kinetic energy of the  $N$  electrons and has the form

$$(2.3) \quad T_N = \sum_{i=1}^N T_i ,$$

where  $T_i$  acts on the coordinate of the  $i^{\text{th}}$  electron. The nonrelativistic choice is  $T = p^2$  with  $\mathbf{p} = -i\nabla$  and  $p^2 = -\Delta$  in appropriate units.

**2.1. Nonrelativistic Stability for Fermions.** The problem of stability of the second kind for nonrelativistic quantum mechanics was recognized in the early days by a few physicists, e.g., Onsager, but not by many. It was not solved until

1967 in one of the most beautiful papers in mathematical physics by Dyson and Lenard [10].

They found that the Pauli principle, i.e., Fermi-Dirac statistics, is essential. Mathematically, this means that the Hilbert space is the subspace of antisymmetric functions, i.e.,  $\mathcal{H}^{\text{phys}} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^2)$ . This is how the Pauli principle is interpreted post-Schrödinger; Pauli invented his principle a year earlier, however!

Their value for  $C$  in (1.2) was rather high, about  $-10^{15}$  eV (electron volts) for  $Z = 1$ . (The ground state energy of a hydrogen atom is  $-13$  eV.) The situation was improved later by Thirring and myself [31] to about  $-20$  eV for  $Z = 1$  by introducing an inequality that holds only for the kinetic energy of fermions (not bosons) in an arbitrary state  $\Psi$ .

$$(2.4) \quad \langle \Psi, T_N \Psi \rangle \geq (\text{const.}) \int_{\mathbb{R}^3} \rho_{\Psi}(\mathbf{x})^{5/3} d^3 \mathbf{x} ,$$

where  $\rho_{\Psi}$  is the one-body density in the (normalized) fermionic wave function  $\Psi$  (of space and spin) given by an integration over  $(N - 1)$  coordinates and  $N$  spins as follows.

$$(2.5) \quad \rho_{\Psi}(\mathbf{x}) = N \sum_{\sigma_1, \dots, \sigma_N} \int_{\mathbb{R}^{3(N-1)}} |\Psi(\mathbf{x}, \mathbf{x}_2, \dots, \mathbf{x}_N; \sigma_1, \dots, \sigma_N)|^2 d^3 \mathbf{x}_2 \cdots d^3 \mathbf{x}_N .$$

Inequality (2.4) allows one simply to reduce the quantum mechanical stability problem to the stability of Thomas-Fermi theory, which was worked out earlier by Simon and myself [30].

The older inequality of Sobolev, mentioned in Sect. 1.1,

$$(2.6) \quad \langle \Psi, T_N \Psi \rangle \geq (\text{const.}) \left( \int_{\mathbb{R}^3} \rho_{\Psi}(\mathbf{x})^3 d^3 \mathbf{x} \right)^{1/3} ,$$

is not as useful as (2.4) for the many-body problem because its right side is proportional to  $N$  instead of  $N^{5/3}$ . It is, however, strong enough to yield the stability of a system, like an atom, that has only a few electrons.

It is amazing that from the birth of quantum mechanics until 1967 none of the luminaries of physics had quantified the fact that electrostatics plus the uncertainty principle *do not suffice* for stability of the second kind, and thereby make thermodynamics possible (although they do suffice for the first kind). See Sect. 2.2. It was noted, however, that the Pauli principle was responsible for the large sizes of atoms and bulk matter (see, e.g., [9, 10]).

**2.2. Nonrelativistic Instability for Bosons.** What goes wrong if we have charged bosons instead of fermions? Stability of the first kind (1.1) holds in the nonrelativistic case, but (1.2) fails. If we assume the nuclei are infinitely massive, as before, and  $N = KZ$  then  $E_0 \sim -N^{5/3}$  [10, 20]. To remedy the situation we can let the nuclei have finite mass (e.g., the same mass as the negative particles). Then, as Dyson showed [9],  $E_0 \leq -(\text{const.})N^{7/5}$ . This calculation was highly non-trivial! Dyson had to construct a variational function with pairing of the Bogolubov type in a rigorous fashion and this took several pages.

Thus, finite nuclear mass improves the situation, but not enough. The question whether  $N^{7/5}$  is the correct power law remained open for many years. A lower bound of this type was needed and that was finally obtained in [6].

The results of this Section 2 can be summarized by saying that stability of the hydrogen atom is one thing but stability of many-body physics is something else !

### 3. Relativistic Kinematics (no magnetic field)

The next step is to try to get some idea of the effects of relativistic kinematics, which means replacing  $p^2$  by  $\sqrt{p^2 + 1}$  in non-quantum physics. (Recall that  $mc^2 = 1$  in our units.) The simplest way to do this is to substitute  $\sqrt{p^2 + 1}$  for  $T$  in (2.3). The Dirac operator will be discussed later on, but for now this choice of  $T$  will suffice. Actually, it was Dirac's choice before he discovered his operator and it works well in some cases. For example, Chandrasekhar used it successfully, and accurately, to calculate the collapse of white dwarfs (and later, neutron stars).

Since we are interested only in stability, we may, and shall, substitute  $|\mathbf{p}| = \sqrt{-\Delta}$  for  $T$ . The error thus introduced is bounded by a constant times  $N$  since  $|\mathbf{p}| < \sqrt{p^2 + 1} < |\mathbf{p}| + 1$  (as an operator inequality). Our Hamiltonian is now  $H_N = \sum_{i=1}^N |\mathbf{p}_i| + \alpha V_c$ .

**3.1. One-Electron Atom.** The touchstone of quantum mechanics is the Hamiltonian for 'hydrogen' which is, in our case,

$$(3.1) \quad H = |\mathbf{p}| - Z\alpha/|\mathbf{x}| = \sqrt{-\Delta} - Z\alpha/|\mathbf{x}| .$$

It is well known (also to Dirac) that the analogous operator with  $|\mathbf{p}|$  replaced by the Dirac operator (5.1) ceases to make sense when  $Z\alpha > 1$ . Something similar happens for (3.1).

$$(3.2) \quad E_0 = \begin{cases} 0 & \text{if } Z\alpha \leq 2/\pi; \\ -\infty & \text{if } Z\alpha > 2/\pi . \end{cases}$$

The reason for this behavior is that both  $|\mathbf{p}|$  and  $|\mathbf{x}|^{-1}$  scale in the same way. Either the first term in (3.1) wins or the second does.

A result similar to (3.2) was obtained in [11] for the free Dirac operator  $D(0)$  in place of  $|\mathbf{p}|$ , but with the wave function  $\Psi$  restricted to lie in the positive spectral subspace of  $D(0)$ . Here, the critical value is  $\alpha Z \leq (4\pi)/(4 + \pi^2) > 2/\pi$ .

The moral to be drawn from this is that relativistic kinematics plus quantum mechanics is a 'critical' theory (in the mathematical sense). This fact will plague any relativistic theory of electrons and the electromagnetic field – primitive or sophisticated.

**3.2. Many Electrons and Nuclei.** When there are many electrons is it true that the condition  $Z\alpha \leq \text{const.}$  is the only one that has to be considered? The answer is no! One *also* needs the condition that  $\alpha$  itself must be small, regardless of how small  $Z$  might be. This fact can be called a 'discovery' but actually it is an overdue realization of some basic physical ideas. It should have been realized shortly after Dirac's theory in 1927, but it does not seem to have been noted until 1983 [8].

The underlying physical heuristics is the following. With  $\alpha$  fixed, suppose  $Z\alpha = 10^{-6} \ll 1$ , so that an atom is stable, but suppose that we have  $2 \times 10^6$  such nuclei. By bringing them together at a common point we will have a nucleus with  $Z\alpha = 2$  and one electron suffices to cause collapse into it. Then (1.1) fails. What prevents this from happening, presumably, is the nucleus-nucleus repulsion energy which goes to  $+\infty$  as the nuclei come together. But this repulsion energy is

proportional to  $(Z\alpha)^2/\alpha$  and, therefore, if we regard  $Z\alpha$  as fixed we see that  $1/\alpha$  must be large enough in order to prevent collapse.

Whether or not the reader believes this argument, the mathematical fact is that there is a fixed, finite number  $\alpha_c \leq 2.72$  ([32]) so that when  $\alpha > \alpha_c$  (1.1) fails for *every* positive  $Z$  and for every  $N \geq 1$  (with or without the Pauli principle).

The open question was whether (1.2) holds for *all*  $N$  and  $K$  if  $Z\alpha$  and  $\alpha$  are both small enough. The breakthrough was due to Conlon [5] who proved (1.2), for fermions, if  $Z = 1$  and  $\alpha < 10^{-200}$ . The situation was improved by Fefferman and de la Llave [13] to  $Z = 1$  and  $\alpha < 0.16$ . Finally, the expected correct condition  $Z\alpha \leq 2/\pi$  and  $\alpha < 1/94$  was obtained in [32]. (This paper contains a detailed history up to 1988.) The situation was further improved in [27]. The multi-particle version of the use of the free Dirac operator, as in Sect. 3.1, was treated in [18].

Finally, it has to be noted that charged bosons are *always* unstable of the first kind (not merely the second kind, as in the nonrelativistic case) for *every* choice of  $Z > 0, \alpha > 0$ . E.g., there is instability if  $Z^{2/3}\alpha N^{1/3} > 36$  ([32]).

We are indeed fortunate that there are no stable, negatively charged bosons.

#### 4. Interaction of Matter with Classical Magnetic Fields

The magnetic field  $\mathbf{B}$  is defined by a vector potential  $\mathbf{A}(\mathbf{x})$  and  $\mathbf{B}(\mathbf{x}) = \text{curl } \mathbf{A}(\mathbf{x})$ . In this section we take a first step (warmup exercise) by regarding  $\mathbf{A}$  as classical, but indeterminate, and we introduce the classical field energy

$$(4.1) \quad H_f = \frac{1}{8\pi} \int_{\mathbb{R}^3} B(\mathbf{x})^2 dx .$$

The Hamiltonian is now

$$(4.2) \quad H_N(\mathbf{A}) = T_N(\mathbf{A}) + \alpha V_c + H_f ,$$

in which the kinetic energy operator has the form (2.3) but depends on  $\mathbf{A}$ . We now define  $E_0$  to be the infimum of  $\langle \Psi, H_N(\mathbf{A})\Psi \rangle$  both with respect to  $\Psi$  and with respect to  $\mathbf{A}$ .

**4.1. Nonrelativistic Matter with Magnetic Field.** The simplest situation is merely ‘minimal coupling’ without spin, namely,

$$(4.3) \quad T(\mathbf{A}) = |\mathbf{p} + \sqrt{\alpha}\mathbf{A}(\mathbf{x})|^2$$

This choice does not change any of our previous results qualitatively. The field energy is not needed for stability. On the one-particle level, we have the ‘diamagnetic inequality’  $\langle \phi, |\mathbf{p} + \mathbf{A}(\mathbf{x})|^2 \phi \rangle \geq \langle |\phi|, p^2 |\phi| \rangle$ . The same holds for  $|\mathbf{p} + \mathbf{A}(\mathbf{x})|$  and  $|\mathbf{p}|$ . More importantly, inequality (2.4) for fermions continues to hold (with the same constant) with  $T(\mathbf{A})$  in place of  $p^2$ . (There is an inequality similar to (2.4) for  $|\mathbf{p}|$ , with  $5/3$  replaced by  $4/3$ , which also continues to hold with minimal substitution [7].)

The situation gets much more interesting if spin is included. This takes us a bit closer to the relativistic case. The kinetic energy operator is the Pauli operator

$$(4.4) \quad T^P(\mathbf{A}) = |\mathbf{p} + \sqrt{\alpha}\mathbf{A}(\mathbf{x})|^2 + \sqrt{\alpha}\mathbf{B}(\mathbf{x}) \cdot \boldsymbol{\sigma} ,$$

where  $\boldsymbol{\sigma}$  is the vector of  $2 \times 2$  Pauli spin matrices and  $L^2(\mathbb{R}^3)$  is replaced by  $L^2(\mathbb{R}^3; \mathbb{C}^3)$

4.1.1. *One-Electron Atom.* The stability problem with  $T^P(\mathbf{A})$  is complicated, even for a one-electron atom. Without the field energy  $H_f$  the Hamiltonian is unbounded below. (For fixed  $\mathbf{A}$  it is bounded but the energy tends to  $-\infty$  like  $-(\log B)^2$  for a homogeneous field [2].) The field energy saves the day, but the result is surprising [14] (recall that we must minimize the energy with respect to  $\Psi$  and  $\mathbf{A}$ ):

$$(4.5) \quad |\mathbf{p} + \sqrt{\alpha} \mathbf{A}(\mathbf{x})|^2 + \sqrt{\alpha} \mathbf{B}(\mathbf{x}) \cdot \boldsymbol{\sigma} - Z\alpha/|\mathbf{x}| + H_f$$

is bounded below if and only if  $Z\alpha^2 \leq C$ , where  $C$  is some constant that can be bounded as  $1 < C < 9\pi^2/8$ .

The proof of instability [33] is difficult and requires the construction of a zero mode (soliton) for the Pauli operator, i.e., a finite energy magnetic field and a square integrable  $\psi$  such that

$$(4.6) \quad T^P(\mathbf{A})\psi = 0 .$$

The usual kinetic energy  $|\mathbf{p} + \mathbf{A}(\mathbf{x})|^2$  has no such zero mode for any  $\mathbf{A}$ , even when 0 is the bottom of its spectrum.

The original magnetic field [33] that did the job in (4.6) is independently interesting, geometrically (many others have been found since then).

$$\mathbf{B}(\mathbf{x}) = \frac{12}{(1 + |\mathbf{x}|^2)^3} [(1 - x^2)\mathbf{w} + 2(\mathbf{w} \cdot \mathbf{x})\mathbf{x} + 2\mathbf{w} \wedge \mathbf{x}]$$

with  $|\mathbf{w}| = 1$ . The field lines of this magnetic field form a family of curves, which, when stereographically projected onto the 3-dimensional unit sphere, become the great circles in what is known as the Hopf fibration.

Thus, we begin to see that nonrelativistic matter with magnetic fields behaves like relativistic matter without fields – to some extent.

The moral of this story is that a magnetic field, which we might think of as possibly self-generated, can cause an electron to fall into the nucleus. The uncertainty principle cannot prevent this, not even for an atom!

4.1.2. *Many Electrons and Many Nuclei.* In analogy with the relativistic (no magnetic field) case, we can see that stability of the first kind fails if  $Z\alpha^2$  or  $\alpha$  is too large. The heuristic reasoning is the same and the proof is similar.

We can also hope that stability of the second kind holds if both  $Z\alpha^2$  and  $\alpha$  are small enough. The problem is complicated by the fact that it is the field energy  $H_f$  that will prevent collapse, but there there is only one field energy while there are  $N \gg 1$  electrons.

The hope was finally realized, however. Fefferman [12] proved stability of the second kind for  $H_N(\mathbf{A})$  with the Pauli  $T^P(\mathbf{A})$  for  $Z = 1$  and “ $\alpha$  sufficiently small”. A few months later it was proved [28] for  $Z\alpha^2 \leq 0.04$  and  $\alpha \leq 0.06$ . With  $\alpha = 1/137$  this amounts to  $Z \leq 1050$ . This very large  $Z$  region of stability is comforting because it means that perturbation theory (in  $\mathbf{A}$ ) can be reliably used for this particular problem.

Using the results in [28], Bugliaro, Fröhlich and Graf [3] proved stability of the same nonrelativistic Hamiltonian – but with an ultraviolet cutoff, quantized magnetic field whose field energy is described below. (Note: No cutoffs are needed for classical fields.)

There is also the very important work of Bach, Fröhlich, and Sigal [4] who showed that this nonrelativistic Hamiltonian with ultraviolet cutoff, quantized field

and with sufficiently small values of the parameters has other properties that one expects. E.g., the excited states of atoms dissolve into resonances and only the ground state is stable. The infrared singularity notwithstanding, the ground state actually exists (the bottom of the spectrum is an eigenvalue); this was shown in [4] for small parameters and in [15], [26] for all values of the parameters. (See Sect. 7.)

## 5. Relativity Plus Magnetic Fields

As a next step in our efforts to understand QED and the many-body problem we introduce relativity theory along with the classical magnetic field.

**5.1. Relativity Plus Classical Magnetic Fields.** Originally, Dirac and others thought of replacing  $T^P(\mathbf{A})$  by  $\sqrt{T^P(\mathbf{A}) + 1}$  but this was not successful mathematically and does not seem to conform to experiment. Consequently, we introduce the Dirac operator for  $T$  in (2.3), (4.2)

$$(5.1) \quad D(\mathbf{A}) = \boldsymbol{\alpha} \cdot \mathbf{p} + \sqrt{\alpha} \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x}) + \beta m ,$$

where  $\boldsymbol{\alpha}$  and  $\beta$  denote the  $4 \times 4$  Dirac matrices and  $\sqrt{\alpha}$  is the electron charge as before. (This notation of  $\boldsymbol{\alpha}$  and  $\alpha$  is historical and is not mine.) The Hilbert space for  $N$  electrons is now changed to

$$(5.2) \quad \mathcal{H} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^4) .$$

The well known problem with  $D(\mathbf{A})$  is that it is unbounded below, and so we cannot hope to have stability of the first kind, even with  $Z = 0$ . Let us imitate QED (but without pair production or renormalization) by restricting the electron wave function to lie in the positive spectral subspace of a Dirac operator.

Which Dirac operator?

There are two natural operators in the problem. One is  $D(0)$ , the free Dirac operator. The other is  $D(\mathbf{A})$  that is used in the Hamiltonian. In almost all formulations of QED the electron is defined by the positive spectral subspace of  $D(0)$ . Thus, we can define

$$(5.3) \quad \mathcal{H}^{\text{phys}} = P^+ \mathcal{H} = \prod_{i=1}^N \pi_i \mathcal{H} ,$$

where  $P^+ = \prod_{i=1}^N \pi_i$ , and  $\pi_i$  is the projector of onto the positive spectral subspace of  $D_i(0) = \boldsymbol{\alpha} \cdot \mathbf{p}_i + \beta m$ , the free Dirac operator for the  $i^{\text{th}}$  electron. We then restrict the allowed wave functions in the variational principle to those  $\Psi$  satisfying

$$(5.4) \quad \Psi = P^+ \Psi \quad \text{i.e., } \Psi \in \mathcal{H}^{\text{phys}} .$$

Another way to say this is that we replace the Hamiltonian (4.2) by  $P^+ H_N P^+$  on  $\mathcal{H}$  and look for the bottom of its spectrum.

It turns out that this prescription leads to disaster! While the use of  $D(0)$  makes sense for an atom, it fails miserably for the many-fermion problem, as discovered in [29] and refined in [16]. The result is:

*For all  $\alpha > 0$  in (5.1) (with or without the Coulomb term  $\alpha V_c$ ) one can find  $N$  large enough so that  $E_0 = -\infty$ .*

In other words, the term  $\sqrt{\alpha} \boldsymbol{\alpha} \cdot \mathbf{A}$  in the Dirac operator can cause an instability that the field energy cannot prevent.

It turns out, however, that the situation is saved if one uses the positive spectral subspace of the Dirac operator  $D(\mathbf{A})$  to define an electron. (This makes the concept of an electron  $\mathbf{A}$  dependent, but when we make the vector potential into a dynamical

quantity in the next section, this will be less peculiar since there will be no definite vector potential but only a fluctuating quantity.) The definition of the physical Hilbert space is as in (5.3) but with  $\pi_i$  being the projector onto the positive subspace of the full Dirac operator  $D_i(\mathbf{A}) = \boldsymbol{\alpha} \cdot \mathbf{p}_i + \sqrt{\alpha} \boldsymbol{\alpha} \cdot \mathbf{A}(\mathbf{x}_i) + \beta m$ . Note that these  $\pi_i$  projectors commute with each other and hence their product  $P^+$  is a projector.

The result [29] for this model ((4.2) with the Dirac operator and the restriction to the positive spectral subspace of  $D(\mathbf{A})$ ) is reminiscent of the situations we have encountered before:

*If  $\alpha$  and  $Z$  are small enough stability of the second kind holds for this model.*

Typical stability values that are rigorously established [29] are  $Z \leq 56$  with  $\alpha = 1/137$  or  $\alpha \leq 1/8.2$  with  $Z = 1$ .

### 6. Quantized Electromagnetic Fields

Let us now try to analyze some of the problems connected with the quantization of the electromagnetic field. The great discovery of Max Planck [34], which was the first step in the new quantum theory, was that the energy of the electromagnetic field came in quantized units. The energy unit of electromagnetic waves of frequency  $\nu$  is  $h\nu$ , and in terms of wave number  $k$  (i.e., the wave is proportional to  $\exp(ik \cdot x)$ ) it is  $\hbar c|k|$  since  $2\pi\nu/|k| = c = \text{speed of light}$ .

We begin with the problem of generalizing the results in the previous subsection to the quantized field.

**6.1. Relativity Plus Quantized Magnetic Field.** The obvious next step is to try to imitate the strategy of Sect. 5.1 but with the quantized  $\mathbf{A}$  field. This was done in [24]. The quantized  $A$  field is described by an operator-valued Fourier transform as

$$(6.1) \quad \mathbf{A}(\mathbf{x}) = \frac{1}{2\pi} \sum_{\lambda=1}^2 \int_{|\mathbf{k}| \leq \Lambda} \frac{\boldsymbol{\varepsilon}_\lambda(\mathbf{k})}{\sqrt{|\mathbf{k}|}} \left[ a_\lambda(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + a_\lambda^*(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] d^3\mathbf{k} ,$$

where  $\Lambda$  is the ultraviolet cutoff on the wave-numbers  $|\mathbf{k}|$ . The operators  $a_\lambda, a_\lambda^*$  satisfy the usual canonical commutation relations

$$(6.2) \quad [a_\lambda(\mathbf{k}), a_\nu^*(\mathbf{q})] = \delta(\mathbf{k} - \mathbf{q}) \delta_{\lambda,\nu} , \quad [a_\lambda(\mathbf{k}), a_\nu(\mathbf{q})] = 0 , \quad \text{etc}$$

and the vectors  $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$  are two orthonormal polarization vectors perpendicular to  $\mathbf{k}$  and to each other.

The field energy  $H_f$  is now given by a normal-ordered version of (4.1)

$$(6.3) \quad H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |\mathbf{k}| a_\lambda^*(\mathbf{k}) a_\lambda(\mathbf{k}) d^3\mathbf{k}$$

The Dirac operator is the same as before, (5.1). Note that  $D_i(\mathbf{A})$  and  $D_j(\mathbf{A})$  still commute with each other (since  $\mathbf{A}(\mathbf{x})$  commutes with  $\mathbf{A}(\mathbf{y})$ ). This is important because it allows us to imitate Sect. 5.1.

In analogy with (5.2) we define

$$(6.4) \quad \mathcal{H} = \wedge^N L^2(\mathbb{R}^3; \mathbb{C}^4) \otimes \mathcal{F} ,$$

where  $\mathcal{F}$  is the Fock space for the photon field. We can then define the *physical* Hilbert space as before

$$(6.5) \quad \mathcal{H}^{\text{phys}} = \Pi \mathcal{H} = \prod_{i=1}^N \pi_i \mathcal{H} ,$$

where the projectors  $\pi_i$  project onto the positive spectral subspace of either  $D_i(0)$  or  $D_i(\mathbf{A})$ .

Perhaps not surprisingly, the former case leads to catastrophe, as before. This is so, even with the ultraviolet cutoff, which we did not have in Sect. 5.1. Because of the cutoff the catastrophe is milder and involves instability of the second kind instead of the first kind. This result relies on a coherent state construction in [16].

The latter case (use of  $D(\mathbf{A})$  to define an electron) leads to stability of the second kind if  $Z$  and  $\alpha$  are not too large. Otherwise, there is instability of the first kind. The rigorous estimates are comparable to the ones in Sect. 5.1.

Clearly, many things have yet to be done to understand the stability of matter in the context of QED. Renormalization and pair production have to be included, for example.

The results of this section suggest, however, that a significant change in the Hilbert space structure of QED might be necessary. We see that it does not seem possible to keep to the current view that the Hilbert space is a simple tensor product of a space for the electrons and a Fock space for the photons. That leads to instability for many particles (or large charge, if the idea of ‘particle’ is unacceptable). The ‘bare’ electron is not really a good physical concept and one must think of the electron as always accompanied by its electromagnetic field. Matter and the photon field are inextricably linked in the Hilbert space  $\mathcal{H}^{\text{phys}}$ .

The following tables [24] summarize some of the results of this and the previous sections

**Electrons defined by projection onto the positive  
subspace of  $D(0)$ , the free Dirac operator**

	Classical or quantized field without cutoff $\Lambda$ $\alpha > 0$ but arbitrarily small.	Classical or quantized field with cutoff $\Lambda$ $\alpha > 0$ but arbitrarily small.
Without Coulomb potential $\alpha V_c$	Instability of the first kind	Instability of the second kind
With Coulomb potential $\alpha V_c$	Instability of the first kind	Instability of the second kind

**Electrons defined by projection onto the positive  
subspace of  $D(\mathbf{A})$ , the Dirac operator with field**

	Classical field with or without cutoff $\Lambda$ or quantized field with cutoff $\Lambda$
Without Coulomb potential $\alpha V_c$	The Hamiltonian is positive
With Coulomb potential $\alpha V_c$	Instability of the first kind when either $\alpha$ or $Z\alpha$ is too large
	Stability of the second kind when both $\alpha$ and $Z\alpha$ are small enough

**6.2. Mass Renormalization.** In both classical and quantum electrodynamics there is a problem of mass renormalization. This means that when a charge is accelerated its accompanying electromagnetic field is also accelerated and acts like an additional mass. The ‘bare mass’ of the particle (which is the mass that appears in the Hamiltonian) must be chosen so that the final, physical mass (as measured in experiments) agrees with the physically measured value.

For a point particle, the additional mass is infinity, classically. For QED it is also infinite, but the divergence is less rapid as the radius of the charge goes to zero. In any case, with a finite ultraviolet cutoff  $\Lambda$  the additional mass is finite, but it is far from clear that, for each  $\Lambda > 0$  one can adjust the bare mass (while keeping it positive) to give the correct physical mass. Opinions differ on this point and very little is known rigorously about the problem outside of perturbation theory. See [17].

There are two ways to define mass renormalization. Take one particle ( $N = 1$ ) and then either

1. Find the bottom of the spectrum of  $T + H_f$  under the condition that the total momentum of particle plus field is  $p$ . Call it  $E(p)$  and write, for small  $p$ ,

$$E(p) = E(p = 0) + p^2/2m_{\text{physical}}$$

or else

2. Compute the binding energy of hydrogen ( $N = 1, K = 1, Z = 1$ ). Call it  $E_0$  and set

$$E_0 = m_{\text{physical}}c^2\alpha^2/2\hbar^2$$

The first way is the usual one; the second is motivated by the earliest experiment in quantum mechanics. These two definitions are not the same. In any case, we [25] can now obtain non-trivial bounds on the binding energy (in the context of the Schrödinger Hamiltonian or the Pauli Hamiltonian interacting with the quantized field) and thereby get some bounds on the renormalized mass using definition 2. For large cutoff  $\Lambda$ , these bounds differ in their  $\Lambda$  dependence from what might be expected from perturbation theory.

## 7. Existence of Atoms in Non-relativistic QED

One of the most recent topics concerns the seemingly trivial question of the existence of atoms. In some sense this question is the opposite of the stability of matter question.

The Hamiltonian we shall use to describe an atom or molecule with  $N$  electrons is

$$(7.1) \quad H_N = \sum_{i=1}^N T_i^P(A) + \alpha V_c + H_f$$

where  $T_i^P(A)$  is the Pauli kinetic energy operator (4.4), but  $A$  is the quantized magnetic field given by (6.1), and  $H_f$  is the energy of the quantized field given by (6.3). As before,  $V_c$  is the Coulomb potential (2.1) of some fixed nuclei whose total nuclear charge is denoted by  $Z = \sum Z_j$ .

To show the existence of stable atoms we need to establish two things about  $H_N$

1. The ground state energy (bottom of the spectrum) of  $H_N$  is lower than that of  $H_{N'}$  i.e., of a system with  $N' < N$  electrons (with the remaining  $N - N'$  electrons being allowed to escape to infinity). This is called *the binding condition*.

2. The bottom of the spectrum of  $H_N$  is actually an eigenvalue, i.e., Schrödinger's equation has a square integrable solution with  $E =$  the bottom of the spectrum.

In the case of the Schrödinger equation without the field, problem 1. was solved by Zhislin in 1960 for the case  $N < Z + 1$ , which includes the neutral molecule. He did this by using a localization technique, whose positive localization energy ( $r^{-2}$ ) is more than offset by the Coulomb attraction ( $-r^{-1}$ ) of a positively charged system ( $Z - N'$ ) to a negatively charged electron. The existence of the ground state (problem 2.) follows from standard arguments because in this case the bottom of the spectrum is negative while the bottom of the essential spectrum (which, in this case, is the bottom of the continuum) starts at zero. Thus, there is a gap in the spectrum and the technique of taking weak limits easily yields a non-zero eigenfunction [23].

When we turn on the interaction with the quantized magnetic field the situation changes significantly. One major difference is that the bottom of the essential spectrum is now the bottom of the spectrum because we can always create photons with arbitrarily small energy (recall that the energy of a photon with momentum  $k$  is  $|k|$ ). Therefore, if a ground state exists it necessarily lies at the bottom of the essential spectrum and is not isolated. Eigenvalues in the continuum are notoriously difficult to handle, even for the simple Schrödinger operator.

A second major difference is that it is necessary to localize the  $A$  field as well as the electrons. This localization costs an energy  $r^{-1}$ , not  $r^{-2}$  as before, essentially because the field energy is proportional to  $|k|$  instead of  $k^2$ . Thus, the field localization competes with the Coulomb attraction.

Problems 1. and 2. were solved in [4] under the condition that  $\alpha$  and  $\Lambda$  are small enough.

The first general result, valid for all values of the various constants, was in [15], where it was shown that 2. holds whenever 1. holds.

Finally, 1. was shown to hold for all values of the constants [26] under the same natural condition as Zhislin's, i.e.,  $N < Z + 1$ .

## References

- [1] This paper is an updated version of E.H. Lieb: *The Stability of Matter and Quantum Electrodynamics*, Milan Jour. Math. **71**, 199-217 (2003). An earlier version appears in the

- proceedings of the Werner Heisenberg Centennial, December, 2001 in Munich. Springer Verlag (in press).
- [2] J. Avron, I. Herbst, B. Simon: *Schrödinger operators with magnetic fields III*, Commun. Math. Phys. **79**, 529- 572 (1981).
  - [3] L. Bugliaro, J. Fröhlich, G. M. Graf: *Stability of quantum electrodynamics with nonrelativistic matter*, Phys. Rev. Lett. **77**, 3494-3497 (1996).
  - [4] V. Bach, J. Fröhlich, I. M. Sigal: *Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field*. Comm. Math. Phys. **207** 249-290 (1999).
  - [5] J. G. Conlon: *The ground state of a classical gas*, Commun. Math. Phys. **94**, 439-458 (1984).
  - [6] J. G. Conlon, E. H. Lieb, H.-T. Yau: *The  $N^{7/5}$  law for charged bosons*, Commun. Math. Phys. **116**, 417-448 (1988).
  - [7] I. Daubechies: *An uncertainty principle for fermions with generalized kinetic energy*, Commun. Math. Phys. **90**, 511-520 (1983).
  - [8] I. Daubechies, E. H. Lieb: *One electron relativistic molecules with Coulomb interaction*, Commun. Math. Phys. **90**, 497-510 (1983).
  - [9] F. J. Dyson: *Ground state energy of a finite system of charged particles*, J. Math. Phys. **8**, 1538-1545 (1967).
  - [10] F. J. Dyson, A. Lenard: *Stability of matter I and II*, J. Math. Phys. **8**, 423-434 (1967), **9**, 1538-1545 (1968).
  - [11] W. D. Evans, P. P. Perry, H. Siedentop: *The spectrum of relativistic one-electron atoms according to Bethe and Salpeter*, Commun. Math. Phys. **178**, 733-746 (1996).
  - [12] C. Fefferman: *Stability of Coulomb systems in a magnetic field*, Proc. Nat. Acad. Sci. USA, **92**, 5006-5007 (1995).
  - [13] C. Fefferman, R. de la Llave: *Relativistic stability of matter. I*. Rev. Mat. Iberoamericana **2**, 119-213 (1986).
  - [14] J. Fröhlich, E. H. Lieb, M. Loss: *Stability of Coulomb systems with magnetic fields I. The one-electron atom*, Commun. Math. Phys. **104**, 251-270 (1986). See also E. H. Lieb, M. Loss: *Stability of Coulomb systems with magnetic Fields II. The many-electron atom and the one-electron molecule*, Commun. Math. Phys. **104**, 271-282 (1986).
  - [15] M. Griesemer, E. H. Lieb, M. Loss: *Ground states in non-relativistic quantum electrodynamics*, Invent. Math. **145**, 557-595 (2001).
  - [16] M. Griesemer, C. Tix: *Instability of pseudo-relativistic model of matter with self-generated magnetic field*, J. Math. Phys. **40**, 1780-1791 (1999).
  - [17] C. Hainzl and R. Seiringer: *Mass renormalization and energy level shifts in non-relativistic QED*, Adv. Theor. Math. Phys. **6**, 847-871 (2002).
  - [18] G. Hoever, H. Siedentop: *The Brown-Ravenhall operator*, Math. Phys. Electronic Jour. **5**, no. 6 (1999).
  - [19] E. H. Lieb: *The stability of matter*, Rev. Mod. Phys. **48**, 553-569 (1976).
  - [20] E. H. Lieb: *The  $N^{5/3}$  law for bosons*, Phys. Lett. **70A**, 71-73 (1979).
  - [21] E. H. Lieb: *The stability of matter: From atoms to stars*, Bull. Amer. Math. Soc. **22**, 1-49 (1990).
  - [22] E. H. Lieb, J.L. Lebowitz: *The existence of thermodynamics for real matter with Coulomb forces*, Phys. Rev. Lett. **22**, 631-634 (1969).
  - [23] E. H. Lieb, M. Loss: *Analysis*, American Mathematical Society (1997).
  - [24] E. H. Lieb, M. Loss: *Stability of a Model of Relativistic Quantum Electrodynamics*, Commun. Math. Phys. **228**, 561-588 (2002). arXiv math-ph/0109002, mp\_arc 01-315.
  - [25] E. H. Lieb, M. Loss: *A Bound on Binding Energies and Mass Renormalization in Models of Quantum Electrodynamics*, J. Stat. Phys. **108**, 1057-1069 (2002). arXiv math-ph/0110027.
  - [26] E. H. Lieb, M. Loss: *Existence of Atoms and Molecules in Non-relativistic Quantum Electrodynamics*, Adv. Theor. Math. Phys. (in press). arXiv math-ph/0307046.
  - [27] E. H. Lieb, M. Loss, H. Siedentop: *Stability of relativistic matter via Thomas-Fermi theory*, Helv. Phys. Acta **69**, 974-984 (1996).
  - [28] E. H. Lieb, M. Loss, J. P. Solovej: *Stability of matter in magnetic fields*, Phys. Rev. Lett. **75**, 985-989 (1995).
  - [29] E. H. Lieb, H. Siedentop, J. P. Solovej: *Stability and instability of relativistic electrons in magnetic fields*, J. Stat. Phys. **89**, 37-59 (1997). See also *Stability of relativistic matter with magnetic fields*, Phys. Rev. Lett. **79**, 1785-1788 (1997).
  - [30] E. H. Lieb, B. Simon: *Thomas-Fermi theory revisited*, Phys. Rev. Lett. **31**, 681-683 (1973).

- [31] E. H. Lieb, W. Thirring: *Bound for the kinetic energy of fermions which proves the stability of matter*, Phys. Rev. Lett. **35**, 687-689 (1975). Errata **35**, 1116 (1975).
- [32] E. H. Lieb, H.-T. Yau: *The stability and instability of relativistic matter*, Commun. Math. Phys. **118**, 177-213 (1988). See also *Many-body stability implies a bound on the fine structure constant*, Phys. Rev. Lett. **61**, 1695-1697 (1988).
- [33] M. Loss, H.-T. Yau: *Stability of Coulomb systems with magnetic fields III. Zero energy bound states of the Pauli operator*, Commun. Math. Phys. **104**, 283-290 (1986).
- [34] M. Planck: *Zur Theorie des Gesetzes der Energieverteilung im Normalspektrum*, Verhandlung der Deutschen Physikalischen Gesellschaft **2**, 237-245 (1900).
- [35] G. Zhislin: *A study of the spectrum of the Schrödinger operator for a system of several particles*, Trudy Moscov Mat. Obsc. **9**, 81-120 (1960).

DEPARTMENTS OF MATHEMATICS AND PHYSICS, PRINCETON UNIVERSITY, P.O. BOX 708,  
PRINCETON, NJ 08544-0708, USA

*E-mail address:* `lieb@princeton.edu`

## Titles in This Series

- 477 **Ignacio Luengo, Editor**, Recent Trends in Cryptography, 2008
- 476 **Carlos Villegas-Blas, Editor**, Fourth summer school in analysis and mathematical physics: Topics in spectral theory and quantum mechanics, 2008
- 475 **Jean-Paul Brasselet, José Luis Cisneros-Molina, David Massey, José Seade, and Bernard Teissier, Editors**, Singularities II: Geometric and topological aspects, 2008
- 474 **Jean-Paul Brasselet, José Luis Cisneros-Molina, David Massey, José Seade, and Bernard Teissier, Editors**, Singularities I: Algebraic and analytic aspects, 2008
- 473 **Alberto Farina and Jean-Claude Saut, Editors**, Stationary and time dependent Gross-Pitaevskii equations, 2008
- 472 **James Arthur, Wilfried Schmid, and Peter E. Trapa, Editors**, Representation Theory of Real Reductive Lie Groups, 2008
- 471 **Diego Dominici and Robert S. Maier, Editors**, Special functions and orthogonal polynomials, 2008
- 470 **Luise-Charlotte Kappe, Arturo Magidin, and Robert Fitzgerald Morse, Editors**, Computational group theory and the theory of groups, 2008
- 469 **Keith Burns, Dmitry Dolgopyat, and Yakov Pesin, Editors**, Geometric and probabilistic structures in dynamics, 2008
- 468 **Bruce Gilligan and Guy J. Roos, Editors**, Symmetries in complex analysis, 2008
- 467 **Alfred G. Noël, Donald R. King, Gaston M. N'Guérékata, and Edray H. Goins, Editors**, Council for African American researchers in the mathematical sciences: Volume V, 2008
- 466 **Boo Cheong Khoo, Zhilin Li, and Ping Lin, Editors**, Moving interface problems and applications in fluid dynamics, 2008
- 465 **Valery Alexeev, Arnaud Beauville, C. Herbert Clemens, and Elham Izadi, Editors**, Curves and Abelian varieties, 2008
- 464 **Gestur Ólafsson, Eric L. Grinberg, David Larson, Palle E. T. Jorgensen, Peter R. Massopust, Eric Todd Quinto, and Boris Rubin, Editors**, Radon transforms, geometry, and wavelets, 2008
- 463 **Kristin E. Lauter and Kenneth A. Ribet, Editors**, Computational arithmetic geometry, 2008
- 462 **Giuseppe Dito, Hugo García-Compeán, Ernesto Lupercio, and Francisco J. Turrubiates, Editors**, Non-commutative geometry in mathematics and physics, 2008
- 461 **Gary L. Mullen, Daniel Panario, and Igor Shparlinski, Editors**, Finite fields and applications, 2008
- 460 **Megumi Harada, Yael Karshon, Mikiya Masuda, and Taras Panov, Editors**, Toric topology, 2008
- 459 **Marcelo J. Saia and José Seade, Editors**, Real and complex singularities, 2008
- 458 **Jinho Baik, Thomas Kriecherbauer, Luen-Chau Li, Kenneth D. T-R McLaughlin, and Carlos Tomei, Editors**, Integrable systems and random matrices, 2008
- 457 **Tewodros Amdeberhan and Victor H. Moll, Editors**, Tapas in experimental mathematics, 2008
- 456 **S. K. Jain and S. Parvathi, Editors**, Noncommutative rings, group rings, diagram algebras and their applications, 2008
- 455 **Mark Agranovsky, Daoud Bshouty, Lavi Karp, Simeon Reich, David Shoikhet, and Lawrence Zalcman, Editors**, Complex analysis and dynamical systems III, 2008
- 454 **Rita A. Hirschweiler and Thomas H. MacGregor, Editors**, Banach spaces of analytic functions, 2008
- 453 **Jacob E. Goodman, János Pach, and Richard Pollack, Editors**, Surveys on Discrete and Computational Geometry—Twenty Years Later, 2008

TITLES IN THIS SERIES

- 452 **Matthias Beck, Christian Haase, Bruce Reznick, Michèle Vergne, Volkmar Welker, and Ruriko Yoshida, Editors**, *Integer points in polyhedra*, 2008
- 451 **David R. Larson, Peter Massopust, Zuhair Nashed, Minh Chuong Nguyen, Manos Papadakis, and Ahmed Zayed, Editors**, *Frames and operator theory in analysis and signal processing*, 2008
- 450 **Giuseppe Dito, Jiang-Hua Lu, Yoshiaki Maeda, and Alan Weinstein, Editors**, *Poisson geometry in mathematics and physics*, 2008
- 449 **Robert S. Doran, Calvin C. Moore, and Robert J. Zimmer, Editors**, *Group representations, ergodic theory, and mathematical physics: A tribute to George W. Mackey*, 2007
- 448 **Alberto Corso, Juan Migliore, and Claudia Polini, Editors**, *Algebra, geometry and their interactions*, 2007
- 447 **François Germinet and Peter Hislop, Editors**, *Adventures in mathematical physics*, 2007
- 446 **Henri Berestycki, Michiel Bertsch, Felix E. Browder, Louis Nirenberg, Lambertus A. Peletier, and Laurent Véron, Editors**, *Perspectives in Nonlinear Partial Differential Equations*, 2007
- 445 **Laura De Carli and Mario Milman, Editors**, *Interpolation Theory and Applications*, 2007
- 444 **Joseph Rosenblatt, Alexander Stokolos, and Ahmed I. Zayed, Editors**, *Topics in harmonic analysis and ergodic theory*, 2007
- 443 **Joseph Stephen Verducci and Xiaotong Shen, Editors**, *Prediction and discovery*, 2007
- 442 **Yi-Zhi Huang and Kailash C Misra, Editors**, *Lie algebras, vertex operator algebras and their applications*, 2007
- 441 **Louis H. Kauffman, David E. Radford, and Fernando J. O. Souza, Editors**, *Hopf algebras and generalizations*, 2007
- 440 **Fernanda Botelho, Thomas Hagen, and James Jamison, Editors**, *Fluids and Waves*, 2007
- 439 **Donatella Danielli, Editor**, *Recent developments in nonlinear partial differential equations*, 2007
- 438 **Marc Burger, Michael Farber, Robert Ghrist, and Daniel Koditschek, Editors**, *Topology and robotics*, 2007
- 437 **José C. Mourão, João P. Nunes, Roger Picken, and Jean-Claude Zambrini, Editors**, *Prospects in mathematical physics*, 2007
- 436 **Luchezar L. Avramov, Daniel Christensen, William G Dwyer, Michael A Mandell, and Brooke E Shipley, Editors**, *Interactions between homotopy theory and algebra*, 2007
- 435 **Krzysztof Jarosz, Editor**, *Function spaces*, 2007
- 434 **S. Paycha and B. Uribe, Editors**, *Geometric and topological methods for quantum field theory*, 2007
- 433 **Pavel Etingof, Shlomo Gelaki, and Steven Shnider, Editors**, *Quantum groups*, 2007
- 432 **Dick Canary, Jane Gilman, Juha Heinonen, and Howard Masur, Editors**, *In the tradition of Ahlfors-Bers, IV*, 2007
- 431 **Michael Batanin, Alexei Davydov, Michael Johnson, Stephen Lack, and Amnon Neeman, Editors**, *Categories in algebra, geometry and mathematical physics*, 2007

For a complete list of titles in this series, visit the  
AMS Bookstore at [www.ams.org/bookstore/](http://www.ams.org/bookstore/).

This book consists of three expository articles written by outstanding researchers in Mathematical Physics: Rafael Benguria, Peter Hislop, Elliott Lieb, and Helmut Linde. The articles are based on their lectures at the Fourth Summer School in Analysis and Mathematical Physics, held at the Institute of Mathematics, Universidad Nacional Autónoma de México, Cuernavaca in May 2005.

The main goal of the articles is to link the basic knowledge of a graduate student in Mathematics with three current research topics in Mathematical Physics: Isoperimetric inequalities for eigenvalues of the Laplace Operator, Random Schrödinger Operators, and Stability of Matter, respectively.

These well written articles will guide and introduce the reader to current research topics and will also provide information on recent progress in some areas of Mathematical Physics.

ISBN 978-0-8218-4064-1



**CONM/476**

American Mathematical Society  
[www.ams.org](http://www.ams.org)

Sociedad Matemática Mexicana  
[www.smm.org.mx](http://www.smm.org.mx)